MATH 250: TOPOLOGY I

FALL 2025 LOOSE ENDS

Contents

1. Wednesday, 9/3

1.1. Here I try to dispel some potential confusion about bases.

Let X be any set. Let $\mathcal{B}$ be any collection of subsets of X. A useful general observation:

Lemma 1.1. *For any subset $Y \subseteq X$, the following conditions are equivalent:*

(1) *Y is the union of some elements of $\mathcal{B}$.*
(2) *For any $x \in Y$, there is some $B \in \mathcal{B}$ such that $x \in B \subseteq Y$.*

Now let $\mathcal{T}$ be the collection of all subsets of X that can be written as unions of elements of $\mathcal{B}$. Using the lemma, we see that

$$\mathcal{T} = \left\{ \text{subsets } U \subseteq X \;\middle|\; \begin{array}{l} \text{for any } x \in U \text{, we have some } B \in \mathcal{B} \\ \text{such that } x \in B \subseteq U \end{array} \right\}.$$

Theorem 1.2. *Suppose that $\mathcal{B}$ satisfies the following conditions:*

(I) *Every point of X belongs to some element of $\mathcal{B}$.*
(II) *For any $B, B' \in \mathcal{B}$ and any point x of the intersection $B \cap B'$, we can find some $B'' \in \mathcal{B}$ such that $x \in B'' \subseteq B \cap B'$.*

Then $\mathcal{T}$ is a topology on X.

We proved this theorem at the start of the course, implicitly using Lemma 1.1. The only hard part is checking that finite intersections of elements of $\mathcal{T}$ are still elements of $\mathcal{T}$. To make this easier, I mentioned that it suffices by induction to check intersections between pairs of elements of $\mathcal{T}$.

Any collection $\mathcal{B}$ that satisfies hypotheses (I)–(II) in the theorem above is called a *basis*. In the situation of the theorem, we say that $\mathcal{B}$ *generates* or *induces* the topology $\mathcal{T}$, and that $\mathcal{B}$ is a *basis for $\mathcal{T}$* specifically.

1.2. Separately, if we are given $\mathcal{T}$ to start, then there is a way to check whether a subcollection $\mathcal{C} \subseteq \mathcal{T}$ is a basis that generates $\mathcal{T}$. In Munkres, this is Lemma 13.2.

Theorem 1.3. *Fix a topology $\mathcal{T}$ on X and a subset $\mathcal{C} \subseteq \mathcal{T}$. Suppose that for each $x \in X$ and $U \in \mathcal{T}$, there is some $C \in \mathcal{C}$ such that $x \in C \subseteq \mathcal{C}$. Then $\mathcal{C}$ is a basis, and moreover, the topology it generates is $\mathcal{T}$.*

2. Monday, 9/8

2.1. Let X be a set, and let $d : X \times X \to [0, \infty)$ be a metric on X. For all $x \in X$ and $\delta > 0$, we define the *d-ball* with center x and radius δ to be

$$B_d(x, \delta) = \{y \in X \mid d(x, y) < \delta\}.$$

Below is a cleaner version of a long proof from lecture.

Theorem 2.1. *The set $\{B_d(x, \delta) \mid x \in X$ and $\delta > 0\}$ forms a basis.*

Proof. Let $\mathcal{B}$ denote the set in question. We must check two axioms:

(I) Any point of X is contained in some element of $\mathcal{B}$.
(II) Given any two elements of $\mathcal{B}$ and a point in their intersection, we can find some other element of $\mathcal{B}$ containing that point and contained within the intersection as a subset.

(I) holds because for any $x \in X$, we have $x \in B(x, \delta)$ for any choice of δ.

To show (II): Pick balls $B_d(x, \epsilon)$ and $B_d(x', \epsilon')$ and a point z in their intersection $B_d(x, \epsilon) \cap B_d(x', \epsilon')$. We must exhibit some d-ball that contains z and is contained within the intersection as a subset.

It suffices to find some $\delta > 0$ such that

$$B_d(z, \delta) \subseteq B_d(x, \epsilon) \cap B_d(x', \epsilon').$$

Explicitly, this condition on δ means that

$$\text{if } y \in X \text{ satisfies } d(z, y) < \delta, \quad \text{then } d(x, y) < \epsilon \text{ and } d(x', y) < \epsilon'.$$

(Informally, this means that if y is close enough to z, then it is close enough to x and x' as well.) By drawing a picture of the situation, we get the idea that we need to use the triangle inequality to bound the distance $d(x, y)$ in terms of the distances $d(x, z)$ and $d(z, y)$.

Since $z \in B_d(x, \epsilon)$, we know that $d(x, z) < \epsilon$. Rearranging, $\epsilon - d(x, z) > 0$. So we can pick α such that $\epsilon - d(x, z) > \alpha > 0$. Then $d(x, z) + \alpha < \epsilon$. So if $y \in X$ satisfies $d(z, y) < \alpha$, then it also satisfies

$$\begin{aligned} d(x, y) &\leq d(x, z) + d(z, y) && \text{by the triangle inequality} \\ &< d(x, z) + \alpha && \text{by the hypothesis on } y \\ &< \epsilon. \end{aligned}$$

By analogous arguments, we can pick α' such that $\epsilon' - d(x', z) > \alpha' > 0$, and in this case, if y satisfies $d(z, y) < \alpha'$, then $d(x', y) < \epsilon'$.

Finally, set $\delta = \min(\alpha, \alpha')$. We see that if $y \in X$ satisfies $d(z, y) < \delta$, then we have both $d(x, y) < \epsilon$ and $d(x', y) < \epsilon'$. So we have found the desired δ. □