

MATH 250: TOPOLOGY I

FALL 2025 LOOSE ENDS

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1. WEDNESDAY, 9/3

1.1. Here I try to dispel some potential confusion about bases.

Let X be any set. Let \mathcal{B} be any collection of subsets of X . A useful general observation:

Lemma 1.1. *For any subset $Y \subseteq X$, the following conditions are equivalent:*

- (1) Y is the union of some elements of \mathcal{B} .
- (2) For any $x \in Y$, there is some $B \in \mathcal{B}$ such that $x \in B \subseteq Y$.

Now let \mathcal{T} be the collection of all subsets of X that can be written as unions of elements of \mathcal{B} . Using the lemma, we see that

$$\mathcal{T} = \left\{ \text{subsets } U \subseteq X \left| \begin{array}{l} \text{for any } x \in U, \text{ we have some } B \in \mathcal{B} \\ \text{such that } x \in B \subseteq U \end{array} \right. \right\}.$$

Theorem 1.2. *Suppose that \mathcal{B} satisfies the following conditions:*

- (I) *Every point of X belongs to some element of \mathcal{B} .*
- (II) *For any $B, B' \in \mathcal{B}$ and any point x of the intersection $B \cap B'$, we can find some $B'' \in \mathcal{B}$ such that $x \in B'' \subseteq B \cap B'$.*

Then \mathcal{T} is a topology on X .

We proved this theorem at the start of the course, implicitly using Lemma 1.1. The only hard part is checking that finite intersections of elements of \mathcal{T} are still elements of \mathcal{T} . To make this easier, I mentioned that it suffices by induction to check intersections between pairs of elements of \mathcal{T} .

Any collection \mathcal{B} that satisfies hypotheses (I)–(II) in the theorem above is called a *basis*. In the situation of the theorem, we say that \mathcal{B} *generates* or *induces* the topology \mathcal{T} , and that \mathcal{B} is a *basis for \mathcal{T}* specifically.

1.2. Separately, if we are given \mathcal{T} to start, then there is a way to check whether a subcollection $\mathcal{C} \subseteq \mathcal{T}$ is a basis that generates \mathcal{T} . In Munkres, this is Lemma 13.2.

Theorem 1.3. *Fix a topology \mathcal{T} on X and a subset $\mathcal{C} \subseteq \mathcal{T}$. Suppose that for each $x \in X$ and $U \in \mathcal{T}$, there is some $C \in \mathcal{C}$ such that $x \in C \subseteq U$. Then \mathcal{C} is a basis, and moreover, the topology it generates is \mathcal{T} .*

2. MONDAY, 9/8

2.1. Let X be a set, and let $d : X \times X \rightarrow [0, \infty)$ be a metric on X . For all $x \in X$ and $\delta > 0$, we define the *d-ball* with center x and radius δ to be

$$B_d(x, \delta) = \{y \in X \mid d(x, y) < \delta\}.$$

Below is a cleaner version of a long proof from lecture.

Theorem 2.1. *The set $\{B_d(x, \delta) \mid x \in X \text{ and } \delta > 0\}$ forms a basis.*

Proof. Let \mathcal{B} denote the set in question. We must check two axioms:

- (I) Any point of X is contained in some element of \mathcal{B} .
- (II) Given any two elements of \mathcal{B} and a point in their intersection, we can find some other element of \mathcal{B} containing that point and contained within the intersection as a subset.

(I) holds because for any $x \in X$, we have $x \in B_d(x, \delta)$ for any choice of δ .

To show (II): Pick balls $B_d(x, \delta)$ and $B_d(x', \delta')$ and a point z in their intersection $B_d(x, \delta) \cap B_d(x', \delta')$. We must exhibit some d -ball that contains z and is contained within the intersection as a subset.

It suffices to find some $\epsilon > 0$ such that

$$B_d(z, \epsilon) \subseteq B_d(x, \delta) \cap B_d(x', \delta').$$

Explicitly, this condition on ϵ means that

$$\text{if } y \in X \text{ satisfies } d(z, y) < \epsilon, \text{ then } d(x, y) < \delta \text{ and } d(x', y) < \delta'.$$

(Informally, this means that if y is close enough to z , then it is close enough to x and x' as well.) By drawing a picture of the situation, we get the idea that we need to use the triangle inequality to bound the distance $d(x, y)$ in terms of the distances $d(x, z)$ and $d(z, y)$.

Since $z \in B_d(x, \delta) \cap B_d(x', \delta')$, we know that $d(x, z) < \delta$ and $d(x', z) < \delta'$. Let $\alpha = \delta - d(x, z)$ and $\alpha' = \delta' - d(x', z)$, the respective distances from z to the boundaries of the balls $B_d(x, \delta)$ and $B_d(x', \delta')$. Now observe that if $y \in X$ satisfies $d(z, y) < \alpha$, then y also satisfies

$$\begin{aligned} d(x, y) &\leq d(x, z) + d(z, y) && \text{by the triangle inequality} \\ &< d(x, z) + \alpha && \text{by the hypothesis on } y \\ &= \delta. \end{aligned}$$

An analogous argument shows that if y satisfies $d(z, y) < \alpha'$, then $d(x', y) < \delta'$.

So let $\epsilon = \min(\alpha, \alpha')$. We see that if $y \in X$ satisfies $d(z, y) < \epsilon$, then we have both $d(x, y) < \delta$ and $d(x', y) < \delta'$. So we have found the desired ϵ . \square

3. PROBLEM SET 2, #9

Problem. Let X be arbitrary, and let $d : X \times X \rightarrow [0, \infty)$ be an arbitrary metric. Assume that the function $e : X \times X \rightarrow [0, \infty)$ defined by

$$e(x, y) = \frac{d(x, y)}{1 + d(x, y)}.$$

is a bounded metric. Show that d and e induce the same topology on X .

Solution. Let \mathcal{T}_d and \mathcal{T}_e denote the topologies respectively induced by d and e .

We first show that \mathcal{T}_d is finer than \mathcal{T}_e , meaning $\mathcal{T}_e \subseteq \mathcal{T}_d$. Since elements of \mathcal{T}_e are unions of e -balls, it suffices to check that any e -ball is an element of \mathcal{T}_d . So fix an e -ball $B_e(x, \delta)$. It suffices to show that for $y \in B_e(x, \delta)$, we can find some $\epsilon > 0$ such that $B_d(y, \epsilon) \subseteq B_e(x, \delta)$.

As a warmup, ignore d : Can we find some $\epsilon > 0$ such that $B_e(y, \epsilon) \subseteq B_e(x, \delta)$? Explicitly, for any y satisfying $e(x, y) < \delta$, we have to exhibit some $\epsilon > 0$ such that, if z satisfies $e(y, z) < \epsilon$, then z also satisfies $e(x, z) < \delta$. The argument will be similar to the latter half of the proof of Theorem 2.1. Namely, let $\epsilon = \delta - e(x, y) > 0$, the distance from y to the boundary of the ball $B_e(x, \delta)$. If z satisfies $e(y, z) < \epsilon$, then by the triangle inequality, it satisfies

$$e(x, z) \leq e(x, y) + e(y, z) < e(x, y) + \epsilon = \delta,$$

as needed. So this choice of ϵ does give $B_e(y, \epsilon) \subseteq B_e(x, \delta)$.

Now go back to the original problem involving d . By combining $B_e(y, \epsilon) \subseteq B_e(x, \delta)$ with the following observation, we get $B_d(y, \epsilon) \subseteq B_e(x, \delta)$, as needed.

Lemma 3.1. *For any $y \in X$ and $\epsilon > 0$, we have $B_d(y, \epsilon) \subseteq B_e(y, \epsilon)$.*

Proof. Left as an exercise. □

Now we show the reverse inclusion: $\mathcal{T}_d \subseteq \mathcal{T}_e$. So fix a d -ball $B_d(x, \delta)$. We must show that for any $y \in B_d(x, \delta)$, we can find some $\epsilon > 0$ such that $B_e(x, \epsilon) \subseteq B_d(x, \delta)$. Explicitly, for any y satisfying $d(x, y) < \delta$, we have to exhibit some $\epsilon > 0$ such that, if z satisfies $e(y, z) < \epsilon$, then z also satisfies $d(x, z) < \delta$.

Since the roles of d and e in Lemma 3.1 cannot be switched, we cannot just replicate our earlier argument with d and e switched. But we still expect to use the triangle inequality that $d(x, z) \leq d(x, y) + d(y, z)$. Letting $\alpha = \delta - d(x, y) > 0$ gives us $d(x, y) + \alpha = \delta$. So we just need to exhibit $\epsilon > 0$ such that $e(y, z) < \epsilon$ implies $d(y, z) < \alpha$, because for such ϵ ,

$$e(y, z) < \epsilon \quad \text{will imply} \quad d(x, z) \leq d(x, y) + d(y, z) < d(x, y) + \alpha = \delta,$$

as needed.

Rearranging the identity $e(y, z) = \frac{d(y, z)}{1 + d(y, z)}$ gives $d(y, z) = \frac{e(y, z)}{1 - e(y, z)}$. Moreover, rearranging $e(y, z) < \epsilon$ gives $\frac{e(y, z)}{1 - e(y, z)} < \frac{\epsilon}{1 - \epsilon}$. So the following lemma finishes the argument:

Lemma 3.2. *For any $\alpha > 0$, there is some $\epsilon > 0$ such that $\frac{\epsilon}{1-\epsilon} < \alpha$. (Moreover, we can pick $\epsilon < 1$, so that $\frac{\epsilon}{1-\epsilon}$ is well-defined.)*

Proof. Left as an exercise. *Hint:* If $0 < \epsilon < \frac{1}{2}$, then $\frac{1}{1-\epsilon} < 2$. \square

Since we have shown that \mathcal{T}_d and \mathcal{T}_e contain each other, they coincide. That is, d and e induce the same topology. \square

4. PROBLEM SET 4

Problem. We say that a nonempty space X is *contractible* if and only if its identity map is nullhomotopic: *i.e.*, homotopic to some constant-valued map.

Suppose that X, Y are nonempty spaces. Let $[X, Y]$ be the set of continuous maps from X into Y modulo homotopy. Show that:

- (1) If Y is contractible, then $[X, Y]$ is a singleton.
- (2) If X is contractible and Y is path-connected, then $[X, Y]$ is a singleton.

Solution to (1). Since Y is contractible and nonempty, we can find some point $y_0 \in Y$ and some homotopy from the identity map on Y to the constant map on Y with value y_0 : that is, from id_Y to the map $c_{y_0}: Y \rightarrow Y$ defined by $c_{y_0}(y) = y_0$.

Using this homotopy, we will show that any continuous map $f: X \rightarrow Y$ is homotopic to the constant map on X with value y_0 .

A special case of Problem Set 5, #1 is the following fact: For any spaces X, Y, Z and continuous maps $f: X \rightarrow Y$ and $g, g': Y \rightarrow Z$, if $g \sim g'$, then $g \circ f \sim g' \circ f$. Applying this to our setup, we deduce that

$$\text{id}_Y \circ f \sim c_{y_0} \circ f.$$

Observe that $\text{id}_Y \circ f = f$, while $c_{y_0} \circ f$ is the constant map on X with value y_0 . We have therefore shown that every continuous map from X into Y is homotopic to the latter map. \square

Solution to (2). Since X is contractible and nonempty, we can find some point $x_0 \in X$ and some homotopy from the identity map id_X to the constant map $c_{x_0}: X \rightarrow X$ defined by $c_{x_0}(x) = x_0$.

Pick a continuous map $f: X \rightarrow Y$. Another special case of Problem Set 5, #1 is the following fact: For any spaces X, Y, Z and continuous maps $f, f': X \rightarrow Y$ and $g: Y \rightarrow Z$, if $f \sim f'$, then $g \circ f \sim g \circ f'$. We deduce that

$$f \circ \text{id}_X \sim f \circ c_{x_0}.$$

Observe that $f \circ \text{id}_X = f$, while $f \circ c_{x_0}$ is the constant map $c_{f(x_0)}: X \rightarrow Y$ defined by $c_{f(x_0)}(x) = f(x_0)$.

Now imagine we have another continuous map $f': X \rightarrow Y$. We want to show that $f \sim f'$. By our preceding work,

$$f \sim c_{f(x_0)} \quad \text{and} \quad f' \sim c_{f'(x_0)}.$$

Since Y is path-connected, we can choose a path $\gamma: [0, 1] \rightarrow Y$ from $f(x_0)$ to $f'(x_0)$. Now observe that $h: X \times [0, 1] \rightarrow Y$ defined by $h(x, t) = \gamma(t)$ for all $x \in X$ and $t \in [0, 1]$ is a homotopy from $c_{f(x_0)}$ to $c_{f'(x_0)}$. \square