

2.

Today we discuss Borel subgroups of reductive groups and the corresponding quotients.

2.1.

We have discussed how an algebraic group G over $\bar{\mathbf{F}}_q$, equipped with a Frobenius map $F : G \rightarrow G$, gives rise to a finite group G^F . When G is reductive, the structure of G , *resp.* G^F , closely resembles that of GL_n , *resp.* $\mathrm{GL}_n(\mathbf{F}_q)$.

Even earlier, we discussed the [Bruhat decomposition](#). For now, let k be an arbitrary algebraically closed field. Let $B \subseteq \mathrm{GL}_n$ be the subgroup of upper-triangular matrices, and for all $w \in S_n$, let $\dot{w} \in \mathrm{GL}_n$ be the permutation matrix of w . The Bruhat decomposition on k -points is

$$\mathrm{GL}_n(k) = \coprod_{w \in S_n} B(k)\dot{w}B(k).$$

Its proof is similar to how we used row reduction to establish the Schubert cell decomposition of any Grassmannian. Namely, it suffices to show:

Theorem 2.1. *The coset space $B(k) \backslash \mathrm{GL}_n(k)$ is the disjoint union of the subsets $B(k) \backslash (B(k)\dot{w}B(k))$ for $w \in S_n$.*

Proof. We can identify cosets of $B(k)$ with (complete) flags in k^n via the map $B(k)g \mapsto \vec{V} \cdot g$, where $\vec{V} = (V_i)_i$ is the standard flag in row notation. The rows of g define an ordered basis $(v_i)_i$ such that $V_i \cdot g = \langle v_1, \dots, v_i \rangle$ for all i .

So apply row reduction to g . The result is an upper-triangular matrix $b \in B(k)$. From the algorithm, we also get a permutation $w^{-1} \in S_n$ that only depends on the flag $\vec{V} \cdot g$: the composition of the row swaps (from the left) used to reduce g to b . We have $\vec{V} \cdot g = \vec{V} \cdot \dot{w}b$. \square

Note that the expression $B\dot{w}B$ can be reduced even further. For instance, we can always take b unipotent in the proof above. Another way to see this: Recall that $B = TU$, where T , *resp.* U is the subgroup of diagonal, *resp.* unipotent matrices, and observe that permutation matrices normalize T , meaning $\dot{w}T = T\dot{w}$ for all w .

Something stronger is true. For any algebraic groups $H \subseteq G$, Milne Prop. 1.83 exhibits an algebraic group $N_G(H)$ such that $N_G(H)(R) = N_{G(R)}(H(R))$ for any k -algebra R . It turns out that the connected components $N_{\mathrm{GL}_n}(T)$ are precisely the cosets $\dot{w}T$ for $w \in S_n$. This suggests how Bruhat decomposition ought to generalize beyond GL_n .

Define the [Weyl group](#) of a maximal torus $T \subseteq G$ to be the normalizer $W = W(G, T) := N_G(T)/T$. Note that for any $w \in W$, and any algebraic subgroup $B \subseteq G$ containing T , the notation wB is unambiguous.

Theorem 2.2. *Suppose that G is a reductive algebraic group. Let $B = T \ltimes U \subseteq G$ be a Borel subgroup, where $U = [B, B]$. Then $B(k) \backslash G(k)$ is the disjoint union of the subsets $B(k) \backslash (B(k)wB(k))$ for $w \in W(G, T)$.*

2.2.

As in the first lecture, we switch notation from left-hand quotients back to right-hand quotients. The set $G(k)/B(k)$ is precisely the set of k -points of the fppf sheaf quotient G/B , essentially because $\text{Spec } k$ has no nontrivial fppf covers. However, we can be much more concrete about spaces like this. The key idea is a representation-theoretic characterization of algebraic subgroups:

Theorem 2.3 (Chevalley). *If G is any affine algebraic group with algebraic subgroup G' , then there exist a (finite-dimensional) representation V of G and a subspace $V' \subseteq V$ such that $G'(k) = \{g \in G(k) \mid gV' \subseteq V'\}$. We can even choose V, V' so that V' is a line.*

Proof. Let I be the kernel of the quotient map $k[G] \rightarrow k[G']$. We can pick a finite generating set for I as an ideal. Then we can pick a finite-dimensional $k[G]$ -comodule $V \subseteq k[G]$ containing these generators. (This is similar to the proof of the linearity of affine algebraic groups, except that here, our representation of G is the comodule itself, not its dual.) To get $V' \subseteq V$, we take $V' = V \cap I$.

If $g \in G'(k)$, then $gI \subseteq I$, so $gV' \subseteq V'$. Conversely, if $gV' \subseteq V'$, then g sends every generator of I to another element of I , but g acts on $k[G]$ by algebra automorphisms, so $gI \subseteq I$ and hence I is also the kernel of the quotient map $k[G] \rightarrow k[G'g]$, which forces $g \in G'$.

Finally, once we have such V, V' , we see that the same characterization of G' holds when we replace V, V' by $\bigwedge^d V, \bigwedge^d V'$, respectively, where $d = \dim(V')$. \square

Corollary 2.4 (Chevalley–Plücker). *If G is a smooth affine algebraic group with algebraic subgroup G' , then there is a locally closed, G -equivariant embedding $G/G' \rightarrow \mathbf{P}V$ for some representation V of G . In particular, G/G' is a quasiprojective variety.*

Proof. Take V, V' as in the theorem, with V' a line. Let $G/G' \rightarrow \mathbf{P}V$ be induced by the map from G onto the orbit of $[V']$. The smoothness of G ensures that the latter is faithfully flat, allowing us to identify the orbit with G/G' . \square

2.3.

An algebraic subgroup $P \subseteq G$ is *parabolic* if and only if G/P is projective, not merely quasiprojective. As it turns out, there is a nice characterization of parabolic subgroups. For the proof of the following fixed-point theorem, see Milne Chapter 17.

Theorem 2.5 (Borel). *If B is a connected, smooth, solvable algebraic group acting on a nonempty proper variety X , then X^B is nonempty.*

Corollary 2.6. *Suppose that G is a smooth affine algebraic group.*

- (1) *If B is a Borel subgroup and P a parabolic subgroup of G , then some $G(k)$ -conjugate of B is contained in P .*
- (2) *Conversely, any algebraic subgroup of G that contains a Borel is parabolic.*

Remark 2.7. In the setup above, G must have *some* Borel subgroup, because G has finite dimension and $\{1\}$ is connected, smooth, and solvable.

Proof. (1): By Borel's theorem, the action of B by left multiplication on G/B must have a fixed point gP , in which case $g^{-1}Bg$ is a Borel contained in P .

(2): Since the image of any proper variety is proper, it suffices to show that if $B \subseteq G$ is a Borel, then G/B is proper. We induct on the dimension of G . Pick a faithful representation V of G . The action of G on $\mathbf{P}V$ must have a closed orbit. The stabilizer of any k -point of this orbit is a parabolic subgroup $P \subseteq G$. By (1), it contains some conjugate of B , and without loss of generality, we may replace B with this conjugate. Two cases: Either P is smaller than G , in which case P/B is proper by the inductive hypothesis, and hence G/B is proper, or else $P = G$, in which case V^G contains a line, and we can replace V with $V/(V^G)$ until we either reach $\{0\}$ or reduce to the previous case. \square

Corollary 2.8. *Any two Borels in a smooth affine algebraic group are conjugate.*

Example 2.9. Any Borel of GL_n is conjugate to the subgroup of upper-triangular matrices. Similarly, any parabolic of GL_n is conjugate to some subgroup of *block* upper-triangular matrices.

Recall that by Milne Thm. 16.27, any two maximal tori in a connected, solvable, affine algebraic group are conjugate. This fact combined with Corollary 2.8 proves the Cartan–Lie–Kolchin theorem stated last time.

Another way to state Corollary 2.8 is: The conjugation action of $G(k)$ on the set of Borel subgroups of G is transitive. Milne Thm. 17.48 shows that the stabilizer of any Borel is itself:

Theorem 2.10. *If B is any Borel subgroup of a connected, smooth, affine algebraic group G , then $N_G(B) = B$. Thus the map $gB \mapsto gBg^{-1}$ is a bijection from $(G/B)(k)$ to the set of Borel subgroups of G .*

When we regard G/B as the variety of Borel subgroups of G , we will call it the *flag variety* and denote it by \mathcal{B} . Indeed, for $G = \mathrm{GL}_n$, the above theorem follows from Corollary 2.8 together with the fact that if B is the stabilizer of a flag \vec{V} , then gBg^{-1} is the stabilizer of $g \cdot \vec{V}$.

2.4.

The orbit decomposition of G/B under the left action of B gives rise, on k -points, to the Bruhat decomposition that we discussed at the start. Note that we have not yet proven the precise decomposition for general G . We can sketch the gist modulo the following result. For $G = \mathrm{GL}_n$, it is a byproduct of the argument that proves Jordan–Hölder, via the flag interpretation of $\mathcal{B}(k)$.¹

Theorem 2.11. *If G is reductive, then any two Borel subgroups of G contain a common maximal torus of G .*

Sketch of Bruhat decomposition. We exhibit a map from $B(k) \backslash G(k) / B(k)$ to the Weyl group $W = W(G, T)$. For any $g \in G(k)$, pick a maximal torus $S \subseteq B \cap gBg^{-1}$. By Cartan–Lie–Kolchin, we can write

$$S = bTb^{-1} = (gb'g^{-1})(gTg^{-1})(gb'g^{-1})^{-1}$$

for some $b, b' \in B(k)$. But then $b^{-1}gb'$ normalizes T , so we obtain an element $[b^{-1}gb'] \in W$. One has to check that this element only depends on BgB . \square

2.5.

Henceforth, $k = \bar{\mathbf{F}}_q$. In what follows, recall that we often write X^F in place of $X^F(k)$ when F is a (relative) Frobenius map on X .

Any Frobenius map $F : G \rightarrow G$ that respects the group law and stabilizes an algebraic subgroup $H \subseteq G$ induces an analogous map $F : G/H \rightarrow G/H$. The identification $(G/H)(k) = G(k)/H(k)$ induces an identification

$$(G/H)^F \{F\text{-stable orbits of } H(k) \text{ on } G(k)\}.$$

The action of $G(k)$ on $(G/H)(k)$ restricts to an action of G^F on $(G/H)^F$.

Consider the *standard Frobenius map* $F : \mathrm{GL}_n \rightarrow \mathrm{GL}_n$ given by raising each matrix coordinate to the q th power, so that GL_n^F is the group classically denoted $\mathrm{GL}_n(\mathbf{F}_q)$. Then F stabilizes B and fixes \dot{w} for all w . Hence the Bruhat decomposition of $\mathrm{GL}_n(k)$ into double cosets of $B(k)$ implies an analogous decomposition of GL_n^F into double cosets of B^F . With more work,² one can further show that $((B\dot{w}B)/B)^F = (B^F\dot{w}B^F)/B^F$, and hence, $(\mathrm{GL}_n/B)^F = \mathrm{GL}_n^F/B^F$. What happens for general G, B, F ?

It turns out that the connectedness of B ensures that $(G/B)^F = G^F/B^F$ holds for any F -stable Borel B . On Problem Set 1, you will use the theorem below to deduce (a version of) the first corollary following it:

¹Here I borrow from the answers to <https://mathoverflow.net/q/15438>.

²... than I originally thought was necessary, during the lecture...

Theorem 2.12 (Lang). *Let H be a connected, smooth algebraic group over k and $F : H \rightarrow H$ the Frobenius map for some \mathbf{F}_q -form. Then the [Lang map](#)*

$$h \mapsto h^{-1} F(h) : H \rightarrow H$$

is surjective.

The proof of Lang's theorem is given on Wikipedia. The key idea is to calculate the induced map on Lie algebras, using the fact that the differential of F vanishes to show bijectivity.

Remark 2.13. Note that the Lang map is finite étale, and its fiber over the identity is precisely H^F . For this reason, one can think of the theorem as presenting H as an H^F -principal bundle over itself in the étale topology. This leads to bizarre topological conclusions for, say, $H = \mathbf{G}_a$ and $F(x) = x^q$.

Remark 2.14. In the affine case, Steinberg generalized Lang's theorem from Frobenius maps to any surjective map F with finitely many fixed points. So I sometimes speak of the Lang–Steinberg theorem even where it is overkill.

Corollary 2.15. *Let G be a connected, smooth algebraic group over k with a Frobenius map $F : G \rightarrow G$. Let \mathcal{X} be a set with a $G(k)$ -action and a map $f : \mathcal{X} \rightarrow \mathcal{X}$ such that $f(g \cdot x) = F(g) \cdot f(x)$ for all $g \in G(k)$ and $x \in \mathcal{X}$.³ Then:*

- (1) *Every f -stable $G(k)$ -orbit on \mathcal{X} contains an f -fixed point.*
- (2) *If $\mathcal{X} = G(k)/H(k)$ for some F -stable $H \subseteq G$, and f is induced by F , then $\mathcal{X}^f = G^F/H^F$.*

Corollary 2.16. *A connected, smooth affine algebraic group with Frobenius map F always contains an F -stable Borel pair. In particular, any F -stable Borel contains an F -stable maximal torus.*

Proof. Take \mathcal{X} to be the set of all Borel pairs, and $f : \mathcal{X} \rightarrow \mathcal{X}$ to be defined by $f(B, T) = (F(B), F(T))$. Now Cartan–Lie–Kolchin implies the first statement. Replacing the ambient group with a given F -stable Borel, we deduce the second statement. \square

Remark 2.17. In the setting of a more general field $k = \bar{K}$, a K -form of an algebraic group G is called [quasi-split](#) if and only if there is a Borel subgroup of G that descends to the K -form. In this language, the last result essentially says that \mathbf{F}_q -forms are always quasi-split.

However, a given F -stable maximal torus need not be contained in an F -stable Borel. If it is contained in such a Borel, then it is called [F-maximally split](#).

³This setup generalizes the notion of compatible Frobenius maps defined earlier.

Example 2.18. Let F be the standard Frobenius map on GL_2 . Then the diagonal torus of GL_2 is F -maximally split. At the same time, there is a different F -stable maximal torus $T \subseteq \mathrm{GL}_2$ defined on k -points by

$$T(k) = \left\{ \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \mid a^2 + b^2 \neq 0 \right\}.$$

It descends to an \mathbf{F}_q -form T_1 with the same matrix presentation. If T were contained in an F -stable Borel, then that Borel would have an \mathbf{F}_q -form containing T_1 , which we can rule out by computation.

Note that the \mathbf{F}_q -form of \mathbf{G}_m called $\mathrm{U}(1)$ embeds into T_1 . However, the Frobenius map on \mathbf{G}_m coming from $\mathrm{U}(1)$ is not related to the F above.

Corollary 2.19. *Any two F -stable Borel subgroups of a connected, smooth affine algebraic group G are conjugate under G^F , not just under $G(k)$.*

Proof. Pick an F -stable Borel B . The isomorphism $gB \mapsto gBg^{-1} : G/B \rightarrow G/B$ is F -equivariant. Now use $(G/B)^F = G^F/B^F$. \square

2.6.

We now focus on a connected, smooth affine algebraic group G over $\bar{\mathbf{F}}_q$ with a Frobenius map F corresponding to an \mathbf{F}_q -form and an F -stable Borel pair (B, T) . Recall that $W = N_G(T)/T$. We claim that:

Proposition 2.20. *If $H \subseteq G$ is an F -stable, smooth algebraic subgroup and $N_G(H)/H$ is finite, then $N_G(H)$ is also F -stable. If H is moreover connected, then $N_G(H)^F = N_{G^F}(H^F)$.*

Proof. First, the hypotheses imply that $N := N_G(H)$ is smooth, and hence reduced. So the kernel of the quotient map $k[G] \rightarrow k[N]$ equals its radical. So by the Nullstellensatz, it is determined by $N(k) = \mathrm{Hom}(k[N], k)$ as a subset of $G(k) = \mathrm{Hom}(k[G], k)$ (where these are Hom spaces in the category of k -algebras). We deduce that if $N(k)$ is F -stable, then N is too.

Next, F must induce a self-bijection of $H(k)$. So for all $n \in N(k)$ and $h \in H(k)$, we see that $F(n) \cdot h = F(n \cdot F^{-1}(h)) \in F(H(k)) = H(k)$, where \cdot here denotes the conjugation action. Hence $F(n) \in N(k)$, as needed.

The inclusion $N_G(H)^F \subseteq N_{G^F}(H^F)$ follows from the definitions. For the converse, we must show that if an element of G^F normalizes H^F , then it normalizes H . Rewrite G^F, H^F in terms of the \mathbf{F}_q -points of the corresponding \mathbf{F}_q -forms, then use an argument similar to that in the first paragraph. \square

Corollary 2.21. *In our setup, if G is reductive, then $N_G(T)$ is F -stable. In particular, the F -action on $N_G(T)$ descends to W .*

By Corollary 2.15, we have $W^F = N_G(T)^F / T^F = N_{G^F}(T^F) / T^F$. The Bruhat decomposition of $G(k)$ restricts to

$$G^F = \coprod_{w \in W^F} B^F w B^F.$$

We say that (G, T) is *split* under F if and only if F acts trivially on W .

Example 2.22. Let $(-)^{\tau} : \mathrm{GL}_n \rightarrow \mathrm{GL}_n$ be the “anti-transpose” given by $g^{\tau} = J g^t J$, where g^t is the usual transpose and $J \in \mathrm{GL}_n(k)$ is the matrix with 1’s along the anti-diagonal and 0’s everywhere else. Then

$$F'(g) := (g^{\tau})^{-q} = (g^{-q})^{\tau}$$

is a Frobenius map that differs from the standard one when $n \geq 3$, or when q is odd and $n = 2$. It defines the \mathbf{F}_q -form of GL_n promised earlier, $\mathrm{GU}(n)$. For $n \geq 3$, we can check that F' acts on $W = S_n$ nontrivially.

2.7.

As usual, let $U = [B, B]$. Since B is F -stable, so is U . The G^F -action on G^F / U^F defines a representation of G^F on

$$I = \mathrm{Ind}_{U^F}^{G^F}(1) := \{\mathbf{C}\text{-valued functions on the finite set } G^F / U^F\}.$$

The G^F -stable summands of I are called the *principal series representations* of G^F . Some standard theory shows that $I \simeq \bigoplus_{\theta} I_{\theta}$ as a representation, where the sum runs over characters $\theta : B^F \rightarrow \mathbf{C}^{\times}$ that factor through $T^F \simeq B^F / U^F$, and

$$I_{\theta} = \mathrm{Ind}_{B^F}^{G^F}(\theta)$$

for all θ . To determine the summands of I_{θ} , we analyze $\mathrm{End}_{G^F}(I_{\theta})$.

Here is a very general principle. Suppose that Γ is a finite group and Ξ a finite set with a Γ -action. Let $\mathbf{C}\Xi$ be the representation of Γ formed by the \mathbf{C} -valued functions on Ξ under $[g \cdot f](-) = f(g^{-1} \cdot -)$. Let Γ act on $\Xi \times \Xi$ diagonally, and endow $\mathbf{C}(\Xi \times \Xi)$ with the *convolution* product

$$(f_1 * f_2)(x, y) = \sum_{z \in \Xi} f_1(x, z) f_2(z, y).$$

Note that $\mathbf{C}(\Xi \times \Xi)^{\Gamma}$ forms a subalgebra of $\mathbf{C}(\Xi \times \Xi)$.

Proposition 2.23. *There is an isomorphism of \mathbf{C} -algebras*

$$\mathbf{C}(\Xi \times \Xi)^{\Gamma} \xrightarrow{\sim} \mathrm{End}_{\Gamma}(\mathbf{C}\Xi),$$

$$1_O \mapsto \left(1_x \mapsto \sum_{\substack{y \in \Xi \\ (x, y) \in O}} 1_y \right),$$

where O denotes any Γ -orbit of $\Xi \times \Xi$, and $1_O, 1_x$ refer to indicator functions on $O, \{x\}$.

Above, the image of 1_O in $\text{End}_\Gamma(\mathbf{C}\Xi)$ is called the *Hecke operator* for O . In the case where $\Xi = \Gamma/H$ for some subgroup $H \subseteq \Gamma$, we have a further bijection

$$\begin{aligned} \Gamma \backslash (\Gamma/H \times \Gamma/H) &\xrightarrow{\sim} H \backslash \Gamma/H, \\ (yH, xH) &\mapsto Hy^{-1}xH, \end{aligned}$$

which induces an isomorphism of vector spaces $\mathbf{C}(\Xi \times \Xi)^\Gamma \simeq (\mathbf{C}\Gamma)^{H \times H}$. Taking $\Gamma = G^F$ and $H = U^F, B^F$, we deduce:

Corollary 2.24. *As a vector space, $\text{End}_{G^F}(I)$, resp. $\text{End}_{G^F}(I(1))$, has a basis indexed by $U^F \backslash G^F / U^F$, resp. $B^F \backslash G^F / B^F$. In particular, the latter is also indexed by W^F .*

The arguments above can be pushed further to analyze $\text{Hom}_{G^F}(I_\chi, I_\psi)$ for any χ, ψ . Returning to the abstract setup, let A, B be subgroups of Γ , and let α , resp. β , be an arbitrary \mathbf{C} -valued character of A , resp. B . For all $g \in \Gamma$, we set $A^g = g^{-1}Ag$, so that $\alpha^g(-) := \alpha(g(-)g^{-1})$ is a \mathbf{C} -valued character of A^g .

The following is proved via Frobenius reciprocity in most texts on character theory, such as Serre's book:

Theorem 2.25 (Mackey). *Above, there is an isomorphism of vector spaces*

$$\text{Hom}_\Gamma(\text{Ind}_A^\Gamma(\alpha), \text{Ind}_B^\Gamma(\beta)) \simeq \bigoplus_{g \in A \backslash \Gamma / B} \text{Hom}_{A^g \cap B}(\alpha^g, \beta).$$

Corollary 2.26. *We have*

$$\text{Hom}_{G^F}(I_\theta, I_\eta) \simeq \bigoplus_{w \in W^F} \text{Hom}_{(B^F)^w \cap B^F}(\theta^w, \eta).$$

In particular, I_θ is irreducible if and only if $\theta^w \neq \theta$ as characters of T^F for all $w \neq e$.