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STRONGLY REGULAR GRAPHS AND SPIN MODELS FOR THE KAUFFMAN POLYNOMIAL

ABSTRACT. We study spin models for invariants of links as defined by Jones [22]. We consider the two algebras generated by the weight matrices of such models under ordinary or Hadamard product and establish an isomorphism between them. When these algebras coincide they form the Bose–Mesner algebra of a formally self-dual association scheme. We study the special case of strongly regular graphs, which is associated to a particularly interesting link invariant, the Kauffman polynomial [27]. This leads to a classification of spin models for the Kauffman polynomial in terms of formally self-dual strongly regular graphs with strongly regular subconstituents [7]. In particular we obtain a new model based on the Higman–Sims graph [17].

1. INTRODUCTION

The recent discovery of new invariants of knots and links, the Jones polynomial [20] and its two-variable generalizations, the homfly polynomial ([12], [35]) and the Kauffman polynomial [27], raised considerable interest and activity (see, for instance, the surveys [9], [14], [26], [29], [31]), not only among knot theorists but also among other mathematicians and physicists. In particular it was realized that the new invariants can be defined as partition functions of certain vertex models in the spirit of statistical mechanics (see, for instance, [21], [24], [25], [28], [37], [39]). From the combinatorial point of view, these models provide a satisfactory semantic background for the formal recursive 2-dimensional definition of the invariants. On the other hand, their direct relationship with the quantum group formalism ([11], [37]) could lead to their 3-dimensional interpretation in the light of the work by Witten [40] and subsequent developments (see, for instance, [10]).

But the vertex models do not seem to give a simple explanation of a number of combinatorial properties of the invariants. Let us consider, for instance, the Kauffman polynomial, which can be defined in a combinatorial way on link diagrams. Link diagrams can be represented by plane graphs with signed edges. It turns out that the Kauffman polynomial of a link diagram only depends on the abstract structure of the associated signed graph ([33], see also [19]). However the vertex models of [37] do not exhibit this property. This raises the question of finding models for the Kauffman polynomial, or at least for some of its specializations, in terms of this abstract graph structure. Natural candidates for such models are the following

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generalizations of the dichromatic polynomial [38], called *spin models* (see, for instance, [3], [15], [22]).

A *state* of a graph is an assignment to each vertex of one *spin*, or *color*, chosen in a certain finite set. Then each edge receives a *weight* depending only on its sign and on the unordered pair of colors of its ends (this weight function defines the model), and the weight of a state is the product of weights of the edges. The corresponding graph invariant is the *partition function* of the model, i.e. the sum of weights of states. Spin models which define link invariants can be characterized by a few equations to be satisfied by the weight functions [22], and the Kauffman polynomial will be obtained, provided a single additional equation is satisfied.

The above equations can be easily written in matrix form, using the usual matrix product and the Hadamard (i.e. componentwise) product. In Section 2, after a presentation of the general framework, we associate to any spin model which satisfies these equations a pair of algebras (one for each type of product) and we establish an isomorphism between them. An interesting case is when both algebras coincide with the Bose–Mesner algebra of some association scheme (see, for instance, [1], [5]). Then the above isomorphism acts as a formal duality (a combinatorial abstraction of the discrete Fourier transform). Moreover, the equation which defines the Kauffman polynomial leads to this situation, with the additional constraint that the Bose–Mesner algebra has dimension at most 3. In other words, formally self-dual strongly regular graphs are the natural setting for the study of spin models for the Kauffman polynomial.

We present such a study in Section 3. It turns out that a primitive formally self-dual strongly regular graph will yield a spin model if and only if each of its subconstituents is strongly regular (a property already studied by Cameron *et al.* [7]). This yields a classification of the spin models for specializations of the Kauffman polynomial which unifies the description of previously known models and also exhibits a new one based on the Higman–Sims graph [17].

We conclude in Section 4 with a few questions and perspectives of future research.

2. Link invariants and spin models

2.1. *Link Diagrams and Link Invariants*

A *link* is a finite collection of mutually disjoint simple closed curves in 3-space, each curve being called a *component* of the link. An *oriented link* is a

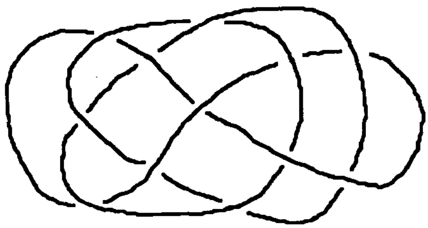


Fig. 1. A diagram.

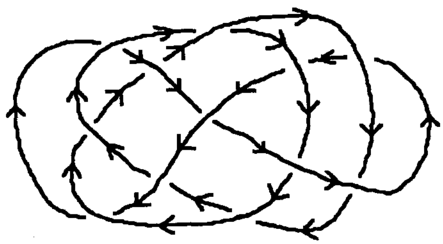


Fig. 1′. An oriented diagram.

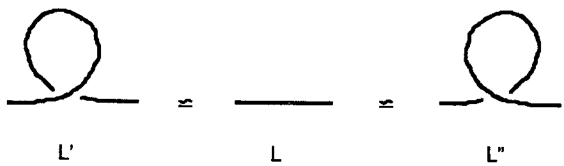
link with an orientation assigned to each component. Two links are said to be *ambient isotopic* if there exists an isotopic deformation of the ambient 3-dimensional space which carries one onto the other. For oriented links, it is required in addition that the isotopic deformation respects the orientation of each component. A link is *tame* if it is ambient isotopic to a link whose components are simple closed polygons. From now on all links will be assumed to be tame.

Every link can be represented by a *diagram.* This is a projection of the link on a plane which has a finite number of multiple points, each of which is a simple crossing. Near each crossing an obvious pictorial convention specifies which segment of the link goes under the other. Moreover, for oriented links the orientation of the components is indicated by arrows in the natural way. Examples are shown on Figures 1 and 1′.

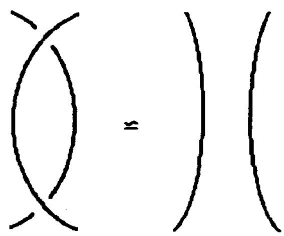
The following result (see, for instance, [6, Ch. 1]) gives a combinatorial reformulation of the ambient isotopy relation for links in terms of diagrams.

REIDEMEISTER'S THEOREM. *Two diagrams represent ambient isotopic links if and only if one can be obtained from the other by a finite sequence of moves of one of the following three types* (*Figure* 2)*:*

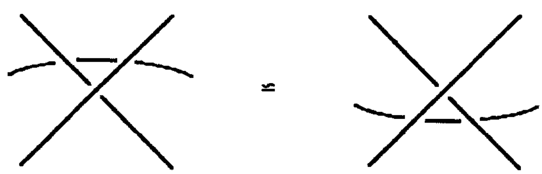
REMARKS. (i) Reidemeister moves are performed by selecting a region inside which the diagram takes one of the forms described in Figure 2, and



Reidemeister moves of type I



Reidemeister moves of type II



Reidemeister moves of type III

Fig. 2.

then replacing this local configuration by an equivalent one without changing the rest of the diagram.

(ii) For oriented diagrams, the same result holds if each move is replaced by a corresponding set of *oriented moves* defined in the obvious way.

Reidemeister's theorem allows the combinatorial construction of invariants of ambient isotopy of links, considered as valuations of diagrams which are invariant under Reidemeister moves. We shall be concerned here with one successful outcome of this approach, the *Kauffman polynomial* [27]. It can take two forms which we shall distinguish by a parameter ε in $\{+1, -1\}$ ($\varepsilon = -1$ corresponds to the so-called *Dubrovnik form*). We shall define it as a mapping F_ε from the class of diagrams to the ring of Laurent polynomials with integer coefficients in two variables z, a, which is invariant under Reidemeister moves of types II and III, takes the value 1 on the diagram consisting of a single component with no crossings, and satisfies the rules:

(1) $$F_\varepsilon(L') = a^{-1}F_\varepsilon(L),\ F_\varepsilon(L'') = aF_\varepsilon(L)$$

(2) $$F_\varepsilon(L^+) + \varepsilon F_\varepsilon(L^-) = z(F_\varepsilon(L^0) + \varepsilon F_\varepsilon(L^\infty)) \quad \textit{(exchange identity)}$$

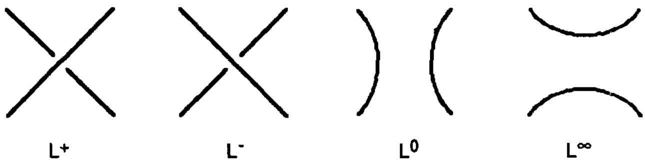


Fig. 3.

whenever the diagrams L, L', L'' (respectively: L^+, L^-, L^0, L^∞) are identical outside some disk and behave as indicated on Figure 2 (moves of type I) (respectively: Figure 3) inside this disk.

It is easy to obtain from F_ε an invariant of oriented links as follows. For an oriented diagram L, define the *sign* of a crossing according to the convention described on Figure 4.

The *writhe* of L, denoted by $w(L)$, is the sum of signs of its crossings. It is easy to check that the writhe is invariant under oriented Reidemeister moves of types II and III. Moreover, for three diagrams L, L', L'' related as shown on Figure 2 with corresponding orientations, $w(L) = w(L') + 1 = w(L'') - 1$. Hence $a^{-w(L)}F_\varepsilon(L)$ defines an invariant of oriented links [27]. One can similarly obtain from F_ε an invariant of unoriented links by using, instead of the writhe, the sum of signs of self-crossings of the different components. See [33], [37].

2.2. *Link Invariants and Spin Models*

The following definition is essentially that of Jones [22] (see also [3], [15]). Let a be a non-zero complex number, n a positive integer, and d one of its square roots. A *spin model with loop variable d and modulus a* is a triple (X, W^+, W^-), where X is a finite set of size $n = d^2$ and W^+, W^- are complex-valued functions on X^2 which satisfy the following properties for all α, β, γ in X:

(3) $$W^+(\alpha, \beta) = W^+(\beta, \alpha), \qquad W^-(\alpha, \beta) = W^-(\beta, \alpha).$$

(4) $$W^+(\alpha, \alpha) = a, \qquad W^-(\alpha, \alpha) = a^{-1}.$$

(5) $$\sum_{x\in X} W^+(\alpha, x) = da^{-1}, \qquad \sum_{x\in X} W^-(\alpha, x) = da.$$

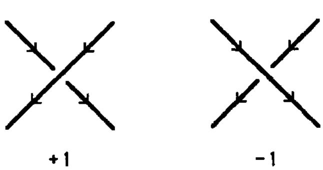


Fig. 4. The sign of a crossing.

(6) $W^+(\alpha, \beta)W^-(\alpha, \beta) = 1.$

(7) $\sum_{x\in X} W^-(\alpha, x)W^+(x, \beta) = n\delta(\alpha, \beta)$ (where δ is the Kronecker symbol).

(8) $\sum_{x\in X} W^+(\alpha, x)W^+(\beta, x)W^-(\gamma, x) = dW^+(\alpha, \beta)W^-(\beta, \gamma)W^-(\gamma, \alpha).$

REMARK. The above properties are not independent (see, for instance, [22, Prop. 2.11]), and we shall take advantage of this fact later.

A spin model (X, W^+, W^-) defines a link invariant as follows. Consider an unoriented diagram L which is connected as a subset of the plane. Color the plane regions delimited by L black and white in such a way that adjacent regions have different colors and the unbounded region is coloured white. Let $b(L)$ be the number of black regions of L. For every crossing v we define $s(v) \in \{+, -\}$ as shown on Figure 5.

Call *state* of L any mapping σ from the set of black regions to X. Given a crossing v and a state σ, write $\langle v|\sigma\rangle$ for $W^{s(v)}(\sigma(f), \sigma(g))$, where f, g are the two (possibly identical) black regions incident with v. Let $\langle L|\sigma\rangle$ be the product of the local weights $\langle v|\sigma\rangle$ over all crossings (this is set as equal to 1 if L has no crossings). We define the *partition function* of the model as $Z(L) = d^{-b(L)} \sum \langle L|\sigma\rangle$, where the summation is taken over all states σ. When the diagram L is not connected, we define $Z(L)$ as equal to the product of the values of Z on its connected components.

The following result appears in [22] (Theorem 2.8 and Proposition 2.11).

PROPOSITION 1. *Z is invariant under Reidemeister moves II and III, takes the value d on the diagram consisting of a single component with no crossings, and satisfies $Z(L') = a^{-1}Z(L)$, $Z(L'') = aZ(L)$ whenever the diagrams L, L', L'' are identical outside some disk and behave as indicated on Figure* 2 (*moves of type* I) *inside this disk.*

Sketch of proof. The behaviour of Z under a Reidemeister move is studied locally by considering a restricted set of states where the values of all black regions which are not modified by the moves are fixed. For the moves of type I there are four cases to consider and they are handled by properties (4), (5) of

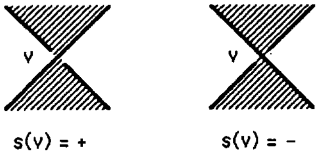


Fig. 5.

$W^{\pm}$. For the moves of type II there are two cases corresponding to properties (6) and (7), and in the case corresponding to (7) the possibility of disconnecting the diagram must be taken into account. Finally property (8) is enough to deal with the invariance under the Reidemeister move of type III for both cases of coloring of the regions concerned by the move. □

It follows from Proposition 1 that Z can be converted into an invariant of oriented or unoriented links by the normalization processes which we have described in the case of the Kauffman polynomial.

REMARKS. (i) More generally a spin model defines an invariant of graphs with signed edges as follows (see, for instance, [3], [15]). States are now mappings from the vertex-set to X, local weights are assigned to the edges according to their signs and to the values of their end-vertices and the partition function is defined as above. Given a connected diagram L, we may represent it by a plane graph, the vertices of which are the black regions of L, with one edge for each crossing v which receives the sign of v and joins the black regions incident to v. Then $Z(L)$ is just the partition function of this graph. Thus link invariants associated to spin models are invariant under transformations of diagrams which preserve the abstract structure of the associated graph. The Kauffman polynomial satisfies this property (see, for instance, [33]) and thus it is natural to look for spin models to describe this invariant.

(ii) As shown in [22], to every spin model is associated a family of representations of the braid groups such that the invariant Z corresponds to the trace in these representations.

2.3. *Spin Models and Association Schemes*

In the sequel I will denote the unit matrix and J the matrix with all entries equal to 1, dimensions being given by the context. Given two matrices M and N of corresponding dimensions, their *Hadamard product* $M \circ N$ is defined by $(M \circ N)_{ij} = M_{ij}N_{ij}$ for all i, j. An *association scheme with m classes* (see [1], [5]) is defined on a finite non-empty set X by a partition $\{R_0, \dots, R_m\}$ of X^2 into $m+1$ non-empty symmetric relations R_i such that R_0 is the diagonal of X^2 and the following property holds: for every i, j, k in $\{0, \dots, m\}$ there exists an integer p_{ij}^k such that for every pair (y, z) in R_k there are exactly p_{ij}^k elements x of X with $(x, y) \in R_i$ and $(x, z) \in R_j$. Note that $p_{ij}^k = p_{ji}^k$ since the R_i are symmetric.

If each relation R_i is represented by its $(0, 1)$ adjacency matrix A_i (with respect to a fixed ordering of X), the above definition translates as follows:

$A_i \neq 0$, A_i is symmetric, $A_i \circ A_j = \delta(i,j)A_i$, $\Sigma_{i=0,\ldots,m} A_i = J$, $A_0 = I$, and $A_i A_j = \Sigma_{k=0,\ldots,m} p_{ij}^k A_k$. It follows that the A_i span an $(m+1)$-dimensional commutative algebra $\mathscr{A}$ of complex symmetric matrices which contains I and J and is closed under the Hadamard product ($\mathscr{A}$ is the *Bose–Mesner algebra* of the association scheme [4]). Conversely, every algebra $\mathscr{A}$ with the above properties is the Bose–Mesner algebra of some association scheme ([5, Th. 2.6.1]). Note that the A_i form a basis of minimal idempotents of $\mathscr{A}$ for the Hadamard product. Similarly $\mathscr{A}$ has a unique basis $\{E_0, \ldots, E_m\}$ of minimal idempotents for the ordinary matrix product, with $E_i E_j = \delta(i,j)E_i$, $\Sigma_{i=0,\ldots,m} E_i = I$, and $E_0 = n^{-1}J$, where $n = |X|$. The *eigenmatrices* P and Q of the association scheme relate these two bases of $\mathscr{A}$ as follows:

$$A_j = \sum_{i=0,\ldots,m} P_{ij} E_i,\ E_j = n^{-1} \sum_{i=0,\ldots,m} Q_{ij} A_i,$$

so that $PQ = QP = nI$. An association scheme is called *formally self-dual* if the idempotents E_i can be ordered in such a way that $P = Q$.

Let us now come back to spin models and identify complex-valued functions on X^2 with n by n matrices indexed by X. Consider two such matrices W^+, W^-. For every (β, γ) in X^2 let $Y_{\beta\gamma}$ be the column vector indexed by X defined by

(9) $\qquad Y_{\beta\gamma}(x) = W^+(\beta, x)W^-(\gamma, x) \quad$ for every x in X.

The following result is an immediate reformulation of properties (3)–(8).

PROPOSITION 2. *(X, W^+, W^-) is a spin model with loop variable d and modulus a iff the following properties hold:*

(10) $\qquad W^+$ *and* W^- *are symmetric.*

(11) $\qquad I \circ W^+ = aI, \qquad I \circ W^- = a^{-1}I.$

(12) $\qquad JW^+ = da^{-1}J, \qquad JW^- = daJ.$

(13) $\qquad W^+ \circ W^- = J.$

(14) $\qquad W^+W^- = nI.$

(15) $\qquad$ *For every (β, γ) in X^2,* $\quad W^+ Y_{\beta\gamma} = dW^-(\beta, \gamma) Y_{\beta\gamma}$.

REMARK. Certain solutions of Equations (13)–(14) have some interest in a different context and have been obtained by methods similar to those of the present paper (see [16], [34]).

Now let us associate to a spin model (X, W^+, W^-) two commutative complex algebras $\mathcal{M}$ (generated by W^+ and J with product the ordinary matrix product) and $\mathcal{H}$ (generated by W^- and I with product the Hadamard product).

Also, let $\mathcal{M}'$ be the subalgebra of $\mathcal{M}$ generated by W^+ and $\mathcal{H}'$ be the subalgebra of $\mathcal{H}$ generated by W^-.

Let $f(x) = \Sigma_{i=0,\ldots,k} c_i x^i$ be the minimal polynomial of W^+ (thus $c_k = 1$). Note that, by (14), $c_0 \neq 0$. Hence

$$I = -c_0^{-1} \sum_{i=1,\ldots,k} c_i (W^+)^i \text{ and } W^- = -c_0^{-1} n \sum_{i=1,\ldots,k} c_i (W^+)^{i-1}$$

both belong to $\mathcal{M}'$.

PROPOSITION 3. (i) *There exists a unique algebra isomorphism ψ from $\mathcal{M}'$ to $\mathcal{H}'$ such that $\psi(W^+) = dW^-$. Moreover, $\psi(I) = J$, $\psi(W^-) = dW^+$, and ψ uniquely extends to an isomorphism from $\mathcal{M}$ to $\mathcal{H}$ (also denoted by ψ) such that $\psi(J) = nI$.*

(ii) *For every M in $\mathcal{M}$,*

$$\psi(M) = d^{-1}a^{-1}W^-(W^+ \circ (W^- M)) = d^{-1}aW^+(W^- \circ (W^+ M)).$$

Proof. (i) Clearly $\{(W^+)^i, i = 1, \ldots, k\}$ is a basis of $\mathcal{M}'$.

Let ψ be the linear map from $\mathcal{M}'$ to $\mathcal{H}'$ defined by $\psi((W^+)^i) = d^i(\circ^i W^-)(i = 1, \ldots, k)$, where $\circ^i W^-$ is obtained from J by i iterations of the Hadamard product with W^-.

For every (β, γ) in X^2, the vector $Y_{\beta\gamma}$ is non-zero by (9), (13).
Hence, by (15), $dW^-(\beta, \gamma)$ is an eigenvalue of W^+ and $f(dW^-(\beta, \gamma)) = 0$. It follows that $c_0 J + \Sigma_{i=1,\ldots,k} c_i d^i(\circ^i W^-) = 0$ and (taking the Hadamard product with W^+), by (13), $c_0 W^+ + \Sigma_{i=1,\ldots,k} c_i d^i(\circ^{i-1} W^-) = 0$.

Then

$$\psi(I) = -c_0^{-1} \sum_{i=1,\ldots,k} c_i d^i(\circ^i W^-) = J$$

and

$$\psi(W^-) = -c_0^{-1} n \sum_{i=1,\ldots,k} c_i d^{i-1}(\circ^{i-1} W^-) = dW^+$$

as required.

It is now clear that $\{\psi((W^+)^i) = d^i(\circ^i W^-), i = 1, \ldots, k\}$ spans the algebra $\mathcal{H}'$. Let us show that this set is a basis of $\mathcal{H}'$.

Otherwise, $\Sigma_{i=1,\ldots,k} c'_i d^i(\circ^i W^-) = 0$ for some c'_i not all zero. Then for every (β, γ) in X^2, $\Sigma_{i=1,\ldots,k} c'_i (dW^-(\beta, \gamma))^i = 0$ and, by (15), $\Sigma_{i=1,\ldots,k} c'_i (W^+)^i Y_{\beta\gamma} = 0$.

For every x in X,

$$\sum_{\beta \in X} Y_{\beta\gamma}(x) = \sum_{\beta \in X} W^+(\beta, x) W^-(\gamma, x) = da^{-1} W^-(\gamma, x)$$

by (12). Since W^- is non-singular by (14), the $Y_{\beta\gamma}$ form a spanning set and hence $\Sigma_{i=1,\ldots,k} c_i'(W^+)^i = 0$, a contradiction.

Since the elements of the basis $\{\psi((W^+)^i) = d^i({}^{\circ i}W^-),\ i = 1, \ldots, k\}$ of $\mathscr{H}'$ obey the same product rule (defined by the equality $c_0 J + \Sigma_{i=1,\ldots,k} c_i d^i({}^{\circ i}W^-) = 0$) as the basis $\{(W^+)^i, i = 1, \ldots, k\}$ of $\mathscr{M}'$, ψ is an isomorphism from $\mathscr{M}'$ to $\mathscr{H}'$.

Now if $\mathscr{M}' = \mathscr{M}$, let us write $J = \Sigma_{i=1,\ldots,k} b_i (W^+)^i$, so that $\psi(J) = \Sigma_{i=1,\ldots,k} b_i d^i({}^{\circ i}W^-)$. Let U be the vector with all components equal to 1. By (9), (14), for every (β, γ) in X^2, $JY_{\beta\gamma} = n\delta(\beta, \gamma)U$, and by (15)

$$JY_{\beta\gamma} = \sum_{i=1,\ldots,k} b_i (W^+)^i Y_{\beta\gamma} = \sum_{i=1,\ldots,k} b_i (dW^-(\beta, \gamma))^i Y_{\beta\gamma}.$$

Noting that $Y_{\beta\beta} = U$ by (9), (13), we obtain $\Sigma_{i=1,\ldots,k} b_i (dW^-(\beta, \gamma))^i = n\delta(\beta, \gamma)$, or equivalently $\psi(J) = nI$. Thus $\mathscr{H}' = \mathscr{H}$ and the proof is complete in this case.

Finally if $\mathscr{M}' \neq \mathscr{M}$, it follows from (12) that $\mathscr{M}' \cup \{J\}$ spans $\mathscr{M}$. We define $\psi(J)$ as equal to nI. Clearly by (11) $\mathscr{H}' \cup \{I\}$ spans $\mathscr{H}$. Let us show that moreover $\mathscr{H}' \neq \mathscr{H}$. Otherwise assume that $nI = \Sigma_{i=1,\ldots,k} b_i' d^i({}^{\circ i}W^-)$. Then for every (β, γ) in X^2,

$$\sum_{i=1,\ldots,k} b_i' (dW^-(\beta, \gamma))^i = n\delta(\beta, \gamma)$$

and by (15)

$$\sum_{i=1,\ldots,k} b_i' (W^+)^i Y_{\beta\gamma} = n\delta(\beta, \gamma) Y_{\beta\gamma}.$$

Since $JY_{\beta\gamma} = n\delta(\beta, \gamma)U = n\delta(\beta, \gamma)Y_{\beta\gamma}$ and the $Y_{\beta\gamma}$ form a spanning set, $\Sigma_{i=1,\ldots,k} b_i'(W^+)^i = J$, a contradiction.

It is now enough to show that the elements of the basis $\{d^i({}^{\circ i}W^-),$ $i = 1, \ldots, k\} \cup \{nI\}$ of $\mathscr{H}$ obey the same product rule as the elements of the basis $\{(W^+)^i,\ i = 1, \ldots, k\} \cup \{J\}$ of $\mathscr{M}$. This follows immediately from properties (11), (12).

(ii) First we observe that for every M in $\mathscr{M}$: for every (β, γ) in X^2, $MY_{\beta\gamma} = [\psi(M)](\beta, \gamma) Y_{\beta\gamma}$. Indeed, this property is easily seen to be closed under linear combinations and ordinary matrix products, it holds for $M = W^+$ by (15) and it also holds for $M = J$ since (as we have just seen) $JY_{\beta\gamma} = n\delta(\beta, \gamma) Y_{\beta\gamma}$.

Recall also from the proof of (i) that for every x in X,

$$\sum_{\beta\in X} Y_{\beta\gamma}(x) = da^{-1}W^{-}(\gamma, x). \text{ Similarly, by (12),}$$

$$\sum_{\gamma\in X} Y_{\beta\gamma}(x) = \sum_{\gamma\in X} W^{+}(\beta, x)W^{-}(\gamma, x) = daW^{+}(\beta, x).$$

Now, for every y in X, let us consider the equality

$$\sum_{\beta\in X} [MY_{\beta\gamma}](y) = \sum_{\beta\in X} [\psi(M)](\beta, \gamma)Y_{\beta\gamma}(y).$$

The left-hand side equals

$$\left[M\sum_{\beta\in X} Y_{\beta\gamma}\right](y) = \sum_{x\in X} M(y, x)\left[\sum_{\beta\in X} Y_{\beta\gamma}\right](x) = \sum_{x\in X} M(y, x)(da^{-1}W^{-}(\gamma, x))$$

$$= da^{-1}\sum_{x\in X} M(y, x)W^{-}(x, \gamma) = da^{-1}[MW^{-}](y, \gamma).$$

The right-hand side equals

$$\sum_{\beta\in X} [\psi(M)](\beta, \gamma)W^{+}(\beta, y)W^{-}(\gamma, y) = W^{-}(\gamma, y)\sum_{\beta\in X} W^{+}(y, \beta)[\psi(M)](\beta, \gamma)$$

$$= W^{-}(y, \gamma)[W^{+}\psi(M)](y, \gamma) = [W^{-}\circ(W^{+}\psi(M))](y, \gamma).$$

Hence $da^{-1}MW^{-} = W^{-}\circ(W^{+}\psi(M))$. Using (13), (14) and the commutativity of $\mathscr{M}$ we easily obtain that $\psi(M) = d^{-1}a^{-1}W^{-}(W^{+}\circ(W^{-}M))$.

The proof of the other formula for $\psi(M)$ is quite similar. We consider the equality

$$\sum_{\gamma\in X} [MY_{\beta\gamma}](y) = \sum_{\gamma\in X} [\psi(M)](\beta, \gamma)Y_{\beta\gamma}(y).$$

The left-hand side is easily converted into $da[MW^{+}](y, \beta)$, whereas the right-hand side equals

$$\sum_{\gamma\in X} [\psi(M)](\beta, \gamma)W^{+}(\beta, y)W^{-}(\gamma, y) = W^{+}(\beta, y)\sum_{\gamma\in X} [\psi(M)](\beta, \gamma)W^{-}(\gamma, y)$$

$$= [W^{+}\circ(\psi(M)W^{-})](\beta, y).$$

The equality $daMW^{+} = W^{+}\circ(\psi(M)W^{-})$ is then transformed into the required result. □

The following result is a corollary of Proposition 3.

PROPOSITION 4. *If $\mathscr{M}$ is closed under the Hadamard product, it is the Bose–Mesner algebra of a formally self-dual association scheme.*

Proof. Since W^- and I belong to $\mathcal{M}$, $\mathcal{H} \subseteq \mathcal{M}$ and equality holds since these two algebras have the same dimension. $\mathcal{H} = \mathcal{M}$ is a commutative algebra of complex symmetric matrices which contains I and J and is closed under the Hadamard product. It is shown in [5, Th. 2.6.1] that every such algebra is the Bose–Mesner algebra of some association scheme. Let $\{E_0, \dots, E_m\}$ (respectively $\{A_0, \dots, A_m\}$) be the unique basis of minimal idempotents of $\mathcal{M}$ for the ordinary (respectively Hadamard) matrix product, with $E_0 = n^{-1}J$, $A_0 = I$. By Proposition 3(i), $\psi(E_0) = A_0$ and we may choose the indices so that $\psi(E_j) = A_j$ for $j = 1, \dots, m$. Using Proposition 3(ii) and (13), (14) it is easy to show that $d^{-1}\psi$ is an involution and hence $\psi(A_j) = nE_j$. Recall that the eigenmatrices P and Q are defined by

$$A_j = \sum_{i=0,\dots,m} P_{ij}E_i, \qquad E_j = n^{-1} \sum_{i=0,\dots,m} Q_{ij}A_i.$$

Hence for $j = 1, \dots, m$, $nE_j = \psi(A_j) = \Sigma_{i=0,\dots,m} P_{ij}A_i$ and this shows that $P = Q$. $\square$

Now we consider a formally self-dual association scheme on the set X. We keep the same notations as above. The duality operator ψ is defined by $\psi(E_j) = A_j = \Sigma_{i=0,\dots,m} P_{ij}E_i$ or equivalently

$$\psi(A_j) = nE_j = \Sigma_{i=0,\dots,m} P_{ij}A_i \qquad (j = 1, \dots, m).$$

We consider an element $W^+ = \Sigma_{i=0,\dots,m} t_iA_i$ of the Bose–Mesner algebra such that the t_i are non-zero and we set $W^- = d^{-1}\psi(W^+)$. Let T (respectively T^-) be the $(m+1)$-dimensional column vector whose components are the t_i (respectively t_i^{-1}). A pair (β, γ) in X^2 will be said to be *of type i* if $A_i(\beta, \gamma) = 1$.

PROPOSITION 5. *(X, W^+, W^-) is a spin model with loop variable d and modulus $a = t_0$ iff the following properties hold;*

(16) $PT = dT^-$.

(17) *For every i, j in $\{1, \dots, m\}$ with $t_i \neq t_j$ and for every pair (β, γ) of type i, $E_jY_{\beta\gamma} = 0$.*

Proof.

$$W^- = d^{-1} \sum_{j=0,\dots,m} t_j\psi(A_j) = d^{-1} \sum_{j=0,\dots,m} t_j \sum_{i=0,\dots,m} P_{ij}A_i$$
$$= \sum_{i=0,\dots,m} \left(d^{-1} \sum_{j=0,\dots,m} P_{ij}t_j\right) A_i.$$

On the other hand, property (13) is equivalent to the fact that $W^- = \Sigma_{i=0,\dots,m} t_i^{-1} A_i$. Thus property (16) is just a reformulation of property

(13). Now clearly $I \circ W^+ = t_0 I$ and $I \circ W^- = t_0^{-1} I$, so that (11) is satisfied with $a = t_0$. Also

$$\psi(JW^+) = \psi(J) \circ \psi(W^+) = (nI) \circ (dW^-) = nd(I \circ W^-) = dt_0^{-1} nI = dt_0^{-1} \psi(J),$$

so that $JW^+ = dt_0^{-1} J$, and similarly $JW^- = dt_0 J$. Thus (12) holds. In the same way, $\psi(W^+ W^-) = \psi(W^+) \circ \psi(W^-) = (dW^-) \circ (dW^+) = nJ = n\psi(I)$, which proves (14).

Finally let us examine (15). Note that

$$W^+ = d^{-1}\psi(W^-) = d^{-1} \Sigma_{j=0,\dots,m} t_j^{-1} \psi(A_j) = \Sigma_{j=0,\dots,m} dt_j^{-1} E_j.$$

Now if (β, γ) is of type i, $W^-(\beta, \gamma) = t_i^{-1}$ and (15) becomes

$$\Sigma_{j=0,\dots,m} dt_j^{-1} E_j Y_{\beta\gamma} = dt_i^{-1} Y_{\beta\gamma}.$$

Two vectors being equal iff they have the same image under each E_j, this equation holds iff $t_j^{-1} E_j Y_{\beta\gamma} = t_i^{-1} E_j Y_{\beta\gamma}$ for every j in $\{0, \dots, m\}$. Note that since $JY_{\beta\gamma} = n\delta(\beta, \gamma)U$, $E_0 Y_{\beta\gamma} = 0$ when (β, γ) is not of type 0. Similarly, since $Y_{\beta\beta} = U = E_0 U$, $E_j Y_{\beta\beta} = 0$ for $j \neq 0$. Thus we have shown the equivalence of (15) and (17). □

REMARK. When the t_i are distinct, property (17) can be reformulated as follows: for $i = 1, \dots, m$ and for every pair (β, γ) of type i, $E_i Y_{\beta\gamma} = Y_{\beta\gamma}$.

3. Spin Models for the Kauffman Polynomial

3.1. *The Exchange Identity and its Consequences*

We would like to find spin models (X, W^+, W^-) such that the associated partition function Z is an evaluation of the Kauffman polynomial up to normalization, that is, for some complex numbers z, a, $Z(L) = dF_\varepsilon(L, z, a)$ for every diagram L. Note that by (1) the modulus of the model will be equal to a. Now all we need to satisfy (2) is that the exchange identity $Z(L^+) + \varepsilon Z(L^-) = z(Z(L^0) + \varepsilon Z(L^\infty))$ holds whenever the diagrams L^+, L^-, L^0, L^∞ are connected and locally related as indicated on Figure 3. (If some of these diagrams are not connected we may apply the same set of Reidemeister moves of type II to the four diagrams to transform them into connected diagrams.) We may assume that the regions are colored as shown on Figure 6. (The other case corresponds to the same equation with both sides multiplied by ε.)

Recall that, when the diagram L is connected, $Z(L)$ is of the form $d^{-b(L)} \Sigma \langle L | \sigma \rangle$, where the summation is over the set of states σ of L, which are

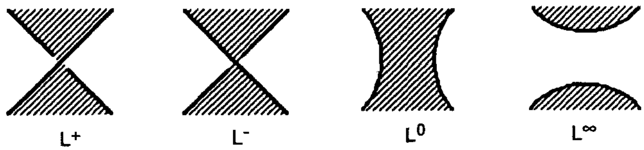


Fig. 6.

valuations of its black regions. Obviously we may consider that L^+, L^-, L^∞ have the same set of black regions and hence the same set of states. Let f_1, f_2 be the two black regions incident to the common crossing of L^+, L^- where they differ. We may identify the states of L^0 with the states of L^+, L^-, or L^∞ such that f_1, f_2 have the same value. Equivalently we shall consider that L^0 has the same set of states as the three other diagrams, assigning weight zero to the states such that f_1, f_2 have different values. Then for any state σ with $\sigma(f_1) = \alpha, \sigma(f_2) = \beta$:

$$\langle L^+|\sigma\rangle = W^+(\alpha, \beta)\langle L^\infty|\sigma\rangle, \ \langle L^-|\sigma\rangle = W^-(\alpha, \beta)\langle L^\infty|\sigma\rangle,$$
$$\langle L^0|\sigma\rangle = \delta(\alpha, \beta)\langle L^\infty|\sigma\rangle.$$

Noting that $b(L^+) = b(L^-) = b(L^\infty) = b(L^0) + 1$ we see that the contribution of the state σ to $Z(L^+) + \varepsilon Z(L^-) - z(Z(L^0) + \varepsilon Z(L^\infty))$ equals its contribution to $Z(L^\infty)$ multiplied by a factor

$$W^+(\alpha, \beta) + \varepsilon W^-(\alpha, \beta) - zd\,\delta(\alpha, \beta) - z\varepsilon.$$

Thus the natural requirement in order that our spin model corresponds to a specialization of the Kauffman polynomial is that this factor vanishes for every α, β, or, equivalently, that

$$(18) \qquad W^+ + \varepsilon W^- = z(dI + \varepsilon J).$$

Let us now compare this equation with properties (11) and (12) of Proposition 2. The Hadamard product by I gives

$$(19) \qquad a + \varepsilon a^{-1} = z(d + \varepsilon)$$

and the ordinary product by J gives an equivalent equation.

From now on we shall exclude the trivial models with $n = 1$. Hence $d + \varepsilon \neq 0$ and Equation (19) will define the Kauffman variable z in terms of the modulus a and the loop variable d of the spin model.

Consider now the algebra $\mathcal{M}$ generated by W^+ and J with product the ordinary matrix product. We already know (by (14)) that $\mathcal{M}$ contains I. By (12), (14), multiplying Equation (18) by W^+ gives $(W^+)^2 + \varepsilon nI = z(dW^+ + \varepsilon da^{-1}J)$, so that the set $\{I, J, W^+\}$ spans $\mathcal{M}$ and $\mathcal{M}$ has dimension 2 or 3. Using (11), (13), the Hadamard product of Equation (18) by W^+ gives

$\circ^2 W^+ + \varepsilon J = z\ (daI + \varepsilon W^+)$. This, together with (11), shows that $\mathcal{M}$ is closed under the Hadamard product. Thus, by Proposition 4, $\mathcal{M}$ is the Bose–Mesner algebra of a formally self-dual association scheme.

3.2. *The 2-Dimensional Case and the Jones Polynomial*

Assume first that $\mathcal{M}$ is 2-dimensional, that is $\{I, J\}$ spans $\mathcal{M}$. The matrices defining the corresponding association scheme are $A_0 = I$, $A_1 = J - I$. The minimal idempotents are $E_0 = n^{-1}J$, $E_1 = I - n^{-1}J$, so that

$$P = \begin{bmatrix} 1 & n-1 \\ 1 & -1 \end{bmatrix}.$$

The self-duality property $P^2 = nI$ is immediate. Equation (16), $PT = dT^-$, is equivalent to the system $t_0 + (n-1)t_1 = dt_0^{-1}, t_0 - t_1 = dt_1^{-1}$. This is easily reduced to $d = -(t_1)^2 - (t_1)^{-2}$, $t_0 = -(t_1)^{-3}$. Moreover property (17) is trivially satisfied since $m = 1$.

Note that

$$W^+ + \varepsilon W^- = (t_0 + \varepsilon t_0^{-1})I + (t_1 + \varepsilon t_1^{-1})(J - I) = (\varepsilon t_1 + t_1^{-1})(dI + \varepsilon J).$$

We have obtained the well-known spin model for Kauffman's bracket polynomial (see, for instance, [22, Example 2.17]), which after multiplication by the suitable writhe factor yields the Jones polynomial of oriented links.

3.3. *The 3-Dimensional Case and Strongly Regular Graphs*

From now on we assume that $\mathcal{M}$ is 3-dimensional. We write

$$(20) \qquad A_1 = kE_0 + sE_1 + rE_2.$$

Since $A_0 = I = E_0 + E_1 + E_2$ and $nE_0 = J = A_0 + A_1 + A_2$, we see that

$$P = \begin{bmatrix} 1 & k & n-k-1 \\ 1 & s & -s-1 \\ 1 & r & -r-1 \end{bmatrix}.$$

Using (20) it is easy to show that

$$A_1^2 = (r + s)A_1 - rsI + (k - r)(k - s)E_0.$$

Setting $\mu = (k - r)(k - s)/n$, we obtain

$$(21) \qquad A_1^2 = (\mu - rs)I + (r + s + \mu)A_1 + \mu A_2.$$

We also have

$$(22) \qquad JA_1 = kJ.$$

The combinatorial interpretation of these equations is as follows (see, for instance, [8], [36]). The matrix A_1 is the adjacency matrix of a simple graph G (on the vertex-set X of size n) which is regular of degree k by (22). Looking at the diagonal elements in (21) we see that $\mu - rs = k$, or equivalently

$$(23) \qquad n(k + rs) = (k - r)(k - s).$$

Moreover, by (21) any two adjacent vertices have exactly $\lambda = r + s + \mu$ common neighbors and any two non-adjacent vertices have exactly μ common neighbors. Thus G is a *strongly regular graph* with parameters (n, k, λ, μ). Our hypothesis that $\mathcal{M}$ is 3-dimensional is equivalent to the requirement that G is non-trivial, that is neither complete nor empty. This is also equivalent to the conditions $k \neq 0$, $k \neq n - 1$. Note that this implies that $n \geqslant 4$, which we assume in the sequel.

The following convention is useful in what follows. We shall say that G has *three eigenvalues* k, r, s, even when these numbers are not distinct. We call an *eigenvector* for the eigenvalue k (respectively s, r) a non-zero vector in the *eigenspace* $\mathrm{Im}(E_0)$ (respectively $\mathrm{Im}(E_1)$, $\mathrm{Im}(E_2)$). Thus, even if k equals r or s, every eigenvector for the eigenvalue k is a multiple of the all-one vector $\boldsymbol{U}$, whereas every eigenvector for r or s is orthogonal to $\boldsymbol{U}$.

REMARK. Since $k > 0$ and $\mathrm{Trace}(A_1) = 0$, A_1 has at least one negative eigenvalue. Also $rs = \mu - k \leqslant 0$. Hence one of r, s is negative and the other is positive or zero. We depart here from the usual convention that s should be negative.

Now it is easily checked that the duality property $P^2 = nI$ is equivalent to the two equations

$$(24) \qquad k = r^2 + r - rs,$$

$$(25) \qquad n = (r - s)^2,$$

which are assumed to hold in the sequel. Then G is said to be *formally self-dual.* Its other parameters are

$$(26) \qquad \lambda = r^2 + 2r + s,$$

$$(27) \qquad \mu = r^2 + r.$$

The non-triviality conditions $k \neq 0$ and $k \neq n - 1$ now reduce to

$r + 1 - s \neq 0$ (which is already guaranteed by (25) and $n \neq 1$), and

(28) $\quad r \neq 0, \quad s \neq -1.$

Once the graph G is given, the structure just described is determined up to the exchange of E_1 and E_2. This exchange will yield a new matrix P which will also satisfy the duality property $P^2 = nI$ if and only if the pair of equations (24), (25) is invariant under the exchanges $k \to n - k - 1$, $s \to -s - 1$, $r \to -r - 1$; that is, iff

$$n - k - 1 = (r + 1)^2 - r - 1 - (r + 1)(s + 1).$$

This reduces to $k = s^2 + s - rs$, or equivalently to $r^2 + r = s^2 + s$. Since $r \neq s$ this holds if and only if $r + s + 1 = 0$. Then G is a *conference graph*, i.e. a strongly regular graph with parameters $n = 4\mu + 1$, $k = 2\mu$ and $\lambda = \mu - 1$. In this case, we shall have to specify which matrix P has been chosen.

Let G' be the complementary graph of G, $A'_0 = A_0$, $A'_1 = A_2$, $A'_2 = A_1$, $E'_0 = E_0$, $E'_1 = E_2$, $E'_2 = E_1$, $k' = n - 1 - k$, $s' = -r - 1$, $r' = -s - 1$, $\lambda' = r'^2 + 2r' + s'$ and $\mu' = r'^2 + r'$. It is easy to see that all properties discussed so far in this section are invariant under the replacement of each object by the 'primed' object. We shall take advantage of this *complementation operation* later to reduce the number of cases to be studied.

REMARK. The formally self-dual strongly regular graphs are usually classified as follows. If $\mu = \rho^2 + \rho$ for a non-negative (respectively negative) eigenvalue ρ in $\{r, s\}$, the graph G is said to be of negative (respectively pseudo) Latin square type. When G is a conference graph $\mu = r^2 + r = s^2 + s$ and G can be considered as belonging to both types.

3.4. *Solution of Equation* (16), $PT = dT^-$

Since $n = (r - s)^2$ by (25), we may write

(29) $\quad d = \varepsilon(r - s)$ with ε in $\{1, -1\}$.

Let us now solve (16), $PT = dT^-$. If this holds, $nT = P^2T = dPT^-$, that is $PT^- = dT$. Then $P(T + \varepsilon T^-) = \varepsilon d(T + \varepsilon T^-)$. Noting that $\mathrm{Trace}(P) = s - r = -\varepsilon d$, we see that the eigenspace of P for the eigenvalue εd has dimension 1. The vector $U + \varepsilon d^{-1}PU$ (where the three components of U are equal to 1) spans this eigenspace and has components $(1 + \varepsilon d, 1, 1)$. It follows that the components of $T + \varepsilon T^-$ are $((1 + \varepsilon d)(t_1 + \varepsilon t_1^{-1}),\ t_1 + \varepsilon t_1^{-1},\ t_1 + \varepsilon t_1^{-1})$.

Recall that $W^+ = \Sigma_{i=0,1,2} t_i A_i$ and $W^- = \Sigma_{i=0,1,2} t_i^{-1} A_i$. The exchange equation

(18) $\quad W^+ + \varepsilon W^- = z(dI + \varepsilon J)$

becomes:

$$\sum_{i=0,1,2} (t_i + \varepsilon t_i^{-1})A_i = z((d + \varepsilon)A_0 + \varepsilon(A_1 + A_2))$$

and will be satisfied for $z = \varepsilon t_1 + t_1^{-1}$. In the sequel we use the more classical notations $t_0 = a$ and $t_1 = \varepsilon t$, so that

(30) $\quad z = t + \varepsilon t^{-1}$.

Note that Equation (19) is satisfied. Since $t_2 + \varepsilon t_2^{-1} = t_1 + \varepsilon t_1^{-1}$ and $t_2 \neq t_1$ (otherwise $\mathcal{M}$ would be 2-dimensional), we have $t_2 = \varepsilon t_1^{-1} = t^{-1}$. Thus

(31) $\quad W^+ = aI + \varepsilon t A_1 + t^{-1} A_2, \qquad W^- = a^{-1} I + \varepsilon t^{-1} A_1 + t A_2.$

Note that by introducing $t' = \varepsilon t^{-1}$, $z' = z$, $(W^+)' = W^+$, $(W^-)' = W^-$, Equations (30) and (31) are invariant under complementation.

Now since $P(T + \varepsilon T^-) = \varepsilon d(T + \varepsilon T^-)$ already holds, Equation (16) reduces to $P(T - \varepsilon T^-) = -\varepsilon d(T - \varepsilon T^-)$.

Let us write this down explicitly, using Equations (24), (25) and (29). The matrix $P + \varepsilon dI$ has rank 1 and we are left with the single equation:

$$a - \varepsilon a^{-1} + (r + s + 1)(\varepsilon t - t^{-1}) = 0.$$

By (19), (29), (30), we also have

$$a + \varepsilon a^{-1} - (r - s + 1)(\varepsilon t + t^{-1}) = 0.$$

The sum and difference of the two above equations are

(32) $\quad a = -s\varepsilon t + (r + 1)t^{-1}, \qquad a^{-1} = -s\varepsilon t^{-1} + (r + 1)t.$

It is easy to check that if we set $a' = a$ these two equations are invariant by complementation.

We now examine the solutions in a and t of Equations (32). The two equations will be compatible (and yield the value of a in terms of t) if and only if

(33) $\quad s^2 + (r + 1)^2 - \varepsilon s(r + 1)(t^2 + t^{-2}) = 1.$

If $s = 0$ we must have $(r + 1)^2 = 1$. By (28) we must have $r = -2$. Then by (24) the graph G is a square. Similarly, if $r = -1$ we must have $s = 1$ and G consists of two independent edges (the complement of a square). In these two cases $r - s = -2$, t is arbitrary and $d = -2\varepsilon$. We note the compatibility condition

(34) $\quad s(r+1) \neq 0 \quad \text{for } n \geqslant 5.$

Finally, if $s(r+1) \neq 0$, Equation (33) has in general four solutions in t, which can be obtained from one of them by inversion or change of sign, and by (32) the four corresponding values of a will be related in the same way. In general for the evaluation of the Kauffman polynomial on a diagram L, the inversion of a and t corresponds to the replacement of L by its mirror image, and the change of sign of a and t can be interpreted as the multiplication by a factor $(-1)^{w(L)}$.

REMARKS. (i) We have only two solutions, with $t^2 = t^{-2} = \varepsilon' \in \{1, -1\}$, when $s^2 + (r+1)^2 - 2\varepsilon\varepsilon' s(r+1) = 1$, or equivalently $(s - \varepsilon\varepsilon'(r+1))^2 = 1$. The case $\varepsilon t^2 = 1$ cannot occur since $t_1 \neq t_2$. Then $\varepsilon t^2 = \varepsilon\varepsilon' = -1$, $r + s + 1 \in \{1, -1\}$ and (32) gives $a = (r+s+1)t^{-1}$. Since $z = 0$ by (30) we can only obtain trivial evaluations of the Kauffman polynomial.

(ii) For every solution (a, t) of Equations (32), $(-ia, it)$ is a solution of the same equations where ε is replaced by $-\varepsilon$. One can check that this is compatible with the relationship between the two versions of the Kauffman polynomial established by Lickorish [30].

For further use we record the following easy consequences of (32), (33):

(35) $\quad at^{-1} + a^{-1}t = \varepsilon(r^2 - s^2 + 2r)/s \quad \text{when } s \neq 0.$

(36) $\quad at^{-1} - a^{-1}t = -(t^2 - t^{-2})(r+1).$

3.5. *Characterization of Spin Models for the Kauffman Polynomial*

We now assume that W^+, W^- are given by (31) and that Equations (32) hold. Since $t_1 \neq t_2$ we may reformulate property (17) as follows:

(37) For every pair (β, γ) of adjacent vertices of G, $\quad E_2 Y_{\beta\gamma} = 0.$

(38) For every pair (β, γ) of adjacent vertices of G', $\quad E'_2 Y_{\beta\gamma} = 0.$

We shall begin with the following reformulation of (37).

PROPOSITION 6. *G satisfies* (37) *iff $n = 4$ or, for $n \geqslant 5$,*

(39) *either $\lambda = 0$ or for every vertex of G the subgraph of G induced by its neighbors is strongly regular with eigenvalues λ, r_1, s, where $r_1 = \lambda/(r - s + 2)$.*

Proof. Given two adjacent vertices β, γ of G, for i, j in $\{0, 1, 2\}$ we shall denote by $G_{ij}(\beta,\gamma)$ the set of vertices x such that (β,x) is of type i and (γ,x) is of type j. Moreover, each subset of X will be identified with its characteristic column n-vector (a component equals 1 if the corresponding vertex belongs to the subset and equals 0 otherwise). Then clearly (9) becomes

$$Y_{\beta\gamma} = \sum_{i,j\in\{0,1,2\}} t_i t_j^{-1} G_{ij}(\beta,\gamma).$$

Now since β, γ are adjacent in G, $G_{00}(\beta,\gamma) = G_{02}(\beta,\gamma) = G_{20}(\beta,\gamma) = \varnothing$, $G_{01}(\beta,\gamma) = \{\beta\}$ and $G_{10}(\beta,\gamma) = \{\gamma\}$. We find

$$Y_{\beta\gamma} = \varepsilon a t^{-1}\{\beta\} + \varepsilon a^{-1} t\{\gamma\} + G_{11}(\beta,\gamma) + G_{22}(\beta,\gamma) + \varepsilon t^2 G_{12}(\beta,\gamma) + \varepsilon t^{-2} G_{21}(\beta,\gamma).$$

The pair of equalities $E_2 Y_{\beta\gamma} = 0$, $E_2 Y_{\gamma\beta} = 0$ is equivalent to the pair $E_2(Y_{\beta\gamma} + Y_{\gamma\beta}) = 0$, $E_2(Y_{\beta\gamma} - Y_{\gamma\beta}) = 0$. Noting that $G_{ij}(\beta,\gamma) = G_{ji}(\gamma,\beta)$ we find that

$$Y_{\beta\gamma} - Y_{\gamma\beta} = \varepsilon(at^{-1} - a^{-1}t)(\{\beta\} - \{\gamma\}) + (\varepsilon t^2 - \varepsilon t^{-2})(G_{12}(\beta,\gamma) - G_{12}(\gamma,\beta)).$$

Observe that

$$A_1\{\beta\} = G_{10}(\beta,\gamma) + G_{11}(\beta,\gamma) + G_{12}(\beta,\gamma)$$

and similarly

$$A_1\{\gamma\} = G_{10}(\gamma,\beta) + G_{11}(\gamma,\beta) + G_{12}(\gamma,\beta).$$

Hence

$$G_{12}(\beta,\gamma) - G_{12}(\gamma,\beta) = A_1\{\beta\} - A_1\{\gamma\} + \{\beta\} - \{\gamma\}.$$

This yields

$$\varepsilon(Y_{\beta\gamma} - Y_{\gamma\beta}) = (at^{-1} - a^{-1}t + t^2 - t^{-2})(\{\beta\} - \{\gamma\}) + (t^2 - t^{-2})A_1(\{\beta\} - \{\gamma\}).$$

Then (36) shows that $Y_{\beta\gamma} - Y_{\gamma\beta}$ is a multiple of $(A_1 - rI)(\{\beta\} - \{\gamma\})$. Now $E_2 A_1 = rE_2$ by (20) and we have proved that $E_2(Y_{\beta\gamma} - Y_{\gamma\beta}) = 0$.

Let us now study the remaining equation, $E_2(Y_{\beta\gamma} + Y_{\gamma\beta}) = 0$. We find

$$\begin{aligned} Y_{\beta\gamma} + Y_{\gamma\beta} &= \varepsilon(at^{-1} + a^{-1}t)(\{\beta\} + \{\gamma\}) + 2(G_{11}(\beta,\gamma) + G_{22}(\beta,\gamma)) \\ &\quad + (\varepsilon t^2 + \varepsilon t^{-2})(G_{12}(\beta,\gamma) + G_{12}(\gamma,\beta)). \end{aligned}$$

Using the identities

$$\begin{aligned} G_{11}(\beta,\gamma) + G_{22}(\beta,\gamma) &= X - (\{\beta\} + \{\gamma\}) - (G_{12}(\beta,\gamma) + G_{12}(\gamma,\beta)), \\ G_{12}(\beta,\gamma) &= A_1\{\beta\} - \{\gamma\} - G_{11}(\beta,\gamma), \end{aligned}$$

and

$$G_{12}(\gamma,\beta) = A_1\{\gamma\} - \{\beta\} - G_{11}(\gamma,\beta),$$

we obtain

$$\begin{aligned} Y_{\beta\gamma} + Y_{\gamma\beta} = 2X &+ \varepsilon(at^{-1} + a^{-1}t - t^2 - t^{-2})(\{\beta\} + \{\gamma\}) \\ &+ (\varepsilon t^2 + \varepsilon t^{-2} - 2)(A_1(\{\beta\} + \{\gamma\}) - 2G_{11}(\beta, \gamma)). \end{aligned}$$

Using $E_2A_1 = rE_2$, $E_2X = 0$ we get

$$\begin{aligned} E_2(Y_{\beta\gamma} + Y_{\gamma\beta}) = (\varepsilon(at^{-1} + a^{-1}t - t^2 - t^{-2}) + r(\varepsilon t^2 + \varepsilon t^{-2} - 2))E_2(\{\beta\} + \{\gamma\}) \\ - 2(\varepsilon t^2 + \varepsilon t^{-2} - 2)E_2G_{11}(\beta, \gamma)). \end{aligned}$$

If $s = 0$, we know from Section 3.4 that $r = -2$ and G is a square. Then $G_{11}(\beta,\gamma) = \varnothing$ and, using the expression of P given in Section 3.3, we find $4E_2 = A_0 - A_1 + A_2$. One easily checks graphically that $E_2(\{\beta\} + \{\gamma\}) = 0$ and hence $E_2(Y_{\beta\gamma} + Y_{\gamma\beta}) = 0$ in this case. Similarly, if $r = -1$, G is the complement of a square and $G_{11}(\beta,\gamma) = \varnothing$. One obtains $4E_2 = 2A_0 - 2A_1$ and again $E_2(Y_{\beta\gamma} + Y_{\gamma\beta}) = 0$. This settles the case $n = 4$ since k must then be equal to 1 or 2. We now assume $n \geqslant 5$.

Since $s(r + 1) \neq 0$ by (34), we may apply (33), (35) to obtain:

$$E_2(Y_{\beta\gamma} + Y_{\gamma\beta}) = \left(\frac{2(r - s)}{s(r + 1)}\right)((r^2 + 2r + s)E_2(\{\beta\} + \{\gamma\}) - (r - s + 2)E_2G_{11}(\beta, \gamma)).$$

Noting that $r - s + 2 \neq 0$ by (25), we write $r_1 = \lambda/(r - s + 2)$. Then using (26) the property $E_2(Y_{\beta\gamma} + Y_{\gamma\beta}) = 0$ is equivalent to

$$(40) \qquad E_2G_{11}(\beta, \gamma) = r_1E_2(\{\beta\} + \{\gamma\}).$$

If $\lambda = r^2 + 2r + s = 0$, we also have $r_1 = 0$ and, since G has no triangles, $G_{11}(\beta,\gamma) = \varnothing$. Thus (40) is satisfied. We now assume that $\lambda \neq 0$.

Suppose first that (40) holds. Let us consider the subgraph $G(\beta)$ of G induced by the vertex-set $A_1\{\beta\}$, and let B be its adjacency matrix. This graph has k vertices $x_1, \ldots, x_k$ and is regular of degree λ. Let $V = \Sigma_{i=1,\ldots,k} v_i\{x_i\}$ be an eigenvector of $G(\beta)$ for an eigenvalue ρ, and assume also that $\Sigma_{i=1,\ldots,k} v_i = 0$. Since the restriction of $A_1\{x_i\}$ to the vertex-set of $G(\beta)$ is $G_{11}(\beta, x_i)$, $BV = \rho V$ is equivalent to $\Sigma_{i=1,\ldots,k} v_iG_{11}(\beta, x_i) = \rho V$. Multiplying by E_2 we obtain $\Sigma_{i=1,\ldots,k} v_ir_1E_2(\{\beta\} + \{x_i\}) = \rho E_2V$ and hence $(r_1 - \rho)E_2V = 0$.

Thus if $\rho \neq r_1, E_2V = 0$ and since we also have $E_0V = n^{-1}JV = 0$, using the equality $I = E_0 + E_1 + E_2$ we obtain $E_1V = V$ and hence $A_1V = sV$ by (20). This equation, restricted to the components of V corresponding to the vertices of $G(\beta)$, then implies that $\rho = s$. Since B commutes with J, the hyperplane of equation $\Sigma_{i=1,\ldots,k} v_i = 0$ has a basis consisting of eigenvectors of B and hence is contained in the kernel of the matrix $(B - r_1I)(B - sI)$. It follows that this matrix is a multiple of J and $G(\beta)$ is strongly regular with eigenvalues λ, r_1, s.

Conversely, assume that for every vertex β of G, the subgraph $G(\beta)$ with vertices $x_1, \ldots, x_k$ is strongly regular with eigenvalues λ, r_1, s. Since $\lambda \neq 0$, $G(\beta)$ is not empty. Since $\mu = r^2 + r$ by (27), $r \neq 0$ by (28), $r \neq -1$ by (34), $\mu \neq 0$ and G is not a disjoint union of cliques, so that $G(\beta)$ is not complete. Thus the definitions of Section 3.3 are relevant for $G(\beta)$. Consider a vertex γ adjacent to β in G, and write $G_{11}(\beta, \gamma) - r_1\{\gamma\} = mV^0 + V^1 + V^2$, where $V^0 = \Sigma_{i=1,\ldots,k}\{x_i\}$, V^1 is an eigenvector of $G(\beta)$ for the eigenvalue r_1 and V^2 is an eigenvector of $G(\beta)$ for the eigenvalue s. Taking the scalar product with V^0 we obtain $m = k^{-1}(\lambda - r_1)$.

The vector $G_{11}(\beta, \gamma) - r_1\{\gamma\}$ represents a column of $B - r_1 I$ and hence is orthogonal to all eigenvectors of $G\{\beta\}$ for the eigenvalue r_1. Hence $V^1 = 0$. Now let us write $A_1 V^2 = sV^2 + V'$, where all components of V' corresponding to vertices of $G(\beta)$ are zero. Then, denoting the scalar product by $\langle , \rangle$:

$$\langle sV^2, sV^2 \rangle + \langle V', V' \rangle = \langle A_1 V^2, A_1 V^2 \rangle = \langle A_1^2 V^2, V^2 \rangle.$$

By (21), $A_1^2 = (r + s)A_1 - rsI + \mu J$. Hence,

$$A_1^2 V^2 = (r+s)A_1 V^2 - rsV^2 = (r+s)(sV^2 + V') - rsV^2 = s^2 V^2 + (r+s)V'.$$

Since $\langle V^2, V' \rangle = 0$ we obtain $\langle A_1^2 V^2, V^2 \rangle = s^2 \langle V^2, V^2 \rangle$ and hence $\langle V', V' \rangle = 0$. It follows that $A_1 V^2 = sV^2$.

Multiplying by E_2 and using (20) we obtain $rE_2 V^2 = sE_2 V^2$, which yields $E_2 V^2 = 0$ since $n = (r - s)^2 \neq 0$. Thus we obtain

$$E_2(G_{11}(\beta, \gamma) - r_1\{\gamma\}) = mE_2 V^0.$$

Since $V^0 = A_1\{\beta\}$, (20) yields $E_2 V^0 = rE_2\{\beta\}$. Now it is easy to check that $mr = k^{-1}(\lambda - r_1)r = r_1$ and (40) follows. □

Let us say that G is *locally strongly regular* if for every vertex β of G, the subgraph $G(\beta)$ is strongly regular.

PROPOSITION 7. *(X, W^+, W^-) is a spin model for the Kauffman polynomial if and only if both G and G' are locally strongly regular.*

Proof. If (37), (38) hold, G and G' are locally strongly regular by Proposition 6. Conversely, assume that G, G' are both locally strongly regular and $n \geqslant 5$ (the case $n = 4$ has already been settled). By Proposition 6, it will be enough to show that both graphs satisfy property (39). This result is essentially contained in [7]. In that paper $G(\beta)$ and $(G'(\beta))'$ are called *subconstituents* of G, and thus G satisfies the property of Proposition 7 iff it has strongly regular subconstituents. The following proof is directly inspired from [7], but we needed a different way of distinguishing the roles of the two eigenvalues. Also the stronger hypotheses allow some shortcuts.

We shall assume without loss of generality that $k \geqslant n - k - 1$.

First case: $k = n - k - 1$. Then, by (24), $(r - s)^2 = n = 2k + 1 = 2(r^2 + r - rs) + 1$. This reduces to $s^2 = (r + 1)^2$. Since $n \neq 1$, $s = r + 1$ is excluded and we have $s = -r - 1$. Thus G is a conference graph, with parameters $n = 4\mu + 1, k = 2\mu, \lambda = \mu - 1$ and $\mu \geqslant 1$. For $\mu = 1$, G and G' are both pentagons and satisfy (39) since $\lambda = \lambda' = 0$. Assume now $\mu \geqslant 2$. For every vertex β of G, the subgraph $G(\beta)$ is strongly regular on 2μ vertices with degree $\mu - 1$. Its adjacency matrix B satisfies the equations $B^2 = (\mu - 1)I + pB + q(J - I - B)$ (for some non-negative integers p, q) and $JB = (\mu - 1)J$. Multiplying the first equation by J and using the second we get:

$$(\mu - 1)^2 = (\mu - 1) + p(\mu - 1) + q(2\mu - 1 - (\mu - 1)),$$

that is, $(\mu - 1)(\mu - 2 - p) = \mu q$. If $q \neq 0$, $\mu - 1$ divides q and $\mu - 2 - p < \mu$, a contradiction. Hence $q = 0$ and $p = \mu - 2$. It easily follows that $G(\beta)$ consists of two disjoint cliques of size μ. Let C be formed by adjoining β to one of these cliques. Since C is a clique and $\lambda = \mu - 1 = |C| - 2$, no vertex of $X - C$ is adjacent to two distinct vertices of C. On the other hand, every vertex of C is adjacent to exactly μ vertices of $X - C$ since $k = 2\mu$. Hence $|X - C| \geqslant \mu|C|$, or equivalently $3\mu \geqslant \mu(\mu + 1)$, which implies $\mu = 2$. There is a unique solution, the *lattice graph* $L_2(3)$ (a Cartesian product of two triangles), which is isomorphic to its complement. Then $\{r, s\} = \{1, -2\}$, and every subgraph $G(\beta)$ consists of two disjoint edges and thus has eigenvalues 1, 1, -1. G can be provided with two matrices P satisfying the duality property $P^2 = nI$, but only one will satisfy property (39). Indeed, to get $r_1 \in \{1, -1\}$ we must take $s = 1$, $r = -2$ and then $r_1 = -1$.

Second case: $k > n - k - 1$. Let us fix a vertex β of G. For every column vector V indexed by X, let $\pi(V)$ be the $(n - k - 1)$-vector obtained from V by deleting all components corresponding to β or to vertices of $G(\beta)$. Since $nE_1 = kA_0 + sA_1 + rA_2$, $\mathrm{Trace}(E_1) = k$ and hence the eigenspace $\mathrm{Im}(E_1)$ is k-dimensional. The inequality $k > n - k - 1$ implies that the space $\mathrm{Im}(E_1) \cap \mathrm{Ker}\ \pi$ is non-zero. Let $V + \rho\{\beta\}$ be a nonzero vector of this space, where all components of V not corresponding to vertices of $G(\beta)$ are zero. We have $J(V + \rho\{\beta\}) = 0$ and $A_1(V + \rho\{\beta\}) = s(V + \rho\{\beta\})$ (by (20)). Comparing the coefficients of $\{\beta\}$ in the second equation yields $\langle U, V\rangle = s\rho$, whereas the first equation gives $\langle U, V\rangle = -\rho$. Since $s \neq -1$ by (28), $\rho = 0$. It follows that V is an eigenvector of $G(\beta)$ for the eigenvalue s. Since $s \neq 0$ by (34), $G(\beta)$ is not empty. And $G(\beta)$ is not a clique, otherwise G would be an union of disjoint cliques and μ would be zero, a contradiction with (27), (28) and (34). Now we may apply to $G(\beta)$ Equation (23) of Section 3.3 to find the third eigenvalue r_1 of $G(\beta)$. This yields $k(\lambda + r_1 s) = (\lambda - r_1)(\lambda - s)$, or equivalently $(\lambda + (k-1)s)r_1 = \lambda(\lambda - s - k)$. Using (24), (26), easy calculations give $\lambda + (k-1)s = r(s+1)(r-s+2)$ and $\lambda - s - k = r(s+1)$. By (28), we obtain $r_1 = \lambda/(r - s + 2)$ as required. Thus G satisfies (39).

Now let V be an eigenvector of $G(\beta)$ for the eigenvalue r_1, so that all components of V not corresponding to vertices of $G(\beta)$ are zero. We write $A_1 V = r_1 V + V'$, where all components of V' corresponding to vertices of $G(\beta)$ are zero. Since $JV = 0$, the component of $A_1 V$ corresponding to β is zero and all non-zero components of V' correspond to vertices of $G'(\beta)$. Note that by (24), (26), the subgraph $G'(\beta)$ has $k' = r'^2 + r' - r's'$ vertices and is regular of degree $\lambda' = r'^2 + 2r' + s'$. We may assume $G'(\beta)$ is non-empty (that is, $\lambda' \neq 0$), otherwise (39) already holds. Also $G'(\beta)$ is not complete otherwise G' would be an union of disjoint cliques, in contradiction with (27), (28) and (34). Now since $A_1^2 = (r+s)A_1 - rsI + \mu J$ by (21) and $JV = 0$,

$$A_1^2 V = (r+s)A_1 V - rsV = (r+s)(r_1 V + V') - rsV = (r_1(r+s) - rs)V + (r+s)V'.$$

It follows that

$$A_1 V' = A_1^2 V - r_1 A_1 V = (r_1(r+s) - rs - r_1^2)V + (r+s-r_1)V'.$$

Note that $JV' = JA_1 V - Jr_1 V = kJV = 0$. Hence

$$\begin{aligned} A_2 V' &= JV' - V' - (r_1(r+s) - rs - r_1^2)V - (r+s-r_1)V' \\ &= -(r_1(r+s) - rs - r_1^2)V - (1 + r + s - r_1)V'. \end{aligned}$$

Also

$$\langle V', V' \rangle = \langle A_1 V, V' \rangle = \langle V, A_1 V' \rangle = (r_1(r+s) - rs - r_1^2)\langle V, V \rangle.$$

Note that $r_1(r+s) - rs - r_1^2 = -(r_1 - s)(r_1 - r)$. The two eigenvalues r_1, s of $G(\beta)$ are distinct (see the first remark of Section 3.3). The equality $r_1 = r$ can be written $r^2 + 2r + s = r(r - s + 2)$, or equivalently $s(r+1) = 0$, which is excluded by (34). Hence $\langle V', V' \rangle \neq 0$. It follows that V' is an eigenvector of $G'(\beta)$ for the eigenvalue $r_1 - r - s - 1$.

Let $r'_1 = \lambda'/(r' - s' + 2)$. It is easy to check that $r_1 - r - s - 1 = r'_1$. To find the third eigenvalue ρ of $G'(\beta)$ we may apply again Equation (23). This yields:

$$k'(\lambda' + r'_1 \rho) = (\lambda' - r'_1)(\lambda' - \rho),$$

or equivalently

$$\rho(\lambda' + (k' - 1)r'_1) = \lambda'(\lambda' - r'_1 - k').$$

We already know from above that $\rho = s'$ is a solution. This solution is unique, provided that $\lambda' + (k' - 1)r'_1 \neq 0$, or equivalently (dividing by $\lambda' \neq 0$) $r' - s' + 2 + k' - 1 \neq 0$. This condition becomes

$$(r' + 1)^2 \neq s'(r' + 1) \quad \text{or} \quad s^2 \neq s(r + 1).$$

This is satisfied since $s \neq 0$ by (34) and $s \neq r + 1$ by (25). Thus G' also satisfies (39). □

3.6. *A Classification of Spin Models for the Kauffman Polynomial*

According to the previous results, spin models for the Kauffman polynomial can be classified (up to simple symmetries) by formally self-dual, locally strongly regular graphs (excluding by (34) disjoint unions of cliques and their complements with more than four vertices). We now review the spin models which we can derive from existing knowledge about such graphs (see [7]). We shall not consider the graphs such that $r + s + 1 \in \{1, -1\}$ which lead to topologically uninteresting models (see Remark (i) of Section 3.4). Also we shall retain only one graph for each pair of complementary graphs, and one value of ε for each graph.

3.6.1. *The complete graphs.* They yield known spin models for Kauffman's bracket polynomial and consequently for the Jones polynomial (see Section 3.2).

3.6.2. *The square.* Then $s = 0, r = -2$, t is arbitrary and $a = -t^{-1}$. We get a spin model for a one-variable specialization of the Kauffman polynomial characterized by the loop value $d = -2\varepsilon$. The existence of this model is mentioned by Jones [22]. Other interesting models exist in this case (see [32] for vertex models, [18] for an IRF model). For $\varepsilon = -1$, the simplest description is given by $F_{-1}(L) = \frac{1}{2}\Sigma(-t^{-1})^{w(L')}$, where the sum is over all orientations L' of L.

3.6.3. *The pentagon.* Then $\{r, s\} = \{-\tau, \tau - 1\}$, where $\tau = (1 + \sqrt{5})/2$ (the golden ratio) and there are two possible choices of matrix P. We choose $s = -\tau, r = \tau - 1$ (the other case will be obtained by changing the sign of $\sqrt{5}$) and $\varepsilon = 1$. We obtain $d = \sqrt{5}$ and $t^2 + t^{-2} = -\tau$. Without loss of generality, we may take $t = e^{2i\pi/5}$, and then we have $a = 1$ and $z = \tau - 1$. The corresponding spin model appears in the work of Jones ([22], [23], see also [13]).

3.6.4. *The lattice graphs.* Let $q \geqslant 2$ be an integer. The *lattice graph* $L_2(q)$ has vertex-set the Cartesian product $\{1,\ldots,q\} \times \{1,\ldots,q\}$, two vertices being adjacent iff they differ in precisely one coordinate. It is strongly regular with parameters $n = q^2$, $k = 2(q-1)$, $\lambda = q-2$ and $\mu = 2$. We have already encountered $L_2(2)$ (the square) and $L_2(3)$ (see the proof of Proposition 7). $L_2(q)$ has eigenvalues k, $q-2$ and -2. The self-duality condition is satisfied with $s = q-2$, $r = -2$ (the choice $s = -2$, $r = q-2$ would also be feasible for $q = 3$, but we have seen that it does not satisfy property (39)). For every vertex β, $G(\beta)$ is a disjoint union of two cliques and $G'(\beta)$ is the complement of the lattice graph $L_2(q-1)$. Taking $\varepsilon = -1$ and assuming $q \geqslant 3$ we obtain $d = q = t^2 + t^{-2} + 2$ and $a = t^3$. This specialization of the Kauffman polynomial is well known to be equivalent (up to a change of variables) to the square of Kauffman's bracket polynomial [30]. In fact we could have derived directly the present 'lattice graph' model from the model of Section 3.2. Indeed, if $Z(L) = d^{-b(L)} \Sigma \langle L|\sigma\rangle$, then $(Z(L))^2 = (d^2)^{-b(L)} \Sigma \langle L|(\sigma,\sigma')\rangle$, where the sum is taken over all pairs of states (σ,σ') and $\langle L|(\sigma,\sigma')\rangle = \langle L|\sigma\rangle\langle L|\sigma'\rangle$. It is then easy to check that if $Z(L)$ is the model of Section 3.2 with parameter t_1, $(Z(L))^2$ is the lattice graph model with parameter $t = t_1^{-2}$.

3.6.5. *The Higman–Sims graph.* This graph was invented by D. G. Higman and C. H. Sims [17] to construct their famous sporadic finite simple group of order 44 352 000. It is strongly regular with parameters $n = 100$, $k = 22$, $\lambda = 0$, $\mu = 6$ and eigenvalues 22, -8, 2. For every vertex β, the vertices non-adjacent to β and distinct from it induce a strongly regular subgraph with parameters $n' = 77$, $k' = 16$, $\lambda' = 0$, $\mu' = 4$ (see for instance [5, p. 394, remark (iv)]). We may take $s = -8$, $r = 2$ to satisfy the self-duality conditions. For $\varepsilon = -1$ we find $d = -10$ and $t^2 + t^{-2} = 3$, or equivalently $(t - t^{-1})^2 = 1$. We shall take $t = \tau$. Then we obtain $a = -5\tau - 3 = -\tau^5$ and $z = 1$. Typical values of this specialization of the Kauffman polynomial are -21 for the Hopf link, $50\tau - 113$ for the right-handed trefoil knot, $50\tau + 63$ for the left-handed trefoil, and 265 for the figure-eight knot.

4. Conclusion

We have obtained a classification of spin models for the Kauffman polynomial in terms of certain strongly regular graphs. These graphs seem to be quite scarce, and it is very unlikely that the corresponding spin models could provide a full construction of the Kauffman polynomial. In fact, it is conjectured in [7] that apart from graphs with $r + s + 1 \in \{1, -1\}$ and the graphs described in 3.6.1–3.6.4, graphs meeting our requirements have, up to complementation, parameters

$$(41) \qquad s = -r^2 - 2r, \quad n = (r^2 + 3r)^2. \qquad k = r(r^2 + 3r + 1), \qquad \lambda = 0, \quad \mu = r^2 + r, \quad r \text{ a positive integer.}$$

For $r = 1$ we have the complement of the Clebsch graph, which satisfies $r + s + 1 = -1$, and for $r = 2$ we have the Higman–Sims graph of 3.6.5. It is known that any graph with parameters (41) has strongly regular subconstituents (see for instance [8, Th. 4.5], and [36, §7]). Moreover, the existence of such a graph is equivalent to the existence of a certain 3-design.

Let us now solve (32) under the assumptions $s = -r^2 - 2r$ and $s(r + 1) \neq 0$. Equation (33) becomes

$$s^2 - s - \varepsilon s(r + 1)(t^2 + t^{-2}) = 0,$$

and hence

$$t^2 + t^{-2} = \frac{s - 1}{\varepsilon(r + 1)} = -\varepsilon(r + 1).$$

Carrying this into (32) yields $a = \varepsilon t^5$. Thus the existence of an infinite family of graphs with parameters (41) is equivalent to the existence of a spin model description for the one-variable specialization of the Kauffman polynomial defined by the equation $a = \varepsilon t^5$.

The present work could be extended in several directions.

It is shown in [23] that any spin model for the Kauffman polynomial can be 'Baxterized', i.e. converted into a solution of the Yang–Baxter equation with spectral parameter. Such a solution corresponds to an integrable model in statistical mechanics. It might be interesting to study in more detail the Baxterization of the rather special 'Higman–Sims' model of 3.6.5.

The results of Section 2 indicate that a good approach to the construction of spin models would be to work in the algebra of a formally self-dual association scheme, first solve Equations (16), and then study the remaining equations (17). In addition to the detailed study of specific schemes (such as metric schemes) which might lead to new link invariants, it would be fruitful to obtain general conditions under which solutions exist. This could provide the still lacking 'machine' for constructing spin models evoked in [22].

Bose–Mesner algebras appear naturally in the approach of [2] to solutions of Yang–Baxter equations compatible with certain 'admissible patterns' (equality constraints on the entries of the weight matrix which are invariant under inversion), as well as more general structures, and in particular *coherent algebras* (a noncommutative generalization of Bose–Mesner algebras). Concerning the study of link invariants, it would be interesting to take into account orientations in spin models. This would allow the introduction of non-commutativity and would impose weaker restrictions on the models, at the cost of a greater complexity. We could then try to find spin models for the Homfly polynomial, or new ones for the Kauffman polynomial.

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