Hypothesis Testing

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Motivation

- Any data analysis algorithm applied to a set of data will produce some result(s)
  - There have been claims that the results reported in more than 50% of published papers are false (Ioannidis, 2005)
- Results may be a result of random variation
  - Any particular data set is a finite sample from a larger population
  - Often significant variation among instances in a data set
  - Unusual events or coincidences do happen, especially when looking at lots of events
  - For this and other reasons, results may not replicate, i.e., generalize to other samples of data
- Data scientists need to help ensure that results of data analysis are not **false discoveries**, i.e., not meaningful or reproducible

Significance & Hypothesis Testing

- Testing approaches are used to help avoid many of these problems
- Ultimate verification lies in the real world

Testing

- Make inferences (decisions) about that validity of a result
- For this, we need two things:
  - A statement that we want to disprove
    - Called the **null hypothesis (H_0)**
    - The null hypothesis is typically a statement that the result is merely due to random variation
    - It is the opposite of what we would like to show
  - A random variable, R, called a **test statistic**
    - The distribution of R under H_0 is called the **null distribution**
    - The value of R is obtained from the result and is typically numeric

Examples of Null Hypotheses

- A coin or a die is a fair coin or die.
- The difference between the means of two samples is 0
- The purchase of a particular item in a store is unrelated to the purchase of a second item, e.g., the purchase of bread and milk are unconnected

Significance Testing

- Significance testing was devised by the famous statistician Ronald Fisher

- For many years, significance testing has been a key approach for justifying the validity of scientific results

- Introduced the concept of **p-value**, which is widely used and misused

How Significance Testing Works

- Analyze the data to obtain a result
  - For example, data could be from flipping a coin 10 times to test its fairness
- The result is expressed as a value of the test statistic, R
  - For example, let R be the number of heads in 10 flips
- Compute the probability of seeing the current value of R or something more extreme
  - This probability is known as the **p-value** of the test statistic

How Significance Testing Works …

- If the p-value is sufficiently small, we say that the result is statistically significant
  - We say we reject the null hypothesis, H_0
  - A threshold on the p-value is called the **significance level,** $\boldsymbol{\alpha}$
    - Often the significance level is 0.01 or 0.05
- If the p-value is **not** sufficiently small, we say that we fail to reject the null hypothesis
  - Sometimes we say that we accept the null hypothesis, but a high p-value does not necessarily imply the null hypothesis is true

p-value $= P(R|H_0) \neq P(H_0|R) = \frac{P(R|H_0)\,P(H_0)}{P(R)}$

Example: Testing a coin for fairness

- H_0: $P(X=1) = P(X=0) = 0.5$
- Define the test statistic R to be the number of heads in 10 flips
- Set the significance level α to be 0.05
- The number of heads R has a binomial distribution
- For which values of R would you reject H_0?

k	$P(S=k)$
0	0.001
1	0.01
2	0.044
3	0.117
4	0.205
5	0.246
6	0.205
7	0.117
8	0.044
9	0.01
10	0.001

Neyman-Pearson Hypothesis Testing

- Devised by statisticians Neyman and Pearson in response to perceived shortcomings in significance testing
  - Explicitly specifies an **alternative hypothesis, H_1**
  - Significance testing cannot quantify how an observed results supports H_1
  - Define an **alternative distribution** which is the distribution of the test statistic if H_1 is true
  - We define a **critical region** for the test statistic R
    - If the value of R falls in the critical region, we reject H_0
    - We may or may not accept H_1 if H_0 is rejected
  - The **significance level, α,** is the probability of the critical region under H_0

Hypothesis Testing …

- **Type I Error (α)**: Error of incorrectly rejecting the null hypothesis for a result.
  - It is equal to the probability of the critical region under H_0, i.e., is the same as the significance level, $\boldsymbol{\alpha}$.
  - Formally, $\alpha = P(R \in \text{Critical Region} \mid H_0)$

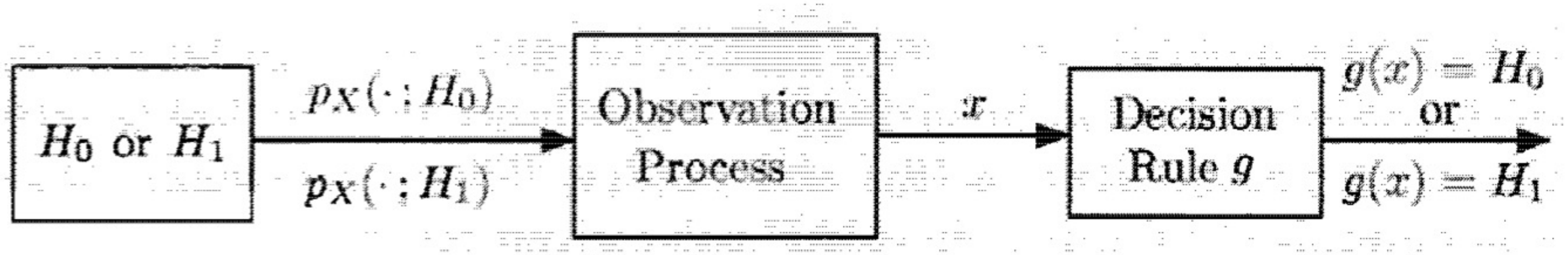
- **Type II Error (β)**: Error of falsely calling a result as not significant when the alternative hypothesis is true.
  - It is equal to the probability of observing test statistic values outside the critical region under H_1
  - Formally, $\beta = P(R \notin \text{Critical Region} \mid H_1)$.

Hypothesis Testing …

- Power: which is the probability of the critical region under H_1, i.e., $1-\beta$.
  - Power indicates how effective a test will be at correctly rejecting the null hypothesis.
  - Low power means that many results that actually show the desired pattern or phenomenon will not be considered significant and thus will be missed.
  - Thus, if the power of a test is low, then it may not be appropriate to ignore results that fall outside the critical region.

Binary Hypothesis Testing

- Binary θ; new terminology:
  - **null hypothesis** H_0:
    $X \sim p_X(x; H_0) \quad$ [or $f_X(x; H_0)$]
  - **alternative hypothesis** H_1:
    $X \sim p_X(x; H_1) \quad$ [or $f_X(x; H_1)$]



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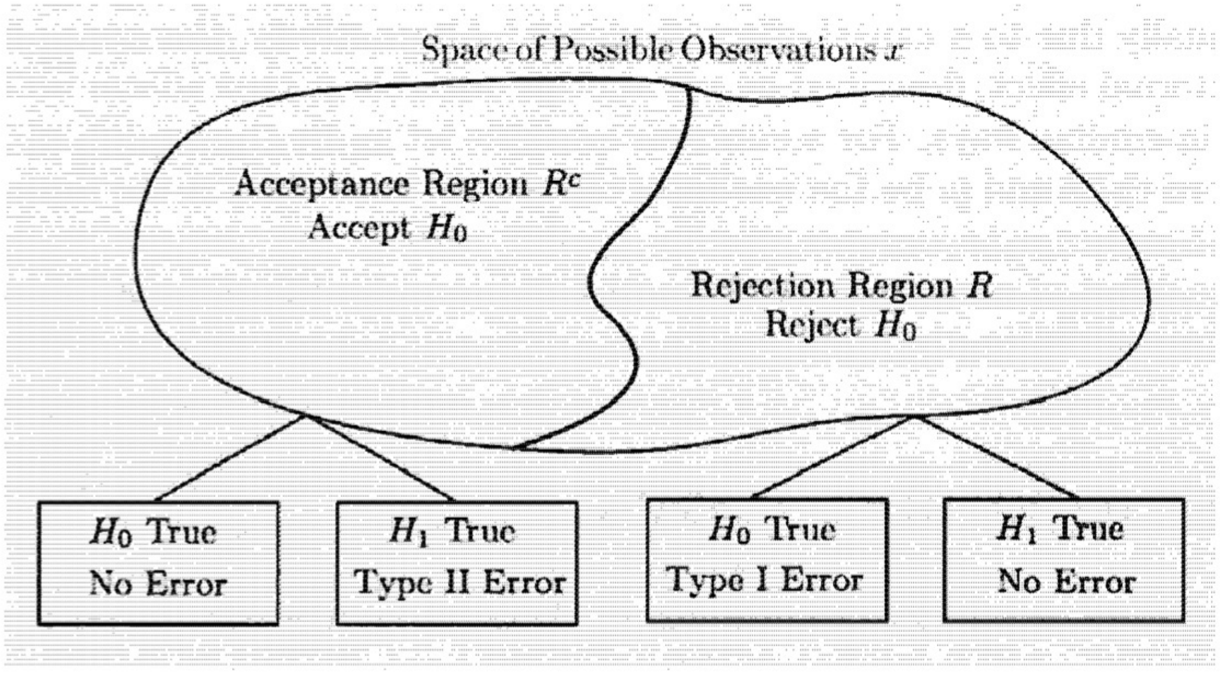
- Partition the space of possible data vectors
  Rejection region R:
  reject H_0 iff data $\in R$



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- Partition the space of possible data vectors
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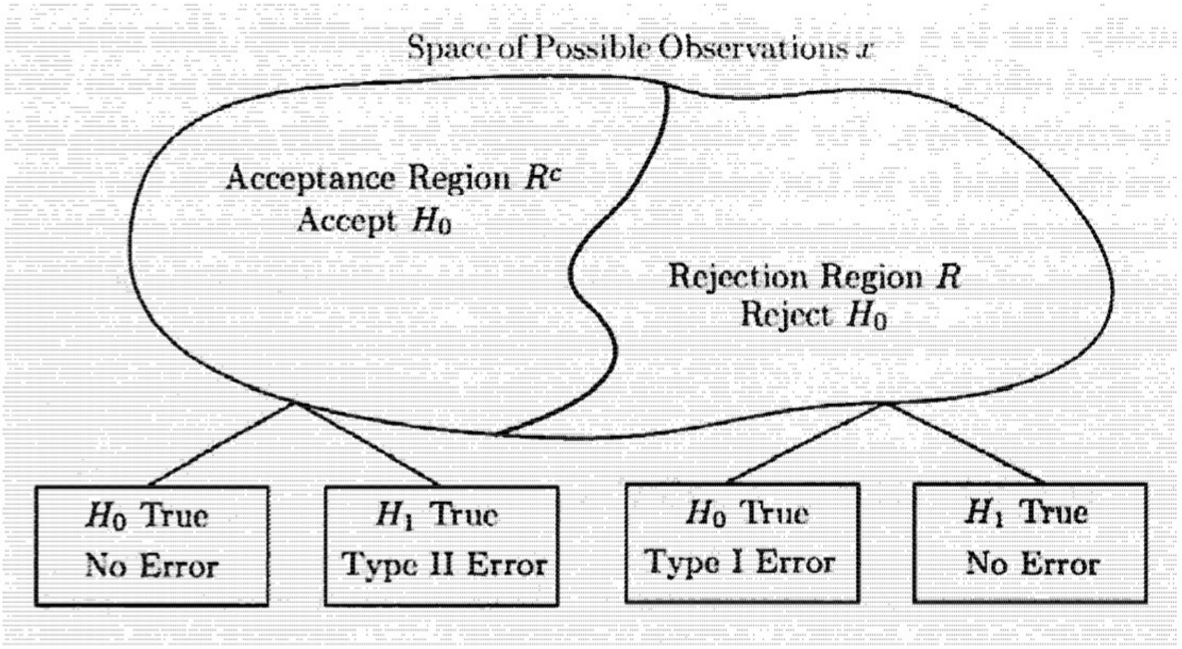
- Types of errors:

  – **Type I** (**false rejection**, false alarm):
  H_0 true, but rejected

$$\alpha(R) = \mathbf{P}(X \in R\,; H_0)$$

  – **Type II** (**false acceptance**, missed detection):
  H_0 false, but accepted



Example: Classifying Medical Results

- The value of a blood test is used as the test statistic, R, to identify whether a patient has a particular disease or not.
  - H_0: For patients **not** having the disease, R has distribution $\mathcal{N}(40, 5)$
  - H_1: For patients having the disease, R has distribution $\mathcal{N}(60, 5)$

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$$\alpha = \int_{50}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(R-u)^2}{2\sigma^2}}\, dR = \int_{50}^{\infty} \frac{1}{\sqrt{50\pi}} e^{-\frac{(R-40)^2}{50}}\, dR = 0.023,\ \mu = 40,\ \sigma = 5$$

$$\beta = \int_{-\infty}^{50} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(R-u)^2}{2\sigma^2}}\, dR = \int_{-\infty}^{50} \frac{1}{\sqrt{50\pi}} e^{-\frac{(R-60)^2}{50}}\, dR = 0.023,\ \mu = 60,\ \sigma = 5$$

Power = 1 - β = 0.977

Bayesian Hypothesis Testing

- Parameter Θ takes one of m values $\{\theta_1, \ldots, \theta_m\}$.

- Hypothesis $H_i \equiv$ event $\{\Theta = \theta_i\}$

- Hypothesis Testing: given observation x, select one of the hypotheses $H_1, \ldots, H_m$

- **MAP Rule**: Select the hypothesis with the largest posterior probability

  - Select H_i if $\mathbb{P}(\Theta = \theta_i | X = x) = p_{\Theta|X}(\theta_i|x)$ is largest

    if $p_\Theta(\theta_i) p_{X|\Theta}(x|\theta_i)$ (if X is discrete)
    if $p_\Theta(\theta_i) f_{X|\Theta}(x|\theta_i)$ (if X is continuous) is largest

Example Ia

- We have two biased coins; coin 1 and coin 2; with biases (i.e. probability of heads) equal to p_1 and p_2 respectively. We choose a coin at random, and want to infer its identity based on the outcome of a single toss. Use the MAP Rule to do so.

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    - Consider random variables Θ, X.
      $\Theta = 1$ indicates coin 1, and $\Theta = 2$ indicates coin 2.
      $X = 1$ is coin flips heads, $X = 0$ for tails.

    - Suppose X = 0 (tails). Then, we select hypothesis Θ = 1 if

$$\begin{aligned} p_\Theta(1) p_{X|\Theta}(0|1) &> p_\Theta(2) p_{X|\Theta}(0|2) \\ (1 - p_1) &> (1 - p_2) \end{aligned}$$

Example Ib

- We have two biased coins; coin 1 and coin 2; with biases (i.e. probability of heads) equal to p_1 and p_2 respectively. We choose a coin at random, and want to infer its identity based on the outcome of **n** coin tosses. Use the MAP Rule to do so.

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  - Consider random variables Θ, X.
    $\Theta = 1$ indicates coin 1, and $\Theta = 2$ indicates coin 2.
    $X = k$ if k heads in the n coin flips.

  - Suppose X = k. Then, we select hypothesis Θ = 1 if

$$\begin{aligned} p_\Theta(1) p_{X|\Theta}(k|1) &> p_\Theta(2) p_{X|\Theta}(k|2) \\ p_1^k (1-p_1)^{n-k} &> p_2^k (1-p_2)^{n-k} \end{aligned}$$

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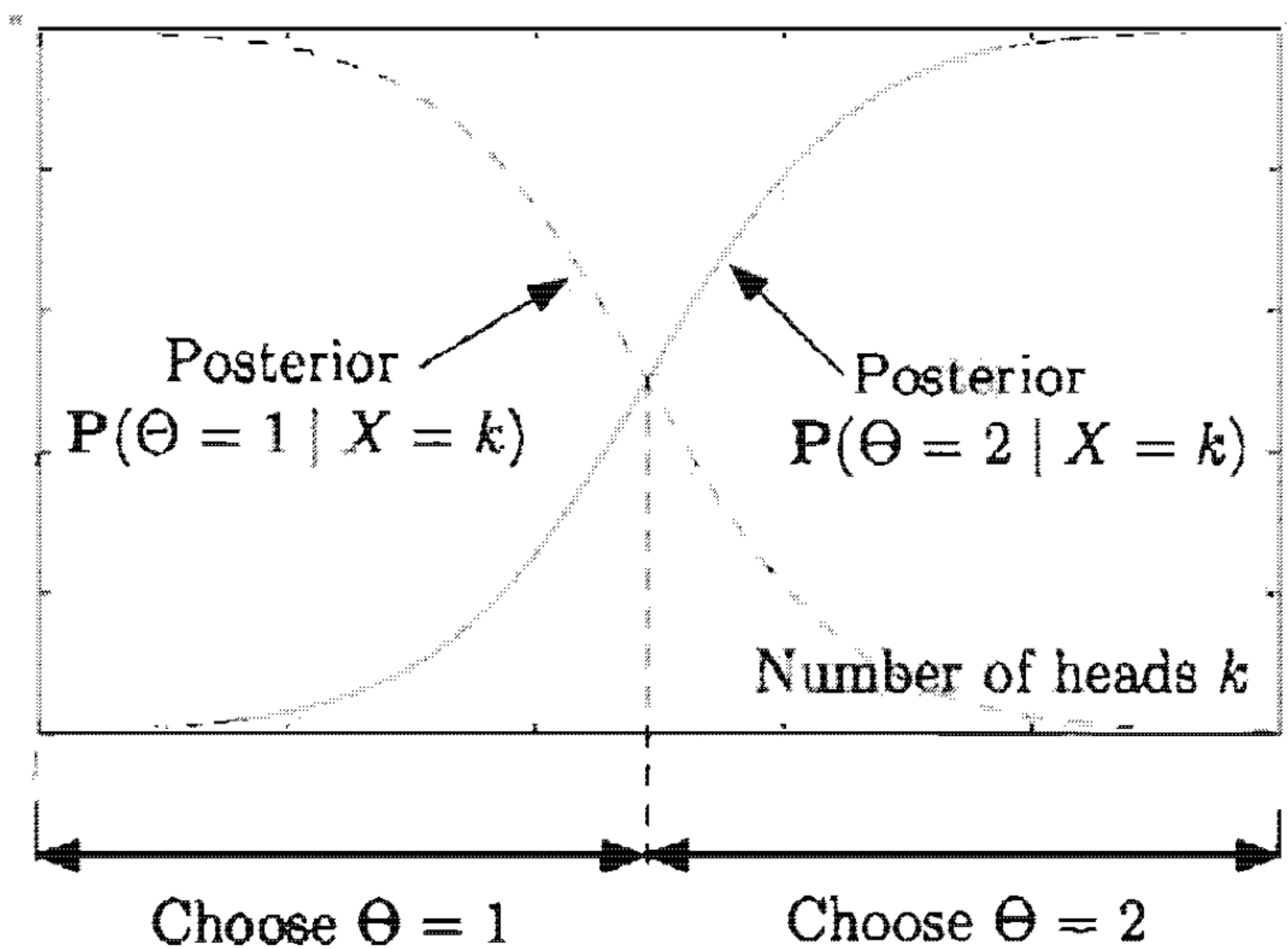

Likelihood Ratio Test (LRT)

- Bayesian case (MAP rule): choose H_1 if:
  $\mathbf{P}(H_1 \mid X = x) > \mathbf{P}(H_0 \mid X = x)$
  or
  $$\frac{\mathbf{P}(X = x \mid H_1)\mathbf{P}(H_1)}{\mathbf{P}(X = x)} > \frac{\mathbf{P}(X = x \mid H_0)\mathbf{P}(H_0)}{\mathbf{P}(X = x)}$$
  or
  $$\frac{\mathbf{P}(X = x \mid H_1)}{\mathbf{P}(X = x \mid H_0)} > \frac{\mathbf{P}(H_1)}{\mathbf{P}(H_0)}$$
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- Nonbayesian version: choose H_1 if
  $$\frac{\mathbf{P}(X = x; H_1)}{\mathbf{P}(X = x; H_0)} > \xi \quad \text{(discrete case)}$$
  $$\frac{f_X(x; H_1)}{f_X(x; H_0)} > \xi \qquad \text{(continuous case)}$$

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- threshold ξ trades off the two types of error
  - choose ξ so that $\mathbf{P}(\text{reject } H_0; H_0) = \alpha$ (e.g., $\alpha = 0.05$)

Example I

We have a six-sided die that we want to test for fairness, and we formulate two hypotheses for the probabilities of the six faces:

$$H_0 \text{ (fair die):} \qquad p_X(x; H_0) = \frac{1}{6}, \qquad x = 1, \ldots, 6,$$

$$H_1 \text{ (loaded die):} \qquad p_X(x; H_1) = \begin{cases} \frac{1}{4}, & \text{if } x = 1, 2, \\ \frac{1}{8}, & \text{if } x = 3, 4, 5, 6. \end{cases}$$

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The likelihood ratio for a single roll x of the die is

$$L(x) = \begin{cases} \dfrac{1/4}{1/6} = \dfrac{3}{2}, & \text{if } x = 1, 2, \\[2ex] \dfrac{1/8}{1/6} = \dfrac{3}{4}, & \text{if } x = 3, 4, 5, 6. \end{cases}$$

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$\xi < \dfrac{3}{4}$: reject H_0 for all x;

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$$\frac{3}{4} < \xi < \frac{3}{2}: \qquad \text{accept } H_0 \text{ if } x = 3, 4, 5, 6; \text{ reject } H_0 \text{ if } x = 1, 2;$$

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\frac{3}{2} < \xi: &\quad \text{accept } H_0 \text{ for all } x.
\end{aligned}
$$

probability of false rejection $\mathbf{P}(\text{Reject } H_0 ; H_0)$

$$
\alpha(\xi) = \begin{cases} 1, & \text{if } \xi < \frac{3}{4}, \\ \mathbf{P}(X = 1, 2; H_0) = \frac{1}{3}, & \text{if } \frac{3}{4} < \xi < \frac{3}{2}, \\ 0, & \text{if } \frac{3}{2} < \xi. \end{cases}
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probability of false acceptance $\mathbf{P}(\text{Accept } H_0 \text{: } H_1)$

$$
\beta(\xi) = \begin{cases} 0, & \text{if } \xi < \frac{3}{4}, \\ \mathbf{P}(X = 3, 4, 5, 6; H_1) = \frac{1}{2}, & \text{if } \frac{3}{4} < \xi < \frac{3}{2}, \\ 1, & \text{if } \frac{3}{2} < \xi. \end{cases}
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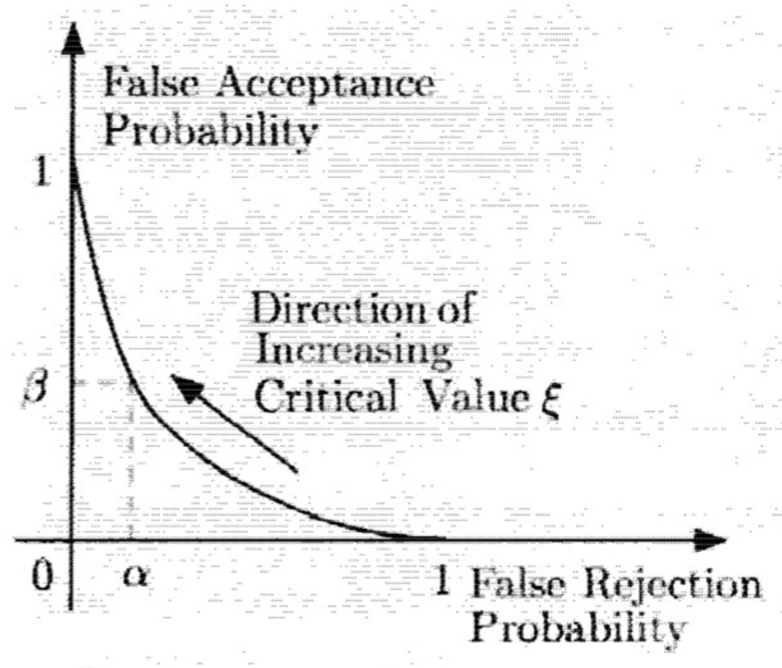
Example I: Tradeoff



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Summary: Likelihood Ratio Test

- Start with a target value α for the false rejection probability.
- Choose a value for ξ such that the false rejection probability is equal to α:

$$\mathbf{P}\big(L(X) > \xi; H_0\big) = \alpha.$$

- Once the value x of X is observed, reject H_0 if $L(x) > \xi$.

Example II

A surveillance camera checks a certain area and record a signal $X = W$, or $X = 1 + W$, depending on whether an intruder is absent or present (hypotheses H_0 or H_1 respectively).

Assume that $W \sim N(0,v)$, for some known $v > 0$. Write out the Likelihood Ratio Test for testing the presence of an intruder (i.e. hypothesis H_0 or H_1), given the camera signal x.

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Assume that W ~ N(0,v), for some known v > 0. Write out the Likelihood Ratio Test for testing the presence of an intruder (i.e. hypothesis H_0 or H_1), given the camera signal x.

$$f_X(x; H_0) = \frac{1}{\sqrt{2\pi v}} \exp\left\{-\frac{x^2}{2v}\right\}, \qquad f_X(x; H_1) = \frac{1}{\sqrt{2\pi v}} \exp\left\{-\frac{(x-1)^2}{2v}\right\},$$

Likelihood Function:

$$L(x) = \frac{f_X(x; H_1)}{f_X(x; H_0)} = \exp\left\{\frac{x^2 - (x-1)^2}{2v}\right\} = \exp\left\{\frac{2x-1}{2v}\right\}.$$

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Likelihood Ratio:

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LRT Test:

$$L(x) > \xi \;\equiv\; x > v \log \xi + \frac{1}{2}$$

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Rejection Region:

$$R = \{x \mid x > \gamma\} \qquad \gamma = v \log \xi + \frac{1}{2};$$

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so that for a specified value of α, we can obtain the value of γ from a table/calculator

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Rejection Region:

$$R = \{x \,|\, x > \gamma\} \qquad \gamma = v \log \xi + \frac{1}{2};$$

False Rejection Probability:

$$\alpha = \mathbf{P}(X > \gamma; H_0) = \mathbf{P}(W > \gamma),$$

so that for a specified value of α, we can obtain the value of γ from a table/calculator

False Acceptance Probability:

$$\beta = \mathbf{P}\big(X \leq \gamma; H_1\big) = \mathbf{P}\,(1 + W \leq \gamma) = \mathbf{P}\,(W \leq \gamma - 1),$$

Example III

A Discrete Example. Consider $n = 25$ independent tosses of a coin. Under hypothesis H_0 (respectively, H_1), the probability of a head at each toss is equal to $\theta_0 = 1/2$ (respectively, $\theta_1 = 2/3$). Let X be the number of heads observed. If we set the false rejection probability to 0.1, what is the rejection region associated with the LRT?

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A Discrete Example. Consider $n = 25$ independent tosses of a coin. Under hypothesis H_0 (respectively, H_1), the probability of a head at each toss is equal to $\theta_0 = 1/2$ (respectively, $\theta_1 = 2/3$). Let X be the number of heads observed. If we set the false rejection probability to 0.1, what is the rejection region associated with the LRT?

We observe that when $X = k$, the likelihood ratio is of the form

$$L(k) = \frac{\binom{n}{k}\theta_1^k(1-\theta_1)^{n-k}}{\binom{n}{k}\theta_0^k(1-\theta_0)^{n-k}} = \left(\frac{\theta_1}{\theta_0}\cdot\frac{1-\theta_0}{1-\theta_1}\right)^k \cdot \left(\frac{1-\theta_1}{1-\theta_0}\right)^n = 2^k\left(\frac{2}{3}\right)^{25}.$$

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$\mathbf{P}(X > \gamma; H_0) \leq 0.1$, or

$$\sum_{i=\gamma+1}^{25} \binom{25}{i} 2^{-25} \leq 0.1.$$

Numerical evaluations yield γ = 16

Example IV

Problem : Artemisia moves to a new house and she is "fifty-percent sure" that the phone number is 2537267. To verify this. she uses the house phone to dial 2537267. she obtains a busy signal. and concludes that this is indeed the correct number. Assuming that the probability of a typical seven-digit phone number being busy at any given time is 1%. what is the probability that Artemisia's conclusion was correct?

H_0 : the phone number is 2537267,

H_1 : the phone number is not 2537267,

$$\mathbf{P}(H_0) = \mathbf{P}(H_1) = 0.5.$$

Under H_0, we expect a busy signal with certainty:

$$\mathbf{P}(B \mid H_0) = 1.$$

Under H_1, the conditional probability of B is

$$\mathbf{P}(B \mid H_1) = 0.01.$$

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$$\mathbf{P}(H_0 \mid B) = \frac{\mathbf{P}(B \mid H_0)\mathbf{P}(H_0)}{\mathbf{P}(B \mid H_0)\mathbf{P}(H_0) + \mathbf{P}(B \mid H_1)\mathbf{P}(H_1)} = \frac{0.5}{0.5 + 0.005} \approx 0.99.$$