MATH 250: TOPOLOGY I PROBLEM SET #2

FALL 2025

Due Wednesday, September 17. Please attempt all of the problems. Six of them will be graded. You may consult books, papers, and websites as long as you cite all sources and write up your solutions in your own words.

Problem 1. Recall the evenly-spaced topology on $\mathbf{Z}$ from Problem Set 1.

(1) Show that if there were finitely many prime numbers, then the set $\{1, -1\} \subseteq \mathbf{Z}$ would be open in this topology. *Note:* You may assume that any integer greater than 1 has a prime divisor.

(2) Deduce that there must be infinitely many primes.

Problem 2 (Munkres 91–92, #1)**.** Let X be a topological space, and let $A \subseteq Y \subseteq X$ be subsets. Endow Y with the subspace topology that it inherits from X. Show that the subspace topology that A inherits from Y is the subspace topology that A inherits from X.

Problem 3. Let X be a topological space, and let $A \subseteq X$ be a subset endowed with the subspace topology.

(1) Give an example where a subset of A is open in A, but not open in X.

(2) Now suppose that A itself is open in X. Prove that a subset of A is open in A if and only if it is open in X.

Problem 4. Show that the following topological spaces are homeomorphic:

$$\mathbf{R}, \qquad (0, \infty), \qquad (0, 1).$$

Above, $\mathbf{R}$ is endowed with the analytic topology; $(0, \infty)$ and $(0, 1)$ are endowed with their subspace topologies. You may assume that differentiable functions are continuous, and that a composition of homeomorphisms is a homeomorphism.

Problem 5. Endow $\mathbf{R}$ with the analytic topology, and

$$X = \{\tfrac{1}{n} \mid n = 1, 2, 3, \ldots\} \cup \{0\}$$

with the subspace topology. Show that:

(1) For all integers $n > 0$, the singleton set $\{\frac{1}{n}\}$ is *clopen*: both closed and open.

(2) $\{0\}$ is closed but not open.

Problem 6 (Munkres 128, #9(c)–(d))**.** Recall that the Euclidean norm on $\mathbf{R}^n$ is given by $\|u\| = \sqrt{u \cdot u}$, where

$$u \cdot v := u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$$

for general $u, v \in \mathbf{R}^n$.

Use the Cauchy–Schwarz inequality $|u \cdot v| \leq \|u\|\|v\|$ to show that $\|u+v\| \leq \|u\|+\|v\|$ for all $u, v \in \mathbf{R}^n$. Conclude that the *Euclidean metric* defined by $d(x, y) = \|x - y\|$ really is a metric on $\mathbf{R}^n$.

Problem 7. Let X be arbitrary, and let $d : X \times X \to [0, \infty)$ be an arbitrary metric. Assume that the function $e : X \times X \to [0, \infty)$ defined by

$$e(x, y) = \frac{d(x, y)}{1 + d(x, y)}.$$

is a bounded metric. Show that d and e induce the same topology on X.

Note: Munkres 129, #11 asks the reader to prove that e itself is a metric, which is harder.

Problem 8. Let us say that metrics $d, d' : X \times X \to [0, \infty)$ are *equivalent* if and only if there are constants $A, B > 0$ such that

$$d(x, y) \leq Ad'(x, y) \text{ and } d'(x, y) \leq Bd(x, y) \text{ for all } x, y \in X.$$

In the setting of Problem 7, show that e is not equivalent to d when $X = \mathbf{R}$ and d is the Euclidean metric.

Problem 9 (Munkres 127, #6)**.** The *uniform topology* on

$$\mathbf{R}^\omega := \{\text{sequences } (a_1, a_2, a_3, \ldots) \text{ with } a_i \in \mathbf{R} \text{ for all } i\}$$

is induced by the *uniform metric* $\bar{\rho}(x, y) = \sup_{i>0} \min\{1, |x_i - y_i|\}$. For all $x \in \mathbf{R}^\omega$ and $0 < \epsilon < 1$, show that:

(1) The following set is not open in the uniform topology:

$$U(x, \epsilon) := (x_1 - \epsilon, x_1 + \epsilon) \times (x_2 - \epsilon, x_2 + \epsilon) \times \ldots$$

(2) Nonetheless, $B_{\bar{\rho}}(x, \epsilon) = \bigcup_{\delta < \epsilon} U(x, \delta)$.

Problem 10. The *box topology* on $\mathbf{R}^\omega$ is defined as follows: U is open if and only if, for all $x \in U$, there is some set of the form $V = (a_1, b_1) \times (a_2, b_2) \times \cdots$ such that $x \in V \subseteq U$.

(1) Show that the box topology really is a topology on $\mathbf{R}^\omega$. *Hint:* Use the Axiom of Choice.
(2) Show that the box topology is strictly finer than the uniform topology. *Hint:* Use Problem 9.

Problem 11. Let $f : X \to S$ be a continuous map. Below, all subsets are given their subspace topologies.

(1) Show that $f|_{f^{-1}(T)} : f^{-1}(T) \to T$ is continuous for any $T \subseteq S$.
(2) Show that $f|_Y : Y \to S$ is continuous for any $Y \subseteq X$.
(3) Use (1)–(2) to show that $f|_Y : Y \to f(Y)$ is continuous for any $Y \subseteq X$.

Problem 12 (Munkres 112, #10)**.** Show that if $f : A \to B$ and $g : C \to D$ are continuous maps, then $(f, g) : A \times C \to B \times D$ defined by

$$(f, g)(a, c) = (f(a), g(c))$$

is continuous with respect to the product topologies.