2.

Notes on affine roots.

2.1. Fix a (reduced, finite) root datum $(X, \mathfrak{R}, X^\vee, \mathfrak{R}^\vee)$. Thus, X and $X^\vee$ are dual lattices of finite rank, while $\mathfrak{R} \subseteq X$ and $\mathfrak{R}^\vee \subseteq X^\vee$ are finite subsets constrained by conditions spelled out in, *e.g.*, Bourbaki. Elements of $\mathfrak{R}$ are called *roots* of the root datum; elements of $\mathfrak{R}$ are called *coroots*.

We regard roots as linear functionals on $X^\vee_{\mathbf{R}} := X^\vee \otimes \mathbf{R}$. We write W for the Weyl group of $\mathfrak{R}$: the group generated by the reflections in $X^\vee_{\mathbf{R}}$ corresponding to the root hyperplanes.

For our purposes, an *affine root* is a function on $X^\vee_{\mathbf{R}}$ of the form $\alpha + k$, where $\alpha \in \mathfrak{R}$ and $k \in \mathbf{Z}$. We set $\mathfrak{R}^{\mathrm{aff}} = \mathfrak{R} \times \mathbf{Z}$, viewed as the set of affine roots. We picture them as affine hyperplanes in $X^\vee_{\mathbf{R}}$ equipped with orientations of their normals. The value of an affine root at a point describes the signed distance from the affine hyperplane to the point, with a positive, *resp.* negative, sign when the normal points toward, *resp.* away from, the point. For all $x \in X^\vee_{\mathbf{R}}$, we set

$$\mathfrak{R}^{\mathrm{aff}}(x) = \{\alpha + k \in \mathfrak{R}^{\mathrm{aff}} \mid \alpha(x) + k = 0\}.$$

Note that the projection $\mathfrak{R}^{\mathrm{aff}}(x) \to \mathfrak{R}$ that sends $\alpha + k \mapsto \alpha$ is a bijection precisely when x belongs to $(\mathbf{Z}\mathfrak{R})^\vee$, the dual lattice to the root lattice. In particular, $X^\vee$ is a sublattice of $(\mathbf{Z}\mathfrak{R})^\vee$.

2.2. Given a point $x \in X^\vee_{\mathbf{R}}$ and a real number $s > 0$, we set

$$\begin{aligned} \mathfrak{R}^{\mathrm{aff}}(x, s) &= \{\alpha + k \in \mathfrak{R}^{\mathrm{aff}} \mid \alpha(x) + k = s\}, \\ \mathfrak{R}^{\mathrm{aff}}_{<}(x, s) &= \{\alpha + k \in \mathfrak{R}^{\mathrm{aff}} \mid 0 \leq \alpha(x) + k < s\}. \end{aligned}$$

Definition 2.1. We define an (x, s)*-bouquet* of the given root datum to be a pair $(\xi, \mathfrak{D})$, where $\xi \in X^\vee$ and $\mathfrak{D} \subseteq \mathfrak{R}^{\mathrm{aff}}(\xi) \cap \mathfrak{R}^{\mathrm{aff}}(x, s)$.

Definition 2.2. For any (x, s)-bouquet $(\xi, \mathfrak{D})$, we define a *coloring* of $(\xi, \mathfrak{D})$ to be a vector $\epsilon \in X^\vee_{\mathbf{R}}$ such that

$$\left.\begin{array}{rl} \text{if } \alpha + k \notin \mathfrak{D}, & \text{then } \alpha(\epsilon) \geq 0, \\ \text{and if } \alpha + k \in \mathfrak{D}, & \text{then } \alpha(\epsilon) < 0, \end{array}\right\} \quad \text{for all } \alpha + k \in \mathfrak{R}^{\mathrm{aff}}(\xi) \cap \mathfrak{R}^{\mathrm{aff}}(x, s).$$

We write $\mathrm{Col}(\xi, \mathfrak{D})$ for the set of colorings of $(\xi, \mathfrak{D})$. For any W-orbit o on $X^\vee_{\mathbf{R}}$, we write $\mathrm{Col}(\xi, \mathfrak{D}, o)$ for the subset of colorings that belong to o.

Definition 2.3. For any $\xi \in X^\vee$ and $\epsilon \in X^\vee_{\mathbf{R}}$, we define the *inversion set* of (ξ, ϵ) to be

$$\mathrm{Inv}_\xi(\epsilon) = \mathrm{Inv}_\xi^{x,s}(\epsilon) := \{\alpha + k \in \mathfrak{R}^{\mathrm{aff}}(\xi) \cap \mathfrak{R}^{\mathrm{aff}}_{<}(x, s) \mid \alpha(\epsilon) < 0\}.$$

Definition 2.4. We define the *LLT function* of an (x, s)-bouquet $(\xi, \mathfrak{D})$ to be the function $\mathrm{LLT}_{\xi,\mathfrak{D}}(t, -) = \mathrm{LLT}^{x,s}_{\xi,\mathfrak{D}}(t, -) : X^\vee_{\mathbf{R}}/W \to \mathbf{Z}[t]$ such that

$$\mathrm{LLT}_{\xi,\mathfrak{D}}(t, o) = \sum_{\epsilon \in \mathrm{Col}(\xi,\mathfrak{D},o)} t^{|\mathrm{Inv}_\xi(\epsilon)|}.$$

Parallel to (1.1), we also set

$$\mathrm{Inv}(\xi) = \{\alpha + k \in \mathfrak{R}^{\mathrm{aff}}_<(x, s) \mid \alpha(\xi) + k < 0\}.$$

Note that $\mathrm{Inv}(\xi)$ is disjoint from $\mathfrak{R}^{\mathrm{aff}}(\xi)$, hence disjoint from $\mathrm{Inv}_\xi(\epsilon)$ for any ϵ.

2.3. The LLT function of a bouquet is constructible with respect to a finite stratification of $X^\vee_{\mathbf{R}}/W$ determined by the roots. To explain how, it is convenient to fix a system of simple roots $\{\alpha_1, \dots, \alpha_r\} \subseteq \mathfrak{R}$. For each subset $J \subseteq \{1, \dots, r\}$, form the locus

$$Z_J = \left\{ x \in X^\vee_{\mathbf{R}} \;\middle|\; \begin{array}{ll} \alpha_i(x) > 0 & \text{for all } i \in J, \\ \alpha_i(x) = 0 & \text{for all } i \notin J \end{array} \right\}.$$

Note that if $J = \emptyset$, then $Z_J = \{0\}$, whereas if $J = \{1, \dots, r\}$, then Z_J is an open region known as the *fundamental Weyl chamber* associated with the system of simple roots.

We declare two subsets of $\{1, \dots, r\}$ to be equivalent whenever the corresponding sets of simple roots are conjugate under W. Let $\mathfrak{J}$ be a full, irredundant set of representatives of the equivalence classes under this relation. Then the disjoint union of the loci Z_J for $J \in \mathfrak{J}$ provides a fundamental domain for the W-action on $X^\vee_{\mathbf{R}}$. The images of the sets Z_J in $X^\vee_{\mathbf{R}}/W$ form a stratification of $X^\vee_{\mathbf{R}}/W$, along which our LLT functions are locally constant.

For this reason, it suffices to evaluate $\mathrm{LLT}_{\xi,\mathfrak{D}}(t, o)$ at orbits o of the form $[\rho^\vee_J]$, where $J \in \mathfrak{J}$ and $\rho^\vee_J$ is defined as follows. Let $\{\omega^\vee_1, \dots, \omega^\vee_r\}$ be the basis of $(\mathbf{Z}\mathfrak{R})^\vee$ dual to the basis of simple roots. The elements $\omega^\vee_i$ are also known as fundamental coweights. Let $\mathfrak{R}_J \subseteq \mathfrak{R}$ be the root subsystem generated by the simple roots α_i with $i \in J$, and let

$$\rho^\vee_J = \sum_{i \in J} \omega^\vee_i.$$

Then $\alpha_i(\rho^\vee_J) > 0$ when $i \in J$, and $\alpha_i(\rho^\vee_J) = 0$ otherwise.

In the case where $J = \{1, \dots, r\}$, we will write $\rho^\vee$ in place of $\rho^\vee_J$. We will also write

$$\mathrm{LLT}^{x,s}_{\xi,\mathfrak{D}}(t, J), \quad \textit{resp.} \quad \mathrm{Col}(\xi, \mathfrak{D}, J)$$

in place of $\mathrm{LLT}^{x,s}_{\xi,\mathfrak{D}}(t, [\rho^\vee_J])$, *resp.* $\mathrm{Col}(\xi, \mathfrak{D}, [\rho^\vee_J])$. We are now ready to explain how the new definitions recover the classical ones.

2.4. Take $(X, \mathfrak{R}, X^\vee, \mathfrak{R}^\vee)$ to be the root datum of the semisimple algebraic group SL_n. We identify $X^\vee$ with the lattice $\mathbf{Z}_0^n = \{x \in \mathbf{Z}^n \mid \sum_i x_i = 0\}$. The roots of SL_n are the functionals $\alpha_{i,j}$, for $1 \le i, j \le n$ with $i \neq j$, defined by

$$\langle \alpha_{i,j}, x \rangle = x_i - x_j$$

for all $x \in X^\vee$. Without loss of generality, we may take the simple roots to be $\alpha_i := \alpha_{i,i+1}$ for $1 \le i \le n-1$. Then the fundamental coweights are given by $\omega_i = \frac{1}{2}(1, \dots, 1, -1, \dots, -1)$, where the sign change occurs between the ith and $(i+1)$st entries.

Each partition $\mu \vdash n$ defines a subset $J \subseteq \{1, \dots, n-1\}$: explicitly,

$$J = \{\mu_1, \mu_1 + \mu_2, \mu_1 + \mu_2 + \mu_3, \dots\} \setminus \{n\}.$$

Writing l for the length of μ, so that $\mu_i = 0$ for $i > l$, we have

$$\rho_J^\vee = (\overbrace{\tfrac{l}{2}, \dots, \tfrac{l}{2}}^{\mu_1 \text{ times}}, \overbrace{\tfrac{l-2}{2}, \dots, \tfrac{l-2}{2}}^{\mu_2 \text{ times}}, \dots, \overbrace{-\tfrac{l}{2}, \dots, -\tfrac{l}{2}}^{\mu_l \text{ times}}).$$

In particular, $\rho^\vee = (\frac{n-1}{2}, \frac{n-3}{2}, \dots, -\frac{n-1}{2})$.

Example 2.5. For any x, s, we set

$$X^\vee(x, s) = \{\xi \in X^\vee \mid \alpha(\xi) + k \ge 0 \text{ for all } \alpha + k \in \mathfrak{R}^{\mathrm{aff}}(x, s)\}.$$

Then the set $X^\vee(x, s)$ generalizes the *Sommers region* studied in the combinatorics literature: See Sommers's paper [S] and the related work of Lusztig–Smelt, Fan, and Sage. Indeed, if d is coprime to n, then $X^\vee(\frac{1}{n}\rho^\vee, \frac{d}{n})$ is precisely the set $\mathbf{Z}_0^n(d)$ used by Hikita, discussed in §1.8.

2.5. Fix an arbitrary real number ϱ. For all $x \in X_{\mathbf{R}}^\vee$ such that $-nx_i + \varrho \in \mathbf{Z}$ for all i, and $\xi \in X^\vee$, let

$$C(\xi) = \{c_1, \dots, c_n\} \quad \text{where } c_i = n\xi_i - nx_i + \varrho \text{ for } 1 \le i \le n.$$

For instance, if $\varrho = \frac{n-1}{2}$ and $x = \frac{1}{n}\rho^\vee$, then this definition of $C(\xi)$ recovers the definition used in §1.8. The following elementary observation is the key:

Proposition 2.6. *For any ϱ, x, ξ as above, and affine root $\alpha_{i,j} + k \in \mathfrak{R}^{\mathrm{aff}}(\xi)$, we have*

$$\tfrac{1}{n}(c_j - c_i) = \alpha_{i,j}(x) + k.$$

Equivalently, for any x, ξ, and real $s > 0$, the map

$$\begin{aligned} \mathfrak{R}^{\mathrm{aff}}(\xi) \cap \mathfrak{R}^{\mathrm{aff}}(x, s) &\to \{(c, c') \in C(\xi)^2 \mid \tfrac{1}{n}(c - c') = s\} \\ \alpha_{i,j} + k &\mapsto (c_j, c_i) \end{aligned}$$

is a bijection.

Corollary 2.7. *For any ϱ, x, ξ as above, and positive integer d, the map*

$$\begin{aligned} \mathfrak{R}^{\mathrm{aff}}(\xi) \cap \mathfrak{R}^{\mathrm{aff}}(x, \tfrac{d}{n}) &\to \{c \in C(\xi) \mid c + d \in C(\xi)\} \\ \alpha_{i,j} + k &\mapsto c_i \end{aligned}$$

induces a bijection between:

- *Subsets $\mathfrak{D} \subseteq \mathfrak{R}^{\mathrm{aff}}(\xi) \cap \mathfrak{R}^{\mathrm{aff}}(x, \frac{d}{n})$.*
- *Subsets $D \subseteq C(\xi)$ such that if $c \in D$, then $c + d \in D$.*

Note that the Weyl group is $W = S_n$, acting on $X^\vee = \mathbf{Z}_0^n$ by permuting entries. Hence, if $J \subseteq \{1, \ldots, n-1\}$ corresponds to $\mu \vdash n$ as in §2.3, then in any vector of the form $\rho_J^\vee \cdot w$ with $J \subseteq \{1, \ldots, n-1\}$ and $w \in W$, the ith largest value among the entries is repeated μ_i times. Below, define

$$T_{\mu,w} : C(\xi) \to \mathbf{N}$$

to send the first μ_1 entries of $(c_1, \ldots, c_n)$ to 1, the next μ_2 entries to 2, *etc.*

Proposition 2.8. *Fix ϱ, x, d, ξ as above. Let $J \subseteq \{1, \ldots, n-1\}$ correspond to $\mu \vdash n$. Then the map $w \mapsto T_{\mu,w}$ induces a bijection between:*

- *The W-orbit of $\rho_J^\vee$.*
- *The set of functions $T : C(\xi) \to \mathbf{N}$ such that $\mu_k = |T^{-1}(k)|$.*

Moreover, we have $\alpha_{i,j}(\epsilon) \geq 0$ if and only if $T(c_j) \leq T(c_i)$.

Corollary 2.9. *Fix ϱ, x, d as above, and an $(x, \frac{d}{n})$-bouquet $(\xi, \mathfrak{D})$. Let $D \subseteq C(\xi)$ correspond to $\mathfrak{D}$, and let $J \subseteq \{1, \ldots, n-1\}$ correspond to $\mu \vdash n$. Suppose that $C(\xi)$ is d-contiguous. Then:*

(1) The map $\rho_{J(\mu)}^\vee \cdot w \mapsto T_{\mu,w}$ is a bijection

$$\mathrm{Col}(\xi, \mathfrak{D}, J) \xrightarrow{\sim} \mathrm{Col}(C(\xi), D, \mu).$$

(2) If $\rho_J^\vee \cdot w \in \mathrm{Col}(\xi, \mathfrak{D})$, then the map $\alpha_{i,j} + k \mapsto (c_j, c_i)$ is a bijection

$$\mathrm{Inv}_\xi(\rho_J^\vee \cdot w) \xrightarrow{\sim} \mathrm{Inv}(T).$$

In particular, $\mathrm{LLT}_{\xi,\mathfrak{D}}^{x,d/n}(t, J) = \langle h_\mu, \mathrm{LLT}_{C(\xi),D}(t)\rangle$.

Conversely, any element (C, D) of the set $\mathbf{A}(d)$ in §1.5, such that $|C| = n$, arises from some ϱ, x and $(x, \frac{d}{n})$-bouquet $(\xi, \mathfrak{D})$. For instance, we can take $\varrho = \frac{1}{n} \sum_i c_i$ and $x_i = -\frac{1}{n}(c_i - \varrho)$ for all i and $\xi = 0$. Invoking Lemma 1.1, we conclude:

Theorem 2.10. *For any d-tuple of ribbons $\vec{\nu}$ of size n, we can find some $x \in X_{\mathbf{R}}^\vee$ for $G = \mathrm{SL}_n$ and $(x, \frac{d}{n})$-bouquet $(\xi, \mathfrak{D})$ such that*

$$\mathrm{LLT}_{\xi,\mathfrak{D}}^{x,d/n}(t, J) = \langle h_\mu, \mathrm{LLT}_{\vec{\nu}}(t)\rangle$$

whenever $J \subseteq \{1, \ldots, n-1\}$ corresponds to $\mu \vdash n$.