Warmup suppose V, W are finite-dim'l

$$\mathrm{Hom}(V, W) = \{\text{linear maps } V \text{ to } W\}$$

Q1 dim Hom(V, W) in terms of dim V, dim W?

A1 $\dim \mathrm{Hom}(V, W) = (\dim V)(\dim W)$

Q2 basis for Hom in terms of bases for V, W?

A2 pick ordered bases $(v_1, \ldots, v_n)$ for V
$(w_1, \ldots, w_m)$ for W

linear iso $\mathrm{Hom}(V, W)$ to $\mathrm{Mat}_{m \times n}(F)$:

$T \mapsto M$ s.t. $Tv_i = \sum_j M_{j, i} w_j$

for all k, ℓ, have a matrix M s.t.

$M_{\ell, k} = 1$

$M_{j, i} = 0$ for all $(j, i) \neq (\ell, k)$

[draw matrix]

[what is the corresponding elt of Hom(V, W)?]

let $\theta_{\ell, k} : V$ to W be def by

$\theta_{\ell, k}(v_k) = w_\ell$

$\theta_{\ell, k}(v_i) = 0$ for all $i \neq k$

then $(\theta_{\ell, k})_{(\ell, k)}$ is a basis for Hom

Q3 take $V = F$
how does everything above simplify?

A3 take the basis for V to be (v_1)
where $v_1 = 1$ in F

then the basis for Hom is $\theta_{1, 1}, \ldots, \theta_{m, 1}$
where $\theta_{\ell, 1}(1) = w_\ell$

in particular, $\dim \mathrm{Hom}(F, W) = m = \dim W$
so $\mathrm{Hom}(F, W)$ and W are linearly isomorphic
in fact, we have an explicit iso: [what is it?]

$\theta_{\ell, 1} \mapsto w_\ell$

<u>Claim</u> this iso is the same for any basis for W
i.e., a <u>coord-indep</u> iso $\mathrm{Hom}(F, W)$ to W

<u>Proof</u> it can be rewritten as:

$\theta \mapsto \theta(1)$ for all θ in $\mathrm{Hom}(F, W)$

<u>Q4</u> similarly, $\dim V = n = \dim \mathrm{Hom}(V, F)$
so V and $\mathrm{Hom}(V, F)$ are linearly isomorphic
we have the explicit iso

$v_k \mapsto \theta_{1, k}$

is this iso the same for any basis for V?

<u>A4</u> no in general
take $n = 2$
let $(v'_1, v'_2) = (v_1, v_1 + v_2)$

if coord-indep, then $v'_k \mapsto \theta'_{1, k}$ where
$\theta'_2(v'_1) = 0$
$\theta'_2(v'_2) = 1$

but we need $\theta'_{1,2} = \theta_{1,1} + \theta_{1,2}$

but $\theta_{1,1}(v'_2) + \theta_{1,2}(v'_2)$

$= \theta_{1,1}(v_1) + \theta_{1,1}(v_2) + \theta_{1,2}(v_1) + \theta_{1,2}(v_2)$

$= 1 + 0 + 0 + 1 = 2$, contradiction

[so let's treat Hom(V, F) as different from V]

(Axler §3F)

<u>Df</u> the dual vector space to V is

$V^\vee := \mathrm{Hom}(V, F)$

$= \{\text{linear maps } \theta : V \text{ to } F\}$

under $(\theta + \theta')(v) = \theta v + \theta' v$

$(a \bullet \theta)v = a\,\theta v$

its elements are called <u>F-linear functionals</u>

<u>Def-Lem</u> if V is finite dim'l, then $\dim V^\vee = \dim V$

in fact:

a basis for V defines a <u>dual basis</u> for V

<u>Def-Pf</u> if $v_1, v_2, \ldots, v_n$ is a basis for V

then for all k, define $v^\vee_k$ in $V^\vee$ by

$v^\vee_k(e_k) = 1$

$v^\vee_k(e_j) = 0$ for all $j \neq k$

[earlier, $v^\vee_k$ was called $\theta_{1,k}$]

<u>Lem</u> if V is infinite-dim'l

then $V^\vee$ has greater cardinality than V

Pf when V = F[x]   [assuming F = R or C]

F[x] = {const’s} cup bigcup_{n > 0} {p | deg(p) = n}
so (cardinality of F[x])
    ≤ (cardinality of F sqcup F^2 sqcup …)
    = (cardinality of F) because F is infinite

claim: (cardinality of F^**N**) ≤ (cardinality of F[x]$^\vee$)
for any f in F^**N**, define θ_f in F[x]$^\vee$ by
    θ_f(x^n) = f(n)
then f mapsto θ_f is an injective map F^**N** to F[x]$^\vee$

by Cantor, (cardinality of F) < (cardinality of F^**N**)

Rem   inj. map F^**N** to F[x]$^\vee$ is actually an iso

Df   suppose T : V to W is a linear map
    its dual $T^\vee$: $W^\vee$ to $V^\vee$ is the map def by

    $T^\vee(\psi) = \psi \circ T$ (as a map from V to F)

Lem $T^\vee$ is also linear

picture: $V \xrightarrow{T} V \xrightarrow{\psi,\, \psi'} V$

Pf   want $T^\vee(\psi + \psi') = T^\vee(\psi) + T^\vee(\psi')$:

$((\psi + \psi') \circ T)v = (\psi + \psi')(Tv) = \psi(Tv) + \psi'(Tv)$
    $= (\psi \circ T)v + (\psi' \circ T)v$

the proof that $T^\vee(a \bullet \theta) = a \bullet T^\vee(\theta)$ is similar

Summary

1) taking duals of vector spaces and lin maps "reverses" the direction of other constructions ["contravariance"]
2) a basis for V defines a dual basis for $V^\vee$
3) if we view elts of F^n as cols, then we also view elts of $(F^n)^\vee$ as rows: $(F^n)^\vee = \mathrm{Hom}(F^n, F) = \mathrm{Mat}_{1 \times n}(F)$

An Application recall:

Thm if $F = \mathbb{C}$ and V is fin. dim. then any lin. op on V has an upper-triangular matrix

earlier, proved via induction on $n = \dim V$
base case $n = 0$

earlier, used eigenline = T-stable subsp. of dim 1
this time, will use:

Thm′ there's a T-stable subsp. of dim $n - 1$

Thm′ implies Thm:

pick ordered basis $(e_1, \ldots, e_{n-1})$ for W s.t.
matrix of $T|_W$ wrt e_i is triangular
extend to ordered basis $(e_1, \ldots, e_n)$ for V
matrix of T is wrt e_i is again triangular

[draw matrix]

Pf of Thm′ since $V^\vee$ is also finite-dim'l, $T^\vee : V^\vee$ to $V^\vee$ has an eigenvector θ say, with eigenval λ

ker(θ) is T-stable:

if v in ker(θ)

then $\theta(Tv) = (T^\vee(\theta))v = (\lambda\theta)v = \lambda(\theta v) = \lambda\mathbf{0} = \mathbf{0}$

so Tv in ker(θ)

claim dim ker(θ) = n – 1

know dim ker(θ) = n – dim im(θ)

but $\theta \neq 0$, so im(θ) = F, so dim im(θ) = 1 □

key step?

(Fθ $T^\vee$-stable in $V^\vee$) implies (ker(θ) T-stable in V)

more generally:

Df for any linear subspace U sub V the annihilator of U is

$\mathrm{Ann}_{V^\vee}(U) = \{\theta \text{ in } V^\vee \mid \theta(u) = 0 \text{ for all } u \text{ in } U\}$

next time: $\mathrm{Ann}_{V^\vee}(U)$ is a linear subspace of $V^\vee$