STAT3100 2024 Fall
Assignment 1 Solution

Due: 11:59pm, Friday September 27. Submit via Gradescope.

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1. Show that

   (a) $\begin{pmatrix} n \\ r \end{pmatrix} = \frac{n-r+1}{r} \cdot \begin{pmatrix} n \\ r-1 \end{pmatrix}$;

   Solution:

$$\begin{aligned} RHS &= \tfrac{n-r+1}{r} \begin{pmatrix} n \\ r-1 \end{pmatrix} \\ &= \tfrac{n-r+1}{r} \tfrac{n!}{(r-1)!(n-r+1)!} \\ &= \tfrac{n!}{r!(n-r)!} \\ &= \begin{pmatrix} n \\ r \end{pmatrix} \\ &= LHS \end{aligned}$$

   (b) $\begin{pmatrix} n \\ r \end{pmatrix} = \frac{n}{n-r} \cdot \begin{pmatrix} n-1 \\ r \end{pmatrix}$;

   Solution:

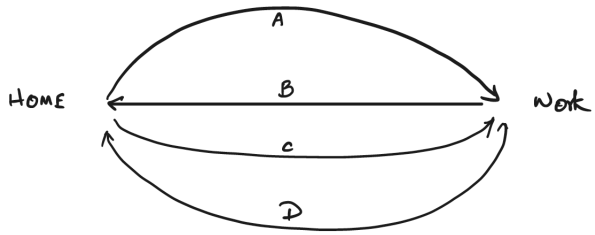
$$\begin{aligned} RHS &= \tfrac{n}{n-r} \tfrac{(n-1)!}{r!(n-1-r)!} \\ &= \tfrac{n!}{r!(n-r)!} \\ &= \begin{pmatrix} n \\ r \end{pmatrix} \\ &= LHS \end{aligned}$$

   (c) $n \begin{pmatrix} n-1 \\ r \end{pmatrix} = (r+1) \begin{pmatrix} n \\ r+1 \end{pmatrix}$;

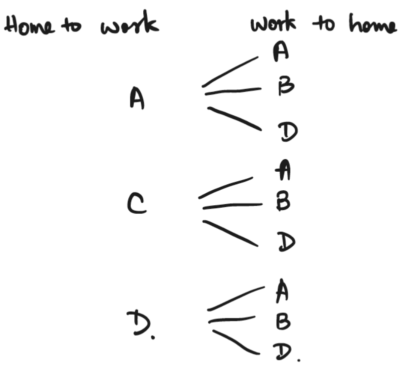
   Solution:

$$\begin{aligned} LHS &= n \begin{pmatrix} n-1 \\ r \end{pmatrix} \\ &= n \tfrac{(n-1)!}{r!(n-1-r)!} \\ &= \tfrac{n!}{r!(n-r-1)!} \\ RHS &= (r+1) \begin{pmatrix} n \\ r+1 \end{pmatrix} \\ &= (r+1) \tfrac{n!}{(r+1)!(n-r-1)!} \\ &= \tfrac{n!}{r!(n-r-1)!} \\ LHS &= RHS \end{aligned}$$

2. There are four routes A, B, C and D between a person's home and the place where he works, but route B is one-way such that he cannot take it on the way to work and route C is one-way such that he cannot take it on the way home.

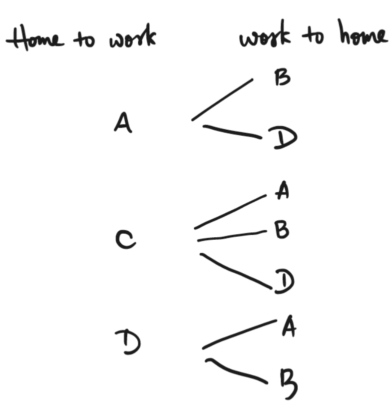


(2a) Draw a tree diagram showing the various ways the person can go to and from work.
Solution:



In total, there are 9 ways to go to and from work.

(2b) Draw a tree diagram showing the various ways the person can go to and from work without taking the same route both ways.
Solution:



In total, there are 7 ways to go to and from work without taking the same route both ways.

3. (a) How many three-digit numbers can be formed from the digits 0, 1, 2, 3, 4, 5 and 6, if each digit can be used only once?

   Solution: Suppose the first digit cannot be 0, then, the first digit can only be selected from 6 numbers, whatever number selected for the first digit can not be selected for the second digit, so, there are 6 numbers left to be selected for the second digit, after that, 5 numbers left to be selected for the third digit. Therefore, there are $6 \times 6 \times 5 = 180$ three-digit numbers can be formed.

   (b) How many of these are odd number?

   Solution: Suppose the third (last) digit can only be selected from odd numbers 1, 3, 5, then, there are 6 numbers left. Given that the first digit cannot be 0, so, only 5 numbers can be selected for the first digit. Whatever number selected for the first and third digits cannot be used again, so, there are 5 numbers left for the second digit. Therefore, there are $3 \times 5 \times 5 = 75$ three-digit numbers can be formed.

   (c) How many are greater than 330?

   Solution:

   Situation 1: $\boxed{3}$ $\boxed{}$ $\boxed{}$

   the first # is 3. | can put 4, 5, 6, here | 5 numbers left to be selected.

   there are $3 \times 5 = 15$ ways.

   Situation 2: $\boxed{}$ $\boxed{}$ $\boxed{}$

   the first # is selected from 4, 5, 6, | $6 \times 5 = 30$ choices for the last 2 digits.

   there are $3 \times 6 \times 5 = 90$ ways

   Therefore, there are $15 + 90 = 105$ 3-digits greater than 330.

4. **Sampling with and without replacement.**
   Sampling with replacement:
   A box contains tickets marked 1, 2, 3, ..., n. A ticket is drawn at random from the box. Then this ticket is replaced in the box and a second ticket is drawn at random. Find the probabilities of the following events:

   (a) the first ticket drawn is number 1 and the second ticket drawn is number 2;

   Solution: $\frac{1}{n} \cdot \frac{1}{n} = \frac{1}{n^2}$.

   (b) the numbers on the two tickets are conservative integers, meaning the first number drawn is one less than the second number drawn.

Solution: $\frac{n-1}{n} \cdot \frac{1}{n} = \frac{n-1}{n^2}$. (The first number cannot be the biggest number n.)

(c) the second number drawn is bigger than the first number drawn.

Solution: $\frac{n-1}{n^2} + \frac{n-2}{n^2} + \cdots + \frac{2}{n^2} + \frac{1}{n^2} = \frac{n(n-1)/2}{n^2} = \frac{n-1}{2n}$.

Sampling without replacement:

(d) Repeat (a) through (c) assuming instead that the first ticket drawn is not replaced, so the second ticket drawn must be different from the first.

Solution:

a) $\frac{1}{n} \cdot \frac{1}{n-1} = \frac{1}{n(n-1)}$

b) $\frac{n-1}{n} \cdot \frac{1}{n-1} = \frac{1}{n}$

c) $\frac{n-1}{n(n-1)} + \frac{n-2}{n(n-1)} + \cdots + \frac{2}{n(n-1)} + \frac{1}{n(n-1)} = \frac{n(n-1)/2}{n(n-1)} = \frac{1}{2}$.

5. How many possible ways of having 2 full houses among two hands of randomly selected cards from a deck of 52 poker cards. (One hand has 5 cards, a full house has three of one rank and two of another rank, e.g., three aces and two kings.)

Solution:

- Number of ways of choosing two different ranks (for 2 set of 3 cards of the same rank) from 13 is given by

$$\binom{13}{2} = \frac{13 \times 12}{2} = 78$$

- Number of ways to select 3 cards of one rank is

$$\binom{4}{3} = 4$$

So, the number of ways to select 2 set of 3 cards of the same rank is

$$\binom{13}{2} \times \binom{4}{3} \times \binom{4}{3} = 78 \times 4 \times 4 = 1248$$

- Number of ways to select 2 sets of 2 cards of another rank is

$$\binom{11}{2} \times \binom{4}{2} \times \binom{4}{2} + \binom{11}{1} = 1991$$

because the 2 set of 2 cards of another rank can be different or the same.

- In total, there are $1248 \times 1991 = 2,484,768$ possible ways of having two full houses among two hands. (There are alternative solutions, but would give the same number of ways.)