Review for vector spaces V and W, have

a vector space $W \otimes V = \mathrm{Bil}(W^\vee, V^\vee)$
called their tensor product
[whose elts are called tensors]

and a bilinear map $W \times V$ to $W \otimes V$ denoted

(w, v) mapsto $w \otimes v$,

explicitly, $w \otimes v : W^\vee \times V^\vee$ to F is def by
$(w \otimes v)(\psi, \theta) = \psi(w)\theta(v)$

Rem the elts of $W \otimes V$ taking the form $w \otimes v$
are called pure tensors
all other elts are called mixed tensors

today: be more concrete, assuming V, W fin. dim'l

Q what is $\dim W \otimes V$ in terms of
dim W and dim V? [pause]

A
$$\begin{aligned}\dim W \otimes V &= \dim \mathrm{Bil}(W^\vee, V^\vee)\\ &= (\dim W^\vee)(\dim V^\vee)\\ &= (\dim W)(\dim V)\end{aligned}$$

Q how to get a basis for $W \otimes V$
of this size? [pause]
in terms of bases of W and V? [pause]

A fix ordered bases $v_1, \ldots, v_n$ for V
$w_1, \ldots, w_m$ for W
consider the elts $w_j \otimes v_i$

Thm $(w_j \otimes v_i)_{j, i}$ is a basis for $W \otimes V$

Pf this set has size $mn = \dim W \otimes V$
so, enough to show it is linearly indep.

suppose $\sum_{j, i} c_{j, i} (w_j \otimes v_i) = \mathbf{0}$
for some $c_{j, i}$ in F

[idea: need to use def of $w_j \otimes v_i$ somehow]
our chosen bases of V and W define dual bases
$\theta_1, \ldots, \theta_n$ for $V^\vee$
$\psi_1, \ldots, \psi_m$ for $W^\vee$
[recall what dual means]
then the def of $w_j \otimes v_i$ becomes:

$$(w_j \otimes v_i)(\psi_\ell, \theta_k) = \begin{cases} 1 & \text{if } (j, i) = (\ell, k) \\ 0 & \text{else} \end{cases}$$

so evaluating $\sum_{j, i} c_{j, i} (w_j \otimes v_i)$
(as a bilinear functional) on (ψ_ℓ, θ_k)

gives $c_{\ell, k} (w_\ell \otimes v_k)(\psi_\ell, \theta_k) = c_{\ell, k}$

so if $\sum_{j, i} c_{j, i} (w_j \otimes v_i) = \mathbf{0}_{W \otimes V}$
then $c_{\ell, k} = 0$ for all ℓ, k

hence the set of $w_j \otimes v_i$'s is linearly indep. □

Q suppose $v = \sum_i a_i v_i$
$w = \sum_j b_j w_j$

what is the expansion of $w \otimes v$ wrt the basis
$(w_j \otimes v_i)_{j, i}$?

simpler questions:

given v, v' in V and w, w' in W and c in F,

how to simplify $(w + w') \otimes v$?

$w \otimes (v + v')$?

$(cw) \otimes v$?

$w \otimes (cv)$?

<u>A</u> [mentioned last time that we would prove:]

<u>Lem</u> the map $B : W \times V$ to $W \otimes V$ def by $B(w, v) = w \otimes v$ is bilinear

i.e., $B(w, –) : V$ to $W \otimes V$

and $B(–, v) : W$ to $W \otimes V$

are linear maps for any w in W, v in V

equivalently: for all v, v', w, w', and c,

1) $(w + w') \otimes v = w \otimes v + w' \otimes v$

2) $w \otimes (v + v') = w \otimes v + w \otimes v'$

3) $(cw) \otimes v = c(w \otimes v) = w \otimes (cv)$

<u>Pf of 1)</u> for any (ψ, θ) in $W^\vee \times V^\vee$:

$((w + w') \otimes v)(\psi, \theta)$ [pause: next?]

$= \psi(w + w')\theta(v)$

$= (\psi(w) + \psi(w'))\theta(v)$

$= \psi(w)\theta(v) + \psi(w')\theta(v)$

$= (w \otimes v)(\psi, \theta) + (w' \otimes v)(\psi, \theta)$

$= (w \otimes v + w' \otimes v)(\psi, \theta)$

hence $(w + w') \otimes v = w \otimes v + w' \otimes v$

<u>Pf of 2)</u> similar

<u>Pf of 3)</u> “left to the reader”

Cor given $v = \sum_i a_i v_i$,
$w = \sum_j b_j w_j$

$w \otimes v = \sum_{j, i} (b_j w_j) \otimes (a_i v_i)$
$= \sum_{j, i} b_j a_i (w_j \otimes v_i)$

Q what is the "simplest" mixed tensor, i.e., tensor not of the form $w \otimes v$?
[pause]
if dim V = 1 or dim W = 1? no dice

take $V = W = F^2$ and (e1, e2) the standard basis

by the thm, $F^2 \otimes F^2$ has the basis
$(e1 \otimes e1, e1 \otimes e2, e2 \otimes e1, e2 \otimes e2)$

any pure tensor will look like

$(b1\, e1 + b2\, e2) \otimes (c1\, e1 + c2\, e2)$
$= b1\, c1\, (e1 \otimes e1) + b1\, c2\, (e1 \otimes e2)$
$+ b2\, c1\, (e2 \otimes e1) + b2\, c2\, (e2 \otimes e2)$

the mixed tensors are the ones that cannot be written this way for any b1, b2, c1, c2
[pause: example?]
e.g., $e1 \otimes e1 + e2 \otimes e2$

Rem so far we've discussed the $\otimes$ operation on vectors

remember that $\otimes$ also denotes a separate operation on vector spaces

Properties of Tensor Products of Vector Spaces

recall that $W \oplus V$ denotes the vector space
formed by $W \times V$
[e.g., $F^m \oplus F^n = F^{m + n}$]

[below, equality signs should be isomphm signs]

1) $W \otimes (V \oplus V') = W \otimes V \oplus W \otimes V'$
2) $(W \oplus W') \otimes V = W \otimes V \oplus W' \otimes V$
3) $(W \otimes V) \otimes U = W \otimes (V \otimes U)$

seem obvious but take more work:
iso's, not equalities, so we must give actual maps

e.g., what's the left-to-right linear iso in 1)

$W \otimes (V \oplus V')$ spanned by pure tensors $w \otimes (v, v')$
so enough to say where they go: [pause: where?]

$w \otimes (v, v')$ mapsto $(w \otimes v, w \otimes v')$

(Axler §9B, 9D) using the associativity 3),
form iterated tensor products:

$V_1 \otimes V_2 \otimes \ldots \otimes V_r$

<u>Df</u> a map $\mu : V_1 \times V_2 \times \ldots \times V_r$ to U
is multilinear iff,

for any index i, and choice of w_j in V_j for all $j \neq i$,
the map V_i to U given by
v mapsto $\mu(\ldots, w_{i - 1}, v, w_{i + 1}, \ldots)$ is linear

U = F: we say μ is a <u>multilinear functional</u>

U = F and V = V_i for all i: it's an <u>r-linear form</u>

let $\mathrm{Mult}(V_1, \ldots, V_r) = \{$multilinear functionals on $V_1 \times V_2 \times \ldots \times V_r\}$

<u>Thm</u> just as $V_1 \otimes V_2$ satisfies
$\mathrm{Bil}(V_1, V_2) = (V_1 \otimes V_2)^\vee$,

so $V_1 \otimes \ldots \otimes V_r$ satisfies
$\mathrm{Mult}(V_1, \ldots, V_r) = (V_1 \otimes \ldots \otimes V_r)^\vee$

[why care?] on Wed:
determinants as multilinear forms