Warmup last time we proved:

Thm if $F = C$ and V is fin. dim. then any linear op on V has an upper-triangular matrix

Q can we remove the hypothesis $F = C$?

no: upper-triangular matrix implies existence of eigenvector

we saw that rotations in R^2 have no eigenvector

Q can we remove the hypothesis V f.d.?

no: same issue of eigenvectors [recall F[x]]

suppose T has upper-triangular matrix M

$T = T' + T''$ T' diagonalizable, T'' nilpotent

corresponds to

$M = D + N$, D diagonal, N upper-triangular with zero diag.

let $\lambda_1, \ldots, \lambda_k$ be the diagonal entries of D, in order, without including repeated entries

they are eigenvals of T'

[are they eigenvals of T? need $\ker(T - \lambda)$ nonzero]

know λ_1 is an eigenval of T; the others, unclear

Ex $\lambda \neq \mu$

$$M1 = \begin{pmatrix} \lambda & x & y \\ & \mu & z \\ & & \mu \end{pmatrix} \qquad M2 = \begin{pmatrix} \lambda & x & y \\ & \lambda & z \\ & & \mu \end{pmatrix}$$

$\ker(M1 - \mu)$, $\ker(M2 - \mu) \neq \{\mathbf{0}\}$?

$$\begin{pmatrix} \lambda-\mu & x & y \\ & 0 & z \\ & & 0 \end{pmatrix} \begin{pmatrix} * \\ * \\ * \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{e.g.} \begin{pmatrix} x \\ -(\lambda - \mu) \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} \lambda-\mu & x & y \\ & \lambda-\mu & z \\ & & 0 \end{pmatrix} \begin{pmatrix} * \\ * \\ * \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{e.g.} \begin{pmatrix} -xz/(\ ) + y \\ z \\ -(\lambda - \mu) \end{pmatrix}$$

we will prove:

Thm if T has an upper-triangular matrix M then every diagonal entry of M is an eigenval of T

and $\dim \ker(T - \lambda) \leq$ # of times λ occurs a.k.a. multiplicity of λ

Ex multiplicity of μ is 2 in both cases below:

$$\begin{pmatrix} \lambda & & \\ & \mu & \\ & & \mu \end{pmatrix} \qquad \begin{pmatrix} \lambda & & \\ & \mu & 1 \\ & & \mu \end{pmatrix}$$

eigenspace: $\{(0, y, z)\}$ $\{(0, y, 0)\}$

to “fix” discrepancy, weaken notion of eigenspace

(Axler §8A–8B) let $T : V \to V$ be arbitrary

Df the generalized λ-eigenspace for T is

$\{v \in V \mid (T - \lambda)^n v = \mathbf{0} \text{ for some } n\}$
$= \bigcup_{n > 0} \ker((T - \lambda)^n)$

its elts are called <u>generalized λ-eigenvectors</u> for T

Lem the gen’lized eigenspace is indeed a linear subspace of V

Pf $\ker(T - \lambda) \subseteq \ker((T - \lambda)^2) \subseteq \ldots$
so $\bigcup_{n > 0} (\ ) = \sum_{n > 0} (\ )$

Lem the generalized λ-eigenspace for T is the largest T-stable lin. sub. $W \subseteq V$ s.t.
$(T - \lambda)|_W$ is nilpotent

Pf $\ker((T - \lambda)^n)$ is T-stable for all n
by stability lem
so $\bigcup_{n > 0} \ker((T - \lambda)^n)$ is stable
and nilpotence condition holds for it

conversely: easy to show that
if W is T-stable and $(T - \lambda)|_W$ is nilpotent,
then $W \subseteq \ker((T - \lambda)^n)$

[i.e., if W is the gen’lized λ-eigenspace,
then $T|_W$ has an upper-triangular matrix
with only λ’s on the diagonal]

even when V is not a sum of eigenspaces for T, we still have:

<u>Thm</u> if $F = \mathbb{C}$ and V is fin. dim.
then V is a direct sum of
gen'lized eigenspaces for T:

there exist a finite list of (pairwise distinct) λ_i s.t.

$$V = \sum_i W_i$$

where W_i is the generalized λ_i-eigenspace for T and this sum is a direct sum

[slightly stronger than triangularity of T]

<u>Pf</u> again, want to induct on dim V

[recall proof of triangularity:] pick an eigenval λ
then $\dim \ker(T - \lambda) > 0$
so $\dim \operatorname{im}(T - \lambda) < \dim V$
want: to replace ker with gen'lized eigensp.
direct-sum structure

solution: since dim V finite, can pick k s.t.

$$\ker((T - \lambda)^k) = \bigcup_{n > 0} \ker((T - \lambda)^n)$$

i.e. $\ker((T - \lambda)^k) = \ker((T - \lambda)^{k + 1}) = \ldots$

<u>Lem</u> for such k,
1) $\ker((T - \lambda)^k) \cap \operatorname{im}((T - \lambda)^k) = \{\mathbf{0}\}$
2) $\ker((T - \lambda)^k)$, $\operatorname{im}((T - \lambda)^k)$ are T-stable

let $W = \mathrm{im}((T - \lambda)^k)$
if 2) holds, then can apply inductive hypothesis
to $T|_W$, a linear op on W
so W is a direct sum of gen'lized eigensp.'s
if 1) holds, then $V = \ker((T - \lambda)^k) + \mathrm{im}((T - \lambda)^k)$
by dim formula (PS3)
and the sum is a direct sum
remains to check λ is not an eigenval of $T|_W$:
follows from lem about maximality □

<u>Pf of Lem</u>

1) pick w in $\ker((T - \lambda)^k)$ cap $\mathrm{im}((T - \lambda)^k)$
then $w = (T - \lambda)^k v$ for some v in V
but $(T - \lambda)^k w = \mathbf{0}$

altogether $\mathbf{0} = (T - \lambda)^k w$
$= (T - \lambda)^{2k} v$
$= (T - \lambda)^k v$ by maximality of k
$= w$

2) $\mathrm{im}((T - \lambda)^k)$ is T-stable by stability lem

suppose v in $\ker((T - \lambda)^k)$
then $(T - \lambda)^k(T v) = T((T - \lambda)^k) v) = \mathbf{0}$
so Tv in $\ker((T - \lambda)^k)$

that is: the stability lem has an analogue for ker's

next time:
analyze structure of T on each gen'lized eigensp.