Warmup R analytic, [0, 1] subspace

Q what subsets of [0, 1] are
open in [0, 1] but closed in R?

[tricky, given only the tools developed so far]

suppose U sub [0, 1] works
since U is closed in R, know R – U is open in R
let V = [0, 1] cap (R – U) = [0, 1] – U
then V is open in [0, 1]
so [0, 1] is the union of disjoint opens U and V

[how can this happen?]
it turns out: must have either U = Ø or V = Ø

(Munkres §23) let X be any topological space

Df a separation of X is
a pair of disjoint nonempty opens U, V
s.t. X = U cup V

we say that X is connected iff it has no separation
i.e.
whenever X = U cup V for disjoint open U, V,
we require either U = Ø or V = Ø

Rem if U, V form a separation
then U, V are not just open, but closed

“clopen” = “both open and closed”

[a limit-pt criterion for connectedness:]

<u>Thm</u> suppose Y sub X and A, B sub Y s.t.
A, B disjoint nonempty and Y = A cup B

then A, B is a separation of Y (in subsp. top)
iff
neither contains limit pt of the other
[i.e., limit cond. implies A, B open in Y]

<u>Pf</u> suppose A, B is a separation of Y

know Cl_Y(A) = A cup {limit pts of A}
but A closed in Y, so Cl_Y(A) = A
so {limit pts of A} sub A disj from B
analogously, {limit pts of B} disj from A

suppose neither A, B contains limit pt of other

then Cl_Y(A) cap B = Ø and Cl_Y(B) cap A = Ø
but A = Y – B and B = Y – A
so Cl_Y(A) sub A and Cl_Y(B) sub B
so Cl_Y(A) = A and Cl_Y(B) = B
so A, B closed (hence also open)

<u>Ex</u> X = analytic R, Y = [0, 1/2) cup (1/2, 1]
A = [0, 1/2), B = (1/2, 1] separates Y

<u>Ex</u> X = analytic R^2, Y = A cup B where
A = {(x, 0) | x in R}, B = {(x, 1/x) | x in R}
[picture]

<u>Ex</u> is the empty set connected? yes!

Ex X = analytic R, Y = Q [set of rationals]

claim: for all a, b in Q s.t. a ≠ b,
have a separation Q = A cup B s.t.
a in A and b in B
[even though Q is not discrete]

pick irrational α in R s.t. a < α < b
A = {x in Q | x < α}, B = {x in Q | x > α}

[if Y = R, then is the analogous claim still true? no]

Rem in general, if Y sub X:

can have X connected but Y disconnected
can also have Y disconnected but X connected

Ex is R^ω connected? [need: in which top?]

in box topology: no!
let U = {bounded sequences x}
[i.e., there is M s.t. |x_i| ≤ M for all i]
V = {unbounded sequences x}

U, V disjoint nonempty and R^ω = U cup V
U is open: if x in U and B is a box around x,
then B sub U
V is open: if x in V and B is a box around x,
then B sub V

by contrast:
it turns out R^ω is connected in the product top

[difficult, so first let's prove smthg easier:]

<u>Thm</u> any finite product of connected spaces is connected

<u>Pf</u> Munkres 118, #4:
X_1 × … × X_n homeo to
(X_1 × … × X_{n – 1}) × X_n

so by induction,
suffices to show: X, Y conn implies
X × Y conn

[draw picture]

if X × Y = Ø then done, so can assume X, Y ≠ Ø

idea: use a "slice" X × {b}

[draw slice inside X × Y]

X × Y = bigcup_{x in X} {x} × Y
= bigcup_{x in X} (X × {b}) cup ({x} × Y)

note: X conn implies X × {b} conn,
Y conn implies {x} × Y conn
bigcap_x (X × {b}) cup ({x} × Y) ≠ Ø

[so it remains to show:]

<u>Lem 1</u> if {Z_i}_i is a collection of sub's of Z,
Z_i connected for all i,
bigcap_i Z_i ≠ Ø,
then bigcup_i Z_i is also connected

Pf   pick p in bigcap_i Z_i

suppose bigcap_i Z_i = C cup D
    where C, D are disjoint
either p in C or p in D
WLOG assume p in C
want to show that Z_i sub C for all i
    because then, D = Ø
    meaning C, D is not a separation

Lem 2   if C, D is a separation of X,
        Y sub X is connected,
    then Y sub C or Y sub D

Pf   otherwise, C cap Y and C cap D
    form a separation of Y

Sorites

- finite product of conn. spaces is conn.
- union of conn.'s w/ point in common is conn.

also:
- image of conn. under cts map is conn.
- if A conn. and A sub B sub Cl_X(A) sub X,
  then B conn.

next time: is R connected? yes but why?