8.

Notes on assigning Quot-scheme-type statistics to cells of GL_n affine Springer fibers.

8.1. Affine Flag Varieties and Springer Fibers

8.1. Throughout, $\mathbf{F}$ is an algebraically closed field of characteristic zero or p. Let $F = \mathbf{F}(\!(\varpi)\!)$ and $\mathcal{O} = \mathbf{F}[\![\varpi]\!]$.

Let G be a connected, reductive algebraic group over $\mathbf{F}$ of Coxeter number $n < p$. Here, we define the Coxeter number to be 1 plus the height of the highest root(s) in any root datum for G. Let LG be the loop group of G.

8.2. Fix a maximal torus $T \subseteq G$ and a Borel $B \subseteq G$ containing T. These data define a root datum $(X, \mathfrak{R}, X^\vee, \mathfrak{R}^\vee)$ and a system of simple roots $\{\alpha_1, \dots, \alpha_r\} \subseteq \mathfrak{R}$. Let W be the Weyl group of $\mathfrak{R}$, and let $W^{\mathrm{aff}} = W \ltimes X^\vee$.

For each $\alpha \in \mathfrak{R}$, we fix a generator ξ_α of the corresponding root subspace of $\mathfrak{g}$. We set $\rho^\vee = \frac{1}{2} \sum_i \alpha_i^\vee \in X^\vee \otimes \mathbf{Q}$, so that $\langle \alpha_i, \rho^\vee \rangle = 1$ for all i.

8.3. For all $x \in X^\vee \otimes \mathbf{R}$, let $P_x \subseteq LG$ be the corresponding parahoric subgroup. Let L_x be the Levi quotient of P_x, and let W_x be the Weyl group of L_x. Writing

$$\mathfrak{g}(F)_{x,\kappa} = \mathbf{F}\langle \varpi^k \xi_\alpha \mid \langle \alpha, x \rangle + k = \kappa \rangle,$$

we know that on the level of points, the Lie algebras of P_x and L_x are

$$\mathfrak{p}_x(\mathbf{F}) = \widehat{\bigoplus_{\kappa \geq 0}} \mathfrak{g}(F)_{x,\kappa},$$
$$\mathfrak{l}_x(\mathbf{F}) = \mathfrak{g}(F)_{x,0}.$$

Let $\mathcal{F}\ell_x = LG/P_x$, the affine flag variety of G of type x.

8.4. Fix an integer d coprime to n. Since $-n < \langle \alpha, \rho^\vee \rangle < n$ for all $\alpha \in \mathfrak{R}$, we know that $I_d := P_{\frac{d}{n}\rho^\vee}$ is an Iwahori subgroup of LG, with Levi quotient T. Let $- \cdot_d -$ be the $\mathbf{G}_m$-action on LG defined by

$$c \cdot_d g(\varpi) = c^{2d\rho^\vee} g(c^{2n}\varpi) c^{-2\rho^\vee}.$$

This is the action used by Oblomkov–Yun. If we take $d = 1$, then we recover the action in the papers of Lusztig–Smelt, C.-K. Fan, and Sommers.

Since $T \subseteq L_x$, we know that $- \cdot_d -$ descends to a $\mathbf{G}_m$-action on $\mathcal{F}\ell_x$. Via the Bruhat-type decomposition

$$\mathcal{F}\ell_x = \coprod_{wW_x \in W^{\mathrm{aff}}/W_x} I_d \dot{w} P_x / P_x,$$

where $\dot{w}$ is any lift of w to $N_{G(F)}(T(\mathcal{O}))/T(\mathcal{O})$, we see that the fixed points of $\mathcal{F}\ell_x$ under $- \cdot_d -$ are precisely the cosets of the form $\dot{w} P_x$ for $wW_x \in W^{\mathrm{aff}}/W_x$.

8.5. Let $L\mathfrak{g}$ be the Lie algebra of LG. The action $- \cdot_d -$ induces a $\mathbf{G}_m$-action on $L\mathfrak{g}$ that we again denote $- \cdot_d -$. Explicitly,

$$c \cdot_d \gamma(\varpi) = \mathrm{Ad}(c^{2d\rho^\vee})\gamma(c^{2n}\varpi).$$

For any $\gamma \in \mathfrak{g}(F)$, the affine Springer fiber over γ of type x is

$$\mathcal{F}\ell_x(\gamma) = \{gP_x \in \mathcal{F}\ell_x \mid \mathrm{Ad}(g^{-1})\gamma \in \mathfrak{p}_x\}.$$

We claim that if γ is an eigenvector under $- \cdot_d -$, then $\mathcal{F}\ell_x(\gamma)$ is stable under $- \cdot_d -$. Indeed, for all $g \in G(F)$, we have

$$\begin{aligned} c \cdot_d (\mathrm{Ad}(g^{-1})\gamma(\varpi)) &= \mathrm{Ad}(c^{2d\rho^\vee} g^{-1})\gamma(c^{2n}\varpi) \\ &= \mathrm{Ad}(c \cdot_d g^{-1})(c \cdot_d \gamma(\varpi)), \end{aligned}$$

which shows that if γ is an eigenvector under $- \cdot_d -$, then

$$\begin{aligned} gP_x \in \mathcal{F}\ell_x(\gamma) &\iff \mathrm{Ad}(g^{-1})\gamma \in \mathfrak{p}_x \\ &\iff c \cdot_d \mathrm{Ad}(g^{-1})\gamma \in \mathfrak{p}_x && \text{because } \mathfrak{p}_x \text{ is stable under } - \cdot_d - \\ &\iff \mathrm{Ad}(c \cdot_d g^{-1})(c \cdot_d \gamma) \in \mathfrak{p}_x \\ &\iff \mathrm{Ad}(c \cdot_d g^{-1})\gamma \in \mathfrak{p}_x \\ &\iff (c \cdot_d g)P_x \in \mathcal{F}\ell_x(\gamma) && \text{because } (c \cdot_d g)^{-1} = c \cdot_d g^{-1}. \end{aligned}$$

8.6. Note that $\mathfrak{g}(F)_{\frac{d}{n}\rho^\vee, \kappa}$ is the eigenspace of $- \cdot_d -$ of weight $2n\kappa$. Indeed,

$$\langle \alpha^\vee, \tfrac{d}{n}\rho^\vee \rangle + k = \kappa \qquad \iff \qquad c \cdot_d \varpi^k \xi_\alpha = c^{2n\kappa}\varpi^k \xi_\alpha.$$

For any integer ℓ, let $e_\ell \in \mathfrak{g}$ be the sum of the elements ξ_α such that $\langle \alpha, \rho^\vee \rangle = \ell$. For all k and ℓ, we see that $\varpi^k e_\ell \in \mathfrak{g}(F)_{\frac{d}{n}\rho^\vee, \frac{\ell d}{n} + k}$. In particular,

$$\gamma_d := e_1 + \varpi^d e_{1-n}$$

is an eigenvector of $- \cdot_d -$ of weight $2d$.

8.7. We parametrize the fixed points of the affine Springer fiber over γ_d as follows:

$$\begin{aligned} (\mathcal{F}\ell_x(\gamma_d))^{\mathbf{G}_m} &= \{\dot{w}P_x \in \mathcal{F}\ell_x \mid \mathrm{Ad}(\dot{w}^{-1})\gamma \in \mathfrak{p}_x\} \\ &= \{\dot{w}P_x \in \mathcal{F}\ell_x \mid \gamma \in \mathfrak{p}_{w \cdot x}\} \\ &= \left\{ \varpi^{w \cdot x} P_x \in \mathcal{F}\ell_x \;\middle|\; \begin{array}{l} \langle \alpha, w \cdot x \rangle \geq 0 \text{ when } \langle \alpha, \rho^\vee \rangle = 1, \\ \langle \alpha, w \cdot x \rangle \geq -d \text{ when } \langle \alpha, \rho^\vee \rangle = 1 - n \end{array} \right\} \\ &\simeq \left\{ \varpi^{y} P_x \in \mathcal{F}\ell_x \;\middle|\; \begin{array}{l} \langle \alpha, y \rangle \geq 0 \text{ when } \langle \alpha, \rho^\vee \rangle = 1, \\ \langle \alpha, y \rangle \leq d \text{ when } \langle \alpha, \rho^\vee \rangle = n - 1 \end{array} \right\}. \end{aligned}$$

Let $D_d(x)$ be the set of $y \in W^{\mathrm{aff}} \cdot x$ satisfying the inequalities in the last expression.

8.8. Recall that if $\gamma = \mathrm{Ad}(h)\gamma'$, then left multiplication by h is an isomorphism of ind-schemes from $\mathcal{F}\ell_x(\gamma')$ onto $\mathcal{F}\ell_x(\gamma)$.

If, moreover, γ' is an eigenvector under $- \cdot_d -$, then this isomorphism transports the induced $\mathbf{G}_m$-action on $\mathcal{F}\ell_x(\gamma')$ to a conjugate action on $\mathcal{F}\ell_x(\gamma)$. Explicitly, the latter is induced by the $\mathbf{G}_m$-action on LG defined by

$$c \cdot_{d,h} g := h(c \cdot_d (h^{-1}g)). \tag{8.1}$$

We deduce that gP_x is fixed under $- \cdot_{d,h} -$ if and only if $gP_x = hg'P_x$ for some g' such that $g'P_x$ is fixed under $- \cdot_d -$.

We are most interested in the case where $h = \varpi^\mu$ for some $\mu \in X^\vee$. Setting

$$\begin{aligned} \gamma_{d,\mu} &= \mathrm{Ad}(\varpi^\mu)\gamma_d, \\ D_{d,\mu}(x) &= D_d(x) + \mu, \end{aligned}$$

we conclude that

$$\begin{aligned} \mathcal{F}\ell_x(\gamma_{d,\mu})^{\mathbf{G}_m} &= \varpi^\mu(\mathcal{F}\ell_x(\gamma_d)^{\mathbf{G}_m}) \\ &= \{\varpi^y P_x \mid y \in D_{d,\mu}(x)\}, \end{aligned}$$

where $\mathcal{F}\ell_x(\gamma_d)^{\mathbf{G}_m}$, *resp.* $\mathcal{F}\ell_x(\gamma_{d,\mu})^{\mathbf{G}_m}$, is defined using $- \cdot_d -$, *resp.* $- \cdot_{d,\varpi^\mu} -$. Also note that here, (8.1) simplifies to $c \cdot_{d,h} g = c^{-\mu}(c \cdot_d g)$.

8.2. Quot Schemes of Curve Singularities

8.9. Now we take $G = \mathrm{GL}_n$ and T to be the subgroup of diagonal matrices. Here we can fix identifications $X^\vee = \mathbf{Z}^n$ and $W = S_n$. Writing δ_i for the ith standard basis vector of $\mathbf{Z}^n$, we can set $\alpha_i^\vee = \delta_i - \delta_{i+1}$ for $i = 1, \ldots, n-1$. Let

$$\begin{aligned} X^\vee_{\geq 0} &= \mathbf{Z}^n_{\geq 0}, \\ W^{\mathrm{aff}}_{\geq 0} &= W \ltimes X^\vee_{\geq 0}. \end{aligned}$$

Let $\mathcal{F}\ell_{x,\geq 0}$ be the union of the loci $P_0\varpi^\mu P_x/P_x \subseteq \mathcal{F}\ell_x$ where $\mu \in X^\vee_{\geq 0}$. That is,

$$\begin{aligned} \mathcal{F}\ell_{x,\geq 0} &= \coprod_{[w] \in W^{\mathrm{aff}}_{\geq 0}/W_x} I_d \dot{w} P_x/P_x \\ &= \coprod_{[\mu] \in X^\vee_{\geq 0}/W_x} P_0\varpi^\mu P_x/P_x. \end{aligned}$$

(Here, note that $P_0 = G(\mathcal{O})$.) For all $\gamma \in \mathfrak{g}(F)$, let

$$\mathcal{F}\ell_{x,\geq 0}(\gamma) = \mathcal{F}\ell_x(\gamma) \cap \mathcal{F}\ell_{x,\geq 0}.$$

In what follows, we will show how ind-schemes of the form $\mathcal{F}\ell_{0,\geq 0}(\gamma_{d,\mu})$ can be interpreted as Quot schemes of plane curve singularities with $\mathbf{G}_m$-actions.

8.10. Let $\mathfrak{g}(F)$ act on F^n by right multiplication. For an arbitrary element $\gamma \in \gamma(F)$, let $\mathcal{M}(\gamma)$ be the ind-scheme over $\mathbf{F}$ that, at the level of points, parametrizes $\mathcal{O}[\gamma]$-submodules $M \subseteq F^n$ that are projective over $\mathcal{O}$ and satisfy $M \otimes F \simeq F^n$. Then there is an isomorphism

$$\begin{array}{rcl} \mathcal{F}\ell_0(\gamma) & \xrightarrow{\sim} & \mathcal{M}(\gamma), \\ gP_0 & \mapsto & \mathcal{O}^n \cdot g^{-1}. \end{array}$$

We write $[M]$ for the $\mathbf{F}$-point of $\mathcal{M}(\gamma)$ corresponding to an $\mathcal{O}[\gamma]$-submodule $M \subseteq F[\gamma]$. Let $\mathcal{M}_{\geq 0}(\gamma) \subseteq \mathcal{M}(\gamma)$ be the sub-ind-scheme defined at the level of points by

$$\mathcal{M}_{\geq 0}(\gamma) = \{[M] \in \mathcal{M}(\gamma) \mid M \subseteq \mathcal{O}^n\}.$$

The preceding isomorphism restricts to an isomorphism $\mathcal{F}\ell_{0,\geq 0}(\gamma) \xrightarrow{\sim} \mathcal{M}_{\geq 0}(\gamma)$.

8.11. Now suppose that $\gamma = \mathrm{Ad}(h)\sigma_a = h\sigma_a h^{-1}$ for some $a \in F^n$ and $h \in G(F)$, where σ_a is the companion matrix

$$\sigma_a = \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ -a_n & -a_{n-1} & \cdots & -a_1 \end{pmatrix}.$$

Let e_i be the ith standard basis vector of F^n. At the same time, let

$$p_a(T) = T^n + a_1 T^{n-1} + \cdots + a_{n-1} T + a_n \in F[T],$$

and let γ act on $F[T]/p_a(T)$ by T. Then, by Cayley–Hamilton, there is a γ-equivariant isomorphism of F-vector spaces

$$\begin{array}{rcl} F^n & \xrightarrow{\sim} & F[T]/p_a(T) \\ e_i \cdot h^{-1} & \mapsto & T^{i-1}. \end{array} \tag{8.2}$$

Writing $h = (h_{i,j})_{i,j}$, we see that (8.2) sends

$$e_i \mapsto f_h^{(i)} := \sum_{j=1}^{n} h_{i,j} T^{j-1}.$$

Let $M(\gamma, h)$ be the $\mathcal{O}$-submodule of $F[\gamma]$ spanned by the elements $f_h^{(i)}$, and let $\mathcal{Q}(\gamma, h)$ be the Quot scheme whose points parametrize $\mathcal{O}[\gamma]$-submodules of $M(\gamma, h)$ of finite colength. Then (8.2) induces an isomorphism

$$\mathcal{M}_{\geq 0}(\gamma) \xrightarrow{\sim} \mathcal{Q}(\gamma, h).$$

8.12. We now study the induced isomorphism

$$\mathcal{F}\ell_{0,\geq 0}(\gamma) \xrightarrow{\sim} \mathcal{Q}(\gamma, h) \tag{8.3}$$

in the special case where

$$\begin{aligned}(a_0, \dots, a_{n-1}, a_n) &= (0, \dots, 0, \varpi^d),\\ h &= \varpi^\mu\end{aligned}$$

for some positive integer d coprime to n and $\mu \in X^\vee$. Thus $\gamma = \gamma_{d,\mu}$.

Note that the general case where $h \in T(F)$ can be reduced to the case where $h = \varpi^\mu$ for some μ, since the $T(F)$-action on $\mathcal{F}\ell_0 = G(F)/G(\mathcal{O})$ factors through $T(F)/T(\mathcal{O}) \simeq \varpi^{X^\vee}$.

Let $\mu = \sum_i \mu_i \delta_i$ be the expansion of μ under $X^\vee = \mathbf{Z}^n$. For the above choice of a and h, we see that (8.2) sends

$$e_i \mapsto f^{(i)}_{\varpi^\mu} = \varpi^{\mu_i} T^{i-1}$$

for all i, and therefore sends

$$e_i \cdot g^{-1} \mapsto f^{(i)}_{g^{-1}\varpi^\mu} := \sum_{j=1}^n g^{-1}_{i,j} \varpi^{\mu_j} T^{j-1}$$

for all $g \in G(F)$, where we have written $g^{-1} = (g^{-1}_{i,j})_{i,j}$.

Recall that $\mathcal{F}\ell_0(\gamma)$ is stable under the $\mathbf{G}_m$-action $- \cdot_{d,h} -$ coming from (8.1). The action further restricts to $\mathcal{F}\ell_{0,\geq 0}(\gamma)$, as as we can check by bootstrapping the $h = 1$ case. The isomorphism (8.3) transports $- \cdot_{d,h} -$ to a $\mathbf{G}_m$-action on $\mathcal{Q}(\gamma, h)$. We now describe the latter.

Lemma 8.1. *Let $\gamma = \gamma_{d,\mu}$ and $h = \varpi^\mu$. Let $\mathbf{G}_m$ act on $F[T]/(T^n - \varpi^d)$, hence on $M(\gamma, h)$, according to*

$$\begin{aligned} c \cdot \varpi &= c^{2n} \varpi,\\ c \cdot T &= c^{2d} T.\end{aligned}$$

Then (8.3) transports the $\mathbf{G}_m$-action $- \cdot_{d,h} -$ on $\mathcal{F}\ell_{0,\geq 0}(\gamma)$ to the $\mathbf{G}_m$-action on $\mathcal{Q}(\gamma, h)$ induced by the action on $M(\gamma, h)$ above.

Proof. First, observe that if $g' = c \cdot_{d,h} g = c^{-2n\mu}(c \cdot_d g)$, then $(g')^{-1} = (c \cdot_d g^{-1})c^{2n\mu}$. We deduce that

$$g'(\varpi)^{-1}_{i,j} = c^{2d(j-i)+2n\mu_j} g(c^{2n}\varpi)^{-1}_{i,j},$$

from which we get

$$
\begin{aligned}
e_i \cdot g'(\varpi)^{-1} \mapsto & \sum_{j=1}^{n} g'(\varpi)^{-1}_{i,j} \varpi^{\mu_j} T^{j-1} \\
&= \sum_{j=1}^{n} c^{2d(j-i)+2n\mu_j} g(c^{2n}\varpi)^{-1}_{i,j} \varpi^{\mu_j} T^{j-1} \\
&= c^{-2d(i-1)} \sum_{j=1}^{n} g(c^{2n}\varpi)^{-1}_{i,j} (c^{2n}\varpi)^{\mu_j} (c^{2d}T)^{j-1} \\
&= c^{-2d(i-1)} (c \cdot f^{(i)}_{g^{-1}\varpi^{\mu}}).
\end{aligned}
$$

Since $c^{-2d(i-1)}$ is just a nonzero scalar, we conclude that the $\mathcal{O}[\gamma]$-submodule of $M(\gamma, h)$ corresponding to $(c \cdot_{d,h} g)P_0 = g'P_0 \in \mathcal{F}\ell_{0,\geq 0}(\gamma)$ is the one generated by the elements $c \cdot f^{(i)}_{g^{-1}\varpi^{\mu}}$, as needed. $\square$

8.3. Cells

8.13. Henceforth, we set $M_{d,\mu} = M(\gamma_{d,\mu}, \varpi^{\mu})$ and $\mathcal{Q}_{d,\mu} = \mathcal{Q}(\gamma_{d,\mu}, \varpi^{\mu})$. It will be convenient to change coordinates via the isomorphism of $\mathbf{F}$-algebras:

$$
\begin{array}{rcl}
F[T]/(T^n - \varpi^d) & \xrightarrow{\sim} & \mathbf{F}[\varrho^n, \varrho^d], \\
\varpi & \mapsto & \varrho^n, \\
T & \mapsto & \varrho^d.
\end{array}
$$

Under this isomorphism, the $\mathbf{G}_m$-action on $F[T]/(T^n - \varpi^d)$ from above corresponds to the $\mathbf{G}_m$-action on $\mathbf{F}[\varrho^n, \varrho^d]$ given by $c \cdot \varrho = c\varrho$. Moreover, $f^{(i)}_{\varpi^{\mu}} = \varpi^{\mu_i} T^{i-1}$ corresponds to $\varrho^{a_i(\mu)}$, where

$$a_i(\mu) = n\mu_i + d(i-1) \qquad \text{for } 1 \leq i \leq n.$$

Observe that at the same time,

$$
\gamma_{d,\mu} = \begin{pmatrix}
0 & \varpi^{b_1} & & \\
 & 0 & \ddots & \\
 & & \ddots & \varpi^{b_{n-1}} \\
\varpi^{b_0} & & & 0
\end{pmatrix}
$$

where the integers $b_i = b_i(\mu)$ are defined by

$$
b_i(\mu) = \begin{cases} \mu_i - \mu_{i+1} & i = 1, \ldots, n-1, \\ d + \mu_n - \mu_1 & i = 0. \end{cases}
$$

Thus, the tuple $a(\mu)$ determines the tuple $b(\mu)$ via the identity

$$n b_i(\mu) = a_i(\mu) + d - a_{i+1}(\mu),$$

where we set $a_0 := a_n$, *i.e.*, interpreting the index i as a residue modulo n.

8.14. To summarize: In the new coordinate ϱ, we have

$$M_{d,\mu} = \mathcal{O}\langle \varrho^{a_1(\mu)}, \ldots, \varrho^{a_n(\mu)} \rangle.$$

We write $[N]$ for the $\mathbf{F}$-point of $\mathcal{Q}_{d,\mu}$ corresponding to an $\mathbf{F}[\varrho^n, \varrho^d]$-submodule $N \subseteq M_{d,\mu}$. Such a submodule is stable under the $\mathbf{G}_m$-action on $M_{d,\mu}$ if and only if it can be generated by a set of monomials in ϱ. In this case, under the isomorphism $\mathcal{Q}_{d,\mu} \simeq \mathcal{F}\ell_{0,\geq 0}(\gamma_{d,\mu})$, the point $[N]$ corresponds to a point of $\mathcal{F}\ell_{0,\geq 0}(\gamma_{d,\mu})$ fixed under the $\mathbf{G}_m$-action (8.1), hence to a point of the form

$$\varpi^\nu P_0 \in \mathcal{F}\ell_{0,\geq 0}(\gamma_{d,\mu})^{\mathbf{G}_m} \qquad \text{for some } \nu \in D_{d,\mu}(0) \cap X^\vee_{\geq 0}.$$

In this case, we set $N_\nu = N$. Writing

$$a_i(\mu + \nu) = n(\mu_i + \nu_i) + d(i-1),$$

similarly to before, we can check that

$$N_\nu = \mathcal{O}\langle \varrho^{a_1(\mu+\nu)}, \ldots, \varrho^{a_n(\mu+\nu)} \rangle.$$

8.15. To each point $\nu \in D_{d,\mu}(0) \cap X^\vee_{\geq 0}$, we assign two numerical invariants, in a way that depends on the pair (d, μ). Namely, let

$$\begin{aligned} \mathsf{a}_{d,\mu}(\nu) &= \dim_{\mathbf{F}}(M_{d,\mu}/N_\nu), \\ \mathsf{d}_{d,\mu}(\nu) &= \dim_{\mathbf{F}}(\mathcal{Q}_{d,\mu}(\nu)), \end{aligned}$$

where, in the second formula,

$$\mathcal{Q}_{d,\mu}(\nu) = \{[N] \in \mathcal{Q}_{d,\mu} \mid c \cdot [N] \to [N_\nu] \text{ as } c \to 0\}.$$

Both of these invariants admit formulas that are explicitly combinatorial. First, since $\varpi = \varrho^n$, we see that

$$\mathsf{a}_{d,\mu}(\nu) = \nu_1 + \cdots + \nu_n.$$

Second, by modifying a formula of Piontkowski,

$$\mathsf{d}_{d,\mu}(\nu) = \sum_{i=1}^{n} \mathsf{gap}_{d,\mu,i}(\nu),$$

where we set

$$\begin{aligned} \mathsf{gap}_{d,\mu,i}(\nu) &= |\{a_i(\mu+\nu) \leq a < a_i(\mu+\nu) + d \mid \varrho^a \in M_{d,\mu} - N_\nu\}| \\ &= |\{0 \leq j < d \mid \varrho^{a_i(\mu+\nu)+j} \in M_{d,\mu} - N_\nu\}|. \end{aligned}$$

To rewrite $\mathsf{gap}_{d,\mu,i}$ solely in terms of d, μ, i, ν, let

$$\sigma_d : \{1, 2, \ldots, n\} \times \mathbf{Z} \to \{1, 2, \ldots, n\}$$

be the map defined by

$$d\sigma_d(i, j) \equiv di + j \pmod{n}.$$

This gives $a_i(\mu + \nu) + j \equiv d(i - 1) + j \equiv a_{\sigma_d(i,j)}(\mu) \pmod{n}$, from which

$$\begin{aligned} \mathsf{gap}_{d,\mu,i}(\nu) &= |\{0 \leq j < d \mid a_{\sigma_d(i,j)}(\mu) \leq a_i(\mu + \nu) + j < a_{\sigma_d(i,j)}(\mu + \nu)\}| \\ &= |\{0 \leq j < d \mid 0 \leq a_i(\mu + \nu) - a_{\sigma_d(i,j)}(\mu) + j < n\nu_{\sigma_d(i,j)}\}|. \end{aligned}$$

Above, $a_i(\mu + \nu) - a_{\sigma_d(i,j)}(\mu) + j$ is a multiple of n, explicitly given by

$$\begin{aligned} &a_i(\mu + \nu) - a_{\sigma_d(i,j)}(\mu) + j \\ &= n(\mu_i + \nu_i) + d(i - 1) + j - n(\mu_{\sigma_d(i,j)} + \nu_{\sigma_d(i,j)}) + d(\sigma_d(i, j) - 1) \\ &= n\langle \alpha_{i,\sigma_d(i,j)}, \mu + \nu \rangle + d(i - \sigma_d(i, j)) + j, \end{aligned}$$

where in general, $\alpha_{i,k} : \mathbf{Z}^n \to \mathbf{Z}$ is the root $\alpha_{i,k}(x_1, \ldots, x_n) = x_i - x_k$.