<u>Warmup</u> in F^2, provisionally define:

a reflection to be a lin op that sends
$e_1, e_2 \mapsto e_1, -e_2$
for some basis e_1, e_2

a shear to be a lin op that sends
$e_1, e_2 \mapsto e_1, e_1 + be_2$
for some basis e_1, e_2

e.g.,
$\begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}$ is a shear matrix that is not $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$

<u>Q</u> reflection matrix that is not diagonal?
shear matrix that is not triangular?

take $e_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $e_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

graphing shows: $e_1, e_2 \mapsto e_1, -e_2$

yields $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ in the std basis

<u>Ex</u> $M = \begin{pmatrix} 5/2 & -3/2 \\ 3/2 & -1/2 \end{pmatrix}$ is a shear matrix wrt e_1, e_2

$Me_1 = e_1$ and

M sends e_2 to $\begin{pmatrix} -4 \\ -2 \end{pmatrix} = \begin{pmatrix} -3 \\ -3 \end{pmatrix} + \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

Idea have: qualitative defns of geometric ops
and examples given by matrices
want: matrix-independent defns

Q how to formalize “matrix-independent”?

fix linear map T : V to W

(v_1, …, v_n) ordered basis of V,
(v_1, …, w_m) ordered basis of W
M matrix of T wrt (v_i)_i, (w_j)_j

(e_1, …, e_n) another ordered basis of V,
(f_1, …, f_m) another ordered basis of W
matrix of T wrt (e_i)_i, (f_j)_j? in terms of M?

	id_V		T		id_W	
V	to	V	to	W	to	W
e_i		v_i		w_j		f_j

let A be the matrix of id_V “from e_i to v_i”
B id_W “from w_j to f_j”

A, B are invertible [since they represent iso’s]
the matrix of T wrt (e_i)_i, (f_j)_j is B • M • A

possibly confusing:
ith col of A expresses e_i as sum_j A_{k, i} v_k
jth col of B w_j as sum_ℓ B_{ℓ, j} f_ℓ

[id_V sends e_i to e_i; id_W sends w_j to w_j]

suppose $W = V$,
$(w_j)_j = (v_i)_i$,
$(f_j)_j = (e_i)_i$

here: T is a linear op on V,
M is its matrix "from $(v_i)_i$ to $(v_i)_i$"

A is the matrix of the id map "from e_i to v_i"
B is the matrix of the id map "from v_i to e_i"
so $A = B^{-1}$

<u>Thm</u>: if M is the matrix of a lin op $T : V$ to V
wrt a basis $(v_i)_i$ for V,
then BMB^{-1} is its matrix
wrt another basis $(e_i)_i$
where $v_j = \sum_i B_{j, i} e_i$

<u>Rem</u>: general case $T : V$ to W also useful,
but the statement is cumbersome –
easier to rederive from picture of maps

<u>Ex</u>: $V = \{p \text{ in } F[x] \mid p = 0 \text{ or } \deg p \leq 3\}$
$T(p) = dp/dx$
$(v1, v2, v3, v4) = (1, x, x^2, x^3)$
$(e1, e2, e3, e4) = (1, x, x^2/2, x^3/6)$

$$M = \begin{matrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{matrix} \qquad B = \begin{matrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 6 \end{matrix}$$

[compute B^(-1)]
[compute BMB^(-1)]

Df matrices M, N are conjugate* iff,
for some matrix P,
$P \cdot M \cdot P^{-1}$ is well-defined,
equals N

* also say “M and N are conjugates of each other”

Rem almost always, we only use this notion
in the context where M, N are square

Df a property of M is conjugation-invariant
iff it is the same for all conjugates of M

Cor if M is the matrix of a lin op T,
then any property of T is
a conjugation-invariant property of M

and conversely!
any conjugation-invariant property of M
only depends on T

Ex entries of M are not conj-invariant

what are the conj-invariant functions Mat_2 to F?

for a 2×2 matrix $M = \begin{matrix} P & Q \\ R & S \end{matrix}$

$\mathrm{tr}(M) = P + S$ and $\det(M) = PS - QR$ are invariant
[any others?]

Thm any invariant poly of the coord fns
on Mat_2 must be a poly in tr and det

why this is hard:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} P & Q \\ R & S \end{pmatrix} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$\begin{pmatrix} aP + bR & aQ + bS \\ cP + dR & cQ + dS \end{pmatrix} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$adP + bdR - acQ - bcS$

$\quad -abP - bbR + aaQ + bbS$

$cdP + ddR - ccQ - cdS$

$\quad -bcP -bdR + acQ + adS$

…we will give a bettter proof later

Ex fix an integer $k > 0$

the property ($M^k = 0_n$) is conj-invariant because

$$(BMB^{(-1)})^k = BM^kB^{(-1)} = 0_n$$

corresponds to having $T \circ \ldots \circ T$ = zero where T is iterated k times

Ex since nilpotence is conj-invariant, unipotence is conj-invariant:

$$\begin{aligned} B(I_n + M)B^{(-1)} &= BI_nB^{(-1)} + BMB^{(-1)} \\ &= I_n + BMB^{(-1)} \end{aligned}$$

above, used the left & right distributive properties
for matrix multiplication

Ex for fixed $k > 0$,
the property $M^k = I_n$ is conj-invariant

M is called an involution iff $M^2 = I_n$

Rem reflections are involutions

observe that for any n,
conjugacy is an equivalence relation on Mat_n

Df the conjugacy classes of Mat_n are
the equivalence classes for conjugacy