Zeta Functions as Knot Invariants

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O. Kivinen, M. Q. Trinh. The Hilb-vs-Quot Conjecture. *J. reine angew. Math. (Crelle)*, (2025). 44 pp.

1 The Riemann Hypothesis

(Euler ~1730s) Proof of the infinitude of primes:

If there were finitely many, then we'd have

$$\prod_{\text{prime } p > 0} \left(1 + \frac{1}{p} + \frac{1}{p^2} + \cdots\right) = \sum_{n=1}^{\infty} \frac{1}{n}.$$

But the series diverges—a contradiction.

He also implicitly studied the *zeta function*

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

For real $s > 1$, we have $\zeta(s) = \prod_{p} \frac{1}{1 - p^{-s}}$.

What if we allow s to be complex?

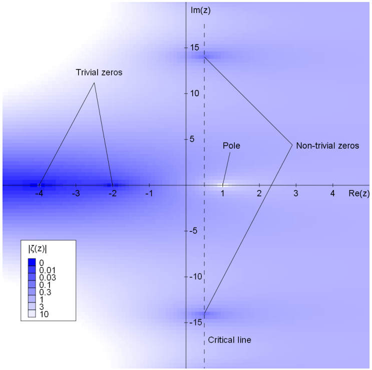
(Riemann 1859) A unique $\mathbf{C}$-valued function ζ that is

- *holomorphic* (complex-differentiable) when $s \neq 1$.
- given by $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^s}$ when $\mathrm{Re}(s) > 1$.

He checked that $\zeta(n) = 0$ for $n = -2, -4, -6, \ldots$ by relating these zeros to poles of the Γ function.

He speculated from examples that all other zeros of ζ live on the *critical line* $\mathrm{Re}(s) = \frac{1}{2}$.

Location of zeros $\leftrightsquigarrow$ distribution of prime numbers.



Wikipedia, "Riemann hypothesis"

(Hardy 1914) Infinitely many zeros on the line.

(Pratt–Robles–Zaharescu–Zeindler 2020) Among zeros s with $0 < \mathrm{Re}(s) < 1$, over five-twelfths on the line.

(Dedekind ~1860s) Generalize the formula

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

by replacing $\mathbf{Z}$ with other *rings* R.

Thus R is a set with operations $+$ and $\cdot$ resembling addition and multiplication.

$$\zeta_R(s) = \sum_{\substack{\text{ideals } I \subseteq R \\ |R/I| < \infty}} \frac{1}{|R/I|^s}$$

An *ideal* I is the collection of all finite linear combos $c_1 \cdot x_1 + \cdots + c_k \cdot x_k$ for some fixed $x_1, x_2, \ldots \in R$.

The *quotient* R/I is the set of translates $y + I \subseteq R$.

Note For ζ_R to make sense, the number of I such that $|R/I| = n$ must be finite for each $n > 0$.

Ex Every ideal of $\mathbf{Z}$ takes the form

$$n\mathbf{Z} = \{\text{multiples of } n\} \quad \text{for some integer } n \geq 0.$$

For instance, the ideal generated by 30 and 2025 is

$$\{c_1 \cdot 30 + c_2 \cdot 2025 \mid c_1, c_2 \in \mathbf{Z}\} = 15\mathbf{Z}.$$

We see that

$$\zeta_{\mathbf{Z}}(s) = \sum_{\substack{n\mathbf{Z} \subseteq \mathbf{Z} \\ n > 0}} \frac{1}{|\mathbf{Z}/n\mathbf{Z}|^s} = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

In other rings, ideals may not simplify so nicely.

Why care?

(Hilbert–Polyá ~1910s) The values e^{it} such that

$$\zeta(\tfrac{1}{2} + it) = 0 \quad \text{and} \quad 0 < \operatorname{Re}(\tfrac{1}{2} + it) < 1$$

behave like the eigenvalues of a random unitary matrix. Maybe this is what forces t to be <u>real</u>?

(Weil ~1940s) There is a class of rings R,

coming from *algebraic geometry* over $\mathbf{F}_p := \mathbf{Z}/p\mathbf{Z}$,

where an analogous fact for ζ_R might be provable.

(Grothendieck–Deligne ~1960s–70s) Yes.

2 Weil's Rosetta Stone

Algebraic geometry studies shapes cut out by polynomial equations.

For simplicity, we'll stick to *(affine) hypersurfaces*

$$V_f = \{\vec{a} = (a_1, \ldots, a_n) \in \bar{\mathbf{F}}_p^n \mid f(\vec{a}) = 0\},$$

cut out by a single polynomial $f \in \mathbf{F}_p[x_1, \ldots, x_n]$.

V_f is *smooth* at a point $\vec{a}$ when $\frac{\partial f}{\partial x_i}(\vec{a}) \neq 0$ for some i. Else, *singular* at $\vec{a}$.

Ex Hypersurfaces in $\bar{\mathbf{F}}_p^2$ are plane curves. Consider

$$f(x, y) = y^2 - x^3 - c \quad \text{for constant } c \in \mathbf{F}_p.$$

For which c is V_f smooth everywhere?

The *ring of polynomial functions* on V_f is

$$R_f := \mathbf{F}_p[x_1, \ldots, x_n]/(f).$$

André Weil, in a letter to his sister Simone, described a dictionary:

$\mathbf{Z}$	R_f	V_f
$n\mathbf{Z}$	ideals	(closed) subvarieties
$p\mathbf{Z}$	maximal ideals	(closed) points

The first and last columns: a Rosetta stone between number theory and algebraic geometry.

Conj (Weil 1948) Assume V_f is smooth everywhere. Then zeros of $\zeta_{R_f}(s)$ have $\mathrm{Re}(s) \in \{\frac{1}{2}, \frac{3}{2}, \ldots, \frac{2n-1}{2}\}$.

Set $\zeta_f(s) := \zeta_{R_f}(s)$ for convenience.

(Grothendieck ~1964) $\zeta_f(s)$ is a rational function in

$$\mathsf{q} := p^{-s}.$$

In fact: polynomials $\phi_0, \phi_1, \ldots, \phi_{2n-2}$ such that

$$\zeta_f(s) = \frac{\phi_1(\mathsf{q}) \cdots \phi_{2n-3}(\mathsf{q})}{\phi_0(\mathsf{q}) \cdot \phi_2(\mathsf{q}) \cdots \phi_{2n-2}(\mathsf{q})}.$$

ϕ_k is the charpoly of a certain operator F on a certain vector space $\mathrm{H}^k(V_f)$.

Reduces Weil's conjecture to a "Hilbert–Polyá" claim:

Conj The eigenvalues of F on $\mathrm{H}^k(V_f)$ all have absolute value* $p^{k/2}$.

* With respect to any embedding into $\mathbf{C}$.

(Deligne 1974) True for all f (assuming V_f smooth).

In fact, Weil conjectured—and Deligne proved—results for all smooth varieties, not just hypersurfaces.

Ex Taking $n = 2$ and $f(x, y) = y^2 - x^3 - c$:

$$\phi_0(t) = 1 - p\mathsf{q}$$

$$\phi_1(t) = 1 - a_p\mathsf{q} + p\mathsf{q}^2 \quad \text{for some integer } a_p,$$

giving $\zeta_f(s) = \dfrac{1 - a_p p^{-s} + p^{1-2s}}{1 - p^{1-s}}$. It turns out:

- $|a_p| \leq 2p^{1/2}$.
- So the two roots of $\phi_1(\mathsf{q})$ satisfy $|\mathsf{q}| = p^{-1/2}$.
- So the zeros of $\zeta_f(s)$ satisfy $\mathrm{Re}(s) = \frac{1}{2}$.

What if V_f has singularities?

Simplest case: V_f has a unique singularity at the origin $(0, \ldots 0)$. It turns out that here,

$$\zeta_f(s) = \zeta_f^\circ(s) \cdot \hat{\zeta}_f(s),$$

where:

- ζ_f° satisfies Weil's Riemann Hypothesis.
- $\hat{\zeta}_f$ is the analogue of ζ_f with the power-series ring

$$\hat{R}_f := \mathbf{F}_p[\![x_1, \ldots, x_n]\!]/(f)$$

in place of R_f.

Does $\hat{\zeta}_f(s) = \displaystyle\sum_{\substack{I \subseteq \hat{R}_f \\ |\hat{R}_f/I| < \infty}} \frac{1}{|\hat{R}_f/I|^s}$ satisfy a RH?

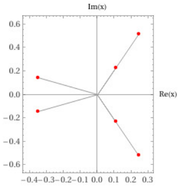
Ex If $f = y^2 - x^3$, then $\hat{\zeta}_f(s) = \dfrac{1 + p\mathsf{q}^2}{1 - \mathsf{q}}$.

Roots in q are $\pm p^{-1/2}$.

Ex If $f = y^3 - x^4$, then

$$\hat{\zeta}_f(s) = \frac{1 + p\mathsf{q}^2 + p^2\mathsf{q}^3 + p^2\mathsf{q}^4 + p^3\mathsf{q}^6}{1 - \mathsf{q}}.$$

Two roots on the circle $|\mathsf{q}| = p^{-1/2}$. The rest <u>not</u> on any circle $|\mathsf{q}| = p^{-k/2}$.



WolframAlpha

3 From Curves to Knots In general, if V_f is a plane curve through the origin, then

$$\hat{\zeta}_f(s) = \frac{P_f(p^{1/2}, p^{-s})}{1 - p^{-s}}$$

for some $P_f(\mathsf{t}, \mathsf{q}) \in \mathbf{Z}\left[\mathsf{t}, \mathsf{q}, \frac{1}{1-\mathsf{q}}\right]$.

The polynomials P_f are remarkably ubiquitous.

(Gorsky–Mazin 2013)

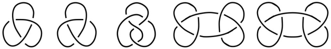
If $f = y^n - x^{n+1}$, then $P_f(1, 1) = \frac{(2n)!}{(n+1)!n!}$, the nth Catalan number.

For instance, if $f = y^3 - x^4$, then

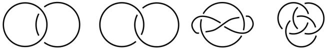
$$P_f(\mathsf{t}, \mathsf{q}) = 1 + \mathsf{t}^2\mathsf{q}^2 + \mathsf{t}^4\mathsf{q}^3 + \mathsf{t}^4\mathsf{q}^4 + \mathsf{t}^6\mathsf{q}^6,$$
$$P_f(1, 1) = 5.$$

The P_f also arise from *knot/link invariants*.

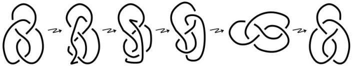
A *knot* is a (tame) embedding of S^1 into $\mathbf{R}^3$ or S^3.



A *link* is similar, but can have multiple circles.



Two links are *isotopic* when they fit into a continuous family of embeddings.



Chmutov–Duzhin–Mostovoy

A *complex* plane curve $V_f \subseteq \mathbf{C}^2$ through $(0,0)$ defines a link

$L_f := V_f \cap S^3$, where S^3 is a 3-sphere around $(0,0)$.

Ex If $f = y^n - x^m$, then L_f is the (m,n) *torus link*. It's a knot when m and n are coprime.

Wolfram, "Explore Torus Knots"

Ex If $f = y^4 - 2x^3y^2 - 4x^5y + x^6 - x^7$, then L_f is the closure of the braid

Cherednik–Danilenko

Conj (Oblomkov–Shende ~2010), Thm (Maulik 2012)

$$P_f(-1, \mathsf{q}) = \lim_{\mathsf{a}\to 0} \left[(\mathsf{aq}^{-1})^{\mu}\, \mathbb{P}_{L_f}(\mathsf{a}, \mathsf{q})\right],$$

where $\mu \in \mathbf{Z}$ and $\mathbb{P}$ is the *HOMFLYPT invariant*, discovered in 1986 and defined by the rules:

(1) $$\mathsf{a}\mathbb{P} - \mathsf{a}^{-1}\mathbb{P} = (\mathsf{q}^{1/2} - \mathsf{q}^{-1/2})\mathbb{P}$$

(2) $$\mathbb{P}_{\bigcirc} = 1$$

Conj (Oblomkov–Rasmussen–Shende ~2013)

$$P_f(\mathsf{t}, \mathsf{q}) = \lim_{\mathsf{a}\to 0} \left[(\mathsf{aq}^{-1})^{\mu}\, \mathbf{P}_{L_f}(\mathsf{a}, \mathsf{t}, \mathsf{q})\right],$$

where $\mathbf{P}$ is a refinement of $\mathbb{P}$, discovered in the mid-2000s by Khovanov–Rozansky.

- The full conjecture incorporates a by refining P_f.
- $\mathbf{P}$ is defined by *categorifying* (1)–(2). Polynomials become graded chain complexes.
- The ORS conjecture is surprising because $\mathbf{P}$ is defined *diagrammatically*, while P_f is *geometric*.

(ORS ~2013) True for $f = y^2 - x^m$ with m odd.

Here, $P_f = 1 + \mathsf{t}^2\mathsf{q}^2 + \cdots + \mathsf{t}^{m-1}\mathsf{q}^{m-1}$.

(Kivinen–T 2025) True for $f = y^3 - x^m$ with $3 \nmid m$.

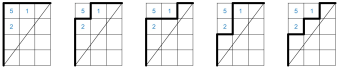
Here, P_f is more complicated.

The full results incorporate a. Our result gives a new closed formula for $\mathbf{P}_{\mathrm{torus}(m,3)}$.

Proof of Kivinen–T (2025)

1 Combinatorial recursions for $\mathbf{P}_{\mathrm{torus}(m,n)}$ due to Mellit and Hogancamp–Mellit.

2 When m and n are coprime, get a formula summing over *$m \times n$ Dyck paths*:



At the same time, $\hat{R}_f \simeq \mathbf{F}_p[\![u^m, u^n]\!]$.

We relate the Dyck paths to the combinatorics of $\hat{R}_f$*-submodules* $M \subseteq \mathbf{F}_p[\![u]\!]$.

3 We relate $\displaystyle\sum_M \frac{1}{|\mathbf{F}[\![u]\!]/M|^s}$ to $\displaystyle\sum_I \frac{1}{|\hat{R}_f/I|^s}$.

Uses *Serre duality*. For now, requires $\min(m,n) \leq 3$.

Big Picture

I'm interested in special functions that appear in

- *algebraic geometry* (*e.g.*, zeta functions)
- *knot theory* (*e.g.*, HOMFLYPT polynomials)
- *combinatorics* (*e.g.*, Dyck-path statistics)

A modern name for the study of such special functions is *representation theory*.

T (2021) If L comes from a positive n-strand braid, then $\mathbf{P}_L(\mathsf{a},\mathsf{t},\mathsf{q})$ can be recovered from a *representation* of S_n on the cohomology of an explicit variety $\mathcal{Z}_L$.

So if $L = L_f$, then $\mathbf{P}_{L_f}$ relates to both V_f and $\mathcal{Z}_{L_f}$.

Any direct relationship between these varieties?

4 Cherednik's New Hypothesis

Recall: For prime p and $f = y^3 - x^4$, the roots of

$$P_f(p^{1/2}, \mathsf{q}) = 1 + p\mathsf{q}^2 + p^2\mathsf{q}^3 + p^2\mathsf{q}^4 + p^3\mathsf{q}^6$$

do not all satisfy $|\mathsf{q}| = p^{-1/2}$.

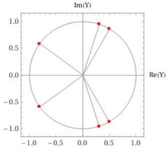
Conj (Cherednik 2018) For any $f(x, y)$,

there's some open interval $\mathfrak{I} \subseteq \mathbf{R}_{>0}$ such that

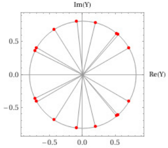
$$\alpha \in \mathfrak{I} \implies \text{all zeros of } P_f(\alpha, \mathsf{q}) \text{ satisfy } |\mathsf{q}| = \alpha^{-1/2}.$$

Cherednik + ORS predicts mysterious arithmetic constraints on the link invariant $\mathbf{P}$.

$f = y^3 - x^4, \quad \alpha = 1$:



$f = y^4 - 2x^3y^2 - 4x^5y + x^6 - x^7, \quad \alpha = 1.51$:



Thank you for listening.