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Cell Decompositions of Hecke Traces and Link Polynomials

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Abstract. For specific trace functions on the Hecke algebra of a finite Weyl group W, we establish formulas for their values at positive braids in terms of point counts of Deodhar-type cells in associated algebraic varieties. For irreducible W, we deduce a uniform enumeration result that interpolates between its rational Catalan and parking combinatorics, generalizing our earlier work with Galashin and Lam. The key is a new relationship between the varieties from that work and the braid Steinberg varieties introduced by Trinh. For $W = S_n$, we prove a similar point-counting formula for each a-degree in the HOMFLYPT polynomial of the link closure of the braid, generalizing work of Shende–Treumann–Zaslow for the "highest" degree.

Keywords: Hecke algebra, link invariant, Coxeter–Catalan combinatorics, Deodhar cell, braid variety, Springer fiber

1 Introduction

In this work, we present a collection of formulas for special values of special traces on the Hecke algebras $H_W(\mathbf{v})$ associated with finite Weyl groups W. These formulas arise from point counting on algebraic varieties over finite fields. Nonetheless, the traces subsume various polynomials studied in combinatorics and knot theory: in the former, polynomials interpolating between the rational q-Catalan and parking numbers of W; in the latter, *arbitrary* a-degrees of the HOMFLYPT polynomials of positive links.

The role of Hecke algebras in both subjects is well-known. Our new contribution is to focus attention on central elements of the form

$$\sum_{w \in X} \sigma_w \sigma_{w^{-1}}, \tag{1.1}$$

where X is a subset of W, and $(\sigma_w)_{w \in W}$ denotes the standard basis of the Hecke algebra in a particular normalization. These central elements interact nicely with certain cell

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decompositions of our algebraic varieties, which generalize and take inspiration from similar decompositions studied by Deodhar [4].

In the rest of this introduction, we review all necessary background. In Section 2, we introduce various algebraic varieties associated with positive elements in the braid group of W, and state our results about their cell decompositions. In Section 3, we state applications to relative norms for Hecke algebras, link invariants, and combinatorial enumeration, as well as some directions of ongoing work. Throughout, we use the standard "q-notations" $[k]_q := 1 + q + \cdots + q^{k-1}$ and $[k]_q! := [k]_q \cdots [2]_q[1]_q$.

1.1 HOMFLYPT and Catalan

The relationship between algebraic geometry, knot theory, and Catalan combinatorics can be traced back to a link invariant discovered in the 80s, now called the *(reduced) HOMFLYPT polynomial* [5]:

$$\mathcal{P} : \{\text{links in } \mathbb{R}^3\}/\text{isotopy} \to \mathbb{Z}[a^{\pm 1}](\mathbf{v}). \tag{1.2}$$

On the one hand, pieces of the HOMFLYPT polynomials of certain links, called torus knots, recover q-analogues of the *rational Catalan numbers* defined by

$$\mathrm{Cat}_{n,p} = \frac{(p+n-1)!}{n!p!} \quad \text{for all coprime } n, p > 0, \tag{1.3}$$

which themselves recover the classical Catalan numbers at $p = n+1$; explicitly, the *rational q-Catalan number* $\mathrm{Cat}_{n,p}(q)$ is defined by replacing each factorial $k!$ above with $[k]_q!$. On the other hand, the HOMFLYPT polynomials of links more general than torus knots can be expressed in terms of the point counts of certain algebraic varieties built from the groups $\mathrm{GL}_n(\mathbb{F}_q)$ and their flag varieties.

The phenomena described above were discovered through a particular construction of HOMFLYPT due to Ocneanu. First recall the fact, due to Alexander, that every link is the closure of some *braid* β up to isotopy. In this case we denote the link isotopy class by L_β. From the braid group on n strands

$$Br_n = \left\langle \sigma_1, \ldots, \sigma_{n-1} \;\middle|\; \begin{array}{cl} \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} & \text{for } 1 \le i \le n-2, \\ \sigma_i\sigma_j = \sigma_j\sigma_i & \text{for } |i-j| > 1 \end{array} \right\rangle, \tag{1.4}$$

or rather, its group algebra over $\mathbb{Z}[\mathbf{v}^{\pm 1}]$, one constructs the *Hecke algebra*

$$H_n(\mathbf{v}) = \frac{\mathbb{Z}[\mathbf{v}^{\pm 1}]Br_n}{\langle (\sigma_i - \mathbf{v})(\sigma_i + \mathbf{v}) \text{ for all } i\rangle}. \tag{1.5}$$

A *trace* on $H_n(\mathbf{v})$ is a $\mathbb{Z}[\mathbf{v}^{\pm 1}]$-linear function that takes the same value on $\alpha\beta$ and $\beta\alpha$ for all $\alpha, \beta \in H_n(\mathbf{v})$. Ocneanu constructed such a trace $\xi_n : H_n(\mathbf{v}) \to \mathbb{Z}[a^{\pm 1}](\mathbf{v})$ for all n, and showed how to construct $\mathcal{P}(L_\beta)$ for all $\beta \in Br_n$ by renormalizing $\xi_n(\beta)$ [10].

The Hecke algebra, in turn, specializes at $\mathbf{v} \to 1$ to the group ring $\mathbb{Z}S_n$. Through the close connection between the representation theories of $H_n(\mathbf{v})$ and S_n, Jones computed the HOMFLYPT polynomial of the *(n, p)-torus knot* $L_{n,p} := L_{(\sigma_1 \cdots \sigma_{n-1})^p}$, for $p > 0$ coprime to n [10]. In modern conventions, the powers of a in this polynomial range between μ and $\mu + 2(\min(n, p) - 1)$, where $\mu = (n - 1)(p - 1)$.

For a general link L, it will be convenient to write $\mathcal{P}_i(L)$ for the $\mathbb{Z}[\mathbf{v}^{\pm 1}]$-coefficient of $a^{\mathrm{low}+i}$ in $\mathcal{P}(L)$, where a^{low} is the lowest power of a occurring in $\mathcal{P}(L)$. Inspection of Jones's formula shows a relationship to rational q-Catalan numbers:

$$\mathbf{v}^{-\mu}\mathrm{Cat}_{S_n,p}(\mathbf{v}^2) = \mathcal{P}_0(L_{n,p}) = \mathcal{P}_{2(n-1)}(L_{n,p+n}). \tag{1.6}$$

1.2 Geometry over $\mathbb{F}_q$

It turns out that the algebras $H_n(\mathbf{v})$ do have an enumerative meaning that involves their relation to the groups $G = \mathrm{GL}_n(\mathbb{F}_q)$. To explain, fix a *Borel subgroup* $B \subseteq G$, such as the subgroup of upper-triangular matrices, and recall the Bruhat decomposition

$$G = \coprod_{w \in S_n} BwB, \tag{1.7}$$

where each element of S_n has been lifted to a permutation matrix in G. By a classical theorem of Iwahori, $H_n(q^{1/2}) := H_n(\mathbf{v})|_{\mathbf{v}=q^{1/2}}$ is isomorphic to a certain convolution algebra formed by $(B \times B^{\mathrm{op}})$-invariant functions on G.

The quotient G/B can be identified with the set of complete flags in $\mathbb{F}_q^n$. Iwahori's result can be rewritten in terms of G/B as follows. First, a pair of cosets (yB, xB) is said to be in *relative position* $w \in S_n$ if and only if $By^{-1}xB = BwB$. In this case, we write $yB \xrightarrow{w} xB$. The stratification of $G/B \times G/B$ by relative position is precisely its stratification into orbits under the diagonal action of G. Thus $H_n(q^{1/2})$ also forms a convolution algebra of G-invariant functions on $G/B \times G/B$. The indicator functions of the G-orbits lift to the elements of a basis for $H_n(\mathbf{v})$ as a free module over $\mathbb{Z}[\mathbf{v}^{\pm 1}]$, called the *standard basis* $\{\mathbf{1}_w\}_{w \in S_n}$.

In our conventions, $\sigma_i = \mathbf{v}^{-1}\mathbf{1}_{s_i}$, where $s_i = (i, i+1) \in S_n$. Fix a word $\vec{s} = (s_{i_1}, \ldots, s_{i_\ell})$ and set $\beta_{\vec{s}} = \sigma_{i_1} \cdots \sigma_{i_\ell} \in Br_n$. In [12], Shende–Treumann–Zaslow observed that

$$\frac{|X(\vec{s})|}{|G|} = \left[\mathbf{v}^{\ell-n}\,\mathcal{P}_{2(n-1)}(L_{\beta_{\vec{s}}})\right]\Big|_{\mathbf{v}\to q^{1/2}}, \tag{1.8}$$

where the left-hand side uses the set

$$X(\vec{s}) = \{(x_1B, \ldots, x_\ell B) \in (G/B)^\ell \mid x_\ell B \xrightarrow{s_{i_1}} x_1 B \xrightarrow{s_{i_2}} \cdots \xrightarrow{s_{i_\ell}} x_\ell B\}. \tag{1.9}$$

Their original proof involved a partition of $X(\vec{s})$ into subsets indexed by so-called rulings of a Legendrian representative of $L_{\beta_{\vec{s}}}$. We now know a more direct proof. The main step is to show that $|X(\vec{s})| = \tau(\beta_{\vec{s}})|_{\mathbf{v}\to q^{1/2}}$, where $\tau : H_n(\mathbf{v}) \to \mathbb{Z}[\mathbf{v}^{\pm 1}]$ is the trace given by

$$\tau(\mathbf{1}_{\mathrm{id}}) = 1 \quad \text{and} \quad \tau(\mathbf{1}_w) = 0 \text{ for } w \neq \mathrm{id}. \tag{1.10}$$

In what follows, let $\mathbf{G}, \mathbf{B}, \mathbf{X}(\vec{s})$ be the algebraic groups and varieties over $\bar{\mathbb{F}}_q$ that recover $G, B, X(\vec{s})$ on Frobenius-fixed points. Here we use the Frobenius map $F : \mathbf{G} \to \mathbf{G}$ given on matrix coordinates $x_{i,j}$ by $F(x_{i,j}) = x_{i,j}^q$. Note that the $\mathbf{G}$-action on $\mathbf{G}/\mathbf{B}$ induces a $\mathbf{G}$-action on $\mathbf{X}(\vec{s})$.

In the mid-2000s, Khovanov–Rozansky discovered a link invariant now called *triply-graded link homology* and denoted HHH, whose (triply)-graded dimension is a refinement of $\mathcal{P}$. Let $\mathrm{HHH}_i \subseteq \mathrm{HHH}$ be the summand corresponding to $\mathcal{P}_i$. In [6], Galashin–Lam strengthened (1.8) for so-called Richardson braids $\beta_{\vec{s}}$, by matching $\mathrm{HHH}_{2(n-1)}(L_{\beta_{\vec{s}}})$ with the weight-graded, $\mathbf{G}$-equivariant compactly-supported cohomology of $\mathbf{X}(\vec{s})$. It is explained there that when $L_{\beta_{\vec{s}}} = L_{n,p}$, the cohomological and weight gradings recover the *rational q,t-Catalan number* $\mathrm{Cat}_{n,p}(q,t)$ studied by Loehr–Warrington, Hikita, and others, via a Dyck-path formula for $\mathrm{HHH}(L_{n,p})$ conjectured by Gorsky–Neguţ and proved by Mellit. These q,t-numbers specialize to our q-numbers.

Later, for general $\vec{s}$, Trinh proved a formula for the entire triply-graded homology $\mathrm{HHH}(L_{\beta_{\vec{s}}})$, in terms of an S_n-action on the weight-graded, $\mathbf{G}$-equivariant cohomology of a larger *Steinberg variety* $\mathbf{Z}(\vec{s})$. Taking the anti-invariant part of the formula gives an extension of the Galashin–Lam result to all $\vec{s}$ [13]. For our purposes, we only need a more elementary construction that produces a *rational* character of S_n depending on q, which we will define in §2.2 and denote by $\chi_{q,\mathbf{Z}(\vec{s})} : \mathbb{Q}S_n \to \mathbb{Q}$.

Theorem 1 (Trinh [13])**.** *For any word $\vec{s}$, we have $\mathcal{P}(L_{\beta_{\vec{s}}})|_{\mathbf{v}\to q^{1/2}} = \sum_k (-a^2)^k \chi_{q,\mathbf{Z}(\vec{s})}(e_{S_n,\Lambda^k})$, where the elements $e_{S_n,\Lambda^k} \in \mathbb{Q}S_n$ are defined in §3.3.*

1.3 From S_n to W

The varieties above generalize beyond $\mathbf{G} = \mathbf{GL}_n$ to any (connected, smooth) reductive algebraic group $\mathbf{G}$ over $\bar{\mathbb{F}}_q$. Any such algebraic group is determined by a root datum $(\Phi \subseteq \mathrm{X}, \Phi^\vee \subseteq \mathrm{X}^\vee)$, consisting of dual lattices $\mathrm{X}, \mathrm{X}^\vee$ and root systems $\Phi, \Phi^\vee$ satisfying certain conditions. In this setting, the symmetric group S_n is replaced by the *Weyl group* W of Φ, a reflection group of the vector space $V := \mathrm{X}^\vee \otimes \mathbb{Q}$. The set of transpositions $\{s_i\}_i \subseteq S_n$ is replaced by a minimal generating set of *simple reflections* $S \subseteq W$. Thus, any word $\vec{s}$ in S gives rise to a $\mathbf{G}$-variety $\mathbf{Z}(\vec{s})$ with a W-action on some version of its cohomology, and to a rational character $\chi_{q,\mathbf{Z}(\vec{s})} : \mathbb{Q}W \to \mathbb{Q}$, also defined in §2.2.

The pair (W, S) forms an example of a finite Coxeter system. This structure gives rise to a group Br_W generalizing Br_n, and to an algebra $H_W(\mathbf{v})$ generalizing $H_n(\mathbf{v})$. The analogue of $\sigma_1 \cdots \sigma_{n-1} \in Br_n$ is an element $\beta_{\vec{c}} \in Br_W$, where $\vec{c}$ is a fixed ordering of S, or

Coxeter word. In general, the elements of Br_W are no longer related to knot theory in the way that classical braids are.

By a classical theorem of Chevalley, the graded ring of invariants $\mathbb{Q}[V]^W$, where V is placed in degree 1, is freely generated by homogeneous elements. Their degrees $d_1 \leq \cdots \leq d_r$, where $r = \dim V$, are called the *(fundamental) degrees* of the W-action on V. If W is *irreducible*, meaning it is not a direct product of two smaller reflection groups, then d_r is the unique largest degree, called the *Coxeter number* of W and denoted h. Coxeter–Catalan combinatorics studies enumerative interpretations of the *rational Catalan numbers* of W, defined in the irreducible cases by

$$\mathrm{Cat}_{W,p} = \prod_{1 \leq i \leq r} \frac{p + d_i - 1}{d_i} \quad \text{for all } p > 0 \text{ coprime to } h. \tag{1.11}$$

The *rational q-Catalan number* $\mathrm{Cat}_{W,p}(q)$ is formed by replacing the ith factor above with $[p + d_i - 1]_q / [d_i]_q$. It turns out that $\mathrm{Cat}_{W,p}(q) \in \mathbb{Z}[q]$. When $W = S_n$, the fundamental degrees are $2, 3, \ldots, n$, giving $\mathrm{Cat}_{S_n,p}(q) = \mathrm{Cat}_{n,p}(q)$.

A major tool in this subject is a finite-dimensional graded representation of W that we will call the *(algebraic) rational parking space* and denote by $\Pi_{W,p} = \bigoplus_i \Pi^i_{W,p}$. Its graded dimension is $[p]_q^r$, the *rational q-parking number*, whereas the graded dimension of its W-invariant subspace is the rational q-Catalan number $\mathrm{Cat}_{W,p}(q)$. By character theory, $\Pi_{W,p}$ is determined up to isomorphism by requiring that

$$\sum_i q^i \operatorname{tr}(w \mid \Pi^i_{W,p}) = \frac{\det(1 - q^p w \mid V)}{\det(1 - qw \mid V)} \quad \text{for all } w \in W. \tag{1.12}$$

As explained in [1], it can be realized as a quotient of the polynomial ring $\mathbb{Q}[V]$ by an ideal depending on p, arising from the representation theory of the so-called rational Cherednik algebra of W. It can also be realized via the Steinberg varieties of [13]:

Theorem 2 (Trinh [13])**.** *If $\vec{c}$ is a Coxeter word for (W, S), and $\vec{c}^p$ is its p-fold concatenation, then $\chi_{q,\mathbf{Z}(\vec{c}^p)}(w)$ matches the expressions in (1.12).*

2 Cell Decompositions

Henceforth, we reserve **boldface** uppercase for algebraic varieties and algebraic groups over $\bar{\mathbb{F}}_q$, and ordinary *italics* for the corresponding sets and groups formed by their F-fixed points, where $F : \mathbf{G} \to \mathbf{G}$ is the Frobenius map arising from a split $\mathbb{F}_q$-form of $\mathbf{G}$. We mention without further comment that some q-identities below require that the characteristic of $\mathbf{F}_q$ not divide $|W|$.

We fix an F-stable Borel subgroup $\mathbf{B} \subseteq \mathbf{G}$ and an F-stable, maximally split maximal torus $\mathbf{T} \subseteq \mathbf{B}$. Once we identify W with $N_{\mathbf{G}}(\mathbf{T})/\mathbf{T}$, we get a Bruhat decomposition of $\mathbf{G}$, *resp.* G, into double cosets $\mathbf{B}w\mathbf{B}$, *resp.* BwB. ("Maximally split" implies that $N_{\mathbf{G}}(\mathbf{T})/\mathbf{T} \simeq N_G(T)/T$.) Note that $r = \dim \mathbf{T}$.

Recall that for any $w \in W$, the *Bruhat length* $\ell(w)$ is the minimal length among words in S that represent w, or equivalently, the dimension of $\mathbf{B}w\mathbf{B}/\mathbf{B}$. The *Bruhat order* on W is the partial order $<$ generated by the relations $w < ws$ for all $w \in W$ and $s \in S$ such that $\ell(w) < \ell(ws)$, and the analogous relations with sw in place of ws. There is a unique, involutive, longest element $w_\circ \in W$; multiplication by $w_\circ$ inverts the Bruhat order.

2.1 Richardson Varieties and Deodhar Cells

As a warm-up, we review a simpler construction from our joint work with Galashin and Lam [7]. For any word $\vec{s} = (s^{(1)}, \dots, s^{(\ell)})$ in S, let

$$\mathbf{O}(\vec{s}) = \{\vec{x}\mathbf{B} = (x_0\mathbf{B}, x_1\mathbf{B}, \dots, x_\ell\mathbf{B}) \in (\mathbf{G}/\mathbf{B})^{1+\ell} \mid x_0\mathbf{B} \xrightarrow{s^{(1)}} x_1\mathbf{B} \xrightarrow{s^{(2)}} \cdots \xrightarrow{s^{(\ell)}} x_1\mathbf{B}\}. \quad (2.1)$$

For any $v \in W$, the *v-twisted (open) Richardson variety* of $\vec{s}$ in [7] is

$$\mathbf{R}^{(v)}(\vec{s}) = \{\vec{x}B \in \mathbf{O}(\vec{s}) \mid x_\ell\mathbf{B} \xrightarrow{vw_\circ} \mathbf{B} = x_0 v w_\circ \mathbf{B}\}. \quad (2.2)$$

These varieties admit cell decompositions of the following form.

Recall that a *subword* of $\vec{s}$ is a sequence $\vec{\omega} = (\omega^{(1)}, \dots, \omega^{(\ell)})$ such that $\omega^{(i)} \in \{\mathrm{id}, s^{(i)}\}$ for all i. It will be convenient to write $\omega_{(i)} := \omega^{(1)} \cdots \omega^{(i)}$ below. For any $v \in W$, a *v-distinguished subword* of $\vec{s}$ is a subword $\vec{\omega}$ such that $v\omega_{(i)} \leq v\omega_{(i-1)}s^{(i)}$ for all i. For any such $\vec{\omega}$, we set

$$\mathbf{d}_{\vec{\omega}} = \{i \mid v\omega_{(i)} < v\omega_{(i-1)}\} \quad \text{and} \quad \mathbf{e}_{\vec{\omega}} = \{i \mid \omega^{(i)} = \mathrm{id}\}. \quad (2.3)$$

Let $\mathcal{D}^{(v)}(\vec{s})$ be the set of v-distinguished subwords $\vec{\omega}$ of $\vec{s}$ for which $\omega_{(\ell)} = \mathrm{id}$, and let $\mathcal{M}^{(v)}(\vec{s}) \subseteq \mathcal{D}^{(v)}(\vec{s})$ be the subset of $\vec{\omega}$ such that $|\mathbf{e}_{\vec{\omega}}| = r$. Then Deodhar essentially observed in [4] that $\mathbf{R}^{(v)}(\vec{s})$ is partitioned by disjoint, $\mathbf{B}$-stable subvarieties $\mathbf{R}^{(v)}(\vec{s}, \vec{\omega})$, now called *Deodhar cells*, for $\vec{\omega}$ running over $\mathcal{D}^{(v)}(\vec{s})$ and

$$\mathbf{R}^{(v)}(\vec{s}, \vec{\omega}) := \{\vec{x}\mathbf{B} \in \mathbf{R}^{(v)}(\vec{s}) \mid \mathbf{B} \xrightarrow{w_\circ v\omega_{(i)}} x_i\mathbf{B} \text{ for all } i\} \quad (2.4)$$

$$\simeq \left\{ \vec{t} \in \mathbf{A}^\ell \;\middle|\; \begin{array}{ll} t_i \neq 0 & \text{for } i \in \mathbf{e}_{\vec{\omega}}, \\ t_i = 0 & \text{for } i \notin \mathbf{d}_{\vec{\omega}} \cup \mathbf{e}_{\vec{\omega}} \end{array} \right\}. \quad (2.5)$$

Above, $\mathbf{A}^\ell$ denotes ℓ-dimensional affine space. In particular,

$$|R^{(v)}(\vec{s}, \vec{\omega})| = q^{|\mathbf{d}_{\vec{\omega}}|}(q-1)^{|\mathbf{e}_{\vec{\omega}}|}. \quad (2.6)$$

This relates the point count $|R^{(v)}(\vec{s})|$ to the trace (1.10), since we showed in [7] that

$$\tau(\beta_{\vec{s}}\sigma_{vw_\circ}\sigma_{(vw_\circ)^{-1}}) = \sum_{\vec{\omega}\in\mathcal{D}^{(v)}(\vec{s})} \mathbf{v}^{2|\mathbf{d}_{\vec{\omega}}|}(\mathbf{v}^2-1)^{|\mathbf{e}_{\vec{\omega}}|}, \tag{2.7}$$

where, for all $w \in W$, we have set $\sigma_w = \mathbf{v}^{-\ell(w)}\mathbf{1}_w$. Indeed, when $W = S_n$, the Deodhar cell decomposition recovers the ruling partition mentioned in *loc. cit.*

2.2 Springer Theory

Recall that $\mathbf{B} = \mathbf{T} \ltimes \mathbf{U}$, where $\mathbf{U}$ is the unipotent radical of $\mathbf{B}$, *i.e.*, its unique maximal, connected, normal unipotent subgroup. The *Springer resolution* is the variety of pairs $(u, x\mathbf{B}) \in \mathbf{G} \times \mathbf{G}/\mathbf{B}$ that satisfy $u \in x\mathbf{U}x^{-1}$, which forms a resolution-of-singularities of the unipotent variety of $\mathbf{G}$. The fibers of the resolution map are called *Springer fibers*. In the 70s, Springer showed that W acts on the cohomology of any Springer fiber $(\mathbf{G}/\mathbf{B})_u$ with $u \in U = \mathbf{U}^F$. This gives rise to a character $\chi_{q,(\mathbf{G}/\mathbf{B})_u} : \mathbb{Q}W \to \mathbb{Q}$:

$$\chi_{q,(\mathbf{G}/\mathbf{B})_u}(w) = \mathrm{tr}(wF \mid \mathrm{H}^*((\mathbf{G}/\mathbf{B})_u)). \tag{2.8}$$

When $W = S_n$, it can be computed in terms of q-Kostka polynomials. We can now define the character $\chi_{q,\mathbf{Z}(\vec{s})}$ mentioned earlier: It is

$$\chi_{q,\mathbf{Z}(\vec{s})} = \frac{1}{|G|}\sum_{u\in U}|O(\vec{s})_u|\chi_{q,(\mathbf{G}/\mathbf{B})_u}, \tag{2.9}$$

where $O(\vec{s})_u$ is the set of Frobenius-fixed points of

$$\mathbf{O}(\vec{s})_u = \{\vec{x}\mathbf{B} \in \mathbf{O}(\vec{s}) \mid ux_\ell\mathbf{B} = x_0\mathbf{B}\}. \tag{2.10}$$

Indeed, at the level of sets, $Z(\vec{s}) = \coprod_{u\in U}(O(\vec{s})_u \times (G/B)_u)$.

We will not actually use the variety $\mathbf{Z}(\vec{s})$ in what follows. It turns out that to obtain clean cell decompositions, we need a "gauged" version

$$\mathbf{Z}_\square(\vec{s}) = \{(u, \vec{x}\mathbf{B}) \in \mathbf{U} \times \mathbf{O}(\vec{s}) \mid ux_\ell\mathbf{B} = x_0\mathbf{B}\}. \tag{2.11}$$

It contains equivalent information, in the sense that $|Z(\vec{s})|/|G| = |Z_\square(\vec{s})|/|B|$, and moreover, the $\mathbf{G}$-action on $\mathbf{Z}(\vec{s})$ restricts to a $\mathbf{B}$-action on $\mathbf{Z}_\square(\vec{s})$.

In [3], Borho–MacPherson studied a generalization of Springer theory depending on a choice of subset $J \subseteq S$. To explain their work, let $W_J \subseteq W$ be the subgroup generated by J, and let $\mathbf{P}_J = \mathbf{B}W_J\mathbf{B}$, so that $\mathbf{P}_J$ forms an example of a *parabolic subgroup* of $\mathbf{G}$. We have $\mathbf{P}_J = \mathbf{L}_J \ltimes \mathbf{U}_J$, where $\mathbf{L}_J$ is a reductive algebraic group containing $\mathbf{T}$, called the *Levi* *factor* of $\mathbf{P}_J$, while $\mathbf{U}_J$ is the unipotent radical of $\mathbf{P}_J$. We can identify W_J with $N_{\mathbf{L}_J}(\mathbf{T})/\mathbf{T}$, so W_J is also a Weyl group. Let

$$e_{W_J,\pm} = \frac{1}{|W_J|} \sum_{w \in W_J} (\pm 1)^{\ell(w)} w, \tag{2.12}$$

so that $e_{W_J,+}$, *resp.* $e_{W_J,-}$, is the symmetrizer, *resp.* anti-symmetrizer, in $\mathbb{Q}W_J$.

The (smaller) *partial Springer resolution* is the variety of pairs $(u, x\mathbf{P}_J) \in \mathbf{G} \times \mathbf{G}/\mathbf{P}_J$ that satisfy $u \in x\mathbf{P}_J x^{-1}$, which forms a resolution-of-singularities of the Zariski closure of a certain unipotent conjugacy class in $\mathbf{G}$ determined by J. Borho–MacPherson's work implies that the point count of the *partial Springer fiber* over $u \in U$ is given by

$$|(G/P_J)_u| = \chi_{q,(\mathbf{G}/\mathbf{B})_u}(e_{W_J,-}). \tag{2.13}$$

We now introduce parabolic generalizations of the varieties $\mathbf{Z}_\square(\vec{s})$. Let $w_{J,\circ}$ be the longest element of W_J, and let

$$\mathbf{Z}_\square^{J,+}(\vec{s}) = \{(v, \vec{y}\mathbf{B}, z\mathbf{B}) \in \mathbf{U}_J \times \mathbf{O}(\vec{s}) \times \mathbf{G}/\mathbf{B} \mid vy_\ell\mathbf{B} \xrightarrow{w_{J,\circ}} z\mathbf{B} \xleftarrow{w_{J,\circ}} y_0\mathbf{B}\}, \tag{2.14}$$

$$\mathbf{Z}_\square^{J,-}(\vec{s}) = \{(v, \vec{y}\mathbf{B}) \in \mathbf{U}_J \times \mathbf{O}(\vec{s}) \mid vy_\ell\mathbf{B} = y_0\mathbf{B}\}. \tag{2.15}$$

Note that $\mathbf{Z}_\square^{\varnothing,+}(\vec{s}) = \mathbf{Z}_\square^{\varnothing,-}(\vec{s}) = \mathbf{Z}_\square(\vec{s})$, whereas $\mathbf{Z}_\square^{S,-} = \mathbf{X}(\vec{s})$. Using (2.13), we show that:

$$|Z_\square^{J,\pm}(\vec{s})|/|P_J| = \chi_{q,\mathbf{Z}(\vec{s})}(e_{W_J,\pm}). \tag{2.16}$$

We can stratify $\mathbf{Z}_\square^{J,\pm}(\vec{s})$ into disjoint $\mathbf{B}$-stable subvarieties $\mathbf{Z}_\square^{[v],\pm}(\vec{s})$, corresponding to the conditions $\mathbf{B}y_\ell^{-1}\mathbf{P}_J = \mathbf{B}v\mathbf{P}_J$, for $[v]$ running over W/W_J. Let $W^{J,+}$ be the set of minimal-length(!) left coset representatives for W_J in W. Let $W^{J,-}$ be the set of maximal-length representatives. Our main geometric result is:

Theorem 3. *If* $v \in W^{J,+}$, resp. $v \in W^{J,-}$, *then* $\mathbf{Z}_\square^{[v],+}(\vec{s})$, resp. $\mathbf{Z}_\square^{[v],-}(\vec{s})$, *forms a* $\mathbf{B}$*-equivariant affine-space bundle over* $\mathbf{R}^{(v)}(\vec{s})$ *in the smooth topology on* $\bar{\mathbb{F}}_q$*-schemes, of relative dimension* $\ell(w_\circ) = \dim \mathbf{U}$, resp. $\ell(w_\circ w_{J,\circ}) = \dim \mathbf{U}_J$, *and moreover,*

$$|Z_\square^{[v],+}(\vec{s})| = |U||R^{(v)}(\vec{s})|, \quad \text{resp.} \quad |Z_\square^{[v],-}(\vec{s})| = |U_J||R^{(v)}(\vec{s})|. \tag{2.17}$$

Corollary 1. *For any subset* $J \subseteq S$ *and word* $\vec{s}$ *in* S, *we have*

$$\chi_{q,\mathbf{Z}(\vec{s})}(e_{W_J,+}) = \frac{1}{|B \cap L_J|} \sum_{v \in W^{J,+}} \sum_{\vec{\omega} \in \mathcal{D}^{(v)}(\vec{s})} q^{|\mathbf{d}_{\vec{\omega}}|}(q-1)^{|\mathbf{e}_{\vec{\omega}}|}, \tag{2.18}$$

$$\chi_{q,\mathbf{Z}(\vec{s})}(e_{W_J,-}) = \frac{1}{|L_J|} \sum_{v \in W^{J,-}} \sum_{\vec{\omega} \in \mathcal{D}^{(v)}(\vec{s})} q^{|\mathbf{d}_{\vec{\omega}}|}(q-1)^{|\mathbf{e}_{\vec{\omega}}|}. \tag{2.19}$$

Proof. Combine (2.16), Theorem 3, and (2.6). □

3 Applications

3.1 Relative Norms

Recall that $\sigma_v = \mathbf{v}^{-\ell(v)}\mathbf{1}_v$. As explained in [9], Hoefsmit–Scott observed that the formula

$$N_J(\beta) := \sum_{v \in W^J} \sigma_v \beta \sigma_{v^{-1}} \tag{3.1}$$

defines an injective $\mathbb{Z}[\mathbf{v}^{\pm 1}]$-linear *relative norm* $N_J : Z(H_{W_J}(\mathbf{v})) \to Z(H_W(\mathbf{v}))$, where we write $Z(H)$ to denote the center of an algebra H. As special cases,

$$N_J(1) = \sum_{v \in W^{J,+}} \sigma_v \sigma_{v^{-1}} \quad \text{and} \quad N_J(\sigma^2_{w_{J,\circ}}) = \sum_{v \in W^{J,-}} \sigma_v \sigma_{v^{-1}}. \tag{3.2}$$

Combining Corollary 1 with (2.7), we deduce:

Corollary 2. *For any word $\vec{s}$, the specialization $\mathbf{v} \to q^{1/2}$ sends*

$$\chi_{q,\mathbf{Z}(\vec{s})}(e_{W_J,+}) \to \frac{1}{|B \cap L_J|}\tau(\beta_{\vec{s}} N_J(\sigma^2_{w_{J,\circ}})) \quad \textit{and} \quad \chi_{q,\mathbf{Z}(\vec{s})}(e_{W_J,-}) \to \frac{1}{|L_J|}\tau(\beta_{\vec{s}} N_J(1)). \tag{3.3}$$

When $W = S_n$, this result recovers formulas that Lascoux proved with symmetric functions: See [14, Prop. 3.8, Thm. 4.1]. In ongoing work, we establish a more general *compatibility* between N_J and parabolic induction of class functions from L_J to G.

3.2 Parabolic Rational Parking Numbers

Henceforth, W is irreducible with Coxeter number h. By varying J, we can define polynomials interpolating between the rational q-Catalan and q-parking numbers of W. Let $d_{J,1}, \ldots, d_{J,r}$ be the degrees of the W_J-action on V, and let $e_{J,1}(V), \ldots, e_{J,r}(V)$ be the degrees in which V occurs as a simple $\mathbb{Q}W_J$-submodule of the *coinvariant module* $\mathbb{Q}[V]/I_J(V)$, where $I_J(V) \subseteq \mathbb{Q}[V]$ is the ideal of W_J-invariants of positive degree. When $J = S$, we have $d_{J,i} = d_i$ and $e_{J,i}(V) = d_i + 1$. For $p > 0$ coprime to h, we define the *parabolic rational parking numbers* by

$$\mathrm{Park}^{J,\pm}_{W,p} = \prod_i \frac{p \pm e_{J,i}(V)}{d_i}. \tag{3.4}$$

We define the *parabolic rational q-parking numbers* $\mathrm{Park}^{J,\pm}_{w,p}(q)$ by replacing the ith factor above with $[p \pm e_{J,i}(V)]_q/[d_i]_q$. Extending work of Bessis–Reiner [2], we prove:

Theorem 4. $\mathrm{Park}^{J,+}_{w,p}(q)$, resp. $\mathrm{Park}^{J,-}_{w,p}(q)$, *is the graded dimension of the subspace of W_J-invariants*, resp. *W_J-anti-invariants, of $\Pi_{W,p}$.*

Corollary 3. *For any $J \subseteq S$, Coxeter word $\vec{c}$, and integer $p > 0$ coprime to h, we have*

$$\mathrm{Park}^{J,+}_{W,p}(q) = \frac{1}{|B \cap L_J|} \sum_{v \in W^{J,+}} \sum_{\vec{\omega} \in \mathcal{D}^{(v)}(\vec{c}^p)} q^{|\mathbf{d}_{\vec{\omega}}|}(q-1)^{|\mathbf{e}_{\vec{\omega}}|}, \tag{3.5}$$

$$\mathrm{Park}^{J,-}_{W,p}(q) = \frac{1}{|L_J|} \sum_{v \in W^{J,-}} \sum_{\vec{\omega} \in \mathcal{D}^{(v)}(\vec{c}^p)} q^{|\mathbf{d}_{\vec{\omega}}|}(q-1)^{|\mathbf{e}_{\vec{\omega}}|}. \tag{3.6}$$

In particular, $\mathrm{Park}^{J,+}_{W,p} = \sum_{v \in W^{J,+}} |\mathcal{M}^{(v)}(\vec{c}^p)|$ *and* $\mathrm{Park}^{J,-}_{W,p} = \frac{1}{|W_J|} \sum_{v \in W^{J,-}} |\mathcal{M}^{(v)}(\vec{c}^p)|$.

Proof. Combine Theorem 2, Corollary 1, and Theorem 4. □

Note that $\mathrm{Park}^{\varnothing,+}_{w,p}(q) = \mathrm{Park}^{\varnothing,-}_{w,p}(q) = [p]^r_q$ and $\mathrm{Park}^{S,+}_{w,p}(q) = \mathrm{Cat}_{W,p}(q)$. Therefore, Corollary 3 interpolates between the parking and Catalan enumeration results of [7].

3.3 HOMFLYPT a-Degrees and Kirkman Numbers

Observe that $W^{J,+}$, *resp.* $W^{J,-}$, consists of those $w \in W$ whose *(right) ascent set* $\mathrm{Asc}(w) := \{s \in S \mid ws > w\}$, *resp.* *descent set* $\mathrm{Des}(w) := \{s \in S \mid ws < w\}$, contains J. Hence, the elements in (3.2) respectively decompose as sums, over supersets $I \supseteq J$, of elements

$$\zeta_I^+ := \sum_{\mathrm{Asc}(v)=I} \sigma_v \sigma_{v^{-1}} \quad \text{and} \quad \zeta_I^- := \sum_{\mathrm{Des}(v)=I} \sigma_v \sigma_{v^{-1}}. \tag{3.7}$$

Note that $\zeta_S^+ = \zeta_\varnothing^- = 1$ and $\zeta_\varnothing^+ = \zeta_S^- = \sigma_{w_\circ}^2$. By inclusion-exclusion on the elements in (3.2), the elements $\zeta_I^\pm$ are again central in $H_W(\mathbf{v})$.

Question 1. *For general W and I, is there a more familiar description of the traces on $H_W(\mathbf{v})$ that send $\beta \mapsto \tau(\beta\zeta_I^\pm)$?*

Henceforth, $W = S_n$. Identifying S with the index set $\{1, \ldots, n-1\}$, we see that $\mathrm{Des}(w)$ consists of the indices $i \in S$ such that w, as a permutation, satisfies $w(i+1) < w(i)$. An analogous statement holds for $\mathrm{Asc}(w)$.

Recall that the irreducible characters of S_n are indexed by partitions $\lambda \vdash n$. The hook partition $(n-k, 1, \ldots, 1)$ corresponds to the character of $\Lambda^k(V)$, the kth exterior power of the reflection representation V. Let $e_{S_n,\Lambda^k} \in \mathbb{Q}S_n$ be the symmetrizers corresponding to these characters; more concretely,

$$\det(1 - tw \mid V) = \sum_k (-t)^k e_{S_n,\Lambda^k}(w) \quad \text{for all } w \in S_n. \tag{3.8}$$

Using work of Isaev–Ogievetsky on central elements in $H_n(\mathbf{v})$ [8], we show:

Theorem 5. *If $I = \{1, 2, \ldots, k\}$, then* $\chi_{q,\mathbf{Z}(\vec{s})}(e_{S_n,\Lambda^{n-1-k}}) = \frac{1}{(q-1)^{n-1}} \tau(\beta_{\vec{s}}\zeta_I^-)|_{\mathbf{v} \to q^{1/2}}$.

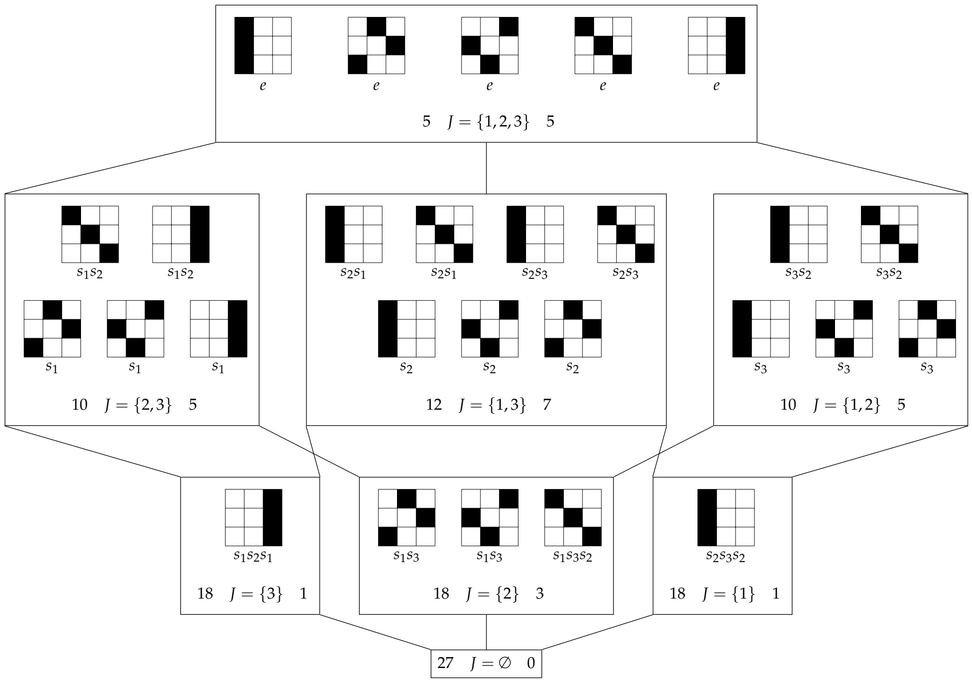
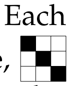


Figure 1: We take $W = S_4$ and $\vec{c} = (s_1, s_2, s_3)$ and $p = 3$. Each box is a set $\mathcal{D}^J(\vec{c}^p) := \coprod_{v \in W^{J,+}} \mathcal{D}^{(v)}(\vec{c}^p)$ for some J. Edges between boxes are containments between J's. Each $\vec{\omega} \in \mathcal{D}^J(\vec{c}^p)$ is drawn as a 3×3 box, with skips $\omega^{(i)} = \mathrm{id}$ in black. For example,  represents $\vec{\omega} = (\mathrm{id}, s_2, s_3, s_1, \mathrm{id}, s_3, s_1, s_2, \mathrm{id})$. In each box, the number left, *resp.* right, of J counts $w \in W$ with $\mathrm{Des}(w) \supseteq J$, *resp.* $\mathrm{Des}(w) = J$. The former is $\mathrm{Park}^{J,+}_{W,p}$. The rightmost number in the $(k+1)$th row is $\mathcal{P}_{2k}(L_{4,3})|_{\mathbf{v}\to 1}$.

By Theorem 2, the values $\chi_{q,\mathbf{Z}(\vec{c}^p)}(e_{S_n,\Lambda^k})$ for Coxeter words $\vec{c}$ and p coprime to n are the *rational q-Kirkman numbers* of [11] in type A. Via (2.7), Theorems 1 and 5 give:

Corollary 4. *For any word $\vec{s} = (s_{i_1}, \ldots, s_{i_\ell})$ and $0 \leq k \leq n-1$, we have*

$$\mathbf{v}^{\ell-n}\mathcal{P}_{2(n-1-k)}(L_{\beta_{\vec{s}}}) = \frac{1}{(\mathbf{v}^2-1)^{n-1}} \sum_{\substack{v \in S_n \\ \mathrm{Des}(v)=\{1,\ldots,k\}}} \sum_{\vec{\omega} \in \mathcal{D}^{(v)}(\vec{s})} \mathbf{v}^{2|\mathbf{d}_{\vec{\omega}}|}(\mathbf{v}^2-1)^{|\mathbf{e}_{\vec{\omega}}|}. \tag{3.9}$$

That is, each a-degree of $\mathcal{P}(L_{\beta_{\vec{s}}})$ is a sum of Deodhar-cell point counts.

Figure 1 illustrates Corollaries 3 and 4 simultaneously. When $k = 0$, the outer sum on the right-hand side of (3.9) collapses to $v = \mathrm{id}$, and we recover the "Legendrian ruling filtration" formula of Shende–Treumann–Zaslow mentioned in §1.2.

It is natural to seek a generalization of Corollary 4 to other W. For example, when $\vec{s} = \vec{c}^{h+1}$, this would recover the f-vectors of the W-associahedron. We have been unable to find such a construction. This may be related to the absence of uniform formulas for q-Kirkman numbers in general. Attractive formulas do exist for *coincidental types*, where the degrees of W form an arithmetic sequence [11, §10].

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References

[1] D. Armstrong, V. Reiner, and B. Rhoades. "Parking Spaces". *Adv. Math.* **269** (2015), pp. 647–706.

[2] D. Bessis and V. Reiner. "Cyclic Sieving of Noncrossing Partitions for Complex Reflection Groups". *Ann. Comb.* **15** (2011), pp. 197–222.

[3] W. Borho and R. MacPherson. "Partial Resolutions of Nilpotent Varieties". *Astérisque* **101–102** (1983), pp. 23–74.

[4] V. V. Deodhar. "On Some Geometric Aspects of Bruhat Orderings. I. A Finer Decompostion of Bruhat Cells". *Invent. math.* **79** (1985), pp. 499–511.

[5] P. Freyd, D. Yetter, J. Hoste, W. B. R. Lickorish, K. Millett, and A. Ocneanu. "A New Polynomial Invariant of Knots and Links". *Bull. Amer. Math. Soc.* **12**.2 (1985), pp. 239–246.

[6] P. Galashin and T. Lam. "Positroids, Knots, and q,t-Catalan Numbers". *Duke Math. J.* **173**.11 (2024), pp. 2117–2195.

[7] P. Galashin, T. Lam, M.-T. Q. Trinh, and N. Williams. "Rational Noncrossing Coxeter–Catalan Combinatorics". *Proc. Lond. Math. Soc.* **129**.4 (2024). e12643, 50 pp.

[8] A. P. Isaev and O. V. Ogievetsky. "On Representations of Hecke Algebras". *Czechoslovak Journal of Physics* **55**.11 (2005), pp. 1433–1441.

[9] L. K. Jones. "Centers of Generic Hecke Algebras". *Trans. Amer. Math. Soc.* **317**.1 (1990), pp. 361–392.

[10] V. F. R. Jones. "Hecke Algebra Representations of Braid Groups and Link Polynomials". *Ann. of Math. (2)* **126**.2 (1987), pp. 335–388.

[11] V. Reiner, A. V. Shepler, and E. Sommers. "Invariant Theory for Coincidental Complex Reflection Groups". *Math. Z.* **298**.1–2 (2021), pp. 787–820.

[12] V. Shende, D. Treumann, and E. Zaslow. "Legendrian Knots and Constructible Sheaves". *Invent. math.* **207** (2017), pp. 1031–1133.

[13] M.-T. Q. Trinh. "From the Hecke Category to the Unipotent Locus". 2021. arXiv:2106.07444.

[14] J. Wan and W. Wang. "Frobenius Map for the Centers of Hecke Algebras". *Trans. Amer. Math. Soc.* **367**.8 (2015), pp. 5507–5520.