Affine Springer Fibers and Level-Rank Duality

Minh-Tâm Quang Trinh

Yale University

1 Springer Theory

2 Deligne–Lusztig Theory

3 Level-Rank Duality

Mainly about joint work with Ting Xue:

`arXiv:2311.17106`

See also the extended abstract on my website, which we have submitted to FPSAC '25.

1 Springer Theory

Work over $\mathbf{C}$.

$\mathbf{G}$ connected reductive group

$\mathbf{B}$ Borel subgroup

An element $\gamma \in \mathbf{g} = \mathrm{Lie}(\mathbf{G})$ is *regular semisimple* iff $\mathbf{G}_\gamma$ is a maximal torus.

In this case, the Springer fiber

$$\mathcal{F}l_\gamma = \{g\mathbf{B} \in \mathbf{G}/\mathbf{B} \mid \gamma \in \mathrm{Lie}(g\mathbf{B}g^{-1})\}$$

is a torsor for the Weyl group W.

That is, $\mathcal{F}l_\gamma$ forms a W-bundle as we vary γ over the regular semisimple locus of $\mathbf{g}$.

$\mathbf{G}(\!(z)\!)$ loop group

$\mathbf{I}$ Iwahori subgroup of $\mathbf{G}[\![z]\!]$

The affine Springer fibers

$$\mathcal{F}l_\gamma = \{g\mathbf{I} \in \mathbf{G}(\!(z)\!)/\mathbf{I} \mid \gamma \in \mathrm{Lie}(g\mathbf{I}g^{-1})\}$$

are *not* locally constant over the regular semisimple locus of $\mathbf{g}(\!(z)\!)$, but only over certain subsets.

Example   Take $\mathbf{G} = \mathbf{SL}_2$.

If $\gamma = \left(\begin{smallmatrix} & 1 \\ z & \end{smallmatrix}\right)$, then $\mathcal{F}l_\gamma$ is a single point.

If $\gamma = \left(\begin{smallmatrix} z & \\ & -z \end{smallmatrix}\right)$, then $\mathcal{F}l_\gamma$ is an *infinite* chain of $\mathbf{P}^1$'s.

Fix a maximal torus $\mathbf{A} \subseteq \mathbf{B}$ and a fraction $\frac{d}{m} > 0$ in lowest terms.

Let $\rho^\vee = \frac{1}{2}\sum_\alpha \alpha^\vee \in \frac{1}{2}X_*(\mathbf{A})$.

$$\mathbf{C}^\times \curvearrowright \mathbf{G}(\!(z)\!) \ : \ c \cdot g(z) = \mathrm{Ad}(c^{d\rho^\vee})g(c^m z).$$

(Oblomkov–Yun) $\mathcal{F}l_\gamma$ is locally constant over

$$\mathbf{g}^{\mathrm{rs}}_{d/m} = \{\gamma \in \mathbf{g}(\!(z)\!)^{\mathrm{rs}} \mid c \cdot \gamma = c^d \gamma\},$$

and $\mathbf{C}^\times \curvearrowright \mathcal{F}l_\gamma$ for such γ.

We say these elements are *homogeneous of slope* $\frac{d}{m}$.

Example   Take $\mathbf{B} \subseteq \mathbf{SL}_2$ upper-triangular.

The preceding examples: slopes $\frac{1}{2}, 1$.

Note that $\mathbf{G}_0 := (\mathbf{G}(\!(z)\!)^{\mathbf{C}^\times})^\circ \curvearrowright \mathbf{g}^{\mathrm{rs}}_{d/m}$.

(Oblomkov–Yun) Take $\mathbf{G}$ simply-connected, simple.

For $\gamma \in \mathbf{g}^{\mathrm{rs}}_{d/m}$ with $\mathcal{F}l_\gamma$ proper:

- A *perverse filtration* P on $\mathrm{H}^*_{\mathbf{C}^\times}(\mathcal{F}l_\gamma)$, arising from a Ngô-type global model.
- An action of a *rational Cherednik algebra* on

$$\mathcal{E}_\gamma := \sum_{i,j} \mathsf{x}^i \mathsf{y}^j \, \mathrm{gr}^{\mathsf{P}}_i \, \mathrm{H}^j_{\mathbf{C}^\times}(\mathcal{F}l_\gamma)^{\pi_0(\mathbf{G}_{0,\gamma})}|_{\epsilon \to 1},$$

where ϵ is a generator of $\mathrm{H}_{\mathbf{C}^\times}(\mathit{point})$.

The rational Cherednik algebra is a deformation of $\mathbf{C}W \ltimes \mathcal{D}(\mathbf{a})$, to be denoted $D^{\mathrm{rat}}_{d/m}$.

$D^{\mathrm{rat}}_{d/m}$	$\mathrm{U}\mathbf{g}$
$\mathbf{C}[\mathbf{a}] \otimes \mathbf{C}W \otimes \mathbf{C}[\mathbf{a}^*]$	$\mathrm{U}\mathbf{n}_- \otimes \mathbf{C}[\mathbf{a}] \otimes \mathrm{U}\mathbf{n}_+$
$\Delta_{d/m}(\chi)$	$\Delta(\lambda)$
$L_{d/m}(\chi)$	$L(\lambda)$

Problem Give a formula for $E_\gamma := \mathcal{E}_\gamma|_{\mathsf{y}=-1}$, the virtual $D^{\mathrm{rat}}_{d/m}$-module formed by collapsing H^*.

Idea Monodromy of E_γ over a certain $\mathbf{c}^{\mathrm{rs}}_{d/m} \subseteq \mathbf{g}^{\mathrm{rs}}_{d/m}$ commutes with the Cherednik action.

Roughly, $\mathbf{c}^{\mathrm{rs}}_{d/m}$ is a transverse slice to $\mathbf{G}_0 \curvearrowright \mathbf{g}^{\mathrm{rs}}_{d/m}$.

The monodromy seems to factor through an algebra from *Deligne–Lusztig theory*.

Deligne–Lusztig studied geometry over *finite fields*.

But up to Tate twist,

$$\mathrm{Gal}(\bar{\mathbf{F}}_q|\mathbf{F}_q) \quad \simeq \quad \hat{\mathbf{Z}} \quad \simeq \quad \mathrm{Gal}(\overline{\mathbf{C}(\!(z)\!)}|\mathbf{C}(\!(z)\!)).$$

Forms of $\mathbf{G}$ are classified by Dynkin automorphisms in the same way over $\mathbf{F}_q$ as over $\mathbf{C}(\!(z)\!)$.

Much of Oblomkov–Yun's setup generalizes from $\mathbf{G}$ to any of its forms $\mathbf{G}_{\mathbf{C}(\!(z)\!)}$.

The tori $\mathbf{A}, \mathbf{G}_\gamma$ generalize to forms $\mathbf{A}_{\mathbf{C}(\!(z)\!)}, \mathbf{G}_{\mathbf{C}(\!(z)\!),\gamma}$.

These have corresponding forms $\mathbf{A}_{\mathbf{F}_q}, \mathbf{T}_{\mathbf{F}_q}$.

2 Deligne–Lusztig Theory

Work over $\bar{\mathbf{F}}_q$ for good q.

Forms of $\mathbf{G}$ over $\mathbf{F}_q$ correspond to Frobenius maps

$$F \curvearrowright \mathbf{G}.$$

We say that $G = \mathbf{G}^F$ is a *finite group of Lie type*.

F-stable Levis $\mathbf{L} \subseteq \mathbf{G}$ correspond to Levis $L \subseteq G$.

Deligne–Lusztig introduced varieties† $Y_{\mathbf{L}}^{\mathbf{G}}$ such that

$$G \quad \curvearrowright \quad \mathrm{H}_c^*(Y_{\mathbf{L}}^{\mathbf{G}}) \quad \curvearrowleft \quad L.$$

Induction map $R_L^G : \mathrm{K}_0(L) \to \mathrm{K}_0(G)$:

$$R_L^G(\lambda) = \textstyle\sum_i (-1)^i \mathrm{H}_c^i(Y_{\mathbf{L}}^{\mathbf{G}})[\lambda].$$

† Actually, $Y_{\mathbf{L}}^{\mathbf{G}}$ depends on a parabolic $\mathbf{P} \supseteq \mathbf{L}$.

(Broué–Malle) For *m-regular* maximal tori $\mathbf{T}$, a specific algebra $H_T^G(\mathsf{q})$ such that

$$H_T^G(\zeta_m) = \bar{\mathbf{Q}} W_T^G, \quad \text{where } W_T^G = N_G(T)/T.$$

They conjecture:

1 $H_T^G(q) \otimes \bar{\mathbf{Q}}_\ell \simeq \mathrm{End}_G(\mathrm{H}_c^*(Y_{\mathbf{T}}^{\mathbf{G}})[1_T])$.

2 As a virtual $(G, H_T^G(q))$-bimodule,

$$R_T^G(1_T) = \sum_{\substack{\rho \in \mathrm{Irr}(G) \\ (\rho, R_T^G(1_T)) \neq 0}} \varepsilon_{T,\rho}(\rho \otimes \chi_{T,\rho,q})$$

where $\varepsilon_{T,\rho} \in \{\pm 1\}$ and $\chi_{T,\rho} \in \mathrm{Irr}(W_T^G)$.

(And $\chi_{T,\rho,q} \in \mathrm{K}_0(H_T^G(q))$ corresponds to $\chi_{T,\rho}$.)

Back to Springer. $(\mathbf{A}_{\mathbf{F}_q}, \mathbf{T}_{\mathbf{F}_q} \leftrightarrow \mathbf{A}_{\mathbf{C}((z))}, \mathbf{G}_{\mathbf{C}((z)),\gamma})$

It turns out that $\mathbf{A}$ and $\mathbf{T}$ are 1- and m-regular.

Moreover, $\pi_1(\mathbf{c}_{d/m}^{\mathrm{rs}})$ is the braid group of W_T^G.

Conjecture (T–Xue)

1 $\pi_1(\mathbf{c}_{d/m}^{\mathrm{rs}}) \curvearrowright \mathcal{E}_\gamma$ factors through $H_T^G(1)$.

2 As a virtual $(D_{d/m}^{\mathrm{rat}}, H_T^G(1))$-bimodule,†

$$E_\gamma = \sum_{\substack{\rho \in \mathrm{Irr}(G) \\ (\rho, R_A^G(1_A)) \neq 0 \\ (\rho, R_T^G(1_T)) \neq 0}} \varepsilon_{T\rho}(\Delta_{d/m}(\chi_{A,\rho}) \otimes \chi_{T,\rho,1}).$$

† In general, $D_{d/m}^{\mathrm{rat}}$ is defined using W_A^G.

Theorem (T–Xue) True in these cases:

- m is the (twisted) Coxeter number of $\mathbf{G}_{\mathbf{C}((z))}$.
- $(\mathbf{G}_{\mathbf{C}((z))}, m) = ({}^2A_2, 2), (C_2, 2), (G_2, 3), (G_2, 2)$.

Under a conjecture of OY, true in further cases.

Example Take $\mathbf{G}_{\mathbf{C}((z))}$ split, m its Coxeter number.

$\chi_{A,\rho}$ runs over characters $\chi_{\wedge^k(\mathbf{a})}$ of W_A^G.

$\chi_{T,\rho}$ runs over *all* characters of $W_T^G = \mathbf{Z}/m\mathbf{Z}$.

In $\mathrm{K}_0(D_{d/m}^{\mathrm{rat}})$,

$$\begin{aligned}[E_\gamma] &= \textstyle\sum_k (-1)^k[\Delta_{d/m}(\chi_{\wedge^k(\mathbf{a})})] \\ &= [L_{d/m}(\chi_{\mathsf{triv}})].\end{aligned}$$

Cf. the BGG resolution of Berest–Etingof–Ginzburg.

3 Level-Rank Duality

Compare E_γ given by

(1) $$\textstyle\sum_\rho \varepsilon_{T,\rho}(\Delta_{d/m}(\chi_{A,\rho}) \otimes \chi_{T,\rho,1})$$

with $R_A^G(1_A) \otimes_{\bar{\mathbf{Q}}_\ell G} R_T^G(1_T)$ given by

(2) $$\textstyle\sum_\rho \varepsilon_{T,\rho}(\chi_{A,\rho,q} \otimes \chi_{T,\rho,q}).$$

The *Knizhnik–Zamolodchik functor*

$$\mathsf{KZ} : \mathsf{Rep}(D_{d/m}^{\mathrm{rat}}) \to \mathsf{Rep}(H_A^G(\zeta_m))$$

sends $\mathsf{KZ}(\Delta_{d/m}(\chi)) = \chi_{\zeta_m}$. Thus an analogy:

$$\boxed{\mathbf{F}_q \;:\; (q, q) \;::\; \mathbf{C}((z)) \;:\; (\zeta_m, 1)}$$

The symmetry between A and T led us to new discoveries about the Harish–Chandra theory of G.

Let $\mathrm{Uch}(G)$ be the set of *unipotent* irreps of G, which occur in $R_T^G(1_T)$ for some maximal torus $\mathbf{T}$.

(Broué–Malle–Michel) Fix a positive integer l.

- $\mathbf{L} \subseteq \mathbf{G}$ is *l-split* iff $\mathbf{L} = Z_{\mathbf{G}}(\mathbf{S})^\circ$, where

$$\mathbf{S} \text{ is a torus with } |S| \text{ a power of } \Phi_l(q).$$

- $\lambda \in \mathrm{Uch}(L)$ is *l-cuspidal* iff $(\lambda, R_M^G(\mu)) = 0$ for any l-split $M \neq L$.

As we run over pairs $(\mathbf{L}, \lambda)$ up to conjugacy,

$$\mathrm{Uch}(G) = \coprod \mathrm{Uch}(G)_{\mathbf{L},\lambda},$$

where $\mathrm{Uch}(G)_{\mathbf{L},\lambda} = \{\rho \mid (\rho, R_L^G(\lambda)) \neq 0\}$.

For $l = 1$, these are classical *Harish-Chandra series*.

Generalizing our discussion for maximal tori:

Broué–Malle define a Hecke algebra $H_{L,\lambda}^G(\mathsf{q})$ such that

$$H_{L,\lambda}^G(\zeta_l) = \bar{\mathbf{Q}} W_{L,\lambda}^G, \text{ where } W_{L,\lambda}^G = N_G(L,\lambda)/L.$$

They conjecture:

1 $H_{L,\lambda}^G(q) \otimes \bar{\mathbf{Q}}_\ell = \mathrm{End}_G(\mathrm{H}_c^*(Y_{\mathbf{L}}^{\mathbf{G}})[\lambda])$.

2 As a virtual $(G, H_{L,\lambda}^G(q))$-bimodule,

$$R_L^G(\lambda) = \sum_{\rho \in \mathrm{Uch}(G)_{\mathbf{L},\lambda}} \varepsilon_{L,\lambda,\rho}(\rho \otimes \chi_{L,\lambda,\rho,q})$$

where $\varepsilon_{L,\lambda,\rho} \in \{\pm 1\}$ and $\chi_{L,\lambda,\rho} \in \mathrm{Irr}(W_{L,\lambda}^G)$.

Via the *decomposition map*

$$\chi \mapsto \chi_{\zeta_m} : \mathrm{Irr}(W^G_{L,\lambda}) \to \mathrm{K}_0(H^G_{L,\lambda}(\zeta_m)),$$

we partition $\mathrm{Irr}(W^G_{L,\lambda})$ into *blocks*, describing how $H^G_{L,\lambda}(\zeta_m)$ fails to be semisimple.

Conjecture (T–Xue) Fix l, m.

Fix an l-cuspidal $(\mathbf{L}, \lambda)$ and m-cuspidal $(\mathbf{M}, \mu)$.

1 The set

$$\begin{aligned} &\{\chi_{L,\lambda,\rho} \mid \rho \in \mathrm{Uch}(G)_{\mathbf{L},\lambda} \cap \mathrm{Uch}(G)_{\mathbf{M},\mu}\}, \\ \textit{resp.}\quad &\{\chi_{M,\mu,\rho} \mid \rho \in \mathrm{Uch}(G)_{\mathbf{L},\lambda} \cap \mathrm{Uch}(G)_{\mathbf{M},\mu}\}, \end{aligned}$$

is a union of $H^G_{L,\lambda}(\zeta_m)$-, *resp.* $H^G_{M,\mu}(\zeta_l)$-blocks.

2 The indexing induces a matching of blocks.

Theorem (T–Xue) (1), (2) are compatible with block sizes for essentially all G, l, m with G exceptional.

Conjecture (T–Xue) In the preceding setup:

3 Via KZ functors, the bijection in (2) lifts to a derived equivalence between category-O blocks of appropriate rational Cherednik algebras.

Theorem (T–Xue) (1), (2), (3) hold for $G = \mathrm{GL}_n$ when l, m are coprime.

Note that $W_{L,\lambda}^{\mathrm{GL}_n} \simeq S_N \ltimes \mathbf{Z}_l^N$ for some N, *etc.*

$$\mathsf{Rep}(H_{L,\lambda}^{\mathrm{GL}_n}(\zeta_m)) \quad \text{and} \quad \mathsf{Rep}(H_{M\mu}^{\mathrm{GL}_n}(\zeta_l))$$

can be interpreted in terms of *higher-level Fock spaces*

$$\bigoplus_{\substack{\vec{s} \in \mathbf{Z}^l \\ |\vec{s}| = s}} \Lambda_{\mathsf{q}}^{\vec{s}} \xleftarrow{\sim} \Lambda_{\mathsf{q}}^{s} \xrightarrow{\sim} \bigoplus_{\substack{\vec{r} \in \mathbf{Z}^m \\ |\vec{r}| = s}} \Lambda_{\mathsf{q}}^{\vec{r}}.$$

Above, $\Lambda_{\mathsf{q}}^{\vec{s}} \simeq \bigoplus_N \mathrm{K}_0(S_N \ltimes \mathbf{Z}_l^N) \otimes \mathbf{Q}(\mathsf{q})$, *etc.*

Level-rank duality of Frenkel, Uglov, Chuang–Miyachi, Rouquier–Shan–Varagnolo–Vasserot. . .

Our conjectures generalize level-rank duality from GL_n to arbitrary G.

Thank you for listening.