20.

Following Webster–Williamson's paper "The Geometry of Markov Traces", I explain how unipotent character sheaves give rise to an algebro-geometric interpretation of the weights of Jones–Ocneanu's HOMFLYPT Markov traces. In fact, they suggest an analogous trace μ_W for any W, first discovered by Y. Gomi and defined in terms of numbers coming from Deligne–Lusztig theory.

20.1.

We keep the usual choices of $k, k_1, G, F, \mathcal{B}, W$, with F acting trivially on W. Recall the Hecke category $\mathsf{H}_W = \mathsf{K}^b(\mathsf{C}_W)$, where C_W is the full additive subcategory of $\mathsf{D}_{G_1}(\mathcal{B}_1 \times \mathcal{B}_1)$ generated by the (pure) objects $E_{w,1}\langle m\rangle$ for $w \in W$ and $m \in \mathbf{Z}$. Recall also M_G, the full additive subcategory of $\mathsf{D}_{G_1}(G_1)$ generated by the objects $E_1\langle m\rangle$ for mixed objects $E_1 \in \mathsf{Perv}_{G_1}(G_1)$ and $m \in \mathbf{Z}$. The character functor

$$\mathsf{CH}_1 := \pi_{1,!}act_1^* = \pi_{1,*}act_1^*, \quad \text{defined via } \mathcal{B} \times \mathcal{B} \xleftarrow{act} G \times \mathcal{B} \xrightarrow{\pi} G,$$

restricts to a functor from C_W into M_G. Recall that we set $\bar{K}_{w,1} = \mathsf{CH}_1(E_{w,1})$. Thus CH_1 sends a general object of H_W to a complex up to homotopy whose terms are direct sums of objects $\bar{K}_{w,1}\langle m\rangle$.

We described a functor $\mathsf{HHH} : \mathsf{H}_W \to \mathsf{Vec}^{\text{3-gr}}_{\bar{\mathbf{Q}}_\ell}$, and a theorem of Webster–Williamson interpreting it as a composition

$$\mathsf{HHH} : \mathsf{H}_W \xrightarrow{\mathrm{gr}^{\mathsf{W}}_* \mathrm{H}^*_G(G\times\mathcal{B}, act^*(-))} \mathsf{K}^b(\mathsf{Vect}^{\text{2-gr}}_{\bar{\mathbf{Q}}_\ell}) \xrightarrow{\mathrm{H}_*} \mathsf{Vect}^{\text{3-gr}}_{\bar{\mathbf{Q}}_\ell},$$

where $\mathsf{W}_{\leq *}$ is the weight filtration defined via the action of Frobenius on hypercohomology. We can rewrite the first arrow in terms of CH_1, since

$$\mathrm{H}^*_G(G \times \mathcal{B}, act^*(-)) = \mathrm{H}^*_G(G, \mathsf{CH}(-)).$$

So in trying to compute the first arrow, we are led to compute $\mathrm{gr}^{\mathsf{W}}_* \mathrm{H}^*_G(G, \bar{K}_w)$ for all w.

Lusztig gave us a formula, relating the multiplicities of the unipotent character sheaves $A_{\chi,1}$ in the perverse cohomology sheaves ${}^p\mathcal{H}^i(\bar{K}_{w,1})$ to multiplicities that essentially appear in Deligne–Lusztig theory. A priori, this is not enough: Outside of the cases where $W = S_n$, the perverse sheaves ${}^p\mathcal{H}^i(\bar{K}_{w,1})$ have Jordan–Hölder factors not of the form $A_{\chi,1}\langle m\rangle$, *a.k.a.* cuspidal factors. However, we are saved by Proposition 8 in Webster–Williamson, derived from a non-equivariant analogue proved by Lusztig:

Theorem 20.1 (Lusztig, Webster–Williamson)**.** *If E is a* cuspidal *(equivariant) unipotent character sheaf, then* $\mathrm{H}^*_G(G, E) \simeq 0$.

We have, in fact, already computed $\mathrm{gr}^{\mathsf{W}}_* \mathrm{H}^*_G(G, A_{\chi,1})$ for all $\chi \in \mathrm{Irr}(W)$: We have seen that the smallness of the Grothendieck alteration $\pi_1 : \tilde{G}_1 \to G_1$ implies that

$$\bar{K}_{e,1} = \pi_{1,*}(\bar{\mathbf{Q}}_\ell)_{\tilde{G}_1} = \bigoplus_{\chi} \chi \otimes A_{\chi,1},$$

and we computed that

$$\mathrm{gr}^{\mathsf{W}}_{i+j} \mathrm{H}^j_G(G, \bar{K}_e) \simeq \mathrm{H}^{j-i}(T, \bar{\mathbf{Q}}_\ell) \otimes \mathrm{H}^i_T(T, \bar{\mathbf{Q}}_\ell)$$

for a fixed F-stable maximal torus $T \subseteq G$ with split k_1-form. It will be convenient to set $V = \mathrm{X}^*(T) \otimes \bar{\mathbf{Q}}_\ell$, so that:

- $\bigwedge^*(V) \simeq \mathrm{H}^*(T, \bar{\mathbf{Q}}_\ell)$.
- $\mathrm{Sym}^*(V) \simeq \mathrm{H}^{2*}_T(T, \bar{\mathbf{Q}}_\ell)$. Here, the superscripts mean that degree j on the left corresponds to degree $2j$ on the right.

All these isomorphisms turn out to be W-equivariant. Hence we obtain

$$\mathrm{gr}^{\mathsf{W}}_{2i+j} \mathrm{H}^j_G(G, A_\chi) \simeq (\textstyle\bigwedge^{j-2i}(V) \otimes \mathrm{Sym}^i(V))[\chi],$$

where as usual, $[\chi]$ means we take the χ-isotypic component of a representation.

20.2.

We can make the right-hand side more explicit using generating functions. First, for any linear operator w on a vector space V, we have the identities:

$$\sum_{i \geq 0} (-\mathsf{z})^i \,\mathrm{tr}(w \mid \textstyle\bigwedge^i(V)) = \det(1 - \mathsf{z}w \mid V),$$

$$\sum_{i \geq 0} \mathsf{z}^i \,\mathrm{tr}(w \mid \mathrm{Sym}^i(V)) = \frac{1}{\det(1 - \mathsf{z}w \mid V)}.$$

(These identities may look familiar if you have studied symmetric functions from the books of Macdonald or Stanley, or if you have studied the comparison between Weil and Kapranov zeta functions.) So we have

$$\begin{aligned}(\chi, \textstyle\bigwedge(V) \otimes \mathrm{Sym}(V))_{W,\mathsf{y},\mathsf{z}} &:= \displaystyle\sum_{l,i} \mathsf{y}^l \mathsf{z}^i \dim\left[(\textstyle\bigwedge^l(V) \otimes \mathrm{Sym}^i(V))[\chi]\right] \\ &= \frac{1}{|W|} \sum_{w \in W} \frac{\chi(w^{-1}) \det(1 - \mathsf{y}w \mid V)}{\det(1 - \mathsf{z}w \mid V)}.\end{aligned}$$

The $\mathsf{y} \to 0$ specialization of this formal series is known as the *Molien series*. We denote it by $(\chi, \mathrm{Sym}(V))_{W,\mathsf{z}}$. For instance, taking χ to be the trivial character $1 = 1_W$ gives

$$(1, \mathrm{Sym}(V))_{W,\mathsf{z}} = \sum_{i \geq 0} \mathsf{z}^i \dim (\mathrm{Sym}^i(V))^W.$$

Chevalley proved that the ring of invariants $\mathbf{Q}[V]^W$ is a polynomial ring, freely generated by r elements of homogeneous degree, where $r = \dim V$. Their degrees are called the *fundamental degrees* of the W-action on V. Writing these degrees as $d_1, \ldots, d_r$, we have

$$(1, \mathrm{Sym}(V))_{W,\mathsf{z}} = \frac{1}{(1-\mathsf{z}^{d_1})\cdots(1-\mathsf{z}^{d_r})}.$$

More generally, for any χ, it turns out that

$$(\chi, \mathrm{Sym}(V))_{W,\mathsf{z}} = \frac{f_\chi(\mathsf{z})}{(1-\mathsf{z}^{d_1})\cdots(1-\mathsf{z}^{d_r})}.$$

for some polynomial $f_\chi(\mathsf{z}) \in \mathbf{Z}[\mathsf{z}]$ known as the *fake degree* of χ. (See §2.5–2.6 in Springer's paper "Regular Elements of Finite Reflection Groups".)

Example 20.2. If $W = S_n$, then the irreducible characters of W correspond to integer partitions $\lambda \vdash n$. Write χ^λ for the character corresponding to λ, and set $f_\lambda = f_{\chi^\lambda}$. In the convention where $\chi^{(n)}$ is the trivial character and $\chi^{(1^n)}$ is the sign character, we have

$$\begin{aligned}
f_{(n)}(\mathsf{z}) &= 1 && \text{for all } n,\\
f_{(1^n)}(\mathsf{z}) &= \mathsf{z}^{\binom{n}{2}} && \text{for all } n,\\
f_{(2,1)}(\mathsf{z}) &= \mathsf{z} + \mathsf{z}^2.
\end{aligned}$$

The fake degrees for $n = 4$ are given in Problem Set 4.

20.3.

Recall the multiplicity formula of Lusztig stated earlier:

$$\sum_j (-1)^j ([{}^p\mathcal{H}^j(\bar{K}_{w,1})] : [A_{\chi,1}])_{\mathsf{x}} = \sum_{\psi \in \mathrm{Irr}(W)} (R_\chi, \rho_\chi)\psi_{\mathsf{x}}(c_w),$$

where $(c_w)_w$ is the Kazhdan–Lusztig basis of $H_W(\mathsf{x})$ and:

- $R_\chi = \frac{1}{|W|}\sum_{w \in W} \chi(w) R_w$, a rational linear combination of Deligne–Lusztig virtual characters.
- ρ_χ is the unipotent principal series character attached to χ.
- $\psi_{\mathsf{x}} : H_W(\mathsf{x}) \to \mathbf{Q}(\mathsf{x})$ is the ($\mathbf{Z}[\mathsf{x}^{\pm 1}]$-linear) deformation of $\psi : \mathbf{Z}W \to \mathbf{Q}$.

Using this formula and Theorem 20.1, we deduce:

$$\begin{aligned}
&\sum_j (-1)^j \mathsf{y}^l \mathsf{z}^i \dim \mathrm{gr}_{4i+l} \mathrm{H}_G^{2i+l}(G, {}^p\mathcal{H}^j(\bar{K}_w)) \\
&\qquad = \sum_\psi (R_\psi, \rho_\chi)_{G^F} \psi_{\mathsf{x}}(c_w)(\chi, \textstyle\bigwedge(V) \otimes \mathrm{Sym}(V))_{W,\mathsf{y},\mathsf{z}}.
\end{aligned} \tag{20.1}$$

Recall that the left-hand side of (20.1) is a decategorification of $\mathsf{HHH}(E_{w,1})$ (where $E_{w,1}$, as an object of H_W, is placed in degree zero). The class $[E_{w,1}] \in [\mathsf{C}_W]_\oplus$ corresponds to the Kazhdan–Lusztig element $c_w \in H_W(\mathsf{x})$. So replacing $\bar{K}_w$ with other objects of H_W on the left corresponds to replacing c_w with other elements of $H_W(\mathsf{x})$ on the right.

For $W = S_n$, several further things happen. First, a result from Lusztig's *Characters of Reductive Groups over a Finite Field*:

Theorem 20.3 (Lusztig)**.** *In the type-A cases where $W = S_n$, we have*

$$(R_\psi, \rho_\chi)_{G^F} = \begin{cases} 1 & \psi = \chi, \\ 0 & \psi \neq \chi. \end{cases}$$

At the same time, when $W = S_n$, the left-hand side of (20.1) should recover the value of the HOMFLYPT Markov trace μ_n on c_w, up to certain grading shifts and substitutions transforming the variables y, z to the HOMFLYPT variables a, x. Therefore, the right-hand side must be expressing how the trace μ_n *decomposes* into the traces ψ_{x}.

20.4.

To sum up: Since HHH categorifies the Markov trace, its decomposition into unipotent character sheaves (after certain semisimplications) specializes to the decomposition of the Markov trace into irreducible characters of $H_W(\mathsf{x})$.

These weights were first calculated by Jones–Ocneanu, who expressed them via combinatorial formulas involving Young diagrams. The Lie-theoretic form on the right-hand side of (20.1) was discovered by Yashushi Gomi, who showed purely algebraically(!), case by case, that it still obeys an inductive rule analogous to that for the Markov traces μ_n. Gomi suggested that for general G, the right-hand side of (20.1) be understood as a canonical generalization of the traces μ_n to other Weyl groups.

For general G, it turns out that the multiplicities (R_ψ, ρ_χ) form entries within a certain square matrix, describing how to transform the *almost-characters* R_ψ into all unipotent irreducible characters, not just the principal series. This matrix is sometimes called Lusztig's *exotic* or *nonabelian Fourier matrix*, as Lusztig found a way to write it in terms of Fourier transforms on the "2-class functions" of certain finite groups smaller than W.[1] These matrices obey a numerology that suggests how to generalize them from finite Weyl groups to arbitrary finite Coxeter groups.

[1]For more, I recommend various notes by Iordan Ganev: *e.g.*, `https://ivganev.github.io/math/files/grps-Lie-type.pdf`.