Review given U sub V
with inclusion map $i : U$ to V
[what structures does it produce?]

- $\mathrm{Ann}_{V^\vee} = \{\theta \text{ in } V^\vee \mid \theta|_U = \mathbf{0}_{V^\vee}\}$
- $i^\vee : V^\vee$ to $U^\vee$ defined by $i^\vee(\theta) = \theta|_U$

Lem $\mathrm{Ann}_{V^\vee}(U) = \ker(i^\vee)$

also:

- quotient map $q : V$ to V/U
- $q^\vee : (V/U)^\vee$ to $V^\vee$

[to finish that discussion:]

Thm $\mathrm{Ann}_{V^\vee}(U) = \mathrm{im}(q^\vee)$

recall: elts of V/U are subsets $v + U$
where v in V

Lem $v + U = v' + U$, as subsets, iff $v - v'$ in U

$(v + U) + (w + U) = (v + w) + U$
$\lambda \bullet (v + U) = \lambda v + U$

use lemma to check that these op's are well-def

Ex $V = F^2$ and $U = \{(x, x) \mid x \text{ in } F\}$
for any (a, b) in V:
$(a, b) + U = \{(a + x, b + x) \mid x \text{ in } F\}$

so elts of V/U are translates of U
[draw picture] [elts of V/U are translates in gen'l]

[what is the zero vector in V/U?]
0_{V/U} = 0 + U = U [the trivial translate of U]

quotient map $q : V$ to V/U defined by $q(v) = v + U$
note that $v + U = U$ iff v in U
therefore:

<u>Lem</u> $U = \ker(q : V$ to $V/U)$

<u>Pf of Thm</u> want $\mathrm{Ann}(U) = \mathrm{im}(q^\vee : (V/U)^\vee$ to $V^\vee)$

[first, what is $q^\vee$?] $q^\vee(\psi) = \psi \circ q$

θ in $\mathrm{im}(q^\vee)$ iff $\theta = \psi \circ q$ for some ψ in $(V/U)^\vee$
θ in $\mathrm{Ann}(U)$ iff $\theta|\_U$ is zero
iff U sub $\ker(\theta)$

so want to show: $\theta = \psi \circ q$ for some ψ
iff U sub $\ker(\theta)$

"only if": $\ker(q)$ sub $\ker(\theta)$ by PS4, #3
$U = \ker(q)$ by lemma

"if": for all $v + U$ in V/U
let $\psi(v + U) = \theta(v)$

claim ψ is a well-def lin map V/U to F
will then have $\theta(v) = \psi(v + U) = (\psi \circ q)(v)$

well-def:
if $v + U = v' + U$
then $v - v'$ in U
so $\psi(v + U) = \psi(v' + (v - v') + U) = \psi(v' + U)$

linearity of θ implies linearity of ψ:

$\psi((v + U) + (w + U)) = \psi((v + w) + U)$
$= \theta(v + w)$
$= \theta(v) + \theta(w)$
$= \psi(v + U) + \psi(w + U)$

and similarly for scalar multiplication □

<u>Summary</u> U sub V gives rise to

[injective]	inclusion i : U to V
[surjective]	quotient q : V to V/U

$$(V/U)^\vee \xrightarrow{q^\vee} V^\vee \xrightarrow{i^\vee} U^\vee$$

s.t. im(i) = U = ker(q)

and dually $\mathrm{im}(q^\vee) = \mathrm{Ann}_{V^\vee}(U) = \ker(i^\vee)$

today–Wed: bilinear pairings, forms (§9A)
tensors (§9D)

let V_1, V_2 be arbitrary vector spaces

recall that

$$V_1 \times V_2 = \{(v_1, v_2) \mid v_i \text{ in } V_i\}$$

forms a vector space under entrywise + and •

<u>Df</u> a <u>bilinear pairing</u> between V_1 and V_2 is a linear map from $V_1 \times V_2$ to F

if $V = V_1 = V_2$

then β is called a <u>bilinear form</u> on V we say that a bilinear form β on V is <u>symmetric</u> iff

$$\beta(w, v) = \beta(v, w) \text{ for all } v, w \text{ in } V$$

<u>Ex</u> for any n, the <u>dot product</u> on F^n def by

$$w \bullet w = w1\ v1 + w2\ v2 + \dots + wn\ vn$$

is a symmetric bilinear form on F^n

[what does bilinearity mean?]

for all w, w′, v, v′ in V and λ in F:

$$(w + w') \bullet v = w \bullet v + w' \bullet v$$

$$w \bullet (v + v') = w \bullet v + w \bullet v'$$

and $(\lambda w) \bullet v = \lambda(w \bullet v) = w \bullet (\lambda v)$

<u>Ex</u> let β : F^2 × F^2 to F be def by

$$\beta((w1, w2), (v1, v2)) = w2\ v1 - w1\ v2$$

we will show later that it is bilinear

is it symmetric? no

in fact, it is <u>anti-symmetric</u>: β(w, v) = –β(v, w) for all w, v

<u>Ex</u> fix an n × n matrix M

let β_M : F^n × F^n be def by

$$\beta\_M(w, v) = \begin{matrix} w1 & \dots & wn \end{matrix} \quad M \quad \begin{matrix} v1 \\ \dots \\ vn \end{matrix}$$

the preceding examples are special cases [why?]:

$M = I$ yields $v \bullet w$

$M = \begin{matrix} 0 & -1 \\ 1 & 0 \end{matrix}$ yields the anti-symmetric ex

claim: β_M is bilinear

key: recall that if cols represent elts of V (wrt some basis) then rows represent elts of $V^\vee$ (wrt same basis)

<u>Lem</u> the <u>evaluation map</u> < , > : $V^\vee \times V$ to F def by <θ, v> = θ(v) is a bilinear pairing

[the notation < , > will help avoid confusion soon]

<u>Pf</u> exercise

<u>Cor</u> for any linear op T : V to V the map α : $V^\vee \times V$ to F defined by α(θ, v) = <θ, Tv> is a bilinear pairing

<u>Cor</u> for any linear ops T : V to V, S : V to $V^\vee$ the map β : V × V to F defined by β(w, v) = <Sw, Tv> is a bilinear pairing

<u>Cor</u> the map β_M : F^n × F^n to F defined by β_M(w, v) = w^t M v is a bilinear pairing