Affine Springer Fibers and Level-Rank Duality

Minh-Tâm Quang Trinh

Yale University

1 Springer Theory

2 Deligne–Lusztig Theory

3 Level-Rank Duality

Mainly about joint work with Ting Xue:

arXiv:2311.17106

See also the extended abstract on my website, which we have submitted to FPSAC ’25.

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Work over $\mathbf{C}$.

$\mathbf{G}$	connected reductive group
$\mathbf{A}$	maximal torus
W	Weyl group

The *rational Cherednik algebra* D_c^{rat} is a deformation of $\mathbf{C}W \ltimes \mathcal{D}(\mathbf{a})$ depending on a parameter $c \in \mathbf{C}$.

D_c^{rat}	$\mathrm{U}\mathbf{g}$
$\mathbf{C}[\mathbf{a}] \otimes \mathbf{C}W \otimes \mathbf{C}[\mathbf{a}^*]$	$\mathrm{U}\mathbf{n}_- \otimes \mathrm{U}\mathbf{a} \otimes \mathrm{U}\mathbf{n}_+$
$\Delta_c(\chi)$	$\Delta(\lambda)$
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For c rational and *positive*, D_c^{rat}-modules from the geometry of *affine Springer fibers*.

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$\mathbf{I} \subseteq \mathbf{G}[\![z]\!]$	Iwahori lifting $\mathbf{B} \subseteq \mathbf{G}$

The affine Springer fiber over $\gamma \in \mathbf{g}(\!(z)\!)$ is

$$\mathcal{F}l_\gamma = \{g\mathbf{I} \in \mathbf{G}(\!(z)\!)/\mathbf{I} \mid \gamma \in \mathrm{Lie}(g\mathbf{I}g^{-1})\}.$$

Note that $\mathbf{G}(\!(z)\!)/\mathbf{I}$ is infinite-dimensional.

We say that γ is *regular semisimple* iff $\mathbf{G}(\!(z)\!)_\gamma^\circ$ is a maximal torus.

Here $\mathcal{F}l_\gamma$ is finite-dimensional!

But it varies wildly over $\mathbf{g}(\!(z)\!)^{\mathrm{rs}} \subseteq \mathbf{g}(\!(z)\!)$.

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Fix rational $c = \frac{d}{m} > 0$ in lowest terms.

Let $\mathbf{C}^\times \curvearrowright \mathbf{G}(\!(z)\!)$ according to

$$\boxed{c \cdot g(z) = \mathrm{Ad}(c^{d\rho^\vee})g(c^m z).} \qquad \left(\rho^\vee = \textstyle\sum_\alpha \omega_\alpha^\vee\right)$$

(Oblomkov–Yun) $\mathcal{F}l_\gamma$ is locally constant over

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Example Take $\mathbf{G} = \mathbf{SL}_2$ and $\mathbf{B}$ upper-triangular.

Then $\begin{pmatrix} 1 & \\ & -1 \end{pmatrix}, \begin{pmatrix} & 1 \\ z & \end{pmatrix}, \begin{pmatrix} z & \\ & -z \end{pmatrix}$ have slopes $0, \frac{1}{2}, 1$.

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- A *perverse filtration* $\mathsf{P}_{\leq *}$ on $\mathrm{H}^*_{\mathbf{C}^\times}(\mathcal{F}l_\gamma)$.

It arises from a Ngô-type global model.

- An action of $D^{\mathrm{rat}}_{d/m}$ on

$$\mathcal{E}_\gamma := \mathrm{gr}^{\mathsf{P}}_* \mathrm{H}^*_{\mathbf{C}^\times}(\mathcal{F}l_\gamma)^{\pi_0(\mathbf{G}_{0,\gamma})}|_{\epsilon \to 1},$$

where $\mathbf{G}_0 = (\mathbf{G}(\!(z)\!)^{\mathbf{C}^\times})^\circ$ and $\epsilon \in \mathrm{H}^2_{\mathbf{C}^\times}(\mathit{point})$.

As a module, $\mathcal{E}_\gamma$ contains $L_{d/m}(\chi_{\mathsf{triv}})$.

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Problem Give a formula for $D^{\mathrm{rat}}_{d/m} \curvearrowright \mathcal{E}_\gamma$ in general.

In practice, too hard. Replace with

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Idea $D^{\mathrm{rat}}_{d/m}$ commutes with monodromy of $\mathcal{E}_\gamma$ over

$$\mathbf{c}^{\mathrm{rs}}_{d/m} \subseteq \mathbf{g}^{\mathrm{rs}}_{d/m},$$

a Kostant-type transverse slice to $\mathbf{G}_0 \curvearrowright \mathbf{g}^{\mathrm{rs}}_{d/m}$.

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Deligne–Lusztig studied groups over *finite fields*. But up to Tate twist,

$$\mathrm{Gal}(\bar{\mathbf{F}}_q|\mathbf{F}_q) \quad \simeq \quad \hat{\mathbf{Z}} \quad \simeq \quad \mathrm{Gal}(\overline{\mathbf{C}(\!(z)\!)}|\mathbf{C}(\!(z)\!)).$$

Forms of $\mathbf{G}$ are classified by Dynkin automorphisms in the same way over $\mathbf{F}_q$ as over $\mathbf{C}(\!(z)\!)$.

Much of Oblomkov–Yun's setup generalizes from $\mathbf{G}$ to any of its forms $\mathbf{G}_{\mathbf{C}(\!(z)\!)}$.

The tori $\mathbf{A}, \mathbf{G}_\gamma$ generalize to forms $\mathbf{A}_{\mathbf{C}(\!(z)\!)}, \mathbf{G}_{\mathbf{C}(\!(z)\!),\gamma}$.

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2 Deligne–Lusztig Theory

Work over $\bar{\mathbf{F}}_q$ for good q.

$$\{\text{forms of } \mathbf{G} \text{ over } \mathbf{F}_q\} \quad \leftrightarrow \quad \{\text{Frobenii } F \curvearrowright \mathbf{G}\}$$

We say that $G = \mathbf{G}^F$ is a *finite group of Lie type*.

F-stable Levis $\mathbf{L} \subseteq \mathbf{G}$ correspond to Levis $L \subseteq G$.

Deligne–Lusztig introduced varieties† $Y_{\mathbf{L}}^{\mathbf{G}}$ such that

$$G \quad \curvearrowright \quad \mathrm{H}_c^*(Y_{\mathbf{L}}^{\mathbf{G}}) \quad \curvearrowleft \quad L.$$

Induction map $R_L^G : \mathrm{K}_0(L) \to \mathrm{K}_0(G)$:

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(Broué–Malle) For *m-regular* maximal tori $\mathbf{T}$, a specific algebra $H_T^G(\mathsf{q})$ such that

$$H_T^G(\zeta_m) = \bar{\mathbf{Q}} W_T^G, \quad \text{where } W_T^G = N_G(T)/T.$$

They conjecture:

1. $H_T^G(q) \otimes \bar{\mathbf{Q}}_\ell \simeq \mathrm{End}_G(\mathrm{H}_c^*(Y_{\mathbf{T}}^{\mathbf{G}})[1_T])$.
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Back to Springer. $(\mathbf{A}_{\mathbf{F}_q}, \mathbf{T}_{\mathbf{F}_q} \leftrightarrow \mathbf{A}_{\mathbf{C}((z))}, \mathbf{G}_{\mathbf{C}((z)),\gamma})$

It turns out that $\mathbf{A}$ and $\mathbf{T}$ are 1- and m-regular.

Moreover, $\pi_1(\mathbf{c}_{d/m}^{\mathrm{rs}})$ is the braid group of W_T^G.

Conjecture (T–Xue)

1. $\pi_1(\mathbf{c}_{d/m}^{\mathrm{rs}}) \curvearrowright \mathcal{E}_\gamma$ factors through $H_T^G(1)$.
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Theorem (T–Xue) True in these cases:

- m is the (twisted) Coxeter number of $\mathbf{G}_{\mathbf{C}(\!(z)\!)}$.
- $(\mathbf{G}_{\mathbf{C}(\!(z)\!)}, m) = ({}^2A_2, 2), (C_2, 2), (G_2, 3), (G_2, 2)$.

Under a conjecture of OY, true in further cases.

Example Take $\mathbf{G}_{\mathbf{C}(\!(z)\!)}$ split, m its Coxeter number.

$\chi_{A,\rho}$ runs over characters $\chi_{\wedge^k(\mathbf{a})}$ of W_A^G.

$\chi_{T,\rho}$ runs over *all* characters of $W_T^G = \mathbf{Z}/m\mathbf{Z}$.

In $\mathrm{K}_0(D_{d/m}^{\mathrm{rat}})$,

$$[E_\gamma] = \sum\nolimits_k (-1)^k [\Delta_{d/m}(\chi_{\wedge^k(\mathbf{a})})]$$
$$= [L_{d/m}(\chi_{\mathsf{triv}})].$$

Cf. the BGG resolution of Berest–Etingof–Ginzburg.

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$$\begin{aligned}[E_\gamma] &= \textstyle\sum_k (-1)^k [\Delta_{d/m}(\chi_{\wedge^k(\mathbf{a})})] \\ &= [L_{d/m}(\chi_{\mathsf{triv}})].\end{aligned}$$

Cf. the BGG resolution of Berest–Etingof–Ginzburg.

Theorem (T–Xue) True in these cases:

- m is the (twisted) Coxeter number of $\mathbf{G}_{\mathbf{C}((z))}$.
- $(\mathbf{G}_{\mathbf{C}((z))}, m) = ({}^2A_2, 2), (C_2, 2), (G_2, 3), (G_2, 2)$.

Under a conjecture of OY, true in further cases.

Example Take $\mathbf{G}_{\mathbf{C}((z))}$ split, m its Coxeter number.

$\chi_{A,\rho}$ runs over characters $\chi_{\wedge^k(\mathbf{a})}$ of W_A^G.

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Compare E_γ given by

$$\sum_\rho \varepsilon_{T,\rho}(\Delta_{d/m}(\chi_{A,\rho}) \otimes \chi_{T,\rho,1})$$

with $R_A^G(1_A) \otimes_{\bar{\mathbf{Q}}_\ell G} R_T^G(1_T)$ given by

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$$\mathsf{KZ} : \mathsf{Rep}(D^{\mathrm{rat}}_{d/m}) \to \mathsf{Rep}(H_A^G(\zeta_m))$$

sends $\mathsf{KZ}(\Delta_{d/m}(\chi)) = \chi_{\zeta_m}$. Thus an analogy:

$$\boxed{\mathbf{F}_q \ : \ (q, q) \ :: \ \mathbf{C}((z)) \ : \ (\zeta_m, 1)}$$

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Let $\mathrm{Uch}(G)$ be the set of *unipotent* irreps of G, which occur in $R_T^G(1_T)$ for some maximal torus $\mathbf{T}$.

(Broué–Malle–Michel) Fix a positive integer l.

- $\mathbf{L} \subseteq \mathbf{G}$ is *l-split* iff $\mathbf{L} = Z_{\mathbf{G}}(\mathbf{S})^\circ$, where

  $\mathbf{S}$ is a torus with $|S|$ a power of $\Phi_l(q)$.

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As we run over pairs $(\mathbf{L}, \lambda)$ up to conjugacy,

$$\mathrm{Uch}(G) = \coprod \mathrm{Uch}(G)_{\mathbf{L},\lambda},$$

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For $l = 1$, these are classical *Harish-Chandra series*.

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Generalizing our discussion for maximal tori:

Broué–Malle define a Hecke algebra $H_{L,\lambda}^G(\mathsf{q})$ such that

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They conjecture:

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$$\chi \mapsto \chi_{\zeta_m} : \mathrm{Irr}(W^G_{L,\lambda}) \to \mathrm{K}_0(H^G_{L,\lambda}(\zeta_m)),$$

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Conjecture (T–Xue) Fix l, m.

Fix an l-cuspidal $(\mathbf{L}, \lambda)$ and m-cuspidal $(\mathbf{M}, \mu)$.

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$$\mathsf{Rep}(H_{L,\lambda}^{\mathrm{GL}_n}(\zeta_m)) \quad \text{and} \quad \mathsf{Rep}(H_{M\mu}^{\mathrm{GL}_n}(\zeta_l))$$

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Thank you for listening.