Warmup

analytic topology $T_{an}$ on $R^n$:

U sub $R^n$ is in $T_{an}$ iff
for all x in U there exists $\delta > 0$ s.t. $B(x, \delta)$ sub U

equivalently (when n = 1)
for all x in U, there exist a, b s.t. x in (a, b) sub U

Df lower-limit topology $T_\ell$ on R:

U sub R is in $T_\ell$ iff
for all x in U, there exists b s.t. [x, b) sub U

Q1 is $T_\ell$ a topology on R? [yes]

Q2 is $T_\ell$ coarser than $T_{an}$? finer?
incomparable?

Prop $T_\ell$ is strictly finer than $T_{an}$

Pf $T_\ell$ is finer than $T_{an}$:
suppose U anlytc open in R
suppose x in U
pick a < b s.t. x in (a, b) sub U
then [x, b) sub (a, b)

strict because [0, 1) in $T_\ell$ but notin $T_{an}$

$\boxed{T_{indisc} < T_f < T_{an} < T_\ell < T_{disc}}$

(Munkres §18, 16) recall from real analysis:

$f : R^n$ to $R^m$ is <u>Bolzano continuous</u> iff

for all $x$ in $R^n$ and $\varepsilon > 0$, there exists $\delta > 0$ s.t.
$|x - x'| < \delta$ implies $|f(x) - f(x')| < \varepsilon$

equivalently
for all $x$ in $R^n$ and $\varepsilon > 0$, there exists $\delta > 0$ s.t.
$x'$ in $B(x, \delta)$ implies $f(x')$ in $B(f(x), \varepsilon)$

equivalently
for all $x$ in $R^n$ and $\varepsilon > 0$, there exists $\delta > 0$ s.t.
$f^{-1}(B(f(x), \varepsilon))$ contains $B(x, \delta)$

<u>Goal</u> generalize this notion

given topological spaces $(X, T_X)$ and $(Y, T_Y)$:

<u>Df</u> a function $f : X$ to $Y$ is <u>continuous</u> iff
$V$ open in $Y$ implies $f^{-1}(V)$ open in $X$

<u>Thm</u> $f : R^n$ to $R^m$ is Bolzano cts
iff
$f$ is cts wrt the analytic topologies

<u>Pf</u>

suppose $f$ cts wrt analytic topologies:

fix $x$ in $R^n$ and $\varepsilon > 0$

$B(f(x), \varepsilon)$ is open in $R^m$

so $f^{-1}(B(f(x), \varepsilon))$ is open in $R^n$

so $B(x, \delta)$ sub $f^{-1}(B(f(x), \varepsilon))$ for some $\delta > 0$

suppose $f$ Bolzano cts:

fix $V$ anlytc open in $R^m$

want $f^{-1}(V)$ anlytc open in $R^n$

suppose $x$ in $f^{-1}(V)$

can find $\varepsilon > 0$ s.t. $B(f(x), \varepsilon)$ sub $V$

then $f^{-1}(B(f(x), \varepsilon))$ sub $f^{-1}(V)$

pick $\delta > 0$s.t. $B(x, \delta)$ sub $f^{-1}(B(f(x), \varepsilon))$

then $x$ in $B(x, \delta)$ sub $f^{-1}(V)$

<u>Ex</u> which maps are continuous?

$f : (R, T_\ell)$ to $(R, T_{an})$, $f(x) = x$ [yes]

$f : (R, T_{an})$ to $(R, T_\ell)$, $f(x) = x$ [no: $[0, 1)$]

$f : (R, T_{an})$ to $(R, T_\ell)$, $f(x) = 31$ [yes]

<u>General Facts</u>

1) S finer than T:

id : (X, S) to (X, T) continuous

2) S strictly coarser than T:

id : (X, S) to (X, T) not continuous

3) constant maps are always continuous

also

4) compositions of cts maps are cts

henceforth omit T from (X, T) when understood

<u>Df</u> cts f : X to Y is a homeomorphism iff
it has a two-sided cts inverse g : Y to X
i.e. g(f(x)) = x for all x in X,
f(g(y)) = y for all y in Y

"what is shape?"

"X and Y have the same shape
when there is a homeo between them"

<u>Ex</u> id : (X, T) to (X, T) is always a homeo
with inverse id

<u>Ex</u> [however:]
a cts bijection need not be a homeo

[we already have an example! which?]
f : (R, T_ℓ) to (R, T_{an}), f(x) = x

<u>Ex</u> more homeo's in the analytic topology:

f : R to R, f(x) = x^3
f : R^2 to R^2 f(x, y) = (x + y, (x – y)^3)

[compositions of homeo's are homeo's]

<u>Q</u> is there a homeo R to R^2? vice versa?

## The Subspace Topology

fix $A$ sub $X$

**Df** the subspace topology on $A$ induced by $X$:

$U$ sub $A$ is open iff there exists $V$ open in $X$ s.t.

$U = V \cap A$

**Ex** $X = \mathbb{R}$ and $A = [0, \infty)$

suppose $0 \leq a < b$

| | |
|---|---|
| $(a, b)$ open in $[0, \infty)$? in $\mathbb{R}$? | [yes, yes] |
| $[0, b)$ open in $[0, \infty)$? in $\mathbb{R}$? | [yes, no] |
| $[a, b)$ open in $[0, \infty)$? | [no, no] |

[moral: to say "U is open", must clarify "where"]

**Prop** the subspace topology on $A$ is the (unique) coarsest topology s.t. the inclusion $i : A \to X$ is continuous

**Pf** easy that $i : A \to X$ cts wrt sub. topology

suppose $i : A \to X$ cts wrt some topology $T$ on $A$

fix $U$ open in subspace topology on $A$

want $U$ in $T$

$U = V \cap A$ for some $V$ open in $X$

so $U = i^{-1}(V)$ in $T$ by continuity of $i$ wrt $T$