Project Summary: Quot Schemes, Link Homology, and q,t-Symmetric Functions

Overview.

Around any singularity of a complex algebraic plane curve, a small-enough 3-sphere intersects the curve in a topological link, well-defined up to isotopy. A decade ago, Oblomkov–Rasmussen–Shende conjectured a remarkable identity relating the Hilbert schemes of the germ of the curve there to the HOMFLYPT/Khovanov–Rozansky (KhR) homology of this link. Recently, the PI has proven results that suggest the ORS conjecture be expanded into several disjoint statements: *(1)* a conjecture relating the Hilbert schemes to other Quot schemes by a motivic change of variables, *(2)* a conjecture relating the latter schemes to KhR and its refinements, and *(3)* a conjecture about the nonabelian Hodge theory of an algebraic variety whose homology explicitly recovers the KhR homology of a link. Conjectures (1)–(2) were proposed in joint work with Kivinen.

The PI will study these conjectures in at least three directions. First, they are most tractable for "toric" singularities and links. Here, the Quot schemes admit affine pavings, with cells indexed by certain points in cocharacter lattices. Then cases of (1) and (2) become doable via the combinatorics of affine hyperplanes and q,t-symmetric functions, respectively, while (3) becomes plausible by analogy with a strategy of Maulik–Shen. Nonetheless these toric cases are enriched by symmetries not present in general: namely, cohomological actions of double affine Hecke algebras (DAHAs). Jointly with Xue, the PI will show that these actions produce bijections between blocks of different DAHA module categories, which generalize bijections of Uglov between bases of higher Fock spaces. Finally, the PI intends to prove (2) in full by matching two incarnations of the Macdonald basis of q,t-symmetric functions: one related to KhR and its construction via character sheaves, the other to Shalika expansions of orbital integrals and their geometrization via affine Springer fibers.

Intellectual Merit.

The merit of this proposal starts from the ORS conjecture itself: It gives an intrinsic, not skein-dependent, meaning to the KhR homology of links of plane curves, and conversely, computable formulas for the point counts of Quot schemes of plane curves over finite fields. The latter also appear in the Langlands program, as generalized orbital integrals for the Lie algebras $\mathfrak{gl}_n(\mathbf{F}_q(\!(t)\!))$. Yet the proposed work would yield many more applications. Conjecture (1) seems to be a new wall-crossing phenomenon for Quot schemes. The DAHA dualities in the project with Xue constrain the Φ-Harish-Chandra theories introduced by Broué–Malle–Michel for finite groups of Lie type, and predict explicit formulas for geometric representations of DAHAs across works of Lusztig–Yun, Oblomkov–Yun, and Vilonen–Xue. The project about Macdonald polynomials would establish two new geometric paradigms for their combinatorics. The varieties in (3) refine the "braid varieties" studied in cluster algebra, encoding all A-degrees of KhR homology and not just the lowest.

Broader Impacts.

The PI has mentored high-school and undergraduate students extensively, both at UChicago and at MIT. Through MIT PRIMES alone, he mentored or co-advised five students on research projects he designed himself. One mentee won Tenth Place in the 2022 Regeneron STS. For the 2023 PROMYS math camp, the PI designed and advised a research project completed by an all-female team of students. Beyond mentoring, the PI cares strongly about accessible exposition. As a graduate student, he coauthored a 21-page chapter of a textbook on Soergel bimodules. As a postdoc, he has given service talks at venues including MIT, Edinburgh, UChicago, and an AIM research community. He has also put online 90 pages of typed notes for an elementary number theory course that he taught in Spring 2023.