naive idea behind Seifert–van Kampen:

if $X$ is covered by open $U_i$'s [draw]
then can compute $\pi_1(X)$ from $\pi_1(U_i)$'s

but only works when all intersections of $U_i$'s are path-connected

<u>Thm</u> (Seifert–van Kampen)

suppose $A, U_1, U_2 \subseteq X$ are open and $x \in A$

s.t. $X = U_1 \cup U_2$,
$A = U_1 \cap U_2$
$A, U_1, U_2$ are all path-connected

with inclusion maps

$$\begin{tikzcd} & U_1 \arrow[dr, "j_1"] & \\ A \arrow[ur, "i_1"] \arrow[dr, "i_2"'] & & X \\ & U_2 \arrow[ur, "j_2"'] & \end{tikzcd}$$

then:

1) $j_{1,*}, j_{2,*}$ induce a surjective hom

$$\pi_1(U_1, x) * \pi_1(U_2, x) \to \pi_1(X, x)$$

2) via this hom, $\pi_1(X, x)$ is the largest quotient of $\pi_1(U_1, x) * \pi_1(U_2, x)$ in which

$$i_{1,*}([\gamma]) \sim i_{2,*}([\gamma]) \text{ for all } [\gamma] \in \pi_1(A, x)$$

Ex X the figure-eight,
$U_1$, $U_2$ open thickenings of the $S^1$'s,
$x$ the intersection point
[draw]

since $\pi_1(A, x)$ is trivial:

$$\begin{aligned}\pi_1(X, x) &\simeq \pi_1(U_1, x) * \pi_1(U_2, x)\\ &\simeq \pi_1(S^1, x) * \pi_1(S^1, x)\\ &\simeq \mathbb{Z} * \mathbb{Z}\end{aligned}$$

Ex $X = S^2$,
$U_1$, $U_2$ open thickenings of opposed hemispheres,
[draw]

here, $\pi_1(A, x) \simeq \pi_1(S^1, x) \simeq \mathbb{Z}$
but $\pi_1(U_1, x)$, $\pi_1(U_2, x)$ trivial
so $\pi_1(X, x)$ also trivial

Rem Seifert–van Kampen alone cannot compute $\pi_1(S^1)$

(Munkres §53–54) let $o = (1, 0)$ in $S^1$

recall: $\omega_n(s) = (\cos(2\pi ns), \sin(2\pi ns))$

Thm $\Phi : \mathbb{Z} \to \pi_1(S^1, o)$ def by $\Phi(n) = [\omega_n]$ is an isomorphism

Pf Step 1. $\Phi$ is a homomorphism
Step 2. $\Phi$ is bijective

Step 1 must show that $\Phi(m + n) = \Phi(m) + \Phi(n)$, meaning $[\omega_{m + n}] = [\omega_m * \omega_n]$

key idea: let $p : \mathbb{R} \to S^1$ be

$$p(x) = (\cos(2\pi x), \sin(2\pi x))$$

Lem for any $a, b$ in $\mathbb{Z}$ s.t. $b - a = n$, we have

$$\omega_n = p \circ \acute{\omega}_{a, b}$$

where $\acute{\omega}_{a, b} : [0, 1] \to \mathbb{R}$ is defined by

$$\acute{\omega}_{a, b}(s) = (1 - s)a + sb$$

[draw]

Lem for any $a, b, c$ in $\mathbb{Z}$, we have

$$[\acute{\omega}_{a, b} * \acute{\omega}_{b, c}] = [\acute{\omega}_{a, c}]$$

[draw]

therefore,
$[(p \circ \acute{\omega}_{a, b}) * (p \circ \acute{\omega}_{b, c})]$
$= [p \circ (\acute{\omega}_{a, b} * \acute{\omega}_{b, c})]$ [Munkres 327]
$= [p \circ \acute{\omega}_{a, c}]$ [PS5, #1]

giving $[\omega_m * \omega_n] = [\omega_{m + n}]$

before Step 2, we introduce the notion of coverin map inspired by $p : \mathbb{R} \to S^1$