Elastic Spin and Orbital Angular Momenta

Edition: 1

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1 Canonical momentum, spin, and orbital angular momentum of elastic waves

It is easily accepted that $T = \frac{1}{2}\rho\dot{\boldsymbol{a}}^2$ represented the density of kinetic energy.

$$W \stackrel{?}{=} \frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{a}}} \cdot \dot{\boldsymbol{a}} - \mathcal{L} = \rho\dot{\boldsymbol{a}} \cdot \dot{\boldsymbol{a}} - \frac{1}{2}\rho\dot{\boldsymbol{a}}^2 + U = T + U$$

$$\boldsymbol{P} \stackrel{?}{=} -\frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{a}}} \cdot (\nabla)\boldsymbol{a} = -\rho\boldsymbol{v} \cdot (\nabla)\boldsymbol{a}$$

$$\boldsymbol{J} \stackrel{?}{=} -\frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{a}}} \cdot (\boldsymbol{r} \times \nabla)\boldsymbol{a} - \frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{a}}} \times \boldsymbol{a} \equiv \boldsymbol{L} + \boldsymbol{S}$$

where

$$\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{P}, \boldsymbol{S} = -\rho\boldsymbol{v} \times \boldsymbol{a}$$

These are all said to be consequences of the Nother's theorem, but how?

Why elastic waves can have both longitudinal and transverse contributions:

$$\boldsymbol{a} = \boldsymbol{a}_L + \boldsymbol{a}_T, \nabla \times \boldsymbol{a}_L = 0, \nabla \cdot \boldsymbol{a}_T = 0$$

$$\boldsymbol{a}(\boldsymbol{r}, t) = \Re[\boldsymbol{a}(\boldsymbol{r})e^{-i\omega t}], \boldsymbol{v}(\boldsymbol{r}, t) = \Re[\boldsymbol{v}(\boldsymbol{r})e^{-i\omega t}], \boldsymbol{v} \stackrel{?}{=} -i\omega\boldsymbol{a}$$

Prove that

$$\bar{\boldsymbol{P}} = \frac{\rho\omega}{2}\Im[\boldsymbol{a}^* \cdot (\nabla)\boldsymbol{a}]$$

$$\begin{aligned}
P_j &= -\rho(v_i\partial_i)a_j \\
&= -\rho(\partial_t\Re[a_i(\boldsymbol{r})e^{-i\omega t}]\partial_i\Re[a_j(\boldsymbol{r})e^{-i\omega t}]) \\
&= -\rho(\Re(-i\omega a_i(\boldsymbol{r})e^{-i\omega t})\Re[\partial_i a_j(\boldsymbol{r})e^{-i\omega t}]) \\
&= -\rho\frac{-i\omega a_i(\boldsymbol{r})e^{-i\omega t} + i\omega a_i^*(\boldsymbol{r})e^{i\omega t}}{2} \cdot \frac{\partial_i a_j(\boldsymbol{r})e^{-i\omega t} + \partial_i a_j^*(\boldsymbol{r})e^{i\omega t}}{2} \\
&= \frac{\rho\omega}{4} i\left(a_i(\boldsymbol{r})\partial_i a_j^*(\boldsymbol{r}) - a_i^*(\boldsymbol{r})\partial_i a_j(\boldsymbol{r})\right) \\
&= -\frac{\rho\omega}{4} i\left(a_i^*(\boldsymbol{r})\partial_i a_j(\boldsymbol{r}) - (a_i^*(\boldsymbol{r})\partial_i a_j(\boldsymbol{r}))^*\right) \\
&= -\frac{\rho\omega}{4} i(2i\Im[a_i^*(\boldsymbol{r})\partial_i a_j(\boldsymbol{r})]) \\
&= \frac{\rho\omega}{2}\Im[a_i^*(\boldsymbol{r})\partial_i a_j(\boldsymbol{r})]
\end{aligned}$$

Why $\bar{\boldsymbol{P}}$ resemble local expectation values of quantum-mechanical momentum $(-i\nabla)$ operator with the "wave function" $\boldsymbol{\psi} = \sqrt{\frac{\rho\omega}{2}}\boldsymbol{a}$, $\boldsymbol{\psi}^* \cdot \boldsymbol{\psi} = 2\bar{T}/\omega$? First, let's check that $\boldsymbol{\psi}^* \cdot \boldsymbol{\psi} = 2\bar{T}/\omega$. $\boldsymbol{\psi}^* \cdot \boldsymbol{\psi} = \frac{\rho\omega}{2}|\boldsymbol{a}|^2 = \frac{2}{\omega}\frac{\rho\omega^2}{4}|\boldsymbol{a}|^2 = 2\bar{T}/\omega$. According to my memory, the expectation value should be $\boldsymbol{\psi}^* \cdot (-i\nabla)\boldsymbol{\psi} = \frac{\rho\omega}{2}\boldsymbol{a}^* \cdot (-i\nabla)\boldsymbol{a} \stackrel{?}{=} \bar{\boldsymbol{P}}$, so how to understand this inconformity?

I think I have done something wrong,

$$\boldsymbol{a}^* \cdot (\nabla)\boldsymbol{a} = (\boldsymbol{a}^* \cdot \nabla)\boldsymbol{a} + \boldsymbol{a}^* \times (\nabla \times \boldsymbol{a})$$

So I have to additionally calculate

$$\begin{aligned}
(\boldsymbol{v} \times (\nabla \times \boldsymbol{a}))_i &= \varepsilon_{ijk} v_j (\nabla \times \boldsymbol{a})_k \\
&= \varepsilon_{ijk} v_j \varepsilon_{kmn} \partial_m a_n \\
&= \varepsilon_{ijk} \varepsilon_{mnk} v_j \partial_m a_n \\
&= (\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}) v_j \partial_m a_n \\
&= v_j (\partial_i a_j - \partial_j a_i) \\
&= \frac{-i\omega a_j(\boldsymbol{r}) e^{-i\omega t} + i\omega a_j^*(\boldsymbol{r}) e^{i\omega t}}{2} \cdot \\
&\left(\frac{\partial_i a_j(\boldsymbol{r}) e^{-i\omega t} + \partial_i a_j^*(\boldsymbol{r}) e^{i\omega t}}{2} - \frac{\partial_j a_i(\boldsymbol{r}) e^{-i\omega t} + \partial_j a_i^*(\boldsymbol{r}) e^{i\omega t}}{2} \right) \\
&= \frac{1}{4} i\omega \left(-a_j \partial_i a_j^* + a_j^* \partial_i a_j + a_j \partial_j a_i^* - a_j^* \partial_j a_i \right) \\
&= \frac{1}{4} i\omega \left(2i\Im[a_j^* \partial_i a_j] - 2i\Im[a_j^* \partial_j a_i] \right) \\
&= \frac{\omega}{2} \Im[a_j^* (\partial_j a_i - \partial_i a_j)] \\
&= -\frac{\omega}{2} \Im[\boldsymbol{a}^* \times (\nabla \times \boldsymbol{a})]
\end{aligned}$$

2 Momentum and angular momentum of cylindrical modes

Why the cylindrical field has the form

$$\boldsymbol{a} = [a_r(r), a_\varphi(r), a_z(r)] e^{i\ell\varphi + ik_z z},$$

where k_z the longitude wave number, and ℓ is the integer azimuthal quantum number? Normally, I don't expect that we have quantum number in a wave problem, why here?

Why use the Cartesian components while change to the associated basis of circular polarizations in the (x, y)-plane: $a^\pm = [(a_x \mp i a_y)/\sqrt{2}] = [(a_r \mp i a_\varphi)/\sqrt{2}] e^{\mp i\varphi}$?

Prove that $\frac{\bar{P}_z}{2\bar{T}} = \frac{k_z}{\omega}$.

$$\begin{aligned}
\bar{P}_z &= \frac{\rho\omega}{2} \Im[a_r^* \partial_r a_z(r) e^{i\ell\varphi + ik_z z} + a_\varphi^* \frac{1}{r} \partial_\varphi a_z(r) e^{i\ell\varphi + ik_z z} + a_z^* \partial_z a_z(r) e^{i\ell\varphi + ik_z z}] \\
&= \frac{\rho\omega}{2} \Im[a_r^*(r) \partial_r a_z(r) + (i\ell) a_\varphi^*(r) \frac{1}{r} a_z(r) + (ik_z)|a_z|^2] \\
&= \frac{\rho\omega}{2} [k_z |a_z|^2 + ?]
\end{aligned}$$

We have more terms:

$$\begin{aligned}
\Im[a_j^* (\partial_z a_j - \partial_j a_z)] &= \Im[a_r^* (\partial_z a_r - \partial_r a_z) + a_\varphi^* (\partial_z a_\varphi - \frac{1}{r} \partial_\varphi a_z) + a_z^* (\partial_z a_z - \partial_z a_z)] \\
&= \Im[a_r^* (\partial_z a_r - \partial_r a_z) + a_\varphi^* (\partial_z a_\varphi - \frac{1}{r} \partial_\varphi a_z)] \\
&= \Im[(ik_z)|a_r|^2 - a_r^* \partial_r a_z + (ik_z)|a_\varphi|^2 - (i\ell) a_\varphi^* \frac{1}{r} a_z]
\end{aligned}$$

Now we get the desired result!

How to change the field in the representation of the circular Cartesian basis?

3 Transverse spin of a Reyleigh wave

How to derive the field of Rayleigh wave propagating along the z axis and the $x = 0$ surface of an isotropic medium ($x < 0$)?

References