STAT3100 2021F
Assignment 4 Solution

Due: 11:59pm, Sunday, November 17, 2024

---

1. A fair coin is tossed five times. Let X be the number of heads that occur, and let Y be the number of heads occurring on the last two tosses. Find the conditional probability distribution of X for all possible values of Y.

   Solution:
   $$f_{X|Y}(x|y) = \begin{pmatrix} 3 \\ x-y \end{pmatrix} \left(\frac{1}{2}\right)^3,$$
   for $0 \leq x - y \leq 3$, $y = 0, 1, 2$, and $x = 0, 1, 2, 3, 4, 5$.

2. Given $\mathrm{E}(X+4) = 10$ and $E[(X+4)^2] = 116$, find the $\mathrm{var}(X+4)$, μ and σ^2.

   Solution:
   $$\mathrm{E}(X+4) = 10 \quad \Rightarrow \quad \mu = \mathrm{E}(X) = 6$$
   $$\begin{aligned} \mathrm{var}(X+4) &= \mathrm{E}[(X+4)^2] - [\mathrm{E}(X+4)]^2 \\ &= 116 - 10^2 \\ &= 16. \end{aligned}$$
   $$\mathrm{var}(X+4) = \mathrm{var}(X) = \sigma^2 = 16.$$

3. Let μ and σ^2 be the mean and the variance of the random variable X. Determine $\mathrm{E}[\frac{X-\mu}{\sigma}]$ and $\mathrm{E}[(\frac{X-\mu}{\sigma})^2]$.

   Solution:
   $$\mathrm{E}\left[\frac{X-\mu}{\sigma}\right] = \frac{1}{\sigma}\mathrm{E}(X-\mu) = \frac{0}{\sigma} = 0,$$
   $$\mathrm{E}\left[\left(\frac{X-\mu}{\sigma}\right)^2\right] = \mathrm{E}\left[\frac{(X-\mu)^2}{\sigma^2}\right] = \frac{1}{\sigma^2}\mathrm{E}[(X-\mu)^2] = \frac{1}{\sigma^2}\sigma^2 = 1.$$

4. Show that if a random variable has the pdf
   $$f(x) = \frac{1}{2}e^{-|x|} \qquad \text{for } -\infty < x < \infty$$
   its moment generating function is given by
   $$M_X(t) = \frac{1}{1-t^2}.$$
   Use this moment generating function to find the mean and the variance of X.

Solution:

$$
\begin{aligned}
M_X(t) &= \mathrm{E}(e^{Xt}) \\
&= \int_{-\infty}^{\infty} e^{xt} f(x)dx \\
&= \int_{-\infty}^{\infty} e^{xt}\frac{1}{2}e^{-|x|}dx \\
&= \frac{1}{2}\left[\int_{-\infty}^{0} e^{xt}e^{x}dx + \int_{0}^{\infty} e^{xt}e^{-x}dx\right] \\
&= \frac{1}{2}\left[\int_{-\infty}^{0} e^{(t+1)x}dx + \int_{0}^{\infty} e^{(t-1)x}dx\right] \\
&= \frac{1}{2}\left\{\left[\frac{1}{t+1}e^{(t+1)x}\right]_{-\infty}^{0} + \left[\frac{1}{t-1}e^{(t-1)x}\right]_{0}^{\infty}\right\} \\
&= \frac{1}{2}\left[\frac{1}{t+1} - \frac{1}{t-1}\right] \\
&= \frac{1}{1-t^2}
\end{aligned}
$$

for $|t| < 1$ is very small such that $t - 1 < 0$. Given the mgf, we can find $\mathrm{E}(X)$ and $E(X^2)$.

$$
\begin{aligned}
\mu = \mathrm{E}(X) &= \left.\frac{dM_X(t)}{dt}\right|_{t=0} \\
&= \left.\frac{d\frac{1}{1-t^2}}{dt}\right|_{t=0} \\
&= \left.2\frac{t}{(1-t^2)^2}\right|_{t=0} \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{E}(X^2) &= \left.\frac{d^2M_X(t)}{dt^2}\right|_{t=0} \\
&= \left.\frac{d^2\frac{t}{(1-t^2)^2}}{dt}\right|_{t=0} \\
&= \left[\frac{2}{(1-t^2)^2} + \frac{8t^2}{(1-t^2)^3}\right]_{t=0} \\
&= 2
\end{aligned}
$$

Then

$$\mathrm{var}(X) = \mathrm{E}(X^2) - [\mathrm{E}(X)]^2 = 2 - 0 = 2.$$

5. If the joint pdf of X and Y is given by

$$f_{X,Y}(x,y) = \frac{1}{3}(x+y) \qquad \text{for } 0 < x < 1, 0 < y < 2,$$

(a) find the E(X), E(Y), and E(XY).

Solution: First, we find the marginal pdf's of X and Y.

$$f_X(x) = \int_0^2 \frac{1}{3}(x+y)dy = \left[\frac{1}{3}xy + \frac{1}{6}y^2\right]_0^2 = \frac{2x}{3} + \frac{2}{3} = \frac{2}{3}(x+1), \quad 0 < x < 1.$$

$$f_Y(y) = \int_0^1 \frac{1}{3}(x+y)dx = \left[\frac{1}{6}x^2 + \frac{1}{3}xy\right]_0^1 = \frac{1}{6} + \frac{2y}{6} = \frac{1}{6}(2y+1), \quad 0 < y < 2.$$

$$\mathrm{E}(X) = \int_0^1 x \cdot \frac{2}{3}(x+1)dx = \frac{2}{3}\int_0^1 (x^2+x)dx = \frac{2}{3}\left[\frac{x^2}{3} + \frac{x^2}{2}\right]_0^1 = \frac{5}{9}$$

$$\mathrm{E}(Y) = \int_0^2 y \cdot \frac{1}{6}(2y+1)dy = \frac{1}{6}\left[\frac{2y^3}{3} + \frac{y^2}{2}\right]_0^2 = \frac{11}{9}$$

$$\begin{aligned}
\mathrm{E}(XY) &= \int_0^2 \int_0^1 xy \cdot \frac{1}{3}(x+y)dxdy \\
&= \int_0^2 \frac{y}{3} \int_0^1 (x^2 + xy)dxdy \\
&= \int_0^2 \frac{y}{3}\left[\frac{x^3}{3} + \frac{x^2 y}{2}\right]_0^1 dy \\
&= \int_0^2 \frac{y}{3}\left(\frac{1}{3} + \frac{y}{2}\right) dy \\
&= \int_0^2 \frac{y}{9} + \frac{y^2}{6} dy \\
&= \left[\frac{y^2}{18} + \frac{y^3}{18}\right]_0^2 = \frac{2}{3}.
\end{aligned}$$

(b) find the mean and variance of $W = 3X + 4Y - 5$,

Solution:

$$\mathrm{E}(W) = \mathrm{E}(3X+4Y-5) = 3\mathrm{E}(X) + 4\mathrm{E}(Y) - 5 = 3\left(\frac{5}{9}\right) + 4\left(\frac{11}{9}\right) - 5 = \frac{14}{9}.$$

$$\mathrm{var}(W) = \mathrm{var}(3X+4Y-5) = 9\mathrm{var}(X) + 16\mathrm{var}(Y) + 24\mathrm{cov}(X,Y)$$

To find the variance of W, we need to find the variance of X and Y,

$$\mathrm{E}(X^2) = \int_0^1 x^2 \cdot \frac{2}{3}(x+1)dx = \frac{2}{3}\int_0^1 (x^3+x^2)ds = \frac{2}{3}\left[\frac{x^4}{4} + \frac{x^3}{3}\right]_0^1 = \frac{7}{18}$$

$$\mathrm{E}(Y^2) = \int_0^2 y^2 \cdot \frac{1}{6}(2y+1)dy = \left[\frac{y^4}{12} + \frac{y^3}{18}\right]_0^2 = \frac{16}{9}$$

$$\mathrm{var}(X) = \mathrm{E}(X^2) - [\mathrm{E}(X)]^2 = \frac{7}{18} - \left(\frac{5}{9}\right)^2 = \frac{13}{162}$$

$$\text{var}(Y) = \text{E}(Y^2) - [\text{E}(Y)]^2 = \frac{16}{9} - \left(\frac{11}{9}\right)^2 = \frac{23}{81}$$

$$\text{cov}(X,Y) = \text{E}(XY) - \text{E}(X)\text{E}(Y) = \frac{2}{3} - \left(\frac{5}{9}\right)\left(\frac{11}{9}\right) = \frac{-1}{81}$$

$$\begin{aligned}\text{var}(W) &= 9\text{var}(X) + 16\text{var}(Y) + 24\text{cov}(X,Y) \\ &= 9\left(\frac{13}{162}\right) + 16\left(\frac{23}{81}\right) + 24\left(\frac{-1}{81}\right) \\ &= \frac{805}{162}\end{aligned}$$

(c) find the conditional mean and conditional variance of Y given $X = \frac{1}{2}$.
Solution: We work out the conditional distribution of Y for given X first,

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{\frac{1}{3}(x+y)}{\frac{2}{3}(x+1)} = \frac{x+y}{2(x+1)}, \quad 0 < x < 1, \quad 0 < y < 2$$

Given $X = \frac{1}{2}$,

$$f_{Y|X}\left(y\Big|\frac{1}{2}\right) = \frac{\frac{1}{2}+y}{2(\frac{1}{2}+1)} = \frac{2y+1}{6}, \quad 0 < y < 2.$$

Then

$$\text{E}\left(Y\Big|X = \frac{1}{2}\right) = \int_{-\infty}^{\infty} y f_{Y|X}\left(y\Big|\frac{1}{2}\right) dy = \int_0^2 y\left(\frac{2y+1}{6}\right) dy = \left[\frac{y^3}{9} + \frac{y^2}{12}\right]_0^2 = \frac{8}{9} + \frac{1}{3} = \frac{11}{9},$$

and

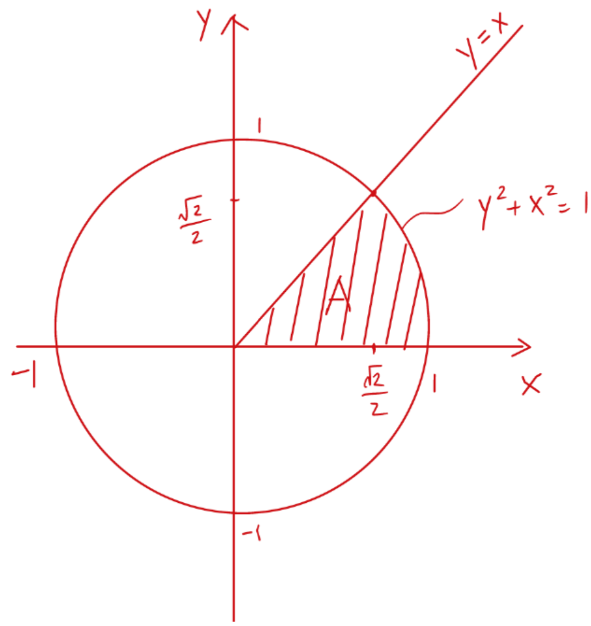
$$\text{E}\left(Y^2|X = \frac{1}{2}\right) = \int_{-\infty}^{\infty} y^2 f_{Y|X}\left(y\Big|\frac{1}{2}\right) dy = \int_0^2 y^2\left(\frac{2y+1}{6}\right) dy = \left[\frac{y^4}{12} + \frac{y^3}{18}\right]_0^2 = \frac{16}{9}$$

$$\Rightarrow \text{var}\left(Y|X = \frac{1}{2}\right) = \text{E}\left(Y^2|X = \frac{1}{2}\right) - \left[\text{E}\left(Y|X = \frac{1}{2}\right)\right]^2 = \frac{16}{9} - \left(\frac{11}{9}\right)^2 = \frac{23}{81}.$$

6. (Ex. 3.100, P123, Recommended textbook of Miller & Miller (2014)) A sharpshooter is aiming at a circular target with radius 1. While setting the origin of a xy-plane at the center of the target, let the X and Y represent the coordinates of the points of impact at the xy-plane. We have the joint pdf of X and Y as

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{\pi} & \text{for } 0 < x^2 + y^2 < 1 \\ 0 & \text{o.w} \end{cases}$$

Find



(a) $P[(X,Y) \in A]$, where A is the sector of the circle in the first quadrant bounded by the lines $Y = 0$ and $Y = X$. (Hint: the area of A is shown in a figure on next page.)

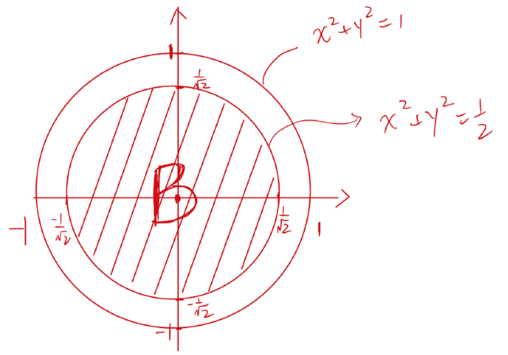
Solution: Suppose the event of interest is when the point (X, Y) fall in the area A as shown in the figure. The area of interest can be expressed by inequalities of, for each point that $0 < y < 1/\sqrt{2}$, we have $y < x$ and $x < \sqrt{1-y^2}$, that is $y < x < \sqrt{1-y^2}$. Therefore,

$$\begin{aligned} P((X,Y) \in A) &= \int_0^{\frac{1}{\sqrt{2}}} \int_y^{\sqrt{1-y^2}} \frac{1}{\pi} dx dy \\ &= \frac{1}{\pi} \int_0^{\frac{1}{\sqrt{2}}} \left(\sqrt{1-y^2} - y \right) dy \\ &= \frac{1}{\pi} \left[\frac{y}{2}\sqrt{1-y^2} + \frac{1}{2} sin^{-1} y - \frac{1}{2} y^2 \right]_0^{\frac{1}{\sqrt{2}}} \\ &= \frac{1}{\pi} \left[\frac{1}{4} + \frac{1}{2} sin^{-1} \frac{1}{\sqrt{2}} - \frac{1}{4} \right] \\ &= \frac{1}{8} \end{aligned}$$

Alternative solution: For $f_{X,Y}(x, y) = \frac{1}{\pi}, 0 < x^2 + y^2 < 1$, that means every point in the circle is equally likely. Therefore

$$P((X,Y) \in A] = \frac{\text{Area } (A)}{\text{Area (circle)}} = \frac{1}{8}.$$

(b) $P(0 < X^2 + Y^2 < \frac{1}{2})$.



Solution: Suppose the event of interest is when the point (X, Y) falls in the area B described by the inequality $0 < X^2 + Y^2 < \frac{1}{2}$ as shown in the figure. For each point (x, y) that $-\frac{1}{\sqrt{2}} < y < \frac{1}{\sqrt{2}}$, we have $0 < x^2 + y^2 < \frac{1}{2}$, that means $x^2 < \frac{1}{2} - y^2$. So, for $-\sqrt{\frac{1}{2} - y^2} < x < \sqrt{\frac{1}{2} - y^2}$ and $-\frac{1}{\sqrt{2}} < y < \frac{1}{\sqrt{2}}$, we have

$$
\begin{aligned}
P(0 < X^2 + Y^2 < \frac{1}{2}) &= \int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} \int_{-\sqrt{\frac{1}{2}-y^2}}^{\sqrt{\frac{1}{2}-y^2}} \frac{1}{\pi}\, dxdy \\
&= \frac{1}{\pi} \int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} x\Big|_{-\sqrt{\frac{1}{2}-y^2}}^{\sqrt{\frac{1}{2}-y^2}}\, dy \\
&= \frac{1}{\pi} \int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} 2\sqrt{\frac{1}{2} - y^2 dy} \\
&= \frac{1}{\pi} \left[y\sqrt{\frac{1}{2} - y^2} + \frac{1}{2}\sin^{-1}(\sqrt{2}y) \right]_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} \\
&= \frac{1}{\pi} \left[\frac{1}{2}\sin^{-1}(1) - \frac{1}{2}\sin^{-1}(-1) \right] \\
&= \frac{1}{\pi} \left(\frac{\pi}{2} \right) \\
&= \frac{1}{2}
\end{aligned}
$$

Or, alternatively, with the uniform distribution of (X, Y) in the circle, the probability that the points satisfy $0 < X^2 + Y^2 < \frac{1}{2}$ is

$$
P((X, Y) \in B] = \frac{\text{Area }(B)}{\text{Area (circle)}} = \frac{\pi r^2}{\pi R^2} = \frac{1/2}{1},
$$

for $r = \sqrt{1/2}$ is the radius of inner circle B and $R = 1$ is the radius of the outer circle.