Review given vector spaces V, W, U:

$$W \times V = \{(w, v) \mid w \text{ in } W, v \text{ in } V\}$$

a map $\beta : W \times V$ to U is bilinear iff
$\beta(w, –), \beta(–, v)$ are linear for all w in W, v in V

β is called a bilinear pairing when $U = F$

$$\mathrm{Bil}(W, V) = \{\text{bilinear pairings on } W \times V\}$$

is a vector space: $(a \cdot \beta + \beta')(\ ,\ ) = a\beta(\ ,\ ) + \beta'(\ ,\ )$

Ex last time: dot products are bilinear
they are also symmetric: $w \cdot v = v \cdot w$

Ex let $\beta : F^2 \times F^2$ to F be def by
$\beta((a, c), (b, d)) = ad – bc$

is it bilinear? yes [how to check?]
is it symmetric? no
in fact, it is anti-symmetric: $\beta((a, c), (b, d)) = –\beta((b, d), (a, c))$

contrast $(W \times V)^\vee = \mathrm{Hom}(W \times V, F)$
with $\mathrm{Bil}(W, V)$

know: $\dim (W \times V)^\vee = \dim W + \dim V$

goal: $\dim \mathrm{Bil}(W, V) = (\dim W)(\dim V)$
[what other spaces have this dim?]

Df the evaluation map $\langle\ ,\ \rangle : V^\vee \times V$ to F
is def by $\langle\theta, v\rangle = \theta(v)$

[the notation $\langle\ ,\ \rangle$ will help avoid confusion soon]

Lem $\langle\ ,\ \rangle$ is a bilinear pairing

Pf for fixed θ, want $\langle\theta, -\rangle$ linear
true by definition of θ

for fixed v, want $\langle -, v\rangle$ linear:

$$\begin{aligned} \langle a\theta + \theta', v\rangle &= (a\theta + \theta')(v) \\ &= a\theta(v) + \theta'(v) \\ &= a\langle\theta, v\rangle + \langle\theta', v\rangle \end{aligned}$$

[idea: can build all other pairings from $\langle\ ,\ \rangle$]

Lem if $S : W$ to $V^\vee$ is linear
then $\beta\_S : W \times V$ to F
def by $\beta\_S(w, v) = \langle Sw, v\rangle$ is bilinear

moreover:
$\Phi : \mathrm{Hom}(W, V^\vee)$ to $\mathrm{Bil}(W, V)$ def by $\Phi(S) = \beta\_S$
is linear

Pf $\beta\_S(w, -) = \langle Sw, -\rangle$ linear by lem
as for $\beta\_S(-, v) = \langle S(-), v\rangle$:
$\langle -, v\rangle$ is linear by lem, so $\langle S(-), v\rangle$ is

$$\begin{aligned} \Phi(a \bullet S + S')(w, v) &= \langle (a \bullet S + S')w, v\rangle \\ &= a\langle Sw, v\rangle + \langle S'w, v\rangle \text{ in Hom} \\ &= a\Phi(S)(w, v) + \Phi(S')(w, v) \\ &= (a \bullet \Phi(S) + \Phi(S'))(w, v) \text{ in Bil} \end{aligned}$$

[something stronger holds:]

<u>Thm</u> Φ is a linear iso $\mathrm{Hom}(W, V^\vee)$ to $\mathrm{Bil}(W, V)$

<u>Pf</u> we give a two-sided inverse

let $\Psi : \mathrm{Bil}(W, V)$ to $\mathrm{Hom}(W, V^\vee)$ be def as follows:

$\Psi(\beta) : W$ to $V^\vee$ is the linear map
def by $\Psi(\beta)(w) = \beta(w, -)$ for all w

$$\begin{aligned}[\Psi(a \bullet \beta + \beta')(w) &= (a \bullet \beta + \beta')(w, -) \\ &= a\beta(w, -) + \beta'(w, -) \text{ in Bil} \\ &= a \bullet \Psi(\beta)(w) + \Psi(\beta')(w) \\ &= (a \bullet \Psi(\beta) + \Psi(\beta'))(w) \text{ in Hom}]\end{aligned}$$

$\Psi(\Phi(S))(w) = \Phi(S)(w, -) = \langle Sw, - \rangle = Sw$
for all w
[where last step uses def of $\langle\ ,\ \rangle$]
so $\Psi(\Phi(S)) = S$

$\Phi(\Psi(\beta))(w, v) = \langle \Psi(\beta)w, v \rangle = \beta(w, v)$
for all w, v
so $\Phi(\Psi(\beta)) = \beta$ □

<u>Rem</u> thm works even if V, W are infin. dim'l

<u>Cor</u> if V, W are fin. dim'l,
then $\dim \mathrm{Bil}(W, V) = \dim \mathrm{Hom}(W, V^\vee) = (\dim W)(\dim V)$

[now make this concrete:]

Ex fix ordered bases $v_1, \ldots, v_n$ for V,
$w_1, \ldots, w_m$ for W

for any $m \times n$ matrix M: let β_M be def by

$$\beta_M(w, v) \quad = \quad b_1 \ldots b_m \quad M \quad \begin{matrix} a_1 \\ \ldots \\ a_n \end{matrix}$$

for all $v = \sum_i a_i v_i$ and $w = \sum_j b_j w_j$

claim that β_M is a bilinear pairing on $W \times V$

in fact: $M \mapsto \beta_M$
is just a basis-dependent version of
$S \mapsto \beta_S$

how?

write elts of V, W as cols wrt given bases

col of a_i's?	v wrt (v_i)
matrix M?	linear map $T : V \to W$
row of b_j's?	w^t wrt (w_j^t)
row-col multiplication?	$\langle\ ,\ \rangle$

so $\beta_M(w, v) = \langle w^t, Tv\rangle$

[want to find S s.t. the right-hand side is $\langle Sw, v\rangle$]

recall: $T : V \to W$ has a dual $T^\vee : W^\vee \to V^\vee$ def by

$$T^\vee(\theta) = \theta \circ T, \quad \text{meaning } \langle T^\vee(\theta), v\rangle = \langle \theta, Tv\rangle$$

let $S : W \to V^\vee$ be def by $Sw = T^\vee(w^t)$

then $\beta_M(w, v) = \langle T^\vee(w^t), v\rangle = \langle Sw, v\rangle$

Rem earlier examples are special cases:

$M = I_n$ yields $v \cdot w$

$M = \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix}$ yields the anti-symmetric ex

[check the latter explicitly]

(Axler §9D) motivation for tensors:

knowing $\mathrm{Bil}(W, V)$ usually differs from $(W \oplus V)^\vee$,
we seek a vector space $W \otimes V$ s.t.

$$\mathrm{Bil}(W, V) = (W \otimes V)^\vee$$

Df Axler defines the tensor product of W and V to be

$$W \otimes V = \mathrm{Bil}(W^\vee, V^\vee)$$

this is often painful in practice [why?]
other authors use other defns

for all (w, v) in $W \times V$, let $w \otimes v$ in $W \otimes V$ be the bilinear pairing on $W^\vee \times V^\vee$ such that

$$(w \otimes v)(\psi, \theta) = \psi(w)\theta(v)$$

for all (ψ, θ) in $W^\vee \times V^\vee$

[next time:]

Lem the map $W \times V$ to $W \otimes V$ that sends
(w, v) mapsto $w \otimes v$ is itself bilinear

Thm for any V, W, U,
every bilinear map $\beta : W \times V$ to U
takes the form

$$\beta(w, v) = [\beta](w \otimes v)$$

for some unique linear map $[\beta] : W \otimes V$ to U

moreover, this gives a linear iso
$\{\text{bilinear maps } W \times V \text{ to } U\} = \mathrm{Hom}(W \otimes V, U)$

Slogan (w, v) mapsto $w \otimes v$ is the “universal”
bilinear map out of $W \times V$