given a covering $p : E \to X$ and x in X:

<u>Thm</u> (Lifting) for all e in $p^{-1}(x)$:

a) injection $\varphi_e : p_*(\pi_1(E, e))\backslash\pi_1(X, x) \to p^{-1}(x)$
defined by
$\varphi_e(p_*(\pi_1(E, e)) * [\gamma]) = e \cdot [\gamma]$

b) if E is path-connected, then φ_e is bijective

<u>Thm</u> (Galois) if X is conn. & locally simply-conn., then a bijection:

$$\{\text{pointed (path-conn.) coverings of } X\}/\sim \;\to \{\text{subgroups of } \pi_1(X, x)\}$$

defined by $(p : E \to X, e)$ mapsto $p_*(\pi_1(E, e))$

why Galois?

H subgp of H'	p extends p'
$H = \pi_1(X, x)$ [?]	p trivial [= id_X]
H trivial [?]	E simply-conn.

<u>Df</u> a covering $p : E \to X$ is universal iff E is path-connected & simply-connected

<u>Thm</u> if X is conn. & locally simply-conn., then: a universal covering exists

idea: want $\pi_1(X, x) \to p^{-1}(x)$ to be a bijection

take $p^{-1}(x) = \pi_1(X, x)$ literally

Df the path space of X is

$$\Pi_X = \{\text{paths } \gamma : [0, 1] \to X\}$$

[space not set?] with the compact-open topology:

the topology with subbasis $B_{K, U}$, where

$$B_{K, U} = \{\gamma \mid \gamma(K) \text{ is inside } U\}$$

for all compact K in [0, 1] and open U in X
[why a subbasis? note that $B_{[0, 1], X} = \Pi_X$]
[how can we get from Π_X back to X?]

Lem the maps $p_0, p_1 : \Pi_X \to X$ defined by $p_0(\gamma) = \gamma(0)$ and $p_1(\gamma) = \gamma(1)$ are cts

Pf $p_i^{-1}(U) = B_{\{i\}, U}$ for i = 0, 1

the based/pointed path space of (X, x) is

$$\Pi_{X, x} = \{\gamma \text{ in } \Pi_X \mid \gamma(0) = x\} = p_0^{-1}(x)$$

consider $p_1 : \Pi_{X, x} \to X$:
- $p_1^{-1}(y)$ = {paths in X from x to y}
- $p_1^{-1}(x)$ = {loops in X based at x}

Lem $\Pi_{X, x}$ is contractible

$$h : \Pi_{X, x} \times [0, 1] \to \Pi_{X, x}$$

def by $h(\gamma, t) = \gamma_t$, where
$\gamma_t(s) = \gamma(s)$ $s \le t$
$\gamma_t(s) = \gamma(t)$ $s \ge t$

almost what we want: [what's left?]

need to quotient by path-homotopy:

if $\gamma \sim_{path} \gamma'$, then $p_1(\gamma) = p_1(\gamma')$

<u>Thm</u> $p_1 : \Pi_{X,\, x} \to X$ induces a universal covering $\Pi_{X,\, x} /\!\sim_{path} \to X$

<u>Rem</u> every other cover is a quotient of this

if $p : E \to X$ is a covering, then self-homeo's $\varphi : E \to E$ s.t. $p \circ \varphi = p$ are called <u>deck transformations</u> of p and form a group Deck(p)

[similar to pointed self-equiv.'s but need not preserve a basept on E]

if p is universal, then Deck(p) is iso to $\pi_1(X)$

[in general, Deck(p) is iso to $N_{\pi\_1(X)}(\pi_1(E))$]

so any subgrp H of $\pi_1(X)$ is iso to one of Deck(p)

$$\pi_1((\Pi_{X,\, x} /\!\sim_{path})/H) = H$$

thus surjectivity in the Galois correspondence

where to go from here?

- homology groups $H^n(X)$
- higher homotopy groups $\pi_n(X)$
- manifold theory

a <u>surface</u> is a manifold of dim 2

the <u>connect sum</u> (<u>#-sum</u>) of connected surfaces
is defined in PS9, #3 [draw]
[any connected manifold is path-connected]

<u>Thm</u> (Classification of Compact Surfaces)

any conn. component of a compact surface S is
the #-sum of a sphere with finitely many

- handles [T: draw]
- cross-handles [K: draw]
- cross-caps [P^2: draw]
- holes (including boundary)

call such a component <u>ordinary</u>

[hardest part of the proof:] S has a triangulation:
it is the quotient of a disjoint union of
finitely many triangles mod “edge zipping”
[edges paired up with orientations]

<u>Zip Proof of Classification from Triangulation</u>
(due to John H. Conway)

induction on zipping relations

base case: before zipping, each triangle *is*
a sphere with a single hole, hence ordinary

<u>Claim</u> every time we zip together ordinary
components [possibly one to itself],
the result is still ordinary

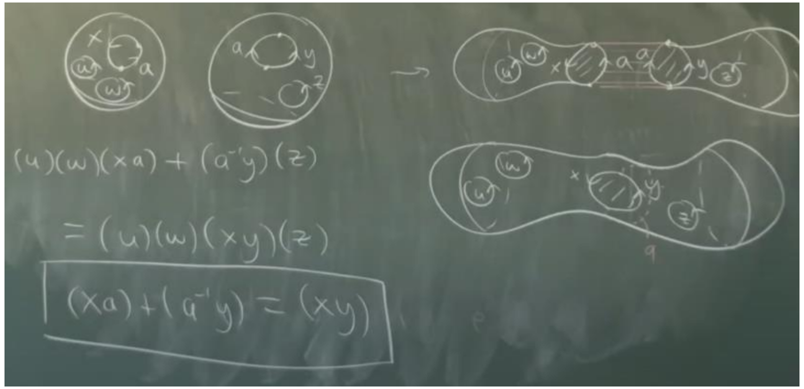
how to label holes with zippers?

(non-reduced) words like abc... up to

cyclic shift ab...yz = zab...y = ...

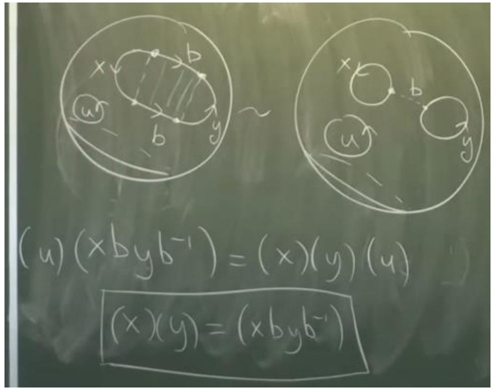
inversion ab...yz = $z^{-1}y^{-1}...b^{-1}a^{-1}$

1) zipping “a” to “a^{-1}” on separate holes:
handle with possible extra hole

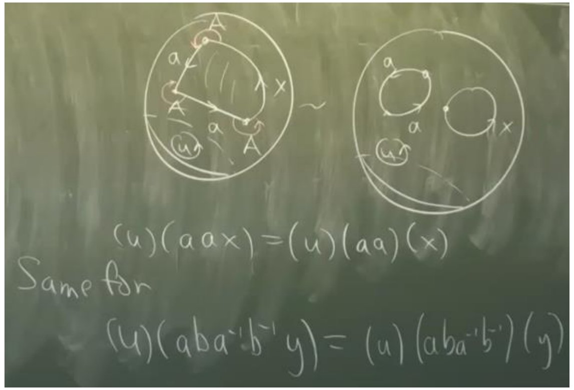


2) zipping “a” to “a” on separate holes:
handle or cross-handle with possible hole

3) zipping “b” to “b^{-1}” on the same hole:
possible extra hole



4) zipping “b” to “b” on the same hole:
cross-cap with possible hole



lastly:

<u>Addendum</u> a cross-handle can be replaced with two cross-caps

see Francis–Weeks, “Conway’s ZIP Proof”