MATH 340: ADVANCED LINEAR ALGEBRA
PROBLEM SET #5

SPRING 2025

Due Wednesday, March 5. You may consult books, papers, and websites as long as you cite all sources and write up your solutions in your own words.

Problem 1. Let $S : F^3 \to F^3$ be a linear operator. Give examples where...

(1) $\ldots \ker(S) \cap \operatorname{im}(S) = \{\vec{0}\}$, but S is not a projection.
(2) $\ldots \ker(S) \cap \operatorname{im}(S) \neq \{\vec{0}\}$, but S is not nilpotent.

Problem 2. Let V be any vector space, and let $A, B : V \to V$ be linear operators such that $A \circ B = B \circ A$.

(1) Show that any eigenspace for A is B-stable.
(2) Suppose that V is finite-dimensional. Using (1), show that if there is a basis of V in which the matrix of A is diagonal with *pairwise distinct* diagonal entries, then it is also a basis in which the matrix of B is diagonal.

Problem 3. Suppose that V is finite-dimensional, and that $v_1, \ldots, v_m \in V$ is a (nonempty) list of vectors. Show that if $v_1, \ldots, v_m$ are eigenvectors of some linear operator on V, with *pairwise distinct* eigenvalues, then $\{v_1, \ldots, v_m\}$ is a linearly independent set. *Hint:* Induction.

Problem 4 (Axler §5A, #19–20)**.** Recall $F^{\mathbf{N}} = \{(z_1, z_2, z_3, \ldots) \mid z_i \in F \text{ for all } i\}$.

(1) Show that the *forward shift operator* T defined by

$$T(z_1, z_2, z_3, \ldots) = (0, z_1, z_2, \ldots)$$

has no eigenvalues.

(2) Find all eigenvectors of the *backward shift operator* S defined by

$$S(z_1, z_2, z_3, \ldots) = (z_2, z_3, z_4, \ldots).$$

Deduce that every element of F occurs as an eigenvalue of S.

Problem 5. Let V be any complex vector space, and let $T : V \to V$ be a linear operator such that $T^n = \mathrm{Id}_V$. Let $\zeta_n = e^{2\pi i/n} \in \mathbf{C}$.

(1) Show that $V = V_0 + V_1 + \cdots + V_{n-1}$, where

$$V_k = \{v \in V \mid Tv = \zeta_n^k v\}.$$

Hint: There are explicit formulas decomposing any $v \in V$ into a sum $v = \sum_{k=0}^{n-1} v^{(k)}$ with $v^{(k)} \in V_k$ for all k. The $n = 2$ case was #4 on Problem Set 4. If you are stuck, start with the $n = 3$ case.

(2) Show that the sum in (1) is a direct-sum decomposition of V.

Problem 6. Fix an integer m, and let $\mathbf{Z}_m$ denote the integers modulo m. The set of $\mathbf{C}$-valued functions on $\mathbf{Z}_m$ forms a complex vector space V under pointwise addition and scaling. Fix a residue $a \in \mathbf{Z}_m$, and let $T_a : V \to V$ be defined by

$$T_a(f)(x) = f(x - a).$$

(1) Show that T_a is a $\mathbf{C}$-linear operator on V that satisfies $T^m = \mathrm{Id}_V$. Hence, by Problem 5, V is a direct sum of eigenspaces for T_a.
(2) Show that each (nonzero) eigenspace of T_1 is one-dimensional. *Hint:* Show that any eigenvector is uniquely determined by its eigenvalue and $f(0)$.
(3) Give an example where $m \geq 2$ and $a \not\equiv 0$, but T_a has (nonzero) eigenspaces of different dimensions.

Problem 7. Let V be a vector space over F. The *projective space over V* is

$$\mathbf{P}V = \{\text{1-dimensional linear subspaces of } V\}.$$

(This implicitly depends on F, too.) Let $T : V \to V$ be a linear operator.

(1) Show that if T is *invertible*, then there is a well-defined map $\bar{T} : \mathbf{P}V \to \mathbf{P}V$ given by $L \mapsto T(L)$, to be called the *projectivization of T*.
(2) Does $\bar{T} = \mathrm{Id}_{\mathbf{P}V}$ imply that $T = \mathrm{Id}_V$?
(3) Suppose that $F = \mathbf{C}$ and $\dim(V) = 2$ and $T^n = \mathrm{Id}_V$ for some $n > 0$. Show that T is invertible, and that one of two cases must hold: either $\bar{T} = \mathrm{Id}_{\mathbf{P}V}$, or $\bar{T}$ fixes exactly two points. *Hint:* Problem 5.

Problem 8. Look up the definition of the ring of *quaternions* $\mathbf{H}$. It forms a 4-dimensional real vector space with basis $\{1, i, j, k\}$:

$$\mathbf{H} = \{a1 + bi + cj + dk \mid a, b, c, d \in \mathbf{R}\}.$$

(1) Show that $\mathbf{H}$ forms a complex vector space in which the vector addition remains the same, but the scalar multiplication is left multiplication by elements of $\mathbf{C} = \{a + bi \mid a, b \in \mathbf{R}\} \subseteq \mathbf{H}$.
(2) Determine a basis for $\mathbf{H}$ as a complex vector space.
(3) Let $H \subseteq \mathrm{Mat}_2(\mathbf{C})$ be the complex linear subspace

$$H = \left\{ \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix} \middle| \, \alpha, \beta \in \mathbf{C} \right\}.$$

Using Problem Set 3, #8 as a guide, give an explicit isomorphism of complex vector spaces $M : \mathbf{H} \to H$ such that $M(z_1 z_2) = M(z_1) \cdot M(z_2)$. *Hint:* Reduce this identity the case where z_2 belongs to the basis from (2).