Jacobs University
Fall 2022

October 25, 2022

Stochastic Methods Lab

Assignment Sheet 6

Due on November 1, 2022

Note: The work is to be submitted via `git`, as discussed in class. The coding language is Python. Please make sure that your code actually runs and produces the requested output. Please make your code readable for the instructor and TA, and include comments wherever necessary. Please submit .py source code, not jupyter notebooks. Theoretical questions may be submitted as a scan of handwritten notes or typed up (e.g., using LaTeX). The submission deadline is midnight of the stated due date.

Problem 1 [4 points]

Compute an ensemble (at least 1000) of standard Brownian paths $W(t)$ over the interval $[0, 1]$ partitioned into $N = 600$ time steps. Plot the empirically determined mean and standard deviation of the ensemble as a function of time. In the same figure, plot 10 sample paths.

Problem 2 [8 points]

(a) Compute an ensemble of geometric Brownian paths (at least $M = 1000$)

$$S(t) = \exp\left(\left(\mu - \frac{1}{2}\,\sigma^2\right)\, t + \sigma\, W(t)\right)$$

as a function of time on the interval $[0, 1]$ partitioned into $N = 500$ time steps. Then plot the empirical mean and standard deviation and 6 sample paths. Use the parameters $\mu = 0.7$ and $\sigma = 0.4$.

(b) Plot the mean and standard deviation of the stock price paths which underlie the binomial tree model (using the risk-neutral probabilities) with $N = 600$ time steps calibrated with the same set of parameters $r = \mu$ and annualized volatility σ as in part (a), together with 6 sample paths. Use the calibration that we discussed in class and in Assignment Sheet 3.

(c) Now plot the results of part (a) and (b) in the same figure and describe what you see.

Problem 3 [4 points]

Use geometric Brownian motion (with $\mu = 0.3$, $\sigma = 0.7$) in a Monte-Carlo valuation of a European call option with strike price $K = 0.8$, time to maturity $T = 1$ and risk free rate $r = \mu$. (The result is about 0.5.) Compare your result against the price obtained from using the Black-Scholes formula by plotting the deviation from the Black-Scholes price against the number of samples in a doubly logarithmic plot.

What is the convergence rate of the Monte-Carlo method as a function of the number of samples?

Problem 4 [4 points]

For some large N, approximate the Itô integral

$$I(T) = \int_0^T X(t)\, dW(t) = \lim_{N\to\infty} \sum_{i=0}^{N-1} X(t_i) \Big(W(t_{i+1}) - W(t_i)\Big)$$

and the Stratonovich integral

$$S(T) = \int_0^T X(t) \circ dW(t) = \lim_{N\to\infty} \sum_{i=0}^{N-1} X\left(\frac{t_{i+1}+t_i}{2}\right) \Big(W(t_{i+1}) - W(t_i)\Big),$$

where $W(t)$ denotes standard Brownian motion, and $t_i = i\,\Delta t$ with $\Delta t = T/N$. As example, choose $X(t) = W(t)$.

(a) Plot one realization of Brownian motion, and the corresponding Itô and Stratonovich integrals.

(b) For some large N, look at the difference between the Itô and Stratonovich integrals and describe what you see.