ANNULAR WEBS AND A CONJECTURE OF HAIMAN

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Abstract. Haiman conjectured that when traces corresponding to monomial symmetric functions are evaluated on the Hecke-algebra elements denoted C'_w by Kazhdan–Lusztig, the resulting polynomials have nonnegative coefficients. We show that recent work on annular webs implies this for permutations w that are 321-hexagon-avoiding. As a byproduct, we observe a precise analogy that matches the two bases in Kazhdan–Lusztig's work with elementary and homogeneous symmetric functions.

1. Introduction

1.1. Let $H_n(x)$ be the Iwahori–Hecke algebra of the symmetric group S_n over $\mathbf{Z}[x^{\pm 1}]$. As a quotient of the group algebra of a braid group, it has a standard basis $\{\sigma_w\}_{w \in S_n}$, consisting of the images of the positive permutation braids.

Kazhdan–Lusztig introduced two new bases for $H_n(x)$ with remarkable properties [KL79]. Taking our x to be their $q^{1/2}$, we will focus on the basis that they denote by $\{C'_w\}_w$, but write b_w in place of C'_w for simplicity. When the elements b_w are expanded in the standard basis, the coefficients are Laurent polynomials in x with nonnegative integer coefficients. Up to rescaling, these are the celebrated Kazhdan–Lusztig polynomials for S_n. Their positivity can be proved through a geometric interpretation of $H_n(x)$ in terms of sheaves on flag varieties.

The representation theory of S_n deforms to that of $H_n(x)$. In particular, each character $\chi : S_n \to \bar{\mathbf{Q}}$ defines a $\mathbf{Z}[x^{\pm 1}]$-linear function $\chi_x : H_n(x) \to \overline{\mathbf{Q}(x)}$ that still enjoys the trace property $\chi(\alpha\beta) = \chi(\beta\alpha)$. At the same time, the *irreducible* characters of S_n are indexed by integer partitions of n. Let χ^λ be the irreducible character indexed by $\lambda \vdash n$. A geometric argument, similar to that used in the positivity of the Kazhdan–Lusztig polynomials, proves that $\chi_x^\lambda(b_w) \in \mathbf{Z}_{\geq 0}[x^{\pm 1}]$ for all w and λ.

Haiman found evidence for a stronger positivity statement. Recall that for all λ, μ, the Kostka number $K_{\lambda,\mu}$ counts semistandard Young tableaux of shape λ and weight μ. The Kostka numbers can be assembled into a unitriangular matrix of nonnegative integers. In particular, this matrix has an inverse with integer entries, so there are functions $\phi_x^\mu : H_n(x) \to \mathbf{Z}[x^{\pm 1}]$ uniquely defined by requiring

$$\chi_x^\lambda = \sum_\mu K_{\lambda,\mu} \phi_x^\mu \quad \text{for all } \lambda \vdash n. \tag{1.1}$$

What follows is the main part of Conjecture 2.1 in [Hai93].

Conjecture 1.1 (Haiman)**.** *$\phi_x^\mu(b_w) \in \mathbf{Z}_{\geq 0}[x^{\pm 1}]$ for all $w \in S_n$ and $\mu \vdash n$.*

Abreu–Nigro observe that Conjecture 1.1 would imply several conjectures about the indifference graphs of Hessenberg functions in algebraic combinatorics: notably, the Stanley–Stembridge conjecture on the e-positivity of their chromatic symmetric functions, and Shareshian–Wachs's generalization of this conjecture to chromatic quasi-symmetric functions [AN24].

1.2. This note will show how recent work of Queffelec–Rose and Gorsky–Wedrich on the diagrammatics of $H_n(x)$ solves some cases of Conjecture 1.1.

For $1 \leq i \leq n-1$, let $b_i = b_{s_i}$, where $s_i \in S_n$ is the transposition that swaps i and $i+1$. The main theorem is:

Theorem 1.2. *$\phi_x^\mu(b_{i_1} \cdots b_{i_\ell}) \in \mathbf{Z}_{\geq 0}[x^{\pm 1}]$ for any sequence of indices $i_1, \ldots, i_\ell$ that range between 1 and $n-1$ inclusive, and any $\mu \vdash n$.*

In what follows, suppose that $w \in S_n$ is given by $w = [w_1 w_2 \cdots w_n]$, meaning it sends i to w_i for $1 \leq i \leq n$. Fix $m \leq n$ and $v = [v_1 v_2 \cdots v_m] \in S_m$. We say that w is *$v_1 \cdots v_m$-avoiding* if and only if the sequence $(w_1, \ldots, w_n)$ does *not* contain a subsequence of size m whose elements have the same relative order as $(v_1, \ldots, v_m)$. More formally, this means we cannot find indices $1 \leq p_1 < \cdots < p_m \leq n$ such that $w_{p_i} < w_{p_j}$ whenever $i < j$ and $v_i < v_j$.

We write $S_n^{v_1 \cdots v_m} \subseteq S_n$ for the set of $v_1 \cdots v_m$-avoiding elements. Following Billey–Warrington, we say that w is *321-hexagon-avoiding* if and only if

$$w \in S_n^{321} \cap S_n^{46718235} \cap S_n^{46781235} \cap S_n^{56718234} \cap S_n^{56781234}.$$

In [BW01], Billey–Warrington prove that the following conditions are equivalent:

(1) w is 321-hexagon-avoiding.
(2) $b_w = b_{i_1} \cdots b_{i_\ell}$ whenever $w = s_{i_1} \cdots s_{i_\ell}$ and ℓ is the minimal length among such expressions.

Via this result, Theorem 1.2 implies:

Corollary 1.3. *Conjecture 1.1 holds when w is 321-hexagon-avoiding.*

1.3. The key observation is that Remark 4.21 of [GW23], a refinement of the annular web evaluation algorithm of [QR18], provides a counterpart to Theorem 1.2 (in fact, a slightly stronger statement) in the setting of MOY webs. The passage from Hecke-algebra traces to web diagrammatics is best explained by assembling the cocenters of all the Hecke algebras into a direct sum that we identify with Macdonald's ring of symmetric functions $\Lambda(x)$ over $\mathbf{Z}[x^{\pm 1}]$, after extending scalars. This defines a universal trace

$$\mathsf{tr} : \bigoplus_n H_n(x) \to \Lambda(x).$$

There is a natural candidate for the diagrammatic counterpart to tr: the map ann that sends a rectangular web to its annular closure.

For any $\beta \in H_n(x)$, the value of $\phi_x^\mu(\beta)$ is just the μ-th coefficient when we expand $\mathsf{tr}(\beta)$ in the basis of complete homogeneous symmetric functions $\{h_\mu\}_\mu$: a fact already noted in [AN24]. Ultimately, we relate Theorem 1.2 to [GW23, Rem.

4.21] through a commutative diagram that relates tr to ann, and assigns simple webs to the b_i and h_μ.

I do not know of any earlier statement of this commutative diagram in the literature, despite closely related results in the work of Aiston, Lukac, Morton, *et al.*: See [MM08] and the references there.

In fact, there is an analogous but inequivalent commutative diagram, where $\{b_w\}_w$ is replaced by Kazhdan–Lusztig's *other* basis for the Hecke algebra, and $\{h_\mu\}_\mu$ is replaced by the basis of elementary symmetric functions $\{e_\mu\}_\mu$. The dichotomy between the two diagrams seems related to the dichotomy between the framed and unframed Khovanov–Rozansky functors in [GW23], though not exactly. We expand on this at the end of the note.

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2. Hecke Algebras and Symmetric Functions

2.1. Recall that S_n forms a Coxeter group, in which the transpositions s_i for $1 \leq i \leq n-1$ (see §1.2) form a system of simple reflections. With respect to this Coxeter presentation, let ℓ_w denote the Bruhat length of $w \in S_n$, and let $<$ be the Bruhat order on S_n [GP00, Ch. 1].

2.2. Formally, we take the *Iwahori–Hecke algebra* of S_n to be the $\mathbf{Z}[x^{\pm 1}]$-algebra $H_n(x)$ spanned as a free module by elements σ_w for $w \in S_n$, modulo the following relations, where we set $\sigma_i := \sigma_{s_i}$:

$$\sigma_w \sigma_i = \begin{cases} \sigma_{ws_i} & ws_i > w, \\ \sigma_{ws_i} + (x - x^{-1})\sigma_w & ws_i < w. \end{cases} \tag{2.1}$$

There is an additive involution $D : H_n(x) \to H_n(x)$ that sends $x \mapsto x^{-1}$ and $\sigma_w \mapsto \sigma_{w^{-1}}^{-1}$ for all $w \in S_n$.

2.3. Let $\mathbf{K} = \mathbf{Q}(x)$. It turns out that $\mathbf{K} \otimes_{\mathbf{Z}[x^{\pm 1}]} H_n(x)$ is split as a $\mathbf{K}$-algebra. At the same time, there is an isomorphism of rings $H_n(x)|_{x \to 1} \simeq \mathbf{Z}S_n$. So by Tits deformation [GP00, Ch. 7], the semisimplicity of $\mathbf{Q}S_n$ implies the semisimplicity of $\mathbf{K} \otimes_{\mathbf{Z}[x^{\pm 1}]} H_n(x)$, and moreover, there is a bijection between isomorphism classes of simple $\mathbf{Q}S_n$-modules and those of simple $(\mathbf{K} \otimes_{\mathbf{Z}[x^{\pm 1}]} H_n(x))$-modules.

This induces the assignment from characters $\chi : S_n \to \bar{\mathbf{Q}}$ to $\mathbf{Z}[x^{\pm 1}]$-linear trace functions $\chi_x : H_n(x) \to \bar{\mathbf{K}}$ described in the introduction. Explicitly, $\chi_x(\beta)$ is the trace of β on the $(\mathbf{K} \otimes_{\mathbf{Z}[x^{\pm 1}]} H_n(x))$-module that corresponds to the $\mathbf{Q}S_n$-module with character χ.

2.4. Kazhdan–Lusztig proved that for all $w \in S_n$, there is a unique D-invariant element $b_w \in H_n(x)$ such that

$$b_w = \sum_{y \leq w} x^{\ell_y - \ell_w} P_{y,w}(x^2) \sigma_y$$

for some $P_{y,w}(q) \in \mathbf{Z}[q]$ satisfying

$$\begin{aligned} P_{w,w}(q) &= 1, \\ \deg P_{y,w}(q) &\leq \tfrac{1}{2}(\ell_w - \ell_y - 1) \quad \text{for all } w, y \in S_n \text{ with } y \leq w. \end{aligned} \tag{2.2}$$

Let j be the additive involution that sends $x \mapsto x^{-1}$ and $\sigma_w \mapsto (-1)^{\ell_w}\sigma_w$. Let $c_w = j(b_w)$. Then c_w is the unique D-invariant element of $H_n(x)$ such that

$$c_w = \sum_{y \leq w} (-1)^{\ell_y} x^{\ell_w - \ell_y} P_{y,w}(x^{-2})\sigma_y$$

for some $P_{y,w}(q) \in \mathbf{Z}[q]$ satisfying (2.2). They turn out to be the same polynomials as before.

The sets $\{b_w\}_{w \in S_n}$ and $\{c_w\}_{w \in S_n}$ form bases for $H_n(x)$ as a free $\mathbf{Z}[x^{\pm 1}]$-module, known as the two *Kazhdan–Lusztig bases* or *canonical bases*. The polynomials $P_{y,w}(q)$ are the *Kazhdan–Lusztig polynomials* for S_n. Note that in [KL79], b_w and c_w are respectively denoted C'_w and $-C_w$.

Henceforth, we will write b_i, c_i in place of b_{s_i}, c_{s_i}. We can check that

$$\begin{aligned} b_i &= x^{-1} + \sigma_i &&= x + \sigma_i^{-1}, \\ c_i &= x - \sigma_i &&= x^{-1} - \sigma_i^{-1}. \end{aligned} \tag{2.3}$$

Just as $\{\sigma_i\}_i$ generates $H_n(x)$ as a $\mathbf{Z}[x^{\pm 1}]$-algebra, so do $\{b_i\}_i$ and $\{c_i\}_i$.

2.5. Let Λ be the graded ring of symmetric functions over $\mathbf{Z}$ in (countably) infinitely many variables. For background on Λ, we refer to [Mac15, Ch. I]. In this note, we will need the following elements of Λ indexed by integer partitions λ:

the Schur functions	s_λ,
the monomial symmetric functions	$m_\lambda = m_{\lambda_1} m_{\lambda_2} \ldots$,
the complete homogeneous symmetric functions	$h_\lambda = h_{\lambda_1} h_{\lambda_2} \ldots$,
the elementary symmetric functions	$e_\lambda = e_{\lambda_1} e_{\lambda_2} \ldots$

Let Λ_n be the degree-n component of Λ. The Schur functions s_λ with $\lambda \vdash n$ form a basis for Λ as a free $\mathbf{Z}$-module; analogous statements hold with m_λ or h_λ or e_λ in place of s_λ.

Recall the Kostka numbers $K_{\lambda,\mu} \in \mathbf{Z}$ from the introduction. As explained in [Mac15, §I.6], they relate the elements s_λ, m_μ, h_μ via the identities

$$\begin{aligned} s_\lambda &= \sum_\mu K_{\lambda,\mu} m_\mu, \\ h_\mu &= \sum_\lambda K_{\lambda,\mu} s_\lambda. \end{aligned} \tag{2.4}$$

The first identity shows that Haiman's character ϕ_x^μ is to the monomial symmetric function μ_μ as the irreducible character χ_x^λ is to the Schur function s_λ.

Note that $K_{\lambda,\lambda} = 1$ for all λ, and that $K_{\lambda,\mu} = 0$ whenever $\mu > \lambda$ in the dominance order on partitions. This makes precise the unitriangularity mentioned earlier.

2.6. Let $\Lambda(x) = \mathbf{Z}[x^{\pm 1}] \otimes \Lambda$ and $\Lambda_n(x) = \mathbf{Z}[x^{\pm 1}] \otimes \Lambda_n$ for all n. The map tr mentioned in §1.3 is the sum of the $\mathbf{Z}[x^{\pm 1}]$-linear maps

$$\mathsf{tr}_n : H_n(x) \to \Lambda_n(x) \quad \text{defined by } \mathsf{tr}_n(\beta) = \sum_{\lambda \vdash n} \chi_x^\lambda(\beta) s_\lambda.$$

By construction, $\mathsf{tr}_n(\alpha\beta) = \mathsf{tr}_n(\beta\alpha)$ for all α, β. So the universal property of the cocenter of $H_n(x)$ defines a $\mathbf{Z}[x^{\pm 1}]$-linear map from the cocenter into $\Lambda(x)$, which turns out to be an isomorphism of $\mathbf{Z}[x^{\pm 1}]$-modules.

Let $\langle -, - \rangle : \Lambda(x) \times \Lambda(x) \to \mathbf{Z}[x^{\pm 1}]$ be the *Hall pairing*: the $\mathbf{Z}[x^{\pm 1}]$-linear pairing under which the Schur functions s_λ form an orthonormal basis. It lets us write

$$\chi_x^\lambda(\beta) = \langle \mathsf{tr}_n(\beta), s_\lambda \rangle \quad \text{for all } \lambda \vdash n.$$

By (1.1) and (2.4), we deduce:

$$\mathsf{tr}_n(\beta) = \sum_{\mu \vdash n} \phi_x^\mu(\beta) h_\mu. \tag{2.5}$$

Altogether, Theorem 1.2 is claiming that for any sequence of indices $i_1, \ldots, i_\ell$ that range between 1 and $n-1$ inclusive, the expansion of $\mathsf{tr}_n(b_{i_1} \cdots b_{i_\ell})$ in the complete homogeneous basis of $\Lambda_n(x)$ will have coefficients in $\mathbf{Z}_{\geq 0}[x^{\pm 1}]$.

3. Webs

3.1. We refer to [GW23, §2] for background on the Murakami–Ohtsuki–Yamada (MOY) web calculus. Note that their q is our x.

3.2. Let $H_n^{\mathsf{moy}}(x)$ be the free $\mathbf{Z}[x^{\pm 1}]$-module generated by strictly upward-oriented web diagrams in a rectangle, connecting n inputs with label 1 at the bottom to n outputs with label 1 at the top, modulo the relations of the MOY bracket. It forms a $\mathbf{Z}[x^{\pm 1}]$-algebra under concatenation of diagrams. Murakami–Ohtsuki–Yamada showed that this algebra is isomorphic to $H_n(x)$ [MOY98].

For $1 \leq i \leq n-1$, let $\mathsf{can}_i \in H_n^{\mathsf{moy}}(x)$ denote the ith merge-split web. The notation can is intended to suggest the adjective *canonical*. The precise statement proved in [MOY98], up to sign and up to use of Schur–Weyl duality, is that there is an isomorphism of $\mathbf{Z}[x^{\pm 1}]$-algebras

$$\Theta_b : H_n(x) \to H_n^{\mathsf{moy}}(x) \quad \text{defined by } \Theta_b(b_i) = \mathsf{can}_i.$$

To explain how this isomorphism appears in [GW23]: Recall that we can identify $H_n(x)$ with the skein algebra of a rectangle with n inputs and n outputs, by sending σ_i to the ith simple twist. Decategorified, rectangular analogues of formulas (16) and (17) in [GW23] define two isomorphisms from this skein algebra to $H_n^{\mathsf{moy}}(x)$. Our map Θ_b corresponds to their *framed* map (16).

3.3. Let $\mathcal{C}^{\mathsf{moy}}(x)$ be the free $\mathbf{Z}[x^{\pm 1}]$-module generated by positively-oriented web diagrams in an annulus. It forms a commutative $\mathbf{Z}[x^{\pm 1}]$-algebra under nesting of diagrams.

For any n and $\mu \vdash n$, let $o^\mu \in \mathcal{C}^{\mathsf{moy}}(x)$ be the diagram consisting of concentric essential circles with labels $\mu_1, \mu_2, \ldots$ Note that by the commutativity of $\mathcal{C}^{\mathsf{moy}}(x)$, the order of these circles does not matter. The *annular web evaluation algorithm* of Queffelec–Rose [QR18, Lem. 5.2] shows that the set $\{o_\mu\}_\mu$ forms a basis for $\mathcal{C}^{\mathsf{moy}}(x)$ as a free $\mathbf{Z}[x^{\pm 1}]$-module. The definition of the MOY bracket then implies, directly, that $\mathcal{C}^{\mathsf{moy}}(x)$ is freely generated *as an algebra* by the elements $o_n := o_{(n)}$ corresponding to single, labeled essential circles. By placing o_n in degree n, we can further endow $\mathcal{C}^{\mathsf{moy}}(x)$ with the structure of a *graded* algebra.

At the same time, display (2.4), *resp.* (2.8), in [Mac15] implies that $\Lambda(x)$ is freely generated as an algebra by the set $\{e_n\}_n$, *resp.* the set $\{h_n\}_n$. Thus, there are isomorphisms of graded $\mathbf{Z}[x^{\pm 1}]$-algebras

$$\Xi_h, \Xi_e : \Lambda(x) \to \mathcal{C}^{\mathsf{moy}}(x) \quad \text{defined by } \Xi_e(e_\mu) = o_\mu \text{ and } \Xi_h(h_\mu) = o_\mu.$$

They differ precisely by the $\mathbf{Z}[x^{\pm 1}]$-algebra involution of $\Lambda(x)$ that swaps h_μ and e_μ.

Prior to the introduction of webs, an analogous isomorphism for the skein algebra of the annulus was first established by Turaev [Tur88].

3.4. As in the literature on skein algebras, there is a $\mathbf{Z}[x^{\pm 1}]$-linear map

$$\mathsf{ann} : \bigoplus_n H_n^{\mathsf{moy}}(x) \to \mathcal{C}^{\mathsf{moy}}(x)$$

called *annular closure*. It is defined graphically, by embedding a rectangle into an annulus as a sector, so that the upward orientation in the rectangle becomes the positive orientation in the annulus, then wrapping the n outputs of the rectangle around the annulus, without crossing, back to the n inputs.

Queffelec–Rose's annular web evaluation algorithm originally treated $\mathcal{C}^{\mathsf{moy}}(x)$ as the triangulated Grothendieck group of the bounded homotopy category of a graded, linear category of annular foams between positively-oriented webs. Gorsky–Wedrich observed that it could be refined, by instead treating $\mathcal{C}^{\mathsf{moy}}(x)$ as the additive Grothendieck group of the *Karoubi* or *idempotent completion* of this foam category [GW23, Rem. 4.21]. The refinement shows:

Theorem 3.1 (Positive Annular Web Evaluation)**.** *The expansion of any single positively-oriented annular web in the basis $\{o_\mu\}_\mu$ for $\mathcal{C}^{\mathsf{moy}}(x)$ will have coefficients in $\mathbf{Z}_{\geq 0}[x^{\pm 1}]$. In particular, this applies to $\mathsf{ann}(\mathsf{can}_{i_1} \cdots \mathsf{can}_{i_\ell})$ for any indices $i_1, \ldots, i_\ell$.*

3.5. Together, (2.5) and Theorem 3.1 reduce Theorem 1.2 to the following result:

Theorem 3.2. *The following diagram commutes:*

$$\begin{tikzcd}
H_n(x) \arrow[r, "\mathsf{tr}"] \arrow[d, "\Theta_b"'] & \Lambda(x) \arrow[d, "\Xi_h"] \\
H_n^{\mathsf{moy}}(x) \arrow[r, "\mathsf{ann}"] & \mathcal{C}^{\mathsf{moy}}(x)
\end{tikzcd}$$

That is: For any $\beta \in H_n(x)$ and $\mu \vdash n$, the value of $\phi_x^\mu(\beta)$ is the coefficient of o_μ when we expand $\mathsf{ann}(\Theta_b(\beta))$ in the basis $\{o_\mu\}_\mu$.

Proof. TODO □

3.6. Theorem 3.1 suggests a categorification of Conjecture 1.1.

Let C be the category denoted $\mathsf{Kar}(A\mathbf{Foam}^+)$ in [GW23]: a Karoubi-complete, graded, linear category of foams between positively-oriented webs. Let H_n be the analogous category where we replace the annulus by a rectangle with n inputs and n outputs. By work of Mackaay–Vaz [MV10], H_n is a diagrammatic presentation of the category of Soergel bimodules for S_n, and hence, categorifies $H_n(x)$. Let $\mathbf{B}_w$ be the indecomposable object of H_n indexed by $w \in S_n$, so that the isomorphism from the Grothendieck group to $H_n(x)$ sends $[\mathbf{B}_w]$ to b_w. Let $\mathbf{O}_\mu$ be the object of C_n underlying the annular web o_μ.

Conjecture 3.3. *For all $w \in S_n$, the annular closure of $\mathbf{B}_w$ is isomorphic in C to a direct sum of objects of the form $\mathbf{O}_\mu$.*

4. The Other Commutative Diagram

4.1. We claim that there is an isomorphism of $\mathbf{Z}[x^{\pm 1}]$-algebras

$$\Theta_c : H_n(x) \to H_n^{\mathsf{moy}}(x) \quad \text{defined by } \Theta_c(c_i) = \mathsf{can}_i,$$

analogous to but distinct from Θ_b. Indeed, the relations (2.1) show that if either $\alpha_i = b_i$ for all i, or $\alpha_i = c_i$ for all i, then $H_n(x)$ is generated by its subset $\{\alpha_i\}_i$ modulo the relations

$$\begin{cases} \alpha_i\alpha_{i+1}\alpha_i - \alpha_i = \alpha_{i+1}\alpha_i\alpha_{i+1} - \alpha_{i+1}, & \\ \alpha_i\alpha_j = \alpha_j\alpha_i & \text{for } |i-j| > 1, \\ \alpha_i^2 = (x + x^{-1})\alpha_i. & \end{cases}$$

We see that Θ_c is the precomposition of Θ_b with the $\mathbf{Z}[x^{\pm 1}]$-algebra involution of $H_n(x)$ that swaps b_i and c_i. Note that this involution is not the map j from §2.4, since j is not $\mathbf{Z}[x^{\pm 1}]$-linear.

4.2. Moreover, we claim the following analogue of Theorem 3.2.

Theorem 4.1. *The following diagram commutes:*

$$\begin{tikzcd} H_n(x) \arrow[r, "\mathsf{tr}"] \arrow[d, "\Theta_c"'] & \Lambda(x) \arrow[d, "\Xi_e"] \\ H_n^{\mathsf{moy}}(x) \arrow[r, "\mathsf{ann}"] & \mathcal{C}^{\mathsf{moy}}(x) \end{tikzcd}$$

Proof. First, recall that the involution of $\Lambda(x)$ that swaps h_μ and e_μ also swaps s_λ and s_{λ^t}, where λ^t is the transpose of λ [Mac15, (3.8)]. So the map

$$\mathsf{tr}_n^t : H_n(x) \to \Lambda_n(x) \quad \text{defined by } \mathsf{tr}_n^t(\beta) = \sum_{\lambda \vdash n} \chi_x^\lambda(\beta) s_{\lambda^t}$$

satisfies $\Xi_e \circ \mathsf{tr}_n = \Xi_h \circ \mathsf{tr}_n^t$. Next, let η be the $\mathbf{Z}[x^{\pm 1}]$-algebra involution of $H_n(x)$ that swaps b_i and c_i. Since $\Theta_c = \Theta_b \circ \eta$, it remains to show that $\mathsf{tr}^t = \mathsf{tr} \circ \eta$, then invoke Theorem 3.2.

Observe that $\mathsf{tr}_n^t(\beta) = \sum_{\lambda \vdash n} \chi_x^{\lambda^t}(\beta) s_\lambda$. So we must show that $\chi_x^{\lambda^t} = \chi_x^\lambda \circ \eta$ for all λ. Since $\chi^{\lambda^t} = \varepsilon\chi^\lambda$, where ε is the sign character of S_n, we are done by the following lemma. □

Lemma 4.2. *$(\varepsilon\chi)_x = \chi_x \circ \eta$ for all characters χ.*

Proof. By Proposition 9.4.1 of [GP00],

$$(\varepsilon\chi^\lambda)_x(\sigma_w) = (-1)^{\ell_w}\chi_x^\lambda(\sigma_w)|_{x\to x^{-1}}.$$

Using (2.3), we deduce that

$$\chi_x^{\lambda^t}(b_{i_1}\cdots b_{i_\ell}) = \chi_x^\lambda(c_{i_1}\cdots c_{i_\ell}) = \chi_x^\lambda(\eta(b_{i_1}\cdots b_{i_\ell}))$$

for any sequence of indices $i_1,\ldots,i_\ell$. But the b_i generate $H_n(x)$ as an algebra, so every element of $H_n(x)$ is a linear combination of elements of the form $b_{i_1}\cdots b_{i_\ell}$. □

Remark 4.3. The sign character ε, the algebra involution η, and Lemma 4.2 all generalize beyond S_n to any finite Coxeter group.

4.3. TODO

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