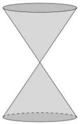
HOMFLYPT Twisting as a Homotopy Equivalence

Minh-Tâm Quang Trinh

Howard University

$G = \mathrm{GL}_n$

$\mathcal{U} = \{\text{unipotent } n \times n \text{ matrices}\} \subseteq G$



<u>Ex</u> For $n = 2$,

$$\mathcal{U} = \left\{ \begin{pmatrix} 1+a & b \\ c & 1-a \end{pmatrix} \middle| \, a^2 + bc = 0 \right\}$$

because u is unipotent $\iff$ $u - I$ is nilpotent
$\iff$ $a + d = ad - bc = 0$.

B upper-triangular subgroup of G

<u>Bruhat, Cartan, Chevalley</u>

$$G = \bigsqcup_{w \in S_n} B\dot{w}B$$

where $\dot{w}$ is the permutation matrix lifting w.

Lets us study G in terms of the *Borel subgroup* B and the *Weyl group* S_n.

What can we say about $\mathcal{U}$ in terms of

$$U = B \cap \mathcal{U}?$$

B_- lower-triangular subgroup of G

<u>Fine–Herstein '58, Steinberg '65</u>

$$\begin{aligned}|\mathcal{U}(\mathbb{F}_q)| &= q^{n(n-1)}\\ &= |U(\mathbb{F}_q)|^2\\ &= |UU_-(\mathbb{F}_q)|.\end{aligned}$$

where $U_- = B_- \cap \mathcal{U}$.

$\mathcal{U}_w = \mathcal{U} \cap B\dot{w}B, \quad \mathcal{V}_w = UU_- \cap B\dot{w}B$

<u>Kawanaka '75</u> For any $w \in S_n$,

$$|\mathcal{U}_w(\mathbb{F}_q)| = |\mathcal{V}_w(\mathbb{F}_q)|.$$

<u>Ex</u> For any n, we have $\mathcal{U}_{\mathrm{id}} = U = \mathcal{V}_{\mathrm{id}}$.

<u>Ex</u> For $n = 3$ and $\dot{w} = \begin{pmatrix} & & 1\\ & 1 & \\ 1 & & \end{pmatrix}$,

$$\mathcal{U}_w \simeq U \times \{(a,b,c,d) \mid a, b \neq 0,\ (1+ab)^3 = abcd\}$$

$$\mathcal{V}_w \simeq U \times \{(a,b,c,d) \mid a, b \neq 0,\ 1+ab = abcd\}$$

These are not even homeomorphic over $\mathbb{C}$.

T diagonal subgroup

$$T \curvearrowright \mathcal{U}_w, \mathcal{V}_w \quad \text{by conjugation.}$$

<u>Thm (T)</u> For any $w \in S_n$,

$$\mathrm{gr}^{\mathsf{W}}_* \mathrm{H}^*_{c,T}(\mathcal{U}_w(\mathbb{C})) \simeq \mathrm{gr}^{\mathsf{W}}_* \mathrm{H}^*_{c,T}(\mathcal{V}_w(\mathbb{C})).$$

$\mathrm{H}^*_{c,T}$ is *equivariant cohomology with compact support*.

$\mathsf{W}_{\leq *}$ is its *weight filtration*.

This implies Kawanaka's identity

$$|\mathcal{U}_w(\mathbb{F}_q)| = |\mathcal{V}_w(\mathbb{F}_q)|$$

via results of Katz.

The map

$$UU_- \to \mathcal{U} \quad \text{given by } xy \mapsto xyx^{-1}$$

is T-equivariant.

<u>Conj (T)</u> It restricts to a homotopy equivalence

$$\mathcal{V}_w(\mathbb{C}) \to \mathcal{U}_w(\mathbb{C}).$$

This would imply the theorem about cohomology.

(The original map is itself a homotopy equivalence: UU_- and $\mathcal{U}$ are both contractible.)

Each $w \in S_n$ lifts to some *braid* $\sigma_w \in Br_n^+$.

For a positive braid $\beta = \sigma_{w_1} \cdots \sigma_{w_k}$, let

$$X_\beta = B\dot{w}_1 B \times^B B\dot{w}_2 B \times^B \cdots \times^B B\dot{w}_k B$$

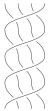
where $\times^B$ means "$\times$ mod antidiagonal B."

For any conjugation-stable set $C \subseteq G$, form

$$\begin{array}{ccc} X_\beta^C & \to & X_\beta \\ \downarrow & & \downarrow \\ C & \to & G \end{array}$$

$\mathcal{U}_w, \mathcal{V}_w$ are special cases of the X_β^C.

The *full twist* π:



Turns out: $\mathcal{U}_w = X_{\sigma_w}^{\mathcal{U}}$ and $\mathcal{V}_w = X_{\sigma_w \pi}^1$.

Thm (T) For any $\beta \in Br_n^+$,

$$|X_\beta^{\mathcal{U}}(\mathbb{F}_q)| = |X_{\beta\pi}^1(\mathbb{F}_q)|,$$

$$\mathrm{gr}_*^{\mathsf{W}} \, \mathrm{H}_{c,T}^*(X_\beta^{\mathcal{U}}(\mathbb{C})) \simeq \mathrm{gr}_*^{\mathsf{W}} \, \mathrm{H}_{c,T}^*(X_{\beta\pi}^1(\mathbb{C})).$$

Surprisingly, the proof uses knot theory.

Previous theorem is the case $\beta = \sigma_w$.

HOMFLYPT poly $P : \{\text{links}\} \to \mathbf{Z}[\![q]\!][a, q^{-1/2}]$,

KhR superpoly $\mathbf{P} : \{\text{links}\} \to \mathbf{Z}[\![q]\!][a, q^{-1/2}, t]$

<u>Kálmán '09</u> Writing $\widehat{\beta}$ for the closure of β,

$$P(\widehat{\beta})[a^{|\beta|-n+1}] = P(\widehat{\beta\pi})[a^{|\beta|+n-1}]$$

where $P[a^i]$ means "q-coefficient of a^i."

<u>Gorsky–Hogancamp–Mellit–Nakagane '19</u>

True with $\mathbf{P}$ in place of P.

We will relate $X_\beta^{\mathcal{U}}, X_\beta^1$ to these pieces of $\mathbf{P}$.

$X_\beta^{\mathcal{U}}, X_\beta^1$ are pieces of a larger variety $\tilde{X}_\beta^{\mathcal{U}}$.

Springer resolution of $\mathcal{U}$:

$$\tilde{\mathcal{U}} = \{(u, gB) \in \mathcal{U} \times G/B \mid ugB = gB\}.$$

Pullback squares:

$$\begin{array}{ccccccc} X_\beta^1 \times G/B & \to & \tilde{X}_\beta^{\mathcal{U}} & \to & X_\beta^{\mathcal{U}} & \to & X_\beta \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \{1\} \times G/B & \to & \tilde{\mathcal{U}} & \to & \mathcal{U} & \to & G \end{array}$$

To relate $X_\beta^{\mathcal{U}}, X_\beta^1$ to pieces of $\mathbf{P}$, relate $\tilde{X}_\beta^{\mathcal{U}}$ to $\mathbf{P}$.

Thm (T) For any $\beta \in Br_n^+$,

$$S_n \curvearrowright \mathrm{gr}_*^{\mathsf{W}} \mathrm{H}_{c,T}^*(\tilde{X}_\beta^{\mathcal{U}}(\mathbb{C})).$$

Moreover,

$$\mathbf{P}(\widehat{\beta}) \;\propto\; (\Lambda^*(\mathbb{V}), \mathrm{gr}_*^{\mathsf{W}} \mathrm{H}_{c,T}^*(\tilde{X}_\beta^{\mathcal{U}}(\mathbb{C})))_{S_n}$$

where $\mathbb{V} = \mathbb{C}^{n-1}$ is the standard irrep of S_n.

The variable a corresponds to the Λ^*-grading.

$$\implies \quad \mathbf{P}(\widehat{\beta})[a^{\mathrm{lo}}] = \dim \mathrm{gr}_*^{\mathsf{W}} \mathrm{H}_{c,T}^*(X_\beta^{\mathcal{U}}(\mathbb{C})),$$

$$\implies \quad \mathbf{P}(\widehat{\beta})[a^{\mathrm{hi}}] = \dim \mathrm{gr}_*^{\mathsf{W}} \mathrm{H}_{c,T}^*(X_\beta^{1}(\mathbb{C})).$$

For P, we can use *symmetric functions* to be more concrete.

Hotta–Springer '77 For $u \in \mathcal{U}$ of Jordan type $\mu \vdash n$,

$$S_n \curvearrowright \mathrm{H}^*(\tilde{\mathcal{U}}_u)$$

is given by the *Hall–Littlewood polynomial* $\tilde{H}_\mu(q)$.

Consequently:

$$P(\widehat{\beta})[a^{\mathrm{lo}+2k}] \propto \sum_{\mu \vdash n} \langle s_{\Lambda^k}, \tilde{H}_\mu(q)\rangle \cdot |X_{\beta,\mu}(\mathbb{F}_q)|.$$

s_{Λ^k} is the *Schur function* of type $(n-k, 1^k)$.

$X_{\beta,\mu} \subseteq X_\beta$ is the preimage of $\{u \in \mathcal{U} \mid \mathrm{type}(u) = \mu\}$.

Why should $\mathcal{U}$ be related to knot invariants?

P arises from traces on *Hecke algebras*.

$\mathbf{P}$ arises from traces on *Hecke categories*.

$$
\begin{aligned}
&\mathsf{D}^b_{mix,G}\mathsf{Perv}(\mathcal{U}) \\
&\quad\simeq \mathsf{D}^b\mathsf{Mod}(\mathbb{C}S_n \ltimes \mathrm{Sym}(\mathbb{V})) &&\text{(Rider)} \\
&\quad\simeq \mathsf{hTr}(\mathsf{Hecke}(S_n)) &&\text{(Gorsky–Wedrich)}
\end{aligned}
$$

Passing from X_β to $\tilde{X}^{\mathcal{U}}_\beta$ is like passing from β to its *annular closure*.

Thank you for listening.