



Hilb vs Quot vs HOMFLYPT

Minh-Tâm Quang Trinh

Howard University

- 1 Knots and Links
- 2 Plane Curve Singularities
- 3 Hilb vs Quot
- 4 Generic Singularities
- 5 A Lie-Theoretic Viewpoint

O. Kivinen, M. Trinh. [The Hilb-vs-Quot conjecture.](#)
J. reine angew. Math., **828** (2025), 83–126.

image credits: Chmutov–Duzhin–Mostovoy, Bar-Natan,
 Cherednik–Danilenko

1 Knots and Links

Some *knots* in \mathbf{R}^3 (or S^3).



Links allow multiple circles.



Knot theory studies *isotopy* invariants of links.

Trade-off between being *strong* and being *practical*.

$\pi_1(S^3 \setminus L)$ is a strong, but impractical, invariant.

The *Alexander polynomial* $\Delta_L(q)$ is more practical.

Built from the monodromy of $H_1(X_L, \mathbf{Z})$ for a certain infinite cyclic cover $X_L \rightarrow S^3 \setminus L$.

By contrast, the *HOMFLYPT polynomial* $\mathbb{P}_L(a, q)$ is defined via skein relations.

$$(1) \quad \mathbb{P}_{\bigcirc} = 1$$

$$(2) \quad a \mathbb{P}_{\nearrow} - a^{-1} \mathbb{P}_{\nwarrow} = (q^{1/2} - q^{-1/2}) \mathbb{P}_{\searrow}$$

In fact, $\mathbb{P}_L(-1, q) = \Delta_L(q)$.

But Δ_L is *intrinsic* to L , whereas \mathbb{P}_L is *diagrammatic*: a priori, it depends on the diagram.

$\Delta_L = 1$ does not imply $L = \bigcirc$.

Unknown whether $\mathbb{P}_L = 1$ implies $L = \bigcirc$.

Khovanov–Rozansky '07 A further refinement

$$\mathbf{P}_L(a, q, t)$$

such that $\mathbf{P}_L(a, q, -1) = \mathbb{P}_L(a, q)$.

The dimension of the *HOMFLYPT homology* of L :
a \mathbf{Z}^3 -graded vector space built by *categorifying* (1)–(2).

Kronheimer–Mrowka '10 $\mathbf{P}_L = 1$ implies $L = \bigcirc$.

But \mathbf{P} is much more painful to compute than \mathbb{P} !

Mellit '16, Elias–Hogancamp–Mellit '15–19

Recursions to compute \mathbf{P} for *torus links*.



\Rightarrow Mellit '16 A closed formula for any torus *knot*.

\Rightarrow Gorsky–Mazin–Vazirani '20 Another formula, valid for any torus *link*.

For the (m, n) torus knot, both formulas sum over (m, n) *Dyck paths*.



Both formulas look like

$$\mathbf{P} \propto \sum_D q^{a(D)} t^{c(D)} \prod_{\square \in S} f_{\bullet, D, \square}(a, q, t).$$

$a(D)$ counts colored \square 's above D ; $c(D)$ is messy.

$$S_{\text{Mellit}} = \{\square \mid \square \nearrow D\},$$

$$S_{\text{GMV}} = \{\square \mid D \swarrow \square \text{ with } \square \text{ colored}\}.$$

Example For the $(3, 4)$ torus knot:

q^a	t^c	$\prod f_{\text{Mellit}}$	$\prod f_{\text{GMV}}$
q^3	t^3	1	$1 + aq^{-1}$
q^2	t^2	$1 + at$	$(1 + aq^{-1})(1 + aq^{-1}t)$
q	t^2	$1 + at$	$1 + aq^{-1}$
q	t	$1 + at$	$1 + aq^{-1}$
1	1	$(1 + at)(1 + at^2)$	1

2 Plane Curve Singularities Let

$$S_\epsilon^3 = \{(x, y) \in \mathbf{C}^2 \mid |x|^2 + |y|^2 = \epsilon\}.$$

Let $C \subseteq \mathbf{C}^2$ be an algebraic curve through $(0, 0)$.

The *link* of C at the origin is

$$L_C = \{(x, y) \in S_\epsilon^3 \mid f(x, y) = 0\},$$

independent of ϵ up to isotopy.

Example For $y^n = x^m$, it's the (m, n) torus link.

In general, *components* of L_C correspond to *branches* of C at the origin.

Example Take C parameterized by

$$(x(u), y(u)) = (u^4, u^6 + u^7).$$

Then L_C is the *link closure* of



In general, the completed local ring of C looks like

$$R_C = R_{C_1} \times \cdots \times R_{C_b},$$

and the branches look like $R_{C_i} \simeq \mathbf{C}[[u^{n_i}, u^{m_i} + \cdots]]$ by Newton–Puiseux.

Components of L_C are iterated *cables* of torus knots, determined by Puiseux exponents.

Oblomkov–Shende proposed a formula for \mathbb{P}_{L_C} in terms of the *intrinsic* ring R_C .

Later, with Rasmussen, they upgraded to \mathbf{P}_{L_C} .

Consider the *Hilbert schemes*

$$\mathcal{H}_C^\ell = \{\text{ideals } I \subseteq R_C \mid \dim(R_C/I) = \ell\}.$$

Conj (ORS '12) The lowest a -degree $\mathbf{P}_{L_C}^{\text{lo}}$ satisfies

$$\boxed{\frac{\mathbf{P}_{L_C}^{\text{lo}}(q, qt)}{1 - q} \propto \sum_{\ell \geq 0} q^\ell \chi(t^{1/2}, \mathcal{H}_C^\ell),}$$

where χ denotes *virtual weight polynomials*.

Recall: $\chi(t^{1/2}, Z) = |Z(\mathbf{F}_t)|$ when t is a prime power and Z is especially nice.

Example We have $\mathbf{P}_{(2,3) \text{ torus}} \propto 1 + qt + at$.

For $C = \{y^2 = x^3\}$, we have

$$\mathcal{H}^0 = pt, \quad \mathcal{H}^1 = pt, \quad \mathcal{H}^\ell = pt \sqcup \mathbf{C} \text{ for } \ell \geq 2,$$

$$\text{giving } 1 + q + \frac{q^2}{1-q}(1+t) = \frac{1}{1-q}(1 + q^2t).$$

Now consider *nested Hilbert schemes*

$$\mathcal{H}_C^{\ell,k} = \{(I, J) \in \mathcal{H}_C^\ell \times \mathcal{H}_C^{\ell+k} \mid I \supseteq J \supseteq \langle x, y \rangle J\}.$$

Conj (ORS '12) The full polynomial satisfies

$$\boxed{\frac{\mathbf{P}_{L_C}(a, q, qt)}{1 - q} \propto \sum_{k, \ell \geq 0} a^k q^\ell t^{\binom{k}{2}} \chi(t^{1/2}, \mathcal{H}_C^{\ell,k}).}$$

Maulik '12 True at the level of $\mathbb{P}_L = \mathbf{P}_L|_{t \rightarrow -1}$.

Key idea is an analogue for \mathbb{P}_L of a *wall-crossing identity* from DT theory.

Unknown how to upgrade to \mathbf{P}_L .

Maulik–Yun, Migliorini–Shende '11

Morally, why are the Hilbert schemes hard?

They know about a *perverse filtration* $\mathbf{P}_{\leq *}$:

$$\sum_{\ell} q^{\ell} \chi(s, \mathcal{H}_C^{\ell}) = \frac{\sum_i q^i \chi(s, \mathbf{gr}_i^{\mathbf{P}} H^*(\bar{\mathcal{P}}_C / \mathbf{Z}^b))}{(1 - q)^b}$$

where $\bar{\mathcal{P}}_C$ is the *compactified Picard* parametrizing full, finitely-gen'd R_C -submodules of $\text{Frac}(R_C)$.

3 Hilb vs Quot Let \tilde{R}_C be the *normalization* of R_C :

$$R_C \hookrightarrow \tilde{R}_C = \mathbf{C}[[u_1]] \times \cdots \times \mathbf{C}[[u_b]].$$

Consider the *Quot schemes*

$$\mathcal{Q}_C^{\ell} = \{R_C\text{-modules } M \subseteq \tilde{R}_C \mid \dim(\tilde{R}_C/M) = \ell\}$$

An initial motivation for these varieties:

Thm A (Kivinen–T '23) We have

$$\sum_{\ell} q^{\ell} \chi(s, \mathcal{Q}_C^{\ell}) = \frac{\sum_i q^i \chi(s, \bar{\mathcal{P}}_C^{(i)} / \mathbf{Z}^b)}{(1 - q)^b}$$

for an explicit \mathbf{Z}^b -stable stratification $\bar{\mathcal{P}}_C = \coprod_i \bar{\mathcal{P}}_C^{(i)}$.

Recall ORS:

$$\frac{\mathbf{P}_{LC}(a, q, qt)}{1 - q} \propto \sum_{k, \ell \geq 0} a^k q^\ell t^{\binom{k}{2}} \chi(t^{1/2}, \mathcal{H}_C^{\ell, k}).$$

Consider, instead, *nested Quot schemes*

$$\mathcal{Q}_C^{\ell, k} = \{(M, N) \in \mathcal{Q}_C^\ell \times \mathcal{Q}_C^{\ell+k} \mid M \supseteq N \supseteq \langle x, y \rangle M\}.$$

“Quot ORS” Conj (Kivinen–T ’23) For any C ,

$$\frac{\mathbf{P}_{LC}(a, q, t)}{1 - q} \propto \sum_{k, \ell \geq 0} a^k q^\ell t^{\binom{k}{2}} \chi(t^{1/2}, \mathcal{Q}_C^{\ell, k}).$$

Thm B (Kivinen–T ’23) Quot ORS holds in full for:

- $y^n = x^m$ with m, n coprime.
- $y^n = x^{nk}$.

“Hilb-vs-Quot” Conj (Kivinen–T ’23) For any C ,

$$\sum_{\ell} q^\ell \chi(t^{1/2}, \mathcal{H}_C^\ell) = \sum_{\ell} q^\ell \chi((qt)^{1/2}, \mathcal{Q}_C^\ell).$$

Remarks on Hilb-vs-Quot

- Should really be an identity in $K_0(\text{Var})$.
- $t \mapsto qt$ because \mathcal{Q}^ℓ is *larger* than \mathcal{H}^ℓ for fixed ℓ .
- Unibranch case was proposed by Cherednik in a different form, without the \mathcal{Q}^ℓ .

Example Take $C = \{y^3 = x^4\}$.

The \mathbf{C}^\times -action on C induces actions on the \mathcal{H}^ℓ , \mathcal{Q}^ℓ .

The \mathbf{C}^\times -orbits form affine pavings.

\mathcal{H}^0	\mathcal{H}^1	\mathcal{H}^2	\mathcal{H}^3	\mathcal{H}^4	\mathcal{H}^5	\mathcal{H}^6	\dots
pt			\mathbf{C}^2	\mathbf{C}^2		\mathbf{C}^3	\dots
		\mathbf{C}	\mathbf{C}		\mathbf{C}^2	\mathbf{C}^2	\dots
	pt			\mathbf{C}^2	\mathbf{C}^2	\mathbf{C}^2	\dots
		pt		\mathbf{C}	\mathbf{C}	\mathbf{C}	\dots
			pt	pt	pt	pt	\dots

The rows classify monomial ideals as R_C -modules.

The colors are \mathcal{Q}^0 , \mathcal{Q}^1 , \mathcal{Q}^2 , \mathcal{Q}^3 , \dots

Similar picture for any $y^n = x^m$ with m, n coprime.

“Hilb ORS” is hard because Hilb-vs-Quot is hard.

Thm C (Kivinen–T ’23) Hilb-vs-Quot holds for

$$y^n = x^m \quad \text{with } m, n \text{ coprime and } n \leq 3.$$

Key idea is that for fixed n , we can compute

$$\lim_{\substack{m \rightarrow \infty \\ \gcd(m, n) = 1}} \sum_{\ell} q^{\ell} \chi(t^{1/2}, \mathcal{H}_{y^n = x^m}^{\ell}),$$

$$\lim_{\substack{m \rightarrow \infty \\ \gcd(m, n) = 1}} \sum_{\ell} q^{\ell} \chi(t^{1/2}, \mathcal{Q}_{y^n = x^m}^{\ell})$$

by combinatorics.

If $n \leq 3$, then Serre duality shows that the limits determine the series for fixed m .

4 Generic Singularities

Gorsky–Mazin–Oblomkov '22 +
Caprau–González–Hogancamp–Mazin '23

Quot ORS holds for the lowest a -degree $\mathbf{P}_{L_C}^{\text{lo}}$, for all *generic* unibranch singularities C .

For a generic unibranch singularity, we have

$$R_C \simeq \mathbf{C}[[u^{nd}, u^{md} + u^{md+1} + \dots]].$$

for some m, n, d with m, n coprime.

Here L_C is a knot: the $(mnd + 1, n)$ -cable of the (m, n) torus knot.

(CGHM) For such knots, there is a generalization of the GMV formula for \mathbf{P} .

Formula is a sum over $md \times nd$ Dyck paths.

(GMO) For such singularities, the Quot schemes \mathcal{Q}^ℓ remain paved by affines.

Indexed by (m -admissible) $\langle md, nd \rangle$ -sets in $\mathbf{Z}_{\geq 0}$.

Example Take $(m, n, d) = (3, 2, 2)$ and

$$R_C \simeq \mathbf{C}[[u^4, u^6 + u^7]].$$

One stratum in \mathcal{Q}^5 consists of all $M \subseteq \widetilde{R}_C$ such that

$$\{\text{ord}_u(f) \mid f \in M\} = \{1, 3, 5, 7, 8, 9\} \cup \mathbf{Z}_{\geq 11}.$$

Stable under addition by $md = 6$ and $nd = 4$.

Recall GMV:

$$\sum_D q^{a(D)} t^{c(D)} \prod_{\square \in S_{\text{GMV}}(D)} f_{\text{GMV}, D, \square}(a, q, t).$$

Gorsky–Mazin–Vazirani '17 Bijection

$$\{\text{admissible sets } \Delta\} \xrightarrow{\sim} \{\text{Dyck paths } D\}.$$

If Δ indexes $\mathcal{Q}_{\Delta}^{\ell}$, then $q^{a(D)} t^{c(D)} = q^{\ell} t^{\dim \mathcal{Q}_{\Delta}^{\ell}}$.

But the factors $\prod_{\square} f_{\text{GMV}, D, \square}$ don't look like

$$\sum_{k, \Delta'} a^k t^{\binom{k}{2}} \chi(t^{1/2}, \mathcal{Q}_{\Delta \supseteq \Delta'}^{\ell, k}) \quad \text{for fixed } \ell, \Delta,$$

so cannot match higher a -degree terms.

Recall Mellit vs GMV:

$\prod f_{\text{Mellit}}$	$\prod f_{\text{GMV}}$
1	$1 + aq^{-1}$
$1 + at$	$(1 + aq^{-1})(1 + aq^{-1}t)$
$1 + at$	$1 + aq^{-1}$
$1 + at$	$1 + aq^{-1}$
$(1 + at)(1 + at^2)$	1

Thm (T in progress)

- 1 Mellit's formula for \mathbf{P} generalizes to the knots of generic unibranch singularities.
- 2 f_{Mellit} does match nested Quot.

Cor (T) Quot ORS holds in full for generic unibranch singularities.

5 A Lie-Theoretic Viewpoint What got me into this?

For any connected group G , subgroup K , and $\gamma \in \text{Lie}(G)$, have a fixed-point set

$$X_K^\gamma = \{gK \in G/K \mid \gamma \in \text{Ad}(g)(\text{Lie}(K))\}$$

called the associated *Springer fiber*.

Laumon '02 Take C a branched n -cover of a line.
Then its compactified Picard

$$\bar{\mathcal{P}}_C = \{\text{full, finitely-gen'd modules } M \subseteq \text{Frac}(R_C)\}$$

is such for $G = \text{GL}_n(\mathbf{C}((z)))$ and $K = \text{GL}_n(\mathbf{C}[[z]])$.

In this case, also called an *affine Springer fiber*.

For instance, suppose that

$$R_C = \mathbf{C}[[u^n, F(u)]] \quad \text{with } \gcd(u^n, F(u)) = u.$$

Setting $u = z^{1/n}$ and fixing

$$\iota : \mathbf{C}((u)) \xrightarrow{\sim} \mathbf{C}((z))^n,$$

can transport $F(u) \curvearrowright \mathbf{C}((u))$ to some $\gamma(\iota) \curvearrowright \mathbf{C}((z))^n$.

We will always have $\bar{\mathcal{P}}_C \simeq X_K^{\gamma(\iota)}$.

For two choices of ι , the *positive truncations*

$$X_K^{\gamma(\iota)} \cap \text{Mat}_n(\mathbf{C}[[z]])/K$$

recover $\bigsqcup_\ell \mathcal{H}_C^\ell$ and $\bigsqcup_\ell \mathcal{Q}_C^\ell$.

Hilb-vs-Quot is a comparison of such truncations.

This viewpoint also suggests:

- 1 Replacing $K = \mathrm{GL}_n(\mathbf{C}[[z]])$ with other *parahorics*.
- 2 Replacing GL_n with other reductive groups.

(1) leads to flagged versions of \mathcal{H}^ℓ , \mathcal{Q}^ℓ that encode the nested versions $\mathcal{H}^{\ell,k}$, $\mathcal{Q}^{\ell,k}$ and more.

(2) leads to conjectures relating affine Springer fibers to analogues of \mathbf{P} for *Artin groups* Br_W .

Thank you for listening.

Thm (T '21) Formulas for such invariants via

$$W \curvearrowright H_{G,c}^*(Z_\beta),$$

where Z_β is the *braid Steinberg variety* of $\beta \in \mathrm{Br}_W^+$.

Hope Nonabelian Hodge relates X_K^γ and Z_β .