目录

第一部分　复数

数论

问题 1. 对于 $b^2, b \in \mathbb{Z}^+$, 找出 $c, d \in \mathbb{Z}^+$ 使得 $b^2 = c^2 + d^2$.

证明. □

球极投影

定义 0.1. [扩展复平面] 集合 $\mathbb{C} \cup \{\infty\}$ 称为扩展复平面，记为 $\mathbb{C}_\infty$. 对于 $z, \in \mathbb{C}_\infty$, 定义度量 $d(z,\infty) = d(\infty, z) = \infty$.

从扩展平面到三维球面 S 有映射 f 定义如下

$$f(z) = \left(\frac{z+\bar{z}}{|z|^2+1}, \frac{-i(z-\bar{z})}{|z|^2+1}, \frac{|z|^2-1}{|z|^2+1}\right) \tag{1}$$

我们来证明球极投影是一个连续的双射，并且其逆映射也连续。

命题 0.1. $f^{-1}\colon S \to \mathbb{C}_\infty$ 存在，而且对任意 $(x_1, x_2, x_3) \in S$, 有

$$f^{-1}(x_1, x_2, x_3) = \frac{x_1 + ix_2}{1 - x_3}$$

.

把 $\mathbb{C}_\infty$ 看作是 $\mathbb{R}^{\not\in}{}_\infty$, 也可以写

$$f^{-1}(x_1, x_2, x_3) = \left(\frac{x_1}{1-x_3}, \frac{x_2}{1-x_3}\right).$$

命题 0.2. $g\colon S \to \mathbb{R}\colon (x_1, x_2, x_3) \mapsto \left(\frac{x_1}{1-x_3}, \frac{x_2}{1-x_3}\right)$ 是连续函数。

要证明定理 0.2，我们有下面更一般的命题。

命题 0.3. 函数 $f\colon Z \to X \times Y$ 连续当，且仅当其分量函数

$$f_1\colon Z \to X$$

和

$$f_2\colon Z \to Y$$

都连续。

忽略对这个一般命题的证明，直接用它来证明

证明. 对于 $x = (x_1, x_2, x_3)$, 设 $S \to \mathbb{R}$ 的函数 $g_1(x) = \frac{x_1}{1-x_3}$, $g_2(x) = \frac{x_2}{1-x_3}$. 那么 $f^{-1}(x) = (g_1(x), g_2(x))$.

- g_1,g_2 都是连续函数.

  这是因为 g_1 可以分解为 $(x_1, x_2, x_3) \mapsto (x_1, 1 - x_3) \mapsto \frac{x_1}{1-x_3}$. g_2 类似。

- 由于 g_1, g_2 都连续，所以，根据 命题 0.3 f^{-1} 也连续。

$\square$

1 The logarithm and inverse functions

Branches.

1.1 Inverse sin functions

$w = \sin^{-1}(z)$ then $z = sin(w) = \frac{e^{iw} - e^{-iw}}{2i}$ from which

1.2 Inverse Hyperbolic functions

$$z = \frac{e^w - e^{-w}}{e^w + e^{-w}}.$$

from which

$$e^{2w} =$$

Solve for e^w and take ln .

Note that $\ln(x)$ has infinite many branches while $\sqrt[5]{x}$ only has 5 branches.

2 Limits

3 Cauchy Riemann Equation

Cauchy Riemann Equation not only provides a way to check whether a complex variabled function is differentiable, but also is a tool to compute the derivative. It also connects the differential of multivariable functions and complex variable fucntions.

4 Harmonics

定义 4.1. Harmonics if φ is twice differentiable on $\mathbb{R}$ with continuous second derivative.

If $\frac{\partial^2}{\partial^2}\varphi + \frac{\partial^2}{\partial y^2}\varphi = 0$. Usually written as $\partial^2\varphi = 0$.
Examples

1. $\varphi(x, y) = x^2y^2$. Then $\frac{\partial^2}{\partial x^2}\varphi(x) + \frac{\partial^2}{\partial y^2}\varphi = 2 + (-2) = 0$.

2. $$\varphi(x, y) = \sin(x)\cosh(y).$$

   Then
   $$\frac{\partial \cos(x)\cosh(y)}{\partial x} \to \sin(x)\cosh(y)$$

定理 4.1. *Holomorphic implies real and imaginary part harmonic.*

Given that we just need to check that u and v satisfies $\nabla^2 v = 0$ and $\nabla^2 v = 0$. Since u and v are the imaginary part of the holomorphic function, they satisfy the cauchy rieman equation. Use cauhy-riman equation to show u and v are harmonic.

定理 4.2 (Reconstruction theorem)**.** *$R \in \mathbb{R}^2$ open set simply-connected set. $u\colon R \to R$ a harmonic function. Then there is a harmonic function $v\colon R \to R$ so that $f(x + yi) = u(x, y) + iv(x, y)$ is a holomophic function on R.*

The function v is called the *harmonic conjugate* of u and it is unique up to the addition of a constant function.

证明. By the cauchy-riemenn theorem for a function $f(x + iy) = u(x, y) + iv(x, y)$ to be holomorphic, we need u and v to satisfy the cauchy riemann equations.

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

and

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x}$$

$G\colon R \to \mathbb{R}^2$ be te vector field with component functions $G_1 = \frac{\partial u}{\partial y}$ and $G_2 = \frac{\partial u}{\partial x}$. $G = (-\frac{\partial u}{\partial y}, \frac{\partial u}{\partial x})$. So C-R equation can be written as $\nabla v = (-\frac{\partial u}{\partial y}, \frac{\partial u}{\partial x}) = G$.

So v exists.

To show $F(x, y) = (u(x, y), v(x, y))$ is differentiable on R it is enough to show that the first partial derivative is u and v exist and are continuous on R.

Since u is harmonic, its second derivatives exist, and therefore the first derivative exist and are continuous. But, by the cr equatons, the first derivatives of v are (up to sign) the first derivative of u. Hence the first derivatives of v are also continuous.

THerefore, F is multivariable differentiable on R and so by the C-R theorem, f is holomophic on R. $\square$