MATH 250: TOPOLOGY I PROBLEM SET #2

FALL 2025

Due Wednesday, September 17. Please attempt all of the problems. <u>Six</u> of them will be graded. You may consult books, papers, and websites as long as you cite all sources and write up your solutions in your own words. **Last update: 9/15**.

Problem 1. Endow $\mathbf{Z}$ with the evenly-spaced topology from Problem Set 1.

(1) Show that if there were finitely many (positive) primes, then $\{1, -1\}$ would be open. You may assume that any integer greater than 1 has a prime divisor.
(2) Deduce that there must be infinitely many primes.

Problem 2 (Munkres 91–92, #1)**.** Let X be a topological space, and let $A \subseteq Y \subseteq X$ be subsets. Endow Y with the subspace topology that it inherits from X. Show that the subspace topology that A inherits from Y is the subspace topology that A inherits from X.

Problem 3. Let X be a topological space, and let $A \subseteq X$ be a subset endowed with the subspace topology.

(1) Give an example where a subset of A is open in A, but not open in X.
(2) Now suppose that <u>A itself is open in X</u>. Prove that a subset of A is open in A if and only if it is open in X.

Problem 4. Endow $\mathbf{R}$ with the analytic topology, and endow

$$X = \{\tfrac{1}{n} \mid n = 1, 2, 3, \ldots\} \cup \{0\}$$

with the subspace topology. Show that:

(1) For all integers $n > 0$, the singleton set $\{\tfrac{1}{n}\}$ is *clopen*: both closed and open.
(2) $\{0\}$ is closed but not open.

Problem 5. Show that the following topological spaces are homeomorphic:

$$\mathbf{R}, \qquad (0, \infty), \qquad (0, 1).$$

Above, $\mathbf{R}$ is endowed with the analytic topology; $(0, \infty)$ and $(0, 1)$ are endowed with their subspace topologies. You may assume that differentiable functions are continuous, and that a composition of homeomorphisms is a homeomorphism.

Problem 6. Let $f : X \to S$ be a continuous map. Below, all subsets are given their subspace topologies.

(1) Show that $f|_{f^{-1}(T)} : f^{-1}(T) \to T$ is continuous for any $T \subseteq S$.
(2) Show that $f|_Y : Y \to S$ is continuous for any $Y \subseteq X$.
(3) Use (1)–(2) to show that $f|_Y : Y \to f(Y)$ is continuous for any $Y \subseteq X$.

Problem 7 (Munkres 112, #10). Show that if $f : A \to B$ and $g : C \to D$ are continuous maps, then $(f, g) : A \times C \to B \times D$ defined by

$$(f, g)(a, c) = (f(a), g(c))$$

is continuous with respect to the product topologies.

Problem 8 (Munkres 128, #9(c)–(d)). Recall that the Euclidean norm on $\mathbf{R}^n$ is given by $\|u\| = \sqrt{u \cdot u}$, where

$$u \cdot v := u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$$

for general $u, v \in \mathbf{R}^n$.

Use the Cauchy–Schwarz inequality $|u{\cdot}v| \leq \|u\|\|v\|$ to show that $\|u+v\| \leq \|u\|+\|v\|$ for all $u, v \in \mathbf{R}^n$. Conclude that the *Euclidean metric* defined by $d(x, y) = \|x - y\|$ really is a metric on $\mathbf{R}^n$.

Problem 9. Let X be arbitrary, and let $d : X \times X \to [0, \infty)$ be an arbitrary metric. Assume that the function $e : X \times X \to [0, \infty)$ defined by

$$e(x, y) = \frac{d(x, y)}{1 + d(x, y)}.$$

is a bounded metric. Show that d and e induce the same topology on X.

Note: Munkres 129, #11 asks the reader to prove that e itself is a metric, which is harder.

Problem 10. Let us say that metrics $d, d' : X \times X \to [0, \infty)$ are *equivalent* if and only if there are constants $A, B > 0$ such that

$$d(x, y) \leq Ad'(x, y) \text{ and } d'(x, y) \leq Bd(x, y) \text{ for all } x, y \in X.$$

In the setting of Problem 9, show that e is not equivalent to d when $X = \mathbf{R}$ and d is the Euclidean metric.

Problem 11 (Munkres 127, #6). The *uniform topology* on

$$\mathbf{R}^\omega := \{\text{sequences } (a_1, a_2, a_3, \ldots) \text{ with } a_i \in \mathbf{R} \text{ for all } i\}$$

is induced by the *uniform metric*

$$\bar{\rho}(x, y) = \sup_{i > 0} \min\{1, |x_i - y_i|\}.$$

For all $x \in \mathbf{R}^\omega$ and $0 < \epsilon < 1$, show that:

    (1) The following set is not open in the uniform topology:

$$U(x, \epsilon) := (x_1 - \epsilon, x_1 + \epsilon) \times (x_2 - \epsilon, x_2 + \epsilon) \times \ldots$$

    (2) Nonetheless, $B_{\bar{\rho}}(x, \epsilon) = \bigcup_{\delta < \epsilon} U(x, \delta)$.

Problem 12. The *box topology* on $\mathbf{R}^\omega$ is defined as follows: U is open if and only if, for all $x \in U$, there are open sets $U_1, U_2, \ldots$ in $\mathbf{R}$ such that $x \in (U_1 \times U_2 \times \cdots) \subseteq U$.

(1) Show that the box topology really is a topology on $\mathbf{R}^\omega$. *Hint:* Use the Axiom of Choice.

(2) Show that the box topology is strictly finer than the uniform topology. *Hint:* Use Problem 11.