

## Warmup

recall:

$R^\omega$  = all seq's  $(a_1, a_2, \dots)$

$R^\infty$  = seq's eventually zero

[why do we use  $R^\infty$  to refer to the latter?]

for all  $n$ , an injective map

$R^n$  to  $R^\omega$  (extend by zero's past  $n$ )

what is the union of their images?  $R^\infty$

moreover:

there is a metric  $\rho : R^\infty \times R^\infty$  to  $[0, \infty)$  def by

$$\rho(x, y) = \max_{\{i = 1, 2, \dots\}} |x_i - y_i|$$

that restricts to the square metric on  $R^n$  for all  $n$

$\rho$  is not well-defined on  $R^\omega$

Df the uniform metric on  $R^\omega$  is defined by  
 $u(x, y) = \sup_i \min\{1, |x_i - y_i|\}$

[is that the same as  $\min\{1, \sup_i |x_i - y_i|\}$ ? no]

is this a metric?

1)  $u(x, y) = 0$  implies  $x = y$ ? [yes]

2)  $u(x, y) = u(y, x)$ ? [yes]

3)  $u(x, y) + u(y, z) \geq u(x, z)$ ?

$$\begin{aligned} & \sup_i \min\{1, |x_i - y_i|\} + \sup_i \min\{1, |y_i - z_i|\} \\ & \geq \sup_i (\min\{1, |x_i - y_i|\} + \min\{1, |y_i - z_i|\}) \\ & \geq \sup_i \min\{1, |x_i - z_i|\} \end{aligned}$$

[thus far, topologies gen'd by metrics]  
[now, topologies gen'd by collections of subsets]

(Munkres §13, 19)

$X$  any set,  
 $\{B_i\}_i$  any collection of subsets of  $X$

Df         $\{B_i\}_i$  is a subbasis for a topology on  $X$   
iff  
 $X = \bigcup_i B_i$

$\{B_i\}_i$  is a basis for a topology on  $X$   
iff  
 $X = \bigcup_i B_i$  and for all  $i, j$ ,  
each  $x$  in  $B_i \cap B_j$  is in a  $B_k$

a subbasis  $\{C_j\}_j$  gives rise to a basis  $\{B_i\}_i$ :

$\{B_i\}_i = \{\text{finite intersections of the } C_j\text{'s, including } \emptyset\}$

a basis  $\{B_i\}_i$  gives rise to a topology  $T$ :

$T = \{\text{arbitrary unions of the } B_i\text{'s, incl. } \emptyset\}$   
 $= \{U \subseteq X \mid \text{for all } x \text{ in } U, \text{ have } x \text{ in } B_i \subseteq U \text{ for some } i\}$   
[why are the two defn's of  $T$  equivalent?]

Rem        different bases can induce  
the same topology

[just like different metrics inducing the same top]

Ex bases in  $\mathbb{R}^n$ :

$\{\text{balls } B(x, \delta) \mid x \in \mathbb{R}^n \text{ and } \delta > 0\}$

but also,  $\{(a_1, b_1) \times \dots \times (a_n, b_n) \mid a_i, b_i\}$

Lem suppose  $T$  is a topology on  $X$ ,  
 $\{C_i\}_i$  any subcollection of  $T$   
s.t. for all  $U$  in  $T$ , and  $x$  in  $U$ ,  
there is some  $i$  s.t.  $x \in C_i \subseteq U$

then  $\{C_i\}_i$  is a basis, and it gives rise to  $T$

Pf  $X \in T$ , so  $X = \bigcup_i C_i$   
 $C_i$ 's in  $T$ , so  $C_i \cap C_j \in T$ ,  
so each  $x \in C_i \cap C_j$  is in a  $C_k$   
immediate by hypothesis that  $\{C_i\}_i$  induces  $T$

Warning a subbasis is not a special kind of basis  
[prefix “sub-” is misleading]  
a subbasis generates a basis,  
and forms a subset of that basis

Warning bases for topologies  $T$   
have nothing to do with  
bases for vector spaces  $V$

if we fix a basis for  $T$ , then an open set in  $T$  can  
be a union of basis open sets in many ways

if we fix a basis for  $V$ , then a vector in  $V$  can be a  
linear combo of basis vectors in only one way

[return to  $R^\omega$ :]

Df the box topology on  $R^\omega$  is gen'd by  
the basis of “boxes”

$(a_1, b_1) \times (a_2, b_2) \times \dots$  [and so on forever]

[is this really a basis?]

Df the product topology on  $R^\omega$  is gen'd by  
the subbasis of sets

$C_{\{i, a, b\}} = \{x = (x_1, x_2, \dots) \mid a < x_i < b\}$   
as we run over  $i > 0$  and  $a < b$

Q how do these topologies compare?

the basis generated by the product subbasis:

$\{\text{finite intersections of the } C_{\{i, a, b\}}\}$

these intersections look like

$B_{\{J, \mathbf{a}, \mathbf{b}\}} = \{x = (x_1, x_2, \dots) \mid a_i < x_i < b_i$   
for  $i$  in  $J\}$

as we run over

finite sets  $J \text{ sub } \{1, 2, \dots\}$

$\mathbf{a} = (a_i)_{\{i \text{ in } J\}}, \mathbf{b} = (b_i)_{\{i \text{ in } J\}}$

conclude:

each product basis open set is a union of  
some box basis open sets

Lem      suppose  $S$  is gen'd by a basis  $\{B_i\}_i$ ,  
                  $T$  is gen'd by a basis  $\{C_j\}_j$ ,  
                 and each  $B_i$  is a union of  $C_j$ 's

then  $T$  is finer than  $S$

Pf          exercise

Cor          the box topology is finer than  
                 the product topology [but not the same?]

why care? [surprisingly, the product topology is  
"better behaved"] we will discuss later:

the product topology is the coarsest topology  
that makes  $\text{pr}_i: \mathbb{R}^\omega \rightarrow \mathbb{R}$  continuous for all  $i$

what about the uniform topology gen'd by

$$u(x, y) = \sup_i \min\{1, |x_i - y_i|\} ?$$

Q1          what's a basis for the uniform topology?

Q2          how does it compare to box, product?

A1          the collection of balls  $B_u(x, \delta)$

suffices to use balls where  $\delta \leq 1$  [why?] for such,  
 $B_u(x, \delta) = \{y \mid \sup_i |x_i - y_i| < \delta\}$  [why?]  
 $\neq \{y \mid |x_i - y_i| < \delta\}$  [why??]

A2          compare bases:  
box supset uniform supset product  
[do any of them coincide?]