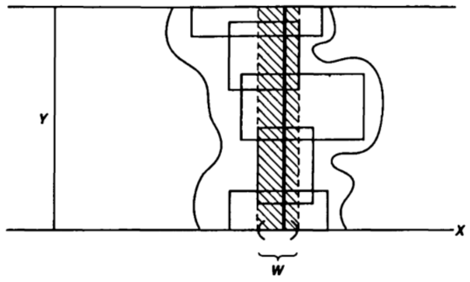
(Munkres §27) [notation change below: N]

Tube Lem fix a in X

for all open N sub X × Y s.t. {a} × Y sub N,
there is open W sub X s.t. {a} × Y sub W × Y
sub N



Pf recall: product top X × Y gen by U × V
for open U sub X and open V sub Y

so N is a union of such sets U × V:
say, {U_i × V_i}_i
then {a} × Y sub N = bigcup_i (U_i × V_i)
so by compactness, there is a finite set I s.t.
{a} × Y sub bigcup_{i in I} (U_i × V_i)

now set W = bigcap_{i in I} U_i
check:

- W is open in X, since I is finite
- a in W, since a in U_i for all i
- W × Y sub N, since…tricky

pick (x, y) in W × Y: want (x, y) in N
know (a, y) in U_i × V_i for some i in I [why?]
then (x, y) in W × V_i sub U_i × V_i sub N □

(Munkres §30, 36) second half of the course:

<u>topological manifolds</u>, esp. <u>2-dim'l</u> manifolds

idea: a space X "locally" homeo to R^n for given n

<u>Ex</u> the analytic circle S^1 sub R^2 is
not homeo to R,
but is a union of subspaces that are

<u>Ex</u> more generally, the unit n-sphere
$\{x \text{ in } R^{(n + 1)} \mid \sum_i x_i^2 = 1\}$

<u>Ex</u> nice graphs, no "weird" intersections:
$\{(x, y, z) \mid z = 1/(x^2 + y^2)\}$

<u>Ex</u> [0, 1] should <u>not</u> be a manifold per se
(maybe: a "manifold with boundary" …)
but {0, 1} <u>should</u> be;
even discrete sets like $\{1/n \mid n = 1, 2, \ldots\}$

<u>Rem</u> why "manifold"?

Riemann (June 10, 1854), trans. Clifford (1873),
"On the hypotheses which underlie geometry":

"Mannigfaltigkeit" to convey a "manifold-ness"
of directions, lengths, etc.

<u>"Df" 1</u> X is a manifold iff, for all x in X,
there is an open nbd U sub X of x s.t.
U is homeo to R^n for some $n \geq 0$

Objection 1 take the line with two origins:

$A = (R \times \{a, b\})/\sim$, where $(x, a) \sim (x, b)$ for $x \neq 0$

each "0" in A has a nbd homeo to (–1, 1),
hence to R [just exclude the other "0"]

want convergent seq's in manifolds to have unique limits

let's say that X is locally Euclidean iff X satisfies the "local" condition in Df 1

"Df" 2 a manifold is a Hausdorff, locally Euclid. topological space

Objection 2 locally R^n can still be globally blah

1) take X an uncountable discrete set
each x has open nbd {x} homeo to R^0
but |X| can be unimaginably large

2) the "long line":

suppose S is a totally ordered set
there is a lexicographic total order on $S \times [0, 1)$:
$(s, x) < (s, y)$ for arbitrary s and $x < y$
$(s, x) < (t, y)$ for $s < t$ and arbitrary x, y

we define the S-line to be

$(S \times [0, 1))$ – {any lex'ically minimum elts} we give it the <u>order</u> topology gen by intervals:

$\{(s, z) \mid x < z < y\}$ for arbitrary s and $x < y$

$\{(u, z) \mid s \leq u \leq t,$ for $s < t$ and arbitrary x, y

$x < z$ if $s = u$,

$z < y$ if $u = t\}$

<u>Thm</u> (Hartogs) there is a minimal uncountable <u>well-ordered</u> set (ω_1, <)

i.e. ω_1 is an uncountable set, < a total order, s.t.

any strict subset of ω_1 is countable

any nonempty subset of ω_1 has a least elt

[apparently does not require the axiom of choice]

the ω_1-line is VERY LONG

<u>Thm</u> the ω_1-line is locally homeo to R, and path-connected, but not homeo to R

<u>Pf</u> see Munkres 158–159, #12

[so here is a way to rule out the long line:]

<u>Df</u> we say X is second-countable iff the top. of X is gen by a countable basis

<u>Rem</u> there's also a notion of first-countable, but Hausdorff + locally Euclid. implies it

<u>Df</u> a manifold is a Hausdorff, second-countable, locally Euclid. space

Rem if X is second-countable
then {conn. comp's of X} is countable
ruling out uncountable discrete sets

but second-countability is stronger:
the ω_1-line is not 2nd-countable
yet has only one conn. comp.

[we mainly picture manifolds as locally Euclidean subsets of R^N for some large N:]

Df an embedding is a cts injective map

Thm if a manifold is compact, then it admits an embedding into R^N for some $N > 0$

Pf Munkres §36

in gen'l, a manifold embedded in R^N is locally homeo to R^n for various $n \leq N$
[that is, n is usually smaller than N]

we say X is an n-manifold iff it is covered by open nbds all homeo to R^n for the same n

in this case, we say n is the dimension of X

Q why not define
manifolds via compact manifolds,
compact manifolds via embedding?

Intrinsic vs Extrinsic

goal of Riemann, Poincaré, Weyl, etc.:
study manifolds without fixing a larger "container"

the universe should be a 4-manifold
(with extra metric structure)

but we don't want to presume it's contained in
some larger R^N that we can't detect

[how to prove the embedding thm?
uses a notion called "partitions of unity"]

[we may or may not get to it]