Last time how many ways to show

R and S^1 are not homeomorphic?

both Hausdorff
both connected
both path-connected

R – {t} is disconnected for any t in R [why?]

Thm S^1 – {q} is connected for any q in S^1

Pf by symmetry, can assume q = (1, 0)

p(t) = (cos 2πt, sin 2πt) is a homeo
from (0, 1) onto S^1 – {q}

R is not compact [why?]

Thm S^1 is compact

Pf [0, 1] is compact by Heine–Borel

p(t) = (cos 2πt, sin 2πt) is a surjective cts map
from [0, 1] onto S^1

cts images of compact spaces are compact

(Munkres §51) even more ways to show
R and S^1 not homeomorphic?

Df suppose f, g : X to Y are cts maps

a <u>homotopy</u> from f to g is a cts map

$$\varphi : X \times [0, 1] \text{ to } Y$$

s.t. for all x in X, we have

$$\varphi(x, 0) = f(x) \text{ and } \varphi(x, 1) = g(x)$$

[a cts "movie" in time, showing f at time $t = 0$ and g at time $t = 1$]

<u>Ex</u> let $f, g : R$ to R be def by

$$f(x) = x \text{ and } g(x) = -x$$

then $\varphi(x, t) = (1 - 2t)x$ is a homotopy from f to g

[draw]

<u>Ex</u> if $f = g$, then there is a homotopy def by $\varphi(x, t) = f(x)$ for all t

<u>Ex</u> if $\varphi : X \times [0, 1]$ to Y is a homotopy, then there is a homotopy $\psi : X \times [0, 1]$ to Y def by $\psi(x, t) = \varphi(x, 1 - t)$

<u>XC</u> suppose $f_1, f_2, f_3 : X$ to Y are all cts

if there are homotopies from f_1 to f_2,
from f_2 to f_3,

then there is a homotopy from f_1 to f_3

Df f, g are homotopic iff
there is some homotopy from f to g

in this case we write f ~ g

Df for any spaces X, Y, we write

Maps(X, Y) = {cts maps from X to Y}

we've proved that ~ is an equivalence relation on Maps(X, Y), so it makes sense to write

[X, Y] = Maps(X, Y)/~

Thm any two cts maps from R to R
are homotopic: i.e., [R, R] is a singleton

Pf 1 pick cts f, g : R to R

then $\varphi(x, t) = (1 - t)f(x) + tg(x)$ is a homotopy

Pf 2 since ~ is an equivalence relation,

enough to show that any f : R to R is
nulhomotopic = homotopic to some constant map

$\varphi(x, t) = (1 - t)f(x) + s$ works, for any s in R

Df a [nonempty] space is contractible iff
its identity map is nulhomotopic

we've shown that R is contractible

XC let $f, g : X$ to Y and $F, G : Y$ to Z be cts

if $f \sim g$ and $F \sim G$, then $F \circ f \sim G \circ g$

as corollaries, $F \circ f \sim F \circ g$ and $F \circ f \sim G \circ f$

XC use the preceding problem to show:

if X, Y are nonempty and Y is contractible,
then any two cts maps from X into Y are
homotopic: i.e., $[X, Y]$ is a singleton

Thm $[S^1, S^1]$ is not a singleton!