1

X complex alg curve, $\quad G$ complex reductive alg group

$$\begin{array}{ccccc} & & \text{nonabelian Hodge} & & \\ \mathcal{M}_{G,\mathrm{B}}(X) & \approx & \mathcal{M}_{G,\mathrm{dR}}(X) & \approx & \mathcal{M}_{G,\mathrm{Dol}}(X) \\ & & & & \updownarrow \quad \text{Langlands} \\ & & & & \mathcal{M}_{G^\vee,\mathrm{Dol}}(X) \end{array}$$

HMS: $\quad \mathrm{Coh}_S(\mathcal{M}_{G,\mathrm{B}}) \xrightarrow{?} \mathrm{Fuk}(\mathcal{M}_{G^\vee,\mathrm{Dol}}) \simeq \mathcal{D}(\mathcal{M}_{G^\vee,\mathrm{Dol}})$

2

Ex 1 $\quad G = \mathrm{GL}_n$

$\mathcal{M}_{\mathrm{B}}$ local systems $\rho : \pi_1(X) \to G$

$\mathcal{M}_{\mathrm{dR}}$ flat connections $(E, \nabla : E \to E \otimes \Omega^1(D))$

$\mathcal{M}_{\mathrm{Dol}}$ Higgs bundles $(E, \theta : E \to E \otimes \Omega^1(D))$

X of genus g, $\quad G = \mathrm{GL}_1$

$$\mathcal{M}_{\mathrm{B}} = (\mathbf{C}^\times)^{2g}, \quad \mathcal{M}_{\mathrm{dR}} = \mathcal{M}_{\mathrm{Dol}} = T^*\mathrm{Jac}(X) \approx \mathbf{C}^g \times (S^1)^{2g}$$

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Ex 2 (BBMY) $\quad X = \mathbf{P}^1 - \{0, \infty\}, \quad \gamma \in \mathfrak{g}[z]$ homogeneous

$\mathcal{M}_{\mathrm{B}} = \{$twisted Stokes local systems$\} =$ “braid variety”

$\mathcal{M}_{\mathrm{Dol}} = \{$twisted wild Higgs bundles with tail $\gamma \frac{dz}{z}$ at $\infty\}$

BBMY–Feng–Le Hung: for $\gamma^\vee$ of “integral slope”, a map

$$\mathrm{K}_0(\mathrm{Coh}(\mathcal{M}_{G,\mathrm{B}})) \to \mathrm{K}_0(\mathrm{Fuk}(\mathcal{M}_{G^\vee,\mathrm{Dol}}))$$

$\approx$ Breuil–Mézard $\quad \mathrm{K}_0(\mathrm{Rep}_{\bar{\mathbf{F}}_p}(G(\mathbf{F}_p))) \to \mathrm{Ch}_{\mathrm{mid}}(\mathcal{X}^{\mathrm{EG}})$

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Geometry of BBMY $\mathcal{M}_{G^\vee,\mathrm{Dol}}$:

— $\mathbf{C}^\times$ action contracting to Lagrangian central fiber $\mathcal{F}l_\gamma$

— $\mathcal{F}l_\gamma$ is an "affine Springer fiber for γ"

— $\mathrm{H}^*_{\mathbf{C}^\times}(\mathcal{F}l_\gamma)$ is a $(\widetilde{W}, \widetilde{W})$-bimodule (for integral slope)

BBMY expect mirror symmetry to be biequivariant

Feng–Le Hung use biequivariance to make their analogy precise

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Theme (joint with T. Xue) conjectural bimodule structure of

$$\mathbb{V}_\gamma := \sum_{i,j} (-1)^i \operatorname{gr}_j \mathrm{H}^i_{\mathbf{C}^\times}(\mathcal{F}l_\gamma)^{T_\gamma}|_{\epsilon\to 1}$$

for homogeneous γ of any "slope", where

— T_γ is the centralizer torus $Z_{G(\!(z)\!)}(\gamma)^\circ$

— $\mathrm{H}_{\mathbf{C}^\times}(\text{point}) = \mathbf{C}[\epsilon]$

Punchline answer uses rep theory of $G(\mathbf{F}_q)$

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$\gamma \in \mathfrak{g}[\![z]\!]$ gives a vector field on (fpqc quotient)

$$\mathcal{F}l := G(\!(z)\!)/I, \quad \text{where } I \subseteq G(\!(z)\!) \text{ lifts } B \subseteq G$$

the affine Springer fiber is the fixed-point locus

$$\mathcal{F}l_\gamma := \{gI \in \mathcal{F}l \mid \gamma \in \operatorname{Lie}(gIg^{-1})\}$$

γ is regular semisimple iff $Z_{G(\!(z)\!)}(\gamma)^\circ$ is a max torus of $G(\!(z)\!)$

Kazhdan–Lusztig: if γ is reg ss, then $\mathcal{F}l_\gamma$ is finite-dim'l

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$G = \mathrm{SL}_n$:   underlying ind-scheme is moduli of Higgs bundles:

$$\mathcal{F}l_\gamma \simeq \left\{ (E, \theta, \iota) \;\middle|\; \begin{array}{l} E \in \mathrm{Bun}_n^0(D),\ \theta \in \mathfrak{gl}(E), \\ \iota : (E, \theta)|_{D^\circ} \xrightarrow{\sim} (E^{\mathrm{triv}}, \gamma)|_{D^\circ} \end{array} \right\}$$

where $D = \mathrm{Spec}\,\mathbf{C}[\![z]\!]$ and $D^\circ = D - \{0\}$

gluing gives $\mathcal{F}_\gamma \hookrightarrow \mathcal{M}_{G,\mathrm{Dol}}$ (actually, just homeo onto image)

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$gI \mapsto xgI$ is an isom $\mathcal{F}l_\gamma \xrightarrow{\sim} \mathcal{F}l_{x\gamma x^{-1}}$

in fact, isom class only depends on $\mathrm{charpoly}_\gamma(t) \in \mathbf{C}[\![z]\!][t]$

<u>Ex</u> $\gamma(z) = \left(\begin{smallmatrix} 0 & 1 \\ z^3 & 0 \end{smallmatrix}\right)$ gives $\mathrm{charpoly}_{\gamma(z)}(t) = t^2 - z^3$

$\{t^2 = z^3\}$ is a curve with $\mathbf{C}^\times$-action: $c \cdot (t, z) = (c^3 t, c^2 z)$

γ is a $\mathbf{C}^\times$-eigenvector: $\mathrm{Ad}\left(\left(\begin{smallmatrix} c^{3/2} & 0 \\ 0 & c^{-3/2} \end{smallmatrix}\right)\right) \gamma(c^2 z) = c^3 \gamma(z)$

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for $d/m \in \mathbf{Q}_+$, let $\mathbf{C}^\times \curvearrowright G(\!(z)\!), \mathfrak{g}(\!(z)\!)$ by

$$c \cdot g(z) = \mathrm{Ad}(c^{d\rho^\vee}) g(c^m z)$$

then $\gamma \in \mathfrak{g}[\![z]\!]$ is homogeneous of slope d/m iff it is an eigenvector of weight d/m