Warmup suppose T : F^2 to F^2 sends

v1 = $\begin{matrix}1\\1\end{matrix}$ to $\begin{matrix}2\\2\end{matrix}$ v2 = $\begin{matrix}-1\\1\end{matrix}$ to $\begin{matrix}1\\-1\end{matrix}$

matrix of T wrt std basis (e1, e2) of F^2?

1) matrix wrt (v1, v2): M = $\begin{matrix}2 & 0\\0 & -1\end{matrix}$

2) v1 = e1 + e2 B = $\begin{matrix}1 & -1\\1 & 1\end{matrix}$
v2 = –e1 + e2

3) det(B) = 2 [B^(-1) = $\begin{matrix}1/2 & 1/2\\-1/2 & 1/2\end{matrix}$]
adj(B) = $\begin{matrix}1 & 1\\-1 & 1\end{matrix}$

BMB^(-1) = (1/2) B M adj(B)
= (1/2) $\begin{matrix}2 & 1\\2 & -1\end{matrix}$ $\begin{matrix}1 & 1\\-1 & 1\end{matrix}$
= (1/2) $\begin{matrix}1 & 3\\3 & 1\end{matrix}$
= $\begin{matrix}1/2 & 3/2\\3/2 & 1/2\end{matrix}$ [works]

(Axler §5A) given a linear op T : V to V:

Df a subspace W sub V is T-stable iff
w in W implies Tw in W

Ex above: Fv1, Fv2 are T-stable lines
Fe1, Fe2 are not T-stable
[matrix wrt (v1, v2) easier to understand]

Ex for any linear op T : V to V,
{**0**} and V are are [trivially] T-stable

Ex suppose T : F^3 to F^3 has the matrix

$$\begin{matrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{matrix}$$ wrt some basis

nontrivial T-stable subspace? W = {(x, y, 0) | x, y}

if instead we have

$$\begin{matrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & * \end{matrix}$$

then U = {(0, 0, z) | z} also T-stable

here, V = W + U; in general:

a block-diag matrix for T with k blocks
corresponds to
a T-stable direct sum V = W_1 + … + W_k

best situation: each W_i is a line
[what means?]
in this case: T has a <u>diagonal</u> matrix

<u>Df</u> for any linear op T : V to V

an <u>eigenline</u> of T is a T-stable line [dim-1 sub.]
an <u>eigenvector</u> of T is v in V s.t. Fv is an eigenline
(which forces v ≠ **0**)
if Tv = λv, then we call λ the <u>eigenvalue</u> of the line

Ex $P : F^2$ to F^2 given by the matrix
$$\begin{pmatrix} 0 & 0 \\ 3 & 1 \end{pmatrix}$$
wrt the std basis

[does P have any eigenvectors?]

$$\begin{pmatrix} \lambda x \\ \lambda y \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 3x + y \end{pmatrix}$$

[take $\lambda \neq 0$:] $\{(0, y)\}$ eigenline with eigenvalue 1
[take $\lambda = 0$:] $\{(x, -3x)\}$ eigenline with eigenvalue 0

Ex $N : F^2$ to F^2 given by the matrix
$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$
wrt the std basis

$$\begin{pmatrix} \lambda x \\ \lambda y \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ x \end{pmatrix}$$

[regardless of λ:] $x = 0$
$\{(0, y)\}$ is the only eigenline with eigenvalue 0
[same regardless of whether $F = R$ or $F = C$]

Ex $H : R^2$ to R^2 given by the matrix
$$\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$
wrt the std basis

$$\begin{pmatrix} \lambda x \\ \lambda y \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x - y \\ x + y \end{pmatrix}$$

messy to solve…

notice: $(1/\sqrt{2})\, H = \begin{matrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{matrix}$

$= \begin{matrix} \cos(\pi/4) & -\sin(\pi/4) \\ \sin(\pi/4) & \cos(\pi/4) \end{matrix}$

so H is the composition of: rotate by $\pi/4$
scale by $\sqrt{2}$

no H-stable lines through **0**

compare to H' : C^2 to C^2 given by

$\begin{matrix} 1 & -1 \\ 1 & 1 \end{matrix}$

claim: $1 + i$ and $1 - i$ are eigenvalues

$(1 + i)x = x - y$
$(1 + i)y = x + y$
imply $ix = -y$ and $iy = x$

so $\{(x, ix)\}$ eigenline with eigenvalue $1 + i$
$\{(x, -ix)\}$ eigenline with eigenvalue $1 - i$

<u>Rem</u> C^2 is iso to the complex'n of R^2

$$\begin{matrix} (R^2)\_C & \xrightarrow{H\_C} & (R^2)\_C \\ = & & \\ C^2 & \xrightarrow[H']{} & C^2 \end{matrix}$$

Moral choice of R vs C affects eigenstuff

Thm suppose $F = C$
V is fin. dim. and not $\{\mathbf{0}\}$
then any linear op on V has an eigenline

key idea: plug linear op T into polynomials

p(z) = sum_k a_k z^k gives
p(T) = sum_k a_k T^k where T^k = kth iterate of T

note: constant term a_0 treated as a_0 id_V

Lem for any F and v in V:
some nonconst p(z) gives $p(T)\, v = \mathbf{0}$

Pf of Thm pick $v \neq \mathbf{0}$

using lemma, pick f(z) of minimal deg s.t.
f is nonconst and $f(T)\, v = \mathbf{0}$

by the fund. thm of algebra, f has a root λ: i.e.,

$f(z) = (z - \lambda)\, g(z)$ for (nonzero) g(z) in C[z]

now $(T - \lambda\, id_V)\, (g(T)\, v) = \mathbf{0}$
notice $T(g(T)v) = \lambda(g(T)v)$
so just need $g(T)\, v \neq \mathbf{0}$
if g nonconst, then done bc deg(g) < deg(f)
if g const, then done bc g nonzero and $v \neq \mathbf{0}$ □

[where did we use F = C? fund. thm of algebra]

Summary

if W sub V is stable under T : V to V
then T restricts to a lin op $T|_W$, easier to study

nicer when W is a line
nicest when V is a sum of eigenlines
i.e., T is a diagonalizable operator

over R T may have no eigenlines
over C T will have some eigenline,
but V need not be sum(eigenlines)

Rem the sum of eigenlines with eigenval λ
is called the λ-eigenspace of T