# Affine Springer Fibers and Level-Rank Duality

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Mainly about joint work with Ting Xue:

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See also the extended abstract on my website, which we have submitted to FPSAC '25.

## 1 Springer Theory

Work over $\mathbf{C}$.

$G$ connected reductive group with Lie algebra $\mathfrak{g}$

$A$ maximal torus

$B$ Borel containing $A$

$W$ Weyl group $(= N_G(A)/A)$

The *Grothendieck alteration* is $\tilde{\mathfrak{g}} \to \mathfrak{g}$, where

$$\tilde{\mathfrak{g}} = \{(\gamma, gB) \in \mathfrak{g} \times G/B \mid \gamma \in g\mathfrak{b}g^{-1}\}$$

Over the *regular semisimple* locus

$$\mathfrak{g}^{\mathrm{rs}} = \{\gamma \in \mathfrak{g} \mid G_\gamma \text{ is a maximal torus}\},$$

it restricts to an unramified $W$-cover.

In fact, $\widetilde{\mathfrak{g}^{\mathrm{rs}}} \to \mathfrak{g}^{\mathrm{rs}}$ is a pullback of $\mathfrak{a}^{\mathrm{rs}} \to \mathfrak{a}^{\mathrm{rs}} /\!\!/ W$.

$W^{\mathrm{aff}} = W \ltimes X_*(A) \qquad (= N_{G((z))}(A[\![z]\!])/A[\![z]\!])$

**(Kazhdan–Lusztig)** A loop or *affine* analogue:

$$\mathfrak{g}((z))^{\mathrm{rs}} = \left\{ \gamma \in \mathfrak{g}((z)) \;\middle|\; \begin{array}{l} G((z))_\gamma \text{ is a possibly} \\ \textit{nonsplit} \text{ maximal torus} \end{array} \right\},$$

$$\widetilde{\mathfrak{g}((z))^{\mathrm{rs}}} = \{(\gamma, gI) \in \mathfrak{g}((z))^{\mathrm{rs}} \times G((z))/I \mid \gamma \in g\mathfrak{I}g^{-1}\},$$

where $I \subseteq G[\![z]\!]$ is the preimage of $B \subseteq G$.

Unlike before, the map $\widetilde{\mathfrak{g}((z))^{\mathrm{rs}}} \to \mathfrak{g}((z))^{\mathrm{rs}}$ is not even locally constant.

Yet the fibers $\mathcal{F}l_\gamma \subseteq G((z))/I$ are finite-dimensional.

For nonempty fibers, $W^{\mathrm{aff}} \curvearrowright \mathrm{H}^*_c(\mathcal{F}l_\gamma)$.

**Example** Suppose that $G = \mathrm{SL}_2$,

$$\mathfrak{g}((z)) = \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \;\middle|\; a, b, c \in \mathbf{C}[\![z]\!] \right\},$$

$$\mathfrak{I} = \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \;\middle|\; a, b \in \mathbf{C}[\![z]\!],\ c \in z\mathbf{C}[\![z]\!] \right\}.$$

$\gamma = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$ $\qquad \mathcal{F}l_\gamma = \{\dot{w}I \mid w \in W^{\mathrm{aff}}\}$

$\gamma = \begin{pmatrix} & 1 \\ z & \end{pmatrix}$ $\qquad \mathcal{F}l_\gamma = \{I\}$

$\gamma = \begin{pmatrix} z & \\ & -z \end{pmatrix}$ $\qquad \mathcal{F}l_\gamma$ is an *infinite* chain of projective lines, intersecting transversely at the $\dot{w}I$

$\gamma = \begin{pmatrix} & z \\ z^2 & \end{pmatrix}$ $\qquad \mathcal{F}l_\gamma$ is a union of *two* transverse projective lines

More complicated components are possible.

At the same time, $\mathbf{C}[\mathfrak{a}] = \mathrm{H}^*_I(\mathit{point}) \curvearrowright \mathrm{H}^*_c(\mathcal{F}l_\gamma)$.

For special $\gamma$, the $(\mathbf{C}W^{\mathrm{aff}} \ltimes \mathbf{C}[\mathfrak{a}])$-action *deforms*.

Let $m$ be a positive integer and $\rho^\vee = \frac{1}{2}\sum_\alpha \alpha^\vee$.

$$\mathbf{C}^\times \curvearrowright \mathfrak{g}(\!(z^{1/2})\!),$$
$$c \cdot_m \gamma(z^{1/2}) = \mathrm{Ad}(z^{\rho^\vee})\gamma(z^{m/2}).$$

We say* that $\gamma$ is *homogeneous* of slope $\frac{d}{m}$ iff

$$c \cdot_m \gamma = c^d \gamma.$$

In this case, $\mathcal{F}l_\gamma$ is stable under $\mathbf{C}^\times \curvearrowright G(\!(z)\!)/I$.

The preceding examples: slopes 0, $\frac{1}{2}$, 1, $\frac{3}{2}$.

* Differs by conjugation from the $\gamma$ in Oblomkov–Yun.

(Oblomkov–Yun) Take $G$ simply-connected, simple.

If $\gamma$ has slope $\frac{d}{m} > 0$, then

$$\mathrm{D}^{\mathrm{trig}}_{d/m} \curvearrowright \mathrm{H}^*_{\mathbf{C}^\times}(\mathcal{F}l_\gamma)^{\pi_0(A_\gamma)}|_{\epsilon \to 1}.$$

- $\epsilon$ is the generator of $\mathrm{H}^*_{\mathbf{C}^\times}(\mathit{point})$.
- $\mathrm{D}^{\mathrm{trig}}_{d/m}$ is a *graded DAHA*, deforming $\mathbf{C}W^{\mathrm{aff}} \ltimes \mathbf{C}[\mathfrak{a}]$.

If $\mathcal{F}l_\gamma$ is projective, then the action degenerates to

$$\mathrm{D}^{\mathrm{rat}}_{d/m} \curvearrowright \mathrm{gr}^{\mathsf{P}}_* \mathrm{H}^*_{\mathbf{C}^\times}(\mathcal{F}l_\gamma)^{\pi_0(A_\gamma)}|_{\epsilon \to 1},$$

- $\mathsf{P}$ is a *perverse filtration* arising from a Ngô-type global model.
- $\mathrm{D}^{\mathrm{rat}}_{d/m}$ is a *rational DAHA*, deforming $\mathbf{C}W \ltimes \mathcal{D}(\mathfrak{a})$.

| | $\mathrm{D}^{\mathrm{rat}}_{d/m}$ | $\mathrm{U}\mathfrak{g}$ |
|---|---|---|
| PBW | $\mathbf{C}[\mathfrak{a}] \otimes \mathbf{C}W \otimes \mathbf{C}[\mathfrak{a}^*]$ | $\mathrm{U}\mathfrak{n}_- \otimes \mathbf{C}[\mathfrak{a}] \otimes \mathrm{U}\mathfrak{n}_+$ |
| Verma | $\Delta_{d/m}(\chi)$ | $\Delta(\lambda)$ |
| simple | $L_{d/m}(\chi)$ | $L(\lambda)$ |
| character | $[M]_{\mathsf{x}} \in \mathrm{K}_0(W)(\!(\mathsf{x})\!)$ | $[M] \in \mathbf{Z}[X^*(A)]$ |

**Problem** Compute the module/character structure of

$$\mathcal{E}_{\gamma,\mathsf{x},\mathsf{y}} = \sum_{i,j} \mathsf{x}^i \mathsf{y}^j \, \mathrm{gr}^{\mathsf{P}}_i \, \mathrm{H}^j_{\mathbf{C}^\times}(\mathcal{F}l_\gamma)^{\pi_0(A_\gamma)}|_{\epsilon \to 1}.$$

**Idea** Over a certain locus $\mathfrak{c}^{\mathrm{rs}}_{d/m} \subseteq \mathfrak{g}(\!(z)\!)^{\mathrm{rs}}$, the $\mathcal{E}_\gamma$ form a local system.

The actions of $\mathrm{D}^{\mathrm{rat}}_{d/m}$ and $\pi_1(\mathfrak{c}^{\mathrm{rs}}_{d/m})$ *commute.*

Oblomkov–Yun showed that $\pi_1(\mathfrak{c}^{\mathrm{rs}}_{d/m})$ is the braid group for some *complex* reflection group $C$.

**Conjecture (T–Xue)** Given $G$, $\frac{d}{m}$, $\gamma$:

1 The monodromy of $\mathcal{E}_\gamma$ factors through a Hecke algebra for $C$ with parameter $\mathsf{q} = 1$.

2 There exist $\{\chi_\rho\}_\rho \subseteq \mathrm{Irr}(W)$, $\{\psi_\rho\}_\rho \subseteq \mathrm{Irr}(C)$, and signs $\varepsilon_\rho$ such that

$$\mathcal{E}_{\gamma,\mathsf{x},-1} = \sum_\rho \varepsilon_\rho \left([\Delta_{d/m}(\chi_\rho)]_{\mathsf{x}} \otimes (\psi_\rho)_{\mathsf{q}=1}\right)$$

as a virtual $(\mathrm{D}^{\mathrm{rat}}_{d/m}, \text{Hecke})$-bimodule.

The rest of this talk is about the explicit recipe for $\rho, \chi_\rho, \psi_\rho, \varepsilon_\rho$, which was surprising to us.

Much of this setup lets us replace $G$ with a *quasi-split* form $G_{\mathbf{C}(\!(z)\!)}$ over $\mathbf{C}(\!(z)\!)$.

Replace $A$ with the maximal torus $A_{\mathbf{C}(\!(z)\!)}$ defined by the Dynkin automorphism.

Observe that, up to Tate twist,

$$\mathrm{Gal}(\bar{\mathbf{F}}_q|\mathbf{F}_q) \quad \simeq \quad \hat{\mathbf{Z}} \quad \simeq \quad \mathrm{Gal}(\overline{\mathbf{C}(\!(z)\!)}|\mathbf{C}(\!(z)\!)).$$

Fix a "good" $q$ for $G$.

$$\begin{aligned} \mathbf{G}_{\mathbf{F}_q} &\quad\leftrightarrow\quad G_{\mathbf{C}(\!(z)\!)} \\ \mathbf{A}_{\mathbf{F}_q} &\quad\leftrightarrow\quad A_{\mathbf{C}(\!(z)\!)} \\ \mathbf{T}_{\mathbf{F}_q} &\quad\leftrightarrow\quad G_{\mathbf{C}(\!(z)\!),\gamma} \end{aligned}$$

Our Hecke algebra will arise from the $\mathbf{F}_q$ side.

## 2 Deligne–Lusztig Theory

$$F = \mathrm{Frob} \otimes \mathrm{id} \quad \curvearrowright \quad \mathbf{G} = \mathbf{G}_{\mathbf{F}_q} \otimes_{\mathbf{F}_q} \bar{\mathbf{F}}_q.$$

$\mathbf{G}^F = \mathbf{G}_{\mathbf{F}_q}(\mathbf{F}_q)$ forms a *finite group of Lie type*.

For any $F$-stable maximal torus $\mathbf{T}$ contained in a Borel $\mathbf{B} = \mathbf{T} \ltimes \mathbf{U}$, set

$$Y_{\mathbf{B}}^{\mathbf{G}} = \{g \in \mathbf{G} \mid g^{-1}F(g) \in F(\mathbf{U})\}/(\mathbf{U} \cap F(\mathbf{U})).$$

Commuting actions:

$$\mathbf{G}^F \quad \curvearrowright \quad \mathrm{H}_c^*(Y_{\mathbf{B}}^{\mathbf{G}}) \quad \curvearrowleft \quad \mathbf{T}^F.$$

For $\theta \in \mathrm{Irr}(\mathbf{T}^F)$, set $R_{\mathbf{T}}^{\mathbf{G}}(\theta) = \sum_i (-1)^i \mathrm{H}_c^i(Y_{\mathbf{B}}^{\mathbf{G}})[\theta]$.

Conjecturally, independent of $\mathbf{B}$ when $q$ is good.

Every irrep of $\mathbf{G}^F$ occurs in $R_{\mathbf{T}}^{\mathbf{G}}(\theta)$ for some $\mathbf{T}, \theta$.

(Broué–Malle) For certain *m-regular* $\mathbf{T}$, a specific algebra $H^{\mathbf{G}}_{\mathbf{T}}(\mathsf{q})$ such that

$$H^{\mathbf{G}}_{\mathbf{T}}(\zeta_m) = \bar{\mathbf{Q}}W^{\mathbf{G}}_{\mathbf{T}}, \quad \text{where } W^{\mathbf{G}}_{\mathbf{T}} = N_{\mathbf{G}^F}(\mathbf{T}^F)/\mathbf{T}^F.$$

They conjecture:

1 $H^{\mathbf{G}}_{\mathbf{T}}(q) \otimes \bar{\mathbf{Q}}_\ell \simeq \mathrm{End}_{\mathbf{G}^F}(\mathrm{H}^*_c(Y^{\mathbf{G}}_{\mathbf{B}})[1_{\mathbf{T}^F}])$.

2 As a virtual $(\mathbf{G}^F, H^{\mathbf{G}}_{\mathbf{T}}(q))$-bimodule,

$$R^{\mathbf{G}}_{\mathbf{T}}(1_{\mathbf{T}^F}) = \sum_{\substack{\rho \in \mathrm{Irr}(\mathbf{G}^F) \\ (\rho, R^{\mathbf{G}}_{\mathbf{T}}(1_{\mathbf{T}^F})) \neq 0}} \varepsilon_{\mathbf{T},\rho} \left(\rho \otimes (\chi_{\mathbf{T},\rho})_{\mathsf{q}=q}\right)$$

where $\chi_{\mathbf{T},\rho} \in \mathrm{Irr}(W^{\mathbf{G}}_{\mathbf{T}})$ and $\varepsilon_{\mathbf{T},\rho} \in \{\pm 1\}$.

Back to $\mathbf{C}(\!(z)\!)$. Assume that $\gcd(d, m) = 1$ and

$$\mathbf{G}_{\mathbf{F}_q}, \mathbf{A}_{\mathbf{F}_q}, \mathbf{T}_{\mathbf{F}_q} \quad \leftrightarrow \quad G_{\mathbf{C}(\!(z)\!)}, A_{\mathbf{C}(\!(z)\!)}, G_{\mathbf{C}(\!(z)\!),\gamma}.$$

The $F$-stable tori $\mathbf{A}$ and $\mathbf{T}$ are 1- and $m$-regular.

The braid group of $W^{\mathbf{G}}_{\mathbf{T}}$ is $\pi_1(\mathfrak{c}^{\mathrm{rs}}_{d/m})$.

Conjecture (T–Xue)

1 The monodromy of $\mathcal{E}_\gamma$ factors through $H^{\mathbf{G}}_{\mathbf{T}}(1)$.

2 Using $W^{\mathbf{G}}_{\mathbf{A}}$ to define $\mathrm{D}^{\mathrm{rat}}_{d/m}$ for nonsplit $G_{\mathbf{C}(\!(z)\!)}$,

$$\mathcal{E}_{\gamma,\mathsf{x},-1} = \sum_{\rho} \varepsilon_{\mathbf{T},\rho} \left([\Delta_{d/m}(\chi_{\mathbf{A},\rho})]_{\mathsf{x}} \otimes (\chi_{\mathbf{T},\rho})_{\mathsf{q}=1}\right)$$

as a virtual $(\mathrm{D}^{\mathrm{rat}}_{d/m}, H^{\mathbf{G}}_{\mathbf{T}}(1))$-bimodule.

Here, require both $(\rho, R^{\mathbf{G}}_{\mathbf{A}}(1_{\mathbf{A}^F})), (\rho, R^{\mathbf{G}}_{\mathbf{T}}(1_{\mathbf{T}^F})) \neq 0$.

**Theorem (T–Xue)** True for $(G_{\mathbf{C}((z))}, m)$ in any of these cases:

- $m$ is the (twisted) Coxeter number of $G_{\mathbf{C}((z))}$.
- $({}^2A_2, 2), (C_2, 2), (G_2, 3), (G_2, 2)$.
- $({}^2A_3, 2), ({}^2A_4, 2), ({}^3D_4, 6), ({}^3D_4, 3)$, assuming a conjecture of Oblomkov–Yun.

**Example** Take $G_{\mathbf{C}((z))}$ split, $m$ its Coxeter number.

$\chi_{\mathbf{A},\rho}$ runs over "wedge" characters of $W$.

$\chi_{\mathbf{T},\rho}$ runs over all characters of $W_{\mathbf{T}}^{\mathbf{G}} = \mathbf{Z}/m\mathbf{Z}$.

The virtual $\mathrm{D}_{d/m}^{\mathrm{rat}}$-module is

$$\sum_{k=0}^{m-1} (-1)^k [\Delta_{d/m}(\chi_{\wedge^k(\mathfrak{a})})]_{\mathsf{x}} = [L_{d/m}(1_W)]_{\mathsf{x}}.$$

## 3 Level–Rank Duality

Compare $\mathcal{E}_{\gamma,\mathsf{x},-1}$ given by

$$(1) \qquad \sum_{\rho} \varepsilon_{\mathbf{T},\rho} \left([\Delta_{d/m}(\chi_{\mathbf{A},\rho})]_{\mathsf{x}} \otimes (\chi_{\mathbf{T},\rho})_{\mathsf{q}=1}\right)$$

with $R_{\mathbf{A}}^{\mathbf{G}}(1_{\mathbf{A}^F}) \otimes R_{\mathbf{T}}^{\mathbf{G}}(1_{\mathbf{T}^F})$ given by

$$(2) \qquad \sum_{\rho} \varepsilon_{\mathbf{T},\rho} \left((\chi_{\mathbf{A},\rho})_{\mathsf{q}=q} \otimes (\chi_{\mathbf{T},\rho})_{\mathsf{q}=q}\right).$$

(Note that $\varepsilon_{\mathbf{A},\rho} = 1$ for all $\rho$ in $R_{\mathbf{A}}^{\mathbf{G}}(1_{\mathbf{A}^F})$.)

**Problem** How to make (1) look more symmetric, the way that (2) is symmetric?

**Idea** Would need to replace $\mathrm{D}_{d/m}^{\mathrm{rat}}$ with $H_{\mathbf{A}}^{\mathbf{G}}(\zeta_m)$.

(Ginzburg–Guay–Opdam–Rouquier) An exact *Knizhnik–Zamolodchikov (KZ) functor*

$$\mathsf{KZ} : \underbrace{\mathsf{Rep}(\mathrm{D}^{\mathrm{rat}}_{d/m})}_{\text{category } \mathcal{O}} \to \mathsf{Rep}(H^{\mathbf{G}}_{\mathbf{A}}(\zeta_m^d))$$

such that $\mathsf{KZ}(\Delta_{d/m}(\chi)) = \chi_{\mathsf{q}=\zeta_m^d}$ for all $\chi$.

| Deligne–Lusztig | affine Springer |
|---|---|
| $\mathbf{F}_q$ | $\mathbf{C}(\!(z)\!)$ |
| $R^{\mathbf{G}}_{\mathbf{A}}(1_{\mathbf{A}^F}) \otimes R^{\mathbf{G}}_{\mathbf{T}}(1_{\mathbf{T}^F})$ | $(\mathsf{KZ} \otimes \mathsf{id})(\mathcal{E}_{\gamma,\mathsf{x},-1})$ |
| $H^{\mathbf{G}}_{\mathbf{A}}(q) \otimes H^{\mathbf{G}}_{\mathbf{T}}(q)^{\mathrm{op}}$ | $H^{\mathbf{G}}_{\mathbf{A}}(\zeta_m^d) \otimes H^{\mathbf{G}}_{\mathbf{T}}(\zeta_1)^{\mathrm{op}}$ |

Note the "reciprocity" between $\zeta_m$ and $\zeta_1$.

Let $\mathrm{Uch}(\mathbf{G}^F)$ be the set of *unipotent* irreps of $\mathbf{G}^F$: the irreps that occur in $R^{\mathbf{G}}_{\mathbf{T}}(1_{\mathbf{T}^F})$ for some $\mathbf{T}$.

(Broué–Malle–Michel) Fix $l > 0$. An *$l$-cuspidal pair* for $\mathbf{G}$ consists of:

- A Levi subgroup $\mathbf{L} = Z_{\mathbf{G}}(\mathbf{S})^\circ$, where $\mathbf{S}$ is a torus with $|\mathbf{S}^F|$ a power of $\Phi_l(q)$.
- $\lambda \in \mathrm{Uch}(\mathbf{L}^F)$ such that $(\lambda, R^{\mathbf{L}}_{\mathbf{M}}(\mu)) = 0$ for any $l$-split $\mathbf{M} \subset \mathbf{L}$ and $\mu$.

As we run over $l$-cuspidal pairs up to conjugacy,

$$\mathrm{Uch}(\mathbf{G}^F) = \coprod_{[\mathbf{L},\lambda]} \mathrm{Uch}(\mathbf{G}^F)_{\mathbf{L},\lambda}$$

where $\mathrm{Uch}(\mathbf{G}^F)_{\mathbf{L},\lambda} = \{\rho \mid (\rho, R^{\mathbf{G}}_{\mathbf{L}}(\lambda)) \neq 0\}$.

For $l = 1$, these are classical *Harish-Chandra series*.

Broué–Malle define a Hecke algebra $H^{\mathbf{G}}_{\mathbf{L},\lambda}(\mathsf{q})$ such that

$$H^{\mathbf{G}}_{\mathbf{L},\lambda}(\zeta_l) = \bar{\mathbf{Q}}W^{\mathbf{G}}_{\mathbf{L},\lambda}, \text{ where } W^{\mathbf{G}}_{\mathbf{L},\lambda} = N_{\mathbf{G}^F}(\mathbf{L}^F, \lambda)/\mathbf{L}^F.$$

For an appropriate Deligne–Lusztig variety $Y^{\mathbf{G}}_{\mathbf{P}}$, they conjecture:

1 $H^{\mathbf{G}}_{\mathbf{L},\lambda}(q) \otimes \bar{\mathbf{Q}}_\ell = \mathrm{End}_{\mathbf{G}^F}(\mathrm{H}^*_c(Y^{\mathbf{G}}_{\mathbf{P}})[\lambda])$.

2 As a virtual $(\mathbf{G}^F, H^{\mathbf{G}}_{\mathbf{L},\lambda}(q))$-bimodule,

$$R^{\mathbf{G}}_{\mathbf{L}}(\lambda) = \sum_{\rho \in \mathrm{Uch}(\mathbf{G}^F)_{\mathbf{L},\lambda}} \varepsilon_{\mathbf{L},\lambda,\rho} \left(\rho \otimes (\chi_{\mathbf{L},\lambda,\rho})_{\mathsf{q}=q}\right)$$

where $\chi_{\mathbf{L},\lambda,\rho} \in \mathrm{Irr}(W^{\mathbf{G}}_{\mathbf{T}})$ and $\varepsilon_{\mathbf{L},\lambda,\rho} \in \{\pm 1\}$.

If $m \neq l$, then $H^{\mathbf{G}}_{\mathbf{L},\lambda}(\zeta_m)$ need not be semisimple. Via the decomposition map

$$\chi \mapsto \chi_{\mathsf{q}=\zeta_m} : \mathrm{Irr}(W^{\mathbf{G}}_{\mathbf{L},\lambda}) \to \mathrm{K}_0(H^{\mathbf{G}}_{\mathbf{L},\lambda}(\zeta_m)),$$

we can define a partition of $\mathrm{Irr}(W^{\mathbf{G}}_{\mathbf{L},\lambda})$ into subsets to be called *$(H^{\mathbf{G}}_{\mathbf{L},\lambda}, m)$-blocks*.

**Conjecture (T–Xue)** Fix $l, m > 0$, an $l$-cuspidal $(\mathbf{L}, \lambda)$, and an $m$-cuspidal $(\mathbf{M}, \mu)$.

1 The images of

$$\mathrm{Uch}(\mathbf{G}^F)_{\mathbf{L},\lambda} \cap \mathrm{Uch}(\mathbf{G}^F)_{\mathbf{M},\mu} \xrightarrow{\chi_{\mathbf{L},\lambda,-}} \mathrm{Irr}(W^{\mathbf{G}}_{\mathbf{L},\lambda})$$
$$\mathrm{Uch}(\mathbf{G}^F)_{\mathbf{L},\lambda} \cap \mathrm{Uch}(\mathbf{G}^F)_{\mathbf{M},\mu} \xrightarrow{\chi_{\mathbf{M},\mu,-}} \mathrm{Irr}(W^{\mathbf{G}}_{\mathbf{M},\mu})$$

are unions of $(H^{\mathbf{G}}_{\mathbf{L},\lambda}, m)$- and $(H^{\mathbf{G}}_{\mathbf{M},\mu}, l)$-blocks.

2 They induce a matching between these blocks.

**Theorem (T–Xue)** (1), (2) are compatible with block sizes for essentially all $\mathbf{G}, l, m$ with $\mathbf{G}$ exceptional.

**Conjecture (T–Xue)** In the preceding setup:

3 The bijection in (2) lifts to a derived equivalence between category-O blocks of appropriate rational DAHAs.

**Theorem (T–Xue)** (1), (2), (3) hold for $\mathbf{G} = \mathbf{GL}_n$ when $l, m$ are coprime.

Note that here, $W^{\mathbf{GL}_n}_{\mathbf{L},\lambda} \simeq S_N \ltimes \mathbf{Z}_l^N$ for some $N$, *etc.*

$$\mathsf{Rep}(H^{\mathbf{GL}_n}_{\mathbf{L},\lambda}(\zeta_m)) \quad \text{and} \quad \mathsf{Rep}(H^{\mathbf{GL}_n}_{\mathbf{M},\mu}(\zeta_l))$$

can be interpreted in terms of *higher-level Fock spaces*

$$\bigoplus_{\substack{\vec{s} \in \mathbf{Z}^l \\ |\vec{s}| = s}} \Lambda_{\mathsf{q}}^{\vec{s}} \xleftarrow{\sim} \Lambda_{\mathsf{q}}^{s} \xrightarrow{\sim} \bigoplus_{\substack{\vec{r} \in \mathbf{Z}^m \\ |\vec{r}| = s}} \Lambda_{\mathsf{q}}^{\vec{r}}.$$

Above, $\Lambda_{\mathsf{q}}^{\vec{s}} \simeq \bigoplus_N \mathrm{K}_0(S_N \ltimes \mathbf{Z}_l^N) \otimes \mathbf{Q}(\mathsf{q})$, *etc.*

Use the *level-rank duality* of Frenkel, Uglov, Chuang–Miyachi, Shan–Varagnolo–Vasserot, ...

In other words: Our conjectures generalize level-rank duality from $\mathbf{GL}_n$ to arbitrary $\mathbf{G}$.

*Thank you for listening.*