Complex Numbers

Note: We expect WOOT students to be already familiar with most of the material in the start of this handout, so you should focus on the material that is new to you. This material is also covered in depth in Art of Problem Solving's *Precalculus* textbook.

1 Definitions

A *complex number* is a number of the form $z = a + bi$, where a and b are real numbers, and $i^2 = -1$. The numbers a and b are the *real* and *imaginary* parts of z, respectively. A number of the form bi is said to be *pure imaginary*. Every complex number can be written uniquely in the form $z = a+bi$, so that if $z = a+bi = c+di$, where a, b, c, and d are real numbers, then $a = c$ and $b = d$.

Given $z = a + bi$, the complex number $\overline{z} = a - bi$ is called the *conjugate* of z. Note that

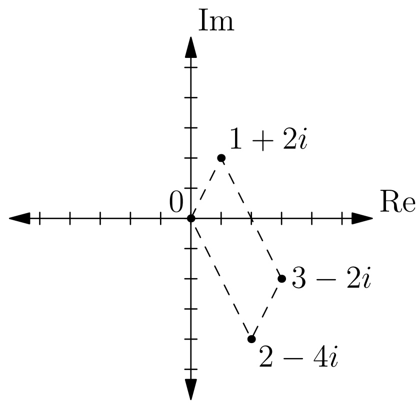
$$z\overline{z} = (a + bi)(a - bi) = a^2 + b^2.$$

Also, $\overline{z} = z$ if and only z is a real number, and $\overline{z} = -z$ if and only if z is pure imaginary. It is easy to verify that $\overline{z + w} = \overline{z} + \overline{w}$ and $\overline{zw} = \overline{z} \cdot \overline{w}$ for all complex numbers z, w.

The *magnitude* of $z = a + bi$ is given by $|z| = \sqrt{a^2 + b^2}$, so we can also write

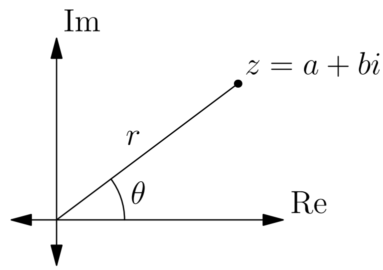
$$z\overline{z} = |z|^2.$$

We can plot complex numbers in the *complex plane*, just as we can plot points in the coordinate plane. In the picture to the right, note that when we add $1+2i$ and $2-4i$, we get $3-2i$. When we plot the complex numbers 0, $1+2i$, $2-4i$, and $3-2i$, we find that they form the vertices of a parallelogram. Thus, even an operation as simple as addition has a geometric interpretation. Many geometric concepts can be elegantly described using complex numbers.



If we plot a complex number $z = a + bi$ in the complex plane, then we can measure its distance from the origin and the angle that z makes with the positive real axis, denoting the distance and the angle by r and θ, respectively. Note that $r = \sqrt{a^2 + b^2} = |z|$. The angle θ is known as the *argument* of z. Thus, the parameters r and θ give us an alternative way of specifying complex numbers.

Complex Numbers



Furthermore, $a = r\cos\theta$ and $b = r\sin\theta$, so we can write

$$\begin{aligned} z &= a + bi \\ &= r\cos\theta + ir\sin\theta \\ &= r(\cos\theta + i\sin\theta). \end{aligned}$$

The expression $\cos\theta + i\sin\theta$ is sometimes abbreviated as $\operatorname{cis}\theta$, so we can also write $z = r\operatorname{cis}\theta$. This form of a complex number is known as *polar form*. (The form $z = a + bi$ is known as *rectangular form*.)

Multiplying two complex numbers in polar form is simple:

$$\begin{aligned} r(\cos\theta + i\sin\theta)\cdot s(\cos\psi + i\sin\psi) &= rs(\cos\theta + i\sin\theta)(\cos\psi + i\sin\psi) \\ &= rs[(\cos\theta\cos\psi - \sin\theta\sin\psi) + i(\cos\theta\sin\psi + \sin\theta\cos\psi)] \\ &= rs[\cos(\theta+\psi) + i\sin(\theta+\psi)]. \end{aligned}$$

Thus, the magnitude of a product is the **product** of the magnitudes, and the argument of a product is the **sum** of the arguments. In particular, if we let $f(\theta) = \cos\theta + i\sin\theta$, then

$$f(\theta)f(\psi) = f(\theta+\psi) \qquad (*)$$

for all angles θ and ψ. By a straightforward induction argument, $f(\theta)^n = f(n\theta)$ for any integer n. This gives us the following result.

Theorem 1.1. (De Moivre's Theorem) For any angle θ and integer n,

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta.$$

The only "nice" functions that satisfy $(*)$ are exponential functions: functions of the form $f(x) = b^x$ for some base b. Indeed, $f(\theta)$ is of this form:

Theorem 1.2. (Euler's Formula) For any angle θ,

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

This gives us another way of expressing complex numbers, known as *exponential form*. Note that De Moivre's Theorem is an immediate consequence of Euler's Formula.

2 Roots of Unity

For a positive integer n, the set of complex numbers z satisfying $z^n = 1$ are known as the n^{th} *roots of unity*. To find these roots of unity, let $z = re^{i\theta}$. Then the equation $z^n = 1$ becomes

$$r^n e^{ni\theta} = 1.$$

Taking the magnitude of both sides, we get $|r^n e^{ni\theta}| = 1$, so $|r^n||e^{ni\theta}| = 1$. But

$$|e^{ni\theta}| = |\cos n\theta + i \sin n\theta| = \sqrt{\cos^2 n\theta + \sin^2 n\theta} = 1,$$

so $|r^n| = 1$. Since r is positive, we have $r^n = 1$, so $r = 1$. Thus we get

$$e^{ni\theta} = 1.$$

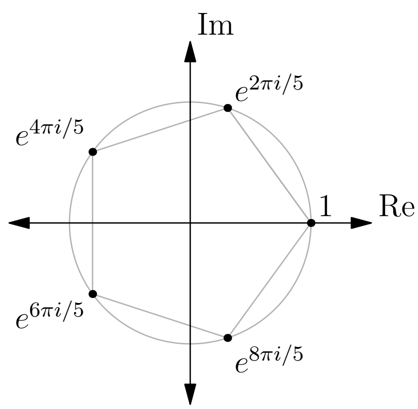
We can rewrite this equation as

$$\cos n\theta + i \sin n\theta = 1.$$

Taking the real and imaginary parts of this equation, we get $\cos n\theta = 1$ and $\sin n\theta = 0$. This means that $n\theta$ is an integer multiple of 2π, so let $n\theta = 2k\pi$ for some integer k. Therefore, $z^n = 1$ if and only if $z = e^{2k\pi i/n}$ for some integer k. But since $e^{2\pi i} = 1$, we can restrict our attention to the values of k where $0 \le k \le n-1$. Hence, the n^{th} roots of unity are

$$1,\ e^{2\pi i/n},\ e^{4\pi i/n},\ \dots,\ e^{2(n-1)\pi i/n}.$$

In the complex plane, for $n \ge 3$, the n^{th} roots of unity form the vertices of a regular n-gon. For example, the 5^{th} roots of unity are shown below.



3 Primitive Roots of Unity and Cyclotomic Polynomials

We say that z is a *primitive* n^{th} root of unity if $z^n = 1$ and $z^m \neq 1$ for $1 \leq m \leq n-1$. (In other words, z is an n^{th} root of unity, but not an m^{th} root of unity for any positive integer m less than n.) For example, 1, i, -1, and $-i$ are the fourth roots of unity, but only i and $-i$ are primitive fourth roots of unity.

Also, note that $e^{2\pi i/n}$ is always a primitive n^{th} root of unity. This is a special case of the following:

Problem 3.1. Prove that a complex number z is a primitive n^{th} root of unity if and only if it can be written in the form $z = e^{2k\pi i/n}$, where k is an integer relatively prime to n.

Solution: We know that z is an n^{th} root of unity if and only if it can be written in the form $z = e^{2k\pi i/n}$ where k is an integer. So it suffices to show that

$$e^{2k\pi i/n} \text{ is a primitive } n^{\text{th}} \text{ root of unity} \quad \Leftrightarrow \quad \gcd(k,n) = 1.$$

It turns out that is easier to prove the equivalent statement about the negations:

$$e^{2k\pi i/n} \text{ is } \textit{not} \text{ a primitive } n^{\text{th}} \text{ root of unity} \quad \Leftrightarrow \quad \gcd(k,n) = d \text{ for some } d > 1.$$

First, suppose $e^{2k\pi i/n}$ is not a primitive n^{th} root of unity. Then there exists some m with $1 \leq m < n$ such that $\left(e^{2k\pi i/n}\right)^m = 1$. But $\left(e^{2k\pi i/n}\right)^m = e^{2km\pi i/n} = 1$, so this means that km is a multiple of n, and since $m < n$, we must have that k and n have a prime factor in common. So k and n are not relatively prime.

Conversely, suppose that $\gcd(k,n) = d$ for some $d > 1$. Let $k = k'd$ and $n = n'd$, and note that $n' < n$. Then

$$\left(e^{2k\pi i/n}\right)^{n'} = \left(e^{2k'd\pi i/n}\right)^{n'} = e^{2k'n'd\pi i/n} = e^{2k'n\pi i/n} = e^{2k'\pi i} = 1.$$

Therefore, $e^{2k\pi i/n}$ is not a primitive n^{th} root of unity, since its n'^{th} power is 1, with $n' < n$. □

It follows from Problem 3.1 that there are $\phi(n)$ primitive n^{th} roots of unity, where $\phi(n)$ is the number of positive integers less than or equal to n that are relatively prime to n.

The n^{th} *cyclotomic polynomial* is the monic polynomial whose roots are the primitive n^{th} roots of unity. It is denoted by Φ_n. Note that by Problem 3.1, we know that Φ_n must have degree $\phi(n)$. The first few cyclotomic polynomials are listed below.

n	$\Phi_n(x)$
1	$x-1$
2	$x+1$
3	x^2+x+1
4	x^2+1
5	$x^4+x^3+x^2+x+1$
6	x^2-x+1

Two perhaps surprising facts about the cyclotomic polynomials are:

Theorem 3.2. For all n, the coefficients of $\Phi_n(x)$ are integers, and $\Phi_n(x)$ is irreducible over the rationals.

The first part of Theorem 3.2—that $\Phi_n(x)$ has integer coefficients—is a consequence of Problem 1(a) below, together with some facts about polynomial algebra: as a challenge, see if you can prove this using the result from Problem 1(a). The irreducibility is much more difficult to prove and requires some rather deep theorems from abstract algebra.

You may notice in the table above that all of the nonzero coefficients of the first six cyclotomic polynomials are 1 or -1. This will actually continue to be true for the first 104 cyclotomic polynomials. However, it is not true in general; $\Phi_{105}(x)$ is the first cyclotomic polynomial with coefficients whose absolute values are greater than 1, and in general the coefficients of $\Phi_n(x)$ can assume arbitrarily high values. Make sure you do not fall in the trap of making assumptions about the magnitude of the coefficients of cyclotomic polynomials.

Review Problems

1. (a) Prove that, for any positive integer n,

$$\prod_{d|n} \Phi_d(x) = x^n - 1,$$

where the product on the left-hand side ranges over all positive integers d dividing n.

(b) Find $\Phi_p(x)$, where p is prime.

2. The sets $A = \{z : z^{18} = 1\}$ and $B = \{w : w^{48} = 1\}$ are both sets of complex roots of unity. The set $C = \{zw : z \in A \text{ and } w \in B\}$ is also a set of complex roots of unity. How many distinct elements are in C? (AIME, 1990)

3. Let P be the product of those roots of $z^6 + z^4 + z^3 + z^2 + 1 = 0$ that have a positive imaginary part. Given

that $P = e^{i\theta}$, where $0 \le \theta < 2\pi$, determine θ. (Based on AIME, 1996)

4. Describe the roots of the polynomial

$$P(x) = (1 + x + x^2 + \cdots + x^{17})^2 - x^{17}.$$

(Based on AIME, 2004)

Challenge Problems

5. Let $w = e^{2\pi i/m}$, where m is an odd positive integer. Prove that

$$\left|w + 2w^2 + 3w^3 + \cdots + mw^m\right|^{-1} = \frac{2}{m}\sin\frac{\pi}{m}.$$

(Based on AHSME, 1984)

6. The equation

$$x^{10} + (13x - 1)^{10} = 0$$

has 10 complex roots $r_1, \overline{r}_1, r_2, \overline{r}_2, r_3, \overline{r}_3, r_4, \overline{r}_4, r_5, \overline{r}_5$. Find the value of

$$\frac{1}{r_1\overline{r}_1} + \frac{1}{r_2\overline{r}_2} + \frac{1}{r_3\overline{r}_3} + \frac{1}{r_4\overline{r}_4} + \frac{1}{r_5\overline{r}_5}.$$

(AIME, 1994)

7. Let ω be an n^{th} root of unity. Prove that ω is a primitive n^{th} root of unity if and only if the set

$$\{1, \omega, \omega^2, \ldots, \omega^{n-1}\}$$

is the set of all of the n^{th} roots of unity.

8. Prove that if n is odd and $n \ge 3$, then $\Phi_{2n}(x) = \Phi_n(-x)$.