MATH 250: Topology I

[intros]

[recap: website
LaTeX
textbook
late hw policy
exam dates]

[initial reading]

I) algebraic topology:

how to tell shapes apart?

Euler: Königsberg (1736), “V – E + F = 2” (1750)
Poincaré: Analysis situs (1890s–1900s)

II) point-set topology:

what IS shape?

Weyl: rigorous defn of “surface” (1913)
Hausdorff: topological spaces (1914)

[we first do (II), then do (I)]

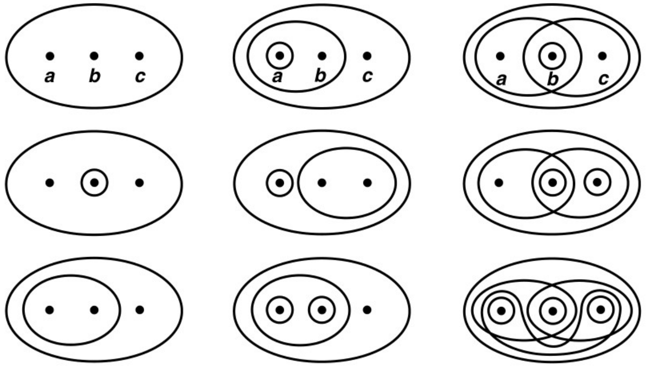
(Munkres §12) fix a set X

Def a topology on X is
a collection T of subsets of X s.t.

1) Ø, X in T
2) if {U_i}_i is a subcollection of T, then the union of the U_i in X is in T
3) if {U_i}_i is a finite subcollection of T, then the intersection of the U_i in X is in T

we say that
(X, T) is a topological space
the elements of T are its open sets

Ex X = {a, b, c}



[in each case, Ø is not depicted]

[example collection of subsets that isn’t a topology?]

Ex X = R^n with |x| = sqrt(sum_i x_i^2)

write B(x, δ) = {y in R^n | |y – x| < δ}

we say U sub R^n is analytically open iff

for all x in U,

there is some δ > 0 s.t. B(x, δ) sub U

Thm {analytically open sets} is a topology on R^n

1) easy
2) suppose U_i anlytc opens, U = bigcup_i U_i

pick x in U

x belongs to x in U_j for some j

have δ > 0 s.t. B(x, δ) sub U_j sub U

3) suppose finitely many i,

U_i analytic opens, V = bigcap_i U_i

pick x in V

for all i, pick δ_i > 0 s.t. B(x, δ_i) sub U_i

[what next?] let δ = min_i δ_i

then B(x, δ) sub B(x, δ_i) sub U_i for all i

therefore B(x, δ) sub V

[observe: 3) wouldn't work for infinite δ_i → 0]

we call this the analytic topology T_{an} on R^n

Rem not the only topology possible:

also discrete and indiscrete topologies

Ex X arbitrary

T_f = {Ø} cup {U sub X s.t. X – U is finite}

Thm T_f is a topology on X

1) easy
2) suppose {U_i}_i is a subcollection of T_f,
    U = bigcup_i U_i
    [what happens if U_i = Ø for all i?]
    [what if U_i ≠ Ø for some i?]
3) suppose finitely many i,
    U_i subcollection of T_f
    V = bigcap_i U_i
    [what happens if U_i = Ø for some i?]
    [what if U_i ≠ Ø for all i?]

we call this the finite complement topology

Df given topologies T, T′ on the same X
s.t. T is a subcollection of T′,

we say that
    T is coarser than T′
    T′ is finer than T

[T′ is more refined: it sees more open sets]

Ex [how do the topologies on R^n compare?
analytic, discrete, indiscrete, finite-comp]

T_{indisc} sub T_f sub T_{an} sub T_{disc}

Rem topologies can be incomparable: think about $X = \{a, b, c\}$

(Munkres §13) $\{B_i\}_{i \in I}$ any collection of subsets of X

Df $\{B_i\}_{i \in I}$ is a basis for a top on X iff

1) $X = \bigcup_{i \in I} B_i$
2) for all i, j in I, and x in $B_i \cap B_j$,
have k in I s.t. $x \in B_k \subseteq B_i \cap B_j$
["$B_i \cap B_j$ is covered by B_k's"]

Rem Munkres also discusses "subbases"

[idea: reduce q's abt topologies to q's abt bases]

any basis generates a topology: take

$$T = \{\bigcup_{i \in I'} B_i \mid \text{any } I' \subseteq I\}$$

[note! includes $I' = \varnothing$]

[is this really a topology?]

Ex $\{B(x, \delta) \mid x \in R \text{ and } \delta > 0\}$ is a basis for the analytic top on R [why?]

[what if we drop the condition $\delta > 0$?]

Rem different bases can generate the same topology

[criterion to check when a subcoll is a basis]

<u>Thm</u> suppose T is a topology on X,

{B_i}_{i in I} a subcollection of T

s.t. for all U in T, and x in U,

there is some k in I s.t. x in B_k sub U

then {B_i}_{i in I} is a basis, and it generates T

<u>Pf</u> 1) X = bigcup_{i in I} B_i [why?]

2) pick i, j, and x in B_i cap B_j

B_i cap B_j in T since B_i, B_j in T

get k s.t. x in B_k sub B_i cap B_k

so {B_i}_i is a basis

finally, T = {bigcup_{i in I′} B_i | I′ sub I} [why?]

<u>Ex</u> recall: {B(x, δ) | x, δ in R} is a basis for the analytic top on R

same as {(a, b) | a, b in R} [why?]

is {(a, b) | a, b in Q} a basis?

is {(a, b) | a, b in Z} a basis?