MATH 430: INTRODUCTION TO TOPOLOGY
PROBLEM SET #6

SPRING 2025

Due Wednesday, March 26. You may consult books, papers, and websites as long as you cite all sources and write up your solutions in your own words.

Problem 1. Let X, Y be (topological) manifolds, and let $f : X \to Y$ be continuous.

(1) Show that the graph $\{(x, f(x)) \mid x \in X\}$ is homeomorphic to X, as a subspace of $X \times Y$.
(2) Deduce from (1) that $S := \{(x, \sin(\frac{1}{x})) \mid x > 0\}$, as a subspace of analytic $\mathbf{R}^2$, forms a manifold. (Be slightly careful: S is not the graph of a function on the x-axis, but only of a function on its positive part.)
(3) Show that the closure of S in $\mathbf{R}^2$, the topologist's sine curve, is not a manifold.

Problem 2. Let $A = (-1, 1)$, and let

$$a * b = \frac{a + b}{1 + ab}$$

for all $a, b \in \mathbf{R}$, assuming the right-hand side is well-defined.

(1) Show that if $a, b \in A$, then $a * b \in A$.
(2) Show that A forms a group under $*$. What is the identity element?

Problem 3. Let $(G, \circ)$ be the group of self-homeomorphisms of $[0, 1]$, where $\circ$ denotes composition of self-maps. Show that this group is not commutative: *i.e.*, that $f \circ g \neq g \circ f$ for some $f, g \in G$. *Hint:* It suffices to work with piecewise-linear homeomorphisms.

Problem 4. For any spaces X, Y and embeddings $f, g : X \to Y$, we define an *isotopy* in Y from f to g to be a continuous map $\phi : X \times [0, 1] \to Y$ such that

$$\phi(-, 0) = f \text{ and } \phi(-, 1) = g \text{ and } \phi(-, t) : X \to Y \text{ is an embedding for all } t.$$

Note how the last condition distinguishes isotopies from homotopies. If such a map ϕ exists, then we say that f and g are *isotopic* in Y. Show that:

(1) Isotopy in Y defines an equivalence relation on the embeddings of X into Y.
(2) The self-homeomorphisms of $[0, 1]$ defined by $f(t) = t$ and $g(t) = 1 - t$ are not isotopic. *Hint:* Intermediate value theorem for $\phi(0, t) - \phi(1, t)$.

Problem 5 (Munkres 330, #2)**.** For any spaces X, Y, let $[X, Y]$ be the set of homotopy classes of maps of X into Y. For clarity, let $I = [0, 1]$. Show that:

(1) If X is nonempty, then $[X, I]$ is a singleton.
(2) If Y is nonempty and path-connected, then $[I, Y]$ is a singleton.

Problem 6 (Munkres 330, #3)**.** Keep the notation of Problem 5. We say that a nonempty space X is *contractible* if and only if its identity map is nulhomotopic: *i.e.*, homotopic to some constant-valued map. Show that:

(1) I and $\mathbf{R}$ are contractible.
(2) Any contractible (nonempty) space is path-connected.
(3) If X, Y are nonempty and Y is contractible, then $[X, Y]$ is a singleton.
(4) If X, Y are nonempty, X is contractible, and Y is path-connected, then $[X, Y]$ is a singleton.

Problem 7 (Munkres 335, #4)**.** Let $A \subseteq X$ be a *retract*, meaning there is a continuous map $r : X \to A$ such that $r(a) = a$ for all $\in A$, also known as a *retraction*. Show that

$$r_* : \pi_1(X, a_0) \to \pi_1(A, a_0)$$

is surjective for any $a_0 \in A$.

Problem 8 (Munkres 335 #5)**.** Let A be a subspace of $\mathbf{R}^n$ for some $n \geq 0$, and let $h : A \to Y$ be continuous. Show that if h is the restriction of a continuous map from $\mathbf{R}^n$ into Y, then for any $a_0 \in A$, the homomorphism

$$h_* : \pi_1(A, a_0) \to \pi_1(Y, y_0) \quad (\text{where } y_0 = f(a_0))$$

is *trivial*: It sends every element to the identity element in the target.