MATH 250: TOPOLOGY I FINAL GUIDE

FALL 2025

The midterm exam will be held in-class on **Monday, December 8, 2025**. It will start at 12:30 pm and end at 3:00 pm (**2.5 hours**).

You will be allowed to look at any notes on paper that you wrote prior to the exam, and at the textbook (Munkres, *Topology*, 2nd Ed.). However, you will not be allowed to use electronic devices of any kind—including phones, computers, tablets, or other visual/audio devices—or any software.

What Could Appear

§12–13. Topologies.

- definitions of the discrete, indiscrete, and finite-complement topologies on any set, and why these are topologies
- how the analytic topology on $\mathbf{R}^n$ compares to the topologies above
- what it means for a basis to generate a given topology on X
- examples where different bases generate the same topology on X

§16–18, 22. Continuous Maps, Subspaces, Quotients.

- examples where some subset of A is open in A, but not in X
- definitions of the interior and closure of a subset A of a given space X
- definitions of the Hausdorff ($= T_2$) and T_1 properties
- what it means for a sequence of points to converge to a given point
- what the Hausdorff property implies for convergence of sequences
- what it means for a map between topological spaces to be continuous, or more strongly, a homeomorphism
- how to check continuity of a map $f\colon X \to Y$ using a basis for the topology on Y
- examples of continuous bijections that are not homeomorphisms
- examples of quotient spaces (*e.g.*, constructed using equivalence relations)

§15, 19–20. Products, Metrics.

- why $\mathbf{R}^n$ and $\mathbf{R}^\omega$ are examples of direct products of sets
- how the box and product topologies compare to each other, for $\mathbf{R}^n$, $\mathbf{R}^\infty$, $\mathbf{R}^\omega$
- how the product topology on $\prod_{i \in I} X_i$ is related to continuity of the various projection maps $\mathrm{pr}_j\colon \prod_{i \in I} X_i \to X_j$
- how the euclidean and square metrics compare to each other on $\mathbf{R}^n$
- why the uniform metric on $\mathbf{R}^\omega$ is a metric

§23–27. Connectedness and Compactness.

- how connected subspaces of X interact with separations of X
- how connectedness interacts with continuous maps and finite products
- spaces that are totally disconnected but not discrete
- why path-connected implies connected
- statement of the intermediate value theorem
- definitions of connected components and path components
- what it means to be locally connected or locally path connected
- examples of spaces that are disconnected but locally connected
- statement of Heine–Borel for subsets of $\mathbf{R}$

§51–52, 54, 58–59. Homotopy, Path Homotopy, Fundamental Groups.

- how homotopy and path homotopy differ
- how homotopies and path homotopies interact with compositions of maps
- why star-convex subsets of $\mathbf{R}^n$ are contractible
- examples of spaces that are homotopy equivalent, but not homeomorphic
- the definition of $\pi_1(X, x)$
- meaning of $f_*\colon \pi_1(X, x) \to \pi_1(Y, f(x))$ for a continuous map $f\colon X \to Y$
- effect of changing the basepoint x on $\pi_1(X, x)$
- an explicit isomorphism $\pi_1(S^1, p) \simeq \mathbf{Z}$ (say, where $p = (1, 0) \in \mathbf{R}^2$)
- $\pi_1(\prod_i X_i, (x_i)_i) \simeq \prod_i \pi_1(X_i, x_i)$

§68–71, ≈ 73. The Seifert–Van Kampen Theorem.

- meaning of a presentation of a group by generators and relations
- meaning of the free product $G_1 * G_2$ for groups G_1, G_2
- $\pi_1(S^1 \vee S^1)$ and similar wedge products
- definition of the homomorphism $\pi_1(U, x) * \pi_1(V, x) \to \pi_1(X, x)$, when $X = U \cup V$ and $x \in U \cap V$
- examples where $X = U \cup V$ and $x \in U \cap V$, but $\pi_1(X, x) \not\simeq \pi_1(U, x) * \pi_1(V, x)$
- statement of Seifert–van Kampen, especially the hypotheses

§53. Coverings.

- examples and non-examples of covering maps
- statement of the path-lifting and homotopy-lifting properties

What We'll Have Covered by Then, But Will Not Appear

- the axiom of choice
- equivalence/inequivalence of metrics (Problem Set 2, #10)
- convergence in the uniform topology on $\mathbf{R}^\omega$
- the topologist's sine curve
- regularity and normality
- inverse limits, like the p-adic integers $\mathbf{Z}_p$
- computing π_1 of a genus-g surface, for $g \geq 2$

- topological groups
- the correspondence between (pointed) covering maps and subgroups of π_1

Practice Problems

Try to hand-write a solution to each problem within 10–15 minutes. Throughout, $\mathbf{R}$ *has the analytic topology unless otherwise specified.*

On the exam, you will not need to use complete sentences, but the clearer your work, the more points you will earn.

Problem 1. Do any practice problems from the midterm study guide

> https://mqtrinh.github.io/math/teaching/howard/math-250/howard_math-250_f-25_midterm-guide.pdf

that you hadn't done earlier.

Problem 2. Give an explicit homeomorphism between $(-1, 1)$ and $\mathbf{R}$.

Problem 3. Show that if X and Y are contractible, then $X \times Y$ is contractible.

Problem 4. Give an example of spaces X and Y that are homotopy equivalent, such that every point of X is a cut point, but some point of Y is not a cut point. Then give another example.

Problem 5. Show explicitly that $\mathbf{R}^2 \setminus \{(0, 0)\}$ is path-connected. That is, for arbitrary points $(a_1, a_2), (b_1, b_2) \neq (0, 0)$, show that there is always an explicit path from one to the other that avoids $(0, 0)$. You may need casework.

Problem 6. There was a typo in Problem #5 on Problem Set 6. Redo the problem with the typo corrected. That is, suppose that X is the union of path-connected open sets $U, V \subseteq X$ whose intersection is a nonempty path-connected subspace A. Give examples where:

(1) X, U, V are simply-connected, but A is not.
(2) X and U are simply-connected, but V is not.
(3) X is simply-connected, but U and V are not.
(4) A is simply-connected, but X, U, V are not.

Problem 7. Recall that $\pi_1(S^1, p) \simeq \mathbf{Z}$ for any basepoint p. Take $p = (1, 0)$.

(1) Give a map $f \colon S^1 \to S^1$ such that $f(p) = p$ and $f_*(\pi_1(S^1, p)) = \pi_1(S^1, p)$, but f is not the identity map.
(2) Give a map $g \colon S^1 \to S^1$ such that $g(p) = p$ and $g_*(\pi_1(S^1, p))$ has index 2025 in $\pi_1(S^1, p)$. *Hint:* Use polar coordinates.

Problem 8. For each of the following, give a simply-connected (and connected) covering space.

(1) S^1.
(2) $S^1 \times S^1 \times S^1$.
(3) $S^1 \vee S^1$.