last time:

- given inner prod. spaces $(V, \langle\ ,\ \rangle)$, $(W, \{\ ,\ \})$,
  $T : V$ to W defines an adjoint $T^* : W$ to V
  characterized by $\{Tv, w\} = \langle v, T^*w\rangle$

- if V, W are finite-dim'l
  then get orthonormal bases for V and W
  i.e., the basis vectors e_i satisfy

  $\langle e_i, e_i\rangle = 1$

  $\langle e_j, e_i\rangle = 0$ if $j \neq i$

let M be the matrix of T wrt the orthonormal bases

Q what is the matrix of T^*?
[let them cook a bit]

A WLOG can take $V = F^n$ and $W = F^m$
under their dot / skew-dot products
and choose the bases to be the std bases

$F = R$: for all v and w, we require

$\{Tv, w\} = (Tv)^t w = v^t T^t w$

$\langle v, T^*w\} = v^t T^*w$

taking $v = e_j$ and $w = e_i$ shows

$T^t_{j, i} = T^*_{j, i}$ for all j, i

so $T^* = T^t$

$F = C$: for all v and w,

$\{Tv, w\} = (Tv)^t \bar{w} = v^t T^t \bar{w}$

$\langle v, T^*w\} = v^t \overline{(T^*w)} = v^t \overline{(T^*)}\, \bar{w}$

so $T^t_{j, i} = \overline{(T^*)}_{j, i}$ for all j, i

so $T^* = \overline{(T^t)}$ [that is:]

Thm wrt orthonormal bases for both V and W, $T : V$ to W and $T^* : W$ to V have matrices that are mutual conjugate transposes [we may write $M^* = \overline{(M^t)}$]

[works for both R and C: conjugation does nothing in the case of R]

[we deduce:]

Properties of Adjoints

- $(aS + T)^* = \bar{a}S^* + T^*$ [for all a in F and S, T]
- $Id^* = Id$
- $(T^*)^* = T$
- $(S \circ T)^* = T^* \circ S^*$
- T^* is invertible iff T is, and in this case, $(T^*)^{-1} = (T^{-1})^*$

[backing up a bit to §6B–6C, we revisit]

Orthogonal Complements

Df given linear U sub V, its orthogonal complement (wrt $\langle\ ,\ \rangle$) is

$$U^\perp = \{v \text{ in } V \mid \langle v, u\rangle = 0 \text{ for all } u \text{ in } U\}$$

- by definiteness of $\langle\ ,\ \rangle$, $U^\perp$ cap $U = \{\mathbf{0}\}$
- [we saw last time:] by Gram–Schmidt, if V is finite-dim'l, then $V = U + U^\perp$

if V is finite-dim'l, then we also have:

- $(U_1 + U_2)^\perp$ = [what?] $(U_1)^\perp$ cap $(U_2)^\perp$ [and why?]
- $(U^\perp)^\perp = U$

[notice similarity to properties of adjoints...]

Q how do adjoints interact with complements?

Thm 1) $\mathrm{im}(T)^{\perp} = \ker(T^*)$ and $\ker(T^*)^{\perp} = \mathrm{im}(T)$
2) $\ker(T)^{\perp} = \mathrm{im}(T^*)$ and $\mathrm{im}(T^*)^{\perp} = \ker(T)$

Pf 2) follows from 1) by swapping T and T*

to show $\mathrm{im}(T)^{\perp} = \ker(T^*)$:

w in ker(T*) iff $T^*w = \mathbf{0}\_V$
iff $\langle v, T^*w\rangle = 0$ for all v in V
iff $\langle Tv, w\rangle = 0$ for all v in V
iff w in $\mathrm{im}(T)^{\perp}$

taking $(\ )^{\perp}$ of both sides, we get $\ker(T^*)^{\perp} = \mathrm{im}(T)$

Cor if V, W are finite-dim'l, then direct sums:

$$V = \ker(T) + \mathrm{im}(T^*)$$
$$W = \mathrm{im}(T) + \ker(T^*)$$

(Axler §7B) now consider a linear op T : V to V

Df [a linear op] T is self-adjoint iff T* = T,
i.e., $\langle Tv', v\rangle = \langle v', Tv\rangle$ for all v, v'

if M is the matrix of T wrt an orthonormal basis,
then T* = T iff M* = M
[where M* denotes the conjugate tranpose]

Prop if T is self-adjoint (over either R or C)
then every eigenval of T is real
[is the converse true? no]

Pf let v be an eigenvec with eigenval λ
[what is $\langle Tv, v\rangle$? pause]

$\langle Tv, v\rangle = \langle \lambda v, v\rangle = \lambda\langle v, v\rangle$
but also $\langle v, Tv\rangle = \langle v, \lambda v\rangle = \bar{\lambda}\langle v, v\rangle$

since $v \neq \mathbf{0}$, we know $\langle v, v\rangle \neq 0$ by definiteness
so $\lambda = \bar{\lambda}$

[here is a slightly weaker notion:]

Df a linear op T is normal iff
$T^* \circ T = T \circ T^*$, i.e., they commute

[thus, any self-adjoint operator is normal]

Ex let $M : F^2$ to F^2 be

$$M = \begin{matrix} 1 & -1 \\ 1 & 1 \end{matrix} \quad \text{so that} \quad M^* = \begin{matrix} 1 & 1 \\ -1 & 1 \end{matrix}$$

then $M^* \neq M$, yet $$M^*M = \begin{matrix} 2 & 0 \\ 0 & 2 \end{matrix} = MM^*$$

Prop T is normal iff $\|Tv\| = \|T^*v\|$ for all v in V

Pf T is normal
iff $T^* \circ T - T \circ T^*$ is zero
iff $\langle (T^* \circ T - T \circ T^*)v, v\rangle = 0$ for all v
iff $\langle T^*Tv, v\rangle = \langle TT^*v, v\rangle$ for all v
iff $\langle Tv, Tv\rangle = \langle T^*v, T^*v\rangle$ for all v

<u>Thm</u> if $T : V$ to V is normal, then:

1) $\ker(T^*) = \ker(T)$

2) $\mathrm{im}(T^*) = \mathrm{im}(T)$

3) $T - \lambda$ is normal for all λ in F

<u>Pf</u> 1) follows from the prop

2) from $\mathrm{im}(T^*) = \ker(T)^\perp = \ker(T^*)^\perp = \mathrm{im}(T)$

3) from $(T - \lambda) \circ (T - \lambda)^*$

$$= (T - \lambda) \circ (T^* - \bar{\lambda})$$
$$= T \circ T^* - \bar{\lambda}T - \lambda T^* + |\lambda|^2$$
$$= T^* \circ T - \bar{\lambda}T - \lambda T^* + |\lambda|^2$$
$$= (T^* - \bar{\lambda}) \circ (T - \lambda)$$
$$= (T - \lambda)^* \circ (T - \lambda)$$

<u>Cor</u> if $T : V$ to V is normal then V is a direct sum $\ker(T) + \mathrm{im}(T)$

<u>Pf</u> V is a direct sum $\ker(T) + \mathrm{im}(T^*)$ but $\mathrm{im}(T^*) = \mathrm{im}(T)$

<u>Cor</u> if $T : V$ to V is normal then for all v in V and λ in F, we have

$$Tv = \lambda v \text{ if and only if } T^*v = \bar{\lambda}v$$

<u>Pf</u> $\ker(T^* - \bar{\lambda}) = \ker((T - \lambda)^*) = \ker(T - \lambda)$

[our goal next time: the] <u>Spectral Thm</u>

if V is finite-dim'l and $T : V$ to V is normal then T is diagonalizable