HW5

Problem1

证明下列积分计算公式具有三次代数精度。

$$\int_a^b f(x)dx \approx \frac{b-a}{b}\left[f(a)+4f\left(\frac{\alpha+b}{2}\right)+f(b)\right]$$

solution

When $f(x)=1$, equation left: $\int_a^b f(x)dx = b-a$ equation right: $\frac{b-a}{6}[1+4+1]=b-a$ left=right

When $f(x)=x$, equation left: $\int_a^b x\,dx = \frac{x^2}{2}\Big|_a^b = \frac{b^2-a^2}{2}$ equation right: $\frac{b-a}{6}[a+4\cdot\frac{a+b}{2}+b]=\frac{b^2-a^2}{2}$

left=right

when $f(x)=x^2$, left: $\int_a^b x^2dx=\frac{x^3}{2}\Big|_a^b=\frac{b^3-a^3}{3}$ right: $\frac{b-a}{6}[a^2+4\cdot(\frac{a+b}{2})^2+b^2]=\frac{b^3-a^3}{3}$ right=left

when $f(x)=x^3$, left: $\int_a^b x^3dx=\frac{b^4-a^4}{4}$ right: $\frac{b-a}{6}[a^3+4\cdot(\frac{a+b}{2})^3+b^3]=\frac{b^4-a^4}{4}$ left=right

when $f(x)=x^4$, left: $\int_a^b x^4dx=\frac{x^5}{5}\Big|_a^b=\frac{b^5-a^5}{5}$

right $\frac{b-a}{6}[a^4+4\cdot(\frac{a+b}{2})^4+b^4]=\frac{b-a}{6}(5a^4+4a^3b+6a^2b^2+4ab^3+5b^4)=\frac{b^5-a^5}{5}$

left=right

So f(x) has cubic algebraic precision.

Problem2

假设 $N(h)$ 对于任意 $h>0$，都是 M 的近似，其中 $M=N(h)+K_1h^2+K_2h^4+K_3h^6+\cdots$，

$K_1,K_2,K_3,\ldots$是常数。请使用 $N(h),N\left(\frac{h}{3}\right),N\left(\frac{h}{9}\right)$来产生 M 的 $O\left(h^6\right)$近似。

Problem3

a. $\displaystyle\int_{-0.25}^{0.25}(\cos x)^2\,dx$

b. $\displaystyle\int_{-0.5}^{0} x\ln(x+1)\,dx$

c. $\displaystyle\int_{0.75}^{1.3}\left((\sin x)^2-2x\sin x+1\right)\,dx$

d. $\displaystyle\int_{e}^{e+1}\frac{1}{x\ln x}\,dx$

Approximate the following integrals using the Trapezoidal rule and Simpson's rule, respectively.

Trapezoidal rule

a

$$\int_{-0.25}^{0.25}(\cos(x))^2dx \approx \frac{(0.25+0.25)}{2}(f(-0.25)+f(0.25))$$

$$\int_{-0.25}^{0.25}(\cos(x))^2dx \approx 0.25(0.938791281+0.938791281) \approx 0.46939564$$

b

$$\int_{-0.5}^{0} x\ln(x+1)dx \approx \frac{(0+0.5)}{2}(f(-0.5)+f(0))$$

$$\int_{-0.5}^{0} x\ln(x+1)dx \approx 0.25(0.34657359+0) \approx 0.0866433975$$

c

$$\int_{0.75}^{1.3} \big((\sin(x))^2 - 2x\sin(x)+1\big)dx \approx \frac{(1.3-0.75)}{2}(f(0.75)+f(1.3))$$

$$\int_{0.75}^{1.3} \Big((\sin(x))^2 - 2x\sin(x)+1\Big)dx \approx 0.275(0.442173259-0.576806905) \approx -0.0370242526$$

d

$$\int_{e}^{e+1} \frac{1}{x\ln(x)}dx \approx \frac{(e+1-e)}{2}(f(e)+f(e+1))$$

$$\int_{e}^{e+1} \frac{1}{x\ln(x)}dx \approx 0.5\left(e^{-1}+0.204788904\right) \approx 0.286334172$$

Simpson's rule

a

$$\int_{-0.25}^{0.25}(\cos(x))^2dx \approx \frac{\left(\frac{1}{4}+\frac{1}{4}\right)}{6}\left(f\left(-\frac{1}{4}\right)+4f(0)+f\left(\frac{1}{4}\right)\right)$$

$$\int_{-0.25}^{0.25}(\cos(x))^2dx \approx \frac{1}{12}(0.938791281+4+0.938791281) \approx 0.489798$$

b

$$\int_{-0.5}^{0} x\ln(x+1)dx \approx \frac{\left(0+\frac{1}{2}\right)}{6}\left(f\left(-\frac{1}{2}\right)+4f\left(-\frac{1}{4}\right)+f(0)\right)$$

$$\int_{-0.5}^{0} x\ln(x+1)dx \approx 0.25(0.34657359+0.287682+0) \approx 0.052854$$

c

$$\int_{0.75}^{1.3}\left((\sin(x))^2-2x\sin(x)+1\right)dx \approx \frac{(1.3-0.75)}{6}(f(0.75)+4f(1.025)+f(1.3))$$

$$\int_{0.75}^{1.3}\left((\sin(x))^2-2x\sin(x)+1\right)dx \approx \frac{11}{120}(0.442173-0.086510-0.576806) \approx -0.020271$$

d

$$\int_{e}^{c+1}\frac{1}{x\ln(x)}dx \approx \frac{(c+1-c)}{6}\left(f(e)+4f\left(\frac{2e+1}{2}\right)+f(e+1)\right)$$

$$\int_{e}^{c+1}\frac{1}{x\ln(x)}dx \approx \frac{1}{6}\left(e^{-1}+1.063354+0.204788\right) \approx 0.272670$$

Problem4

Use Romberg integration to compute $R_{3,3}$ for the following integrals.

a. $\int_{-1}^{1}(\cos x)^2\,dx$

b. $\int_{-0.75}^{0.75} x\ln(x+1)\,dx$

c. $\int_{1}^{4}\left((\sin x)^2-2x\sin x+1\right)\,dx$

d. $\int_{e}^{2e}\frac{1}{x\ln x}\,dx$

a

The function is even and we observe the integral of an even function on a symmetric interval, our integral will be equal to

$$2\int_0^1(\cos x)^2\mathrm{d}x.$$

So we define

$$f(x)=2(cosx)^2$$

Now let's find $R_{1,1}$ on the interval $[0,1]$:

$$\begin{aligned}R_{1,1}&=\frac{1}{2}\cdot(2+0.5838531636)\\&=1.2919265818\end{aligned}$$

We need to find

$$R_{3,3}=R_{3,2}+\frac{1}{15}\cdot(R_{3,2}-R_{2,2}).$$

$$R_{2,2} = R_{2,1} + \frac{1}{3}(R_{2,1} - R_{1,1}).$$

$$\begin{aligned} R_{2,1} &= \frac{1}{2}\Big(R_{1,1} + h_1 f\Big(\frac{1}{2}\Big)\Big) \\ &= \frac{1}{2}\Big(1.2919265818 + 1.5403023059\Big) \\ &= 1.41611444385. \end{aligned}$$

$$\begin{aligned} R_{2,2} &= 1.41611444385 + \frac{1}{3}(1.41611444385 - 1.2919265818) \\ &= 1.4575103979. \end{aligned}$$

Let's see what we need to calculate $R_{3,2}$

$$R_{3,2} = R_{3,1} + \frac{1}{3}(R_{3,1} - R_{2,1}).$$

So we see that we need $R_{3,1}$, which we will calculate using the Chomposite Trapezoidal rule

$$\begin{aligned} R_{3,1} &= \frac{1}{2}(R_{2,1} + h_2(f(a + h_3) + f(a + 3h_3))) \\ &= \frac{1}{2}\left(1.41611444385 + \frac{1}{2}\Big(f\big(0.25\big) + f\big(0.75\big)\Big)\right) \\ &= \frac{1}{2}\left(1.41611444385 + \frac{1}{2}\Big(1.8775825619 + 1.0707372017\Big)\right) \\ &= 1.445137162825 \end{aligned}$$

$$\begin{aligned} R_{3,2} &= 1.445137162825 + \frac{1}{3}(1.445137162825 - 1.41611444385) \\ &= 1.45481140248. \end{aligned}$$

$$\begin{aligned} R_{3,3} &= 1.45481140248 + \frac{1}{15} \cdot (1.45481140248 - 1.4575103979) \\ &= 1.454631. \end{aligned}$$

b

Let's define the function

$$f(x) = x\ln(x + 1).$$

Now let's find $R_{1,1}$ on the interval [-0.75,0.75]:

$$R_{1,1} = 0.75 \cdot (-0.75\ln 0.25 + 0.75\ln 1.75) = 1.0945744589.$$

In the problem we need to find $R_{3,3}$ so let's see what we need to calculate it

$$R_{3,3} = R_{3,2} + \frac{1}{15} \cdot (R_{3,2} - R_{2,2}).$$

We see that we need $R_{3,2}$ and $R_{2,2}$. To calculate

$$R_{2,2} = R_{2,1} + \frac{1}{3}(R_{2,1} - R_{1,1}).$$

we need $R_{2,1}$, so let's calculate it

$$\begin{aligned} R_{2,1} &= \frac{1}{2}\Big(R_{1,1} + h_1 f(0)\Big) \\ &= \frac{1}{2}\Big(1.0945744589 + 1.5 \cdot 0\Big) \\ &= 0.54728722945. \end{aligned}$$

Based on the previous calculation, we get

$$R_{2,2} = 0.54728722945 + \frac{1}{3}(0.54728722945 - 1.0945744589) = 0.364858152967.$$

Let's see what we need to calculate $R_{3,2}$

$$R_{3,2} = R_{3,1} + \frac{1}{3}(R_{3,1} - R_{2,1}).$$

So we see that we need $R_{3,1}$, which we will calculate using the Chomposite Trapezoidal rule

$$\begin{aligned} R_{3,1} &= \frac{1}{2}(R_{2,1} + h_2(f(a + h_3) + f(a + 3h_3))) \\ &= \frac{1}{2}\big(0.54728722945 + 0.75\,(f\big(-0.375\big) + f\big(0.375\big))\big) \\ &= \frac{1}{2}\Bigg(0.54728722945 + 0.75\Big(0.176251361 + 0.1194201492\Big)\Bigg) \\ &= 0.38452043105. \end{aligned}$$

Based on the previous calculation, we can calculate $R_{3,2}$

$$R_{3,2} = 0.38452043105 + \frac{1}{3}(0.38452043105 - 0.54728722945) = 0.3302648316.$$

Finally, let's calculate $R_{3,3}$

$$R_{3,3} = 0.3302648316 + \frac{1}{15} \cdot (0.3302648316 - 0.364858152967) = 0.3279586.$$

c

Let's define the function

$$f(x) = (\sin x)^2 - 2x\sin x + 1.$$

Now let's find $R_{1,1}$ on the interval [1,4]:

$$\begin{aligned}R_{1,1} &= \frac{3}{2}\cdot\left((\sin 1)^2 - 2\sin 1 + 1 + (\sin 4)^2 - 8\sin 4 + 1\right)\\ &= 11.478452142.\end{aligned}$$

In the problem we need to find $R_{3,3}$ so let's see what we need to calculate it

$$R_{3,3} = R_{3,2} + \frac{1}{15}\cdot(R_{3,2} - R_{2,2}).$$

We see that we need $R_{3,2}$ and $R_{2,2}$. To calculate

$$R_{2,2} = R_{2,1} + \frac{1}{3}(R_{2,1} - R_{1,1}).$$

we need $R_{2,1}$, so let's calculate it

$$\begin{aligned}R_{2,1} &= \frac{1}{2}(R_{1,1} + h_1 f(a+h_2))\\ &= \frac{1}{2}\left(11.478452142 + 3f\left(\frac{5}{2}\right)\right)\\ &= \frac{1}{2}(11.478452142 - 3\cdot 1.63419\\ &= 3.28793835105.\end{aligned}$$

Based on the previous calculation, we get

$$R_{2,2} = 3.28793835105 + \frac{1}{3}(3.28793835105 - 11.478452142) = 0.5577670874.$$

Let's see what we need to calculate $R_{3,2}$

$$R_{3,2} = R_{3,1} + \frac{1}{3}(R_{3,1} - R_{2,1}).$$

So we see that we need $R_{3,1}$, which we will calculate using the Chomposite Trapezoidal rule

$$\begin{aligned}R_{3,1} &= \frac{1}{2}(R_{2,1} + h_2(f(a+h_3) + f(a+3h_3)))\\ &= \frac{1}{2}\left(3.28793835105 + 1.5\left(f(1.75) + f(3.25)\right)\right)\\ &= \frac{1}{2}(3.28793835105 + 1.5(-1.4757225 + 1.7149747)\\ &= 1.823408326.\end{aligned}$$

Based on the previous calculation, we can calculate $R_{3,2}$

$$R_{3,2} = 1.823408326 + \frac{1}{3}(1.823408326 - 3.28793835105) = 1.33523165098.$$

$$
\begin{aligned}
R_{3,3} &= 1.33523165098 + \frac{1}{15} \cdot (1.33523165098 - 0.5577670874) \\
&= 1.387062622.
\end{aligned}
$$

d

Let's define the function

$$f(x) = \frac{1}{x \ln x}.$$

Now let's find $R_{1,1}$ on the interval $[e, 2e]$:

$$R_{1,1} = \frac{e}{2}\left(\frac{1}{e \ln e} + \frac{1}{2e \ln(2e)}\right) = 0.6476540273.$$

In the problem we need to find $R_{3,3}$ so let's see what we need to calculate it

$$R_{3,3} = R_{3,2} + \frac{1}{15} \cdot (R_{3,2} - R_{2,2}).$$

We see that we need $R_{3,2}$ and $R_{2,2}$. To calculate

$$R_{2,2} = R_{2,1} + \frac{1}{3}(R_{2,1} - R_{1,1}).$$

we need $R_{2,1}$, so let's calculate it

$$
\begin{aligned}
R_{2,1} &= \frac{1}{2}(R_{1,1} + h_1 f(a + h_2)) \\
&= \frac{1}{2}\left(0.6476540273 + ef\left(\frac{3e}{2}\right)\right) \\
&= \frac{1}{2}(0.6476540273 + e \cdot 0.174499 \\
&= 0.5609964257.
\end{aligned}
$$

Based on the previous calculation, we get

$$R_{2,2} = 0.5609964257 + \frac{1}{3}(0.5609964257 - 0.6476540273) = 0.5321105585.$$

Let's see what we need to calculate $R_{3,2}$

$$R_{3,2} = R_{3,1} + \frac{1}{3}(R_{3,1} - R_{2,1}).$$

So we see that we need $R_{3,1}$, which we will calculate using the Chomposite Trapezoidal rule

$$
\begin{aligned}
R_{3,1} &= \frac{1}{2}(R_{2,1} + h_2(f(a+h_3) + f(a+3h_3))) \\
&= \frac{1}{2}\left(0.5609964257 + \frac{e}{2}\left(f\left(\frac{5e}{4}\right) + f\left(\frac{7e}{4}\right)\right)\right) \\
&= \frac{1}{2}(0.5609964257 + \frac{e}{2}(0.240612439 + 0.13478 \\
&= 0.5356089692.
\end{aligned}
$$

Based on the previous calculation, we can calculate $R_{3,2}$

$$
R_{3,2} = 0.5356089692 + \frac{1}{3}(0.5356089692 - 0.5609964257) = 0.5271464837.
$$

$$
\begin{aligned}
R_{3,3} &= 0.5271464837 + \frac{1}{15} \cdot (0.5271464837 - 0.5321105585) \\
&= 0.5268155.
\end{aligned}
$$

Problem5

Use Euler's method to approximate the solutions for each of the following initial-value problems.

a. $y' = y/t - (y/t)^2, \quad 1 \le t \le 2, \quad y(1) = 1,$ with $h = 0.1$

b. $y' = 1 + y/t + (y/t)^2, \quad 1 \le t \le 3, \quad y(1) = 0,$ with $h = 0.2$

a

We have

$$
w_0 = y(t_0) = y(a) = y(1) = 1
$$

From the step size

$$
\begin{aligned}
h &= \frac{b-a}{N} = \frac{t_N - t_0}{N} = \frac{2-1}{N} \\
N &= \frac{1}{h} = \frac{1}{0.1} = 10
\end{aligned}
$$

With

$$
\begin{aligned}
t_i = t_0 + ih = a + ih = 1 + i(0.1) \\
t_i = 1 + 0.1i.
\end{aligned}
$$

Then

$$
\begin{aligned}
f(t,y) = \left[\frac{y}{t} - \left(\frac{y}{t}\right)^2\right] &\Rightarrow f(t_i, y_i) = \left[\frac{y_i}{t_i} - \left(\frac{y_i}{t_i}\right)^2\right] \\
&\Rightarrow f(t_i, w_i) = \left[\frac{w_i}{t_i} - \left(\frac{w_i}{t_i}\right)^2\right]
\end{aligned}
$$

From equation(1), we get

$$f\left(t_i, w_i\right)=\frac{w_i}{1+0.1 i}-\left(\frac{w_i}{1+0.1 i}\right)^2$$

Using Euler's difference equation

$$w_{i+1}=w_i+h f\left(t_i, w_i\right), \quad i=0,1, \cdots, N-1.$$

Then from equation (2), we have

$$w_{i+1}=w_i+(0.1)\left[\frac{w_i}{1+0.1 i}-\left(\frac{w_i}{1+0.1 i}\right)^2\right],$$

t_i	$=t_0+ih=1+0.1i,$	$i=0,1,\cdots,N=0,1,2,3,4,5,6,7,8,9,10$
i	t_i	
0	$t_0=1$	$w_0=y\left(t_0\right)=y(1)=1$
1	$t_1=1.1$	$w_1\approx y\left(t_1\right)\approx y(1.1)=1.0000$
2	$t_2=1.2$	$w_2\approx y\left(t_2\right)\approx y(1.2)=1.00826$
3	$t_3=1.3$	$w_3\approx y\left(t_3\right)\approx y(1.3)=1.02169$
4	$t_4=1.4$	$w_4\approx y\left(t_4\right)\approx y(1.4)=1.03852$
5	$t_5=1.5$	$w_5\approx y\left(t_5\right)\approx y(1.5)=1.05767$
6	$t_6=1.6$	$w_6\approx y\left(t_6\right)\approx y(1.6)=1.07846$
7	$t_7=1.7$	$w_7\approx y\left(t_7\right)\approx y(1.7)=1.07846$
8	$t_8=1.8$	$w_8\approx y\left(t_8\right)\approx y(1.8)=1.12326$
9	$t_9=1.9$	$w_9\approx y\left(t_9\right)\approx y(1.9)=1.14672$
10	$t_{10}=2$	$w_{10}\approx y\left(t_{10}\right)\approx y(2)=1.17065$

b

We have

$$w_0=y\left(t_0\right)=y(a)=y(1)=0$$

From the step size

$$\begin{gathered}h=\frac{b-a}{N}=\frac{t_N-t_0}{N}=\frac{3-1}{N} \\ N=\frac{2}{h}=\frac{2}{0.2}=10\end{gathered}$$

With

$$\begin{gathered}t_i=t_0+ih=a+ih=1+i(0.2) \\ t_i=1+0.2 i.\end{gathered}$$

Then

$$f(t, y) = \left[1 + \frac{y}{t} + \left(\frac{y}{t}\right)^2\right] \Rightarrow f(t_i, y_i) = \left[1 + \frac{y_i}{t_i} + \left(\frac{y_i}{t_i}\right)^2\right]$$
$$\Rightarrow f(t_i, w_i) = \left[1 + \frac{w_i}{t_i} + \left(\frac{w_i}{t_i}\right)^2\right]$$

From equation (1), we get

$$f(t_i, w_i) = \left[1 + \frac{w_i}{1 + 0.2i} + \left(\frac{w_i}{1 + 0.2i}\right)^2\right].$$

Using Euler's difference equation

$$w_{i+1} = w_i + hf(t_i, w_i), \quad i = 0, 1, \cdots, N-1.$$

Then from equation (2), we have

$$\begin{aligned} w_{i+1} &= w_i + (0.2)\left[1 + \frac{w_i}{1 + 0.2i} + \left(\frac{w_i}{1 + 0.2i}\right)^2\right], \\ i &= 0, 1, \cdots, N-1 \\ &= 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 \end{aligned}$$

Result

$t_i = t_0 + ih = 1 + 0.2i, \quad i = 0, 1, \cdots, N = 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10$

i	t_i	w_i
0	$t_0 = 1$	$w_0 = y(t_0) = y(1) = 0$
1	$t_1 = 1.2$	$w_1 \approx y(t_1) \approx y(1.2) = 0.20000$
2	$t_2 = 1.4$	$w_2 \approx y(t_2) \approx y(1.4) = 0.438889$
3	$t_3 = 1.6$	$w_3 \approx y(t_3) \approx y(1.6) = 0.721243$
4	$t_4 = 1.8$	$w_4 \approx y(t_4) \approx y(1.8) = 1.05204$
5	$t_5 = 2$	$w_5 \approx y(t_5) \approx y(2) = 1.43725$
6	$t_6 = 2.2$	$w_6 \approx y(t_6) \approx y(2.2) = 1.88426$
7	$t_7 = 2.4$	$w_7 \approx y(t_7) \approx y(2.4) = 2.40227$
8	$t_8 = 2.6$	$w_8 \approx y(t_8) \approx y(2.6) = 3.00284$
9	$t_9 = 2.8$	$w_9 \approx y(t_0) \approx y(2.8) = 3.7006$
10	$t_{10} = 3$	$w_{10} \approx y(t_{10}) \approx y(3) = 4.51428$