MATH 430: INTRODUCTION TO TOPOLOGY
PROBLEM SET #2

SPRING 2025

Due Wednesday, January 29. You may consult books, papers, and websites as long as you cite all sources and write up your solutions in your own words. **Updated on 1/24, in red.**

Problem 1. Endow $\mathbf{R}^2$ with the analytic topology. How is the subspace topology on $\mathbf{R}$, viewed as the x-axis of $\mathbf{R}^2$, related to the analytic topology on $\mathbf{R}$?

Problem 2. Endow $\mathbf{R}$ with the analytic topology, and

$$X = \{\tfrac{1}{n} \mid n = 1, 2, 3, \ldots\} \cup \{0\}$$

with the subspace topology.

(1) Show that for all integers $n > 0$, the singleton set $\{\frac{1}{n}\}$ is *clopen*: both closed and open.
(2) Show that $\{0\}$ is closed but not open.

Problem 3 (Munkres 128, #9(c)–(d))**.** Recall that the Euclidean norm on $\mathbf{R}^n$ is given by $\|u\| = \sqrt{u \cdot u}$, where

$$u \cdot v := u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$$

for general $u, v \in \mathbf{R}^n$.

(1) Use the Cauchy–Schwarz inequality $|u \cdot v| \leq \|u\|\|v\|$ to show that $\|u + v\| \leq \|u\| + \|v\|$ for all $u, v \in \mathbf{R}^n$.
(2) Conclude that the *Euclidean metric* $d(x, y) = \|x - y\|$ really is a metric $d : \mathbf{R}^n \times \mathbf{R}^n \to [0, \infty)$.

Problem 4 (Munkres 129, #11)**.** Let X be arbitrary, and let $d : X \times X \to [0, \infty)$ be an arbitrary metric. Let $e : X \times X \to [0, \infty)$ be defined by

$$e(x, y) = \frac{d(x, y)}{1 + d(x, y)}.$$

Show that:

(1) e is a (bounded) metric. *Hint:* The mean value theorem shows that if $f(x) = \frac{x}{1+x}$, then $f(a + b) - f(b) \leq f(a)$ for all $a, b \geq 0$.
(2) d and e induce the same topology on X.

Problem 5. Recall that metrics $d, d' : X \times X \to [0, \infty)$ are *equivalent* if and only if there are constants $A, B > 0$ such that

$$d(x, y) \leq Ad'(x, y) \text{ and } d'(x, y) \leq Bd(x, y) \text{ for all } x, y \in X.$$

In the setting of Problem 4, show that e is not equivalent to d when $X = \mathbf{R}$ and d is the Euclidean metric.

Problem 6 (Munkres 127, #6)**.** The *uniform topology* on

$$\mathbf{R}^\omega := \{\text{sequences } (a_1, a_2, a_3, \ldots) \text{ with } a_i \in \mathbf{R} \text{ for all } i\}$$

is induced by the *uniform metric* $\bar{\rho}(x, y) = \sup_{i>0} \min\{1, |x_i - y_i|\}$. For all $x \in \mathbf{R}^\omega$ and $\delta > 0$, show that:

(1) The set $U(x, \delta) := (x_1 - \delta, x_1 + \delta) \times (x_2 - \delta, x_2 + \delta) \times \cdots$ is not open in the uniform topology.
(2) Nonetheless, $B_{\bar{\rho}}(x, \epsilon) = \bigcup_{\delta<\epsilon} U(x, \delta)$ for any $\epsilon \leq 1$.

Problem 7. The *box topology* on $\mathbf{R}^\omega$ is defined as follows: U is open if and only if, for all $x \in U$, there is some set of the form $V = (a_1, b_1) \times (a_2, b_2) \times \cdots$ such that $x \in V \subseteq U$. In what follows, assume the Axiom of Choice.

(1) Show that the box topology really is a topology on $\mathbf{R}^\omega$.
(2) Use Problem 6 to verify that the box topology is strictly finer than the uniform topology.

Problem 8 (Munkres 127, #5)**.** Let $\mathbf{R}^\infty \subseteq \mathbf{R}^\omega$ be the subset of sequences $(a_i)_{i>0}$ such that $a_i \neq 0$ for only finitely many i. Determine the closure of $\mathbf{R}^\infty$ in the uniform topology on $\mathbf{R}^\omega$.