HW4

Problem1

Construct the Lagrange interpolating polynomials for the following functions, and find a bound for the absolute error on the interval $[x_0, x_n]$.

a. $f(x) = e^{2x} \cos 3x, \quad x_0 = 0, x_1 = 0.3, x_2 = 0.6, n = 2$

b. $f(x) = \sin(\ln x), \quad x_0 = 2.0, x_1 = 2.4, x_2 = 2.6, n = 2$

solution

a.

The second degree Lagrange interpolating polynomial is given as

$$P_2(x) = L_0(x)f(x_0) + L_1(x)f(x_1) + L_2(x)f(x_2)$$

when n = 2 as

$$L_0(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}$$

$$L_1(x) = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}$$

$$L_2(x) = \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}$$

The nodes are $x_0 = 0, x_1 = 0.3, x_2 = 0.6$. Substitute into the expressions for L_k above to obtain

$$L_0(x) = \frac{(x-0.3)(x-0.6)}{(0-0.3)(0-0.6)} = \frac{(x-0.3)(x-0.6)}{0.18} = \frac{50}{9}x^2 - 5x + 1$$

$$L_1(x) = \frac{(x-0)(x-0.6)}{(0.3-0)(0.3-0.6)} = \frac{x(x-0.6)}{0.09} = -\frac{100}{9}x^2 + \frac{20}{3}x$$

$$L_2(x) = \frac{(x-0)(x-0.3)}{(0.6-0)(0.6-0.3)} = \frac{x(x-0.3)}{0.18} = \frac{50}{9}x^2 - \frac{5}{3}x$$

We also need to evaluate $f(x_k)$ for $k = 0, 1, 2$ as follows

$$f(x_0) = f(0) = e^{2.0}cos(3 \times 0) = 1$$

$$f(x_1) = f(0.3) = e^{2.0 \times 0.3}cos(3 \times 0.3) = 1.13264721$$

$$f(x_2) = f(0.6) = e^{2.0 \times 0.6}cos(3 \times 0.6) = -0.75433752$$

Now determine $P_2(x)$as follows

$$P_2(x) = L_0(x)f(x_0) + L_1(x)f(x_1) + L_2(x)f(x_2)$$

$$= (\frac{50}{9}x^2 - 5x + 1)\times f(0) + (-\frac{100}{9}x^2 + \frac{20}{3}x)\times f(0.3) + (\frac{50}{9}x^2 - \frac{5}{3}x)\times f(0.6)$$

$$= -11.220177x^2 + 3.808211x + 1$$

According to Theorem 3.3, the absolute is

$$|f(x) - P_n(x)| = |\frac{f^{(n+1)(\xi(x))}}{(n+1)!}\prod_{k=0}^{n}(x - x_k)|$$

where $\xi(x) \in [0, 0.6]$.when n=2, the absolute error is

$$|f(x) - P_2(x)| = |\frac{f^{(3)(\xi(x))}}{(3)!}\prod_{k=0}^{2}(x - x_k)| = |\frac{f^{(3)(\xi(x))}}{6}(x - 0)(x - 0.3)(x - 0.6)|$$

The product $x(x - 0.3)(x - 0.6)$ is a third degree polynomial with extreme points $x_1 = 0.1268$ and $x_2 = 0.4732$which we find as follows.

$$p(x) = x(x - 0.3)(x - 0.6) = x^3 - 0.9x^2 + 0.18x$$

$$p'(x) = 3x^2 - 1.8x + 0.18$$

$$p'(x) = 0 \quad \Longrightarrow x_{1,2} = \frac{1.8\pm\sqrt{1.8^2 - 4\times 3\times 0.18}}{2\times 3}$$

Therefore, the extreme values of the product on $[0, 0.6]$ are $p(0.1268) = 0.01039$ and $(0.4732) = -0.01039$ hence the maximum absolute value of the product on that interval is 0.01039.
The error $|f(x) - P_2(x)|$also depends on the third derivative of f. Let's find that derivate

$$f(x) = e^{2x}\cos 3x$$

$$f'(x) = 2e^{2x}\cos 3x - e^{2x}\cdot\sin 3x\cdot 3 = e^{2x}(2\cos 3x - 3\sin 3x)$$

$$f\prime\prime(x) = 2e^{2x}(2\cos 3x - 3\sin 3x) + e^{2x}(-6\sin 3x - 9\cos 3x) = e^{2x}(-5cos3x - 12sin3x)$$

$$f'''(x) = 2e^{2x}(-5\cos 3x - 12\sin 3x) + e^{2x}(15\sin 3x - 36\cos 3x) = e^{2x}(-46\cos 3x - 9\sin 3x)$$

The plot below reveals that the maximum absolute value of the third derivative on $[0, 0.6]$ is for $f\prime\prime\prime(0.2604) = -65.6522$. Substitute the information we found into the error bound formula as shown below.

$$|f(x) - P_2(x)| = |\frac{f^{(3)(\xi(x))}}{(3)!}\prod_{k=0}^{2}(x - x_k)| = |\frac{f^{(3)(\xi(x))}}{6}(x - 0)(x - 0.3)(x - 0.6)| \leq \frac{65.6522}{6}\times 0.($$

b.

$$P_2(x) = L_0(x)f(x_0) + L_1(x)f(x_1) + L_2(x)f(x_2)$$

when n = 2 as

$$L_0(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}$$

$$L_1(x) = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}$$

$$L_2(x) = \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}$$

The nodes are $x_0 = 2, x_1 = 2.4, x_2 = 2.6$. Substitute into the expressions for L_k above to obtain

$$L_0(x) = \frac{(x-2.4)(x-2.6)}{(2-2.4)(2-2.6)} = \frac{(x-2.4)(x-2.6)}{0.24} = \frac{25}{6}x^2 - \frac{125}{6}x + 26$$

$$L_1(x) = \frac{(x-2)(x-2.6)}{(2.4-2)(2.4-2.6)} = -\frac{(x-2)(x-2.6)}{0.08} = -12.5x^2 + 57.5x - 65$$

$$L_2(x) = \frac{(x-2)(x-2.4)}{(2.6-2)(2.6-2.4)} = \frac{(x-2)(x-2.4)}{0.12} = \frac{25}{3}x^2 - \frac{110}{3}x + 40$$

We also need to evaluate $f(x_k)$ for $k = 0, 1, 2$ as follows

$$f(x_0) = f(2) = \sin(ln2) = 0.63896127$$

$$f(x_1) = f(2.4) = \sin(ln2.4) = 0.76784388$$

$$f(x_2) = f(2.6) = \sin(ln2.6) = 0.81660905$$

Now determine $P_2(x)$as follows

$$P_2(x) = L_0(x)f(x_0) + L_1(x)f(x_1) + L_2(x)f(x_2)$$

$$= (\frac{25}{6}x^2 - \frac{125}{6}x + 26) \cdot f(2) + (-12.5x^2 + 57.5x - 65) \cdot f(2.4) + (\frac{25}{3}x^2 - \frac{110}{3}x + 40) \cdot f(2.6$$

$$= -0.130634x^2 + 0.896998x - 0.632497$$

According to Theorem 3.3, the absolute is

$$|f(x) - P_n(x)| = |\frac{f^{(n+1)(\xi(x))}}{(n+1)!}\prod_{k=0}^{n}(x - x_k)|$$

where $\xi(x) \in [2, 2.6]$.when n=2, the absolute error is

$$|f(x) - P_2(x)| = |\frac{f^{(3)(\xi(x))}}{(3)!}\prod_{k=0}^{2}(x - x_k)| = |\frac{f^{(3)(\xi(x))}}{6}(x-2)(x-2.4)(x-2.6)|$$

The product $(x-2)(x-2.4)(x-2.6)$ is a third degree polynomial with extreme points $x_1 = 2.5097$ and

$x_2 = 2.1569$which we find as follows.

$$p(x) = (x-2)(x-2.4)(x-2.6) = x^3 - 7x^2 + 16.24x - 12.48$$

$$p'(x) = 3x^2 - 14x + 16.24$$

$$p'(x) = 0 \quad \Longrightarrow x_{1,2} = \frac{14 \pm \sqrt{14^2 - 4 \times 3 \times 16.24}}{2 \times 3}$$

Therefore, the extreme values of the product on $[2, 2.6]$ are $p(2.5097) = -0.005$ and $p(2.1569) = 0.0169$ hence the maximum absolute value of the product on that interval is 0.0169.
The error $|f(x) - P_2(x)|$ also depends on the third derivative of f.Let's find that derivative.

$$f(x) = \sin(\ln x)$$

$$f'(x) = \cos(\ln x) \cdot \frac{1}{x}$$

$$f''(x) = \frac{-sin(lnx)\frac{1}{x}x - cos(lnx)}{x^2} = \frac{-cos(lnx) - sin(lnx)}{x^2}$$

$$f'''(x) = \frac{(sin(lnx)\frac{1}{x} - cos(lnx)\frac{1}{x})x^2 - (-cos(lnx) - sin(lnx))2x}{x^4} = \frac{3sin(lnx) + cos(lnx)}{x^3}$$

The plot below reveals that the maximum absolute value of the third derivative on $[2, 2.6]$ is for $f'''(2) = 0.335765$.Substitute the information we found into the error bound formula as shown below.

$$|f(x) - P_2(x)| = |\frac{f^{(3)}(\zeta(x))}{6}(x-2)(x-2.4)(x-2.6) \leq \frac{0.335765}{6} \times 0.0169 = 9.46 \times 10^{-4}$$

Problem2

Let $P_3(x)$ be the interpolating polynomial for the data $(0, 0)$, $(0.5, y)$, $(1, 3)$, and $(2, 2)$. The coefficient of x^3 in $P_3(x)$ is 6. Find y.

Solution

$$P_3(x) = L_0(x)f(x_0) + L_1(x)f(x_1) + L_2(x)f(x_2) + L_3(x)f(x_3)$$

when n = 3 as

$$L_0(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}$$

$$L_1(x) = \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}$$

$$L_2(x) = \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}$$

$$L_3(x) = \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}$$

The nodes are $x_0 = 0, x_1 = 0.5, x_2 = 1, x3 = 2$. Substitute into the expressions for L_k above to obtain

$$L_0(x) = \frac{(x-0.5)(x-1)(x-2)}{(0-0.5)(0-1)(0-x_2)} = \frac{x^3 - \frac{7}{2}x^2 + \frac{7}{2}x - 1}{-1} = -x^3 + \frac{7}{2}x^2 - \frac{7}{2}x + 1$$

$$L_1(x) = \frac{(x-0)(x-1)(x-2)}{(0.5-0)(0.5-1)(0.5-2)} = \frac{8}{3}x^3 - 8x^2 + \frac{16}{3}x$$

$$L_2(x) = \frac{(x-0)(x-0.5)(x-2)}{(1-0)(1-0.5)(1-2)} = -2x^3 + 5x^2 - 2x$$

$$L_3(x) = \frac{(x-0)(x-0.5)(x-1)}{(2-0)(2-0.5)(2-1)} = \frac{1}{3}x^3 - \frac{1}{2}x^2 + \frac{1}{6}x$$

Thus,

$$\begin{aligned} P_3(x) &= L_0(x)f(x_0) + L_1(x)f(x_1) + L_2(x)f(x_2) + L_3(x)f(x_3) \\ &= L_1(x)\cdot y + L_2(x)\cdot 3 + L_3(x)\cdot 2 \\ &= (\frac{8y-16}{3})x^3 + (-8y+14)x^2 + (\frac{16y-17}{3})x \end{aligned}$$

The coefficient of x^3 to be equal to 6,

$$\frac{8y-16}{3} = 6 \implies y = 4.25$$

Then, the polynomial becomes

$$P_3(x) = 6x^3 - 20x^2 + 17x$$

So

$$P_3(0) = 0, P_3(0.5) = 4.25, P_3(1) = 3, P_3(2) = 2$$

Problem3

Neville's method is used to approximate $f(0.4)$, giving the following table.

$x_0 = 0$	$P_0 = 1$			
$x_1 = 0.25$	$P_1 = 2$	$P_{01} = 2.6$		
$x_2 = 0.5$	P_2	$P_{1,2}$	$P_{0,1,2}$	
$x_3 = 0.75$	$P_3 = 8$	$P_{2,3} = 2.4$	$P_{1,2,3} = 2.96$	$P_{0,1,2,3} = 3.016$

Determine $P_2 = f(0.5)$.

solution
According to the Neville's method,

$$P_{0,1,\ldots,k}(x)=\frac{(x-x_j)P_{0,1,\ldots,j-1,j+1,\ldots,k}(x)-(x-x_i)P_{0,1,\ldots,i-1,i+1,\ldots,k}(x)}{(x_i-x_j)}$$

We have nodes:

$$f(x_0)=f(0)=P_0=1$$

$$f(x_1)=f(0.5)=P_1=2$$

$$f(x_2)=f(0.5)=P_2$$

$$f(x_3)=f(0.75)=P_3=8$$

$P_{2,3}$ can be constructed from P_2 and P_3 as:

$$P_{2,3}=\frac{(x-x_2)P_3-(x-x_3)P_2}{x_3-x_2}$$

Hence, substitute the data we have:

$$P_2=4$$

Suppose $x_j=j$, for $j=0,1,2,3$ and it is known that

$$P_{0,1}(x)=2x+1,\quad P_{0,2}(x)=x+1,\quad \text{and}\quad P_{1,2,3}(2.5)=3.$$

Find $P_{0,1,2,3}(2.5)$.

solution
According to the Neville's method,

$$P_{0,1,\ldots,k}(x)=\frac{(x-x_j)P_{0,1,\ldots,j-1,j+1,\ldots,k}(x)-(x-x_i)P_{0,1,\ldots,i-1,i+1,\ldots,k}(x)}{(x_i-x_j)}$$

So we have:

$$P_{0,1,2,3}(x)=\frac{(x-x_0)P_{1,2,3}(x)-(x-x_3)P_{0,1,2}(x)}{x_3-x_0}$$

As we see from the equation, we need to know the value of the polynomial $P_{0,1,2}$
So,

$$P_{0,1,2}(x)=\frac{(x-x_0)P_{1,2}(x)-(x-x_2)P_{0,1}(x)}{x_2-x_0}$$

So,

$$P_{0,1,2,3}(x) = \frac{(x-x_0)P_{1,2,3}(x)}{x_3-x_0} - \frac{x-x_3}{x_3-x_0}\left[\frac{(x-x_0)P_{1,2}(x)-(x-x_2)P_{0,1}(x)}{x_2-x_0}\right]$$

$$= \frac{(x-x_0)P_{1,2,3}(x)}{x_3-x_0} - \frac{x-x_3}{x_3-x_0}\left[\frac{(x-x_0)(x+1)-(x-x_2)(2x+1)}{x_2-x_0}\right]$$

Then,

$$P_{0,1,2,3}(2.5) = 2.9792$$

Problem4

For a function f, the forward-divided differences are given by

$x_0 = 0.0$	$f[x_0]$		
		$f[x_0, x_1]$	
$x_1 = 0.4$	$f[x_1]$		$f[x_0, x_1, x_2] = \frac{50}{7}$
		$f[x_1, x_2] = 10$	
$x_2 = 0.7$	$f[x_2] = 6$		

Determine the missing entries in the table.

The k-th divided difference relative to $x_0, x_1, \ldots, x_k$ is definied as:

$$f\left[x_0, x_1, \ldots, x_k\right] = \frac{f\left[x_1, x_2, \ldots, x_k\right] - f\left[x_0, x_1, \ldots, x_k - 1\right]}{x_k - x_0} ----(1) k = 0, 1, \ldots, n$$

if $k = 0, 1, 2$ we have the following divided differences:

- Zero'th divided differences:

$$f[x_k] = f(x_k) ----(2)$$

- First divied difference:

$$f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0} ----(3)$$

$$f[x_1, x_2] = \frac{f(x_2) - f(x_1)}{x_2 - x_1} ----(4)$$

- Second divided difference:

$$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0} ----(5)$$

From equation (4),

$$\frac{6 - f(x_1)}{0.7 - 0.4} = 10 \Longrightarrow f(x_1) = 3$$

From equation (5),

$$\frac{10 - f[x_0, x_1]}{0.7 - 0.0} = \frac{50}{7} \Longrightarrow f[x_0, x_1] = 5$$

From equation (3),

$$\frac{6 - f(x_0)}{0.4 - 0.0} = 5 \Longrightarrow f(x_0) = 4$$

Problem5

Determine the natural cubic spline S that interpolates the data $f(0) = 0$, $f(1) = 1$, and $f(2) = 2$.
Determine the clamped cubic spline s that interpolates the data $f(0) = 0$, $f(1) = 1$, $f(2) = 2$ and satisfies $s'(0) = s'(2) = 1$.

solution

a

if f is a function of the variable x defined on the closed interval $[a, b]$,and we have a set of nodes $a = x_0 < x_1 < \cdots < x_n = b \; in[a, b]$,then a natural cubic spline interpolant $S(x)$ is a third order polynomial that satisfy the following conditions:

- $S_j(x)$, defined on the subinterval $[x_j, x_{j+1}]$ for each $j = 0, 1, \ldots, n-1$, have the general from:

$$S_j(x) = a_j + b_j(x - x_j) + c_j(x - x_j)^2 + d_j(x - x_j)^3$$

- For each $j = 0, 1, \ldots, n-1$ we have:

$$S_j(x_j) = f(x_j)$$

$$S_j(x_{j+1}) = f(x_{j+1})$$

- For each $j = 0, 1, \ldots, n-2$ we have:

$$S_{j+1}(x_{j+1}) = S_j(x_{j+1})$$

- For each $j = 0, 1, \ldots, n-2$ we have:

$$S'_{j+1}(x_{j+1}) = S'_j(x_{j+1})$$

$$S''_{j+1}(x_{j+1}) = S''_j(x_{j+1})$$

- Natural conditions:

$$c_0 = S''(x_0) = 0$$

$$c_n = S''(x_n) = 0$$

To find $S(x)$,we define $h_j = x_{j+1} - x_j$ and compute the coefficient c_j by solving the matrix equation $Ax = b$, where the matrix A is defined as:

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ h_0 & 2(h_0 + h_1)) & h_1 & 0 & 0 \\ 0 & h_1 & 2(h_1 + h_2)h_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & h_{n-2} & 2(h_{n-2} - h_{n-1}) & h_{n-1} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

The matrix b is given by:

$$b = \begin{bmatrix} 0 \\ \frac{3}{h_1}(a_2 - a_1) - \frac{3}{h_0}(a_1 - a_0) \\ \vdots \\ \frac{3}{h_{n-1}}(a_n - a_{n-1}) - \frac{3}{h_{n-2}}(a_{n-1} - a_{n-2}) \\ 0 \end{bmatrix}$$

and

$$x = \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_n \end{bmatrix}$$

Finally, the coefficients a_j, b_j, d_j are given:

$$a_j = f(x_j), \quad j = 0, 1, \dots, n$$

$$b_j = \frac{a_{j+1} - a_j}{h_j} - \frac{h_j(c_{j+1} + 2c_j)}{3}, \quad j = 0, 1, \dots, n-1$$

$$a_j = \frac{c_{j+1} - c_j}{3h_j}, \quad j = 0, 1, \dots, n-1$$

we have:

$$h_0 = x_1 - x_0 = 1, h_1 = x_2 - x_1 = 1$$

So substitute the data into the equations:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 4 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$b = \begin{bmatrix} 0 \\ \frac{3}{h_1}(a_2 - a_1) - \frac{3}{h_0}(a_1 - a_0) \\ 0 \end{bmatrix}$$

From $Ax = b$:

$$c_0 = 0$$

$$c_0 + 4c_1 + c_2 = \frac{3}{h_1}(a_2 - a_1) - \frac{3}{h_0}(a_1 - a_0)$$

$$c_2 = 0$$

To find c_1, we need to find a_j:

$$a_0 = f(0) = 0, a_1 = f(1) = 1, a_2 = f(2) = 2 \Longrightarrow c_1 = 0$$

Then :

$$b_0 = 1, b_1 = 1, d_0 = 0, d_1 = 0$$

Finally,

$$S_0(x) = x, S_1(x) = x$$

b
The same as **a**
we have:

$$h_0 = x_1 - x_0 = 1, h_1 = x_2 - x_1 = 1$$

So substitute the data into the equations:

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

$$b = \begin{bmatrix} \frac{3}{h_0}(a_1 - a_0) - 3f'(0) \\ \frac{3}{h_1}(a_2 - a_1) - \frac{3}{h_0}(a_1 - a_0) \\ 3f'(2) - \frac{3}{h_1}(a_2 - a_1) \end{bmatrix}$$

From $Ax = b$:

$$2c_0 + c_1 = \frac{3}{h_0}(a_1 - a_0) - 3f'(0)$$

$$c_0 + 4c_1 + c_2 = \frac{3}{h_1}(a_2 - a_1) - \frac{3}{h_0}(a_1 - a_0)$$

$$c_1 + 2c_2 = 3f'(2) - \frac{3}{h_1}(a_2 - a_1)$$

To find c_1, we need to find a_j:

$$a_0 = f(0) = 0, a_1 = f(1) = 1, a_2 = f(2) = 2 \Longrightarrow c_0 = c_1 = c_2 = 0$$

Then :

$$b_0 = 1, b_1 = 1, d_0 = 0, d_1 = 0$$

Finally,

$$S_0(x) = x, S_1(x) = x$$

Problem6

Use the most accurate three-point formula to determine each missing entry in the following tables.

a.

x	$f(x)$	$f'(x)$
1.1	9.025013	
1.2	11.02318	
1.3	13.46374	
1.4	16.44465	

b.

x	$f(x)$	$f'(x)$
8.1	16.94410	
8.3	17.56492	
8.5	18.19056	
8.7	18.82091	

Three-point endpoint formula allows us to compute $f'(x_0)$, where x_0 is an endpoint of the interval $[x_0, x_2]$ as:

$$f'(x_0) = \frac{1}{2h}[-3f(x_0) + 4f(x_0 + h) - f(x_0 + 2h)] + E(f), - - - - (1)$$

where $x_1 = x_0 + h, x_2 = x_0 + 2h$, and:

$$h = \frac{x_2 - x_0}{2} = x_1 - x_0 = x_2 - x_1.$$

In the above expression, $E(f) = O(h^2)$ is the error in the approximation.

If x_0 is the right-endpoint of the interval $[x_0 - 2h, x_0]$, to find $f\prime(x_0)$ we use Eq.(1)(1) with $-h$ instead of h
.
Also recall that, if x_0 is the midpoint of the interval $[x_0 - h, x_0 + h]$, we can approximate $f\prime(x_0)$ by the three-poirt midpoint formula:

$$f'(x_0) = \frac{1}{2h}[f(x_0 + h) - f(x_0 - h)] - E_M(f), E_M(f) = O(h^2) - - - -(2)$$

To use Eq.(1), we note the fact that $x_1 = 1.2$ and $x_2 = 1.3$, so:

$h = x_1 - x_0 = x_2 - x_1 = 0.1.$

Hence, Eq.(1) gives for $f'(x_0) = f'(1.1)$:

$$
\begin{aligned}
f'(1.1) &= \frac{1}{2(0.1)}[-3f(1.1) + 4f(1.2) - f(1.3)] \\
&= \frac{1}{0.2}[-3(9.025013) + 4(11.02318) - 13.46374] \\
&= \frac{1}{0.2}(-27.075039 + 44.09272 - 13.46374) \\
&= \frac{3.553941}{0.2} \\
&= 17.769705.
\end{aligned}
$$

Now consider the computation of $f'(1,2)$. We see that we have two options for the calculation. The first is to consider the interval from $x_0 = 1.2$ to $x_2 = 1.4$, so that $x_0 = 1,2$ is the left-endpoint of this interval, and apply the three-point endpoint formula of Eq.(1), with $h = 0.1$.

The other option is to consider the interval from $x_0 - h = x_0 - 0.1 = 1.1$ to $x_0 + h = 1.3$ so that $x_0 = 1.2$ isthe midpoint of this interval, and then use the three-point midpoint formula given in Eq.(2).

As is discussed in the text, the second method is more accurate, because uses values of f on both sides of x_0. So, in what follows we will use this method.

Hence

$$
\begin{aligned}
f(1.2) &= \frac{1}{2(0.1)}[f(1.3) - f(1.1)] \\
&= \frac{1}{0.2}(13.46374 - 9.025013) \\
&= \frac{4.438727}{0.2} \\
&= 22.193635.
\end{aligned}
$$

Similarly, to compute $f'(1.3)$ we use the three-point midpoint formula because wé have the values of f at $x_0 - h = 1.2$ and $x_0 + h = 1.4$.

$$
\begin{aligned}
f'(1.3) &= \frac{1}{2(0.1)}[f(1.4) - f(1.2)] \\
&= \frac{1}{0.2}(16.44465 - 11.02318) \\
&= \frac{5.42147}{0.2} \\
&= 27.107350.
\end{aligned}
$$

Finally, to compute $f'(1.4)$, we cannot use the three-point midpoint formula because this point is the right endpoint of the interval [1.2,1.4]. So, we must use the three-point endpoint formula, with $x_0 = 1.4, h = -h$, and$h = 0.1$ (so $x_0 - h = 1.3$, and $x_0 - 2h = 1.2$).

$$
\begin{aligned}
f'(1.4) &= \frac{1}{2(-h)}[-3f(x_0) + 4f(x_0 - h) - f(x_0 - 2h)] \\
&= \frac{1}{2(-0.1)}[-3f(1.4) + 4f(1.3) - f(1.2)] \\
&= -\frac{1}{0.2}[-3(16.44465) + 4(13.46374) - 11.02318] \\
&= -\frac{1}{0.2}(-49.33395 + 53.85496 - 11.02318) \\
&= \frac{6.50217}{0.2} \\
&= 32.510850.
\end{aligned}
$$

Hence

x	$f(x)$	$f\prime(x)$
1.1	9.025013	17.769705
1.2	11.02318	22.193635
1.3	13.46374	27.107350
1.4	16.44465	32.510850

(b)
To approximate $f'(x)$ at $x = 8.1$. Use Three-Point Endpoint Formula,

$$f'(x_0) = \frac{1}{2h}\Big[-3f(x_0) + 4f(x_0 + h) - f(x_0 + 2h)\Big]$$

By taking $x_0 = 8.1 \quad \text{and} \quad h = 0.2$, substitute the values

$$f'(8.1) = \frac{1}{2(0.2)}\Big[-3f(8.1) + 4f(8.1 + 0.2) - f(8.1 + 2(0.2))\Big]$$

$$= \frac{1}{0.4}\Big[-3f(8.3) \Big]$$

$$= \frac{1}{0.4}\Big[-3\big(16.94410\big) + 4\big(17.56492\big) - 18.19056\Big]$$

$$= \frac{1}{0.4}[1.23682] = 3.092050$$

To approximate $f'(x)$ at $x = 8.3$ use Three-Point Midpoint Formula

$$f'(x_0) = \frac{1}{2h}\Big[f(x_0 + h) - f(x_0 - h)\Big]$$

By taking $x_0 = 8.3 \quad \text{and} \quad h = 0.2$, substitute the values

$$f'(8.3)=\frac{1}{2(0.2)}\Big[f(8.3+0.2)-f(8.3-0.2)\Big]$$

$$=\frac{1}{0.4}\Big[f(8.5)-f(8.1)\Big]$$

$$=\frac{1}{0.4}[18.19056-16.94410]$$

$$=\frac{1}{0.4}[1.24646]=3.116150$$

To approximate $f'(x)$ at $x=8.5$ use Three-Point Midpoint Formula

$$f'(x_0)=\frac{1}{2h}\Big[f(x_0+h)-f(x_0-h)\Big]$$

By taking $x_0=8.5 \quad \text{and} \quad h=0.2$, substitute the values

$$f'(8.5)=\frac{1}{2(0.2)}\Big[f(8.5+0.2)-f(8.5-0.2)\Big]$$

$$=\frac{1}{0.4}\Big[f(8.7)-f(8.3)\Big]$$

$$=\frac{1}{0.4}[18.82091-17.56492]$$

$$=\frac{1}{0.4}[1.25599]=3.139975$$

To approximate $f'(x)$ at $x=8.7$ use Three-Point Endpoint Formula

$$f'(x_0)=\frac{1}{2h}\Big[-3f(x_0)+4f(x_0+h)-f(x_0+2h)\Big]$$

By taking $x_0=8.7 \quad and \quad h=-0.2$,substitute the values

$$f'(8.7)=\frac{1}{2(-0.2)}\Big[-3f(8.7)+4f(8.7+(-0.2))-f(8.7+2(-0.2))\Big]$$
$$=\frac{1}{-0.4}\Big[-3f(8.7)+4f(8.5)-f(8.7-0.4)\Big]$$
$$=-\frac{1}{0.4}\Big[-3f(8.7)+4f(8.5)-f(8.3)\Big]$$
$$=-\frac{1}{0.4}\Big[-3\Big(18.82091\Big)+4\Big(18.19056\Big)-17.56492\Big]$$
$$=3.163525$$

x	$f(x)$	$f\prime(x)$
8.1	16.94410	3.092050

x	$f(x)$	$f\prime(x)$
8.3	17.56492	3.116150
8.5	18.19056	3.139975
8.7	18.82091	3.163525

Problem7

Suppose that $N(h)$ is an approximation to M for every $h > 0$ and that

$$M = N(h) + K_1h^2 + K_2h^4 + K_3h^6 + \cdots,$$

for some constants $K_1, K_2, K_3, \ldots$. Use the values $N(h)$, $N\left(\frac{h}{3}\right)$, and $N\left(\frac{h}{9}\right)$ to produce an $O(h^6)$ approximation to M.

solution
We have

$$\begin{aligned}
\mathrm{M} &= N(h) + K_1h^2 + K_2h^4 + K_3h^6 + \cdots \\
\mathrm{M} &= N(h/3) + K_1\frac{h^2}{9} + K_2\frac{h^4}{81} + K_3\frac{h^6}{729} + \cdots \\
\mathrm{M} &= N(h/9) + K_1\frac{h^2}{81} + K_2\frac{h^4}{6561} + K_3\frac{h^6}{531441} + \cdots
\end{aligned}$$

Multiply the first equation by A, the second by B, and the third by C. Adding and canceling the K_1 and K_2 terms yields the equations

$$\begin{aligned}
A + \frac{B}{9} + \frac{C}{81} &= 0 \\
A + \frac{B}{81} + \frac{C}{6561} &= 0
\end{aligned}$$

Subtracting gives $\frac{8B}{81} + \frac{80C}{6561} = 0$. Multiply by 6561, and we have $648B + 80C = 0$, or $81B + 10C = 0$ Set $C = -81, B = 10$, and $A = -\frac{1}{9}$. Therefore,

$$\left(-\frac{1}{9} + 10 - 81\right)M = -\frac{N(h)}{9} + 10N(h/3) - 81N(h/9) + O(h^6),$$

or

$$M = \frac{1}{640}(729N(h/9) - 90N(h/3) + N(h)) + O(h^6).$$

Problem8

Consider the following data:

i	x_i	y_i
1	0	6
2	2	8
3	4	14
4	5	20

1. Compute the linear least squares polynomial approximation for this data.
2. Compute the error E of the above approximation.

$$a_0 = \frac{\sum_{i=1}^{4} x_i^2 \sum_{i=1}^{4} y_i - \sum_{i=1}^{4} x_i y_i \sum_{i=1}^{4} x_i}{4(\sum_{i=1}^{4} x_i^2) - (\sum_{i=1}^{4} x_i)^2}$$

$$= \frac{45 * 48 - 172 * 11}{4 * 45 - 11^2} = 4.5423728813559325$$

$$a_1 = \frac{4 \sum_{i=1}^{4} x_i y_i - \sum_{i=1}^{4} x_i \sum_{i=1}^{4} y_i}{4(\sum_{i=1}^{4} x_i^2) - (\sum_{i=1}^{4} x_i)^2}$$

$$= \frac{4 * 172 - 11 * 48}{4 * 45 - 11^2} = 2.711864406779661$$

So

$$y = 2.711864406779661x + 4.5423728813559325$$

$$E = \sum_{i=1}^{m} [y_i - (a_1 x_i + a_0)]^2 = 11.525423728813566$$