



# Skein Relations from Quantum Mechanics

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- 1 Quantum Mechanics
- 2 The Algebra of Coupled Momenta
- 3 Skeins
- 4 Hecke Algebras

Themes of this talk:

- Solving problems in quantum mechanics = studying relations among linear operators.
- The algebras generated by these operators can be abstracted to other settings.
- A particular algebra governing quantum angular momentum also shows up in knot theory.
- Not a coincidence: Representation theory predicts a hierarchy of algebras of broad importance.

## 1 Quantum Mechanics

classical

An observable is a *function*  $f : \mathcal{M} \rightarrow \mathbf{R}$  on a state space  $\mathcal{M}$ .

A measurement in  $E \subseteq \mathbf{R}$  lets us infer a state in  $f^{-1}(E) \subseteq \mathcal{M}$ .

quantum

A state is a line in a Hilbert space  $\mathcal{H}$ .

An observable is a *projection-valued measure*. It assigns a projection  $\pi_E : \mathcal{H} \rightarrow \mathcal{H}$  to each  $E \subseteq \mathbf{R}$ .

The probability of a measurement in  $E$  is

$$\langle \varphi, \pi_E(\varphi) \rangle, \quad \text{for a state with unit vector } \varphi \in \mathcal{H}.$$

The expectation of the observable, given  $\varphi$ , is

$$\langle \varphi, J(\varphi) \rangle, \quad \text{where } J(\varphi) = \int_{\mathbf{R}} \lambda d\pi_{\lambda}(\varphi).$$

We often say that  $J : \mathcal{H} \rightarrow \mathcal{H}$  “is” the observable.

We’ll focus on (total) *quantum angular momentum*:

$$J_x, \quad J_y, \quad J_z.$$

In experiment, the product of the variances of the observables has a strictly positive lower bound.

Heisenberg ( $\sim 1925$ ) Can *derive* this mathematically from the identities

$$[J_x, J_y] = i\hbar J_z, \quad [J_y, J_z] = i\hbar J_x, \quad [J_z, J_x] = i\hbar J_y,$$

where  $[A, B] = A \circ B - B \circ A$ .

Let  $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ ,  $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ ,  $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ :

$$[\sigma_x, \sigma_y] = 2i\sigma_z, \quad [\sigma_y, \sigma_z] = 2i\sigma_x, \quad [\sigma_z, \sigma_x] = 2i\sigma_y.$$

The actions  $J_x, J_y, J_z \curvearrowright \mathcal{H}$  define a *representation* of the Lie algebra

$$\mathfrak{sl}_2 = \mathbf{C}\sigma_x + \mathbf{C}\sigma_y + \mathbf{C}\sigma_z \subseteq \text{Mat}_2(\mathbf{C}).$$

Classic example where the algebra underlying QM has broader importance.

Our main topic is a fancier, more modern example, arising from *coupled* momenta.

## 2 The Algebra of Coupled Momenta

The  $\mathfrak{sl}_2$ -action puts a lot of structure on  $\mathcal{H}$ .

The action must respect a direct-sum decomposition

$$\mathcal{H} = \bigoplus_{s=0, \frac{1}{2}, 1, \frac{3}{2}, \dots} V_s^{\oplus m_s}, \quad \text{where } \dim V_s = 2s + 1.$$

Above,  $s$  is called the *spin number* of  $V_s$ .

Elementary particles have fixed spin numbers.

A system of particles with spins  $s_1, s_2, \dots$  has a state space given by a tensor product:

$$\mathcal{H} = V_{s_1} \otimes V_{s_2} \otimes \dots$$

How to decompose the  $\otimes$  product back into a  $\oplus$  sum?

### two-body problem

$V_c$  occurs in  $V_a \otimes V_b$  if and only if  $a, b, c$  form the sides of a triangle and  $a + b + c \in \mathbf{Z}$ .

In this case, the embedding is unique up to scaling.

$\implies \text{Hom}_{\mathfrak{sl}_2}(V_c, V_a \otimes V_b)$  is one-dimensional.

### three-body problem

$V_d$  can occur in  $V_a \otimes V_b \otimes V_c$  more than once.

$\text{Hom}(V_d, V_a \otimes V_b \otimes V_c)$  has two bases  $(\Phi_e)_e, (\Psi_f)_f$ :

$$\mathbf{C}\Phi_e = \text{Hom}(V_d, \textcolor{red}{V}_e \otimes V_c) \otimes \text{Hom}(\textcolor{red}{V}_e, V_a \otimes V_b),$$

$$\mathbf{C}\Psi_f = \text{Hom}(V_d, V_a \otimes \textcolor{red}{V}_f) \otimes \text{Hom}(\textcolor{red}{V}_f, V_b \otimes V_c).$$

The *6j symbols* are the entries of the change-of-basis matrix from  $(\Phi_e)_e$  to  $(\Psi_f)_f$ :

$$\begin{Bmatrix} a & b & e \\ c & d & f \end{Bmatrix} \in \mathbf{C}.$$

Using self-duality of  $V_a$ , *etc.*, we can show that the symbol is invariant under permutations of  $a, b, c, d$ .

Regge (1958) A more surprising symmetry

$$\begin{Bmatrix} a & b & e \\ c & d & f \end{Bmatrix} = \begin{Bmatrix} p-a & p-b & e \\ p-c & p-d & f \end{Bmatrix},$$

where  $p = \frac{a+b+c+d}{2}$ .

Granovskii–Zhedanov (1988) Conceptual proof of these symmetries via a new operator algebra.

On any  $\mathcal{H}$ , have  $\textcolor{red}{J}^2 := J_x^2 + J_y^2 + J_z^2$  commuting with  $J_x, J_y, J_z$ .

Any nonzero vector in  $V_s$  is an eigenvector of  $J^2$  with eigenvalue  $\hbar^2 s(s+1)$ . Thus,  $J^2$  distinguishes spins.

The  $6j$  symbols arise from two ways to parenthesize:

$$\textcolor{red}{V}_e \otimes V_c \rightarrow (\textcolor{red}{V}_a \otimes \textcolor{red}{V}_b) \otimes V_c,$$

$$V_a \otimes \textcolor{red}{V}_f \rightarrow V_a \otimes (\textcolor{red}{V}_b \otimes \textcolor{red}{V}_c).$$

From  $J_{12}^2 \curvearrowright V_a \otimes V_b$  and  $J_{23}^2 \curvearrowright V_b \otimes V_c$ , we form

$$J_{12}^2 \otimes 1, \quad 1 \otimes J_{23}^2 \quad \curvearrowright \quad V_a \otimes V_b \otimes V_c.$$

The nontriviality of the  $6j$  symbols is the failure of  $J_{12}^2 \otimes 1$  and  $1 \otimes J_{23}^2$  to commute.

3 Skeins We set  $\textcolor{red}{K}_{13} = [K_{12}, K_{23}]$ , where

$$\textcolor{red}{K}_{12} = J_{12}^2 \otimes 1, \quad \textcolor{red}{K}_{23} = 1 \otimes J_{23}^2.$$

The other commutation relations look like

$$[K_{23}, K_{13}] = 2(\eta_1 + \theta K_{23} - \{K_{12}, K_{23}\} - K_{23}^2),$$

$$[K_{13}, K_{12}] = 2(\eta_2 + \theta K_{12} - \{K_{12}, K_{23}\} - K_{12}^2),$$

where  $\{A, B\} = A \circ B + B \circ A$ .

Here,  $\eta_1, \eta_2, \theta$  are polynomial functions of  $a, b, c$ .

Now consider these relations on abstract *variables*.

Berest–Samuelson (2018) These relations arise in knot theory, from Kauffman’s construction of the *Jones polynomial*.

$\Sigma$  = 4-punctured sphere = 3-punctured plane



The *Kauffman skein module* of  $\Sigma$  is

$$\begin{aligned} \text{Sk}_{\Sigma}(q) &= \frac{\mathbf{C}[q^{\pm 1}]\langle \text{unoriented link diagrams in } \Sigma \rangle}{(\text{skein relations})} \\ &= \mathbf{C}[q^{\pm 1}]\langle \Gamma_1, \Gamma_2, \Gamma_3, \Gamma_{12}, \Gamma_{23}, \Gamma_{13}, \Gamma_{123} \rangle \end{aligned}$$

where  $\Gamma_I$  is the loop encircling the punctures in  $I$ .

We make  $\text{Sk}_{\Sigma}(q)$  into a ring by declaring:  $\Gamma \cdot \Gamma'$  is the diagram where we put  $\Gamma'$  on top of  $\Gamma$ .

$\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_{123}$  then belong to the center.

$$\begin{aligned} \text{X} &= q \text{ (cup) } + q^{-1} \text{ (cap) } \\ \bigcirc &= -q^2 - q^{-2} \end{aligned}$$

Berest–Samuelson (2018), Fig. 2

Bullock–Przytycki (1999)  $\mathbf{C}(q) \otimes \text{Sk}_{\Sigma}(q)$  is generated by the elements  $\kappa_{ij} := \frac{\Gamma_{ij} - [2]_q}{(q - q^{-1})^2}$  modulo

$$\begin{aligned} [\kappa_{12}, \kappa_{23}]_q &= \kappa_{13}, \\ [\kappa_{23}, \kappa_{13}]_q &= [2]_q(\eta_{1,q} + \theta_q \kappa_{23} - \{\kappa_{12}, \kappa_{23}\} - \kappa_{23}^2), \\ [\kappa_{13}, \kappa_{12}]_q &= [2]_q(\eta_{2,q} + \theta_q \kappa_{12} - \{\kappa_{12}, \kappa_{23}\} - \kappa_{12}^2), \end{aligned}$$

and one more relation. ( $[A, B]_q = qAB - q^{-1}BA$ .)

$\eta_{1,q}, \eta_{2,q}, \theta_q$  are functions of  $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_{123}$ .

## 4 Hecke Algebras

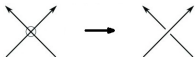
**Problem** Explain the coincidence of identities.

An analogue for *oriented* links should be easier:

$\text{Rep}(\mathfrak{sl}_2)$  has a deformation  $\text{Rep}(U_q(\mathfrak{sl}_2))$  involving a Hopf algebra  $U_q(\mathfrak{sl}_2)$ . The swap maps

$$\begin{array}{ccc} V \otimes U & v \otimes u \\ \uparrow & \uparrow \\ U \otimes V & u \otimes v \end{array}$$

deform to maps that behave like *braidings*.



Elements in an oriented analogue of  $\text{Sk}_\Sigma(q)$  encode diagrams of maps in  $\text{Rep}(U_q(\mathfrak{sl}_2))$ .

$\mathfrak{g}$	$\text{End}_{U_q(\mathfrak{g})}(V^{\otimes n})$	skeins
$\mathfrak{sl}_k$	<i>Hecke algebra</i>	$[0, 1] \times [0, 1]$
$\widehat{\mathfrak{sl}}_k \supset \mathfrak{sl}_k((z))$	<i>affine Hecke algebra</i>	$S^1 \times [0, 1]$

Frenkel ( $\sim 1990$ ) There should be a third row for a “double affine” theory.

Cherednik (1992) The “right” definition of a *double affine Hecke algebra*, with two parameters  $q_1, q_2$ .

$\approx$  Frohman–Gelca (2000) The spherical DAHA for  $\mathfrak{sl}_2$  is a quotient of  $\text{Sk}_\Sigma(q)$ .

$\approx$  Bullock–Przytycki (2000) The  $q_1 = q_2$  limit is a quotient of  $\text{Sk}_T(q)$ , where  $T$  is the torus  $S^1 \times S^1$ .



Spin in QM is explained by the classification

$$\{\text{finite-dim. irreps of } \mathfrak{sl}_2\} = \{V_s\}_{s=0, \frac{1}{2}, 1, \frac{3}{2}, \dots}$$

Analogous classifications known for  $\text{Sk}(q)$ , DAHAs...

**Problem** Construct the irreps from knot theory, *etc.*

Cherednik, Berest–Etingof–Ginzburg (2000s)

Can classify finite-dim. irreps of DAHAs by way of *rational* degenerations.

Gorsky–Oblomkov–Rasmussen–Shende (2014)

Spherical rational DAHA irreps from *triply-graded Khovanov–Rozansky homology* of torus knots.

KhR polynomial in  $a, q, t \rightsquigarrow$  Jones polynomial in  $q$ .



Wolfram, “Torus Knots”

Trinh (2021) Uniform character formula generalizing to torus *links* (and beyond):

$$\sum_{\lambda \vdash n} \text{Deg}_{\lambda}(e^{2\pi i/n})[\Delta_{m/n}(\chi^{\lambda})]_q.$$

**Problem** Explain it using  $\text{Sk}_T(q) \rightarrow \text{sDAHA}$ .

**Problem** Lift theory from DAHA to  $\text{Sk}_T(q), \text{Sk}_{\Sigma}(q)$ .

*Thank you for listening.*