2.

Notes on using GAP to calculate character formulas.

2.1. Fix a finite Coxeter group W with reflection representation $V = V_{\mathsf{refl}}$ and a regular elliptic slope $\nu \in \mathbf{Q}_{>0}$. For any character χ, we write:

- χ' for its sign twist.
- Feg_χ for the fake degree of χ.
- Deg_χ for the generic degree of the unipotent principal series of χ.
- $\kappa(\chi) = \frac{1}{\chi(e)} \sum_t \chi(t)$, where e is the identity of W and the sum runs over all reflections in W. Sometimes I call this the *content* of χ.

Note that $N := \kappa(\chi_{\mathsf{triv}})$ is the total number of reflections. We will study

$$\Omega := q^{N\nu} \sum_{\chi \in \mathrm{Irr}(W)} \mathrm{Deg}_\chi(e^{2\pi i\nu}) q^{\kappa(\chi)\nu} \chi' \cdot \chi_{\mathrm{Sym}(V)},$$

where $\chi_{\mathrm{Sym}(V)} := \sum_i q^i \chi_{\mathrm{Sym}^i(V)}$. Explicitly,

$$\chi_{\mathrm{Sym}(V)} = \frac{\sum_\psi \mathrm{Feg}_\psi(q)\psi}{\prod_i (1 - q^{d_i})},$$

where the product runs over the fundamental degrees of the W-action on V.

2.2. First, for all χ, we have

$$\begin{aligned} \mathrm{Deg}_\chi(e^{2\pi i\nu}) &= (-1)^{2N\nu} \mathrm{Deg}_{\chi'}(e^{2\pi i\nu}), \\ \kappa(\chi) &= -\kappa(\chi'). \end{aligned}$$

Therefore,

$$\begin{aligned} \Omega &= (-1)^{2N\nu} q^{N\nu} \sum_{\chi \in \mathrm{Irr}(W)} \mathrm{Deg}_\chi(e^{2\pi i\nu}) q^{-\kappa(\chi)\nu} \chi \cdot \mathsf{sym}(V) \\ &= \frac{(-1)^{2N\nu} q^{N\nu}}{\prod_i (1 - q^{d_i})} \sum_{\chi, \psi \in \mathrm{Irr}(W)} \mathrm{Deg}_\chi(e^{2\pi i\nu}) \mathrm{Feg}_\psi(q) q^{-\kappa(\chi)\nu} \chi \cdot \psi. \end{aligned}$$

2.3. Sample GAP code for $W = W(D_4)$ and $\nu = \frac{1}{4}$. Note that `qexp` also has to be tweaked when W is not of simply-laced type.

2.3.1. Input data:

```
W:=CoxeterGroup("D",4); N:=12; refl:=4; m:=4; d:=1;
```

2.3.2. Setup:

```
q:=Indeterminate(Rationals); q.name:="q"; Wt:=CharTable(W);
irr:=Wt.irreducibles; len:=Length(irr);
fdeg:=FakeDegrees(W,q);
udeg:=UnipotentDegrees(W,q);
qexp:=List([1..len], i->(N*d/m)*(1 - irr[i][refl]/irr[i][1]));
udegval:=List([1..len], i->Value(udeg[i], E(m)));
```

2.3.3. The summation:

```
coeff:=List([1..len], i->List([1..len], j->udegval[i]*fdeg[j]
    *q^Int(qexp[i]) ));
tens:=List([1..len], i->MatScalarProducts(Wt, irr, Tensored(
    [irr[i]], irr)));
f:=0;
for i in [1..len] do for j in [1..len] do
    f:=f+coeff[i][j]*tens[i][j]; od; od;
```

2.3.4. Output:

```
denom:=1;
for d in ReflectionDegrees(W) do denom:=denom*(1-q^d); od;
for i in [1..len] do Print(i, "\t", f[i]/denom, "\n"); od;
```

2.4. Some example outputs where $d = 1$, meaning $\nu = \frac{1}{m}$. The lists of regular elliptic numbers come from Varagnolo–Vasserot.

W	`N`	`refl`	{reg. ell. nums}	m	Ω
D_4	12	4	$2, \mathbf{4}, 6$	4	$(1+q^2)\chi_{\mathrm{triv}} + q\chi_V$
				2	$(1+q^2-q^4+q^6+q^8)\chi_{\mathrm{triv}} + (q^2+q^4+q^6)\chi_{12} + (q-q^3-q^5+q^7)\chi_V + q^4\chi_{10} + (q^2+q^4+q^6)(\chi_8+\chi_9) + (q^3+q^5)\chi_6 - 2q^4\chi_5$
E_6	36	16	$3, 6, \mathbf{9}, 12$	9	$(1+q^2)\chi_{\mathrm{triv}} + q\chi_V$
				6	$(1+q^2+q^3+q^4+q^6)\chi_{\mathrm{triv}} + (q+q^2+q^3+q^4+q^5)\chi_V + q^3\chi_7 + (q^2+q^4)\chi_{11} + q^3\chi_{15}$
E_7	63	32	$2, 6, \mathbf{14}, 18$	14	$(1+q^2)\chi_{\mathrm{triv}} + q\chi_V$