STAT3100 2024F
Assignment 5 Solution

Due: 11:59pm, Friday, December 1, 2024

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1. A geometric distribution has the probability distribution function

$$f(x;p) = p(1-p)^{x-1}, \qquad \text{for } x = 1, 2, 3, ...$$

(a) Find the moment generating function for the geometric distribution and use it to find mean and variance of the distribution.
Solution:

$$\begin{aligned} M_X(t) &= \mathrm{E}(e^{xt}) \\ &= \sum_{x=1}^{\infty} e^{xt} p(1-p)^{x-1} \\ &= p(1-p)^{-1} \sum_{x=1}^{\infty} [e^t(1-p)]^x \\ &= p(1-p)^{-1}[1-e^t(1-p)]^{-1}[e^t(1-p)] \sum_{x=1}^{\infty} [1-e^t(1-p)][e^t(1-p)]^{x-1} \\ &= \frac{e^t p}{1-e^t(1-p)} \end{aligned}$$

Given the above moment generating function, we can find the first and second moment as follows

$$\begin{aligned} \mu = \mathrm{E}(X) &= \left.\frac{dM_x(t)}{dt}\right|_{t=0} \\ &= \left.\frac{pe^t[1-e^t(1-p)] - pe^t[-(1-p)e^t]}{[1-e^t(1-p)]^2}\right|_{t=0} \\ &= \left.\frac{pe^t}{[1-e^t(1-p)]^2}\right|_{t=0} \\ &= \frac{p}{[1-(1-p)]^2} \\ &= \frac{1}{p} \end{aligned}$$

$$\begin{aligned} \mathrm{E}(X^2) &= \left.\frac{d^2M_x(t)}{dt^2}\right|_{t=0} \\ &= \left.\frac{d\, \frac{pe^t}{[1-e^t(1-p)]^2}}{dt}\right|_{t=0} \end{aligned}$$

$$
\begin{aligned}
&= \left.\frac{pe^t[1-e^t(1-p)]^2 - pe^t(2)[1-e^t(1-p)][-e^t(1-p)]}{[1-e^t(1-p)]^4}\right|_{t=0} \\
&= \left.\frac{pe^t[1+e^t(1-p)]}{[1-e^t(1-p)]^3}\right|_{t=0} \\
&= \frac{p(2-p)}{p^3} \\
&= \frac{2-p}{p^2}
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{var}(X) &= \mathrm{E}(X^2) - [\mathrm{E}(X)]^2 \\
&= \frac{2-p}{p^2} - \left(\frac{1}{p}\right)^2 \\
&= \frac{1-p}{p^2} = \frac{1}{p}\left(\frac{1}{p} - 1\right)
\end{aligned}
$$

(b) Differentiating with respect to p on the both sides of the equation

$$\sum_{x=1}^{\infty} f(x;p) = 1$$

to find the mean of the geometric distribution.

Solution:

$$
\begin{aligned}
\frac{d1}{dp} &= \frac{d\ \sum_{x=1}^{\infty} f(x;p)}{dp} \\
0 &= \frac{d\ \sum_{x=1}^{\infty} p(1-p)^{x-1}}{dp} \\
0 &= \sum_{x=1}^{\infty} \frac{d\ p(1-p)^{x-1}}{dp} \\
0 &= \sum_{x=1}^{\infty} \left[(1-p)^{x-1} - p(x-1)(1-p)^{x-2}\right] \\
0 &= \sum_{x=1}^{\infty} (1-p)^{x-1} - \sum_{x=1}^{\infty} p(x-1)(1-p)^{x-2} \\
0 &= \frac{1}{p}\sum_{x=1}^{\infty} p(1-p)^{x-1} - \sum_{x=2}^{\infty} p(x-1)(1-p)^{x-2} \\
0 &= \frac{1}{p} - \sum_{x=1}^{\infty} xp(1-p)^{x-1} \\
0 &= \frac{1}{p} - \mathrm{E}(X) \\
\mathrm{E}(X) &= \frac{1}{p}
\end{aligned}
$$

(c) Show that $P(X = x + n | X > n) = P(X = x)$.

Solution:

$$\begin{aligned}
LHS &= \frac{P(X = x+n, X > n)}{P(X > n)} \\
&= \frac{P(X = x+n)}{P(X > n)} \\
&= \frac{p(1-p)^{x+n-1}}{1 - P(X \le n)} \\
&= \frac{p(1-p)^{x+n-1}}{1 - \sum_{x=1}^{n} p(1-p)^{x-1}} \\
&= \frac{p(1-p)^{x+n-1}}{1 - p \cdot \frac{1-(1-p)^n}{1-(1-p)}} \\
&= \frac{p(1-p)^{x+n-1}}{(1-p)^n} \\
&= p(1-p)^{x-1} \\
&= P(X = x) \\
&= RHS
\end{aligned}$$

2. Flaws on a certain type of drapery material appear on the average of one in 150 square feet. Assume flaws appear according to a Poisson distribution.

   (a) Find the probability of at most one flaw appearing in 225 square feet of drapery.

   Solution: Given that on average 1 flaw in 150 ft^2, the mean flaw rate for 225ft^2 is 1.5 flaws. Let X be the number of flaws appearing in 225 ft^2, $X \sim$Poisson (λ), with $\lambda = 1.5$.

$$\begin{aligned}
P(X \le 1) &= P(X = 0) + P(X = 1) \\
&= e^{-\lambda} + e^{-\lambda}\lambda \\
&= e^{-1.5}(1 + 1.5) \\
&= 0.5578254
\end{aligned}$$

   (b) Find the probability that no flaws appear in 75 square feet of drapery.

   Solution: The mean flaw rate for a 75ft^2 area is 0.5 flaws. Let X be the number of flaws appearing in 75 ft^2, $X \sim$Poisson (λ), with $\lambda = 0.5$.

$$P(X = 0) = e^{-\lambda} = e^{-0.5} = 0.6065307.$$

3. Gamma(α, β) distribution has the pdf

$$f(x:\alpha,\beta) = \frac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta}, \quad x > 0.$$

(a) Find the rth moment, $\mathrm{E}(X^r)$, of the distribution.
Solution:

$$\begin{aligned}
\mathrm{E}(X^r) &= \int_0^\infty x^r \frac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta} dx \\
&= \frac{1}{\beta^\alpha \Gamma(\alpha)} \int_0^\infty x^{\alpha+r-1} e^{-x/\beta} dx \\
&= \frac{1}{\beta^\alpha \Gamma(\alpha)} \beta^{\alpha+r} \Gamma(\alpha+r) \int_0^\infty \frac{1}{\beta^{\alpha+r}\Gamma(\alpha+r)} x^{\alpha+r-1} e^{-x/\beta} dx \\
&= \frac{\beta^r \Gamma(\alpha+r)}{\Gamma(\alpha)}
\end{aligned}$$

(b) Find the moment generating function of the Gamma distribution and use it to find mean and variance of the distribution.
Solution:

$$\begin{aligned}
M_X(t) &= \mathrm{E}(e^{Xt}) \\
&= \int_0^\infty e^{xt} \frac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta} dx \\
&= \int_0^\infty \frac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-x(\frac{1}{\beta}-t)} dx \\
&= \int_0^\infty \frac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-x(\frac{1-\beta t}{\beta})} dx \\
&= \int_0^\infty \frac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-x/\frac{\beta}{1-\beta t}} dx \\
&= \frac{\left(\frac{\beta}{1-\beta t}\right)^\alpha}{\beta^\alpha} \int_0^\infty \frac{1}{\left(\frac{\beta}{1-\beta t}\right)^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-x/\frac{\beta}{1-\beta t}} dx \\
&= \frac{1}{(1-\beta t)^\alpha}
\end{aligned}$$

$$\begin{aligned}
\mu = \mathrm{E}(X) &= \left.\frac{dM_X(t)}{dt}\right|_{t=0} \\
&= \left.\frac{d\frac{1}{(1-\beta t)^\alpha}}{dt}\right|_{t=0} \\
&= (-\alpha)(1-\beta t)^{-\alpha-1}(-\beta)\Big|_{t=0} \\
&= \alpha\beta
\end{aligned}$$

$$
\begin{aligned}
\mathrm{E}(X^2) &= \left.\frac{d^2M_X(t)}{dt^2}\right|_{t=0} \\
&= \left.\frac{d\alpha\beta(1-\beta t)^{-\alpha-1}}{dt}\right|_{t=0} \\
&= \left.\alpha\beta(-\alpha-1)(1-\beta t)^{-\alpha-2}(-\beta)\right|_{t=0} \\
&= \alpha(\alpha+1)\beta^2
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{var}(X) &= \mathrm{E}(X^2) - [E(X)]^2 \\
&= \alpha(\alpha+1)\beta^2 - \alpha^2\beta^2 \\
&= \alpha\beta^2
\end{aligned}
$$

4. Let X and Y be random variables for the location of each point in the unit circle $\left\{(x,y)\ :\ x^2+y^2<\frac{1}{2}\right\}$. Given that each point is equally likely to be selected in the unit circle with the joint pdf of X and Y given as

$$
f_{X,Y}(x,y) = \begin{cases} \frac{1}{\pi} & \text{for } 0 < x^2+y^2 < 1 \\ 0 & \text{o.w} \end{cases}
$$

Suppose 2000 points are selected independently and randomly from the unit circle. Let W be the number of points that fall into $A = \left\{(x,y)\ :\ x^2+y^2<\frac{1}{2}\right\}$. What is the distribution of W? Find its mean and variance.
Solution: From previous work in Assignment 4, we know that for each point randomly selected from the circle, the probability that the point falls in the area A is 0.5. We let this probability be p. Then, we random selected points for 2000 times, then W, the number of points falling in A, would follow a Binomial (2000, p) distribution. The mean and variance of W are then given by

$$
\mathrm{E}(W) = np = 2000(0.5) = 1000
$$

$$
\mathrm{var}(W) = np(1-p) = 2000(0.5)(0.5) = 500.
$$

5. During one stage in the manufacture of integrated circuit chips, a coating must be applied. If 70% of chip receive a thick enough coating, find the probability that, among 15 chip,

   (a) at least 12 will have thick enough coatings;
   Solution: Let X be the number of chips that have thick enough coatings, then X ~Binomial (15, p).

$$
\begin{aligned}
P(X \geq 12) &= \sum_{x=12}^{15} P(X=x) \\
&= \binom{15}{12} p^{12}(1-p)^3 + \binom{15}{13} p^{13}(1-p)^2 + \binom{15}{14} p^{14}(1-p) + \binom{15}{15} p^{15}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{15 \cdot 14 \cdot 13}{3 \cdot 2}(0.7)^{12}(0.3)^3 + \frac{15 \cdot 14}{2}(0.7)^{13}(0.3)^2 + 15(0.7)^{14}(0.3) + (0.7)^{15} \\
&= 0.1700402 + 0.09156011 + 0.03052004 + 0.004747562 \\
&= 0.2968679
\end{aligned}
$$

or use R we found

```
> 1-pbinom(11, 15, 0.7)
[1] 0.2968679
```

and so,

$$P(X \geq 12) = 1 - P(X \leq 11) = 0.2968679$$

(b) at most 6 will have thick enough coatings;
Solution: Again, use R, we have

```
> pbinom(6, 15, 0.7)
[1] 0.01524253
```

and so $P(X \leq 6) = 0.01524253$.

(c) exactly 10 will have thick enough coating.
We can use R to find

```
> dbinom(10, 15, 0.7)
[1] 0.2061304
```

and so $P(X = 10) = 0.26061304$.

6. Use the gamma function, we can write

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{2}\int_0^\infty e^{-\frac{1}{2}z^2}dz$$

and $\Gamma(\frac{1}{2}) = \sqrt{\pi}$. If $X \sim N(\mu, \sigma^2)$, then show that the function

$$f(x; \mu, \sigma^2) = \frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

is a probability density.
Solution:

(1) $f(x; \mu, \sigma^2) \geq 0$ for all $x \in \mathcal{R}$.

(2) Let $Z = \frac{x-\mu}{\sigma}$, then for $-\infty < x < \infty$, $-\infty < z < \infty$, and $\frac{dz}{dx} = \frac{1}{\sigma}$ and so that $dz = \frac{1}{\sigma}dx$. Use substitution method, we have

$$
\begin{aligned}
\int_{-\infty}^{\infty} f(x; \mu, \sigma^2)dx &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(x-\mu)^2}{2\sigma^2}}dx \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}dz
\end{aligned}
$$

$$\begin{aligned}
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{z^2}{2}} dz \\
&= \frac{1}{\sqrt{2\pi}} 2 \int_{0}^{\infty} e^{-\frac{z^2}{2}} dz \\
&= \frac{1}{\sqrt{2\pi}} 2 \sqrt{\frac{\pi}{2}} \\
&= 1
\end{aligned}$$

Therefore, the $f(x;\mu,\sigma)$ is a proper pdf.

7. (Bonus question) A grocery store is running a sales promotion where a customer will receive one of the letters A, E, L, S, U and V for each purchase. The letters are given away randomly by cashier. If a customer collects all six letters ("VALUES"), he/she will get a coupon of \$10 that can be used in the store. What is the expected number of purchases need to get a coupon. (Hint: Let X be the number of purchases need to collect all six letters. Let X_i be the number of purchases to get the ith letter, for $i = 1, \ldots, 6$. So, $X = X_1 + X_2 + \ldots + X_6$.)
Solution: With one purchase, a customer will get a ticket with whatever letter that becomes the first letter. Therefor $\mathrm{E}(X_1) = 1$. To get the second letter, i.e, a different letter which is not the same as the first one, the number of purchase X_2 has the pdf as

$$f_{X_2}(x) = P(X_2 = x) = \left(\frac{1}{6})^{x-1}(\frac{5}{6}\right), \quad x = 1, 2, 3, ...$$

In fact, $X_2 \sim geometric(p)$ with $p = \frac{5}{6}$, and

$$\mathrm{E}(X_2) = \frac{1}{p} = \frac{6}{5}$$

To get the third letter, i.e, a different letter which is not the same as the first and second letters, the number of purchase X_3 has the pdf as

$$f_{X_3}(x) = P(X_3 = x) = \left(\frac{2}{6}\right)^{x-1} \left(\frac{4}{6}\right), \quad x = 1, 2, 3, ...$$

and $X_3 \sim geometric(p)$ with $p = \frac{4}{6}$, such that

$$\mathrm{E}(X_3) = \frac{1}{p} = \frac{6}{4}$$

Similarly, we have

$$\mathrm{E}(X_4) = \frac{6}{3}, \quad \mathrm{E}(X_5) = \frac{6}{2}, \quad \mathrm{E}(X_5) = \frac{6}{1}.$$

As X is the number of purchases to collect all 6 letters,

$$X = X_1 + X_2 + X_3 + X_4 + X_5 + X_6$$

then

$$\begin{aligned}
\mathrm{E}(X) &= \mathrm{E}(X_1 + X_2 + X_3 + X_4 + X_5 + X_6) \\
&= \mathrm{E}(X_1) + \mathrm{E}(X_2) + \mathrm{E}(X_3) + \mathrm{E}(X_4) + \mathrm{E}(X_5) + \mathrm{E}(X_6) \\
&= 1 + \frac{6}{5} + \frac{6}{4} + \frac{6}{3} + \frac{6}{2} + \frac{6}{1} \\
&= 14.7.
\end{aligned}$$