MATH 250: TOPOLOGY I PROBLEM SET #3

FALL 2025

Due Wednesday, October 1. Please attempt all of the problems. Six of them will be graded. You may consult books, papers, and websites as long as you cite all sources and write up your solutions in your own words.

Problem 1 (Munkres 128, #4(1))**.** Consider the product, uniform, and box topologies on $\mathbf{R}^\omega$. In which topologies are the following functions continuous?

$$f(t) = (t, 2t, 3t, \ldots), \qquad g(t) = (t, t, t, \ldots), \qquad h(t) = (t, \tfrac{1}{2}t, \tfrac{1}{3}t, \ldots).$$

Problem 2 (Munkres 128, #4(2))**.** Same setup as Problem 1. In which topologies do the following sequences converge?

$$\begin{aligned} (w_i)_i \text{ where } w_1 &= (1, 1, 1, 1, \ldots), \\ w_2 &= (0, 2, 2, 2, \ldots), \\ w_3 &= (0, 0, 3, 3, \ldots), \\ &\ldots \end{aligned} \qquad \begin{aligned} (x_i)_i \text{ where } x_1 &= (1, 1, 1, 1, \ldots), \\ x_2 &= (0, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \ldots) \\ x_3 &= (0, 0, \tfrac{1}{3}, \tfrac{1}{3}, \ldots), \\ &\ldots \end{aligned}$$

$$\begin{aligned} (y_i)_i \text{ where } y_1 &= (1, 0, 0, 0, \ldots), \\ y_2 &= (\tfrac{1}{2}, \tfrac{1}{2}, 0, 0, \ldots), \\ y_3 &= (\tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}, 0, \ldots), \\ &\ldots \end{aligned} \qquad \begin{aligned} (z_i)_i \text{ where } z_1 &= (1, 1, 0, 0, \ldots), \\ z_2 &= (\tfrac{1}{2}, \tfrac{1}{2}, 0, 0, \ldots) \\ z_3 &= (\tfrac{1}{3}, \tfrac{1}{3}, 0, 0, \ldots), \\ &\ldots \end{aligned}$$

Problem 3 (Munkres 118, #7)**.** Let $\mathbf{R}^\infty \subseteq \mathbf{R}^\omega$ be the subset of sequences $(a_i)_{i>0}$ such that $a_i \neq 0$ for only finitely many i. What is the closure of $\mathbf{R}^\infty \ldots$

(1) …in the box topology on $\mathbf{R}^\omega$?
(2) …in the product topology on $\mathbf{R}^\omega$?

Problem 4 (Munkres 118, #8)**.** Fix sequences $(a_1, a_2, \ldots), (b_1, b_2, \ldots) \in \mathbf{R}^\omega$ such that $a_i > 0$ for all i. Let $h : \mathbf{R}^\omega \to \mathbf{R}^\omega$ be defined by

$$h(x_1, x_2, \ldots) = (a_1 x_1 + b_1, a_2 x_2 + b_2, \ldots)$$

(1) Show that in the product topology, h is a self-homeomorphism of $\mathbf{R}^\omega$.
(2) What happens in the box topology?

Problem 5 (Munkres 127, #7)**.** Now consider the map h in Problem 4 in the uniform topology on $\mathbf{R}^\omega$. Under what conditions on $(a_i)_i$ and $(b_i)_i$ is $h \ldots$

(1) …continuous?
(2) …a homeomorphism?

Problem 6 (Munkres 92, #3)**.** Endow $[-1,1]$ with the analytic topology: *i.e.*, the subspace topology it inherits from analytic $\mathbf{R}$. Determine which of the following sets are open in $[-1,1]$, and which are open in $\mathbf{R}$.

$$A = \{x \mid \tfrac{1}{2} < |x| < 1\}, \qquad B = \{x \mid \tfrac{1}{2} < |x| \leq 1\},$$
$$C = \{x \mid \tfrac{1}{2} \leq |x| < 1\}, \qquad D = \{x \mid \tfrac{1}{2} \leq |x| \leq 1\},$$
$$E = \{x \mid 0 < |x| < 1 \text{ and } \tfrac{1}{x} \notin \mathbf{Z}_+]\}.$$

Problem 7 (Munkres 92, #6)**.** Show that the countable collection

$$\{(a,b) \times (c,d) \mid a < b \text{ and } c < d \text{ and } a,b,c,d \text{ are rational}\}$$

is a basis for $\mathbf{R}^2$. You may assume Problem 5 from Problem Set 1.

Problem 8 (Munkres 101, #11–13)**.** Show that:

(1) A product of two Hausdorff spaces is Hausdorff (in the product topology).
(2) A subspace of a Hausdorff space is Hausdorff (in the subspace topology).
(3) X is Hausdorff if and only if its *diagonal* $\Delta_X = \{(x,x) \mid x \in X\}$ is closed in (the product topology on) $X \times X$.

Problem 9 (Munkres 118, #6)**.** Let $(X_\alpha)_\alpha$ be an arbitrary collection of topological spaces, and let $x^{(1)}, x^{(2)}, \ldots$ be a sequence of points in $\prod_\alpha X_\alpha$. Observe that each point $x^{(i)}$ has the form

$$x^{(i)} = (x_\alpha^{(i)})_\alpha, \quad \text{where } x_\alpha^{(i)} \in X_\alpha \text{ for all } \alpha.$$

(1) Show that in the product topology, the sequence converges to a point $x = (x_\alpha)_\alpha$ if and only if, for all α, the sequence $x_\alpha^{(1)}, x_\alpha^{(2)}, \ldots$ converges to x_α.
(2) What happens if we replace the product topology with the box topology?

Problem 10 (Munkres 144, #2)**.** Let $p : X \to Y$ be a continuous map.

(1) Show that if there is a continuous map $f : Y \to X$ such that $p \circ f$ is the identity map on Y, then p is a quotient map.
(2) A *retraction* from X onto a subset A is a continuous map $r : X \to A$ such that $r(a) = a$ for all $a \in A$. Deduce from (1) that retractions are quotient maps.

Problem 11 (Munkres 152, #2)**.** Let $(A_n)_{n=1}^\infty$ be a sequence of connected subspaces of X, such that $A_n \cap A_{n+1} \neq \emptyset$ for all n. Show that $\bigcup_{n=1}^\infty A_n$ is connected.

Problem 12 (Munkres 152, #9)**.** Let X, Y be connected, and let $A \subseteq X$ and $B \subseteq Y$ be proper subsets. Show that

$$(X \times Y) - (A \times B)$$

is a connected subspace of $X \times Y$.

Problem 13 (Munkres 152, #11)**.** Let $p : X \to Y$ be a quotient map. Show that if Y is connected and each subspace $p^{-1}(y) \subseteq X$ is connected, then X is connected.

Problem 14 (Munkres 152, #5)**.** We say that X is *totally disconnected* if and only if its only nonempty connected subspaces are one-point sets.

(1) Show that if X is discrete, then X is totally disconnected.
(2) Show that the set of rational numbers $\mathbf{Q}$, as a subspace of (analytic) $\mathbf{R}$, is totally disconnected, but not discrete.