MATH 250: TOPOLOGY I MIDTERM

FALL 2025

You have till 2:00 pm to do these problems in any order. You do not need to write in complete sentences, but please show your work. Throughout the exam, **$\mathbf{R}$ has the analytic topology unless otherwise specified.**

You are allowed to look at the textbook (Munkres, *Topology*, 2nd Ed.) and any notes on paper that you wrote prior to the exam. **You may omit proofs for any claims proved in the textbook.** You are not allowed to use electronic devices of any kind, such as phones, computers, tablets, or audio devices. If you need to use the bathroom, please give any phones or other electronics to the proctor first.

Problem 1 (3 points)**.** Let d and e be metrics on a set X. Fix $x \in X$ and $\delta > 0$. Show that

$$\text{if } d(x,y) \leq 2025\, e(x,y) \text{ for all } y \in X, \quad \text{then } B_e(x,\delta) \subseteq B_d(x, 2025\,\delta).$$

Solution. Pick $y \in B_e(x,\delta)$. Then $e(x,y) < \delta$. Then

$$d(x,y) \leq 2025\, e(x,y) < 2025\,\delta.$$

So $d(x,y) < 2025\,\delta$, meaning $y \in B_d(x, 2025\,\delta)$. □

Problem 2 (5 points)**.** Let $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ respectively denote sequences of points in spaces X and Y. Show that if $a_n \to a$ and $b_n \to b$, where $\to$ means convergence, then $(a_n, b_n) \to (a,b)$ in the product topology on $X \times Y$.

Solution. We must show that for any open $U \subseteq X \times Y$ containing (a,b), there is some N such that $(a_n, b_n) \in U$ for all $n \geq N$.

By the definition of the product topology, there exist some open $V_1 \subseteq X$ and $V_2 \subseteq Y$ such that $(a,b) \subseteq V_1 \times V_2 \subseteq U$. Then $a \in V_1$ and $b \in V_2$. Since $a_n \to a$ in X, we can find M_1 such that $a_n \in V_1$ for all $n \geq M_1$. Similarly, since $b_n \to b$ in Y, we can find M_2 such that $b_n \in V_2$ for all $n \geq M_2$.

Let $N = \max\{M_1, M_2\}$. Then $(a_n, b_n) \in V_1 \times V_2$ for all $n \geq N$. Therefore, $(a_n, b_n) \in U$ for all $n \geq N$, as needed. □

Problem 3 (5 points)**.** Let $\mathcal{B}$ be the collection of rays of the form $(\alpha, \infty) \subseteq \mathbf{R}$, for $\alpha \in \mathbf{R}$. Show that:

(1) $\mathcal{B}$ does not generate the analytic topology on $\mathbf{R}$.
(2) $\mathcal{B}$ is still a basis (for some topology).

Solution. (1) We know that the interval $(-1, 1)$ is open in the analytic topology. We also see that $0 \in (-1, 1)$. We claim that there is no element $B \in \mathcal{B}$ such that $0 \in B \subseteq (-1, 1)$. By Munkres Lemma 13.2, this will prove that $\mathcal{B}$ does not generate the analytic topology.

Indeed, $B = (\alpha, \infty)$ for some $\alpha \in \mathbf{R}$. Regardless of α, we deduce that B contains numbers larger than 1, so $B \not\subseteq (-1, 1)$.

(2) We must check the axioms. First, $\mathbf{R} = \bigcup_{\alpha \in \mathbf{R}} (\alpha, \infty)$, because any number $\beta \in \mathbf{R}$ belongs to $(\beta - 1, \infty)$. Next, suppose that we are given two elements of $\mathcal{B}$, say (α, ∞) and (α', ∞). Then

$$(\alpha, \infty) \cap (\alpha', \infty) = (\max(\alpha, \alpha'), \infty).$$

So it is certainly true that for all $x \in (\alpha, \infty) \cap (\alpha', \infty)$, we can find $B \in \mathcal{B}$ such that $x \in B \subseteq (\alpha, \infty) \cap (\alpha', \infty)$: The intersection itself is our B. $\square$

Problem 4 (5 points)**.**

(1) Show that in the product topology, $\mathbf{R}^\omega$ is path-connected. *Hint:* Munkres Theorem 19.6.
(2) Name (without proof) two elements $x, y \in \mathbf{R}^\omega$ such that there is no path from x to y in the box topology on $\mathbf{R}^\omega$.

In (1), you may assume that linear maps from $[0, 1]$ to $\mathbf{R}$ are continuous.

Solution. (1) We must show that for all $x, y \in \mathbf{R}^\omega$, there's a path from x to y in the product topology. That is, we must give a map $\gamma : [0, 1] \to \mathbf{R}^\omega$ that is continuous for this topology and satisfies $\gamma(0) = x$ and $\gamma(1) = y$.

Write $x = (x_n)_n$ and $y = (y_n)_n$. We choose the map

$$\gamma(t) = ((1 - t)x_n + ty_n)_n.$$

We immediately get $\gamma(0) = (x_n)_n$ and $\gamma(1) = (y_n)_n$. Moreover, for all n, the map $\mathrm{pr}_n \circ \gamma : [0, 1] \to \mathbf{R}$ is continuous because it's a linear function of t: namely, $(\mathrm{pr}_n \circ \gamma)(t) = (1 - t)x_n + ty_n = (y_n - x_n)t + x_n$. By Munkres Theorem 19.6, we deduce that γ is continuous for the product topology on $\mathbf{R}^\omega$.

(2) Just take x bounded, say, $x = (0, 0, 0, \ldots)$, and y unbounded, say, $y = (1, 2, 3, \ldots)$. $\square$

Problem 5 (4 points)**.** Let $p : \mathbf{R} \to S^1$ be defined by $p(t) = (\cos(t), \sin(t))$, and let

$$U = \{(x, y) \in S^1 \mid x > 0\}.$$

(1) Express $p^{-1}(U)$ in terms of intervals in $\mathbf{R}$.
(2) Is $p^{-1}(U)$ connected? (Yes/No)
(3) Is U open in the quotient topology induced by p? (Yes/No)

Solution. (1) Explicitly, $p^{-1}(U)$ is the set of $t \in \mathbf{R}$ such that $\cos(t) > 0$. We have $\cos(t) > 0$ if and only if $t \in (-\frac{\pi}{2}, \frac{\pi}{2}) + 2\pi\mathbf{Z}$. That is,

$$p^{-1}(U) = \bigcup_{n \in \mathbf{Z}} (2\pi n - \tfrac{\pi}{2}, 2\pi n + \tfrac{\pi}{2}).$$

(2) No. [For instance, $\{t \in p^{-1}(U) \mid t < \pi\}$ and $\{t \in p^{-1}(U) \mid t > \pi\}$ constitute a separation of $p^{-1}(U)$.]

(3) Yes. [$p^{-1}(U)$ is a union of open intervals in $\mathbf{R}$, so it is open in $\mathbf{R}$, so U is open in S^1.] $\square$

Problem 6 (2 points)**.** View $A = \{0\} \cup \{\frac{1}{n} \mid n \text{ is a positive integer}\}$ as a subspace of $\mathbf{R}$. Show one of the following statements:

(1) A is compact.
(2) Any open set of A containing 0 is disconnected.

Solution. (1) Let $\mathcal{U}$ be an open cover of A. Pick $U_0 \in \mathcal{U}$ such that $0 \in U_0$. By the definition of the subspace topology, $U_0 = A \cap V$ for some V open in $\mathbf{R}$. Pick $\delta > 0$ such that $(-\delta, \delta) \subseteq V$. Using the archimedean principle, pick a positive integer n large enough that $\frac{1}{n} < \delta$.

Now observe that the only elements of A larger than $\frac{1}{n}$ are $1, \frac{1}{2}, \ldots, \frac{1}{n-1}$. So we can pick finitely many elements of $\mathcal{U}$ to cover these elements: say, $U_1, \ldots, U_k$. We claim that the finite subset $\{U_0, U_1, \ldots, U_k\} \subseteq \mathcal{U}$ remains a cover of A. Indeed, if $m \geq n$, then $\frac{1}{m} \in U_0$, while if $m < n$, then $\frac{1}{m} \in U_i$ for some i with $1 \leq i \leq k$.

(2) By the definition of the subspace topology, $U = A \cap V$ for some V open in $\mathbf{R}$. Pick $\delta > 0$ such that $(-\delta, \delta) \subseteq V$. Using the archimedean principle, pick a positive integer n large enough that $\frac{1}{n} < \delta$.

Let $U_1 = \{x \in U \mid x < \frac{1}{n}\}$ and $U_2 = \{x \in U \mid x > \frac{1}{n+1}\}$. Note that $0 \in U_1$ and $\frac{1}{n} \in U_2$, so both sets are nonempty. Moreover, they are disjoint and their union is U, because there is no element of A strictly between $\frac{1}{n}$ and $\frac{1}{n+1}$. It remains to show that they are both open in A. Indeed,

$$U_1 = U \cap (-\infty, \tfrac{1}{n}) = A \cap (V \cap (-\infty, \tfrac{1}{n})),$$
$$U_2 = U \cap (\tfrac{1}{n+1}, \infty) = A \cap (V \cap (\tfrac{1}{n+1}, \infty)),$$

as needed. $\square$