5.

Throughout, $A = \mathbf{Z}[q^{\pm\frac{1}{2}}]$ and $Q = \mathbf{Q}(q^{\frac{1}{2}})$.

Fix an integer $n \geq 1$. Let H_n be the Iwahori–Hecke algebra of S_n over A. Let $\{T_w\}_{w \in S_n}$ be the standard basis of H_n, normalized so that

$$T_w T_s = \begin{cases} T_{ws} & ws > s \\ qT_{ws} + (q-1)T_w & ws < s \end{cases}$$

for any simple reflection s, where $<$ is the Bruhat order on S_n. Here, a simple reflection is an element of the form $s_i = (i\ i+1)$ in cycle notation for some $1 \leq i \leq n-1$. Let c_w be the canonical or Kazhdan–Lusztig (KL) basis of H_n, so that

$$q^{\frac{|w|}{2}} c_w = \sum_{y \leq w} P_{y,w}(q) T_y,$$

where $|w|$ is the Bruhat length of w and $P_{y,w}(q) \in 1 + q\mathbf{Z}_{\geq 0}[q]$ is the KL polynomial of (y, w).

For any integer partition $\lambda \vdash n$, let $\chi^\lambda : S_n \to \mathbf{Q}$ be the corresponding irreducible character, with the convention that $\chi^{(n)}$ is the trivial character and $\chi^{(1,\ldots,1)}$ the sign character. Let $\chi_q^\lambda : Q \otimes_A H_n \to Q$ be the Q-linear trace that arises from χ^λ via Tits deformation. It turns out, *e.g.*, by Starkey's rule, that χ^λ takes values in $\mathbf{Z}$, and that $\chi_q^\lambda|_{H_n}$ takes values in A. Henceforth, we abbreviate $\chi_q^\lambda|_{H_n}$ to χ_q^λ. We also set

$$\chi^{\mathsf{triv}} = \chi^{(n)} \quad \text{and} \quad \chi^{\mathsf{sgn}} = \chi^{(1,\ldots,1)}$$

for convenience.

For any partition $\mu = (\mu_1, \ldots, \mu_m) \vdash n$, let $S_\mu = S_{\mu_1} \times \cdots \times S_{\mu_\ell} \subseteq S_n$ be the parabolic or Young subgroup defined by μ. The Kostka numbers $K_{\lambda,\mu} \in \mathbf{Z}$ are uniquely determined by setting

$$\mathrm{Ind}_{S_\mu}^{S_n}(\chi^{\mathsf{triv}} \otimes \cdots \otimes \chi^{\mathsf{triv}}) = \sum_{\lambda \vdash n} K_{\lambda,\mu} \chi^\lambda.$$

With respect to the dominance order on partitions, the matrix $(K_{\lambda,\mu})_{\lambda,\mu}$ is unipotent upper-triangular, so its inverse is also defined over $\mathbf{Z}$. We can thus define a collection of A-linear traces $m_q^\mu : H_n \to A$ by setting

$$\chi_q^\lambda = \sum_{\mu \vdash n} K_{\lambda,\mu} m_q^\mu.$$

In this formula, note that the subscripts on the Kostka numbers remain the same, but the roles of λ and μ are swapped from their roles in the previous formula.

Example 5.1. Take $n = 4$. With λ along rows and μ along columns, the Kostka matrix and its inverse are

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ & 1 & 1 & 2 & 3 \\ & & 1 & 1 & 2 \\ & & & 1 & 3 \\ & & & & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & -1 & 0 & 1 & -1 \\ & 1 & -1 & -1 & 2 \\ & & 1 & -1 & 1 \\ & & & 1 & -3 \\ & & & & 1 \end{pmatrix},$$

respectively.

The letter m in the notation m_q^μ is meant to suggest "monomial". For, under the Frobenius character map from the vector space of Q-linear traces on $Q \otimes_A H_n$ into the ring of symmetric functions over Q, the element m_q^μ is sent to the monomial symmetric function for μ, just as the element χ_q^λ is sent to the Schur function for λ.

Conjecture 5.2 (Haiman 1993). *For all $w \in S_n$ and $\mu \vdash n$, the trace $m_q^\mu(c_w) \in A$ has only nonnegative coefficients as a Laurent polynomial in $q^{\frac{1}{2}}$.*

For $k \geq 2$ and $v = [v_1 v_2 \cdots v_k] \in S_k$, we say that $w = [w_1 w_2 \cdots w_n] \in S_n$ is $v_1 v_2 \cdots v_k$-*containing* iff there exist indices $1 \leq p_1 < \cdots < p_k \leq n$ such that for all $i < j$ with $v_i < v_j$, we have $w_{p_i} < w_{p_j}$. Informally: w is $v_1 v_2 \cdots v_k$-containing iff $(w_1, \ldots, w_n)$ contains a subsequence of size k whose elements have the same relative order as $(v_1, \ldots, v_k)$.

Otherwise, we say that w is $v_1 v_2 \cdots v_k$-*avoiding*. We write $S_n^{v_1 v_2 \cdots v_k} \subseteq S_n$ for the subset of $v_1 v_2 \cdots v_k$-avoiding elements. It turns out that

$$w \in S_n^{312} \implies w \in S_n^{3412} \cap S_n^{4231} \iff P_{1,w}(q) = 1.$$

Above, the $\iff$ statement is a 1990 result of Lakshmibai–Sandhya.

Example 5.3. Take $n = 4$. Write the elements of S_4 using *right-to-left* composition of permutations, and abbreviate s, t, u for s_1, s_2, s_3, respectively. In order of increasing length, the elements of S_4 are:

0	id	[1234]	3	sts	[3214]	4	$stsu$	[3241]
1	s	[2134]		stu	[2341]		$stut$	[2431]
	t	[1324]		sut	[2413]		$suts$	[4213]
	u	[1243]		tsu	[3142]		$tsut$	[3412]
2	st	[2314]		tut	[1432]		$tuts$	[4132]
	ts	[3124]		uts	[4123]	5	$stsut$	[3421]
	su	[2143]					$stuts$	[4231]
	tu	[1342]					$tsuts$	[4312]
	ut	[1423]				6	$stsuts$	[4321]

Blue means 321-avoiding, red means 312-avoiding, and green means $P_{1,w}(q) \neq 1$.

Conjecture 5.4 (Haiman 1993)**.** *For all $w \in S_n$, there exists a subset $X \subseteq S_n^{312}$, not necessarily unique, such that*

$$\chi_q^\lambda(c_w) = \sum_{x \in X} \chi_q^\lambda(c_x)$$

for all $\lambda \vdash n$. As a consequence, $P_{1,w}(q) = \sum_{x \in X} q^{\frac{|w|-|x|}{2}}$.

Remark 5.5. More generally, it is true that for any coefficients $a_x \in Q$, we have

$$\chi_q^\lambda(c_w) = \sum_{x \in S_n} a_x \chi_q^\lambda(c_x) \text{ for all } \lambda \implies P_{1,w}(q) = \sum_{x \in S_n} q^{\frac{|w|-|x|}{2}} a_x P_{1,x}(q)$$

To see this, recall that the usual symmetrizing trace on H_n sends $c_x \mapsto q^{-\frac{|w|}{2}} P_{1,x}(q)$ for all x, and this trace is a Q-linear combination of the χ_q^λ.

Theorem 5.6 (Abreu–Nigro 2022)**.** *Conjecture 5.4 fails for $n = 8$ and $w = [62754381]$. In this case, $P_{1,w}(q) = 1 + q$.*

5.1.

For any A-algebra H, we write $[H]$ to denote the Q-vector space of Q-linear traces on $Q \otimes_A H$. Thus there is a universal trace $H \to [H]$ through which every other trace factors, and the sets $\{\chi_q^\lambda\}_\lambda$, $\{m_q^\mu\}_\mu$ form two bases of $[H]$, with transition matrix $(K_{\lambda,\mu})_{\lambda,\mu}$.

There is a diagrammatic presentation of H_n in which the elements of H_n are depicted by planar graphs called MOY graphs. Using it, we will introduce a new basis for $[H]$ that we call the *circlet basis*. Its elements are again indexed by partitions $\mu \vdash n$, and will be denoted o_q^μ.

Conjecture 5.7. *We have $o_q^\mu = m_q^\mu$ for all $\mu \vdash n$.*

Following Billey–Warrington, we say that $w \in S_n$ is 321-*hexagon-avoiding* iff it belongs to

$$S_n^{321\text{hex}} := S_n^{321} \cap S_n^{46718235} \cap S_n^{46781235} \cap S_n^{56718234} \cap S_n^{56781234}.$$

In a 2001 paper, Billey–Warrington prove that the following conditions are equivalent:

(1) $w \in S_n^{321\text{hex}}$.
(2) $c_w = c_{s_{i_1}} \cdots c_{s_{i_\ell}}$ whenever $(s_{i_1}, \cdots, s_{i_\ell})$ is a reduced expression for w.
(3) The Bott–Samelson resolution of the Schubert variety attached to w is a small morphism of varieties.

We will show:

Theorem 5.8. *If Conjecture 5.7 holds, then Conjecture 5.2 holds under the added hypothesis that $w \in S_n^{321\text{hex}}$.*

5.2.

Let $TL_{n,\delta}$ be the Temperley–Lieb algebra on n strands over $\mathbf{Z}[\delta]$. We turn A into a $\mathbf{Z}[\delta]$-algebra via the assignment $\delta \mapsto q^{\frac{1}{2}} + q^{-\frac{1}{2}}$, and we set $TL_n = A \otimes_{\mathbf{Z}[\delta]} TL_{n,\delta}$. Then, by 1985 work of Jones, there is a quotient morphism of A-algebras

$$\Theta : H_n \to TL_n.$$

By a 1997 result of Fan–Green, the kernel of θ is precisely the A-linear span of the elements c_w such that w is 321-containing.

Let $b_w = \Theta(c_w)$ for all w. Then b_w is nonzero if and only if $w \in S_n^{321}$, and furthermore, $\{b_w\}_{w \in S_n^{321}}$ forms a basis for TL_n. In fact, this set also forms a $\mathbf{Z}[\delta]$-basis for $TL_{n,\delta}$, as it coincides with the diagram basis of $TL_{n,\delta}$ studied by Jones and Kauffman.

The morphism Θ induces a quotient morphism of A-modules

$$\Theta : [H_n] \to [TL_n].$$

Let $\bar{\chi}_q^\lambda = \Theta(\chi_q^\lambda) \neq 0$, and let $\bar{m}_q^\mu$, $\bar{o}_q^\mu$ be defined similarly. Moreover, write $\lambda \Vdash n$ iff every part of λ has size ≤ 2. In Jones's 1987 Annals paper, he asserts without proof that $\bar{\chi}_q^\lambda \neq 0$ if and only if $\lambda \Vdash n$. In particular, the set $\{\bar{\chi}_q^\lambda\}_{\lambda \Vdash n}$ forms a basis for $[TL_n]$. A more detailed exposition of this folklore result can be found in the 2008 senior thesis of Anne Moore at Macalester College.

Due to the unitriangularity of the Kostka matrix $(K_{\lambda,\mu})_{\lambda,\mu}$, it follows that $\{\bar{m}_q^\mu\}_{\mu \Vdash n}$ also forms a basis for $[TL_n]$. We will show:

Theorem 5.9. *We have $\bar{o}_q^\mu = \bar{m}_q^\mu$ for all $\mu \vdash n$.*

For any $J \subseteq \{1, 2, \ldots, n-1\}$, written in increasing order as $i_1 < \cdots < i_{|J|}$, let $v_J \in S_n$ be defined by $v_J = s_{i_1} \cdots s_{i_{|J|}}$. One can check that

$$\{v_J\}_J = S_n^{312} \cap S_n^{321}.$$

The following result shows that a version of Conjecture 5.4 holds at the level of the Temperley–Lieb quotient.

Theorem 5.10. *For all $w \in S_n^{321}$, there exists some $J \subseteq \{1, 2, \ldots, n-1\}$ such that*

$$\bar{o}_q^\mu(w) = \bar{o}_q^\mu(v_J)$$

for all $\mu \vdash n$. In particular, the same conclusion holds with $\{\bar{\chi}_q^\lambda\}_\lambda$ in place of $\{\bar{o}_q^\mu\}_\mu$.

5.3.

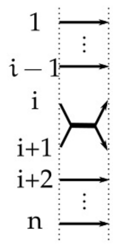
We now review the calculus of MOY graphs, define the circlet basis $\{o_q^\mu\}_\mu$, and prove Theorem 5.8.

MOY stands for Murakami–Ohtsuki–Yamada. In 1998, they showed how to compute the HOMFLYPT polynomial of a link L by first converting a planar diagram for L into an A-linear combination of colored, directed planar graphs, via local relations, then simplifying those graphs via further local relations. A byproduct of their work is a diagrammatic presentation of H_n where the elements c_s for simple reflections s, rather than the elements T_s, are represented by pure diagrams.

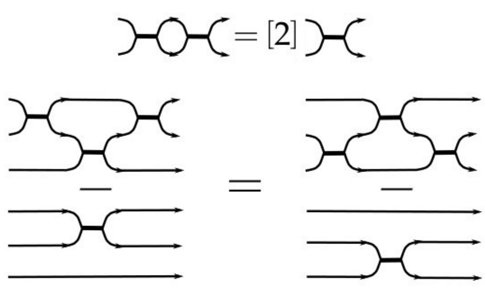
The MOY calculus was rediscovered by Cautis–Kamnitzer–Morrison in a "dual" form, the duality in question being Schur–Weyl duality. Namely, they give a diagrammatic version, called the spider category $\mathsf{Sp}(\mathrm{SL}_k)$, of the full subcategory of $\mathsf{Rep}(\mathrm{U}_q(\mathfrak{sl}_k))$ formed by the tensor powers of the fundamental irreducible representations. Objects in the spider category are ordered sequences of signed integers, depicted as columns of colored vertices, and morphisms are A-linear combinations of MOY graphs connecting a source column to a parallel target column in the plane, such that the orientations of the edges are compatible with the signs of any adjacent boundary vertices.

In particular, if we take $k \geq n$, then by quantum Schur–Weyl, the endomorphisms in $\mathsf{Sp}(\mathrm{SL}_k)$ of the nth tensor power of the standard representation form an A-algebra isomorphic to H_n after base change to Q. This observation defines the diagrammatic presentation of H_n. The pure diagrams arising from elements of H_n are known as (n, n)-*webs*; we will just call them *webs*.

We use images from Rasmussen's 2021 PCMI exposition. The web for c_{s_i} is



where the boldface represents a doubled edge. Multiplication in H_n corresponds to horizontal concatenation of webs. The local relations defining H_n are

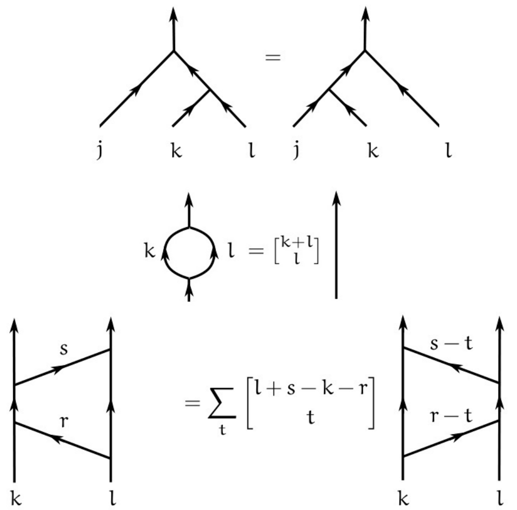


where $[2] = \delta = q^{\frac{1}{2}} + q^{-\frac{1}{2}}$, and in general:

$$[n] = \begin{cases} \frac{q^{\frac{n}{2}} - q^{-\frac{n}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}} & n > 0 \\ 0 & n \leq 0 \end{cases}$$

These relations imply further local relations

(5.1)



where we set

$$\begin{bmatrix} M \\ N \end{bmatrix} = \frac{[M]!}{[N]![M-N]!},$$

$$[N]! = [N] \cdots [2][1],$$

and it is implicitly understood that at any vertex, the sum of the labels on the inflowing edges is equal to that on the outflowing edges.

The map $H_n \to [H_n]$ corresponds to an operation on webs that we call *annular closure*. Starting from a (pure) diagram in a rectangle in the plane that joins n inputs on the left side to n outputs on the right side, we draw an annulus for which the rectangle is the interval between two cross-sections, then wrap the strands from the output vertices all the way around the annulus, joining them up end-to-end with the corresponding input vertices. We need to check:

Lemma 5.11. *If two concentric loops (colored by different numbers) occur in an annular closure, then the outer one can be swapped with the inner, without changing the element of $[H_n]$ being represented.*

For any $\mu = (\mu_1, \ldots, \mu_m) \vdash n$, let o_q^{μ} be the annular diagram consisting of m concentric loops that encircle the puncture, labeled by $\mu_1, \ldots, \mu_m$. By Lemma 5.11, this *circlet diagram* represents a well-defined element of $[H_n]$. Next, we must check:

Lemma 5.12. *Any annular closure can be simplified to an A-linear combination of the diagrams o_q^{μ}, solely by replacing the left-hand sides of the local relations in* (5.1) *with the respective right-hand sides. In particular, $\{o_q^{\mu}\}_{\mu}$ is a basis for $[H_n]$.*

Proof of Theorem 5.8. Let $w \in S_n^{321\mathrm{hex}}$. By Billey–Warrington, we can write $c_w = c_{s_{i_1}} \cdots c_{s_{i_\ell}}$ for some sequence of indices $i_1, \ldots, i_\ell$. Therefore, in the MOY presentation of H_n, the element c_w can be depicted by a pure diagram, *i.e.*, a single web.

By Lemma 5.12, this web can be simplified solely by replacing the left-hand sides of the relations in (5.1) with their right-hand sides. But the latter have q-binomial coefficients, which have only nonnegative coefficients as Laurent polynomials in $q^{\frac{1}{2}}$. $\square$