

Today (modified) Jordan Separation Thm:

Thm if $C \subset S^2$ is a simple closed curve
then $S^2 - C$ is not path connected

Pf assume that $S^2 - C$ is path connected

fix a homeo $C \simeq S^1$

let $a, b, A_1, A_2 \subset C$ correspond to
 $(1, 0), (-1, 0), \{y \geq 0\}, \{y \leq 0\} \subset S^1$

let $U_1 = S^2 - A_1$ and $U_2 = S^2 - A_2$ [draw]

then $U_1 \cap U_2 = S^2 - A_1 - A_2 = S^2 - C$
so it is path connected

but $U_1 = (U_1 \cap U_2) \cup A_2$, etc.
so U_1, U_2 are also path connected

finally, $U_1 \cup U_2 = S^2 - \{a, b\}$ [draw]

fix x in $U_1 \cap U_2 = S^2 - C$

Seifert–van Kampen gives a surjective hom

$\pi_1(U_1, x) * \pi_1(U_2, x)$ to $\pi_1(S^2 - \{a, b\}, x)$

induced by the inclusion maps $i_j : U_j$ to S^2
will show that $(i_1)_*, (i_2)_*$ are trivial hom's
but $\pi_1(S^2 - \{a, b\}) \simeq \pi_1(R^2 - \{(0, 0)\}) \simeq \mathbb{Z}$
so contradiction

enough to show that $(i_1)_*$ is trivial

pick a loop $y : [0, 1]$ to U_1 based at x
lift to cts $\beta : S^1$ to U_1 s.t. $\beta((1, 0)) = x$
get cts $i_1 \circ \beta : S^1$ to $S^2 - \{a, b\}$ [draw]

note that a, b both lie in A_1

Lem 1 suppose K is compact,
 $f : K$ to $S^2 - \{a, b\}$ is cts

if a, b lie in the same path comp of $S^2 - f(K)$,
then f is nulhomotopic

Lem 2 for any X and cts $f : S^1$ to X , TFAE:
1) f is nulhomotopic
2) f extends to cts map on the closed disk
3) f_* is trivial

[Lem 2 is the more memorable and important one]

Pf of Lem 1 fix a homeo $S^2 - \{b\} \simeq R^2$
s.t. a corresponds to $\mathbf{0} = (0, 0)$

then $f : K$ to $S^2 - \{a, b\}$ corresponds to
some $g : L$ to $R^2 - \{\mathbf{0}\}$
since L compact, have $r > 0$ s.t. L sub $B(\mathbf{0}, r)$

fix a path from a to b in $S^2 - K$,
it corresponds to a path from $\mathbf{0}$ “to infty” in $R^2 - L$

fix p in $R^2 - B(\mathbf{0}, r)$
fix a path α from $\mathbf{0}$ to p in $R^2 - L$ [draw]

let $h : L \times [0, 1] \rightarrow \mathbb{R}^2 - \{\mathbf{0}\}$ be def by

$$h(x, t) = g(x) - \alpha(t) \text{ [why well-defined?]}$$

let $H : L \times [0, 1] \rightarrow \mathbb{R}^2 - \{\mathbf{0}\}$ be def by

$$H(x, t) = tg(x) - p \text{ [why well-defined?]}$$

h is a homotopy from $g(x)$ to $g(x) - p$

H is a homotopy from a const map to $g(x) - p$

so g is nulhomotopic

so f is too