Zeta Functions as Knot Invariants

Minh-Tâm Quang Trinh

Howard University

O. Kivinen, M. Q. Trinh. The Hilb-vs-Quot Conjecture.
J. reine angew. Math. (Crelle), (2025). 44 pp.

1 The Riemann Hypothesis

(Euler ~1730s) Proof of the infinitude of primes:

If there were finitely many, then we'd have

$$\prod_{\text{prime } p > 0} \left(1 + \frac{1}{p} + \frac{1}{p^2} + \cdots\right) = \sum_{n=1}^{\infty} \frac{1}{n}.$$

But the series diverges—a contradiction.

He also implicitly studied the *zeta function*

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

For $s > 1$, we have $\zeta(s) = \displaystyle\prod_p \left(1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \cdots\right)$.

What if we allow s to be complex?

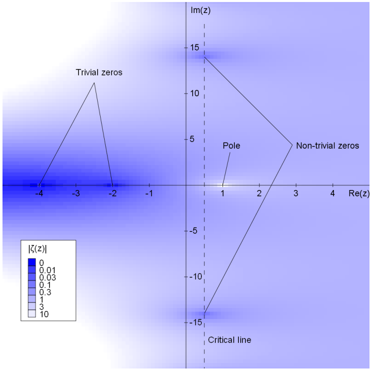
(Riemann 1859) A unique $\mathbf{C}$-valued function ζ that is

- *holomorphic* (complex-differentiable) when $s \neq 1$.
- given by $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^s}$ when $\mathrm{Re}(s) > 1$.

He checked that $\zeta(n) = 0$ for $n = -2, -4, -6, \ldots$ by relating these zeros to poles of the *gamma function*.

He speculated from examples that all other zeros of ζ live on the *critical line* $\mathrm{Re}(s) = \frac{1}{2}$.

Location of zeros $\leftrightsquigarrow$ distribution of prime numbers.



Wikipedia

(Hardy 1914) Infinitely many zeros on the line.

(Pratt–Robles–Zaharescu–Zeindler 2020) Among zeros s with $0 < \mathrm{Re}(s) < 1$, over five-twelfths on the line.

(Dedekind ~1860s) Generalize the formula

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

by replacing $\mathbf{Z}$ with other *rings* R.

Thus R is a set with operations $+$ and $\cdot$ resembling addition and multiplication.

$$\zeta_R(s) = \sum_{\substack{\text{ideals } I \subseteq R \\ |R/I| < \infty}} \frac{1}{|R/I|^s}$$

An *ideal* I is the collection of all finite linear combos $c_1 \cdot x_1 + \cdots + c_k \cdot x_k$ for some fixed $x_1, x_2, \ldots \in R$.

The *quotient* R/I is the set of translates $y + I \subseteq R$.

Note For ζ_R to make sense, the number of I such that $|R/I| = n$ must be finite for each $n > 0$.

Ex Every ideal of $\mathbf{Z}$ takes the form

$$n\mathbf{Z} = \{\text{multiples of } n\} \quad \text{for some integer } n \geq 0.$$

For instance, the ideal generated by 30 and 2025 is

$$\{c_1 \cdot 30 + c_2 \cdot 2025 \mid c_1, c_2 \in \mathbf{Z}\} = 15\mathbf{Z}.$$

We see that

$$\zeta_{\mathbf{Z}}(s) = \sum_{\substack{n\mathbf{Z} \subseteq \mathbf{Z} \\ n > 0}} \frac{1}{|\mathbf{Z}/n\mathbf{Z}|^s} = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

In other rings, ideals may not simplify so nicely.

Why care?

(Hilbert–Polyá ~1910s) The values e^{it} such that

$$\zeta(\tfrac{1}{2} + it) = 0 \quad \text{and} \quad 0 < \mathrm{Re}(\tfrac{1}{2} + it) < 1$$

behave like the eigenvalues of a generic unitary matrix.

$$\begin{aligned} \text{RH} &\iff t \text{ always real} \\ &\iff e^{it} \text{ always on the unit circle.} \end{aligned}$$

(Weil ~1940s) There is a class of rings R,

coming from *algebraic geometry* over $\mathbf{F}_p := \mathbf{Z}/p\mathbf{Z}$,

where analogous facts for ζ_R might be provable.

(Grothendieck–Deligne ~1960s–70s) Yes.

2 Weil's Rosetta Stone

Algebraic geometry studies *varieties*: shapes cut out by polynomial equations.

For simplicity, we'll stick to *(affine) hypersurfaces*

$$V_f = \{\vec{a} = (a_0, a_1, \ldots, a_d) \in \bar{\mathbf{F}}_p^{d+1} \mid f(\vec{a}) = 0\},$$

cut out by a single polynomial $f \in \mathbf{F}_p[x_0, x_1, \ldots, x_d]$.

V_f is *smooth* at a point $\vec{a}$ when $\frac{\partial f}{\partial x_i}(\vec{a}) \neq 0$ for some i. Else, *singular* at $\vec{a}$.

Ex 1-dim. hypersurfaces are plane curves. Consider

$$f(x, y) = y^2 - x^3 - c \quad \text{for constant } c \in \mathbf{F}_p.$$

For which c is V_f smooth everywhere?

The *ring of polynomial functions* on V_f is

$$R_f := \mathbf{F}_p[x_0, x_1, \ldots, x_d]/(f).$$

In a letter to his sister, Weil described a dictionary:

$\mathbf{Z}$	R_f	V_f
$n\mathbf{Z}$	ideals	(closed) subvarieties
$p\mathbf{Z}$	maximal ideals	(closed) points

The first and last columns: a Rosetta stone between number theory and algebraic geometry.

Conj (Weil 1948) Assume V_f is smooth everywhere. Then zeros of $\zeta_{R_f}(s)$ have $\mathrm{Re}(s) \in \{\frac{1}{2}, \frac{3}{2}, \ldots, \frac{2d-1}{2}\}$.

Thm (Weil) True for $f = c_0 x_0^{n_0} + \cdots + c_d x_d^{n_d} - c$.

Set $\zeta_f(s) := \zeta_{R_f}(s)$ for convenience.

(Grothendieck ~1964) $\zeta_f(s)$ is a rational function in

$$\mathsf{q} := p^{-s}.$$

In fact: polynomials $\phi_0, \phi_1, \ldots, \phi_{2d}$ such that

$$\zeta_f(s) = \frac{\phi_1(\mathsf{q}) \cdots \phi_{2d-1}(\mathsf{q})}{\phi_0(\mathsf{q}) \cdot \phi_2(\mathsf{q}) \cdots \phi_{2d}(\mathsf{q})}.$$

ϕ_k is the charpoly of a certain operator F on a certain vector space $\mathrm{H}^k(V_f)$.

Reduces Weil's conjecture to a "Hilbert–Polyá" claim:

Conj The F-eigenvalues on $\mathrm{H}^k(V_f)$ all have complex absolute value* $p^{k/2}$.

* With respect to any embedding into $\mathbf{C}$.

(Deligne 1974) True for all (smooth) f.

In fact, Weil conjectured—and Deligne proved—results for all smooth varieties, not just hypersurfaces.

Ex Taking $d = 1$ and $f(x, y) = y^2 - x^3 - c$:

$$\phi_0(t) = 1 - p\mathsf{q}$$

$$\phi_1(t) = 1 - a_p\mathsf{q} + p\mathsf{q}^2 \quad \text{for some integer } a_p,$$

giving $\zeta_f(s) = \dfrac{1 - a_p p^{-s} + p^{1-2s}}{1 - p^{1-s}}$. It turns out:

- $|a_p| \leq 2p^{1/2}$.
- So the two roots of $\phi_1(\mathsf{q})$ satisfy $|\mathsf{q}| = p^{-1/2}$.
- So the zeros of $\zeta_f(s)$ satisfy $\mathrm{Re}(s) = \frac{1}{2}$.

What if V_f has singularities?

Simplest case: V_f has a unique singularity at the origin $(0, \ldots, 0) \in \bar{\mathbf{F}}_p^N$. It turns out that here,

$$\zeta_f(s) = \zeta_f^\circ(s) \cdot \hat{\zeta}_f(s),$$

where:

- ζ_f° satisfies Weil's Riemann Hypothesis.
- $\hat{\zeta}_f$ is the analogue of ζ_f with the power-series ring

$$\hat{R}_f := \mathbf{F}_p[\![x_0, x_1, \ldots, x_d]\!]/(f)$$

in place of R_f.

Does $\hat{\zeta}_f(s) = \displaystyle\sum_{\substack{I \subseteq \hat{R}_f \\ |\hat{R}_f/I| < \infty}} \frac{1}{|\hat{R}_f/I|^s}$ satisfy a RH?

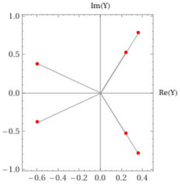
Ex If $f = y^2 - x^3$, then $\hat{\zeta}_f(s) = \dfrac{1+p\mathsf{q}^2}{1-\mathsf{q}}$.

Roots are $\mathsf{q} = \pm p^{-1/2}$.

Ex If $f = y^3 - x^4$, then

$$\hat{\zeta}_f(s) = \frac{1 + p\mathsf{q}^2 + p^2\mathsf{q}^3 + p^2\mathsf{q}^4 + p^3\mathsf{q}^6}{1-\mathsf{q}}.$$

Two roots on the circle $|\mathsf{q}| = p^{-1/2}$. The rest not on any circle $|\mathsf{q}| = p^{-k/2}$.



WolframAlpha

3 From Curves to Knots

Fix $f \in \mathbf{Z}[x, y]$ cutting out a plane curve through the origin.

It turns out we have $P_f(\mathsf{t}, \mathsf{q}) \in \mathbf{Z}\left[\mathsf{t}, \mathsf{q}, \frac{1}{1-\mathsf{q}}\right]$ such that

$$\hat{\zeta}_{f \bmod p}(s) = \frac{P_f(p, p^{-s})}{1 - p^{-s}} \quad \text{for almost all } p.$$

The polynomials P_f are remarkably ubiquitous.

(Gorsky–Mazin 2013)

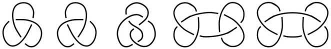
If $f = y^n - x^{n+1}$, then $P_f(1,1) = \frac{(2n)!}{(n+1)!n!}$, the nth Catalan number.

For instance, if $f = y^3 - x^4$, then

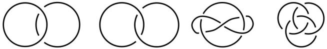
$$P_f(\mathsf{t}, \mathsf{q}) = 1 + \mathsf{t}\mathsf{q}^2 + \mathsf{t}^2\mathsf{q}^3 + \mathsf{t}^2\mathsf{q}^4 + \mathsf{t}^3\mathsf{q}^6,$$
$$P_f(1,1) = 5.$$

The P_f also arise from *knot/link invariants*.

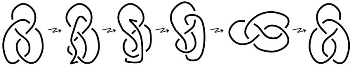
A *knot* is a (tame) embedding of S^1 into $\mathbf{R}^3$ or S^3.



A *link* is similar, but can have multiple circles.



Two links are *isotopic* when they fit into a continuous family of embeddings.



Chmutov–Duzhin–Mostovoy

A *complex* plane curve $V_f \subseteq \mathbf{C}^2$ through $(0,0)$ defines a link (for any small ϵ):

$$L_f := V_f \cap S^3, \quad \text{where } S^3 = \{|x|^2 + |y|^2 = \epsilon\}.$$

Ex If $f = y^n - x^m$, then L_f is the (m,n) *torus link*. It's a knot when m and n are coprime.

Wolfram Language

Ex If $f = y^4 - 2x^3y^2 - 4x^5y + x^6 - x^7$, then L_f is the closure of the braid

Cherednik–Danilenko

Conj (Oblomkov–Shende ~2010)

$$P_f(1, \mathsf{q}^2) = \lim_{\mathsf{a}\to 0} \left[(\mathsf{q}/\mathsf{a})^{\mu}\, \mathbb{P}_{L_f}(\mathsf{a}, \mathsf{q})\right],$$

where $\mu \in \mathbf{Z}$ and $\mathbb{P}$ is the *HOMFLYPT invariant*, discovered in 1986 and defined by the rules:

(1) $$\mathsf{a}\mathbb{P}_{\times} - \mathsf{a}^{-1}\mathbb{P}_{\times} = (\mathsf{q} - \mathsf{q}^{-1})\mathbb{P}_{)(}$$

(2) $$\mathbb{P}_{\bigcirc} = 1$$

Surprising, since P_f is *intrinsic* to f, while $\mathbb{P}$ is defined *diagrammatically*.

Full statement incorporates a by upgrading P_f.

(Maulik 2012) True for all plane curves.

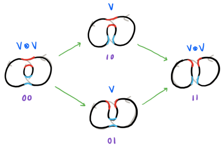
Proof sketch Blow up the singularity; control P_f via wall-crossing and L_f via skein algebra.

Conj (Oblomkov–Rasmussen–Shende ~2013)

$$P_f(\mathsf{t}^2, \mathsf{q}^2) = \lim_{\mathsf{a}\to 0} \left[(\mathsf{q}/\mathsf{a})^{\mu}\, \mathbf{P}_{L_f}(\mathsf{a}, \mathsf{t}, \mathsf{q})\right],$$

where $\mathbf{P}$ is a refinement of $\mathbb{P}$, discovered in the 2000s by Khovanov–Rozansky.

$\mathbf{P}$ is defined by *categorifying* (1)–(2). Unknown how to categorify Maulik's proof.



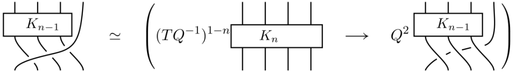
Melissa Zhang

(Kivinen–T 2025) True for $f = y^3 - x^m$ with $3 \nmid m$.

Cor (Kivinen–T) New closed formula for $\mathbf{P}_{\mathrm{torus}(m,3)}$.

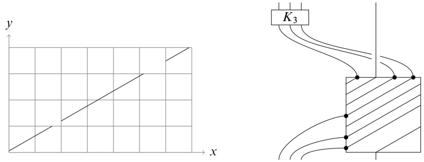
Proof Sketch

1 Recursions that compute $\mathbf{P}_{\mathrm{torus}(m,n)}(\mathsf{a}, \mathsf{t}, \mathsf{q})$, due to Elias–Hogancamp–Mellit.



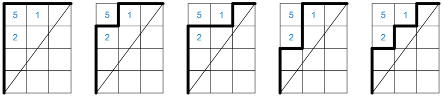
Elias–Hogancamp

Encoding of braids into grid diagrams, due to Mellit.



Mellit

2 For coprime m and n, arrive at a formula summing over *$m \times n$ Dyck paths*.



At the same time, $\hat{R}_{f \bmod p} \simeq \mathbf{F}_p[\![u^m, u^n]\!]$.

We relate Dyck paths to $\hat{R}_f$*-submodules* $M \subseteq \mathbf{F}_p[\![u]\!]$.

3 We then relate

$$\sum_M \frac{1}{|\mathbf{F}_p[\![u]\!]/M|^s} \quad \text{and} \quad \sum_I \frac{1}{|\hat{R}_{f \bmod p}/I|^s}$$

Uses *Serre duality*. For now, requires $\min(m,n) \leq 3$.

Big Picture

I'm interested in special functions that appear in

- *algebraic geometry* (*e.g.*, zeta functions)
- *knot theory* (*e.g.*, HOMFLYPT polynomials)
- *combinatorics* (*e.g.*, Dyck-path statistics)

One modern way to study such special functions is called *representation theory*.

T (2021) If L comes from a positive n-strand braid, then $\mathbf{P}_L(\mathsf{a}, \mathsf{t}, \mathsf{q})$ can be recovered from a *representation* of S_n on the cohomology of an explicit variety $\mathcal{Z}_L$.

So if $L = L_f$, then $\mathbf{P}_{L_f}$ relates to both V_f and $\mathcal{Z}_{L_f}$.

Any direct relationship between these varieties?

4 Cherednik's New Hypothesis

Recall: For $f = y^3 - x^4$ and prime p, the roots of

$$P_f(p, \mathsf{q}) = 1 + p\mathsf{q}^2 + p^2\mathsf{q}^3 + p^2\mathsf{q}^4 + p^3\mathsf{q}^6$$

do not all satisfy $|\mathsf{q}| = p^{-1/2}$.

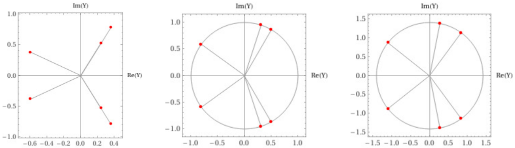
Conj (Cherednik 2018) For any plane curve f:

$$0 < t \leq \tfrac{1}{2} \implies \text{all roots of } P_f(t, \mathsf{q}) \text{ satisfy } |\mathsf{q}| = t^{-1/2}.$$

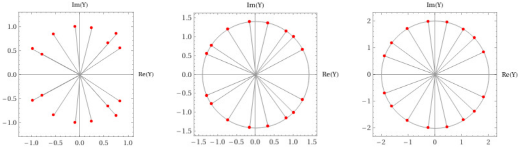
Would imply *arithmetic* constraints on $\mathbf{P}_L(\mathsf{a}, \mathsf{t}, \mathsf{q})$.

Dream (Cherednik) Related to the Lee–Yang theorem on zeros of *partition functions* in statistical mechanics.

$f = y^3 - x^4 \qquad t \in \{2,\ 1,\ \frac{1}{2}\}$:



$f = y^4 - 2x^3y^2 - 4x^5y + x^6 - x^7 \qquad t \in \{1,\ \frac{1}{2},\ \frac{1}{4}\}$:



Thank you for listening.