MATH 430: INTRODUCTION TO TOPOLOGY
PROBLEM SET #4

SPRING 2025

Due Wednesday, February 12. You may consult books, papers, and websites as long as you cite all sources and write up your solutions in your own words.

Problem 1 (Munkres 127–128, #8(c))**.** Recall from Problem Set 3, #8, the set

$$X = \{x \in \mathbf{R}^\omega \mid \textstyle\sum_{i>0} x_i^2 \text{ converges}\}$$

and its ℓ^2 topology. Let H be the *Hilbert cube*

$$H = [0,1] \times [0, \tfrac{1}{2}] \times [0, \tfrac{1}{3}] \times \cdots \subseteq X.$$

Compare the box, ℓ^2, uniform, and product topologies that H inherits from X.

Problem 2 (Munkres 101, #11–13)**.** Show that:

(1) A product of two Hausdorff spaces is Hausdorff (in the product topology).
(2) A subspace of a Hausdorff space is Hausdorff (in the subspace topology).
(3) X is Hausdorff if and only if its *diagonal* $\Delta_X = \{(x,x) \mid x \in X\}$ is closed in (the product topology on) $X \times X$.

Problem 3 (Munkres 118, #6)**.** Let $(X_\alpha)_\alpha$ be an arbitrary collection of topological spaces, and let $x^{(1)}, x^{(2)}, \ldots$ be a sequence of points in $\prod_\alpha X_\alpha$. (Each takes the form $x^{(i)} = (x_\alpha^{(i)})_\alpha$.)

(1) Show that in the product topology, the sequence converges to a point $x = (x_\alpha)_\alpha$ if and only if, for all α, the sequence $x_\alpha^{(1)}, x_\alpha^{(2)}, \ldots$ converges to x_α.
(2) Does (1) remain true if we replace the product topology with the box topology?

Problem 4 (Munkres 144, #2)**.** Let $p : X \to Y$ be a continuous map.

(1) Show that if $p \circ f$ is the identity map on Y for some continuous map $f : Y \to X$, then p is a quotient map.
(2) A *retraction* from X onto a subset A is a continuous map $r : X \to A$ such that $r(a) = a$ for all $a \in A$. Deduce from (1) that retractions are quotient maps.

Problem 5 (Munkres 145, #6)**.** Endow $\mathbf{R}$ with the *K-topology*: the topology generated by the basis of sets of the form $(a,b) - K$, where $a, b \in \mathbf{R}$ and

$$K = \{\tfrac{1}{n} \mid n = 1, 2, 3, \ldots\}.$$

Let Y be the quotient space obtained from $\mathbf{R}$ by collapsing K to a point, and let $p : \mathbf{R} \to Y$ be the resulting map.

(1) Show that Y is not Hausdorff, but satisfies the T_1 *condition*: For all $x, y \in Y$, we can find an open set containing x but not y.
(2) Show that $(p, p)^{-1}(\Delta_Y)$ is closed in $\mathbf{R} \times \mathbf{R}$. Hence, by Problem 2(3), the product and quotient topologies on $Y \times Y$ must differ.

Problem 6 (Munkres 152, #2)**.** Let $(A_n)_{n=1}^{\infty}$ be a sequence of connected subspaces of X, such that $A_n \cap A_{n+1} \neq \emptyset$ for all n. Show that $\bigcup_{n=1}^{\infty} A_n$ is connected.

Problem 7 (Munkres 152, #9)**.** Let X, Y be connected, and let $A \subseteq X$ and $B \subseteq Y$ be proper subsets. Show that

$$(X \times Y) - (A \times B)$$

is a connected subspace of $X \times Y$.

Problem 8 (Munkres 152, #11)**.** Let $p : X \to Y$ be a quotient map. Show that if Y is connected and each subspace $p^{-1}(y) \subseteq X$ is connected, then X is connected.