# STAT3100 2024F
# Assignment 3 Solution

Due: 11:59pm, Friday, October 25, 2024

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1. Let a random experiment be the cast of a pair of fair dice.

   (a) Let $X$ be the smaller of the numbers of dots between two dice if they are different and the common value if they are equal. Find the probability mass function, $f(x)$, of $X$.

   **Solution:** Suppose we have the sample space as follow:

$$S = \begin{bmatrix} (1,1) & (1,2) & (1,3) & (1,4) & (1,5) & (1,6) \\ (2,1) & (2,2) & (2,3) & (2,4) & (2,5) & (2,6) \\ (3,1) & (3,2) & (3,3) & (3,4) & (3,5) & (3,6) \\ (4,1) & (4,2) & (4,3) & (4,4) & (4,5) & (4,6) \\ (5,1) & (5,2) & (5,3) & (5,4) & (5,5) & (5,6) \\ (6,1) & (6,2) & (6,3) & (6,4) & (6,5) & (6,6) \end{bmatrix}$$

   Let $w$ be an outcome of $S$. We know $X = 1, 2, 3, 4, 5, or, 6$. The pmf of $X$ is given as:

$$\begin{aligned} f(1) &= P(X = 1) \\ &= P(w \in \{(1,1), (1,2), (1,3), (1,4), (1,5), (1,6), (2,1), (3,1), (4,1), (5,1), (6,1)\}) \\ &= \tfrac{11}{36} \\ f(2) &= P(X = 2) \\ &= P(w \in \{(2,2), (2,3), (2,4), (2,5), (2,6), (3,2), (4,2), (5,2), (6,2)\}) \\ &= \tfrac{9}{36} \end{aligned}$$

   Similarly, we have $f(3) = P(X = 3) = \frac{7}{36}, f(4) = P(X = 4) = \frac{5}{36}$, $f(5) = P(X = 5) = \frac{3}{36}, f(6) = P(X = 6) = \frac{1}{36}$ and their sum equal 1 as

$$\sum_{x=1}^{6} f(x) = f(1)+f(2)+f(3)+f(4)+f(5)+f(6) = \frac{11}{36}+\frac{9}{36}+\frac{7}{36}+\frac{5}{36}+\frac{3}{36}+\frac{1}{36} = 1$$

   (b) Let $Y$ be the absolute value of the difference of the numbers of dots between two dice . Find the probability mass function, $g(y)$, of $Y$.

   **Solution:** From the sample space $S$, we know that the possible values for $Y$

is 0, 1, 2, 3, 4, 5. Let $g(y) = P(Y = y)$ be the pmf of $Y$.

$$
\begin{aligned}
g(0) &= P(Y=0) \\
&= P(w \in \{(1,1),(2,2),(3,3),(4,4),(5,5),(6,6)\}) \\
&= \tfrac{6}{36} \\
g(1) &= P(Y=1) \\
&= P(w \in \{(1,2),(2,3),(3,4),(4,5),(5,6),(2,1),(3,2),(4,3),(5,4),(6,5)\}) \\
&= \tfrac{10}{36} \\
g(2) &= P(Y=2) \\
&= P(w \in \{(1,3),(2,4),(3,5),(4,6),(3,1),(4,2),(5,3),(6,4)\}) \\
&= \tfrac{8}{36} \\
g(3) &= P(Y=3) \\
&= P(w \in \{(1,4),(2,5),(3,6),(4,1),(5,2),(6,3)\}) \\
&= \tfrac{6}{36}
\end{aligned}
$$

Similarly, $g(4) = \frac{4}{36}$, $g(5) = \frac{2}{36}$, and

$$\sum_{y=0}^{5} g(y) = g(0)+g(1)+g(2)+g(3)+g(4)+g(5) = \frac{6}{36}+\frac{10}{36}+\frac{8}{36}+\frac{6}{36}+\frac{4}{36}+\frac{2}{36} = 1.$$

2. Suppose the cumulative distribution of a discrete random variable $X$ is given by

$$F(x) = \begin{cases} 0 & x < 0 \\ \frac{1}{2} & 0 \le x < 1 \\ \frac{3}{5} & 1 \le x < 2 \\ \frac{4}{5} & 2 \le x < 3 \\ \frac{9}{10} & 3 \le x < 3.5 \\ 1 & x \ge 3.5 \end{cases}$$

(a) Calculate the probability mass function of $X$.

**Solution:** As we can see that $X = 0, 1, 2, 3, 3.5$ with the pmf $f(x) = P(X = x)$ is given by

$$f(x) = P(X = x) = \begin{cases} \frac{1}{2}, & x = 0 \\ \frac{1}{10}, & x = 1, 3, 3.5 \\ \frac{1}{5}, & x = 2 \\ 0, & \text{otherwise.} \end{cases}$$

(b) Find $P(X > 1 \mid X < 3)$.

**Solution:** We know that the conditional probability is given by

$$\begin{aligned} P(X>1|X<3) &= \tfrac{P(X>1, X<3)}{P(X<3)} \\ &= \tfrac{P(1<X<3)}{P(X<3)} \\ &= \tfrac{P(X=2)}{F(2)} \\ &= \tfrac{1/5}{4/5} \\ &= \tfrac{1}{4} \end{aligned}$$

3. The probability density function of random variable $X$, the lifetime of a certain type of electronic device (measured in hours), is given by

$$f(x) = \begin{cases} cx^{-2} & x > 10 \\ 0 & x \le 10 \end{cases}$$

   (a) Find $c$.

   **Solution:** By Theorem 3.3.5, the pdf $f(x) \ge 0$, such that $c \ge 0$. In addition

$$\begin{aligned} 1 &= \textstyle\int_{-\infty}^{\infty} f(x)dx \\ &= \textstyle\int_{10}^{\infty} cx^{-2}dx \\ &= -cx^{-1}|_{10}^{\infty} \\ &= \tfrac{c}{10} \end{aligned}$$

$$\Rightarrow c = 10$$

   (b) Find $P(X > 20)$.

   **Solution:**

$$P(X > 20) = \int_{20}^{\infty} 10x^{-2}dx = -10x^{-1}|_{20}^{\infty} = 0 + 10/20 = 0.5$$

   (c) Find the cumulative distribution function of $X$.

   **Solution:** The cdf of X is given by

$$F(x) = P(X \le x) = \int_{-\infty}^{x} f(t)dt = \int_{10}^{x} 10t^{-2}dt = -10t^{-1}|_{10}^{x} = 1 - \frac{10}{x}$$

4. If the joint probability distribution of $X$ and $Y$ is given by

$$f(x,y) = c(x^2 + y^2) \qquad \text{for } x = -1, 0, 1, 3; \quad y = -1, 2, 3$$

   (a) Find the value of $c$.

**Solution:** First all, $c$ must be non-negative to have $f(x,y) \geq 0$ for all possible $(x,y)$'s. Then

$$\begin{aligned}
1 &= \textstyle\sum_{\text{all x}} \sum_{\text{all y}} f(x,y) \\
&= c \textstyle\sum_{\text{all x}} \sum_{\text{all y}} (x^2+y^2) \\
&= c\,[((-1)^2+(-1)^2)+((-1)^2+2^2)+((-1)^2+3^2)+(0^2+(-1)^2) \\
&\quad +(0^2+2^2)+(0^2+3^2)+(1^2+(-1)^2)+(1^2+2^2)+(1^2+3^2) \\
&\quad +(3^2+(-1)^2)+(3^2+2^2)+(3^2+3^2)] \\
&= c(2+5+10+1+4+9+2+5+10+10+13+18) \\
&= 89c
\end{aligned}$$

$$\Rightarrow c = 1/89$$

(b) Find $P(X=0, Y \leq 2)$.

**Solution:**

$$\begin{aligned}
P(X=0, Y\leq 2) &= P(X=0,Y=-1)+P(X=0,Y=2) \\
&= \tfrac{1}{89}(0^2+(-1)^2)+\tfrac{1}{89}(0^2+(2)^2) \\
&= \tfrac{5}{89}
\end{aligned}$$

(c) Find $P(X+Y>2)$.

**Solution:**

$$\begin{aligned}
P(X+Y>2) &= P(X=0,Y=3)+P(X=1,Y=2)+P(X=1,Y=3) \\
&\quad +P(X=3,Y=2)+P(X=3,Y=3) \\
&= \tfrac{1}{89}[(0^2+3^2)+(1^2+2^2)+(1^2+3^2)+(3^2+2^2)+(3^2+3^2)] \\
&= \tfrac{55}{89}
\end{aligned}$$

(d) Find $f_X(x), f_Y(y)$ the marginal distribution of $X$ and $Y$.

**Solution:**

$$\begin{aligned}
f_X(x) &= \textstyle\sum_{\text{all } y} f_{X,Y}(x,y) \\
&= f_{X,Y}(x,-1)+f_{X,Y}(x,2)+f_{X,Y}(x,3) \\
&= \tfrac{x^2+1}{89}+\tfrac{x^2+4}{89}+\tfrac{x^2+9}{89} \\
&= \tfrac{3x^2+14}{89}, \quad \text{for } x=-1,0,1,3, \text{or otherwise, } f_X(x)=0.
\end{aligned}$$

$$\sum_{\text{all } x} f_X(x) = f_X(-1)+f_X(0)+f_X(1)+f_X(3) = \frac{17}{89}+\frac{14}{89}+\frac{17}{89}+\frac{41}{89} = 1.$$

$$\begin{aligned}
f_Y(y) &= \textstyle\sum_{\text{all } x} f_{X,Y}(x,y) \\
&= f_{X,Y}(-1,Y)+f_{X,Y}(0,y)+f_{X,Y}(1,y)+f_{X,Y}(3,y) \\
&= \tfrac{1+y^2}{89}+\tfrac{y^2}{89}+\tfrac{1+y^2}{89}+\tfrac{9+y^2}{89} \\
&= \tfrac{11+4y^2}{89},, \quad \text{for } y=-1,2,3, \text{or otherwise, } f_Y(y)=0.
\end{aligned}$$

$$\sum_{\text{all } y} f_Y(y) = f_Y(-1) + f_Y(2) + f_Y(3) = \frac{15}{89} + \frac{27}{89} + \frac{47}{89} = 1.$$

5. Let $X$ and $Y$ have the joint pdf

$$f_{X,Y}(x,y) = 2e^{-(x+y)}, \qquad 0 < x < y, \quad 0 < y.$$

Find $P(Y < 3X)$.

**Solution:** Given that $0 < x < y$ such that the joint pdf is positive, we are looking at the probability in the area that $Y < 3X$, that is $0 < \frac{Y}{3} < X < Y$. Therefor we have

$$\begin{aligned} P(Y < 3X) &= \int_0^\infty \int_{\frac{y}{3}}^{y} 2e^{-(x+y)} dx dy \\ &= \int_0^\infty -2e^{-(x+y)}|_{\frac{y}{3}}^{y} dy \\ &= \int_0^\infty (-2e^{-2y} + 2e^{-4y/3}) dy \\ &= [e^{-2y} - \frac{3}{2}e^{-4y/3}]_0^\infty \\ &= \frac{1}{2} \end{aligned}$$

6. If $X$ and $Y$ have joint pdf

$$f_{X,Y}(x,y) = cxy, \qquad 0 \le x \le 2, \quad 0 \le y \le 2,$$

(a) find the value of $c$,

**Solution:** First of all, $c > 0$ and

$$\begin{aligned} 1 &= \int_0^2 \int_0^2 cxydxdy \\ &= \int_0^2 \frac{c}{2}x^2y|_0^2 dy \\ &= \int_0^2 2cydy \\ &= \frac{2cy^2}{2}|_0^2 \\ &= 4c \\ &\Rightarrow c = \frac{1}{4} \end{aligned}$$

(b) find $P(0 \leq X \leq \frac{1}{2}, 0 \leq Y \leq \frac{3}{2})$.

**Solution:**

$$\begin{aligned} P(0 \leq x \leq \frac{1}{2}, 0 \leq y \leq \frac{3}{2}) &= \int_0^{\frac{3}{2}} \int_0^{\frac{1}{2}} \frac{xy}{4} dx dy \\ &= \int_0^{\frac{3}{2}} \frac{y}{32} dy \\ &= \frac{9}{256} \end{aligned}$$

(c) are $X$ and $Y$ independent?

**Solution:**

$$\begin{aligned} f_X(x) &= \int_0^2 \frac{xy}{4} dy \\ &= \frac{xy^2}{8} |_0^2 \\ &= \frac{x}{2} \\ f_Y(y) &= \int_0^2 \frac{xy}{4} dx \\ &= \frac{xy^2}{8} |_0^2 \\ &= \frac{y}{2} \end{aligned}$$

$$f_{X,Y}(x,y) = \frac{xy}{4} = f_X(x) f_Y(y)$$

for $0 < x < 2,\ 0 < y < 2$. So, $X$ and $Y$ are independent.

(d) find $P(X < 1 | Y = \frac{3}{2})$.

**Solution:** Because $X$ and $Y$ are independent so that,

$$f_{X|Y}(x|y) = f_X(x)$$

and therefore

$$\begin{aligned} P(X < 1 | Y = \frac{3}{2}) &= \int_0^1 f_{X|Y}(x|\frac{3}{2}) dx \\ &= \int_0^1 f_X(x) dx \\ &= \int_0^1 \frac{x}{2} dx \\ &= \frac{x^2}{4} |_0^1 \\ &= \frac{1}{4} \end{aligned}$$