8.

A summary of the "new" level-rank duality.

We will explain the evidence for a family of surprising coincidences within the representation theory of a finite reductive group G. For the groups $G = \mathrm{GL}_n(\mathbf{F}_q)$, these coincidences can be expressed very concretely, in terms of the combinatorics of partitions, and the whole story is equivalent to an avatar of Frenkel's level-rank duality studied by Uglov, Chuang–Miyachi, and others.

8.1. Partitions

8.1. Let Π be the set of all integer partitions. We will regard a partition as a weakly decreasing sequence $\pi = (\pi_1, \pi_2, \ldots)$ such that $\pi_i = 0$ for all i large enough; in practice, we write a partition like $\pi = (2, 2, 1, 0, \ldots)$ using a shorthand like $(2^2, 1)$. We define the size of π to be $|\pi| := \pi_1 + \pi_2 + \ldots$ and the length of π to be the number of nonzero entries. For any positive integer m, we define an *m-partition* to be an m-tuple of partitions

$$\vec{\pi} = (\pi^{(0)}, \pi^{(1)}, \ldots, \pi^{(m-1)}),$$

indexed from 0 through $m-1$. We will regard these indices as elements of $\mathbf{Z}_m := \mathbf{Z}/m\mathbf{Z}$. We define the size of $\vec{\pi}$ to be $|\vec{\pi}| := |\pi^{(0)}| + \cdots + |\pi^{(m-1)}|$.

8.2. There is an analogue, for partitions, of the notion of long division by m. To state it, recall that a partition is an *m-core* if and only if it contains no hook lengths divisible by m. For instance, the only 1-core is the empty partition; the 2-cores are the "staircase" partitions 1, $(2, 1)$, $(3, 2, 1)$, *etc.*; but m-cores for $m \geq 3$ are more complicated. Let $\Pi_{m\text{-}cor} \subseteq \Pi$ the subset of m-cores. Then the analogue of the map sending an integer to its quotient and remainder modulo m is a bijection of the form

$$\Pi \xrightarrow{\sim} \Pi^m \times \Pi_{m\text{-}cor},$$

called the *core-quotient bijection at level m*.

To define it, let $\mathbf{B}$ be the collection of subsets $\beta \subseteq \mathbf{Z}$ with the property that $n \in \beta$ when n is a sufficiently negative integer, but $n \notin \beta$ when n is sufficiently positive. There is a bijection $\Pi \times \mathbf{Z} \xrightarrow{\sim} \mathbf{B}$ that takes a pair $(\pi, s) \in \Pi \times \mathbf{Z}$ to the set

$$\beta_{\pi,s} = \{\pi_i - i + s \mid i = 1, 2, \ldots\}.$$

There is also a bijection $\upsilon_m = (\upsilon_m^{(0)}, \upsilon_m^{(1)}, \ldots, \upsilon_m^{(m-1)}) : \mathbf{B} \xrightarrow{\sim} \mathbf{B}^m$, defined by setting

$$\upsilon_m^{(r)}(\beta) = \{q \in \mathbf{Z} \mid mq + r \in \beta\} \quad \text{for } r = 0, 1, \ldots, m-1.$$

The composition

$$\Upsilon_m : \Pi \times \mathbf{Z} \xrightarrow{\sim} \mathbf{B} \xrightarrow{\upsilon_m} \mathbf{B}^m \xrightarrow{\sim} (\Pi \times \mathbf{Z})^m = \Pi^m \times \mathbf{Z}^m$$

determines the core-quotient map as follows: The m-quotient of π is the m-partition formed by the Π^m-component of $\Upsilon_m(\pi, 0)$, while the *m-core* of π is the unique partition μ such that $\Upsilon_m(\mu, s)$ and $\Upsilon_m(\pi, s)$ have the same $\mathbf{Z}^m$-component for any s.

Note that μ is any m-core if and only if the Π^m-component of $\Upsilon_m(\mu, s)$ is the empty m-partition for some (or any) s. Also, note that there is a graphical way to calculate the core-quotient bijection, purely in terms of Young diagrams.

In the literature, elements of $\Pi^m \times \mathbf{Z}^m$ are called *charged m-partitions* and written with the ket notation $|\vec{\pi}, \vec{s}\rangle$ of quantum mechanics. The vector $\vec{s}$ is called the *m-charge*. Elements of $\mathbf{B}^m$ are sometimes called *m-runner abacus configurations*, or *m-abaci*. They can be pictured as subsets of $\mathbf{Z} \times \mathbf{Z}_m$, or more vividly, as configurations of beads on a horizontal abacus with m runners. Under the identification $\Pi^m \times \mathbf{Z}^m \simeq \mathbf{B}^m$, the operation $|\vec{\pi}, \vec{s}\rangle \mapsto |\vec{\emptyset}, \vec{s}\rangle$ corresponds to sliding all beads as far left as possible.

Example 8.1. Let $\pi = (8, 6, 1)$ and $s = 2$. Then $\beta_{\pi,s} \in \mathbf{B}$ is represented by the following 1-abacus:

$$\cdots \quad -4 \quad -3 \quad -2 \quad - \quad 0 \quad - \quad - \quad - \quad - \quad - \quad 6 \quad - \quad - \quad 9$$

So $\Upsilon_3(\pi, s) \in \mathbf{B}^3$ is represented by the following 3-abacus, where we have labeled each position (q, r) with the value of $3q + r$. Note that we write r increasing *downward*.

$$\begin{matrix} \cdots & -12 & -9 & -6 & -3 & 0 & - & 6 & 9 \\ \cdots & -11 & -8 & -5 & -2 & - & - & - & \\ \cdots & -10 & -7 & -4 & - & - & - & - & \end{matrix}$$

Sliding the beads at 6 and 9 as far left as possible gives the 3-abacus of $\Upsilon_3(\mu, s)$, where $\mu = (5, 3, 1)$. So the 3-core of π is $(5, 3, 1)$.

8.3. We will discuss generalizations of υ_m and Υ_m that involve fixing another integer $l > 0$. Let $\upsilon_m^l = (\upsilon_m^{l,(0)}, \upsilon_m^{l,(1)}, \ldots, \upsilon_m^{l,(m-1)}) : \mathbf{B}^l \xrightarrow{\sim} \mathbf{B}^m$ be defined by setting

$$\upsilon_m^{l,(r)}(\vec{\beta}) = \left\{ lq + r' \;\middle|\; \begin{matrix} (q, r') \in \mathbf{Z} \times \mathbf{Z}_l, \\ mq + r \in \beta^{(r')} \end{matrix} \right\} \quad \text{for } r = 0, 1, \ldots, m-1.$$

It is also possible to rewrite υ_m^l as a composition $\mathbf{B}^l \xrightarrow{\upsilon_l^{-1}} \mathbf{B} \xrightarrow{\upsilon_m^*} \mathbf{B}^m$, where υ_m^* is some modified version of υ_m. The composition

$$\Upsilon_m^l : \Pi^l \times \mathbf{Z}^l = \mathbf{B}^l \xrightarrow{\upsilon_m^l} \mathbf{B}^m = \Pi^m \times \mathbf{Z}^m$$

recovers Υ_m when $l = 1$.

Example 8.2. The bijection Υ_4^3 takes the 4-abacus

$$\begin{matrix} \cdots & -12 & -8 & -4 & 0 \\ \cdots & -11 & -7 & -3 & - \\ \cdots & -10 & -6 & -2 & 2 \\ \cdots & -9 & -5 & - & 3 \end{matrix}$$

to the 3-abacus in Example 8.1.

Here is a graphical interpretation of Υ_4^3: If we subdivide the 4-abacus above into 3×4 arrays of points (q_4, r_4), according to the quotient of q_4 modulo 3, and subdivide the

3-abacus in Example 8.1 into 4×3 arrays of points (q_3, r_3), according to the quotient of q_3 modulo 4, then Υ^3_4 amounts to reflecting and rotating each 3×4 array onto a corresponding 4×3 array. In our example:

$$\begin{array}{ccc|ccc} \bullet & \bullet & \bullet & \bullet & - & - \\ \bullet & \bullet & \bullet & - & - & - \\ \bullet & \bullet & \bullet & \bullet & - & - \\ \bullet & \bullet & - & \bullet & - & - \end{array} \quad \mapsto \quad \begin{array}{cccc|cccc} \bullet & \bullet & \bullet & \bullet & \bullet & - & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & - & - & - & - \\ \bullet & \bullet & \bullet & - & - & - & - & - \end{array}$$

In the following two sections, we will explain how Υ^l_m holds meaning in the representation theory of the wreath-product groups

$$S_{N,m} := S_N \ltimes \mathbf{Z}_m^N,$$

or rather, associated Hecke algebras at roots of unity.

8.2. Representations

8.4. Recall that S_N is a reflection group generated by the system of simple reflections $s_1, s_2, \ldots, s_{N-1}$, where s_i transposes letters i and $i+1$. Writing Br_N for the braid group on N strands, we have a quotient map $Br_N \to S_N$, sending the ith positive simple twist σ_i to the transposition s_i. It descends to a quotient map of rings $H_N(x) \to \mathbf{C}S_N$, where the domain is the *Hecke algebra*

$$H_N(x) := \frac{\mathbf{C}[x^{\pm 1}]Br_N}{\langle (\sigma_i - 1)(\sigma_i + x) \mid i = 1, \ldots, N-1 \rangle}.$$

For any nonzero complex number ξ, we set

$$H_N(\xi) = \mathbf{C} \otimes_{\mathbf{Z}[x^{\pm 1}]} H_N(x), \quad \text{where } \mathbf{Z}[x^{\pm 1}] \to \mathbf{C} \text{ sends } x \mapsto \xi.$$

For sufficiently generic $\xi \in \mathbf{C}^\times$, there is an isomorphism $H_N(\xi) \simeq \mathbf{C}S_N$, and hence, a bijection between simple $H_N(\xi)$-modules up to isomorphism and irreducible characters of S_N. By work of Frobenius, Schur, and Young, the latter are also in bijection with integer partitions of size N. But $H_N(\xi)$ may not even be semisimple at special ξ.

8.5. Now fix $m \geq 1$. The group $S_{N,m}$ is a *complex* reflection group, generated by S_N and a pseudo-reflection of order m. The map $Br_N \to S_N = S_{N,1}$ generalizes to a map $Br_{N,m} \to S_{N,m}$, where

$$Br_{N,m} = \begin{cases} Br_N & m = 1, \\ \langle Br_N, \tau \mid \tau\sigma_1\tau\sigma_1 = \sigma_1\tau\sigma_1\tau \text{ and } \tau\sigma_j = \sigma_j\tau \text{ for } j \neq 1 \rangle. & m > 1. \end{cases}$$

The Hecke algebra generalizes to an algebra

$$H_{N,m}(\vec{x}) = \frac{\mathbf{Z}[\vec{x}^{\pm 1}]Br_{N,m}}{\left\langle \begin{array}{l} (\sigma_i - 1)(\sigma_i + x_\sigma) \text{ for } i = 1, \ldots, N-1, \\ (\tau - 1)(\tau - x_{\tau,1}) \cdots (\tau - x_{\tau,m-1}) \end{array} \right\rangle},$$

where $\vec{x}$ denotes a collection of indeterminates $(x_\sigma, x_{\tau,1}, \dots, x_{\tau,m-1})$, and $\vec{x}^{\pm 1}$ means we also adjoin their inverses. For any $\vec{\xi} \in (\mathbf{C}^\times)^m$, we define $H_{N,m}(\vec{\xi})$ similarly to how we defined $H_N(\xi)$.

Like before, there is an isomorphism $H_{N,m}(\vec{\xi}) \simeq \mathbf{C}S_{N,m}$ at generic $\vec{\xi}$, inducing a bijection between simple $H_{N,m}(\vec{\xi})$-modules up to isomorphism and irreducible characters of $S_{N,m}$. By work of Clifford, the latter are indexed by m-partitions of size N. But again like before, $H_{N,m}(\vec{\xi})$ can degenerate at special $\vec{\xi}$.

We are especially interested in parameters of the form

$$\vec{\xi} = (\zeta^m, \zeta^{\vec{s}}) := (\zeta^m, \zeta^{s_1}, \dots, \zeta^{s_{m-1}}) \quad \text{for a root of unity } \zeta \text{ and } m\text{-charge } \vec{s}.$$

Note that if ζ is a primitive mth root of unity, then $H_{N,m}(\zeta^m, \zeta^{\vec{s}}) \simeq \mathbf{C}S_{N,m}$. For more general ζ, it turns out that the representation theory of $H_{N,m}(\zeta^m, \zeta^{\vec{s}})$ encodes and is encoded by the combinatorics of m-cores and m-quotients.

8.6. Given an associative algebra H, we write $\mathscr{K}_0(H)$ for the Grothendieck group of the category of finitely-generated H-modules, and $\mathscr{K}_0^+(H) \subseteq \mathscr{K}_0(H)$ for its non-negative part. Let $H^\circ_{N,m}(\vec{x}) = \mathbf{K} \otimes_{\mathbf{Z}[\vec{x}^{\pm 1}]} H_{N,m}(\vec{x})$ for some field $\mathbf{K} \supseteq \mathbf{Z}[\vec{x}^{\pm 1}]$ such that $H^\circ_{N,m}(\vec{x}) \simeq \mathbf{K}S_{N,m}$, and inside which the subring $\mathbf{Z}[\vec{x}^{\pm 1}]$ is integrally closed.

The specialization map $H_{N,m}(\vec{x}) \to H_{N,m}(\vec{\xi})$ induces a *decomposition map*

$$\mathrm{d}_{\vec{\xi}} : \mathscr{K}_0^+(H^\circ_{N,m}(\vec{x})) \to \mathscr{K}_0^+(H_{N,m}(\vec{\xi})). \tag{8.1}$$

At the same time, every character of $S_{N,m}$ corresponds to a module over $H^\circ_{N,m}(\vec{x})$ up to isomorphism. Altogether, we get maps

$$\{\vec{\pi} \in \Pi^m \mid |\vec{\pi}| = N\} \xrightarrow{\sim} \mathrm{Irr}(S_{N,m}) \to \mathscr{K}_0^+(H^\circ_{N,m}(\vec{x})) \xrightarrow{\mathrm{d}_{\vec{\xi}}} \mathscr{K}_0^+(H_{N,m}(\vec{\xi})).$$

The elements of the image of the composition are the classes of the so-called *Specht modules*. We say that two m-partitions of size N, or irreducible characters of $S_{N,m}$, belong to the same *$\vec{\xi}$-block* if and only if the corresponding Specht modules fit into a finite sequence of $H_{N,m}(\vec{\xi})$-modules in which consecutive modules always share a Jordan–Hölder factor.

Note that $H_{N,m}(\vec{\xi})$ is semisimple if and only if each block is a singleton. In general, $\vec{\xi}$-blocks can be more complicated. The *decomposition matrix* of a $\vec{\xi}$-block is a matrix with rows indexed by isomorphism classes of simple $H_{N,m}(\vec{\xi})$-modules, columns indexed by m-partitions of N, and entries given by the Jordan–Hölder multiplicities of the rows in the Specht $H_{N,m}(\vec{\xi})$-modules corresponding to the columns.

8.7. Let $\zeta_m \in \mathbf{C}^\times$ be a primitive mth root of unity. For any m-charge $\vec{s}$, we will write $|\vec{s}| = s_1 + \cdots + s_m$ in what follows.

Fix $s \in \mathbf{Z}$. In a slogan, the map Υ^l_m turns out to relate $(\zeta^l_m, \zeta^{\vec{r}}_m)$-blocks of l-partitions, as we run over l-charges $\vec{r}$ with $|\vec{r}| = s$, and $(\zeta^m_l, \zeta^{\vec{s}}_l)$-blocks of m-partitions, as we run over m-charges $\vec{s}$ with $|\vec{s}| = s$. To make this precise, we define a *rank-m, level-l Uglov datum* to be a triple $(N, \vec{s}, \mathbf{b})$, where $N \geq 0$ is any integer, $\vec{s}$ is an m-charge, and $\mathbf{b}$ is a $(\zeta^m_l, \zeta^{\vec{s}}_l)$-block of the m-partitions of size N.

Theorem 8.3 (Uglov)**.** *Any rank-l, level-m Uglov datum $(K, \vec{r}, \mathbf{a})$ defines a rank-m, level-l Uglov datum $(N, \vec{s}, \mathbf{b})$ such that $|\vec{r}| = |\vec{s}|$ and:*

(1) This assignment gives a bijection between rank-l, level-m Uglov data and rank-m, level-l Uglov data.

(2) For any $s \in \mathbf{Z}$, the bijection Υ^l_m restricts to a bijection

$$\{|\vec{\pi}, \vec{r}\rangle \in \Pi^l \times \mathbf{Z}^l \mid |\vec{\pi}| = K,\ \vec{\pi} \in \mathbf{a}\} \xrightarrow{\sim} \{|\vec{\varpi}, \vec{s}\rangle \in \Pi^m \times \mathbf{Z}^m \mid |\vec{\varpi}| = N,\ \vec{\varpi} \in \mathbf{b}\}.$$

In fact, this property uniquely determines the bijection in (1).

(3) A precise inversion formula relates the decomposition matrices of $\mathbf{a}$ *and* $\mathbf{b}$*.*

Theorem 8.3 is remarkable because there is no direct relationship between the algebras $H_{M,l}(\zeta^l_m, \zeta^{\vec{r}}_m)$ and $H_{N,m}(\zeta^m_l, \zeta^{\vec{s}}_l)$ beyond the numerics of their parameters.

8.8. We now sketch how Theorem 8.3 is related to Frenkel's *level-rank duality* between two affine Lie algebras, or rather, quantized versions $\mathrm{U}_v(\widehat{\mathfrak{sl}}_l)_{\mathbf{Q}}$ and $\mathrm{U}_v(\widehat{\mathfrak{sl}}_m)_{\mathbf{Q}}$ over the field $\mathbf{Q}(v)$.

For any integer $l > 0$ and m-charge $\vec{s}$, there is a module over the quantum affine algebra $\mathrm{U}_v(\widehat{\mathfrak{sl}}_m)_{\mathbf{Q}}$ called the *Fock space of level l and charge $\vec{s}$*. Its underlying vector space is a formal span of charged m-partitions:

$$\Lambda^{\vec{s}}_v = \bigoplus_{\vec{\pi} \in \Pi^m} \mathbf{Q}(v)|\vec{\pi}, \vec{s}\rangle.$$

It controls the representation theory of the algebras $H_{N,m}(\zeta^m_l, \zeta^{\vec{s}}_l)$ through a theorem of Ariki, identifying the sum of the Grothendieck groups $\mathscr{K}_0(H_{N,m}(\zeta^m_l, \zeta^{\vec{s}}_l))$ for $N \geq 0$ with an irreducible, highest-weight submodule of $\Lambda^{\vec{s}}_v$. For fixed $s \in \mathbf{Z}$, Uglov's commuting actions arise from the isomorphisms of vector spaces

$$\bigoplus_{|\vec{r}|=s} \Lambda^{\vec{r}}_v \xleftarrow{\upsilon_l} \Lambda^s_v \xrightarrow{\upsilon^*_m} \bigoplus_{|\vec{s}|=s} \Lambda^{\vec{s}}_v$$

induced by the maps υ_l and υ^*_m such that $\upsilon^l_m = \upsilon^*_m \circ \upsilon^{-1}_l$. (See the end of §8.1.)

8.3. Categories

8.9. In the decade after Uglov, several teams of authors worked to categorify Theorem 8.3(3) to a statement about highest-weight covers of the module categories of the Hecke algebras $H_{N,m}(\vec{x})$. To explain this, it is convenient to generalize from the groups $S_{N,m}$ to any finite complex reflection group C with reflection representation V.

Generalizing $Br_{N,m}$, the *braid group* of C is the fundamental group

$$Br_C := \pi_1(V^\circ/C),$$

where $V^\circ \subseteq V$ is the open locus where C acts freely. Generalizing $H_{N,m}(\vec{x})$, the *Hecke algebra* $H_C(\vec{x})$ is a certain quotient of $\mathbf{C}[\vec{x}^{\pm 1}][Br_C]$, where $\vec{x}$ is now a collection of indeterminates indexed in terms of the reflection hyperplanes in V and the orders of certain corresponding complex reflections. Specializing $\vec{x}$ to a vector of nonzero complex numbers $\vec{\xi}$, we obtain an algebra $H_C(\vec{\xi})$ a decomposition map $\mathrm{d}_{\vec{\xi}}$ generalizing the map in (8.1), and a notion of $\vec{\xi}$-blocks. The specialization from $H_C(\vec{x})$ to $H_C(\vec{\xi})$ may be viewed as a choice of monodromy condition over V°/C.

The *rational double affine Hecke algebra* or *rational Cherednik algebra* of C, at a vector of complex numbers $\vec{\nu}$ with the same indices as $\vec{\xi}$, is an algebra of deformed polynomial *differential operators* over $V /\!\!/ C$. It takes the form

$$D_C^{rat}(\vec{\nu}) = (\mathbf{C}C \ltimes (\mathrm{Sym}(V) \otimes \mathrm{Sym}(V^*)))/I(\vec{\nu})$$

for some ideal $I(\vec{\nu})$ deforming the Heisenberg–Weyl relations $xy - yx = \langle x, y\rangle$ for $x \in V$ and $y \in V^*$. The rational Cherednik algebra shares many features with the universal enveloping algebras of semisimple Lie algebras: It has a triangular decomposition, where $\mathbf{C}C$ plays the role of the Cartan subalgebra, and an analogue of the Bernstein–Gelfand–Gelfand category O, which we will denote $\mathsf{O}_C(\vec{\nu})$. In particular, $\mathsf{O}_C(\vec{\nu})$ is a highest-weight category whose simple objects are indexed by $\mathrm{Irr}(C)$. For any $\chi \in \mathrm{Irr}(C)$, we write $\Delta_{\vec{\nu}}(\chi)$ to denote the *Verma module* with the corresponding simple quotient.

When $\vec{\xi}$ is related to $\vec{\nu}$ by a certain exponential formula, modules over $H_C(\vec{\xi})$ and $D_C^{rat}(\vec{\nu})$ are related through a Riemann–Hilbert correspondence. Namely, localizing a $D_C^{rat}(\vec{\nu})$-module to V°/W, then taking monodromy along a Knizhnik–Zamolodchikov-type connection, defines a functor

$$\mathsf{KZ} : \mathsf{O}_C(\vec{\nu}) \to \mathsf{Mod}(H_C(\vec{\xi})). \tag{8.2}$$

When C is a *real* reflection group, the formula is just $\vec{\xi} = \exp(2\pi i\vec{\nu})$.

It turns out that the KZ functor sends Verma modules to Specht modules, in the sense that $[\mathsf{KZ}(\Delta_{\vec{\nu}}(\chi))] = \mathrm{d}_{\vec{\xi}}(\chi)$ in $\mathscr{K}_0(H_C(\vec{\xi}))$. Moreover, KZ induces a bijection between the set of block *subcategories* of $\mathsf{O}_C(\vec{\nu})$ and that of $\mathsf{Mod}(H_C(\vec{\xi}))$.

8.10. Let $(K, \vec{r}, \mathbf{a})$ and $(N, \vec{s}, \mathbf{b})$ be Uglov data corresponding to each other in the setup of Theorem 8.3. The following result was essentially conjectured by Chuang–Miyachi, and proved by Rouquier–Shan–Varagnolo–Vasserot.

Theorem 8.4 (Categorical Level-Rank Duality)**.** *For such $(K, \vec{r}, \mathbf{a})$ and $(N, \vec{s}, \mathbf{b})$, the bijection in part (2) of Theorem 8.3 is categorified by a Koszul duality equivalence*

$$\mathsf{O}_{S_{K,l}}(\vec{\nu}_l)_{\mathbf{a}} \simeq \mathsf{O}_{S_{N,m}}(\vec{\nu}_m)_{\mathbf{b}},$$

where the Cherednik parameters $\vec{\nu}_l \in \mathbf{C}^l$ and $\vec{\nu}_m \in \mathbf{C}^m$ respectively lift the Hecke parameters $(\zeta_m^l, \zeta_m^{\vec{r}})$ and $(\zeta_l^m, \zeta_l^{\vec{s}})$ by way of (8.2), and the subscripts $\mathbf{a}, \mathbf{b}$ indicate the block subcategories *of the Cherednik categories O lifting the appropriate block subcategories of the Hecke module categories.*

The proof uses further equivalences between the Cherednik categories and truncated parabolic categories O of the Lie algebras $\widehat{\mathfrak{sl}}_m$, proved independently by Losev, Rouquier–Shan–Varagnolo–Vasserot, and Webster. These in turn rely on the inversion formulas in part (3) of Theorem 8.3.

8.4. Finite Reductive Groups

8.11. We propose a generalization of Theorem 8.3, replacing the groups $S_{N,m}$ with complex reflection groups arising from the representation theory of finite groups of Lie type. Theorem 8.3 will be the *general linear* case of our story.

Fix a prime power q and a (connected, smooth) reductive algebraic group $\mathbf{G}$ over $\bar{\mathbf{F}}_q$, split over $\mathbf{F}_q$, equipped with a q-Frobenius map $F : \mathbf{G} \to \mathbf{G}$. This defines a finite reductive group $G = \mathbf{G}^F$. Throughout, we reserve **boldface** uppercase letters for spaces over $\bar{\mathbf{F}}_q$, and ordinary *italics* for their loci of F-fixed points. We again use $\mathscr{K}_0$ to denote Grothendieck groups.

8.12. By the work of Deligne and Lusztig, the irreducible representations of G can all be obtained from certain induction maps

$$R_{\mathbf{M}}^{\mathbf{G}} : \mathscr{K}_0(M) \to \mathscr{K}_0(G), \quad \text{where } \mathbf{M} \text{ runs over } F\text{-stable Levi subgroups of } \mathbf{G},$$

and $M = \mathbf{M}^F$. In more detail, $R_{\mathbf{M}}^{\mathbf{G}}$ arises from commuting actions of G and M on the compactly-supported étale cohomology of algebraic varieties $\mathbf{Y}_{\mathbf{P}}^{\mathbf{G}}$ over $\bar{\mathbf{F}}_q$, now called *Deligne–Lusztig varieties*. The precise definition depends on a choice of parabolic subgroup $\mathbf{P} \subseteq \mathbf{G}$ containing $\mathbf{M}$. For $\mathbf{F}_q$ of sufficiently large characteristic, the map $R_{\mathbf{M}}^{\mathbf{G}}$ is independent of $\mathbf{P}$.

In fact, to construct the irreducibles of G, it suffices to run over maximal tori, not Levis. An irreducible character is *unipotent* if and only if it occurs in $R_{\mathbf{T}}^{\mathbf{G}}(1)$ for some maximal torus $\mathbf{T}$, where 1 denotes the trivial character. Following Lusztig, we set

$$\mathrm{Uch}(G) = \{\text{unipotent irreducible characters of } G\}.$$

Lusztig shows that $\mathrm{Uch}(G)$ can be indexed in a way independent of q, depending only on the Weyl group W of $\mathbf{G}$ itself.

8.13. An irreducible character $\mu \in \mathrm{Irr}(M)$ is *cuspidal* if and only if it does not occur in $R_{\mathbf{L}}^{\mathbf{M}}(\lambda)$ for some strictly smaller F-stable Levi $\mathbf{L} \subsetneq \mathbf{M}$ and irreducible $\lambda \in \mathrm{Irr}(L)$. For a unipotent cuspidal $\mu \in \mathrm{Uch}(M)$, we define the *Harish-Chandra series* associated with (M, μ) to be the set

$$\mathrm{Uch}(G)_{M,\mu} = \{\rho \in \mathrm{Uch}(G) \mid (\rho, R_{\mathbf{M}}^{\mathbf{G}}(\mu))_G \neq 0\}.$$

Harish-Chandra observed that the sets $\mathrm{Uch}(G)_{M,\mu}$ are pairwise disjoint and partition $\mathrm{Uch}(G)$ as we run over $\mathbf{G}$-conjugacy classes of such *cuspidal pairs* (M, μ) for which the underlying Levi $\mathbf{M} \subseteq \mathbf{G}$ is F-maximally split. This last condition means that $\mathbf{M} = Z_{\mathbf{G}}(\mathbf{T})^\circ$ for some F-stable torus $\mathbf{T}$, maximally split over $\mathbf{F}_q$.

Motivated by ideas from the ℓ-modular representation theory of G at large primes ℓ, Broué–Malle–Michel discovered that Harish–Chandra's theory is the $m = 1$ case of a theory that exists for any positive integer m. (To obtain a relation between ℓ-modular representation theory and what follows, one takes m to be the multiplicative order of q modulo ℓ.) To state their results cleanly, they introduce the notion of a *generic finite reductive group* $\mathbb{G}$ that interpolates the groups G as we keep the root datum and its

Frobenius automorphism fixed, but vary the prime power q. For simplicity, we will avoid this formalism in our discussion below.

Let Φ_m denote the mth cyclotomic polynomial. It defines a class of F-stable tori $\mathbf{T} \subseteq \mathbf{G}$, not necessarily maximal, called *Φ_m-tori*: The corresponding groups $T \subseteq G$ are characterized by the property that

$$|T| \text{ is a power of } \Phi_m(q).$$

An F-stable Levi subgroup $\mathbf{M} \subseteq \mathbf{G}$ is called *Φ_m-split* if and only if it takes the form $\mathbf{M} = Z_{\mathbf{G}}(\mathbf{T})^\circ$ for some Φ_m-torus $\mathbf{T}$. For such $\mathbf{M}$, we say that $\mu \in \mathrm{Uch}(M)$ is *Φ_m-cuspidal* if and only if it does not occur in $R_{\mathbf{L}}^{\mathbf{M}}(\lambda)$ for any smaller Φ_m-split Levi $\mathbf{L}$ and $\lambda \in \mathrm{Uch}(L)$. In this case, we say that (M, μ) is a *Φ_m-cuspidal pair*.

The absolute Weyl group of $\mathbf{G}$ acts by conjugation on the set of Φ_m-cuspidal pairs. By taking $m = 1$, we recover the usual notions of maximally split tori, maximally split Levis, and cuspidal pairs.

Theorem 8.5 (Broué–Malle–Michel)**.** *For any fixed integer $m > 0$, the Harish-Chandra series $\mathrm{Uch}(G)_{M,\mu}$ are disjoint and partition $\mathrm{Uch}(G)$, as (M, μ) runs over a full set of representatives for the absolute Weyl orbits of Φ_m-cuspidal pairs.*

Moreover, if $\rho \in \mathrm{Uch}(G)$ satisfies $\mathrm{Deg}_\rho(\zeta_m) \neq 0$, then $\rho \in \mathrm{Uch}(\mathbb{G})_{T,1}$, where T represents the Φ_m-split Levis that are maximal tori.

8.14. In the classical case where $m = 1$, Lusztig observed that each Harish-Chandra series $\mathrm{Uch}(G)_{M,\mu}$ has a parametrization of the form

$$\chi_{M,\mu} : \mathrm{Uch}(G)_{M,\mu} \xrightarrow{\sim} \mathrm{Irr}(W_{M,\mu}^G),$$

where $W_{M,\mu}^G$ is the stabilizer of μ under the action of $W_M^G := N_G(M)/M$. We say that W_M^G, *resp.* $W_{M,\mu}^G$, is the *relative Weyl group* of M, *resp.* (M, μ), in G. These form real reflection groups; for most choices of μ, they coincide.

The bijection $\chi_{M,\mu}$ arises from comparing the G-commutant of the cohomology of $Y_{\mathbf{P}}^{\mathbf{G}}$ to the group algebra of $W_{M,\mu}^G$. More precisely, Lusztig shows that there is a $\bar{\mathbf{Q}}[x^{\pm 1/\infty}]$-algebra $H_{M,\mu}^G(x)$, a specialization of the algebra $H_{W_{M,\mu}^G}(\vec{x})$ discussed earlier, such that for generic $\xi \in \mathbf{C}^\times$, we have

$$\mathbf{C} \otimes_{\bar{\mathbf{Q}}[\vec{x}^{\pm 1/\infty}]} H_{M,\mu}^G(\xi) \simeq \mathbf{C}W_{M,\mu}^G, \quad \text{where } x \mapsto \xi, \tag{8.3}$$

but at the same time,

$$\bar{\mathbf{Q}}_\ell \otimes_{\bar{\mathbf{Q}}[\vec{x}^{\pm 1/\infty}]} H_{M,\mu}^G(x) \simeq \mathrm{End}_G(\mathrm{H}_c^*(Y_{\mathbf{P}}^{\mathbf{G}}, \bar{\mathbf{Q}}_\ell)), \quad \text{where } x \mapsto q. \tag{8.4}$$

Then $\chi_{M,\mu}$ arises from a composition of isomorphisms

$$\mathscr{K}_0(G) \xrightarrow{\sim} \mathscr{K}_0(H_{M,\mu}^G(q)) \xleftarrow{\sim} \mathscr{K}_0(H_{M,\mu}^G(x)) \xrightarrow{\sim} \mathscr{K}_0(W_{M,\mu}^G),$$

where the first arrow arises from the double-centralizer theorem and the latter two arise from Tits deformation. In fact, these isomorphisms are isometries with respect to natural multiplicity pairings.

Lusztig observed similar results in the cases where the underlying Levi is not F-maximally split, but a maximal torus of Coxeter type. This condition implies that it is Φ_h-split for some integer h that can be viewed as the (twisted) Coxeter number of G. In these cases, the relative Weyl group is cyclic of order h.

Broué–Malle observed that for general m, the relative Weyl groups arising from Φ_m-split Levis and Φ_m-cuspidal pairs are always *complex* reflection groups. Generalizing Lusztig's results, they define an algebra $H^G_{M,\mu}(x)$ for any G, m, M, μ, using case-by-case formulas, such that (8.3) still holds, and various numerical properties of $H^G_{M,\mu}(x)$ are consistent with (8.4).

Conjecture 8.6 (Broué–Malle)**.** (8.4) *holds for all* G, m, M, μ.

Although few cases of this conjecture are known beyond the ones that Lusztig established, Broué–Malle–Michel were still able to generalize his parametrizations $\chi_{M,\mu}$ to all G, m, M, μ, by direct and case-by-case constructions.

Theorem 8.7 (Broué–Malle–Michel)**.** *For any fixed integer* $m > 0$ *and* m*-cuspidal pair* (M, μ)*, there is a map*

$$(\varepsilon_{M,\mu}, \chi_{M,\mu}) : \mathrm{Uch}(G)_{M,\mu} \to \{\pm 1\} \times \mathrm{Irr}(W^G_{M,\mu})$$

such that $\varepsilon_{M,\mu}\chi_{M,\mu} : \mathbf{Z}\mathrm{Uch}(G)_{M,\mu} \to \mathscr{K}_0(W^G_{M,\mu})$ *defines an isometry with respect to the natural multiplicity pairings. In particular,* $\chi_{M,\mu}$ *is bijective. Moreover:*

(1) These maps are compatible with inclusions of m*-split Levis. They intertwine the maps* $R^M_{\mathbb{L}}$ *with the ordinary induction maps between relative Weyl groups.*
(2) The collection of maps $(\varepsilon_{M,\mu}, \chi_{M,\mu})$ *is stable under the action of the absolute Weyl group of* G *on* Φ_m*-cuspidal pairs.*
(3) If $(M, \mu) = (T, 1)$ *for some* Φ_m*-split maximal torus* T*, then*

$$\mathrm{Deg}_\rho(\zeta_m) = \varepsilon_{T,1}(\rho) \deg \chi_{T,1}(\rho).$$

8.15. Take $\mathbf{G} = \mathrm{GL}_n$ under the standard Frobenius map, so that $G = \mathrm{GL}_n(\mathbb{F}_q)$. Then there is the following dictionary between the notions from §8.1 and the notions we have just introduced.

First, recall that every unipotent irreducible character of $\mathrm{GL}_n(\mathbb{F}_q)$ arises from a principal series attached to an irreducible character of its Weyl group S_n. Restated in the language of Harish-Chandra series: For any maximally split maximal torus $\mathbf{A} \subseteq \mathbf{G}$, we have $\mathrm{Uch}(G)_{A,1} = \mathrm{Uch}(G)$ and $W^G_{A,1} \simeq S_n$. At the same time, $\mathrm{Irr}(S_n)$ is indexed by partitions of size n. Thus, we may regard $\chi_{A,1}$ as a bijection:

$$\mathrm{Uch}(G) \simeq \{\rho \in \Pi \mid |\rho| = n\}. \tag{8.5}$$

Henceforth, we abuse notation by writing ρ to denote both characters of G and the corresponding partitions.

Next, for any $m \geq 1$, it turns out that every Φ_m-split Levi subgroup of $\mathbf{G}$ takes the form $\mathbf{M} \simeq \mathrm{GL}_{n-mN} \times \mathbf{S}^N_m$, where $S \simeq \mathbf{F}^\times_{q^m}$. For such $\mathbf{M}$, the analogue of (8.5) says

$$\{m\text{-cuspidals } \mu \in \mathrm{Uch}(M)\} \simeq \{\mu \in \Pi_{m\text{-}cor} \mid |\mu| = n - mN\}.$$

Henceforth, we abuse notation by writing μ to denote both cuspidal characters of M and the corresponding m-core partitions. For a fixed μ, (8.5) restricts to a bijection

$$\mathrm{Uch}(G)_{M,\mu} \simeq \{\rho \in \Pi \mid |\rho| = n \text{ and } m\text{-core}(\rho) = \mu\}. \tag{8.6}$$

At the same time, it turns out that $W^G_{M,\mu} \simeq W^G_M \simeq S_{N,m}$, giving a bijection

$$\mathrm{Irr}(W^G_{M,\mu}) \simeq \{\vec{\pi} \in \Pi^m \mid |\vec{\pi}| = N\}.$$

Under these identifications, $\chi_{M,\mu} : \mathrm{Uch}(G)_{M,\mu} \xrightarrow{\sim} \mathrm{Irr}(W^G_{M,\mu})$ is essentially the map sending a partition to its m-quotient. To be precise: $\chi_{M,\mu}(\rho)$ is the "shifted" m-quotient $\Upsilon_m(\rho, \ell_\mu)$, where ℓ_μ denotes the length of μ as a partition.

8.16. We can now provide the explicit definition of the Broué–Malle algebra $H^G_{M,\mu}(x)$ for $G = \mathrm{GL}_n(\mathbf{F}_q)$. Recall that it is a specialization of $H_{W^G_{M,\mu}}(\vec{x})$, where $W^G_{M,\mu} \simeq S_{N,m}$ for some N.

Let $(b^{(0)}_m, b^{(1)}_m, \ldots, b^{(m-1)}_m)$ be the $\mathbf{Z}^m$-component of $\Upsilon_m(\rho, \ell_\mu) \in \Pi^m \times \mathbf{Z}^m$. Then, in the notation of §8.2, the quotient map $H_{S_{N,m}}(\vec{x}) \to H^G_{M,\mu}(x)$ sends

$$x_\sigma \mapsto x^m \quad \text{and} \quad x_{\tau,j} \mapsto x^{a^{(j)}_m(\mu)} \text{ for all } j,$$

where $a^{(j)}_m(\mu) = mb^{(j)}_m(\mu) + j$. Note that we always have $a^{(0)}_m(\mu) = b^{(0)}_m(\mu) = 0$.

8.5. Level-Rank Dualities for Finite Reductive Groups

8.17. Keep $q, \mathbf{G}, G$ as above. We now fix two positive integers l and m. Let (L, λ) be a Φ_l-cuspidal pair and let (M, μ) be a Φ_m-cuspidal pair. Let $\mathrm{Irr}(W^G_{L,\lambda})_{M,\mu}$ and $\mathrm{Irr}(W^G_{M,\mu})_{L,\lambda}$ be the images of the maps

$$\mathrm{Irr}(W^G_{L,\lambda}) \xleftarrow{\chi_{L,\lambda}} \mathrm{Uch}(G)_{L,\lambda} \cap \mathrm{Uch}(G)_{M,\mu} \xrightarrow{\chi_{M,\mu}} \mathrm{Irr}(W^G_{M,\mu}).$$

We define *$H^{G,\circ}_{M,\mu}(\zeta_l)$-blocks* of $\mathrm{Irr}(W^G_{M,\mu})$ by analogy with the $\vec{\xi}$-blocks in §8.2, via an analogous composition of maps

$$\mathrm{Irr}(W^G_{M,\mu}) \to \mathscr{K}^+_0(H^\circ_{W^G_{M,\mu}}(\vec{x})) \xrightarrow{\mathrm{d}_\zeta} \mathscr{K}^+_0(H^G_{M,\mu}(\zeta_l)).$$

Our main conjecture, like Theorem 8.3, has three parts.

Conjecture 8.8. *With the setup above:*

1. *(1)* $\mathrm{Irr}(W^G_{L,\lambda})_{M,\mu}$ *and* $\mathrm{Irr}(W^G_{M,\mu})_{L,\lambda}$ *are unions of* $H^G_{L,\lambda}(\zeta_m)$*-blocks and* $H^G_{M,\mu}(\zeta_l)$*-blocks, respectively.*
2. *(2) The bijection* $\mathrm{Irr}(W^G_{L,\lambda})_{M,\mu} \xrightarrow{\sim} \mathrm{Irr}(W^G_{M,\mu})_{L,\lambda}$ *induced by* $\chi_{L,\lambda}$ *and* $\chi_{M,\mu}$ *respects blocks. That is, it descends to a bijection* $\chi^{L,\lambda}_{M,\mu}$ *of the form*

$$\{H^G_{L,\lambda}(\zeta_m)\text{-blocks in } \mathrm{Irr}(W^G_{L,\lambda})_{M,\mu}\} \xrightarrow{\sim} \{H^G_{M,\mu}(\zeta_l)\text{-blocks in } \mathrm{Irr}(W^G_{M,\mu})_{L,\lambda}\}.$$

(3) *When* $\chi^{L,\lambda}_{M,\mu}(\mathbf{a}) = \mathbf{b}$*, the decomposition matrices of* $\mathbf{a}$ *and* $\mathbf{b}$ *are related by an inversion formula generalizing Uglov's.*

Moreover, if we fix highest-weight covers of the underlying module categories, then the bijection between $\mathbf{a}$ *and* $\mathbf{b}$ *themselves is categorified by a bounded derived equivalence between their highest-weight covers: that is, an equivalence*

$$\mathsf{D}^b(\mathsf{O}_{W^G_{L,\lambda}}(\vec{v}_l)_{\mathbf{a}}) \simeq \mathsf{D}^b(\mathsf{O}_{W^G_{M,\mu}}(\vec{v}_m)_{\mathbf{b}})$$

for any vectors $\vec{v}_l, \vec{v}_m$ *related to* $\mathbf{a}, \mathbf{b}$ *by certain numerical conditions.*

8.18. The proof of the following theorem relies on the abacus combinatorics discussed in §8.1, as well as the classification of $\vec{\xi}$-blocks due to Lyle–Mathas.

Theorem 8.9. *Suppose that* $\mathbf{G} = \mathrm{GL}_n$ *(under the standard Frobenius) and* ℓ, m *are coprime. Then all three parts of Conjecture 8.8 hold. Moreover:*

(1) $\mathrm{Irr}(W^G_{L,\lambda})_{M,\mu}$ *and* $\mathrm{Irr}(W^G_{M,\mu})_{L,\lambda}$ *each consist of a single block.*
(2) *The bijections in part (2) of Conjecture 8.8 are essentially the bijections* Υ^l_m*, in the sense that there is a commutative diagram:*

$$\begin{tikzcd}
\mathrm{Irr}(W^G_{L,\lambda}) \arrow[d] & \mathrm{Uch}(G)_{L,\lambda} \cap \mathrm{Uch}(G)_{M,\mu} \arrow[l, "\chi_{L,\lambda}"'] \arrow[r, "\chi_{M,\mu}"] \arrow[d] & \mathrm{Irr}(W^G_{M,\mu}) \arrow[d] \\
\Pi^l \times \mathbf{Z}^l \arrow[d] & \Pi \arrow[l, "\Upsilon_l(\rho, \ell_\lambda) \mapsfrom \rho"'] \arrow[r, "\rho \mapsto \Upsilon_m(\rho, \ell_\mu)"] & \Pi^m \times \mathbf{Z}^m \arrow[d] \\
\Pi^l \times \mathbf{Z}^l \arrow[rr, "\Upsilon^l_m"] & & \Pi^m \times \mathbf{Z}^m
\end{tikzcd}$$

Above, ℓ_λ, ℓ_μ *refer to the lengths of* λ *and* μ *as partitions. The vertical arrows in the bottom half arise from affine permutations acting on abaci.*

We expect that the coprimality condition can be removed, though in this generality, part (1) of the theorem will no longer hold.

Moreover, we have checked in almost all cases where $\mathbf{G}$ has exceptional type that Conjecture 8.8(1) is compatible with the sizes of the sets $\mathrm{Uch}(G)_{L,\lambda} \cap \mathrm{Uch}(G)_{M,\mu}$ and the sizes of the blocks contained in $\mathrm{Irr}(W^G_{L,\lambda})_{M,\mu}$ and $\mathrm{Irr}(W^G_{M,\mu})_{L,\lambda}$, where these are known or can be deduced.

Example 8.10. Let $G = \mathrm{GL}_8(\mathbf{F}_q)$ and $l = 4$ and $m = 3$. Let $\lambda = (2^2)$, a 4-core, and $\mu = (2)$, a 3-core. (8.6) gives

$$\mathrm{Uch}(G)_{L,\lambda} \xleftrightarrow{\sim} \{(6,2), (5,3), (2^3, 1^2), (2^2, 1^4)\},$$
$$\mathrm{Uch}(G)_{M,\mu} \xleftrightarrow{\sim} \{(8), (5,3), (5,2,1), (5,1^3), (4,3,1), (3^2,1^2), (2^4), (2^2,1^4), (2,1^6)\}.$$

Therefore, $\mathrm{Uch}(G)_{L,\lambda} \cap \mathrm{Uch}(G)_{M,\mu} \xleftrightarrow{\sim} \{(5,3), (2^2,1^4)\}$.

Let $\pi = (5,3)$. The abaci that represent $\Upsilon_l(\pi, \ell_\lambda) = \Upsilon_4((5,3),2)$ and $\Upsilon_m(\pi, \ell_\mu) = \Upsilon_3((5,3),1)$ are below.

$$\begin{array}{cccccc} \cdots & -12 & -8 & -4 & - & - \\ \cdots & -11 & -7 & -3 & - & - \\ \cdots & -10 & -6 & -2 & - & 6 \\ \cdots & -9 & -5 & -1 & 3 & \end{array} \qquad \begin{array}{ccccccc} \cdots & -12 & -9 & -6 & -3 & - & - \\ \cdots & -11 & -8 & -5 & -2 & - & - \\ \cdots & -10 & -7 & -4 & - & 2 & 5 \end{array}$$

Under appropriate affine permutations of the abaci, they become the abaci discussed in Examples 8.1–8.2.

$$\begin{array}{ccccc} \cdots & -12 & -8 & -4 & 0 \\ \cdots & -11 & -7 & -3 & - \\ \cdots & -10 & -6 & -2 & 2 \\ \cdots & -9 & -5 & - & 3 \end{array} \qquad \begin{array}{cccccccc} \cdots & -12 & -9 & -6 & -3 & 0 & - & 6 & 9 \\ \cdots & -11 & -8 & -5 & -2 & - & - & - & \\ \cdots & -10 & -7 & -4 & - & - & - & - & \end{array}$$

We saw in Example 8.2 that Υ_4^3 transforms the left abacus into the right one.

8.19. A particularly exciting aspect that our discussion has omitted: When $\mathbf{L}, \mathbf{M}$ are maximal tori, we expect novel *geometric* incarnations of the duality in Conjecture 8.8, via bimodules constructed from the cohomology of algebraic varieties. They suggest analogies between these varieties and certain F-twisted Steinberg varieties related to Deligne–Lusztig theory, but with roots of unity replacing the prime power q.

One family of incarnations is of Dolbeault nature: The (ind-)varieties are *affine Springer fibers* that have appeared in work of Oblomkov–Yun, Boixeda-Alvarez–Losev–Kivinen, and others. Another family is of Betti nature: The varieties are *braid Steinberg varieties* that first appeared in my work, and more recently, in work of Bezrukavnikov–Boixeda-Alvarez–McBreen–Yun.