CORE CONNECTIONS ALGEBRA
Parent Guide with Extra Practice

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DESCRIBING FUNCTIONS 1.1.3 through 1.2.2

In addition to introducing students to the classroom norms of problem-based learning, the main objective of these lessons is for students to be able to fully describe the key elements of the graph of a function. To fully describe the graph of a function, students should respond to these graph investigation questions:

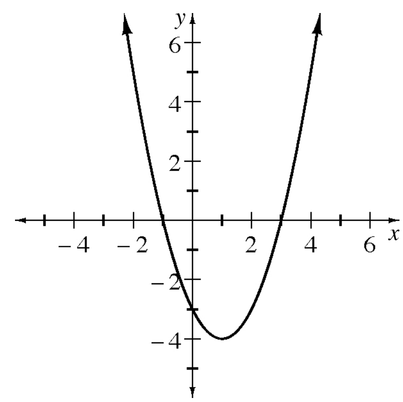
Graph Investigation Question	Sample Summary Statement
What does the graph look like?	*The graph looks like half of a parabola on its side.*
Is the graph increasing or decreasing (reading left to right)?	*As x gets bigger, y gets bigger.*
What are the *x*- and *y*-intercepts?	*The graph intersects both the x- and y-axes at (0, 0).*
Are there any limitations on the inputs (domain) of the equation?	*Only positive values of x and zero are possible.*
Are there any limitations on the outputs (range) of the equation? (Is there a maximum or minimum *y*-value?)	*The smallest y-value is 0. There is no maximum y-value.*
Are there any special points?	*The graph has a "starting" point at (0,0).*
Does the graph have any symmetry? If so, where?	*This graph has no symmetry.*

The more formal concepts of function and domain and range are addressed in Lessons 1.2.4 and 1.2.5.

For more information, see the Math Notes boxes in Lessons 1.1.1, 1.1.2, and 1.1.3. Student responses to the Learning Log in Lesson 1.1.1 (problem 1-32), if it was assigned, can also be helpful.

Example 1

For the function $y = x^2 - 2x - 3$, make an $x \rightarrow y$ table, draw a graph, and fully describe the features.



At this point there is no way to know how many points are sufficient for the $x \rightarrow y$ table. Add more points as necessary until you are convinced of shape and location.

x	–2	–1	0	1	2	3
y	5	0	–3	–4	–3	0

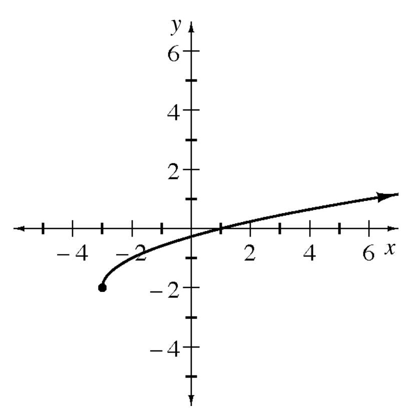
Be careful with substitution and Order of Operations when calculating values. For example if $x = –2$, $y = (-2)^2 - 2(-2) - 3 = 4 + 4 - 3 = 5$.

The graph is a parabola; it opens upward. The x-intercepts are $(–1, 0)$ and $(3, 0)$. The y-intercept is $(0, –3)$. Reading from left to right, the graph decreases until $x = 1$ and then increases. The minimum (lowest) point on the graph (called the vertex) is $(1, –4)$. The vertical line $x = 1$ is a line of symmetry. There are no limitations on inputs to the function. Outputs can be any value greater than or equal to –4.

Example 2

For the function $y = \sqrt{x+3} - 2$, make an $x \rightarrow y$ table, draw a graph, and describe the features.

Note that the smallest possible number for the $x \rightarrow y$ table is $x = -3$. Anything smaller will require the square root of a negative, which is not a real number.



x	–4	–3	0	1	3	6
y		–2	–0.3	0	0.4	1

The graph is half a parabola. It starts at $(–3, –2)$ and has x-intercept $(1, 0)$ and y-intercept $(0, \approx –0.3)$. The graph increases from left to right. The inputs are limited to values of –3 or greater, and the outputs are limited to –2 or greater. There is no line of symmetry.

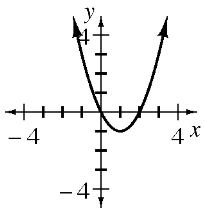
Problems

For each function, make an $x \rightarrow y$ table, draw a graph, and describe the features.

1.	$y = x^2 - 2x$	2.	$y = x^2 + 2x - 3$	3.	$y = \sqrt{x-2}$
4.	$y = 4 - x^2$	5.	$y = x^2 + 2x + 1$	6.	$y = -\sqrt{x} + 3$
7.	$y = -x^2 + 2x - 1$	8.	$y = \lvert x+2 \rvert$	9.	$y = 2\sqrt[3]{x} - 1$

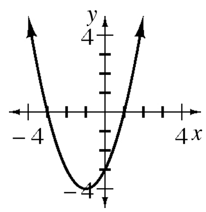
Answers

1.



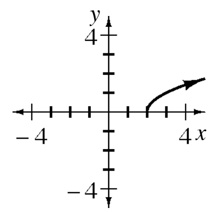
Parabola; intercepts $(0,0)$, $(2,0)$; decreasing until $x = 1$ then increasing; minimum value at $(1,-1)$; $x = 1$ is a line of symmetry. Inputs can be any real number. Outputs are greater than or equal to -1.

2.



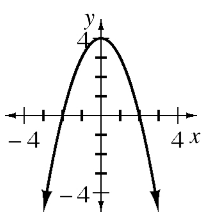
Parabola; intercepts $(-3,0)$, $(1,0)$ and $(0,-3)$; decreasing until $x = -1$, then increasing; minimum value at $(-1,-4)$; $x = -1$ is a line of symmetry. Inputs can be any real number. Outputs are greater than or equal to -4.

3.



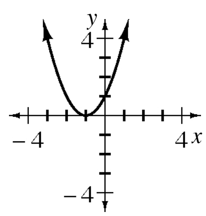
Half-parabola; starting point, intercept and minimum point $(2,0)$; increasing for $x > 2$. Inputs can be any number greater than or equal to 2. Outputs are greater than or equal to 0.

4.



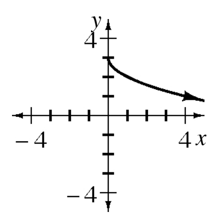
Parabola; intercepts $(-2,0)$, $(2,0)$ and $(0,4)$; increasing for $x < 0$, decreasing for $x > 0$; maximum value at $(0,4)$; $x = 0$ is a line of symmetry. Inputs can be any real number. Outputs are less than or equal to 4.

5.



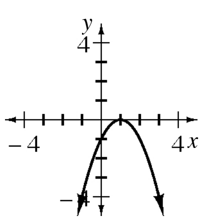
Parabola; intercept $(-1,0)$; decreasing for $x < -1$, increasing for $x > -1$; minimum value at $(-1,0)$; $x = -1$ is a line of symmetry. Inputs can be any real number. Outputs are greater than or equal to 0.

6.



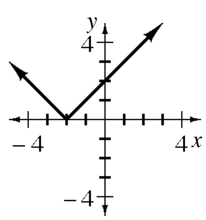
Half-parabola; starting point, intercept and maximum point $(0,3)$; decreasing for $x > 0$. Inputs can be any number greater than or equal to 0. Outputs are less than or equal to 3.

7.



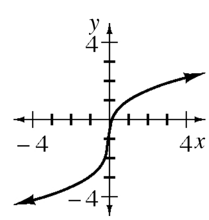
Parabola; intercepts $(1,0)$ and $(0,-1)$; increasing for $x < 1$, decreasing for $x > 1$; maximum value at $(1,0)$; $x = 1$ is a line of symmetry. Inputs can be any real number. Outputs are less than or equal to 0.

8.



V-shape; intercepts $(-2,0)$ and $(0,2)$; decreasing for $x < -2$, increasing for $x > -2$; minimum value at $(-2,0)$; $x = -2$ is a line of symmetry. Inputs can be any real number. Outputs are greater than or equal to 0.

9.



S-shape; intercepts $(0,-1)$ and $(0.125,0)$; increasing for all x from left to right. Inputs and outputs can be any real number. There is no line of symmetry.

FUNCTIONS 1.2.4 through 1.2.5

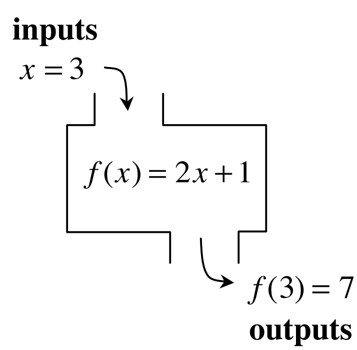
A relationship between the input values (usually x) and the output values (usually y) is called a function if for each input value, there is no more than one output value. Functions can be represented with an illustration of an input–output "machine," as shown in Lesson 1.2.3 of the textbook and in the diagram in Example 1 below.

The set of all possible inputs of a relation is called the **domain**, while the set of all possible outputs of a relation is called the **range**.

For additional information about functions, function notation, and domain and range, see the Math Notes box in Lesson 1.2.5.

Example 1

Numbers, represented by a letter or symbol such as x, are input into the function machine labeled f one at a time, and then the function performs the operation on each input to determine each output, $f(x)$. For example, when $x = 3$ is put into the function f at right, the machine multiplies 3 by 2 and adds 1 to get the output, $f(x)$ which is 7. The notation $f(3) = 7$ shows that the function named f connects the input $x = 3$ with the output 7. This also means the point (3, 7) lies on the graph of the function.



Example 2

a. If $f(x) = \sqrt{x-2}$ then $f(11) = ?$ $\qquad f(11) = \sqrt{11-2} = \sqrt{9} = 3$

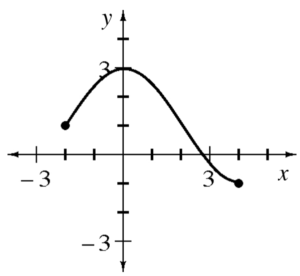
b. If $g(x) = 3 - x^2$ then $g(5) = ?$ $\qquad g(5) = 3 - (5)^2 = 3 - 25 = -22$

c. If $f(x) = \dfrac{x+3}{2x-5}$ then $f(2) = ?$ $\qquad f(2) = \dfrac{2+3}{2 \cdot 2 - 5} = \dfrac{5}{-1} = -5$

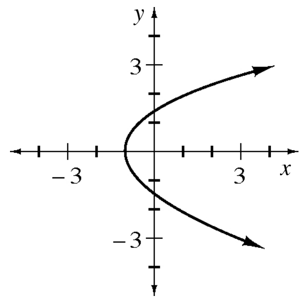
Example 3

A relation in which each input has only one output is called a **function**.

$g(x)$



$f(x)$



$g(x)$ is a function: each input (x) has only one output (y).
$g(-2)=1$, $g(0)=3$, $g(4)=-1$, and so on.

$f(x)$ is not a function: each input greater than –1 has two y-values associated with it.
$f(1)=2$ and $f(1)=-2$.

Example 4

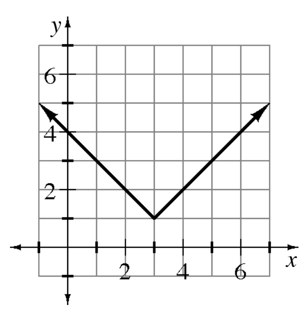
The set of all possible inputs of a relation is called the **domain**, while the set of all possible outputs of a relation is called the **range**.

In Example 3 above, the domain of $g(x)$ is $-2 \le x \le 4$, or "all numbers between –2 and 4." The range is $-1 \le y \le 3$ or "all numbers between –1 and 3."

The domain of $f(x)$ in Example 3 above is $x \ge -1$ or "any real number greater than or equal to –1," since the graph starts at –1 and continues forever to the right. Since the graph of $f(x)$ extends in both the positive and negative y directions forever, the range is "all real numbers."

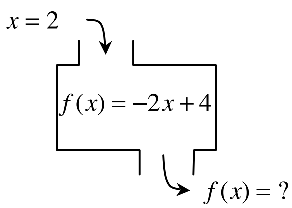
Example 5

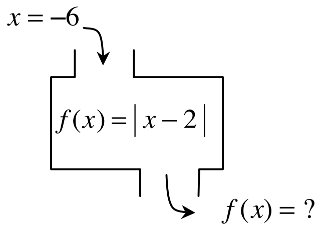
For the graph at right, since the x-values extend forever in both directions the domain is "all real numbers." The y-values start at 1 and go higher so the range is $y \ge 1$ or "all numbers greater or equal to 1."



Problems

Determine the outputs for the given inputs of the following functions.

1.


2.


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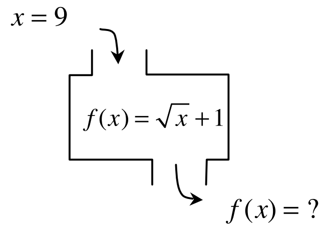


4. $f(x) = (5 - x)^2$
$f(8) = ?$

5. $g(x) = x^2 - 5$
$g(-3) = ?$

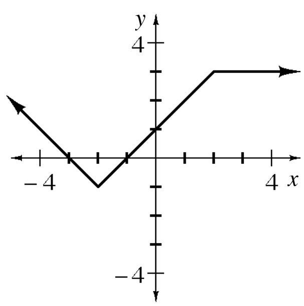
6. $f(x) = \frac{2x+7}{x^2-9}$
$f(3) = ?$

7. $h(x) = 5 - \sqrt{x}$
$h(9) = ?$

8. $h(x) = \sqrt{5 - x}$
$h(9) = ?$

9. $f(x) = -x^2$
$f(4) = ?$

Determine if each relation is a function. Then state its domain and range.

10.


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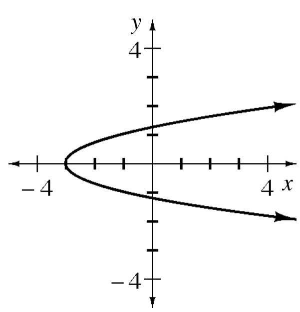

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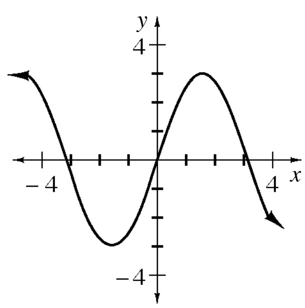

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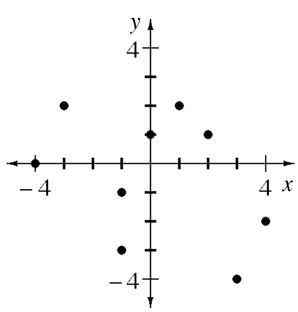
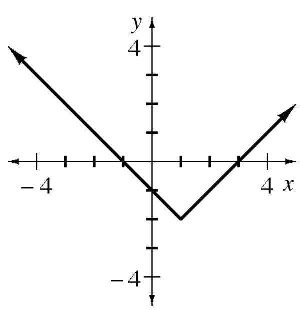
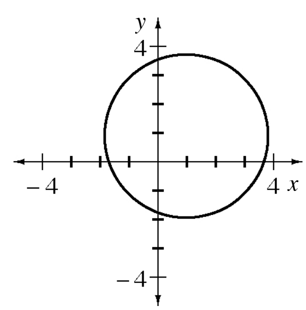


14.


15.


Answers

1.	$f(2) = 0$	2.	$f(-6) = 8$	3.	$f(9) = 4$
4.	$f(8) = 9$	5.	$g(-3) = 4$	6.	not possible
7.	$f(9) = 2$	8.	not possible	9.	$f(4) = -16$
10.	Yes, each input has one output; domain is all numbers, range is $-1 \le y \le 3$	11.	No, for example $x = 0$ has two outputs; domain is $x \ge -3$, range is all numbers	12.	Yes; domain all numbers, range is $-3 \le y \le 3$
13.	No; $x = -1$ has two outputs; domain is $-4, -3, -1, 0, 1, 2, 3, 4$, range is $-4, -3, -2, -1, 0, 1, 2$	14.	Yes; domain is all numbers, range is $y \ge -2$	15.	No, many inputs have two outputs; domain is $-2 \le x \le 4$ range is $-2 \le y \le 4$

SLOPE—A MEASURE OF STEEPNESS 2.1.2 through 2.1.4

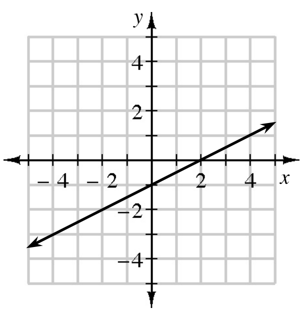
Students used the equation $y = mx + b$ to graph lines and describe patterns in previous courses. Lesson 2.1.1 is a review. When the equation of a line is written in $y = mx + b$ form, the coefficient m represents the slope of the line. Slope indicates the direction of the line and its steepness. The constant b is the y-intercept, written $(0, b)$, and indicates where the line crosses the y-axis.

For additional information about slope, see the Math Notes box in Lesson 2.1.4.

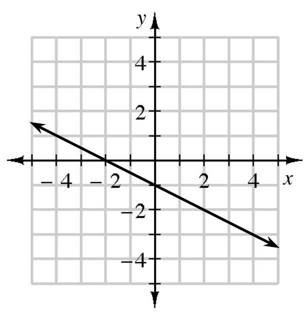
Example 1

If m is positive, the line goes upward from left to right. If m is negative, the line goes downward from left to right. If $m = 0$ then the line is horizontal. The value of b indicates the y-intercept.

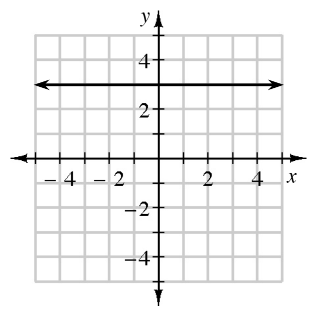
$y = \frac{1}{2}x - 1$



$y = -\frac{1}{2}x - 1$



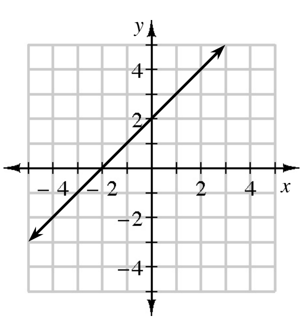
$y = 0x + 3$ or $y = 3$



Example 2

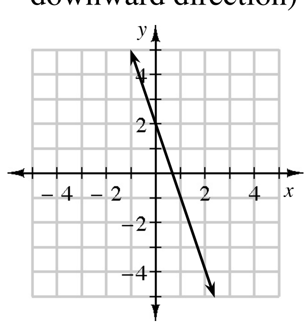
When $m = 1$, as in $y = x$, the line goes upward by one unit each time it goes over one unit to the right. Steeper lines have a larger m-value, that is, $m > 1$ or $m < -1$. Flatter lines have an m-value that is between –1 and 1, often in the form of a fraction. All three examples below have $b = 2$.

$y = x + 2$



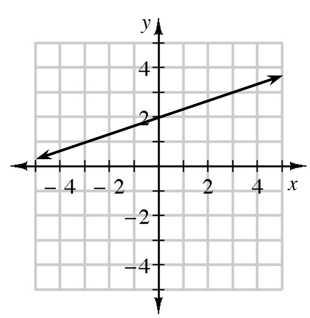
$y = -3x + 2$

(steeper and in the downward direction)



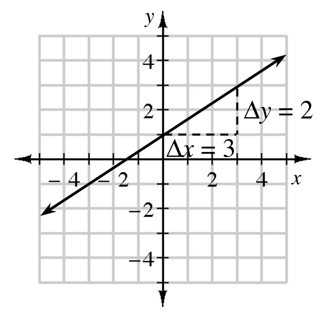
$y = \frac{1}{3}x + 2$

(less steep)

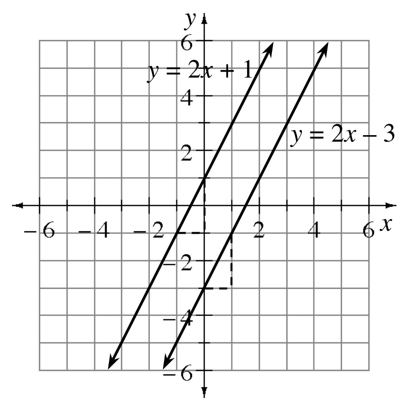


Example 3

If a line is drawn on a set of axes, a *slope triangle* can be drawn between any two convenient points (usually where grid lines cross), as shown in the graph at right. Count the vertical distance (labeled Δy) and the horizontal distance (labeled Δx) on the dashed sides of the slope triangle. Write the distances in a ratio: slope $= m = \frac{\Delta y}{\Delta x} = \frac{2}{3}$. The symbol Δ means change. The order in the fraction is important: the numerator (top of the fraction) must be the vertical distance and the denominator (bottom of the fraction) must be the horizontal distance. The slope of a line is constant, so the slope ratio is the same for any two points on the line.



Parallel lines have the same steepness and direction, so they have the same slope, as shown in the graph at right.

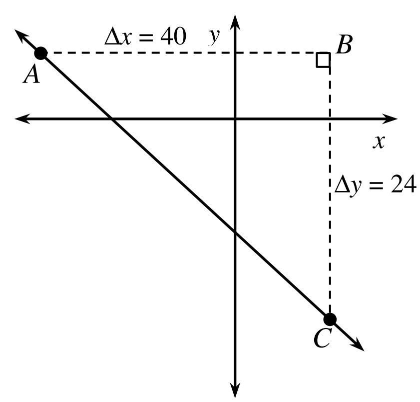


If $\Delta y = 0$, then the line is horizontal and has a slope of zero, that is, $m = 0$. If $\Delta x = 0$, then the line is vertical and its slope is undefined, so we say that it has no slope.

Example 4

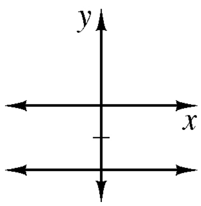
When the vertical and horizontal distances are not easy to determine, you can find the slope by drawing a generic slope triangle and using it to find the lengths of the vertical Δy and horizontal (Δx) segments. The figure at right shows how to find the slope of the line that passes through the points (–21, 9) and (19, –15). First graph the points on unscaled axes by approximating where they are located, and then draw a slope triangle. Next find the distance along the vertical side by noting that it is 9 units from point B to the x-axis then 15 units from the x-axis to point C, so Δy is 24. Then find the distance from point A to the y-axis (21) and the distance from the y-axis to point B (19). Δx is 40. This slope is negative because the line goes downward from left to right, so the slope is

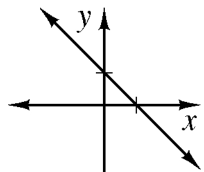
$m = \frac{\Delta y}{\Delta x} = -\frac{24}{40} = -\frac{3}{5}$.

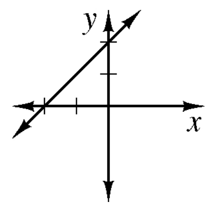


Problems

Is the slope of each line negative, positive, or zero?

1. 

2. 

3. 

Identify the slope in each equation. State whether the graph of the line is steeper or flatter than $y = x$ or $y = -x$, whether it goes up or down from left to right, or if it is horizontal or vertical.

4.	$y = 3x + 2$	5.	$y = -\frac{1}{2}x + 4$	6.	$y = \frac{1}{3}x - 4$
7.	$4x - 3 = y$	8.	$y = -2 + \frac{1}{2}x$	9.	$3 + 2y = 8x$
10.	$y = 2$	11.	$x = 5$	12.	$6x + 3y = 8$

Without graphing, find the slope of each line based on the given information.

13.	$\Delta y = 27 \;\; \Delta x = -8$	14.	$\Delta x = 15 \;\; \Delta y = 3$	15.	$\Delta y = 7 \;\; \Delta x = 0$
16.	Horizontal $\Delta = 6$ Vertical $\Delta = 0$	17.	Between (5, 28) and (64, 12)	18.	Between (–3, 2) and (5, –7)

Answers

1.	zero	2.	negative	3.	positive
4.	Slope = 3, steeper, up	5.	Slope = $-\frac{1}{2}$, flatter, down	6.	Slope = $\frac{1}{3}$, flatter, up
7.	Slope = 4, steeper, up	8.	Slope = $\frac{1}{2}$, flatter, up	9.	Slope = 4, steeper, up
10.	Slope = 0, horizontal	11.	Slope is undefined, vertical	12.	Slope = –2, steeper, down
13.	$-\frac{27}{8}$	14.	$\frac{3}{15} = \frac{1}{5}$	15.	undefined
16.	0	17.	$-\frac{16}{59}$	18.	$-\frac{9}{8}$

WRITING AN EQUATION GIVEN THE SLOPE AND A POINT ON THE LINE 2.3.1

In earlier work students used substitution in equations like $y = 2x + 3$ to find x and y pairs that make the equation true. Students recorded those pairs in a table, and then used them as coordinates to graph a line. Every point (x, y) on the line makes the equation true.

Later, students used the patterns they saw in the tables and graphs to recognize and write equations in the form of $y = mx + b$. The "b" represents the y-intercept of the line, the "m" represents the slope, while x and y represent the coordinates of any point on the line. Each line has a unique value for m and a unique value for b, but there are infinite (x, y) values for each linear equation.

The slope of the line is the same between any two points on that line. We can use this information to write equations without creating tables or graphs.

For additional information, see the Math Notes boxes in Lessons 2.2.2 and 2.2.3.

Example 1

What is the equation of the line with a slope of 2 that passes through the point (10, 17)?

Write the general equation of a line.

$$y = mx + b$$

Substitute the values we know: m, x, and y.

$$17 = 2(10) + b$$

Solve for b.

$$17 = 20 + b$$
$$-3 = b$$

Write the complete equation using the values $m = 2$ and $b = -3$.

$$y = 2x - 3$$

Example 2

This algebraic method can help us write equations of parallel lines. Parallel lines never intersect or meet. They have the *same* slope, m, but *different* y-intercepts, b.

What is the equation of the line parallel to $y = 3x - 4$ that goes through the point (2, 8)?

Write the general equation of a line.	$y = mx + b$
Substitute the values we know: m, x, and y. Since the lines are parallel, the slopes are equal.	$8 = 3(2) + b$
Solve for b.	$8 = 6 + b$ $2 = b$
Write the complete equation.	$y = 3x + 2$

Problems

Write the equation of the line with the given slope that passes through the given point.

1. slope = 5, (3, 13)
2. slope = $-\frac{5}{3}$, (3, –1)
3. slope = –4, (–2, 9)
4. slope = $\frac{3}{2}$, (6, 8)
5. slope = 3, (–7, –23)
6. slope = 2, ($\frac{5}{2}$, –2)

Write the equation of the line *parallel* to the given line that goes through the given point.

7. $y = \frac{3}{5}x + 2$ (0, 0)
8. $y = 4x - 1$ (–2, –6)
9. $y = -2x + 5$ (–4, –2)
10. $y = 4x + 5$ (–6, –28)
11. $y = \frac{1}{3}x - 1$ (6, 9)
12. $y = 3x + 8$ (0, $\frac{1}{2}$)

Answers

1. $y = 5x - 2$
2. $y = -\frac{5}{3}x + 4$
3. $y = -4x + 1$
4. $y = \frac{3}{2}x - 1$
5. $y = 3x - 2$
6. $y = 2x - 7$
7. $y = \frac{3}{5}x$
8. $y = 4x + 2$
9. $y = -2x - 10$
10. $y = 4x - 4$
11. $y = \frac{1}{3}x + 7$
12. $y = 3x + \frac{1}{2}$

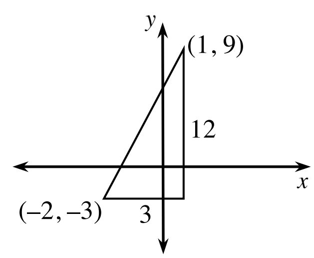
WRITING THE EQUATION OF A LINE GIVEN TWO POINTS 2.3.2

Students now have all the tools they need to find the equation of a line passing through two given points. Recall that the equation of a line requires a slope and a y-intercept in $y = mx + b$. Students can write the equation of a line from two points by creating a slope triangle and calculating $\frac{\Delta y}{\Delta x}$ as explained in Lessons 2.1.2 through 2.1.4.

For additional information, see the Math Notes box in Lesson 3.3.2. For additional examples and more practice, see the Checkpoint 5B materials.

Example 1

Write the equation of the line that passes through the points (1, 9) and (–2, –3).



Position the two points approximately where they belong on coordinate axes—you do not need to be precise. Draw a generic slope triangle.

Calculate slope = $\frac{\Delta y}{\Delta x} = \frac{12}{3} = 4$ using the given values of the two points.

Write the general equation of a line. Substitute m and either one of the points into the equation. For example, use $(x, y) = (1, 9)$ and $m = 4$.

$$y = mx + b$$
$$9 = 4(1) + b$$

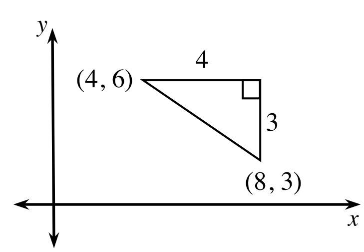
$$\frac{\Delta y}{\Delta x} = \frac{12}{3}$$
$$m = 4$$

Solve for b.
$5 = b$

Write the complete equation.
$y = 4x + 5$

Example 2

Write the equation of the line that passes through the points (8, 3) and (4, 6).



Draw a generic slope triangle located approximately on coordinate axes. Approximate the locations of the given points.

Calculate $m = \frac{\Delta y}{\Delta x} = -\frac{3}{4}$. The slope is negative since the line goes down left to right.

Substitute m and either one of the points, for example (8, 3), into the general equation for a line.

$$y = mx + b$$
$$3 = -\frac{3}{4}(8) + b$$

$$\frac{\Delta y}{\Delta x} = \frac{-3}{4}$$
$$m = -\frac{3}{4}$$

Solve for b.
$9 = b$

Write the complete equation.
$y = -\frac{3}{4}x + 9$

Problems

Write the equation of the line containing each pair of points.

1.	(1, 1) and (0, 4)	2.	(5, 4) and (1, 1)	3.	(1, 3) and (–5, –15)
4.	(–2, 3) and (3, 5)	5.	(2, –1) and (3, –3)	6.	(4, 5) and (–2, –4)
7.	(1, –4) and (–2, 5)	8.	(–3, –2) and (5, –2)	9.	(–4, 1) and (5, –2)

Answers

1.	$y = -3x + 4$	2.	$y = \frac{3}{4}x + \frac{1}{4}$	3.	$y = 3x$
4.	$y = \frac{2}{5}x + 3\frac{4}{5}$	5.	$y = -2x + 3$	6.	$y = \frac{3}{2}x - 1$
7.	$y = -3x - 1$	8.	$y = -2$	9.	$y = -\frac{1}{3}x - \frac{1}{3}$

LAWS OF EXPONENTS 3.1.1 and 3.1.2

In general, to simplify an expression that contains exponents means to eliminate parentheses and negative exponents if possible. The basic **laws of exponents** are listed here.

(1) $x^a \cdot x^b = x^{a+b}$ Examples: $x^3 \cdot x^4 = x^7$; $2^7 \cdot 2^4 = 2^{11}$

(2) $\frac{x^a}{x^b} = x^{a-b}$ Examples: $\frac{x^{10}}{x^4} = x^6$; $\frac{2^4}{2^7} = 2^{-3}$

(3) $(x^a)^b = x^{ab}$ Examples: $(x^4)^3 = x^{12}$; $(2x^3)^5 = 2^5 \cdot x^{15} = 32x^{15}$

(4) $x^0 = 1$ Examples: $2^0 = 1$; $(-3)^0 = 1$; $\left(\frac{1}{4}\right)^0 = 1$

(5) $x^{-n} = \frac{1}{x^n}$ Examples: $x^{-3} = \frac{1}{x^3}$; $y^{-4} = \frac{1}{y^4}$; $4^{-2} = \frac{1}{4^2} = \frac{1}{16}$

(6) $\frac{1}{x^{-n}} = x^n$ Examples: $\frac{1}{x^{-5}} = x^5$; $\frac{1}{x^{-2}} = x^2$; $\frac{1}{3^{-2}} = 3^2 = 9$

(7) $x^{m/n} = \sqrt[n]{x^m}$ Examples: $x^{2/3} = \sqrt[3]{x^2}$; $y^{1/2} = \sqrt{y}$

In all expressions with fractions we assume the denominator does not equal zero.

For additional information, see the Math Notes box in Lesson 3.1.2. For additional examples and practice, see the Checkpoint 5A problems in the back of the textbook.

Example 1

Simplify: $(2xy^3)(5x^2y^4)$

Reorder: $2 \cdot 5 \cdot x \cdot x^2 \cdot y^3 \cdot y^4$

Using law (1): $10x^3y^7$

Example 2

Simplify: $\frac{14x^2y^{12}}{7x^5y^7}$

Separate: $\left(\frac{14}{7}\right) \cdot \left(\frac{x^2}{x^5}\right) \cdot \left(\frac{y^{12}}{y^7}\right)$

Using laws (2) and (5): $2x^{-3}y^5 = \frac{2y^5}{x^3}$

Example 3

Simplify: $(3x^2y^4)^3$

Using law (3): $3^3 \cdot (x^2)^3 \cdot (y^4)^3$

Using law (3) again: $27x^6y^{12}$

Example 4

Simplify: $(2x^3)^{-2}$

Using law (5): $\dfrac{1}{(2x^3)^2}$

Using law (3): $\dfrac{1}{2^2 \cdot (x^3)^2}$

Using law (3) again: $\dfrac{1}{4x^6}$

Example 5

Simplify: $\dfrac{10x^7y^3}{15x^{-2}y^3}$

Separate: $\left(\dfrac{10}{15}\right) \cdot \left(\dfrac{x^7}{x^{-2}}\right) \cdot \left(\dfrac{y^3}{y^3}\right)$

Using law (2): $\dfrac{2}{3}x^9y^0$

Using law (4): $\dfrac{2}{3}x^9 \cdot 1 = \dfrac{2}{3}x^9 = \dfrac{2x^9}{3}$

Problems

Simplify each expression. Final answers should contain no parentheses or negative exponents.

1. $y^5 \cdot y^7$
2. $b^4 \cdot b^3 \cdot b^2$
3. $8^6 \cdot 8^{-2}$
4. $(y^5)^2$
5. $(3a)^4$
6. $\frac{m^8}{m^3}$
7. $\frac{12m^8}{6m^{-3}}$
8. $(x^3y^2)^3$
9. $\frac{(y^4)^2}{(y^3)^2}$
10. $\frac{15x^2y^5}{3x^4y^5}$
11. $(4c^4)(ac^3)(3a^5c)$
12. $(7x^3y^5)^2$
13. $(4xy^2)(2y)^3$
14. $\left(\frac{4}{x^2}\right)^3$
15. $\frac{(2a^7)(3a^2)}{6a^3}$
16. $\left(\frac{5m^3n}{m^5}\right)^3$
17. $(3a^2x^3)^2(2ax^4)^3$
18. $\left(\frac{x^3y}{y^4}\right)^4$
19. $\left(\frac{6x^8y^2}{12x^3y^7}\right)^2$
20. $\frac{(2x^5y^3)^3(4xy^4)^2}{8x^7y^{12}}$
21. x^{-3}
22. $2x^{-3}$
23. $(2x)^{-3}$
24. $(2x^3)^0$
25. $5^{1/2}$
26. $\left(\frac{2x}{3}\right)^{-2}$

Answers

1.	y^{12}	2.	b^9	3.	8^4
4.	y^{10}	5.	$81a^4$	6.	m^5
7.	$2m^{11}$	8.	x^9y^6	9.	y^2
10.	$\frac{5}{x^2}$	11.	$12a^6c^8$	12.	$49x^6y^{10}$
13.	$32xy^5$	14.	$\frac{64}{x^6}$	15.	a^6
16.	$\frac{125n^3}{m^6}$	17.	$72a^7x^{18}$	18.	$\frac{x^{12}}{y^{12}}$
19.	$\frac{x^{10}}{4y^{10}}$	20.	$16x^{10}y^5$	21.	$\frac{1}{x^3}$
22.	$\frac{2}{x^3}$	23.	$\frac{1}{8x^3}$	24.	1
25.	$\sqrt{5}$	26.	$\frac{9}{4x^2}$		

EQUATIONS ↔ ALGEBRA TILES 3.2.1

An Equation Mat can be used together with algebra tiles to represent the process of solving an equation. For assistance with Lesson 3.2.1, see Lessons A.1.1 through A.1.9 in this *Parent Guide with Extra Practice*.

See the Math Notes box in Lesson A.1.8 (in the Appendix chapter of the textbook) and in Lesson 3.2.1 for a list of all the "legal" moves and their corresponding algebraic equivalents. Also see the Math Notes box in Lesson A.1.9 (in the Appendix chapter of the textbook) for checking a solution.

For additional examples and practice, see the Checkpoint 1 materials at the back of the textbook.

MULTIPLYING BINOMIALS 3.2.2 through 3.2.4

Two ways to find the area of a rectangle are: as a product of the (height) · (base) or as the sum of the areas of individual pieces of the rectangle. For a given rectangle these two areas must be the same, so **area as a product = area as a sum**. Algebra tiles, and later, generic rectangles, provide area models to help multiply expressions in a visual, concrete manner.

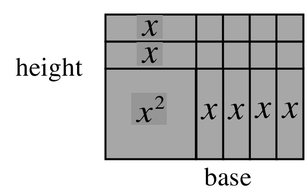
For additional information, see the Math Notes boxes in Lessons 3.2.2, 3.2.3, and 3.3.3. For additional examples and practice, see the Checkpoint 6B materials at the back of the textbook.

Example 1: Using Algebra Tiles

The algebra tile pieces $x^2 + 6x + 8$ are arranged into a rectangle as shown at right. The area of the rectangle can be written as the **product** of its base and height or as the **sum** of its parts.

$$\underbrace{(x+4)}_{\text{base}}\underbrace{(x+2)}_{\text{height}} = \underbrace{x^2+6x+8}_{\text{area}}$$

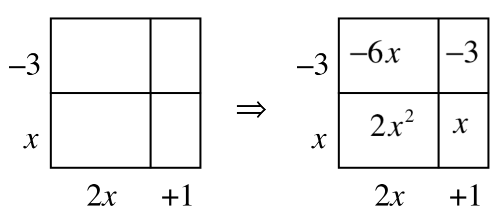
area as a **product** area as a **sum**



Example 2: Using Generic Rectangles

A generic rectangle allows us to organize the problem in the same way as the first example without needing to draw the individual tiles. It does not have to be drawn accurately or to scale.

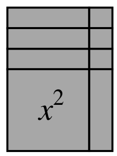
Multiply $\underbrace{(2x+1)}_{\text{base}}\underbrace{(x-3)}_{\text{height}}$.

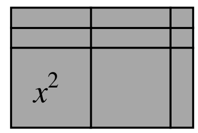


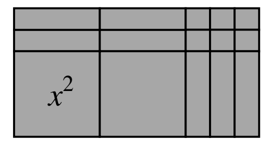
$\Rightarrow \underset{\text{area as a product}}{(2x + 1)(x - 3)} = \underset{\text{area as a sum}}{2x^2 - 5x - 3}$

Problems

Write a statement showing **area as a product** equals **area as a sum**.

1. 

2. 

3. 

4.

x	x^2	$3x$
-5	$-5x$	-15
	x	$+3$

5.

6	$18y$	$-12x$
	$3y$	$-2x$

6.

$3y$	$3xy$	$12y$
-2	$-2x$	-8
	x	$+4$

Multiply.

7.	$(3x + 2)(2x + 7)$	8.	$(2x - 1)(3x + 1)$	9.	$(2x)(x - 1)$
10.	$(2y - 1)(4y + 7)$	11.	$(y - 4)(y + 4)$	12.	$(y)(x - 1)$
13.	$(3x - 1)(x + 2)$	14.	$(2y - 5)(y + 4)$	15.	$(3y)(x - y)$
16.	$(3x - 5)(3x + 5)$	17.	$(4x + 1)^2$	18.	$(x + y)(x + 2)$
19.	$(2y - 3)^2$	20.	$(x - 1)(x + y + 1)$	21.	$(x + 2)(x + y - 2)$

Answers

1.	$(x + 1)(x + 3) = x^2 + 4x + 3$	2.	$(x + 2)(2x + 1) = 2x^2 + 5x + 2$
3.	$(x + 2)(2x + 3) = 2x^2 + 7x + 6$	4.	$(x - 5)(x + 3) = x^2 - 2x - 15$
5.	$6(3y - 2x) = 18y - 12x$	6.	$(x + 4)(3y - 2) = 3xy - 2x + 12y - 8$

7.	$6x^2 + 25x + 14$	8.	$6x^2 - x - 1$	9.	$2x^2 - 2x$
10.	$8y^2 + 10y - 7$	11.	$y^2 - 16$	12.	$xy - y$
13.	$3x^2 + 5x - 2$	14.	$2y^2 + 3y - 20$	15.	$3xy - 3y^2$
16.	$9x^2 - 25$	17.	$16x^2 + 8x + 1$	18.	$x^2 + 2x + xy + 2y$
19.	$4y^2 - 12y + 9$	20.	$x^2 + xy - y - 1$	21.	$x^2 + xy + 2y - 4$

SOLVING EQUATIONS WITH MULTIPLICATION OR ABSOLUTE VALUE 3.3.1

To **solve an equation with multiplication**, first use the Distributive Property or a generic rectangle to rewrite the equation without parentheses, then solve in the usual way. For additional information, see the Math Notes box in Lesson 3.3.1. For additional examples and practice, see the Checkpoint 6B materials at the back of the textbook.

To **solve an equation with absolute value**, first break the problem into two cases since the quantity inside the absolute value can be positive or negative. Then solve each part in the usual way.

Example 1

Solve $6(x + 2) = 3(5x + 1)$

Use the Distributive Property. $6x + 12 = 15x + 3$

Subtract $6x$. $12 = 9x + 3$

Subtract 3. $9 = 9x$

Divide by 9. $1 = x$

Example 2

Solve $x(2x - 4) = (2x + 1)(x + 5)$

Rewrite the equation using the Distributive Property on the left side of the equal sign and a generic rectangle on the right side.

$$2x^2 - 4x = \begin{array}{r|c|c|} \cline{2-3} +5 & 10x & 5 \\ \cline{2-3} x & 2x^2 & x \\ \cline{2-3} \multicolumn{1}{r}{} & \multicolumn{1}{c}{2x} & \multicolumn{1}{c}{+1} \end{array}$$

$$2x^2 - 4x = 2x^2 + 11x + 5$$

Subtract $2x^2$ from both sides. $-4x = 11x + 5$

Subtract $11x$ from both sides. $-15x = 5$

Divide by -15. $x = \frac{5}{-15} = -\frac{1}{3}$

Example 3

Solve $|2x-3|=7$

Separate into two cases.	$2x-3=7$	or	$2x-3=-7$
Add 3.	$2x=10$	or	$2x=-4$
Divide by 2.	$x=5$	or	$x=-2$

Problems

Solve each equation.

1. $3(c+4)=5c+14$
2. $x-4=5(x+2)$
3. $7(x+7)=49-x$
4. $8(x-2)=2(2-x)$
5. $5x-4(x-3)=8$
6. $4y-2(6-y)=6$
7. $2x+2(2x-4)=244$
8. $x(2x-4)=(2x+1)(x-2)$
9. $(x-1)(x+7)=(x+1)(x-3)$
10. $(x+3)(x+4)=(x+1)(x+2)$
11. $2x-5(x+4)=-2(x+3)$
12. $(x+2)(x+3)=x^2+5x+6$
13. $(x-3)(x+5)=x^2-7x-15$
14. $(x+2)(x-2)=(x+3)(x-3)$
15. $\frac{1}{2}x(x+2)=\left(\frac{1}{2}x+2\right)(x-3)$
16. $|3x+2|=11$
17. $|5-x|=9$
18. $|3-2x|=7$
19. $|2x+3|=-7$
20. $|4x+1|=10$

Answers

1. $c=-1$
2. $x=-3.5$
3. $x=0$
4. $x=2$
5. $x=-4$
6. $y=3$
7. $x=42$
8. $x=2$
9. $x=0.5$
10. $x=-2.5$
11. $x=-14$
12. all numbers
13. $x=0$
14. no solution
15. $x=-12$
16. $x=3$ or $-\frac{13}{3}$
17. $x=-4$ or 14
18. $x=-2$ or 5
19. no solution
20. $x=\frac{9}{4}$ or $-\frac{11}{4}$

REWRITING MULTI-VARIABLE EQUATIONS 3.3.2

Rewriting equations with more than one variable uses the same "legal" moves process as solving an equation with one variable in Lessons 3.2.1, A.1.8, and A.1.9. The end result is often not a number, but rather an algebraic expression containing numbers and variables.

For "legal" moves, see the Math Notes box in Lesson 3.2.1. For additional examples and more practice, see the Checkpoint 6A materials at the back of the textbook.

Example 1

Solve for y	$3x-2y=6$
Subtract $3x$	$-2y=-3x+6$
Divide by –2	$y=\frac{-3x+6}{-2}$
Simplify	$y=\frac{3}{2}x-3$

Example 2

Solve for y	$7+2(x+y)=11$
Subtract 7	$2(x+y)=4$
Distribute the 2	$2x+2y=4$
Subtract $2x$	$2y=-2x+4$
Divide by 2	$y=\frac{-2x+4}{2}$
Simplify	$y=-x+2$

Example 3

Solve for x	$y=3x-4$
Add 4	$y+4=3x$
Divide by 3	$\frac{y+4}{3}=x$

Example 4

Solve for t	$I=prt$
Divide by pr	$\frac{I}{pr}=t$

Problems

Solve each equation for the specified variable.

1. Solve for y: $5x+3y=15$
2. Solve for x: $5x+3y=15$
3. Solve for w: $2l+2w=P$
4. Solve for m: $4n=3m-1$
5. Solve for a: $2a+b=c$
6. Solve for a: $b-2a=c$
7. Solve for p: $6-2(q-3p)=4p$
8. Solve for x: $y=\frac{1}{4}x+1$
9. Solve for r: $4(r-3s)=r-5s$

Answers (Other equivalent forms are possible.)

1.	$y=-\frac{5}{3}x+5$	2.	$x=-\frac{3}{5}y+3$	3.	$w=-l+\frac{P}{2}$
4.	$m=\frac{4n+1}{3}$	5.	$a=\frac{c-b}{2}$	6.	$a=\frac{c-b}{-2}$ *or* $\frac{b-c}{2}$
7.	$p=q-3$	8.	$x=4y-4$	9.	$r=\frac{7s}{3}$

WRITING EQUATIONS 4.1.1

In this lesson, students translate written information, often modeling everyday situations, into algebraic symbols and linear equations. Students use "let" statements to specifically define the meaning of each of the variables they use in their equations.

For additional examples and more problems, see the Checkpoint 7A problems at the back of the textbook.

Example 1

The perimeter of a rectangle is 60 cm. The length is 4 times the width. Write one or more equations that model the relationships between the length and width.

Start by identifying what is unknown in the situation. Then define variables, using "let" statements, to represent the unknowns. When writing "let" statements, the units of measurement must also be identified. This is often done using parentheses, as shown in the "let" statements below. In this problem, length and width are unknown.

Let w represent the width (cm) of the rectangle, and let l represent the length (cm).

In this problem there are two variables. To be able to find unique solutions for these two variables, two unique equations need to be written.

From the first sentence and our knowledge about rectangles, the equation $P = 2l + 2w$ can be used to write the equation $60 = 2l + 2w$. From the sentence "the length is 4 times the width" we can write $l = 4w$.

A system of equations is two or more equations that use the same set of variables to represent a situation. The system of equations that represent the situation is:

Let w represent the width (cm) of the rectangle, and let l represent the length (cm).

$$l = 4w$$

$$2l + 2w = 60$$

Note that students who took a CPM middle school course may recall a method called the 5-D Process. This 5-D Process is not reviewed in this course, but it is perfectly acceptable for students to use it to help write and solve equations for word problems.

Using a 5-D table:

	Define		Do	Decide
	Width	Length	Perimeter	$P = 60$?
Trial 1:	10	4(10)	2(40) + 2(10) = 100	too big
Trial 2:	5	4(5)	2(20) + 2(5) = 50	too small
	l	$4(l)$	$2(4w) + 2w = 60$	

Example 2

Mike spent \$11.19 on a bag containing red and blue candies. The bag weighed 11 pounds. The red candy costs \$1.29 a pound and the blue candy costs \$0.79 a pound. How much red candy did Mike have?

Start by identifying the unknowns. The question in the problem is a good place to look because it often asks for something that is unknown. In this problem, the amount of red candy and the amount of blue candy are unknown.

Let r represent the amount of red candy (lb), and b represent the amount of blue candy (lb).

Note how the units of measurement were defined. If we stated "r = red candy" it would be very easy to get confused as to whether r represented the *weight* of the candy or the *cost* of the candy.

From the statement "the bag weighed 11 pounds" we can write $r + b = 11$. Note that in this equation the units are lb + lb = lb, which makes sense.

The cost of the red candy will be \$1.29/pound multiplied by its weight, or $1.29r$. Similarly, the cost of the blue candy will be $0.79b$. Thus $1.29r + 0.79b = 11.19$.

Let r represent the weight of the red candy (lb), and let b represent the weight of the blue candy (lb).

$$r + b = 11$$
$$1.29r + 0.79b = 11.19$$

Problems

Write an equation or a system of equations that models each situation. Do not solve your equations.

1. A rectangle is three times as long as it is wide. Its perimeter is 36 units. Find the length of each side.

2. A rectangle is twice as long as it is wide. Its area is 72 square units. Find the length of each side.

3. The sum of two consecutive odd integers is 76. What are the numbers?

4. Nancy started the year with $425 in the bank and is saving $25 a week. Seamus started the year with $875 and is spending $15 a week. When will they have the same amount of money in the bank?

5. Oliver earns $50 a day and $7.50 for each package he processes at Company A. His paycheck on his first day was $140. How many packages did he process?

6. Dustin has a collection of quarters and pennies. The total value is $4.65. There are 33 coins. How many quarters and pennies does he have?

7. A one-pound mixture of raisins and peanuts costs $7.50. The raisins cost $3.25 a pound and the peanuts cost $5.75 a pound. How much of each ingredient is in the mixture?

8. An adult ticket at an amusement park costs $24.95 and a child's ticket costs $15.95. A group of 10 people paid $186.50 to enter the park. How many were adults?

9. Katy weighs 105 pounds and is gaining 2 pounds a month. James weighs 175 pounds and is losing 3 pounds a month. When will they weigh the same amount?

10. Harper Middle School has 125 fewer students than Holmes Junior High. When the two schools are merged there are 809 students. How many students attend each school?

Answers (Other equivalent forms are possible.)

	One Variable Equation	System of Equations	Let Statement
1.	$2w + 2(3w) = 36$	$l = 3w$ $2w + 2l = 36$	Let l = length, w = width
2.	$w(2w) = 7$	$l = 2w$ $lw = 72$	Let l = length, w = width
3.	$m + (m + 2) = 76$	$m + n = 76$ $n = m + 2$	Let m = the first odd integer and n = the next consecutive odd integer
4.	$425 + 25x = 875 - 15x$	$y = 425 + 25x$ $y = 875 - 15x$	Let x = the number of weeks and y = the total money in the bank
5.	$50 + 7.5p = 140$		Let p = the number of packages Oliver processed
6.	$0.25q + 0.01(33 - q) = 4.65$	$q + p = 33$ $0.25q + 0.01p = 4.65$	Let q = number of quarters, p = number of pennies
7.	$3.25r + 5.75(1 - r) = 7.5$	$r + p = 1$ $3.25r + 5.75p = 7.5(1)$	Let r = weight of raisins and p = weight of peanuts
8.	$24.95a + 15.95(10 - a) = 186.5$	$a + c = 10$ $24.95a + 15.95c = 186.5$	Let a = number of adult tickets and c = number of child's tickets
9.	$105 + 2m = 175 - 3m$	$w = 105 + 2m$ $w = 175 - 3m$	Let m = the number of months and w = the weight of each person
10.	$x + (x - 125) = 809$	$x + y = 809$ $y = x - 125$	Let x = number of Holmes students and y = number of Harper students

SOLVING SYSTEMS BY SUBSTITUTION 4.1.1 and 4.2.1

A system of equations has two or more equations with two or more variables. In the previous course, students were introduced to solving a system by looking at the intersection point of the graphs. They also learned the algebraic **Equal Values Method** of finding solutions. Graphing and the Equal Values Method are convenient when both equations are in $y = mx + b$ form (or can easily be rewritten in $y = mx + b$ form).

The **Substitution Method** is used to change a two-variable system of equations to a one-variable equation. This method is useful when one of the variables is isolated in one of the equations in the system. Two substitutions must be made to find both the x- and y-values for a complete solution.

For additional information, see the Math Notes boxes in Lessons 4.1.1, 4.2.2, 4.2.3, and 4.2.5. For additional examples and more problems solving systems using multiple methods, see the Checkpoint 7A and 7B problems at the back of the textbook.

The equation $x + y = 9$ has infinite possibilities for solutions: $(10, -1)$, $(2, 7)$, $(0, 9)$, …, but if $y = 4$ then there is only one possible value for x. That value is easily seen when we replace (substitute for) y with 4 in the original equation: $x + 4 = 9$, so $x = 5$ when $y = 4$. Substitution and this observation are the basis for the following method to solve systems of equations.

Example 1 (Equal Values Method)

Solve:
$$\begin{aligned} y &= 2x \\ x + y &= 9 \end{aligned}$$

When using the Equal Values Method, start by rewriting both equations in $y = mx + b$ form. In this case, the first equation is already in $y = mx + b$ form. The second equation can be rewritten by subtracting x from both sides:

$$\begin{array}{rcl} x + y &=& 9 \\ -x & & -x \\ y &=& 9 - x \end{array}$$

Since y represents equal values in both the first equation, $y = 2x$, and the second equation, $y = 9 - x$, you can write: $2x = 9 - x$.

Solving for x:
$$\begin{array}{rcl} 2x &=& 9 - x \\ +x & & +x \\ 3x &=& 9 \\ x &=& 3 \end{array}$$

Then, either equation can be used to find y. For example, use the first equation, $y = 2x$, to find y:

Since $x = 3$, and $y = 2x$:

$y = 2(3)$

$y = 6$

Example continues on next page →

Example continued from previous page.

The solution to this system of equation is $x = 3$ and $y = 6$. That is, the values $x = 3$ and $y = 6$ make both of the original equations true. When graphing, the point (3, 6) is the intersection of the two lines.

Always check your solution by substituting the solution back into *both* of the original equations to verify that you get true statements:

For $y = 2x$ at (3, 6),

$$6 \stackrel{?}{=} 2(3)$$

$6 = 6$ is a true statement.

For $x + y = 9$ at (3, 6),

$$3 + 6 \stackrel{?}{=} 9$$

$9 = 9$ is a true statement.

Example 2 (Substitution Method)

Solve: $y = 2x$
$x + y = 9$

The same system of equations in Example 1 can be solved using the Substitution Method. From the first equation, y is equivalent to $2x$. Therefore you can replace the y in the second equation with $2x$:

$$x + y = 9$$

Replace y with $2x$, and solve.

$$x + (2x) = 9$$
$$3x = 9$$
$$x = 3$$

Then, either equation can be used to find y. For example, use the first equation, $y = 2x$, to find y:

$y = 2x$

Since $x = 3$,

$y = 2(3)$

$y = 6$

The solution to this system of equation is $x = 3$ and $y = 6$. That is, the values $x = 3$ and $y = 6$ make both of the original equations true. When graphing, the point (3, 6) is the intersection of the two lines.

Always check your solution by substituting the solution back into *both* of the original equations to verify that you get true statements (see Example 1).

Example 3 (Substitution Method)

Look for a convenient substitution when using the Substitution Method; look for a variable that is by itself on one side of the equation.

Solve: $4x + y = 8$
$x = 5 - y$

Since x is equivalent to $5 - y$, replace x in the first equation with $5 - y$. Solve as usual.

$$\begin{aligned} 4x + y &= 8 \\ 4(5 - y) + y &= 8 \\ 20 - 4y + y &= 8 \\ 20 - 3y &= 8 \\ -3y &= -12 \\ y &= 4 \end{aligned}$$

To find x, use either of the original two equations. In this case, we will use $x = 5 - y$.

$$x = 5 - y$$

Since $y = 4$,

$$x = 5 - (4)$$
$$x = 1$$

The solution is $(1, 4)$. Always check your solution by substituting the solution back into *both* of the original equations to verify that you get true statements.

Example 4

Not all systems have a solution. If the substitution results in an equation that is not true, then there is **no solution**. The graph of a system of two linear equations that has no solutions is two parallel lines; there is no point of intersection. See the Math Notes box in Lesson 4.2.5 for additional information.

Solve: $y = 7 - 3x$
$3x + y = 10$

Replace y in the second equation with $7 - 3x$.

The resulting equation is never true.
There is no solution to this system of equations.

$$\begin{aligned} 3x + y &= 10 \\ 3x + (7 - 3x) &= 10 \\ 3x - 3x + 7 &= 10 \\ 7 &\neq 10 \end{aligned}$$

Example 5

There may also be an **infinite number of solutions**. This graph would appear as a single line for the two equations.

Solve: $y = 4 - 2x$
$-4x - 2y = -8$

Substitute $4 - 2x$ in the second equation for y.

This statement is always true. There are infinite solutions to this system of equations.

$$\begin{aligned} -4x - 2y &= -8 \\ -4x - 2(4 - 2x) &= -8 \\ -4x - 8 + 4x &= -8 \\ -8 &= -8 \end{aligned}$$

Example 6

Sometimes you need to solve one of the equations for x or y to use the Substitution Method.

Solve: $y - 2x = -7$
$3x - 4y = 8$

Solve the first equation for y to get $y = 2x - 7$.
Then substitute $2x - 7$ in the second equation for y.

$$\begin{aligned} 3x - 4(2x - 7) &= 8 \\ 3x - 8x + 28 &= 8 \\ -5x + 28 &= 8 \\ -5x &= -20 \\ x &= 4 \end{aligned}$$

Then find y.

$y = 2x - 7$
Since $x = 4$,
$y = 2(4) - 7$
$y = 1$

Check: $1 - 2(4) = -7$ and $3(4) - 4(1) = 8$.

The solution is (4, 1). This would be the intersection point of the two lines. Always check your solution by substituting the solution back into *both* of the original equations to verify that you get true statements.

Problems

1. $y = -3x$ $4x + y = 2$	2. $y = 7x - 5$ $2x + y = 13$	3. $x = -5y - 4$ $x - 4y = 23$
4. $x + y = 10$ $y = x - 4$	5. $y = 5 - x$ $4x + 2y = 10$	6. $3x + 5y = 23$ $y = x + 3$
7. $y = -x - 2$ $2x + 3y = -9$	8. $y = 2x - 3$ $-2x + y = 1$	9. $x = \frac{1}{2}y + \frac{1}{2}$ $2x + y = -1$
10. $a = 2b + 4$ $b - 2a = 16$	11. $y = 3 - 2x$ $4x + 2y = 6$	12. $y = x + 1$ $x - y = 1$

Answers

1. (2, –6)	2. (2, 9)	3. (11, –3)
4. (7, 3)	5. (0, 5)	6. (1, 4)
7. (3, –5)	8. No solution	9. (0, –1)
10. (–12, –8)	11. Infinite solutions	12. No solution

ELIMINATION METHOD 4.2.3 through 4.2.5

In previous work with systems of equations, one of the variables was usually alone on one side of one of the equations. In those situations, it is convenient to rewrite both equations in $y = mx + b$ form and use the Equal Values Method, or to substitute the expression for one variable into the other equation using the Substitution Method.

Another method for solving systems of equations is the **Elimination Method**. This method is particularly convenient when both equations are in standard form (that is, $ax + by = c$). To solve this type of system, we can rewrite the equations by focusing on the coefficients. (The coefficient is the number in front of the variable.)

See problem 4-56 in the textbook for an additional explanation of the Elimination Method.

For additional information, see the Math Notes boxes in Lessons 4.2.4 and 5.1.1. For additional examples and more problems solving systems using multiple methods, see the Checkpoint 7B materials in the back of the textbook.

Example 1

Solve: $x - y = 2$
$2x + y = 1$

Recall that you are permitted to add the same expression to both sides of an equation. Since $x - y$ is equivalent to 2 (from the first equation), you are permitted to add $x - y$ to one side of the second equation, and 2 to the other side. Then solve.

$$\begin{aligned} 2x + y &= 1 \\ +x - y \quad &\;\; +2 \\ 3x \quad &= 3 \\ x &= 1 \end{aligned}$$

Note that this was an effective way to eliminate y and find x because $-y$ and y were opposite terms; $y + (-y) = 0$.

Now substitute the value of x in either of the original equations to find the y-value.

$$\begin{aligned} 2x + y &= 1 \\ \text{Since } x &= 1, \\ 2(1) + y &= 1 \\ 2 + y &= 1 \\ y &= -1 \end{aligned}$$

The solution is $(1, -1)$, since $x = 1$ and $y = -1$ make both of the original equations true. On a graph, the point of intersection of the two original lines is $(1, -1)$. Always check your solution by substituting the solution back into *both* of the original equations to verify that you get true statements.

See problem 4-56 in the textbook for an additional explanation of the Elimination Method.

Example 2

Solve: $3x + 6y = 24$

$3x + y = -1$

Notice that both equations contain a $3x$ term. We can rewrite $3x + y = -1$ by multiplying both sides by -1, resulting in $-3x + (-y) = 1$. Now the two equations have terms that are opposites: $3x$ and $-3x$. This will be useful in the next step because $-3x + 3x = 0$.

Since $-3x + (-y)$ is equivalent to 1, we can add $-3x + (-y)$ to one side of the equation and add 1 to the other side.

$$\begin{aligned} 3x + 6y &= 24 \\ -3x + (-y) &\quad +1 \\ 5y &= 25 \\ y &= 5 \end{aligned}$$

Notice how the two opposite terms, $3x$ and $-3x$, eliminated each other, allowing us to solve for y.

Then substitute the value of y into either of the original equations to find x.

$$\begin{aligned} 3x + 6(5) &= 24 \\ 3x + 30 &= 24 \\ 3x &= -6 \\ x &= -2 \end{aligned}$$

The solution is $(-2, 5)$. It makes both equations true, and names the point where the two lines intersect on a graph. Always check your solution by substituting the solution back into *both* of the original equations to verify that you get true statements.

A more detailed explanation of this method can be found in the following example.

Example 3

To use the Elimination Method, one of the terms in one of the equations needs to be opposite of the corresponding term in the other equation. One of the equations can be multiplied to make terms opposite. For example, in the system at right, there are no terms that are opposite. However, if the first equation is multiplied by -4, then the two equations will have $4x$ and $-4x$ as opposites. The first equation now looks like this: $-4(x + 3y = 7) \rightarrow -4x + (-12y) = -28$. When multiplying an equation, be sure to multiply all the terms on *both* sides of the equation. With the first equation rewritten, the system of equations now looks like this:

$$\begin{aligned} x + 3y &= 7 \\ 4x - 7y &= -10 \end{aligned}$$

$$\begin{aligned} -4x + (-12y) &= -28 \\ 4x - 7y &= -10 \end{aligned}$$

Since $4x - 7y$ is equivalent to -10, they can be added to both sides of the first equation:

$$\begin{array}{rr} & -4x + (-12y) = -28 \\ + & 4x - 7y \quad\; -10 \\ \hline & -19y \quad = -38 \\ & y \quad = 2 \end{array}$$

Now any of the equations can be used to find x:

Since $4x - 7y = -10$ and $y = 2$,

$$\begin{aligned} 4x - 7(2) &= -10 \\ 4x - 14 &= -10 \\ 4x &= 4 \\ x &= 1 \end{aligned}$$

The solution to the system of equations is $(1, 2)$.

Example 4

If multiplying one equation by a number will not make it possible to eliminate a variable, multiply both equations by different numbers to get coefficients that are the same or opposites.

Solve: $8x - 7y = 5$
$3x - 5y = 9$

One possibility is to multiply the first equation by 3 and the second equation by –8. The resulting terms 24*x* and –24*x* will be opposites, setting up the Elimination Method.

$$3(8x - 7y = 5) \Rightarrow 24x - 21y = 15$$
$$-8(3x - 5y = 9) \Rightarrow -24x + 40y = -72$$

The system of equations is now:

$$\begin{aligned} 24x - 21y &= 15 \\ -24x + 40y &= -72 \end{aligned}$$

This system can be solved by adding equivalent expressions (from the second equation) to the first equation:

$$\begin{array}{rrl} & 24x - 21y & = 15 \\ + & -24x + 40y & -72 \\ \hline & 19y & = -57 \\ & y & = -3 \end{array}$$

Then solving for *x*, the solution is (–2, –3).

For an additional example like this one (where both equations had to be multiplied to create opposite terms), see Example 2 in the Checkpoint 7B materials at the back of the textbook.

Example 5

The special cases of "no solution" and "infinite solutions" can also occur. See the Math Notes box in Lesson 4.2.5 for additional information.

Solve: $4x + 2y = 6$
$2x + y = 3$

Multiplying the second equation by 2 produces $4x + 2y = 6$. The two equations are identical, so when graphed there would be one line with *infinite* solutions because the same ordered pairs are true for both equations.

Solve: $2x + y = 3$
$4x + 2y = 8$

Multiplying the first equation by 2 produces $4x + 2y = 6$. There are no numbers that could make $4x + 2y$ equal to 6, and $4x + 2y$ equal to 8 at the same time. The lines are parallel and there is *no solution*, that is, no point of intersection.

SUMMARY OF METHODS TO SOLVE SYSTEMS

Method	This Method is Most Efficient When	Example
Equal Values	Both equations in y-form.	$y = x - 2$ $y = -2x + 1$
Substitution	One variable is alone on one side of one equation.	$y = -3x - 1$ $3x + 6y = 24$
Elimination: Add to eliminate one variable.	Equations in standard form with opposite coefficients.	$x + 2y = 21$ $3x - 2y = 7$
Elimination: Multiply one equation to eliminate one variable.	Equations in standard form. One equation can be multiplied to create opposite terms.	$x + 2y = 3$ $3x + 2y = 7$
Elimination: Multiply both equations to eliminate one variable.	When nothing else works. In this case you could multiply the first equation by 3 and the second equation by –2, then add to eliminate the opposite terms.	$2x - 5y = 3$ $3x + 2y = 7$

Problems

1. $2x + y = 6$
   $-2x + y = 2$

2. $-4x + 5y = 0$
   $-6x + 5y = -10$

3. $2x - 3y = -9$
   $x + y = -2$

4. $y - x = 4$
   $2y + x = 8$

5. $2x - y = 4$
   $\frac{1}{2}x + y = 1$

6. $-4x + 6y = -20$
   $2x - 3y = 10$

7. $6x - 2y = -16$
   $4x + y = 1$

8. $6x - y = 4$
   $6x + 3y = -16$

9. $2x - 2y = 5$
   $2x - 3y = 3$

10. $y - 2x = 6$
    $y - 2x = -4$

11. $4x - 4y = 14$
    $2x - 4y = 8$

12. $3x + 2y = 12$
    $5x - 3y = -37$

Answers

1. $(1, 4)$
2. $(5, 4)$
3. $(-3, 1)$
4. $(0, 4)$
5. $(2, 0)$
6. Infinite
7. $(-1, 5)$
8. $(-\frac{1}{6}, -5)$
9. $(4.5, 2)$
10. No solution
11. $(3, -\frac{1}{2})$
12. $(-2, 9)$

INTRODUCTION TO SEQUENCES 5.1.1 through 5.1.3

In Chapter 5, students investigate sequences by looking for patterns and rules. Initially in the chapter, students concentrate on arithmetic sequences (sequences generated by adding a constant to the previous term), and then later in the chapter (and in Chapter 7) they consider geometric sequences (sequences generated by multiplying the previous term by a constant).

In Lessons 5.1.1 through 5.1.3 students are introduced to the two types of sequences, arithmetic and geometric, and their graphs, in everyday situations.

For additional examples and explanations, see the next section of this *Parent Guide with Extra Practice*, "Equations for Sequences." For additional information, see the first half of the Math Notes box in Lesson 5.3.2.

Example 1

Peachy Orchard Developers are preparing land to create a large subdivision of single-family homes. They have already built 15 houses on the site. Peachy Orchard plans to build six new homes every month. Create a table of values that will show the number of houses in the Peachy Orchard subdivision over time. Write an equation relating the number of houses over time. Graph the sequence.

Since the subdivision initially has 15 homes, 15 is the number of homes at time $t = 0$. After one month, there will be six more, or 21 homes. After the second month, there will be 27 homes. After each month, we add six homes to the total number of homes in the subdivision. Because we are adding a constant amount after each time period, this is an **arithmetic sequence**.

n, the number of months	$t(n)$, the total number of homes
0	15
1	21
2	27
3	33
4	39

We can find the equation for this situation by noticing that this is a linear function: the growth is constant. All arithmetic sequences are linear.

One way to write the equation that models this situation is to notice that the slope (growth) = 6 homes/month, and the y-intercept (starting point) = 15. Then in $y = mx + b$ form, the equation is $y = 6x + 15$.

Another way to find the equation of a line, especially in situations more complex than this one, is to use two points on the line, calculate the slope (m) between the two points, and then solve for the y-intercept (as in Lesson 2.3.2). This method is shown in the following steps:

Example continues on next page →

Example continued from previous page.

Choose (1, 21) and (4, 39)

$\text{slope} = \frac{\Delta y}{\Delta x} = \frac{\text{change in } y}{\text{change in } x}$

$m = \frac{39-21}{4-1}$

$m = \frac{18}{3}$

$m = 6$

$y = mx + b$

at $(x, y) = (1, 21)$ and $m = 6$,

$21 = 6(1) + b$

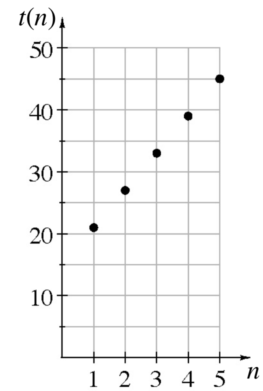
$b = 15$

$y = 6x + 15$

We write the equation as $t(n) = 6n + 15$ to show that this is an arithmetic sequence (as opposed to the linear function $y = mx + b$ or $f(x) = mx + b$) that will find the term t, for any number n. Let $t(n)$ represent the number of houses, and n the number of months.

The sequence would be written: 21, 27, 33, 39, …. Note that sequences usually begin with the first term (in this case, the term for the first month, $n = 1$).

The graph for the sequence is shown at right. Note that it is linear, and that it starts with the point (1, 21).



Example 2

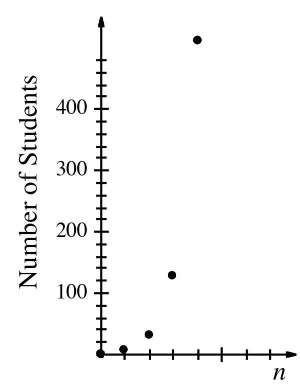
When Rosa tripped and fell into a muddy puddle at lunch (she was so embarrassed!), she knew exactly what would happen: within ten minutes, the two girls who saw her fall would each tell four people what they had seen. Within the next ten minutes, those eight students would each tell four more people. Rosa knew this would continue until everyone in the entire school was talking about her muddy experience. If there are 2016 students in the school, how many "generations" of gossiping would it take before everyone in the school was talking about Rosa? How many minutes would it take? Graph the situation.

At time $t = 0$, only two people see Rosa trip and fall. After ten minutes, each of those two people would tell four people; there are eight students gossiping about Rosa. After another ten minutes, each of those eight students will gossip with four more students; there will be $8 \times 4 = 32$ students gossiping. For the third increment of ten minutes, each of the 32 students will gossip with 4 students; $32 \times 4 = 128$ students gossiping.

Each time we multiply the previous number of students by four to get the next number of students. This is an example of a **geometric sequence**, and the multiplier is four. We record this in a table as shown at right, with n being the number of ten minute increments since Rosa's fall, and $t(n)$ is the number of students discussing the incident at that time. By continuing the table, we note that at time $t = 6$, there will be 2048 students discussing the mishap. Since there are only 2016 students in the school, everyone is gossiping by the sixth ten-minute increment of time. Therefore, just short of 60 minutes, or a little before one hour, everyone knows about Rosa's fall in the muddy puddle.

n, Number of Ten Minute Increments	Number of Students
0	2
1	8
2	32
3	128
4	512
5	1024
6	2048

The graph is shown at right. A geometric relationship is not linear; it is exponential. In future lessons, students would write the sequence as 8, 32, 128, … . Note that sequences usually begin with the first term (in this case, the term for the first month, $n = 1$).



Problems

1. Find the missing terms for this arithmetic sequence and an equation for $t(n)$.

$$\_\_, 15, 11, \_\_, 3$$

2. For this sequence each term is $\frac{1}{5}$ of the previous one. Work forward and backward to find the missing terms.

$$\_\_, \_\_, \tfrac{2}{3}, \_\_, \_\_$$

3. The 30^{th} term of a sequence is 42. If each term in the sequence is four greater than the previous number, what is the first term?

4. The microscopic length of a crystalline structure grows so that each day it is 1.005 times as long as the previous day. If on the third day the structure was 12.5 nm long, write a sequence for how long it was on the first five days. (nm stands for nanometer, or 1×10^{-9} meters.)

5. Davis loves to ride the mini-cars at the amusement park but riders must be no more than 125 cm tall. If on his fourth birthday he is 94 cm tall and grows approximately 5.5 cm per year, at what age will he no longer be able to go on the mini-car ride?

Answers

1. 19 and 8; $t(n) = 23 - 4n$

2. $\frac{50}{3}, \frac{10}{3}, \frac{2}{3}, \frac{2}{15}, \frac{2}{75}$

3. $42 - 29(4) = -74$

4. $\approx 12.38, \approx 12.44, 12.5, \approx 12.56, \approx 12.63, \ldots$

5. $t(n) = 5.5n + 94$, so solve $5.5n + 94 \leq 125$. $n \approx 5.64$. At $\approx 4 + 5.64 = 9.64$ he will be too tall. Davis can continue to go on the ride until he is about $9\frac{1}{2}$ years old.

EQUATIONS FOR SEQUENCES 5.2.1 through 5.2.3

In these lessons, students learn multiple representations for sequences: as a string of numbers, as a table, as a graph, and as an equation. Read more about writing equations for sequences in the Math Notes box in Lesson 7.2.3.

In addition to the ways to write explicit equations for sequences, as explained in the Math Notes box in Lesson 7.2.3, equations for sequences can also be written recursively.
An explicit formula tells exactly how to find any specific term in the sequence. A recursive formula names the first term (or any other term) and how to get from one term to the next. For an explanation of recursive sequences, see the Math Notes box in Lesson 5.3.2.

Example 1

This is the same scenario as in Example 1 of the previous section, *Introduction to Sequences*.

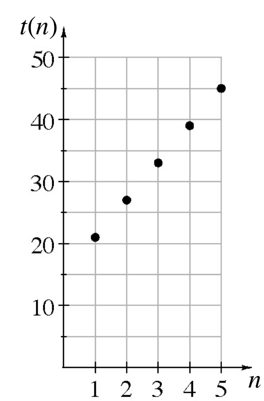
Peachy Orchard Developers are preparing land to create a large subdivision of single-family homes. They have already built 15 houses on the site. Peachy Orchard plans to build six new homes every month. Write a sequence for the number of houses built, then write an equation for the sequence. Fully describe a graph of this sequence.

The sequence is 21, 27, 33, 39, …. Note that sequences usually begin with the first term, where the number of months is $n = 1$.

The common difference is $m = 6$, and the zeroth term is $b = 15$. The equation can be written $t(n) = mn + b = 6n + 15$. Note that for a sequence, $t(n) =$ is used instead of $y =$.
$t(n) =$ indicates the equation is for a discrete sequence, as opposed to a continuous function. Students compared sequences to functions in Lesson 5.3.3.

The equation could also have been written as $a_n = 6n + 15$.

The graph of the sequence is shown at right. There are no x- or y-intercepts. There is no point at (0, 15) because sequences are usually written starting with the *first* term where $n = 1$. The domain consists of *integers* (whole numbers) greater than or equal to one. The range consists of the y-values of the points that follow the rule $t(n) = 6n + 15$ when $n \geq 1$. There are no asymptotes. The graph is linear and is shown at right. *This graph is discrete* (separate points). (Note: The related function, $y = 6x + 15$, would have the domain of all real numbers (including fractions and negatives) and the graph would be a continuous connected line.)



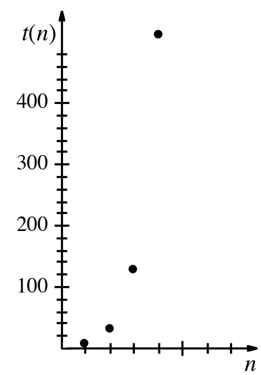
Example 2

This is the same scenario as in Example 2 of the previous section, *Introduction to Sequences*.

When Rosa tripped and fell into a muddy puddle at lunch (she was so embarrassed!), she knew exactly what would happen: within ten minutes, the two girls who saw her fall would each tell four people what they had seen. Within the next ten minutes, those eight students would each tell four more people. Rosa knew this would continue until everyone in the entire school was talking about her muddy experience. Write a sequence for the number of people who knew about Rosa mishap in ten-minute intervals, then write an equation for the sequence. Fully describe a graph of this sequence.

The multiplier is $b = 4$, and the zeroth term is $a = 2$. The equation can be written $t(n) = ab^n = 2 \cdot 4^n$. The equation could also have been written as $a_n = 2 \cdot 4^n$. (Later, in Chapter 7, students will also learn "first term" notation for sequences, $a_n = 8 \cdot 4^{(n-1)}$.)

The sequence is: 8, 32, 128, 512, … . Note that the sequence is written starting with $n = 1$.



The graph of the sequence is to the right. There are no x- or y-intercepts. There is no point at (0, 2) because sequences are usually written starting with the *first* term where $n = 1$. The domain consists of *integers* (whole numbers) greater than or equal to one. The range consists of the y-values of the points that follow the rule $t(n) = 2(4)^n$ when $n \geq 1$. The graph is exponential and is shown at right. There is no symmetry. This graph is discrete (separate points). (Note: The related function, $y = 2 \cdot 4^n$ would have the domain of all real numbers (including fractions and negatives) and the graph would be a connected curve.)

Example 3

Consider the two sequences:

A: $-8, \quad -5, \quad -2, \quad 1, \ldots$
B: $256, \ 128, \ 64, \ldots$

a. For each sequence, is it arithmetic, geometric, or neither? How can you tell? Explain completely.

b. What are the zeroth term and the generator for each sequence?

c. For each sequence, write an equation representing the sequence.

d. Is 378 a term of sequence A? Justify your answer.

e. Is $\frac{1}{4}$ a term of sequence B? Justify your answer.

To determine the type of sequence for A and B above, we have to look at the growth of each sequence.

```
A:   –8,   –5,   –2,   1,    …
       \   / \   / \   /
        +3    +3    +3
```

Sequence A is made (generated) by adding three to each term to get the next term. When each term has a **common difference** (in this case, "+3") the sequence is **arithmetic**.

Sequence B, however, is different. The terms do not have a common difference.

```
B:   256,  128,  64,   …
       \   / \   /
       –128   –64
```

Instead, these terms have a **common ratio** (multiplier). A sequence with a common ratio is a **geometric sequence**.

```
B:   256,  128,  64,   …
       \   / \   /
        ·1/2  ·1/2
```

The first term for sequence A is –8, and has a generator or common difference of +3. Therefore the zeroth term is –11 (because $-11 + 3 = -8$). An arithmetic sequence has an equation of the form $t(n) = mn + b$ (or $a_n = mn + a_0$) where m is the common difference, and b is the initial value. For sequence A, the equation is $t(n) = 3n - 11$, for $n = 1, 2, 3, \ldots$

Example continues on next page →

Example continued from previous page.

For sequence B, the first term is 256 with a generator or common ratio of $\frac{1}{2}$. Therefore the zeroth term is 512, because $512 \cdot \frac{1}{2} = 256$. The general equation for a geometric sequence is $t(n) = ab^n$ where a is the zeroth term, and b is the common ratio (multiplier). For sequence B, the equation is $t(n) = 512\left(\frac{1}{2}\right)^n$ for $n = 1, 2, 3, \ldots$

To check if 378 is a term in sequence A, we could list the terms of the sequence out far enough to check, but that would be time consuming. Instead, we will check if there is an integer n that solves $t(n) = 3n - 11 = 378$.

$$\begin{aligned} 3n - 11 &= 378 \\ 3n &= 389 \\ n &= \tfrac{389}{3} = 129\tfrac{2}{3} \end{aligned}$$

When we solve, n is not a whole number, therefore 378 cannot be a term in the sequence.

Similarly, to check if $\frac{1}{4}$ is a term in sequence B, we need to solve $t(n) = 512\left(\frac{1}{2}\right)^n = \frac{1}{4}$, and look for a whole number solution.

$$\begin{aligned} 512\left(\tfrac{1}{2}\right)^n &= \tfrac{1}{4} \\ \tfrac{1}{512} \cdot 512\left(\tfrac{1}{2}\right)^n &= \tfrac{1}{512} \cdot \tfrac{1}{4} \\ \left(\tfrac{1}{2}\right)^n &= \tfrac{1}{2048} \\ \left(\tfrac{1}{2}\right)^n &= \tfrac{1}{2^{11}} \\ \tfrac{1}{2^n} &= \tfrac{1}{2^{11}} \\ n &= 11 \end{aligned}$$

Although solving an equation like this is probably new for most students, they can solve this problem by using guess-and-check. Also, by writing both sides as a power of 2, students can see the solution easily.

Since the equation has a whole number solution, $\frac{1}{4}$ is a term of sequence B.

That is, when $n = 11$, $t(n) = \frac{1}{4}$.

EXAMPLES FOR ARITHMETIC SEQUENCES

List the first five terms of each arithmetic sequence.

Example 4 (An explicit formula)

$$t(n) = 5n + 2$$

$t(1) = 5(1) + 2 = 7$
$t(2) = 5(2) + 2 = 12$
$t(3) = 5(3) + 2 = 17$
$t(4) = 5(4) + 2 = 22$
$t(5) = 5(5) + 2 = 27$

The sequence is: 7, 12, 17, 22, 27, …

Example 5 (A recursive formula)

$$t(1) = 3,\ t(n+1) = t(n) - 5$$

$t(1) = 3$
$t(2) = t(1) - 5 = 3 - 5 = -2$
$t(3) = t(2) - 5 = -2 - 5 = -7$
$t(4) = t(3) - 5 = -7 - 5 = -12$
$t(5) = t(4) - 5 = -12 - 5 = -17$

The sequence is: 3, –2, –7, –12, –17, …

Example 6

Find an explicit and a recursive formula for the sequence: $-2, 1, 4, 7, \ldots$

Explicit: $m = 3$, $b = -5$, so the equation is: $t(n) = mn + b = 3n - 5$

Recursive: $t(1) = -2$, $t(n + 1) = t(n) + 3$

EXAMPLES FOR GEOMETRIC SEQUENCES

List the first five terms of each geometric sequence.

Example 7 (An explicit formula)

$$t(n) = 3 \cdot 2^{n-1}$$

$t(1) = 3 \cdot 2^{1-1} = 3 \cdot 2^0 = 3$

$t(2) = 3 \cdot 2^{2-1} = 3 \cdot 2^1 = 6$

$t(3) = 3 \cdot 2^{3-1} = 3 \cdot 2^2 = 12$

$t(4) = 3 \cdot 2^{4-1} = 3 \cdot 2^3 = 24$

$t(5) = 3 \cdot 2^{5-1} = 3 \cdot 2^4 = 48$

The sequence is: $3, 6, 12, 24, 48, \ldots$

Example 8 (A recursive formula)

$$t(1) = 8,\ t(n+1) = t(n) \cdot \tfrac{1}{2}$$

$t(1) = 8$

$t(2) = t(1) \cdot \frac{1}{2} = 8 \cdot \frac{1}{2} = 4$

$t(3) = t(2) \cdot \frac{1}{2} = 4 \cdot \frac{1}{2} = 2$

$t(4) = t(3) \cdot \frac{1}{2} = 2 \cdot \frac{1}{2} = 1$

$t(5) = t(4) \cdot \frac{1}{2} = 1 \cdot \frac{1}{2} = \frac{1}{2}$

The sequence is: $8, 4, 2, 1, \frac{1}{2}, \ldots$

Example 9

Find an explicit and a recursive formula for the sequence: $81, 27, 9, 3, \ldots$

Explicit: $a_1 = 81$, $b = \frac{1}{3}$ so the a_0 (the zeroth term) is found by $a_0 = 81 \div \frac{1}{3} = 243$ and the equation is: $a_n = a_0 \cdot b^n = 243 \cdot \left(\frac{1}{3}\right)^n$, or alternatively, $t(n) = 243 \cdot \frac{1}{3}^n$

Recursive: $t(1) = 81$, $t(n+1) = t(n) \cdot \frac{1}{3}$

Problems

Each of the functions listed below defines a sequence. List the first five terms of the sequence, and state whether the sequence is arithmetic, geometric, both, or neither.

1. $t(n) = 5n + 2$
2. $s_n = 3 - 8n$
3. $u(n) = 9n - n^2$
4. $t(n) = (-4)^n$
5. $s(n) = \left(\frac{1}{4}\right)^n$
6. $u(n) = n(n + 1)$
7. $t(n) = 8$
8. $s_n = \frac{3}{4}n + 1$

Identify each of the following sequences as arithmetic or geometric. Then write the equation that gives the terms of the sequence.

9. $48, 24, 12, 6, 3, \ldots$
10. $-4, 3, 10, 17, 24, \ldots$
11. $43, 39, 35, 31, 27, \ldots$
12. $\frac{2}{3}, \frac{1}{2}, \frac{3}{8}, \frac{9}{32}, \frac{27}{128}, \ldots$
13. $5, -5, 5, -5, 5, \ldots$
14. $10, 1, 0.1, 0.01, 0.001, \ldots$

Graph the following two sequences on the same set of axes.

15. $t(n) = -6n + 20$
16. $1, 4, 16, 64, \ldots$

17. Do the two sequences of the last two problems have any terms in common? Explain how you know.
18. Every year since 1548, the average height of a human male has increased slightly. The new height is 100.05% of what it was the previous year. If the average male's height was 54 inches in 1548, what was the average height of a male in 2008?
19. Davis has \$5.40 in his bank account on his fourth birthday. If his parents add \$0.40 to his bank account every week, when will he have enough to buy the new Smokin' Derby race car set which retails for \$24.99?
20. Fully describe the graph of the sequence $t(n) = -4n + 18$.

Arithmetic Sequences

List the first five terms of each arithmetic sequence.

21. $t(n) = 5n - 2$
22. $t(n) = -3n + 5$
23. $t(n) = -15 + \frac{1}{2}n$
24. $t(n) = 5 + 3(n - 1)$
25. $t(1) = 5, t(n + 1) = t(n) + 3$
26. $t(1) = 5, t(n + 1) = t(n) - 3$
27. $t(1) = -3, t(n + 1) = t(n) + 6$
28. $t(1) = \frac{1}{3}, t(a + 1) = t(n) + \frac{1}{2}$

Find the 30^{th} term of each arithmetic sequence.

29. $t(n) = 5n - 2$
30. $t(n) = -15 + \frac{1}{2}n$
31. $t(31) = 53, d = 5$
32. $t(1) = 25, t(n + 1) = t(n) - 3$

For each arithmetic sequence, find an explicit and a recursive formula.

33. 4, 8, 12, 16, 20, …
34. –2, 5, 12, 19, 26, …
35. 27, 15, 3, –9, –21, …
36. $3, 3\frac{1}{3}, 3\frac{2}{3}, 4, 4\frac{1}{3}, \ldots$

Sequences are graphed using points of the form: (term number, term value).
For example, the sequence 4, 9, 16, 25, 36, … would be graphed by plotting the points (1, 4), (2, 9), (3, 16), (4, 25), (5, 36), …. Sequences are graphed as points and not connected.

37. Graph the sequences from problems 21 and 22 above and determine the slope of each line.
38. How does the slope of the line found in the previous problem relate to the sequence?

Geometric Sequences

List the first five terms of each geometric sequence.

39. $t(n) = 5 \cdot 2^n$

40. $t(n) = -3 \cdot 3^n$

41. $t(n) = 40\left(\frac{1}{2}\right)^{n-1}$

42. $t(n) = 6\left(-\frac{1}{2}\right)^{n-1}$

43. $t(1) = 5, t(n + 1) = t(n) \cdot 3$

44. $t(1) = 100, t(n + 1) = t(n) \cdot \frac{1}{2}$

45. $t(1) = -3, t(n + 1) = t(n) \cdot (-2)$

46. $t(1) = \frac{1}{3}, t(n + 1) = t(n) \cdot \frac{1}{2}$

Find the 15^{th} term of each geometric sequence.

47. $t(14) = 232, r = 2$

48. $t(16) = 32, r = 2$

49. $t(14) = 9, r = \frac{2}{3}$

50. $t(16) = 9, r = \frac{2}{3}$

Find an explicit and a recursive formula for each geometric sequence.

51. 2, 10, 50, 250, 1250, …

52. $16, 4, 1, \frac{1}{4}, \frac{1}{16}, \ldots$

53. 5, 15, 45, 135, 405, …

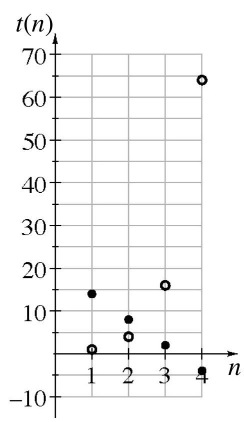
54. 3, –6, 12, –24, 48, …

55. Graph the sequences from problems 39 and 52. Remember the note before problem 37 about graphing sequences.

56. How are the graphs of geometric sequences different from arithmetic sequences?

Answers

1. 7, 12, 17, 22, 27, arithmetic, common difference is 5.
2. –5, –13, –21, –29, –37, arithmetic, common difference is –8.
3. 8, 14, 18, 20, 20, neither
4. –4, 16, –64, 256, –1024, geometric, common ratio is –4.
5. $\frac{1}{4}, \frac{1}{16}, \frac{1}{64}, \frac{1}{256}, \frac{1}{1024}$, geometric, common ratio is $\frac{1}{4}$.
6. 2, 6, 12, 20, 30, neither
7. 8, 8, 8, 8, 8, both, common difference is 0, common ratio is 1.
8. $\frac{7}{4}, \frac{5}{2}, \frac{13}{4}, 4, \frac{19}{4}$, arithmetic, common difference is $\frac{3}{4}$.
9. geometric, $t(n) = 96\left(\frac{1}{2}\right)^n$
10. arithmetic, $t(n) = 7n - 11$
11. arithmetic, $t(n) = -4n + 47$
12. geometric, $t(n) = \left(\frac{8}{9}\right)\left(\frac{3}{4}\right)^n$
13. geometric, $t(n) = -5(-1)^n$
14. geometric, $t(n) = 100\left(\frac{1}{10}\right)^n$
15. See dots on graph.
16. See circles on graph.



17. No, they do not. The graph is discrete, or just points, which are the terms of each sequence. Since they do not share a common point, they do not have any terms in common.
18. In 2008, $54(1.0005)^{460} \approx 67.96$ inches.
19. $t(n) = 0.4n + 5.4$, so solve $0.4n + 5.4 \geq 24.99$. $n = 48.975$. In 49 weeks he will have \$25. If he must also cover tax, he will need another three or four weeks.
20. This is a function, it represents an arithmetic sequence, the graph is discrete but the points are linear. The domain is the positive integers: 1, 2, 3, …. The range is the sequence itself: 14, 10, 6, 2, –2, …. There are no asymptotes.

21. 3, 8, 13, 18, 23

22. 2, –1, –4, –7, –10

23. $-14\frac{1}{2}, -14, -13\frac{1}{2}, -13, -12\frac{1}{2}$

24. 5, 8, 11, 14, 17

25. 5, 8, 11, 14, 17

26. 5, 2, –1, –4, –7

27. –3, 3, 9, 15, 21

28. $\frac{1}{3}, \frac{5}{6}, 1\frac{1}{3}, 1\frac{5}{6}, 2\frac{1}{3}$

29. 148

30. 0

31. 48

32. –62

33. $t(n) = 4n$; $t(1) = 4, t(n + 1) = t(n) + 4$

34. $t(n) = 7n - 9$; $t(1) = -2, t(n + 1) = t(n) + 7$

35. $t(n) = 39 - 12n$; $t(1) = 27$,
$t(n + 1) = t(n) - 12$

36. $a_n = \frac{1}{3}n + 2\frac{2}{3}$; $a_1 = 3, a_{n+1} = a_n + \frac{1}{3}$

37. Graph (21): (1, 3), (2, 8), (3, 13), (4, 18), (5, 23) slope = 5

Graph (22): (1, 2), (2, –1), (3, –4), (4, –7), (5, –10) slope = –3

38. The slope of the line containing the points is the same as the common difference of the sequence.

39. 10, 20, 40, 80, 160

40. –9, –27, –81, –243, –729

41. 40, 20, 10, 5, $\frac{5}{2}$

42. $6, -3, \frac{3}{2}, -\frac{3}{4}, \frac{3}{8}$

43. 5, 15, 45, 135, 405

44. 100, 50, 25, $\frac{25}{2}, \frac{25}{4}$

45. –3. 6, –12, 24, –48

46. $\frac{1}{3}, \frac{1}{6}, \frac{1}{12}, \frac{1}{24}, \frac{1}{48}$

47. 464

48. 16

49. 6

50. $\frac{27}{2}$

51. $t(n) = \frac{2}{5} \cdot 5^n$; $t(1) = 2, t(n + 1) = t(n) \cdot 5$

52. $t(n) = 64 \cdot \left(\frac{1}{4}\right)^n$; $t(1) = 16$,
$t(n + 1) = t(n) \cdot \frac{1}{4}$

53. $t(n) = \frac{5}{3} \cdot 3^n$; $t(1) = 5, t(n + 1) = t(n) \cdot 3$

54. $t(n) = \frac{-3}{2} \cdot (-2)^n$; $t(1) = 3$,
$t(n + 1) = t(n) \cdot (-2)$

55. Graph (39): (1, 10), (2, 20), (3, 40), (4, 80), (5, 160)

Graph (52): (1, 16), (2, 4), (3, 1), $(4, \frac{1}{4}), (5, \frac{1}{16})$

56. Arithmetic sequences are linear and geometric sequences are curved (exponential).

PATTERNS OF GROWTH IN TABLES AND GRAPHS 5.3.1

To recognize if a function is linear, exponential, or neither, look at the differences of the y-values between consecutive integer x-values. If the difference is constant, the graph is linear. If the difference is not constant, look at the pattern in the y-values. If a constant multiplier can be used to move from one y-value to the next, then the function is exponential. (Note that the same multiplier can be used to move from difference to difference in an exponential function.)

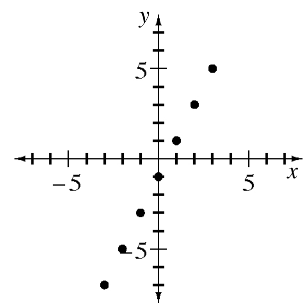
Examples

Based on each table, identify the shape of the graph.

Example 1

x	–3	–2	–1	0	1	2	3
y	–7	–5	–3	–1	1	3	5

2 2 2 2 2 2

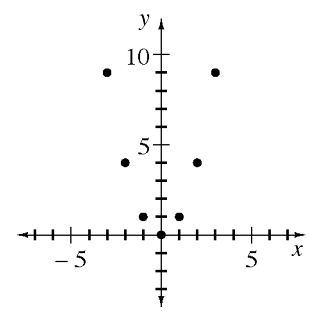


The difference in y-values is always two, a constant.
The function is linear; the graph at right confirms this.

Example 2

x	–3	–2	–1	0	1	2	3
y	9	4	1	0	1	4	9

–5 –3 –1 1 3 5

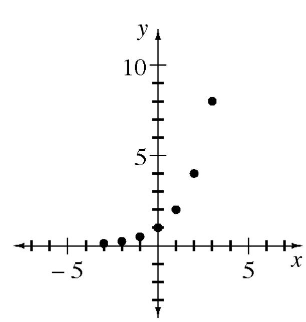


The first difference in y-values is not constant, and there is not a constant multiplier in moving from one y-value to the next. The function is neither linear nor exponential; the graph at right confirms this.

Example 3

x	–3	–2	–1	0	1	2	3
y	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8

$\frac{1}{8}$ $\frac{1}{4}$ $\frac{1}{2}$ 1 2 4



The y-values have a constant multiplier of 2. (Also the differences in y-values have a constant multiplier of 2.)
The function is exponential; the graph at right confirms this.

Problems

Based on the growth (the difference in y-values) shown in the tables, identify the corresponding graph as linear, exponential, or neither.

1.

x	–3	–2	–1	0	1	2	3
y	14	10	6	2	–2	–6	–10

2.

x	–3	–2	–1	0	1	2	3
y	$\frac{1}{2}$	1	2	4	8	16	32

3.

x	–3	–2	–1	0	1	2	3
y	21	12	5	0	–3	–4	–3

4.

x	–3	–2	–1	0	1	2	3
y	–16	–13	–10	–7	–4	–1	2

5.

x	–3	–2	–1	0	1	2	3
y	–14	–9	–4	1	6	11	16

6.

x	–3	–2	–1	0	1	2	3
y	–18	–6	–2	0	2	6	18

7.

x	–3	–2	–1	0	1	2	3
y	4	8	16	32	64	128	256

8.

x	–3	–2	–1	0	1	2	3
y	$\frac{1}{27}$	$\frac{1}{9}$	$\frac{1}{3}$	1	3	9	27

9.

x	–3	–2	–1	0	1	2	3
y	30	20	12	6	2	0	0

10.

x	–3	–2	–1	0	1	2	3
y	11	9	7	5	3	1	–1

11.

x	–3	–2	–1	0	1	2	3
y	$\frac{1}{9}$	$\frac{1}{3}$	1	3	9	27	81

12.

x	–3	–2	–1	0	1	2	3
y	–27	–9	–3	0	3	9	27

13.

x	–3	–2	–1	0	1	2	3
y	0	5	8	9	8	5	0

14.

x	–3	–2	–1	0	1	2	3
y	3	0	–1	0	3	8	15

15.

x	–3	–2	–1	0	1	2	3
y	1	0	–1	–2	–1	0	1

16.

x	–3	–2	–1	0	1	2	3
y	$\frac{9}{8}$	$\frac{9}{4}$	$\frac{9}{2}$	9	18	36	72

Answers

1. linear
2. exponential
3. neither
4. linear
5. linear
6. neither
7. exponential
8. exponential
9. neither
10. linear
11. exponential
12. neither
13. neither
14. neither
15. neither
16. exponential

LINE OF BEST FIT 6.1.1

The line of best fit can model an association between two variables. In this lesson, lines of best fit are estimated by sketching them on a scatterplot. Technology is used in lessons later in this chapter.

Once the line of best fit is established, a slope and y-intercept can be established. In statistical analyses, the slope is often written as the amount of change one could expect in the dependent variable (Δy), when the independent variable is changed by one unit ($\Delta x = 1$). The y-intercept is the predicted value of the dependent variable when the independent variable is zero. Be careful. In statistical scatterplots, the vertical axis is often not drawn at the origin, so the y-intercept can be someplace other than where the line of best fit crosses the vertical axis of the scatterplot.

Associations between two variables are often described by their form, direction, strength of association, and outliers. Form refers to the shape of the pattern in the scatterplot: linear, some type of curve, or perhaps no pattern at all. Direction refers to whether the pattern is in general increasing (from left to right) or decreasing. For linear relationships, the slope can be used to describe the steepness in addition to the direction. The strength of the relationship indicates how closely the data conforms to the established form. Outliers are data points far removed from the bulk of the data.

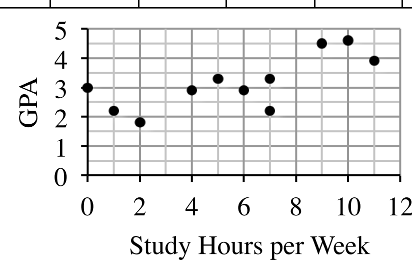
For additional information, see the Math Notes boxes in Lessons 4.1.1, 4.2.1, and 6.1.2, and the narrative in Checkpoint 8 at the back of the textbook. (Note that the problems below do not use a graphing calculator, while the problems in Checkpoint 8 assume a graphing calculator is available. Graphing calculators are introduced for statistical calculations in Lesson 6.1.4.)

Example

It seems reasonable that there would be a relationship between the amount of time a student spends studying and their GPA. Suppose you were interested in predicting a student's GPA based on the hours they study per week. You were able to randomly select 12 students and obtain this information from each student.

Hours of Study	4	5	11	1	15	2	10	6	7	0	7	9
GPA	2.9	3.3	3.9	2.2	4.1	1.8	4.6	2.9	2.2	3	3.3	4.5

a. Without a calculator, make a scatterplot.

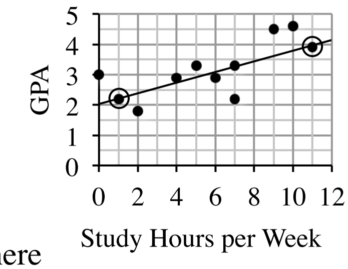


Example continues on next page →

Example continued from previous page.

b. Estimate a line of best fit, draw it with a ruler, and determine its equation.

If it is reasonable, draw the line of best fit through a lattice point (where grid lines intersect) or a data point. That way, the equation will be easier to write. In this problem, the data points (1, 2.2) and (11, 3.9) seem to be on a line that "fits" the data.



Using those two points, and techniques from Lesson 2.3.2, the equation for the estimated line of best fit is $y = 2.03 + 0.17x$, where y is the predicted GPA and x is the number of hours per week the student studies. Note that the variables and their units were defined.

c. Interpret the slope and y-intercept in terms of study hours and GPA.

The slope indicates that a student's GPA is expected to increase by 0.17 points for every additional hour of studying per week. The y-intercept predicts that students who do no studying at all have a GPA of 2.03.

d. Describe the association.

To describe an association means to describe the form, direction, strength, and outliers. The form looks to be linear; it does not appear to be curved nor simply a collection of randomly scattered points. The direction is positive; as we find students who study more, we also find higher GPAs. A student's GPA is expected to increase by 0.17 points for every additional hour of studying per week. The strength is moderate: it is strong enough to easily see its form, but there is scatter about the line. There do not appear to be any outliers.

In summary one could say there is a moderately strong linear relationship between study hours and GPA for students with no outliers in our data.

Problems

1. It seems reasonable that the power of a car is related to its gas mileage. Suppose a random sample of 10 car models is selected and the engine power and city gas mileage is recorded for each one.

Power (hp)	197	170	166	230	381	438	170	326	451	290
Mileage (mpg)	16	24	19	15	13	20	21	11	10	15

a. Create a scatterplot of the data.

b. Estimate a line of best fit and determine its equation.

c. Interpret the slope and y-intercept in the context of horsepower and mpg.

d. Describe the association.

2. Many people believe that students who are strong in music are also strong in mathematics. But the principal at University High School wonders if that same connection exists between music students and English students. The principal went through the records for the past year and found 10 students who were enrolled in both Advanced Placement Music and Advanced Placement English. He compared their final exam scores.

   a. Create a scatterplot of the data.

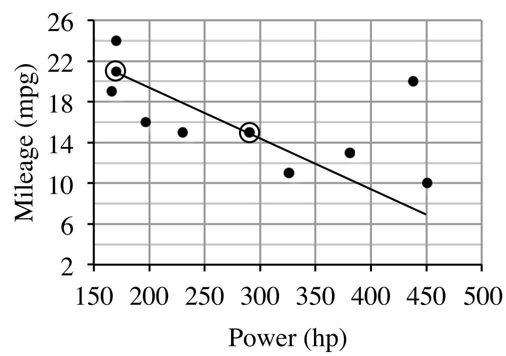
   b. Estimate a line of best fit and determine its equation.

   c. Interpret the slope and y-intercept in context.

   d. Describe the association.

Final Exam Scores	
AP Music	AP English
88	63
74	96
82	86
64	90
97	68
90	90
82	78
72	74
78	96
62	79
checksum 789	*checksum 820*

Answers



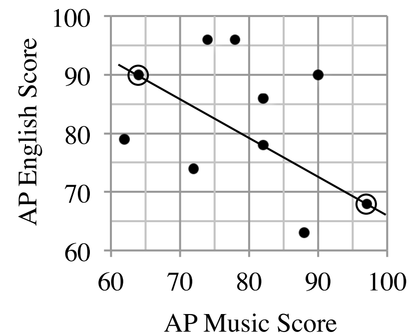
1. a. See plot at right.

   b. Answers can vary. If you ignore the possible outlier at (438, 20), a reasonable line could pass through the points (170, 21) and (290, 15). From techniques in Lesson 2.3.2, the line of best fit has equation $y = 29.5 - 0.05x$.

   c. For every increase of one horsepower, gas mileage is expected to decrease by 0.05 mpg. The y-intercept means that a 0 horsepower car would get 29.5 mpg, which does not make sense. The y-axis is far outside of the data, so it represents an extrapolation. Prediction models of all types are unreliable when you extrapolate them.

   d. When the outlier at (438, 20) is removed, there appears to be a strong, negative, linear relationship. If the outlier is not removed, the association is more moderate (not as strong). For every increase of one horsepower, gas mileage is expected to decrease by about 0.05 mpg.

2. a. See plot at right.



b. Answers can vary. A reasonable line could pass through the points (64, 90) and (97, 68). From techniques in Lesson 2.3.2, the line of best fit has equation $y = 132.7 - 0.67x$.

c. An increase of 1 point for the music score results in a predicted decrease of 0.67 points for the English score. The *y*-intercept would mean that a student who scored 0 on the music test is expected to score 133 on the English test. This does not make sense in the given context. The *y*-axis is far outside of the data and so represents an extrapolation. Prediction models of all types are unreliable when you extrapolate them.

d. There is a lot of scatter around the line of best fit. The linear association is weak. An increase of 1 point on the music test results in a predicted decrease of 0.67 points on the English test. There are no apparent outliers.

RESIDUALS AND UPPER AND LOWER BOUNDS 6.1.2 and 6.1.3

Residuals are a measure of how far the actual data points are from the line of best fit. Residuals are measured in the vertical (y) direction from the data point to the line. A single residual is calculated by:

residual = actual y-value – expected y-value.

A positive residual means that the actual value is greater than the predicted value; a negative residual means that the actual value is less than the predicted value.

A prediction made from a line of best fit gives no indication of the variability in the original data. Upper and lower boundary lines are parallel lines above and below the line of best fit. They give upper and lower limits (a "margin of error") to predictions made by the line of best fit. For example, predicting a test score of 87 is useful. But a prediction of 87 ± 1 is very different from a prediction of 87 ± 10. The upper and lower boundary lines help us put limits like these on predictions.

The most commonly used techniques for finding upper and lower boundary lines are beyond the scope of this course. However, the bounds can be reasonably approximated by finding the residual with the greatest distance, and then adding and subtracting that distance from the prediction.

For additional information, see the Math Notes box in Lesson 6.1.4. For additional examples and more practice on the topics from this chapter, see the Checkpoint 8 materials. Note that the problems below do not use a graphing calculator, while the problems in Checkpoint 8 assume a graphing calculator is available. Graphing calculators are introduced for statistical calculations in Lesson 6.1.4.

Example

It seems reasonable that there would be a relationship between the amount of time a student spends studying and their GPA. Suppose you were interested in predicting a student's GPA based on the hours they study per week. You were able to randomly select 12 students and obtain this information from each student. An estimated line of best fit could be: $y = 2.03 + 0.17x$. (Note that this is the same data as in the Example in Lesson 6.1.1 of this *Parent Guide with Extra Practice*.)

Hours	4	5	11	1	15	2	10	6	7	0	7	9
GPA	2.9	3.3	3.9	2.2	4.1	1.8	4.6	2.9	2.2	3	3.3	4.5

a. Find the residual for the student who studied 10 hours, and interpret it in context. Would a student prefer a positive or negative residual?

Example continues on next page →

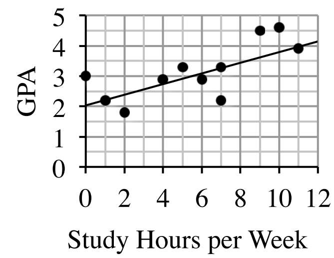
Example continued from previous page.

From the best-fit line, a student who studies 10 hours is predicted to earn a GPA of $y = 2.03 + 0.17(10) = 3.73$. But this student actually received a 4.6 GPA. The residual is:

$$\text{residual} = \text{actual } y\text{-value} - \text{expected } y\text{-value}$$
$$\text{residual} = 4.6 - 3.73 = 0.87$$

The student who studied 10 hours earned a GPA that was 0.87 points higher than predicted. A student would prefer a positive residual because it means that he/she earned a higher GPA than was predicted from the number of hours spent studying.

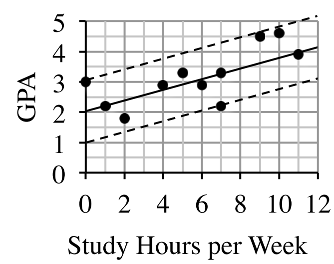
b. Write the equations for the upper and lower boundary lines, and show the boundary lines on a scatterplot.



Looking at the scatterplot, the largest residual (the largest vertical distance, Δy) seems to be at the point (7, 2.2). The predicted GPA for a student who studies 7 hours is $y = 2.03 + 0.17(7) = 3.22$. The residual for this point is:

$$\text{residual} = \text{actual } y\text{-value} - \text{expected } y\text{-value}$$
$$\text{residual} = 2.2 - 3.22 = -1.02$$

The bounds are lines parallel to the best-fit line, but a distance of 1.02 above and below it. Parallel lines have the same slope as the best-fit line. But the y-intercept of the boundary lines will be 1.02 more or less than the y-intercept of the best-fit lines.



Therefore, the upper boundary line is:
$y = (2.03 + 1.02) + 0.17x$ or $y = 3.05 + 0.17x$
and the lower boundary line is:
$y = (2.03 - 1.02) + 0.17x$ or $y = 1.01 + 0.17x$.

c. Predict the upper and lower bound of the GPA for a student who studies 8 hours per week. Is your prediction useful?

The lower bound is $y = 1.01 + 0.17(8) = 2.37$, and the upper bound is $y = 3.03 + 0.17(8) = 4.41$. We predict that a student who studies 8 hours per week has a GPA between 2.37 and 4.41. Due to the large amount of variability in the data collected, this is a large range. The prediction is not particularly useful.

Problems

1. It seems reasonable that the power of a car is related to its gas mileage. Suppose a random sample of 10 car models is selected and the engine power and city gas mileage is recorded for each one. An outlier was removed and a line of best fit was estimated to be $y = 29.5 - 0.05x$. (Note that this is the same data as in Problem 1 in Lesson 6.1.1 of this *Parent Guide with Extra Practice*.)

Power (hp)	197	170	166	230	381	170	326	451	290
Mileage (mpg)	16	24	19	15	13	21	11	10	15

   a. Find the residual for the vehicle that had 381 horsepower, and interpret it in context. Would a car owner prefer a positive or negative residual?

   b. Remove the outlier from your data set. Write the equations for the upper and lower boundary lines, and show the boundary lines on a scatterplot.

   c. Predict the upper and lower bound of the gas mileage for a car with 300 horsepower. Is your prediction useful?

2. Many people believe that students who are strong in music are also strong in mathematics. But the principal at University High School wonders if that same connection exists between music students and English students. The principal went through the records for the past year and found 10 students who were enrolled in both Advanced Placement Music and Advanced Placement English. He compared their final exam scores. A line of best fit was estimated to be $y = 132.7 - 0.67x$. (Note that this is the same data as in Problem 2 in Lesson 6.1.1 of this *Parent Guide with Extra Practice*.)

Final Exam Scores	
AP Music	AP English
88	63
74	96
82	86
64	90
97	68
90	90
82	78
72	74
78	96
62	79
checksum 789	*checksum 820*

   a. The principal checked the records of a student who just entered the school. She had a perfect score of 100 on the English final and her residual was 30 points. What was her predicted English score? What was her music score?

   b. Write the equations for the upper and lower boundary lines, and show the boundary lines on a scatterplot. Do not add the new student from part (a) to your scatterplot.

   c. Predict the upper and lower bound of the AP English score for a student with a perfect score of 100 on the music final. Is your prediction useful?

Answers

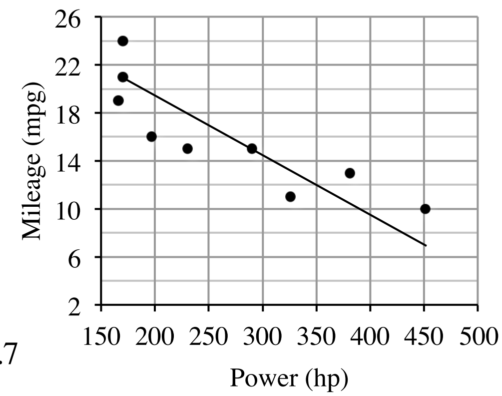
1. a. From the best-fit line, a vehicle with 381 horsepower is predicted to have a mileage of $y = 29.5 - 0.05(381) \approx 10.5$. But this vehicle actually got 13 mpg. The residual is:

$$\text{residual} = \text{actual } y\text{-value} - \text{expected } y\text{-value}$$
$$\text{residual} = 13 - 10.5 = 2.5$$

The vehicle with 381 horsepower got 2.5 mpg more than was predicted. A car owner would prefer a positive residual—a positive residual means the car is getting more miles per gallon than was predicted from its power.

b. Looking at the scatterplot without the outlier, the largest residual (the largest vertical distance, Δy) seems to be at the point (197, 16). The predicted mileage for a car with 197 horsepower is $y = 29.5 - 0.05(197) = 19.7$ mpg. The residual for this point is:

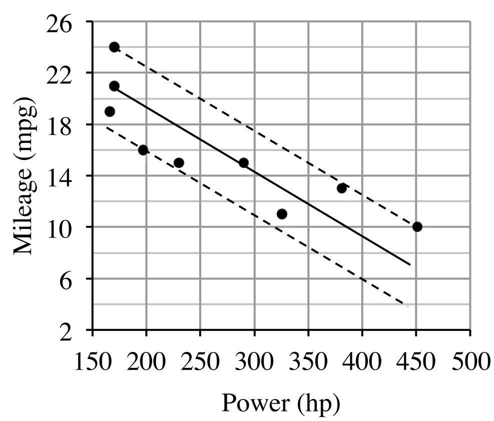
$$\text{residual} = \text{actual } y\text{-value} - \text{expected } y\text{-value}$$
$$\text{residual} = 16 - 19.7 = -3.7$$



The bounds are lines parallel to the best-fit line, but a distance of 3.7 above it and below it. Parallel lines have the same slope as the best-fit line. But the y-intercept of the boundary lines will be 3.7 more or less than the y-intercept of the best-fit lines.

Therefore, the upper boundary line is: $y = (29.5 + 3.7) - 0.05x$ or $y = 33.2 - 0.05x$, and the lower boundary line is: $y = (29.5 - 3.7) - 0.05x$ or $y = 25.8 - 0.05x$.

c. The upper bound is $y = 33.2 - 0.05(300) = 18.2$, and the lower bound is $y = 25.8 - 0.05(300) = 10.8$. We predict that a car with 300 horsepower will have gas mileage between 10.8 and 18.2 mpg. The prediction could be useful: it indicates that this is a car with low gas mileage.



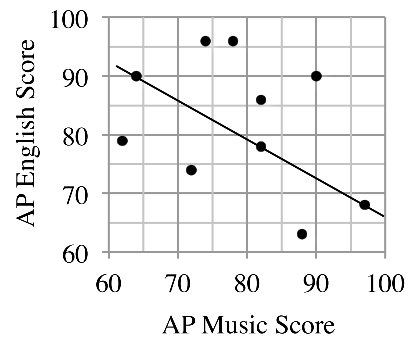
2. a. residual = actual y-value − expected y-value

$30 = 100 -$ expected y-value

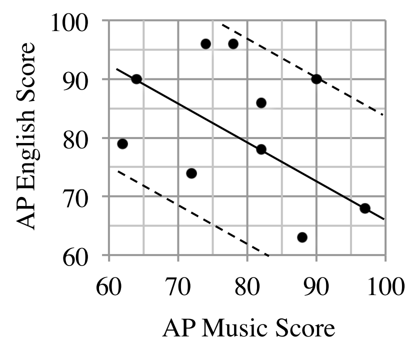
expected y-value $= 100 - 30 = 70$

From the best-fit line we can find the music score: $70 = 132.7 - 0.67x$ or $x \approx 94$. Her predicted English score was 70 and her music score was 94.

b. Looking at the scatterplot, the largest residual (the largest vertical distance, Δy) seems to be at the point (90, 90). The predicted English score for a student with 90 on the music test is $y = 132.7 - 0.67(90) = 72.4$. The residual for this point is $90 - 72.4 = 17.6$.
Therefore, the upper boundary line is:
$y = (132.7 + 17.6) - 0.67x$ or $y = 150.3 - 0.67x$,
and the lower boundary line is:
$y = (132.7 - 17.6) - 0.67x$ or $y = 115.1 - 0.67x$.



c. The upper bound is $y = 150.3 - 0.67(100) = 83.3$, and the lower bound is $y = 115.1 - 0.67(100) = 48.1$. We predict that a student who scores 100 on the AP Music final exam will earn a score between 48 and 83 on the AP English final exam. This range is much too large to be useful. There is a lot of variability in the data the principal collected, making the margin of error on the prediction very large.



LEAST SQUARES REGRESSION LINE 6.1.4

A **least squares regression line** (LSRL) is a line of best fit that minimizes the sum of the square of the residuals.

For additional information, see the Math Notes box in Lesson 6.2.1 and the narrative in Checkpoint 8 at the back of the textbook. Also see problem 6-34 in the textbook for a demonstration of finding the least squares regression line. For additional examples and more practice on the topics from this chapter, see the Checkpoint 8 problems.

CALCULATORS
Statistical calculators or software can find the LSRL and residuals with ease. Students are welcome to use a TI-83/84+ calculator if they have one available. See details about using calculators in the Introduction section of this *Parent Guide with Extra Practice*.

An eTool is also available in your eBook and at studenthelp.cpm.org for students who do not have a graphing calculator.

TI-83/84+ NOTES
More detailed instructions for using a TI-83/84+ calculator can be found in your eBook and at studenthelp.cpm.org. Basic instructions follow.

For part (a) of the example:

- Enter the data points in two lists.
- For data tables in which a *checksum* is given, the *checksum* is used to verify that data has been entered into the calculator correctly. Use 1-Var Stats and verify that Σx on the calculator screen matches the "checksum" given at the bottom of the table in certain problems. Σx stands for "the sum of *x*" or the sum of all the values in the list.
- Use 1-Var Stats to find the maximum and the minimum of both lists, then set the window appropriately.
- Use [2nd] [STAT PLOT] to set up a graph, and then press [GRAPH].

For part (b) of the example:

- Find the LSRL with [STAT] "CALC" "8:LinReg(a+bx)" [2nd] [L1] [,] [2nd] [L2] [,] [VARS] "Y-VARS" "1:Function" "1:Y_1" [ENTER].
- Use [2nd] [WINDOW] to set the scale of the graph, then press [GRAPH].

For part (c) of the example:

Each time you perform a LSRL regression, the calculator will automatically store the residuals in a list named "RESID". You can display the residuals in List 3 as follows.

- Highlight the label for List 3 and press [ENTER].
- Press [2nd] [LIST] "RESID" and then [ENTER] to copy the list.

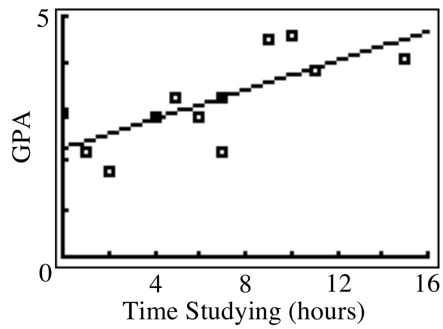
Example

It seems reasonable that there would be a relationship between the amount of time a student spends studying and their GPA. Suppose you were interested in predicting a student's GPA based on the hours they study per week. You were able to randomly select 12 students and obtain this information from each student.

Hours	4	5	11	1	15	2	10	6	7	0	7	9
GPA	2.9	3.3	3.9	2.2	4.1	1.8	4.6	2.9	2.2	3	3.3	4.5

a. Use your calculator to graph the scatterplot. Choose a reasonable viewing window on your calculator, then make a sketch of the scatterplot on your paper and label the axes.

Since the "hours" range from 0 to 15, an appropriate window might have an x-axis scale from 0 to 16 with an interval of 2. The GPA data ranges from 1.8 to 4.6, so an appropriate y-axis might range from 0 to 5 with an interval of 1. Note that the axes are labeled with both the variable name and with the units.

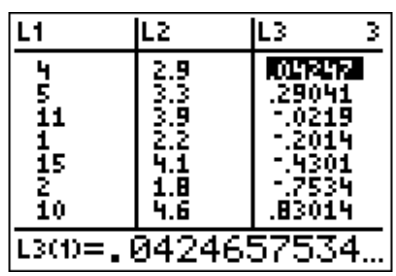


b. Use your calculator to find the equation of the LSRL.

From the calculator, $y = 2.249 + 0.152x$. It is essential when giving an LSRL equation to define the variables and units. In this case, x is the numbers of hours spent studying each week, and y is the predicted GPA for that student.

c. Use your calculator to create a residual list. What is the residual for the student who spent 10 hours studying? Interpret the residual in context.

A partial residual follows (in List L3):



From the residual list, the residual for the student who studied 10 hours is 0.83 grade points. That means the student earned a GPA that was 0.83 points higher than was predicted by the least squares regression line.

Problems

1. It seems reasonable that the power of a car is related to its gas mileage. Suppose a random sample of 9 car models is selected and the engine power and city gas mileage is recorded for each one.

Power (hp)	197	170	166	230	381	170	326	451	290
Mileage (mpg)	16	24	19	15	13	21	11	10	15

   a. Use your calculator to graph the scatterplot. Choose a reasonable viewing window on your calculator, then make a sketch of the scatterplot on your paper and label the axes.

   b. Use your calculator to find the equation of the LSRL. What do you predict the gas mileage will be for a car with 400 horsepower?

   c. Use your calculator to create a residual list. What is the residual for the 230 horsepower car?

2. Many people believe that students who are strong in music are also strong in mathematics. But the principal at University High School wonders if that same connection exists between music students and English students. The principal went through the records for the past year and found 10 students who were enrolled in both Advanced Placement Music and Advanced Placement English. He compared their final exam scores.

Final Exam Scores	
AP Music	AP English
88	63
74	96
82	86
64	90
97	68
90	90
82	78
72	74
78	96
62	79
checksum 789	*checksum 820*

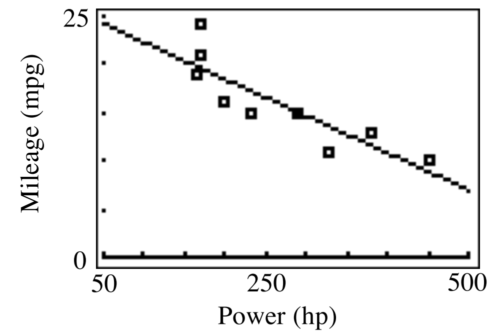
   a. Use your calculator to graph the scatterplot. Choose a reasonable viewing window on your calculator, then make a sketch of the scatterplot on your paper and label the axes. Did you remember to use the *checksum* to verify you entered the data correctly?

   b. Use your calculator to find the equation of the LSRL.

   c. Use your calculator to create a residual list. Would a student prefer a positive or a negative residual in this situation?

Answers

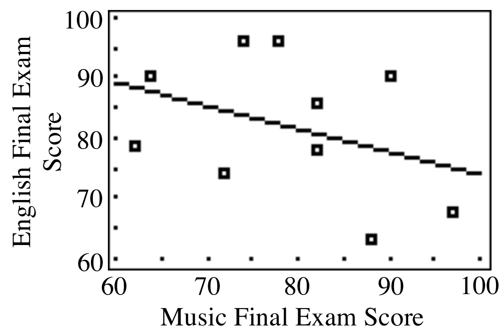
1. a. See scatterplot at right.



   b. $y = 26.11 - 0.0382x$. y is the predicted gas mileage (in mpg) and x is the power (in hp). A car with 400 horsepower is predicted to have gas mileage of $26.11 - 0.0382(400) = 10.8$ mpg.

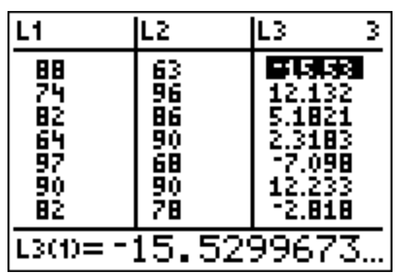
   c. The car with 230 horsepower had residual of –2.3 mpg. That means the car had gas mileage that was 2.3 mpg less than was predicted by the LSRL.

2. a. See scatterplot at right.



   b. $y = 112.1 - 0.3813x$ where y is the predicted AP English final exam score, and x is the AP Music final exam score.

   c. Residual list follows in List L3 below. A student would prefer a positive residual. That means their AP English score is higher than predicted by the LSRL.



L1	L2	L3 3
88	63	-15.53
74	96	12.132
82	86	5.1821
64	90	2.3183
97	68	-7.098
90	90	12.233
82	78	-2.818

L3(1)=-15.5299673…

RESIDUAL PLOTS AND CORRELATION 6.2.1, 6.2.2, and 6.2.4

A **residual plot** is created in order to analyze the appropriateness of a best-fit model. See the Math Notes box in Lesson 6.2.3 for more on residual plots.

The **correlation coefficient**, r, is a measure of how much or how little data is scattered around the LSRL; it is a measure of the strength of a linear association. R-squared can be interpreted as the percentage of the change in the dependent variable (y) which is accounted for or explained by the change in the independent variable (x). See the Math Notes box in Lesson 6.2.4.

For additional information, see the narrative in Checkpoint 8 at the back of the textbook. For additional examples and more practice on the topics from this chapter, see the Checkpoint 8 problems.

CALCULATORS
Statistical calculators or software can make statistical computations with ease. See details about using calculators in the Introduction section of this *Parent Guide with Extra Practice*. See the previous section for additional information about using a TI-83/84+ calculator. More detailed instructions for using a TI-83+/84+ calculator can be found in your eBook or at studenthelp.cpm.org.

When you calculate the LSRL, the calculator reports the correlation coefficient on the same screen as it reports the slope and y-intercept. If your TI calculator does not calculate r, press [2nd] [CATALOG] "DiagnosticOn" [ENTER] [ENTER] and try again.

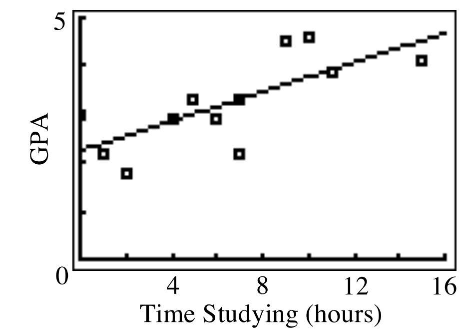
Example

It seems reasonable that there would be a relationship between the amount of time a student spends studying and their GPA. Suppose you were interested in predicting a student's GPA based on the hours they study per week. You were able to randomly select 12 students and obtain this information from each student.

Hours	4	5	11	1	15	2	10	6	7	0	7	9
GPA	2.9	3.3	3.9	2.2	4.1	1.8	4.6	2.9	2.2	3	3.3	4.5

a. Graph the data and the LSRL.

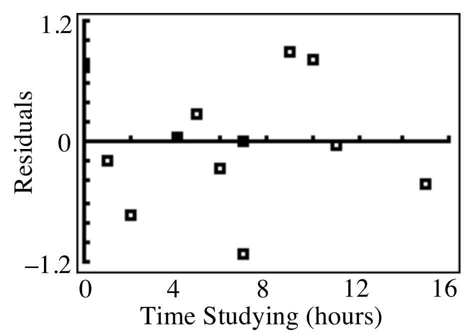
See scatterplot at right.



b. Create and interpret a residual plot.

See residual plot below right.

There is no apparent pattern in the residual plot—it looks like randomly scattered points—which means a linear model (instead of a curve) is the most appropriate way to model the relationship.



c. Find the coefficient of correlation r and R-squared. Interpret the meaning of R-squared in context.

From the calculator, $r \approx 0.7338$, R-squared $\approx 53.8\%$. About 54% of the variability in GPA can be explained by a linear relationship with study hours per week.

d. Fully describe the association between GPA and hours studied.

For a description of how to fully describe an association, see the narrative in Checkpoint 8 at the back of the textbook. *When describing an association, the form direction, strength, and outliers should be described.*

From the calculator, the least squares regression line is $y = 2.249 + 0.152x$, where y is the predicted GPA, and x is the number of hours studied.

The *form* in the scatterplot appears to be linear; it does not appear to be curved nor simply a collection of randomly scattered points. The residual plot shows random scatter, confirming that a linear model was an appropriate choice.

The *direction* is positive; as we find students who study more, we also find higher GPAs. From the slope, a student's GPA is expected to increase by 0.15 points for every additional hour of studying per week.

Example continues on next page →

Example continued from previous page.

The *strength* is moderate: it is strong enough to easily see its form, but there is scatter about the line. The correlation coefficient is 0.73, confirming a moderately strong linear association. From *R*-squared, 54% of the variability in GPA can be explained by the variability in study hours per week.

There are no apparent *outliers*.

Problems

1. It seems reasonable that the power of a car is related to its gas mileage. Suppose a random sample of 9 car models is selected and the engine power and city gas mileage is recorded for each one.

Power (hp)	197	170	166	230	381	170	326	451	290
Mileage (mpg)	16	24	19	15	13	21	11	10	15

   a. Graph the data and the LSRL.

   b. Create and interpret a residual plot.

   c. Find the coefficient of correlation *r* and *R*-squared. Interpret the meaning of *R*-squared in context.

   d. Fully describe the association.

2. Many people believe that students who are strong in music are also strong in mathematics. But the principal at University High School wonders if that same connection exists between music students and English students. The principal went through the records for the past year and found 10 students who were enrolled in both Advanced Placement Music and Advanced Placement English. He compared their final exam scores.

   a. Graph the data and the LSRL.

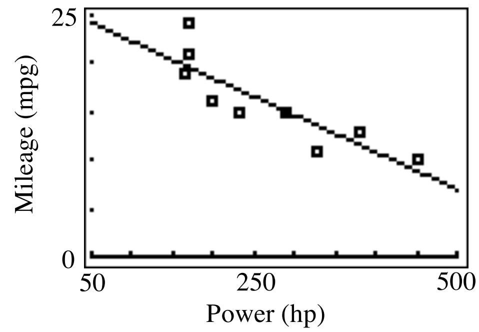
   b. Create and interpret a residual plot.

   c. Find the coefficient of correlation *r* and *R*-squared. Interpret the meaning of *R*-squared in context.

   d. Fully describe the association.

Final Exam Scores	
AP Music	AP English
88	63
74	96
82	86
64	90
97	68
90	90
82	78
72	74
78	96
62	79
checksum 789	*checksum 820*

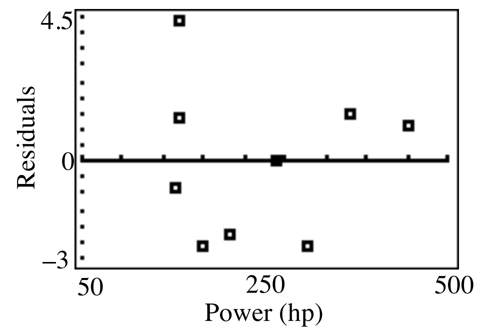
Answers



1. a. See scatterplot at right.

   b. See residual plot below right.

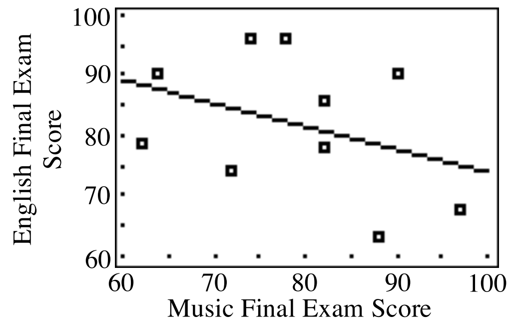
   There is an apparent U-shaped pattern in the residual plot, which suggests a curved model (instead of a line) would be the most appropriate way to model the relationship. Nonetheless, by visual examination of the scatterplot and LSRL, a linear model is not completely inappropriate. We will continue with the analysis of the linear model.



   c. $r \approx -0.8602$, R-squared ≈ 0.7400. About 74% of the variability in gasoline mileage can be explained by a linear relationship with horsepower.

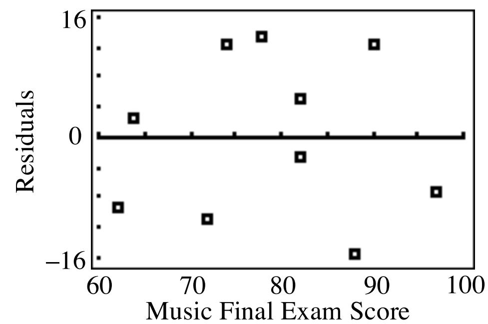
   d. From the calculator, the least squares regression line is $y = 26.11 - 0.0382x$, where y is the predicted mileage (in mpg) and x is the power (in hp). The residual plot indicates a curved model might fit the data better, but by observation of the scatterplot, the LSRL models the data well enough to proceed. The association is negative: the model predicts that for every increase of one horsepower, the mileage is expected to decrease by 0.04 mpg. The association is fairly strong with a correlation coefficient of nearly –0.9. About 74% of the variability in gasoline mileage can be explained by a linear relationship with power. There are no apparent outliers. (However, if you have been following this problem since Lesson 6.1.1 in this *Parent Guide with Extra Practice*, you will recall that the data set initially had an apparent outlier at 438 horsepower and 20 mpg. That point was not considered in creating the best-fit line, and subsequently the data point was dropped from the analysis. If that data point is included, the correlation coefficient will be closer to 0.)

2. a. See scatterplot at right.



b. See residual plot below right.

There is no apparent pattern in the residual plot—it looks like randomly scattered points—which means a linear model (instead of a curve) is the most appropriate way to model the relationship.



c. From the calculator, $r \approx -0.3733$, R-squared $\approx 13.9\%$. About 14% of the variability in AP English scores can be explained by a linear relationship with AP Music scores.

d. The LSRL is $y = 112.1 - 0.3813x$, where y is the predicted AP English score, and x is the AP Music score. The residual plot confirms that a linear model is appropriate. The association is negative: the model predicts that for every increase of one in the AP Music score, the AP English score will drop by 0.38 points. The association is very weak, with a correlation coefficient of approximately –0.37. Since R-squared is approximately 14%, about 14% of the variability in AP English scores can be explained by a linear relationship with AP Music scores. There are no apparent outliers in the data collected.

ASSOCIATION AND CAUSATION 6.2.3

Considering all of the statistical tools available to measure the correlation between two variables it is very tempting to believe that a strong **association** between two variables is evidence of **causation**. The fact that two variables move together in a predictable manner does not show that one causes the other. Cause can usually only be shown using carefully controlled experiments, and usually not by just observing the relationship between variables.

Example

A city is proposing a new bus route. A community activist who owns a residence along the route does some research which shows a strong association between the number of bus stops and the crime rate of a community. She presents her findings to the city council claiming, "Bus stops cause crime, so please vote no." Another community activist who owns a business along the proposed bus route also does research and finds that bus stops are highly associated with the number of restaurants in a neighborhood. He presents his findings to the city council claiming, "Bus stops increase restaurants. Please vote yes."

A city council member (who also teaches statistics) informs them that association is not the same as causation because of the many thousands of variables not included (lurking) in a study. The council member said, "Bus stops are generally an indication of large population centers so bus stops will be found along with other things associated with urban living. This does not mean that bus stops *cause* things found in cities."

a. List three desirable variables that are likely associated with bus stops but not necessarily caused by them.

   Numbers of Museums, Theaters, Stadiums.

b. List three undesirable variables that are likely associated with bus stops but not necessarily caused by them.

   Quantity of litter, air pollution, noise.

Problems

For each faux news headline, accept that the association is true and indicate a hidden or lurking variable that may be the actual cause of the relationship.

a. "BOOKS IN THE HOME ASSOCIATED WITH HIGHER TEST SCORES"
"Students buying truckloads of books before taking national exams."

b. "CHESS CLUB BOASTS HIGHEST IVY LEAGUE ADMISSION RATES"
"Club president now flooded with membership applications."

c. "STUDENTS WHO LIVE OFF CAMPUS AT HIGHER RISK FOR AUTO ACCIDENTS"
"State Police Warn: Play it safe, live on campus."

Answers

a. A person who does a lot of reading would likely have a larger number of books in their home and higher test scores. So reading regularly is likely the cause. Just buying books does not cause higher test scores.

b. Chess is game of strategy and an extracurricular activity. Those who play regularly may be more inclined to apply to Ivy League schools than non-chess players.

c. Students who live off campus probably commute to school and may spend more time driving to and from school than students who live on campus. The additional hours on the road may be the cause of more accidents.

CURVED BEST-FIT MODELS 6.2.5

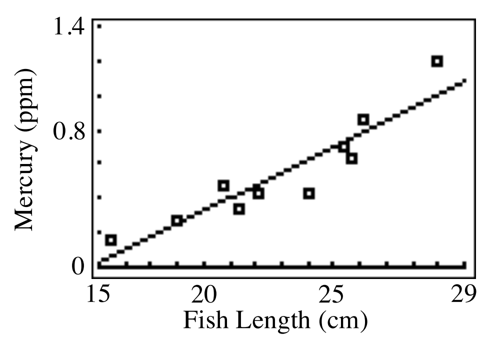
Many relationships are modeled better by a curve than by a line. For additional information, see the Math Notes box in Lesson 6.2.5.

Example

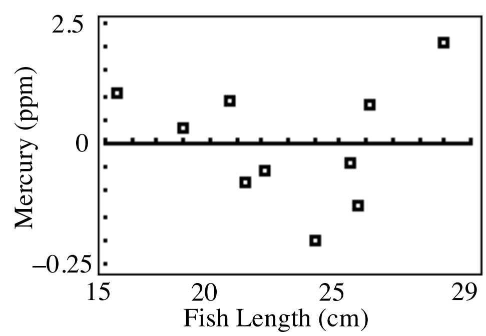
Methyl mercury is a neurotoxin found in many fish species. Plankton absorb mercury from contaminated runoff water, then small fish eat the plankton and the methyl mercury travels up the food web to larger predators. In general, the larger the fish, the higher the concentration of methyl mercury in its flesh. Assume a biologist is sampling white bass from a lake known to have mercury in the water in an effort to find a relationship between the length of the fish and the concentration of methyl mercury in the fish. The following data was collected from a sample of 10 white bass (ppm is "parts per million"):

Length, cm	24.4	23.0	21.1	24.7	25.1	20.3	18.0	27.9	19.8	15.4
Mercury, ppm	0.689	0.428	0.425	0.629	0.863	0.340	0.281	1.20	0.474	0.164

The LSRL was calculated and the following scatterplot and residual plots were made:



The curved pattern in the residual plot indicates that a non-linear model would be a better fit.

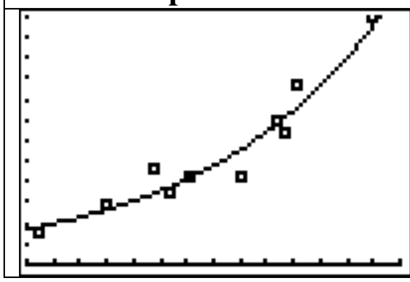
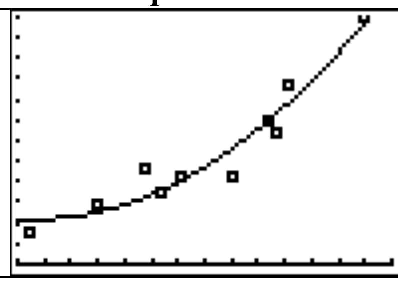
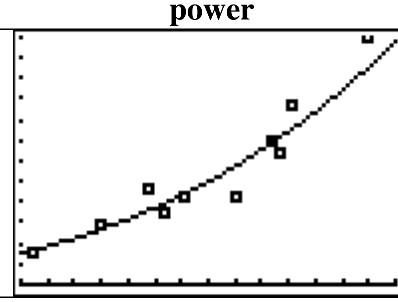


Example continues on next page →

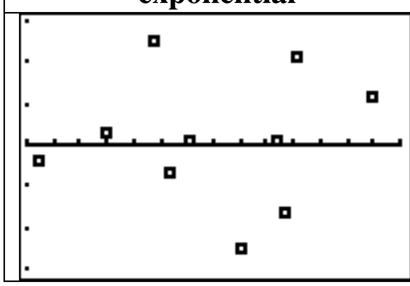
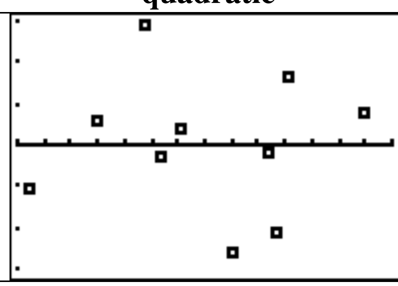
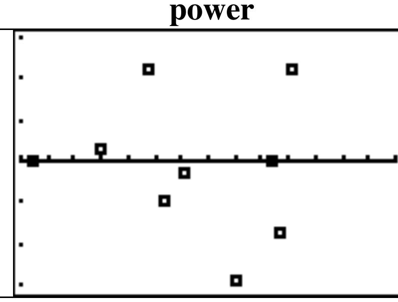
Example continued from previous page.

a. Perform exponential, quadratic, and power model regressions for the data and make the associated scatter and residual plots. These regressions are all available on the TI-83/84+ calculator on the same screen as the linear regression.

SCATTERPLOTS

exponential	**quadratic**	**power**
		

RESIDUAL PLOTS

exponential	**quadratic**	**power**
		

b. Use the residual plots to select the best non-linear model.

Try to find the model with the least patterning or the most “random”-looking scatter. The power model still has a U-shaped pattern; the power model is not a good choice.

Both the exponential and quadratic look fairly random; either would be an appropriate choice based on the plots. But successive growth like this—in this case, the buildup of mercury as the fish ages and thus gets longer—is more likely explained by an exponential relationship than a quadratic one. The exponential model makes a good choice to describe the association between length of the fish and the concentration of mercury.

c. Use your model to predict the amount of mercury in a 20-cm fish.

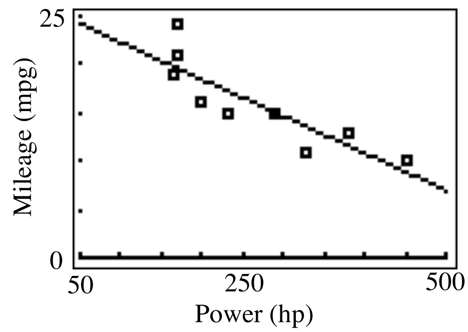
$y = 0.0187 \cdot 1.1588^{x}$ where x is the length of the fish in cm, and y is the concentration of mercury in parts per million. If $x = 20$ cm, then $y = 0.356$. A 20-cm fish is expected to contain 0.356 ppm of mercury.

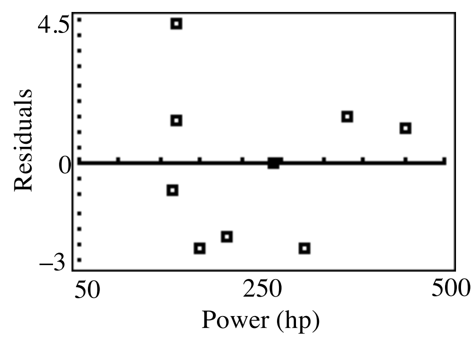
Problem

It seems reasonable that the power of a car is related to its gas mileage. Suppose a random sample of 9 car models is selected and the engine power and city gas mileage is recorded for each one.

Power (hp)	197	170	166	230	381	170	326	451	290
Mileage (mpg)	16	24	19	15	13	21	11	10	15

The LSRL was calculated and the following scatter and residual plots were made. (These plots were also created in Lesson 6.2.1 of this *Parent Guide with Extra Practice*.)





The curved pattern in the residual plot indicates that a non-linear model would be a better fit.

a. Perform exponential, quadratic, and power model regressions for the data and make the associated scatter and residual plots. These regressions are all available on the TI-83/84+ calculator on the same screen as the linear regression.

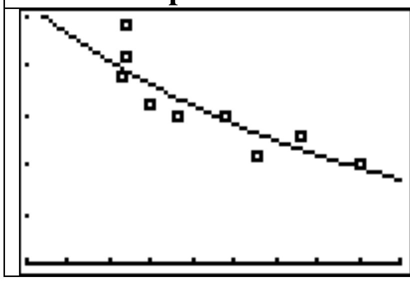
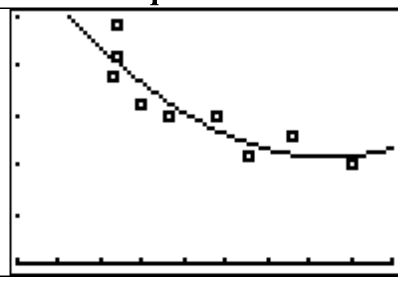
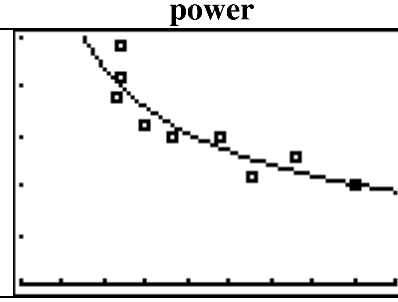
b. Use the residual plots to select the best non-linear model.

c. What do you predict the gas mileage will be for a car with 400 horsepower?

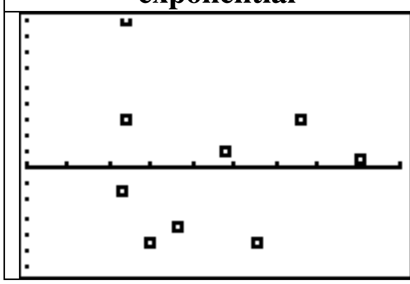
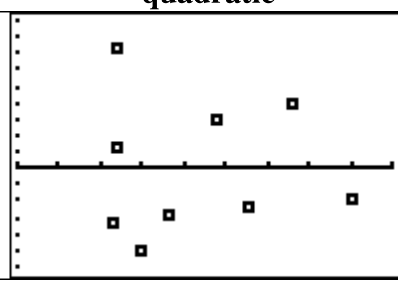
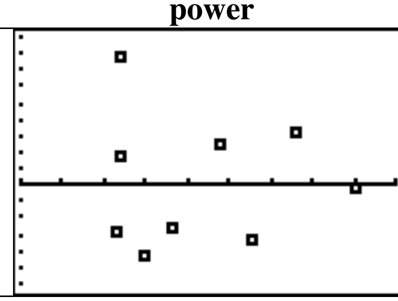
Answer

a.

SCATTERPLOTS

exponential	**quadratic**	**power**
		

RESIDUAL PLOTS

exponential	**quadratic**	**power**
		

b. The exponential residual plot still seems a little U-shaped. Both the quadratic and power models appear to have more randomly scattered residual plots. However we have some very serious concerns about the quadratic model: the quadratic model turns back up, indicating that for slightly higher power the gas mileage increases. That makes no sense. We will use the power model to represent this data.

c. For the power model, $y = 732.94 \cdot x^{-0.7004}$, where x is the power (in hp) and y is the gas mileage (in mpg). If $x = 400$, then $y = 11.03$. We predict that a car with 400 horsepower will get about 11 miles per gallon.

EXPONENTIAL FUNCTIONS 7.1.1 through 7.1.6

In these sections, students generalize what they have learned about geometric sequences to investigate exponential functions. Students study exponential functions of the form $y = ab^x$. Students look at multiple representations of exponential functions, including graphs, tables, equations, and context. They learn how to move from one representation to another. Students learn that the value of a is the "starting value" of the function—a is the y-intercept or the value of the function at $x = 0$. b is the growth (multiplier). If $b > 1$ then the function increases; if b is a fraction between 0 and 1 (that is, $0 < b < 1$), then the function decreases (decays). In this course, values of $b < 0$ are not considered.

For additional information, see the Math Notes boxes in Lesson 7.1.3 and 7.2.3. For additional examples and more practice, see the Checkpoint 9 and Checkpoint 10A materials at the back of the student textbook.

Example 1

LuAnn has $500 with which to open a savings account. She can open an account at Fredrico's Bank, which pays 7% interest, compounded monthly, or Money First Bank, which pays 7.25%, compounded quarterly. LuAnn plans to leave the money in the account, untouched, for ten years. In which account should she place the money? Justify your answer.

Solution: The obvious answer is that she should put the money in the account that will pay her the most interest over the ten years, but which bank is that? At both banks the principal (the initial value) is $500. Fredrico's Bank pays 7% compounded monthly, which means the interest rate is $\frac{0.07}{12} \approx 0.00583$ each month. If LuAnn puts her money into Fredrico's Bank, after one month she will have:

$$500 + 500(0.00583) = 500(1.00583) \approx \$502.92.$$

To calculate the amount at the end of the second month, we must multiply by 1.00583 again, making the amount:

$$500(1.00583)^2 \approx \$505.85.$$

At the end of three months, the balance is:

$$500(1.00583)^3 \approx \$508.80.$$

This will happen every month for ten years, which is 120 months. At the end of 120 months, the balance will be:

$$500(1.00583)^{120} \approx \$1004.43.$$

Note that this last equation is an exponential function in the form $y = ab^x$, where y is the amount of money in the account and x is the number of months (in this case, 120 months). $a = 500$ is the starting value (at 0 months), and $b = 1.00583$ is the multiplier or growth rate for the account each month.

Example continues on next page →

Example continued from previous page.

A similar calculation is performed for Money First Bank. Its interest rate is higher, 7.25%, but it is only calculated and compounded quarterly. (Quarterly means four times each year, or every three months.) Hence, every quarter the bank calculates $\frac{0.0725}{4} = 0.018125$ interest. At the end of the first quarter, LuAnn would have:

$$500(1.018125) \approx \$509.06.$$

At the end of ten years (40 quarters) LuAnn would have:

$$500(1.018125)^{40} \approx \$1025.69.$$

Note that this last equation is an exponential function in the form $y = ab^x$, where y is the amount of money in the account and x is the number of quarters (in this case, 40 quarters). $a = 500$ is the starting value (at 0 quarters), and $b = 1.018125$ is the multiplier or growth rate for the account each quarter.

Since Money First Bank would pay her approximately \$21 more in interest than Fredrico's Bank, she should put her money in Money First Bank.

Example 2

Most homes appreciate in value, at varying rates, depending on the home's location, size, and other factors. But if a home is used as a rental, the Internal Revenue Service allows the owner to assume that it will depreciate in value. Suppose a house that costs \$150,000 is used as a rental property, and depreciates at a rate of 8% per year. What is the multiplier that will give the value of the house after one year? What is the value of the house after one year? What is the value after ten years? When will the house be worth half of its purchase price? Draw a graph of this situation.

Solution: Unlike interest, which increases the value of the house, depreciation takes value away. After one year, the value of the house is $150000 - 0.08(150000)$ which is the same as $150000(0.92)$. Therefore the multiplier is 0.92. After one year, the value of the house is $150000(0.92) = \$138{,}000$. After ten years, the value of the house will be $150000(0.92)^{10} \approx \$65{,}158.27$.

This last equation is an exponential function in the form $y = ab^x$, where y is the value of the house and x is the number of years. $a = 150000$ is the starting value (at 0 years), and $b = 0.92$ is the multiplier or growth factor (in this case, decay) each year.

To find when the house will be worth half of its purchase price, we need to determine when the value of the house reaches \$75,000. We just found that after ten years, the value is below \$75,000, so this situation occurs in less than ten years. To help answer this question, list the house's values in a table to see the depreciation.

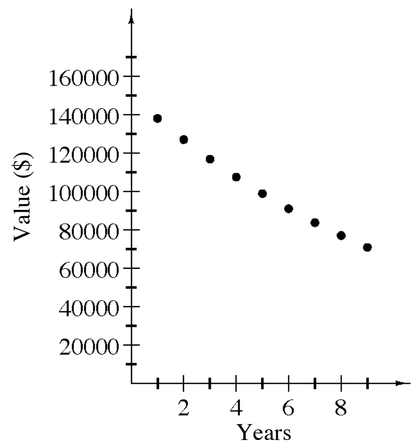
Example continues on next page →

Example continued from previous page.

From the table or graph, you can see that the house will be worth half its purchase price after 8 years.

Note: You can write the equation $75000 = 150000 \cdot 0.92^x$, but you will not have the mathematics to solve this equation for x until a future course. However, you can use the equation to find a more exact value: try different values for x in the equation, so the y-value gets closer and closer to \$75,000. At about 8.313 years the house's value is close to \$75,000.

# Years	House's value
1	138000
2	126960
3	116803.20
4	107458.94
5	98862.23
6	90953.25
7	83676.99
8	76982.83
9	70824.20



Example 3

Write an equation that represents the function in this table.

Week	**Weight of Bacterial Culture (g)**
1	756.00
2	793.80
3	833.49

The exponential function will have the form $y = ab^x$, where y is the weight of the bacterial culture, and x is the number of weeks. The multiplier, b, for the weight of the bacterial culture is 1.05 (because $793.80 \div 756 = 1.05$ and $833.49 \div 793.80 = 1.05$, etc.). The starting point, a is not given because we are not given the weight at Week 0. However, since the growth is 1.05 every week, we know that $(1.05) \cdot$ (weight at Week 0) = 756.00g. The weight at Week 0 is 720g, thus $a = 720$. We can now write the equation:

$$y = 720 \cdot 1.05^x,$$

where y is the weight of the bacterial culture (g), and x is the time (weeks).

Problems

1. In seven years, Seta's son Stu is leaving home for college. Seta hopes to save \$8000 to pay for his first year. She has \$5000 now and has found a bank that pays 7.75% interest, compounded daily. At this rate, will she have the money she needs for Stu's first year of college? If not, how much more does she need?

2. Eight years ago, Rudi thought that he was making a sound investment by buying \$1000 worth of Pro Sports Management stock. Unfortunately, his investment depreciated steadily, losing 15% of its value each year. How much is the stock worth now? Justify your answer.

3. Based on each table below, find the equation of the exponential function $y = ab^x$.

a.

x	$f(x)$
0	1600
1	2000
2	2500
3	3125

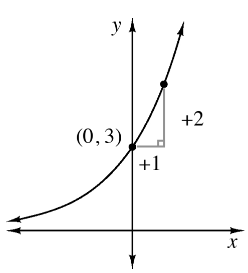
b.

x	y
1	40
2	32
3	25.6

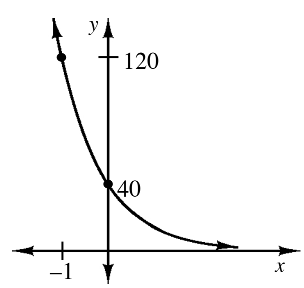
4. The new Bamo Super Ball has a rebound ratio of 0.97. If you dropped the ball from a height of 125 feet, how high will it bounce on the tenth bounce?

5. Based on each graph below, find the equation of the exponential function $y = ab^x$.

a.



b.



6. Fredrico's Bank will let you decide how often your interest will be computed, but with certain restrictions. If your interest is compounded yearly you can earn 8%. If your interest is compounded quarterly, you earn 7.875%. Monthly compounding earns a 7.75% interest rate, while weekly compounding earns a 7.625% interest rate. If your interest is compounded daily, you earn 7.5%. What is the best deal? Justify your answer.

7. Fully investigate the graph of the function $y = \left(\frac{3}{4}\right)^x + 4$. See Describing Functions (Lessons 1.1.3 through 1.2.2) in this *Parent Guide with Extra Practice* for information on how to fully describe the graph of a function.

Answers

1. Yes, she will have about \$8601.02 by then. The daily rate is $\frac{0.0775}{365} \approx 0.000212329$. Seven years is 2555 days, so we have $\$5000(1.000212329)^{2555} \approx \$8601.02$.

2. It is now only worth about \$272.49.

3. a. $y = 1600(1.25)^x$ b. $y = 50(0.8)^x$

4. About 92.18 feet.

5. a. $y = 3\left(\frac{5}{3}\right)^x$ b. $y = 40\left(\frac{1}{3}\right)^x$

6. The best way to do this problem is to choose any amount, and see how it grows over the course of one year. Taking \$100, after one year, 8% compounded yearly will yield \$108. 7.875% compounded quarterly yields \$108.11. 7.75% compounded monthly yields \$108.03. 7.625% compounded weekly yields \$107.91. 7.5% compounded daily yields \$107.79. Quarterly is the best.

7. This is a function that it is continuous and nonlinear (curved). It has a y-intercept of $(0, 5)$, and no x-intercepts. The domain is all real values of x, and the range is all real values of $y > 4$. This function has a horizontal asymptote of $y = 4$, and no vertical asymptotes. It is an exponential function.

FRACTIONAL EXPONENTS 7.2.1

A fractional exponent is equivalent to an expression with roots or radicals.

For $x \neq 0$, $x^{a/b} = (x^a)^{1/b} = \sqrt[b]{x^a}$ or $x^{a/b} = (x^{1/b})^a = (\sqrt[b]{x})^a$.

Fractional exponents may also be used to solve equations containing exponents. For additional information, see the Math Notes box in Lesson 7.2.2.

Example 1

Rewrite each expression in radical form and simplify if possible.

a. $16^{5/4}$

b. $(-8)^{2/3}$

Solution:

a.

$$\begin{aligned} &16^{5/4} \\ &= (16^{1/4})^5 \\ &= (\sqrt[4]{16})^5 \\ &= (2)^5 \\ &= 32 \end{aligned}$$

OR

$$\begin{aligned} &16^{5/4} \\ &= (16^5)^{1/4} \\ &= (1{,}048{,}576)^{1/4} \\ &= \sqrt[4]{1{,}048{,}576} \\ &= 32 \end{aligned}$$

b.

$$\begin{aligned} &(-8)^{2/3} \\ &= \left((-8)^{1/3}\right)^2 \\ &= \left(\sqrt[3]{-8}\right)^2 \\ &= (-2)^2 \\ &= 4 \end{aligned}$$

OR

$$\begin{aligned} &(-8)^{2/3} \\ &= \left((-8)^2\right)^{1/3} \\ &= (64)^{1/3} \\ &= \sqrt[3]{64} \\ &= 4 \end{aligned}$$

Example 2

Simplify each expression. Each answer should contain no parentheses and no negative exponents.

a. $(144x^{-12})^{1/2}$ b. $\left(\frac{8x^7y^3}{x}\right)^{-1/3}$

Using the Power Property of Exponents, the Property of Negative Exponents, and the Property of Fractional Exponents:

a. $$(144x^{-12})^{1/2} = \left(\frac{144}{x^{12}}\right)^{1/2} = \sqrt{\frac{144}{x^{12}}} = \frac{12}{x^6}$$

b. $$\left(\frac{8x^7y^3}{x}\right)^{-1/3} = \left(\frac{x}{8x^7y^3}\right)^{1/3} = \left(\frac{1}{8x^6y^3}\right)^{1/3} = \sqrt[3]{\frac{1}{8x^6y^3}} = \frac{1}{2x^2y}$$

Example 3

Solve the following equations for x.

a. $x^7 = 42$ b. $3x^{12} = 132$

Solution: As with many equations, we need to isolate the variable (get the variable by itself), and then eliminate the exponent. This will require one of the Laws of Exponents, namely $(x^a)^b = x^{ab}$.

a.
$$\begin{aligned} x^7 &= 42 \\ (x^7)^{1/7} &= (42)^{1/7} \\ x^{7\cdot(1/7)} &= (42)^{1/7} \\ x^1 &= (42)^{1/7} \\ x &\approx 1.706 \end{aligned}$$

b.
$$\begin{aligned} 3x^{12} &= 132 \\ \frac{3x^{12}}{3} &= \frac{132}{3} \\ x^{12} &= 44 \\ (x^{12})^{1/12} &= (44)^{1/12} \\ x &= (44)^{1/12} \\ x &\approx \pm 1.371 \end{aligned}$$

The final calculation takes the seventh root of 42 in part (a) and the twelfth root of 44 in part (b). Notice that there is only one answer for part (a), where the exponent is odd, but there are two answers (±) in part (b) where the exponent is even. Even roots always produce two answers, a positive and a negative. Be sure that if the problem is a real-world application that both the positive and the negative results make sense before stating both as solutions. You may have to disregard one solution so that the answer is feasible.

Problems

Change each expression to radical form and simplify.

1. $(64)^{2/3}$
2. $16^{-1/2}$
3. $(-27)^{1/3}$

Simplify the following expressions as much as possible.

4. $(16a^8b^{12})^{3/4}$
5. $\frac{144^{1/2}x^{-3}}{(16^{3/4}x^7)^0}$
6. $\frac{a^{2/3}b^{-3/4}c^{7/8}}{a^{-1/3}b^{1/4}c^{1/8}}$

Solve the following equations for x.

7. $x^8 = 65{,}536$
8. $-5x^{-3} = \frac{25}{40}$
9. $3^{5x} = 9^{x-1}$
10. $\left(\frac{2}{3}\right)^x = 1$
11. $2(3x-5)^4 = 512$
12. $2^{4x-1} = 4$

Find the error in each of the following solutions. Then give the correct solution.

13.
$$\begin{aligned} 4(x+7)^6 &= 1392 \\ (x+7)^6 &= 348 \\ x+7 &= 58 \\ x &= 51 \end{aligned}$$

14.
$$\begin{aligned} 5^{4x+2} &= 10^{3x-1} \\ 5^{4x+2} &= 2 \cdot 5^{3x-1} \\ 4x+2 &= 2(3x-1) \\ 4x+2 &= 6x-1 \\ 3 &= 2x \\ x &= 1.5 \end{aligned}$$

Answers

1. $(\sqrt[3]{64})^2 = 16$
2. $\frac{1}{\sqrt{16}} = \frac{1}{4}$
3. $\sqrt[3]{-27} = -3$
4. $8a^6b^9$
5. $\frac{12}{x^3}$
6. $\frac{ac^{3/4}}{b}$
7. $x = 4$
8. $x = -2$
9. $x = -\frac{2}{3}$
10. $x = 0$
11. $x = 3$
12. $x = \frac{3}{4}$

13. Both sides need to be raised to the $\frac{1}{6}$ (or the 6th root taken), not divided by six. $x \approx -4.35$.

14. Since 5 and 10 cannot be written as the power of the same number, the only way to solve the equation now is by guess and check. $x \approx 11.75$. If you did not get this answer, do not worry about it now. The point of the problem is to spot the error. There is a second error: the 2 was not distributed in the fourth line.

Students write an equation of the form $y = ab^x$ that goes through two given points. (Equations of this form have an asymptote at $y = 0$.)

For additional information, see the Math Notes box in Lesson 9.3.1. For additional examples and more practice, see the Checkpoint 9 and Checkpoint 10A materials.

Example 1

Find a possible equation for an exponential function that passes through the points (0, 8) and $(4, \frac{1}{2})$.

Solution: Substitute the *x*- and *y*-coordinates of each pair of points into the general equation. Then solve the resulting system of two equations to determine *a* and *b*.

$y = ab^x$
Since $(x, y) = (0, 8)$
$8 = ab^0$
Since $b^0 = 1$,
$8 = a(1)$, or
$a = 8$.

$y = ab^x$
Since $(x, y) = (4, \frac{1}{2})$,
$\frac{1}{2} = ab^4$

Substituting $a = 8$ from the first equation into $\frac{1}{2} = ab^4$ from the second equation,

$$\frac{1}{2} = ab^4$$
But, $a = 8$,
$$\frac{1}{2} = 8b^4$$
$$\frac{1}{16} = b^4$$
$$\sqrt[4]{\frac{1}{16}} = \sqrt[4]{b^4}$$
$$\frac{1}{2} = b$$

Since *a* and *b* have been determined, we can now write the equation:

$$y = 8(\tfrac{1}{2})^x$$

Example 2

In the year 2000, Club Leopard was first introduced on the Internet. In 2004, it had 14,867 "leopards" (members). In 2007, the leopard population had risen to 22,610. Model this data with an exponential function and use the model to predict the leopard population in the year 2012.

Solution: We can call the year 2000 our time zero, or $x = 0$. Then 2004 is $x = 4$, and the year 2007 will be $x = 7$. This gives us two data points, (4, 14867) and (7, 22610).

To model with an exponential function we will use the equation $y = ab^x$ and substitute both coordinate pairs to obtain a system of two equations.

$$y = ab^x \qquad y = ab^x$$
$$14867 = ab^4 \qquad 22610 = ab^7$$

Preparing to use the Equal Values Method to solve the system of equations, we rewrite both equations in "$a =$" form:

$$14867 = ab^4 \qquad 22610 = ab^7$$
$$a = \frac{14867}{b^4} \qquad a = \frac{22610}{b^7}$$

Then by the Equal Values Method,

$$\frac{14867}{b^4} = \frac{22610}{b^7}$$
$$14867b^7 = 22610b^4$$
$$\frac{b^7}{b^4} = \frac{22610}{14867}$$
$$b^3 = \frac{22610}{14867}$$
$$\sqrt[3]{b^3} = \sqrt[3]{\frac{22610}{14867}}$$
$$b \approx 1.15$$

From the equations above,

$$a = \frac{14867}{b^4}$$

Since $b \approx 1.15$,

$$a = \frac{14867}{(1.15)^4}$$
$$a \approx 8500$$

Since $a \approx 8500$ and $b \approx 1.15$ we can write the equation: $y = 8500 \cdot 1.15^x$, where y represents the number of members, and x represents the number of years since 2000.

We will use the equation with $x = 12$ to predict the population in 2012.

$$y = 8500(1.15)^x$$
$$y = 8500(1.15)^{12}$$
$$y \approx 45477$$

Assuming the trend continues to the year 2012 as it has in the past, we predict the population in 2012 to be 45,477.

Problems

For each of the following pairs of points, find the equation of an exponential function with an asymptote $y = 0$ that passes through them.

1. $(0, 6)$ and $(3, 48)$
2. $(1, 21)$ and $(2, 147)$
3. $(-1, 72.73)$ and $(3, 106.48)$
4. $(-2, 351.5625)$ and $(3, 115.2)$
5. On a cold wintry day the temperature outside hovered at 0°C. Karen made herself a cup of cocoa, and took it outside where she would be chopping some wood. However, she decided to conduct a mini science experiment instead of drinking her cocoa, so she placed a thermometer in the cocoa and left it sitting next to her as she worked. She wrote down the time and the reading on the thermometer as shown in the table below.

Time since 1st reading	5	10	12	15
Temp (°C)	24.41°	8.51°	5.58°	2.97°

Find the equation of an exponential function of the form $y = ab^x$ that models this data.

Answers

1. $y = 6(2)^x$
2. $y = 3(7)^x$
3. $y = 80(1.1)^x$
4. $y = 225(0.8)^x$
5. Answers will vary, but should be close to $y = 70(0.81)^x$.

FACTORING QUADRATICS 8.1.1 through 8.1.4

Chapter 8 introduces students to rewriting quadratic expressions and solving quadratic equations. Quadratic functions are functions which can be rewritten in the form $y = ax^2 + bx + c$ (where $a \neq 0$) and when graphed, create a U-shaped curve called a parabola.

There are multiple methods that can be used to solve quadratic equations. One of them requires factoring the quadratic expression first. In Lessons 8.1.1 through 8.1.4, students factor quadratic expressions.

In previous chapters, students used algebra tiles to build “generic rectangles” of quadratic expressions. In the figure below, the length and width of the rectangle are $(x + 2)$ and $(x + 4)$. Since the area of a rectangle is given by (base)(height) = area, the area of the rectangle in the figure below can be expressed as a *product*, $(x + 2)(x + 4)$. But the small pieces of the rectangle also make up its area, so the area can be expressed as a *sum*, $4x + 8 + x^2 + 2x$, or $x^2 + 6x + 8$. Thus students wrote $(x + 2)(x + 4) = x^2 + 6x + 8$.

In the figure at right, the length and width of the rectangle, which are $(x + 2)$ and $(x + 4)$, are *factors* of the quadratic expression $x^2 + 6x + 8$, since $(x + 2)$ and $(x + 4)$ multiply together to produce the quadratic expression $x^2 + 6x + 8$. Notice that the $4x$ and the $2x$ are located diagonally from each other. They are like terms and can be combined and written as $6x$.

+ 4	$4x$	8
x	x^2	$2x$
	x +	2

The factors of $x^2 + 6x + 8$ are $(x + 2)$ and $(x + 4)$.

The ax^2 term and the c term are always diagonal to one another in a generic rectangle. In this example, the ax^2 term is $(1x^2)$ and the c term is the constant 8; the product of this diagonal is $1x^2 \cdot 8 = 8x^2$. The two x-terms make up the other diagonal and can be combined into a sum since they are like terms. The b of a quadratic expression is the *sum* of the coefficients of these factors: $2x + 4x = 6x$, so $b = 6$. The product of this other diagonal is $(2x)(4x) = 8x^2$. *Note that the products of the two diagonals are always equivalent*. In the textbook, students may nickname this rule “Casey’s Rule,” after the fictional character Casey in problem 8-4.

To factor a quadratic expression, students need to identify the coefficients of the two x-terms so that the products of the two diagonals are equivalent, and also the sum of the two x-terms is b. Students can use a “diamond problem” to help organize their sums and products. For more information on using a diamond problem and generic rectangle to factor quadratic expressions, see the Math Notes box in Lesson 8.1.4.

For additional information, see the Math Notes boxes in Lessons 8.1.1 through 8.1.4. For additional examples and more practice, see the Checkpoint 10B materials at the back of the student textbook.

Example 1

Factor $x^2 + 7x + 12$.

Sketch a generic rectangle.

Place the x^2 and the 12 along one diagonal.

	12
x^2	

Find two terms whose product is $12x^2$ and whose sum is $7x$. In this case, $3x$ and $4x$. (Students are familiar with this situation as a "diamond problem" from Chapter 1.)

$\mathbf{3x}$	12
x^2	$\mathbf{4x}$

Write these terms along the other diagonal. Either term can go in either diagonal space.

Determine the base and height of the large outer rectangle by using the areas of the small pieces and finding the greatest common factor of each row and column.

$+\ 3$	$3x$	12
x	x^2	$4x$
	x	$+\ 4$

Write the sum as a product (factored form).

$$x^2 + 7x + 12 = (x + 3)(x + 4)$$

Example 2

Factor $x^2 + 7x - 30$.

Sketch a generic rectangle.
Place the x^2 and the -30 along one diagonal.

	-30
x^2	

Find two terms whose product is $-30x^2$ and whose sum is $7x$. In this case, $-3x$ and $10x$.

$\mathbf{-3x}$	-30
x^2	$\mathbf{10x}$

Write these terms along the other diagonal. Either term can go in either diagonal space.

Determine the base and height of the large outer rectangle by using the areas of the small pieces and finding the greatest common factor of each row and column.

-3	$-3x$	-30
x	x^2	$10x$
	x	$+\ 10$

Write the sum as a product (factored form).

$$x^2 + 7x - 30 = (x - 3)(x + 10)$$

Example 3

Factor $x^2 - 15x + 56$.

Sketch a generic rectangle.
Place the x^2 and the 56 along one diagonal.

	56
x^2	

Find two terms whose product is $56x^2$ and whose sum is $-15x$. Write these terms as the other diagonal.

$\mathbf{-8x}$	56
x^2	$\mathbf{-7x}$

Determine the base and height of the large outer rectangle by using the areas of the small pieces and finding the greatest common factor of each row and column.

-8	$-8x$	56
x	x^2	$-7x$
	x	-7

Write the sum as a product (factored form).

$x^2 - 15x + 56 = (x - 7)(x - 8)$

Example 4

Factor $12x^2 - 19x + 5$.

Sketch a generic rectangle.

Place the $12x^2$ and the 5 along one diagonal.

$\mathbf{-15x}$	5
$12x^2$	$\mathbf{-4x}$

$\longrightarrow$

-5	$-15x$	5
$4x$	$12x^2$	$-4x$

Find two terms whose product is $60x^2$ and whose sum is $-19x$. Write these terms as the other diagonal.

Find the base and height of the rectangle. Check the signs of the factors.

Write the sum as a product (factored form). $(3x - 1)(4x - 5) = 12x^2 - 19x + 5$

Example 5

Factor $3x^2 + 21x + 36$.

Note: If a common factor appears in all the terms, it should be factored out first.
For example, $3x^2 + 21x + 36 = 3(x^2 + 7x + 12)$.

Then $x^2 + 7x + 12$ can be factored in the usual way, as in Example.
$x^2 + 7x + 12 = (x + 3)(x + 4)$.

Then, since the expression $3x^2 + 21x + 36$ has a factor of 3,
$3x^2 + 21x + 36 = 3(x^2 + 7x + 12) = 3(x + 3)(x + 4)$.

Problems

1. $x^2 + 5x + 6$	2. $2x^2 + 5x + 3$	3. $3x^2 + 4x + 1$	4. $3x^2 + 30x + 75$
5. $x^2 + 15x + 44$	6. $x^2 + 7x + 6$	7. $2x^2 + 22x + 48$	8. $x^2 + 4x - 32$
9. $4x^2 + 12x + 9$	10. $24x^2 + 22x - 10$	11. $x^2 + x - 72$	12. $3x^2 - 20x - 7$
13. $x^3 - 11x^2 + 28x$	14. $2x^2 + 11x - 6$	15. $2x^2 + 5x - 3$	16. $x^2 - 3x - 10$
17. $4x^2 - 12x + 9$	18. $3x^2 + 2x - 5$	19. $6x^2 - x - 2$	20. $9x^2 - 18x + 8$

Answers

1. $(x + 2)(x + 3)$	2. $(x + 1)(2x + 3)$	3. $(3x + 1)(x + 1)$	4. $3(x + 5)(x + 5)$
5. $(x + 11)(x + 4)$	6. $(x + 6)(x + 1)$	7. $2(x + 8)(x + 3)$	8. $(x + 8)(x - 4)$
9. $(2x + 3)(2x + 3)$	10. $2(3x - 1)(4x + 5)$	11. $(x - 8)(x + 9)$	12. $(x - 7)(3x + 1)$
13. $x(x - 4)(x - 7)$	14. $(x + 6)(2x - 1)$	15. $(x + 3)(2x - 1)$	16. $(x - 5)(x + 2)$
17. $(2x - 3)(2x - 3)$	18. $(3x + 5)(x - 1)$	19. $(2x + 1)(3x - 2)$	20. $(3x - 4)(3x - 2)$

FACTORING SHORTCUTS 8.1.5

Although most factoring problems can be done with generic rectangles, there are two special factoring patterns that, if recognized, can be done by sight. The two patterns are known as the **Difference of Squares** and **Perfect Square Trinomials**. The general patterns are as follows:

Difference of Squares: $a^2x^2 - b^2y^2 = (ax + by)(ax - by)$
Perfect Square Trinomial: $a^2x^2 + 2abxy + b^2y^2 = (ax + by)^2$

Example 1

Difference of Squares	**Perfect Square Trinomials**
$x^2 - 49 = (x + 7)(x - 7)$	$x^2 - 10x + 25 = (x - 5)^2$
$4x^2 - 25 = (2x - 5)(2x + 5)$	$9x^2 + 12x + 4 = (3x + 2)^2$
$x^2 - 36 = (x + 6)(x - 6)$	$x^2 - 6x + 9 = (x - 3)^2$
$9x^2 - 1 = (3x - 1)(3x + 1)$	$4x^2 + 20x + 25 = (2x + 5)^2$

Example 2

Sometimes removing a common factor reveals one of the special patterns:

$8x^2 - 50y^2 \Rightarrow 2(4x^2 - 25y^2) \Rightarrow 2(2x + 5y)(2x - 5y)$

$12x^2 + 12x + 3 \Rightarrow 3(4x^2 + 4x + 1) \Rightarrow 3(2x + 1)^2$

Problems

Factor each difference of squares.

1. $x^2 - 16$
2. $x^2 - 25$
3. $64m^2 - 25$
4. $4p^2 - 9q^2$
5. $9x^2y^2 - 49$
6. $x^4 - 25$
7. $64 - y^2$
8. $144 - 25p^2$
9. $9x^4 - 4y^2$

Factor each perfect square trinomial.

10. $x^2 + 4x + 4$
11. $y^2 + 8y + 16$
12. $m^2 - 10m + 25$
13. $x^2 - 8x + 16$
14. $a^2 + 8ab + 16b^2$
15. $36x^2 + 12x + 1$
16. $25x^2 - 30xy + 9y^2$
17. $9x^2y^2 - 6xy + 1$
18. $49x^2 + 1 + 14x$

Factor completely.

19. $9x^2 - 16$
20. $9x^2 + 24x + 16$
21. $9x^2 - 36$
22. $2x^2 + 8xy + 8y^2$
23. $x^2y + 10xy + 25y$
24. $8x^2 - 72$
25. $4x^3 - 9x$
26. $4x^2 - 8x + 4$
27. $2x^2 + 8$

Answers

1. $(x + 4)(x - 4)$
2. $(x + 5)(x - 5)$
3. $(8m + 5)(8m - 5)$
4. $(2p + 3q)(2p - 3q)$
5. $(3xy + 7)(3xy - 7)$
6. $(x^2 + 5)(x^2 - 5)$
7. $(8 + y)(8 - y)$
8. $(12 + 5p)(12 - 5p)$
9. $(3x^2 + 2y)(3x^2 - 2y)$
10. $(x + 2)^2$
11. $(y + 4)^2$
12. $(m - 5)^2$
13. $(x - 4)^2$
14. $(a + 4b)^2$
15. $(6x + 1)^2$
16. $(5x - 3y)^2$
17. $(3xy - 1)^2$
18. $(7x + 1)^2$
19. $(3x + 4)(3x - 4)$
20. $(3x + 4)^2$
21. $9(x + 2)(x - 2)$
22. $2(x + 2y)^2$
23. $y(x + 5)^2$
24. $8(x + 3)(x - 3)$
25. $x(2x + 3)(2x - 3)$
26. $4(x - 1)^2$
27. $2(x^2 + 4)$

USING THE ZERO PRODUCT PROPERTY 8.2.2 and 8.2.3

The graph of a quadratic function, a parabola, is a symmetrical curve. Its highest or lowest point is called the vertex. The graph is created by using the equation $y = ax^2 + bx + c$. Students have been graphing parabolas by substituting values for x and solving for y. This can be a tedious process, especially if an appropriate range of x-values is not known. If only a quick sketch of the parabola is needed, one possible method is to find the x-intercepts first, then find the vertex and/or the y-intercept. To find the x-intercepts, substitute 0 for y and solve the quadratic equation, $0 = ax^2 + bx + c$. Students will learn multiple methods to solve quadratic equations in this chapter and in Chapter 9. One method to solve quadratic equations uses the Zero Product Property, that is, solving by factoring. This method uses two ideas:

(1) When the product of two or more numbers is zero, then one of the numbers must be zero.

(2) Some quadratic expressions can be factored into the product of two binomials.

For additional information see the Math Notes box in Lesson 8.2.2.

Example 1

Find the x-intercepts of $y = x^2 + 6x + 8$.

The x-intercepts are located on the graph where $y = 0$, so write the quadratic expression equal to zero, then solve for x.	$x^2 + 6x + 8 = 0$
Factor the quadratic expression.	$(x + 4)(x + 2) = 0$
Set each factor equal to 0.	$(x + 4) = 0$ or $(x + 2) = 0$
Solve each equation for x.	$x = -4$ or $x = -2$

The x-intercepts are $(-4, 0)$ and $(-2, 0)$.

You can check your answers by substituting them into the original equation.

$(-4)^2 + 6(-4) + 8 \Rightarrow 16 - 24 + 8 \Rightarrow 0$
$(-2)^2 + 6(-2) + 8 \Rightarrow 4 - 12 + 8 \Rightarrow 0$

Example 2

Solve $2x^2 + 7x - 15 = 0$.

Factor the quadratic expression. $(2x - 3)(x + 5) = 0$

Set each factor equal to 0. $(2x - 3) = 0$ or $(x + 5) = 0$

Solve for each x. $2x = 3$ or $x = -5$

$x = \frac{3}{2}$ or $x = -5$

Example 3

If the quadratic equation does not equal 0, rewrite it algebraically so that it does, then use the Zero Product Property.

Solve $2 = 6x^2 - x$.

Set the equation equal to 0.

$$2 = 6x^2 - x$$
$$0 = 6x^2 - x - 2$$

Factor the quadratic expression.

$$0 = (2x + 1)(3x - 2)$$

Solve each equation for x.

$$(2x+1) = 0 \quad \text{or} \quad (3x-2) = 0$$
$$2x = -1 \quad \text{or} \quad 3x = 2$$
$$x = -\tfrac{1}{2} \qquad x = \tfrac{2}{3}$$

Example 4

Solve $9x^2 - 6x + 1 = 0$.

Factor the quadratic expression.

$$9x^2 - 6x + 1 = 0$$
$$(3x - 1)(3x - 1) = 0$$

Solve each equation for x. Notice the factors are the same so there will be only one solution.

$$(3x - 1) = 0$$
$$3x = 1$$
$$x = \tfrac{1}{3}$$

Problems

Solve for x.

1. $x^2 - x - 12 = 0$	2. $3x^2 - 7x - 6 = 0$	3. $x^2 + x - 20 = 0$
4. $3x^2 + 11x + 10 = 0$	5. $x^2 + 5x = -4$	6. $6x - 9 = x^2$
7. $6x^2 + 5x - 4 = 0$	8. $x^2 - 6x + 8 = 0$	9. $6x^2 - x - 15 = 0$
10. $4x^2 + 12x + 9 = 0$	11. $x^2 - 12x = 28$	12. $2x^2 + 8x + 6 = 0$
13. $2 + 9x = 5x^2$	14. $2x^2 - 5x = 3$	15. $x^2 = 45 - 4x$

Answers

1. $x = 4$ or -3	2. $x = -\frac{2}{3}$ or 3	3. $x = -5$ or 4
4. $x = -\frac{5}{3}$ or -2	5. $x = -4$ or -1	6. $x = 3$
7. $x = -\frac{4}{3}$ or $\frac{1}{2}$	8. $x = 4$ or 2	9. $x = -\frac{3}{2}$ or $\frac{5}{3}$
10. $x = -\frac{3}{2}$	11. $x = 14$ or -2	12. $x = -1$ or -3
13. $x = -\frac{1}{5}$ or 2	14. $x = -\frac{1}{2}$ or 3	15. $x = 5$ or -9

In Lesson 8.2.3, students found that if the equation of a parabola is written in **graphing form**: $f(x) = (x - h)^2 + k$ then the vertex can easily be seen as (h, k). For example, for the parabola $f(x) = (x + 3)^2 - 1$ the vertex is $(-3, -1)$. Students can then set the function equal to zero to find the x-intercepts: solve $0 = (x + 3)^2 - 1$ to find the x-intercepts. For help in solving this type of equation, see the Lesson 8.2.3 Resource Page, available at cpm.org. Students can set $x = 0$ to find the y-intercepts: $y = (0 + 3)^2 - 1$.

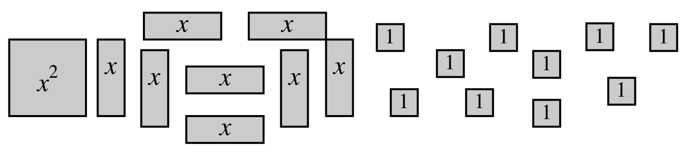
When the equation of the parabola is given in standard form: $f(x) = x^2 + bx + c$, then using the process of **completing the square** can be used to convert standard form into graphing form. Algebra tiles are used to help visualize the process.

For additional examples and practice with graphing quadratic functions, see the Checkpoint 11 materials at the back of the student textbook.

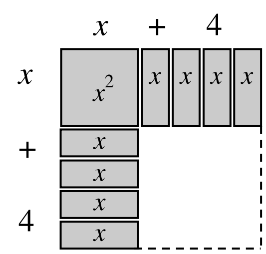
Example 1 (Using algebra tiles)

Complete the square to change $f(x) = x^2 + 8x + 10$ into graphing form, identify the vertex and y-intercept, and draw a graph.

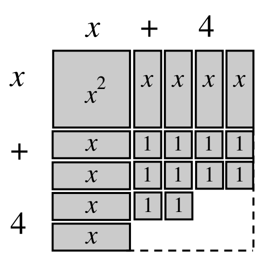
$f(x) = x^2 + 8x + 10$ would look like this:



Arrange the tiles as shown in the picture at right to make a square.



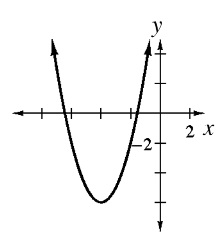
16 small unit tiles are needed to fill in the corner, but only ten unit tiles are available. Show the 10 small square tiles and draw the outline of the whole square.



The **complete square** would have length and width both equal to $(x + 4)$, so the complete square can be represented by the quadratic expression $(x + 4)^2$. But the tiles from $x^2 + 8x + 10$ do not form a complete square—the expression $x^2 + 8x + 10$ has six fewer tiles than a complete square. So $x^2 + 8x + 10$ is a complete square minus 6, or $(x + 4)^2$ minus 6. That is,

$$x^2 + 8x + 10 = (x + 4)^2 - 6.$$

From graphing form, the vertex is at $(h, k) = (-4, -6)$. The y-intercept is where $x = 0$, so $y = (0 + 4)^2 - 6 = 10$. The y-intercept is $(0, 10)$, and the graph is shown at right.



Example 2 (Using the general process)

Complete the square to change $f(x) = x^2 + 5x + 2$ into graphing form. Identify the vertex and y-intercept, and draw a graph.

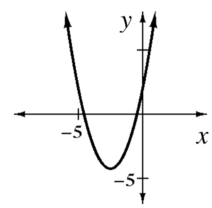
Rewrite the expression as: $f(x) = x^2 + 5x + 2$
$$f(x) = (x^2 + 5x + ?) + 2$$

Make $(x^2 + 5x + ?)$ into a perfect square by taking half of the x-term coefficient and squaring it: $(\frac{5}{2})^2 = \frac{25}{4}$. Then $(x^2 + 5x + \frac{25}{4})$ is a perfect square trinomial.

We need to write an equivalent function with $+\frac{25}{4}$, but if we add $\frac{25}{4}$, we also need to subtract it:

$f(x) = x^2 + 5x + 2$ which can be rewritten as: $f(x) = (x + \frac{5}{2})^2 - \frac{17}{4}$.

$f(x) = (x^2 + 5x + \frac{25}{4}) + 2 - \frac{25}{4}$



The function is now in graphing form.
The vertex is $(-\frac{5}{2}, -\frac{17}{4})$ or $(-2.5, -4.25)$.
The y-intercept is where $x = 0$.
Thus, $y = (0 + 2.5)^2 - 4.25 = 2$ and the y-intercept is $(0, 2)$.

Alternatively, if the process above is unclear, draw a generic rectangle of $x^2 + 5x + 2$ and imagine algebra tiles.

$2.5x$	
x^2	$2.5x$

There should be $(-2.5)^2 = 6.25$ tiles in the upper right corner to complete the square. But the expression $x^2 + 5x + 2$ only provides 2 unit tiles. So there are 4.25 unit tiles missing. Thus, 4.25 is missing from the rectangle below:

$+2.5$	$2.5x$	
x	x^2	$2.5x$
	x	$+2.5$

That is, $(x + 2.5)^2$ minus 4.25 unit tiles is the equivalent of $x^2 + 5x + 2$, or,
$x^2 + 5x + 2 = (x + 2.5)^2 - 4.25$.

Problems

Complete the square to write each equation in graphing form. Then state the vertex.

1. $f(x) = x^2 + 6x + 7$
2. $f(x) = x^2 + 4x + 11$
3. $f(x) = x^2 + 10x$
4. $f(x) = x^2 + 7x + 2$
5. $f(x) = x^2 - 6x + 9$
6. $f(x) = x^2 + 3$
7. $f(x) = x^2 - 4x$
8. $f(x) = x^2 + 2x - 3$
9. $f(x) = x^2 + 5x + 1$
10. $f(x) = x^2 - \frac{1}{3}x$

Answers

1. $f(x) = (x + 3)^2 - 2$; $(-3, -2)$
2. $f(x) = (x + 2)^2 + 7$; $(-2, 7)$
3. $f(x) = (x + 5)^2 - 25$; $(-5, -25)$
4. $f(x) = (x + 3.5)^2 - 10.25$; $(-3.5, -10.25)$
5. $f(x) = (x - 3)^2$; $(3, 0)$
6. $f(x) = x^2 + 3$; $(0, 3)$
7. $f(x) = (x - 2)^2 - 4$; $(2, -4)$
8. $f(x) = (x + 1)^2 - 4$; $(-1, -4)$
9. $f(x) = (x + \frac{5}{2})^2 - \frac{21}{4}$; $(-\frac{5}{2}, -\frac{21}{4})$
10. $f(x) = (x - \frac{1}{6})^2 - \frac{1}{36}$; $(\frac{1}{6}, -\frac{1}{36})$

USING THE QUADRATIC FORMULA 9.1.2 and 9.1.3

When a quadratic equation is not factorable, another method is needed to solve for x. The Quadratic Formula can be used to calculate the roots of a quadratic function, that is, the x-intercepts of the parabola. The Quadratic Formula can be used with any quadratic equation, factorable or not. There may be two, one, or no solutions, depending on whether the parabola intersects the x-axis twice, once, or not at all.
The solution(s) to any quadratic equation $ax^2 + bx + c = 0$ are:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

The $\pm$ symbol is read as "plus or minus." It is shorthand notation that tells you to calculate the formula twice, once with + and again with – to get both x-values.

To use the formula, the quadratic equation must be written in *standard form*: $ax^2 + bx + c = 0$. This is necessary to correctly identify the values of a, b, and c. Once the equation is in standard form and equal to 0, a is the coefficient of the x^2-term, b is the coefficient of the x-term and c is the constant term.

For additional information, see the Math Notes boxes in Lessons 9.1.1 through 9.1.4 and 10.2.4.

Example 1

Solve $2x^2 - 5x - 3 = 0$.

Identify a, b, and c. Watch your signs carefully.	$a = 2$, $b = -5$, $c = -3$
Write the Quadratic Formula.	$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
Substitute a, b, and c into the formula and do the initial calculations.	$x = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(2)(-3)}}{2(2)}$
	$x = \frac{5 \pm \sqrt{25 - (-24)}}{4}$
Simplify the $\sqrt{\ }$.	$x = \frac{5 \pm \sqrt{49}}{4}$
Calculate both values of x.	$x = \frac{5+7}{4} = \frac{12}{4} = 3$ or $x = \frac{5-7}{4} = \frac{-2}{4} = -\frac{1}{2}$

The solutions are $x = 3$ or $x = -\frac{1}{2}$.

Example 2

Solve $3x^2 + 5x + 1 = 0$.

Identify a, b, and c. $a = 3,\ b = 5,\ c = 1$

Write the Quadratic Formula. $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Substitute a, b, and c into the formula and do the initial calculations.

$$x = \frac{-(5) \pm \sqrt{(5)^2 - 4(3)(1)}}{2(3)}$$

$$x = \frac{-5 \pm \sqrt{25 - 12}}{6}$$

Simplify the $\sqrt{\ }$. $x = \frac{-5 \pm \sqrt{13}}{6}$

The solutions are $x = \frac{-5 + \sqrt{13}}{6} \approx -0.23$ or $x = \frac{-5 - \sqrt{13}}{6} \approx -1.43$.

Example 3

Solve $25x^2 - 20x + 4 = 0$.

Identify a, b, and c. $a = 25,\ b = -20,\ c = 4$

Write the Quadratic Formula. $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Substitute a, b, and c into the formula and do the initial calculations.

$$x = \frac{-(-20) \pm \sqrt{(-20)^2 - 4(25)(4)}}{2(25)}$$

$$x = \frac{20 \pm \sqrt{400 - 400}}{50}$$

Simplify the $\sqrt{\ }$. $x = \frac{20 \pm \sqrt{0}}{50}$

This quadratic equation has only one solution: $x = \frac{2}{5}$.

Example 4

Solve $x^2 + 4x + 10 = 0$.

Identify a, b, and c. $a = 1$, $b = 4$, $c = 10$

Write the Quadratic Formula. $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Substitute a, b, and c into the formula and do the initial calculations.

$$x = \frac{-(4) \pm \sqrt{(4)^2 - 4(1)(10)}}{2(1)}$$

$$x = \frac{-4 \pm \sqrt{16 - 40}}{2}$$

Simplify the $\sqrt{}$.

$$x = \frac{-4 \pm \sqrt{-24}}{2}$$

It is impossible to take the square root of a negative number; therefore this quadratic equation has no real solutions.

Example 5

Solve $(3x + 1)(x + 2) = 1$.

Rewrite the equation in standard form. That is, rewrite the product as a sum and then set the equation equal to zero.

$$(3x + 1)(x + 2) = 1$$

$$3x^2 + 7x + 2 = 1$$

$$3x^2 + 7x + 1 = 0$$

Identify a, b, and c. $a = 3$, $b = 7$, $c = 1$

Write the Quadratic Formula. $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Substitute a, b, and c into the formula and do the initial calculations.

$$x = \frac{-(7) \pm \sqrt{(7)^2 - 4(3)(1)}}{2(3)}$$

$$x = \frac{-7 \pm \sqrt{49 - 12}}{6}$$

Simplify.

$$x = \frac{-7 \pm \sqrt{37}}{6}$$

The solutions are $x = \frac{-7 \pm \sqrt{37}}{6}$, or, $x \approx -0.15$ or $x \approx -2.18$.

Example 6

Solve $3x^2+6x+1=0$.

Identify a, b, and c. $a=3$, $b=6$, $c=1$

Write the Quadratic Formula. $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$

Substitute a, b, and c into the formula and do the initial calculations.

$$x=\frac{-(6)\pm\sqrt{(6)^2-4(3)(1)}}{2(3)}$$

$$x=\frac{-6\pm\sqrt{36-12}}{6}$$

Simplify. $x=\frac{-6\pm\sqrt{24}}{6}$

The solutions are $x=\frac{-6\pm\sqrt{24}}{6}$, or, $x\approx-1.82$ or $x\approx-0.18$.

The Math Notes box in Lesson 9.1.4 describes another form of the expression $\frac{-6\pm\sqrt{24}}{6}$ that can be written by simplifying the square root. The result is equivalent to the exact values above.

Factor the $\sqrt{24}$, then simplify by taking the square root of 4. $\sqrt{24}=\sqrt{4}\sqrt{6}=2\sqrt{6}$

Simplify the fraction by dividing every term by 2.

$$x=\frac{-6\pm2\sqrt{6}}{6}$$

$$x=\frac{-3\pm\sqrt{6}}{3}$$

Problems

Solve each equation by using the Quadratic Formula.

1. $x^2 - x - 2 = 0$
2. $x^2 - x - 3 = 0$
3. $-3x^2 + 2x + 1 = 0$
4. $-2 - 2x^2 = 4x$
5. $7x = 10 - 2x^2$
6. $-6x^2 - x + 6 = 0$
7. $6 - 4x + 3x^2 = 8$
8. $4x^2 + x - 1 = 0$
9. $x^2 - 5x + 3 = 0$
10. $0 = 10x^2 - 2x + 3$
11. $x(-3x + 5) = 7x - 10$
12. $(5x + 5)(x - 5) = 7x$

Answers

1. $x = 2$ or -1
2. $x = \frac{1 \pm \sqrt{13}}{2}$ ≈ 2.30 or -1.30
3. $x = -\frac{1}{3}$ or 1
4. $x = -1$
5. $x = \frac{-7 \pm \sqrt{129}}{4}$ ≈ 1.09 or -4.59
6. $x = \frac{1 \pm \sqrt{145}}{-12}$ ≈ -1.09 or 0.92
7. $x = \frac{4 \pm \sqrt{40}}{6} = \frac{2 \pm \sqrt{10}}{3}$ ≈ 1.72 or -0.39
8. $x = \frac{-1 \pm \sqrt{17}}{8}$ ≈ 0.39 or -0.64
9. $x = \frac{5 \pm \sqrt{13}}{2}$ ≈ 4.30 or 0.70
10. no solution
11. $x = \frac{2 \pm \sqrt{124}}{-6} = \frac{1 \pm \sqrt{31}}{-3}$ ≈ -2.19 or 1.52
12. $x = \frac{27 \pm \sqrt{1229}}{10}$ ≈ 6.21 or -0.81

SOLVING ONE-VARIABLE INEQUALITIES 9.2.1 and 9.2.2

To **solve an inequality in one variable,** first change it to an equation and solve. Place the solution, called a "boundary point," on a number line. This point separates the number line into two regions. The boundary point is included in the solution for situations that involve $\geq$ or $\leq$, and excluded from situations that involve strictly $>$ or $<$. On the number line graph, boundary points that are included in the solutions are shown with a solid filled-in circle, and excluded solutions are shown with an open circle. Next, choose a number from within each region separated by the boundary point, and check if the number is true or false in the original inequality. If it is true, then every number in that region is a solution to the inequality. If it is false, then no number in that region is a solution to the inequality.

For additional information, see the Math Notes boxes in Lessons 9.2.1 and 9.3.2.

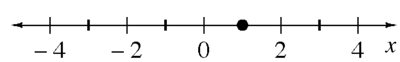
Example 1

Solve: $3x - (x + 2) \geq 0$

Change to an equation and solve.

$$3x - (x+2) = 0$$
$$3x - x - 2 = 0$$
$$2x = 2$$
$$x = 1$$

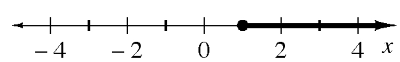
Place the solution (boundary point) on the number line. Because $x = 1$ is also a solution to the inequality ($\geq$), we use a solid dot.



Test a number from each side of the boundary point in the original inequality.

Test $x = 0$	Test $x = 3$
$3 \cdot 0 - (0+2) \geq 0$	$3 \cdot 3 - (3+2) \geq 0$
$-2 \geq 0$	$4 \geq 0$
False	True

The solution is $x \geq 1$.



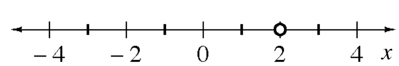
Example 2

Solve: $-x + 6 > x + 2$

Change to an equation and solve.

$$-x + 6 = x + 2$$
$$-2x = -4$$
$$x = 2$$

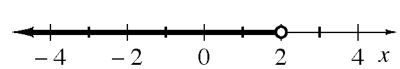
Place the solution (boundary point) on the number line. Because the original problem is a strict inequality ($>$), $x = 2$ is not a solution, so we use an open dot.



Test a number from each side of the boundary point in the original inequality.

Test $x = 0$	Test $x = 4$
$-0 + 6 > 0 + 2$	$-4 + 6 > 4 + 2$
$6 > 2$	$2 > 6$
True	False

The solution is $x < 2$.



Problems

Solve each inequality.

1. $4x-1\geq 7$	2. $2(x-5)\leq 8$	3. $3-2x<x+6$
4. $\frac{1}{2}x>5$	5. $3(x+4)>12$	6. $2x-7\leq 5-4x$
7. $3x+2<11$	8. $4(x-6)\geq 20$	9. $\frac{1}{4}x<2$
10. $12-3x>2x+1$	11. $\frac{x-5}{7}\leq -3$	12. $3(5-x)\geq 7x-1$
13. $3y-(2y+2)\leq 7$	14. $\frac{m+2}{5}<\frac{2m}{3}$	15. $\frac{m-2}{3}\geq\frac{2m+1}{7}$

Answers

1. $x\geq 2$	2. $x\leq 9$	3. $x>-1$
4. $x>10$	5. $x>0$	6. $x\leq 2$
7. $x<3$	8. $x\geq 11$	9. $x<8$
10. $x<\frac{11}{5}$	11. $x\leq -16$	12. $x\leq 1.6$
13. $y\leq 9$	14. $m>\frac{6}{7}$	15. $m\geq 17$

To graph the solutions to an inequality in two variables, first graph the corresponding equation. This graph is the boundary line (or curve), since all points that make the inequality true lie on one side or the other of the line. Before you graph the equation, decide whether the line or curve is part of the solution or not, that is, whether it is solid or dashed. If the inequality symbol is either ≤ or ≥, then the boundary line is part of the inequality and it must be solid. If the inequality symbol is either < or >, then the boundary line is dashed.

Next, decide which side of the boundary line must be shaded to show the part of the graph that represents all values that make the inequality true. Choose a point not on the boundary line. Substitute this point into the *original* inequality. If the inequality is true for the test point, then shade the graph on this side of the boundary line. If the inequality is false for the test point, then shade the opposite side of the line.

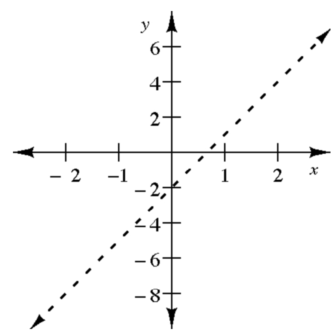
The shaded portion represents all the solutions to the original inequality.

Caution: If you need to rearrange the inequality in order to graph it, such as putting it in slope-intercept form, always use the *original* inequality to test a point, not the rearranged form.

For additional information, see the Math Notes box in Lesson 9.4.1.

Example 1

Graph the solutions to the inequality $y > 3x - 2$.

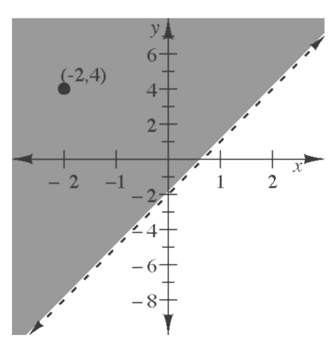


First, graph the line $y = 3x - 2$, but draw it dashed since > means the boundary line is not part of the solution. For example, the point (0, –2) is on the boundary line, but it is not a solution to the inequality because $-2 \not> 3(0) - 2$ or $-2 \not> -2$.

Next, test a point that is not on the boundary line.
For this example, use the point (–2, 4).

$4 > 3(-2) - 2$, so $4 > -8$ which is a true statement.

Since the inequality is true for this test point, shade the region containing the point (–2, 4). All of the coordinate pairs that are solutions lie in the shaded region.



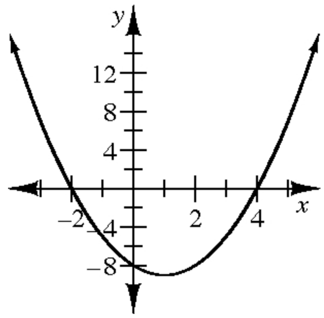
Example 2

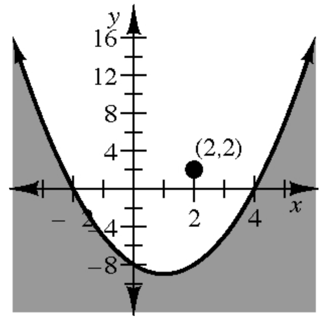
Graph the solutions to the inequality $y \le x^2 - 2x - 8$.

First, graph the parabola $y = x^2 - 2x - 8$ and draw it solid, since $\le$ means the boundary curve is part of the solution.

Next, test the point (2, 2) above the boundary curve.

$2 \le 2^2 - 2 \cdot 2 - 8$, so $2 \le -8$





Since the inequality is false for this test point above the curve, shade below the boundary curve. The solutions are the shaded region.

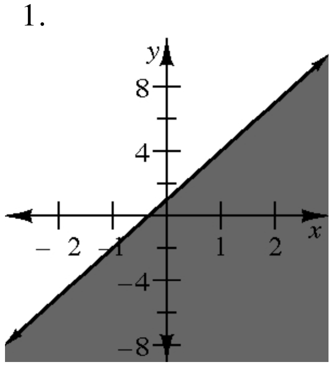
Problems

Graph the solutions to each of the following inequalities on a separate set of axes.

1. $y \le 3x + 1$	2. $y \ge -2x + 3$	3. $y > 4x - 2$
4. $y < -3x - 5$	5. $y \le 3$	6. $x > 1$
7. $y > \frac{2}{3}x + 8$	8. $y < -\frac{3}{5}x - 7$	9. $3x + 2y \ge 7$
10. $-4x + 2y < 3$	11. $y \ge x^2 - 3$	12. $y \le x^2 + 2x$
13. $y < 4 - x^2$	14. $y \le \lvert x + 2 \rvert$	15. $y \ge -\lvert x \rvert + 3$

Answers

1.



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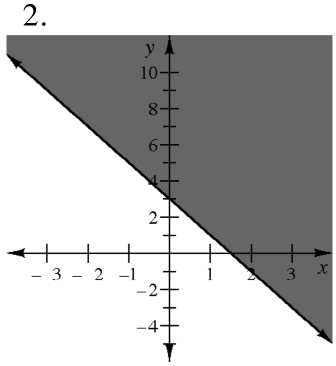

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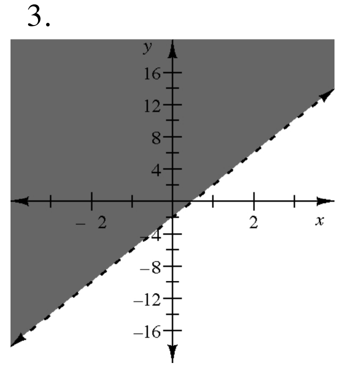

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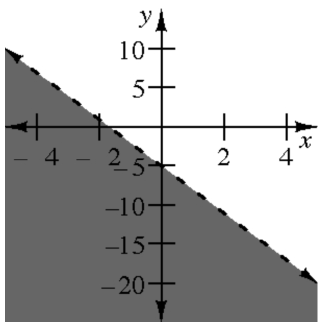

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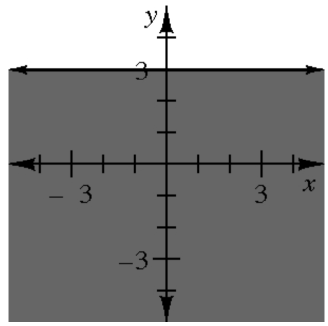

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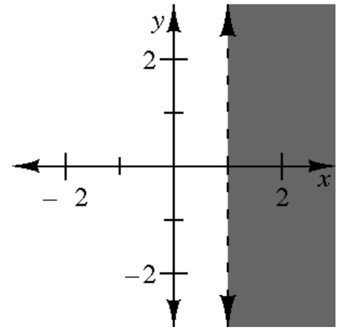

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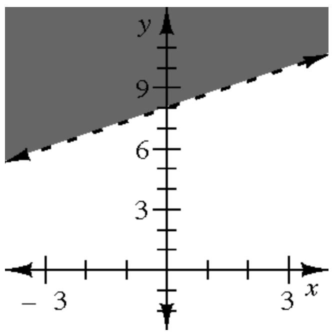

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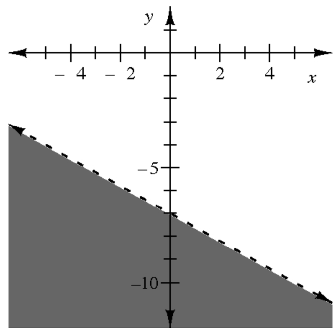

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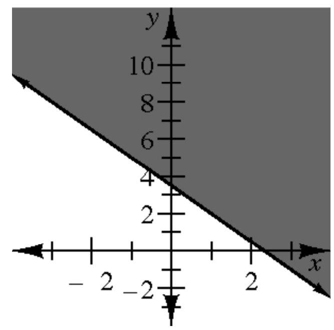

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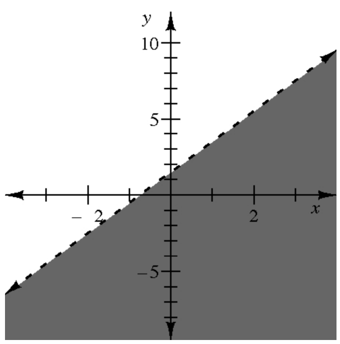

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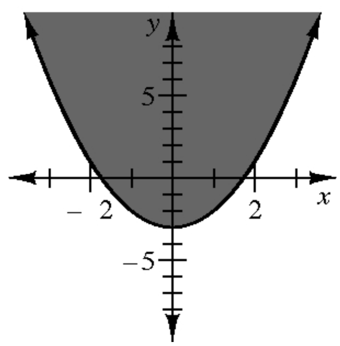

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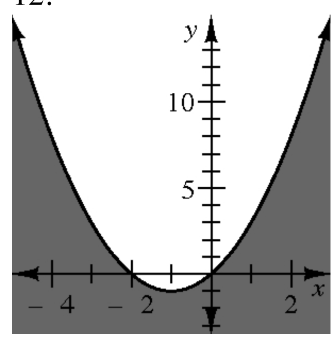

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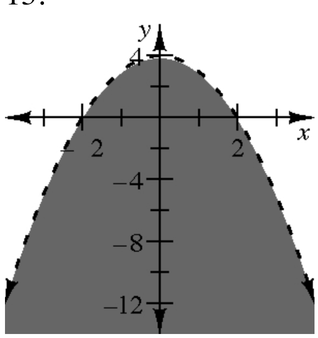

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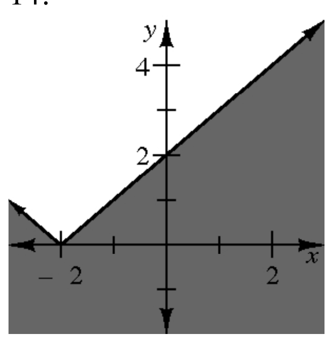

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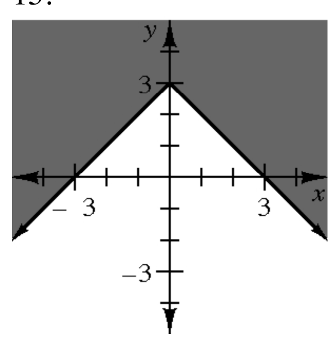


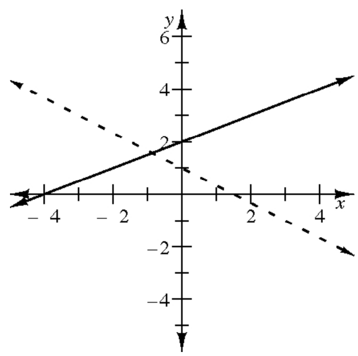
SYSTEMS OF INEQUALITIES 9.4.1 through 9.4.3

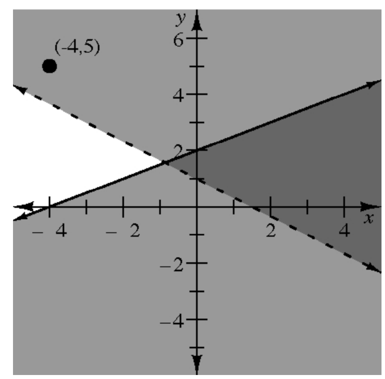
To **graph the solutions to a system of inequalities**, follow the same procedure outlined in the previous section but do it twice—once for each inequality. The solution to the system of inequalities is the **overlap** of the shading from the individual inequalities.

Example 1

Graph the solutions to the system $y \le \frac{1}{2}x + 2$ and $y > -\frac{2}{3}x + 1$.

Graph the lines $y = \frac{1}{2}x + 2$ and $y = -\frac{2}{3}x + 1$. The first is solid and the second is dashed. Test the point (–4, 5) in the first inequality.





$5 \le \frac{1}{2}(-4) + 2$, so $5 \le 0$

This inequality is false, so shade on the opposite side of the boundary line from (–4, 5), that is, below the line.

$5 > -\frac{2}{3}(-4) + 1$, so $5 > \frac{11}{3}$

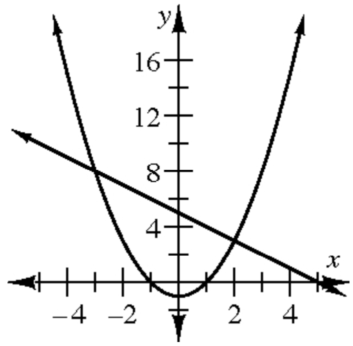
Test the same point in the second inequality. This inequality is true, so shade on the same side of the boundary line as (–4, 5), that is, above the line.

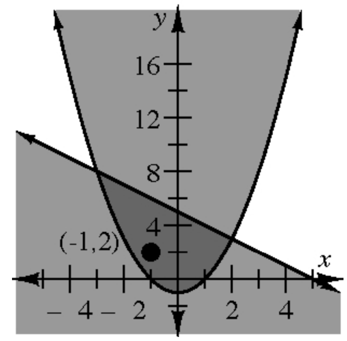
The solutions are represented by the overlap of the two shaded regions shown by the darkest shading in the second graph above right.

Example 2

Graph the solutions to the system $y \le -x + 5$ and $y \ge x^2 - 1$.

Graph the line $y = -x + 5$ and the parabola $y = x^2 - 1$ with a solid line and curve.





$2 \le -(-1) + 5$, so $2 \le 6$

Test the point $(-1, 2)$ in the first inequality. This inequality is true, so shade on the same side of the boundary line as $(-1, 2)$, that is, below the line.

$2 \ge (-1)^2 - 1$, so $2 \ge 0$

Test the same point in the second inequality. This inequality is also true, so shade on the same side of the boundary curve as $(-1, 2)$, that is, inside the curve.

The solutions are in the overlap of the two shaded regions shown by the darkest shading in the second graph above right.

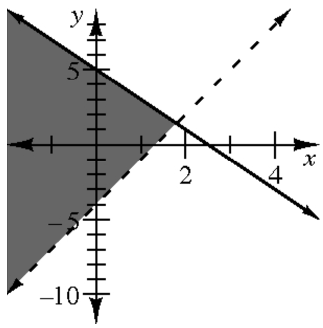
Problems

Graph the solutions to each of the following pairs of inequalities on the same set of axes.

1. $y > 3x - 4$ and $y \le -2x + 5$
2. $y \ge -3x - 6$ and $y > 4x - 4$
3. $y < -\frac{3}{5}x + 4$ and $y < \frac{1}{3}x + 3$
4. $y < -\frac{3}{7}x - 1$ and $y > \frac{4}{5}x + 1$
5. $y < 3$ and $y > \frac{1}{2}x + 2$
6. $x \le 3$ and $y < \frac{3}{4}x - 4$
7. $y \le 2x + 1$ and $y \ge x^2 - 4$
8. $y < -x + 5$ and $y \ge x^2 + 1$
9. $y < -x + 6$ and $y \ge |x - 2|$
10. $y < -x^2 + 5$ and $y \ge |x| - 1$

Answers

1.



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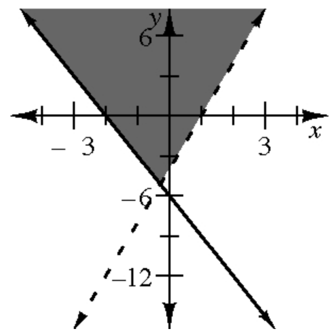

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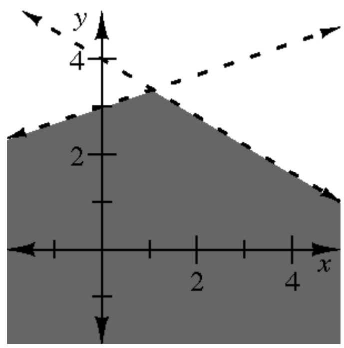

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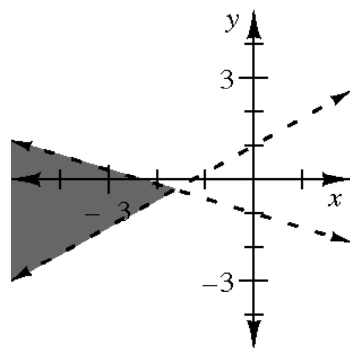

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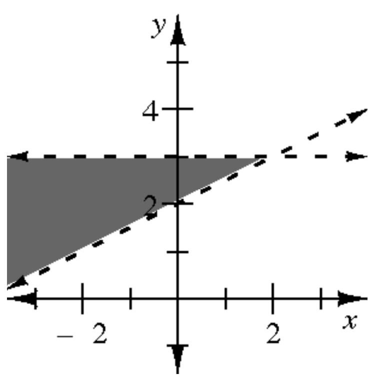

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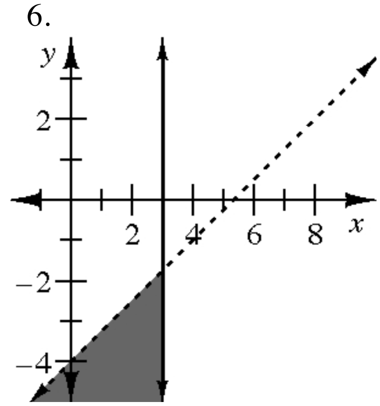

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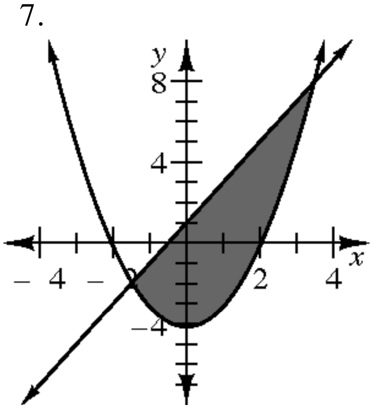

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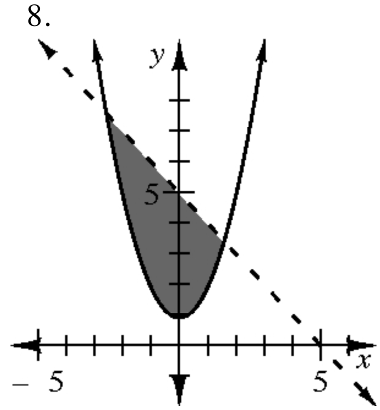

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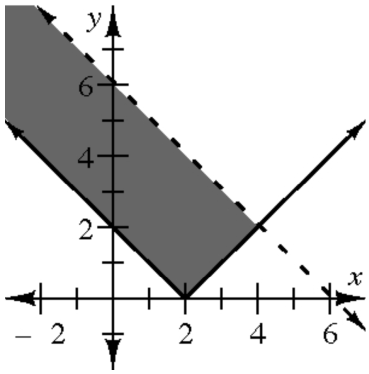

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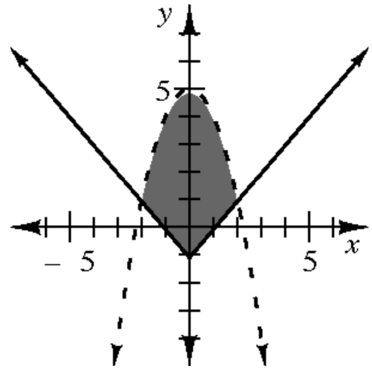


ASSOCIATION IN A TWO-WAY TABLE 10.1.1

Data based on measurements such as height, speed, and temperature is *numerical*. In Chapter 6 you described associations between two numerical variables. Data can also be counts or percentages in named categories such as gender, religion, or blood type. This kind of non-numerical data is called categorical. Categorical data cannot be shown on a scatterplot but there are other techniques to look for relationships (an association) between categorical variables including Venn diagrams, bar charts, and two-way tables.

Example 1

Is there an association between a voter's political party and whether the voter supported the ballot proposition or not? The following data was collected:

	Republican	Democrat
Supported proposition	**234**	**286**
Did not support proposition	**162**	**198**
Undecided	**54**	**66**

Solution

Determine the row and column totals:

	Republican	Democrat	
Supported proposition	**234**	**286**	520
Did not support proposition	**162**	**198**	360
Undecided	**54**	**66**	120
	450	550	1000

Now calculate the percents down the columns. 234/450 = 52% of Republicans supported the proposition. 162/450 = 36% of Republicans did not support the proposition, and 54/450 = 12% were undecided. 286/550 = 52% of Democrats supported the proposition, 198/550 = 36% did not, and 66/550 = 12% of Democrats were undecided. Make a two-way relative frequency table:

	Republican	Democrat
Supported proposition	52%	52%
Did not support proposition	36%	36%
Undecided	12%	12%
	100%	100%

Compare the percentages across the rows. They are all the same, indicating that there is no association between political party and preference for the ballot proposition. 52% of voters supported the ballot proposition regardless of whether they were a Democrat or Republican.

Example 2

A survey of 155 recent high school graduates found that 130 had driver's licenses and 58 had jobs. Twenty-one said they had neither a driver's license nor a job. Is there an association between having a driver's license and a job among the recent graduates?

Solution

Make a two-way table to display the counts (the number of graduates) in each category.
Set up the table and enter the given information.

	Job	No Job	
License			**130**
No Lic.		**21**	
	58		**155**

Add or subtract to fill in the remaining counts.

	Job	No Job	
License	54	76	**130**
No Lic.	4	**21**	25
	58	97	**155**

Investigate: Of the 58 graduates who have jobs, find the percentages of those who have, and do not have, a driver's license. Then, of the 97 graduates who do not have jobs, find the percentages of those who have, and do not have, a driver's license. Fill in the table at right.

	Job	No Job
License	93%	78%
No Lic.	7%	22%
	100%	100%

Compare the percentages across the rows. In this case, the percentages across rows are quite different, indicating an association between having a job and a driver's license. People with jobs tend have a higher percentage of licenses (93%) that those without a job (78%). There is an association between being employed and having a driver's license.

Problems

1. In a survey of 200 pet owners, 76 claimed to not own a cat and 63 indicated they did not own a dog. Eighty-two responded that they own both a cat and a dog. Make a two-way table displaying the counts in each category. Is there an association between cat and dog ownership?

2. Is there an association between Parker's pitching style and Brandon's chance of getting to base when batting in softball games? Brandon batted 32 times when Parker was pitching. Brandon got to base 8 times. Parker used his fastball 12 times total, but Brandon got to base only three times when Parker used his fastball. Make a two-way table and decide whether there is an association between pitching style and getting to base.

Answers

1.

	Dog	No Dog	
Cat	**82**		
No Cat			**76**
		63	**200**

	Dog	No Dog	
Cat	**82**	42	124
No Cat	55	21	**76**
	137	**63**	**200**

	Dog	No Dog
Cat	60%	67%
No Cat	40%	33%
	100%	100%

There appears to be a weak association between cat and dog ownership. The probability of owning a cat is slightly lower if you own a dog. Only 60% of dog owners have a cat, while 67% of non-dog owners have a cat. That is, you are more likely to own a cat if you are not a dog owner. So there is an association. The association is weak—there is not a big difference.

Alternatively, it is also possible to set up the table as follows. The conclusion that there is a weak association is the same. You are less likely to own a dog if you own a cat.

	Cat	No Cat
Dog	66%	72%
No Dog	34%	28%
	100%	100%

2.

	Parker		
	Fastball	Other Pitch	
Brandon Got To Base	**3**	5	**8**
Brandon Struck Out	9	15	24
	12	20	**32**

	Parker	
	Fastball	Other Pitch
Brandon Got To Base	25%	25%
Brandon Struck Out	75%	75%
	100%	100%

There is no association between Parker's pitch and Brandon's ability to get on base. The probability Brandon gets to base is 25%, regardless of whether Parker uses his fastball pitch or not.

Alternatively, it is also possible to set up the table as follows. The conclusion is the same: there is no association. Parker used a fastball 37.5% of the time, regardless of whether Brandon got to base or not.

	Brandon	
	Got To Base	Struck Out
Parker Fastball	37.5%	37.5%
Parker Other Pitch	62.5%	62.5%
	100%	100%

SOLVING BY REWRITING: FRACTION BUSTERS 10.2.1 and 10.2.2

Equations with fractions and/or decimals can be converted into equivalent equations without fractions and/or decimals and then solved in the usual manner. Equations can also be made simpler by factoring a common numerical factor out of each term. Fractions can be eliminated from an equation by multiplying BOTH sides (and all terms) of an equation by the common denominator. If you cannot easily determine the common denominator, then multiply the entire equation by the product of all of the denominators. We call the term used to eliminate the denominators a **fraction buster**. Also remember to check your answers.

For additional information, see the Math Note boxes in Lessons 10.2.1 and 10.2.2.

Example 1

Solve: $0.12x + 7.5 = 0.2x + 3$

Multiply to remove the decimals.

$$100 \cdot (0.12x + 7.5 = 0.2x + 3)$$
$$12x + 750 = 20x + 300$$

Solve in the usual manner.

$$-8x = -450$$
$$x = 56.25$$

Example 2

Solve: $25x^2 + 125x + 150 = 0$

Divide each term by 25 (a common factor).

$$x^2 + 5x + 6 = 0$$

Solve in the usual manner.

$$(x + 2)(x + 3) = 0$$
$$x = -2 \text{ or } x = -3$$

Example 3

Solve: $\frac{x}{2} + \frac{x}{6} = 7$

Multiply both sides of the equation by 6, the common denominator, to remove the fractions.

$$6 \cdot \left(\frac{x}{2} + \frac{x}{6}\right) = 6(7)$$

(Multiplying both sides by 12 would also have been acceptable.)

Distribute and solve as usual.

$$6 \cdot \frac{x}{2} + 6 \cdot \frac{x}{6} = 6 \cdot 7$$
$$3x + x = 42$$
$$4x = 42$$
$$x = \frac{42}{4} = \frac{21}{2} = 10.5$$

Example 4

Solve: $\frac{5}{2x} + \frac{1}{6} = 8$

Multiply both sides of the equation by $6x$, the common denominator, to remove the fractions.

$$6x \cdot \left(\frac{5}{2x} + \frac{1}{6}\right) = 6x(8)$$

(Multiplying both sides by $12x$ would also have been acceptable.)

Distribute and solve as usual.

$$6x \cdot \frac{5}{2x} + 6x \cdot \frac{1}{6} = 6x \cdot 8$$
$$15 + x = 48x$$
$$15 = 47x$$
$$x = \frac{15}{47} \approx 0.32$$

Problems

Rewrite each equation in a simpler form and then solve the new equation.

1. $\frac{x}{3}+\frac{x}{2}=5$
2. $3000x+2000=-1000$
3. $0.02y-1.5=17$
4. $\frac{x}{2}+\frac{x}{3}-\frac{x}{4}=12$
5. $50x^2-200=0$
6. $\frac{x}{9}+\frac{2x}{5}=3$
7. $\frac{3x}{10}+\frac{x}{10}=\frac{15}{10}$
8. $\frac{3}{2x}+\frac{5}{x}=\frac{13}{6}$
9. $x^2-2.5x+1=0$
10. $\frac{2}{3x}-\frac{1}{x}=\frac{1}{36}$
11. $0.002x=5$
12. $10+\frac{5}{x}+\frac{3}{3x}=11$
13. $0.3(x+7)=0.2(x-2)$
14. $x+\frac{x}{2}+\frac{3x}{5}=21$
15. $32\cdot 3x-32\cdot 1=32\cdot 8$
16. $5+\frac{2}{x}+\frac{5}{4x}=\frac{73}{12}$
17. $\frac{17}{2x+1}=\frac{17}{5}$
18. $2+\frac{6}{x}+\frac{6}{3x}=3$
19. $2.5x^2+3x+0.5=0$
20. $\frac{x}{x-2}=\frac{7}{x-2}$

Answers

1. $x=6$
2. $x=-1$
3. $y=925$
4. $x=\frac{144}{7}\approx 20.57$
5. $x=\pm 2$
6. $x=\frac{135}{23}\approx 5.87$
7. $x=\frac{15}{4}=3.75$
8. $x=3$
9. $x=\frac{1}{2}$ or 2
10. $x=-12$
11. $x=2500$
12. $x=6$
13. $x=-25$
14. $x=10$
15. $x=3$
16. $x=3$
17. $x=2$
18. $x=8$
19. $x=-\frac{1}{5}$ or -1
20. $x=7$ Note: x cannot be 2.

MULTIPLE METHODS FOR SOLVING EQUATIONS 10.2.3 and 10.2.6

Equations may be solved in a variety of ways. These sections in the textbook use three methods that allow students to find the solutions to sometimes complex equations in more efficient ways. The methods are called **Rewriting**, **Looking Inside**, and **Undoing**.

Example 4 is a reminder that when solving an equation it is necessary to find all of the solutions. Problems with squares may have two solutions, one solution, or no solution.

$x^2 = 144$	$(x-3)^2 = 0$	$(2x+1)^2 = -7$
$x = \pm 12$	$x = 3$	no real number squared is negative

For additional information, see the Math Notes boxes in Lessons 10.2.3 and 10.2.6.

Example 1 Rewriting

Solve: $\frac{x}{2} + \frac{x}{5} = 3$

This problem can be rewritten without fractions using fraction busters.

$$\begin{aligned} 10 \cdot \left(\tfrac{x}{2} + \tfrac{x}{5}\right) &= 10(3) \\ 5x + 2x &= 30 \\ 7x &= 30 \\ x &= \tfrac{30}{7} \approx 4.29 \end{aligned}$$

Example 2 Undoing

Solve: $\sqrt{x} + 1 = 6$

This problem has a square root and a sum, so subtracting and then squaring both sides of the equation can undo it.

$$\begin{aligned} \sqrt{x} + 1 - 1 &= 6 - 1 \\ \sqrt{x} &= 5 \\ \left(\sqrt{x}\right)^2 &= 5^2 \\ x &= 25 \end{aligned}$$

Example 3 Looking Inside

Solve: $\frac{x+1}{2} = -6$

Looking inside the fraction can solve this problem. Since $\frac{\text{something}}{2} = -6$, the numerator must be -12.

$$\begin{aligned} x + 1 &= -12 \\ x &= -13 \end{aligned}$$

Example 4 Looking Inside

Solve: $|2x+3| = 11$

Looking inside the absolute value can solve this problem. Since $|11| = |-11| = 11$,

$$2x + 3 = 11 \text{ or } 2x + 3 = -11 .$$

Solving both equations yields two answers:

$$\begin{aligned} 2x = 8 \quad &\text{or } 2x = -14 \\ x = 4 \quad &\text{or } x = -7 \end{aligned}$$

Problems

Solve each equation. Find all solutions.

1. $\frac{3x+1}{2} = 5$
2. $4(x-1) + 3 = 15$
3. $\sqrt{2x+5} = 10$
4. $10 - (x+7) = 5$
5. $3(2x-7) = -21$
6. $8 + (\frac{y}{2}) = 10$
7. $\sqrt{x} - 3 = 7$
8. $(x+1)^2 = 81$
9. $\frac{x}{2} - \frac{x}{5} = 3$
10. $|x-2| = 5$
11. $2\sqrt{x-3} = 8$
12. $4(x-1) = 16$
13. $|2x+1| = 9$
14. $\frac{y+7}{3} = 10$
15. $\frac{m}{3} - \frac{2m}{5} = \frac{1}{5}$
16. $x^2 + 5 = 4$
17. $\frac{3y-1}{3} = 10$
18. $2(3x-1) + 7 = -13$
19. $(y-1)^2 = 9$
20. $20 - (3x) = 10$

Answers

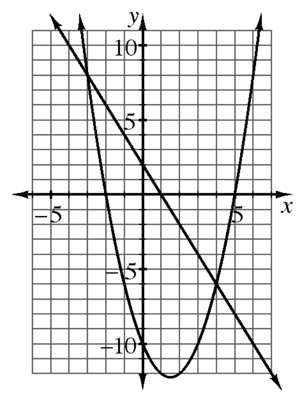
1. $x = 3$
2. $x = 4$
3. $x = \frac{95}{2} = 47.5$
4. $x = -2$
5. $x = 0$
6. $y = 4$
7. $x = 100$
8. $x = 8$ or -10
9. $x = 10$
10. $x = 7$ or -3
11. $x = 19$
12. $x = 5$
13. $x = 4$ or -5
14. $y = 23$
15. $m = -3$
16. no solution
17. $y = \frac{31}{3} = 10.\overline{3}$
18. $x = -3$
19. $y = 4$ or -2
20. $x = \frac{10}{3}$

INTERCEPTS AND INTERSECTIONS 10.3.1

> **Intercepts** and **intersections** both represent a point at which two paths cross, but an intercept specifies where a curve or line crosses an *axis*, whereas an intersection refers to any point where the graphs of two equations cross. When the graph is given, the points may be estimated from the graph. For more accuracy, intercepts and intersections may be found using algebra in most cases.

Example 1

Use the graphs of the parabola $y = x^2 - 3x - 10$ and the line $y = -2x + 2$ at right. The graph shows that the parabola has two x-intercepts $(-2, 0)$ and $(5, 0)$ and one y-intercept $(0, -10)$. The line has x-intercept $(1, 0)$ and y-intercept $(0, 2)$. The parabola and the line cross each other twice yielding two points of intersection: $(-3, 8)$ and $(4, -6)$.



Recall that x-intercepts are always of the form $(x, 0)$ so they can be found by making $y = 0$ and solving for x. Similarly, y-intercepts are always of the form $(0, y)$ so they can be found by making $x = 0$ and solving for y. To find the points of intersection, solve the system of equations. See the algebraic solutions below.

Intercepts

To find the x-intercepts of the parabola, make $y = 0$ and solve for x.

$$0 = x^2 - 3x - 10$$

Factor and use the Zero Product Property.

$$0 = (x - 5)(x + 2)$$

$$x = 5 \text{ or } x = -2,$$

so $(5, 0)$ and $(-2, 0)$ are the x-intercepts.

To find the y-intercepts of the parabola, make $x = 0$ and solve for y.

$$y = 0^2 - 3 \cdot 0 - 10 = -10$$

so $(0, -10)$ is the y-intercept.

To find the x- and y-intercepts of the line follow the same procedure.

If $x = 0$, then $y = -2 \cdot 0 + 2 = 2$ so $(0, 2)$ is the y-intercept.

If $y = 0$, then $0 = -2x + 2$, and $x = 1$ so $(1, 0)$ is the x-intercept.

Intersection

To find the point(s) of intersection of the system of equations use the Equal Values Method or substitution.

$$x^2 - 3x - 10 = -2x + 2$$

Make one side equal to zero, factor, and use the Zero Product Property to solve for x. (The Quadratic Formula is also possible.)

$$x^2 - x - 12 = 0$$

$$(x - 4)(x + 3) = 0$$

$$x = 4 \text{ or } x = -3$$

Substituting $x = 4$ into either equation yields $y = -6$, so $(4, -6)$ is a point of intersection.

Substituting $x = -3$ into either equation yields $y = 8$, so $(-3, 8)$ is a point of intersection.

Example 2

Given the lines $y = \frac{1}{3}x + 8$ and $y = \frac{1}{2}x - 3$, determine their intercepts and intersection without graphing. Using the same methods as shown in Example 1:

Intercepts

For the line $y = \frac{1}{3}x + 8$:

If $x = 0$, then $y = \frac{1}{3} \cdot 0 + 8 = 8$, so (0, 8) is the y-intercept.

If $y = 0$, $0 = \frac{1}{3}x + 8$, and $x = -24$, so (–24, 0) is the x-intercept.

For the line $y = \frac{1}{2}x - 3$:

If $x = 0$, then $y = \frac{1}{2} \cdot 0 - 3 = -3$, so (0, –3) is the y-intercept.

If $y = 0$, then $0 = \frac{1}{2}x - 3$, and $x = 6$, so (6, 0) is the x-intercept.

Intersection

$$\frac{1}{3}x + 8 = \frac{1}{2}x - 3$$

$$6 \cdot \left(\frac{1}{3}x + 8 = \frac{1}{2}x - 3\right)$$

$$2x + 48 = 3x - 18$$

$$66 = x$$

Substituting $x = 66$ into either original equation yields $y = 30$.

The point of intersection is (66, 30).

Problems

For each equation, determine the x- and y-intercepts.

1. $y = 3x - 2$
2. $y = \frac{1}{4}x - 2$
3. $3x + 2y = 12$
4. $-x + 3y = 15$
5. $2y = 15 - 3x$
6. $y = x^2 + 5x + 6$
7. $y = x^2 - 3x - 10$
8. $y = 4x^2 - 11x - 3$
9. $y = x^2 + 3x - 2$

Determine the point(s) of intersection

10. $y = 5x + 1$; $y = -3x - 15$
11. $y = x + 7$; $y = 4x - 5$
12. $x = 7 + 3y$; $x = 4y + 5$
13. $x = -\frac{1}{2}y + 4$; $8x + 3y = 31$
14. $4x - 3y = -10$; $x = \frac{1}{4}y - 1$
15. $2x - y = 6$; $4x - y = 12$
16. $y = x^2 - 3x - 8$; $y = 2$
17. $y = x^2 - 7$; $y = 8 + 2x$
18. $y = x^2 + 2x + 8$; $y = -4x - 1$

Answers

1.	$(0.\overline{6}, 0), (0, -2)$	2.	$(8, 0), (0, -2)$	3.	$(4, 0), (0, 6)$
4.	$(-15, 0), (0, 5)$	5.	$(5, 0), (0, 7.5)$	6.	$(-2, 0), (-3, 0), (0, 6)$
7.	$(5, 0), (-2, 0)$ $(0, -10)$	8.	$(-0.25, 0), (3, 0)$ $(0, -3)$	9.	$(0.56, 0), (-3.56, 0)$ $(0, -2)$
10.	$(-2, -9)$	11.	$(4, 11)$	12.	$(13, 2)$
13.	$(3.5, 1)$	14.	$(-0.25, 3)$	15.	$(3, 0)$
16.	$(5, 2), (-2, 2)$	17.	$(5, 18), (-3, 2)$	18.	$(-3, 11)$

SOLVING QUADRATIC AND ABSOLUTE VALUE INEQUALITIES 10.3.3

There are several methods for solving absolute value and quadratic inequalities, but one method that works for all kinds of inequalities is to change the inequality to an equation, solve it, and then graph the solution(s) on a number line. The solution(s), called "boundary point(s)," divide the number line into regions. Check any point within each region in the *original* inequality. If that point is true, then all the points in that region are solutions. If that point is false, then none of the points in that region are solutions. The boundary points are included in (≥ or ≤) or excluded from (> or <) the solution depending on the inequality sign.

Solving an absolute value or quadratic inequality is very similar to solving a linear inequality in one variable, except that there are often three or more solution regions on the number line instead of just two. For more on solving linear inequalities see the section "Solving One-Variable Inequalities (9.2.1 and 9.2.2)" of this *Parent Guide with Extra Practice*.

Example 1

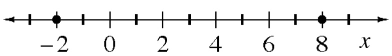
Solve: $|x-3| \le 5$

Change to an equation and solve.

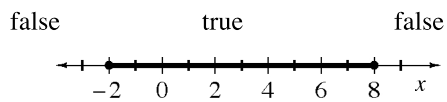
$$|x-3| = 5$$

$$x - 3 = 5 \text{ or } x - 3 = -5$$

$$x = 8 \text{ or } x = -2 \text{ (the boundary points)}$$



Choose $x = -3$, $x = 0$, and $x = 9$ to test in the original inequality. $x = -3$ is false, $x = 0$ is true, and $x = 9$ is false.



The solution is all numbers greater than or equal to –2 and less than or equal to 8, written as $-2 \le x \le 8$.

Example 2

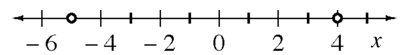
Solve: $|2y+1| > 9$

Change to an equation and solve.

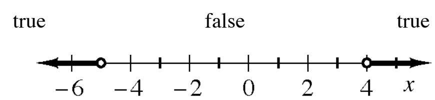
$$|2y+1| = 9$$

$$2y + 1 = 9 \text{ or } 2y + 1 = -9$$

$$y = 4 \text{ or } y = -5 \text{ (the boundary points)}$$



Choose $y = -6$, $y = 0$, and $y = 5$ to test in the original inequality. $y = -6$ is true, $y = 0$ is false, and $y = 5$ is true.



The solution is all numbers less than –5 or greater than 4, written as $y < -5$ or $y > 4$.

Example 3

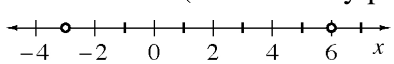
Solve: $x^2 - 3x - 18 < 0$

Change to an equation and solve.

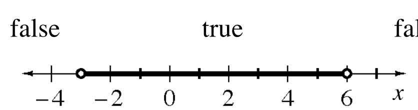
$$x^2 - 3x - 18 = 0$$

$$(x-6)(x+3) = 0$$

$$x = 6 \text{ or } x = -3 \text{ (the boundary points)}$$



Choose $x = -4$, $x = 0$, and $x = 7$ to test in the original inequality. $x = -4$ is false, $x = 0$ is true, and $x = 7$ is false.



The solution is all numbers greater than –3 and less than 6, written as $-3 < x < 6$.

Example 4

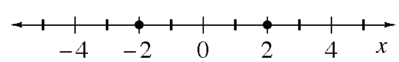
Solve: $m^2 - 3 \ge 1$

Change to an equation and solve.

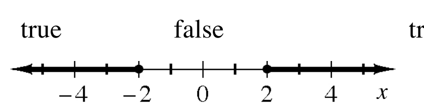
$$m^2 - 3 = 1$$

$$m^2 = 4$$

$$m = \pm 2 \text{ (the boundary points)}$$



Choose $m = -3$, $m = 0$, and $m = 3$ to test in the original inequality. $m = -3$ is true, $m = 0$ is false, and $m = 3$ is true.



The solution is all numbers less than or equal to –2 or greater than or equal to 2, written as $m \le -2$ or $m \ge 2$.

Problems

Solve each inequality.

1.	$\|x+4\| \ge 7$	2.	$x^2+6x+8<0$	3.	$y^2-5y>0$
4.	$\|x\|-5 \le 8$	5.	$\|x-5\| \le 8$	6.	$y^2-5y<0$
7.	$\|4r-2\| > 8$	8.	$x^2-3x-4<0$	9.	$\|3x\| \le 12$
10.	$-x^2-9x-14<0$	11.	$\|1-3x\| \le 13$	12.	$y^2 \le 16$
13.	$\|2x-3\| > 15$	14.	$3x^2+7x-6 \ge 0$	15.	$\|5x\| > -15$
16.	$-2\|x-3\|+6<-4$	17.	$x^2+4x-8<4$	18.	$y^2+6y+9>0$
19.	$\|4-d\| \le 7$	20.	$x(7x-26) \le 8$		

Answers

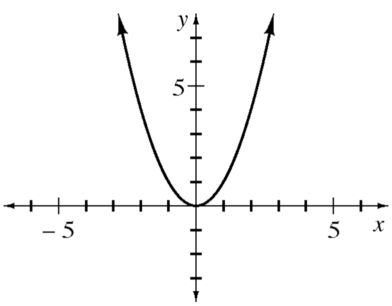
1. $x \geq 3$ or $x \leq -11$	2. $-4 < x < -2$	3. $y < 0$ or $y > 5$
4. $-13 \leq x \leq 13$	5. $-3 \leq x \leq 13$	6. $0 < y < 5$
7. $r < -\frac{3}{2}$ or $r > \frac{5}{2}$	8. $-1 < x < 4$	9. $-4 \leq x \leq 4$
10. $x < -7$ or $x > -2$	11. $-4 < x < \frac{14}{3}$	12. $-4 \leq y \leq 4$
13. $x < -6$ or $x > 9$	14. $x \leq -3$ or $x \geq \frac{2}{3}$	15. all numbers
16. $x > 8$ or $x < -2$	17. $-6 < x < 2$	18. all numbers except $y = -3$
19. $-3 \leq d \leq 11$	20. $-\frac{2}{7} \leq x \leq 4$	

TRANSFORMATIONS OF FUNCTIONS 11.1.1

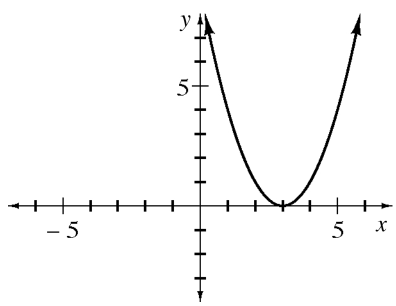
The simplest equation of one shape (e.g., line, parabola, absolute value) is called a parent equation. Changing a parent equation by addition or multiplication moves and changes the size and orientation of the parent graph but does not change the basic shape. These changes are called **transformations**. The first set of examples shows how the parent graph of a parabola can be moved on an *xy*-coordinate system. In later courses, you will learn how to make the parabola wider or narrower. Transformations of other functions are done in a similar manner.

Examples

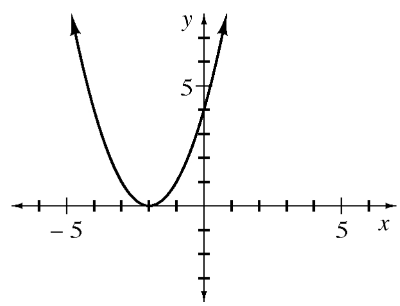
$f(x) = x^2$
the parent graph



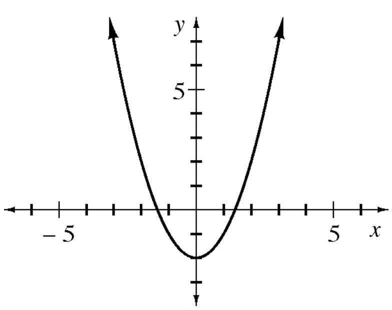
$f(x-3) = (x-3)^2$
right 3 units



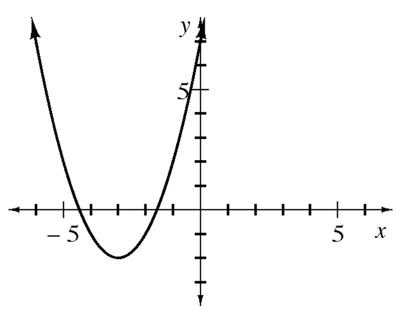
$f(x+2) = (x+2)^2$
left 2 units



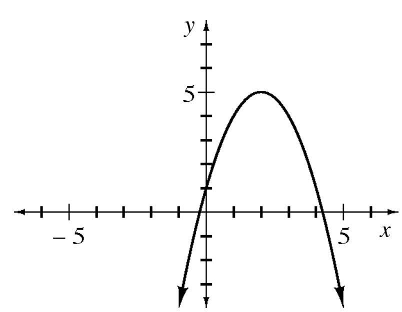
$f(x)-2 = x^2-2$
down 2 units



$f(x+3)-2 = (x+3)^2-2$
left 3 and down 2 units



$-f(x-2)+5 = -(x-2)^2+5$
right 2 and up 5 units, and reflected across *x*-axis.



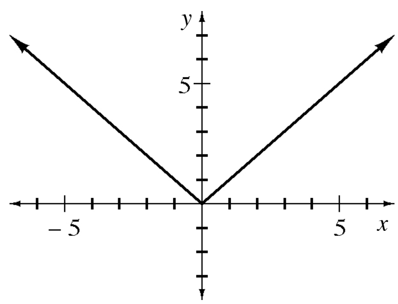
Problems

Predict how each parabola is different from the parent graph. For each of these problems, the parent graph is $f(x) = x^2$.

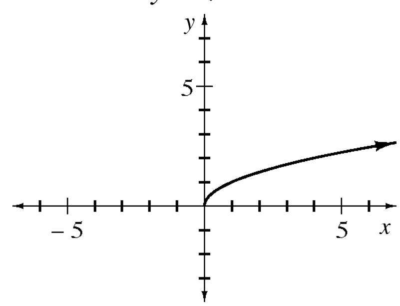
1. $y = f(x + 5)$
2. $y = f(x) + 5$
3. $y = -f(x)$
4. $y = f(x) - 4$
5. $y = f(x - 5)$
6. $y = -f(x + 2)$
7. $y = f(x - 5) - 3$
8. $y = -f(x + 2) + 1$
9. $y = f(x - 3) - 5$

The parent graphs for absolute value and square root are shown at right. They are transformed exactly the same way as parabolas. Predict how the graph of each equation below is different from the parent graph.

$y = |x|$



$y = \sqrt{x}$



10. $f(x - 4)$ where $f(x) = |x|$
11. $f(x) + 5$ where $f(x) = \sqrt{x}$
12. $f(x) + 3$ where $f(x) = |x|$
13. $f(x + 2)$ where $f(x) = \sqrt{x}$
14. $-f(x) + 5$ where $f(x) = |x|$
15. $-f(x)$ where $f(x) = \sqrt{x}$
16. $-f(x + 3)$ where $f(x) = \sqrt{x}$
17. $-f(x - 5)$ where $f(x) = |x|$
18. $-f(x - 2) + 5$ where $f(x) = \sqrt{x}$
19. $f(x) + 3$ where $f(x) = \sqrt{x}$
20. $f(x - 7)$ where $f(x) = \sqrt{x}$
21. $-f(x + 3) + 7$ where $f(x) = |x|$

Answers

1. left 5
2. up 5
3. reflected across x-axis
4. down 4
5. left 5
6. left 2, reflected across x-axis
7. right 5, down 3
8. left 2, up 1, reflected across x-axis
9. right 3, down 5
10. right 4
11. up 5
12. up 3
13. left 2
14. up 5, reflected across x-axis
15. reflected across x-axis
16. left 3, reflected across x-axis
17. right 5, reflected across x-axis
18. right 2, up 5, reflected across x-axis
19. up 3
20. right 7
21. left 3, up 7, reflected across x-axis

COMPARING AND REPRESENTING DATA 11.2.1 through 11.2.3

Data distributions can be represented graphically with histograms and boxplots. Boxplots are described in the Math Notes box in Lesson 11.2.1. For assistance with histograms, see the *Parent Guide with Extra Practice* for Lesson 1.1.4 of *Core Connections, Course 1*, which is available free of charge at cpm.org.

Two distributions of data can be compared by comparing their center, shape, spread, and outliers.

The center, or "typical" value, of a data distribution can be described by the median. If the distribution is symmetric and has no outliers, the mean can be used to describe the center.

Common shapes of data distributions can be found in the Math Notes box in Lesson 11.2.2.

The spread of a distribution can be described with the interquartile range (IQR), described in the Math Notes boxes in Lesson 11.2.1, or the standard deviation, described in the Math Notes box in Lesson 11.2.3. Since the standard deviation is based upon the mean, it should be used only to describe the spread of distributions that are symmetric and without outliers.

An outlier is a data value that is far away from the bulk of the data.

Example 1

University professors are complaining that the English Literature classes at community colleges are not demanding enough. Specifically, the university professors claim that community college literature courses are not assigning enough novels to read. A community college statistics student collected the following data from 42 universities and community college literature courses in the state. Compare the number of novels read in the two types of colleges.

Number of novels assigned in community college literature courses:
13, 10, 15, 12, 14, 9, 11, 15, 12, 14, 9, 10, 13, 15, 12, 9, 11, 15, 12, 10, 15, 14 *checksum 270*

Number of novels assigned in university literature courses:
11, 8, 14, 13, 25, 11, 7, 13, 8, 16, 11, 10, 20, 7, 8, 13, 14, 16, 18, 10 *checksum 253*

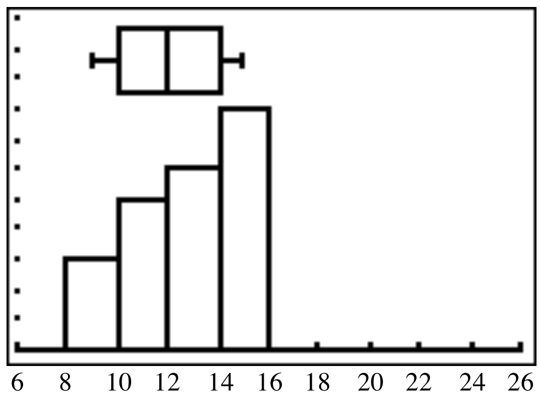
Solution follows on next page →

Solution

Any analysis of data distributions should begin with a graphical representation of the data. A bin width of two was chosen for the histograms that follow. So that the distributions can be compared, both graphs have the same scale on the *x*-axis, and the graphs are stacked on top of each other. For assistance with using a TI-83+/84+ calculator, refer to your eBook or visit studenthelp.cpm.org.

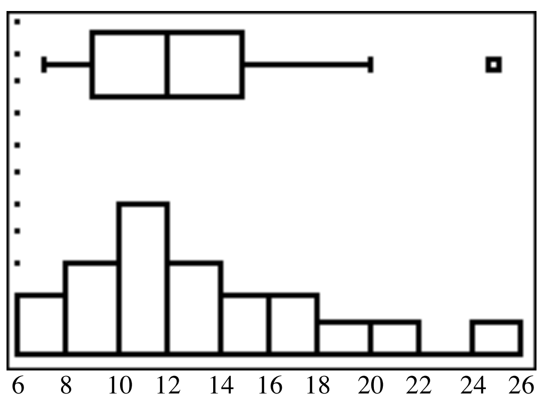
The *checksum* is used to verify that data has been entered into the graphing technology correctly. The sum of the data set, as determined by the statistical functions of the calculator, should match the given *checksum* value.

Community College



Five Number Summary
(9, 10, 12, 14, 15)

University



Five Number Summary
(7, 9, 12, 15, 25)

When comparing the distributions, the center, shape, spread, and outliers should be considered. Since neither of the distributions is nearly symmetric, and one of the distributions has an outlier, it would not be appropriate to use the means or standard deviations to compare. The five number summaries (see the Math Notes box in Lesson 11.2.1) are shown to the right of each graph.

Center: Both types of colleges assign the same median of 12 novels.

Solution continues on next page →

Solution continued from previous page.

Shape: The distribution for community colleges is skewed, with a low of 8 to 9 novels and increasing to a peak at 14 to 15 novels. The distribution at universities is skewed in the other direction, with a peak at 10 to 11 novels.

Spread: The variability in the number of novels assigned at the community college level is much less than the variability between courses at the university level. The IQR for community colleges is 4 novels (14 – 10 = 4), while the university IQR of 6 (15 – 9 = 6) is one-and-a-half times as wide.

Outliers: One course at a university is an outlier; 25 books are assigned in that course. Twenty-five books is far away from the bulk of the university courses. The TI-83/84+ calculator can mark an outlier on a boxplot with dots.

Conclusions: The university professors claim that their courses are more demanding because they assign more novels. However that data does not bear this claim out. 25% of university courses assign more novels than any of the community college courses (the right "whisker," or the top 25% of the courses, for universities is beyond the entire boxplot for community colleges). But just as dramatically, 25% of the university classes assign *fewer* books than any of the community colleges (the left "whisker," or lowest 25%, for universities is *below* the entire boxplot for community colleges). Furthermore, the median number of novels assigned at the two universities is the same—12 books. Community colleges are more consistent from course-to-course in the number of novels they assign (IQR is 4) than are the universities (IQR is 6).

Example 2

A rabbit breeder kept track of the number of offspring from five does (female rabbits) this year. The does had: 243, 215, 184, 280, and 148 kits (baby rabbits) respectively. Show how to calculate the mean and standard deviation of the number of kits per doe without using the statistical functions of a calculator.

The mean is $\frac{243+215+184+280+148}{5} = 214$ kits.

The standard deviation is the square root of the average of the distances to the mean, after the distances have been made positive by squaring. To find the standard deviation, first find the distance each doe is from the mean:
$243-214=29$, $215-214=1$, $184-214=-30$, $280-214=66$, $148-214=-66$.

Find each of the distances squared: $29^2 = 841$, $1^2 = 1$, $(-30)^2 = 900$, $66^2 = 4356$, $(-66)^2 = 4356$.

The mean distance-squared is: $\frac{841+1+900+4356+4356}{5} = 2090.8$

The square root is 45.725. Since the precision of the original measurements was an integer, the final result should also be an integer. The mean number of kits per doe is 214 with a standard deviation of 46 kits.

Problems

1. Different types of toads tend to lay different numbers of eggs. The following data was collected from two different species. Compare the number of eggs laid by American toads to the number laid by Fowler toads. Is it appropriate to summarize the distributions by using mean and standard deviation? Use a bin width of 250 eggs.

   American toads: 9100, 8700, 10300, 9500, 7800, 8900, 9200, 9300, 8900, 8300, 9400, 8000, 9000, 8400, 9700, 10000, 8600, 8900, 9900, 9300 *checksum 181,200*.

   Fowler Toads: 9500, 9100, 9400, 8800, 9000, 8400, 9200, 9200, 8900, 9100, 8600, 9200, 8700, 9800, 9300, 8800, 9200, 9300, 9000, 9100 *checksum 181,600*.

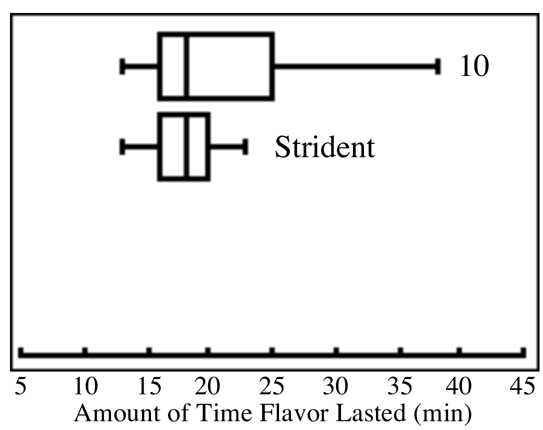
2. Without using the statistical functions on your calculator, find the standard deviation of the number of eggs laid by each of the first five American toads.

3. Do low birth weight babies start crawling at a later age than babies born at an average weight? A psychologist collected the following data for the age at which children started crawling:

   low weight babies: 10, 12, 11, 11, 7, 13, 10, 12, 11, 13, 10, 11, 15, 11, 14, 10 months *checksum 181*.

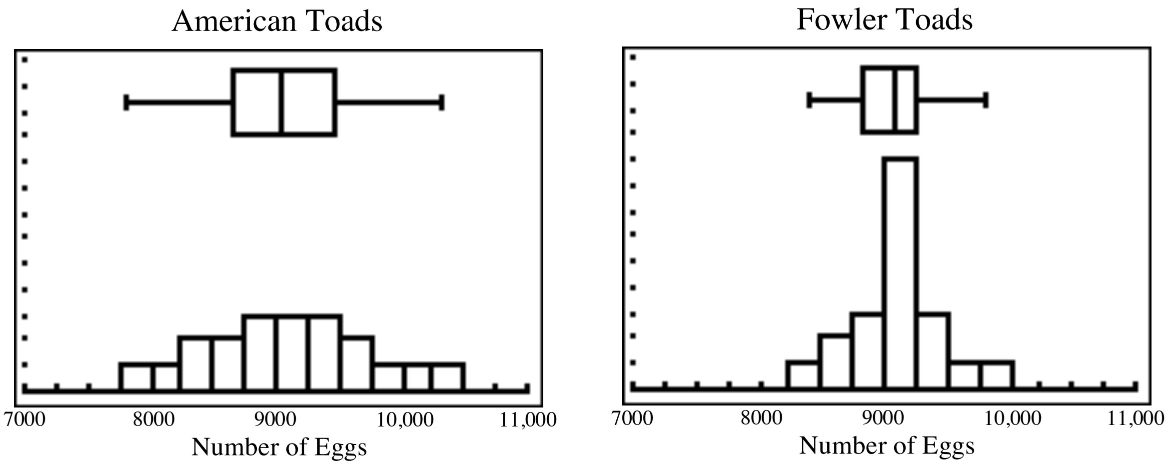
   average weight babies: 7, 6, 13, 9, 8, 7, 5, 7, 9, 8, 10, 8, 11, 7, 7, 10, 6, 8, 7, 6, 12, 8, 7 months *checksum 186*.

4. Compare the amount of time the flavor lasted for people chewing brand "10" chewing gum to the amount of time the flavor lasted in "Strident" chewing gum. See the graph at right. Estimate the mean for each type of gum.



Answers

1. American Toads / Fowler Toads

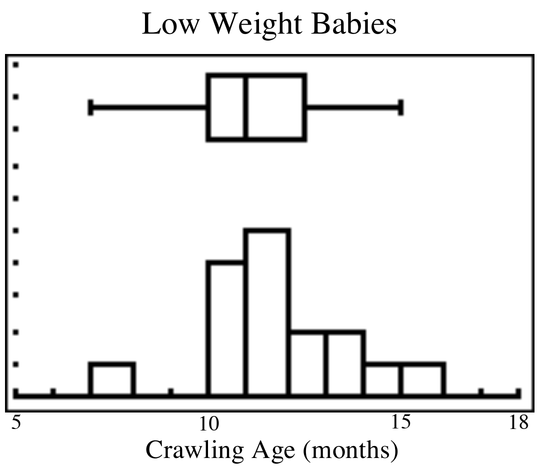


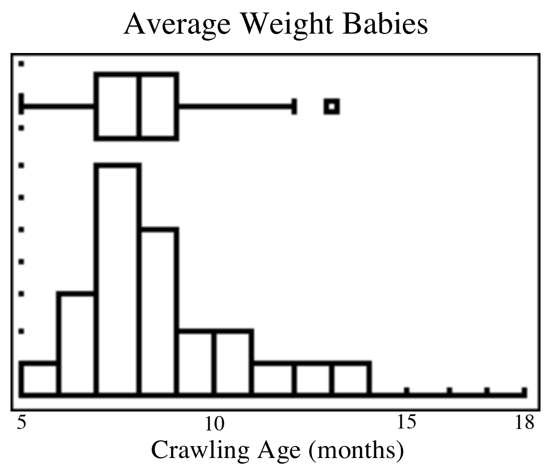
Mean and standard deviation are appropriate statistics because both distributions are fairly symmetric with no outliers.

Both types of toads lay a mean of between 9000 and 9100 eggs. Both distributions are single-peaked and symmetric with no apparent outliers. However there is much greater variability in the number of eggs that American toads lay. The standard deviation for American toads is about 637 while the standard deviation for Fowler toads is only half as much, about 316 eggs.

2. $\sqrt{\frac{20^2+(-380)^2+1220^2+420^2+(-1280)^2}{5}} \approx 830$ eggs

3.





The median and IQR will be used to compare statistics since mean and standard deviation are not appropriate—both distributions are skewed and one has an outlier.

The median age at which low-weight babies start crawling is 11 months, while the median age for average-weight babies is 8 months.

Both distributions are single-peaked and skewed. The low-weight babies appear to have an outlier at 7 months, although the calculator does not identify it as a true outlier. The average-weight babies have an outlier at 13 months.

The variability in the crawling age is roughly the same for low-weight babies (IQR is 2.5 months) as for average-weight babies (IQR is 2 months).

Low-weight babies have their development delayed by about 3 months. About 75% of low-weight babies have not started crawling at the age when 75% of average-weight babies are already crawling.

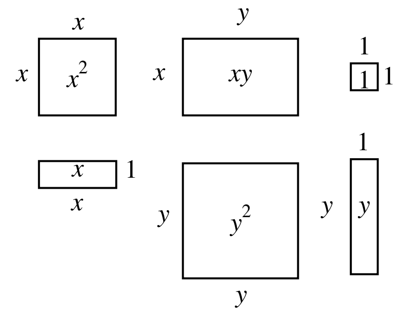
4. The median for both types of gum was about 18 minutes of flavor time. The times for “10” were skewed, while the times for Strident were symmetric. The lower half of the distributions for both gums was the same. But there was much more variability in the upper half of people chewing “10” than in the upper half of Strident. Indeed, more than 25% of “10” chewers reported flavor lasting longer than any of the Strident chewers. Neither gum had outliers in flavor time.

There was more variability in flavor time for “10”—the IQR was about 9 minutes (25 – 16 = 9). The IQR of 4 minutes (20 – 16 = 4) for Strident was less than half that of “10”. That variability is an advantage. If you chew “10,” you will probably be no worse off than chewing Strident, and you could have much longer flavor.

The mean for Strident is about the same as the median since the distribution is symmetric—about 18 minutes. But the mean for “10” is longer than 18 minutes due to the skew in the shape—maybe 22 minutes or so.

COMBINING LIKE TERMS A.1.1 and A.1.2

Algebra tiles are an important part of this course and will be used throughout. The tiles provide students with the opportunity to "see" abstract algebraic expressions and equations with two variables. Regular use of algebra tiles will help students access abstract concepts through the use of concrete physical representations.



In the figures at right, the dimensions of each tile are shown along its sides, and the area is shown on the tile itself. Algebra tiles are named by their areas. For example, the x^2-tile is in the upper left corner; it has an area of x^2.

In diagrams with combinations of tiles, as in Examples 1 to 3 that follow, the perimeter is the distance around the exterior of a figure. The area of a figure is the sum of the areas of the individual tiles.

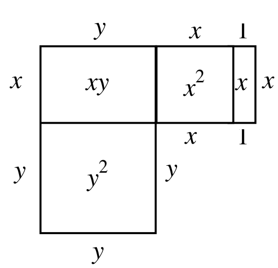
In algebraic expressions, as in Examples 4 to 6 below, combining terms that have the same area to write a simpler expression is called **combining like terms**.

For additional information, refer to the Math Notes boxes in Lessons A.1.1, A.1.2, and A.1.5.

Additional Math Notes boxes in Lesson A.1.3 and A.1.4 can help with evaluating expressions. Students evaluate expressions in homework and Checkpoint 2.

A resource page for creating algebra tiles for home use is available at the CPM website, cpm.org. Go to the student section, and download the Lesson A.1.1 Resource Page.

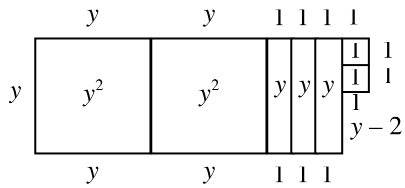
Example 1



Perimeter = $y + x + 1 + x + 1 + x + y + y + y + x = 4x + 4y + 2$
(going clockwise from upper left)

Area = $x^2 + xy + y^2 + x$

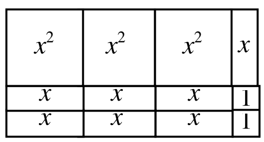
Example 2



Perimeter = $y + y + 7 + (y - 2) + 3 + y + y + y = 6y + 8$
(going clockwise from upper left)

Area = $y^2 + y^2 + y + y + y + 1 + 1 = 2y^2 + 3y + 2$

Example 3



Perimeter = $8x + 6$

Area = $3x^2 + 7x + 2$

Example 4

The expression $3x^2 + 7x + 2 + x^2 + 5 + 2x$ can be rewritten as $3x^2 + x^2 + 7x + 2x + 2 + 5$ and simplified to $4x^2 + 9x + 7$ by combining like terms.

Example 5

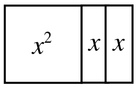
The expression $x^2 + xy + y^2 + 3xy + y^2 + 7$ can be rewritten as $x^2 + xy + 3xy + y^2 + y^2 + 7$ and simplified to $x^2 + 4xy + 2y^2 + 7$ by combining like terms.

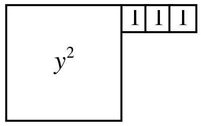
Example 6

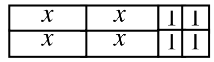
$3x^2 - 4x + 3 + -x^2 + 3x - 7$
$= 3x^2 - x^2 - 4x + 3x + 3 - 7$
$= 2x^2 - x - 4$

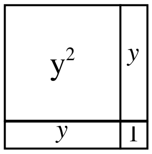
Problems

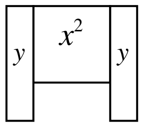
Determine the area and the perimeter of each figure.

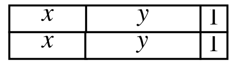
1. 

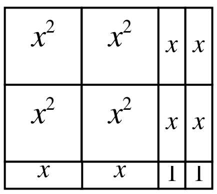
2. 

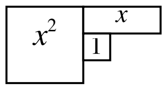
3. 

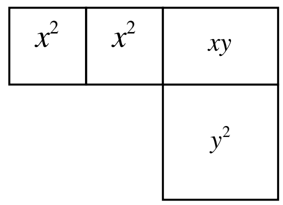
4. 

5. 

6. 

7. 

8. 

9. 

Simplify each expression by combining like terms.

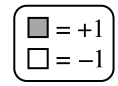
10. $2x^2 + x + 3 + 4x^2 + 3x + 5$
11. $y^2 + 2y + x^2 + 3y^2 + x^2$
12. $x^2 - 3x + 2 + x^2 + 4x - 7$
13. $y^2 + 2y - 3 - 4y^2 - 2y + 3$
14. $4xy + 3x + 2y - 7 + 6xy + 2x + 7$
15. $x^2 - y^2 + 2x + 3y + x^2 + y^2 + 3y$
16. $(4x^2 + 4x - 1) + (x^2 - x + 7)$
17. $(y^2 + 3xy + x^2) + (2y^2 + 4xy - x^2)$
18. $(7x^2 - 6x - 9) - (9x^2 + 3x - 4)$
19. $(3x^2 - 8x - 4) - (5x^2 + x + 1)$

Answers

1. P = $4x + 4$
   A = $x^2 + 2x$
2. P = $4y + 6$
   A = $y^2 + 3$
3. P = $4x + 8$
   A = $4x + 4$
4. P = $4y + 4$
   A = $y^2 + 2y + 1$
5. P = $4y + 4$
   A = $x^2 + 2y$
6. P = $2x + 2y + 6$
   A = $2x + 2y + 2$
7. P = $8x + 6$
   A = $4x^2 + 6x + 2$
8. P = $6x$
   A = $x^2 + x + 1$
9. P = $6x + 4y$
   A = $2x^2 + xy + y^2$
10. $6x^2 + 4x + 8$
11. $4y^2 + 2y + 2x^2$
12. $2x^2 + x - 5$
13. $-3y^2$
14. $10xy + 5x + 2y$
15. $2x^2 + 2x + 6y$
16. $5x^2 + 3x + 6$
17. $3y^2 + 7xy$
18. $-2x^2 - 9x - 5$
19. $-2x^2 - 9x - 5$

SIMPLIFYING ALGEBRAIC EXPRESSIONS A.1.3 through A.1.5

An Expression Mat is a physical representation of an algebraic expression. The upper half of an Expression Mat is the positive (addition) region and the lower half is the negative (subtraction) region. Positive algebra tiles are shaded and negative tiles are blank. (The illustration to the right reminds you that shaded tiles are positive.) A matching pair of tiles with one tile shaded and the other tile blank represents two opposites—with a value of zero. We refer to them as "zero pairs."



On an Expression Mat, tiles may be removed or moved in one of two "legal" ways:

(1) Flip tiles and move them from the negative region to the positive region. That is, change subtraction to adding the opposite.

(2) Remove an equal number of opposite tiles (one shaded and one not shaded) that are within the same region. These pairs of opposite tiles have a value of zero.

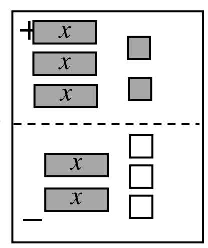
(3) Group tiles that are alike together. That is, combine like terms.

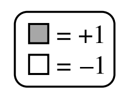
For additional information, see the Math Notes box in Lesson A.1.6.

Example 1

Simplify $3x + 2 - (2x - 3)$.

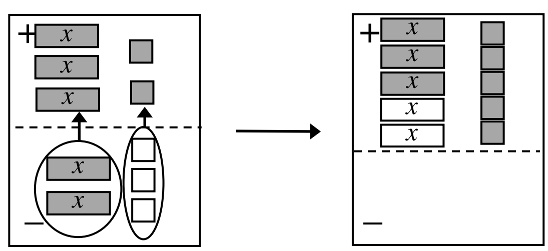
Create an Expression Mat:





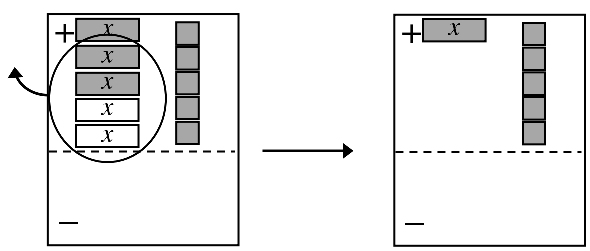
$3x + 2 - (2x - 3)$

Flip tiles in subtraction region to addition region:



$3x + 2 - 2x + 3$

Remove zero pairs:

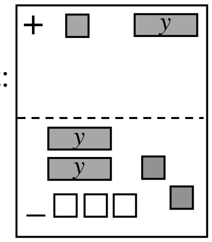


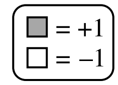
$x + 5$

Therefore, $3x + 2 - (2x - 3)$ simplifies to $x + 5$

Example 2

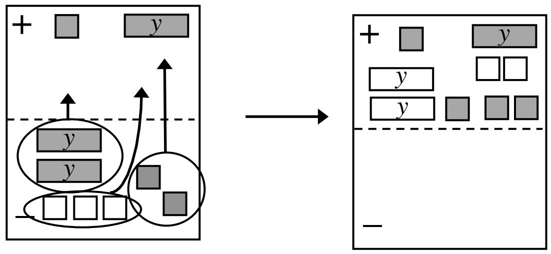
Simplify $1 - (2y - 3) + y - 2$. Create an Expression Mat:





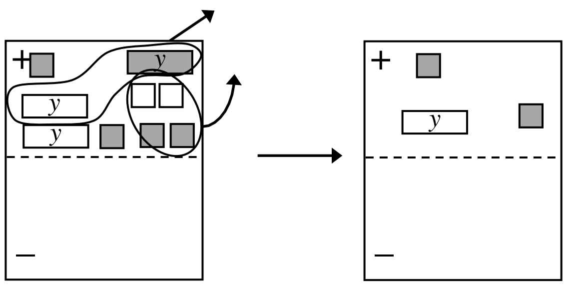
$1-(2y-3)+y-2$

Flip tiles in subtraction region to addition region:



$1+(-2y)+y+(-2)+3$

Remove zero pairs:

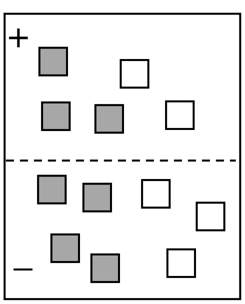


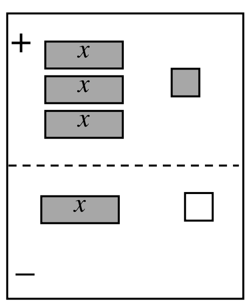
$2+(-y)$

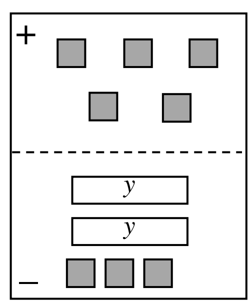
Therefore, $1 - (2y - 3) + y - 2$ simplifies to $2 + (-y)$ or $2 - y$.

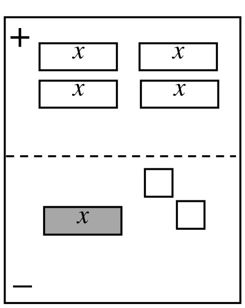
Problems

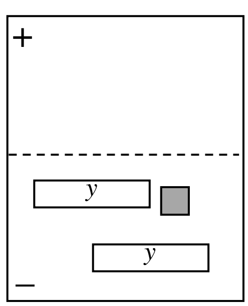
Write the algebraic expression, then simplify.

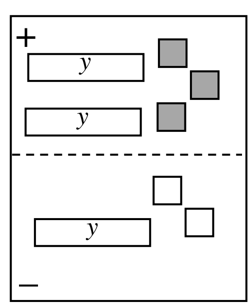
1. 

2. 

3. 

4. 

5. 

6. 

Use algebra tiles to simplify each expression.

7. $3 + 5x - 4 - 7x$
8. $-x - 4x - 7$
9. $-(-x + 3)$
10. $4x - (x + 2)$
11. $5x - (-3x + 2)$
12. $x - 5 - (2 - x)$
13. $1 - 2y - 2y$
14. $-3x + 5 + 5x - 1$
15. $3 - (y + 5)$
16. $-(x + y) + 4x + 2y$
17. $3x - 7 - (3x - 7)$
18. $-(x + 2y + 3) - 3x + y$

Answers

1. $3 + (-2) - 4 - (-3)$
   $= 0$
2. $3x + 1 - x - (-1)$
   $= 2x + 2$
3. $5 - (-2y) - (3)$ or
   $5 - (-2y + 3)$
   $= 2y + 2$
4. $-4x - x - (-2)$
   $= -5x + 2$
5. $-(-2y) - 1$ or
   $-(-2y + 1)$
   $= 2y - 1$
6. $3 + (-2y) - (-y) - (-2)$
   $= -y + 5$
7. $-2x - 1$
8. $-5x - 7$
9. $x - 3$
10. $3x - 2$
11. $8x - 2$
12. $2x - 7$
13. $-4y + 1$
14. $2x + 4$
15. $-y - 2$
16. $3x + y$
17. 0
18. $-4x - y - 3$

COMPARING ALGEBRAIC EXPRESSIONS A.1.6 and A.1.7

Expressions on two side-by-side Expression Mats can be compared to determine which expression is greater.

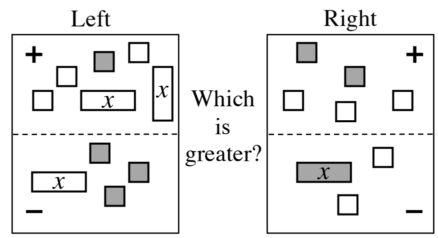
To compare two expressions, represent each expression with tiles on its own separate Expression Mat. Work on each Expression Mat separately to simplify the expression by moving or removing tiles using "legal" moves:

- Flip tiles and move them from the negative region to the positive region. That is, change subtraction to adding the opposite.
- Remove an equal number of opposite tiles (one shaded and one not shaded) that are within the same region. That is, remove the zero pairs.
- Place or remove the same tiles on or from *both* Expression Mats. That is, add or subtract the same value from both sides.
- Group tiles that are alike within the same region together. That is, combine like terms.

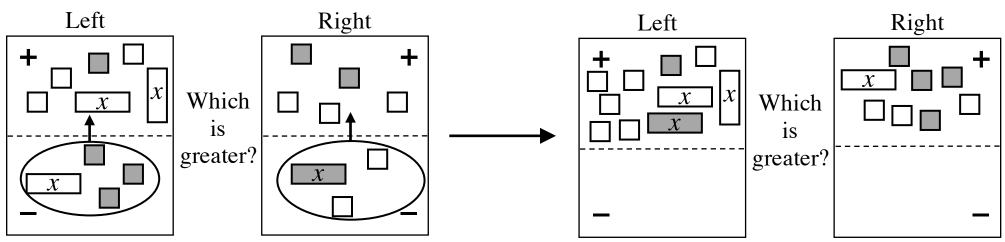
Continue to make "legal" moves, in any order, until the expressions cannot be simplified any more. Compare the expression on the left with the one on the right to determine which expression is greater. If there are variable tiles remaining after simplifying, you do not have enough information to tell which side is greater—depending on what number the variable tile represents, either Expression Mat could be larger than the other.

Example 1

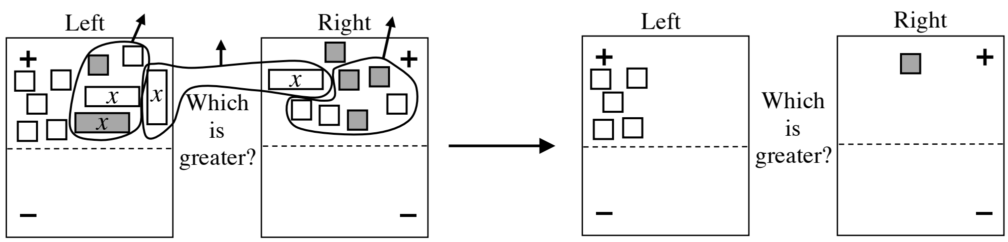
The Expression Comparison Mat at right represents the expressions $-2x - 3 + 1 - (-x + 3)$ and $2 - 3 - (x - 2)$. Use legal moves to simplify and determine which side is greater.



Solution:



Flip tiles and move them from the negative region to the positive region.



Remove an equal number of opposite tiles (one shaded and one not shaded) that are within the same region. Also remove the same tiles from *both* Expression Mats.

$-5 < 1$; The right side is greater.

Students are also asked to record their steps. Different teachers have different expectations but here are two possible ways to record the steps. The steps may also be done in a different order.

Recording the steps symbolically:

Left Expression | **Right Expression**

$$-2x+1-3-\overset{\text{flip}}{(-x+3)} \stackrel{?}{=} 2-3-\overset{\text{flip}}{(x-2)}$$

$$-2x+1-3+x-3 \stackrel{?}{=} 2-3-x+2$$

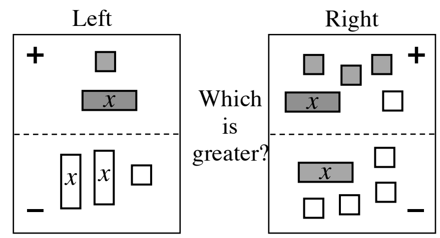
$$\begin{array}{rcl} -x-5 & \stackrel{?}{=} & 1-x \\ -(-x) & & -(-x) \\ \cline{1-1}\cline{3-3} -5 & \stackrel{?}{=} & 1 \end{array}$$

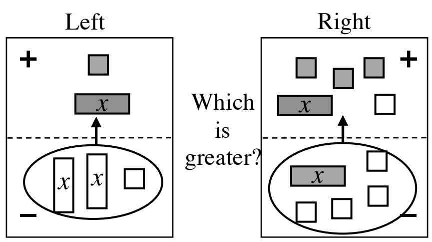
Recording the steps with a written explanation.

Left Expression	**Right Expression**	**Explanation**
$-2x+1-3-(-x+3)$	$2-3-(x-2)$	Starting expressions
$-2x+1-3+x-3$	$2-3-(x-2)$	Flip $-x + 3$ from "–" to "+"
$-x-5$	$2-3-(x-2)$	Combine like terms
$-x-5$	$2-3-x+2$	Flip $x - 2$ from "–" to "+"
-5	$2-3+2$	Remove $-x$ from both sides
-5	1	Combine like terms

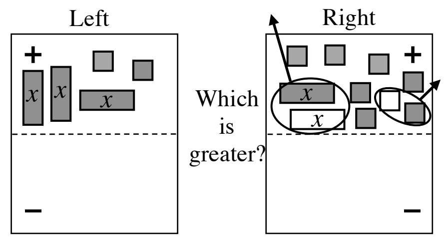
Example 2

Create the expressions $x + 1 - (-1 - 2x)$ and $3 + x - 1 - (x - 4)$ and then use legal moves to simplify and determine which side is greater.

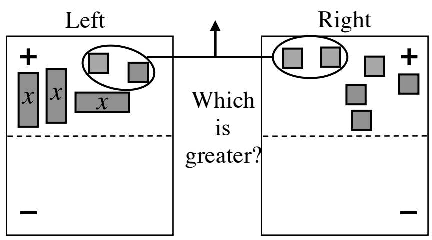




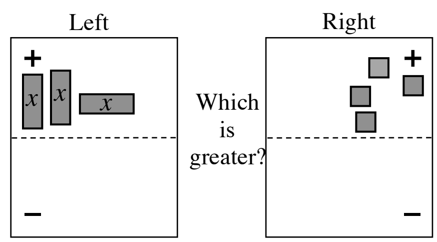
(1) Flip tiles and move them from the negative region to the positive region.



(2) Remove an equal number of opposite tiles that are within the same region.



(3) Remove the same tiles from *both* Expression Mats.

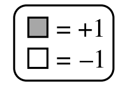


(4) Compare the Expression Mats.

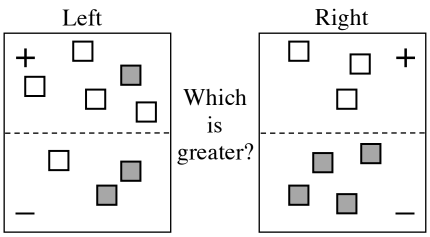
$3x \quad ? \quad 4$

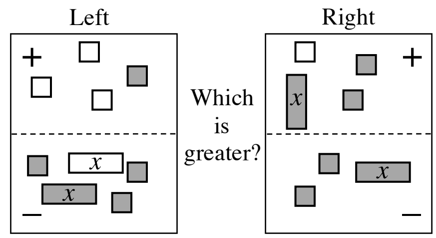
Since we do not know the value of x, it is not possible to determine the greater side.

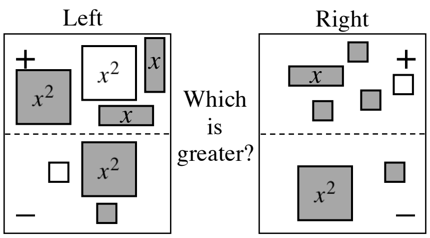
Problems

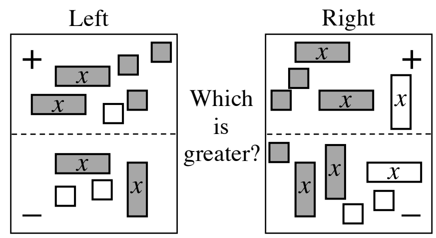


Write an expression for each side of the Expression Comparison Mat. Use legal moves to simplify and determine which side is greater. Carefully record your steps as you go.

1. 

2. 

3. 

4. 

In problems 5 through 10, record your steps as you use legal moves to simplify each expression and determine which side has the greater value.

5. Which is greater: $6 - (2x - 4) - 3$ or $-x - (1 + x) + 4$?
6. Which is greater: $3x - (2 - x) + 1$ or $-5 + 4x + 3$?
7. Which is greater: $-1 + 6x - 2 + 4y - 2x$ or $y + 5x - (-2 + x) + 3y - 2$?
8. Which is greater: $x^2 - 2x + 6 - (-3x)$ or $-(3 - x^2) + 5 + 2x$?
9. Which is greater: $x + 2 - (2 - 2x)$ or $4 + x - 2 - (x - 4)$?
10. Which is greater: $2x + 4 - x - (-2) + x^2$ or $3 + x^2 + 4x - (-3 + 3x)$?

Answers (expressions and explanations will vary)

1. left side is greater
2. right side is greater
3. not enough information
4. left side is greater
5. left side is greater
6. left side is greater
7. right side is greater
8. not enough information
9. not enough information
10. both sides equal

SOLVING EQUATIONS A 1.8 and A 1.9

An Equation Mat can be used to represent the process of solving an equation. An Equation Mat is created by putting two Expression Mats side by side—one for each side of the equal sign.

See the Math Notes boxes in Lessons A.1.8 and 3.2.1 for a list of all the "legal" moves and their corresponding algebraic language. Also see the Math Notes box in Lesson A.1.9 for checking a solution.

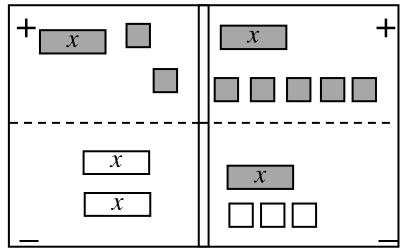
When the process of solving an equation ends with different numbers on each side of the equal sign (for example, $2 = 4$), there are *no real-number solutions* to the problem. When the result is the same expression or number on each side of the equation (for example, $x + 2 = x + 2$) it means that *all real numbers* are solutions.

For additional examples and practice, see the Checkpoint 1 materials.

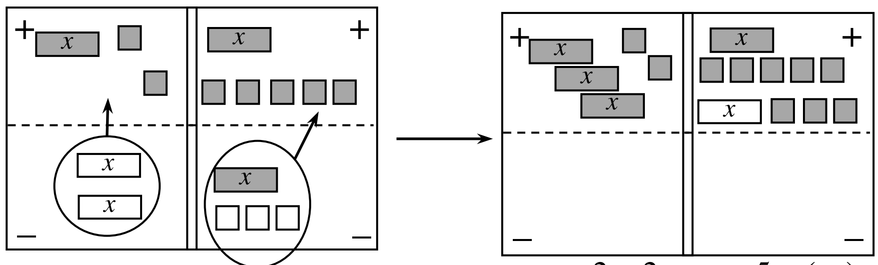
Example 1

Solve $x + 2 - (-2x) = x + 5 - (x - 3)$.

First, build the equation on an Equation Mat. See the Math Notes box in Lesson A.1.8.

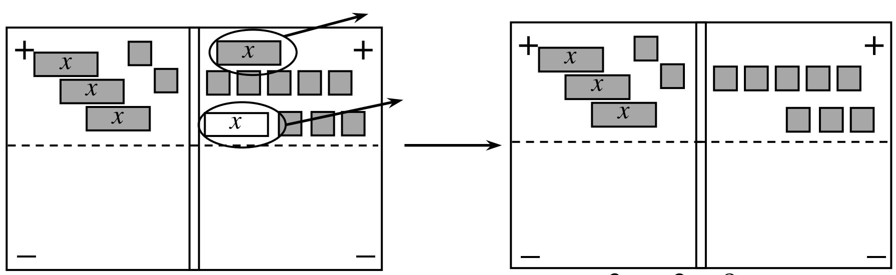


Second, flip the tiles in the subtraction region to the addition region (change subtraction to adding the opposite).



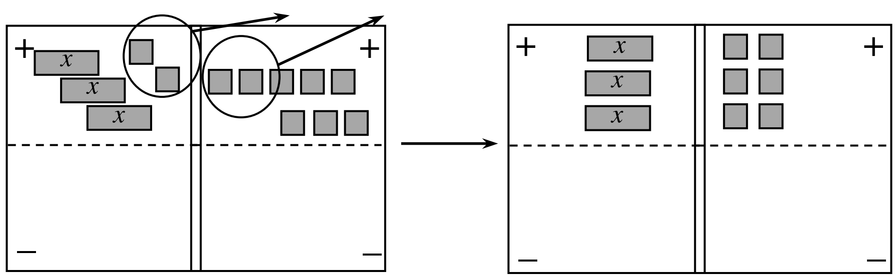
$x + 2 + 2x = x + 5 + (-x) + 3$

Continue to simplify using the "legal" moves, in any order, from the Math Notes box in Lesson 3.2.1. For example, remove zero pairs.



$3x + 2 = 8$

Isolate x-terms on one side and non-x-terms on the other by placing or removing matching tiles from both sides of the Equation Mat. Remove zero pairs again if needed.



$3x = 6$

Finally, arrange tiles into equal-sized groups on both sides. Since both sides of the equation are equal, determine the value of x. In this case, the tiles can be arranged into three groups, resulting in $x = 2$.

Example 2

Solve $3x + 3x - 1 = 4x + 9$

$3x + 3x + (-1) = 4x + 9$	Flip all tiles from subtraction region to addition region.
$6x + (-1) = 4x + 9$	Combine like terms.
$6x = 4x + 10$	Add 1 to each side, remove zero pairs.
$2x = 10$	Remove $4x$ from each side.
$x = 5$	Arrange into two groups.

Example 3

Solve $-2x + 1 - (-3x + 3) = -4 + (-x - 2)$

$-2x + 1 + 3x + (-3) = -4 + (-x) + (-2)$	Flip all tiles from subtraction region to addition region.
$x + (-2) = -x + (-6)$	Combine like terms.
$x = -x + (-4)$	Add 2 to each side, remove zero pairs.
$2x = -4$	Add x to both sides, remove zero pairs.
$x = -2$	Arrange into two groups.

Problems

Solve each equation.

1. $2x - 3 = -x + 3$
2. $1 + 3x - x = x - 4 + 2x$
3. $4 - 3x = 2x - 6$
4. $3 + 3x - (x - 2) = 3x + 4$
5. $-(x + 3) = 2x - 6$
6. $-4 + 3x - 1 = 2x + 1 + 2x$
7. $-x + 3 = 10$
8. $5x - 3 + 2x = x + 7 + 6x$
9. $4y - 8 - 2y = 4$
10. $9 - (1 - 3y) = 4 + y - (3 - y)$
11. $2x - 7 = -x - 1$
12. $-2 - 3x = x - 2 - 4x$
13. $-3x + 7 = x - 1$
14. $1 + 2x - 4 = -3 - (-x)$
15. $2x - 1 - 1 = x - 3 - (-5 + x)$
16. $-4x - 3 = x - 1 - 5x$
17. $10 = x + 6 + 2x$
18. $-(x - 2) = x - 5 - 3x$
19. $6 - x - 3 = 4x - 8$
20. $0.5x - (-x + 3) = x - 5$

Answers

1.	$x = 2$	2.	$x = 5$	3.	$x = 2$	4.	$x = 1$	5.	$x = 1$
6.	$x = -6$	7.	$x = -7$	8.	no solution	9.	$y = 6$	10.	$y = -7$
11.	$x = 2$	12.	all numbers	13.	$x = 2$	14.	$x = 0$	15.	$x = 2$
16.	no solution	17.	$x = 1\frac{1}{3}$	18.	$x = -7$	19.	$x = 2\frac{1}{5}$	20.	$x = -4$

