KIW 52 (Dec. 2025)

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$G = \mathrm{GL}_n$

$\mathcal{U} = \{\text{unipotent } n \times n \text{ matrices}\} \subseteq G$

<u>Ex</u> for $n = 2$, have $\mathcal{U} \simeq \{x^2 + yz = 0\}$

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B upper-triangular subgroup of G

<u>Bruhat, Chevalley</u> understand G via

$$G = \bigsqcup_{w \in S_n} B\dot{w}B, \quad \text{where } \dot{w}\text{'s are permutation matrices}$$

understand $\mathcal{U}$ via $U := B \cap \mathcal{U}$?

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<u>Thm</u> (Fine–Herstein '58, Steinberg '65)

$|\mathcal{U}(\mathbb{F}_q)| = |U(\mathbb{F}_q)|^2 \ (= q^{n(n-1)})$

<u>Thm</u> (Kawanaka '75)

$$|(\overbrace{\mathcal{U} \cap B\dot{w}B}^{\mathcal{U}_w})(\mathbb{F}_q)| = |(\overbrace{UU_- \cap B\dot{w}B}^{\mathcal{V}_w})(\mathbb{F}_q)| \quad \text{for any } w$$

where $U_- \subseteq B_-$ are lower-triangular

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<u>Ex</u> for any n, have $\mathcal{U}_{\mathrm{id}} = U = \mathcal{V}_{\mathrm{id}}$

<u>Ex</u> for $n = 3$ and $\dot{w} = \begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix}$, not even homeo over $\mathbb{C}$

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diagonal $T \curvearrowright \mathcal{U}_w, \mathcal{V}_w$ by conjugation

Thm (T) $\quad \mathrm{gr}^{\mathsf{W}}_* \mathrm{H}^*_{c,T}(\mathcal{U}_w(\mathbb{C})) \simeq \mathrm{gr}^{\mathsf{W}}_* \mathrm{H}^*_{c,T}(\mathcal{V}_w(\mathbb{C}))$

where $\mathsf{W}_{\leq *}$ is the weight filtration on $\mathrm{H}^*_{c,T}$

implies (Kawanaka) via results of Katz

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Conj (T) $\quad$ the T-equivariant map

$$UU_- \to \mathcal{U} \quad \text{given by } xy \mapsto xyx^{-1}$$

restricts to a homotopy equivalence $\mathcal{V}_w(\mathbb{C}) \to \mathcal{U}_w(\mathbb{C})$

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for $\beta = \sigma_{w_1} \cdots \sigma_{w_k} \in Br_n^+$,

$$\begin{array}{ccc} X_\beta^C & \to & X_\beta := B\dot{w}_1 B \times^B B\dot{w}_2 B \times^B \cdots \times^B B\dot{w}_k B \\ \downarrow & & \downarrow \\ C & \to & G \end{array}$$

$\mathcal{U}_w = X_{\sigma_w}^{\mathcal{U}}$ and $\mathcal{V}_w = X_{\sigma_w \pi}^1$, where $\pi := \sigma_{w_\circ}^2$

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Thm (Kálmán '09) $\quad$ writing $\widehat{\beta}$ for the closure of β,

$$P(\widehat{\beta})[a^{|\beta|-n+1}] = P(\widehat{\beta\pi})[a^{|\beta|+n-1}]$$

Thm (Gorsky–Hogancamp–Mellit–Nakagane '19)

true with KhR superpoly $\mathbb{P}$ in place of P

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<u>Thm (T)</u> if $\beta \in Br_n^+$, then

$$|X_\beta^{\mathcal{U}}(\mathbb{F}_q)| = |X_{\beta\pi}^1(\mathbb{F}_q)|,$$
$$\mathrm{gr}_*^{\mathsf{W}} \mathrm{H}_{c,T}^*(X_\beta^{\mathcal{U}}(\mathbb{C})) \simeq \mathrm{gr}_*^{\mathsf{W}} \mathrm{H}_{c,T}^*(X_{\beta\pi}^1(\mathbb{C}))$$

deduced from Kálmán, GHMN

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Springer resolution $\tilde{\mathcal{U}} = \{(u, gB) \in \mathcal{U} \times G/B \mid ugB = gB\}$

$$\begin{array}{ccccccc} X_\beta^1 \times G/B & \to & \tilde{X}_\beta^{\mathcal{U}} & \to & X_\beta^{\mathcal{U}} & \to & X_\beta \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \{1\} \times G/B & \to & \tilde{\mathcal{U}} & \to & \mathcal{U} & \to & G \end{array}$$

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<u>Thm (T)</u> if $\beta \in Br_n^+$, then $S_n \curvearrowright \mathrm{gr}_*^{\mathsf{W}} \mathrm{H}_{c,T}^*(\tilde{X}_\beta^{\mathcal{U}})$ and

$$\mathbb{P}(\hat{\beta}) \;\propto\; \mathrm{Hom}_{S^n}(\Lambda^*(\mathbb{C}^{n-1}), \mathrm{gr}_*^{\mathsf{W}} \mathrm{H}_{c,T}^*(\tilde{X}_\beta^{\mathcal{U}}(\mathbb{C})))$$

$$\implies \quad \mathbb{P}(\widehat{\beta})[a^{|\beta|-n+1}] = \mathrm{gr}_*^{\mathsf{W}} \mathrm{H}_{c,T}^*(X_\beta^{\mathcal{U}}(\mathbb{C}))$$
$$\implies \quad \mathbb{P}(\widehat{\beta})[a^{|\beta|+n-1}] = \mathrm{gr}_*^{\mathsf{W}} \mathrm{H}_{c,T}^*(X_\beta^1(\mathbb{C}))$$

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just as P arises from traces on Hecke algebras,

$$\begin{aligned} \mathsf{D}^b_{mix,G}\mathsf{Perv}(\mathcal{U}) &\simeq \mathsf{D}^b\mathsf{Mod}(\mathbb{C}S_n \ltimes \mathrm{Sym}) && \text{(Rider)} \\ &\simeq \mathsf{hTr}(\mathsf{Hecke}(S_n)) && \text{(Gorsky–Wedrich)} \end{aligned}$$