KIW 52 (Dec. 2025)

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$G = \mathrm{GL}_n$

$\mathcal{U} = \{\text{unipotent } n \times n \text{ matrices}\} \subseteq G$

<u>Ex</u> for $u \in \mathrm{GL}_2$, write $u - I = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$\begin{aligned} u \text{ is unipotent} &\iff u - I \text{ is nilpotent} \\ &\iff a + d = ad - bc = 0 \end{aligned}$$

so for $n = 2$, have $\mathcal{U} = \left\{ \begin{pmatrix} 1+a & b \\ c & 1-a \end{pmatrix} \middle| \, a^2 + bc = 0 \right\}$

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B upper-triangular subgroup of G

<u>Bruhat, Cartan, Chevalley</u> understand G via

$$G = \bigsqcup_{w \in S_n} B\dot{w}B, \quad \text{where } \dot{w}\text{'s are permutation matrices}$$

how to understand $\mathcal{U}$ via $U := B \cap \mathcal{U}$?

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<u>Thm</u> (Steinberg '65) $|\mathcal{U}(\mathbb{F}_q)| = |U(\mathbb{F}_q)|^2 \ (= q^{n(n-1)})$

<u>Thm</u> (Kawanaka '75)

$$|(\overbrace{\mathcal{U} \cap B\dot{w}B}^{\mathcal{U}_w})(\mathbb{F}_q)| = |(\overbrace{UU_- \cap B\dot{w}B}^{\mathcal{V}_w})(\mathbb{F}_q)| \quad \text{for any } w$$

where $U_- \subseteq B_-$ are lower-triangular

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<u>Ex</u> for any n, have $\mathcal{U}_{\mathrm{id}} = U = \mathcal{V}_{\mathrm{id}}$

<u>Ex</u> for $n = 3$ and $\dot{w} = \begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix}$,

$$\mathcal{U}_w \simeq U \times \{(a,b,c,d) \mid a, b \neq 0,\ (1+ab)^3 = abcd\}$$
$$\mathcal{V}_w \simeq U \times \{(a,b,c,d) \mid a, b \neq 0,\ 1+ab = abcd\}$$

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$B = T \ltimes U$ and $B_- = T \ltimes U_-$, where T is diagonal

$$\begin{array}{lcll} T & \curvearrowright & \mathcal{U}_w, & t \cdot u = tut^{-1} \\ T & \curvearrowright & \mathcal{V}_w, & t \cdot xy = (txt^{-1})(tyt^{-1}) \end{array}$$

<u>Thm (T)</u>

$$\mathrm{gr}^{\mathsf{W}}_* \mathrm{H}^*_{c,T}(\mathcal{U}_w(\mathbb{C})) \simeq \mathrm{gr}^{\mathsf{W}}_* \mathrm{H}^*_{c,T}(\mathcal{V}_w(\mathbb{C})) \quad \text{for all } w$$

where $\mathsf{W}_{\leq *}$ is the weight filtration on $\mathrm{H}^*_{c,T}$
implies (Kawanaka) via results of Katz

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a T-equivariant map

$$\Phi \colon UU_- \to \mathcal{U} \quad \text{def by } \Phi(xy) = xyx^{-1}$$

<u>Conj (T)</u> Φ restricts to a homotopy equivalence

$$\mathcal{V}_w(\mathbb{C}) \to \mathcal{U}_w(\mathbb{C}) \quad \text{for any } w$$

would imply thm about $\mathcal{U}_w, \mathcal{V}_w$

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recall: $w \in S_n$ lifts to $\sigma_w \in Br_n^+$

for $\beta = \sigma_{w_1} \cdots \sigma_{w_k} \in Br_n^+$,

$$\begin{array}{ccc} X_\beta^C & \to & X_\beta := B\dot{w}_1 B \times^B B\dot{w}_2 B \times^B \cdots \times^B B\dot{w}_k B \\ \downarrow & & \downarrow \\ C & \to & G \end{array}$$

$\mathcal{U}_w = X_{\sigma_w}^{\mathcal{U}}$ and $\mathcal{V}_w = X_{\sigma_w \pi}^1$, where $\pi := \sigma_{w_\circ}^2$

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HOMFLYPT poly P: $\{\text{links in } \mathbb{R}^3\} \to \mathbf{Z}[\![q]\!][a, q^{-1/2}]$

KhR superpoly $\mathbb{P} : \{\text{links in } \mathbb{R}^3\} \to \mathbf{Z}[\![q]\!][a, q^{-1/2}, t]$

satisfying $\mathbb{P}|_{t \to -1} = P$

<u>Thm (Kálmán '09)</u> writing $\widehat{\beta}$ for the closure of β,

$$P(\widehat{\beta})[a^{|\beta|-n+1}] = P(\widehat{\beta\pi})[a^{|\beta|+n-1}]$$

<u>Thm (GHMN '19)</u> true with $\mathbb{P}$ in place of P

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<u>Thm (T)</u> if $\beta \in Br_n^+$, then

$$|X_\beta^{\mathcal{U}}(\mathbb{F}_q)| = |X_{\beta\pi}^1(\mathbb{F}_q)|,$$

$$\mathrm{gr}_*^{\mathsf{W}} \mathrm{H}_{c,T}^*(X_\beta^{\mathcal{U}}(\mathbb{C})) \simeq \mathrm{gr}_*^{\mathsf{W}} \mathrm{H}_{c,T}^*(X_{\beta\pi}^1(\mathbb{C}))$$

will deduce from Kálmán and GHMN respectively

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$X_\beta^{\mathcal{U}}, X_\beta^1$ are pieces of a larger variety

recall the Springer resolution

$$\tilde{\mathcal{U}} = \{(u, gB) \in \mathcal{U} \times G/B \mid u \in gBg^{-1}\}$$

pullback squares:

$$\begin{array}{ccccc} \tilde{X}_\beta^{\mathcal{U}} & \to & X_\beta^{\mathcal{U}} & \to & X_\beta \\ \downarrow & & \downarrow & & \downarrow \\ \tilde{\mathcal{U}} & \to & \mathcal{U} & \to & G \end{array}$$

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<u>Thm (T)</u> if $\beta \in Br_n^+$, then $S_n \curvearrowright \mathrm{gr}_*^{\mathsf{W}} \mathrm{H}_{c,T}^*(\tilde{X}_\beta^{\mathcal{U}})$ and

$$\mathbb{P}(\hat{\beta}) \;\propto\; (\mathrm{gr}_*^{\mathsf{W}} \mathrm{H}_{c,T}^*(\tilde{X}_\beta^{\mathcal{U}}(\mathbb{C})))[{\textstyle\bigwedge}^*], \quad \text{where } {\textstyle\bigwedge}^i = {\textstyle\bigwedge}^i(\mathrm{std})$$

Hotta–Springer: if $\mathrm{Jordan}(u) = \lambda$, then $\mathrm{H}^*(\tilde{\mathcal{U}}_u) \simeq \mathrm{Ind}_{S_\lambda}^{S_n}(1)$

<u>Cor (T)</u>

1) $\mathbb{P}(\widehat{\beta})[a^{|\beta|-n+1}] = \mathrm{gr}_*^{\mathsf{W}} \mathrm{H}_{c,T}^*(\tilde{X}_\beta^{\mathcal{U}}(\mathbb{C}))[{\textstyle\bigwedge}^0] = \mathrm{gr}_*^{\mathsf{W}} \mathrm{H}_{c,T}^*(X_\beta^{\mathcal{U}}(\mathbb{C}))$

2) $\mathbb{P}(\widehat{\beta})[a^{|\beta|+n-1}] = \mathrm{gr}_*^{\mathsf{W}} \mathrm{H}_{c,T}^*(\tilde{X}_\beta^{\mathcal{U}}(\mathbb{C}))[{\textstyle\bigwedge}^{n-1}] = \mathrm{gr}_*^{\mathsf{W}} \mathrm{H}_{c,T}^*(X_\beta^1(\mathbb{C}))$

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why?

recall: P arises from Markov traces on Hecke algebras

$$\begin{aligned} \mathsf{D}_{mix,G}^b\mathsf{Perv}(\mathcal{U}) &\simeq \mathsf{D}^b\mathsf{Mod}(\mathbb{C}S_n \ltimes \mathrm{Sym}) && \text{(Rider)} \\ &\simeq \mathsf{hTr}(\mathsf{Hecke}(S_n)) && \text{(Gorsky–Wedrich)} \end{aligned}$$