<u>last time</u> V, W fin. dim. vector spaces over F

given $(v_1, \ldots, v_n)$ an ordered basis for V,
$(w_1, \ldots, w_m)$ an ordered basis for W

any linear map $T : V$ to W rep by $m \times n$ matrix M:

in "column" notation,
entry $M_{j, i}$ in the jth row and ith col is def by

$$Tv_i = \sum_j M_{j, i} w_j$$

so ith col of M lists the coeffs of Tv_i wrt $(w_j)_j$

[draw]

<u>Q</u> let $P_3 = \{p \text{ in } F[x] \mid p = 0 \text{ or } \deg p \leq 3\}$
$D(p)(x) = dp(x)/dx$ is an operator on P_3
[why?]

$(1, x, x^2, x^3)$ is an ordered basis for P_3
what is the matrix of D wrt this ordered basis?

$D(x^n) = nx^{n-1}$ so

$$\begin{matrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{matrix}$$

in general
an operator T is <u>nilpotent</u> iff some power of T is 0

<u>Q</u> $N : P_3$ to P_3 def by

$N(1) = x$
$N(x) = x^2$
$N(x^2) = x^3$
$N(x^3) = 0$

matrix of N wrt $(1, x, x^2, x^3)$?

$$\begin{matrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{matrix}$$

[this is also nilpotent: why?]

U another vector space over F, $T' : W$ to U,
can always form $T' \circ T : V$ to U

given $(u_1, \ldots, u_\ell)$ an ordered basis for U:

if T rep by M wrt $(v_i)_i$, $(w_j)_j$,
T' rep by M' wrt $(w_j)_j$, $(u_k)_k$,

then $T' \circ T$ rep by $M' \bullet M$

matrices of $N \circ D$ and $D \circ N$ wrt $(1, x, x^2, x^3)$?

$$\begin{matrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{matrix} \quad \neq \quad \begin{matrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{matrix}$$

note: $D \circ N - N \circ D$ is almost the id matrix…

Q what are im(D) and im(N)?

im(D) = span(1, x, x^2)

im(N) = span(x, x^2, x^3)

[visible in the matrices]

Df for any m × n matrix M

span(cols of M) is a lin. subsp. of F^m

column rank of M is dim span(cols of M)

iso F^m to W : jth std basis vec to w_j

ith col of M to Tv_i

[= sum_M_{j, i}w_j]

restricts to

iso span(cols of M) to im(T)

thus column rank of M = dim im(T)

Rem analogous notion of row rank

turns out that column rank = row rank

[we will defer the proof]

Principle lin maps : abstract :: matrices : explicit

[another example:]

Mat_{m × n}(F) = {m × n matrices}

is a vector space under:

(M + M′)v = Mv + M′v,

(aM)v = a • Mv

0_{Mat_{m × n}} = zero matrix

[what is the corresponding structure for lin maps?]

Df the hom space from V to W is

$\mathrm{Hom}(V, W) = \{\text{linear maps from } V \text{ to } W\}$

which forms a [basis-indep] vector space under:

$(T + T')v = Tv + T'v$,

$(a \cdot T)v = a \cdot Tv$

$\mathbf{0}_{\mathrm{Hom}(V, W)}$ = zero map

Prop if $\dim V = n$ and $\dim W = m$
then a choice of bases $(v_i)_i$, $(w_j)_j$
defines a linear iso
$\mathrm{Hom}(V, W)$ to $\mathrm{Mat}_{m \times n}(F)$

Pf send T to its matrix wrt $(v_i)_i$, $(w_j)_j$
check axioms

Rem Hom(V, W) exists even when
V, W are not finite-dimensional

Specific Linear Maps as Specific Matrices

suppose that $V = W = F^n$ (in the std basis)

what is the matrix of

$T(v) = v$?	identity $I = I_n$
$T(v) = \mathbf{0}_V$?	zero 0_n

take $n = 2$

what is the matrix of

scaling y-axis by 2?

reflect across $y = -x$?

projection onto x-axis?

note: these all depend on the axes chosen

want: basis-indep defns of geometric ops

Df a matrix is <u>invertible</u> iff it reps a lin iso

Ex any $M = \begin{matrix} a & b \\ c & d \end{matrix}$ s.t. $ad - bc \neq 0$

[inverse?] $M^{(-1)} = (1/\det(M))\ \mathrm{adj}(M)$

where $\det(M) = ad - bc$

$\mathrm{adj}(M) = \begin{matrix} d & -b \\ -c & a \end{matrix}$

Df a matrix is a projection iff $M^2 = M$

Q which projections are invertible?

only the identity I

Ex a matrix is unipotent iff it takes the form I + N, where N is nilpotent

e.g., shears in F^2 like $\begin{matrix} 1 & b \\ 0 & 1 \end{matrix}$