HW3

Problem1

The following linear systems $A\mathbf{x} = \mathbf{b}$ have $\mathbf{x}$ as the actual solution and $\tilde{\mathbf{x}}$ as an approximate solution. Compute $\|\mathbf{x} - \tilde{\mathbf{x}}\|_\infty$ and $\|A\tilde{\mathbf{x}} - \mathbf{b}\|_\infty$.

a.
$$x_1 + 2x_2 + 3x_3 = 1,$$
$$2x_1 + 3x_2 + 4x_3 = -1,$$
$$3x_1 + 4x_2 + 6x_3 = 2,$$
$$\mathbf{x} = (0, -7, 5)^t,$$
$$\tilde{\mathbf{x}} = (-0.2, -7.5, 5.4)^t.$$

b.
$$x_1 + 2x_2 + 3x_3 = 1,$$
$$2x_1 + 3x_2 + 4x_3 = -1,$$
$$3x_1 + 4x_2 + 6x_3 = 2,$$
$$\mathbf{x} = (0, -7, 5)^t,$$
$$\tilde{\mathbf{x}} = (-0.33, -7.9, 5.8)^t.$$

solution

a.

Let's first write our system of equations in the form $Ax = b$:

$$\underbrace{\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 6 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}}_{b}$$

Now,

$$A\tilde{x} - b = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 6 \end{bmatrix} \begin{bmatrix} -0.2 \\ -7.5 \\ 5.4 \end{bmatrix} - \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ -0.3 \\ -0.2 \end{bmatrix}$$

So

$$||A\tilde{x} - b||_\infty = max\{|0|, |-0.3|, |-0.2|\} = 0.3$$

We also have:

$$||x - \tilde{x}||_\infty = \left\| \begin{bmatrix} 0 \\ -7 \\ 5 \end{bmatrix} - \begin{bmatrix} -0.2 \\ -7.5 \\ 5.4 \end{bmatrix} \right\|_\infty = 0.5$$

a.

Let's first write our system of equations in the form $Ax = b$:

$$\underbrace{\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 6 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}}_{b}$$

Now,

$$A\tilde{x} - b = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 6 \end{bmatrix} \begin{bmatrix} -0.33 \\ -7.9 \\ 5.8 \end{bmatrix} - \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0.27 \\ -0.16 \\ 0.21 \end{bmatrix}$$

So

$$||A\tilde{x} - b||_\infty = max\{|0.27|, |-0.16|, |0.21|\} = 0.27$$

We also have:

$$||x - \tilde{x}||_\infty = \left\| \begin{bmatrix} 0 \\ -7 \\ 5 \end{bmatrix} - \begin{bmatrix} -0.33 \\ -7.9 \\ 5.8 \end{bmatrix} \right\|_\infty = 0.9$$

Problem2

solution

Show that if A is symmetric, then $||A||_2 = \rho(A)$.

If A is symmetric, so A is a n $\times$ n matrix, and $A^T = A$
We also have that $\|A\|_2 = [\rho(A^TA)]^{1/2} = [\rho(A^2)]^{1/2}$
Suppose λ is the eigenvalue of A
so λ^2 is the eigenvalue of A^2
so,

$$\|A\|_2 = [\rho(A^TA)]^{1/2}$$

Problem3

Implement the algorithm of Gaussian elimination with scaled partial pivoting, and solve the following linear systems.

a. $0.03x_1 + 58.9x_2 = 59.2,$
$5.31x_1 - 6.10x_2 = 47.0.$
Actual solution $[10, 1]$.

b. $3.03x_1 - 12.1x_2 + 14x_3 = -119,$
$-3.03x_1 + 12.1x_2 - 7x_3 = 120,$
$6.11x_1 - 14.2x_2 + 21x_3 = -139.$
Actual solution $[0, 10, \frac{1}{7}]$.

solution

a.

The first pivot element, $a_{11}^{(1)} = 0.03$

$m_{21} = \frac{5.31}{0.03} = 177$

$\Longrightarrow \begin{cases} 0.03x_1 + 58.9x_2 = & 59.2 \\ \quad -10431x_2 = & 10431 \end{cases}$

$\Longrightarrow \begin{cases} x_1 = & 10 \\ x_2 = & 1 \end{cases}$

b.

We have a coefficient matrix:

$$\begin{pmatrix} 3.03 & 12.1 & 14 & -119 \\ -3.03 & 12.1 & -7 & 120 \\ 6.11 & -14.2 & 21 & -139 \end{pmatrix}$$

Performing $E_2 + E_1 -> E_2$

$$\begin{pmatrix} 3.03 & 12.1 & 14 & -119 \\ 0 & 0 & 7 & 1 \\ 6.11 & -14.2 & 21 & -139 \end{pmatrix}$$

Performing $E_1 - 2 \times E_2 -> E_1, E_3 - 3 \times E_2 -> E_3$

$$\begin{pmatrix} 3.03 & 12.1 & 14 & -121 \\ 0 & 0 & 7 & 1 \\ 6.11 & -14.2 & 21 & -142 \end{pmatrix}$$

The first pivot element, $a_{11}^{(1)} = 3.03$

$m_{31} = \frac{6.11}{3.03} = 2$

Performing $E_3 - m_{31}E_1$

$\Longrightarrow \begin{cases} 3.03x_1 - 12.1x_2 + 14x_3 = & -121 \\ \quad -28.2x_2 - 7x_3 = & 282 \end{cases}$

$$x_1 = 0, x_2 = 10, x_3 = \frac{1}{7}$$

Problem4

Implement the Jacobi iterative method and list the first three iteration results when solving the following linear systems, using $X^{(0)} = 0$

a. $\begin{aligned} 4x_1 + x_2 - x_3 &= 5, \\ -x_1 + 3x_2 + x_3 &= -4, \\ 2x_1 + 2x_2 + 5x_3 &= 1. \end{aligned}$

b. $\begin{aligned} -2x_1 + x_2 + \tfrac{1}{2}x_3 &= 4, \\ x_1 - 2x_2 - \tfrac{1}{2}x_3 &= -4, \\ x_2 + 2x_3 &= 0. \end{aligned}$

solution

a.

First, let's express x_1 in the first equation, x_2 in the second equation, x_3 in the third equation:

$x_1 = -\frac{1}{4}x_2 + \frac{1}{4}x_3 + \frac{5}{4}$

$x_2 = \frac{1}{3}x_1 - \frac{1}{3}x_3 - \frac{4}{3}$

$x_3 = -\frac{2}{5}x_1 - \frac{2}{5}x_2 + \frac{1}{5}$

We have that $x^{(0)}=0$ so set $x_1^{(0)}=x_2^{(0)}=x_3^{(0)}=0$ and the iterations follow the formula
$x_1^{(k)}=-\frac{1}{4}x_2^{(k-1)}+\frac{1}{4}x_3^{(k-1)}+\frac{5}{4}$
$x_2^{(k)}=\frac{1}{3}x_1^{(k-1)}-\frac{1}{3}x_3^{(k-1)}-\frac{4}{3}$
$x_3^{(k)}=-\frac{2}{5}x_1^{(k-1)}-\frac{2}{5}x_2^{(k-1)}+\frac{1}{5}$
For the first iteration set $k=1:$
$x_1^{(1)}=-\frac{1}{4}x_2^{(0)}+\frac{1}{4}x_3^{(0)}+\frac{5}{4}=\frac{5}{4}$
$x_2^{(1)}=\frac{1}{3}x_1^{(0)}-\frac{1}{3}x_3^{(0)}-\frac{4}{3}=-\frac{4}{3}$
$x_3^{(1)}=-\frac{2}{5}x_1^{(0)}-\frac{2}{5}x_2^{(0)}+\frac{1}{5}=\frac{1}{5}$
For the second iteration set $k=2:$
$x_1^{(2)}=-\frac{1}{4}x_2^{(1)}+\frac{1}{4}x_3^{(1)}+\frac{5}{4}=\frac{49}{30}$
$x_2^{(2)}=\frac{1}{3}x_1^{(1)}-\frac{1}{3}x_3^{(1)}-\frac{4}{3}=-\frac{59}{60}$
$x_3^{(2)}=-\frac{2}{5}x_1^{(1)}-\frac{2}{5}x_2^{(1)}+\frac{1}{5}=\frac{7}{30}$
For the third iteration set $k=3:$
$x_1^{(3)}=-\frac{1}{4}x_2^{(2)}+\frac{1}{4}x_3^{(2)}+\frac{5}{4}=\frac{373}{240}$
$x_2^{(3)}=\frac{1}{3}x_1^{(2)}-\frac{1}{3}x_3^{(2)}-\frac{4}{3}=-\frac{13}{15}$
$x_3^{(3)}=-\frac{2}{5}x_1^{(2)}-\frac{2}{5}x_2^{(2)}+\frac{1}{5}=-\frac{3}{50}$

b.

First, let's express x_1 in the first equation, x_2 in the second equation, x_3 in the third equation:
$x_1=\frac{1}{2}x_2+\frac{1}{4}x_3-2$
$x_2=\frac{1}{2}x_1-\frac{1}{4}x_3+2$
$x_3=-\frac{1}{2}x_2$
We have that $x^{(0)}=0$ so set $x_1^{(0)}=x_2^{(0)}=x_3^{(0)}=0$ and the iterations follow the formula
$x_1^{(k)}=-\frac{1}{2}x_2^{(k-1)}+\frac{1}{4}x_3^{(k-1)}-2$
$x_2^{(k)}=\frac{1}{2}x_1^{(k-1)}-\frac{1}{4}x_3^{(k-1)}+2$
$x_3^{(k)}=-\frac{1}{2}x_2^{(k-1)}$
For the first iteration set $k=1:$
$x_1^{(1)}=-\frac{1}{2}x_2^{(0)}+\frac{1}{4}x_3^{(0)}-2=-2$
$x_2^{(1)}=\frac{1}{2}x_1^{(0)}-\frac{1}{4}x_3^{(0)}+2=2$
$x_3^{(1)}=-\frac{1}{2}x_2^{(0)}=0$
For the second iteration set $k=2:$
$x_1^{(2)}=-\frac{1}{2}x_2^{(1)}+\frac{1}{4}x_3^{(1)}-2=-1$
$x_2^{(2)}=\frac{1}{2}x_1^{(1)}-\frac{1}{4}x_3^{(1)}+2=1$
$x_3^{(2)}=-\frac{1}{2}x_2^{(1)}=-1$
For the third iteration set $k=3:$
$x_1^{(3)}=-\frac{1}{2}x_2^{(2)}+\frac{1}{4}x_3^{(2)}-2=-\frac{9}{4}$
$x_2^{(3)}=\frac{1}{2}x_1^{(2)}-\frac{1}{4}x_3^{(2)}+2=\frac{7}{4}$
$x_3^{(3)}=-\frac{1}{2}x_2^{(2)}=-\frac{1}{2}$

Problem5

Use the Jacobi method and Gauss-Seidel method to solove the following linear systems, with TOL = 0.001 in the L_∞ norm.

a. $$\begin{aligned} 3x_1 - x_2 + x_3 &= 1, \\ 3x_1 + 6x_2 + 2x_3 &= 0, \\ 3x_1 + 3x_2 + 7x_3 &= 4. \end{aligned}$$

b. $$\begin{aligned} 10x_1 - x_2 &= 9, \\ -x_1 + 10x_2 - 2x_3 &= 7, \\ - 2x_2 + 10x_3 &= 6. \end{aligned}$$

solution

a

k	x_1^k	x_2^k	x_3^k
1	0.333333333	-0.166666667	0.500000000
2	0.111111111	-0.222222222	0.619047619
3	0.052910053	-0.232804233	0.648526077
4	0.039556563	-0.235953641	0.655598747
5	0.036149204	-0.236607518	0.657339277
6	0.035351068	-0.236788627	0.657758954

b.

k	x_1^k	x_2^k	x_3^k
1	0.900000000	0.790000000	0.758000000
2	0.979000000	0.949500000	0.789900000
3	0.994950000	0.957475000	0.791495000
4	0.995747500	0.957873750	0.791574750

Reference Code

MATLAB

```
% Gauss-Seidel
clear;
% 输入值
A = [3, -1, 1; 3,6, 2; 3, 3, 7];
b = [1; 0; 4];
tol = 1e-3;
N = 100;
x = [0; 0; 0];
x_backup = [0; 0; 0];
y = [0; 0; 0];
%
A_ = A;
for i = 1 : length(A)
    A_(i,i) = 0;
end
disp('Gauss_Seidel Methods')
disp('---------------------------------------------------')
disp(' k       x_1^k          x_2^k          x_3^k        ')
disp('---------------------------------------------------')
formatSpec = '%2d    %.9f    %.9f    %.9f    \n';
for i = 0 : N

    for j = 1 : length(A)
        y(j,1) = (b(j) - A_(j,:) * x) / A(j,j);
        x_backup(j) = x(j);   % 备份“老值”
        x(j) = y(j);     % “新值” 替换 “老值”

    end
    fprintf(formatSpec,[i+1,y(1),y(2),y(3)]) % Printing output

    if (max(abs(x_backup - y)) < tol)
        fprintf('迭代次数: %d\n', i);
        fprintf('方程组的根: %10.8f\n', y);

        break;
    end

end
if i == N
    fprintf('迭代方法失败\n');
end
```

Problem6

Prove: If A is a matrix and $\rho_1, \rho_2, \ldots, \rho_k$ are distinct eigenvalues of A with associated eigenvectors $x_1, x_2, \ldots, x_k$, then $\{x_1, x_2, \ldots, x_k\}$ linearly independent set.

solution

Assume that these eigenvectors are linearly dependent, exist n constants that are not all zero(c_i):

$$c_1x_1 + c_2x_2 + .. + c_nx_n = 0 - - - - (1)$$

Using the matrix A left-multiplication, according to $Ax_i = \rho_i x_i$

$$c_1\rho_1x_1 + c_2\rho_2x_2 + \ldots + c_n\rho_nx_n = 0 - - - - (2)$$

Using (2) subtract $\rho_n \times (1)$:

$$c_1(\rho_1 - \rho_n)x_1 + c_2(\rho_2 - \rho_n)x_2 + \ldots + c_{n-1}(\rho_{n-1} - \rho_n)x_{n-1} - - - - (3)$$

Now make substitution $d_i - > c_i(\rho_i - \rho_n)$

$$d_1x_1 + d_2x_2 + \ldots + d_{n-1}x_{n-1} = 0 - - - - (4)$$

Perform the same treatment to (4):

$$d_1(\rho_1 - \rho_{n-1})x_1 + d_2(\rho_2 - \rho_{n-1})x_2 + \ldots + d_{n-2}(\rho_{n-2} - \rho_{n-1})x_{n-2} = 0 - - - - (5)$$

Perform the same thing $n - 2$ times:

$$m_1(\rho_1 - \rho_2)x_1 + m_2(\rho_2 - \rho_3)x_2 = 0$$

Make substitution, $n_1 = m_1(\rho_1 - \rho_3), n_2 = m_2(\rho_2 - \rho_3)$
Perform the same thing last time:

$$n_1(\rho_1 - \rho_2)x_1 = 0$$

So $n_1 = 0, m_1 = 0$
Itreate to the last

$$c_i = 0 \ for \ i = 1, 2, \ldots, n$$

So if A is a matrix and $\rho_1, \rho_2, \ldots, \rho_k$ are distinct eigenvalues of A with associated eigenvectors $x_1, x_2, \ldots, x_k$, then $\{x_1, x_2, \ldots, x_k\}$ linearly independent set.

Problem7

Prove that a strictly diagonally dominant matrix is invertible.

solution

Suppose that A isn't invertible, then we have that $det(A) = 0$
So $AX = 0$ have non-zero solution, let $X = (x_1, x_2, \ldots, x_n)^T, |x_k| = max\{|x_i|\}$
Now we have that $\sum_{j=1}^{n} a_{kj}x_j = 0$
Hence $|a_{kk}||x_k| = |\sum_{j \neq k} a_{kj}x_j| - - - - (1)$

We have that A a strictly diagonally dominant matrix
$|a_{kk}||x_k| \geq |x_k| \sum_{j \neq k} |a_{kj}| > \sum_{j \neq k} |a_{kj}||x_j| \geq |\sum_{j \neq k} a_{kj} x_j| ----(2)$
Find the contradiction between (1) and (2) . Therefore, A is reversible.