Recall π_1(one-holed solid donut) $\simeq \mathbb{Z}$

one-holed solid donut $= S^1 \times (a, b)$
$\sim S^1$

Q what is π_1(two-holed solid donut)?

two-hold solid donut = figure-eight

Df given nonempty spaces X, Y
x in X and y in Y,

the wedge sum of (X, x) and (Y, y) is the quotient

$X \vee Y = (X \sqcup Y)/\sim$ where $x \sim y$
but no other $\sim$'s

thus the figure-eight is $S^1 \vee S^1$
[here the basepoints don't matter]

Q in general, how is $\pi_1(X \vee Y)$ related to $\pi_1(X, x)$ and $\pi_1(Y, y)$?

Ex consider "generating" loops γ_1, γ_2 for the two circles in the figure-eight

[draw]

then $\gamma_1 * \gamma_2 \neq \gamma_2 * \gamma_1$ [though hard to prove]

Ex by contrast, consider β, γ generating $\pi_1(S^1 \times S^1) \simeq \mathbb{Z}^2$

then β, γ correspond to (1, 0), (0, 1) in Z^2
so $\beta * \gamma = \gamma * \beta$

another (independent) proof:

$S^1 = [0, 1]/\sim$
similarly, $S^1 \times S^1 = ([0, 1] \times [0, 1])/\sim$

[draw]

can draw β and γ in terms of $[0, 1] \times [0, 1]$

then draw $\beta * \gamma$ and $\gamma * \beta$

[draw]

Thm $\pi_1(X \times Y)$ is iso to
the direct product of $\pi_1(X)$ and $\pi_1(Y)$

Df the direct product of $\{G_i\}_i$ is
a group $\prod_i G_i$

where elts are $(g_i)_i$'s, the law is coordinate-wise

Thm $\pi_1(X \vee Y)$ is iso to
the free product of $\pi_1(X)$ and $\pi_1(Y)$

Df the free product of $\{G_i\}_i$ is
a group $\bigast_i G_i$ [\bigast]

where elts are words over $\coprod_i G_i$,
the law is concatenation mod rels of each G_i

<u>Ex</u> π_1(two-holed solid donut)

$\simeq \quad \pi_1(S^1 \vee S^1)$

$\simeq \quad Z \ast Z$

$Z \ast Z$ = { words in a, b, a^{-1}, b^{-1} }

<u>Df</u> the free group on n letters is

$F_n = Z \ast \ldots \ast Z$

with n copies of Z

<u>Ex</u> $\pi_1(S^1 \vee \ldots \vee S^1) \simeq F_n$

when there are n copies of S^1

<u>Analogy</u>

products	Top	product space $\prod_i X_i$
	Grp	direct product $\prod_i G_i$
	Mod	direct product $\prod_i M_i$
coproducts	Top	wedge sum $\bigvee_i X_i$
	Grp	free product $\ast_i G_i$
	Mod	direct sum $\bigoplus_i M_i$

<u>Df</u> given spaces A, X ,Y,

maps $i : A \to X$ and $j : A \to Y$,

the <u>gluing</u> of X and Y along A is the quotient

$(X \sqcup Y)/\sim$ where $i(a) \sim j(a)$ for all a in A gluing generalizes wedge sum

Ex let $A = \{a, b\}$ and $X = [0, 1]$ and $Y = [1, 2]$
let $i(a) = 0$, $i(b) = 1$, $j(a) = 1$, $j(b) = 2$

[draw]

the gluing is homeo to S^1

Ex let $A = S^1 \times (-\varepsilon, \varepsilon)$
let X, Y be two copies of D^2
let i, j be the boundary inclusions

[draw]

the gluing is homeo to S^2

Q how is π_1 of the gluing related to the π_1's of A, X, Y?