

ANNULAR WEBS AND A CONJECTURE OF HAIMAN

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ABSTRACT. Haiman conjectured that when traces corresponding to monomial symmetric functions are evaluated on the Hecke-algebra elements denoted C'_w by Kazhdan–Lusztig, the resulting polynomials have nonnegative coefficients. We show that recent work on annular webs implies this for permutations w that are 321-hexagon-avoiding.

1. INTRODUCTION

1.1. Let $H_n(x)$ be the Iwahori–Hecke algebra of the symmetric group S_n over $\mathbf{Z}[x^{\pm 1}]$. As a quotient of the group algebra of a braid group, it has a standard basis $\{\sigma_w\}_{w \in S_n}$, consisting of the images of the positive permutation braids.

Kazhdan–Lusztig introduced two new bases for $H_n(x)$ with remarkable properties [KL79]. Taking our x to be their $q^{1/2}$, we will focus on the basis that they denote by $\{C'_w\}_w$, but write b_w in place of C'_w for simplicity. When the elements b_w are expanded in the standard basis, the coefficients are Laurent polynomials in x with nonnegative integer coefficients. Up to rescaling, these are the celebrated Kazhdan–Lusztig polynomials for S_n . Their positivity can be proved through a geometric interpretation of $H_n(x)$ in terms of sheaves on flag varieties.

The representation theory of S_n deforms to that of $H_n(x)$. In particular, each character $\chi : S_n \rightarrow \bar{\mathbf{Q}}$ defines a $\mathbf{Z}[x^{\pm 1}]$ -linear function $\chi_x : H_n(x) \rightarrow \bar{\mathbf{Q}}(x)$ that still enjoys the trace property $\chi(\alpha\beta) = \chi(\beta\alpha)$. At the same time, the *irreducible* characters of S_n are indexed by integer partitions of n . Let χ^λ be the irreducible character indexed by $\lambda \vdash n$. A geometric argument, similar to that used in the positivity of the Kazhdan–Lusztig polynomials, proves that $\chi_x^\lambda(b_w) \in \mathbf{Z}_{\geq 0}[x^{\pm 1}]$ for all w and λ .

Haiman found evidence for a stronger positivity statement. Recall that for all λ, μ , the Kostka number $K_{\lambda, \mu}$ counts semistandard Young tableaux of shape λ and weight μ . The Kostka numbers can be assembled into a unitriangular matrix of nonnegative integers. In particular, this matrix has an inverse with integer entries, so there are functions $\phi_x^\mu : H_n(x) \rightarrow \mathbf{Z}[x^{\pm 1}]$ uniquely defined by requiring

$$(1.1) \quad \chi_x^\lambda = \sum_{\mu} K_{\lambda, \mu} \phi_x^\mu \quad \text{for all } \lambda \vdash n.$$

What follows is the main part of Conjecture 2.1 in [Hai93].

Conjecture 1.1 (Haiman). $\phi_x^\mu(b_w) \in \mathbf{Z}_{\geq 0}[x^{\pm 1}]$ for all $w \in S_n$ and $\mu \vdash n$.

Abreu–Nigro observe that Conjecture 1.1 would imply several conjectures about the indifference graphs of Hessenberg functions in algebraic combinatorics: notably,

the Stanley–Stembridge conjecture on the e -positivity of their chromatic symmetric functions, and Shareshian–Wachs’s generalization of this conjecture to chromatic quasi-symmetric functions [AN24].

1.2. This note will show how recent work of Queffelec–Rose and Gorsky–Wedrich on the diagrammatics of $H_n(x)$ solves some cases of Conjecture 1.1.

For $1 \leq i \leq n-1$, let $b_i = b_{s_i}$, where $s_i \in S_n$ is the transposition that swaps i and $i+1$. The main theorem is:

Theorem 1.2. $\phi_x^\mu(b_{i_1} \cdots b_{i_\ell}) \in \mathbf{Z}_{\geq 0}[x^{\pm 1}]$ for any sequence of indices i_1, \dots, i_ℓ that range between 1 and $n-1$ inclusive, and any $\mu \vdash n$.

In what follows, suppose that $w \in S_n$ is given by $w = [w_1 w_2 \cdots w_n]$, meaning it sends i to w_i for $1 \leq i \leq n$. Fix $m \leq n$ and $v = [v_1 v_2 \cdots v_m] \in S_m$. We say that w is $v_1 \cdots v_m$ -avoiding if and only if the sequence (w_1, \dots, w_n) does not contain a subsequence of size m whose elements have the same relative order as (v_1, \dots, v_m) . More formally, this means we cannot find indices $1 \leq p_1 < \cdots < p_m \leq n$ such that $w_{p_i} < w_{p_j}$ whenever $i < j$ and $v_i < v_j$.

We write $S_n^{v_1 \cdots v_m} \subseteq S_n$ for the set of $v_1 \cdots v_m$ -avoiding elements. Following Billey–Warrington, we say that w is 321-hexagon-avoiding if and only if

$$w \in S_n^{321} \cap S_n^{46718235} \cap S_n^{46781235} \cap S_n^{56718234} \cap S_n^{56781234}.$$

In [BW01], Billey–Warrington prove that the following conditions are equivalent:

- (1) w is 321-hexagon-avoiding.
- (2) $b_w = b_{i_1} \cdots b_{i_\ell}$ whenever $w = s_{i_1} \cdots s_{i_\ell}$ and ℓ is the minimal length among such expressions.

Via this result, Theorem 1.2 implies:

Corollary 1.3. Conjecture 1.1 holds when w is 321-hexagon-avoiding.

1.3. The key observation is that Remark 4.21 of [GW23], a refinement of the annular web evaluation algorithm of [QR18], provides a counterpart to Theorem 1.2 (in fact, a slightly stronger statement) in the setting of Murakami–Ohtsuki–Yamada (MOY) webs. The passage from Hecke-algebra traces to web diagrammatics is best explained by assembling the cocenters of all the Hecke algebras into a direct sum that we identify with Macdonald’s ring of symmetric functions $\Lambda(x)$ over $\mathbf{Z}[x^{\pm 1}]$, after extending scalars. There is a universal trace

$$\mathrm{tr} : \bigoplus_n H_n(x) \rightarrow \Lambda(x).$$

There is also a natural candidate for the diagrammatic counterpart to tr : the map ann that sends a rectangular web to its annular closure.

For any $\beta \in H_n(x)$, the value of $\phi_x^\mu(\beta)$ is just the μ -th coefficient when we expand $\mathrm{tr}(\beta)$ in the basis of complete homogeneous symmetric functions $\{h_\mu\}_\mu$: a fact already noted in [AN24]. Ultimately, we relate Theorem 1.2 to [GW23, Rem. 4.21] through a commutative diagram that relates tr to ann , and assigns simple webs to the b_i and h_μ .

In fact, there is another, inequivalent commutative diagram, where $\{b_w\}_w$ is replaced by Kazhdan–Lusztig’s *other* basis for the Hecke algebra, and $\{h_\mu\}_\mu$ is replaced by the basis of elementary symmetric functions $\{e_\mu\}_\mu$. We present both diagrams together in Theorem 4.3. Their existence follows almost tautologically from the defining properties of two, mutually dual versions of the MOY web calculus: h_μ corresponds to the *symmetric* version where strand labels are symmetric powers, while e_μ corresponds to the *anti-symmetric* version where they are exterior powers. The original formalism of [MOY98] is the anti-symmetric one.

We have not found any explicit prior statement of Theorem 4.3 in the literature, though it seems to be folklore. Lemma 4.25 and Remark 4.26 of [GW23] establish closely related statements. We show that either of our commutative diagrams can be deduced from the other via a more general statement, Proposition 5.1, that holds for any finite Coxeter group. In particular, our treatment is *not* compatible with [RW20], where the skein relations for the symmetric and anti-symmetric web calculi are inconsistent.

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2. HECKE ALGEBRAS

2.1. This section reviews background that applies to any finite Coxeter group W with system of simple reflections S . Let $<$ denote the Bruhat order on W , and for any $w \in W$, let ℓ_w denote the Bruhat length of w [GP00, Ch. 1].

2.2. Formally, we define the *Iwahori–Hecke algebra* of W to be the $\mathbf{Z}[x^{\pm 1}]$ -algebra $H_W(x)$ spanned as a free module by elements σ_w for $w \in W$, modulo the following relations:

$$(2.1) \quad \sigma_w \sigma_s = \begin{cases} \sigma_{ws} & ws > w, \\ \sigma_{ws} + (x - x^{-1})\sigma_w & ws < w. \end{cases}$$

Let D be the additive involution of $H_W(x)$ that sends $x \mapsto x^{-1}$ and $\sigma_w \mapsto \sigma_{w^{-1}}^{-1}$ for all $w \in W$.

2.3. Let $\mathbf{K} = \mathbf{F}(x)$, where $\mathbf{F} \supseteq \mathbf{Q}$ is a splitting field for W . If $W = S_n$, then we can take $\mathbf{F} = \mathbf{Q}$.

It turns out that $\mathbf{K}H_W(x) := \mathbf{K} \otimes_{\mathbf{Z}[x^{\pm 1}]} H_W(x)$ is split as a \mathbf{K} -algebra [GP00, Thm. 9.3.5]. At the same time, there is an isomorphism of rings $H_W(x)|_{x \rightarrow 1} \simeq \mathbf{Z}W$. So by Tits deformation [GP00, §7.4], the semisimplicity of $\mathbf{F}W$ implies the semisimplicity of $\mathbf{K}H_W(x)$, and moreover, there is a bijection between isomorphism classes of simple $\mathbf{K}H_W(x)$ -modules and those of simple $\mathbf{F}W$ -modules, compatible with the $x \rightarrow 1$ specialization from $H_W(x)$ to $\mathbf{Z}W$.

This induces the assignment from characters χ of W to $\mathbf{Z}[x^{\pm 1}]$ -linear trace functions $\chi_x : H_W(x) \rightarrow \bar{\mathbf{K}}$ mentioned in the introduction. Explicitly, $\chi_x(\beta)$ is the trace of β on the $\mathbf{K}H_W(x)$ -module that corresponds to the $\mathbf{F}W$ -module with character χ .

2.4. Kazhdan–Lusztig proved that for all $w \in W$, there is a unique D -invariant element $b_w \in H_W(x)$ such that

$$b_w = \sum_{y \leq w} x^{\ell_y - \ell_w} P_{y,w}(x^2) \sigma_y$$

for some $P_{y,w}(q) \in \mathbf{Z}[q]$ satisfying

$$(2.2) \quad \begin{aligned} P_{w,w}(q) &= 1, \\ \deg P_{y,w}(q) &\leq \frac{1}{2}(\ell_w - \ell_y - 1) \quad \text{for all } w, y \in W \text{ with } y \leq w. \end{aligned}$$

Let j be the additive involution of $H_W(x)$ that sends $x \mapsto x^{-1}$ and $\sigma_w \mapsto (-1)^{\ell_w} \sigma_w$. Let $c_w = j(b_w)$. Then c_w is the unique D -invariant element of $H_W(x)$ such that

$$c_w = \sum_{y \leq w} (-1)^{\ell_y} x^{\ell_w - \ell_y} P_{y,w}(x^{-2}) \sigma_y$$

for some $P_{y,w}(q) \in \mathbf{Z}[q]$ satisfying (2.2). They turn out to be the same polynomials as before.

The sets $\{b_w\}_{w \in W}$ and $\{c_w\}_{w \in W}$ form bases for $H_W(x)$ as a free $\mathbf{Z}[x^{\pm 1}]$ -module, known as the two *Kazhdan–Lusztig bases* or *canonical bases*. The polynomials $P_{y,w}(q)$ are the *Kazhdan–Lusztig polynomials* for W . Note that in [KL79], b_w and c_w are respectively denoted C'_w and $-C_w$. (Also note the minus sign.)

2.5. For any $s \in S$, we have

$$(2.3) \quad \begin{aligned} b_s &= x^{-1} + \sigma_s &= x &+ \sigma_s^{-1}, \\ c_s &= x - \sigma_s &= x^{-1} - \sigma_s^{-1}. \end{aligned}$$

Just as $\{\sigma_s\}_{s \in S}$ generates $H_W(x)$ as a $\mathbf{Z}[x^{\pm 1}]$ -algebra, so do $\{b_s\}_s$ and $\{c_s\}_s$. In fact, one can check using (2.1) and (2.3) that whether we take $\gamma_s = b_s$ for all s or take $\gamma_s = c_s$ for all s , the defining relations of $H_n(x)$ with respect to the generating set $\{\gamma_s\}_s$ remain the same.

Let η be the involution of $H_n(x)$ as a $\mathbf{Z}[x^{\pm 1}]$ -algebra defined by swapping b_s and c_s for all $s \in S$. This is different from j , since j is not $\mathbf{Z}[x^{\pm 1}]$ -linear.

2.6. In the next two sections, we will focus on $W = S_n$. Here, we will always take $S = \{s_1, s_2, \dots, s_{n-1}\}$, where $s_i = (i, i+1)$ as in the introduction.

We will also write $H_n(x)$ in place of $H_{S_n}(x)$, and write σ_i, b_i, c_i in place of $\sigma_{s_i}, b_{s_i}, c_{s_i}$. Whether we take $\gamma_i = b_i$ or $\gamma_i = c_i$, the defining relations of $H_n(x)$ with respect to the generating set $\{\gamma_i\}_i$ are:

$$\left\{ \begin{aligned} \gamma_i \gamma_{i+1} \gamma_i - \gamma_i &= \gamma_{i+1} \gamma_i \gamma_{i+1} - \gamma_{i+1}, \\ \gamma_i \gamma_j &= \gamma_j \gamma_i && \text{for } |i - j| > 1, \\ \gamma_i^2 &= (x + x^{-1}) \gamma_i. \end{aligned} \right.$$

3. SYMMETRIC FUNCTIONS

3.1. Let Λ be the graded ring of symmetric functions over \mathbf{Z} in (countably) infinitely many variables. For background on Λ , we refer to [Mac15, Ch. I]. In this note, we

will need the following elements of Λ indexed by integer partitions λ :

the Schur functions	s_λ ,
the monomial symmetric functions	$m_\lambda = m_{\lambda_1} m_{\lambda_2} \dots$,
the complete homogeneous symmetric functions	$h_\lambda = h_{\lambda_1} h_{\lambda_2} \dots$,
the elementary symmetric functions	$e_\lambda = e_{\lambda_1} e_{\lambda_2} \dots$.

Let Λ_n be the degree- n component of Λ . The Schur functions s_λ with $\lambda \vdash n$ form a basis for Λ as a free \mathbf{Z} -module; analogous statements hold with m_λ or h_λ or e_λ in place of s_λ .

3.2. Recall the Kostka numbers $K_{\lambda,\mu} \in \mathbf{Z}$ from the introduction. As explained in [Mac15, §I.6], they relate the elements s_λ, m_μ, h_μ via the identities

$$(3.1) \quad s_\lambda = \sum_{\mu} K_{\lambda,\mu} m_\mu,$$

$$(3.2) \quad h_\mu = \sum_{\lambda} K_{\lambda,\mu} s_\lambda.$$

Comparing (1.1) to (3.1) shows the analogy: Haiman's character ϕ_x^μ is to m_μ as the irreducible character χ_x^λ is to s_λ .

Note that $K_{\lambda,\lambda} = 1$ for all λ , and that $K_{\lambda,\mu} = 0$ whenever $\mu > \lambda$ in the dominance order on partitions. This makes precise the unitriangularity mentioned earlier.

3.3. Let $\Lambda(x) = \mathbf{Z}[x^{\pm 1}] \otimes \Lambda$ and $\Lambda_n(x) = \mathbf{Z}[x^{\pm 1}] \otimes \Lambda_n$ for all n . The map tr mentioned in §1.3 is the sum of the $\mathbf{Z}[x^{\pm 1}]$ -linear maps

$$\text{tr}_n : H_n(x) \rightarrow \Lambda_n(x) \quad \text{defined by } \text{tr}_n = \sum_{\lambda \vdash n} s_\lambda \chi_x^\lambda.$$

By construction, $\text{tr}_n(\alpha\beta) = \text{tr}_n(\beta\alpha)$ for all α, β . So the universal property of the cocenter of $H_n(x)$ defines a $\mathbf{Z}[x^{\pm 1}]$ -linear map from the cocenter into $\Lambda(x)$, which turns out to be an isomorphism of $\mathbf{Z}[x^{\pm 1}]$ -modules.

Let $\langle -, - \rangle : \Lambda(x) \times \Lambda(x) \rightarrow \mathbf{Z}[x^{\pm 1}]$ be the *Hall pairing*: the $\mathbf{Z}[x^{\pm 1}]$ -linear pairing under which the Schur functions s_λ are orthonormal. By (1.1) and (3.2), we deduce:

Lemma 3.1. $\text{tr}_n = \sum_{\mu \vdash n} h_\mu \phi_x^\mu.$

Altogether, Theorem 1.2 is claiming that for any sequence of indices i_1, \dots, i_ℓ that range between 1 and $n-1$ inclusive, the expansion of $\text{tr}_n(b_{i_1} \cdots b_{i_\ell})$ in the complete homogeneous basis of $\Lambda_n(x)$ will have coefficients in $\mathbf{Z}_{\geq 0}[x^{\pm 1}]$.

4. WEBS

4.1. Let $H_n^{\text{web}}(x)$ be the free $\mathbf{Z}[x^{\pm 1}]$ -module generated by strictly upward-oriented web diagrams in a rectangle, connecting n inputs with label 1 at the bottom to n outputs with label 1 at the top, modulo the relations of the MOY bracket. It forms a $\mathbf{Z}[x^{\pm 1}]$ -algebra under concatenation of diagrams. Via quantum Schur–Weyl duality, the work of Murakami–Ohtsuki–Yamada shows that this algebra is isomorphic to $H_n(x)$ [MOY98]. Note that their q is our x .

For $1 \leq i \leq n-1$, let $\text{can}_i \in H_n^{\text{web}}(x)$ denote the i th merge-split web. The notation can is intended to suggest the adjective *canonical*. The precise result implied by [MOY98] is an isomorphism of $\mathbf{Z}[x^{\pm 1}]$ -algebras

$$\Theta_c : H_n(x) \xrightarrow{\sim} H_n^{\text{web}}(x) \quad \text{defined by } \Theta_c(c_i) = \text{can}_i,$$

By precomposition with the $\mathbf{Z}[x^{\pm 1}]$ -algebra involution η from §2.5, we obtain an isomorphism of $\mathbf{Z}[x^{\pm 1}]$ -algebras

$$\Theta_b : H_n(x) \xrightarrow{\sim} H_n^{\text{web}}(x) \quad \text{defined by } \Theta_b(b_i) = \text{can}_i.$$

Remark 4.1. We point out that Θ_b , but not Θ_c , appears in [GW23]. Recall that we can identify $H_n(x)$ with the skein algebra of a rectangle with n inputs and n outputs, by sending σ_i to the i th simple twist. Decategorified, rectangular analogues of formulas (16) and (17) in [GW23] define two isomorphisms from this skein algebra to $H_n^{\text{web}}(x)$. Our map Θ_b corresponds to their *framed* map (16). By contrast, Θ_c does not quite correspond to the *unframed* map (17), even up to rescaling.

4.2. Let $\mathcal{C}^{\text{web}}(x)$ be the free $\mathbf{Z}[x^{\pm 1}]$ -module generated by positively-oriented web diagrams in an annulus. It forms a commutative $\mathbf{Z}[x^{\pm 1}]$ -algebra under nesting of diagrams.

For any n and $\mu \vdash n$, let $o^\mu \in \mathcal{C}^{\text{web}}(x)$ be the diagram consisting of concentric essential circles with labels μ_1, μ_2, \dots . Note that by the commutativity of $\mathcal{C}^{\text{web}}(x)$, the order of these circles does not matter. The *annular web evaluation algorithm* of Queffelec–Rose [QR18, Lem. 5.2] shows that the set $\{o_\mu\}_\mu$ forms a basis for $\mathcal{C}^{\text{web}}(x)$ as a free $\mathbf{Z}[x^{\pm 1}]$ -module. The definition of the MOY bracket then implies, directly, that $\mathcal{C}^{\text{web}}(x)$ is freely generated *as an algebra* by the elements $o_n := o_{(n)}$ corresponding to single, labeled essential circles.

At the same time, display (2.4), *resp.* (2.8), in [Mac15] implies that $\Lambda(x)$ is freely generated as an algebra by the set $\{e_n\}_n$, *resp.* the set $\{h_n\}_n$. Thus, there are isomorphisms of $\mathbf{Z}[x^{\pm 1}]$ -algebras

$$\Xi_h, \Xi_e : \Lambda(x) \xrightarrow{\sim} \mathcal{C}^{\text{web}}(x) \quad \text{defined by } \Xi_e(e_\mu) = o_\mu \text{ and } \Xi_h(h_\mu) = o_\mu.$$

They differ precisely by the $\mathbf{Z}[x^{\pm 1}]$ -algebra involution of $\Lambda(x)$ that swaps h_μ and e_μ .

Prior to the introduction of webs, an analogous isomorphism for the skein algebra of the annulus was first established by Turaev [Tur88].

4.3. Queffelec–Rose’s annular web evaluation algorithm originally treated $\mathcal{C}^{\text{web}}(x)$ as the triangulated Grothendieck group of the bounded homotopy category of a graded, linear category of foams between positively-oriented annular webs. Gorsky–Wedrich observed that it could be refined, by instead treating $\mathcal{C}^{\text{web}}(x)$ as the additive Grothendieck group of the *Karoubi* or *idempotent completion* of this foam category [GW23, Rem. 4.21]. The refinement shows:

Theorem 4.2 (Queffelec–Rose + Gorsky–Wedrich). *The expansion of any annular web in the basis $\{o_\mu\}_\mu$ for $\mathcal{C}^{\text{web}}(x)$ will have coefficients in $\mathbf{Z}_{\geq 0}[x^{\pm 1}]$.*

4.4. There is a $\mathbf{Z}[x^{\pm 1}]$ -linear map

$$\text{ann} : \bigoplus_n H_n^{\text{web}}(x) \rightarrow \mathcal{C}^{\text{web}}(x)$$

called *annular closure*. It is defined graphically, by embedding a rectangle into an annulus as a sector, so that the upward orientation in the rectangle becomes the positive orientation in the annulus, then wrapping the n outputs of the rectangle around the annulus, without crossing, back to the n inputs. For more on annular closure in the skein literature, see [MM08] and the references there.

Via Lemma 3.1 and Theorem 4.2, we conclude that Theorem 1.2 follows from the commutativity of diagram (I) below.

Theorem 4.3. *The following diagrams commute:*

$$(I) \quad \begin{array}{ccc} H_n(x) & \xrightarrow{\text{tr}} & \Lambda(x) \\ \Theta_b \downarrow & & \downarrow \Xi_h \\ H_n^{\text{web}}(x) & \xrightarrow{\text{ann}} & \mathcal{C}^{\text{web}}(x) \end{array}$$

$$(II) \quad \begin{array}{ccc} H_n(x) & \xrightarrow{\text{tr}} & \Lambda(x) \\ \Theta_c \downarrow & & \downarrow \Xi_e \\ H_n^{\text{web}}(x) & \xrightarrow{\text{ann}} & \mathcal{C}^{\text{web}}(x) \end{array}$$

Below, we handle (I) and (II) in parallel. As an alternative, we show in §5.2 how the commutativity of one implies that of the other.

Proof. For convenience, set $(\star, \diamond, \heartsuit) \in \{(I, b, h), (II, c, e)\}$.

Step 1. First, we reduce to checking specific central elements of $H_n(x)$. In what follows, $Z(-)$ and $[-]$ denote center and cocenter, respectively. Let $\mathbf{K} = \mathbf{Q}(x)$. The map Θ_\diamond induces \mathbf{K} -linear isomorphisms

$$\Theta_\diamond : Z(\mathbf{K}H_n(x)) \xrightarrow{\sim} Z(\mathbf{K}H_n^{\text{web}}(x)) \quad \text{and} \quad [\Theta_\diamond] : [\mathbf{K}H_n(x)] \xrightarrow{\sim} [\mathbf{K}H_n^{\text{web}}(x)].$$

It is enough to show the commutativity of diagram (\star) where we extend scalars to \mathbf{K} and replace $H_n(x)$, $H_n^{\text{web}}(x)$, Θ_\diamond with $[\mathbf{K}H_n(x)]$, $[\mathbf{K}H_n^{\text{web}}(x)]$, $[\Theta_\diamond]$. But the cocenters of $\mathbf{K}H_n(x)$ and $\mathbf{K}H_n^{\text{web}}(x)$ are isomorphic to their centers. So it remains to show the commutativity of

$$\begin{array}{ccc} Z(\mathbf{K}H_n(x)) & \xrightarrow{\text{tr}} & \mathbf{K}\Lambda(x) \\ \Theta_\diamond \downarrow & & \downarrow \Xi_\heartsuit \\ Z(\mathbf{K}H_n^{\text{web}}(x)) & \xrightarrow{\text{ann}} & \mathbf{K}\mathcal{C}^{\text{web}}(x) \end{array}$$

where $\mathbf{K}\Lambda_n(x)$, $\mathbf{K}\mathcal{C}^{\text{web}}(x)$ are the \mathbf{K} -linear extensions of $\Lambda(x)$, $\mathcal{C}^{\text{web}}(x)$. As the top arrow is bijective, the basis $\{\heartsuit_\mu\}_\mu$ for $\mathbf{K}\Lambda(x)$ lifts to a basis $\{\heartsuit_\mu^\vee\}_\mu$ for $Z(\mathbf{K}H_n(x))$. To show the commutativity of (\star) , it remains to show that the annular closure of $\Theta_\diamond(\heartsuit_\mu^\vee)$ is o_μ .

Step 2. Next, we reduce to the case where μ is a trivial partition. Observe that for general $\mu = (\mu_1, \mu_2, \dots, \mu_\ell)$, we have \mathbf{K} -linear maps

$$(4.1) \quad \mathbf{K}H_{\mu_1}(x) \times \cdots \times \mathbf{K}H_{\mu_\ell}(x) \rightarrow \mathbf{K}H_n(x),$$

$$(4.2) \quad \mathbf{K}H_{\mu_1}^{\text{web}}(x) \times \cdots \times \mathbf{K}H_{\mu_\ell}^{\text{web}}(x) \rightarrow \mathbf{K}H_n^{\text{web}}(x)$$

compatible with Θ_\diamond . The first sends $(\heartsuit_{\mu_1}^\vee, \dots, \heartsuit_{\mu_\ell}^\vee) \mapsto \heartsuit_\mu^\vee$. Moreover, as we run over n and μ , the maps of the first, *resp.* second, kind endow $\bigoplus_n \mathbf{Z}(\mathbf{K}H_n(x))$, *resp.* $\bigoplus_n \mathbf{Z}(\mathbf{K}H_n^{\text{web}}(x))$, with the structure of a commutative algebra, in such a way that

$$\bigoplus_n \mathbf{Z}(\mathbf{K}H_n(x)) \xrightarrow{\text{tr}} \mathbf{K}\Lambda(x) \quad \text{and} \quad \bigoplus_n \mathbf{Z}(\mathbf{K}H_n^{\text{web}}(x)) \xrightarrow{\text{ann}} \mathbf{K}C^{\text{web}}(x)$$

become isomorphisms of algebras. Recall from §4.2 that Ξ_e is one, too. So if ann sends $\Theta_\diamond(\heartsuit_{\mu_i}^\vee) \mapsto o_{\mu_i}$ for all i , then it also sends $\Theta_\diamond(\heartsuit_\mu^\vee) \mapsto o_\mu$.

Step 3. Let $\tilde{z}_n \in H_n^{\text{web}}(x)$ be the web that merges all n strands into a single strand labeled n , then splits them up again. We see that \tilde{z}_n is quasi-idempotent in $\mathbf{Z}(\mathbf{K}H_n^{\text{web}}(x))$ (viewed as a subring of $\mathbf{K}H_n^{\text{web}}(x)$). Let z_n be the idempotent rescaling of \tilde{z}_n . Then $\text{ann}(z_n) = o_\mu$. It remains to show that $\Theta_\diamond(\heartsuit_n^\vee) = z_n$.

For any free \mathbf{K} -module V of finite rank, let $\mathbf{K}H_n^V(x)$ be the commutant of the $U_x(\mathfrak{gl}(V))$ -action on $V^{\otimes n}$. These algebras admit analogues of (4.1)–(4.2):

$$(4.3) \quad \mathbf{K}H_{\mu_1}^V(x) \times \cdots \times \mathbf{K}H_{\mu_\ell}^V(x) \rightarrow \mathbf{K}H_n^V(x).$$

Quantum Schur–Weyl duality defines homomorphisms $\Psi_n : \mathbf{K}H_n(x) \rightarrow \mathbf{K}H_n^V(x)$ that intertwine (4.1) with (4.3) for all μ , and are faithful for n less than or equal to the rank of V . Writing π_n^- and π_n^+ for the elements of $\mathbf{K}H_n^V(x)$ respectively defined by projection onto the n th exterior and symmetric powers of V , we have $\Psi_n(e_n^\vee) = \pi_n^-$ and $\Psi_n(h_n^\vee) = \pi_n^+$.

Meanwhile, the anti-symmetric web calculus of [MOY98] shows that there is a collection of homomorphisms $\Psi_n^{\text{web}, -} : \mathbf{K}H_n^{\text{web}}(x) \rightarrow \mathbf{K}H_n^V(x)$ uniquely characterized by intertwining (4.2) with (4.3) for all μ and the condition that $\Psi_2^{\text{web}, -}(z_2) = \pi_2^-$. Moreover, $\Psi_n^{\text{web}, -}(z_n) = \pi_n^-$ for all n . The symmetric analogue of [MOY98] shows the same statements with $+$ in place of $-$ everywhere. Now we do the crucial calculations¹

$$e_2^\vee = \frac{1}{x+x^{-1}} c_1, \quad h_2^\vee = \frac{1}{x+x^{-1}} b_1, \quad z_2 = \frac{1}{x+x^{-1}} \tilde{z}_2 = \frac{1}{x+x^{-1}} \text{can}_1,$$

which show that $\Psi_n(\Theta_c^{-1}(z_2)) = \pi_2^-$ and $\Psi_n(\Theta_b^{-1}(z_2)) = \pi_2^+$. Therefore,

$$\Psi_n \circ \Theta_c^{-1} = \Psi_n^{\text{web}, -} \quad \text{and} \quad \Psi_n \circ \Theta_b^{-1} = \Psi_n^{\text{web}, +}.$$

This forces $\Theta_c^{-1}(z_n) = h_n^\vee$ and $\Theta_b^{-1}(z_n) = e_n^\vee$ for all n . □

4.5. Theorem 4.2 also suggests a categorification of Conjecture 1.1.

Let \mathbf{C} be the category denoted $\mathbf{Kar}(A\mathbf{Foam}^+)$ in [GW23]: the Karoubi completion of a graded, linear category of foams between positively-oriented annular webs. Let \mathbf{H}_n be the analogous category where we replace the annulus by a rectangle with

¹Compare to the proof of Lemma 4.25 in [GW23].

n inputs and n outputs. By work of Mackaay–Vaz (in the \mathfrak{sl} , rather than \mathfrak{gl} , setting) [MV10], H_n is a diagrammatic presentation of the category of Soergel bimodules for S_n , and hence, categorifies $H_n(x)$.

Let \mathbf{B}_w be the indecomposable object of H_n indexed by $w \in S_n$, so that the isomorphism from the Grothendieck group to $H_n(x)$ sends $[\mathbf{B}_w]$ to b_w . Let \mathbf{O}_μ be the object of C_n underlying the annular web o_μ .

Conjecture 4.4. *For all $w \in S_n$, the annular closure of \mathbf{B}_w is isomorphic in C to a direct sum of objects of the form \mathbf{O}_μ .*

5. INTERTWINING DUALITIES

5.1. We return to the generality of a finite Coxeter group W . Let ε be its sign character, defined by $\varepsilon(w) = (-1)^{\ell_w}$. The following result should be very well-known, but we have not found an explicit reference.

Proposition 5.1. *For any irreducible character χ of W , we have*

$$(\varepsilon\chi)_x = \chi_x \circ \eta$$

as functions on $H_W(x)$, where η is the $\mathbf{Z}[x^{\pm 1}]$ -algebra involution from §2.5 that swaps the Kazhdan–Lusztig bases.

Proof. By Proposition 9.4.1 of [GP00],

$$(\varepsilon\chi)_x(\sigma_w) = (-1)^{\ell_w} \chi_x(\sigma_w)|_{x \rightarrow x^{-1}} \quad \text{for all } w \in W.$$

Using (2.3), we deduce that

$$\chi_x^{\lambda^t}(b_{s(1)} \cdots b_{s(\ell)}) = \chi_x^\lambda(c_{s(1)} \cdots c_{s(\ell)}) = \chi_x^\lambda(\eta(b_{s(1)} \cdots b_{s(\ell)}))$$

for any sequence of elements $s^{(1)}, \dots, s^{(\ell)} \in S$. But $H_W(x)$ is generated as an algebra by $\{b_s\}_{s \in S}$, so every element of $H_W(x)$ is a linear combination of elements of the form $b_{s(1)} \cdots b_{s(\ell)}$. \square

5.2. Using Proposition 5.1, we can show that the commutativity of either diagram (I) in Theorem 4.3 implies that of the other.

Proof. First, recall that the involution of $\Lambda(x)$ that swaps h_μ and e_μ also swaps s_λ and s_{λ^t} , where λ^t is the transpose of λ [Mac15, (3.8)]. So the map

$$\mathrm{tr}_n^t : H_n(x) \rightarrow \Lambda_n(x) \quad \text{defined by } \mathrm{tr}_n^t = \sum_{\lambda \vdash n} s_{\lambda^t} \chi_x^\lambda$$

satisfies $\Xi_e \circ \mathrm{tr}_n = \Xi_h \circ \mathrm{tr}_n^t$. Next, observe that

$$\mathrm{tr}_n^t = \sum_{\lambda \vdash n} s_\lambda \chi_x^{\lambda^t} = \sum_{\lambda \vdash n} s_\lambda (\varepsilon \chi^\lambda)_x.$$

So by Proposition 5.1, $\mathrm{tr}_n^t = \mathrm{tr}_n \circ \eta$.

Altogether, $\Xi_e \circ \mathrm{tr}_n = \Xi_h \circ \mathrm{tr}_n \circ \eta$, whereas $\Theta_c = \Theta_b \circ \eta$. So by the involutivity of η , we have $\mathrm{ann} \circ \Theta_c = \Xi_e \circ \mathrm{tr}_n$ if and only if $\mathrm{ann} \circ \Theta_b = \Xi_h \circ \mathrm{tr}_n$. \square

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