The Gen and Cogen Formulas for Torus Knot Homology

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(results joint with Oscar Kivinen)

triply-graded Khovanov–Rozansky homology:

$$\mathrm{HHH} : \{\text{links}\}/\text{isotopy} \to \mathrm{Vect}_{\text{3-gr}}$$

KhR polynomial: $\mathbf{P}_L(a, q, t) = \sum_{i,j,k} a^i q^j t^k \, \mathrm{HHH}^{i,j,k}(L)$

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(Elias–Hogancamp–Mellit) computed $\mathbf{P}_{m,n} := \mathbf{P}_{\mathrm{torus}(m,n)}$ for all $m, n > 0$

$$\mathrm{torus}(m, n) = \widehat{\beta_n^m}, \quad \text{where } \beta_n = \sigma_1\sigma_2 \cdots \sigma_{n-1} \in \mathrm{Br}_n.$$

<u>Ex</u> $\mathbf{P}_{2,1} = \mathbf{P}_{\text{unknot}} = 1$ <u>Ex</u> $\mathbf{P}_{2,2} = 1 + \frac{qt}{1-q} + \frac{at}{1-q}$

<u>Ex</u> $\mathbf{P}_{2,3} = 1 + qt + at$

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<u>Thm</u> $$\sum_{\Delta \in D_{m,n}} q^{g_\Delta} t^{h_\Delta} f_\Delta^{\text{Gen}} \overset{\text{KT}}{=} \mathbf{P}_{m,n} \overset{\text{GMV}}{=} \sum_{\Delta \in D_{m,n}} q^{g_\Delta} t^{h_\Delta} f_\Delta^{\text{Cogen}}$$

where the LHS requires m, n coprime

<u>Ex</u> $\mathbf{P}_{4,3}$ has 11 terms:

$q^{g_\Delta} t^{h_\Delta}$	Gen$^+$		Cogen	
q^3t^3	$\emptyset$	1	$\{5\}$	$1 + aq^{-1}$
q^2t^2	$\{5\}$	$1 + at$	$\{1, 2\}$	$(1 + aq^{-1})(1 + aq^{-1}t)$
qt^2	$\{1\}$	$1 + at$	$\{2\}$	$1 + aq^{-1}$
qt	$\{2\}$	$1 + at$	$\{1\}$	$1 + aq^{-1}$
1	$\{1, 2\}$	$(1 + at)(1 + at^2)$	$\emptyset$	1

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$\mathbf{N}_0 = \{0, 1, 2, \ldots\}$

$$D_{m,n} = \{\Delta \subseteq \mathbf{N}_0 \mid \Delta + m, \Delta + n \subseteq \Delta \text{ and } 0 \in \Delta\}.$$

$g_\Delta = |\mathbf{N}_0 - \Delta|$
$h_\Delta = \sum_{k \in \mathrm{Gen}_n(\Delta)} |\{k < j < k + n \mid j \notin \Delta\}|$, where

$$\mathrm{Gen}_n(\Delta) = \{k \in \Delta \mid k - n \notin \Delta\}.$$

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$\mathrm{Gen}(\Delta) = \{k \in \Delta \mid k - m, k - n \notin \Delta\}$
$\mathrm{Gen}^+(\Delta) = \mathrm{Gen}(\Delta) - \{0\}$
$\mathrm{Cogen}(\Delta) = \{k \in \mathbf{N}_0 - \Delta \mid k + m, k + n \in \Delta\}$

$$\begin{aligned} f_\Delta^{\mathrm{Gen}} &= \prod_{k \in \mathrm{Gen}^+} (1 + at^{|\{j \in \mathrm{Gen}_n \mid k - m < j < k\}|}), \\ f_\Delta^{\mathrm{Cogen}} &= \prod_{k \in \mathrm{Cogen}} (1 + aq^{-1}t^{|\{j \in \mathrm{Gen}_n \mid k + n < j < k + n + m\}|}) \end{aligned}$$

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<u>Ex</u> take $\Delta = \{0, 3, 4, 5, 6, \ldots\} = \mathbf{N}_0 - \{1, 2\} \qquad \in D_{4,3}$

$\mathrm{Gen}_3(\Delta) = \{0, 4, 5\}, \quad \mathrm{Gen}^+(\Delta) = \{5\}, \quad \mathrm{Cogen}(\Delta) = \{1, 2\},$

$$\begin{aligned} f_\Delta^{\mathrm{Gen}} &= 1 + at^{|\{j \in \mathrm{Gen}_3 \mid 5 - 4 < j < 5\}|} \\ &= 1 + at \\ f_\Delta^{\mathrm{Cogen}} &= (1 + aq^{-1}t^{|\{j \in \mathrm{Gen}_3 \mid 1 + 3 < j < 1 + 7\}|}) \\ &\qquad \cdot (1 + aq^{-1}t^{|\{j \in \mathrm{Gen}_3 \mid 2 + 3 < j < 2 + 7\}|}) \\ &= (1 + aq^{-1}t)(1 + aq^{-1}) \end{aligned}$$

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Gen formula from Mellit, <u>G&T</u> (2022)

Cogen formula from Hogancamp–Mellit, arXiv:1909.00418, via Gorsky–Mazin–Vazirani (2020)

Both rely on recursions from Elias–Hogancamp (2019)

In turn starts from Khovanov (2008):

$$\text{braid } \beta \quad \rightsquigarrow \quad \text{bimodule complex } F_\beta \quad \rightsquigarrow \quad \mathrm{HHH}(\hat{\beta})$$

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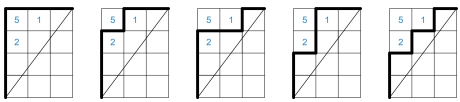
Mellit's recursion decomposes β_n^m into a sum indexed by m/n Dyck paths:



bijection $D_{m,n} \xrightarrow{\sim} \{m/n \text{ Dyck paths } \pi\}$

role of Dyck paths predicted by Gorsky–Neguţ (2016)

start with diagonal and "sweep up" applying local rules from Mellit, <u>Duke</u> (2021)

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$$\nu_*(\pi) = \{\text{bottom right corners of squares} \searrow \pi\}$$
$$\nu^*(\pi) = \{\text{top left corners of squares} \nwarrow \pi\}$$

<u>Lem</u> if $\Delta \mapsto \pi$, then:

(1) $\{j \in \mathrm{Gen}_n(\Delta) \mid k - m < j < k\} \xrightarrow{\sim} \nu_*(\pi)$

(2) $f_\Delta^{\mathrm{Gen}} = \prod_{p \in \nu_*(\pi)} (1 + at^{|\kappa_\pi(p)|}) = \frac{1}{1+a} \prod_{p \in \nu^*(\pi)} (1 + at^{|\kappa_\pi(p)|})$

where $\kappa_\pi(p) = \{\text{horizontal steps of } \pi \text{ meeting } l_{m/n}(p)\}$

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we used the Gen formula to show:

<u>Thm</u> for fixed $n > 0$,

$$\lim_{\substack{m \to \infty \\ m,\, n \text{ coprime}}} \mathbf{P}_{m,n}(a, q, t) = \prod_{2 \leq j \leq n} \frac{1 + at^{j-1}}{1 - qt^{j-1}}$$

<u>Cor</u> formula for $\mathbf{P}_{m,n}$ when $n \leq 3$ and m, n coprime

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<u>Pf</u> $\mathbf{P}_{m,n}(a, q, qt^2)$ matches $\mathbf{P}_{\infty,n}$ up to q-degree m

but $\mathbf{P}_{m,n}(a, q, qt^2)$ is determined by its terms up to degree $\frac{1}{2}(m-1)(n-1)$, by palindromicity wrt $q^{-1} \leftrightarrow qt^2$

<u>Cor</u> the ORS conjecture for $y^n = x^m$:

$$\frac{\mathbf{P}_{m,n}(a, q, qt^2)}{1 - q} = \sum_{k,\ell} a^k q^\ell t^{k^2 - k} \chi(t, \underbrace{\mathrm{Hilb}^\ell_{k\text{-nest}}(y^n = x^m)}_{\subseteq \mathrm{Hilb}^\ell \times \mathrm{Hilb}^{\ell+k}})$$

<u>Pf</u> ORS established a matching formula on the Hilb side