naive idea behind Seifert–van Kampen:

if X is covered by open U_i’s [draw]
then can compute $\pi_1(X)$ from $\pi_1(U_i)$’s

but only works when all intersections of U_i’s are path-connected

<u>Thm</u> (Seifert–van Kampen)

suppose A, U_1, U_2 sub X are open and x in A

s.t. $X = U_1$ cup U_2,
$A = U_1$ cap U_2
A, U_1, U_2 are all path-connected

with inclusion maps

$$\begin{array}{ccccccc} & i_1 & & U_1 & & j_1 & \\ A & & & & & & X \\ & i_2 & & U_2 & & j_2 & \end{array}$$

then:

1) $j_{\{1, *\}}$, $j_{\{2, *\}}$ induce a surjective hom

$\pi_1(U_1, x) * \pi_1(U_2, x)$ to $\pi_1(X, x)$

2) via this hom, $\pi_1(X, x)$ is the largest quotient of $\pi_1(U_1, x) * \pi_1(U_2, x)$ in which

$i_{\{1, *\}}([\gamma]) \sim i_{\{2, *\}}([\gamma])$ for all $[\gamma]$ in $\pi_1(A, x)$

Ex X the figure-eight,
U_1, U_2 open thickenings of the S^1's,
x the intersection point
[draw]

since $\pi_1(A, x)$ is trivial:

$$\pi_1(X, x) \simeq \pi_1(U_1, x) * \pi_1(U_2, x) \simeq \pi_1(S^1, x) * \pi_1(S^1, x) \simeq \mathbb{Z} * \mathbb{Z}$$

Ex $X = S^2$,
U_1, U_2 open thickenings of opposed hemispheres,
[draw]

here, $\pi_1(A, x) \simeq \pi_1(S^1, x) \simeq \mathbb{Z}$
but $\pi_1(U_1, x)$, $\pi_1(U_2, x)$ trivial
so $\pi_1(X, x)$ also trivial

Rem Seifert–van Kampen alone cannot compute $\pi_1(S^1)$

(Munkres §53–54) let $o = (1, 0)$ in S^1

recall: $\omega_n(s) = (\cos(2\pi ns), \sin(2\pi ns))$

Thm $\Phi : \mathbb{Z} \to \pi_1(S^1, o)$ def by $\Phi(n) = [\omega_n]$ is an isomorphism

Pf Step 1. Φ is a homomorphism
Step 2. Φ is bijective

Step 1 must show that $\Phi(m + n) = \Phi(m) + \Phi(n)$, meaning $[\omega_{m + n}] = [\omega_m * \omega_n]$

key idea: let $p : \mathbb{R} \to S^1$ be

$$p(x) = (\cos(2\pi x), \sin(2\pi x))$$

Lem for any a, b in Z s.t. $b - a = n$, we have

$$\omega_n = p \circ \acute{\omega}_{a, b}$$

where $\acute{\omega}_{a, b} : [0, 1] \to \mathbb{R}$ is defined by

$$\acute{\omega}_{a, b}(s) = (1 - s)a + sb$$

[draw]

Lem for any a, b, c in Z, we have

$$[\acute{\omega}_{a, b} * \acute{\omega}_{b, c}] = [\acute{\omega}_{a, c}]$$

[draw]

therefore,

$[(p \circ \acute{\omega}_{a, b}) * (p \circ \acute{\omega}_{b, c})]$

$= [p \circ (\acute{\omega}_{a, b} * \acute{\omega}_{b, c})]$ [Munkres 327]

$= [p \circ \acute{\omega}_{a, c}]$ [PS5, #1]

giving $[\omega_m * \omega_n] = [\omega_{m + n}]$

before Step 2, some definitions inspired by the map $p : \mathbb{R} \to S^1$:

suppose $p : E$ to X is cts

<u>Df</u> p is a <u>covering map</u> iff, for all x in X, we have an open nbd x in U sub X s.t.

1) $p^{-1}(U)$ is homeo to a union of disjoint copies of U
2) p restricts to a homeo from each copy onto U

here, we say U is <u>evenly covered</u> by p

<u>Ex</u> let $p : (–1, 1)$ to $[0, 1)$ be squaring

if $0 < a < b$, then (a, b) is evenly covered
but $[0, b)$ is never evenly covered

<u>Ex</u> let $p : (–1, 1)$ to S^1 be

$$p(x) = (\cos(2\pi x), \sin(2\pi x))$$

open nbd's of 0 are not evenly covered

but!

$p : \mathbb{R}$ to S^1 defined by the same formula is a covering map