MATH 340: ADVANCED LINEAR ALGEBRA
PROBLEM SET #9

SPRING 2025

Due Friday, April 25. You may consult books, papers, and websites as long as you cite all sources and write up your solutions in your own words.

Problem 1. Recall that in the two-element field $\mathbf{F}$ introduced on Problem Set 2, #8, we have $-1 = 1$: that is, 1 is its own additive inverse.

(1) Show that if $F \in \{\mathbf{R}, \mathbf{C}\}$, then a bilinear form on a vector space over F is alternating if and only if it is antisymmetric.
(2) What goes wrong in (1) when $F = \mathbf{F}$?

In the remainder of this problem set, we exclude this field, returning to our usual assumption that all vector spaces are defined over $\mathbf{R}$ *or* $\mathbf{C}$*.*

Problem 2. Let $\mathbf{E} \subseteq \mathfrak{sl}(2, \mathbf{C})$ be the <u>real</u> vector space of trace-zero, Hermitian 2×2 complex matrices. (See Problem Set 3, #6, and Problem Set 7, #2.)

(1) Using part (2) of Problem Set 3, #6, check that E_1, E_2, E_3 form a basis for $\mathbf{E}$, where

$$E_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad E_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

(2) Let $S : \mathbf{E} \xrightarrow{\sim} \mathbf{R}^3$ be the linear isomorphism that sends (E_1, E_2, E_3) onto the standard basis. Show that

$$S(X) \cdot S(Y) = \frac{1}{2} \operatorname{tr}(XY)$$

for all $X, Y \in V$, where $- \cdot -$ is the dot product. Thus, the right-hand side defines an inner product on $\mathbf{E}$.

Problem 3. On a complex inner product space, operators T such that $\langle Tu, Tv \rangle = \langle u, v \rangle$ for all vectors u, v are usually called *unitary*, rather than orthogonal. Let

$$\mathrm{SO}(2) = \{2 \times 2 \text{ real matrices } M \mid \det(M) = 1 \text{ and } M^t M = I\},$$
$$\mathrm{SU}(2) = \{2 \times 2 \text{ complex matrices } M \mid \det(M) = 1 \text{ and } M^* M = I\},$$

where $M^*_{j,i} = \bar{M}_{i,j}$. The notations SO and SU stand for *special orthogonal* and *special unitary*. Using Problem Set 3, #8 and Problem Set 5, #8, give bijections

$$M : \{z \in \mathbf{C} \mid |z| = 1\} \xrightarrow{\sim} \mathrm{SO}(2), \qquad M : \{z \in \mathbf{H} \mid |z| = 1\} \xrightarrow{\sim} \mathrm{SU}(2)$$

such that $M(z_1 z_2) = M(z_1) \cdot M(z_2)$ in both cases. Above, the absolute value of a quaternion is given by $|a1 + bi + cj + dk| = \sqrt{a^2 + b^2 + c^2 + d^2}$.

Problem 4. Keep the notation of Problems 2–3. Show that if $M \in \mathrm{SU}(2)$, then

$$T_M : \mathbf{E} \to \mathbf{E} \quad \text{defined by } T_M(X) = MXM^{-1}$$

is a well-defined, linear, and orthogonal with respect to $\langle X, Y\rangle := \frac{1}{2}\operatorname{tr}(XY)$.

Together, Problem 3 and this problem encapsulate W. R. Hamilton's formalism for describing 3-dimensional rotations via quaternions.

Problem 5. Recall, from Problem Set 8, #4, the inner product on $\mathbf{C}[t]$ defined by

$$\langle f, g\rangle = \int_{-1}^{1} f(t)\overline{g(t)}\,dt.$$

(1) Show explicitly that the linear operator $D(p(t)) = \frac{d}{dt}p(t)$ is not self-adjoint with respect to $\langle -, -\rangle$.
(2) Is the linear operator $T(p(t)) = tp(t)$ self-adjoint?

Problem 6. Let T be an orthogonal linear operator on an inner product space V of dimension n. In the language of Problem Set 8, #7, the problem below proves the *Cartan–Dieudonné theorem* that T is a composition of $\leq n$ reflections.

(1) Show that for all $v, w \in V$ such that $\|v\| = \|w\|$, the reflection $S_{v-w} : V \to V$ swaps v and w.
(2) Let $(e_1, \ldots, e_n)$ be a basis for V, and let $f_i = Te_i$ for all i. Suppose that for some k, there is an orthogonal linear operator $T_k : V \to V$ such that $T_k e_i = f_i$ for all $i \leq k$. Using Problem Set 8, #8(1), show that

$$\begin{aligned} \|T_k e_{k+1}\| &= \|f_{k+1}\|, \\ \|T_k e_{k+1} - f_i\| &= \|f_{k+1} - f_i\| \qquad \text{for all } i < k+1. \end{aligned}$$

Hint: Orthogonal operators preserve the norms of vectors.
(3) Let $T_0 = \mathrm{Id}_V$. and $T_{k+1} = S_{T_k e_{k+1} - f_{k+1}} \circ T_k$ for all $k \geq 0$. Use (1)–(2) to show that $T_{k+1} e_i = f_i$ for all $i \leq k+1$. Thus, $T_n = T$.
Hint: In the case where $i < k+1$, expand the second display from (2), then apply the first display to arrive at $\langle T_k e_{k+1}, f_i\rangle = \langle f_{k+1}, f_i\rangle$. This shows that f_i is orthogonal to $T_k e_{k+1} - f_{k+1}$.

Problem 7. Show that if $M \in \mathrm{Mat}_n(\mathbf{C})$ is invertible, then $M = QR$ for some unitary Q and invertible upper-triangular R. This is called the *QR decomposition* of M. *Hint:* Interpret the Gram–Schmidt process for the columns of M in terms of right multiplication by another matrix.

Problem 8. We say that $A \in \mathrm{Mat}_n(\mathbf{C})$ is *positive-definite* if and only if $\langle u, v\rangle := u^t A\bar{v}$ is positive-definite.

(1) Show that if $A \in \mathrm{Mat}_n(\mathbf{C})$ is Hermitian and positive-definite, then $A = BB^*$ for some invertible B. *Hint:* The spectral theorem.
(2) Use Problem 6 to deduce that $A = R^*R$ for some invertible and upper-triangular R. This is called the *Cholesky decomposition* of A. It is a special case of *singular value decomposition*. (The square roots of the eigenvalues of A are the *singular values* of R.)