1.

On interpreting Haiman's monomial-character conjecture in terms of webs, also known as MOY graphs. Based on `2205_05`.

1.1.

Haiman conjectured a new positivity property of the Kazhdan–Lusztig bases of the Iwahori–Hecke algebras of the symmetric groups.

1.1. For any positive integer N, let S_N be the symmetric group on N letters. It is generated by the transpositions $s_i = (i, i+1)$ for $1 \leq i \leq N-1$. We take the *Iwahori–Hecke algebra* of S_N to be the $\mathbf{Z}[x^{\pm 1}]$-algebra $H_N(x)$ generated by elements σ_i for $1 \leq i \leq N-1$, modulo the relations

$$\begin{aligned} \sigma_i\sigma_{i+1}\sigma_i &= \sigma_{i+1}\sigma_i\sigma_{i+1}, \\ \sigma_i\sigma_j &= \sigma_j\sigma_i && \text{for } |i-j| > 1, \\ \sigma_i^2 &= 1 + (x - x^{-1})\sigma_i. \end{aligned}$$

The last relation is equivalent to requiring that σ_i be invertible with $\sigma_i - \sigma_i^{-1} = x - x^{-1}$. Hence, there is a ring anti-automorphism $D : H_N(x) \to H_N(x)$ that sends $x \mapsto x^{-1}$ and $\sigma_i \mapsto \sigma_i^{-1}$ for all i.

1.2. Note that $H_N(x)$ is a deformation of the group ring $\mathbf{Z}S_N$, in the sense that there is a ring isomorphism $H_N(x)/(x-1) \simeq \mathbf{Z}S_N$.

Let $\mathbf{K} = \mathbf{Q}(x)$. It turns out that $\mathbf{K}H_N(x) := \mathbf{K} \otimes_{\mathbf{Z}[x^{\pm 1}]} H_N(x)$ is split as a $\mathbf{K}$-algebra. Hence, by Tits deformation, the semisimplicity of $\mathbf{Q}S_N$ implies the semisimplicity of $\mathbf{K}H_N(x)$, and moreover, there is a bijection between isomorphism classes of simple $\mathbf{Q}S_N$-modules and isomorphism classes of simple $\mathbf{K}H_N(x)$-modules. In particular, each character $\chi : S_N \to \mathbf{Q}$ defines a $\mathbf{K}$-linear trace function $\chi_x : \mathbf{K}H_N(x) \to \mathbf{K}$.

Recall that the irreducible characters of S_N are indexed by integer partitions of N. We write χ^λ for the irreducible character indexed by $\lambda \vdash N$.

1.3. Kazhdan–Lusztig discovered two remarkable D-invariant bases for $H_N(x)$ as a free $\mathbf{Z}[x^{\pm 1}]$-module. To define them, view S_N as a Coxeter group, in which $\{s_i\}_i$ is a fixed system of simple reflections. Let ℓ_w denote the Bruhat length of $w \in S_N$, and let $<$ be the Bruhat order on S_N. Then for all $w \in S_N$, there is a unique element b_w of $H_N(x)$ such that:

(1) $D(b_w) = b_w$.
(2) $b_w = \sum_{y \leq w} P_{y,w}(x^2) x^{\ell_y - \ell_w} \sigma_y$ for some $P_{y,w}(q) \in \mathbf{Z}[q]$ such that

$$\begin{aligned} \deg P_{y,w}(q) &\leq \tfrac{1}{2}(\ell_w - \ell_y - 1), \\ P_{w,w}(q) &= 1 \end{aligned} \tag{1.1}$$

for all w, y.

Let $j : H_N(x) \to H_N(x)$ be the ring automorphism that sends $x \mapsto x^{-1}$ and $\sigma_i \mapsto -\sigma_i$ for all i. Let $c_w = j(b_w)$. Then c_w is the unique element of $H_N(x)$ such that:

(1) $D(c_w) = c_w$.
(2) $c_w = -\sum_{y \leq w} P_{y,w}(x^{-2})(-x)^{\ell_w - \ell_y}\sigma_y$ for some $P_{y,w}(q) \in \mathbf{Z}[q]$ satisfying (1.1). (They turn out to be the same polynomials as before.)

The $P_{y,w}(q)$ are now called *Kazhdan–Lusztig polynomials*. Note that in Kazhdan–Lusztig's notation, our b_w and c_w respectively correspond to their C'_w and $-C_w$. It will be convenient to write b_i, c_i in place of b_{s_i}, c_{s_i}. We can check that

$$\begin{aligned} b_i &= x^{-1} + \sigma_i &&= x + \sigma_i^{-1}, \\ c_i &= x - \sigma_i &&= x^{-1} - \sigma_i^{-1}. \end{aligned}$$

Thus, $\{b_i\}_i$ and $\{c_i\}_i$ form alternative generating sets for $H_N(x)$ as a $\mathbf{Z}[x^{\pm 1}]$-algebra. At the same time, the sets $\{b_w\}_{w \in S_N}$ and $\{c_w\}_{w \in S_N}$ form bases for $H_N(x)$ as a free $\mathbf{Z}[x^{\pm 1}]$-module.

1.4. There is a geometric interpretation of the Iwahori–Hecke algebra, in terms of mixed perverse sheaves on flag varieties. The standard basis $\{\sigma_w\}_w$ corresponds to the sheaves obtained by extension-by-zero from constant sheaves on Bruhat orbits. The bases $\{b_w\}_w$ and $\{c_w\}_w$ respectively correspond to intersection cohomology (IC) sheaves and tilting complexes. This interpretation of the b_w shows that Kazhdan–Lusztig polynomials have nonnegative coefficients.

A similar argument, using the interpretation of the trace functions χ_x^λ in terms of mixed perverse sheaves on the algebraic groups GL_N, shows that $\chi_x^\lambda(b_w)$, a priori an element of $\mathbf{K} = \mathbf{Q}(x)$, has nonnegative, integral coefficients for all $w \in S_N$ and $\lambda \vdash N$. That is, $\chi_x^\lambda(b_w) \in \mathbf{Z}_{\geq 0}[x^{\pm 1}]$. No analogous property holds for the values $\chi_x^\lambda(c_w)$.

1.5. Haiman's conjecture is about a collection of trace functions ϕ_x^μ such that the transition matrix from the ϕ_x^λ to the χ_x^λ is nonnegative, but the inverse transition matrix can have negative entries.

Let Λ be the graded ring of symmetric functions in variables $X_1, X_2, \ldots$ Recall that its degree-N summand $\Lambda_N \subseteq \Lambda$ admits the following bases as a $\mathbf{Z}$-module:

$\{s_\lambda\}_{\lambda \vdash N}$,	where the s_λ are Schur functions,
$\{m_\lambda\}_{\lambda \vdash N}$	where the $m_\lambda = m_{\lambda_1} m_{\lambda_2} \cdots$ are monomial symmetric functions,
$\{h_\lambda\}_{\lambda \vdash N}$	where the $h_\lambda = h_{\lambda_1} h_{\lambda_2} \cdots$ are complete homogeneous symmetric functions,
$\{e_\lambda\}_{\lambda \vdash N}$	where the $e_\lambda = e_{\lambda_1} e_{\lambda_2} \cdots$ are elementary symmetric functions.

We set aside the e_λ for now.

There is a unipotent triangular matrix of integers $K = \{K_{\lambda,\mu}\}_{\lambda \geq \mu}$ such that

$$s_\lambda = \sum_{\mu \leq \lambda} K_{\lambda,\mu} m_\mu \quad \text{and} \quad h_\mu = \sum_{\lambda \geq \mu} K_{\lambda,\mu} s_\lambda. \tag{1.2}$$

The integers $K_{\lambda,\mu}$ are known as the *Kostka numbers*. They admit a purely combinatorial definition via Young diagrams.

For any $\mathbf{K}$-algebra H, let $\mathscr{C}(H)$ denote the vector space of $\mathbf{K}$-valued trace functions on H. Then $\mathscr{C}(H_N(x))$ is spanned by the deformed irreducible characters χ_x^λ. Writing

$\Lambda_N(x) = \mathbf{Z}[x^{\pm 1}] \otimes_{\mathbf{Z}} \Lambda_N$, we obtain an isomorphism of vector spaces

$$\mathrm{ch} : \mathscr{C}(H_N(x)) \xrightarrow{\sim} \mathbf{K} \otimes_{\mathbf{Z}[x^{\pm 1}]} \Lambda_N(x) \quad \text{defined by } \mathrm{ch}(\chi_x^\lambda) = s_\lambda,$$

known as the *deformed Frobenius characteristic*. Let $\phi_x^\mu = \mathrm{ch}^{-1}(m_\mu)$, so that

$$\chi_x^\lambda = \sum_{\mu \leq \lambda} K_{\lambda,\mu} \phi_x^\mu. \tag{1.3}$$

Note that, since the matrix of integers K is unipotent triangular, its inverse also has integral entries. Hence the integrality of $\chi_x^\lambda(b_w)$ for all λ implies the integrality of $\phi_x^\mu(b_w)$ for all μ. However, the inverse matrix to K will generally have negative entries, making the following expectation surprising:

Conjecture 1.1 (Haiman)**.** *$\phi_x^\mu(b_w)$ has nonnegative coefficients for all w and μ.*

1.2.

We claim that Conjecture 1.1 has an especially simple meaning in the web description of Iwahori–Hecke algebras.

1.6. Let $\Lambda(x) = \bigoplus_N \Lambda_N(x)$. The point is to interpret the $\mathbf{Z}[x^{\pm 1}]$-linear map

$$\mathrm{tr} : \bigoplus_N H_N(x) \to \Lambda(x) \quad \text{defined by } \mathrm{tr}(\beta) = \sum_{\lambda \vdash N} \chi_x^\lambda(\beta) s_\lambda \text{ for all } \beta \in H_N(x)$$

using webs. *Nota bene* that this is not a ring homomorphism. It should instead be viewed as a cocenter map for the direct sum of the Iwahori–Hecke algebras: that is, as as a universal trace.

Let $\langle -, - \rangle : \Lambda(x) \times \Lambda(x) \to \mathbf{Z}[x^{\pm 1}]$ be the *Hall pairing*: the $\mathbf{Z}[x^{\pm 1}]$-linear pairing under which the Schur functions s_λ form an orthonormal basis. It lets us write

$$\chi_x^\lambda(\beta) = \langle \mathrm{tr}(\beta), s_\lambda \rangle \quad \text{for all } \beta \in H_N(x) \text{ and } \mu \vdash N.$$

By (1.3) and (1.2), we deduce:

$$\mathrm{tr}(\beta) = \sum_{\mu \vdash N} \phi_x^\mu(\beta) h_\mu \quad \text{for all } \beta \in H_N(x). \tag{1.4}$$

1.7. We refer to Gorsky–Wedrich, "Evaluations. . . " for background on webs. Note that their q is our x.

Let $H_N^{\mathrm{MOY}}(x)$ be the free $\mathbf{Z}[x^{\pm 1}]$-module generated by strictly upward-oriented web diagrams in a rectangle, connecting N inputs with label 1 on the left to N outputs with label 1 on the right, modulo the relations of the MOY bracket. It forms a $\mathbf{Z}[x^{\pm 1}]$-algebra under concatenation of diagrams. The work of Murakami–Ohtsuki–Yamada (MOY) implies that this algebra is isomorphic to $H_N(x)$.

Let $\mathscr{C}^{\mathrm{MOY}}(x)$ be the free $\mathbf{Z}[x^{\pm 1}]$-module generated by positively-oriented web diagrams in an annulus. It forms a commutative $\mathbf{Z}[x^{\pm 1}]$-algebra under nesting of diagrams. By the work of Turaev, this algebra is isomorphic to $\Lambda(x)$.

As in the work of Morton *et al.* on skein algebras, there is a $\mathbf{Z}[x^{\pm 1}]$-linear map

$$\mathsf{ann} : \bigoplus_N H_N^{\mathrm{MOY}}(x) \to \mathscr{C}^{\mathrm{MOY}}(x)$$

called *annular closure*. It is defined graphically, by embedding a rectangle into an annulus as a sector, so that the upward orientation in the rectangle becomes the positive orientation in the annulus, then wrapping the n outputs of the rectangle around the annulus, without crossing, back to the n inputs.

1.8. For $1 \leq i \leq N-1$, let $\mathsf{can}_i \in H_N^{\mathrm{MOY}}(x)$ denote the ith merge-split web. The notation can is intended to suggest *canonical*.

The precise statement proved in [MOY], up to sign and up to use of Schur–Weyl duality, is that there is an isomorphism of $\mathbf{Z}[x^{\pm 1}]$-algebras

$$\Theta_b : H_N(x) \to H_N^{\mathrm{MOY}}(x) \quad \text{defined by } \Theta_b(b_i) = \mathsf{can}_i.$$

We claim that similarly, there is an isomorphism of $\mathbf{Z}[x^{\pm 1}]$-algebras

$$\Theta_c : H_N(x) \to H_N^{\mathrm{MOY}}(x) \quad \text{defined by } \Theta_c(c_i) = \mathsf{can}_i.$$

Indeed, the structure constants of multiplication in $H_N(x)$ with respect to the bases $\{b_w\}_w$ and $\{c_w\}_w$ are the same.

Recall that we can identify $H_N(x)$ with the skein algebra of the rectangle, by sending σ_i to the ith simple twist. Formulas (16) and (17) in [GW] implicitly define two homomorphisms from this skein algebra to $H_N^{\mathrm{MOY}}(x)$, upon their decategorification. The map Θ_b corresponds to the *framed* map (16), while the map Θ_c almost but does not quite correspond to the *unframed* map (17).

For any $N > 0$ and $\mu \vdash N$, let $o^\mu \in \mathscr{C}^{\mathrm{MOY}}(x)$ be the diagram consisting of concentric essential circles with labels $\mu_1, \mu_2, \ldots$ Note that by the commutativity of $\mathscr{C}^{\mathrm{MOY}}(x)$, the order of these circles does not matter. Queffelec–Rose proved that the set $\{o_\mu\}_\mu$ forms a basis for $\mathscr{C}^{\mathrm{MOY}}(x)$ as a free $\mathbf{Z}[x^{\pm 1}]$-module. Gorsky–Wedrich implicitly prove the following statements in [GW, §4.7]:

Theorem 1.2. *(1) There is an isomorphism $\Xi_e : \Lambda(x) \to \mathscr{C}^{\mathrm{MOY}}(x)$ defined by $\Xi_e(e_\mu) = o_\mu$. The following diagram commutes:*

$$\begin{tikzcd}
\bigoplus_N H_N(x) \arrow[r, "\mathsf{tr}"] \arrow[d, "\Theta_c"'] & \Lambda(x) \arrow[d, "\Xi_e"] \\
\bigoplus_N H_N^{\mathrm{MOY}}(x) \arrow[r, "\mathsf{ann}"] & \mathscr{C}^{\mathrm{MOY}}(x)
\end{tikzcd}$$

(2) There is an isomorphism $\Xi_h : \Lambda(x) \to \mathscr{C}^{\mathrm{MOY}}(x)$ defined by $\Xi_h(h_\mu) = o_\mu$. The following diagram commutes:

$$\begin{tikzcd}
\bigoplus_N H_N(x) \arrow[r, "\mathsf{tr}"] \arrow[d, "\Theta_b"'] & \Lambda(x) \arrow[d, "\Xi_h"] \\
\bigoplus_N H_N^{\mathrm{MOY}}(x) \arrow[r, "\mathsf{ann}"] & \mathscr{C}^{\mathrm{MOY}}(x)
\end{tikzcd}$$

We deduce from (1.4) that for any $\beta \in H_N(x)$ and $\mu \vdash N$, the value of $\phi_x^\mu(\beta)$ is the coefficient of o_μ when we expand $\mathsf{ann}(\Theta_b(\beta))$ in the basis $\{o_\mu\}_\mu$. Now let

$$\mathsf{can}_w = \Theta_b(b_w) = \Theta_c(c_w) \quad \text{for all } N \text{ and } w \in S_N.$$

Taking $\beta = b_w$, we conclude:

Corollary 1.3. *For any N and $w \in S_N$, Conjecture 1.1 for w is equivalent to claiming that the expansion of $\mathsf{ann}(\mathsf{can}_w)$ in the basis $\{o_\mu\}_\mu$ has nonnegative coefficients.*

1.3.

We would like to prove Conjecture 1.1 for nice w: namely, for w such that can_w can be written as a single web. Below, we write $w = [w_1 w_2 \cdots w_N]$ to mean that w sends i to w_i for $1 \leq i \leq N$.

1.9. Fix $2 \leq M \leq N$ and $v = [v_1 v_2 \cdots v_M] \in S_M$. We say that $w = [w_1 w_2 \cdots w_N] \in S_N$ is *$v_1 v_2 \cdots v_M$-containing* if and only if there exist indices $1 \leq p_1 < \cdots < p_M \leq N$ such that for all $i < j$ with $v_i < v_j$, we have $w_{p_i} < w_{p_j}$. Informally: w is $v_1 v_2 \cdots v_M$-containing if and only if the sequence $(w_1, \ldots, w_N)$ contains a subsequence of size M whose elements have the same relative order as $(v_1, \ldots, v_M)$.

Otherwise, we say that w is *$v_1 v_2 \cdots v_M$-avoiding*. We write $S_N^{v_1 v_2 \cdots v_M} \subseteq S_N$ for the subset of $v_1 v_2 \cdots v_k$-avoiding elements. It turns out that

$$w \in S_N^{312} \implies w \in S_N^{3412} \cap S_n^{4231} \iff P_{1,w}(q) = 1.$$

The biconditional statement is a 1990 result of Lakshmibai–Sandhya.

1.10. Following Billey–Warrington, we say that $w \in S_N$ is *321-hexagon-avoiding* if and only if it belongs to

$$S_N^{321\text{hex}} := S_N^{321} \cap S_N^{46718235} \cap S_N^{46781235} \cap S_N^{56718234} \cap S_N^{56781234}.$$

Billey–Warrington prove that the following conditions are equivalent:

(1) $w \in S_N^{321\text{hex}}$.
(2) $b_w = b_{s_{i_1}} \cdots b_{s_{i_\ell}}$ whenever $(s_{i_1}, \cdots, s_{i_\ell})$ is a reduced expression for w.
(3) The Bott–Samelson resolution of the Schubert variety attached to w is a small morphism of varieties.

It seems that Remark 4.21 of [GW], building on [QR], proves the following.

Theorem 1.4. *For any sequence of indices $i_1, i_2, \ldots, i_\ell$, the expansion of*

$$\mathsf{ann}(\mathsf{can}_{i_1} \mathsf{can}_{i_2} \cdots \mathsf{can}_{i_\ell})$$

in the basis $\{o_\mu\}_\mu$ has nonnegative coefficients. Hence, Conjecture 1.1 holds in the cases where w is 321-hexagon-avoiding.