

目录

第一部分 复数	1
1 The logarithm and inverse functions	3
1.1 Inverse sin functions	3
1.2 Inverse Hyperbolic functions	3
2 Limits	4
3 Cauchy Riemann Equation	5
4 Harmonics	6

第一部分 复数

数论

问题 1. 两个平方和乘积也是一个平方和。

$$(a^2 + b^2)(c^2 + d^2) = (())^2 + (())^2$$

证明.

□

球极投影

定义 0.1. 集合 $\mathbb{C} \cup \{\infty\}$ 称为扩展复平面, 记为 \mathbb{C}_∞ . 对于 $z, \in \mathbb{C}_\infty$, 定义度量 $d(z, \infty) = d(\infty, z) = \infty$.

从扩展平面到三维球面 S 有映射 f 定义如下

$$f(z) = \left(\frac{z + \bar{z}}{|z|^2 + 1}, \frac{-i(z - \bar{z})}{|z|^2 + 1}, \frac{|z|^2 - 1}{|z|^2 + 1} \right) \quad (1)$$

我们来证明球极投影是一个连续的双射, 并且其逆映射也连续。

命题 0.1. $f^{-1}: S \rightarrow \mathbb{C}_\infty$ 存在, 而且对任意 $(x_1, x_2, x_3) \in S$, 有

$$f^{-1}(x_1, x_2, x_3) = \frac{x_1 + ix_2}{1 - x_3}$$

.

把 \mathbb{C}_∞ 看作是 \mathbb{R}_∞^2 , 也可以写

$$f^{-1}(x_1, x_2, x_3) = \left(\frac{x_1}{1 - x_3}, \frac{x_2}{1 - x_3} \right).$$

命题 0.2. $g: S \rightarrow \mathbb{R}: (x_1, x_2, x_3) \mapsto \left(\frac{x_1}{1 - x_3}, \frac{x_2}{1 - x_3} \right)$ 是连续函数。

要证明定理 0.2, 我们有下面更一般的命题。

命题 0.3. 函数 $f: Z \rightarrow X \times Y$ 连续当，且仅当其分量函数

$$f_1: Z \rightarrow X$$

和

$$f_2: Z \rightarrow Y$$

都连续。

忽略对这个一般命题的证明，直接用它来证明

证明. 对于 $x = (x_1, x_2, x_3)$, 设 $S \rightarrow \mathbb{R}$ 的函数 $g_1(x) = \frac{x_1}{1-x_3}$, $g_2(x) = \frac{x_2}{1-x_3}$. 那么 $f^{-1}(x) = (g_1(x), g_2(x))$.

- g_1, g_2 都是连续函数.

这是因为 g_1 可以分解为 $(x_1, x_2, x_3) \mapsto (x_1, 1 - x_3) \mapsto \frac{x_1}{1-x_3}$. g_2 类似。

- 由于 g_1, g_2 都连续，所以，根据 **命题 0.3** f^{-1} 也连续。

□

1 The logarithm and inverse functions

Branches.

1.1 Inverse sin functions

$w = \sin^{-1}(z)$ then $z = \sin(w) = \frac{e^{iw} - e^{-iw}}{2i}$ from which

1.2 Inverse Hyperbolic functions

$$z = \frac{e^w - e^{-w}}{e^w + e^{-w}}.$$

from which

$$e^{2w} =$$

Solve for e^w and take \ln .

Note that $\ln(x)$ has infinite many branches while $\sqrt[5]{x}$ only has 5 branches.

2 Limits

3 Cauchy Riemann Equation

Cauchy Riemann Equation not only provides a way to check whether a complex variable function is differentiable, but also is a tool to compute the derivative. It also connects the differential of multivariable functions and complex variable functions.

4 Harmonics

定义 4.1. Harmonics if φ is twice differentiable on \mathbb{R} with continuous second derivative.

If $\frac{\partial^2}{\partial x^2}\varphi + \frac{\partial^2}{\partial y^2}\varphi = 0$. Usually written as $\partial^2\varphi = 0$.

Examples

1. $\varphi(x, y) = x^2y^2$. Then $\frac{\partial^2}{\partial x^2}\varphi(x) + \frac{\partial^2}{\partial y^2}\varphi = 2 + (-2) = 0$.

2.

$$\varphi(x, y) = \sin(x) \cosh(y).$$

Then

$$\frac{\partial \cos(x) \cosh(y)}{\partial x} \rightarrow \sin(x) \cosh(y)$$

定理 4.1. Holomorphic implies real and imaginary part harmonic.

Given that we just need to check that u and v satisfies $\nabla^2 v = 0$ and $\nabla^2 u = 0$. Since u and v are the imaginary part of the holomorphic function, they satisfy the cauchy rieman equation. Use cauchy-rieman equation to show u and v are harmonic.

定理 4.2 (Reconstruction theorem). $R \in \mathbb{R}^2$ open set simply-connected set. $u: R \rightarrow \mathbb{R}$ a harmonic function. Then there is a harmonic function $v: R \rightarrow \mathbb{R}$ so that $f(x + yi) = u(x, y) + iv(x, y)$ is a holomorphic function on R .

The function v is called the *harmonic conjugate* of u and it is unique up to the addition of a constant function.

证明. By the cauchy-riemenn theorem for a function $f(x + iy) = u(x, y) + iv(x, y)$ to be holomorphic, we need u and v to satisfy the cauchy riemann equations.

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

and

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x}$$

$G: R \rightarrow \mathbb{R}^2$ be te vector field with component functions $G_1 = \frac{\partial u}{\partial y}$ and $G_2 = \frac{\partial u}{\partial x}$. $G = (-\frac{\partial u}{\partial y}, \frac{\partial u}{\partial x})$. So C-R equation can be written as $\nabla v = (-\frac{\partial u}{\partial y}, \frac{\partial u}{\partial x}) = G$.

So v exists.

To show $F(x, y) = (u(x, y), v(x, y))$ is differentiable on R it is enough to show that the first partial derivative is u and v exist and are continuous on R .

Since u is harmonic, its second derivatives exist, and therefore the first derivative exist and are continuous. But, by the cr equatons, the first derivatives of v are (up to sign) the first derivative of u . Hence the first derivatives of v are also continuous.

Therefore, F is multivariable differentiable on R and so by the C-R theorem, f is holomorphic on R . \square