<u>Recall</u> given paths β, γ s.t. $\beta(1) = \gamma(0)$,

$\beta * \gamma$ is a new path $(\beta * \gamma)(s) = \beta(2s)$ $\quad s \leq 1/2$

$(\beta * \gamma)(s) = \gamma(2s - 1)$ $\quad s \geq 1/2$

$\gamma \sim_p \gamma'$ means there's a path homotopy from γ to γ'

Q suppose $\beta(1) = \gamma(0) = \gamma'(0)$ and $\gamma \sim_p \gamma'$
do we have $\beta * \gamma \sim_p \beta * \gamma'$?

Q suppose $\beta(1) = \beta'(1) = \gamma(0)$ and $\beta \sim_p \beta'$
do we have $\beta * \gamma \sim_p \beta' * \gamma$?

<u>Thm</u> if $\beta(1) = \beta'(1) = \gamma(0) = \gamma'(0)$
and $\beta \sim_p \beta'$ and $\gamma \sim_p \gamma'$,

then $\beta * \gamma \sim_p \beta' * \gamma'$

<u>Pf</u> pick a path homotopy h from β to β'
pick a path homotopy j from γ to γ'

then $h(1, t) = \beta(1) = \beta'(1) = \gamma(0) = \gamma'(0) = j(0, t)$

set $k(s, t) = h(2s, t)$ $\quad s \leq 1/2$
$k(s, t) = j(2s - 1, t)$ $\quad s \geq 1/2$

<u>Df</u> write $[\gamma]$ for the $\sim_p$ equiv class of γ

for any paths β, γ s.t. $\beta(1) = \gamma(0)$,
take $[\beta] * [\gamma]$ to be the equiv class $[\beta * \gamma]$

by thm, $*$ is a well-def operation on equiv classes

(Munkres §52) now focus on loops

if β, γ are loops in X at the same <u>basepoint</u> x
then $\beta * \gamma$ is also a loop at x

get a binary operation on equiv classes of loops:
$[\beta] * [\gamma] = [\beta * \gamma]$

Thm 1   if α, β, γ are paths s.t. $\alpha(1) = \beta(0)$
$\beta(1) = \gamma(0)$

then $(\alpha * \beta) * \gamma \sim_p \alpha * (\beta * \gamma)$

so $([\alpha] * [\beta]) * [\gamma] = [\alpha] * ([\beta] * [\gamma])$

<u>Thm 2</u> write $e_x : [0, 1]$ to X for the <u>constant</u> path $e_x(s) = x$

then $e_x * \gamma \sim_p \gamma$ for all paths γ starting at x

$\beta \sim_p \beta * e_x$ for all paths β ending at x

so $[e_x] * [\gamma] = [\gamma]$

$[\beta] * [e_x] = [\beta]$

<u>Thm 3</u> write $\bar{\gamma}(s) = \gamma(1 - s)$ for the <u>reverse</u> path

then $\gamma * \bar{\gamma} \sim_p e_x$ for γ starting at x

$\bar{\gamma} * \gamma \sim_p e_y$ for γ ending at y

so $[\gamma] * [\bar{\gamma}] = [e_x]$

$[\bar{\gamma}] * [\gamma] = [e_y]$

$\underline{\text{Pf of Thm 3}}$ want $[\gamma] * [\bar{\gamma}] = [e_x]$ for γ starting at x

need path homotopy $h : [0, 1] \times [0, 1]$ to X

s.t. for all s in $[0, 1]$,

$$h(s, 0) = e_x(s) = x$$
$$h(s, 1) = (\gamma * \bar{\gamma})(s)$$

for fixed t, the path $h(s, t)$ "freezes" at $\gamma(t)$, then rewinds

$h(s, t)$:

x to $\gamma(t)$	s in $[0, t/2]$
stay at $\gamma(t)$	s in $[t/2, 1 - t/2]$
$\gamma(t)$ back to x	s in $[1 - t/2, 1]$

$$h(s,t) = \begin{cases} \gamma(2s) & s \text{ in } [0, t/2] \\ \gamma(t) & s \text{ in } [t/2, 1 - t/2] \\ \gamma(2 - 2s) = \bar{\gamma}(2s) & s \text{ in } [1 - t/2, 1] \end{cases}$$

Cor   for loops in X based at a point x:

1) $([\alpha] * [\beta]) * [\gamma] = [\alpha] * ([\beta] * [\gamma])$
2) $[\gamma] * [e_x] = [\gamma] = [e_x] * [\gamma]$
3) $[\gamma] * [\bar{\gamma}] = [e_x] = [\bar{\gamma}] * [\gamma]$

Df   the *fundamental group* of X based at x is

$$\pi_1(X, x) = \{[\gamma] \mid \text{loops } \gamma \text{ in } X \text{ based at } x\}$$

under the operation * on $\sim_p$ equiv classes

Q   how much does $\pi_1(X, x)$ depend on X and x?

[Thm suppose $f : X$ to Y is cts

1) if γ, γ' are paths in X s.t. $\gamma \sim_p \gamma'$
then $f \circ \gamma, f \circ \gamma'$ are paths in Y s.t. $f \circ \gamma \sim_p f \circ \gamma'$

2) if β, γ are paths in X s.t. $\beta(1) = \gamma(0)$,
then $f \circ (\beta * \gamma) = (f \circ \beta) * (f \circ \gamma)$

Cor suppose $f : X$ to Y is cts and $f(x) = y$

1) well-def map $f_* : \pi_1(X, x)$ to $\pi_1(Y, y)$ s.t.
$f_*([\gamma]) = [f \circ \gamma]$

2) f_* is a group homomorphism:
$f_*([\beta] * [\gamma]) = f_*([\beta]) * f_*([\gamma])$]