Bull. London Math. Soc. 48 (2016) 1–11
doi:10.1112/blms/bdv065

Space and spaces

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The idea of space is central in the way we think. We organize our perceptions in physical space, we think of time strung out as a line, and we carry spatial notions over to any number of our conceptual constructs. Nevertheless, in some sense space is 'our' technology, wonderfully evolved for dealing with our experience of the physical world. It is probably only an approximation to reality.

Long ago, in an account of quantum field theory written for the 'general reader' by Freeman Dyson, one of my favourite writers about science, I came upon a passage where he said that there are well-understood mathematical reasons why when we quantize a wave-like physical system such as the electromagnetic field the result is best described in terms of particles, but that, unfortunately, the reasons cannot be explained in non-mathematical terms. I felt rather let down, for I don't like the idea that mathematics is an arcane mystery in which even the basic ideas can only be explained to initiates. At the same time, I wondered which well-understood mathematical reasons Dyson had in mind. I have thought about the question a lot since then, but haven't come up with anything likely to be of much use to non-mathematicians, and I leave it as a challenge. It seems to me a challenge even to give a clear mathematical account. I suppose that Dyson was thinking of the traditional statement that a free field can be regarded as a system of independent harmonic oscillators — I remember how mystifying that seemed when I first encountered it as an undergraduate. My belief now is that the essential point involves thinking carefully about how we use the concept of space. That is the theme of this talk.

Among our tools for making sense of the world the idea of a one-dimensional continuum — not only in space or time, but also of colour, sound, taste, or smell, and even of pleasure, pain, beauty, or moral excellence — is perhaps even more basic than that of space. The greatest triumph of mathematics is surely the invention of the real numbers — a model of the one-dimensional continuum constructed from the idea of counting — and the slow elucidation of the paradoxes the modelling involves. The real numbers encode perfectly all our intuitions of how a continuum should be.[†] But my subject today is the concept of a 'topological space', which captures a more general intuitive idea. It too seems to me a triumph of mathematics, if not quite on the level of the real numbers. Primarily, a space is a *set* with a notion of *proximity*, which tells us when a point of the space is moving continuously as a function of time, and tells us when a real-valued function on the space is continuous. A feature of the structure is that different regions in the space can in some sense be studied independently. The formal definition of a topological space does not involve the real numbers, though, as you see, my account of the intuition involves them fundamentally.

Received 29 July 2015; published online 22 October 2015.

Presidential Address to the London Mathematical Society delivered at its Annual General Meeting in London on 15 November 2013.

[†]Once when I was young and innocent I tried to explain to an eminent philosopher in my Oxford college why mathematicians are no longer puzzled about how Achilles can overtake a tortoise, and the resulting explosion has left me wary of transgressing cultural boundaries.

But spaces also have a non-local aspect. The crudest way to classify topological spaces is by their homotopy-type. If we start with topological spaces and do not distinguish between maps which are homotopic — which can be deformed continuously to each other — then we get the *homotopy category*. In this category, homotopy-equivalent spaces become isomorphic, and coffee cups are doughnuts. Effectively, the homotopy category is very much smaller than the category of topological spaces. It is the receptacle for the *non-local* properties of spaces: it records how they are connected-up globally — what we still need to know when we know everything about the local structure near every point of the space.

We have strong intuitions about homotopy-types, at least in low dimensions. No-one ever doubted that a real polynomial equation of odd degree has a real root, long before there was any framework to prove the intermediate-value theorem. The same immediate conviction attaches to the two-dimensional analogue of the intermediate-value theorem, which gives us the winding-number proof that a polynomial equation with complex coefficients has a complex root — the so-called 'fundamental theorem of algebra'†.

The non-local nature of the homotopy category gives it a life of its own. Though in principle its objects are sets, we often come upon objects in the homotopy category for which there is no specially singled-out topological space which represents them. The London tube map is obviously well-defined up to homotopy, but what, exactly, is the underlying set? There are many more extreme examples: in mathematical logic and computer science it can be useful to think of a 'proof' as a 'path', defined up to homotopy, from one proposition to another. Evidently these paths are not maps from an interval of the real numbers into a definite topological space.

One area where the independent life of the homotopy category is important is algebraic geometry. An algebraic variety over the complex numbers is a space, but not just any space: when a space is defined by algebraic equations its homotopy-type is very strongly constrained, and has an elaborate additional structure. For the moment I'd just like to point out that the homotopy-type of an algebraic variety plays a mysterious role in unexpected contexts. I first became interested in mathematics through physics, and I was incurious about number-theoretical questions like Fermat's last theorem. I felt more interested when the theorem was restated to me as the fact that the real plane curve with equation $x^n + y^n = 1$ somehow gets from $(1, 0)$ to $(0, 1)$ without going through any points with rational coordinates. But I was only truly gripped when I heard of Mordell's conjecture, proved by Faltings in 1983, which gives a criterion for *any* plane curve defined by a polynomial equation with rational coefficients to have at most *finitely* many rational points. The criterion is in terms of the homotopy-type of the *surface* formed by the *complex* points of the curve: if the surface has genus greater than 1, that is, has more than one 'handle', then there are at most finitely many rational points. What can the global topology of the complex variety — or even of the subspace of points with real coordinates — have to do with the rational points? An unromantic answer is that a property of the algebraic equations happens to control both; but the topological type of the surface leaps out at one, while the genus is far from being a salient feature of the *algebraic* description of the curve.

More magical still is the case of algebraic varieties defined over a finite field by equations with integer coefficients. According to the Weil conjectures, proved by Grothendieck and Deligne, the number of points of such a variety is again related to the homotopy-type of the associated *complex* variety. The essential idea of the Weil conjectures is that an algebraic variety over a finite field, though it is not in any ordinary sense a topological space, nevertheless has a

†Our intuition is of course patchy and unreliable. In proving the fundamental theorem of algebra we imagine a polynomial of degree n as taking a large disc of cloth and laying it down on a circular plate so that the edge of the cloth runs n times around the boundary of the plate. We rightly feel certain that the cloth must cover the plate completely, but it is less immediately evident that performing the act with ordinary cloth in ordinary space is impossible.

homotopy-type which is defined by its algebraic structure. Grothendieck's construction of this homotopy-type began from the observation that to know the homotopy-type of an ordinary space one does not need to know the space as a point-set: all one needs to know is the way its contractible open subsets are fitted together combinatorially. There is a simple construction, making essential use of the real numbers, which associates a space[†] $|\mathcal{C}|$, and hence a homotopy-type, to any *category* $\mathcal{C}$. The set $\mathcal{U}_X$ of contractible open subsets of a space X is the set of objects of a category, with just the inclusions as morphisms. Surprisingly, the homotopy-type of $|\mathcal{U}_X|$ is precisely the homotopy-type of the space X. What really makes the homotopy-type of a category significant is that it depends on the category only up to equivalence of categories: the category of 'all' finite sets and bijections has the same homotopy type as the countable subcategory whose objects are just the sequence of particular sets $\{1, 2, \ldots, n\}$ for $n \geqslant 0$, and whose morphisms are the symmetric groups. Grothendieck defined a category of 'generalized open subsets' for an arbitrary algebraic variety, and (oversimplifying slightly) the homotopy-type of the variety is defined as that of this category.

When we prove the fundamental theorem of algebra by homotopy theory we are applying our intuitions to our best model of physical space. But we often use the language of spaces and geometry just as an analogy. We say, for instance, that a module is projective if it is 'locally free', or speak of an 'infinitesimal deformation' of an algebraic variety over an arbitrary field. In these cases, there is no genuine space around[‡] of the sort to which our intuitions apply, but the analogies are nonetheless very powerful tools. It is something of a surprise, therefore, that the homotopy-types we assign to algebraic varieties over arbitrary fields *are* those of genuine spaces, even though the points of the spaces are not the points of the variety. Over the last half-century the assignment of homotopy-types to groups, rings, and all manner of other algebraic objects, has become steadily more pervasive in mathematics, since its beginnings when it was called 'homological algebra'. I am as far as possible from being a mathematical logician, but I am intrigued that when I began research fifty years ago one of the avant-garde ideas in logic was Lawvere's proposal that categories and not sets were the right starting point for the foundations of mathematics, while nowadays we have Voevodsky's 'univalent foundations' project — a theory of 'types' — which bases the foundations on homotopy theory.

In the same half-century the subject called 'noncommutative geometry' has sprung up. It begins from the rough correspondence — contravariant — between topological spaces and commutative algebras over $\mathbb{C}$. In one direction, we associate to a space X the algebra $C(X)$ of continuous complex-valued functions on X, and, in the other, to a commutative algebra $\mathcal{A}$ we associate the space $\operatorname{Spec}(\mathcal{A})$ of algebra homomorphisms $\mathcal{A} \to \mathbb{C}$, or, equivalently, the space of irreducible $\mathcal{A}$-modules.

Why might a mathematician want to extend this correspondence to include noncommutative algebras? One kind of reason is that the properties of a commutative algebra, even if one's interest in it is purely algebraic, are unquestionably illuminated by thinking in terms of the space it defines, and one can aim for similar illumination about noncommutative rings.

A quite different *kind* of reason is that we encounter mathematical objects which we feel intuitively are 'spaces', but whose space-like properties cannot be captured by the usual concept of a topological space. Standard examples are the space of orbits of a group which

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[†]Roughly: one takes a point for each object of the category, adds a path for each morphism in the category, then fills in a 2-simplex with edges f, g, $g \circ f$ for each composable pair f, g of morphisms, a 3-simplex for each composable triple of morphisms, and so on.

[‡]We can, of course, define the Zariski topology on the set of points of an algebraic variety over any field. This is a topological space of a sort; but it is a weird space from any ordinary point of view, and when we use it I feel we are deploying powers of analogy rather than of spatial intuition.

acts ergodically on a space — for example, the group of integers $\mathbb{Z}$ acting on the circle $\mathbb{T}$ by an irrational rotation — and, more generally, the space of leaves of a foliated manifold when each leaf is dense in the manifold. These badly behaved quotients of ordinary spaces are the examples which Connes uses to motivate the study of noncommutative geometry, for, even though every scalar-valued continuous function of the leaf or orbit is necessarily constant, nevertheless there are non-constant *operator-valued* functions, and the leaves or orbits precisely parametrize the irreducible representations of a noncommutative algebra $\mathcal{A}$ naturally associated[†] to the foliation or group action. It is useful to think of $\mathcal{A}$ as playing the role of the functions on the space. Another surprise, in these quotient-space examples, is that — as we shall see — there is an ordinary homotopy-type naturally associated to the noncommutative algebra.

These examples may seem rather pathological, but the idea can be seen in much simpler situations. The prime examples of quotient spaces are spaces of *isomorphism classes* of objects of topological categories[‡]. Spaces of isomorphism classes are not quite ordinary spaces. For a simple example of the extra subtlety, let us think of the space of 'dumbbells' of weight 1. By a dumbbell, I mean a massless rod of unit length with a weight a attached to one end and a weight b attached to the other. Thus a dumbbell of weight 1 is described by a pair (a, b) of positive numbers such that $a + b = 1$, but we have the isomorphism $(a, b) \cong (b, a)$. This means that the space of isomorphism classes of dumbbells is the quotient of the open interval $(0, 1)$ by the relation $a \sim 1 - a$, that is, it is the half-open interval $(0, \frac{1}{2}]$. But the situation is more complicated. Is the space simply connected? Consider the closed path of dumbbells from $(\frac{1}{3}, \frac{2}{3})$ to $(\frac{2}{3}, \frac{1}{3})$ which makes the weight trickle from right to left. We might first think we can contract this through closed paths to the constant path at the point $(\frac{1}{2}, \frac{1}{2})$. But when we went around the original path in the space of dumbbells the ends of the dumbbell changed places, and the path looked like the boundary of a Möbius band[§]. We have a strong intuition that we cannot get rid of the twist in the path just by making the weights at the ends equal, and that the 'true' fundamental group of the space is of order 2. The space of the *category* of dumbbells does indeed have a fundamental group of order 2. Noncommutative geometry handles this situation very satisfactorily. Any topological category defines an algebra, and hence a noncommutative space, by a straightforward generalization of the twisted group-algebra which described the orbits of a group action, and equivalent topological categories define the same noncommutative space[¶].

[†]When a discrete group Γ acts on a compact Hausdorff space X, the natural algebra $\mathcal{A}$ is the *twisted group-algebra* $C(X)[\Gamma]$. The representation of $\mathcal{A}$ associated to an orbit ω of Γ in X is on the Hilbert space $\ell^2(\omega)$, on which Γ acts by translation and a function $f \in C(X)$ acts by multiplication by the restriction of f to ω. A potentially confusing point, if one wants to think of elements of $\mathcal{A}$ as operator-valued functions on the space of orbits, is that the space $\ell^2(\omega)$ on which the operator acts changes with the orbit: that is essential, for a continuous function from the orbits to the operators in a *fixed* space would have to be constant for the same reason as a scalar-valued function.

[‡]A topological category is one whose sets of objects and of morphisms have a topology. When one has an equivalence relation on a space X one can think of the points of X as the objects of a category, and an equivalence $x \sim x'$ as a morphism from x to x', so that the set of morphisms of the category is a subspace of $X \times X$. Just like a discrete category, a topological category $\mathcal{C}$ defines a space $|\mathcal{C}|$, and hence a homotopy-type, and this is the homotopy-type of the noncommutative quotient-space which was just mentioned.

[§]To be clear what is meant by deforming a closed path in the space of isomorphism classes of the category: the path consists of a continuous path $\{P_t\}_{0 \leqslant t \leqslant 1}$ in the space of objects of the category (in this case the interval $(0, 1)$) together with a definite isomorphism between the objects P_0 and P_1. To deform the path continuously, as a closed path, one must move not only the objects P_t but also the isomorphism $P_0 \to P_1$ continuously.

[¶]For example, the topological category describing the action of the cyclic group $\mathbb{Z}$ on the circle $\mathbb{T}$ by multiplication by $e^{i\theta}$ is a subcategory of that describing the foliation of the torus $\mathbb{T} \times \mathbb{T}$ by lines of slope θ. The inclusion of topological categories is an equivalence, so the noncommutative spaces are the same. The homotopy-type is that of a torus: this is more plausible from the foliation description, because the leaves of the foliation of the torus are contractible.

Why do we care about spaces of isomorphism classes? One of the striking discoveries of twentieth-century physics was that *all* of the state-spaces of fundamental physics are of this kind: in the language of physics, they are *gauge theories*. In the nineteenth century a state of the electromagnetic field, in the absence of particles, was described by its electric and magnetic field strengths, which together form a 6-component tensor field F on space-time satisfying Maxwell's equations. It was already known, nevertheless, that the 'mathematical artifice' of the (not uniquely defined) 'vector potential' A was mysteriously indispensable in discussing the field. Then, in the twentieth century, the Bohm–Aharonov experiment showed that in a non-simply-connected region an electromagnetic field is *not* completely described by its field strength, and that a more subtle idea is needed: an electromagnetic field is not a function on space-time but rather is an 'object', in fact a pair (L, A) consisting of a complex hermitian line-bundle L on space-time equipped with a well-defined connection A which is the 'vector potential'. Isomorphic pairs (L, A) define the same field. The classical field strength is the curvature F of the connection. Isomorphic pairs have the same curvature, and the curvature determines the pair up to isomorphism if the region is simply connected, but not otherwise.

Similarly, in general relativity it is crucial that a state of the gravitational field is not a metric tensor g_{ij} (satisfying Einstein's equations) on a fixed space-time manifold $\hat{M}$, but rather is an isomorphism class of pairs $(\hat{M}, g_{ij})$. This seems to have been a point which held Einstein up in arriving at his gravitational equation, as he was concerned that the equation did not prescribe a unique time-evolution for the tensor field g_{ij}.

The idea of space enters physics in at least two different ways. In classical (nonrelativistic) physics, apart from the space and time in which we live, we encounter a new space whenever we single out a part of the world to study: the space Y of *states* of the system we are considering. Usually the state is described by the instantaneous configuration of the system, which is a point x of a *configuration manifold* X, together with its instantaneous rate of change, which is a tangent vector to X at x. Thus Y is the tangent bundle TX of X. Physics tells us that Y is a Poisson manifold (that is, it comes with a 'bracket' operation

$$\{\ ,\ \} : \mathrm{C}^\infty(Y) \times \mathrm{C}^\infty(Y) \longrightarrow \mathrm{C}^\infty(Y)$$

on the vector space $\mathrm{C}^\infty(Y)$ of smooth real-valued functions on Y which makes $\mathrm{C}^\infty(Y)$ a Lie algebra), and there is a function $H : Y \to \mathbb{R}$, the *energy* or *Hamiltonian*, such that the time-evolution of the states in Y is given by

$$\frac{d}{dt} f = \{H, f\},$$

where f is any element of $\mathrm{C}^\infty(Y)$, such as a coordinate-function on Y.

Passing from classical to quantum physics forces the state-spaces Y to be something more general than ordinary topological spaces. To study quantum gravity we probably need to rethink space-time itself, but we shall not go down that road in this talk: we shall stick to quantum field theory, which is the approximation to physics in which the dynamics of space-time is ignored. In classical physics space-time[†] $M \times \mathbb{R}$ is given to us, and the state-space $Y = Y_M$ of the world is TX_M, where a point of the configuration space X_M consists of a finite subset σ of M — the positions of particles — together with some 'fields', which are smooth functions[‡] defined only in the complement $M \setminus \sigma$ of the particles. The worst complication, ultimately fatal for classical physics, lies in the difficulty of prescribing how the fields behave in the neighbourhood of the particles.

[†]For simplicity, I shall assume here that space-time $\hat{M}$ is the product of a space manifold M with the time-axis $\mathbb{R}$.

[‡]More accurately, sections of some bundle on M.

In one way quantum field theory is enormously simpler than its classical counterpart, for there are no particles: the manifold of configurations X_M is simply a space of smooth fields on M. In exchange for this simplification, the state-space Y_M can no longer be interpreted as an ordinary space: it is an object of noncommutative geometry, and that is why when we look at the world we sometimes see particles and sometimes see fields. But before coming to that I must emphasize a philosophical point. A classical dynamical system (Y, H) consisting of a Poisson manifold Y and a Hamiltonian function $H : Y \to \mathbb{R}$ is not recognizable as a description of the world: if Y is 60-dimensional it might equally well describe ten particles moving in $\mathbb{R}^3$ or a single particle moving in $\mathbb{R}^{30}$. The physics of the situation lies in the prescription or *functor* which tells us how $Y = Y_M$ is constructed from the physical space manifold M; in other words, it lies in the prescription telling us that the world consists of particles and fields. A quantum system is a noncommutative analogue of a pair (Y, H) consisting of a Poisson manifold and a Hamiltonian, and it has a similar lack of relation to what we see. Quantum field theory, on the other hand, is the description of the functor $M \mapsto Y_M$ which tells us how to construct a quantum system Y_M out of a space-time M. It *does* aim to describe the world.

I believe quantum theory forces us to accept that the truest description of the physical world is in terms of noncommutative spaces. Nonrelativistically — and our intuitions are certainly nonrelativistic — the structure which Nature seems to provide is a noncommutative topological† $*$-algebra $\mathcal{A}$ of observables. We form our picture of the world by recognizing the noncommutative algebra as a small deformation‡ $\mathcal{A} = \mathcal{A}_h$ of a commutative algebra $\mathcal{A}_0$ which we can think of as the functions on a classical state-space. (The deformation parameter h is Planck's constant.) More precisely, out of the vast torrent of observables which the world presents to us we select a subalgebra $\mathcal{A}_h$ which we can recognize as a very small deformation of a commutative algebra. Then we can identify $\mathcal{A}_h$ with $\mathcal{A}_0$ as a vector space, and can define a Poisson bracket on $\mathcal{A}_0$ as the departure of $\mathcal{A}_h$ from commutativity, to leading order in h, that is,

$$fg - gf = ih\{f, g\} + \mathcal{O}(h^2),$$

where the product on the left is that of $\mathcal{A}_h$. This gives us a classical picture: a space of states Y defined by the commutative algebra $\mathcal{A}_0$, a Poisson structure on Y, and a time-evolution which arises from the noncommutativity and a 'Hamiltonian' which is an element H of $\mathcal{A}_h$.

Let us quickly review some of the ways in which noncommutative geometry is different.

Passing from commutative to noncommutative geometry blurs the idea of points — for the 'points' of a noncommutative algebra are its irreducible representations, which do not usually form a reasonable space — and also, in general, the idea that different parts of the space can be studied independently. At first it might seem that we have lost geometry altogether. That is not the case, but it is true that the category of noncommutative spaces has some features resembling the homotopy category more than the category of spaces. Indeed, speaking sociologically, much of noncommutative geometry is about the homotopy-types of the objects; it is focussed on algebraic-topological ideas like cyclic homology and K-theory.

Another 'ungeometrical' feature of noncommutative geometry arises because in the usual interpretation the actual geometric object is not the noncommutative ring or algebra $\mathcal{A}$ itself,

† To an algebraist we are here letting in topology by the back door; but quantum theory does require us to know when *observables* are close, otherwise a self-adjoint operator would not have a spectrum. I shall return to this point later.

‡ The meaning of this needs care. In the typical examples, the isomorphism class of $\mathcal{A}_h$ is independent of h if $h \neq 0$, but jumps when $h = 0$.

but rather the *category* of left $\mathcal{A}$-modules: two algebras define the same space if their module categories are equivalent. The algebra $\mathrm{Mat}_n(\mathbb{C})$ of $n \times n$ complex matrices is just a point in the eyes of noncommutative geometry, whatever the value of n, and for any algebra $\mathcal{A}$ the algebra $\mathrm{Mat}_n(\mathcal{A})$ defines the same noncommutative space as $\mathcal{A}$. Because the modules form an additive category this means we can *add* morphisms of noncommutative spaces. In geometrical language, we automatically include multi-valued maps along with ordinary maps, and, in my view, this is a fundamental reason that quantum field theory deals with assemblies of identical particles rather than single particles. There is a best-possible way of adjoining maps to the homotopy category so as to make it additive. The result is the *stable homotopy* category, about which I shall say a little more presently; and it does indeed turn out that noncommutative spaces have stable homotopy-types rather than the usual sort.

Yet another perspective on the passage from commutative to noncommutative geometry focusses on inadequacies of the standard notion of a topological space. Topological spaces fall short of our needs in two dual respects. I have already mentioned the problem of 'bad' quotient constructions, but there is a dual inadequacy in dealing with 'bad' subspaces.

The prime examples of 'bad' subspaces are spaces of solutions of systems of equations. The simplest illustration is a root of multiplicity $m > 1$ of a polynomial equation in one variable, where a single point needs somehow to remember that it is potentially m distinct points. Spaces where the points have this kind of additional structure can be handled very simply in terms of algebra, and in this case only commutative algebra is needed. If $f \in \mathbb{C}[x]$ is a polynomial with distinct roots, then the quotient ring $\mathcal{A} = \mathbb{C}[x]/(f)$ is the algebra of complex-valued functions on the set of roots of f. If the roots are not distinct, the ring $\mathbb{C}[x]/(f)$ has nilpotent elements and is no longer the algebra of functions on a set, but, following Grothendieck, one can still think of it as defining a space in which the points have multiplicities: it is an 'infinitesimal thickening' of the actual set of roots. (The generalization of this situation which arises in quantum field theory is the zero-set of the classical equations of motion on the space of fields: this is the zero-set of a smooth section of an infinite-dimensional vector bundle on an infinite-dimensional smooth manifold.)

Whereas the 'bad' quotient spaces which were mentioned above do have well-defined homotopy-types in the usual sense, 'bad' subspaces have only *stable* homotopy-types, in the sense that one can at best say what the homotopy-type becomes after iterated suspension[†]. To understand why this is so, let us think of the 'overdetermined' case when X, though nonempty, is defined as $f^{-1}(0)$, where f is a proper smooth immersion $f : U \to \mathbb{R}^n$ of a manifold U of dimension $m < n$. Then, morally, X has the negative dimension $m - n$, for if f is perturbed a little it will disappear completely, and one has to probe with an $(n - m)$-dimensional family to find it.

Both deficiencies of the notion of topological space are relevant in quantum theory. I have said enough about quotients. We meet noncommutative geometry more directly when we look for the origin of the Poisson bracket which describes the time-evolution of functions on the classical state-space. From the quantum perspective it is the residual vestige of noncommutativity, but in classical physics it arises because the evolution of the state is governed by a *variational principle*: the 'principle of least action'. In this picture the classical trajectories sit as a subspace — the solutions of the Euler–Lagrange equation for the action functional — in the much larger space of trajectories which need not satisfy the equations of motion. In this talk I can give only the barest idea of the relation between the noncommutativity and the variational principle. One way to pass from classical to quantum physics is to contemplate *all* trajectories

[†]The n-fold suspension of a compact space X is the one-point compactification of $X \times \mathbb{R}^n$.

in the classical configuration space and to weight them according to their action (so that the trajectories obeying the classical equations of motion provide the leading contribution). From that viewpoint the semi-classical approximation is the study of an infinitesimal thickening of the classical state-space. More circumstantially, suppose we measure the product of the position $x(t)$ and the velocity $v(t)$ at time t of a particle moving along a line with trajectory given by $x : \mathbb{R} \to \mathbb{R}$. If we measure the position first, we are considering the limit of $(x(t+\varepsilon) - x(t))x(t)/\varepsilon$ as $\varepsilon \to 0$. If we make the measurements in the reverse order we are considering the limit of $x(t+\varepsilon)(x(t+\varepsilon) - x(t))/\varepsilon$. The two expressions differ by $(x(t+\varepsilon) - x(t))^2/\varepsilon$. If x is a smooth path, this difference tends to 0 as $\varepsilon \to 0$. But if x is a random 'Brownian'-type path then the expected value of $|x(t+\varepsilon) - x(t)|$ is of order $\varepsilon^{1/2}$ rather than ε, and so when we average over random paths we expect $x(t)v(t) - v(t)x(t)$ to be nonzero. (The complex-valued weighting of the quantum random paths makes this nonzero value the purely imaginary constant ih.)

As I have said, quantum theory suggests looking at a noncommutative space in terms of the commutative space of which it is a small deformation. This is a somewhat different picture from the usual one of noncommutative geometry, but it is helpful for understanding the wave-particle duality of quantum field theory. I shall spend the remainder of this talk describing how noncommutative geometry can make a space of fields look like an assembly of particles. This is the 'well-understood mathematical fact' that Dyson was referring to in the words I quoted at the beginning of my talk.

The simplest kind of field on a compact Riemannian manifold M is described by a smooth real-valued function on M, and so the configuration space X_M is the infinite-dimensional vector space $\mathrm{C}^\infty(M)$, and the classical state-space is its tangent bundle

$$Y_M = \mathrm{C}^\infty(M) \oplus \mathrm{C}^\infty(M).$$

We take the energy of a state $(\phi, \dot{\phi}) \in Y_M$ to be given by the quadratic form

$$\frac{1}{2}\int_M \{\dot{\phi}(x)^2 + \|\nabla\phi(x)\|^2 + m^2\phi(x)^2\}\, dx.$$

What we want to explain is how noncommutativity makes the algebra of functions on the vector space Y_M look like the functions on the tangent bundle of the configuration space

$$X_M^{\mathrm{part}} = \coprod_{n\geqslant 0} (M^n/\mathrm{Symm}_n)$$

of an indefinite number of indistinguishable particles moving in M.

The first point is that quantum field theory tells us that the appropriate commutative topological algebra $\mathcal{A}_0$ of smooth functions on the infinite-dimensional space Y_M is more subtle than the schematic description $\mathrm{C}^\infty(Y_M)$ suggests. It does *not* contain the smooth function $(\phi, \dot{\phi}) \mapsto \phi(x)$ given by evaluation at a point x of M. Traditionally, this is expressed by saying that $\phi(x)$ is 'fluctuating too fast' to have an 'expected value'. On the other hand it does contain the 'smeared-out' function

$$(\phi, \dot{\phi}) \longmapsto \phi_f = \int_M f(x)\phi(x)\, dx,$$

for every smooth test-function f with compact support on M. This, together with the corresponding smearings of $\dot{\phi}$, gives us a linear map $Y \to \mathcal{A}_0$ which extends to a map of algebras

$$S(Y) \longrightarrow \mathcal{A}_0$$

from the symmetric algebra of Y.

This map of algebras is a dense embedding. Nevertheless, there are still many different topologies in which one might complete the symmetric algebra, and quantum field theory picks out a topological algebra $\mathcal{A}_0$ of functions on Y_M which is intimately related to the geometry of the manifold M. For example, one might guess that the appropriate completion of the symmetric square $S^2(\mathrm{C}^\infty(M))$ should contain all the symmetric functions in $\mathrm{C}^\infty(M \times M)$, but in fact it also contains smooth delta-functions along the diagonal in $M \times M$, that is, the functions obtained by smearing $\phi(x)^2$, and more generally, all functions of $(\phi, \dot{\phi})$ obtained by smearing any differential polynomial in $\phi(x)$ and $\dot{\phi}(x)$, such as the Hamiltonian density itself.

Let us now see how the existence of a noncommutative deformation of the algebra $\mathcal{A}_0$ completely changes the geometric picture.

The crucial example which gives the idea is the abstract polynomial algebra $\mathbb{C}[a]$ in a single variable. This is a dense subalgebra of the algebra $\mathrm{C}^\infty(\mathbb{R})$ of functions on the real line, and also of the algebra $\mathrm{C}(\mathbb{N})$ of functions[†] on the discrete space $\mathbb{N}$ of positive integers. But $\mathrm{C}(\mathbb{N})$ has a quite different algebraic structure from either $\mathbb{C}[a]$ or $\mathrm{C}^\infty(\mathbb{R})$, being generated by idempotents e_k for $k \geqslant 0$ such that $e_k e_m = 0$ when $k \neq m$.

Whether the algebra $\mathbb{C}[a]$ should be regarded as consisting of functions on the line $\mathbb{R}$ or on a closed subset Σ of it — that is, determining the *spectrum* Σ of the operator a — is prescribed by the *topology* on the algebra, which is an essential part of the quantum-mechanical picture. The question is: for which points $\lambda \in \mathbb{R}$ is the evaluation-map $f \mapsto f(\lambda)$ continuous[‡]. There is a simple algebraic mechanism, basic to quantum mechanics, which can force us to see $\mathbb{C}[a]$ as an algebra of functions on the discrete space $\mathbb{N}$. Suppose that $\mathbb{C}[a]$ arises as a subalgebra of a noncommutative topological $*$-algebra $\mathcal{A}$ generated by an element b such that $b^*b - bb^* = 1$, and that a is the self-adjoint element bb^*. The traditional way of stating the result is

THEOREM. *If a $*$-action of $\mathbb{C}[a]$ on a Hilbert space $\mathcal{H}$ extends to an action of $\mathcal{A}$, then the spectrum of a is $\mathbb{N}$.*

This theorem describes the quantum harmonic oscillator. It is a version of the Stone–von Neumann theorem on the uniqueness of the irreducible representation of the Heisenberg algebra. I have stated it in the traditional not-very-precise physicists' way involving unbounded operators in Hilbert space. But the essential point is the following brutal algebraic observation which does not involve Hilbert spaces at all. It is an argument no geometer will like, but it seems to be vital for quantum theory.

THEOREM. *If $\theta : \mathcal{A} \to \mathbb{C}$ is a linear map whose restriction to $\mathbb{C}[a]$ is a homomorphism of algebras, and which also satisfies $\theta(f^*) = \overline{\theta(f)}$ and $\theta(ff^*) \geqslant 0$ for all $f \in \mathcal{A}$, then $\theta(a)$ is a positive integer.*

Proof. Suppose $\theta(a) = \lambda$. Then $a^2 - a = bb^*bb^* - bb^* = b^2(b^2)^*$, so $\lambda^2 - \lambda = \theta(b^2(b^2)^*) \geqslant 0$. Continuing by induction, we have

$$\lambda(\lambda - 1)(\lambda - 2)\ldots(\lambda - k + 1) = \theta(b^k(b^k)^*) \geqslant 0.$$

If this is true for all k, then λ must be a positive integer. $\square$

[†]For any closed $\Sigma \subset \mathbb{R}$, we give $C(\Sigma)$ the topology of uniform convergence on compact subsets.

[‡]Operator algebraists like to hide the topology in algebra by encoding it in the way the algebra is completed, but I feel that obscures the real issue.

The physical interpretation is that if we are presented with a system represented by the algebra $\mathcal{A}$ whose Hamiltonian is a multiple of a, say $h\omega a$, then we might happen to be interested in the self-adjoint elements

$$q = (b + b^*)/2k, \quad p = m\omega(b - b^*)/2ik$$

(where $k = (m\omega/2h)^{1/2}$) such that $pq - qp = ih$. These generate $\mathcal{A}$, and we might see $\mathcal{A}$ as a small deformation of the commutative algebra $\mathbb{C}[p, q]$. Then we would see a particle oscillating on a line, with position q and momentum p and Hamiltonian $(1/2m)(p^2 + m^2\omega^2 q^2) - \frac{1}{2}h\omega$. But (depending on the scales of the elements we choose to observe) the whole algebra $\mathcal{A}$ may not be sufficiently commutative, and our sluggish perceptions may 'see' only the commutative subalgebra $\mathbb{C}[a]$, which describes a stationary system with a discrete sequence of possible states. Notice that the algebra $\mathcal{A}$ is $\mathbb{Z}$-graded, with b and b^* of degrees $+1$ and -1, respectively, and $\mathbb{C}[a]$ is the subalgebra of elements of degree 0.

Quantum field theory starts from a far-reaching (but not much harder to prove) generalization of the preceding theorem which explains why we see particles and not waves when we look at the quantum algebra of the state-space of fields on a manifold M. The preceding result is the case when M is a single point.

THEOREM. *If $\mathcal{A}_h$ is the standard Heisenberg deformation of the commutative algebra*[†] $\mathcal{A}_0$ *of functions on the symplectic vector space*

$$Y_M = \mathrm{C}^\infty(M) \oplus \mathrm{C}^\infty(M),$$

then there are commuting elements $a_f \in \mathcal{A}_h$ for $f \in \mathrm{C}^\infty(M)$ which generate an algebra isomorphic to $\mathrm{C}^\infty(X_M^{\mathrm{part}})$, where

$$X_M^{\mathrm{part}} = \coprod_{n \geqslant 0} (M^n/\mathrm{Symm}_n).$$

Here a_f corresponds to the symmetric function $(x_1, \ldots, x_n) \mapsto \sum f(x_i)$ on $M \times \cdots \times M$, though as a function on Y_M it is obtained by smearing the quadratic function $a(x)$ which takes $(\phi, \dot{\phi})$ to

$$\tfrac{1}{2}\{\phi(\triangle + m^2)^{\frac{1}{2}}\phi + \dot{\phi}(\triangle + m^2)^{-\frac{1}{2}}\dot{\phi}\}(x)$$

with the function f on M. In particular, a_1 is the function which counts the number of particles present, and it commutes with the Hamiltonian H. The commutant $\mathcal{A}_h^0$ of a_1 is generated by the elements a_f and their time-derivatives $\dot{a}_f = ih^{-1}[H, a_f]$. It is the usual quantum deformation of the algebra of functions on the tangent bundle $Y_M^{\mathrm{part}} = TX_M^{\mathrm{part}}$. It does not contain the operators ϕ_f and $\dot{\phi}_f$ which describe the actual classical fields: they do not commute with the number operator a_1, but are the analogues of the elements q, p in the usual Stone–von Neumann theorem. They are not part of the 'picture' in which one sees particles, though of course from a different point of view in different circumstances we might see the whole algebra $\mathcal{A}_h$ as approximately commutative, giving us a picture of fields rather than of particles.

Just like the simplest Heisenberg algebra $\mathbb{C}\langle b, b^*\rangle/(b^*b - bb^* = 1)$, the algebra $\mathcal{A}_h$ is $\mathbb{Z}$-graded, with the 'particle' algebra $\mathcal{A}_h^0$ as its degree 0 part.

[†]This is a little disingenuous, in that I am assuming that the topology of $\mathcal{A}_h$ is sufficiently fine to allow us to form the operators a_f. A more satisfactory formulation of the theorem would need a fuller account of quantum field theory.

References

This talk is aimed at too many kinds of readers for a conventional list of references to be appropriate. Any amount of material about the topics I have mentioned, at every level, can be found on-line, so I shall make just a few remarks.

The canonical account of noncommutative geometry is Connes's book [**1**], written in the language of operator algebras, which is well-adapted to quantum theory. But there is another approach focussed on the module categories which stands to Connes's work in the way that algebraic varieties do to topological spaces. A central reference is [**3**], but so far the approach lacks an account like Connes's.

The idea of the homotopy-type of a category goes back at least to the invention of Čech cohomology, and was greatly exploited, if rather inexplicitly, by Grothendieck and his school. I tried to popularize the idea, especially for topological categories, in [**5**, **6**], and it was developed much further in Quillen's work [**4**] on algebraic K-theory, and subsequently by many other people.

The basic reference for the relation between noncommutativity and variational principles, due to Dirac, Wheeler, and Feynman, is [**2**].

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