(Munkres §70) suppose X has basept x

U, V open in X,

X = U cup V,

x in U cap V

Q how are π_1(X, x), π_1(U, x), π_1(V, x), π_1(U cap V, x) related?

Ex suppose that X = B v C with wedge pt x

suppose that B sub U and C sub V s.t.

- U, V are open in X,
- B is a def retract of U,
- C is a def retract of V,
- {x} is a def retract of U cap V [draw]

then π_1(U) ≃ π_1(B),

π_1(V) ≃ π_1(C),

π_1(U, x) * π_1(V, x) ≃ π_1(X, x)

namely, given a word [γ_1] [γ_2] … [γ_k]

where each γ_i is a loop at x in either U or V,

we see that γ_1 * γ_2 * … * γ_k is a loop at x in X

so we send the word to [γ_1 * γ_2 * … * γ_k]

Ex suppose that p ≠ q in D^2

X = D^2

U = D^2 – {p}

V = D^2 – {q}

U cap V = D^2 – {p, q}

here, $\pi_1(X, x)$ is trivial,

but $\pi_1(U, x) \simeq Z$ and $\pi_1(V, x) \simeq Z$

<u>Moral</u> in general,
$\pi_1(U, x) * \pi_1(V, x)$ to $\pi_1(X, x)$
might be surjective but not bijective

<u>Idea</u> in last example,

$\pi_1(U \cap V, x) \simeq Z * Z$
$\simeq \pi_1(U, x) * \pi_1(V, x)$

so maybe necessary to use $\pi_1(U \cap V, x)$ somehow

<u>Ex</u> $X = S^2$
U, V overlapping open hemispheres
$U \cap V$ open annulus

here, π_1 is trivial for X, U, V, but not for $U \cap V$

<u>Idea</u> don't use $\pi_1(U \cap V, x)$ itself

inclusions $i : U \cap V$ to U and $j : U \cap V$ to V

$i_* : \pi_1(U \cap V, x)$ to $\pi_1(U, x)$
$j_* : \pi_1(U \cap V, x)$ to $\pi_1(V, x)$

use $i_*(\pi_1(U \cap V, x))$ and $j_*(\pi_1(U \cap V, x))$:

in last example, these images are trivial

Thm (Seifert–Van Kampen) suppose that

U, V are open in X,
X = U cup V,
x in U cap V,
U, V, U cap V are all path-connected

embed

i_*(π_1(U cap V, x)) sub π_1(U, x)
sub π_1(U, x) * π_1(V, x),
j_*(π_1(U cap V, x)) sub π_1(V, x)
sub π_1(U, x) * π_1(V, x)

let N sub π_1(U, x) * π_1(V, x) be
the smallest normal subgrp containing

{i_*([γ]) * j_*([γ])^{-1} | [γ] in π_1(U cap V, x)}

then (π_1(U, x) * π_1(V, x))/N ≃ π_1(X, x)

LHS is the largest quotient where i_*([γ]) = j_*([γ])
for all [γ] in π_1(U cap V, x)

works for X = D^2
B = D^2 – {p}
C = D^2 – {q}

works for X = S^2
B, C overlapping hemispheres

Q π_1 of the hollow two-holed donut?