MATH 665 PROBLEM SET 4

FALL 2024

Due Thursday, December 5. You may consult books, papers, and websites as long as you cite all sources and write up your solutions in your own words.

Problem 1. Let $w_\circ \in S_n$ be the longest element in Bruhat length. The word $w_\circ = (s_1 s_2 \cdots s_{n-1})(s_1 s_2 \cdots s_{n-2}) \cdots (s_1 s_2)(s_1)$ is *reduced*—that is, of minimal length—from which

$$\sigma_\circ := \sigma_{w_\circ} = (\sigma_1 \sigma_2 \cdots \sigma_{n-1})(\sigma_1 \sigma_2 \cdots \sigma_{n-2}) \cdots (\sigma_1 \sigma_2)(\sigma_1).$$

The figure above[1] depicts the braid $\sigma_\circ$ when $n = 5$.

(1) Show that $\sigma_\circ^2 = (\sigma_1 \sigma_2 \cdots \sigma_{n-1})^n$. This elements is called the *full twist*.
(2) Using (1), give a proof by picture that $\sigma_\circ^2$ is central in Br_n.
(3) By Lemma 9.3 in Jones's "Hecke Algebra Representations of Braid Groups and Link Polynomials", $\sigma_\circ^2$ acts on any simple H_{S_n}-module by an explicit monomial, depending only on the partition $\lambda \vdash n$ that indexes the module. For $n = 4$, use Jones's paper to compute these monomials (in our conventions) for all λ. Note that his σ_i is our $\mathsf{x}\sigma_i$, and his q is our x^2.

Problem 2. The table below describes the characters $\chi_{\mathsf{x}}^\lambda : H_{S_4} \to \mathbf{Z}[\mathsf{x}^{\pm 1}]$ that correspond to the irreducible characters $\chi^\lambda : S_4 \to \mathbf{Z}$ under Tits deformation, for all $\lambda \vdash 4$.

	1	σ_1	$\sigma_{s_1 s_3} = \sigma_1 \sigma_3$	$\sigma_{s_1 s_2} = \sigma_1 \sigma_2$	$\sigma_{s_1 s_2 s_3} = \sigma_1 \sigma_2 \sigma_3$
$\chi^{(4)}$	1	x	x^2	x^2	x^3
$\chi^{(3,1)}$	3	$2\mathsf{x} - \mathsf{x}^{-1}$	$\mathsf{x}^2 - 2$	$\mathsf{x}^2 - 1$	$-\mathsf{x}$
$\chi^{(2,2)}$	2	$\mathsf{x} - \mathsf{x}^{-1}$	$\mathsf{x}^2 + \mathsf{x}^{-2}$	-1	0
$\chi^{(2,1^2)}$	3	$-2\mathsf{x}^{-1} + \mathsf{x}$	$\mathsf{x}^{-2} - 2$	$\mathsf{x}^{-2} - 1$	x^{-1}
$\chi^{(1^4)}$	1	$-\mathsf{x}^{-1}$	x^{-2}	x^{-2}	$-\mathsf{x}^{-3}$

Using this table and Problem 1, we will compute a piece of $\mathbf{P}(\hat{\beta})$, where

$$\beta = (\sigma_2 \sigma_1 \sigma_3 \sigma_2)^3 \sigma_1^7 \in Br_4.$$

This knot is also called the $(2, 13)$-cable of the trefoil in the blackboard framing.

(1) Show that β is conjugate to $\sigma_\circ^3 \sigma_1$ in Br_4. *Hint:* Draw pictures.
(2) Use (1) and Problem 1 to compute $\chi_{\mathsf{x}}^\lambda(\beta)$ for all $\lambda \vdash 4$. *Hint:* You will still need to deal with $\sigma_\circ \sigma_1$. Use conjugacy and Hecke relations.

[1] From Chmutov–Duzhin–Mostovoy's book.

(3) By Gomi, the "extremal" a-degrees of $\mu_4(\beta)$, hence of $\mathbf{P}(\hat{\beta})$, are given by

$$\frac{(-\mathsf{x})^3}{(1-\mathsf{x}^4)(1-\mathsf{x}^6)(1-\mathsf{x}^8)}\sum_\lambda \chi_{\mathsf{x}}^\lambda(\beta) f_\lambda(\mathsf{x}^2) \quad\text{and}\quad \frac{(-\mathsf{x})^3}{(1-\mathsf{x}^4)(1-\mathsf{x}^6)(1-\mathsf{x}^8)}\sum_\lambda \chi_{\mathsf{x}}^\lambda(\beta) f_{\lambda^t}(\mathsf{x}^2), \qquad \text{where } \begin{cases} f_{(4)}(\mathsf{q}) = 1, \\ f_{(3,1)}(\mathsf{q}) = \mathsf{q}+\mathsf{q}^2+\mathsf{q}^3, \\ f_{(2,2)}(\mathsf{q}) = \mathsf{q}^2+\mathsf{q}^4, \\ f_{(2,1^2)}(\mathsf{q}) = \mathsf{q}^3+\mathsf{q}^4+\mathsf{q}^5, \\ f_{(1^4)}(\mathsf{q}) = \mathsf{q}^6 \end{cases}$$

and λ^t is the transpose of λ. The f_λ are the *generic degrees* of S_4.

Compute these expressions and compare to §7 of Oblomkov–Shende's "The Hilbert Scheme of a Plane Curve Singularity..." Their z is our $\mathsf{x}-\mathsf{x}^{-1}$.

Problem 3 (O. Dudas)**.** Let G be a connected smooth reductive algebraic group over $\bar{\mathbf{F}}_q$. Fix a Frobenius map $F : G \to G$ and an F-stable Borel B. Assume that F acts trivially on the Weyl group W. For all $w, s \in W$ with s simple, let

$$X_w = \{xB \in G/B \mid xB \xrightarrow{w} F(x)B\},$$
$$X_{w,s} := \{(xB, yB) \in (G/B)^2 \mid xB \xrightarrow{w} yB \xrightarrow{s} F(x)B\},$$
$$X_{w,s,s} := \{(xB, yB, zB) \in (G/B)^3 \mid xB \xrightarrow{w} yB \xrightarrow{s} zB \xrightarrow{s} F(x)B\}.$$

We will show $R_{X_{w,s,s}} = R_{X_w}$, where $R_{(-)}(g) := \sum_i (-1)^i \operatorname{tr}(g \mid \mathrm{H}^i_c(-))$ for $g \in G^F$.

(1) Let $Z = \{(xB, yB, zB) \in X_{w,s,s} \mid yB = F(x)B\}$. Show that Z forms a G^F-equivariant line bundle over X_w, locally trivial in the smooth topology. Deduce that $R_Z = R_{X_w}$.

(2) Let $Y = \{(xB, yB, zB) \in X_{w,s,s} \mid yB \xrightarrow{s} F(x)B\}$. Show that Y forms the complement to the zero section in a G^F-equivariant line bundle over $X_{w,s}$, locally trivial in the smooth topology. Deduce that $R_Y = 0$.

Hints: Use $\mathrm{H}^*_c(L) \simeq \mathrm{H}^*_c(X)[-2]$ for a line bundle $L \to X$ in the smooth topology. In (2), use additivity of Lefschetz number.

Problem 4. Take $G = \mathrm{PGL}_2$ and standard F in the setup of the previous problem, so that $G/B \simeq \mathbf{P}^1$. Let $\mathcal{U} \subseteq G$ be the unipotent locus. For $m \geq 1$, let

$$O_m = \{(x_1B, \ldots, x_mB) \in (G/B)^m \mid x_1B \xrightarrow{s} \cdots \xrightarrow{s} x_mB\},$$
$$\mathcal{U}_m = \{(\vec{x}B, u) \in O_m \times \mathcal{U} \mid x_mB \xrightarrow{s} ux_1B\},$$
$$\mathcal{X}_m = \{\vec{x}B \in O_m \mid x_mB \xrightarrow{s} x_1B\}.$$

Show that $|\mathcal{X}^F_{m+2}| \neq |\mathcal{X}^F_m|$, but $|\mathcal{X}^F_{m+2}| = |\mathcal{U}^F_m|$.

In Problem 5, we use the notation and conventions of the "active learning" notes on Soergel bimodules, taking $W = S_2 = \{e, s\}$.

Problem 5. For $W = S_2$, the Koszul resolution of R over $\tilde{R} = R \otimes_K R^{\mathrm{op}}$ can be normalized to $\tilde{R}\langle -2\rangle \xrightarrow{d} \underline{\tilde{R}}$, where $d(1 \otimes 1) = \frac{1}{2}(1 \otimes \alpha - \alpha \otimes 1)$. Using this:

(1) Compute $\mathsf{HH}_*(\mathbf{B}_e)$ and $\mathsf{HH}_*(\mathbf{B}_s)$.

(2) *Hard:* Compute $\mathsf{HH}_*(\mathbf{B}_s \xrightarrow{\epsilon} \mathbf{B}_e)$. Deduce that $\mathsf{HHH}_*(\Delta_s)$ is one-dimensional.

(3) *Harder:* Use #5 on Problem Set 3 to compute HHH_* on the complexes $\mathcal{R}^+_s * \mathcal{R}^+_s$ and $\mathcal{R}^+_s * \mathcal{R}^+_s * \mathcal{R}^+_s$. Compare to $\mu_2(\sigma_1^2)$ and $\mu_2(\sigma_1^3)$.