3.

Based on `2402_02`.

For any root datum, we introduce functions that count roots obeying certain linear conditions. Building on ideas of Goresky–Kottwitz–MacPherson and Hikita, we show that these functions interpolate between objects of combinatorics and algebraic geometry: LLT polynomials for tuples of shifted ribbons, and generating functions for certain dimensions arising in the study of equivalued affine Springer fibers of arbitrary type. We deduce a geometric interpretation of Haglund–Haiman–Loehr's formula for Macdonald polynomials. We also obtain a notion of LLT polynomials for general root data that is quite different from Grojnowski–Haiman's.

3.1.

3.1. Fix a (reduced, finite) root datum $(X, \mathfrak{R}, X^\vee, \mathfrak{R}^\vee)$. Thus, X and $X^\vee$ are dual lattices of finite rank, while $\mathfrak{R} \subseteq X$ and $\mathfrak{R}^\vee \subseteq X^\vee$ are finite subsets constrained by conditions spelled out in, *e.g.*, Bourbaki.

Elements of $\mathfrak{R}$ are called *roots*. We regard them as linear functionals on

$$X^\vee_{\mathbf{R}} := X^\vee \otimes \mathbf{R}.$$

We write W for the Weyl group of $\mathfrak{R}$, regarded as a reflection group of $X^\vee_{\mathbf{R}}$.

3.2. For any $x \in X^\vee_{\mathbf{R}}$ and real number $\varrho > 0$, we set

$$\mathfrak{R}_{x,\varrho} = \{\alpha \in \mathfrak{R} \mid \alpha(x) = \varrho\},$$
$$\mathfrak{R}_{x,[0,\varrho)} = \{\alpha \in \mathfrak{R} \mid 0 < \alpha(x) < \varrho\}.$$

Definition 3.1. For any $\epsilon \in X^\vee_{\mathbf{R}}$:

(1) We define the *(x, ϱ)-descent set* of ϵ to be

$$\mathsf{Des}(\epsilon) = \mathsf{Des}_{x,\varrho}(\epsilon) := \{\alpha \in \mathfrak{R}_{x,\varrho} \mid \alpha(\epsilon) < 0\}.$$

(2) We define the *(x, ϱ)-inversion set* of ϵ to be

$$\mathsf{Inv}(\epsilon) = \mathsf{Inv}_{x,\varrho}(\epsilon) := \{\alpha \in \mathfrak{R}_{x,[0,\varrho)} \mid \alpha(\epsilon) < 0\}.$$

Definition 3.2. We define the *LLT function* of a subset $\mathfrak{Q} \subseteq \mathfrak{R}_{x,\varrho}$ to be the function $F_{\mathfrak{Q}} = F^{\varrho}_{x,\mathfrak{Q}} : X^\vee_{\mathbf{R}}/W \to \mathbf{Z}[t]$ such that

$$F_{\mathfrak{Q}}(o) = \sum_{\substack{\epsilon \in o \\ \mathsf{Des}(\epsilon) = \mathfrak{Q}}} t^{|\mathsf{Inv}(\epsilon)|}.$$

3.3. The function $F_{\mathfrak{Q}}$ is constructible with respect to the (finite) stratification of $X^\vee_{\mathbf{R}}/W$ determined by the roots. To explain how, we fix a system of simple roots $\{\alpha_i\}_{i\in I} \subseteq \mathfrak{R}$. For each subset $J \subseteq I$, let

$$\mathfrak{C}_J = \left\{ x \in X^\vee_{\mathbf{R}} \,\middle|\, \begin{array}{ll} \alpha_i(x) > 0 & \text{for all } i \in J, \\ \alpha_i(x) = 0 & \text{for all } i \notin J \end{array} \right\}.$$

Note that $\mathfrak{C}_\emptyset = \{0\}$, whereas $\mathfrak{C}_I$ is the open *fundamental Weyl chamber* associated with the system of simple roots.

We declare two subsets of I to be equivalent whenever the corresponding sets of simple roots are conjugate under W. Fix a full set of representatives J for this equivalence relation. The union of the sets $\mathfrak{C}_J$ for $J \in$ J is a fundamental domain for the W-action on $X^\vee_{\mathbf{R}}$. The images of these sets in $X^\vee_{\mathbf{R}}/W$ form a stratification of $X^\vee_{\mathbf{R}}/W$, along which our LLT functions are locally constant.

For this reason, it suffices to evaluate $F_{\mathfrak{Q}}$ at orbits o of the form

$$o_J := [\rho^\vee_J]$$

where $J \in$ J and $\rho^\vee_J$ is defined as follows. For all $i \in I$, let $\omega^\vee_i \in (\mathbf{Z}\mathfrak{R})^\vee$ be the *fundamental coweight* dual to α_i. Let

$$\rho^\vee_J = \sum_{i\in J} \omega^\vee_i.$$

Then $\alpha_i(\rho^\vee_J) > 0$ when $i \in J$, and $\alpha_i(\rho^\vee_J) = 0$ otherwise. For convenience, we also set $\rho^\vee = \rho^\vee_I$.

Example 3.3. Take $(X, \mathfrak{R}, X^\vee, \mathfrak{R}^\vee)$ to be the root datum of the general linear group GL_N. We identify the lattice $X^\vee$ with $\mathbf{Z}^N$, and the group W with S_N. The roots of GL_N are the functionals $\alpha_{i,j}$, for $1 \leq i, j \leq N$ with $i \neq j$, defined by

$$\langle \alpha_{i,j}, x\rangle = x_i - x_j \quad \text{for all } x \in X^\vee.$$

Without loss of generality, we can take $I = \{1, \ldots, n-1\}$ and the simple roots to be $\alpha_i := \alpha_{i,i+1}$ for all i. Then

$$\omega_i = (\overbrace{\tfrac{1}{2}, \ldots, \tfrac{1}{2}}^{i \text{ times}}, -\tfrac{1}{2}, \ldots, -\tfrac{1}{2}).$$

Each partition $\mu \vdash N$ defines a subset $J \subseteq I$: explicitly,

$$J = \{\mu_1, \mu_1 + \mu_2, \mu_1 + \mu_2 + \mu_3, \ldots\} \setminus \{N\}.$$

We set $o_\mu = o_J$. Writing l for the length of μ, we have

$$\rho^\vee_J = (\overbrace{\tfrac{l}{2}, \ldots, \tfrac{l}{2}}^{\mu_1 \text{ times}}, \overbrace{\tfrac{l-2}{2}, \ldots, \tfrac{l-2}{2}}^{\mu_2 \text{ times}}, \ldots, \overbrace{-\tfrac{l}{2}, \ldots, -\tfrac{l}{2}}^{\mu_l \text{ times}}).$$

In particular, $\rho^\vee = (\frac{N-1}{2}, \frac{N-3}{2}, \ldots, -\frac{N-1}{2})$.

3.2.

We now show that the functions $F^{\varrho}_{x,\mathfrak{Q}}$ subsume the LLT polynomials attached to tuples of ribbons, possibly with shifted contents.

3.4. Fix a positive integer d. It will be convenient to write $\mathbf{Z}_d = \{0, 1, \ldots, d-1\}$. Let $\mathbf{c} : \mathbf{Z}_d \times \mathbf{Z} \xrightarrow{\sim} \mathbf{Z}$ be the bijection

$$\mathbf{c}(i, j) = i - dj.$$

For any tuples $\nu = (\nu_0, \nu_1, \ldots, \nu_{d-1}) \in \mathbf{Z}^d_{\geq 0}$ and $s = (s_0, s_1, \ldots, s_{d-1}) \in \mathbf{Z}^d$, let

$$C_{\nu,s} = \bigcup_{i \in \mathbf{Z}_d} C^{(i)}_{\nu,s}, \quad \text{where } C^{(i)}_{\nu,s} = \{\mathbf{c}(i, j) \mid s_i \leq j < \nu_i + s_i\}.$$

The following kinds of data are in bijection:

- triples (ν, s, D), where $\nu \in \mathbf{Z}^d_{\geq 0}$ and $s \in \mathbf{Z}^d$ and D is an arbitrary subset of $C_{\nu,s}$;
- d-tuples of ribbons (*a.k.a.* border strips or rim hooks) labeled with content shifts.

Explicitly, (ν, s, D) corresponds to the tuple where the ith ribbon has length ν_i, its contents are offset by s_i, and and its descents correspond to the values of j such that $\mathbf{c}(i, j) \in D$.

3.5. Let $C \subseteq \mathbf{Z}$ be any subset. For all $c \in C$, we set

$$\mathsf{attk}_C(u) = C \cap \{c + 1, \ldots, c + d - 1\}.$$

A *filling* of C is a function $f : C \to \mathbf{Z}_{>0}$. A *descent* of such a function f is an element $c \in C$ such that $c + d \in C$ and $f(c) > f(c + d)$. An *inversion* of f is a pair $(c, b) \in C^2$ such that $b \in \mathsf{attk}_C(c)$ and $f(c) > f(b)$. We write $\mathsf{Des}(f)$ and $\mathsf{Inv}(f)$ for the collections of descents and inversions of f, respectively.

3.6. Let $\Lambda = \varprojlim_N \mathbf{Z}[x_1, x_2, \ldots, x_N]^{S_N}$, the ring of symmetric functions.

For any finite subset $C \subseteq \mathbf{Z}$ and filling $f : C \to \mathbf{Z}_{>0}$, we set

$$x^f = \prod_{k>0} x_k^{|f^{-1}(k)|}.$$

For all $\nu \in \mathbf{Z}^d_{\geq 0}$ and $s \in \mathbf{Z}^d$ and $D \subseteq C_{\nu,s}$, the LLT polynomial of the tuple of ribbons associated with (ν, s, D) is

$$F_{(\nu,s),D} = \sum_{\substack{f : C_{\nu,s} \to \mathbf{Z}_{>0} \\ \mathsf{Des}(f) = D}} x^f t^{|\mathsf{Inv}(f)|} \in \Lambda[t].$$

Compare to [HHL05, Def. 3.2]. Up to normalization, this is the special case of formula (76) in [HHLRU], where each skew shape is a ribbon. Note that our convention for $\mathbf{c}$ is inconsistent with [HHL08].

3.7. We use the notation of Example 3.3 in what follows. Keeping d, ν, s as above, let $N = \sum_i \nu_i$. Take $(X, \mathfrak{R}, X^\vee, \mathfrak{R}^\vee)$ to be the root datum of GL_N, and identify $X^\vee_{\mathbf{R}}$ with $\mathbf{R}^N$. We say that $x \in \mathbf{R}^N$ is *adapted to (ν, s)* if and only if its entries x_i are pairwise distinct and some affine map $\mathbf{t}_{\nu,s} : \mathbf{R} \to \mathbf{R}$ of strictly negative slope restricts to a bijection

$$\mathbf{t}_{\nu,s} : \{x_i \mid 1 \leq i \leq N\} \xrightarrow{\sim} C_{\nu,s}.$$

Together, (ν, s) and x uniquely determine $\mathbf{t}_{\nu,s}$. If $\mathbf{t}_{\nu,s}$ has slope $-m$, then we say that x is adapted to (ν, s) *with slope m*. Unpacking definitions, we deduce:

Lemma 3.4. *If $x \in \mathbf{R}^N$ is adapted to (ν, s) with slope m, then the map*

$$\begin{aligned} \mathfrak{R} &\to C^2_{\nu,s} \\ \alpha_{i,j} &\mapsto (\mathbf{t}_{\nu,s}(x_j), \mathbf{t}_{\nu,s}(x_i)) \end{aligned} \tag{3.1}$$

restricts to bijections

$$\mathfrak{R}_{x,d/m} \xrightarrow{\sim} \{(c, b) \in C^2_{\nu,s} \mid b = c + d\}, \tag{3.2}$$

$$\mathfrak{R}_{x,[0,d/m)} \xrightarrow{\sim} \{(c, b) \in C^2_{\nu,s} \mid b \in \mathsf{attk}_{C_{\nu,s}}(c)\}. \tag{3.3}$$

Example 3.5. Suppose that $N \leq d$. Take $\nu = (1, \ldots, 1, 0, \ldots, 0)$, where 1 occurs N times, and $s = (0, 0, \ldots, 0)$. Then for any $m > 0$, the vectors in the S_N-orbit of $\frac{1}{m}\rho^\vee$ are adapted to (ν, s) with slope m.

3.8. Next, we compare fillings, descents, and inversions. For any partition μ, we set

$$x^\mu = \prod_k x_k^{\mu_k}.$$

Recall the usual bases for Λ as a $\mathbf{Z}$-module: the monomial symmetric functions $\{m_\mu\}_\mu$, the homogeneous symmetric functions $\{h_\mu\}_\mu$, the Schur functions $\{s_\mu\}_\mu$, *etc*. We write $\langle -, - \rangle$ for the Hall pairing on Λ, under which $\{m_\mu\}_\mu$ and $\{h_\mu\}_\mu$ form dual bases, and the basis $\{s_\mu\}_\mu$ is orthonormal. We extend $\langle -, - \rangle$ to $\Lambda[t]$ by linearity. Then $\langle h_\mu, F_{(\nu,s),D}\rangle \in \mathbf{Z}[t]$ is the coefficient of x^μ in $F_{(\nu,s),D}$: equivalently,

$$\langle h_\mu, F_{(\nu,s),D}\rangle = \sum_{\substack{f : C_{\nu,s} \to \mathbf{Z}_{>0} \\ \mathsf{Des}(f) = D \\ |f^{-1}(k)| = \mu_k \text{ for all } k}} t^{|\mathsf{Inv}(f)|}.$$

Here, we set $\mu_k = 0$ for all k beyond the length of μ. In what follows, let $\mathsf{Fill}_{\nu,s}(\mu)$ be the set of fillings $f : C_{\nu,s} \to \mathbf{Z}_{>0}$ such that $|f^{-1}(k)| = \mu_k$ for all k.

Suppose that $\mu \vdash N$, as in the setup of Example 3.3. In any vector $\epsilon \in o_\mu$, the kth largest value among the entries of ϵ occurs exactly μ_k times. Since S_N acts transitively on the orbit, every possible permutation of the entries occurs. Therefore, each $\epsilon \in o_\mu$ determines and is determined by the function

$$f_\epsilon : \{1, \ldots, N-1\} \to \mathbf{Z}_{>0}$$

such that $f_\epsilon(i) = k$ if and only if ϵ_i is the kth largest value among the entries of ϵ. Next, suppose that $x \in \mathbf{R}^N$ is adapted to (ν, s). Let

$$f_{\epsilon,x} : C_{\nu,s} \to \mathbf{Z}_{>0}$$

be the filling such that $f_{\epsilon,x}(\mathbf{t}_{\nu,s}(x_i)) = f_\epsilon(i)$ for all i. By construction, $f_{\epsilon,x} \in \mathsf{Fill}_{\nu,s}(\mu)$, and every element of $\mathsf{Fill}_{\nu,s}(\mu)$ arises this way. Moreover, for fixed x, we can recover f_ϵ and hence ϵ from $f_{\epsilon,x}$ This proves part (1) of the following. Parts (2)–(3) amount to unpacking definitions.

Lemma 3.6. *If $x \in \mathbf{R}^N$ is adapted to (ν, s), then:*

(1) The map $o_\mu \to \mathsf{Fill}_{\nu,s}(\mu)$ that sends $\epsilon \mapsto f_{\epsilon,x}$ is a bijection.

If it is adapted to (ν, s) with slope m, then:

(2) (3.2) induces a bijection $\mathsf{Des}(\epsilon) \xrightarrow{\sim} \mathsf{Des}_{x,d/m}(f_{\epsilon,x})$.

(3) (3.3) induces a bijection $\mathsf{Inv}(\epsilon) \xrightarrow{\sim} \mathsf{Inv}_{x,d/m}(f_{\epsilon,x})$.

Proposition 3.7. *If $x \in \mathbf{R}^N$ is adapted to (ν, s) with slope m, and (3.2) transports $\mathfrak{Q} \subseteq \mathfrak{R}_{x,\varrho}$ bijectively onto $D \subseteq C_{\nu,s}$, then*

$$F^{d/m}_{x,\mathfrak{Q}}(o_\mu) = \langle h_\mu, F_{(\nu,s),D}\rangle \quad \textit{for all } \mu \vdash N.$$

3.3.

We interpret $F^{\varrho}_{x,\mathfrak{Q}}$ in terms of affine roots and parahorics.

3.9. An *affine root* is a function on $X^\vee_{\mathbf{R}}$ of the form $\alpha + k$, where $\alpha \in \mathfrak{R}$ and $k \in \mathbf{Z}$. We picture them as affine hyperplanes in $X^\vee_{\mathbf{R}}$ equipped with oriented normals. The value of an affine root at a point describes the signed distance from the affine hyperplane to the point, with a positive, *resp.* negative, sign when the normal points toward, *resp.* away from, the point. We regard $\mathfrak{R}^{\mathrm{aff}} := \mathfrak{R} \times \mathbf{Z}$ as the set of affine roots.

3.10. For all $x \in X^\vee_{\mathbf{R}}$ and $\varrho \in \mathbf{R}$, let

$$\begin{aligned}
\mathfrak{R}^{\mathrm{aff}}_{x,\varrho} &= \{(\alpha, k) \in \mathfrak{R}^{\mathrm{aff}} \mid \alpha(x) + k = \varrho\},\\
\mathfrak{R}^{\mathrm{aff}}_{x,[0,\varrho)} &= \{(\alpha, k) \in \mathfrak{R}^{\mathrm{aff}} \mid 0 < \alpha(x) + k < \varrho\},\\
\mathfrak{R}^{\mathrm{aff}}_{x,<0} &= \{(\alpha, k) \in \mathfrak{R}^{\mathrm{aff}} \mid \alpha(x) + k < 0\}.
\end{aligned}$$

Note that $\mathfrak{R}^{\mathrm{aff}}_{0,0} = \mathfrak{R}$. For any $\xi \in X^\vee_{\mathbf{R}}$, the translation $(\alpha, k) \mapsto (\alpha, k - \alpha(\xi))$ restricts to bijections

$$\mathfrak{R}^{\mathrm{aff}}_{y,\varsigma} \xrightarrow{\sim} \mathfrak{R}^{\mathrm{aff}}_{\xi+y,\varsigma} \quad \text{for all } y \in X^\vee_{\mathbf{R}} \text{ and } \varsigma \in \mathbf{R}.$$

Hence, for any $\xi \in X^\vee_{\mathbf{R}}$ and $\epsilon \in X^\vee_{\mathbf{R}}$, we have bijections

$$\begin{aligned}
\mathsf{Des}_{x,\varrho}(\epsilon) &\xrightarrow{\sim} \mathfrak{R}^{\mathrm{aff}}_{\xi,0} \cap \mathfrak{R}^{\mathrm{aff}}_{\xi+\epsilon,<0} \cap \mathfrak{R}^{\mathrm{aff}}_{\xi+x,\varrho},\\
\mathsf{Inv}_{x,\varrho}(\epsilon) &\xrightarrow{\sim} \mathfrak{R}^{\mathrm{aff}}_{\xi,0} \cap \mathfrak{R}^{\mathrm{aff}}_{\xi+\epsilon,<0} \cap \mathfrak{R}^{\mathrm{aff}}_{\xi+x,[0,\varrho)}.
\end{aligned}$$

From the latter, we deduce:

Lemma 3.8. *Fix $\epsilon \in X^\vee_{\mathbf{R}}$ such that $|\alpha(\epsilon)| < \min_{\xi \in X^\vee} |\alpha(\xi)|$ for all $\alpha \in \mathfrak{R}$. Then*

$$|\mathfrak{R}^{\mathrm{aff}}_{\xi,<0} \cap \mathfrak{R}^{\mathrm{aff}}_{\xi+x,[0,\varrho)}| + |\mathsf{Inv}_{x,\varrho}(\epsilon)| = |\mathfrak{R}^{\mathrm{aff}}_{\xi+\epsilon,<0} \cap \mathfrak{R}^{\mathrm{aff}}_{\xi+x,[0,\varrho)}|$$

for all $\xi \in X^\vee$.

Proof. For such ϵ, we see that $\mathfrak{R}^{\mathrm{aff}}_{\xi+\epsilon,<0} = \mathfrak{R}^{\mathrm{aff}}_{\xi,<0} \cup (\mathfrak{R}^{\mathrm{aff}}_{\xi,0} \cap \mathfrak{R}^{\mathrm{aff}}_{\xi+\epsilon,<0})$ is a disjoint union. Intersecting with $\mathfrak{R}^{\mathrm{aff}}_{\xi+x,[0,\varrho)}$ gives the result. □

3.11. In what follows, we work over an algebraically closed field of good characteristic. Let G be the reductive algebraic group defined by the root datum. Fix a maximal torus $T \subseteq G$. Let $\mathfrak{g}$ and $\mathfrak{t}$ be the Lie algebras of G and T, respectively. For all $\alpha \in \mathfrak{R}$, let $\mathfrak{g}_\alpha \subseteq \mathfrak{g}$ be the corresponding root subspace.

Let $G(\!(z)\!)$ be the loop group of G, and let $\mathfrak{g}(\!(z)\!)$ be its Lie algebra. For all $x \in X^\vee_{\mathbf{R}}$ and $\varrho \in \mathbf{R}$, let

$$\begin{aligned}
\mathfrak{g}(\!(z)\!)_{x,\varrho} &= \bigoplus_{(\alpha,k) \in \mathfrak{R}^{\mathrm{aff}}_{x,\varrho}} z^k \mathfrak{g}_\alpha \oplus \begin{cases} z^\varrho \mathfrak{t} & \varrho \in \mathbf{Z}, \\ 0 & \varrho \notin \mathbf{Z}, \end{cases} \\
\mathfrak{g}(\!(z)\!)_{x,\geq\varrho} &= \widehat{\bigoplus_{\varsigma \geq 0}} \mathfrak{g}(\!(z)\!)_{x,\varsigma}.
\end{aligned}$$

If $\varrho \geq 0$, then $\mathfrak{g}(\!(z)\!)_{x,\geq\varrho}$ forms a Lie subalgebra of $\mathfrak{g}(\!(z)\!)$. The parahoric subalgebras of $\mathfrak{g}(\!(z)\!)$ containing $\mathfrak{t}[[z]]$ are precisely the Lie subalgebras of the form

$$\mathfrak{K}_x := \mathfrak{g}(\!(z)\!)_{x,\geq 0}.$$

Let K_x be the parahoric subgroup of $G(\!(z)\!)$ with Lie algebra $\mathfrak{K}_x$.

3.12. For any bounded-below subset $C \subseteq \mathbf{Z}$ and $c \in C$, let

$$\begin{aligned} \mathsf{tail}_C(c) &= C \cap \{c - dk \mid k \in \mathbf{Z}_{>0}\}, \\ \mathsf{wing}_C(c) &= \{b \in \mathsf{attk}_C(c) \mid |\mathsf{tail}_C(b)| \leq |\mathsf{tail}_C(c)|\}. \end{aligned}$$

Note that $\mathsf{tail}_C(c)$ and $\mathsf{wing}_C(c)$ are respectively what [HHL08] would call the leg and arm of c. However, if $C = C_{\nu,s}$ for some (ν, s), and we view ν as an integer composition of N, then these terms are reversed from most conventions: *e.g.*, the English or French conventions for the diagram of ν. In [HHL08], the authors signal their nonstandard terminology by marking their diagram $\mathrm{dg}'(\mu)$ instead of $\mathrm{dg}(\mu)$, but we prefer to use different terms altogether.

For any $\nu \in \mathbf{Z}^d_{\geq 0}$ and $s \in \mathbf{Z}^d$, and any x that is adapted to (ν, s) with slope m in the sense of §3.7, let $\mathfrak{R}^{\mathsf{t}}_{x,d/m}, \mathfrak{R}^{\mathsf{w}}_{x,d/m} \subseteq \mathfrak{R}$ be defined by requiring that

$$\left.\begin{aligned} \mathfrak{R}^{\mathsf{t}}_{x,d/m} &\xrightarrow{\sim} \{(c,b) \in C^2_{\nu,s} \mid b \in \mathsf{tail}_{C_{\nu,s}}(c)\}, \\ \mathfrak{R}^{\mathsf{w}}_{x,d/m} &\xrightarrow{\sim} \{(c,b) \in C^2_{\nu,s} \mid b \in \mathsf{wing}_{C_{\nu,s}}(c)\} \end{aligned}\right\} \quad \text{under (3.1)}.$$

Note that $\mathfrak{R}^{\mathsf{w}}_{x,d/m} \subseteq \mathfrak{R}_{x,[0,d/m)}$ because $\mathsf{wing}_C(c) \subseteq \mathsf{attk}_C(c)$ for any C, c. The notation is justified by:

Lemma 3.9. *Suppose that $x \in \mathbf{R}^N$ is adapted to (ν, s) with slope m, for some d and $\nu \in \mathbf{Z}^d_{\geq 0}$ and $s \in \mathbf{Z}^d$. Then $\mathfrak{R}^{\mathsf{t}}_{x,d/m}$ and $\mathfrak{R}^{\mathsf{w}}_{x,d/m}$ only depend on x and d/m, not on $d, (\nu, s), m$.*

Proof. Suppose that x is also adapted to (ν', s') with slope m' for some $\nu' \in \mathbf{Z}^{d'}_{\geq 0}$ and $s' \in \mathbf{Z}^{d'}$, where d' is another positive integer. For some permutation $w \in S_N$, there is a bijection $C_{\nu,s} \xrightarrow{\sim} C_{\nu',s'}$ that sends $\mathbf{t}_{\nu,s}(x_i) \mapsto \mathbf{t}_{\nu',s'}(x_{w(i)})$ for all i. If $d/m = d'/m'$, then this bijection transports the tails and wings of any element of $C_{\nu,s}$ bijectively onto the tails and wings of its image in $C_{\nu',s'}$. $\square$