Last time (X, x) gives π_1(X, x)
f gives f_*

if f is a homeo, then f_* is an iso

Ex if f = id_X, then f_* = id_{π_1(X, x)}

Q is the converse true?

Ex R and {0} are not homeomorphic [why?]
but π_1(R, 0) = π_1({0}, 0)

Thm if X is a convex subspace of R^n
then for any x and loop γ based at x,
have γ $\sim_p$ e_x
thus π_1(X, x) = {[e_x]}

Pf recall the homotopy
h(s, t) = (1 – t)*x + t*γ(s)

is it a path homotopy?

h(0, t) = (1 – t)*x + t*γ(0) = (1 – t)*x + t*x = x
h(1, t) = (1 – t)*x + t*γ(1) = (1 – t)*x + t*x = x

Moral convex subspaces of R^n need not be homeomorphic
but their π_1's are always isomorphic

but recall:

any nonempty convex X is contractible:
there is x_0 in X s.t. id_X ~ (constant map at x_0)

(Munkres §58)

Df a homotopy equivalence btw X and Y is a pair of cts $f : X \to Y$ and $g : Y \to X$

s.t. $g \circ f \sim id_X$ and $f \circ g \sim id_Y$

here we say X and Y are homotopy equivalent

Thm if $f : X \to Y$ and $g : Y \to X$ form a homotopy equivalence

then $f_* : \pi_1(X, x) \to \pi_1(Y, f(x))$,
$g_* : \pi_1(Y, y) \to \pi_1(X, g(y))$

are isomorphisms for any x in X and y in Y

Ex suppose X is contractible
pick x_0 in X s.t. $id_X \sim$ (const map at x)

$Y = \{x_0\}$

$f : X \to Y$ $f(x) = x_0$, the constant map
$g : Y \to X$ $g(x_0) = x_0$, the inclusion

then $(g \circ f)(x) = x_0$, so $g \circ f \sim id_X$
while $f \circ g = id_Y \sim id_Y$

so X is homotopy equivalent to $\{x_0\}$

Ex $X = R^2 - \{(0, 0)\}$ and $Y = S^1$

$r : X \to Y$ $r(x, y) = (x, y)/|(x, y)|$
$i : Y \to X$ $i(x, y) = (x, y)$

then $(i \circ r)(x, y) = (x, y)/|(x, y)|$, so $i \circ r \sim id_X$
via $h((x, y), t) = ((1 - t)/|(x, y)| + t)*(x, y)$
while $(r \circ i) = id_Y$

so $R^2 - \{(0, 0)\}$ is homotopy equivalent to S^1

Rem these examples are a specific kind of homotopy equivalence called a deformation retract

Pf of Thm suppose $f : X$ to Y and $g : Y$ to X
s.t. $g \circ f \sim id_X$ and $f \circ g \sim id_Y$

will show that $f_* : \pi_1(X, x)$ to $\pi_1(Y, f(x))$
is an iso for any x in X [argument for g_* is similar]

three lemmas:

1) if $f : X$ to Y and $g : Y$ to Z are cts maps
then $(g \circ f)_* = g_* \circ f_*$ [from last time]

2) if $\varphi : G$ to H and $\psi : H$ to K are maps
s.t. $\psi \circ \varphi$ is bijective
then φ is injective and ψ is surjective

3) if α is a path in X from x_0 to x_1
then $\breve{\alpha} : \pi_1(X, x_0)$ to $\pi_1(X, x_1)$ def by

$$\breve{\alpha}([\gamma]) = [\bar{\alpha} * \gamma * \alpha]$$

is an isomorphism

by 1), $g_* \circ f_* = (g \circ f)_*$,
$f_* \circ g_* = (f \circ g)_*$

so by 2), just need $(g \circ f)_*$ and $(f \circ g)_*$ to be isos

pick homotopies j from $g \circ f$ to id_X,
k from $f \circ g$ to id_Y

can use 3) to show: if $f, f' : A$ to X are cts,
h a homotopy from f to f',
a in A,

then $f'_* = \breve{\alpha}_h \circ f_* : \pi_1(A, a)$ to $\pi_1(X, f'(a))$

where $\alpha_h(s) = h(a, s)$, a path from $f(a)$ to $f'(a)$

apply to j and k:

$$(g \circ f)_* = \breve{\alpha}_j \circ id_{\{X, *\}} = \breve{\alpha}_j$$
$$(f \circ g)_* = \breve{\alpha}_k \circ id_{\{Y, *\}} = \breve{\alpha}_k$$

but by 3), $\breve{\alpha}_j$ and $\breve{\alpha}_k$ are isos