(Munkres §54)   wrap-up of:

<u>Thm</u>   let $p : E \to X$ be a covering, e in E

1) for any path $\gamma : [0, 1] \to X$ s.t.
      $p(e) = \gamma(0)$,
    a <u>unique</u> $\Gamma : [0, 1] \to E$ s.t.
      $\Gamma(0) = e$ and $\gamma = p \circ \Gamma$,
    which we call the <u>lift</u> of γ to E

2) for any homotopy $h : [0, 1]^2 \to X$ s.t.
      $p(e) = h(0, 0)$,
    a <u>unique</u> homotopy $H : [0, 1]^2 \to E$ s.t.
      $H(0, 0) = e$ and $h = p \circ H$

3) in 2), if h is a path homotopy, then so is H

last time, proved 1) via [what lemma?]
the Lebesgue number lemma

<u>Pf of 2)</u>   as last time:
      pick an open cover $\{U_\alpha\}_\alpha$ s.t.
      each U_α is evenly covered by p

by double application of Lebesgue, can find

$$0 = s_0 \quad < \quad s_1 \quad < \quad \ldots \quad < \quad s_n = 1$$
$$0 = t_0 \quad < \quad t_1 \quad < \quad \ldots \quad < \quad t_m = 1$$

s.t. $h : [0, 1]^2 \to X$ maps each rectangle
      $[s_i, s_{i + 1}] \times [t_j, t_{j + 1}]$ into a single U_α at a time

again, build H : [0, 1]² → E inductively:
    set H(0, 0) = e
    order the rectangles lexicographically
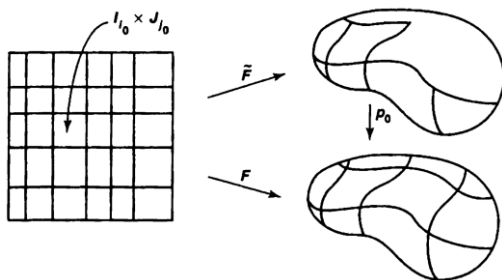   for a given (i, j),
        let $R_{i, j} = [s_i, s_{i + 1}] \times [t_j, t_{j + 1}]$
        let $A_{i, j}$ be the union of [0, 1] × {0},
            {0} × [0, 1], all rectangles prior to R

note that $A_{i, j}$ cap $R_{i, j}$ is a subset of $\partial A_{i, j}$
want to extend $H|_{A\_\{i, j\}}$ to $H|_{A\_\{i, j\} \, cup \, R\_\{i, j\}}$

similarly to last time:
    h maps $R_{i, j}$ into a single U_α,
    U_α is evenly covered by p

the key difference from last time:
we seek a lift of $h|_{R\_\{i, j\}}$ that extends
    the subspace $H(A_{i, j}$ cap $R_{i, j})$ of $p^{-1}(U_\alpha)$
    not just a single point $\Gamma(s_i)$ in $p^{-1}(U_\alpha)$

[what saves us?]
$H(A_{i, j}$ cap $R_{i, j})$ is connected
so inside $p^{-1}(U_\alpha)$, it must be contained in
    a single homeomorphic copy of U_α

now the rest of the proof is analogous to
the proof of 1) □

<u>Pf of 3)</u>   suppose h is a path homotopy between
                  two paths from x to y in X

then $H(0, t)$ in $p^{-1}(x)$ for all t in [0, 1]
but $p^{-1}(x)$ is discrete, since p is a covering
so $H(0, t)$ is constant for all t

similarly, $H(1, t)$ is constant for all t
so H is a path homotopy in E

<u>Cor</u>   if $\gamma_0 \sim_{path} \gamma_1$ in X and e in E s.t.
             $p(e) = \gamma_0(0) = \gamma_1(0)$,

           then γ_0, γ_1 have unique lifts Γ_0, Γ_1 in E
               starting at e
           and $\Gamma_0 \sim_{path} \Gamma_1$

<u>Applications</u>

<u>Cor 1</u>   if $p : E \to X$ is a covering and $p(e) = x$,
             then $p_* : \pi_1(E, e) \to \pi_1(X, x)$ is injective

<u>Pf</u>   let Γ_0, Γ_1 be loops in E based at e s.t.
             $p_*([\Gamma_0]) = p_*([\Gamma_1])$

by construction, $p_*(\Gamma_i) = [p \circ \Gamma_i]$
     and Γ_0, Γ_1 are lifts of $p \circ \Gamma_0, p \circ \Gamma_1$
so $[\Gamma_0] = [\Gamma_1]$

<u>Ex</u>   recall that for any integer n > 0
             there is an n-fold covering $p_n : S^1 \to S^1$

under $\pi_1(S^1) = \mathbb{Z}$, we have $im(p_{n, *}) = n\mathbb{Z}$ [draw]

each [γ] in $\pi_1(X, x)$ defines a permutation of $p^{-1}(x)$ as follows:

> each e in $p^{-1}(x)$ is the <u>start</u> of a unique lift of γ
> the permutation sends $e \mapsto e \cdot [γ]$, where
>> $e \cdot [γ]$ is the <u>end</u> of that lift
> [draw solenoid]

<u>Cor 2</u> (Lifting Correspondence)   given e in $p^{-1}(x)$:

a) $e \cdot [γ] = e$ if and only if [γ] in $p_*(\pi_1(E, e))$
    thus, an injective map

$$\varphi_e : p_*(\pi_1(E, e))\backslash\pi_1(X, x) \to p^{-1}(x)$$

    defined by $\varphi_e(p_*(\pi_1(E, e)) * [γ]) = e \cdot [γ]$

b)   if E is path-connected, then φ_e is bijective

c)   if E is path-connected and simply-connected,
    then φ_e is a bijection $\pi_1(X, x) \to p^{-1}(x)$

<u>Pf of a)</u>   [γ] = $p_*([Γ])$ for some [Γ] in $\pi_1(E, e))$
        iff the lift of γ to e is a loop (namely, Γ)
        iff $e \cdot [γ] = e$

<u>Pf of b)</u>   for any e' in $p^{-1}(x)$,
        we can pick a path Γ from e to e' in E
        let $γ = p \circ Γ$
        then Γ is the lift of γ to e, so $e \cdot [γ] = e'$

<u>Pf of c)</u>   immediate from b)

Ex   recall that $RP^2 = S^2/\sim$

             where $\sim$ is antipodal identification

S^2 is simply-conn & $S^2 \to RP^2$ is a 2-fold covering

so $|\pi_1(RP^2)| = 2$

so new proof that $\pi_1(RP^2) = Z/2Z$

Df   a pointed covering is a pair (p, e) s.t.

           $p : E \to X$ is a <u>path-conn.</u> covering

           e in E

if $(p : E \to X, e)$ and $(p' : E' \to X, e')$ are

    pointed coverings of the same X,

then a pointed equivalence from p to p' is

    a homeo $f : (E, e) \to (E', e')$ s.t. $p = p' \circ f$

for such f, we write $(E, e) \sim (E', e')$

we also see that f_* is an isomorphism

        $\pi_1(E, e) \to \pi_1(E', e')$

thus $p_*(\pi_1(E, e)) = p'_*(f_*(\pi_1(E, e))) = p'_*(E', e')$

[we've proven well-def'ness, but not bijectivity, in:]
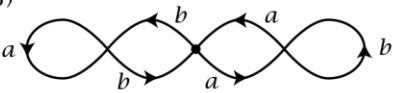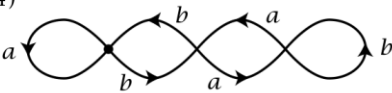
<u>Cor 3</u> (Galois Correspondence)
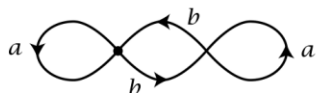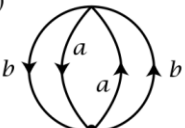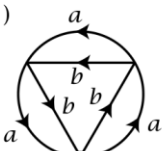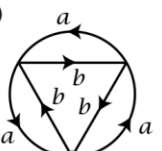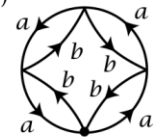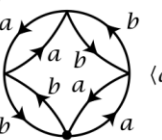
if X is conn. & <u>locally simply-conn.</u>,

then the map $(p : E \to X, e) \mapsto p_*(\pi_1(E, e))$ is

a bijection   {pointed coverings of X}/$\sim$

                $\to$ {subgroups of $\pi_1(X, x)$}

<u>Rem</u>   if E = E' but e ≠ e', then (E, e), (E', e')

           might not be equivalent

Ex   coverings of the figure-eight
          and corresponding subgroups of its π_1:

(1) $\langle a, b^2, bab^{-1} \rangle$	(2) $\langle a^2, b^2, ab \rangle$
(3) $\langle a^2, b^2, aba^{-1}, bab^{-1} \rangle$	(4) $\langle a, b^2, ba^2b^{-1}, baba^{-1}b^{-1} \rangle$
(5) $\langle a^3, b^3, ab^{-1}, b^{-1}a \rangle$	(6) $\langle a^3, b^3, ab, ba \rangle$
(7) $\langle a^4, b^4, ab, ba, a^2b^2 \rangle$	(8) $\langle a^2, b^2, (ab)^2, (ba)^2, ab^2a \rangle$