MATH 250: TOPOLOGY I PROBLEM SET #1

FALL 2025

Due Wednesday, September 3. Please attempt all of the problems. Six of them will be graded. You may consult books, papers, and websites as long as you cite all sources and write up your solutions in your own words. **Last update: 8/28**.

Problem 1. Let $f : X \to Y$ be an arbitrary map between sets.

(1) Let $\{X_\alpha\}_\alpha$ be an arbitrary collection of subsets of X. Show that

$$f\left(\bigcup_\alpha X_\alpha\right) = \bigcup_\alpha f(X_\alpha) \quad \text{and} \quad f\left(\bigcap_\alpha X_\alpha\right) \subseteq \bigcap_\alpha f(X_\alpha).$$

(2) In the setup of (1), give an example where

$$f\left(\bigcap_\alpha X_\alpha\right) \neq \bigcap_\alpha f(X_\alpha).$$

(3) Let $\{Y_\beta\}_\beta$ be an arbitrary collection of subsets of Y. Show that

$$f^{-1}\left(\bigcup_\beta Y_\beta\right) = \bigcup_\beta f^{-1}(Y_\beta) \quad \text{and} \quad f^{-1}\left(\bigcap_\beta Y_\beta\right) = \bigcap_\beta f^{-1}(Y_\beta).$$

Solution. (1) To show that $f\left(\bigcup_\alpha X_\alpha\right) = \bigcup_\alpha f(X_\alpha)$, we must show the following claims: (a) $f\left(\bigcup_\alpha X_\alpha\right) \subseteq \bigcup_\alpha f(X_\alpha)$, and (b) $f\left(\bigcup_\alpha X_\alpha\right) \supseteq \bigcup_\alpha f(X_\alpha)$.

To show (a): If $x \in \bigcup_\alpha X_\alpha$, then $x \in X_{\alpha'}$ for some specific α' among the indices α. Thus, $f(x) \in f(X_{\alpha'})$. Thus, $f(x) \in \bigcup_\alpha f(X_\alpha)$.

To show (b): If $y \in \bigcup_\alpha f(X_\alpha)$, then $y \in f(X_{\alpha'})$ for some index α'. Thus, $y = f(x)$ for some $x \in X_{\alpha'}$. Note that x is also an element of $\bigcup_\alpha X_\alpha$. Therefore, $y \in f(\bigcup_\alpha X_\alpha)$.

To show that $f(\bigcap_\alpha X_\alpha) \subseteq \bigcap_\alpha X_\alpha$: If $x \in \bigcap_\alpha X_\alpha$, then $x \in X_\alpha$ for all indices α. Thus, $f(x) \in f(X_\alpha)$ for all α. Thus, $f(x) \in \bigcap_\alpha f(X_\alpha)$.

(2) Let $X = \{a, b\}$ and $Y = \{c\}$. Let $f : X \to Y$ be defined by $f(a) = f(b) = c$. Now let $X_1 = \{a\}$ and $X_2 = \{b\}$. Then

$$f(X_1 \cap X_2) = f(\emptyset) = \emptyset, \quad \text{but} \quad f(X_1) \cap f(X_2) = \{c\}.$$

Therefore, $f(X_1 \cap X_2) \neq f(X_1) \cap f(X_2)$. (Other solutions are possible.)

(3) To show that $f^{-1}\left(\bigcup_\beta Y_\beta\right) = \bigcup_\beta f^{-1}(Y_\beta)$, we must show: (a) $f^{-1}\left(\bigcup_\beta Y_\beta\right) \subseteq \bigcup_\beta f^{-1}(Y_\beta)$, and (b) $f^{-1}\left(\bigcup_\beta Y_\beta\right) \supseteq \bigcup_\beta f^{-1}(Y_\beta)$.

To show (a): If $x \in f^{-1}\left(\bigcup_\beta Y_\beta\right)$, then $f(x) \in \bigcup_\beta Y_\beta$. Thus, $f(x) \in Y_{\beta'}$ for some specific β' among the indices β. Thus, $x \in f^{-1}(Y_{\beta'})$. Thus, $x \in \bigcup_\beta f^{-1}(Y_\beta)$.

To show (b): If $x \in \bigcup_\beta f^{-1}(Y_\beta)$, then $x \in f^{-1}(Y_{\beta'})$ for some index β'. Thus, $f(x) \in Y_{\beta'}$. Thus, $f(x) \in \bigcup_\beta Y_\beta$. Thus, $x \in f^{-1}\left(\bigcup_\beta Y_\beta\right)$.

To show that $f^{-1}\left(\bigcap_\beta Y_\beta\right) = \bigcap_\beta f^{-1}(Y_\beta)$: The proof is analogous to the preceding one, once we replace $\bigcup_\beta$ with $\bigcap_\beta$ everywhere, and replace the phrase "for some [specific] index β" with the phrase "for all indices β" in two places. $\square$

Problem 2 (Munkres 83, #1)**.** Let X be a topological space, and let A be a subset of X. Suppose that for each $x \in A$, there is an open set U containing x such that $U \subseteq A$. Show that A is also open.

Solution. For each $x \in A$, the hypotheses of the problem give us some U_x open in X such that $x \in U_x \subseteq A$. We claim that $A = \bigcup_{x \in A} U_x$. This will show that A is a union of open sets, and hence, is also open.

Indeed: If $a \in A$, then $a \in U_a \subseteq \bigcup_{x \in A} U_x$, which shows that $A \subseteq \bigcup_{x \in A} U_x$. Conversely, $U_x \subseteq A$ for all $x \in A$, so we must have $\bigcup_{x \in A} U_x \subseteq A$ as well. $\square$

Problem 3 (Munkres 83, #3)**.** Let X be any set. Show that the collection

$$\{\emptyset\} \cup \{U \subseteq X \mid X - U \text{ countable}\}$$

always forms a topology on X. Does

$$\{\emptyset, X\} \cup \{U \subseteq X \mid X - U \text{ is infinite}\}$$

always form a topology on X?

Solution. Let's write $\mathcal{T}$ to denote the first collection. To show that $\mathcal{T}$ forms a topology:

(1) By definition, $\emptyset \in \mathcal{T}$. Since $X - X = \emptyset$, which is countable, $X \in \mathcal{T}$.

(2) Suppose that $\{U_\alpha\}_\alpha$ is a subset of $\mathcal{T}$. To show that $\bigcup_\alpha U_\alpha \in \mathcal{T}$, we consider two cases:
- If $U_\alpha = \emptyset$ for all indices α, then $\bigcup_\alpha U_\alpha = \emptyset \in \mathcal{T}$.
- If instead, $U_\alpha \neq \emptyset$ for some index α, then we require $X - U_\alpha$ to be countable for that choice of α. Thus, $X - \bigcup_\alpha U_\alpha$, being a subset of the previous set, is also countable. Thus, $\bigcup_\alpha U_\alpha \in \mathcal{T}$.

(3) Suppose that $\{U_\alpha\}_\alpha$ is a *finite* subset of $\mathcal{T}$. To show that $\bigcap_\alpha U_\alpha \in \mathcal{T}$, we again consider two cases:

- If $U_\alpha = \emptyset$ for some index α, then $\bigcap_\alpha U_\alpha = \emptyset \in \mathcal{T}$.
- If instead, $U_\alpha \neq \emptyset$ for all indices α, then $X - U_\alpha$ is countable for all α. Thus, $X - \bigcap_\alpha U_\alpha = \bigcup_\alpha (X - U_\alpha)$ is a *finite* union of countable sets, and hence, still countable. Thus, $\bigcap_\alpha U_\alpha \in \mathcal{T}$.

Let's write $\mathcal{S}$ to denote the second collection in the problem. We will give an example where it does not form a topology. Namely take $X = \mathbf{R}$. Let Y and Z be the subsets of positive and negative real numbers, respectively. Then $Y, Z \in \mathcal{S}$, because $X - Y$ and $X - Z$ are infinite. But $Y \cup Z \notin \mathcal{S}$, because $X - (Y \cup Z) = \{0\}$, which is finite. (There are solutions using other choices of infinite X.) □

Problem 4 (Munkres 83, (c))**.** Suppose that $X = \{a, b, c\}$ and

$$\mathcal{T}_1 = \{\emptyset, X, \{a\}, \{a, b\}\} \quad \text{and} \quad \mathcal{T}_2 = \{\emptyset, X, \{a\}, \{b, c\}\}.$$

Find the smallest topology containing $\mathcal{T}_1$ and $\mathcal{T}_2$ as subsets, and the largest topology that is contained in both $\mathcal{T}_1$ and $\mathcal{T}_2$ as a subset.

Solution. Any topology containing $\mathcal{T}_1$ and $\mathcal{T}_2$ as subsets must at least contain the set

$$\{\emptyset, X, \{a\}, \{a, b\}, \{b, c\}\}.$$

But in order to satisfy the definition of a topology, it must be closed under unions and finite intersections of its elements. (Note that the word "finite" is unnecessary here, because X has finitely many subsets in the first place.) If we consider all possible unions or intersections of the sets in the collection above, the only new set that we can form is $\{b\} = \{a, b\} \cap \{b, c\}$. And once we append this set to our collection, we cannot make any new ones. So the smallest topology containing $\mathcal{T}_1$ and $\mathcal{T}_2$ is

$$\{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{b, c\}\}.$$

Next, any topology contained in both $\mathcal{T}_1$ and $\mathcal{T}_2$ as a subset must also be a subset of

$$\{\emptyset, X, \{a\}\}.$$

But we see that the set above satisfies all the axioms of a topology, so it is, itself, the largest topology contained in both $\mathcal{T}_1$ and $\mathcal{T}_2$. □

Problem 5 (Munkres 83, #8(a))**.** Using Munkres Lemma 13.2, show that the countable collection

$$\mathcal{B} = \{(a, b) \mid a < b \text{ and } a, b \text{ are rational}\}$$

forms a basis for the analytic topology on $\mathbf{R}$.

Solution. Observe that every element of $\mathcal{B}$ is analytically open: In other words, $\mathcal{B}$ is a subset of the analytic topology on $\mathbf{R}$. So by Munkres Lemma 13.2, it suffices to show that for any analytically open set U in $\mathbf{R}$, and element $x \in U$, we can find an element of $\mathcal{B}$—that is, an interval (a, b) with $a < b$ and a, b rational—such that

$$x \in (a, b) \subseteq U.$$

Since U is analytically open, we can at least find some $\delta > 0$ such that $B(x, \delta) = (x - \delta, x + \delta) \subseteq U$. Then we can find rational numbers a, b such that $x - \delta < a < x$ and $x < b < x + \delta$.[note] These numbers give us the desired interval (a, b). $\square$

[note] In more detail: Let us prove that for any real numbers $X < Y$, there is a rational strictly between X and Y. By the archimedean principle, *cf.* Munkres p. 33, there is some positive integer n such that $\frac{1}{n} < Y - X$. Thus $nY > nX + 1$. Again by the archimedean principle, there are integers greater than nX, so by the well-ordering principle, there is a minimal integer m greater than nX. Now $nX < m$, and by minimality, $m \leq nX + 1$. Therefore $nX < m < nY$, from which $X < \frac{m}{n} < Y$.

Problem 6. Let $a\mathbf{Z} + b = \{aq + b \mid q \in \mathbf{Z}\}$, for any integers a and b. Let

$$\mathcal{B} = \{a\mathbf{Z} + b \mid a, b \in \mathbf{Z} \text{ with } a \neq 0\}.$$

Show that $\mathcal{B}$ forms a basis for some topology on $\mathbf{Z}$. *Hint:* If $x \in a\mathbf{Z} + b$, then $a\mathbf{Z} + b = a\mathbf{Z} + x$.

Solution. We must check the axioms for a basis.

(1) To show that $\mathbf{Z}$ is the union of the elements of $\mathcal{B}$: The inclusion $\supseteq$ is clear because $a\mathbf{Z} + b \subseteq \mathbf{Z}$ for all $a, b \in \mathbf{Z}$. For the inclusion $\subseteq$, observe that by setting $a = 1$ and $b = 0$, we get $a\mathbf{Z} + b = 1\mathbf{Z} + 0 = \mathbf{Z}$.

(2) We claim that for any two elements of $\mathcal{B}$—say, $a\mathbf{Z} + b$ and $c\mathbf{Z} + d$, where $a, b, c, d \in \mathbf{Z}$ and $a, c \neq 0$—and any x in their intersection, we can find $m, n \in \mathbf{Z}$ with $m \neq 0$ such that

$$x \in m\mathbf{Z} + n \subseteq (a\mathbf{Z} + b) \cap (c\mathbf{Z} + d).$$

Using the hint, we have $a\mathbf{Z} + b = a\mathbf{Z} + x$ and $c\mathbf{Z} + d = c\mathbf{Z} + x$. Observe that $ac\mathbf{Z} \subseteq a\mathbf{Z}$ and $ac\mathbf{Z} \subseteq c\mathbf{Z}$, from which

$$ac\mathbf{Z} + x \subseteq (a\mathbf{Z} + x) \cap (c\mathbf{Z} + x) = (a\mathbf{Z} + b) \cap (c\mathbf{Z} + d).$$

Therefore, taking $m = ac$ and $n = x$ concludes the proof. (We could have also used, for instance, $\operatorname{lcm}(a, c)$ in place of ac above.) $\square$

Problem 7. Endow $\mathbf{R}$ with the analytic topology. Give an example of a <u>continuous</u>, <u>non-constant</u> map $f : \mathbf{R} \to \mathbf{R}$ and an open set $U \subseteq \mathbf{R}$ such that $f(U)$ is *not* open. *Hint:* There is a solution where f is a quadratic polynomial. You may assume that polynomial maps are continuous.

Solution. Let f be defined by $f(x) = x^2$. Then f is a polynomial map, hence continuous, and non-constant. Let U be $\mathbf{R}$ itself. We see that $f(U) = \{x^2 \mid x \in \mathbf{R}\}$ is the set of *nonnegative* real numbers. We claim that this set, which we will denote $\mathbf{R}_{\geq 0}$, is not open in the analytic topology. Indeed, it contains 0, but there is no $\delta > 0$ such that $B(0, \delta) = (-\delta, \delta) \subseteq \mathbf{R}_{\geq 0}$. (Other solutions may use other choices of f.) □

Problem 8. Let X, Y be topological spaces, and let $f : X \to Y$ be a continuous bijection. Show that if $f(U)$ is open in Y for every open set U in X, then f is a homeomorphism.

Solution. We must show that the inverse map $f^{-1} : Y \to X$ is continuous. That is, we must show that if $U \subseteq X$ is open, then its preimage

$$(f^{-1})^{-1}(U) = \{y \in Y \mid f^{-1}(y) \in U\}$$

is open in Y. But $f^{-1}(y) \in U$ if and only if $y \in f(U)$, so we have

$$(f^{-1})^{-1}(U) = f(U).$$

Since U is open in X, the hypotheses of the problem show that $f(U)$ is open in Y, as needed. □

Problem 9. Recall the notion of a *group* from the initial reading. Show that:

(1) $\mathbf{R}$ forms a group under the law of addition.
(2) $\mathbf{R}$ does not form a group under the law of multiplication.
(3) The set of positive real numbers $\mathbf{R}_+$ forms a group under multiplication.
(4) The set of positive integers $\mathbf{Z}_+$ does not form a group under multiplication.

Solution.
(1) Addition is associative, 0 is an identity element because $x + 0 = 0 + x = x$ for all $x \in \mathbf{R}$, and we have inverse elements because for all real x, the number $-x$ is real and satisfies $x + (-x) = (-x) + x = 0$.
(2) If $e \in \mathbf{R}$ is an identity element for multiplication, then $e = e \cdot 1 = 1$: That is, the only possible such identity element is the number 1. But there is no number $x \in \mathbf{R}$ such that $0 \cdot x = 1$. So we have found that 0 has no inverse in $\mathbf{R}$ under the multiplication law.
(3) Multiplication is associative, 1 is an identity element because $x \cdot 1 = 1 \cdot x = 1$ for all $x \in \mathbf{R}_+$, and we have inverse elements because for all positive real x, the number x^{-1} is also positive real and satisfies $x \cdot x^{-1} = x^{-1} \cdot x = 1$.
(4) The solution is similar to that in (2): If a is an integer larger than 1, then there is no positive integer b such that $a \cdot b = 1$. □

Problem 10. For part (3), recall or look up the notion of a *subgroup*.

(1) Show that for any set X, the set of bijections from X to itself forms a group under the law of composition (*i.e.*, $g \circ f$ defined by $(g \circ f)(x) = g(f(x))$). This group is usually denoted $\mathrm{Sym}(X)$.

(2) Give two elements $f, g \in \mathrm{Sym}(\{a, b, c\})$ such that $g \circ f \neq f \circ g$.
(3) Suppose that X is endowed with a topology. Show that the set of homeomorphisms from X to itself forms a subgroup of $\mathrm{Sym}(X)$. This subgroup is usually denoted $\mathrm{Homeo}(X)$.

Solution. (1) Composition of functions is associative. To show that the identity map $\mathrm{id} : X \to X$ defined by $\mathrm{id}(x) = x$ is an identity element for the operation $\circ$, observe that $\mathrm{id}(f(x)) = f(x) = f(\mathrm{id}(x))$ for all x, which means $\mathrm{id} \circ f = f = f \circ \mathrm{id}$.

Finally, we have inverse elements because any bijection f has an inverse map f^{-1} satisfying $f \circ f^{-1} = \mathrm{id} = f^{-1} \circ f$.[note]

(2) Let f be defined by

$$f(a) = b, \quad f(b) = a, \quad f(c) = c.$$

Let g be defined by

$$g(a) = a, \quad g(b) = c, \quad f(c) = b.$$

Then $g(f(a)) = c$ whereas $f(g(a)) = b$.

(3) It suffices to check that $\mathrm{Homeo}(X)$ contains the identity element for $\mathrm{Sym}(X)$, and that it is closed under $\circ$. Indeed, identity maps are always continuous, and compositions of continiuius maps are continuous, by parts (b) and (c) of Munkres Theorem 18.2. $\square$

[note] In more detail: Suppose that $f : X \to X$ is bijective. Then for all $y \in X$, the preimage $f^{-1}(y)$ is nonempty, by surjectivity of f, and contains at most one element, by injectivity of f, so altogether, $f^{-1}(y)$ contains exactly one element. So there is a well-defined map sending each $y \in X$ to the single element of $f^{-1}(y)$, which we will denote by $f^{-1} : X \to X$. We check that $f(f^{-1}(y)) = y$ and $f^{-1}(f(x)) = x$, so $f \circ f^{-1} = \mathrm{id} = f^{-1} \circ f$.