Frequentist Data Analysis II

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Bernoulli distribution

Data, D = 

- P(Heads) = θ, P(Tails) = 1-θ

- Flips are **i.i.d.**:
  - **Independent** events
  - **Identically distributed** according to Bernoulli distribution

<u>Choose θ that maximizes the probability of observed data</u>

Probability of one coin flip

Let's say we observe a coin flip $X \in \{0, 1\}$.

The probability of this coin flip,
given a Bernoulli distribution with parameter p:

$$p^X (1 - p)^{1-X}.$$

Equal to p when $X = 1$, and equal to $(1 - p)$ when $X = 0$.

Probability of Multiple Coin Flips

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\end{aligned}$$

…probability of a Bernoulli sample

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$$\ldots\; p^a\, p^b = p^{a+b}$$

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&= p^{n_h}\, (1-p)^{n-n_h}.
\end{aligned}
$$

where n_h is the number of heads,
n is the total number of coin flips

Maximum Likelihood Estimator (MLE)

The MLE solution is then given by solving the following problem:

$$\begin{aligned}
\widehat{p} &= \arg\max_{p} \mathbb{P}(X_1, \ldots, X_n; p) \\
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&= \arg\max_p \left\{n_h \log p + (n - n_h) \log(1-p)\right\}
\end{aligned}
$$

…argmax_x f(x) = argmax_x log f(x)

MLE for coin flips

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$$\implies \frac{n_h}{\widehat{p}} - \frac{n - n_h}{1 - \widehat{p}} = 0$$

$$\implies \widehat{p} = \frac{n_h}{n}.$$

Maximum Likelihood Estimation

Choose θ that maximizes the probability of observed data

$$\widehat{\theta}_{MLE} = \arg\max_{\theta} \; P(D \mid \theta)$$

$$\widehat{\theta}_{MLE} = \frac{\alpha_H}{\alpha_H + \alpha_T} = 3/5$$

"Frequency of heads"

MLE of probability of head:

How many flips do I need?

$$\hat{\theta}_{MLE} = \frac{\alpha_H}{\alpha_H + \alpha_T}$$

- Billionaire says: I flipped 3 heads and 2 tails.
- You say: $\theta = 3/5$, it is the MLE!
- He says: What if I flipped 30 heads and 20 tails?
- You say: Same answer, it is the MLE!
- **He says: If you get the same answer,** would you prefer to flip 5 times or 50 times**?**
- You say: Hmm… The more the merrier???
- He says: Is this why I am paying you the big bucks???

SO FAR:
THE MLE IS A CLASS OF ESTIMATORS THAT ESTIMATE MODEL FROM DATA

KEY QUESTION: HOW GOOD IS THE MLE (OR ANY OTHER ESTIMATOR)?

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By the Law of Large Numbers!

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…linearity of expectation:

E(a X + b Y) = a E(X) + b E(Y)

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&= \mathbb{E}X_1 \\
&= p.
\end{aligned}
$$

Summary: Classical/Frequentist Data Analysis

Parameter θ

Observation X

θ is a deterministic (i.e. not random) but unknown quantity

X is random, with distribution

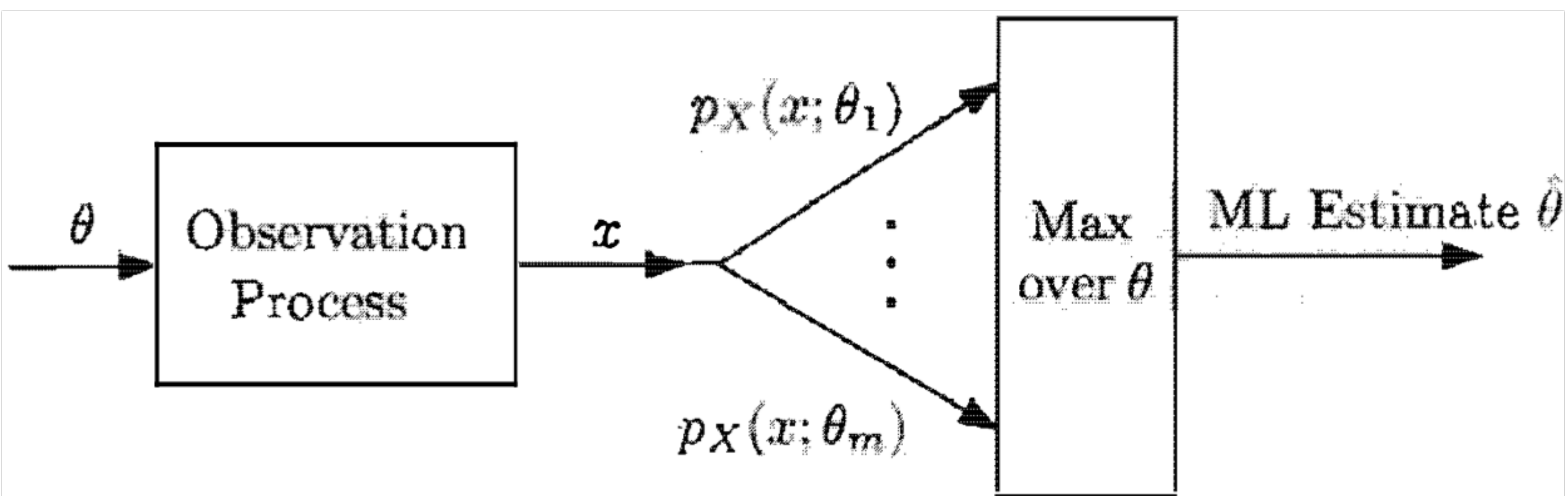
$$p_X(x;\theta) \text{ (if } X \text{ is discrete), or } f_X(x;\theta) \text{ (if } X \text{ is continuous)}$$

- These are NOT conditional probabilities;
  θ is NOT random
  - mathematically: many models,
    one for each possible value of θ

Summary: Maximum Likelihood Estimation

- Model, with unknown parameter(s): $X \sim p_X(x;\theta)$
- Pick θ that "makes data most likely"

$$\widehat{\theta}_{\mathsf{ML}} = \arg\max_{\theta} p_X(x;\theta)$$



Likelihood Function

- We refer to $p_X(x;\theta)$ [or $f_X(x;\theta)$ if X is continuous] as the **likelihood function**.
  - Note that this is a function of θ
- If the observations X_i are independent, the likelihood function takes the form

$$p_X(x_1,\ldots,x_n;\theta) = \prod_{i=1}^{n} p_{X_i}(x_i;\theta)$$

- Log-likelihood function:

$$\log p_X(x_1,\ldots,x_n;\theta) = \log \prod_{i=1}^{n} p_{X_i}(x_i;\theta) = \sum_{i=1}^{n} \log p_{X_i}(x_i;\theta),$$

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Log-Likelihood

- Log-likelihood function

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- It might be analytically more convenient to maximize log-likelihood rather than likelihood --- though either would yield the same answer

$$\arg\max_{\theta}\{f(\theta)\} = \arg\max_{\theta}\{\log f(\theta)\}$$

Example

X is said to have exponential distribution with param. θ if

$$f_X(x;\theta) = \begin{cases} \theta e^{-\theta x} & x \geq 0 \\ 0 & x < 0. \end{cases}$$

$X_1, \ldots, X_n$: i.i.d., exponential(θ)

What is the ML Estimate of θ?

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$$\boxed{\hat{\Theta}_n = \frac{n}{X_1 + \cdots + X_n}}$$

Desirable Properties of Estimators I

- **Unbiased:** $\mathbf{E}[\hat{\Theta}_n] = \theta$
  - exponential example, with $n = 1$:
    $\mathbf{E}[1/X_1] = \infty \neq \theta$
    (biased)

- **Bias**: $\mathbf{E}[\hat{\Theta}_n] - \theta$
  - Unbiased: bias equals zero

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Properties:

- $\mathbf{E}[\hat{\Theta}_n] = \theta$ (unbiased)
- WLLN: $\hat{\Theta}_n \to \theta$ (consistency)

Example

- Recall the Romeo and Juliet example, where Juliet was late on any date by a random amount X, uniformly distributed over the interval [0,θ] where θ is unknown. Unlike in the previous case, let us assume here that θ is deterministic (i.e. not random). What is the ML estimate of θ, if Juliet is late by an amount x on their first date?

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ML estimate of θ is x