9.

Having constructed the HOMFLY-PT link polynomial from Markov traces on the Hecke algebras of the symmetric groups, we analyze general traces from a character-theoretic viewpoint, in the setting of general symmetric algebras. The main reference is Chapter 7 of Geck–Pfeiffer's book.

9.1.

Fix a commutative ring A with unity. Given any A-module E, we write $E^\vee$ to denote the module dual to E.

Fix an associative algebra H over A. Let $Z(H)$ denote its center. We will always assume that $A \subseteq Z(H)$.

For any A-module E, an *E-valued trace* on H is a A-linear map $\tau : H \to E$ such that $\tau(xy) = \tau(yx)$ for all $x, y \in H$.

Example 9.1. Any H-module M that is free of finite rank over A defines an A-valued trace χ_M called its *character*: $\chi_M(x) = \mathrm{tr}_A(x \mid M)$ for all $x \in H$.

Let $[H, H]$ be the additive subgroup of H generated by all commutators $[x, y] := xy - yx$. It is a $Z(H)$-module, hence an A-module. The quotient $H/[H, H]$ is called the *cocenter* of H. By construction, an A-linear map out of H is a trace if and only if it factors through the map $H \to H/[H, H]$, which could be called the *universal trace* on H.

Remark 9.2. Our $[H, H]$ is not the commutator ideal of H, which some texts denote by the same notation. The quotient of H by its commutator ideal is its abelianization, which is usually smaller than its cocenter.

9.2.

Henceforth, we assume that H is free of finite rank as an A-module. We say that an A-valued trace τ on H is *symmetrizing*, and that (H, τ) forms a *symmetric algebra* over A, if and only if the symmetric bilinear pairing

$$\begin{aligned} H \otimes H &\to A \\ (x, y) &\mapsto \tau(xy) \end{aligned} \tag{9.1}$$

is nondegenerate. Explicitly, this means: If $\tau_x \in H^\vee$ denotes the functional

$$\tau_x(y) = \tau(xy),$$

then the map that sends $x \mapsto \tau_x$ is an isomorphism of modules $H \xrightarrow{\sim} H^\vee$.

For convenience, let $\mathcal{T}(H) \subseteq H^\vee$ denote the module of A-valued traces on H. Unwinding the definitions, we see:

Proposition 9.3. *If $\tau : H \to A$ is symmetrizing, then:*

(1) The pairing (9.1) descends to a nondegenerate pairing

$$Z(H) \otimes H/[H, H] \to A.$$

(2) The map $x \mapsto \tau_x$ restricts to an isomorphism of modules $Z(H) \xrightarrow{\sim} \mathcal{T}(H)$.

To describe the inverse to the map in (2), let $(e_i)_i, (f_i)_i$ be ordered A-linear bases for H that are dual under (9.1). This means $\tau(e_i f_j)$ equals 1 when $i = j$ and 0 when $i \neq j$.

Proposition 9.4. *For any $x \in H$, we have $x = \sum_i \tau(xe_i) f_i$.*

Proof. If y denotes the right-hand side, then $\tau_x(e_i) = \tau_y(e_i)$ for all i, whence $\tau_x = \tau_y$, whence $x = y$. □

Corollary 9.5. *The inverse to the map in Proposition 9.3(2) sends a trace χ to the element $z_\chi := \sum_i \chi(e_i) f_i$.*

Observe that for any traces $\chi, \psi \in \mathcal{T}(H)$, we have

$$\psi(z_\chi) = \sum_i \chi(e_i)\psi(f_i) = \sum_i \psi(f_i)\chi(e_i) = \chi(z_\psi).$$

This leads us to consider the symmetric bilinear pairing

$$(-,-)_\tau : \mathcal{T}(H) \otimes \mathcal{T}(H) \to A$$

for which $(\chi, \psi)_\tau$ is the element above. It turns out that we have all seen an example of this pairing before.

Example 9.6. Let Γ be any finite group, and take $H = A\Gamma$, its group algebra over A. Let $e \in \Gamma$ be the identity. Then there is a symmetrizing trace τ on H defined by $\tau(e) = 1$ and $\tau(g) = 0$ for all $g \neq e$. If $(g_i)_i$ is any ordering of the elements of Γ, then $(g_i^{-1})_i$ is the dual ordered basis under (9.1). Therefore,

$$(\chi, \psi)_\tau = \sum_{g \in \Gamma} \chi(g)\psi(g^{-1}).$$

We conclude that when A is a field whose characteristic does not divide $|\Gamma|$, then $(-,-)_\tau$ is a rescaling of the usual pairing $(-,-)_\Gamma$ on class functions on Γ.

9.3.

Based on the last example, we might hope that the representations of symmetric algebras are as clean as those of finite groups. It turns out that if $(-,-)_\tau$ is nondegenerate, then they are, in fact, semisimple.

In what follows, we keep A, H, τ, and the dual (ordered) bases $(e_i)_i, (f_i)_i$ as above. We will write the H-action on H-modules as a right action, both to be consistent with Geck–Pfeiffer and because we will later take H to be an Iwahori–Hecke algebra, which previously acted on $R_{e,1}$ from the right.

To start, there is a version of Weyl's unitarization trick for symmetric algebras: namely, Geck–Pfeiffer Lem. 7.1.10.

Proposition 9.7. *For any H-modules M, M', there is an A-linear map*

$$I = I_{M,M'} : \mathrm{Hom}_A(M, M') \to \mathrm{Hom}_H(M, M').$$

Explicitly, $I(\phi)(m) = \sum_i \phi(m \cdot e_i) \cdot f_i$ for all $m \in M$. Moreover, $I_{M,M'}$ is independent of the choice of $(e_i)_i, (f_i)_i$.

Proof of the first claim. We must show that for all $m \in M$ and $x \in H$, we have $I(\phi)(m \cdot x) = I(\phi)(m) \cdot x$. Let $a_{i,j} \in A$ be the unique scalars such that $xe_i = \sum_j a_{i,j} e_j$ for all i. By Proposition 9.4,

$$f_j x = \sum_i \tau(f_j x e_i) f_i = \sum_{i,k} a_{i,k} \tau(f_j e_k) f_i = \sum_i a_{i,j} f_i \quad \text{for all } j.$$

Therefore,

$$\sum_i \phi(m \cdot xe_i) \cdot f_i = \sum_{i,j} \phi(m \cdot e_j) \cdot a_{i,j} f_i = \sum_j \phi(m \cdot e_j) \cdot f_j x,$$

as desired. □

9.4.

Using the "averaging" operators $I_{M,M'}$, it is possible to generalize much of classical character theory from finite groups to symmetric algebras. To save time, we will omit proofs, merely pointing out the classical parallels. Henceforth:

- We assume that A is an integral domain with field of fractions K. We set $KH = K \otimes_A H$.
- We only consider KH-modules that have finite dimension over K.

Extending τ to a K-valued trace on KH, we see that it defines a symmetrizing trace on KH as well.

We now focus on KH. The following result, Geck–Pfeiffer Lemma 7.1.11, generalizes Maschke's theorem for a finite group Γ, since $I(\mathrm{id}_V) = |\Gamma| \, \mathrm{id}_V$ for any representation V of Γ.

Theorem 9.8 (Gaschütz–Ikeda)**.** *Let V be a KH-module. Then V is projective over KH if and only if $\mathrm{id}_V = I(\phi)$ for some $\phi \in \mathrm{End}_K(V)$.*

Schur's lemma says that if V is a simple KH-module, then $\mathrm{End}_{KH}(V)$ is a division algebra over K. Recall that such a module V is *split* over K if and only if $\mathrm{End}_{KH}(V) \simeq K \, \mathrm{id}_V$. The following result, combining Geck–Pfeiffer Theorem 7.2.1 and Corollary 7.2.2, generalizes Schur orthogonality for matrix coefficients.

Theorem 9.9. *If V is a simple KH-module split over K, then there is a (unique) element $\mathbf{s}_V$ such that*

$$I(\phi) = \mathbf{s}_V \operatorname{tr}(\phi) \operatorname{id}_V \quad \textit{for all } \phi \in \operatorname{End}_K(V).$$

It only depends on the isomorphism class of V as a KH-module.

In particular, if V' is another such KH-module and $\rho : KH \to \operatorname{Mat}_n(K)$, resp. *$\rho' : KH \to \operatorname{Mat}_{n'}(K)$ is the action on V,* resp. *V' in a fixed basis, then*

$$\sum_i \rho(e_i)_{k,l}\rho'(f_i)_{k',l'} = \begin{cases} \mathbf{s}_V & V = V' \textit{ and } (k,l) = (l',k'), \\ 0 & \textit{else}. \end{cases}$$

By Geck–Pfeiffer Exercise 7.4, a simple KH-module V split over K is determined by its character χ_V. The following result, Geck–Pfeiffer Corollary 7.2.4, generalizes Schur orthogonality for characters.

Corollary 9.10. *Let V, V' be simple KH-modules split over K. Then*

$$(\chi_V, \chi_{V'})_\tau = \begin{cases} \mathbf{s}_V \dim(V) & V \simeq V' \textit{ as } KH\textit{-modules}, \\ 0 & \textit{else}. \end{cases}$$

In particular, $\mathbf{s}_V = \frac{1}{\dim(V)} \sum_i \chi_V(e_i)\chi_V(f_i)$.

The following result, combining Geck–Pfeiffer Theorem 7.2.6 and Proposition 7.2.7, describes when KH is semisimple, and recovers Artin–Wedderburn in this case. To state it, recall that KH is *split* over K if and only if every simple KH-module is split over K.

Corollary 9.11. *A simple KH-module V split over K is projective if and only if $\mathbf{s}_V \neq 0$. In particular, if H is split over K, then:*

(1) The following are equivalent:
- *(a) KH is semisimple as an algebra.*
- *(b) $\mathbf{s}_V \neq 0$ for all simple KH-modules V.*
- *(c) The pairing $(-,-)_\tau$ on $\mathcal{T}(KH)$ is nondegenerate.*

(2) In the situation of (1), we have

$$\tau = \sum_V \frac{1}{\mathbf{s}_V}\chi_V \textit{ in } \mathcal{T}(KH),$$

$$1 = \sum_V e_V \textit{ in } KH, \qquad \textit{where } e_V = \frac{1}{\mathbf{s}_V}\sum_i \chi_V(e_i) f_i$$

and the sums run over isomorphism classes of simple KH-modules. The e_V are primitive orthogonal idempotents of KH.

Example 9.12. Fix a finite Coxeter group W. Fix an integral domain A' containing $\mathbf{Z}[q^{-1/2}]$, with field of fractions K_0. Take $A = A'[\mathsf{x}^{\pm 1}]$, so that $K = K'(\mathsf{x})$, and take

$$H = A' \otimes_{\mathbf{Z}} H_W(\mathsf{x}) = A \otimes_{\mathbf{Z}[\mathsf{x}^{\pm 1}]} H_W(\mathsf{x}),$$

where $H_W(\mathsf{x})$ is the Hecke algebra of W over $\mathbf{Z}[\mathsf{x}^{\pm 1}]$. Then H has the A-linear basis $(\sigma_w)_{w \in W}$. There is a symmetrizing trace τ on H defined by $\tau(\sigma_e) = 1$ and $\tau(\sigma_w) = 0$ for all $w \neq e$. Under this trace, $(\sigma_{w^{-1}})_w$ is the ordered basis dual to $(\sigma_w)_w$. We deduce that for any simple KH-module V, we have

$$\mathbf{s}_V = \frac{1}{\dim(V)} \sum_w \chi_V(\sigma_w)\chi_V(\sigma_{w^{-1}}).$$

It remains to describe when KH is semisimple, and in this case, to classify its simple modules.

9.5.

To conclude, we explain the general form of the Tits deformation theorem giving a criterion for: (1) KH_W to be semisimple, and (2) the existence of a bijection between irreducible characters of W and characters of simple modules over KH_W. For this we need some machinery that is usually presented in the setting of modular representation theory.

Let $R(KH)$ be the Grothendieck group of (finite-dimensional) KH-modules, and let $R^+(KH) \subseteq R(KH)$ be the semiring of classes represented by actual, not virtual, modules. There is a map

$$\mathsf{p}_K : R^+(KH) \to (1 + \mathrm{t}K[\mathrm{t}])^H$$

that sends $[V]$ to the collection of characteristic polynomials for elements of H acting on V:

$$\mathsf{p}_K(V) = (\det_K(1 - \mathrm{t}x \mid V))_{x \in H}.$$

Lemma 9.13 (Brauer–Nesbitt)**.** *If the characters χ_V form a linearly independent subset of $\mathcal{T}(KH)$ as we run over simple KH-modules V, then p_H is injective.*

Lemma 9.14. *If A is integrally closed, then p_K factors through* $\mathrm{Maps}(H, A[\mathrm{t}])$.

Let B be another integral domain, say with field of fractions L, and let $\varphi : A \to B$ be a surjective ring homomorphism. Then we can form $BH = B \otimes_A H$ and $LH = L \otimes_B BH$. Let $R(LH), R^+(LH), \mathsf{p}_L$ be defined similarly to $R(KH), R^+(KH), \mathsf{p}_K$.

Theorem 9.15. *Suppose that A is integrally closed and LH is split over L. Then there is a unique additive map $\mathsf{d}_\varphi : R^+(KH) \to R^+(LH)$ such that the following diagram commutes:*

$$\begin{array}{ccc} R^+(KH) & \xrightarrow{\mathsf{p}_K} & (1+\mathsf{t}A[\mathsf{t}])^H \\ {\scriptstyle \mathsf{d}_\varphi}\downarrow & & \downarrow{\scriptstyle \varphi} \\ R^+(LH) & \xrightarrow{\mathsf{p}_L} & (1+\mathsf{t}L[\mathsf{t}])^H \end{array}$$

Explicitly, if $\mathcal{O} \subseteq K$ is a valuation ring such that $\mathcal{O}$, *resp.* its maximal ideal, contains A, *resp.* $\ker(\varphi)$,[1] then there is an embedding of L into the residue field of $\mathcal{O}$. Let k be this residue field. If LH is split over L, then the map $R^+(LH) \to R^+(kH)$ given by extension of scalars is an isomorphism. The map d_φ above sends $[V]$ to the class of the image in k of any H-stable, full $\mathcal{O}$-lattice in V, viewed as an element of $R^+(LH)$.

Theorem 9.16 (Tits Deformation)**.** *In the setup above, suppose furthermore that KH is split and LH is semisimple. Then:*

(1) KH is also semisimple.
(2) d_φ induces a bijection between simple KH-modules up to isomorphism and simple LH-modules up to isomorphism.

Example 9.17. Keep the setup of Example 9.12. Let $B = A'$, so that $L = K'$, and let $\varphi : A \to B$ be the map that sends $\mathsf{x} \mapsto q^{1/2}$.

If A' is integrally closed, then so is A by Gauss's lemma. So if A' is integrally closed and $K'W$ is split semisimple as a K'-algebra, then Tits's deformation theorem applies, giving us a bijection between $\mathrm{Irr}(W)$ and the set of simple KH-modules up to isomorphism.

It turns out that if W is crystallographic, then we can take $K' = \mathbf{Q}$. If W is merely a finite Coxeter group, then we can take K' to be the totally real number field generated by the character values of all characters of W.

[1] Thank-you to Vlad for finding the correct statement here, during the lecture.