



Knots, Plethysms, and the Riordan Group

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1 Fruit

“You can’t add together apples and oranges.”

Well, not in real life.

But in mathematics, you can make up a new world where this is possible.

The *free vector space* on $X = \{\text{apple}, \text{orange}, \text{pear}\}$:

$$\mathbf{C}\langle X \rangle = \{a \cdot \text{apple} + b \cdot \text{orange} + c \cdot \text{pear} \mid a, b, c \in \mathbf{C}\}.$$

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Maybe dumb, because the sum of the vectors “apple” and “orange” is just “apple + orange”.

But there’s a vector space where it simplifies further.

(1) Start with some relations like

$$\text{pear} \sim \text{apple} + \text{orange}, \quad \text{orange} \sim 2 \cdot \text{apple}.$$

(2) Let Rel be the span of “ $\text{pear} - \text{apple} - \text{orange}$ ” and “ $\text{orange} - 2 \cdot \text{apple}$ ”.

(3) Extend \sim to an equivalence relation on $\mathbf{C}\langle X \rangle$:

$$v \sim v' \iff v - v' \in Rel.$$

The set of equivalence classes is a new vector space $\mathbf{C}\langle X \rangle / Rel$, in which \sim defines equality.

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2 Knots and Links I'm interested in *knots* and *links*.

Knot diagrams:



Links allow multiple circles:



An *oriented link diagram* is a link diagram where we give each circle a direction/orientation.

Also interested in diagrams that are strictly inside a given region $\Omega \subseteq \mathbf{R}^2$.

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We will treat two diagrams in Ω as equal as long as they are *isotopic*:

We can deform one into the other within Ω , without tearing any circles.

Let \mathcal{L}_Ω be the set of all oriented link diagrams in Ω , including the empty diagram.

$$\mathbf{C}\langle \mathcal{L}_\Omega \rangle = \{\text{finite linear combos of elements of } \mathcal{L}_\Omega\}$$

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To study it, try to find equivalence relations on it that give us smaller, more tractable vector spaces.

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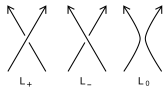
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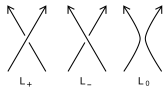
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Fix constants $a \neq 0$ and $q \neq 0, 1$.

It turns out that the following local *skein relations* are especially interesting.

$$\begin{aligned} \text{(crossing)} - \text{(crossing)} &= (q - q^{-1}) \text{(parallel strands)} \\ \text{(circle)} &= \frac{a - a^{-1}}{q - q^{-1}} \text{(empty disk)} , \quad \text{(loop)} = -a^{-1} \text{(vertical strand)} \end{aligned}$$

When $\Omega \neq \mathbf{R}^2$, we will be a bit more restrictive:

We will only apply the relations when the drawings take place inside an open disk inside Ω .

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The relations give us a linear subspace $\text{Rel}_\Omega \subseteq \mathbf{C}\langle \mathcal{L}_\Omega \rangle$.

The *HOMFLYPT skein module* of Ω is

$$\text{Sk}_\Omega = \mathbf{C}\langle \mathcal{L}_\Omega \rangle / \text{Rel}_\Omega.$$

Thm (\approx HOMFLYPT 1986)

$\text{Sk}_{\mathbf{R}^2} = \mathbf{C}.$

That is, any diagram in \mathbf{R}^2 is a scalar multiple of the empty diagram modulo the skein relations.

The *HOMFLYPT invariant* of a link is the scalar that we get from any diagram of the link.

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$$\begin{aligned} \text{Crossing} &- \text{Crossing} = (q - q^{-1}) \text{Two Vertical Lines} \\ \text{Circle} &= \frac{a - a^{-1}}{q - q^{-1}} \text{Empty Circle}, \quad \text{Loop} = -a^{-1} \text{Three Lines} \end{aligned}$$

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Example Consider the following element in $\mathbf{C}\langle \mathcal{L}_{\mathbf{R}^2} \rangle$:



Modulo the “crossing” rule,

$$L = \text{(red circle with arrow)} \text{(blue circle with arrow)} + (q - q^{-1}) \text{(empty circle with arrow)}$$

Modulo $\bigcirc = \frac{a-a^{-1}}{q-q^{-1}} \cdot \emptyset$,

$$L = \left(\frac{a - a^{-1}}{q - q^{-1}} \right)^2 \cdot \emptyset + a - a^{-1} \cdot \emptyset.$$

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Example Consider the following element in $\mathbf{C}\langle\mathcal{L}_{\mathbf{R}^2}\rangle$:

$$L = \text{Diagram of two overlapping circles, one red and one blue, with arrows indicating a crossing.}$$

Modulo the “crossing” rule,

$$L = \text{Diagram of two separate circles, one red and one blue, with arrows} + (q - q^{-1}) \text{Diagram of a single circle with an arrow.}$$

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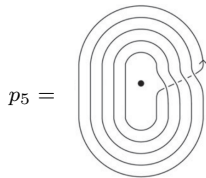
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For $\Omega = \mathbf{R}^2 \setminus \mathbf{0}$, what happens?

Cannot simplify circles in $\mathbf{R}^2 \setminus \mathbf{0}$ that go around $\mathbf{0}$.

In fact: pairwise distinct diagrams p_n for all $n \in \mathbf{Z}$.



($n > 0$ is counterclockwise, $n < 0$ clockwise.)

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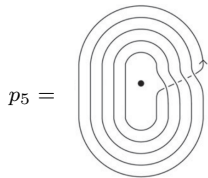
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If we have two diagrams L and L' , then we can put L around L' to get a new diagram

$$L \cdot L'.$$

Note that $L \cdot L'$ and $L' \cdot L$ are isotopic.

Extend this to a binary operation on $\text{Sk}_{\mathbf{R}^2 \setminus \mathbf{0}}$, by making it distribute over addition.

Think of this as a multiplication law, which turns $\text{Sk}_{\mathbf{R}^2 \setminus \mathbf{0}}$ into a *ring*.

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The collection of all monomials in the p_n 's is a basis for $\text{Sk}_{\mathbf{R}^2 \setminus \mathbf{0}}$ as a vector space.

Corollary As a *ring*,

$$\boxed{\text{Sk}_{\mathbf{R}^2 \setminus \mathbf{0}} = \mathbf{C}[p_0, p_{\pm 1}, p_{\pm 2}, \dots].}$$

Remark

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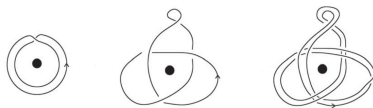
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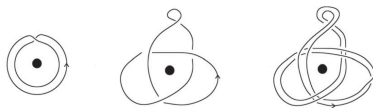
The first diagram above is p_2 . Call the middle one L .

The last diagram is the *plethysm* $L \circ p_2$.

If L had multiple knot components, then we would form $L \circ p_2$ by inserting p_2 into each component, following their orientations.

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It is fun to check that:

$$(1) \quad p_m \circ p_n = p_{mn} \text{ for any } m, n.$$

How to define $L \circ K$ for any K and L ?

Since any element of $\text{Sk}(\mathbf{R}^2 \setminus \mathbf{0})$ is a polynomial in the p_n 's, it is enough to require:

$$(2) \quad - \circ K \text{ distributes over } + \text{ and } \cdot, \text{ for all } K.$$

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Thm (1)–(3) define a binary operation on $\text{Sk}_{\mathbf{R}^2 \setminus \mathbf{0}}$.

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By comparison, the composition operation

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4 Riordan Revisited

In combinatorics, we like to study number sequences c_0, c_1, c_2, \dots through *generating functions*

$$c(t) = c_0 + c_1 t + c_2 t^2 + \dots,$$

a.k.a. *formal power series*. They form a ring $\mathbf{C}[[t]]$.

The word “formal” means we don’t worry about whether $c(t)$ converges at any given value of t .

Any polynomial is a power series:

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Example Take $c(t) = 1 + t + t^2 + \dots$

$c(1+t)$ is not well-defined. By contrast,

$$\begin{aligned} c(t+t^2) &= 1 + (t+t^2) + (t+t^2)^2 + (t+t^2)^3 + \dots \\ &= \begin{cases} 1 \\ + t + t^2 \\ + t^2 + 2t^3 + t^4 \\ + t^3 + 3t^4 + \dots \\ + t^4 + \dots \end{cases} \\ &= 1 + t + 2t^2 + 3t^3 + 5t^4 + \dots \end{aligned}$$

In general, can do $\circ : \mathbf{C}[[t]] \times t\mathbf{C}[[t]] \rightarrow \mathbf{C}[[t]]$, where

$$t\mathbf{C}[[t]] = \{c_1 t + c_2 t^2 + \dots\}$$

is the subset of power series with zero constant term.

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$$\begin{aligned} c(t + t^2) &= 1 + (t + t^2) + (t + t^2)^2 + (t + t^2)^3 + \dots \\ &= \begin{cases} 1 \\ + t + t^2 \\ + t^2 + 2t^3 + t^4 \\ + t^3 + 3t^4 + \dots \\ + t^4 + \dots \end{cases} \\ &= 1 + t + 2t^2 + 3t^3 + 5t^4 + \dots \end{aligned}$$

In general, can do $\circ : \mathbf{C}[[t]] \times t\mathbf{C}[[t]] \rightarrow \mathbf{C}[[t]]$, where

$$t\mathbf{C}[[t]] = \{c_1 t + c_2 t^2 + \dots\}$$

is the subset of power series with zero constant term.

Let $\mathbf{C}[[t]]^\circ$ be the further subset of power series with zero constant term and nonzero linear term.

Thm Any element of $\mathbf{C}[[t]]^\circ$ has an inverse under \circ .
Thus $\mathbf{C}[[t]]^\circ$ forms a group under \circ with identity t .

If you think about what I've covered, you'll realize:

There is an analogous group where we replace

$$\mathbf{C}[[t]] \supseteq \mathbf{C}[t]$$

with a certain containment

$$\widehat{\text{Sk}}_{\mathbf{R}^2 \setminus \mathbf{0}} \supseteq \text{Sk}_{\mathbf{R}^2 \setminus \mathbf{0}}$$

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Let me sketch a proof of the theorem relating it to the Riordan group.

Pf sketch For any $f \in \mathbf{C}[[t]]^\circ$, let M_f be the infinite matrix whose columns record the powers of f :

$$M_f = \begin{pmatrix} 1 & & & \\ 0 & c_{1,1} & & \\ 0 & c_{2,1} & c_{2,2} & \\ \vdots & \vdots & & \ddots \end{pmatrix},$$

where the $c_{i,j}$ are given by $f(t)^j = \sum_{i \geq 0} c_{i,j} t^i$.

For example, $M_z = I$, the identity matrix.

In general, we can recover f from M_f by looking at the second column.

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The condition $f \in \mathbf{C}[[t]]^\circ$ implies that M_f is always *lower-triangular* with nonzero diagonal entries.

Thus it is *invertible*.

Now check the following facts:

- (1) M_f^{-1} takes the form M_g for some $g \in \mathbf{C}[[t]]^\circ$.
- (2) $M_{f_1 \circ f_2} = M_{f_1} \cdot M_{f_2}$.

We deduce that for any f , there's some g s.t.

$$M_{g \circ f} = M_g \cdot M_f = M_f^{-1} \cdot M_f = I = M_z.$$

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Recall that the set $\mathbf{C}[[t]]^\times$ of power series with *nonzero* constant term forms a group under \times .

The map $f \mapsto M_f$ can be extended to an embedding

$$\begin{aligned} \mathbf{C}[[t]]^\times \rtimes \mathbf{C}[[t]]^\circ &\hookrightarrow \mathbf{GL}_\infty, \\ (u, f) &\mapsto M_{u,f}. \end{aligned}$$

Shapiro's *Riordan group* is the image.

This also has an analogue in the world of plethysm.

Thank you for listening.