Review given vec. spaces $V_1, V_2, \ldots, V_r, U$,

$\mu : V_1 \times V_2 \times \ldots \times V_r$ to U is multilinear
iff
for any index i, and choice of w_j in V_j for all $j \neq i$,
the map V_i to U def by
$v \mapsto \mu(\ldots, w_{i-1}, v, w_{i+1}, \ldots)$
is linear

μ is called
a multilinear functional when $U = F$
an r-linear form when $U = F$ and $V_1 = \ldots = V_r$

previously saw: $\beta : F^2 \times F^2$ to F def by
$\beta((a, c), (b, d)) = ad - bc$
is a bilinear form on F^2

moreover, β has antisymmetry: $\beta(w, v) = -\beta(v, w)$
[pause: where have we seen $ad - bc$ before?]

$$ad - bc = \det \begin{matrix} a & b \\ c & d \end{matrix}$$

today: generalize this example to higher dim's,
using multilinear forms

(Axler §9B, cont.) fix r, finite-dim'l V, and
an r-linear form $\mu : V^r$ to F

Df we say μ is an alternating r-form iff
$\mu(v_1, \ldots, v_r) = 0$ for any $v_1, \ldots, v_r$
that include a repeated vector:
i.e., $v_i = v_j$ for some distinct i and j

Ex $\quad \beta((a, c), (b, d)) = ad - bc$ is alternating
[check on board]

is the dot product on F^2 alternating? [wait…] no

Ex $\quad$ for any V and r, there is a silly example of an alternating form [what is it?]:
the zero r-form

Q $\quad$ given V, r, can we find other examples?
[turns out there is a constraint:]

Prop $\quad$ if $r > \dim V$, then
the only alternating r-form on V is
the zero form

[did we see the condition $r > \dim V$ before? wait]
recall that if $r > \dim V$, then a list of r vectors
cannot form a linearly independent set

so the prop follows from:

Lem $\quad$ if $v_1, \ldots, v_r$ is a linearly dependent set of vectors in V
then $\mu(v_1, \ldots, v_r) = 0$ for any alternating r-form on V

Pf $\quad$ by the dependence, we can write
some v_i as a lin combo of the others:
say, $v_i = \sum_{j \neq i} c_j v_j$

so $\mu(\ldots, v_i, \ldots) = \sum_{j \neq i} \mu(\ldots, c_j v_j, \ldots) = 0$

so alt. r-forms only interesting for $0 < r \le \dim V$

<u>Rem</u> recall that multilinear functionals on $V_1 \times \dots \times V_r$ form a vector space

[under $(a \cdot \mu + \mu')(\dots) = a\mu(\dots) + \mu'(\dots)$]

so r-linear forms on V form a vector space

and alternating r-forms form a linear subsp. that we will denote $\mathrm{Alt}^r(V)$

[the main thm of today:]

<u>Thm</u> let $n = \dim V$

let $e_1, \dots, e_n$ be an ordered basis

then an alternating n-form μ on V is determined by the value $\mu(e_1, \dots, e_n)$

<u>Lem</u> for any permutation $\sigma = (i_1, \dots, i_n)$ of $(1, \dots, n)$

[i.e., we have $1 \le i_j \le n$ for $j = 1, \dots, n$, and the numbers i_j have no repeats]

$$\mu(e_{i_1}, \dots, e_{i_n}) = (-1)^{\mathrm{Inv}(\sigma)} \mu(e_1, \dots, e_n)$$

where $\mathrm{Inv}(\sigma) = |\{(j, k) \mid 1 \le j < k \le n \text{ but } i_j > i_k\}|$

a.k.a. the <u>inversion number</u> of σ

[thus: $\mu(e_{i_1}, \dots)$ is determined by $\mu(e_1, \dots)$]

<u>Ex</u> let $n = 3$ and $\sigma = (2, 1, 3)$, $\sigma' = (3, 1, 2)$

[give them time to compute]

$\mathrm{Inv}(\sigma) = 1$, $\mathrm{Inv}(\sigma') = 2$

<u>Pf Sketch</u> we assume the following fact:

> every permutation σ is the result of applying a finite sequence of <u>transpositions</u>
> [i.e., pick two indices $j < k$, then swap them]

by induction, can show

$$\mathrm{Inv}(\sigma) \equiv \#\text{ of transpositions needed (mod 2)}$$

so it remains to show:
$\mu(\ldots, e_k, \ldots, e_j, \ldots) = -\mu(\ldots, e_j, \ldots, e_k, \ldots)$
[pause: what's next?]
expand $\mu(\ldots, e_j + e_k, \ldots, e_j + k, \ldots)$ into
four terms [do explicitly on board]
then apply the alternating property

<u>Pf of Thm</u> suppose μ in $\mathrm{Alt}^n(V)$

want to show: for all $v_1, \ldots, v_n$ in V,
$\mu(v_1, \ldots, v_n)$ is determined by $\mu(e_1, \ldots, e_n)$

since $(e_i)_i$ is a basis, have $a_{j,i}$ in F s.t.

$$v_i = \sum_i a_{j, i}\, e_j \qquad \text{for all } i$$

[what next?] substituting and using multilinearity,

$$\mu(v_1, \ldots, v_n) = \sum_\sigma a_{1, i_1} \ldots a_{n, i_n}\, \mu(e_{i_1}, \ldots)$$

$$\boxed{= \sum_\sigma (-1)^{\mathrm{Inv}(\sigma)}\, a_{1, i_1} \ldots a_{n, i_n}\, \mu(e_1, \ldots)}$$

in fact, the proof gives more than the thm:
it gives a formula for $\mu(v_1, \dots, v_n)$ [in the box],
for any list $v_1, \dots, v_n$

<u>Cor 1</u> if the matrix $(a_{j, i})_{j, i}$ is upper-triangular, then

$$\mu(v_1, \dots, v_n) = a_{1,1} \dots a_{n,n} \, \mu(e_1, \dots, e_n)$$

<u>Pf</u> upper-triangular means
for all $j > i$, we have $a_{j, i} = 0$

so in the sum formula for $\mu(v_1, \dots, v_n)$,
any term where $\mathrm{Inv}(\sigma) > 0$ must vanish
so only $\sigma = (1, 2, \dots, n)$ contributes

<u>Cor 2</u> if $n = \dim V$, then the space $\mathrm{Alt}^n(V)$ of alternating n-forms on V is 1-dim'l

<u>Pf</u> pick a basis $e_1, \dots, e_n$
any μ is determined by $\mu(e_1, \dots, e_n)$

in particular: if $\mu'(e_1, \dots, e_n) = \lambda\, \mu(e_1, \dots, e_n)$
then we must have $\mu'(\dots) = \lambda\mu(\dots)$

<u>Motivation</u> using $\mathrm{Alt}^n(V)$, we can interpret determinants as "scaling factors"

for any lin. op $T : V$ to V, alt. form μ in $\mathrm{Alt}^n(V)$,
let $T^*\mu$ in $\mathrm{Alt}^n(V)$ be def by

$$T^*\mu(v_1, \dots, v_n) = \mu(Tv_1, \dots, Tv_n)$$

Rem   Axler writes μ_T instead of $T^*\mu$

Thm   [in this setup:] $T^*\mu(\ldots) = \det(T)\, \mu(\ldots)$

Pf   since $\dim \mathrm{Alt}^n(V) = 1$ [by Cor 1],
suffices to prove it for a fixed nonzero μ

pick a basis $e_1, \ldots, e_n$ s.t. the matrix of T
wrt $(e_i)_i$ is a Jordan canonical form matrix
pick μ s.t. $\mu(e_1, \ldots, e_n) = 1$

write the matrix as $(a_{j, i})_{j, i}$
then $\det(T) = a_{1,1} \ldots a_{n,n}$ [by earlier class]
$= \mu(Te_1, \ldots, Te_n)$ [by Cor 1]
$= T^*\mu(e_1, \ldots, e_n)$ □

[if out of time, just state last cor without proof]

Cor $\det(T) = a_{1,1} \ldots a_{n,n}$ holds
for any T upper-triangular, not just JCF

Cor $\det(S \circ T) = \det(S) \det(T)$

Pf   pick μ nonzero and observe:

$(S \circ T)^*\mu = \det(S \circ T)\, \mu$
but also
$(S \circ T)^*\mu = S^*(T^*\mu) = \det(S) \det(T)\, \mu$