Warmup

analytic topology T_{an} on R^n:

U sub R^n is in T_{an} iff
for all x in U there exists δ > 0 s.t. B(x, δ) sub U

equivalently (when n = 1)
for all x in U, there exist a, b s.t. x in (a, b) sub U

Df lower-limit topology T_ℓ on R:

U sub R is in T_ℓ iff
for all x in U, there exists b s.t. [x, b) sub U

Q1 is T_ ℓ really a topology on R? [yes]
Q2 how does T_ ℓ compare to T_{an}?

Prop T_ℓ is strictly finer than T_{an}

Pf T_ ℓ is finer than T_{an}:
suppose U anlytc open in R
suppose x in U
pick a < b s.t. x in (a, b) sub U
then [x, b) sub (a, b)

strict because [0, 1) in T_ℓ but notin T_{an}

T_{indisc} < T_f < T_{an} < T_ℓ < T_{disc}

(Munkres §18, 16) recall from real analysis:

$f : \mathbb{R}^n \to \mathbb{R}^m$ is <u>Bolzano continuous</u> iff

for all x in $\mathbb{R}^n$ and $\varepsilon > 0$, there exists $\delta > 0$ s.t.
$|x - x'| < \delta$ implies $|f(x) - f(x')| < \varepsilon$

equivalently
for all x in $\mathbb{R}^n$ and $\varepsilon > 0$, there exists $\delta > 0$ s.t.
x' in $B(x, \delta)$ implies $f(x')$ in $B(f(x), \varepsilon)$

equivalently
for all x in $\mathbb{R}^n$ and $\varepsilon > 0$, there exists $\delta > 0$ s.t.
$f^{-1}(B(f(x), \varepsilon))$ contains $B(x, \delta)$

<u>Goal</u> generalize this notion

given topological spaces (X, T_X) and (Y, T_Y):

<u>Df</u> a function $f : X \to Y$ is <u>continuous</u> iff
V open in Y implies $f^{-1}(V)$ open in X

<u>Thm</u> $f : \mathbb{R}^n \to \mathbb{R}^m$ is Bolzano cts
iff
f is cts wrt the analytic topologies

<u>Pf</u>

suppose f cts wrt analytic topologies:

fix x in R^n and $\varepsilon > 0$

  $B(f(x), \varepsilon)$ is open in R^m

  so $f^{-1}(B(f(x), \varepsilon))$ is open in R^n

  so $B(x, \delta)$ sub $f^{-1}(B(f(x), \varepsilon))$ for some $\delta > 0$

suppose f Bolzano cts:

fix V anlytc open in R^m

want $f^{-1}(V)$ anlytc open in R^n

  suppose x in $f^{-1}(V)$

  can find $\varepsilon > 0$ s.t. $B(f(x), \varepsilon)$ sub V

  then $f^{-1}(B(f(x), \varepsilon))$ sub $f^{-1}(V)$

  pick $\delta > 0$s.t. $B(x, \delta)$ sub $f^{-1}(B(f(x), \varepsilon))$

  then x in $B(x, \delta)$ sub $f^{-1}(V)$

<u>Ex</u> which maps are continuous?

$f : (R, T_\ell)$ to (R, T_{an}), $f(x) = x$ [yes]

$f : (R, T_{an})$ to (R, T_ℓ), $f(x) = x$ [no: [0, 1)]

$f : (R, T_{an})$ to (R, T_ℓ), $f(x) = 31$ [yes]

<u>General Facts</u>

1) S finer than T:

  id : (X, S) to (X, T) continuous

2) S strictly coarser than T:

  id : (X, S) to (X, T) not continuous

3) constant maps are always continuous

also

4) compositions of cts maps are cts

henceforth omit T from (X, T) when understood

<u>Df</u> cts f : X to Y is a homeomorphism iff
it has a two-sided cts inverse g : Y to X
i.e. g(f(x)) = x for all x in X,
f(g(y)) = y for all y in Y

“what is shape?”

“X and Y have the same shape
when there is a homeo between them”

<u>Ex</u> id : (X, T) to (X, T) is always a homeo
with inverse id

<u>Ex</u> [however:]
a cts bijection need not be a homeo

[we already have an example! which?]
f : (R, T_ℓ) to (R, T_{an}), f(x) = x

<u>Ex</u> more homeo’s in the analytic topology:

f : R to R, f(x) = x^3
f : R^2 to R^2 f(x, y) = (x + y, (x – y)^3)

[compositions of homeo’s are homeo’s]

<u>Q</u> is there a homeo R to R^2? vice versa?

The Subspace Topology

fix A sub X

Df the subspace topology on A induced by X:

U sub A is open iff there exists V open in X s.t.
$U = V \cap A$

Ex $X = \mathbb{R}$ and $A = [0, \infty)$

suppose $0 \leq a < b$
(a, b) open in $[0, \infty)$? in $\mathbb{R}$? [yes, yes]
$[a, b)$ open in $[0, \infty)$? [depends]

[here, must clarify where U is open]

Ex if A is open in X, and U sub A, then
U open in A iff U open in X

Prop the subspace topology on A is the (unique) coarsest topology s.t. the inclusion $i : A \to X$ is continuous

Pf easy that $i : A \to X$ cts wrt sub. topology

suppose $i : A \to X$ cts wrt some topology T on A
fix U open in subspace topology on A
$U = A \cap V$ for some V open in X
so $U = i^{-1}(V)$ open in T by continuity wrt T