Last time

Thm if $f : X \to Y$ and $g : Y \to X$ form a homotopy equivalence

then $f_* : \pi_1(X, x) \to \pi_1(Y, f(x))$
$g_* : \pi_1(Y, y) \to \pi_1(Y, g(y))$

are isomorphisms for all x in X and y in Y

easier special cases:

Ex for any X and x,

$(id_X)_* = id_{\pi_1(X, x)}$
as maps $\pi_1(X, x) \to \pi_1(X, x)$

Ex if $f : X \to Y$ is a homeo
then $f_* : \pi_1(X, x) \to \pi_1(Y, f(x))$ is an isomorphism

Lem 1 given $f : X \to Y$,
$g : Y \to Z$,

$(g \circ f)_* = g_* \circ f_*$
as maps $\pi_1(X, x) \to \pi_1(Z, g(f(x)))$

so if f and g are two-sided inverses of each other,

$g_* \circ f_* = (id_X)_* = id_{\pi_1(X, x)}$
$f_* \circ g_* = (id_Y)_* = id_{\pi_1(Y, f(x))}$

[Pf of Thm — will show that f_* is an iso, as argument for g is similar

by Lem 1,
$$g_* \circ f_* = (g \circ f)_*$$
$$f_* \circ g_* = (f \circ g)_*$$

Lem 2 — if $s : G \to H$ and $r : H \to K$ s.t. $r \circ s$ is bijective, then s is injective and r is surjective

so to show f_*, g_* bijective,
enough to show $(g \circ f)_*$ and $(f \circ g)_*$ bijective

will show that $(g \circ f)_*$ is bijective
as argument for $f \circ g$ is similar

pick homotopy φ from $g \circ f$ to id_X

Lem 3 — if α is a path in X from x to x'
then iso $\hat{\alpha} : \pi_1(X, x) \to \pi_1(X, x')$
def by $\hat{\alpha}([\gamma]) = [\bar{\alpha} * \gamma * \alpha]$

further: if $F, G : A \to X$ are cts,
φ is a homotopy from F to G,
a in A,

then $\alpha_\varphi = \varphi(a, -)$ is a path from $F(a)$ to $G(a)$ s.t.

$$G_* = \hat{\alpha_\varphi} \circ F_*$$

now: $(g \circ f)_* = \hat{\alpha_\varphi} \circ (\mathrm{id}_X)_* = \hat{\alpha_\varphi}$
so by Lem 3, $(g \circ f)_*$ is an iso]

(Munkres §59) [more examples of π_1's:]

Thm $\pi_1(S^n)$ is trivial for all $n \geq 2$

Df in general, a space is simply-connected iff it is path-connected with trivial π_1

"Pf of Thm" given x in S^n and a loop γ at x,
pick p not in the image of γ
$S^n - \{p\}$ is homeomorphic to R^n

R^n is simply-connected
i.e., any loop in R^n is $\sim_p$ constant loop
so $\gamma \sim_p$ constant loop at x within $S^n - \{p\}$
so $\gamma \sim_p$ constant loop at x within S^n

but the map $\gamma : [0, 1]$ to X could be surjective!

to fix: show that $\gamma \sim_p$ some non-surjective loop [somewhat hard]

Thm for any x in X and y in Y,

$$\pi_1(X \times Y, (x, y)) \simeq \pi_1(X, x) \times \pi_1(Y, y)$$

Cor $\pi_1(S^1 \times S^1) \simeq Z \times Z$ under $+$

(Munkres §70) by contrast:

$\pi_1(\text{figure-eight}) \simeq Z * Z$ [free group]

special cases of the Seifert–van Kampen Thm

Thm (Seifert–van Kampen)

given open A, U_1, U_2 sub X and x in A s.t.

$X = U_1 \cup U_2$,

$A = U_1 \cap U_2$

A, U_1, U_2 are all path-connected,

with inclusion maps

$$\begin{array}{ccccccc} & i_1 & & U_1 & & j_1 & & \\ A & & & & & & X & : \\ & i_2 & & U_2 & & j_2 & & \end{array}$$

1) $j_{1,*}$, $j_{2,*}$ induce a surjective hom

$$\pi_1(U_1, x) * \pi_1(U_2, x) \to \pi_1(X, x)$$

2) via this hom, $\pi_1(X, x)$ is the largest quotient of $\pi_1(U_1, x) * \pi_1(U_2, x)$ in which

$$i_{1,*}([\gamma]) \sim i_{2,*}([\gamma]) \text{ for all } [\gamma] \text{ in } \pi_1(A, x)$$

Ex X the figure-eight,
U_1, U_2 open thickenings of the S^1's,
x the intersection point

[draw]

since $\pi_1(A, x)$ is trivial:

$$\begin{aligned} \pi_1(X, x) &\simeq \pi_1(U_1, x) * \pi_1(U_2, x) \\ &\simeq \pi_1(S^1, x) * \pi_1(S^1, x) \\ &\simeq \mathbb{Z} * \mathbb{Z} \end{aligned}$$

Ex $X = S^2$,
U_1, U_2 open thickenings of opposed hemispheres,

[draw]

here, $\pi_1(A, x) \simeq \pi_1(S^1, x) \simeq Z$
but $\pi_1(U_1, x)$, $\pi_1(U_2, x)$ trivial
so $\pi_1(X, x)$ also trivial