11.

Today, we define the Temperley–Lieb quotient of the Iwahori–Hecke algebra using the Kazhdan–Lusztig basis and cell theory. There are many references for the latter, including Geck–Pfeiffer Chapter 11, the RSME book on Soergel bimodules, and the book by Björner–Brenti. In the last two references, their v is our x^{-1}.

I also like Geordie Williamson's bachelor's thesis. However his exposition is based around Kazhdan–Lusztig's basis $(C_w)_w$, whereas we mostly discuss their basis $(C'_w)_w$.

11.1.

As motivation: Let A be a commutative ring with unity. Let H be an associative A-algebra that is free over A with a fixed basis $(\alpha_w)_{w \in W}$. We would like to know: In which cases do subsets of the basis generate nontrivial two-sided ideals of H? Such ideals give rise to nontrivial quotients of H already equipped with an A-linear basis: namely, the images of the α_w that are nonzero.

Let $\leq'_L$ be the following relation on W: For all $v, w \in W$, we set $v \leq'_L w$ if there exists $h \in H$ such that α_v appears in the expansion of $h\alpha_w$ in the given basis. In a slogan,

"we can cause α_w to see α_v by multiplying from the left".

Let $\leq_L$ be the transitive closure of $\leq'_L$.

Example 11.1. If $\alpha_w \in H^\times$, then w is a maximal element for $\leq_L$. In particular, if $\alpha_e = 1$ for some $e \in W$, then e is a maximal element.

We see that a subset of the basis generates a left ideal $I \subseteq H$ if and only if the subset is downward-closed in $\leq_L$. The following useful fact is Proposition 4.1.1 in Williamson:

Lemma 11.2. *Suppose that S is a subset of W such that $(\alpha_s)_{s \in S}$ generates H as an A-algebra. Then $v \leq_L w$ if and only if there is a sequence of elements $v = v_0, v_1, \ldots, v_\ell = w$ in W such that, for all i, there exists $s \in S$ such that α_{v_i} appears in the expansion of $\alpha_s \alpha_{v_{i+1}}$.*

Let $\sim_L$ be the equivalence relation where $v \sim_L w$ if and only if $v \leq_L w$ and $w \geq_L v$ both hold. We define a *left cell* with respect to $(\alpha_w)_w$ to be an equivalence class of W under $\sim_L$. Tautologically, $\leq_L$ descends to a partial order on the set of left cells.

The standard basis of an Iwahori–Hecke algebra fails to give an interesting cell structure. But other bases are possible.

Example 11.3. Take $A = \mathbf{Z}[\mathsf{x}^{\pm 1}]$ and $H = H_W$. If we set $\alpha_w = \sigma_w$, giving the standard basis, then every basis element is invertible, so W forms a single left cell.

Example 11.4. Take $A = \mathbf{Z}[\mathsf{x}^{\pm 1}]$ and $H = H_W$ and $W = S_2 = \{1, s\}$. Let $c_e = \sigma_e = 1$ and $c_s = \sigma_s + \mathsf{x}^{-1} = \sigma_s^{-1} + \mathsf{x}$. Then

$$c_s^2 = (\sigma_s + \mathsf{x}^{-1})(\sigma_s^{-1} + \mathsf{x}) = \mathsf{x}\sigma_s + 2 + \mathsf{x}^{-1}\sigma_s^{-1} = (\mathsf{x} + \mathsf{x}^{-1})c_s.$$

In contrast to $\sigma_s^2 = (\mathsf{x} - \mathsf{x}^{-1})\sigma_s + 1$, the last expression has no constant term. That is, taking $\alpha_w = c_w$ gives a basis where $\{e\}$ and $\{s\}$ are separate left cells.

We define $\leq'_R, \leq_R, \sim_R$, and *right cells* in an analogous way, but using right multiplication in place of left multiplication. We define $\leq_{LR}$ to be the transitive closure of the relation generated by $\leq_L$ and $\leq_R$, and define $\sim_{LR}$ similarly. We define *two-sided cells* to be the equivalence classes of W under $\sim_{LR}$. In the last example, $\{e\}$ and $\{s\}$ are also separate as right cells, so they form separate two-sided cells as well.

Just as $\leq_L$-downward-closed subsets of W give rise to left ideals of H, so do $\leq_R$-downward-closed, *resp.* $\leq_{LR}$-downward-closed subsets give rise to right, *resp.* two-sided ideals.

11.2.

It turns out that the last example generalizes. To explain how, we point out another feature of the basis there: It is fixed under the ring anti-automorphism $D : H_W \to H_W$ defined for a general, even infinite, Coxeter group W by

$$\begin{aligned} D(\mathsf{x}) &= \mathsf{x}^{-1}, \\ D(\sigma_w) &= \sigma_{w^{-1}}^{-1} && \text{for all } w \in W. \end{aligned}$$

In what follows, recall that $S \subseteq W$ denotes the set of simple reflections, and $\ell, \leq$ denote the Bruhat length function and Bruhat partial order on W. Explicitly, $\ell(w)$ is the minimum length among words in S that represent w. The order $\leq$ is generated from requiring that $sw < w$ when $\ell(sw) = \ell(w) - 1$ and $sw > w$ when $\ell(sw) = \ell(w) + 1$, for all $s \in S$ and $w \in W$.

Theorem 11.5 (Kazhdan–Lusztig)**.** *The algebra H_W admits a unique $\mathbf{Z}[\mathsf{x}^{\pm 1}]$-linear basis $\{c_w\}_{w \in W}$ such that:*

$$\begin{aligned} D(c_w) &= c_w, \\ c_w &= \sum_v D(p_{v,w})\sigma_v \end{aligned}$$

for some polynomials $p_{v,w} \in \mathbf{Z}[\mathsf{x}]$ satisfying:

(1) $p_{v,w} = 0$ *for all* $v \not\leq w$.
(2) $p_{w,w} = 1$.
(3) $p_{v,w} \in \mathsf{x}\mathbf{Z}[\mathsf{x}]$ *for all* $v < w$.

Moreover, if $v < w$*, then*

$$p_{v,w}(\mathsf{x}) = \mathsf{x}^{\ell(w)-\ell(v)} P_{v,w}(\mathsf{x}^{-2})$$

for some $P_{v,w}(\mathsf{q}) \in \mathbf{Z}[\mathsf{q}]$ *such that* $\deg P_{v,w} \leq \frac{1}{2}(\ell(w) - \ell(v) - 1)$.

As preparation: For all $v, y \in W$, let $r_{y,v} \in \mathbf{Z}[\mathsf{x}^{\pm 1}]$ be defined by

$$D(\sigma_v) = \sum_y r_{y,v} \sigma_y.$$

Induction on $\ell(v)$ shows that the sum above is finite. In fact, $r_{y,v} = 0$ when $y \not\leq v$.

Our conventions are related to Kazhdan–Lusztig's by $\mathsf{x} = q^{1/2}$ and $\sigma_w = \mathsf{x}^{\ell(w)} T_w$ and $c_w = C'_w$. Consequently our polynomials are related to theirs by

$$p_{v,w} = \mathsf{x}^{\ell(w)-\ell(v)} D(P_{v,w}) \quad \text{and} \quad r_{y,v} = \mathsf{x}^{\ell(v)-\ell(y)} D(R_{y,v}).$$

The $P_{v,w}$ and $R_{y,v}$ are called the *Kazhdan–Lusztig P-* and *R-polynomials*.

Proof. If the basis $\{c_w\}_w$ exists, then we must have

$$\sum_x D(p_{v,w}) \sigma_x = \sum_{v,y} p_{v,w} r_{y,v} \sigma_y = \sum_{v,y} p_{y,w} r_{v,y} \sigma_v, \tag{11.1}$$

from which $D(p_{v,w}) = \sum_y r_{v,y} p_{y,w}$. But we know that $r_{v,y} = 0$ for $y \not\leq v$, and $r_{v,v} = 1$. So we must have

$$D(p_{v,w}) - p_{v,w} = \sum_{y > v} r_{v,y} p_{y,w}.$$

Conversely, any family of polynomials $p_{v,w}$ satisfying (11.1) and conditions (1)–(3) will determine a basis of the kind we want.

To show its existence and uniqueness, we take these conditions as a definition of the $p_{v,w}$, and induct downward in the order $<$ on the set of elements $v < w$. At step v, the inductive hypothesis says that the right-hand side (=: r.h.s.) of (11.1) is fixed. If we could show $D(\text{r.h.s.}) = -\text{r.h.s.}$, then r.h.s. would take the form $D(p_{v,w}) - p_{v,w}$ for a unique $p_{v,w} \in \mathsf{x}\mathbf{Z}[\mathsf{x}]$, completing the induction. We compute

$$
\begin{aligned}
& D\left(\sum_{y>v} r_{v,y}\, p_{y,w}\right) \\
& = \sum_{y>v} D(r_{v,y}) D(p_{y,w}) \\
& = \sum_{y>v} D(r_{v,y}) \sum_{z\geq y} r_{y,z}\, p_{z,w} && \text{by the induct. hyp.} \\
& = \sum_{\substack{y,z \\ z\geq y\geq v}} D(r_{v,y}) r_{y,z}\, p_{z,w} - \sum_{z\geq v} r_{v,z}\, p_{z,w}. \qquad (11.2)
\end{aligned}
$$

Observe that $\sigma_z = D^2(\sigma_z) = \sum_{v\leq y\leq z} D(r_{y,z}) r_{v,y} \sigma_v$. Therefore,

$$
\sum_{\substack{y \\ v\leq y\leq z}} D(r_{v,y}) r_{y,z} = \sum_{y} r_{v,y} D(r_{y,z}) = \begin{cases} 1 & v = z, \\ 0 & v \neq z. \end{cases}
$$

Therefore (11.2) simplifies to $-\sum_{z>v} r_{v,z}\, p_{z,w}$ as desired. □

Remark 11.6. The story above can be generalized from $\mathbf{Z}[\mathsf{x}^{\pm 1}]$ and W to other rings A and (ranked) posets Λ equipped with an involution on the free A-module generated by Λ. See Proudfoot's article, "The Algebraic Geometry of Kazhdan–Lusztig–Stanley Polynomials".

11.3.

The basis $(c_w)_w$ is called the *Kazhdan–Lusztig basis* of H_W. For $W = S_2$ and $W = S_3$, the polynomials $P_{v,w}$ are all 0 or 1, but for general W, they become much more chaotic.[1] The first nontrivial Kazhdan–Lusztig P-polynomials occur for $W = S_4$: *e.g.*,

$$
P_{s_1, s_2 s_1 s_3 s_2}(\mathsf{q}) = P_{s_1 s_3, s_1 s_2 s_3 s_2 s_1}(\mathsf{q}) = 1 + \mathsf{q}.
$$

(This also implies the nontriviality of $P_{v, s_2 s_1 s_3 s_2}$, $P_{v, s_1 s_2 s_3 s_2 s_1}$ for lower v in the Bruhat order.) So in general it is difficult to compute the $P_{v,w}$, meaning it is difficult to compute the Kazhdan–Lusztig basis as well.

But to calculate the left cells with respect to the Kazhdan–Lusztig basis, we need to know their products. By Lemma 11.2, it suffices to compute the products $c_s c_w$ for $s \in S$ and $c \in W$. Note that $c_s = \sigma_s + \mathsf{x}^{-1}$ for all s.

Following Kazhdan–Lusztig, we will write $v \prec w$ if and only if $v < w$ and $\deg P_{v,w} = \frac{1}{2}(\ell(w) - \ell(v) - 1)$. In this case, we set $\mu(v, w)$ to be the leading coefficient of $P_{v,w}$. The following is Theorem 3.6.1 in Williamson:

[1] See the tables on Mark Goresky's website: `https://www.math.ias.edu/~goresky/preprints.html`

Theorem 11.7. *For any $s \in S$ and $w \in W$, we have*

$$\sigma_s c_w = \begin{cases} \mathsf{x} c_w & sw < w, \\ c_{sw} - \mathsf{x}^{-1} c_w + \sum_{\substack{v \prec w \\ sv < v}} \mu(v,w) c_v & sw > w. \end{cases}$$

Equivalently,

$$c_s c_w = \begin{cases} (\mathsf{x} + \mathsf{x}^{-1}) c_w & sw < w \\ c_{sw} + \sum_{\substack{v \prec w \\ sv < v}} \mu(v,w) c_v & sw > w \end{cases}$$

Example 11.8. Take $W = S_3$ and write $S = \{s, t\}$ for convenience. Since the P-polynomials are trivial in this setting, we have $v \prec w$ if and only if $v < w$ and $\ell(w) - \ell(v) \in \{1, 2\}$. We obtain the following tables of products $\sigma_s c_w$:

	c_e	c_s	c_t
σ_s	$c_s - \mathsf{x}^{-1}$	$\mathsf{x} c_s$	$c_{st} - \mathsf{x}^{-1} c_t$
σ_t	$c_t - \mathsf{x}^{-1}$	$c_{ts} - \mathsf{x}^{-1} c_s$	$\mathsf{x} c_t$

	c_{st}	c_{ts}	c_{sts}
σ_s	$\mathsf{x} c_{st}$	$c_{sts} - \mathsf{x}^{-1} c_{ts} + c_s$	$\mathsf{x} c_{sts}$
σ_t	$c_{sts} - \mathsf{x}^{-1} c_{st} + c_t$	$\mathsf{x} c_{st}$	$\mathsf{x} c_{sts}$

Equivalently:

	c_e	c_s	c_t
c_s	c_s	$(\mathsf{x} + \mathsf{x}^{-1}) c_s$	c_{st}
c_t	c_t	c_{ts}	$(\mathsf{x} + \mathsf{x}^{-1}) c_t$

	c_{st}	c_{ts}	c_{sts}
c_s	$(\mathsf{x} + \mathsf{x}^{-1}) c_{st}$	$c_{sts} + c_s$	$(\mathsf{x} + \mathsf{x}^{-1}) c_{sts}$
c_t	$c_{sts} + c_t$	$(\mathsf{x} + \mathsf{x}^{-1}) c_{st}$	$(\mathsf{x} + \mathsf{x}^{-1}) c_{sts}$

We deduce that the left cells of S_3 are:

$$\{e\}, \{s, ts\}, \{t, st\}, \{sts\}.$$

In a similar way, one checks that the right cells are

$$\{e\}, \{s, st\}, \{t, ts\}, \{sts\}.$$

Thus the two-sided cells are $\{e\}, \{s, t, st, ts\}, \{sts\}$.

The union of the second and third two-sided cells generates a two-sided ideal of H_W. The corresponding quotient ring is $\mathbf{Z}[\mathsf{x}^{\pm 1}]$. The third two-sided cell generates a smaller, principal two-sided ideal. The corresponding quotient is a little more interesting.

11.4.

Below we state some facts about the left, right, and two-sided cells for the Kazhdan–Lusztig basis. For the proofs, we refer to Björner–Brenti.

(1) $\{e\}$ and $\{w_\circ\}$ always form two-sided cells, where $w_\circ \in W$ denotes the unique *longest element* with respect to Bruhat length.
(2) The elements w of a left cell have the same right *descent sets* $\{s \in S \mid ws < w\}$ with respect to the Bruhat order. That is, the set of elements with a fixed right descent set is a union of left cells. Analogous statements hold with the words left and right reversed.
(3) The set of elements $w \in S_n$ for which there exist indices $1 \leq i < j < k \leq n$ such that $w(i) > w(j) > w(k)$ is a $\leq_{LR}$-downward-closed subset of $W = S_n$ for the Kazhdan–Lusztig basis. Note that this condition is equivalent to the existence of a word for w, with respect to the standard generating set $S = \{s_i \mid 1 \leq i \leq n-1\}$, that contains a subword of the form $s_i s_{i+1} s_i$ for some i.

    We say that w is *fully commutative*, or 321-*avoiding*, if and only if these properties do not hold. For instance, in S_3, the only element that is not 321-avoiding is $s_1 s_2 s_1$, corresponding to the permutation (321) itself.

Kazhdan–Lusztig's original motivation to study cells was to construct interesting H_W-modules with explicit bases, via quotients of the form $I_{\leq C}/I_{<C}$, where C is a fixed left cell, $I_{\leq C}$ is the ideal generated by C and all left cells below it, and $I_{<C}$ is the subideal generated by left cells strictly below C. This forms an H_W-module called the *left cell module* generated by C. It turns out that when $W = S_n$, all simple H_W-modules arise this way.

Something that is still mysterious here is why the involution D should have anything to do with a $\mathbf{Z}[\mathsf{x}^{\pm 1}]$-basis of H_W producing interesting cells.

11.5.

We define the *Temperley–Lieb algebra* TL_n over $\mathbf{Z}[\mathsf{x}^{\pm 1}]$ to be the algebra generated by elements $e_1, \ldots, e_{n-1}$ subject to these relations:

$$\begin{aligned} e_i e_{i+1} e_i = e_i e_{i-1} e_i &= e_i, \\ e_i e_j &= e_j e_i && \text{for } |i-j| > 1, \\ e_i^2 &= (\mathsf{x} + \mathsf{x}^{-1}) e_i. \end{aligned}$$

Set $H_n = H_{S_n}$ as usual; also, set $\sigma_i = \sigma_{s_i}$ and $c_i = c_{s_i}$. What follows is part of Corollary 3.11 from a bachelor's thesis at the University of Amsterdam by Tim Weelinck.

Theorem 11.9 (Jones)**.** *There is a surjective homomorphism of* $\mathbf{Z}[\mathsf{x}^{\pm 1}]$*-algebras* $H_n \to TL_n$ *that sends* $c_i \mapsto e_i$ *for all* i*. It is an isomorphism for* $n = 1, 2.$

For $n \geq 3$, the kernel is generated by the elements c_w where $w \in S_n$ is not 321-*avoiding.*

The key point is that the relations $e_i e_{i+1} e_i = e_i e_{i-1} e_i = e_i$ come from the braid relations. The braid relations on the generators $\sigma_i \in H_n$ can be rewritten in terms of the Kazhdan–Lusztig elements c_i as

$$c_i c_{i+1} c_i - c_i = c_{i+1} c_i c_{i+1} - c_{i+1}.$$

Using our calculations for S_3, we can check that

$$c_{s_i s_{i+1} s_i} = c_i c_{i+1} c_i - c_i.$$

In the Temperley–Lieb quotient, the right-hand side becomes $e_i e_{i+1} e_i - e_i$, while the left-hand side vanishes.

Let $S_n^{fc} \subseteq S_n$ be the subset of 321-avoiding elements. For all $w \in S_n^{fc}$, let $e_w \in TL_n$ be the image of $c_w \in H_n$. We deduce:

Corollary 11.10. $(e_w)_{w \in S_n^{fc}}$ *is a* $\mathbf{Z}[\mathsf{x}^{\pm 1}]$*-linear basis for* TL_n*.*

In the notation of the previous lecture, $kTL_n \simeq k \otimes_{\mathbf{Z}[\mathsf{x}^{\pm 1}]} TL_n$, meaning TL_n really is an integral form of kTL_n.

11.6.

One more mystery for now: Consider the ring automorphism $\kappa : H_W \to H_W$ defined for general W by

$$\begin{aligned} \kappa(\mathsf{x}) &= -\mathsf{x}^{-1}, \\ \kappa(\sigma_w) &= \sigma_w. \end{aligned}$$

Then we can check that $\kappa D = D\kappa$. Therefore, $(\kappa(c_w))_w$ is another D-invariant basis of H_W, which turns out to be different from $(c_w)_w$. This is the basis denoted $(C_w)_w$ in Kazhdan–Lusztig's conventions.

This means there are *two* distinct surjective $\mathbf{Z}[\mathsf{x}^{\pm 1}]$-algebra homomorphisms from H_n to TL_n. For $w \in S_n^{fc}$, one sends $c_w \mapsto e_w$, while the other sends $\kappa(c_w) \mapsto -e_w$.

Jones did not work with H_n initially, but with TL_n, which had been discovered earlier in the context of statistical mechanics(!), and which Jones had studied in the context of von Neumann algebras. He introduced traces on the Temperley–Lieb algebras that gave rise to the link invariant now called the Jones polynomial. Later, when the HOMFLYPT polynomial was discovered, it was observed that Ocneanu's Markov trace on H_n was a refinement of Jones's trace on TL_n.

But the *two* distinct ways to express TL_n as a quotient of H_n leads to two distinct traces on TL_n that are specializations of the same trace on H_n.