An introduction to Manifolds

Edition: 1

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1 Smooth Functions on a Euclidean Space

How many partial derivatives we need to examine according to the definition of C^k for a real-valued function? Since

$$\frac{\partial^j f}{\partial x^{i_1} \cdots \partial x^{i_j}}$$

for all $j \leq k$ must be examined, for each j we have n^j choices, and the total choice is:

$$\sum_{j=0}^{k} n^j = \frac{1 - n^{k+1}}{1 - n}$$

So every increase in k will increase exponentially the work.

What is the idea behind the example 1.3, where we try to find a C^∞ function which is not real-analytic? We want this function's derivatives of all order to be zero, at a point, say 'flat', but does that means no function but a constant function can achieve that at this flat point? The example gives a counterexample:

$$f(x) = \begin{cases} e^{-1/x} \text{ for } x > 0 \\ 0 \text{ for } x \leq 0 \end{cases}$$

We first show by induction that for $x > 0$ and $k \geq 0$, the kth derivative $f^{(k)}(x)$ is of the form $p_{2k}(1/x)e^{-1/x}$ for some polynomial $p_{2k}(y)$ of degree $2k$ in y. Suppose that the property is valid for k, we show that it is also valid for $k+1$.

$$f^{(k+1)} = e^{-\frac{1}{x}} \left[\frac{1}{x^2} \left(p_{2k}(\frac{1}{x}) - p_{2k}^{'}(\frac{1}{x}) \right) \right]$$

We can identify $\frac{1}{x^2} \left(p_{2k}(\frac{1}{x}) - p_{2k}^{'}(\frac{1}{x}) \right)$ as a polynomial $p_{2(k+1)}(\frac{1}{x})$. Then since

$$\lim_{x \to 0^+} p_{2k}(1/x)e^{-1/x} = 0$$

for all k, we have $f^k(0) = 0$.

References