

Last time

Thm if $f : X$ to Y and $g : Y$ to X form
a homotopy equivalence

then $f_* : \pi_1(X, x)$ to $\pi_1(Y, f(x))$
 $g_* : \pi_1(Y, y)$ to $\pi_1(X, g(y))$

are isomorphisms for all x in X and y in Y

easier special cases:

Ex for any X and x ,

$(\text{id}_X)_* = \text{id}_{\pi_1(X, x)}$
as maps $\pi_1(X, x)$ to $\pi_1(X, x)$

Ex

if $f : X$ to Y is a homeo
then $f_* : \pi_1(X, x)$ to $\pi_1(Y, f(x))$ is
an isomorphism

Lem 1 given $f : X$ to Y ,
 $g : Y$ to Z ,

$(g \circ f)_* = g_* \circ f_*$
as maps $\pi_1(X, x)$ to $\pi_1(Z, g(f(x)))$

so if f and g are two-sided inverses of each other,

$g_* \circ f_* = (\text{id}_X)_* = \text{id}_{\pi_1(X, x)}$
 $f_* \circ g_* = (\text{id}_Y)_* = \text{id}_{\pi_1(Y, f(x))}$

[Pf of Thm will show that f_* is an iso
as argument for g is similar

by Lem 1, $g_* \circ f_* = (g \circ f)_*$
 $f_* \circ g_* = (f \circ g)_*$

Lem 2 if $s : G$ to H and $r : H$ to K
s.t. $r \circ s$ is bijective,
then s is injective and r is surjective

so to show f_* , g_* bijective,
enough to show $(g \circ f)_*$ and $(f \circ g)_*$ bijective

will show that $(g \circ f)_*$ is bijective
as argument for $f \circ g$ is similar

pick homotopy φ from $g \circ f$ to id_X

Lem 3 if α is a path in X from x to x'
then iso $\hat{\alpha} : \pi_1(X, x)$ to $\pi_1(X, x')$
def by $\hat{\alpha}([\gamma]) = [\bar{\alpha} * \gamma * \alpha]$

further: if $F, G : A$ to X are cts,
 φ is a homotopy from F to G ,
 a in A ,

then $\alpha_\varphi = \varphi(a, -)$ is a path from $F(a)$ to $G(a)$ s.t.

$$G_* = \hat{\alpha}_\varphi \circ F_*$$

now: $(g \circ f)_* = \hat{\alpha}_\varphi \circ (\text{id}_X)_* = \hat{\alpha}_\varphi$
so by Lem 3, $(g \circ f)_*$ is an iso]

(Munkres §59) [more examples of π_1 's:]

Thm $\pi_1(S^n)$ is trivial for all $n \geq 2$

Df in general, a space is simply-connected
iff it is path-connected with trivial π_1

“Pf of Thm” given x in S^n and a loop γ at x ,
pick p not in the image of γ
 $S^n - \{p\}$ is homeomorphic to R^n

R^n is simply-connected

i.e., any loop in R^n is \sim_p constant loop

so $\gamma \sim_p$ constant loop at x within $S^n - \{p\}$

so $\gamma \sim_p$ constant loop at x within S^n

but the map $\gamma : [0, 1]$ to X could be surjective!

to fix: show that $\gamma \sim_p$ some non-surjective loop
[somewhat hard]

Thm for any x in X and y in Y ,

$$\pi_1(X \times Y, (x, y)) \cong \pi_1(X, x) \times \pi_1(Y, y)$$

Cor $\pi_1(S^1 \times S^1) \cong \mathbb{Z} \times \mathbb{Z}$ under $+$

(Munkres §70) by contrast:

$$\pi_1(\text{figure-eight}) \cong \mathbb{Z} * \mathbb{Z} \text{ [free group]}$$

special cases of the Seifert–van Kampen Thm