from the HW:

if {W_i}_{i in I} is a collection of linear sub.'s of V,
with I possibly infinite,
then their sum is defined to be [what?]

sum_{i in I} W_i
= {sum_{i in J} w_i | J sub I finite,
w_i in W_i for all i}
( = {sum_i w_i | w_i in W_i for all i,
w_i = **0** for all but fin many i } )

[will use second version today]

Prop sum_{i in I} W_i is the minimal lin. sub. containing W_i for all i

Df sum_{i in I} W_i is a direct sum
iff [what?]
every elt has a unique expression
sum_{i in I} w_i
s.t. w_i in W_i for all i
(and w_i = **0** for all but fin many i)

[what does unique mean?]

for any sets {w_i}_i, {w'_i}_i
s.t. w_i, w'_i in W_i for all i
and w_i, w'_i = **0** for all but fin many i,

sum_i w_i = sum_i w'_i implies (w_i = w'_i for all i)

Prop suppose the uniqueness holds for **0**:

for any set $\{w_i\}_i$

s.t. $w_i$ in $W_i$ for all i

(and $w_i$ = **0** for all but fin many i),

$\sum_{i \in I} w_i = \mathbf{0}$ implies ($w_i = \mathbf{0}$ for all i)

then the uniqueness holds in general: i.e.,

$\sum_{i \in I} W_i$ is a direct sum

Pf suppose that

$\sum_i w_i = v = \sum_i w'_i$

then $\sum_i (w_i - w'_i) = \mathbf{0}$

so $w_i - w'_i = \mathbf{0}$ for all i

(Axler §2A) $\{v_i\}_{i \in I}$ any set of vectors in V

Df $\{v_i\}_i$ is said to be a linearly independent set of vectors iff either of these equivalent cond's:

I) **0** has a unique expression as $\sum_i a_i v_i$:

for any set $\{a_i\}_i$

s.t. $a_i$ in F for all i,

$a_i = 0$ for all but fin many i,

$\sum_i a_i v_i = \mathbf{0}$ implies ($a_i = 0$ for all i)

II) $\sum_i F v_i$ is a direct sum

else we say $\{v_i\}_i$ is a linearly dependent set

Lem $\{v_i\}_i$ is linearly dependent iff
there exist finite subset $\{v_j\}_{j \in J}$,
$i \notin J$
s.t. $v_i = \sum_{j \in J} a_j v_j$

in this case, we say:

$v_i$ is a linear combination of the $v_j$'s for $j$ in $J$,
with coeffs $a_j$'s

also say:

$v_i$ is linearly dependent upon the $v_j$'s

[motivates next defn:]

Df the span of $\{v_i\}_i$ is (simultaneously)

1) $\{\sum_i a_i v_i \mid a_i \in F$ for all $i$,
$a_i = 0$ for all but fin many $i\}$
2) $\sum_{i \in I} Fv_i$, where $Fv_i = \{av_i \mid a \in F\}$
3) the minimal linear subspace of $V$ containing $v_i$ for all $i$

i.e. 1), 2), 3) are all the same
and $\{v_i\}_i$ is said to span [verb] it

Ex in $F[x]$ = {set of polynomials in $x$ over $F$}:

$\{x^k \mid k \geq 0\} = \{1, x, x^2, x^3, \ldots\}$ spans $F[x]$

[why? every polynomial is a sum of monomials]

<u>Ex</u> let $\mathbf{N} = \{1, 2, 3, \dots\}$
in $F^{\mathbf{N}}$ = {functions from N into F}:
let $e_i : \mathbf{N}$ to F be the function
$e_i(i) = 1$,
$e_i(j) = 0$ for $j \neq i$

$\{e_i \mid i \text{ in } \mathbf{N}\}$ does not span $F^{\mathbf{N}}$

[why?] consider the function f s.t. $f(i) = 1$ for all i

[most striking thm thus far:]

<u>Thm</u> (Steinitz Exchange) if
$\{v_1, \dots, v_k\}$ is a lin. independent set in V,
$\{e_1, \dots, e_n\}$ spans V

then $k \leq n$

[crucially, both sets of vectors are finite]

<u>Cor</u> if V is spanned by n vectors,
then any set with > n vectors
has some linear dependence

<u>Cor</u> if there is a linearly independent set
of k vectors in V,
then any set with < k vectors
cannot span V

<u>Pf of Thm</u> let $S_0 = \{e_1, \dots, e_n\}$

will prove that for $\ell = 1, \dots, k$,
we can construct $S_\ell$ from $S_{\ell - 1}$ s.t.

1) $S_\ell$ still spans V
2) $S_\ell$ has one more $v_i$ and one fewer $e_j$
than $S_{\ell - 1}$
thus $\ell \leq n$ at each step [and $k \leq n$ at the last step]

WLOG reindex the $v_i$'s and $e_j$'s s.t.
$S_{\ell - 1} = \{v_1, \ldots, v_{\ell - 1}, e_\ell, \ldots, e_n\}$
since $S_{\ell - 1}$ spans V,
$v_\ell = \sum_{i = 1}^{\ell - 1} a_i v_i$
$+ \sum_{j = \ell}^n b_j e_j$
with some coeff nonzero
if $b_j = 0$ for all j, then $\{v_i\}_i$ lin. dep.
so we can pick j s.t. $b_j \neq 0$
so $e_j = (1/b_j)(v_\ell - \text{other stuff})$

build $S_\ell$ by appending $v_\ell$ and removing $e_j$ □

(Axler §2B–2C)

<u>Df</u> a basis for V is a set of vectors $\{v_i\}_i$
s.t. 1) $\{v_i\}_i$ spans V
2) $\{v_i\}_i$ is a linearly independent set

<u>Cor</u> if V has a finite basis of size r,
then any basis for V has size r

<u>Pf</u> if $\{e_1, \ldots, e_r\}$ is a basis,
and $\{f_1, \ldots, f_s\}$ is another:

$r \leq s$ because
$\{e_i\}_i$ is lin. indep. and $\{f_j\}_j$ is spanning
$s \geq r$ because
$\{f_i\}_i$ is lin. indep. and $\{e_j\}_j$ is spanning

Df if $V$ has a finite basis,
then we define the dimension of $V$ to be

$$\dim(V) = \text{size of any basis for } V$$

else we say $V$ is infinite-dimensional

“the Good, the Bad, and the Ugly”

V has finite dimension

V has infinite dimension, yet has an explicit basis

e.g., $F[x]$

V has infinite dimension and no explicit basis

e.g., $F^{\mathbf{N}}$