7.

Today we focus on the G^F-equivariant endomorphisms of $\mathbf{R}_{w,1} = \mathrm{H}_c^*(X_w)$, especially $\mathbf{R}_{e,1}$.

7.1.

We have discussed a formula of Deligne–Lusztig that expresses the values of the virtual characters $R_{w,\theta}$ in terms of those where $\theta = 1$, after replacing G^F with (the identity component of) the centralizer of some semisimple element. This motivates us to focus on the virtual characters $R_{w,1}$.

Following Lusztig, an irreducible character of G^F is *unipotent* if and only if it occurs with nonzero multiplicity in $R_{w,1}$ for some $w \in W$. Let

$$\mathrm{Uch}(G^F) \subseteq \mathrm{Irr}(G^F)$$

be the set of unipotent irreducible characters of G^F. Determining $\mathrm{Uch}(G^F)$ for all G, F was possibly the hardest part of Lusztig's work on the representations of finite groups of Lie type. It seems that the complete details appeared for the first time in his 1984 "orange book", 8 years after the Deligne–Lusztig paper. The main theorem is:

Theorem 7.1 (Lusztig)**.** *We can index* $\mathrm{Uch}(G^F)$ *by a set that that depends only on W and the automorphism of W determined by F: not on G or even q. Moreover, if G is of type A, meaning $W = S_n$, then* $\mathrm{Uch}(G^F)$ *consists solely of summands of* $\mathbf{R}_{e,1}$*: there are no* cuspidal *unipotent irreducibles.*

The first statement should be surprising because, as we already saw with $\mathrm{SL}_2(\mathbf{F}_q)$, the Deligne–Lusztig varieties X_w are highly sensitive to q.

The second statement already fails to generalize to $G^F = \mathrm{Sp}_4(\mathbf{F}_q)$. The irreducible characters of this group were classified by Srinivasan by hand in 1968, well before Deligne–Lusztig's work. She found an irreducible that, in her paper, was labeled θ_{10}, and which, in Lusztig's language, is the unique cuspidal unipotent irreducible character of $\mathrm{Sp}_4(\mathbf{F}_q)$.[1] The degree of this character is $\frac{1}{2}q(q-1)^2$.

Suppose that F acts trivially on W. Here Lusztig found that for $\chi \in \mathrm{Irr}(W)$, the virtual characters

$$R_\chi := \frac{1}{|W|} \sum_{w \in W} \chi(w) R_{w,1}$$

come close to being irreducible characters of G^F. To make this assertion more precise, we need to discuss the irreducible summands of $\mathbf{R}_{e,1}$ in more detail.

[1] Srinivasan's label is still used—see Wikipedia—even though it conflicts with our use of θ for characters of maximal tori.

7.2.

Fix an F-stable Borel pair (B, T), so that $\mathbf{R}_{e,1} = \mathrm{Ind}_{B^F}^{G^F}(1)$. Note that the right-hand side is independent of étale cohomology, so we can generalize its coefficients to any ring K. Throughout what follows, we assume that K is of characteristic zero. We define the *Iwahori–Hecke algebra*, or just *Hecke algebra*, of (G^F, B^F) over K to be

$$KH = KH_{T^F}^{G^F}(1) := \mathrm{End}_{KG^F}(\mathrm{Ind}_{KB^F}^{KG^F}(1)).$$

Let us review a result discussed earlier: that H admits a K-linear basis indexed by W^F.

For any $w \in W$, let $O_w \subseteq G/B \times G/B$ be the G-orbit of pairs (yB, xB) such that $By^{-1}xB = BwB$. We say that such a pair is in *relative position w* and write $yB \xrightarrow{w} xB$. If w is fixed by F, then O_w is F-stable. Let $h_w \in KH$ be the *Hecke operator*

$$h_w(1_{xB^F}) = \sum_{\substack{yB^F \in G^F/B^F \\ yB \xrightarrow{w} xB}} 1_{yB^F}.$$

Then we have a sequence of bijections

$$\begin{array}{ccccc} W^F & \xrightarrow{\sim} & B^F \backslash G^F / B^F & \xrightarrow{\sim} & G^F \backslash (G^F/B^F \times G^F/B^F), \\ w & \mapsto & B^F w B^F & \mapsto & O_w^F, \end{array}$$

and an isomorphism of K-vector spaces

$$\begin{aligned} K[G^F \backslash (G^F/B^F \times G^F/B^F)] &\xrightarrow{\sim} KH, \\ 1_{O_w^F} &\mapsto h_w. \end{aligned}$$

The latter is an isomorphism of K-algebras, being the restriction of the map

$$K[G^F/B^F \times G^F/B^F] \xrightarrow{\sim} \mathrm{End}_K(\mathrm{Ind}_{KB^F}^{KG^F}(1))$$

that takes the convolution $(f_1 * f_2)(\bar{y}, \bar{x}) = \sum_{\bar{z}} f_1(\bar{y}, \bar{z}) f_2(\bar{z}, \bar{x})$ on the left-hand side to the composition of endomorphisms on the right-hand side.

7.3.

To go further, we relate the multiplication in KH to the structure of W^F. First, we consider W. Below, let $U = [B, B]$, so that $B = T \ltimes U$.

Recall that every parabolic subgroup of G contains some Borel. Consider the parabolics that contain B with codimension one. Given such a parabolic P, we see that P/B must be a proper, rational smooth curve, hence isomorphic to $\mathbf{P}^1$, the flag variety of SL_2. One can show that $P = L_P \ltimes U_P$, where

$U_P \subseteq U$ is the unipotent radical of P, and the *Levi quotient* L_P is a reductive group for which $[L_P, L_P] \simeq \mathrm{SL}_2$. Then T forms a maximal torus of L_P, and $N_{L_P}(T)/T = N_P(T)/T \subseteq W$ is isomorphic to the Weyl group of SL_2, hence generated by an element s. Conversely, s and B together determine P, since we can check that $P = B \cup BsB$, just like in the Bruhat decomposition for SL_2. In particular, there are only finitely many possibilities for P.

Recall that W acts on the mutually dual *character* and *cocharacter lattices*:

$$\mathrm{X}(T) = \mathrm{Hom}(T, \mathbf{G}_m) \quad \text{and} \quad \mathrm{X}^\vee(T) = \mathrm{Hom}(\mathbf{G}_m, T).$$

As it turns out, the elements s above act on $\mathrm{X}(T)$ and $\mathrm{X}^\vee(T)$ by reflections, and together generate W. Explicitly, since $[L_P, L_P] \simeq \mathrm{SL}_2$, the inclusion $T \subseteq L_P$ defines two opposed root vectors in $\mathrm{X}(T)$, *resp.* coroot vectors in $\mathrm{X}^\vee(T)$. Then s acts on $\mathrm{X}^\vee(T)$, *resp.* $\mathrm{X}(T)$, by reflection across the dual hyperplane. As we run over all P, the roots, *resp.* coroots, that arise in this way are called the *simple roots*, *resp.* simple coroots, of G in $\mathrm{X}(T)$. This is the start of a rich formalism in Lie theory—of root systems, coroot systems, and root data—which we will only sketch here. For the roots of $G = \mathrm{Sp}_4$, see #2 on Problem Set 0.

We can tensor up the lattice $\mathrm{X}^\vee(T)$ from $\mathbf{Z}$ to any field of characteristic zero, such as $\mathbf{R}$, to obtain a representation of W over that field. Altogether, W is a reflection group on a real vector space, and it preserves a lattice of full rank within this vector space, meaning it is *crystallographic*. The s are called its *simple reflections*.

The subgroup $W^F \subseteq W$ remains a crystallographic real reflection group, since it acts on the sublattice of F-invariants $\mathrm{X}^\vee(T)^F$. The dual to this sublattice is the quotient lattice of F-coinvariants $\mathrm{X}(T)_F := \mathrm{X}(T)/\langle F - 1\rangle$. It turns out that the set of simple roots in $\mathrm{X}(T)$ is F-stable, and that its image in $\mathrm{X}(T)_F$ classifies its F-orbits. In this way, one shows that W^F is generated by reflections indexed by F-orbits of simple roots.

7.4.

Let $S \subseteq W$ be the set of simple reflections, which is tautologically indexed by the simple roots. Coxeter showed that W admits a presentation of the form

$$W = \langle S \mid (st)^{m_{s,t}} = e \text{ for all } s, t \in S\rangle,$$

where the $m_{s,t}$ are a collection of positive integers such that $m_{s,s} = 1$ and $m_{s,t} = m_{t,s}$ for all s, t. Any group W with a presentation of this form, where S is finite, is called a *Coxeter group*, and the presentation is called its *Coxeter presentation*. The pair (W, S) is also called a *Coxeter system*.

Example 7.2. In the zeroth lecture, we gave a Coxeter presentation for the symmetric group S_n. Namely, S is the set of simple transpositions $s_i = (i, i + 1)$

for $1 \leq i \leq n-1$; we have $m_{s_i,s_j} = 2$ for $|i-j| > 1$, whereas $m_{s_i,s_j} = 3$ for $|i-j| = 1$.

Example 7.3. For $m \geq 3$, the dihedral group of the regular m-gon has a Coxeter presentation: namely,

$$\langle s, t \mid s^2 = t^2 = (st)^m = e \rangle.$$

Tits observed that for all $w \in W$ and $s \in S$,

$$\text{either} \quad BwBsB = BwsB \quad \text{or} \quad BwBsB = BwsB \cup BwB,$$

and similarly with the order of w, s switched everywhere. In the first case, we set $ws > w$; in the second case, we set $ws < w$. These relations generate a partial order on W, which turns out to be a more explicit description of the closure order on Bruhat cells $BwB \subseteq G$, or equivalently, orbits $O_w \subseteq G/B \times G/B$.

In particular, the relations above show how to describe the multiplication on the Hecke algebra, up to structure constants. We will need the refinement to Bruhat decomposition, given in Theorem 21.80 in Milne, which states:

Theorem 7.4 (Bruhat)**.** *For all $w \in W$, the maps*

$$\begin{aligned} U/(wUw^{-1} \cap U) \to UwB/B &\xrightarrow{\text{id}} BwB/B \\ xwUw^{-1} &\mapsto xwB \end{aligned}$$

are isomorphisms of varieties for all $w \in W$. (Note that wUw^{-1} is independent of how we lift w to $N_G(T)$, because T normalizes U.)

The *Bruhat length* of w, denoted $\ell(w)$, is the common dimension of the varieties above. If w is fixed by F, then wUw^{-1} is F-stable like U, and since $wUw^{-1} \cap U$ is connected, we deduce by Lang that

$$|U^F/(wUw^{-1} \cap U)^F| = |(U/(wUw^{-1} \cap U))^F| = q^{\ell(w)}$$

Let $S_F \subseteq W^F$ be the generating set indexed by the F-orbits of the simple roots, so that (W^F, S_F) is also a Coxeter system. Via the convolution description of the Hecke algebra, Iwahori found:

Theorem 7.5 (Iwahori)**.** *For all $w \in W^F$ and $s \in S_F$, we have the* quadratic relations

$$h_w h_s = \begin{cases} h_{ws} & ws > w, \\ q^{\ell(s)} h_{ws} + (q^{\ell(s)} - 1) h_w & ws < w. \end{cases} \tag{7.1}$$

Writing $(m_{s,t})_{s,t}$ for the integers in the Coxeter presentation of W^F, we also have the braid relations

$$\overbrace{h_s h_t \cdots}^{m_{s,t}\ \text{terms}} = \overbrace{h_t h_s \cdots}^{m_{s,t}\ \text{terms}} \tag{7.2}$$

for all $s, t \in S_F$. Together, (7.1) *and* (7.2) *generate all relations in the Hecke algebra KH.*

Example 7.6. The simplest example: If $G = \mathrm{SL}_2$ and F is standard, so that $W = \{e, s\}$, then the Iwahori–Hecke relation

$$h_s^2 = qh_e + (q-1)h_s$$

amounts to this: The space of G^F-equivariant functions on $(\mathbf{P}^1)^F \times (\mathbf{P}^1)^F$ is spanned by $1_{O_e^F}$, the indicator function on the diagonal, and $1_{O_s^F}$, the indicator function on the complement; and

$$\begin{aligned}\sum_{z \in (\mathbf{P}^1)^F} 1_{O_s^F}(y,z) 1_{O_s^F}(z,x) &= |\{z \in (\mathbf{P}^1)^F \mid z \neq x, y\}| \\ &= \begin{cases} q & x = y, \\ q-1 & x \neq y \end{cases} \\ &= q \cdot 1_{O_e^F}(y,x) + (q-1) \cdot 1_{O_s^F}(y,x).\end{aligned}$$

7.5.

Crucially, the role of q in Iwahori's theorem is generic. We are led to introduce a version of the algebra where q is replaced by a genuine formal variable.

In what follows, (W, S) is abstract: It could be the Coxeter system (W, S) attached to G, B, the smaller Coxeter system (W^F, S_F) attached to G^F, B^F, or even something infinite or non-crystallographic. We define the *generic Iwahori–Hecke algebra* of (W, S) in one parameter to be the $\mathbf{Z}[\mathsf{x}]$-algebra

$$H_W = H_W(\mathsf{x}) := \frac{\mathbf{Z}[\mathsf{x}][\sigma_w \mid w \in W]}{I_W},$$

where I_W is the two-sided ideal generated by

$$\left.\begin{array}{rr} \sigma_w \sigma_s - \sigma_{ws} & ws > w, \\ \sigma_w \sigma_s - h_{ws} - (\mathsf{x}^{\ell(s)} - \mathsf{x}^{-\ell(s)}) h_w & ws < w \end{array}\right\} \quad \text{for all } w \in W \text{ and } s \in S,$$

$$\overbrace{\sigma_s \sigma_t \cdots}^{m_{s,t}\ \text{terms}} = \overbrace{\sigma_t \sigma_s \cdots}^{m_{s,t}\ \text{terms}} \quad \text{for all } s, t \in S.$$

Here we can make sense of the $\mathsf{x} \to 1$ limit:

$$H_W(\vec{x}) / \langle \mathsf{x} - 1 \rangle \simeq \mathbf{Z}W. \tag{7.3}$$

At the same time:

$$H_{W^F}(\mathsf{x})/\langle \mathsf{x} - q^{1/2}\rangle \simeq \mathbf{Z}[q^{\pm 1/2}]H_{B^F}^{G^F}(1)$$

via the map that sends $\sigma_w \mapsto q^{-\ell(w)/2}h_w$ for all w. The reason we prefer this renormalization that introduces square roots is that for general W, the following theorem only works after x, not just x^2, is adjoined.

Theorem 7.7 (Benson–Curtis–Lusztig)**.** *Suppose that W is finite. Then the ring isomorphism* (7.3) *lifts to a $\mathbf{Q}(\mathsf{x})$-algebra isomorphism*

$$\mathbf{Q}(\mathsf{x}) \otimes H_W \simeq \mathbf{Q}(\mathsf{x})W.$$

Remark 7.8. Outside of the Weyl groups of the almost-simple algebraic groups of types E_7 and E_8, one can indeed make the isomorphism work over the smaller scalar field $\mathbf{Q}(\mathsf{x}^2)$.

Corollary 7.9. *If W is finite, then the isomorphism classes of simple modules over $\mathbf{Q}(\mathsf{x}) \otimes H_W$ are in bijection with the irreducible characters of W.*

7.6.

Now we return to G^F. Suppose that K is a field containing $\mathbf{Z}[q^{\pm 1/2}]$. Assume for now the following facts:

(1) $KG^F = \mathrm{End}_{KH}(\mathrm{Ind}_{KB^F}^{KG^F}(1))$.
(2) If $\chi \in \mathrm{Irr}(W)$, then the corresponding simple $(\mathbf{Q}(\mathsf{x}) \otimes H_W)$-module takes the form $\mathbf{Q}(\mathsf{x}) \otimes_{\mathbf{Z}[\mathsf{x}^{\pm 1}]} E_\chi$, where E_χ is a simple H_W-module. Let $KE_{\chi,q} = K \otimes_{\mathbf{Z}[q^{\pm 1/2}]} E_\chi|_{\mathsf{x} \to q^{1/2}}$.

Then the double centralizer theorem yields

$$\mathrm{Ind}_{KB^F}^{KG^F}(1) \simeq \bigoplus_{\chi \in \mathrm{Irr}(W^F)} V_\chi \otimes_K KE_\chi$$

as a (KG^F, KH)-bimodule, where each V_χ is either zero or simple over KG^F. It turns out that V_χ is *always* nonzero. So these are precisely the irreducible unipotent principal series characters of G^F. Let ρ_χ be the character of V_χ.

Theorem 7.10 (Lusztig)**.** *Suppose that F acts trivially on W. If $W = S_n$ for some n, then we have $R_\chi = \rho_\chi$ for all $\chi \in$* Irr(W)*. Else, there is a block-diagonal matrix with small blocks that takes the vector of virtual characters R_χ, possibly extended by some zeroes, to the vector of all irreducible unipotent characters of G^F, including cuspidals.*

When F acts nontrivially on W, Lusztig proved a similar, but more complicated statement. See Secion 7.3 in Carter's survey paper.