HM1

Problem 1

Suppose $f \in C[a,b]$, that x_1 and x_2 are in $[a,b]$.

a. Show that a number ξ exists between x_1 and x_2 with

$$f(\xi) = \frac{f(x_1) + f(x_2)}{2} = \frac{1}{2}f(x_1) + \frac{1}{2}f(x_2).$$

b. Suppose that c_1 and c_2 are positive constants. Show that a number ξ exists between x_1 and x_2 with

$$f(\xi) = \frac{c_1 f(x_1) + c_2 f(x_2)}{c_1 + c_2}.$$

c. Give an example to show that the result in part **b.** does not necessarily hold when c_1 and c_2 have opposite signs with $c_1 \neq -c_2$.

solotion

a.

if $f(x_1)$ = $f(x_2)$, so $\exists\ \xi = x_1$ or $\xi = x_2$

let $f(\xi) = \frac{f(x_1)+f(x_2)}{2}$

if $f(x_1) \neq f(x_2)$, without loss of generality, let $f(x_2) > f(x_1)$

so $f(x_1) < \frac{f(x_1)+f(x_2)}{2} < f(x_2)$

according to the *intermediate value theorem*

$\exists \xi \in [x_1, x_2]$, $f(\xi) = \frac{f(x_1)+f(x_2)}{2}$ = $\frac{1}{2}f(x_1)$ + $\frac{1}{2}f(x_2)$

b.

if $f(x_1) = f(x_2)$, $\exists \xi = x_1$ or $\xi = x_2$, makes $f(\xi) = \frac{c_1 f(x_1)+c_2 f(x_2)}{c_1+c_2}$

if $f(x_1) \neq f(x_2)$, without loss of generality, let $f(x_2) > f(x_1)$,

so $f(x_1) < \frac{c_1 f(x_1)+c_2 f(x_2)}{c_1+c_2} < f(x_2)$

according to the *intermediate value theorem*

$\exists \xi \in [x_1, x_2]$, $f(\xi) = \frac{c_1 f(x_1)+c_2 f(x_2)}{c_1+c_2}$

c.

let $f(x) = x, x_1 = 1, x_2 = 2, c_1 = 1, c_2 = -2$,

so $\frac{c_1 f(x_1)+c_2 f(x_2)}{c_1+c_2}$ = 3

there's no $\xi \in [1,2] \quad makes \quad f(\xi) = 3$

Problem 2

Let $f \in C[a,b]$ be a function whose derivative exists on (a,b). Suppose f is to be evaluated at x_0 in (a,b), but instead of computing the actual value $f(x_0)$, the approximate value, $\tilde{f}(x_0)$, is the actual value of f at $x_0+\epsilon$, that is, $\tilde{f}(x_0) = f(x_0+\epsilon)$.

a. Use the Mean Value Theorem 1.8 to estimate the absolute error $|f(x_0) - \tilde{f}(x_0)|$ and the relative error $|f(x_0) - \tilde{f}(x_0)|/|f(x_0)|$, assuming $f(x_0) \neq 0$.

b. If $\epsilon = 5 \times 10^{-6}$ and $x_0 = 1$, find bounds for the absolute and relative errors for

  i. $f(x) = e^x$

  ii. $f(x) = \sin x$

c. Repeat part (b) with $\epsilon = (5 \times 10^{-6})x_0$ and $x_0 = 10$.

solotion

a.

According to Mean Value Theorem

$|f(x_0) - \tilde{f}(x_0)|$ = $f(x_0) - f(x_0+\varepsilon)$=$f'(\xi)\varepsilon$, $\xi \in (x_0, x_0+\varepsilon)$

when $\varepsilon \to 0$, $|f'(\xi)| \to |f'(x_0)|$

so the absolute error and the relative error are

$$|f(x_0) - \tilde{f}(x_0)| = f'(x_0)\varepsilon$$

$$\frac{|f(x_0) - \tilde{f}(x_0)|}{f(x_0)} = \frac{f'(x_0)\varepsilon}{f(x_0)}$$

b.

i.

From $f(x) = e^x$, we have that $f'(x) = e^x, f'(x_0) = e, \varepsilon = 5 \times 10^{-6}$

so absolute error is $5e \times 10^{-6}$

relative error is 5×10^{-6}

ii.From $f(x) = sinx$, we have that $f'(x) = cosx, f'(x_0) = cos1, \varepsilon = 5 \times 10^{-6}$

so absolute error is $5cos1 \times 10^{-6}$

relative error is $5cot1 \times 10^{-6}$

c.

i.

From $f(x) = e^x$, we have that $f'(x) = e^10, f'(x_0) = e, \varepsilon = 5 \times 10^{-5}$

so absolute error is $5e^{10} \times 10^{-5}$

relative error is 5×10^{-5}

ii.

From $f(x) = sinx$, we have that $f'(x) = cosx, f'(x_0)cos10e, \varepsilon = 5 \times 10^{-5}$

so absolute error is $5cos10 \times 10^{-5}$
relative error is $5cot10 \times 10^{-5}$

Problem 3

Perform the following computations (i) exactly, (ii) using three-digit chopping arithmetic, and (iii) using three-digit rounding arithmetic. (iv) Compute the relative errors in parts (ii) and (iii).

a. $\frac{4}{5} + \frac{1}{3}$

b. $\left(\frac{1}{3} + \frac{3}{11}\right) - \frac{3}{20}$

solution

We have

	a	b
Exact	$\frac{17}{15}$	$\frac{301}{660}$
3-digit chopping	1.13	0.445
Relative error	0.003	0.00233
3-digit rounding	1.13	0.456
Relative error	0.003	0.000133

Proble 4

Suppose that as x approaches zero,

$$F_1(x) = L_1 + O(x^\alpha) \quad \text{and} \quad F_2(x) = L_2 + O(x^\beta).$$

Let c_1 and c_2 be nonzero constants, and define

$$F(x) = c_1F_1(x) + c_2F_2(x) \quad \text{and}$$

$$G(x) = F_1(c_1x) + F_2(c_2x).$$

Show that if $\gamma = \text{minimum } \{\alpha, \beta\}$, then as x approaches zero,

a. $F(x) = c_1L_1 + c_2L_2 + O(x^\gamma)$

b. $G(x) = L_1 + L_2 + O(x^\gamma)$.

solution

a.

$F(x) = c_1F_1(x) + c_2F_2(x) = c_1L_1 + c_1O(x^\alpha) + c_2L_2 + c_2O(x^\beta)$
because $\gamma = min\{\alpha, \beta\}$
so $c_1O(x^\alpha) + c_2O(x^\beta) = O(x^\gamma) + o(x^\gamma)$
when $x \to 0, o(x^\gamma) = 0$

hence

$$F(x) = c_1 L_1 + c_2 L_2 + O(x^\gamma)$$

b.

$G(x) = F_1(c_1 x) + F_2(c_2 x) = L_1 + L_2 + O(c_1^\alpha x^\alpha) + O(c_2^\beta x^\beta)$

The same as **a.**,$O(c_1^\alpha x^\alpha) + O(c_2^\beta x^\beta) = O(x^\gamma) + o(x^\gamma)$

so

$$G(x) = L_1 + L_2 + O(x^\gamma)$$

Problem 5

Implement the Bisection method in C or matlab and find solutions accurate to within 10^{-5} for the following problems. **(List the midpoints in each iteration as well)**. **【请在作业中上交程序代码】**

a. $e^x - x^2 + 3x - 2 = 0$ for $0 \le x \le 1$

b. $x \cos x - 2x^2 + 3x - 1 = 0$ for $0.2 \le x \le 0.3$ and $1.2 \le x \le 1.3$

solution

a.

n	a_n	b_n	p_n	f(p_n)
1	0.000000000	1.000000000	0.500000000	0.898721271
2	0.000000000	0.500000000	0.250000000	-0.028474583
3	0.250000000	0.500000000	0.375000000	0.439366415
4	0.250000000	0.375000000	0.312500000	0.206681691
5	0.250000000	0.312500000	0.281250000	0.089433196
6	0.250000000	0.281250000	0.265625000	0.030564234
7	0.250000000	0.265625000	0.257812500	0.001066368
8	0.250000000	0.257812500	0.253906250	-0.013698684
9	0.253906250	0.257812500	0.255859375	-0.006314807
10	0.255859375	0.257812500	0.256835938	-0.002623882
11	0.256835938	0.257812500	0.257324219	-0.000778673
12	0.257324219	0.257812500	0.257568359	0.000143868
13	0.257324219	0.257568359	0.257446289	-0.000317397

n	a_n	b_n	p_n	f(p_n)
14	0.257446289	0.257568359	0.257507324	-0.000086763
15	0.257507324	0.257568359	0.257537842	0.000028553
16	0.257507324	0.257537842	0.257522583	-0.000029105
17	0.257522583	0.257537842	0.257530212	-0.000000276

so x=0.257530212

b.

$0.2 \leq x \leq 0.3$

n	a_n	b_n	p_n	f(p_n)
1	0.200000000	0.300000000	0.250000000	-0.132771895
2	0.250000000	0.300000000	0.275000000	-0.061583071
3	0.275000000	0.300000000	0.287500000	-0.027112719
4	0.287500000	0.300000000	0.293750000	-0.010160959
5	0.293750000	0.300000000	0.296875000	-0.001756232
6	0.296875000	0.300000000	0.298437500	0.002428306
7	0.296875000	0.298437500	0.297656250	0.000337524
8	0.296875000	0.297656250	0.297265625	-0.000708983
9	0.297265625	0.297656250	0.297460938	-0.000185637
10	0.297460938	0.297656250	0.297558594	0.000075967
11	0.297460938	0.297558594	0.297509766	-0.000054829
12	0.297509766	0.297558594	0.297534180	0.000010570
13	0.297509766	0.297534180	0.297521973	-0.000022129
14	0.297521973	0.297534180	0.297528076	-0.000005779

so x=0.297528076

$1.2 \leq x \leq 1.3$

n	a_n	b_n	p_n	f(p_n)
1	1.200000000	1.300000000	1.250000000	0.019152953
2	1.250000000	1.300000000	1.275000000	-0.054585352
3	1.250000000	1.275000000	1.262500000	-0.017224892
4	1.250000000	1.262500000	1.256250000	0.001086892

n	a_n	b_n	p_n	f(p_n)
5	1.256250000	1.262500000	1.259375000	-0.008038288
6	1.256250000	1.259375000	1.257812500	-0.003468020
7	1.256250000	1.257812500	1.257031250	-0.001188644
8	1.256250000	1.257031250	1.256640625	-0.000050396
9	1.256250000	1.256640625	1.256445313	0.000518368
10	1.256445313	1.256640625	1.256542969	0.000234016
11	1.256542969	1.256640625	1.256591797	0.000091818
12	1.256591797	1.256640625	1.256616211	0.000020713
13	1.256616211	1.256640625	1.256628418	-0.000014841
14	1.256616211	1.256628418	1.256622314	0.000002936

so x=1.256622314

MATLAB

```matlab
fun1 = @(x) exp(x)-x^2+3*x-2;
fun2 = @(x) x*cos(x)-2*x^2+3*x-1;
tol = 1E-5; %
maxIt = 40; %  Max iteration time
[p, flag] = bisect(fun1, 0, 1, tol, maxIt);

[p, flag] = bisect(fun2, 0.2, 0.3, tol, maxIt);

[p, flag] = bisect(fun2, 1.2, 1.3, tol, maxIt);

function [p, flag] = bisect(fun, a, b, tol, maxIt)
flag = 0;   % Use a flag to tell if the output is reliable
if fun(a) * fun(b)> 0   %   Check f(a) and f(b) have different
sign
    error('f(a) and f(b) must have different signs');
end
disp('Bisection Methods')
disp('-----------------------------------------------------------------
---')
disp(' n         a_n             b_n             p_n
f(p_n)')
disp('-----------------------------------------------------------------
---')
formatSpec = '%2d    %.9f    %.9f    %.9f    %.9f\n';
for n = 1:maxIt
    p = (a+b)/2;
    FA = fun(a);
    FP = fun(p);
    fprintf(formatSpec,[n,a,b,p,fun(p)]) % Printing output
    if abs(FP) <= 10^(-15) || (b-a)/2<tol
        flag = 1;
        break;
    else
        if(FA*FP>0)
            a = p;
        else
            b = p;
```

```
            end
        end
    end
    end
```

Problem 6

Implement the fixed-point iteration method and find solutions accurate to within 10^{-2} for the following problems. **(List pn in each iteration as well)**. **【请在作业中上交程序代码】**

a. $2\sin\pi x + x = 0$ on [1, 2], use $p_0 = 1$

b. $3x^2 - e^x = 0$

solution

a.

$x = \frac{1}{\pi} \times arcsin\ (\frac{-1}{2} \times x) + 2$

n	p	f(p_n)
0	1.000000000	1.833333333
1	1.833333333	1.630869246
2	1.630869246	1.696498005
3	1.696498005	1.677657062
4	1.677657062	1.683240993
5	1.683240993	1.681602013

so x = 1.683240993

b.

$x = ln(3 \times x^2)$

n	p	f(p_n)
0	3.000000000	3.295836866
1	3.295836866	3.483932521
2	3.483932521	3.594935670
3	3.594935670	3.657664482

n	p	f(p_n)
4	3.657664482	3.692261937
5	3.692261937	3.711090811
6	3.711090811	3.721263994
7	3.721263994	3.726739076
8	3.726739076	3.729679507

so x= 3.726739076

```matlab
fun1 =@(x) (1/pi)*asin((-1/2)*x)+2
fun2 = @(x) log(3*x^2)
tol  = 1E-2;
maxIt = 40;
[p, flag] = fixedpoint(fun1, 1, tol, maxIt);
[p, flag] = fixedpoint(fun2, 3, tol, maxIt);
function [p, flag] = fixedpoint(fun, p0, tol, maxIt)
n = 1;
flag = 0;
disp('Fixed Pointed Iteration')
disp('----------------------------------')
disp(' n          p           f(p_n)')
disp('----------------------------------')
formatSpec = '%2d    %.9f    %.9f    \n';
fprintf(formatSpec, [n-1, p0, fun(p0)])
while n <= maxIt
    p = fun(p0);
    fprintf(formatSpec, [n, p, fun(p)])
    if abs(p-p0) < tol
        flag = 1;
        break;
    else
        n = n+1;
        p0 = p;
    end
end
end
```

Problem 7

Let $g \in C^1[a,b]$ and p be in (a, b) with g(p) = p and $|g'(p)|>1$. Show that there exists a $\delta>0$ such that if $0<|p_0-p|<\delta$, then $|p_0-p|<|p_1-p|$. Thus, no matter how close the initial approximation p_0 is to p, the next iterate p_1 is father away, so the fixed-point iteration does not converge if $p_0 \neq p$.

solution

$|p_1-p| = |g(p_0)-g(p)| = |p_0-p||g'(\xi)|,\ \xi\in(p_0,p)$

cause $|g'(p)|>1$, that $\exists\delta>0, when\ 0<|p_0-p|<\delta,\ that|g'(\xi)|>1$

so $\exists\delta>0, when\ 0<|p_0-p|<\delta,\ that\ |p_1-p| = |p_0-p||g'(\xi)| > |p_0-p|$