Recall given paths β, γ s.t. $\beta(1) = \gamma(0)$,

$\beta * \gamma$ is a new path $(\beta * \gamma)(s) = \beta(2s)$ $s \leq 1/2$
$(\beta * \gamma)(s) = \gamma(2s - 1)$ $s \geq 1/2$

[draw]

given paths γ, γ' s.t. $\gamma(0) = \gamma'(0)$
$\gamma(1) = \gamma'(1)$

$\gamma \sim_p \gamma'$ means there's a path homotopy from γ to γ'

[draw]

Q suppose $\beta(1) = \gamma(0) = \gamma'(0)$ and $\gamma \sim_p \gamma'$
do we have $\beta * \gamma \sim_p \beta * \gamma'$?

Q suppose $\beta(1) = \beta'(1) = \gamma(0)$ and $\beta \sim_p \beta'$
do we have $\beta * \gamma \sim_p \beta' * \gamma$?

[draw]

Thm if $\beta(1) = \beta'(1) = \gamma(0) = \gamma'(0)$
and $\beta \sim_p \beta'$ and $\gamma \sim_p \gamma'$
then $\beta * \gamma \sim_p \beta' * \gamma'$

Pf pick a path homotopy h from β to β'
pick a path homotopy j from γ to γ'

then $h(1, t) = \beta(1) = \beta'(1) = \gamma(0) = \gamma'(0) = j(0, t)$

set $k(s, t) = h(2s, t)$ $s \leq 1/2$
$k(s, t) = j(2s - 1, t)$ $s \geq 1/2$

Df write $[\gamma]$ for the $\sim_p$ equiv class of γ

for any paths β, γ s.t. $\beta(1) = \gamma(0)$,
take $[\beta] * [\gamma]$ to be the equiv class $[\beta * \gamma]$

by thm, $*$ is a well-def operation on equiv classes

(Munkres §52) now focus on loops:

if β, γ are loops in X at the same basepoint x
then $\beta * \gamma$ is also a loop at x

get a binary operation on equiv classes of loops:

$$[\beta] * [\gamma] = [\beta * \gamma]$$

Thm 1 if α, β, γ are paths s.t. $\alpha(1) = \beta(0)$
$\beta(1) = \gamma(0)$

[draw]

then $(\alpha * \beta) * \gamma \sim_p \alpha * (\beta * \gamma)$
so $([\alpha] * [\beta]) * [\gamma] = [\alpha] * ([\beta] * [\gamma])$

Thm 2 write $e_x : [0, 1]$ to X for
the constant path $e_x(s) = x$

then $e_x * \gamma \sim_p \gamma$ for all paths γ starting at x
$\beta \sim_p \beta * e_x$ for all paths β ending at x

so $[e_x] * [\gamma] = [\gamma]$
$[\beta] * [e_x] = [\beta]$

Thm 3 write $\bar{\gamma}(s) = \gamma(1 - s)$ for the reverse path

then $\gamma * \bar{\gamma} \sim_p e\_x$ for γ starting at x
$\bar{\gamma} * \gamma \sim_p e\_y$ for γ ending at y

so $[\gamma] * [\bar{\gamma}] = [e\_x]$
$[\bar{\gamma}] * [\gamma] = [e\_y]$

[proofs of Thms 1–2 are long and tedious]

[to show that $[\gamma] * [\bar{\gamma}] = [e\_x]$ for γ starting at x:
need path homotopy $h : [0, 1] \times [0, 1]$ to X

s.t. for all s in $[0, 1]$, $h(s, 0) = e\_x(s) = x$
$h(s, 1) = (\gamma * \bar{\gamma})(s)$

for fixed t, the path $h(s, t)$ should "freeze"
when it hits $\gamma(t)$, then go back

$h(s, t)$: x to $\gamma(t)$ s in $[0, t/2]$
stay at $\gamma(t)$ s in $[t/2, 1 - t/2]$
$\gamma(t)$ back to x s in $[1 - t/2, 1]$

$h(s, t) = \gamma(2s)$ s in $[0, t/2]$
$= \gamma(t)$ s in $[t/2, 1 - t/2]$
$= \gamma(2 - 2s) = \bar{\gamma}(2s)$ s in $[1 - t/2, 1]$]

Cor for loops in X based at a point x:
1) $([\alpha] * [\beta]) * [\gamma] = [\alpha] * ([\beta] * [\gamma])$
2) $[\gamma] * [e\_x] = [\gamma] = [e\_x] * [\gamma]$
3) $[\gamma] * [\bar{\gamma}] = [e\_x] = [\bar{\gamma}] * [\gamma]$

Df the fundamental group of X based at x is

$$\pi_1(X, x) = \{[\gamma] \mid \text{loops } \gamma \text{ in } X \text{ based at } x\}$$

under the operation * on $\sim_p$ equiv classes

Q how much does $\pi_1(X, x)$ depend on X and x?