



# Character Formulas from Lusztig Varieties and Affine Springer Fibers

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This talk is about...

- 1 Braids
- 2 Lusztig Varieties
- 3 Springer Fibers
- 4 Affine Springer Fibers

1 Braids The braid group  $Br_n =$

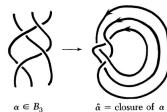
$$\left\langle \sigma_1, \dots, \sigma_{n-1}, \left| \begin{array}{l} \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \\ \sigma_i \sigma_j = \sigma_j \sigma_i \text{ when } |i - j| > 1 \end{array} \right. \right\rangle$$

appears in knot theory and representation theory.

$$\begin{array}{c} \sigma_i \\ \left[ \dots \right] \underset{i}{\times} \underset{i+1}{\left[ \dots \right]} \end{array}$$

A *link* is a collection of circles (tamely) embedded in  $\mathbf{R}^3$ . Knot theory is about isotopy invariants of links.

(Alexander) Every link is the *closure* of some braid.



Let  $G = \mathrm{SL}_n$  and  $B$  its upper-triangular subgroup.

Let  $R(q) = \{\mathbf{Z}\text{-valued functions on } G(\mathbf{F}_q)/B(\mathbf{F}_q)\}.$

(Iwahori) There is a surjective homomorphism

$$\mathbf{Z}Br_n \twoheadrightarrow H_n(q) := \mathrm{End}_{G(\mathbf{F}_q)}(R(q)).$$

To describe it, recall the Bruhat decomposition

$$G = \bigsqcup_{w \in S_n} B\dot{w}B.$$

Let  $h_w \curvearrowright R(q)$  be the *Hecke operator*

$$h_w(\mathbf{1}_{xB(\mathbf{F}_q)}) = \sum_{y^{-1}x \in B\dot{w}B} \mathbf{1}_{yB(\mathbf{F}_q)}.$$

Iwahori sends  $\sigma_i \mapsto h_{s_i}$ , where  $s_i = (i, i+1).$

The *Hecke algebra*

$$H_n := \frac{\mathbf{Z}[\mathbf{q}]Br_n}{\langle \sigma_i^2 - (\mathbf{q} - 1)\sigma_i - \mathbf{q} \rangle}$$

is a generic version:  $H_n|_{\mathbf{q} \rightarrow q} = H_n(q).$

Jones–Ocneanu used traces  $H_n \xrightarrow{\mu_n} \mathbf{Q}(\mathbf{q})[a^{\pm 1}]$  to construct the HOMFLYPT *link invariant*.

If  $L = \hat{\beta}$  for some  $n$  and  $\beta \in Br_n$ , then

$$\mathrm{HOMFLYPT}(\hat{\beta}) = (-a)^{e(\beta)} \mu_n(\beta),$$

where  $e : Br_n \rightarrow \mathbf{Z}$  is the *writhe* map  $\sigma_i \mapsto 1.$

Surprisingly, special values of HOMFLYPT are famous polynomials in combinatorics:  $\mathbf{q}$ -Catalan numbers,  $\mathbf{q}$ -Kirkman numbers, *etc*.

2 Lusztig Varieties We can geometrize Iwahori.

Fix a *positive* braid  $\beta = \sigma_{i_1} \cdots \sigma_{i_\ell}$ .

(Deligne) The variety

$$O(\beta) = \left\{ (g_0 B, g_1 B, \dots, g_\ell B) \left| \begin{array}{l} g_{j-1}^{-1} g_j \in B \dot{s}_{i_j} B \\ \text{for } j = 1, \dots, \ell \end{array} \right. \right\}$$

only depends on  $\beta$ , not  $(i_1, i_2, \dots, i_\ell)$ , up to isomorphisms that fix  $g_0 B$  and  $g_\ell B$ .

In fact, if we fix  $\bar{g}_0, \bar{g}_\ell$  such that  $\bar{g}_0^{-1} \bar{g}_\ell \in B \dot{s} B$ , then

$$\left| \left\{ \bar{g} B \in O(\beta)(\mathbf{F}_q) \left| \begin{array}{l} g_0 B = \bar{g}_0 B, \\ g_\ell B = \bar{g}_\ell B \end{array} \right. \right\} \right|$$

is the coefficient of  $h_w$  when we expand  $\beta$  in the Hecke operator basis of  $H_n(q)$ .

For any  $x \in G(\mathbf{F}_q)$ , form the *braid Lusztig variety*

$$\mathcal{B}(\beta)_x = \{\bar{g} B \in O(\beta) \mid g_\ell B = x g_0 B\}.$$

(Shende–Treumann–Zaslow) Up to a monomial in  $q$ ,

$$\frac{|\mathcal{B}(\beta)_1(\mathbf{F}_q)|}{|\mathrm{PGL}_n(\mathbf{F}_q)|}$$

is the “highest”  $a$ -degree of  $\mathrm{HOMFLYPT}(\hat{\beta})$  at  $\mathbf{q} \rightarrow q$ .

**Example** Let  $n = 2$  and  $\beta = \sigma_1^3 \in Br_2$ .

$$\begin{aligned} O(\beta) &\simeq \{\bar{g} \in (\mathbf{P}^1)^4 \mid g_0 \neq g_1 \neq g_2 \neq g_3\}, \\ \mathcal{B}(\beta)_1 &\simeq \{\bar{g} \in (\mathbf{P}^1)^3 \mid g_1 \neq g_2 \neq g_3\}. \end{aligned}$$

$\mathrm{PGL}_2$  acts simply transitively on the latter.

Indeed,  $\mathrm{HOMFLYPT}(\hat{\sigma}_1^3) = a^2(\mathbf{q} + \mathbf{q}^{-1}) - a^4$ .

### 3 Springer Fibers      How to access other $a$ -degrees?

One way uses Springer theory. Observe that

$$\mathcal{B}_x := \mathcal{B}(1)_x = \{gB \mid gB = xgB\}$$

is the usual Springer fiber over  $x$ , whose cohomology defines a character of  $S_n$ : namely,

$$Q_x(w) := \sum_i q^{i \operatorname{tr}(w \mid H^{2i}(\mathcal{B}_x))}.$$

Most interesting over the unipotent variety  $\mathcal{U} \subseteq G$ .

Thm 1 (T)    Let

$$Q_\beta(w) = \frac{1}{|\operatorname{PGL}_n(\mathbf{F}_q)|} \sum_{u \in \mathcal{U}(\mathbf{F}_q)} |\mathcal{B}(\beta)_u(\mathbf{F}_q)| Q_u(w).$$

Then  $(\chi_{(n-k, 1, \dots, 1)}, Q_\beta)_{S_n}$  sees the  $k$ th  $a$ -degree.

Example    Again, let  $n = 2$  and  $\beta = \sigma_1^3 \in Br_2$ .

$$|\mathcal{B}(\beta)_u| = \begin{cases} |\operatorname{PGL}_2| & u = 1, \\ q^3 & u \neq 1, \end{cases}$$

$$Q_u = \begin{cases} 1 + q \operatorname{sgn} & u = 1, \\ 1 & u \neq 1. \end{cases}$$

Moreover,  $\operatorname{PGL}_2(\mathbf{F}_q) \curvearrowright \mathcal{U}(\mathbf{F}_q) - \{1\}$  transitively, with stabilizer of size  $q$ .

$$\begin{aligned} Q_\beta &= \frac{|\operatorname{PGL}_2|}{|\operatorname{PGL}_2|} \cdot (1 + q \operatorname{sgn}) + \frac{q^3}{q} \cdot 1 \\ &= 1 + q^2 + q \operatorname{sgn}. \end{aligned}$$

Thm 2 (T)    The cohomology of  $\mathcal{U}(\beta) \times_{\mathcal{U}} \mathcal{U}(1)$  sees finer invariants of  $\hat{\beta}$ , where

$$\mathcal{U}(\beta) = \{(u, \vec{g}B) \mid u \in \mathcal{U}, \vec{g}B \in \mathcal{B}(\beta)_u\}.$$

The *full twist*  $\pi = (\sigma_1 \cdots \sigma_{n-1})^n$  generates  $Z(Br_n)$ .



Thm 3 (T) Suppose  $\beta^m = \pi^d$  for some  $d, m > 0$ .

Then up to a monomial,  $Q_\beta(w)$  equals

$$\frac{\text{sgn}(w)}{\det(1 - qw \mid \mathfrak{h})} \sum_{\lambda \vdash n} q^{c(\lambda)d/m} D_\lambda(e^{2\pi i d/m}) \chi_\lambda(w)$$

where:

- $\mathfrak{h}$  is the *reflection representation*.
- $c(\lambda)$  is the sum of *contents* of  $\lambda$ .
- $D_\lambda(t) = K_{\lambda, (1^n)}(t)$  is the *fake degree* of  $\lambda$ .

Proof uses the character theory of  $S_n$  and  $H_n$ .

Thm 3 generalizes to any reductive  $G$ .

Replace  $S_n$  with the Weyl group  $W$ .

Replace  $c$  with  $c(\chi) = \sum_{t \text{ refl.}} \frac{\chi(t)}{\chi(1)}$  and fake degrees with *generic degrees*:

$$D_\chi(t) \in \mathbb{Q}[t] \quad \text{such that} \quad R(q) = \bigoplus_{\chi \in \text{Irr}(W)} \chi_q^{\oplus D_\chi(q)}.$$

When  $m = n$  and  $\gcd(d, n) = 1$ , the formula simplifies:

$$(\text{monomial}) \cdot \left. \frac{\det(1 - q^d w \mid \mathfrak{h})}{\det(1 - qw \mid \mathfrak{h})} \right\} =: \Pi_q^{(d)}.$$

$\Pi_q^{(d)}$  is the character of a *rational parking space*.

$(\text{triv}, \Pi_q^{(d)})_W$  is a *rational  $q$ -Catalan number*.

#### 4 Affine Springer Fibers      Now work over $\mathbf{C}$ .

Rational parking spaces form modules over *rational Cherednik algebras* = *rational DAHAs*:

$$\mathfrak{H}_W = \frac{\mathbf{C}W \ltimes (\mathrm{Sym}(\mathfrak{h}) \otimes \mathrm{Sym}(\mathfrak{h}^\vee))}{\langle \text{relations} \rangle}.$$

finite Springer	affine Springer
$G$	$G((z))$
$G/B$	$G((z))/I$
$W$	$\widetilde{W} = W \ltimes X^\vee$
$\mathbf{C}W \ltimes \mathrm{Sym}(\mathfrak{h})$	$\mathbf{C}\widetilde{W} \ltimes \mathrm{Sym}(\mathfrak{h})$ or $\mathfrak{H}_W$

Above:

- $G((z))$  is the loop group  $G((z))(R) := G(R((z)))$ .
- $I$  is the preimage of  $B$  in  $G[[z]]$ .
- $X^\vee$  is the cocharacter lattice of  $B$ .

Henceforth, we consider Springer fibers over the Lie algebras  $\mathfrak{g}$  and  $\mathfrak{g}((z))$ , not the groups.

$$x : \quad \mathcal{B}_x = \{gB \in G/B \mid g^{-1}xg \in \mathfrak{b}\},$$

$$\gamma = \gamma(z) : \quad \mathcal{B}_\gamma^{\mathrm{aff}} = \{gI \in G((z))/I \mid g^{-1}\gamma g \in \mathfrak{I}\}.$$

The table hides some key differences:

In the **finite** case,  $\mathcal{B}_x$  is **interesting** for  $x$  nilpotent, and eas(ier) for  $x$  regular semisimple.

In the **affine** case,  $\mathcal{B}_\gamma^{\mathrm{aff}}$  is *terribly infinite* for  $\gamma = \gamma(z)$  nilpotent, but **interesting** for  $\gamma(z)$  regular semisimple.

**Example** If  $G = \mathrm{SL}_2$  and  $\gamma(z) = \begin{pmatrix} 0 & 1 \\ z^3 & 0 \end{pmatrix}$ , then

$$\mathcal{B}_\gamma^{\mathrm{aff}} \simeq \mathbf{P}^1 \sqcup_{\mathrm{pt}} \mathbf{P}^1.$$

Fixing  $\nu = d/m > 0$  in lowest terms,  $\mathbf{C}^\times \curvearrowright \mathfrak{g}((z))$ :

$$c \cdot_\nu \gamma(z) = c^{2d\rho^\vee} \gamma(c^{2m} z) c^{-2d\rho^\vee},$$

where  $2\rho^\vee = \sum_{\alpha \in \Phi^+} \alpha^\vee$ .

Let  $\mathfrak{g}((z))_{\nu,k}$  be the **weight- $2k$  eigenspace**.

**Lemma** If  $\gamma$  is an **eigenvector** for  $\cdot_\nu$ , then the induced action on  $G((z))/I$  fixes  $\mathcal{B}_\gamma^{\text{aff}}$ .

**Lemma**  $\mathfrak{g}((z))_{\nu,0}$  is the Lie algebra of a connected reductive group  $L_\nu$ . Moreover,

$$(G((z))/I)^{\mathbf{C}^\times} = \bigsqcup_{w \in W_\nu \setminus \widetilde{W}} L_\nu w I / I,$$

where  $W_\nu$  is the Weyl group of  $L_\nu$ .

Henceforth,  $\gamma \in \mathfrak{g}((z))_{\nu,d}$ .

By Springer,  $\widetilde{W} \curvearrowright H_c^*(\mathcal{B}_\gamma^{\text{aff}}), H_{c,\mathbf{C}^\times}^*(\mathcal{B}_\gamma^{\text{aff}})$ .

(Sommers) If  $m$  is the Coxeter number of  $W$ , then:

- $L_\nu$  is a torus, so  $(\mathcal{B}_\gamma^{\text{aff}})^{\mathbf{C}^\times} \hookrightarrow \widetilde{W}$ .
- Writing  $H_{\mathbf{C}^\times}^*(\text{pt}) = \mathbf{C}[\epsilon]$ , we have

$$H_c^*(\mathcal{B}_\gamma^{\text{aff}}) \simeq H_{c,\mathbf{C}^\times}^*(\mathcal{B}_\gamma^{\text{aff}})|_{\epsilon \rightarrow 1} \simeq H^0((\mathcal{B}_\gamma^{\text{aff}})^{\mathbf{C}^\times}).$$

- For  $w \in W$ , we have

$$\text{tr}(w \mid H_c^*(\mathcal{B}_\gamma^{\text{aff}})) = \lim_{q \rightarrow 1} \frac{\det(1 - q^d w \mid \mathfrak{h})}{\det(1 - qw \mid \mathfrak{h})}.$$

**Example** In the previous  $\text{SL}_2$  example,  $\gamma \in \mathfrak{g}((z))_{\nu,3}$ .

Recall  $\mathcal{B}_\gamma^{\text{aff}} = \mathbf{P}^1 \sqcup_{\text{pt}} \mathbf{P}^1$ . It turns out  $|(\mathcal{B}_\gamma^{\text{aff}})^{\mathbf{C}^\times}| = 3$ .



(Goresky–Kottwitz–MacPherson) For general  $\nu$ ,

$$(\mathcal{B}_\gamma^{\text{aff}})^{\mathbf{C}^\times} = \bigsqcup_{w \in W_\nu \setminus \widetilde{W}} \text{Hess}_{\gamma,w},$$

a disjoint union of *partial Hessenberg varieties*

$$\text{Hess}_{\gamma,w} = \{gP_{\nu,w} \in L_\nu/P_{\nu,w} \mid g^{-1}\gamma g \in \mathfrak{P}_{\nu,w}\},$$

where  $P_{\nu,w} := L_\nu \cap \dot{w}I\dot{w}^{-1}$  and  $\mathfrak{P}_{\nu,w} = \text{Lie}(P_{\nu,w})$ .

They are smooth. They can be empty.

If  $\text{Hess}_{\gamma,w} \neq \emptyset$ , then its codim in  $L_\nu/P_{\nu,w}$  is

$$\left| \left\{ \begin{array}{l} \text{hyperplanes } H \\ \text{in } X^\vee \otimes \mathbf{R} \text{ between} \\ \nu\rho^\vee \text{ and } w \cdot \frac{1}{n}\rho^\vee \end{array} \left| \begin{array}{l} H(\xi) = \langle \alpha, \xi \rangle + k, \\ \langle \alpha, \nu\rho^\vee \rangle = \nu, \\ \alpha \in \Phi, k \in \mathbf{Z} \end{array} \right. \right\} \right|.$$

Proof uses *Moy–Prasad theory*.

Conj (T) For general  $\nu$ , the representation

$$W \curvearrowright H_{c, \mathbf{C}^\times}^*(\mathcal{B}_\gamma^{\text{aff}})|_{\epsilon \rightarrow 1}$$

contains a summand whose character is the  $q \rightarrow 1$  limit of our earlier formula:

$$\frac{\text{sgn}(w)}{\det(1 - qw \mid \mathfrak{h})} \sum_{\chi \in \text{Irr}(W)} q^{c(\chi)\nu} D_\chi(e^{2\pi i\nu}) \chi(w).$$

Dream For certain choices  $\gamma \leftrightarrow \beta$ ,

$$\mathcal{B}_\gamma^{\text{aff}} \quad \text{and} \quad [(\mathcal{U}(\beta) \times_{\mathcal{U}} \tilde{\mathcal{U}})/G]$$

have the “same” Springer theory.

Thank you for listening.