6.

More on Mellit's Annals paper.

6.1.

6.1. Let k be a field of characteristic either zero or greater than n. Let $F = k(\!(\varpi)\!)$ and $\mathcal{O} = k[\![\varpi]\!]$. Let $G = \mathrm{GL}_n$ and $\mathfrak{g} = \mathfrak{gl}_n$. Let $\mathcal{N} \subseteq \mathfrak{g}$ be the nilpotent cone. We take the convention that elements of G and $\mathfrak{g}$ act on F^n from the left, *i.e.*, column notation. For any $\gamma \in \mathfrak{g}(F)$, we write $\ker(\gamma)$ to denote the kernel of γ acting on F^n.

6.2. Let $\theta \in \mathcal{N}(\mathcal{O})$, and let $g \in G(F)$. The action of g on F^n restricts to an isomorphism of vector spaces:

$$g : \ker(\theta) \xrightarrow{\sim} \ker(x\theta x^{-1}). \tag{6.1}$$

Following [M20, Def. 3.6], we say that g is *θ-kernel-strict* iff (6.1) further restricts to an isomorphism of $\mathcal{O}$-modules:

$$g : \ker(\theta) \cap \mathcal{O}^n \xrightarrow{\sim} \ker(g\theta g^{-1}) \cap \mathcal{O}^n.$$

In general, it is always true that (6.1) restricts to an isomorphism $\ker(\theta) \cap \mathcal{O}^n \xrightarrow{\sim} \ker(g\theta g^{-1}) \cap g\mathcal{O}^n$. Therefore, g is θ-kernel-strict if and only if

$$\ker(g\theta g^{-1}) \cap g\mathcal{O}^n = \ker(g\theta g^{-1}) \cap \mathcal{O}^n,$$

or equivalently,

$$\ker(\theta) \cap \mathcal{O}^n = \ker(\theta) \cap g^{-1}\mathcal{O}^n.$$

We deduce that this condition only depends on $g^{-1}\mathcal{O}^n$, not on g itself.

6.3. Let $G(F)_\theta$ be the centralizer of θ in $G(F)$, and let

$$J_\theta = G(F)_\theta \cap G(\mathcal{O}).$$

Then J_θ stabilizes both $\ker(\theta)$ and $\ker_{\mathcal{O}^n}(\theta)$, so g being θ-kernel-strict actually only depends on the orbit of $g^{-1}\mathcal{O}^n$ under left multiplication by J_θ, *i.e.*, on the double coset $[g^{-1}] \in J_\theta \backslash G(F)/G(\mathcal{O})$.

Example 6.1. An element $g \in G(F)$ is 0-kernel-strict if and only if $\mathcal{O}^n = g^{-1}\mathcal{O}^n$, which in turn occurs if and only if $g \in G(\mathcal{O})$. Note that here, $J_\theta = G(\mathcal{O})$.

Example 6.2. Take $n = 2$ and $\theta \in \{\left(\begin{smallmatrix} 0 & F^\times \\ & 0 \end{smallmatrix}\right)\}$, so that $\ker(\theta) = F \oplus 0$ and

$$\ker(\theta) \cap \mathcal{O} = \mathcal{O} \oplus 0.$$

For any $g \in G(F)$ with inverse $g^{-1} = \left(\begin{smallmatrix} x_1 & x_2 \\ y_1 & y_2 \end{smallmatrix}\right)$, we compute:

$$\ker(\theta) \cap g^{-1}\mathcal{O}^n = \left\{\left(\begin{smallmatrix} ax_1+bx_2 \\ 0 \end{smallmatrix}\right) \,\middle|\, a, b \in \mathcal{O} \text{ such that } ay_1 + by_2 = 0\right\}.$$

That is, g is θ-kernel-strict if and only if

$$(x_1, x_2) \cdot (\ker(y_1, y_2) \cap \mathcal{O}^2) = \mathcal{O},$$

where we write (x_1, x_2) for the corresponding linear functional on F^2, and similarly with (y_1, y_2). Note that here, $J_\theta = \mathcal{O}^\times \cdot \left(\begin{smallmatrix} 1 & \mathcal{O} \\ & 1 \end{smallmatrix}\right)$.

In particular, if g is upper-triangular, then $y_1 = 0$ and $x_1, y_2 \in F^\times$. This yields $\ker(y_1, y_2) \cap \mathcal{O}^2 = \mathcal{O} \oplus 0$. So if g is upper-tringular, then it is θ-kernel-strict if and only if $x_1 \in \mathcal{O}^\times$. This recovers a special case of Mellit's claim in the last sentence of [M20, Def. 3.6].

6.2.

6.4. Let $\mathcal{V}$ be the universal family of orbital ind-varieties:

$$\mathcal{V} = \{(\eta, \theta) \in \mathcal{N}(\mathcal{O}) \times \mathcal{N}(\mathcal{O}) \mid \eta \sim_{\mathrm{Ad}(G(F))} \theta\}.$$

Let $\mathcal{T} \to \mathcal{V}$ be the fibration defined by

$$\begin{aligned} \mathcal{T} &= \{(\eta, \theta, g) \in \mathcal{V} \times G(F) \mid g^{-1}\eta g = \theta\} \\ &= \{(\eta, \theta, g) \in \mathcal{V} \times G(F) \mid \eta = g\theta g^{-1}\}. \end{aligned}$$

Finally, let

$$\begin{aligned} \mathcal{T}_{KS} &= \{(\eta, \theta, g) \in \mathcal{T} \mid g \text{ is } \theta\text{-kernel-strict}\} \\ &= \{(\eta, \theta, g) \in \mathcal{T} \mid g^{-1} \text{ is } \eta\text{-kernel-strict}\} \\ &= \{(\eta, \theta, g) \in \mathcal{T} \mid g \text{ defines an } \mathcal{O}\text{-module isomorphism } \ker(\theta) \xrightarrow{\sim} \ker(\eta)\}. \end{aligned}$$

Note that $G(\mathcal{O}) \times G(\mathcal{O})$ acts on $\mathcal{T}$ from the left according to

$$(x, y) \cdot (\eta, \theta, g) = (x\eta x^{-1}, y\theta y^{-1}, xgy^{-1}).$$

By our discussion above, this action restricts to a fiberwise action

$$(G(\mathcal{O}) \times J \to \mathcal{N}(\mathcal{O})) \quad \curvearrowright \quad (\mathcal{T}_{KS} \xrightarrow{\theta} \mathcal{N}(\mathcal{O})),$$

where $J \to \mathcal{N}(\mathcal{O})$ is the group scheme with fiber J_θ above θ.

6.5. For fixed $\eta \in \mathcal{N}(\mathcal{O})$, we write $\mathcal{V}^\eta \subseteq \mathcal{V}$ to denote the corresponding fiber, and define $\mathcal{T}^\eta, \mathcal{T}^\eta_{KS}$ similarly. That is,

$$\begin{aligned}\mathcal{T}^\eta &= \{g \in G(F) \mid g^{-1}\eta g \in \mathcal{N}(\mathcal{O})\},\\ \mathcal{T}^\eta_{KS} &= \{g \in G(F) \mid g^{-1}\eta g \in \mathcal{N}(\mathcal{O}) \text{ and } g^{-1} \text{ is } \eta\text{-kernel-strict}\}.\end{aligned}$$

For all $x \in G(\mathcal{O})$, we see that the isomorphism $\mathcal{T}^\eta \xrightarrow{\sim} \mathcal{T}^{x\eta x^{-1}}$ defined by $g \mapsto xg$ restricts to an isomorphism

$$\mathcal{T}^\eta_{KS} \xrightarrow{\sim} \mathcal{T}^{x\eta x^{-1}}_{KS}.$$

In particular, the G_η-action on $\mathcal{T}^\eta$ restricts to a J_η-action on $\mathcal{T}^\eta_{KS}$.

For fixed $\theta \in \mathcal{N}(\mathcal{O})$, we write $\mathcal{V}^{\eta,\theta} \subseteq \mathcal{V}^\eta$ to denote the corresponding fiber, and define $\mathcal{T}^{\eta,\theta}, \mathcal{T}^{\eta,\theta}_{KS}$ similarly. That is,

$$\begin{aligned}\mathcal{T}^{\eta,\theta} &= \{g \in G(F) \mid \eta = g\theta g^{-1}\},\\ \mathcal{T}^{\eta,\theta}_{KS} &= \{g \in G(F) \mid \eta = g\theta g^{-1} \text{ and } g^{-1} \text{ is } \eta\text{-kernel-strict}\}.\end{aligned}$$

We see that the G_θ-action on $\mathcal{T}^{\eta,\theta}$ defined by $y \cdot g = gy^{-1}$ restricts to a J_θ-action on $\mathcal{T}^{\eta,\theta}_{KS}$, commuting with the J_η-action.

Note that $\mathcal{T}^{\eta,\theta}$ is a torsor for $G(F)_\eta$, or equivalently, for $G(F)_\theta$. Consequently, the J_η- and J_θ-actions on $\mathcal{T}^{\eta,\theta}_{KS}$ are individually, though not jointly, free [*cf.* M20, Prop. 3.12], and moreover, elements of $\mathcal{T}^{\eta,\theta}_{KS}$ belong to the same orbit of left, *resp.* right multiplication by $G(\mathcal{O})$ only if they belong to the same J_η, *resp.* J_θ, orbit. In other words, the maps

$$\begin{aligned}J_\eta\backslash\mathcal{T}^{\eta,\theta}_{KS} &\to G(\mathcal{O})\backslash G(F),\\ \mathcal{T}^{\eta,\theta}_{KS}/J_\theta &\to G(F)/G(\mathcal{O})\end{aligned}$$

are injective.

6.6. For all $g = (g_{i,j})_{i,j} \in G(F)$, let

$$\mathrm{depth}_\varpi(g) = \min_{i,j} \mathrm{val}_\varpi(g_{i,j}).$$

For all $d, N \in \mathbf{Z}$, let

$$\begin{aligned}G(F)_d &= \{g \in G(F) \mid \mathrm{val}_\varpi \det(g) = d\},\\ G(F)_d^{\geq N} &= \{g \in G(F)_d \mid \mathrm{depth}_\varpi(g) \geq N\}.\end{aligned}$$

Note that the subsets $G(F)_d$ are the underlying sets of the connected components of the loop group of G, and that the subsets $G(F)_d^{\geq N}$ form a filtration of $G(F)_d$ increasing in N. We can now rewrite [M20, Lem. 3.7] as follows:

Lemma 6.3 (Mellit)**.** *Suppose that $\eta \in \mathcal{N}(k) \subseteq \mathcal{N}(\mathcal{O})$. Then the map*

$$\mathcal{T}_{KS}^{\eta} \to \mathcal{V}^{\eta}$$

is surjective. For all $\theta \in \mathcal{V}^{\eta}$, the fiber $\mathcal{T}_{KS}^{\eta,\theta}$ is contained in $G(F)_d^{\geq N}$ for some $d = d(\eta, \theta)$ and $N = N(\eta, \theta)$. Moreover, $d \geq 0$.

Following [M20, Def. 3.8], we define the *degree* of (η, θ) to be the integer $d(\eta, \theta)$ in the lemma above.

6.7. For all $\eta \sim_{\mathrm{Ad}(G(F))} \theta$, let

$$\mathcal{T}_{KS,min}^{\eta,\theta} = \{g \in \mathcal{T}_{KS}^{\eta,\theta} \mid g \text{ maximizes } \mathrm{depth}_{\varpi} \text{ in } \mathcal{T}_{KS}^{\eta,\theta}\}.$$

Note that the claim of [M20, §3.4] that this set is stable under $J_\eta \times J_\theta$ is incorrect.

Following *loc. cit.*, we define a *classification datum* for (η, θ) to be a choice of double coset

$$M_{\eta,\theta} := J_\eta g_{\eta,\theta} J_\theta \subseteq \mathcal{T}_{KS}^{\eta,\theta}.$$

Recall from Lemma 6.3 that for some d and N, we have embeddings

$$\begin{aligned} J_\eta \backslash M_{\eta,\theta} &\to G(\mathcal{O}) \backslash G(F)_d^{\geq N}, \\ M_{\eta,\theta}/J_\theta &\to G(F)_d^{\geq N}/G(\mathcal{O}), \end{aligned}$$

which shows that $J_\eta \backslash M_{\eta,\theta}$ and $M_{\eta,\theta}/J_\theta$ are (the underlying sets of) finite-dimensional projective varieties. We define the *motivic weight* of (η, θ) to be the formal ratio

$$\mathrm{wt}(\eta, \theta) = \frac{[M_{\eta,\theta}/J_\theta]}{[J_\eta \backslash M_{\eta,\theta}]}$$

of elements of the Grothendieck ring of varieties over k. It turns out a posteriori that (for k finite, the point count of) this ratio is independent of the choice of $M_{\eta,\theta}$.

Example 6.4. Take $n = 2$ and

$$\eta = \begin{pmatrix} 0 & 1 \\ & 0 \end{pmatrix} \in \mathcal{N}(\mathcal{O}).$$

For all positive integers i, let

$$\begin{aligned} g_i &= \begin{pmatrix} 1 & \\ & \varpi^i \end{pmatrix} \in G(F), \\ \theta_i &= \begin{pmatrix} 0 & \varpi^i \\ & 0 \end{pmatrix} \in \mathcal{N}(\mathcal{O}). \end{aligned}$$

Then $g_i^{-1}\eta g_i = \theta_i$, and in fact, the θ_i form a full set of representatives for the adjoint action of $G(\mathcal{O})$ on $\mathcal{V}^\eta$. That is,

$$\mathcal{V}^\eta = \coprod_{i\geq 0} \theta_i \cdot \mathrm{Ad}(G(\mathcal{O})).$$

Note that

$$\begin{aligned} G(F)_\eta = G(F)_{\theta_i} &= \left\{\begin{pmatrix} 1 & F \\ & 1 \end{pmatrix}\right\}, \\ J_\eta = J_{\theta_i} &= \left\{\begin{pmatrix} 1 & \mathcal{O} \\ & 1 \end{pmatrix}\right\}. \end{aligned}$$

Using Example 6.2, we compute:

$$\begin{aligned} \mathcal{T}^{\eta,\theta_i} = \mathcal{T}^{\eta,\theta_i}_{KS} = G(F)_\eta g_i = g_i G(F)_{\theta_i} &= \left\{\begin{pmatrix} 1 & F \\ & \varpi^i \end{pmatrix}\right\} \subseteq G(F)_i^{\geq i}, \\ \mathcal{T}^{\eta,\theta_i}_{KS,min} &= \left\{\begin{pmatrix} 1 & \varpi^i\mathcal{O} \\ & \varpi^i \end{pmatrix}\right\}. \end{aligned}$$

The latter display shows that for any $g_{\eta,\theta_i} \in \mathcal{T}^{\eta,\theta_i}_{KS,min}$, we have

$$\begin{aligned} M_{\eta,\theta_i} &= \left\{\begin{pmatrix} 1 & \mathcal{O} \\ & \varpi^i \end{pmatrix}\right\}, \\ M_{\eta,\theta_i}/J_{\theta_i} &\simeq pt, \\ J_\eta \backslash M_{\eta,\theta_i} &\simeq \mathbf{A}^i(k). \end{aligned}$$

We deduce that

$$\begin{aligned} \deg(\eta,\theta_i) &= i, \\ \mathrm{wt}(\eta,\theta_i) &= \frac{1}{[\mathbf{A}^i]}. \end{aligned}$$

6.3.

6.8. Now suppose that k is a finite field of cardinality q. Fix η. Our main interest is the volume of the stack $[\mathcal{V}^\eta/_{\mathrm{Ad}}G(\mathcal{O})]$.

To be more precise, let vol be a suitable fiberwise measure on $J \to \mathcal{N}(\mathcal{O})$. Let $|-|$ be the function on k-varieties that counts k-points. We want to compare

$$\sum_{[\theta]\in\mathcal{V}^\eta/_{\mathrm{Ad}}G(\mathcal{O})} \mathrm{vol}(J_\theta)^{-1}$$

to the following (q,t)-series studied by Mellit:

$$\sum_{[\theta]\in\mathcal{V}^\eta/_{\mathrm{Ad}}G(\mathcal{O})} t^{\deg(\eta,\theta)}|\mathrm{wt}(\eta,\theta)|.$$

The forms of the sums lead us to compare, directly,

$$\mathrm{vol}(J_\theta)^{-1} \quad \text{and} \quad t^{\deg(\eta,\theta)}\frac{|M_{\eta,\theta}/J_\theta|}{|J_\eta\backslash M_{\eta,\theta}|}. \tag{6.2}$$

6.9. Henceforth, we assume that $\eta\in\mathcal{N}(k)\subseteq\mathcal{N}(\mathcal{O})$. Using Iwasawa decomposition, we can control the form of our choices of representatives for the adjoint action of $G(\mathcal{O})$ on $\mathcal{V}^\eta$. (Compare to `2100_08`.)

Namely, extend η to an $\mathfrak{sl}_2$-triple in $\mathfrak{g}$. Let τ be the Cartan element, and let $\mathfrak{g}_n\subseteq\mathfrak{g}$ be the n-eigenspace of the adjoint action of τ, so that $\eta\in\mathfrak{g}_2$. Let

$$\mathfrak{g}_{\geq n}=\bigoplus_{i\geq n}\mathfrak{g}_i \quad \text{and} \quad \mathfrak{g}_{>n}=\bigoplus_{i>n}\mathfrak{g}_i.$$

Let $P\subseteq G$ be the parabolic subgroup with Lie algebra $\mathfrak{g}_{\geq 0}$. One might say that P is the *Jacobson–Morozov parabolic* attached to η. It may differ from the Richardson parabolic of η. It is known that

$$P(F)\supseteq G(F)_\eta.$$

Iwasawa gives

$$G(F)=P(F)G(\mathcal{O}).$$

Let $P=L\ltimes U$ be the Levi decomposition of P.

Let $Z_\eta(F)=\eta\cdot\mathrm{Ad}(L(F))\subseteq\mathfrak{g}_2(F)$, and let $Z_\eta(\mathcal{O})=Z_\eta\cap\mathfrak{g}_2(\mathcal{O})$. Then Lemma 1 of [R72] says

$$\eta\cdot\mathrm{Ad}(P)=Z_\eta(F)+\mathfrak{g}_{>2}(F).$$

So in our study of (6.2), it suffices to take

$$\theta\in Z_\eta(\mathcal{O})+\mathfrak{g}_{>2}(\mathcal{O}).$$

This in turn yields the containment $\mathcal{T}^{\eta,\theta}\subseteq P(F)$.

6.10. Now suppose that η is associated to the integer partition $\lambda\vdash n$ in the following sense. Writing $\lambda=(\lambda_1\geq\lambda_2\geq\cdots\geq\lambda_\ell)$ with $\sum_i\lambda_i=n$, let $I_\lambda\subseteq\{0,1,\ldots,n-1\}$ be the set of indices of the form $\lambda_1+\cdots+\lambda_j$ for some $j\geq 0$. Writing $(e_1,\ldots,e_n)$ for the standard basis of F^n, take $\eta=\eta_\lambda$ to be the operator

$$\eta\cdot e_{i+1}=\begin{cases}0 & i\in I_\lambda,\\ e_i & \text{else.}\end{cases}$$

Then $\ker(\eta) = F\langle e_{i+1} \mid i \in I_\lambda \rangle$.

Let $\lambda' \vdash n'$ be the partition obtained from λ by removing all parts λ_i of size 1. Thus $I'_\lambda := I_{\lambda'}$ is a subset of I_λ. Let $G' = \mathrm{GL}_{n'}$, and let P', L', U' be the respective analogues of P, L, U with λ', G' in place of λ, G. We calculate directly that P' is a Borel subgroup of G', whence L' is a torus, and that

$$L \simeq L' \times \mathrm{GL}_{n-n'}.$$

Unfortunately, U is larger than U' in a complicated way.

6.11. Henceforth, suppose that λ has at most one part of size 1, so that either $n' = n$ or $n' = n - 1$. In this case, P is a Borel subgroup of G. Since L is now a torus, $\mathfrak{V}$ is contained in the sum of the root spaces that support η.

For all $g \in P(F)$, we know that g^{-1} is η-kernel-strict if and only if

$$\mathcal{O}\langle e_{i+1} \mid i \in I_\lambda \rangle = F\langle e_{i+1} \mid i \in I_\lambda \rangle \cap g^{-1}\mathcal{O}^n.$$

Since P is now a Borel, this occurs if and only if the following condition holds: The columns of g^{-1} with indices $i+1$, as we run over $i \in I_\lambda$, belong to $\mathcal{O}^n$, and their span contains e_{i+1} for all $i \in I_\lambda$. Equivalently, the corresponding cofactor of g^{-1} must be invertible. This condition on g defines a certain subgroup $P(F)_{KS} \subseteq P(F)$. Compare to Example 6.2.

Altogether, we see that for η of the form η_λ for some $\lambda \vdash n$ with at most one part of size 1, and $\theta \in Z_\eta(\mathcal{O}) + \mathfrak{g}_{>2}(\mathcal{O})$, we have

$$\begin{aligned}
\mathcal{T}_{KS}^{\eta,\theta} &= \{g \in P(F) \mid \theta = g^{-1}\eta g \text{ and } g^{-1} \text{ is } \eta\text{-kernel-strict}\} \\
&= \{g \in P(F)_{KS} \mid \theta = g^{-1}\eta g\}.
\end{aligned}$$

Example 6.5. Take $n = 5$ and $\lambda = (3, 2)$. Here, the Jacobson–Morozov grading is given by the following matrix:

$$\begin{pmatrix}
0 & 2 & 4 & 1 & 3 \\
-2 & 0 & 2 & -1 & 1 \\
-4 & -2 & 0 & -3 & -1 \\
-1 & 1 & 3 & 0 & 2 \\
-3 & -1 & 1 & -2 & 0
\end{pmatrix}.$$

Thus L is the maximal diagonal torus and

$$U = \left\{\begin{pmatrix}
1 & * & * & * & * \\
 & 1 & * & & * \\
 & & 1 & & \\
 & * & * & 1 & * \\
 & & * & & 1
\end{pmatrix}\right\}, \quad P(F)_{KS} = \left\{\begin{pmatrix}
\mathcal{O}^\times & * & * & \mathcal{O} & * \\
 & F^\times & * & & * \\
 & & F^\times & & \\
 & * & * & \mathcal{O}^\times & * \\
 & & * & & F^\times
\end{pmatrix}\right\}.$$