8.

We explain how the knots and links of geometric topology are related to braid groups, hence to the Hecke algebras of symmetric groups. The main reference is Jones's 1987 paper, "Hecke Algebra Representations of Braid Groups and Link Polynomials". Most of the diagrams in these notes are stolen from the textbook on arXiv by Chmutov–Duzhin–Mostovoy.

8.1.

A *knot*, *resp.* *link*, is the image of a continuous embedding of a circle, *resp.* a finite disjoint union of circles, into a fixed (topological) 3-manifold.[1] Note that finite disjoint unions of circles are the same as closed 1-manifolds. We almost always take the 3-manifold to be $\mathbf{R}^3$, although we will sometimes need the 3-sphere S^3 or a thickened annulus $D^2 \times S^1$.

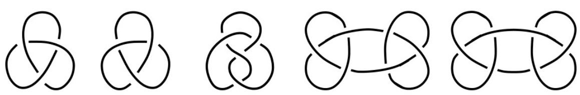
If M, N are manifolds and $u, v : N \to M$ are two continuous embeddings, then an *isotopy* from u to v is a continuous map $\phi : N \times [0,1] \to M$ such that:

(1) $\phi_t = \phi(-,t) : N \to M$ is a continuous embedding for all t.
(2) $\phi_0 = u$ and $\phi_1 = v$.

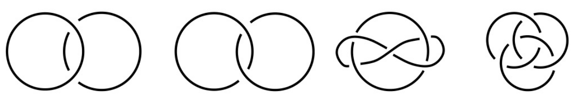
If such an isotopy exists, then we say that u and v are *isotopic*. Note that it is possible to construct examples of non-isotopic maps with the same image. Consequently, if M_1, M_2 are submanifolds of M, then we say that M_1 and M_2 are *isotopic* in M if and only if we can find *some* N and *some* $u_i : N \to M_i$ with image M_i for $i = 1, 2$ such that u_1 and u_2 are isotopic.

A link is *tame* if and only if it is isotopic to the image of a piecewise linear embedding with finitely many singular points. Henceforth, we only deal with tame knots and links, and suppress the adjective.

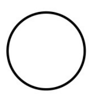
A *knot/link diagram* is a drawing of a projection of a knot/link in $\mathbf{R}^3$ onto a plane, but keeping track of over- and undercrossings. Some knot diagrams:



Some link diagrams:

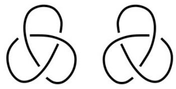


The simplest knot/link is the *unknot*:

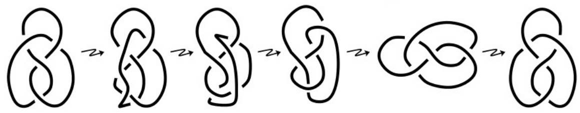


[1] What mathematicians call a knot is what sailors and climbers would call a *grommet*.

Roughly, knot theory is the study of how to tell whether two links are isotopic. For instance, it turns out *trefoils* are not isotopic to their mirrors.

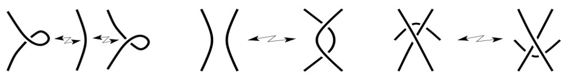


By contrast, any *figure-eight knot* is isotopic to its mirror.



As it turns out, there is a special trick—tricolorability—that shows why the mirror trefoils are not isotopic. But on general links, tricolorability is too weak of an isotopy invariant. At the other extreme, the following classical theorem gives a completely discrete characterization of isotopy between links, but is not directly useful in practice.

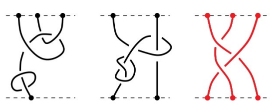
Theorem 8.1 (Reidemeister)**.** *Two links are isotopic if and only if they admit diagrams that differ by some finite sequence of operations consisting of the following* local moves*:*



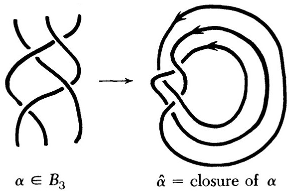
8.2.

What makes it difficult to apply Reidemeister's theorem systematically is that, within a given link diagram, the local pictures above can appear in so many different configurations relative to each other. It would be easier if we could impose some sort of linear order on the positions of the pictures.

In this way, we are led to study braids. Informally, a *braid on n strands* is like a link, but connects n ordered inputs at one end of a box or cylinder to n ordered outputs at the other end, without trackbacks. Below, only the red diagram depicts a braid: specifically, on 3 strands.



Given a diagram of a braid β, we can draw strands going from its outputs back to its inputs in the same order, without any further crossings. The result is a diagram of a link in $\mathbf{R}^3$, well-defined up to isotopy, which we call the *link closure* $\hat{\beta}$ of the braid β. Figure 1.4 from Jones's paper illustrates this operation.



Alternately, we can fix a point next to the braid, then require that all of the strands wind once around it before they join to the braid inputs. This produces a diagram of a link in the thickened annulus $D^2 \times S^1$, sometimes called the *annular closure* of β. We will denote it by β°, though there is no standard notation.

The following theorem is proved in Alexander's 1923 paper " A lemma on systems of knotted curves", which is roughly two pages long.

Theorem 8.2 (Alexander)**.** *Every link in $\mathbf{R}^3$ is isotopic to the closure of some braid.*

Proof sketch. The idea is to pick a point O inside the diagram, away from any strands, then modify the diagram by Reidemeister moves until every component of the link is a circle winding around O with a consistent direction. Since we assume that the link is tame, we can reduce to the case where every component is piecewise-linear or *polygonal*. When an edge of the polygon backtracks with respect to the direction we have chosen, there is a trick that lets us replace it with two consecutive edges in the direction we want.[2] □

8.3.

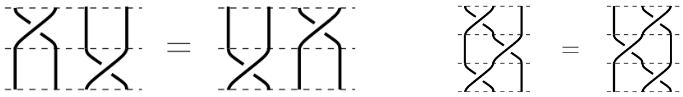
Henceforth, we will conflate braids with their isotopy classes, as the latter form well-behaved groups miraculously related to Hecke algebras. For any $n \geq 1$, let the *braid group on n strands* be defined by the following presentation from the zeroth lecture:

$$Br_n = \left\langle \sigma_1, \ldots, \sigma_{n-1} \;\middle|\; \begin{array}{l} \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}, \\ \sigma_i\sigma_j = \sigma_j\sigma_i \text{ for } |i-j| > 1 \end{array} \right\rangle. \tag{8.1}$$

The elements of Br_n correspond to actual braids: σ_i, also called the ith *simple twist*, is the following diagram:

The group law is bottom-to-top concatenation of diagrams, which is completely described by these relations:

[2]See, the pictures on these slides of Manturov for BIMSA: `https://bimsa.net/doc/notes/31803.pdf`



For any n, we view Br_n as a subgroup of Br_{n+1}, as suggested by our notation. This increasing sequence of groups allows us to make two crucial observations:

(1) Two braids on the same number of strands have the same annular link closures if and only if they are conjugate. That is, $\beta, \beta' \in Br_n$ satisfy $\beta^\circ = (\beta')^\circ$ if and only if $\beta' = \alpha\beta\alpha^{-1}$ for some $\alpha \in Br_n$.
(2) If β has n strands, then it has the same link closure (in $\mathbf{R}^3$) as $\beta\sigma_n$, a braid on n strands. That is, for all $\beta \in Br_n$, we have $\hat{\beta} = \widehat{\beta\sigma_n}$.

We say that β' is related to β by the *first*, *resp.* *second Markov move* if and only if $\beta' = \alpha\beta\alpha^{-1}$ for some α, *resp.* $\beta' = \beta\sigma_n$ with $\beta, \beta' \in Br_n$. The following theorem, proved in Markov's 1936 paper "Über die freie Äquivalenz der geschlossenen Zöpfe",[3] says that these moves are as strong as Reidemeister's:

Theorem 8.3 (Markov)**.** *For all $n, n' \geq 1$ and $\beta \in Br_n$ and $\beta' \in Br_{n'}$, we have $\hat{\beta} = \hat{\beta}'$ if and only if β, β' differ by a finite sequence of operations consisting of the two Markov moves.*

8.4.

As before, we note the close resemblance to the Coxeter presentation of the symmetric group on n letters:

$$S_n = \left\langle s_1, \ldots, s_{n-1} \;\middle|\; \begin{array}{l} s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, \\ s_i s_j = s_j s_i \text{ for } |i-j| > 1, \\ s_i^2 = e \end{array} \right\rangle,$$

Notably, the surjective group homomorphism $Br_n \to S_n$ that sends the simple twist σ_i to the simple reflection s_i *factors through the Iwahori–Hecke algebra* of S_n: More precisely, we have surjective ring homomorphisms

$$\mathbf{Z}[\mathsf{x}]Br_n \to H_{S_n}(\mathsf{x}) \to \mathbf{Z}S_n.$$

That is, we can rewrite the Hecke algebra as a quotient of $\mathbf{Z}[\mathsf{x}]Br_n$:

$$H_n = H_{S_n}(\mathsf{x}) := \frac{\mathbf{Z}[\mathsf{x}]Br_n}{\langle \sigma_i^2 - (\mathsf{x} - \mathsf{x}^{-1})\sigma_i - 1 \mid 1 \leq i \leq n-1 \rangle}$$

This raises the possibility of using the increasing sequence of algebras H_n to simplify the study of the increasing sequence of groups Br_n.

For instance, taking inspiration from Markov's theorem, we might try to construct a link invariant by defining a function f_n on Br_n for all n, such that:

[3] Available here: https://www.mathnet.ru/php/archive.phtml?wshow=paper&jrnid=sm&paperid=5359

(A) The f_n are class functions.
(B) We have $f_{n+1}(\beta\sigma_n) = f_n(\beta)$ for all $\beta \in Br_n$.

It turns out to be difficult to find a truly nontrivial yet computable family of such functions. When the f_n take values in a ring R, one option is to weaken (B) to:

(B′) We have $f_{n+1}(\beta\sigma_n) = cf_n(\beta)$ for all $\beta \in Br_n$, where $c \in R^\times$ is fixed.

Then one might try to make the f_n computable through induction on n, and correct for the repeated factors of c by multiplying the result by a further factor at the very end. This makes (B) easier, but not (A): f_n still needs to be some interesting class function on Br_n.

In general, traces of representations provide interesting class functions. The key is that instead of constructing representations of Br_n directly, we can obtain them by pullback from H_n, and we have reason to believe that the representation theory of H_n is simpler, being closer to that of S_n. Considerations like these led Ocneanu, building on work of Jones, to discover the following result.

Theorem 8.4 (Jones–Ocneanu). *There is a family of* $\mathbf{Z}[\mathsf{x}^{\pm 1}]$*-linear functions*

$$\mu_n : H_n \to \mathbf{Z}[\mathsf{x}^{\pm 1}, \tfrac{1}{\mathsf{x}-\mathsf{x}^{-1}}][\mathsf{a}^{\pm 1}]$$

uniquely determined for all $n \geq 1$ by these properties:

(1) $\mu_n(\alpha\beta) = \mu_n(\beta\alpha)$ *for all n and $\alpha, \beta \in Br_n$.*
(2) $\mu_{n+1}(\beta\sigma_n^{\pm 1}) = -\mathsf{a}^{\mp 1}\mu_n(\beta)$ *for all $\beta \in Br_n$.*
(3) $\mu_1(1) = 1$.

In short, the functions μ_n satisfy analogues of properties (A) and (B′), but are defined on the algebras H_n rather than the groups Br_n. In the literature, such family of functions is called a family of *Markov traces*. Up to normalization, the functions in the theorem are also known as the *Jones–Ocneanu traces*. Note that the quadratic Hecke relation and property (2) together imply that

$$\mu_{n+1}(\beta) = \frac{\mathsf{a} - \mathsf{a}^{-1}}{\mathsf{x} - \mathsf{x}^{-1}} \cdot \mu_n(\beta) \quad \text{for all } \beta \in Br_n$$

in our conventions.

Remark 8.5. (1) The correspondence $\sigma_n^{\pm 1} \leftrightarrow -\mathsf{a}^{\mp 1}$ is just a convention, but turns out to simplify formulas for positive braids later.
(2) Sometimes, people prefer to normalize the Jones–Ocneanu traces so that $\mu_{n+1}|_{H_n} = \mu_n$ instead. This is the convention that Jones uses in his 1987 Annals paper. Note too that our element σ_i corresponds to Jones's element $q^{-1/2}g_i$ under $\mathsf{x} \to q^{1/2}$, not to g_i itself.

We define the *writhe* of a braid β to be the sum $e(\beta)$ of the exponents in any word in the generators σ_i that represents β. This integer only depends on β, not the word; in fact, $e : Br_n \to \mathbf{Z}$ is a group homomorphism.

Corollary 8.6 (Ocneanu). *For any $n \geq 1$ and $\beta \in Br_n$, the Laurent polynomial*

$$\mathbf{P}(\hat{\beta}) = (-\mathsf{a})^{e(\beta)}\mu_n(\beta) \in \mathbf{Z}[\mathsf{x}^{\pm 1}, \tfrac{1}{\mathsf{x}-\mathsf{x}^{-1}}][\mathsf{a}^{\pm 1}]$$

is an isotopy invariant of the link closure $\hat{\beta}$, not just of the braid β.

Above, $\mathbf{P}(\hat{\beta})$ is called the *reduced HOMFLY-PT polynomial* of $\hat{\beta}$, after its discoverers. The "O" stands for Ocneanu. The adjective "reduced" means that $\mathbf{P}(\text{unknot}) = 1$. In some contexts, it is important to work with an *unreduced* version $\bar{\mathbf{P}}$ defined by

$$\bar{\mathbf{P}}(\hat{\beta}) = \frac{\mathsf{a}-\mathsf{a}^{-1}}{\mathsf{x}-\mathsf{x}^{-1}} \cdot \mathbf{P}(\hat{\beta}).$$

Remark 8.7. In the literature, HOMFLY-PT is often written in variables a and q. Our a is usually the same as a; our x is usually either $q^{1/2}$ or q.

8.5.

The relation between S_n, Br_n, and H_n generalizes to other finite Coxeter groups W. Namely, we can always define a group Br_W by an *Artin–Tits presentation* analogous to (8.1), such that there are surjective ring homomorphisms

$$\mathbf{Z}[\mathsf{x}]Br_W \to H_W(\mathsf{x}) \to \mathbf{Z}W.$$

The deep reason for this is a theorem of Brieskorn, matching the *Artin–Tits group* Br_W defined by generators and relations with the *fundamental group* of V^{reg}/W, where V is the reflection representation of W over $\mathbf{C}$, and $V^{\mathrm{reg}} \subseteq V$ is the locus where W acts freely.