Review given U sub V
with inclusion map $i : U$ to V
[what structures does it produce?]

- $\mathrm{Ann}_{V^\vee} = \{\theta \text{ in } V^\vee \mid \theta|_U = \mathbf{0}_{V^\vee}\}$
- $i^\vee : V^\vee$ to $U^\vee$ defined by $i^\vee(\theta) = \theta|_U$

Lem $\mathrm{Ann}_{V^\vee}(U) = \ker(i^\vee)$

also:

- quotient map $q : V$ to V/U
- $q^\vee : (V/U)^\vee$ to $V^\vee$

[to finish that discussion:]

Thm $\mathrm{Ann}_{V^\vee}(U) = \mathrm{im}(q^\vee)$

recall: elts of V/U are subsets $v + U$
where v in V

Lem $v + U = v' + U$, as subsets, iff $v - v'$ in U

$(v + U) + (w + U) = (v + w) + U$
$\lambda \bullet (v + U) = \lambda v + U$

use lemma to check that these op's are well-def

Ex $V = F^2$ and $U = \{(x, x) \mid x \text{ in } F\}$
for any (a, b) in V:
$(a, b) + U = \{(a + x, b + x) \mid x \text{ in } F\}$

so elts of V/U are translates of U
[draw picture] [elts of V/U are translates in gen'l]

[what is the zero vector in V/U?]
0_{V/U} = 0 + U = U [the trivial translate of U]

quotient map q : V to V/U defined by q(v) = v + U
note that v + U = U iff v in U
therefore:

<u>Lem</u> U = ker(q : V to V/U)

<u>Pf of Thm</u> want Ann(U) = im(q^v : $(V/U)^v$ to V^v)

[first, what is q^v?] $q^v(\psi) = \psi \circ q$

θ in im(q^v) iff θ = ψ ∘ q for some ψ in $(V/U)^v$
θ in Ann(U) iff θ|_U is zero
iff U sub ker(θ)

so want to show: θ = ψ ∘ q for some ψ
iff U sub ker(θ)

"only if": ker(q) sub ker(θ) by PS4, #3
U = ker(q) by lemma

"if": for all v + U in V/U
let ψ(v + U) = θ(v)

claim ψ is a well-def lin map V/U to F
will then have θ(v) = ψ(v + U) = (ψ ∘ q)(v)

well-def:
if v + U = v′ + U
then v – v′ in U
so ψ(v + U) = ψ(v′ + (v – v′) + U) = ψ(v′ + U)

linearity of θ implies linearity of ψ:

$\psi((v + U) + (w + U)) = \psi((v + w) + U)$
$= \theta(v + w)$
$= \theta(v) + \theta(w)$
$= \psi(v + U) + \psi(w + U)$

and similarly for scalar multiplication □

<u>Summary</u> U sub V gives rise to

[injective] inclusion i : U to V
[surjective] quotient q : V to V/U

$$(V/U)^\vee \xrightarrow{q^\vee} V^\vee \xrightarrow{i^\vee} U^\vee$$

s.t. im(i) = U = ker(q)
and dually $\mathrm{im}(q^\vee) = \mathrm{Ann}\_\{V^\vee\}(U) = \ker(i^\vee)$

today – 3/31: bilinear maps, forms (§9A)
tensors (§9D)

let V, W be arbitrary vector spaces
recall that

$$W \times V = \{(w, v) \mid w \text{ in } W \text{ and } v \text{ in } V\}$$

forms a vector space under entrywise + and •

<u>Goal</u> contrast
linear functionals on W × V
with
bilinear functionals on W × V

[first, the basic properties of W × V:]

what is dim $W \times V$? dim W + dim V

[why?] e.g., $F^m \times F^n$ is isomorphic to $F^{m + n}$
[proof without choosing bases?]

<u>Lem</u> let $S : W$ to $W \times V$,
$T : V$ to $W \times V$
be def by $S(w) = (w, \mathbf{0}_V)$,
$T(v) = (\mathbf{0}_W, v)$

then $W \times V = \mathrm{im}(S) + \mathrm{im}(T)$, and this sum is direct

<u>Pf</u> any elt of $W \times V$ looks like (w, v)
for some w in W and v in V
then $(w, v) = (w, \mathbf{0}_V) + (\mathbf{0}_W, v)$
so $W \times V = \mathrm{im}(S) + \mathrm{im}(T)$

[why is the sum is direct?]
$\mathrm{im}(S)$ cap $\mathrm{im}(T) = \{(\mathbf{0}_W, \mathbf{0}_V)\}$
and $(\mathbf{0}_W, \mathbf{0}_V)$ is the zero vector of $W \times V$

<u>Df</u> due to the lemma,
we write $W \oplus V$ in place of $W \times V$
to emphasize the <u>vec. space</u> structure
on the product

$W \oplus V$ is also called an <u>external (direct) sum</u>

<u>Cor</u> dim $W \oplus V$ = dim W + dim V

<u>Rem</u> note: dim $(W \oplus V)^\vee$ = dim $W^\vee$ + dim $V^\vee$
in fact, $(W \oplus V)^\vee$ is isomphc to $W^\vee \oplus V^\vee$

Df for all V, W, U:
a bilinear map from $W \times V$ to U is
a map $\beta : W \times V$ to U s.t.

for all w in W, $\beta(w, –) : V$ to U is linear
for all v in V, $\beta(–, v) : W$ to U is linear

U = F: β is called a bilinear functional/pairing

U = F and W = V: β is called a bilinear form

Ex for any n, the dot product on F^n def by

$$\delta((b1, \ldots, bn), (c1, \ldots, cn)) = \text{sum\_i } bi\ ci$$

is a bilinear form on F^n

[what does bilinearity mean?]

for all w, w′, v, v′ in V and λ in F:

$$\delta(w + w', v) = \delta(w, v) + \delta(w', v)$$
$$\delta(w, v + v') = \delta(w, v) + \delta(w, v')$$
$$\delta(\lambda w, v) = \lambda\delta(w, v) = \delta(w, \lambda v)$$

usually people write $w \bullet v$ instead of $\delta(w, v)$

Crucial Point

I) bilinear pairings on $W \times V$ usually not linear
II) linear funct'ls on $W \oplus V$ usually not bilinear

Ex δ isn't linear: e.g., we would need
$\delta(aw, av) = \delta(a \bullet (w, v)) = a\delta(w, v)$
but $(aw) \bullet (av) \neq a(w \bullet v)$ in general

[already for $n = 1$]

<u>Ex</u> let $\theta : F^2 = F \oplus F$ to F be $\theta(b, c) = b + c$
then θ is linear, but usually not bilinear
[similar to PS6, #8]

<u>Df</u> $\mathrm{Bil}(W, V) = \{$bilinear pairings $W \times V$ to $F\}$

<u>Lem</u> $\mathrm{Bil}(W, V)$ forms a vector space under

$$(\beta + \beta')(w, v) = \beta(w, v) + \beta'(w, v)$$

$$(a \cdot \beta)(w, v) = a\beta(w, v)$$

what is $\dim \mathrm{Bil}(W, V)$?