9.

Trying to generalize Haiman's conjecture beyond type A.

9.1.

First, we review Haiman's conjecture.

9.1. Fix $n \geq 1$. For any partitions $\lambda, \mu \vdash n$, let $\tilde{K}_{\lambda,\mu}(x) \in \mathbf{Z}[x]$ be the modified Kostka polynomial for (λ, μ). Let $k_{\lambda,\mu} = \tilde{K}_{\lambda,\mu}(1)$. The matrix $k := (k_{\lambda,\mu})_{\lambda,\mu}$ is nonnegative and unitriangular. For $n = 2, 3, 4, 5$, it looks like this, with rows indexed by λ:

$$\begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 \\ & 1 & 2 \\ & & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ & 1 & 1 & 2 & 3 \\ & & 1 & 1 & 2 \\ & & & 1 & 3 \\ & & & & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ & 1 & 1 & 2 & 2 & 3 & 4 \\ & & 1 & 1 & 2 & 3 & 5 \\ & & & 1 & 1 & 3 & 6 \\ & & & & 1 & 2 & 5 \\ & & & & & 1 & 4 \\ & & & & & & 1 \end{pmatrix}.$$

Let $l = (l_{\mu,\lambda})_{\mu,\lambda}$ denote the inverse matrix. For $n = 2, 3, 4, 5$, it looks like this:

$$\begin{pmatrix} 1 & -1 \\ & 1 \end{pmatrix}, \begin{pmatrix} 1 & -1 & 1 \\ & 1 & -2 \\ & & 1 \end{pmatrix}, \begin{pmatrix} 1 & -1 & 0 & 1 & -1 \\ & 1 & -1 & -1 & 2 \\ & & 1 & -1 & 1 \\ & & & 1 & -3 \\ & & & & 1 \end{pmatrix}, \begin{pmatrix} 1 & -1 & 0 & 1 & 0 & -1 & 1 \\ & 1 & -1 & -1 & 1 & 1 & -2 \\ & & 1 & -1 & -1 & 2 & -2 \\ & & & 1 & -1 & -1 & 3 \\ & & & & 1 & -2 & 3 \\ & & & & & 1 & -4 \\ & & & & & & 1 \end{pmatrix}.$$

9.2. Let $H_n = H_n(v)$ be the Hecke algebra of S_n. For all $w \in S_n$, let $c_w \in H_n$ be the associated Kazhdan–Lusztig element (denoted C'_w in their original paper). For all $\lambda \vdash n$, let $\chi^\lambda : S_n \to \mathbf{Z}$ be the irreducible character indexed by λ, and let $\chi_v^\lambda : H_n \to \mathbf{Z}[v^{\pm 1}]$ be the trace of the simple H_n-module corresponding to χ^λ.

Conjecture 9.1 (Haiman, 1993)**.** *For any $w \in S_n$ and $\mu \vdash n$, the v-polynomial*

$$\xi_{\mu,w} := \sum_{\lambda \vdash n} l_{\mu,\lambda}\, \chi_v^\lambda(c_w)$$

has nonnegative coefficients.

Let $\xi = (\xi_{\mu,w})_{\mu,w}$. Haiman's conjecture is saying that the nonnegativity of the character values $\chi_v^\lambda(c_w)$ arises from the simultaneous nonnegativity of k and ξ.

Example 9.2. Take $n = 4$. Let $z = v + v^{-1}$ for convenience. Here, Haiman's conjecture amounts to the nonnegativity of

$$\begin{pmatrix} 1 & -1 & 0 & 1 & -1 \\ & 1 & -1 & -1 & 2 \\ & & 1 & -1 & 1 \\ & & & 1 & -3 \\ & & & & 1 \end{pmatrix} \begin{pmatrix} 1 & z & z^2 & z^2 & z^3 & z^3 - z & z^4 & z^4 - z^2 & z^5 - 2z^3 + z & z^5 - z^3 & z^6 - 3z^4 + 2z^2 \\ 3 & 2z & z^2 + 1 & z^2 & 2z & z^3 - z & 2z^2 & z^2 & z & z^3 & \\ 2 & z & 1 & z^2 & z & 1 & z^2 & & & & \\ 3 & z & 1 & & & & & & & & \\ 1 & & & & & & & & & & \end{pmatrix}$$

$$= \begin{pmatrix} & & & & z^3 - 2z & & z^4 - 2z^2 & z^4 - 2z^2 & z^5 - 2z^3 & z^5 - 2z^3 & z^6 - 3z^4 + 2z^2 \\ & & z^2 - 1 & & z & z^3 - z - 1 & z^2 & z^2 & z & z^3 & \\ & & & z^2 & z & 1 & z^2 & & & & \\ & z & 1 & & & & & & & & \\ 1 & & & & & & & & & & \end{pmatrix}.$$

Above, the columns correspond to the following ordered subset of W:

$$e,\ s_1,\ s_1s_2,\ s_1s_3,\ s_1s_2s_3,\ s_1s_2s_1,\ s_2s_1s_3s_2,\ s_2s_3s_2s_1,\ s_2s_1s_3s_2s_1,\ s_1s_2s_3s_2s_1,\ w_\circ,$$

where s_i is the transposition of i and $i + 1$.

9.2.

We now generalize S_n to the Weyl group W of a complex semisimple algebraic group G.

9.3. Let $H_W = H_W(v)$ be the Hecke algebra of W. As before, we write $(c_w)_w$ for its Kazhdan–Lusztig basis. Let $\mathrm{Irr}(W)$ be the set of irreducible characters of W, and for each $\chi \in \mathrm{Irr}(W)$, let $\chi_v : H_W \to \mathbf{Z}[v^{\pm 1}]$ be the trace of the corresponding simple H_W-module.

In generalizing Haiman's conjecture to W, we immediately face the problem that $\chi_v(c_w) \in \mathbf{Z}[v^{\pm 1}]$ can now have negative coefficients.

Example 9.3. Take W of type BC_2. It has a Coxeter presentation with generators s, t and relations $s^2 = t^2 = stst = e$. Meanwhile, it has four characters of degree 1: the trivial character, the sign character, and two others that we can label χ_1 and χ_2. We can check using GAP that $\chi_{1,v}(c_{sts}) = \chi_{2,v}(c_{tst}) = -v - v^{-1}$ in a certain labeling.

9.4. If we look at Haiman's Lemma 1.1, asserting the nonnegativity of $\chi_v(c_w)$ when $W = S_n$, we see that the key idea is to realize χ_v as the trace of a left cell module for H_n. So in place of irreducible characters, we might try to use characters arising from cell modules.

For any left cell $C \subseteq W$, let $\psi_v^C : H_W \to \mathbf{Z}[v^{\pm 1}]$ be the trace of the corresponding module. We now face the issue that, even in type A, different left cells C may yield the same function ψ_v^C. Namely, if $W = S_n$ and $\lambda \vdash n$, then it turns out that there are $\chi^\lambda(1)$ many choices of left cell $C \subseteq S_n$ such that $\psi_v^C = \chi_v^\lambda$.

In general type, the functions ψ_v^C are nonnegative integral combinations of irreducible characters of H_W that can overlap with each other. Namely, each left cell is contained in some two-sided cell, and to each two-sided cell $\mathfrak{C}$, Lusztig assigns a unique *special* character $\chi^{\mathfrak{C}} \in \mathrm{Irr}(W)$ such that for any left cell $C \subseteq \mathfrak{C}$, the decomposition of ψ_v^C into irreducibles must contain $\chi_v^{\mathfrak{C}}$ with multiplicity one. Lusztig showed that the map $\mathfrak{C} \mapsto \chi^{\mathfrak{C}}$ from the set of two-sided cells into $\mathrm{Irr}(W)$ is injective. Let $\mathrm{Irr}(W)^\dagger \subseteq \mathrm{Irr}(W)$ be its image: the subset of special characters. If $W = S_n$, then

- $\mathrm{Irr}(S_n)^\dagger = \mathrm{Irr}(S_n)$,
- there are $\chi^{\mathfrak{C}}(1)$ many left cells inside $\mathfrak{C}$, and
- $\psi_v^C = \chi_v^{\mathfrak{C}}$ for any such left cell C.

These facts are closely related to the Robinson–Schensted correspondence.

In a 2018 paper, Lusztig shows a particular case of the following statement:

Conjecture 9.4. *$\chi_v^{\mathfrak{C}}(c_w)$ has nonnegative v-coefficients for any two-sided cell $\mathfrak{C}$ and $w \in W$.*

If the conjecture holds, then it suggests that in place of the cell characters ψ_v^C, we might use the special characters $\chi_v^{\mathfrak{C}}$. Note that if $W = S_n$, then every irreducible character of W takes the form $\chi^{\mathfrak{C}}$ for some unique $\mathfrak{C} \subseteq S_n$.

9.5. Next, we need to generalize the modified Kostka polynomials.

In general, let $\mathfrak{g}$ be the Lie algebra of G. When G is of type A_{n-1}, we can interpret the modified Kostka polynomials for $W = S_n$ in terms of the geometry of G:

$$\tilde{K}_{\lambda,\mu}(x) = \sum_i x^i (\chi^\lambda, \mathrm{H}^{2i}(\mathscr{B}_\mu))_{S_n},$$

where $\mathscr{B}_\mu$ is the Springer fiber above any element of the nilpotent orbit $\mathscr{O}_\mu \subseteq \mathfrak{g}$ indexed by μ, and $\mathrm{H}^*(\mathscr{B}_\mu)$ denotes its Betti cohomology.

Beyond type A, there are more nilpotent orbits of $\mathfrak{g}$ than two-sided cells of W. However, the Springer correspondence provides an injective map from two-sided cells to nilpotent orbits.

Recall that Springer assigns to each $\chi \in \mathrm{Irr}(W)$ a pair $(\mathscr{O}(\chi), \rho(\chi))$, where $\mathscr{O}(\chi) \subseteq \mathfrak{g}$ is a nilpotent orbit and $\rho(\chi)$ is an irreducible character of the finite component group

$$A_u := Z_G(u)/Z_G(u)^\circ \quad \text{for a fixed } u \in \mathscr{O}(\chi).$$

Let Σ_G be the collection of pairs $(\mathscr{O}, \rho)$ such that $\mathscr{O} \subseteq \mathfrak{g}$ is a nilpotent orbit and $\rho \in \mathrm{Irr}(A_u)$ for some $u \in \mathscr{O}$. Explicitly, $(\mathscr{O}(\chi), \rho(\chi))$ is uniquely determined by the property that the multiplicity space $\mathrm{H}^{\mathrm{top}}(\mathscr{B}_u)[\rho]$, where $V[\rho] := \mathrm{Hom}_{A_u}(\rho, V)$, is a nonzero sum of copies of χ as a representation of W. In particular, the map

$$\mathrm{Irr}(W) \to \Sigma_G$$

is injective. In type A, the groups A_u are trivial, so Σ_G reduces to the set of nilpotent orbits; the Springer correspondence sends $\chi^\lambda \mapsto \mathscr{O}_\lambda$.

In general, let $\Sigma_G^\dagger \subseteq \Sigma_G$ be the image of $\mathrm{Irr}(W)^\dagger$ under Springer's map, and let $\Sigma_G^1 \subseteq \Sigma_G$ be the subset of pairs of the form $(\mathscr{O}, 1_{A_u})$. It turns out that

$$\Sigma_G^\dagger \subseteq \Sigma_G^1.$$

We thus obtain an injective map

$$\begin{aligned} \{\text{two-sided cells of } W\} &\to \{\text{nilpotent orbits of } \mathfrak{g}\}, \\ \mathfrak{C} &\mapsto \mathscr{O}(\chi^{\mathfrak{C}}). \end{aligned}$$

The elements of the image are called *special* nilpotent orbits.

For any class function ψ on W and nilpotent orbit $\mathscr{O} \subseteq \mathfrak{g}$, let

$$k_{\psi,\mathscr{O}} = (\psi, \mathrm{H}^*(\mathscr{B}_u))_W = \sum_i (\psi, \mathrm{H}^{2i}(\mathscr{B}_u))_W \quad \text{for a fixed } u \in \mathscr{O}.$$

For any two-sided cell $\mathfrak{C} \subseteq W$, let

$$k_{\mathfrak{C},\mathscr{O}} = k_{\chi^{\mathfrak{C}},\mathscr{O}}.$$

Finally, let $k = (k_{\mathfrak{C},\mathscr{O}})_{\mathfrak{C},\mathscr{O}}$ be the matrix where $\mathfrak{C}$ runs over all two-sided cells, whereas $\mathscr{O}$ runs over special nilpotent orbits. This is a matrix of nonnegative integers, and in type A, it recovers the matrix of modified Kostka numbers from earlier.

9.6. Let $a : \mathrm{Irr}(W) \to \mathbf{Z}_{\geq 0}$ be Lusztig's a-function. It turns out that:

- a defines a total order on the subset of special characters $\mathrm{Irr}(W)^\dagger$.
- For any $\chi^{\mathfrak{C}} \in \mathrm{Irr}(W)^\dagger$ and $u \in \mathscr{O}(\chi^{\mathfrak{C}})$, the multiplicity of $\chi^{\mathfrak{C}}$ in $\mathrm{H}^{\mathrm{top}}(\mathscr{B}_u)[1_{A_u}]$, hence in all of $\mathrm{H}^{\mathrm{top}}(\mathscr{B}_u)$, is exactly one, and zero in $\mathrm{H}^i(\mathscr{B}_u)$ for all $i \neq \mathrm{top}$.

In this sense, the matrix k can be made unitriangular. In particular, it is invertible. Let $l = (l_{\mathscr{O},\mathfrak{C}})_{\mathscr{O},\mathfrak{C}}$ denote the inverse matrix.

Conjecture 9.5. *For any $w \in W$ and special nilpotent orbit $\mathscr{O} \subseteq \mathfrak{g}$, the v-polynomial*

$$\xi_{\mathscr{O},w} := \sum_{\mathfrak{C}} l_{\mathscr{O},\mathfrak{C}} \chi_v^{\mathfrak{C}}(c_w)$$

has nonnegative coefficients.

Parallel to our earlier discussion, let $\xi = (\xi_{\mathscr{O},w})_{\mathscr{O},w}$. The new conjecture is saying that the nonnegativity of the special character values $\chi_v^C(c_w)$ arises from the simultaneous nonnegativity of k and ξ.

9.3.

9.7. We consider G of type C_n. Again, let $z = v + v^{-1}$ for convenience.

When $\chi \in \mathrm{Irr}(W)$, work of D. Kim gives a closed formula for $k_{\chi,\mathscr{O}}$ in terms of Kostka polynomials at $x = -1$. Namely, if $\mathscr{O}$ is an orbit of Jordan type $\mu \vdash 2n$ and $\chi \in \mathrm{Irr}(W)$ is indexed by a partition $\lambda \vdash 2n$ with empty 2-core, then

$$k_{\chi,\mathscr{O}} = \varepsilon_\lambda \tilde{K}_{\lambda,\mu}(-1), \quad \text{where } \varepsilon_\lambda = (-1)^{\sum_i \lambda_{2i}}.$$

(Above, we have fixed a sign typo.)

9.8. Take $n = 2$. We compute $(\chi_v(c_w))_{\chi,w}$ to be

$$\begin{array}{c} (2,-) \\ (1^2,-) \\ (1,1) \\ (-,2) \\ (-,1^2) \end{array} \begin{pmatrix} 1 & z & z & z^2 & z^2 & z^3-z & z^3-z & z^4-2z^2 \\ 1 & & z & & & & -z & \\ 2 & z & z & 2 & 2 & z & z & \\ 1 & z & & & & -z & & \\ 1 & & & & & & & \end{pmatrix},$$

where the left-hand labels are GAP's labels. (We have changed the order of the characters from GAP's original order.) Here is the GAP code:

```
v:=X(Rationals); v.name:="v";
W:=CoxeterGroup("B", 2); H:=Hecke(W, [v^2, v^2]); b:=Basis(H, "C'");
for w in Elements(W) do
    AppendTo("b2.txt", b(w), "\n");
    vals:=HeckeCharValues(b(w));
    for i in [1..Length(vals)] do
        AppendTo("b2.txt", i, "\t", vals[i], "\n"); od;
    AppendTo("b2.txt", "\n"); od; quit;
```

Labeling the characters from top to bottom by $\chi_1, \ldots, \chi_5$, the cell characters ψ^C are

$$\overbrace{\chi_1,}\quad \overbrace{\chi_2+\chi_3, \quad \chi_3+\chi_4,}\quad \overbrace{\chi_5,}$$

each occurring once, the braces indicating the two-sided cells. Thus the special characters are χ_1, χ_3, χ_5. We arrive at

$$(\chi_v^{\mathfrak{C}}(c_w))_{\mathfrak{C},w} = \begin{pmatrix} 1 & z & z & z^2 & z^2 & z^3-z & z^3-z & z^4-2z^2 \\ 2 & z & z & 2 & 2 & z & z & \\ 1 & & & & & & & \end{pmatrix},$$

Meanwhile, to compute k, we take the modified Kostka matrix

$$(\tilde{K}_{\lambda,\mu}(x))_{\lambda,\mu} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ & x & x & x+x^2 & x+x^2+x^3 \\ & & x^2 & x^2 & x^2+x^4 \\ & & & x^3 & x^3+x^4+x^5 \\ & & & & x^6 \end{pmatrix},$$

and specialize it to

$$(\tilde{K}_{\lambda,\mu}(-1))_{\lambda,\mu} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ & -1 & -1 & 0 & -1 \\ & & 1 & 1 & 2 \\ & & & -1 & -1 \\ & & & & 1 \end{pmatrix}.$$

The partitions indexing nilpotent orbits are $\mu = (4), (2^2), (2, 1^2), (1^4)$; of these, the ones indexing special orbits are $(4), (2^2), (1^4)$. Under the core-quotient bijection, this ordering of special orbits matches our ordering of special characters: χ_1, χ_3, χ_5. We arrive at

$$k = \begin{pmatrix} 1 & 1 & 1 \\ & 1 & 2 \\ & & 1 \end{pmatrix},$$

whence

$$l = \begin{pmatrix} 1 & -1 & 1 \\ & 1 & -2 \\ & & 1 \end{pmatrix}.$$

Therefore,

$$
\begin{aligned}
l \cdot (\chi_v^{\mathfrak{C}}(c_w))_{\mathfrak{C},w} &= \begin{pmatrix} 1 & -1 & 1 \\ & 1 & -2 \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & z & z & z^2 & z^2 & z^3 - z & z^3 - z & z^4 - 2z^2 \\ 2 & z & z & 2 & 2 & z & z & \\ 1 & & & & & & & \end{pmatrix} \\
&= \begin{pmatrix} & & & z^2 - 2 & z^2 - 2 & z^3 - 2z & z^3 - 2z & z^4 - 2z^2 \\ & z & z & 2 & 2 & & & \\ 1 & & & & & & & \end{pmatrix}.
\end{aligned}
$$

9.9. Take $n = 3$. For simplicity, we will restrict w to the set of minimal-length representatives for the conjugacy classes of W, along with three further elements with nontrivial Kazhdan–Lusztig polynomials. Explicitly, in the Coxeter presentation of W with generators t, s_1, s_2 and relations $t^2 = s_1^2 = s_2^2 = ts_1ts_1 = s_1s_2s_1 = ts_2 = e$, our ordered list of choices for w is

$$e,\ t,\ s_1,\ ts_1,\ ts_2,\ s_1s_2,\ ts_1s_2,\ ts_1ts_1,\ \underline{ts_1s_2t},\ ts_1ts_1s_2,\ \underline{ts_1ts_2s_1},\ \underline{ts_1ts_2s_1t},\ w_\circ,$$

where the elements with nontrivial Kazhdan–Lusztig polynomials are underlined, and the longest element $w_\circ$ is explicitly $ts_1ts_1s_2s_1ts_1s_2$. Using GAP, we compute $(\chi_v(c_w))_{\chi,w}$ to be

$$
\begin{array}{c}
(1^3, -) \\ (1^2, 1) \\ (1, 1^2) \\ (-, 1^3) \\ (21, -) \\ (1, 2) \\ (2, 1) \\ (-, 21) \\ (3, -) \\ (-, 3)
\end{array}
\begin{pmatrix}
1 & z & & & & & & & & & & & \\
3 & 2z & z & 2 & z^2 & 1 & z & & z^2 & & z & z^2 & \\
3 & z & z & 2 & & 1 & & & & & & & \\
1 & & & & & & & & & & & & \\
2 & 2z & z & z^2 & z^2 & 1 & z & z^4 - 2z^2 & z^2 & z^3 - 2z & z^3 - z & z^2 & \\
3 & z & 2z & 2 & z^2 & z^2 + 1 & 2z & & 2z^2 & & 2z & z^2 & \\
3 & 2z & 2z & z^2 + 2 & z^2 & z^2 + 1 & 3z & z^4 - 2z^2 & 3z^2 & z^3 & z^3 + 3z & 3z^2 & \\
2 & & z & & & 1 & & & & & & & \\
1 & z & z & z^2 & z^2 & z^2 & z^3 & z^4 - 2z^2 & z^4 & z^5 - 2z^3 & z^5 - z^3 & z^6 - 3z^4 + 3z^2 & z^9 - 6z^7 + 11z^5 - 6z^3 \\
1 & & z & & & z^2 & & & & & & &
\end{pmatrix}.
$$

(Compare to Table 10.3 in Geck–Pfeiffer, which gives the table of character values on the standard basis of H_W.) Write $\phi_{\lambda',\lambda''}$ for the character with label (λ', λ''). Again using GAP, we compute the characters ψ^C to be

$$
\begin{gathered}
\overbrace{\phi_{3,-}},\quad \overbrace{\phi_{1^2,1} \times 3},\quad \overbrace{\phi_{1,2} \times 3},\quad \overbrace{\phi_{-,1^3}}, \\
\overbrace{\phi_{21,-} + \phi_{2,1}, \quad (\phi_{2,1} + \phi_{-,3}) \times 2}, \\
\overbrace{\phi_{1^3,-} + \phi_{1,1^2}, \quad (\phi_{1,1^2} + \phi_{-,21}) \times 2},
\end{gathered}
$$

where the braces indicate two-sided cells, as before, and the symbols $\times\,(-)$ indicate multiplicities. Here is the GAP code:

```
W:=CoxeterGroup("B", 3); cells:=LeftCells(W);
for i in [1..Length(cells)] do
    Print(Character(cells[i])); od;
```

We relabel the characters $\chi_1, \ldots, \chi_{10}$, as follows:

χ_1	χ_2	χ_3	χ_4	χ_5	χ_6	χ_7	χ_8	χ_9	χ_{10}
$\phi_{3,-}$	$\phi_{21,-}$	$\phi_{1^3,-}$	$\phi_{2,1}$	$\phi_{1^2,1}$	$\phi_{1,2}$	$\phi_{1,1^2}$	$\phi_{-,3}$	$\phi_{-,21}$	$\phi_{-,1^3}$

With the characters $\chi_1, \dots, \chi_{10}$ listed from top to bottom, the preceding matrix becomes

$$\begin{pmatrix}
1 & z & z & z^2 & z^2 & z^2 & z^3 & z^4-2z^2 & z^4 & z^5-2z^3 & z^5-z^3 & z^6-3z^4+3z^2 & z^9-6z^7+11z^5-6z^3 \\
2 & 2z & z & z^2 & z^2 & 1 & z & z^4-2z^2 & z^2 & z^3-2z & z^3-z & z^2 & \\
1 & z & & & & & & & & & & & \\
3 & 2z & 2z & z^2+2 & z^2 & z^2+1 & 3z & z^4-2z^2 & 3z^2 & z^3 & z^3+3z & 3z^2 & \\
3 & 2z & z & 2 & z^2 & 1 & z & & z^2 & & z & z^2 & \\
3 & z & 2z & 2 & z^2 & z^2+1 & 2z & & 2z^2 & & 2z & z^2 & \\
3 & z & z & 2 & & 1 & & & & & & & \\
1 & & z & & & z^2 & & & & & & & \\
2 & & z & & & 1 & & & & & & & \\
1 & & & & & & & & & & & &
\end{pmatrix},$$

The characters ψ^C are now

$$\overbrace{\chi_1}, \quad \overbrace{\chi_5}, \quad \overbrace{\chi_6}, \quad \overbrace{\chi_{10}}, \quad \overbrace{\chi_2+\chi_4, \quad \chi_4+\chi_8}, \quad \overbrace{\chi_3+\chi_7, \quad \chi_7+\phi_9}.$$

Thus the special characters are $\chi_1, \chi_4, \chi_5, \chi_6, \chi_7, \chi_{10}$. We arrive at:

$$\begin{array}{c}
(3,-) \\ (2,1) \\ (1^2,1) \\ (1,2) \\ (1,1^2) \\ (-,3)
\end{array}
\begin{pmatrix}
1 & z & z & z^2 & z^2 & z^2 & z^3 & z^4-2z^2 & z^4 & z^5-2z^3 & z^5-z^3 & z^6-3z^4+3z^2 & z^9-6z^7+11z^5-6z^3 \\
3 & 2z & 2z & z^2+2 & z^2 & z^2+1 & 3z & z^4-2z^2 & 3z^2 & z^3 & z^3+3z & 3z^2 & \\
3 & 2z & z & 2 & z^2 & 1 & z & & z^2 & & z & z^2 & \\
3 & z & 2z & 2 & z^2 & z^2+1 & 2z & & 2z^2 & & 2z & z^2 & \\
3 & z & z & 2 & & 1 & & & & & & & \\
1 & & & & & & & & & & & &
\end{pmatrix}.$$

Meanwhile, we take the modified Kostka matrix $(\tilde{K}_{\lambda,\mu}(x))_{\lambda,\mu}$ given by

$$\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
 & x & x & x+x^2 & x & x+x^2 & x+x^2+x^3 & x+x^2 & x+x^2+x^3 & x+x^2+x^3+x^4 & x+x^2+x^3+x^4+x^5 \\
 & & x^2 & x^2 & x^2 & x^2+x^3 & x^2+x^3+x^4 & x^2+x^3+x^4 & x^2+x^3+2x^4 & x^2+x^3+2x^4+x^5+x^6 & x^2+x^3+2x^4+x^5+2x^6+x^7+x^8 \\
 & & & x^3 & & x^3 & x^3+x^4+x^5 & x^3 & x^3+x^4+x^5 & x^3+x^4+2x^5+x^6+x^7 & x^3+x^4+2x^5+2x^6+2x^7+x^8+x^9 \\
 & & & & x^3 & x^3 & x^3 & x^3 & x^3+x^5 & x^3+x^5+x^6 & x^3+x^5+x^6+x^7+x^9 \\
 & & & & & x^4 & x^4+x^5 & x^4+x^5 & x^4+2x^5+x^6 & x^4+2x^5+2x^6+2x^7+x^8 & x^4+2x^5+2x^6+3x^7+3x^8+2x^9+2x^{10}+x^{11} \\
 & & & & & & x^6 & & x^6 & x^6+x^7+x^8+x^9 & x^6+x^7+2x^8+2x^9+2x^{10}+x^{11}+x^{12} \\
 & & & & & & & x^6 & x^6 & x^6+x^8 & x^6+x^8+x^9+x^{10}+x^{12} \\
 & & & & & & & & x^7 & x^7+x^8+x^9 & x^7+x^8+2x^9+x^{10}+2x^{11}+x^{12}+x^{13} \\
 & & & & & & & & & x^{10} & x^{10}+x^{11}+x^{12}+x^{13}+x^{14} \\
 & & & & & & & & & & x^{15}
\end{pmatrix},$$

and specialize it to

$$(\tilde{K}_{\lambda,\mu}(-1))_{\lambda,\mu} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ & -1 & -1 & & -1 & & -1 & & -1 & & -1 \\ & & 1 & 1 & 1 & & 1 & 1 & 2 & 2 & 3 \\ & & & -1 & & -1 & -1 & -1 & -1 & -2 & -2 \\ & & & & -1 & -1 & -1 & -1 & -2 & -1 & -3 \\ & & & & & 1 & & & & & \\ & & & & & & 1 & & 1 & & 2 \\ & & & & & & & 1 & 1 & 2 & 3 \\ & & & & & & & & -1 & -1 & -3 \\ & & & & & & & & & 1 & 1 \\ & & & & & & & & & & -1 \end{pmatrix}.$$

The partitions corresponding to orbits are

$$\mu = (6), (4,2), (4,1^2), (3^2), (2^3), (2^2,1^2), (2^2,1^4), (1^6).$$

Of these, the ones corresponding to special orbits are[2]

$$(6), (4,2), (3^2), (2^3), (2^2,1^2), (1^6).$$

We arrive at

$$k = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ & 1 & 1 & 1 & 2 & 3 \\ & & 1 & 1 & 2 & 3 \\ & & & 1 & 1 & 3 \\ & & & & 1 & 3 \\ & & & & & 1 \end{pmatrix},$$

whence

$$l = \begin{pmatrix} 1 & -1 & & & 1 & -1 \\ & 1 & -1 & & & \\ & & 1 & -1 & -1 & 3 \\ & & & 1 & -1 & \\ & & & & 1 & -3 \\ & & & & & 1 \end{pmatrix}.$$

Note that this time, under the core-quotient bijection, our ordering of special orbits corresponds to the ordering of special characters $\chi_1, \chi_4, \chi_6, \chi_5, \chi_7, \chi_{10}$. After we swap

[2]See https://ymsc.tsinghua.edu.cn/__local/B/D1/16/18ABEC4147AEAF436087215CF67_3710E259_1DD26F.pdf.

the middle two rows in our matrix $(\chi_v^{\mathfrak{C}}(c_w))_{\mathfrak{C},w}$, we compute $l \cdot (\chi_v^{\mathfrak{C}}(c_w))_{\mathfrak{C},w}$ to be

$$
\begin{pmatrix}
1 & -1 & & & 1 & -1 \\
 & 1 & -1 & & & \\
 & & 1 & -1 & -1 & 3 \\
 & & & 1 & -1 & \\
 & & & & 1 & -3 \\
 & & & & & 1
\end{pmatrix}
\cdot
\begin{pmatrix}
1 & z & z & z^2 & z^2 & z^2 & z^3 & z^4-2z^2 & z^4 & z^5-2z^3 & z^5-z^3 & z^6-3z^4+3z^2 & z^9-6z^7+11z^5-6z^3 \\
3 & 2z & 2z & z^2+2 & z^2 & z^2+1 & 3z & z^4-2z^2 & 3z^2 & z^3 & z^3+3z & 3z^2 & \\
3 & z & 2z & 2 & z^2 & z^2+1 & 2z & & 2z^2 & & 2z & z^2 & \\
3 & 2z & z & 2 & z^2 & 1 & z & & z^2 & & z & z^2 & \\
3 & z & z & 2 & & 1 & & & & & & & \\
1 & & & & & & & & & & & &
\end{pmatrix}
$$

$$
= \begin{pmatrix}
 & & & & & & z^3-3z & & z^4-3z^2 & z^5-3z^3 & z^5-2z^3+3z & z^6-3z^4 & z^9-6z^7+11z^5-6z^3 \\
 & z & & z^2 & & & z & z^4-2z^2 & z^2 & z^3 & z^3+z & 2z^2 & \\
 & \boxed{-2z} & & \boxed{-2} & & z^2-1 & z & & z^2 & & z & & \\
 & z & & & z^2 & & z & & z^2 & & z & z^2 & \\
 & z & z & 2 & & 1 & & & & & & & \\
1 & & & & & & & & & & & &
\end{pmatrix}
$$

The boxes indicate the failure of Conjecture 9.4. With some more GAP work, we find the full list of elements w that fail on the two-sided cell indexed by (3^2) is

$$t, \quad ts_1 \sim s_1t, \quad s_1ts_1, \quad s_1ts_1s_2 \sim s_2s_1ts_1, \quad s_2s_1ts_1s_2.$$