(Munkres §25) recall that for any X:

the connected components of X are
the equiv. classes where x ~ y iff
there is a conn. subsp. containing x and y

the path components of X are
the equiv. classes where x ↔ y iff
there is a path between x and y in X

<u>Lem</u> the conn., resp. path components
are the <u>maximal</u> nonempty
conn., resp. path-conn. subspaces

[e.g., if A sub X conn, then A sub a conn. comp]

<u>Pf</u> by the def of an equiv. relation…

<u>Q</u> are conn. components open? closed?

<u>Ex</u> we say X is totally disconnected iff
all nonempty conn. subspaces of X
are singletons

PS5, #1 says Q is totally disconnected in R
singletons in Q are closed, but not open

<u>Thm</u> if A sub X is conn, then Cl_X(A) is too

more generally:
any B s.t. A sub B sub Cl_X(A) is too

Cor conn. components of X are closed
[by maximality]

Pf of Thm suppose U, V is a separation of B

by Feb 5 lecture (also M. Lem. 23.2), know:
since A is connected, either A sub U or A sub V

then Cl_B(A) sub Cl_B(U) = U
but by Feb 3 lecture (also M. Thm. 17.4),
Cl_B(A) = Cl_X(A) cap B (= B here)
so B sub U
contradicts V being nonempty □

Cor if X has finitely many conn. components
then they are also open [thus clopen]

Q are path components open?
closed?

Ex in the topologist's sine curve
Š = S cup A,
where S = {(x, sin(1/x)) | x in (0, 1]}
A = {(0, y) | y in [–1, 1]}

we can check that S, A are the path components
S is open but not closed
A is closed but not open

Rem since path-connected implies connected

each path component of X is contained in
some conn. component of X

Q when can we ensure
{conn. components of X}
=
{path components of X}?

[helps to weaken our notions of conn., path-conn.]

Df X is locally connected at x iff,
for all open U containing x,
there is a conn. open V s.t. x in V sub U

similar def for locally path-connected,
replacing conn. V with path-conn. V

Ex [0, 1) cup (1, 2] is loc. conn but not conn
Q is not locally connected

Thm if X is locally path-connected
then its conn. comp's are path comp's

Lem if X is locally path-connected
then for any open U sub X,
path comp. P of U,
P is also open in X

[similar lem for locally connected,
replacing path comp. P with conn. comp.]

Pf pick x in P
since X is locally path-connected,
have path-conn. open V sub U s.t.
x in V sub U
by maximality of P, have V sub P

Pf of Thm   pick a conn. component C of X

C ≠ Ø bc it's an equiv. class
so pick x in C
let P be the path component containing x
then P sub C; we want P = C

any path component of X intersecting C
    is conn., hence contained in C [by max'lty]
let Q be the union of these except for P
then C = P cup Q

claim that if Q ≠ Ø, then a contradiction: [why?]
    by lemma, each path comp. of X is open in X
    hence P, Q are open in X, hence in C
    hence P, Q would form a separation of C □

(Munkres §26)   X arbitrary top space

in analysis, two notions of compactness:
    "sequential" compactness
    "finite-cover" compactness
in topology, the latter is the standard notion

Df   an open cover of X is
    a collection of open sets {U_i}_i in X s.t.
        X = bigcup_i U_i

[similar to a subbasis, but need not generate the topology]

    a subcover of {U_i}_i is
    a subcollection that remains a cover

we say that X is <u>compact</u> iff
every cover of X admits a finite subcover

<u>Ex</u> anlytc R not anlytc compact:
take $U_n = (n - 1, n + 1)$ for n in Z

<u>Ex</u> $K = \{1/n \mid n = 1, 2, 3, \ldots\}$ not compact:
[what cover?]
take singletons $\{1/n\}$ for n = 1, 2, 3, …

<u>Ex</u> $Cl_R(K) = \{0\} \cup \{1/n \mid n\}$ is compact:

any cover must include U s.t. 0 in U
then Z – U is a finite set

<u>Ex</u> (Heine–Borel) anlytc [0, 1] is compact

[plays well with subspaces:]

<u>Lem</u> TFAE for Y sub X:

1) Y is compact as a subspace of X
2) for any collection of opens $\{U_i\}_i$ s.t.
$Y \subseteq \bigcup_i U_i$,
there is a finite subcollection $\{U_j\}_{j \in J}$ s.t.
$Y \subseteq \bigcup_{j \in J} U_j$

<u>Pf</u> boring

<u>Thm</u> closed subspaces of compact spaces are compact

thus Heine–Borel implies compactness of $Cl_R(K)$

Pf let X be compact and Y closed

given collection of opens {U_i}_i in X s.t.
Y sub bigcup_i U_i
consider {X – Y} cup {U_i}_i
this is an open cover of X, since X – Y is open
so it has a finite subcover
even if we remove X – Y from this subcover,
it remains finite and its union contains Y

[tell the Sorensen story?]