

# MATH 665 PROBLEM SET 4

FALL 2024

**Due Thursday, December 5.** You may consult books, papers, and websites as long as you cite all sources and write up your solutions in your own words.



**Problem 1.** Let  $w_o \in S_n$  be the longest element in Bruhat length. The word  $w_o = (s_1 s_2 \cdots s_{n-1})(s_1 s_2 \cdots s_{n-2}) \cdots (s_1 s_2)(s_1)$  is *reduced*—that is, of minimal length—from which

$$\sigma_o := \sigma_{w_o} = (\sigma_1 \sigma_2 \cdots \sigma_{n-1})(\sigma_1 \sigma_2 \cdots \sigma_{n-2}) \cdots (\sigma_1 \sigma_2)(\sigma_1).$$

The figure above<sup>1</sup> depicts the braid  $\sigma_o$  when  $n = 5$ .

- (1) Show that  $\sigma_o^2 = (\sigma_1 \sigma_2 \cdots \sigma_{n-1})^n$ . This elements is called the *full twist*.
- (2) Using (1), give a proof by picture that  $\sigma_o^2$  is central in  $Br_n$ .
- (3) By Lemma 9.3 in Jones’s “Hecke Algebra Representations of Braid Groups and Link Polynomials”,  $\sigma_o^2$  acts on any simple  $H_{S_n}$ -module by an explicit monomial, depending only on the partition  $\lambda \vdash n$  that indexes the module. For  $n = 4$ , use Jones’s paper to compute these monomials (in our conventions) for all  $\lambda$ . Note that his  $\sigma_i$  is our  $x\sigma_i$ , and his  $q$  is our  $x^2$ .

**Problem 2.** The table below describes the characters  $\chi_x^\lambda : H_{S_4} \rightarrow \mathbf{Z}[x^{\pm 1}]$  that correspond to the irreducible characters  $\chi^\lambda : S_4 \rightarrow \mathbf{Z}$  under Tits deformation, for all  $\lambda \vdash 4$ .

	1	$\sigma_1$	$\sigma_{s_1 s_3} = \sigma_1 \sigma_3$	$\sigma_{s_1 s_2} = \sigma_1 \sigma_2$	$\sigma_{s_1 s_2 s_3} = \sigma_1 \sigma_2 \sigma_3$
$\chi^{(4)}$	1	$x$	$x^2$	$x^2$	$x^3$
$\chi^{(3,1)}$	3	$2x - x^{-1}$	$x^2 - 2$	$x^2 - 1$	$-x$
$\chi^{(2,2)}$	2	$x - x^{-1}$	$x^2 + x^{-2}$	$-1$	0
$\chi^{(2,1^2)}$	3	$-2x^{-1} + x$	$x^{-2} - 2$	$x^{-2} - 1$	$x^{-1}$
$\chi^{(1^4)}$	1	$-x^{-1}$	$x^{-2}$	$x^{-2}$	$-x^{-3}$

Using this table and Problem 1, we will compute a piece of  $\mathbf{P}(\hat{\beta})$ , where

$$\beta = (\sigma_2 \sigma_1 \sigma_3 \sigma_2)^3 \sigma_1^7 \in Br_4.$$

This knot is also called the (2, 13)-cable of the trefoil in the blackboard framing.

- (1) Show that  $\beta$  is conjugate to  $\sigma_o^3 \sigma_1$  in  $Br_4$ . *Hint:* Draw pictures.
- (2) Use (1) and Problem 1 to compute  $\chi_x^\lambda(\beta)$  for all  $\lambda \vdash 4$ . *Hint:* You will still need to deal with  $\sigma_o \sigma_1$ . Use conjugacy and Hecke relations.

<sup>1</sup>From Chmutov–Duzhin–Mostovoy’s book.

- (3) By Gomi and/or Webster–Williamson, the “extremal”  $a$ -degrees of  $\mu_4(\beta)$ , hence of  $\mathbf{P}(\hat{\beta})$ , are given by

$$\frac{(-x)^3}{(1-x^4)(1-x^6)(1-x^8)} \sum_{\lambda} \chi_x^{\lambda}(\beta) f_{\lambda}(x^2) \quad \text{and} \quad \frac{(-x)^3}{(1-x^4)(1-x^6)(1-x^8)} \sum_{\lambda} \chi_x^{\lambda}(\beta) f_{\lambda^t}(x^2),$$

where

$$\begin{cases} f_{(4)}(\mathbf{q}) = 1, \\ f_{(3,1)}(\mathbf{q}) = \mathbf{q} + \mathbf{q}^2 + \mathbf{q}^3, \\ f_{(2,2)}(\mathbf{q}) = \mathbf{q}^2 + \mathbf{q}^4, \\ f_{(2,1^2)}(\mathbf{q}) = \mathbf{q}^3 + \mathbf{q}^4 + \mathbf{q}^5, \\ f_{(1^4)}(\mathbf{q}) = \mathbf{q}^6 \end{cases}$$

and  $\lambda^t$  is the transpose of  $\lambda$ . Compute these expressions and compare to §7 of Oblomkov–Shende’s “The Hilbert Scheme of a Plane Curve Singularity...” Their  $z$  is our  $x - x^{-1}$ .

**Problem 3** (O. Dudas). Let  $G$  be a connected smooth reductive algebraic group over  $\bar{\mathbf{F}}_q$ . Fix a Frobenius map  $F : G \rightarrow G$  and an  $F$ -stable Borel  $B$ . Assume that  $F$  acts trivially on the Weyl group  $W$ . For all  $w, s \in W$  with  $s$  simple, let

$$\begin{aligned} X_w &= \{xB \in G/B \mid xB \xrightarrow{w} F(x)B\}, \\ X_{w,s} &:= \{(xB, yB) \in (G/B)^2 \mid xB \xrightarrow{w} yB \xrightarrow{s} F(x)B\}, \\ X_{w,s,s} &:= \{(xB, yB, zB) \in (G/B)^3 \mid xB \xrightarrow{w} yB \xrightarrow{s} zB \xrightarrow{s} F(x)B\}. \end{aligned}$$

We will show  $R_{X_{w,s,s}} = R_{X_w}$ , where  $R_{(-)}(g) := \sum_i (-1)^i \text{tr}(g \mid H_c^i(-))$  for  $g \in G^F$ .

- (1) Let  $Z = \{(xB, yB, zB) \in X_{w,s,s} \mid yB = F(x)B\}$ . Show that  $Z$  forms a  $G^F$ -equivariant line bundle over  $X_w$ , locally trivial in the smooth topology. Deduce that  $R_Z = R_{X_w}$ .
- (2) Let  $Y = \{(xB, yB, zB) \in X_{w,s,s} \mid yB \xrightarrow{s} F(x)B\}$ . Show that  $Y$  forms the complement to the zero section in a  $G^F$ -equivariant line bundle over  $X_{w,s}$ , locally trivial in the smooth topology. Deduce that  $R_Y = 0$ .

*Hints:* Use  $H_c^*(L) \simeq H_c^*(X)[-2]$  for a line bundle  $L \rightarrow X$  in the smooth topology. In (2), use additivity of Lefschetz number.

**Problem 4.** Take  $G = \text{PGL}_2$  and standard  $F$  in the setup of the previous problem, so that  $G/B \simeq \mathbf{P}^1$ . Let  $\mathcal{U} \subseteq G$  be the unipotent locus. For  $m \geq 1$ , let

$$\begin{aligned} O_m &= \{(x_1B, \dots, x_mB) \in (G/B)^m \mid x_1B \xrightarrow{s} \dots \xrightarrow{s} x_mB\}, \\ \mathcal{U}_m &= \{(\vec{x}B, u) \in O_m \times \mathcal{U} \mid x_mB \xrightarrow{s} ux_1B\}, \\ \mathcal{X}_m &= \{\vec{x}B \in O_m \mid x_mB \xrightarrow{s} x_1B\}. \end{aligned}$$

Show that  $|\mathcal{X}_{m+2}^F| \neq |\mathcal{X}_m^F|$ , but  $|\mathcal{X}_{m+2}^F| = |\mathcal{U}_m^F|$ .

In Problem 5, we use the notation and conventions of the “active learning” notes on Soergel bimodules, taking  $W = S_2 = \{e, s\}$ .

**Problem 5.** For  $W = S_2$ , the Koszul resolution of  $R$  over  $\tilde{R} = R \otimes_K R^{\text{op}}$  can be normalized to  $\tilde{R}\langle -2 \rangle \xrightarrow{d} \tilde{R}$ , where  $d(1 \otimes 1) = \frac{1}{2}(1 \otimes \alpha - \alpha \otimes 1)$ . Using this:

- (1) Compute  $\text{HH}_*(\mathbf{B}_e)$  and  $\text{HH}_*(\mathbf{B}_s)$ .
- (2) *Hard:* Compute  $\text{HH}_*(\mathbf{B}_s \xrightarrow{e} \mathbf{B}_e)$ . Deduce that  $\text{HHH}_*(\Delta_s)$  is one-dimensional.
- (3) *Harder:* Use #5 on Problem Set 3 to compute  $\text{HHH}_*$  on the complexes  $\mathcal{R}_s^+ * \mathcal{R}_s^+$  and  $\mathcal{R}_s^+ * \mathcal{R}_s^+ * \mathcal{R}_s^+$ . Compare to  $\mu_2(\sigma_1^2)$  and  $\mu_2(\sigma_1^3)$ .