NUMERICAL SCIENTIFIC COMPUTING

MINI PROJECT

Jimmy J. Nielsen and Ondrej Franek

February 7, 2022

1 | The mini-project

This mini-project relates to the Mandelbrot set (after the mathematician Benoit Mandelbrot), which leads to compelling two-dimensional fractal patterns. A fractal contains elements of self-similarity when plotted. The Mandelbrot set is a quadratic complex mapping of the type:

$$z_{i+1} \;=\; z_i^2 + c, \quad i = 0, 1, \ldots, I-1 \tag{1.1}$$

where $c \in \mathbb{C}$ is a point in the complex plane, and $z_i \in \mathbb{C}$ for $i = 0, 1, \ldots, I$. This provides us with z_0, z_1, ..., z_I where the initial condition is $z_0 = 0 + j \cdot 0$ and $z_1, \ldots, z_I$ are referred to as iteratively achieved outputs. In Equation (1.1) the iteration number is $i+1 \in \{1, 2, \ldots, I\}$ and the total number of iterations is I. For each observed complex point c we then compute I iterations of Equation (1.1). For each complex point c we then determine:

$$\imath(c) \;=\; \min \mathbb{T}, \quad \mathbb{T} = \big\{ i \;\big|\; |z_i| > T,\, i = 1, 2, \ldots, I \big\} \cup \{I\} \tag{1.2}$$

where T is a threshold value and the initial condition is always chosen such that $|z_0| \leq T$. From Equation (1.2) we see that $1 \leq \imath(c) \leq I$. So, if $|z_{i+1}| \leq T$, $\forall i = 0, \ldots, I-1$ we obtain $\mathbb{T} = \emptyset \cup \{I\} = \{I\}$ meaning that $\imath(c) = I$. Obviously, in a computational implementation we need not compute all I iterations if an $i+1 < I$ leads to $|z_{i+1}| > T$. We then just set $\imath(c)$ to the smallest $i+1$ that leads to $|z_{i+1}| > T$. For plotting purposes we then form a simple linear mapping as:

$$\mathcal{M}(c) \;=\; \frac{\imath(c)}{I}, \quad 0 < \mathcal{M}(c) \leq 1 \tag{1.3}$$

The smaller $\mathcal{M}(c)$ value we have, the faster that specific complex point c makes $|z_{i+1}|$ in Equation (1.1) increase. In a case with the number of iterations $I = 100$ a value of $0 < \mathcal{M}(c) \lesssim 0.1$ indicates an extremely 'active' (unstable) point whereas a value of $\mathcal{M}(c) = 1$ indicates a stable point. A point c is said to belong to the Mandelbrot set if $|z_{n+1}|$ remains bounded (finite) for $n \to \infty$.

In this mini-project, the computational task is then to determine $\mathcal{M}(c)$ for a c-mesh, which we limit c-wise to $-2 \leq \Re\{c\} \leq 1$ and $-1.5 \leq \Im\{c\} \leq 1.5$. We then select a certain number of points for each of $\Re\{c\}$ and $\Im\{c\}$ as p_{re} and p_{im}, respectively. The complex plane limited by the mesh is described by a complex matrix $\mathbf{C}$ as:

$$\mathbf{C} \;=\; \begin{bmatrix} -2.0 & \cdots & 1.0 \\ \vdots & & \vdots \\ -2.0 & \cdots & 1.0 \end{bmatrix} + j \cdot \begin{bmatrix} 1.5 & \cdots & 1.5 \\ \vdots & & \vdots \\ -1.5 & \cdots & -1.5 \end{bmatrix} \in \mathbb{C}^{p_{\text{im}} \times p_{\text{re}}} \tag{1.4}$$

You should select p_{re} and p_{im} according to the computational resources you have available and the desired resolution. A likely number could be something like $p_{\text{re}} = 5000$ and $p_{\text{im}} = 5000$. We use a threshold of $T = 2$. An example is shown in Figure 1.1.

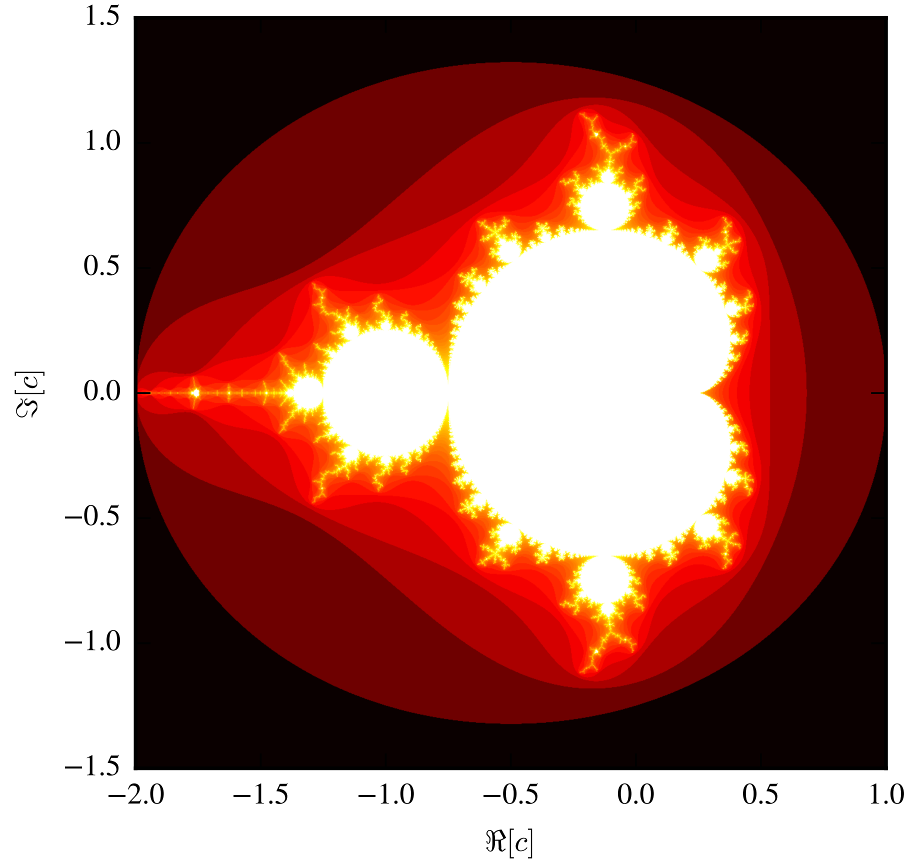

Figure 1.1:: Mandelbrot set plotting $\mathcal{M}(c)$ with $p_{\text{re}} = 5000$ and $p_{\text{im}} = 5000$ using the `matplotlib.pyplot.cm.hot` colour mapping in Pythons Matplotlib.