Review recall F_X = free group on X

given a group G:

- S sub G is a generating set
  iff no smaller subgroup of G contains S
  iff the homomorphism F_S to G is surjective

in this case:

- R sub F_S is a set of relations wrt S
  iff ker(F_S to G) is the smallest kernel, i.e.,
  normal subgp of F_S, containing R

then we can speak of a presentation of G
by generators and relations: $G = \langle S \mid R \rangle$
if $R = \varnothing$, then $G = F_S$ and we write $G = \langle S \rangle$

Rem any G has an "obvious" gen'ting set S:
[pause: what is it?]
take $S = G$ itself
[usually we prefer to study smaller S]

Ex take $G = Z$
what is a one-elt gen'ting set? [pause]
$S = \{1\}$ works [but also another:]
$S = \{-1\}$ also works

what is a two-elt gen'ting set without ± 1? [pause]
[e.g.] $S = \{2, 3\}$

Ex last time, saw that if $G = \{e, s\}$
then $G = \langle s \mid s^2 \rangle$
[abusing notation: s should be $\{s\}$, etc.]

Ex let $G = \mathbb{Z}^2$ [under coordinate-wise +]
what is a generating set? [pause]
$S = \{(1, 0), (0, 1)\}$ works

write $a = (1, 0)$ and $b = (0, 1)$

what is $\ker(F_S \to \mathbb{Z}^2)$? [pause]

elts of F_S are words in a, b, a^{-1}, b^{-1}
if such a word contains
- M a's,
- N b's,
- M' a^{-1}'s,
- N' b^{-1}'s

then it is mapped to $(M - M', N - N')$ in $\mathbb{Z}^2$, so

$\ker(F_S \to \mathbb{Z}^2)$ = {words where the net exponent of a & the net exponent of b are both zero}

e.g., for any w, v in F_S, it contains
the commutator $[w, v] := wvw^{-1}v^{-1}$
[here w^{-1} means the group inverse to w]

Fact ([follows from] Munkres 69.3–69.4)
- $\{[w, v] \mid w, v \text{ in } F_S\}$ is a generating set for $\ker(F_S \to \mathbb{Z}^2)$
- the kernel is the smallest normal subgp containing $[a, b]$

[defer proof for now]

altogether, get the presentation

$\mathbb{Z}^2 = \langle a = (1, 0),\ b = (0, 1) \mid aba^{-1}b^{-1} \rangle$

<u>Free Products</u> [goal: Seifert–van Kampen:]
given groups
$G_1 = \langle S_1 \mid R_1 \rangle$,
$G_2 = \langle S_2 \mid R_2 \rangle$:

<u>Df 1</u> the free product of G_1 and G_2 is

$$G_1 * G_2 = \langle S_1 \cup S_2 \mid R_1 \cup R_2 \rangle$$

<u>Problem</u> a priori, $G_1 * G_2$ could depend on how we present G_1 and G_2
[to solve this issue, new defn:]

<u>Df 2</u> a free product of G_1, G_2 is a group G with maps $\iota_1 : G_1 \to G$, $\iota_2 : G_2 \to G$
s.t., for any group K, we have a bijection

$$\{\text{pairs of hom's } \varphi_1 : G_1 \to K,\ \varphi_2 : G_2 \to K\}$$
$$=$$
$$\{\text{hom's } \Phi : G \to K\}$$

given explicitly by $\varphi_1 = \Phi \circ \iota_1$ and $\varphi_2 = \Phi \circ \iota_2$

<u>Thm</u> the free product in definition #2 is unique up to iso [in fact, "unique iso"]

<u>Pf</u> suppose (G, ι_1, ι_2), (G', ι'_1, ι'_2) both work

taking $\varphi_k = \iota'_k$ above gives a hom $\Phi : G$ to G'
s.t. $\iota'_k = \Phi \circ \iota_k$
taking $\varphi_k = \iota_k$ above gives a hom $\Phi' : G'$ to G
s.t. $\iota_k = \Phi' \circ \iota'_k$

substituting, $\iota_k = \Phi' \circ \Phi \circ \iota_k$
so under the defining bijection for G,
id_G and $\Phi' \circ \Phi$ both correspond to (ι_1, ι_2)
[pause: what next?] so $id_G = \Phi' \circ \Phi$
similarly, $id_{G'} = \Phi \circ \Phi'$
so Φ and Φ' are each other's two-sided inverses □

[thm + proof illustrate "category-theoretic" ideas]

<u>Lem</u> $G_1 * G_2$ in defn #1 satisfies defn #2

<u>Pf</u> left as exercise

<u>Ex</u> the free group F_2 is isomorphic to $Z * Z$

more generally, $*$ is associative:
F_n is isomorphic to $Z * Z * \ldots * Z$ with n copies

<u>Ex</u> let $G = \{e, s\}$, the two-elt group
how to write down elts of $G * G$? [pause]

need to distinguish two copies of s: say, "s" and "t"

$G * G = \{e, s, t, st, ts, sts, tst, \ldots\}$

(Munkres §70) [but slightly changed notation]

Thm (Seifert–van Kampen) take open inclusions
j_1 : U_1 to X,
j_2 : U_2 to X

s.t. X = U_1 cup U_2,
U_1 and U_2 are path connected,
U := U_1 cap U_2 is path-connected

let i_1 : U to U_1 and i_2 : U to U_2 be inclusion
then for any x in U:

1) the homomorphism
π_1(U_1, x) ∗ π_1(U_2, x) to π_1(X, x)
arising from (j_{1,*}, j_{2,*}) via the defn
of free product is surjective

2) the kernel of the homomorphism is
the smallest normal subgp of the domain
containing the elts of the form
i_{1,*}([γ])^{-1} i_{2,*}([γ])
as we run over elts [γ] in π_1(U, x)
[above, i_{k,*}([γ]) in π_1(U_k, x), but then
we implicitly embed it into the free product]

Cor π_1(X, x) is generated by the union of
π_1(U_1, x) and π_1(U_2, x)

Cor if there are open U_1, U_2 sub X s.t.
U_1, U_2 are simply-connected,
X = U_1 cap U_2,
U_1 cap U_2 is path-connected,
then X is simply-connected

Ex take a figure-eight:

[draw]

take open U_1, U_2 s.t.
they deformation retract onto the two loops
$U_1 \cap U_2$ def. retracts onto the middle pt

[draw]

then $\pi_1(U_1, x) = \pi_1(U_2, x) = \pi_1(S^1) = \mathbb{Z}$
but $\pi_1(U_1 \cap U_2, x)$ is trivial

so $\pi_1(\text{figure-eight}, x) = \mathbb{Z} * \mathbb{Z} = F_2$