Bayesian Data Analysis

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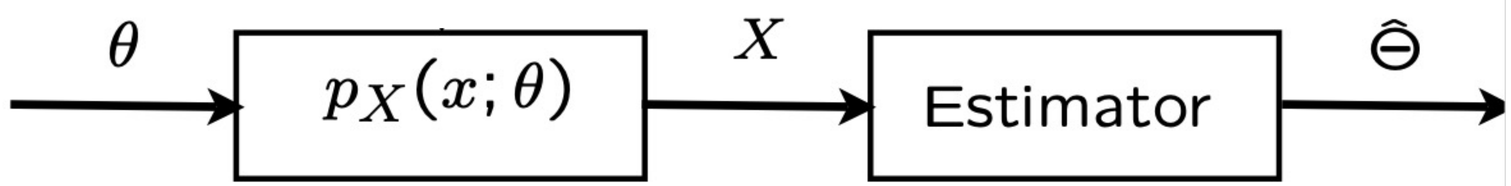
Recall: Frequentist Data Analysis

- Frequentist/Classical Data Analysis



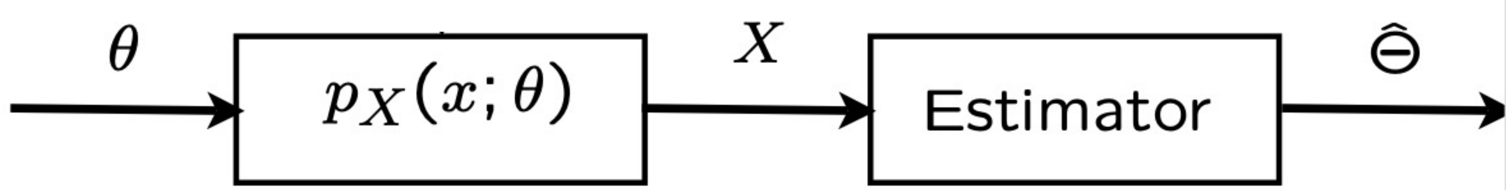
θ: unknown parameter (not a r.v.)

- E.g., $\theta =$ mass of electron

Example: we observe 10 coin flips from a biased coin, what is the bias of the coin θ?

Types of Data Analysis

- Frequentist/Classical Data Analysis



θ: unknown parameter (not a r.v.)

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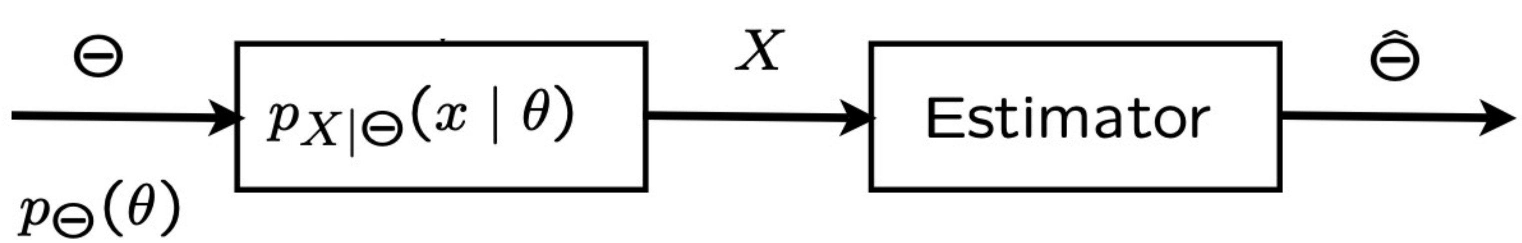
- Bayesian Data Analysis



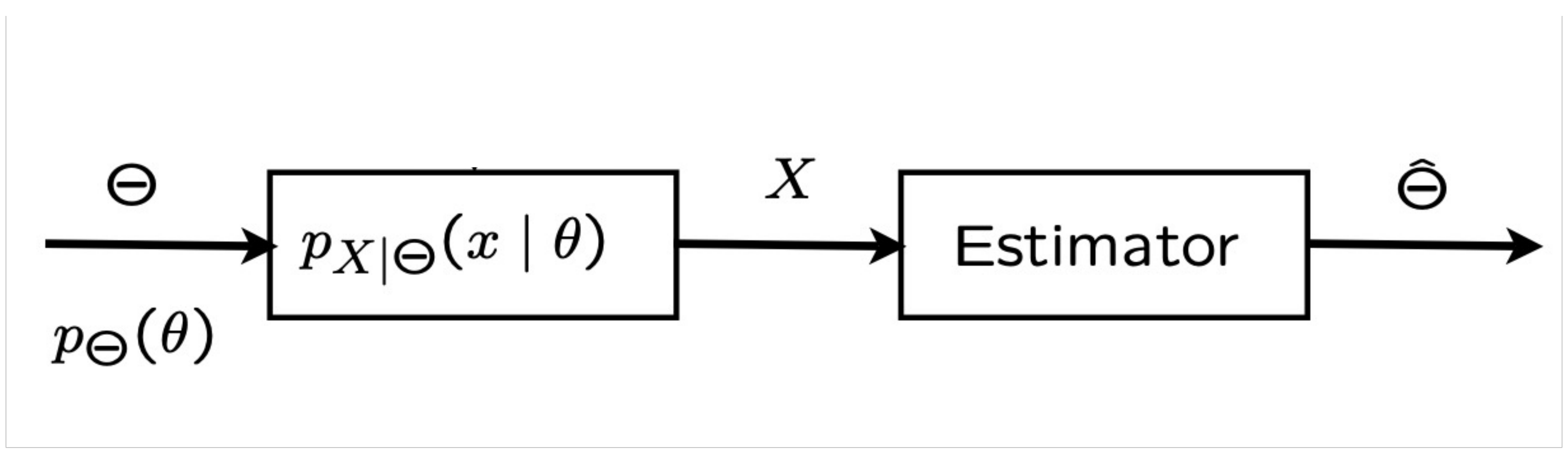
Parameter itself is random!

Bayesian vs Classical

- Classical: Say we have a model with mass of electron as parameter. We might not know the value, but it is nonetheless a constant.
- Bayesian: If we do not know its value completely, use a prior distribution reflecting what we do know.
- Classical: Prior Distribution seems arbitrary.
- Bayesian: Every statistical method makes some choice* ; might as well use a prior to codify these choices.
- Classical: Bayesian methods too difficult to compute (practical considerations)

Bayesian Inference

- We start with a prior distribution p_Θ or f_Θ for the unknown random variable Θ.
- We have a model $p_{X|\Theta}$ or $f_{X|\Theta}$ of the observation vector X.
- After observing the value x of X, we form the posterior distribution of Θ, using the appropriate version of Bayes' rule.



Recall: Your first consulting job

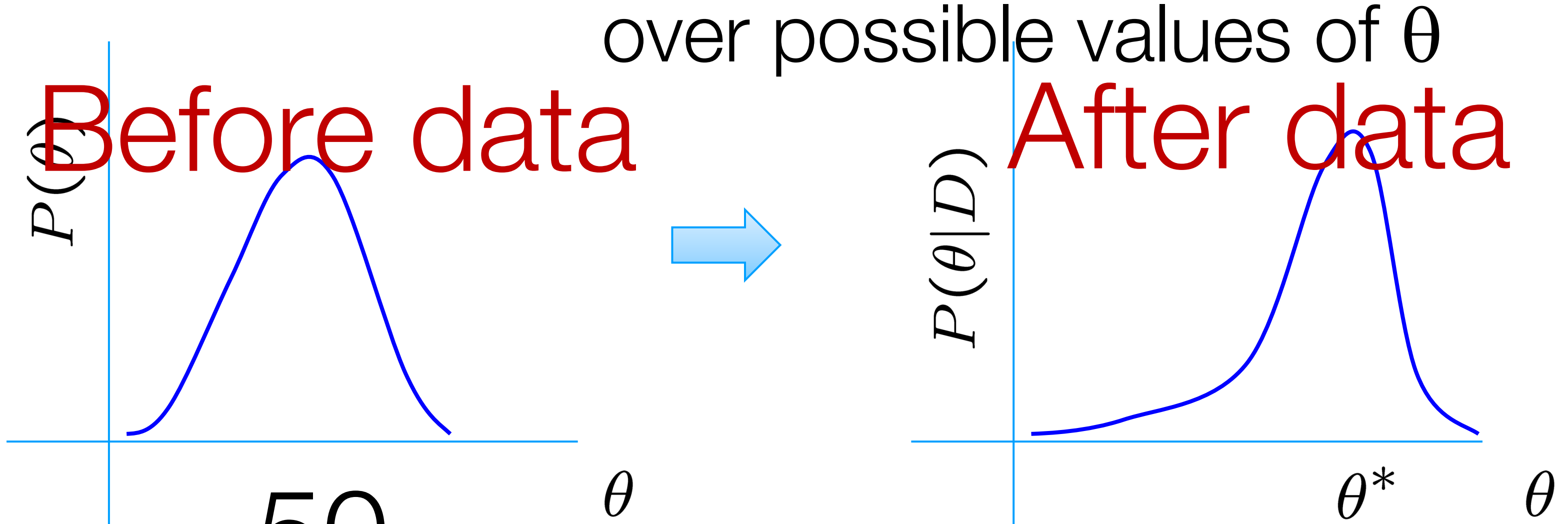
- A billionaire from the suburbs of Seattle asks you a question:
  - He says: I have a coin, if I flip it, what's the probability it will fall with the head up?
  - You say: Please flip it a few times:

  - You say: The probability is: **3/5** because… frequency of heads in all flips
  - **He says: But can I put money on this estimate?**
  - You say: ummm…. Maybe not.
    - Not enough flips (less than sample complexity)

What about prior knowledge?

- Billionaire says: Wait, I know that the coin is “close” to 50-50. What can you do for me now?
  - You say: I can learn it the Bayesian way…

- Rather than estimating a single θ, we obtain a distribution over possible values of θ



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50

Bayesian Learning

- Use Bayes rule:

likelihood prior

$$P(\theta \mid \mathcal{D}) = \frac{P(\mathcal{D} \mid \theta)P(\theta)}{P(\mathcal{D})}$$

Para
met
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Data

Bayes, Thomas (1763) An essay towards solving a problem in the doctrine of chances. *Philosophical Transactions of the Royal Society of London*, **53:370-418**

Bayesian Learning

- Use Bayes rule:

$$P(\theta \mid \mathcal{D}) = \frac{P(\mathcal{D} \mid \theta) P(\theta)}{P(\mathcal{D})}$$

- Or equivalently:

$$P(\theta \mid \mathcal{D}) \propto P(\mathcal{D} \mid \theta) P(\theta)$$

posterior likelihood prior

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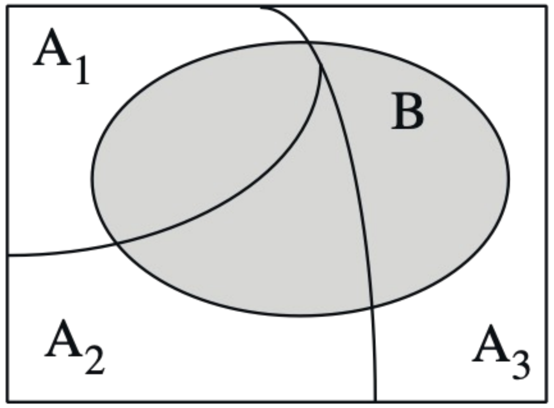
Bayes Rule

- Partition of sample space into A_1, A_2, A_3 :: "Prior Beliefs" $P(A_1)$, $P(A_2)$, $P(A_3)$

- We know $\mathbf{P}(B \mid A_i)$ for each i (for some event B)
- Wish to compute $\mathbf{P}(A_i \mid B)$
  - revise "beliefs", given that B occurred



$$\mathbf{P}(A_i \mid B) = \frac{\mathbf{P}(A_i \cap B)}{\mathbf{P}(B)}$$

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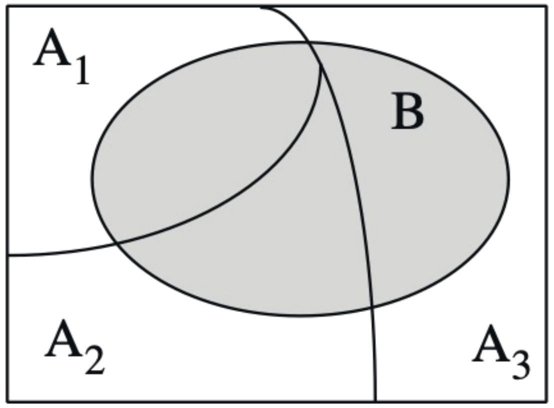

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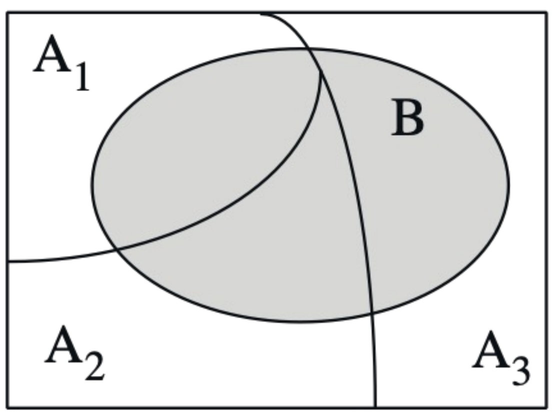

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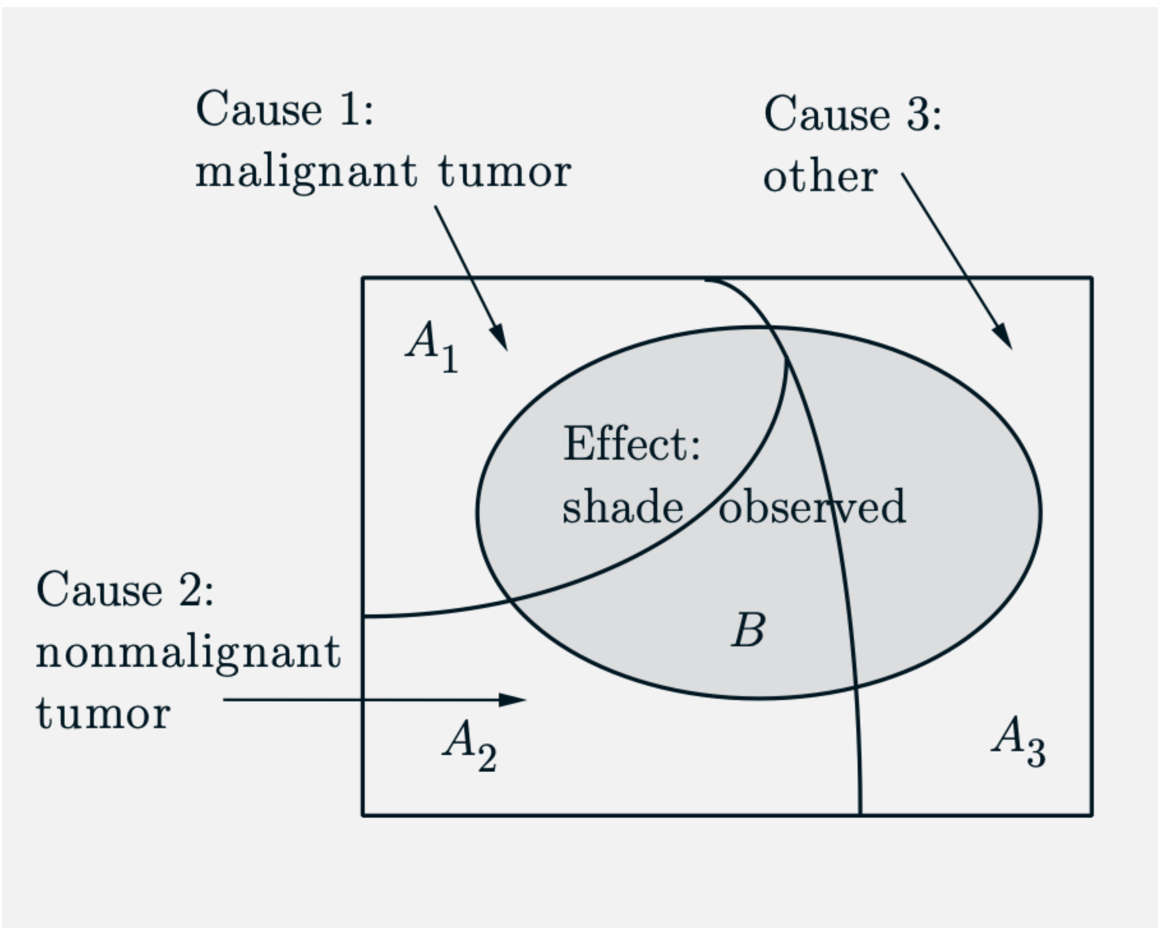

$$\begin{aligned}
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\end{aligned}$$

Given: (a) prior probabilities of causes,
(b) probability of effect given any of the causes

Obtain: probability of any of the causes given effect.

Example I

- A test for a certain rare disease is assumed to be correct 95% of the time: if a person has the disease, the test results are positive with probability 0.95, and if the person does not have the disease, the test results are negative with probability 0.95.

  A random person drawn from a certain population has probability 0.001 of having the disease. Given that the person just tested positive, what is the probability of having the disease?

  - A is the "cause" event that the person has the disease,
    B is the "effect" event that the test results are positive
  - Given :: $P(A)$, $P(A^c)$, and $P(B|A)$, $P(B|A^c)$
    Desired probability :: $P(A \mid B)$, "probability of cause given effect"

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  - A is the "cause" event that the person has the disease,
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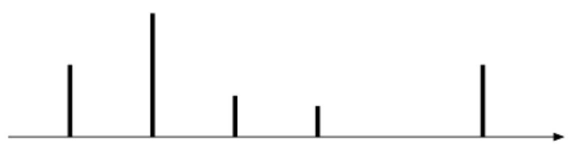
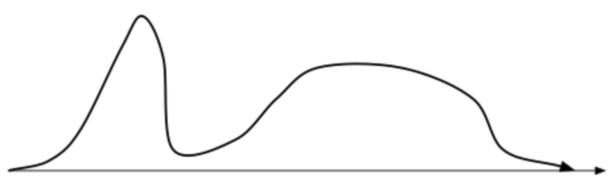
$$
\begin{aligned}
\mathbf{P}(A \mid B) &= \frac{\mathbf{P}(A)\mathbf{P}(B \mid A)}{\mathbf{P}(A)\mathbf{P}(B \mid A) + \mathbf{P}(A^c)\mathbf{P}(B \mid A^c)} \\
&= \frac{0.001 \cdot 0.95}{0.001 \cdot 0.95 + 0.999 \cdot 0.05} \\
&= 0.0187.
\end{aligned}
$$

Output of Bayesian Inference

- Posterior distribution:
  - pmf $p_{\Theta|X}(\cdot \mid x)$ or pdf $f_{\Theta|X}(\cdot \mid x)$





Example Ia

Romeo and Juliet start dating, but Juliet will be late on any date by a random amount X, uniformly distributed over the interval $[0, \theta]$. The parameter θ is unknown and is modeled as the value of a random variable Θ, uniformly distributed between zero and one hour. Assuming that Juliet was late by an amount x on their first date, how should Romeo use this information to update the distribution of Θ?

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Prior PDF

$$f_\Theta(\theta) = \begin{cases} 1, & \text{if } 0 \le \theta \le 1, \\ 0, & \text{otherwise,} \end{cases}$$

Conditional PDF

$$f_{X|\Theta}(x \mid \theta) = \begin{cases} 1/\theta, & \text{if } 0 \le x \le \theta, \\ 0, & \text{otherwise.} \end{cases}$$

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Bayes' Rule

$$f_{\Theta|X}(\theta \,|\, x) = \frac{f_\Theta(\theta)\, f_{X|\Theta}(x \,|\, \theta)}{\displaystyle\int_0^1 f_\Theta(\theta')\, f_{X|\Theta}(x \,|\, \theta')\, d\theta'} = \frac{1/\theta}{\displaystyle\int_x^1 \frac{1}{\theta'}\, d\theta'} = \frac{1}{\theta \cdot |\log x|}, \quad \text{if } x \le \theta \le 1,$$

Example Ib

Consider now a variation involving the first n dates. Assume that Juliet is late by random amounts $X_1, \dots, X_n$, which given $\Theta = \theta$, are uniformly distributed in the interval $[0, \theta]$, and conditionally independent.

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Let $X = (X_1, \ldots, X_n)$ and $x = (x_1, \ldots, x_n)$.

$$f_{X|\Theta}(x \,|\, \theta) = \begin{cases} 1/\theta^n, & \text{if } \overline{x} \le \theta \le 1, \\ 0, & \text{otherwise,} \end{cases}$$

Conditional PDF

$$\overline{x} = \max\{x_1, \ldots, x_n\}.$$

Bayes' Rule

$$f_{\Theta|X}(\theta \,|\, x) = \begin{cases} \dfrac{c(\overline{x})}{\theta^n}, & \text{if } \overline{x} \le \theta \le 1, \\ 0, & \text{otherwise,} \end{cases} \qquad c(\overline{x}) = \frac{1}{\displaystyle\int_{\overline{x}}^{1} \frac{1}{(\theta')^n}\, d\theta'}.$$

Example II

Example : Spam Filtering. An email message may be "spam" or "legitimate." We introduce a parameter Θ, taking values 1 and 2, corresponding to spam and legitimate, respectively, with given probabilities $p_\Theta(1)$ and $p_\Theta(2)$. Let $\{w_1, \ldots, w_n\}$ be a collection of special words (or combinations of words) whose appearance suggests a spam message. For each i, let X_i be the Bernoulli random variable that models the appearance of w_i in the message ($X_i = 1$ if w_i appears and $X_i = 0$ if it does not). We assume that the conditional probabilities $p_{X_i|\Theta}(x_i \,|\, 1)$ and $p_{X_i|\Theta}(x_i \,|\, 2)$, $x_i = 0, 1$, are known. For simplicity we also assume that conditioned on Θ, the random variables $X_1, \ldots, X_n$ are independent.

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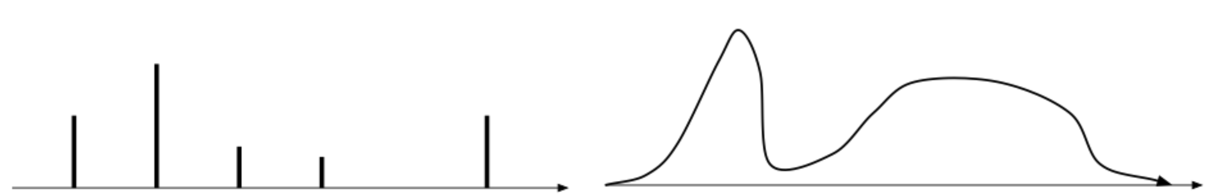
Bayes' Rule

$$\mathbf{P}(\Theta = m \mid X_1 = x_1, \ldots, X_n = x_n) = \frac{p_\Theta(m) \prod_{i=1}^{n} p_{X_i|\Theta}(x_i \mid m)}{\sum_{j=1}^{2} p_\Theta(j) \prod_{i=1}^{n} p_{X_i|\Theta}(x_i \mid j)}, \qquad m = 1, 2.$$

Recall: Bayesian Inference

- We start with a prior distribution p_Θ or f_Θ for the unknown random variable Θ.
- We have a model $p_{X|\Theta}$ or $f_{X|\Theta}$ of the observation vector X.
- After observing the value x of X, we form the posterior distribution of Θ, using the appropriate version of Bayes' rule.

- Posterior distribution:
  - pmf $p_{\Theta|X}(\cdot \mid x)$ or pdf $f_{\Theta|X}(\cdot \mid x)$



Output of Bayesian Inference

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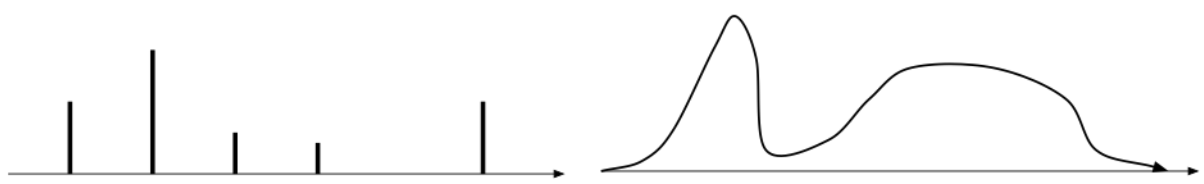

- If interested in a single answer?

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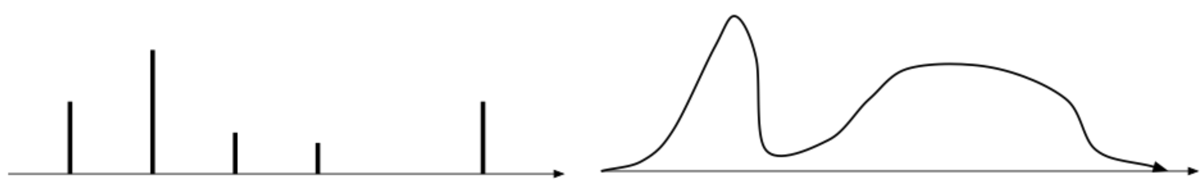

- If interested in a single answer:
  - Maximum a posteriori probability (MAP):
    - $p_{\Theta|X}(\theta^* \mid x) = \max_\theta p_{\Theta|X}(\theta \mid x)$
      minimizes probability of error;
      often used in hypothesis testing
    - $f_{\Theta|X}(\theta^* \mid x) = \max_\theta f_{\Theta|X}(\theta \mid x)$
  - Conditional expectation:

$$\mathbf{E}[\Theta \mid X = y] = \int \theta f_{\Theta|X}(\theta \mid x)\, d\theta$$

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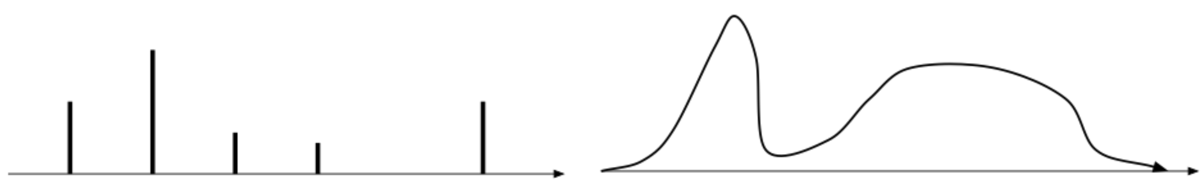

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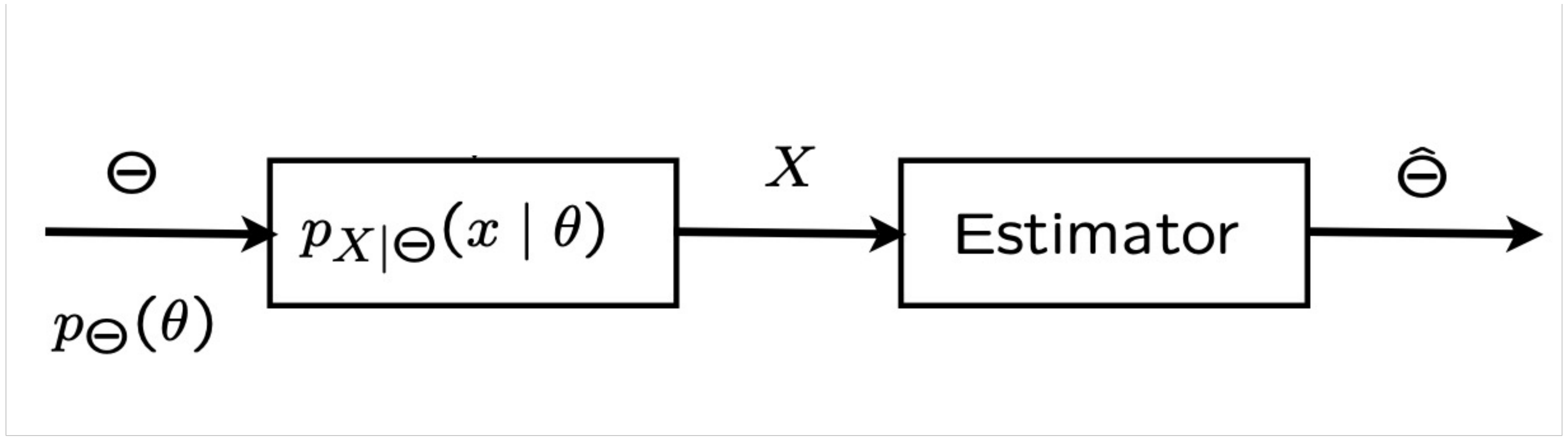
- Single answers can be misleading!

Point Estimation

$$\text{Parameter}: \ \Theta$$
$$\text{Observation}: \ X$$

Given observed value x of X, we want a single numerical value that represents our best guess of Θ

This single numerical value $\widehat{\Theta}$ is called a **point estimate.**



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Point Estimator: $\widehat{\Theta}(X) = g(X)$

Note: a random variable

Point Estimators

- Note that $\widehat{\Theta}(X) = 1$ is an estimator as well!
  - Though not a particularly good one.
- In today's class:

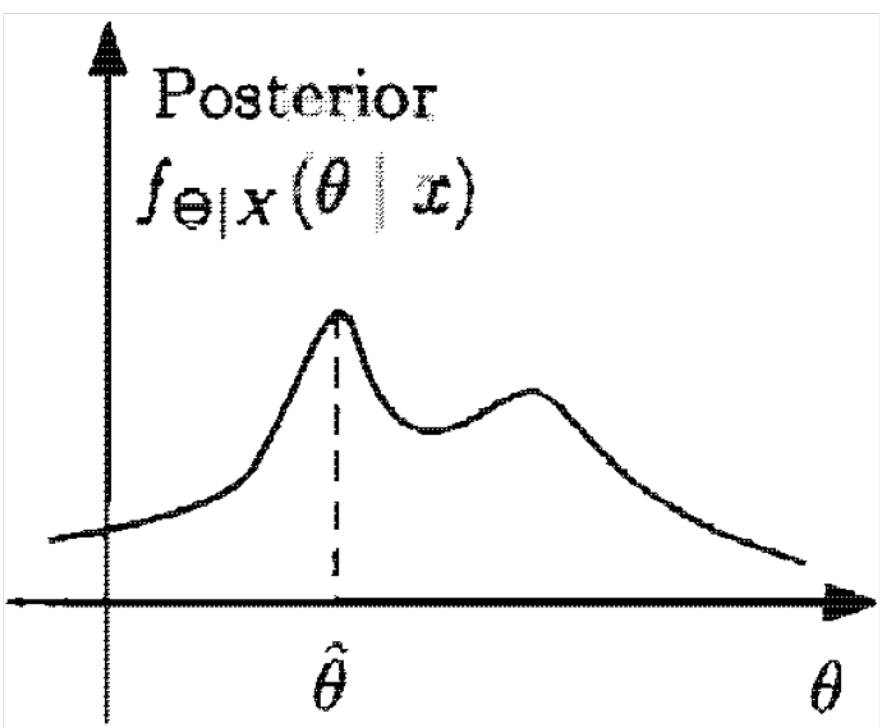
$$\text{MAP Estimator}: \ \widehat{\Theta}(X) = \arg\max_{\Theta} p_{\Theta}(\Theta|X)$$

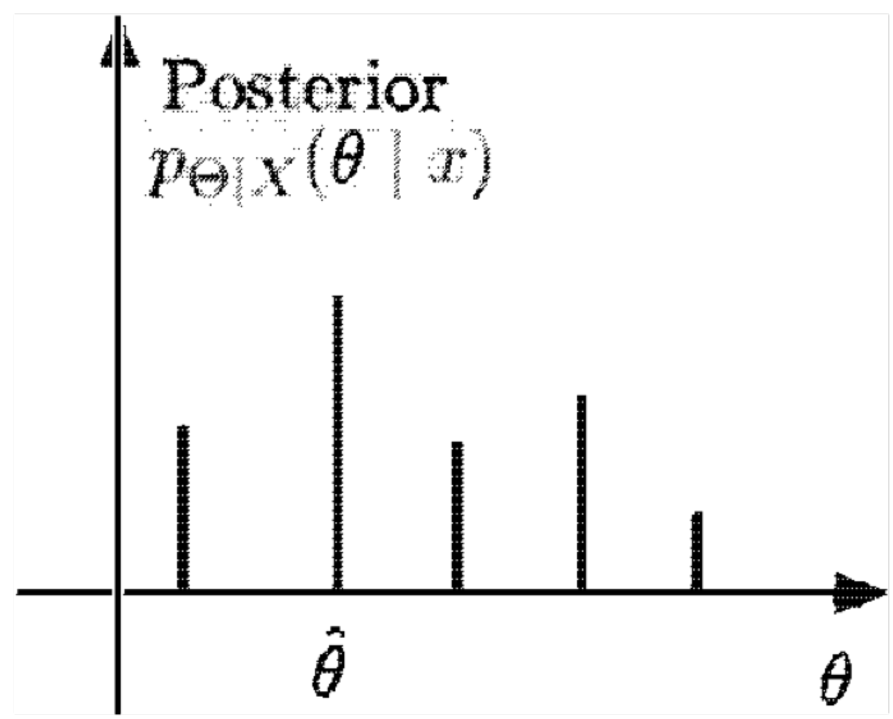
$$\text{Conditional Expectation}: \ \widehat{\Theta}(X) = \mathbb{E}(\Theta|X)$$

MAP Estimator

$$\hat{\theta} = \arg\max_{\theta} p_{\Theta|X}(\theta \mid x), \qquad (\Theta \text{ discrete}),$$

$$\hat{\theta} = \arg\max_{\theta} f_{\Theta|X}(\theta \mid x), \qquad (\Theta \text{ continuous}).$$





Optimality Properties

- Theorem: When Θ is discrete, the MAP estimator maximizes the probability of correct decision, given the observation x.

$$\mathbb{P}(\Theta = g(X)|X) \leq \mathbb{P}(\Theta = g_{\mathrm{map}}(X)|X)$$

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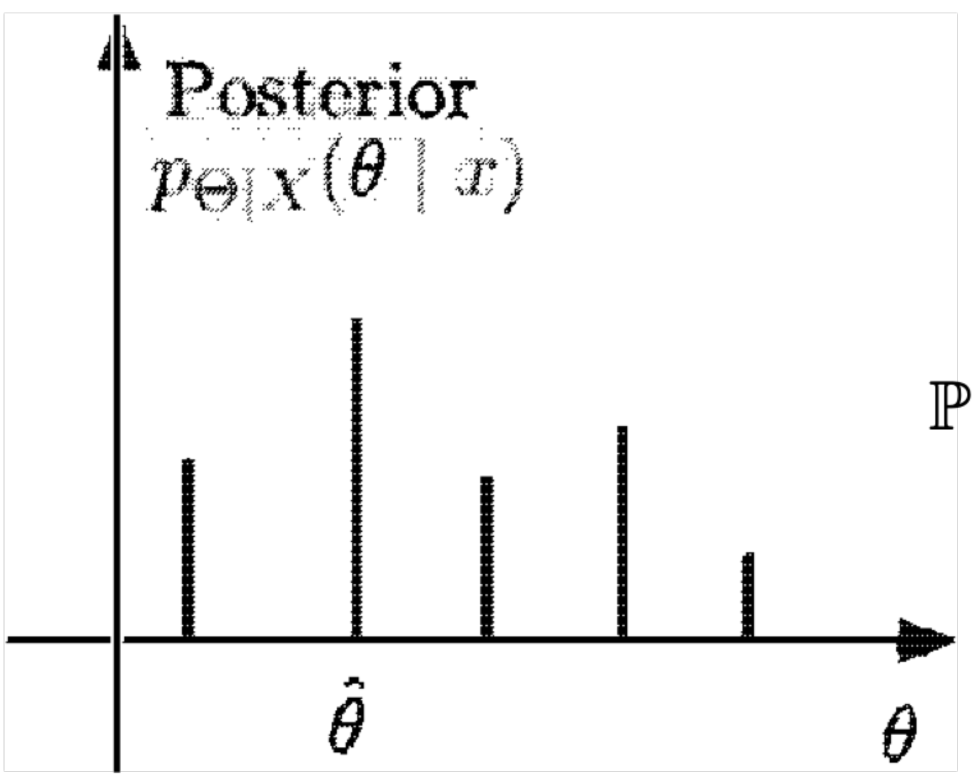

$$\mathbb{P}(\Theta = g(X)|X = x) \leq \mathbb{P}(\Theta = g_{\text{map}}(X)|X = x)$$

MAP Rule

- Given the observation value x, the MAP rule selects a value $\hat{\theta}$ that maximizes over θ the posterior distribution $p_{\Theta|X}(\theta \,|\, x)$ (if Θ is discrete) or $f_{\Theta|X}(\theta \,|\, x)$ (if Θ is continuous).
- Equivalently, it selects $\hat{\theta}$ that maximizes over θ:

$$
\begin{array}{ll}
p_\Theta(\theta) p_{X|\Theta}(x \,|\, \theta) & \text{(if } \Theta \text{ and } X \text{ are discrete)}, \\
p_\Theta(\theta) f_{X|\Theta}(x \,|\, \theta) & \text{(if } \Theta \text{ is discrete and } X \text{ is continuous)}, \\
f_\Theta(\theta) p_{X|\Theta}(x \,|\, \theta) & \text{(if } \Theta \text{ is continuous and } X \text{ is discrete)}, \\
f_\Theta(\theta) f_{X|\Theta}(x \,|\, \theta) & \text{(if } \Theta \text{ and } X \text{ are continuous)}.
\end{array}
$$

... since the denominator depends only on x, and is the same for all \theta

Example I

Example : Consider example prev. where Juliet is late on the first date by a random amount X. The distribution of X is uniform over the interval $[0, \Theta]$, and Θ is an unknown random variable with a uniform prior PDF f_Θ over the interval $[0, 1]$. In that example, we saw that for $x \in [0, 1]$, the posterior PDF is

$$f_{\Theta|X}(\theta \,|\, x) = \begin{cases} \dfrac{1}{\theta \cdot |\log x|}, & \text{if } x \le \theta \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

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$$f_{\Theta|X}(\theta \,|\, x) = \begin{cases} \dfrac{1}{\theta \cdot |\log x|}, & \text{if } x \le \theta \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

For a given x, $f_{\Theta|X}(\theta \,|\, x)$ is decreasing in θ, over the range $[x, 1]$ of possible values of Θ. Thus, the MAP estimate is equal to x. Note that this is an "optimistic" estimate. If Juliet is late by a small amount on the first date ($x \approx 0$), the estimate of future lateness is also small.

Example II

Example : Here the parameter Θ takes values 1 and 2, corresponding to spam and legitimate messages, respectively, with given probabilities $p_\Theta(1)$ and $p_\Theta(2)$, and X_i is the Bernoulli random variable that models the appearance of w_i in the message ($X_i = 1$ if w_i appears and $X_i = 0$ if it does not). We have calculated the posterior probabilities of spam and legitimate messages as

$$\mathbf{P}(\Theta = m \mid X_1 = x_1, \ldots, X_n = x_n) = \frac{p_\Theta(m) \prod_{i=1}^{n} p_{X_i|\Theta}(x_i \mid m)}{\sum_{j=1}^{2} p_\Theta(j) \prod_{i=1}^{n} p_{X_i|\Theta}(x_i \mid j)}, \qquad m = 1, 2.$$

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Suppose we want to classify a message as spam or legitimate based on the corresponding vector $(x_1, \ldots, x_n)$. Then, the MAP rule decides that the message is spam if

Example II

Example : Here the parameter Θ takes values 1 and 2, corresponding to spam and legitimate messages, respectively, with given probabilities $p_\Theta(1)$ and $p_\Theta(2)$, and X_i is the Bernoulli random variable that models the appearance of w_i in the message ($X_i = 1$ if w_i appears and $X_i = 0$ if it does not). We have calculated the posterior probabilities of spam and legitimate messages as

$$\mathbf{P}(\Theta = m \mid X_1 = x_1, \ldots, X_n = x_n) = \frac{p_\Theta(m) \prod_{i=1}^{n} p_{X_i|\Theta}(x_i \mid m)}{\sum_{j=1}^{2} p_\Theta(j) \prod_{i=1}^{n} p_{X_i|\Theta}(x_i \mid j)}, \qquad m = 1, 2.$$

Suppose we want to classify a message as spam or legitimate based on the corresponding vector $(x_1, \ldots, x_n)$. Then, the MAP rule decides that the message is spam if

$$\mathbf{P}(\Theta = 1 \mid X_1 = x_1, \ldots, X_n = x_n) > \mathbf{P}(\Theta = 2 \mid X_1 = x_1, \ldots, X_n = x_n),$$

or equivalently, if

$$p_\Theta(1) \prod_{i=1}^{n} p_{X_i|\Theta}(x_i \mid 1) > p_\Theta(2) \prod_{i=1}^{n} p_{X_i|\Theta}(x_i \mid 2).$$

MAP Estimate with Uniform priors

- Suppose we are doing Bayesian inference, and want to compute the MAP estimate.
- Suppose the prior is $p_\Theta(\theta)$, and the conditional PMF of data, $p_{X|\Theta}(x|\theta)$.
- Then the MAP Estimate:

$$\hat{\theta}_{\text{map}} = \arg\max_{\theta} \left\{ p_\Theta(\theta)\, p_{X|\Theta}(x|\theta) \right\}$$

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- This is exactly the ML estimate, if we were doing classical statistics,

$$\hat{\theta}_{\text{ML}} = \arg\max_\theta \left\{ p_X(x;\theta) \right\}$$

Compare: MLE and MAP

- Model, with unknown parameter(s): $X \sim p_X(x;\theta)$
- Pick θ that "makes data most likely"

$$\widehat{\theta}_{\mathsf{ML}} = \arg\max_{\theta} p_X(x;\theta)$$

- Compare to Bayesian MAP estimation:

$$\widehat{\theta}_{\mathsf{MAP}} = \max_{\theta} \frac{p_{X|\Theta}(x|\theta)p_{\Theta}(\theta)}{p_X(x)}$$

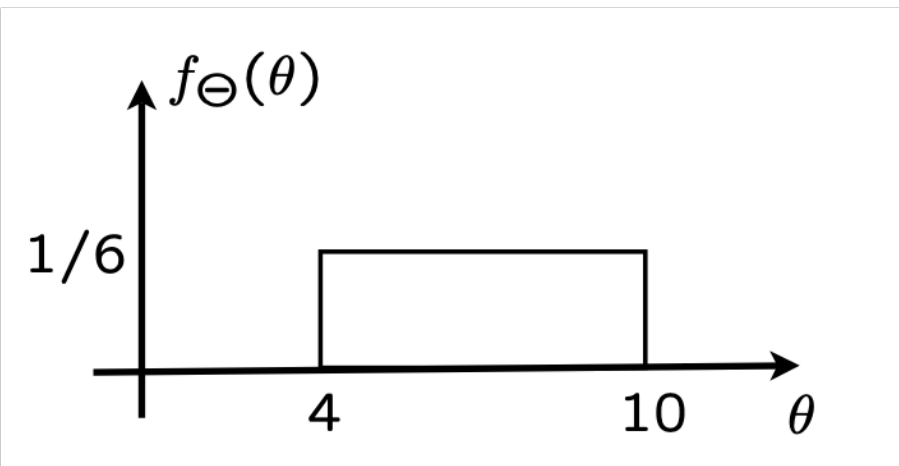
MLE vs MAP

You are no good when sample is small

You give a different answer for different priors

Least Mean Squares Estimation

- Estimation in the absence of information



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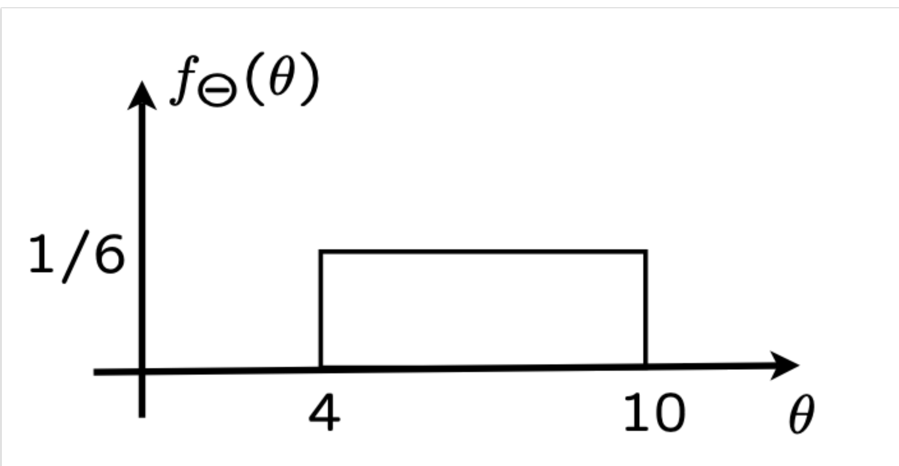

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$$\text{minimize } \mathbf{E}\left[(\Theta - c)^2\right]$$

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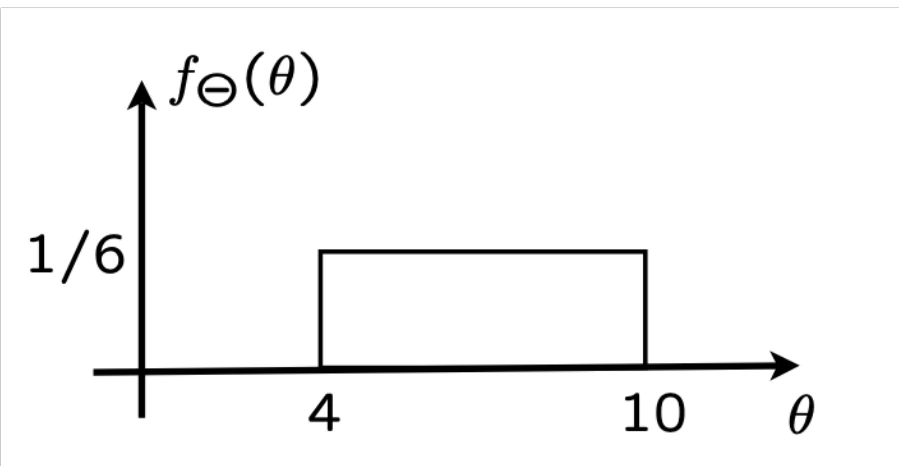

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- Optimal estimate: $c = \mathbf{E}[\Theta]$

- Optimal mean squared error:

$$\mathbf{E}\left[(\Theta - \mathbf{E}[\Theta])^2\right] = \mathrm{Var}(\Theta)$$

LMS Estimation

- Two r.v.'s Θ, X
- we observe that $X = x$
  - new universe: condition on $X = x$
- $\mathbf{E}\left[(\Theta - c)^2 \mid X = x\right]$ is minimized by

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$\boxed{\mathbf{E}[\Theta \mid X] \text{ minimizes } \mathbf{E}\left[(\Theta - g(X))^2\right] \text{ over all estimators } g(\cdot)}$

Example I

Example : Consider example prev. where Juliet is late on the first date by a random amount X. The distribution of X is uniform over the interval $[0, \Theta]$, and Θ is an unknown random variable with a uniform prior PDF f_Θ over the interval $[0, 1]$. In that example, we saw that for $x \in [0, 1]$, the posterior PDF is

$$f_{\Theta|X}(\theta \,|\, x) = \begin{cases} \dfrac{1}{\theta \cdot |\log x|}, & \text{if } x \leq \theta \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

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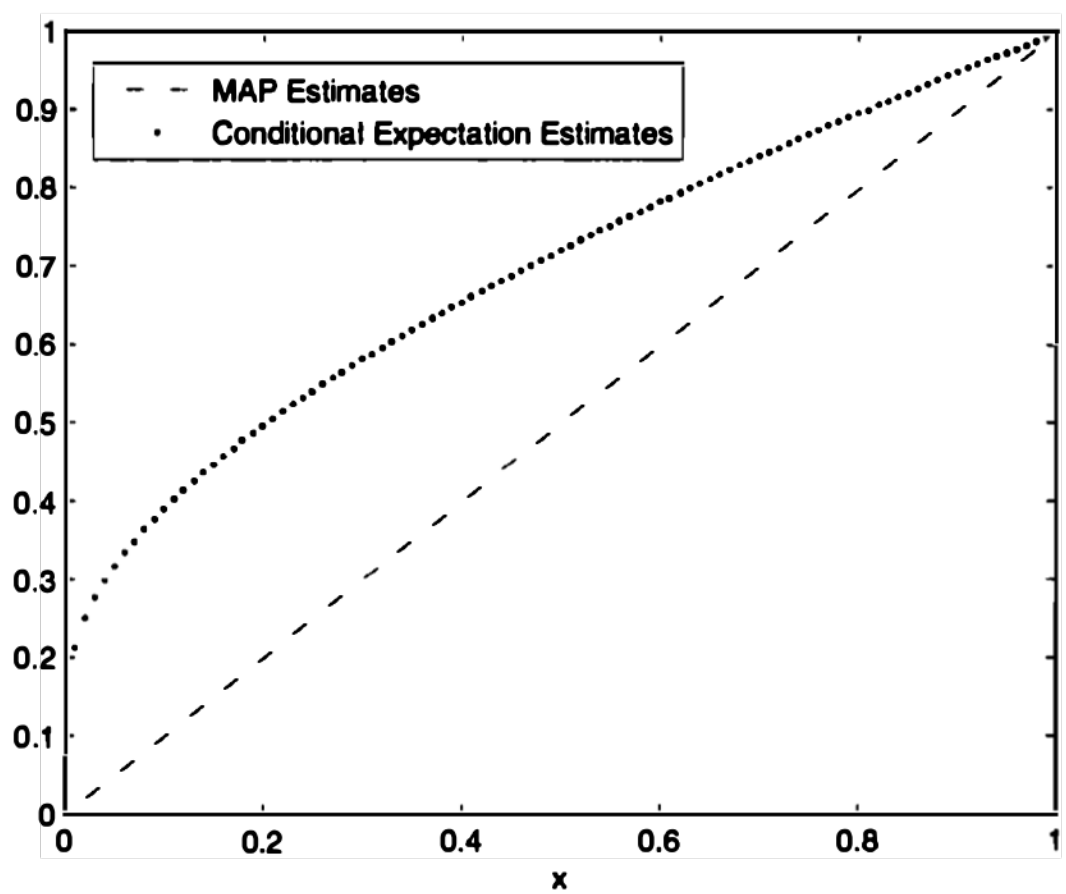

$$\widehat{\Theta}_{\text{MAP}}(x) = x$$

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