2.

We rely on `2302_08`, `2303_06`, `2303_09`.

2.1. Throughout, $\mathbf{F}$ is an algebraically closed field of characteristic zero or $p > 2$. Let $F = \mathbf{F}(\!(\varpi)\!)$ and $\mathcal{O} = \mathbf{F}[\![\varpi]\!]$.

2.2. Let G be a connected reductive algebraic group over $\mathbf{F}$ with Lie algebra $\mathfrak{g}$. Fix a maximal torus $T \subseteq G$ and a Borel subgroup of G containing T. These data define a root datum $(X, \mathfrak{R}, X^\vee, \mathfrak{R}^\vee)$ and a system of simple roots $\{\alpha_1, \dots, \alpha_r\} \subseteq \mathfrak{R}$. Let W be the Weyl group of this root datum, and let $\widetilde{W} = W \ltimes X^\vee$. We set $\rho^\vee = \frac{1}{2}\sum_i \alpha_i^\vee \in \frac{1}{2}X^\vee$, so that $\langle \alpha_i, \rho^\vee \rangle = 1$ for all i.

2.3. For all $x \in X^\vee \otimes \mathbf{R}$, let $P_x \subseteq LG$ be the corresponding parahoric subgroup and $\mathfrak{p}_x$ its Lie algebra. Note that $P_x(\mathbf{F})$ contains $T(\mathcal{O})$. Writing

$$\begin{aligned} \mathfrak{g}(F)_{x,s} &= \bigoplus_{\substack{(\alpha,k) \in \mathfrak{R} \times \mathbf{Z} \\ \langle \alpha, x \rangle + k = s}} \varpi^k \mathfrak{g}_\alpha(\mathbf{F}) \oplus \begin{cases} \varpi^s \mathfrak{t}(\mathbf{F}) & s \in \mathbf{Z}, \\ 0 & \text{else,} \end{cases} \\ \mathfrak{g}(F)_{x,\geq r} &= \widehat{\bigoplus_{s \geq r}} \mathfrak{g}(F)_{x,s}, \end{aligned}$$

we know that $\mathfrak{p}_x(\mathbf{F}) = \mathfrak{g}(F)_{x,\geq 0}$.

Let $\mathcal{F}\ell_x = P_x \backslash LG$, the affine flag variety of G of type x. Note that we are defining $\mathcal{F}\ell_x$ in terms of right cosets, not left cosets, contrary to convention. For any $\gamma \in \mathfrak{g}(F)$, the affine Springer fiber over γ of type x is

$$\mathcal{F}\ell_x^\gamma = \{P_x g \in \mathcal{F}\ell_x \mid \gamma \in \mathfrak{p}_x \cdot_{\mathrm{Ad}} g\}.$$

2.4. Let W_x be the Weyl group of the Levi quotient of P_x. For all $x, y \in X^\vee \otimes \mathbf{R}$, we have a Bruhat-type decomposition

$$\mathcal{F}\ell_x = \coprod_{[w] \in W_x \backslash \widetilde{W} / W_y} \mathcal{F}\ell_{x,y}(w), \quad \text{where } \mathcal{F}\ell_{x,y}(w) = P_x \backslash (P_x w P_y).$$

Above, we have used the identification $\widetilde{W} = N_{G(F)}(T(\mathcal{O}))/T(\mathcal{O})$. Let

$$\mathcal{F}\ell_{x,y}^\gamma(w) = \mathcal{F}\ell_x^\gamma \cap \mathcal{F}\ell_{x,y}(w).$$

Consider the following sets of affine roots:

$$\begin{aligned} \mathfrak{R}^{\mathrm{aff}}_{\geq 0}(y, s) &= \{(\alpha, k) \in \mathfrak{R} \times \mathbf{Z} \mid 0 \leq \langle \alpha, y \rangle + k < s\}, \\ \mathfrak{R}^{\mathrm{aff}}_{< 0}(z) &= \{(\alpha, k) \in \mathfrak{R} \times \mathbf{Z} \mid \langle \alpha, z \rangle + k < 0\}. \end{aligned}$$

The following formulas are implied by [GKM, §4.3–4.8].

Lemma 2.1 (GKM). *We have*

$$\begin{aligned}\dim \mathcal{F}\ell_{x,y}(w) &= \dim \mathfrak{p}_y/(\mathfrak{p}_y \cap \mathfrak{p}_{x\cdot w})\\ &= |\mathfrak{R}^{\mathrm{aff}}_{\geq 0}(y,\infty) \cap \mathfrak{R}^{\mathrm{aff}}_{<0}(x\cdot w)|.\end{aligned}$$

If γ is a regular semisimple element of $\mathfrak{g}(F)_{x,s}$, and $\mathcal{F}\ell^{\gamma}_{x,y}(w) \neq \emptyset$, then we have

$$\begin{aligned}\dim \mathcal{F}\ell^{\gamma}_{x,y}(w) &= \dim \mathfrak{p}_y/(\mathfrak{p}_y \cap \mathfrak{p}_{x\cdot w}) - \dim \mathfrak{g}_{y,\geq s}/(\mathfrak{g}_{y,\geq s} \cap \mathfrak{p}_{x\cdot w})\\ &= |\mathfrak{R}^{\mathrm{aff}}_{\geq 0}(y,s) \cap \mathfrak{R}^{\mathrm{aff}}_{<0}(x\cdot w)|.\end{aligned}$$

Proof. As noted in [GKM, §4.3], the first step is to reduce to the case where w is the identity of W^{aff}, using the fact that the isomorphism $\mathcal{F}\ell_x \xrightarrow{\sim} \mathcal{F}\ell_{x\cdot w}$ that sends $P_x g \mapsto P_{x\cdot w} w^{-1} g$ restricts to isomorphisms

$$\begin{aligned}\mathcal{F}\ell_{x,y}(w) = P_x\backslash(P_x w P_y) &\xrightarrow{\sim} P_{x\cdot w}\backslash(P_{x\cdot w} P_y) = \mathcal{F}\ell_{x\cdot w,y}(1),\\ \mathcal{F}\ell^{\gamma}_x &\xrightarrow{\sim} \mathcal{F}\ell^{\gamma}_{x\cdot w},\end{aligned}$$

the latter because $\gamma \in \mathfrak{p}_x \cdot_{\mathrm{Ad}} g$ means $\gamma \in \mathfrak{p}_{x\cdot w} \cdot_{\mathrm{Ad}} \dot{w}^{-1} g$ for any lift $\dot{w}$ of w.

Henceforth, take $w = 1$. The image of $\mathcal{F}\ell^{\gamma}_{x,y}(1)$ under the isomorphism $\mathcal{F}\ell_{x,y}(1) = P_x\backslash(P_x P_y) \xrightarrow{\sim} (P_y \cap P_x)\backslash P_y$ is

$$Z^{\gamma}_{x,y} := \{(P_x \cap P_y)g \in (P_y \cap P_x)\backslash P_y \mid \gamma \in \mathfrak{p}_x \cdot_{\mathrm{Ad}} g\}.$$

Let $E_{x,y} \to (P_y \cap P_x)\backslash P_y$ be the vector bundle whose fiber over $(P_x \cap P_y)g$ is given by $\mathfrak{g}_{y,\geq s}/(\mathfrak{g}_{y,\geq s} \cap \mathfrak{p}_x \cdot_{\mathrm{Ad}} g)$. Then γ defines a section of $E_{x,y}$, whose zero locus is $Z^{\gamma}_{x,y}$. Since γ is regular semsimple, it is transverse to the zero section, giving $Z^{\gamma}_{x,y} = \dim \mathfrak{p}_y/(\mathfrak{p}_y \cap \mathfrak{p}_x) - \dim \mathfrak{g}_{y,\geq s}/(\mathfrak{g}_{y,\geq s} \cap \mathfrak{p}_x)$. □

2.5. Fix $0 < n < p$. Let $G = \mathrm{GL}_n$ and let T be the subgroup of diagonal matrices. We fix identifications $X^\vee = \mathbf{Z}^n$ and $W = S_n$. For $1 \leq i, j < n$ with $i \neq j$, let $\alpha_{i,j} \in \mathfrak{R}$ be defined by $\alpha_{i,j}(\xi) = \xi_i - \xi_j$, and set $\alpha_i = \alpha_{i,i+1}$.

We will write the lattice part of the extended affine Weyl group multiplicatively: $\widetilde{W} = S_n \ltimes \varpi^{\mathbf{Z}^n}$. For any nonzero integer d coprime to n, we set $I_{(d)} = P_{\frac{d}{n}\rho^\vee}$. This is an Iwahori subgroup of G; it is the standard Iwahori when $d = 1$.

Recall that in this setting, the positive part of $\mathcal{F}\ell_0 = P_0\backslash LG$ is defined by

$$\mathcal{F}\ell_0^+ = \coprod_{x \in \mathbf{Z}^n_{\geq 0}} P_0\backslash(P_0 \varpi^x I_{(1)}).$$

Noting that $P_0 = \coprod_{w \in S_n} I_{(1)} w I_{(1)}$,