

Last time defn of a metric $d : X \times X$ to $[0, \infty)$

for all x, y, z in X :

- 1) $d(x, y) = 0$ if and only if $x = y$
- 2) $d(x, y) = d(y, x)$
- 3) $d(x, y) + d(y, z) \geq d(x, z)$

Warmup do the following define a metric?
 $\eta(x, x) = 0$, and $\eta(x, y) = 1$ for $x \neq y$

1) and 2) easy

3) [how many cases to check?]

if $x = z$: $\eta(x, y) + \eta(y, z) \geq 0 = \eta(x, z)$

if $x \neq z$: either $y \neq x$ or $y \neq z$

so $\eta(x, y) + \eta(y, z) \geq 1 = \eta(x, z)$

what are the balls $B_\eta(x, \delta)$? [X , and $\{x\}$ for all x]

so this metric generates the discrete topology
so we call it the discrete metric

Ex [draw $B_e(\mathbf{0}, 1)$, $B_\rho(\mathbf{0}, 1)$, $B_e(\mathbf{0}, \sqrt{2})$]

euclidean metric:

$$e(x, y) = \sqrt{(x_1 - y_1)^2 + \dots + (x_n - y_n)^2}$$

square metric:

$$\rho(x, y) = \max\{|x_1 - y_1|, \dots, |x_n - y_n|\}$$

$$B_e(\mathbf{0}, 1) \subset B_\rho(\mathbf{0}, 1) \subset B_e(\mathbf{0}, \sqrt{2})$$

Lem for all x and δ ,
 1) $B_e(x, \delta) \text{ sub } B_\rho(x, \delta)$
 2) $B_\rho(x, \delta) \text{ sub } B_e(x, \sqrt{n} \cdot \delta)$

Pf of 2) want: $\rho(x, y) < \delta$ implies $e(x, y) < \sqrt{n} \cdot \delta$
 enough: $e(x, y) \leq \sqrt{n} \cdot \rho(x, y)$
 [note: reversed positions of e and ρ]

$$e(x, y) \leq \sqrt{(n \max_i \{(x_i - y_i)^2\})} = \sqrt{n} \cdot \rho(x, y)$$

Lem let d , resp. d' , be a metric on X ,
 generating a topology T , resp. T'

suppose that for all x in X and $\delta > 0$,
 we have $\delta' > 0$ s.t. $B_{d'}(x, \delta') \text{ sub } B_d(x, \delta)$
 then T' is finer than T

Pf pick U in T

want U in T' , i.e.,

for all x in U , some δ' s.t. $B_{d'}(x, \delta') \text{ sub } U$
 since U in T , have some δ s.t. $B_d(x, \delta) \text{ sub } U$
 thus some δ' s.t. $B_{d'}(x, \delta') \text{ sub } B_d(x, \delta)$

Thm Euclidean and square metrics
 induce the same topology on \mathbb{R}^n
 [the analytic topology]

Rem d, d' are called equivalent
 iff
 there are fixed $A, B > 0$ s.t. for all x, y ,
 $d(x, y) \leq A d'(x, y)$,
 $d'(x, y) \leq B d(x, y)$ [uniformly]

if two metrics are equivalent,
then they generate the same topology

converse is false: see PS2, #9–10

Q what about infinite-dim'l space?

$$R^\omega = \{(x_1, x_2, \dots) \mid x_i \in R \text{ for all } i\}$$

$$R^\infty = \{(x_1, x_2, \dots) \mid x_i \neq 0 \text{ for only fin. many } i\}$$

Euclidean and square metrics don't work on R^ω

Q do they still work on R^∞ ? [yes]

Q are there other metrics on R^ω ? [yes]
see PS2, #11

(Munkres §15, 19) given $\{X_i\}_{i \in I}$,

$$\prod_{i \in I} X_i$$

$$:= \{\text{sequences } (x_i)_{i \in I} \mid x_i \in X_i \text{ for all } i\}$$

Q what if $X_i = \emptyset$ for some i ?
[then $\prod_i X_i = \emptyset$]

Q is R^n of this form?
 R^ω ?
 R^∞ ?
[yes, yes, no]

Q if each X_i has a topology, do we get
a natural topology on $\prod_{i \in I} X_i$?

Df the box topology on $\prod_{i \in I} X_i$:
generated by the basis
 $\{\prod_i U_i \mid U_i \text{ is open in } X_i \text{ for all } i\}$

basis contains the collection of square balls

$$(a, b) \times (a, b) \times \dots \times (a, b)$$

Df the product topology on $\prod_{i \in I} X_i$:
generated by the basis
 $\{\prod_i U_i \mid U_i \text{ is open in } X_i \text{ for all } i, \\ U_i \neq X_i \text{ for fin many } i\}$

so finer than analytic top
conversely, prod's of analytic opens are analytic open

Rem if the indexing set I is finite
then box = product

Q take $I = \mathbb{Z}_+$ and $X_i = \mathbb{R}$ for all i
then $\prod_i X_i = \mathbb{R}^\omega$
do we have box = product here?

Q take $I = \{1, 2, \dots, n\}$ and $X_i = \mathbb{R}$ for all i
then $\prod_i X_i = \mathbb{R}^n$
what is the box/product topology here?

$(-1, 1) \times (-1, 1) \times \dots$ is box-open,
but not product-open

the box top can be finer than the product top