MATH 340: ADVANCED LINEAR ALGEBRA
PROBLEM SET #7

SPRING 2025

Due Wednesday, April 9. You may consult books, papers, and websites as long as you cite all sources and write up your solutions in your own words. **Updated on 3/26 at 4 pm, in red.**

Problem 1. Look up the definition of an integer partition. Let $p(n)$ be the number of partitions of an integer $n > 0$. Using Jordan canonical form, show that $p(n)$ is also the number of conjugacy classes of nilpotent $n \times n$ matrices over $\mathbf{C}$.

Problem 2. Let

$$\mathfrak{sl}(n, F) = \{n \times n \text{ matrices over } F \text{ of trace } 0\}.$$

The notation $\mathfrak{sl}$ stands for *special linear*.

(1) Verify that $\mathfrak{sl}(n, F)$ is a vector space over F of dimension $n^2 - 1$.
(2) Show that an element of $\mathfrak{sl}(2, F)$ is nilpotent if and only if its determinant is zero.
(3) Using a suitable basis to identify $\mathfrak{sl}(2, \mathbf{R})$ with $\mathbf{R}^3$, sketch the subset of nilpotent matrices. (No need to prove the basis is a basis.)

In principle, the following two problems are solved in Axler's text. But it may be easier to think about them from scratch, than to start with Axler.

Problem 3. Let V, W be finite-dimensional vector spaces and $T : V \to W$ a linear map. Show that the kernel $\ker(T^\vee)$ and the annihilator $\mathrm{Ann}_{W^\vee}(\mathrm{im}(T))$ are the same subspace of $W^\vee$.

Problem 4. Keep the setup of Problem 3.

(1) Show that

$$\dim W - \dim \ker(T^\vee) = \dim V - \dim \ker(T).$$

Hint: You'll need Problem 3, a dimension formula relating U and $\mathrm{Ann}_{W^\vee}(U)$ for some $U \subseteq W$, and a dimension formula relating $\ker(T)$ and $\mathrm{im}(T)$.

(2) Deduce from (1) that

$$\dim \mathrm{im}(T^\vee) = \dim \mathrm{im}(T).$$

(3) Using (2), show that the column rank and row rank of any square matrix M agree.

You may use the fact (Axler §3.132) that if M represents a linear operator $T : V \to V$ in some basis for V, then the *transpose* matrix M^t defined by $(M^t)_{j,i} = M_{i,j}$ represents $T^\vee : V^\vee \to V^\vee$ in the dual basis.

Problem 5. Let V be a vector space over F, possibly infinite-dimensional. In each case below, show that T is a linear isomorphism <u>without</u> picking an explicit basis for V. You may still use the fact that a tensor product of vector spaces is spanned by pure tensors.

(1) $T : \{\vec{0}\} \to \{\vec{0}\} \otimes V$ defined in the only possible way.

(2) $T : V^{\oplus n} \to F^n \otimes V$, where $V^{\oplus n}$ is the n-fold direct sum of V for some $n > 0$, defined by

$$T(v^{(1)}, \ldots, v^{(n)}) = \sum_i e_i \otimes v^{(i)},$$

where $e_1, \ldots, e_n$ is an ordered basis for F^n. *Hint:* To show injectivity, use the definition of $e_i \otimes v^{(i)}$ as a bilinear functional and an ordered basis dual to $(e_i)_i$.

Problem 6. Let V, W, U be vector spaces. The set

$$\mathrm{Bil}(W, V \mid U) = \{\text{bilinear maps from } W \times V \text{ into } U\}$$

forms a vector space under $(\beta + \beta')(w, v) = \beta(w, v) + \beta'(w, v)$ and $(a \cdot \beta)(w, v) = a \cdot \beta(w, v)$. It recovers $\mathrm{Bil}(W, V)$ when $U = F$.

For all linear $T : W \otimes V \to U$, let $\beta_T : W \times V \to U$ be the bilinear map such that $\beta_T(w, v) = T(w \otimes v)$. Show that the map

$$\mathrm{B} : \mathrm{Hom}(W \otimes V, U) \to \mathrm{Bil}(W, V \mid U) \quad \text{defined by } \mathrm{B}(T) = \beta_T$$

is linear and injective, <u>without</u> picking explicit bases for the vector spaces involved. *Hint:* Again, pure tensors span $W \otimes V$.

It turns out that B *is an isomorphism, but starting from Axler's definition of* $W \otimes V$*, this is difficult to show without picking explicit bases for* V *and* W*.*

Problem 7. A bilinear form $\beta : V \times V \to F$ is *degenerate* if and only if there is some nonzero $v \in V$ such that either $\beta(v, -)$ or $\beta(-, v)$ is the zero functional on V. It is *nondegenerate* otherwise. Now set $V = F[x]$. Show that:

(1) If $\beta(p, q) = \int_0^1 p(x)q(x)\,dx$, then β is nondegenerate.

(2) If $\beta(p, q) = p(1)q(1)$, then β is degenerate.

Problem 8. Show that for all $n \geq 2$, there is a bilinear form β on F^n such that

$$\beta(w, v) \neq 0 \text{ for some } w, v \in F^n, \quad \text{but } \beta(v, v) = 0 \text{ for all } v.$$

Hint: Take $\beta(w, v) = w^t M v$ for some carefully chosen $n \times n$ matrix M.