HM2

Problem 1

Let $f(x) = -x^3 - \cos x$ and $p_0 = -1$. Use Newton's method to find p_2. Could $p_0 = 0$ be used?

solution

$f(x) = -x^3 - cosx$

$f'(x) = -3x^2 + sinx$

so $P_{n+1} = p_n - \frac{f(p_n)}{f'(p_n)}$

$p_0 = -1, p_1 = -0.880333, p_2 = -0.865684$

$p_0 = 0$ could't be uesd, cause f'(0)=0

Problem 2

Assume that we wish to use the Newton-Raphson method to approximate the root $\frac{1}{b}$ of the nonlinear equation

$$f(x) = b - \frac{1}{x} = 0,$$

where we assume $b > 0$.

(i) Show that $|\epsilon_{k+1}| = \epsilon_k^2$ where ϵ_k is the relative error in x_k at the k-th iteration, given by

$$\epsilon_k = \frac{\frac{1}{b} - x_k}{\frac{1}{b}}.$$

(ii) Show that the Newton-Raphson iterations will converge to $\frac{1}{b}$ for any starting value x_0 provided that

$$0 < x_0 < \frac{2}{b}.$$

solution

(i)

$\epsilon_{k+1} = \frac{\frac{1}{b} - x_{k+1}}{\frac{1}{b}} = 1 - bx_{k+1}$

$\frac{f(x)=b-1}{x}$ so $f'(x) = \frac{1}{x^2}$

hence $x_{k+1} = x_k - \frac{f(x_k)}{f(x_{k+1})} = 2x_k - bx_k^2$

so $|\epsilon_{k+1}| = |1 - b(2x_k - bx_k^2)| = (1 - bx_k)^2 = \epsilon_k^2$

(ii)

$x_{k+1} = 2x_k - bx_k^2$

$cause \quad x_{k+1} - \frac{1}{b} = 2x_k - bx_k^2 - \frac{1}{b} = -b(x_k - \frac{1}{b})^2 = -b^{2k+1}(x_0 - \frac{1}{b})^{2k+2} \quad converge$

$so |bx_0 - 1| < 1 \rightarrow 0 < x_0 < \frac{2}{b}$

Problem 3

Use Newton's method with $\mathbf{x}^{(0)} = \mathbf{0}$ to compute $\mathbf{x}^{(2)}$ for each of the following nonlinear systems.

a.
$$3x_1 - \cos(x_2x_3) - \frac{1}{2} = 0,$$
$$4x_1^2 - 625x_2^2 + 2x_2 - 1 = 0,$$
$$e^{-x_1x_2} + 20x_3 + \frac{10\pi - 3}{3} = 0.$$

b.
$$x_1^2 + x_2 - 37 = 0,$$
$$x_1 - x_2^2 - 5 = 0,$$
$$x_1 + x_2 + x_3 - 3 = 0.$$

solution

a.

$x_1 = \frac{1}{3}cos(x_2x_3) + \frac{1}{6}$,
$x_2 = \frac{1}{9}\sqrt{x_1^2 + sinx_3 + 1.06} - 0.1$,
$x_3 = -\frac{1}{20}e^{-x_1x_2} - \frac{10\pi-3}{60}$.
$g_1(x_1, x_2, x_3) = \frac{1}{3}cos(x_2x_3) + \frac{1}{6}$,
$g_2(x_1, x_2, x_3) = \frac{1}{9}\sqrt{x_1^2 + sinx_3 + 1.06} - 0.1$,
$g_3(x_1, x_2, x_3) = -\frac{1}{20}e^{-x_1x_2} - \frac{10\pi-3}{60}$.
$x_1^{(k)} = \frac{1}{3}cos(x_2^{(k-1)}x_3^{(k-1)}) + \frac{1}{6}$,
$x_2^{(k)} = \frac{1}{9}\sqrt{(x_1^{(k)})^2 + sinx_3^{(k-1)} + 1.06} - 0.1$,
$x_3^{(k)} = -\frac{1}{20}e^{-x_1^{(k)}x_2^{(k)}} - \frac{10\pi-3}{60}$.

so $X^{(2)} = \begin{pmatrix} 0.500167 \\ 0.250804 \\ -0.517387 \end{pmatrix}$

b.

$x_1 = \sqrt{-x_2 + 37}$
$x_2 = \sqrt{x_1 - 5}$
$x_3 = -x_1 - x_2 + 3$
$g_1(x_1, x_2, x_3) = \sqrt{-x_2 + 37}$
$g_2(x_1, x_2, x_3) = \sqrt{x_1 - 5}$
$g_3(x_1, x_2, x_3) = -x_1 - x_2 + 3$

so $X^{(2)} = \begin{pmatrix} 4.350877 \\ 18.491228 \\ -19.842105 \end{pmatrix}$

Problem 4

Use the method of Steepest Descent with $TOL = 0.05$ to approximate the solutions of the following nonlinear systems.

a.
$$15x_1 + x_2^2 - 4x_3 = 13,$$
$$x_1^2 + 10x_2 - x_3 = 11,$$
$$x_2^3 - 25x_3 = -22.$$

b.
$$10x_1 - 2x_2^2 + x_2 - 2x_3 - 5 = 0,$$
$$8x_2^2 + 4x_3^2 - 9 = 0,$$
$$8x_2x_3 + 4 = 0.$$

solution

a.

$$f_1 = 15x_1 + x_2^2 - 4x_3 - 13 = 0$$

$$f_2 = x_1^2 + 10x_2 - x_3 - 11 = 0$$

$$f_3 = x_2^3 - 25x_3 + 22 = 0;$$

Jcaobian Matrix

$$J = \begin{pmatrix} 2.0x_1 & 1.0 & 0.0 \\ 1.0 & -2.0x_2 & 0.0 \\ 1.0 & 1.0 & 1.0 \end{pmatrix}$$

Initial Guess

$$x^{(0)} = (0, 0, 0)^t$$

so the function values

$$F^{(0)} = \begin{pmatrix} -37 \\ -5 \\ -3 \end{pmatrix}$$

Jacobian Matrix

$$J^{(0)} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

Solve the system of linear equations

$$y^{(0)} = -[J^{(0)}]^{-1}F^{(0)}$$

$$y^{(0)} = \left[\begin{pmatrix} 15 & 0 & -4 \\ 0 & 10 & -1 \\ 0 & 0 & -25 \end{pmatrix}\right]^{-1} \begin{pmatrix} -13 \\ -11 \\ 22 \end{pmatrix}$$

$$y^{(0)} = \begin{pmatrix} 1.1013 \\ 1.188 \\ 0.88 \end{pmatrix}$$

So, after the first iteration, the values are

$$x^{(1)} = x^{(0)} + y^{(0)}$$

$$x^{(1)} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 1.1013 \\ 1.188 \\ 0.88 \end{pmatrix}$$

Iteration-2
Second Guess

$$x^{(1)} = \begin{pmatrix} 1.1013 \\ 1.188 \\ 0.88 \end{pmatrix}$$

So, the function values

$$F^{(1)} = \begin{pmatrix} 1.4113 \\ 1.2129 \\ 1.6767 \end{pmatrix}$$

Jacobian Matrix

$$J^{(1)} = \begin{pmatrix} 15 & 2.367 & -4 \\ 2.2027 & 10 & -1 \\ 0 & 4.234 & -25 \end{pmatrix}$$

Solve the system of linear equations

$$y^{(1)} = -[J^{(1)}]^{-1} F^{(1)}$$

$$y^{(1)} = \left[\begin{pmatrix} 15 & 2.376 & -4 \\ 2.2027 & 10 & -1 \\ 0 & 4.234 & -25 \end{pmatrix}\right]^{-1} \begin{pmatrix} 1.4113 \\ 1.2129 \\ 1.6767 \end{pmatrix}$$

$$y^{(1)} = \begin{pmatrix} -0.064646 \\ -0.10208 \\ 0.049779 \end{pmatrix}$$

So, after the second iteration, the values are

$$x^{(2)} = x^{(1)} + y^{(1)}$$

$$x^{(2)} = \begin{pmatrix} -0.064646 \\ -0.10208 \\ 0.049779 \end{pmatrix} + \begin{pmatrix} 1.1013 \\ 1.188 \\ 0.88 \end{pmatrix}$$

$$x^{(2)} = \begin{pmatrix} 1.0367 \\ 1.0859 \\ 0.92978 \end{pmatrix}$$

After Two Iterations the solution is

$$x^{(2)} = \begin{pmatrix} 1.0367 \\ 1.0859 \\ 0.92978 \end{pmatrix}$$

b.

$$f_1 = 10x_1 - 2x_2^2 + x_2 - 2x_3 - 5 = 0$$

$$f_2 = 8x_2^2 + 4x_3^2 - 9 = 0$$

$$f_3 = 8x_2 x_3 + 4 = 0$$

Jacobian Matrix

$$J = \begin{pmatrix} 10 & 1.0 - 4.0x_2 & -2.0 \\ 0.0 & 16.0x_2 & 8.0x_3 \\ 0.0 & 8.0x_3 & 8.0x_2 \end{pmatrix}$$

Initial Guess

$$x^{(0)} = (0, 0, 0)^t$$

so the function values

$$F^{(0)} = \begin{pmatrix} -5 \\ -9 \\ 4 \end{pmatrix}$$

Jacobian Matrix

$$J^{(0)} = \begin{pmatrix} 10 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Solve the system of linear equations
We can see that the above Jacobian matrix has all elements in the second and third rows as zero, so a Solution is not possible.

Problem 5

The nonlinear system

$$x_1^2 - 10x_1 + x_2^2 + 8 = 0, x_1x_2^2 + x_1 - 10x_2 + 8$$

can be transformed into the fixed-point problem

$$x_1 = g_1(x_1, x_2) = \frac{x_1^2 + x_2^2 + 8}{10}, x_2 = g_2(x_1, x_2) = \frac{x_1x_2^2 + x_1 + 8}{10}$$

(a)Show that $G = (g_1, g_2)^t$ mapping $D \subset R^2$ into R^2 has a unique fixed point in $D = \{(x_1, x_2)^t | 0 \leq x_1, x_2 \leq 1.5\}$
(b)Let $x^{(0)} = [0, 1]^t$, and perform two steps of the fixed point iteration to find $X^{(2)}$
solution
(a)
Use the Theorem 10.6
$x_1 = g_1(x_1, x_2) = \frac{x_1^2+x_2^2+8}{10}$
$x_2 = g_2(x_1, x_2) = \frac{x_1x_2^2+x_1+8}{10}$
$\frac{\partial g_1}{\partial x_1} = \frac{x_1}{5}$
$\frac{\partial g_1}{\partial x_2} = \frac{x_2}{5}$
$\frac{\partial g_2}{\partial x_1} = \frac{1}{10}(x_2^2 + 1)$
$\frac{\partial g_2}{\partial x_2} = \frac{x_2}{5}$

so in $D = \{(x_1, x_2)^t | 0 \leq x_1, x_2 \leq 1.5\}$
$\frac{\partial g_1}{\partial x_1} = \frac{x_1}{5} \leq \frac{0.6}{2}$
$\frac{\partial g_2}{\partial x_1} = \frac{1}{10}(x_2^2 + 1) \leq \frac{0.65}{2}$
$\frac{\partial g_1}{\partial x_2} = \frac{x_2}{5} \leq \frac{0.6}{2}$
$\frac{\partial g_2}{\partial x_2} = \frac{x_2}{5} \leq \frac{0.6}{2}$

Therefore, in the domin $D = \{(x_1, x_2)^t | 0 \leq x_1, x_2 \leq 1.5\}$

$$|\frac{\partial g_i}{\partial x_j}| \leq \frac{K}{2}; i, j = 1, 2$$

Hence, the mapping $D \subset R^2 \rightarrow R^2$ has a unique fixed point.

(b)

$$x_1^k = \frac{1}{10}\{(x_1^{k-1})^2 + (x_2^{k-1})^2 + 8\}$$

$$x_2^k = \frac{1}{10}\{(x_1^{k-1})^2 + (x_2^{k-1})^2 + 8\}$$

$$let X^{(0)} = [0, 1]^t$$

so

$$k = 0, x_1^k = 1.0000, x_2^k = 0.0000$$

$$k = 1, x_1^k = 0.9000, x_2^k = 0.8000$$

$$k = 2, x_1^k = 0.9450, x_2^k = 0.9476$$