# MATH 430: INTRODUCTION TO TOPOLOGY
# PROBLEM SET #4

SPRING 2025

**Due Wednesday, February 12.** You may consult books, papers, and websites as long as you cite all sources and write up your solutions in your own words. **Updated on 2/6, in bold.**

**Problem 1** (Munkres 127–128, #8(c))**.** Recall from Problem Set 3, #8, the set

$$X = \{x \in \mathbf{R}^\omega \mid \sum_{i>0} x_i^2 \text{ converges}\}$$

and its $\ell^2$ topology. Let $H$ be the *Hilbert cube*

$$H = [0, 1] \times [0, \tfrac{1}{2}] \times [0, \tfrac{1}{3}] \times \cdots \subseteq X.$$

Compare the box, $\ell^2$, uniform, and product topologies that $H$ inherits from $X$.

**Problem 2** (Munkres 101, #11–13)**.** Show that:

(1) A product of two Hausdorff spaces is Hausdorff (in the product topology).
(2) A subspace of a Hausdorff space is Hausdorff (in the subspace topology).
(3) $X$ is Hausdorff if and only if its *diagonal* $\Delta_X = \{(x, x) \mid x \in X\}$ is closed in (the product topology on) $X \times X$.

**Problem 3** (Munkres 118, #6)**.** Let $(X_\alpha)_\alpha$ be an arbitrary collection of topological spaces, and let $x^{(1)}, x^{(2)}, \ldots$ be a sequence of points in $\prod_\alpha X_\alpha$. (Each takes the form $x^{(i)} = (x_\alpha^{(i)})_\alpha$.)

(1) Show that in the product topology, the sequence converges to a point $x = (x_\alpha)_\alpha$ if and only if, for all $\alpha$, the sequence $x_\alpha^{(1)}, x_\alpha^{(2)}, \ldots$ converges to $x_\alpha$.
(2) Does (1) remain true if we replace the product topology with the box topology?

**Problem 4** (Munkres 144, #2)**.** Let $p : X \to Y$ be a continuous map.

(1) Show that if $p \circ f$ is the identity map on $Y$ for some continuous map $f : Y \to X$, then $p$ is a quotient map.
(2) A *retraction* from $X$ onto a subset $A$ is a continuous map $r : X \to A$ such that $r(a) = a$ for all $a \in A$. Deduce from (1) that retractions are quotient maps.

**Problem 5** (Munkres 145, #6)**.** Endow $\mathbf{R}$ with the *$K$-topology*: the topology generated by the basis consisting of the open intervals $(a, b)$ as well as the sets $(a, b) - K$, where $a, b \in \mathbf{R}$ and

$$K = \{\tfrac{1}{n} \mid n = 1, 2, 3, \ldots\}.$$

Let $Y$ be the quotient space obtained from $\mathbf{R}$ by collapsing $K$ to a point, and let $p : \mathbf{R} \to Y$ be the resulting map.

(1) Show that $Y$ is not Hausdorff, but satisfies the *$T_1$ condition*: For all $x, y \in Y$, we can find an open set containing $x$ but not $y$.
(2) Show that $(p, p)^{-1}(\Delta_Y)$ is closed in $\mathbf{R} \times \mathbf{R}$. Hence, by Problem 2(3), the product and quotient topologies on $Y \times Y$ must differ.

**Problem 6** (Munkres 152, #2)**.** Let $(A_n)_{n=1}^{\infty}$ be a sequence of connected subspaces of $X$, such that $A_n \cap A_{n+1} \neq \emptyset$ for all $n$. Show that $\bigcup_{n=1}^{\infty} A_n$ is connected.

**Problem 7** (Munkres 152, #9)**.** Let $X, Y$ be connected, and let $A \subseteq X$ and $B \subseteq Y$ be proper subsets. Show that

$$(X \times Y) - (A \times B)$$

is a connected subspace of $X \times Y$.

**Problem 8** (Munkres 152, #11)**.** Let $p : X \to Y$ be a quotient map. Show that if $Y$ is connected and each subspace $p^{-1}(y) \subseteq X$ is connected, then $X$ is connected.