(Munkres §25) recall that for any X:

the connected components of X are
 the equiv. classes where x ~ y iff
 there is a conn. subsp. containing x and y

the path components of X are
 the equiv. classes where x ↔ y iff
 there is a path between x and y in X

Lem the conn., resp. path components
are the maximal nonempty
conn., resp. path-conn. subspaces

[e.g., if A sub X conn, then A sub a conn. comp]

Pf by the def of an equiv. relation…

Q are conn. components open? closed?

Ex we say X is totally disconnected iff
all nonempty conn. subspaces of X
are singletons

PS5, #1 says Q is totally disconnected in R
singletons in Q are closed, but not open

Thm if A sub X is conn, then Cl_X(A) is too

more generally:
any B s.t. A sub B sub Cl_X(A) is too

Cor conn. components of X are closed
[by maximality]

Pf of Thm suppose U, V is a separation of B

by Feb 5 lecture (also M. Lem. 23.2), know:
since A is connected, either $A \subseteq U$ or $A \subseteq V$

then $\mathrm{Cl}_B(A) \subseteq \mathrm{Cl}_B(U) = U$
but by Feb 3 lecture (also M. Thm. 17.4),
$\mathrm{Cl}_B(A) = \mathrm{Cl}_X(A) \cap B$ ($= B$ here)
so $B \subseteq U$
contradicts V being nonempty □

Cor if X has finitely many conn. components
then they are also open [thus clopen]

Q are path components open?
closed?

Ex in the topologist's sine curve
$\check{S} = S \cup A$,
where $S = \{(x, \sin(1/x)) \mid x \in (0, 1]\}$
$A = \{(0, y) \mid y \in [-1, 1]\}$

we can check that S, A are the path components
S is open but not closed
A is closed but not open

Rem since path-connected implies connected

each path component of X is contained in
some conn. component of X

Q when can we ensure
{conn. components of X}
=
{path components of X}?

[helps to weaken our notions of conn., path-conn.]

Df X is locally connected at x iff,
for all open U containing x,
there is a conn. open V s.t. x in V sub U

similar def for locally path-connected,
replacing conn. V with path-conn. V

Ex [0, 1) cup (1, 2] is loc. conn but not conn
Q is not locally connected

Thm if X is locally path-connected
then its conn. comp's are path comp's

Lem if X is locally path-connected
then for any open U sub X,
path comp. P of U,
P is also open in X

[similar lem for locally connected,
replacing path comp. P with conn. comp.]

Pf pick x in P
since X is locally path-connected,
have path-conn. open V sub U s.t.
x in V sub U
by maximality of P, have V sub P

<u>Pf of Thm</u> pick a conn. component C of X

C ≠ Ø bc it's an equiv. class
so pick x in C
let P be the path component containing x
then P sub C; we want P = C

any path component of X intersecting C
is conn., hence contained in C [by max'lty]
let Q be the union of these except for P
then C = P cup Q

claim that if Q ≠ Ø, then a contradiction: [why?]
by lemma, each path comp. of X is open in X
hence P, Q are open in X, hence in C
hence P, Q would form a separation of C □

(Munkres §26) X arbitrary top space

in analysis, two notions of compactness:
"sequential" compactness
"finite-cover" compactness
in topology, the latter is the standard notion

<u>Df</u> an open cover of X is
a collection of open sets {U_i}_i in X s.t.
X = bigcup_i U_i

[similar to a subbasis, but need not generate the topology]

a subcover of {U_i}_i is
a subcollection that remains a cover

we say that X is <u>compact</u> iff
every cover of X admits a finite subcover

<u>Ex</u> anlytc R not anlytc compact:
take U_n = (n – 1, n + 1) for n in Z

<u>Ex</u> anlytc {1/n | n = 1, 2, 3, …} not compact:
[what cover?]
take singletons {1/n} for n = 1, 2, 3, …

<u>Ex</u> K = {0} cup {1/n | n} is compact:

any cover must include U s.t. 0 in U
then K – U is a finite set

<u>Ex</u> (Heine–Borel) anlytc [0, 1] is compact

[plays well with subspaces:]

<u>Lem</u> TFAE for Y sub X:

1) Y is compact as a subspace of X
2) for any collection of opens {U_i}_i s.t.
Y sub bigcup_i U_i,
there is a finite subcollection {U_j}_{j in J} s.t.
Y sub bigcup_{j in J} U_j

<u>Pf</u> boring

<u>Thm</u> closed subspaces of compact spaces
are compact

thus Heine–Borel implies compactness of K

Pf let X be compact and Y closed

given collection of opens {U_i}_i in X s.t.
Y sub bigcup_i U_i
consider {X – Y} cup {U_i}_i
this is an open cover of X, since X – Y is open
so it has a finite subcover
even if we remove X – Y from this subcover,
it remains finite and its union contains Y

[tell the Sorensen story?]