# THE FUNCTOR OF POINTS

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**Spoiler Alert:** We solve some exercises in [EH].

## 1. The Yoneda Lemma

Let $\mathsf{C}$ be a locally small category. If $X \in \mathsf{C}$, then the *(covariant) hom functor* of $X$ is the functor $h^X : \mathsf{C} \to \mathsf{Set}$ that sends $Y \mapsto \mathrm{Hom}(X, Y)$, and sends $g : Y_1 \to Y_2$ to the operation of postcomposition with $g$.

**Lemma 1.1** (Yoneda)**.** *Let $\mathcal{F} : \mathsf{C} \to \mathsf{Set}$ be arbitrary, and let $X \in \mathsf{C}$. Then the map*

$$\Phi \mapsto \Phi_X(\mathrm{id}_X) \tag{1}$$

*is a natural bijection* $\mathrm{Hom}(h^X, \mathcal{F}) \to \mathcal{F}(X)$.

*Proof.* For all $Y \in \mathsf{C}$ and $f \in \mathrm{Hom}(X, Y)$, there is a commutative diagram:

$$\begin{tikzcd} \mathrm{Hom}(X, X) \arrow[r, "h^X f"] \arrow[d, "\Phi_X"] & \mathrm{Hom}(X, Y) \arrow[d, "\Phi_Y"] \\ \mathcal{F}(X) \arrow[r, "\mathcal{F} f"] & \mathcal{F}(Y) \end{tikzcd} \tag{2}$$

In particular, $(\mathcal{F} f)(\Phi_X(\mathrm{id}_X)) = \Phi_Y(f)$, which shows that $\Phi$ is uniquely determined by $a = \Phi_X(\mathrm{id}_X)$ and any choice of $a \in \mathcal{F}(X)$ is possible. $\square$

In the notation of 1.1, let $a \in \mathcal{F}(X)$. We say that the pair $(X, a)$ *represents* $\mathcal{F}$ iff the natural transformation $h^X \to \mathcal{F}$ corresponding to $a$ is an isomorphism. If $\mathcal{F}$ is representable, then 1.1 implies the representing object is unique up to unique isomorphism.

**Corollary 1.2.** *The functor $X \mapsto h^X$ is a full and faithful embedding $\mathsf{C} \to \mathsf{Fun}(\mathsf{C}, \mathsf{Set})$.*

Recall that a *presheaf* on $\mathsf{C}$ is another name for a contravariant functor $\mathsf{C} \to \mathsf{Set}$. We write $\mathsf{PSh}(\mathsf{C}) = \mathsf{Fun}(\mathsf{C}, \mathsf{Set}^{\mathrm{op}})$ for the category of presheaves on $\mathsf{C}$. The *contravariant hom functor* of $X$ is the presheaf $h_X : \mathsf{C} \to \mathsf{Set}$ that sends $Y \mapsto \mathrm{Hom}(Y, X)$, and sends $g : Y_1 \to Y_2$ to precomposition with $g$. Thus, the dual of 1.2 says that the map $X \mapsto h_X$ is a full and faithful embedding $\mathsf{C} \to \mathsf{PSh}(\mathsf{C})$. The maps $X \mapsto h^X$ and $X \mapsto h_X$ are called the *Yoneda embeddings*.

The utility of the Yoneda embedding is this: First, it lets us generalize properties of objects of $\mathsf{C}$ in a natural way to properties of functors or presheaves on $\mathsf{C}$. Second, the category of functors from $\mathsf{C}$ to $\mathsf{Set}$ or $\mathsf{Set}^{\mathrm{op}}$ is often more well-behaved than $\mathsf{C}$ itself: We may be able to perform constructions in the larger category that we could not do in $\mathsf{C}$. For example, $\mathsf{C}$ may not contain fiber products, whereas the new category does.

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## 2. $R$-Functors

Fix a ring $R$. Take $\mathsf{C} = \mathsf{Sch}/R$, the category of schemes over $\operatorname{Spec} R$, so that if $R = \mathbb{Z}$, then $\mathsf{Sch}/R = \mathsf{Sch}$. In this setting, we say that $h_X$ is the *functor of points of* $X$, and that the elements of $h_X(Y)$ are the $Y$*-valued points of* $X$. If $A$ is a ring, then we abbreviate $h_X(A) = h_X(\operatorname{Spec} A)$.

We can sharpen 1.2 slightly: It turns out that the functor of points of an $R$-scheme is completely determined by how it behaves on the spectra of $R$-algebras. Let $\mathsf{Ring}(R)$ be the category of $R$-algebras, and for all $X \in \mathsf{Sch}/R$, let

$$h_X^* = h_X \circ \operatorname{Spec} : \mathsf{Ring}(R) \to \mathsf{Set} \tag{3}$$

be the (covariant!) functor that sends $A \mapsto h_X(A) = \operatorname{Hom}_{\mathsf{Sch}/R}(\operatorname{Spec} A, X)$.

**Corollary 2.1.** *If $R$ is a ring, then the functor $X \mapsto h_X^*$ is a full and faithful embedding $\mathsf{Sch}/R \to \mathsf{Fun}(\mathsf{Ring}(R), \mathsf{Set})$.*

*Proof.* Let $X, Y \in \mathsf{Sch}/R$. We must show that every natural transformation $\Phi : h_Y^* \to h_X^*$ comes from a unique $\mathsf{Sch}/R$ morphism $Y \to X$. Take an affine open cover $\{V_j\}$ of $Y$, and let $\iota_j : V_j \to Y$ be the inclusion map. Then $\Phi(\iota_j)$ is a $\mathsf{Sch}/R$ morphism $V_j \to X$, and by compatibility, the family $\{\Phi(\iota_j)\}$ determines a unique $\mathsf{Sch}/R$ morphism on $Y$. □

We will now generalize our topological notions about $R$-schemes to notions about functors $\mathsf{Ring}(R) \to \mathsf{Set}$, which we call *$R$-functors*. First, check that if $\Phi_1 : \mathcal{F}_1 \to \mathcal{G}$ and $\Phi_2 : \mathcal{F}_2 \to \mathcal{G}$ are $R$-functor morphisms, then the fiber product $\mathcal{F}_1 \times_{\mathcal{G}} \mathcal{F}_2$ in the category $\mathsf{Fun}(\mathsf{Ring}(R), \mathsf{Set})$ is realized by the functor that sends

$$A \mapsto \{(a_1, a_2) \in \mathcal{F}_1(A) \times \mathcal{F}_2(A) : \Phi_{1,A}(a_1) = \Phi_{2,A}(a_2)\}. \tag{4}$$

It follows that $X \mapsto h_X^*$ preserves fiber products: For all $X, Y_1, Y_2 \in \mathsf{Sch}/R$,

$$h^*_{Y_1 \times_X Y_2} = h^*_{Y_1} \times_{h_X^*} h^*_{Y_2}. \tag{5}$$

Let $\mathcal{F}$ be an $R$-subfunctor of $\mathcal{G}$, meaning there is a morphism $\mathcal{F} \to \mathcal{G}$ such that $\mathcal{F}(A) \to \mathcal{G}(A)$ is injective for all $A$. We say that $\mathcal{F}$ is *open (resp., closed) in* $\mathcal{G}$ iff, for all $R$-schemes $X$ and $R$-functor morphisms $h_X^* \to \mathcal{G}$, the base change $\mathcal{F} \times_{\mathcal{G}} h_X^*$ is isomorphic to $h_Y^*$ for some open (resp., closed) subscheme $Y$ of $X$.

**Lemma 2.2.** *Let $R$ be a ring. If $X \in \mathsf{Sch}/R$, then the open (resp., closed) $R$-subfunctors of $h_X^*$ are the functors of the form $h_Y^*$ for some open (resp., closed) subscheme $Y$ of $X$.*

*Proof.* By (5) and the compatibility of open (resp., closed) immersions with respect to base change, every functor of the form $h_Y^*$ with $Y$ an open (resp., closed) subscheme of $X$ is an open (resp., closed) subfunctor of $h_X^*$. The reverse inclusion follows from considering the trivial base change to $h_X^*$ itself. □

**Proposition 2.3.** *Let $R$ be a ring, and let $A \in \mathsf{Ring}(R)$.*

(1) *The open $R$-subfunctors of $h^*_{\operatorname{Spec} A}$ are the functors of the form*

$$B \mapsto \{f \in h^*_{\operatorname{Spec} A}(B) : f^{\#}_{\operatorname{Spec} A}(\mathfrak{a})B = B\} \tag{6}$$

*for some $\mathfrak{a} \lhd A$.*

(2) *The closed $R$-subfunctors of $h^*_{\operatorname{Spec} A}$ are the functors of the form*

$$B \mapsto \{f \in h^*_{\operatorname{Spec} A}(B) : f^{\#}_{\operatorname{Spec} A}(\mathfrak{a})B \subseteq \operatorname{nil} B\} \tag{7}$$

*for some $\mathfrak{a} \lhd A$.*

*Proof.*

(1) The open subsets of $\operatorname{Spec} A$ take the form $D(\mathfrak{a}) = \operatorname{Spec} A \setminus V(\mathfrak{a})$ for some $\mathfrak{a} \lhd A$. By 2.2, it suffices to prove that if $\phi : A \to B$ is a $\mathsf{Ring}(R)$ morphism, then $\phi(\mathfrak{a})B = B$ if and only if $f = \operatorname{Spec}(\phi) : \operatorname{Spec} B \to \operatorname{Spec} A$ has image in $D(\mathfrak{a})$. The latter occurs if and only if $\mathfrak{a} \not\subseteq \phi^{-1}(\mathfrak{q})$ for all $\mathfrak{q} \in \operatorname{Spec} B$, meaning there exists $b \in \phi(\mathfrak{a}) \setminus \mathfrak{q}$ for all such $\mathfrak{q}$. In particular, this happens if and only if $\phi(\mathfrak{a})$ contains a unit, as we see from considering maximal $\mathfrak{q}$; this means $\phi(\mathfrak{a})B$ contains 1.

(2) Let $\mathfrak{a} \lhd A$. By 2.2, it suffices to prove that if $\phi : A \to B$ is a $\mathsf{Ring}(R)$ morphism, then $\phi(\mathfrak{a})B \subseteq \operatorname{nil} B$ if and only if $f = \operatorname{Spec}(\phi) : \operatorname{Spec} B \to \operatorname{Spec} A$ has image in $V(\mathfrak{a})$. The latter occurs if and only if $\mathfrak{a} \subseteq \phi^{-1}(\mathfrak{q})$ for all $\mathfrak{q} \in \operatorname{Spec} B$, meaning $\phi(\mathfrak{a}) \subseteq \bigcap_{\mathfrak{q} \in \operatorname{Spec} B} = \operatorname{nil} B$.

□

We write $U_{\mathfrak{a}}$ and $Z_{\mathfrak{a}}$ for the functors defined in (6) and (7), respectively. Note that in general, $U_{\mathfrak{a}}(B) \cup Z_{\mathfrak{a}}(B) \neq h^*_{\operatorname{Spec} A}(B)$; however, equality holds if, for example, $B$ is a DVR or a field.

Suppose $\{\mathcal{U}_i\}$ is a family of open $R$-subfunctors of $\mathcal{F}$. We say that $\{\mathcal{U}_i\}$ is an *open cover of* $\mathcal{F}$ iff, for all $R$-schemes $X$ and $R$-functor morphisms $h^*_X \to \mathcal{F}$ such that $\mathcal{U}_i \times_{\mathcal{F}} h^*_X = h^*_{U_i}$, the family $\{U_i\}$ is an open cover of $X$.

**Example 2.4.** It is false that if $\{\mathcal{U}_i\}$ is an open cover of $\mathcal{F}$, then $\mathcal{F}(A) = \bigcup_i \{\mathcal{U}_i(A)\}$ for all $A \in \mathsf{Ring}(R)$. For example, take $R = \mathbb{Z}$. If $p, q$ are distinct primes, then

$$\{h^*_{\operatorname{Spec} \mathbb{Z}[1/p]}, h^*_{\operatorname{Spec} \mathbb{Z}[1/q]}\} \tag{8}$$

is a cover of $h^*_{\operatorname{Spec} \mathbb{Z}}$, but $h^*_{\operatorname{Spec} \mathbb{Z}}(\operatorname{Spec} \mathbb{Z}) = \{\mathrm{id}_{\operatorname{Spec} \mathbb{Z}}\}$, whereas

$$h^*_{\operatorname{Spec} \mathbb{Z}[1/p]}(\operatorname{Spec} \mathbb{Z}) = h^*_{\operatorname{Spec} \mathbb{Z}[1/q]}(\operatorname{Spec} \mathbb{Z}) = \varnothing \tag{9}$$

[EH, p. 255].

**Proposition 2.5.** *Let $\{\mathcal{U}_i\}$ be a family of open $R$-subfunctors of $\mathcal{F}$. Then $\{\mathcal{U}_i\}$ is an open cover of $\mathcal{F}$ if and only if $\mathcal{F}(\mathbb{F}) = \bigcup_i \mathcal{U}_i(\mathbb{F})$ for all fields $\mathbb{F} \in \mathsf{Ring}(R)$.*

*Proof.* To prove the "only if" direction: Suppose $\{\mathcal{U}_i\}$ is an open cover of $\mathcal{F}$. Let $\mathbb{F} \in \mathsf{Ring}(R)$ be a field. We have $h^*_{\operatorname{Spec} \mathbb{F}} = h^{\mathbb{F}}$, so by the Yoneda Lemma, it suffices to show $\operatorname{Hom}(h^*_{\operatorname{Spec} \mathbb{F}}, \mathcal{F}) \subseteq \bigcup_i \operatorname{Hom}(h^*_{\operatorname{Spec} \mathbb{F}}, \mathcal{U}_i)$.

Fix $\Phi \in \operatorname{Hom}(h^*_{\operatorname{Spec} \mathbb{F}}, \mathcal{F})$. For all $i$, the base change $\mathcal{U}_i \times_{\mathcal{F}} h^*_{\operatorname{Spec} \mathbb{F}}$ with respect to $\Phi$ is either $h^*_{\varnothing}$ or $h^*_{\operatorname{Spec} \mathbb{F}}$, which forces $\mathcal{U}_j \times_{\mathcal{F}} h^*_{\operatorname{Spec} \mathbb{F}} = h^*_{\operatorname{Spec} \mathbb{F}}$ for some $j$. Explicitly, this means $\Phi$ defines a morphism $h^*_{\operatorname{Spec} \mathbb{F}} \to \mathcal{U}_j$, as needed.

To prove the "if" direction: Suppose $\mathcal{F}(\mathbb{F}) = \bigcup_i \mathcal{U}_i(\mathcal{F})$ for all fields $\mathbb{F}$. Let $X \in \mathsf{Sch}/R$. By the transitivity of base change, we can assume $X = \operatorname{Spec} A$ for some $A \in \mathsf{Ring}(R)$. If $\mathfrak{p} \in X$, then we write $k(\mathfrak{p}) = A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$. For all $\mathfrak{p}$, there exists $i$ such that $\mathcal{U}_i \times_{\mathcal{F}} h^*_{k(\mathfrak{p})} = h^*_{k(\mathfrak{p})}$ by the Yoneda Lemma. Therefore, writing $h^*_{U_i} = \mathcal{U}_i \times_{\mathcal{F}} h^*_X$,

$$\begin{aligned} h^*_{k(\mathfrak{p})} &= \mathcal{U}_i \times_{\mathcal{F}} h^*_{k(\mathfrak{p})} \times_{h^*_X} h^*_X \\ &= h^*_{U_i} \times_{h^*_X} h^*_{k(\mathfrak{p})} \\ &= h^*_{U_i \times_X k(\mathfrak{p})}. \end{aligned} \tag{10}$$

This can happen only if $\mathfrak{p} \in U_i$. Therefore, $\{U_i\}$ is an open cover of $X$. □

## References

[EH] D. Eisenbud & J. Harris. *The Geometry of Schemes*. Springer-Verlag (2000).

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