Last time

Thm if $f : X \to Y$ and $g : Y \to X$ form a homotopy equivalence

then $f_* : \pi_1(X, x) \to \pi_1(Y, f(x))$
$g_* : \pi_1(Y, y) \to \pi_1(Y, g(y))$

are isomorphisms for all x in X and y in Y

easier special cases:

Ex for any X and x,

$(\mathrm{id}_X)_* = \mathrm{id}_{\pi_1(X, x)}$
as maps $\pi_1(X, x) \to \pi_1(X, x)$

Ex if $f : X \to Y$ is a homeo
then $f_* : \pi_1(X, x) \to \pi_1(Y, f(x))$ is an isomorphism

Lem 1 given $f : X \to Y$,
$g : Y \to Z$,

$(g \circ f)_* = g_* \circ f_*$
as maps $\pi_1(X, x) \to \pi_1(Z, g(f(x)))$

so if f and g are two-sided inverses of each other,

$g_* \circ f_* = (\mathrm{id}_X)_* = \mathrm{id}_{\pi_1(X, x)}$
$f_* \circ g_* = (\mathrm{id}_Y)_* = \mathrm{id}_{\pi_1(Y, f(x))}$

[<u>Pf of Thm</u> will show that f_* is an iso
as argument for g is similar

by Lem 1, $g_* \circ f_* = (g \circ f)_*$
$f_* \circ g_* = (f \circ g)_*$

<u>Lem 2</u> if $s : G \to H$ and $r : H \to K$
s.t. $r \circ s$ is bijective,
then s is injective and r is surjective

so to show f_*, g_* bijective,
enough to show $(g \circ f)_*$ and $(f \circ g)_*$ bijective

will show that $(g \circ f)_*$ is bijective
as argument for $f \circ g$ is similar

pick homotopy φ from $g \circ f$ to id_X

<u>Lem 3</u> if α is a path in X from x to x'
then iso $\hat{\alpha} : \pi_1(X, x) \to \pi_1(X, x')$
def by $\hat{\alpha}([\gamma]) = [\bar{\alpha} * \gamma * \alpha]$

further: if $F, G : A \to X$ are cts,
φ is a homotopy from F to G,
a in A,

then $\alpha_\varphi = \varphi(a, -)$ is a path from $F(a)$ to $G(a)$ s.t.

$$G_* = \hat{\alpha}_\varphi \circ F_*$$

now: $(g \circ f)_* = \hat{\alpha}_\varphi \circ (\mathrm{id}_X)_* = \hat{\alpha}_\varphi$
so by Lem 3, $(g \circ f)_*$ is an iso]

other examples of π_1's:

Thm $\pi_1(S^n)$ is trivial for all $n \geq 2$

Df in general, a space is simply-connected iff it is path-connected with trivial π_1

"Pf of Thm" given x in S^n and a loop γ at x,
pick p not in the image of γ
$S^n - \{p\}$ is homeomorphic to R^n

R^n is simply-connected
i.e., any loop in R^n is $\sim_p$ constant loop
so $\gamma \sim_p$ constant loop at x within $S^n - \{p\}$
so $\gamma \sim_p$ constant loop at x within S^n

but the map $\gamma : [0, 1]$ to X could be surjective!

to fix: show that $\gamma \sim_p$ some non-surjective loop [somewhat hard]

Thm for any x in X and y in Y,

$$\pi_1(X \times Y, (x, y)) \simeq \pi_1(X, x) \times \pi_1(Y, y)$$

Cor $\pi_1(S^1 \times S^1) \simeq Z^2$ under $+$

by contrast:

Thm $\pi_1(\text{figure-eight}) \simeq Z * Z$ [free group]

Df given path-connected A, B,
a in A and b in B,
the wedge sum of A and B at (a, b) is

$$A \vee B = (A \sqcup B)/(a \sim b)$$

Thm $\pi_1(A \vee B) \simeq \pi_1(A) * \pi_1(B)$
via inclusion maps