## 6.

Notes on the relation between root valuation data and braids.

### *6.1.*

First, we review affine Springer fibers.

Let $G$ be a complex semisimple group of rank $r$, and let $\mathfrak{g}$ be its Lie algebra. Let $\mathfrak{t}$ be the Cartan algebra and $W$ the Weyl group. We set

$$\mathfrak{c} := \mathfrak{t} /\!\!/ W \simeq \mathfrak{g} /\!\!/ G.$$

We know that $\mathfrak{c}$ is an affine space of dimension $r$. The branched cover $\mathfrak{t} \to \mathfrak{c}$ transports the scaling action of $\mathbf{G}_m$ on $\mathfrak{t}$ to a weighted action of $\mathbf{G}_m$ on $\mathfrak{c}$.

**Example 6.1.** If $\mathfrak{g} = \mathfrak{sl}_n$, then $W = S_n$ and $\mathfrak{t}$ is the irreducible representation of $S_n$ of dimension $n - 1$. We can view $\mathfrak{t}(\mathbf{C})$ as the vector space

$$\{z \in \mathbf{C}^n : z_1 + \cdots + z_n = 0\},$$

endowed with the $S_n$-action that permutes coordinates. Passing from a point $z \in \mathfrak{t}(\mathbf{C})$ to its $S_n$-orbit is equivalent to passing from the ordered tuple $(z_1, \ldots, z_n)$ to the underlying unordered multiset. The latter is equivalent to the monic polynomial whose roots are $z_1, \ldots, z_n$. So we can view $\mathfrak{c}(\mathbf{C})$ as the vector space of polynomials

$$\{t^n + a_2 t^{n-2} + \cdots + a_{n-1} t + a_n \in \mathbf{C}[t]\}.$$

The map $\mathfrak{t}(\mathbf{C}) \to \mathfrak{c}(\mathbf{C})$ sends $z \mapsto \prod_j (t - z_j)$. Thus the $\mathbf{G}_m$-action on $\mathfrak{c}$ has weight $i$ on the coordinate $a_i$.

Let $F = \mathbf{C}(\!(t)\!)$ and $\mathcal{O} = \mathbf{C}[\![t]\!]$. The loop group of $G$ is the group $G(F)$, and the *affine Grassmannian* of $G$ is the homogeneous space $G(F)/G(\mathcal{O})$. Every element $\gamma \in \mathfrak{g}(F)$ induces a vector field on $G(F)/G(\mathcal{O})$, whose fixed-point set is called the associated *affine Springer fiber* $X_\gamma$.

Let $\mathfrak{c}^\circ \subseteq \mathfrak{c}$ be the regular locus, i.e., the complement of the branch locus of $\mathfrak{t} \to \mathfrak{c}$. We say that $\gamma$ is *regular semisimple* iff it maps into $\mathfrak{c}^\circ(F)$ under

$$\chi : \mathfrak{g} \to \mathfrak{c}.$$

Kazhdan–Lusztig observed that if $\gamma$ is regular semisimple, then $X_\gamma$ is finite-dimensional over $\mathbf{C}$ and its isomorphism class only depends on $a = \chi(\gamma) \in \mathfrak{c}^\circ(F)$. Moreover, it is nonempty if and only if $a \in \mathfrak{c}(\mathcal{O})$. In this setting, we will write $X_a := X_\gamma$. Altogether, we obtain a family of affine Springer fibers above $\mathfrak{c}^\circ(F) \cap \mathfrak{c}(\mathcal{O})$.

*6.2.*

In “Codimensions of Root Valuation Strata” [GKM], Goresky–Kottwitz–MacPherson introduced an explicit stratification of $\mathfrak{c}^\circ(F)$ and (roughly) conjectured that the topology of affine Springer fibers is constant along its strata.

Let $\Phi$ be the root system of $G$. Consider the set of pairs $(w, r)$ in which $w \in W$ and $r$ is a $\mathbf{Q}$-valued function on $\Phi$. The group $W$ acts on this set according to

$$x \cdot (w, r) = (xwx^{-1}, x \cdot r),$$

where $(x \cdot r)(\alpha) = r(x^{-1}\alpha)$. A *root valuation datum* is an orbit of this action, i.e., an equivalence class $[w, r]$. We now explain, following [GKM], how each point $a \in \mathfrak{c}^\circ(F)$ gives rise to such a datum.

Let $\bar{F} = \mathbf{C}(\!(t^{1/\infty})\!)$, an algebraic closure of $F$. Let $\mathfrak{t}^\circ \subseteq \mathfrak{t}$ be the preimage of $\mathfrak{c}^\circ \subseteq \mathfrak{c}$, and let $\mathfrak{t}^\circ_a$ be the preimage of $a$ along the map $\mathfrak{t}^\circ(\bar{F}) \to \mathfrak{c}^\circ(\bar{F})$. Then $\mathfrak{t}^\circ_a$ is a (simply transitive) $W$-orbit of $\mathfrak{t}^\circ(\bar{F})$. Each point $\xi \in \mathfrak{t}^\circ_a$ defines a function $\boldsymbol{r}_\xi : \Phi \to \mathbf{Q}$, viz.,

$$\boldsymbol{r}_\xi(\alpha) = \mathrm{ord}_t\, \alpha(\xi),$$

once we view $\alpha$ as a linear functional $\mathfrak{t}(\bar{F}) \to \bar{F}$.

Let $\Gamma$ be the Galois group of $\bar{F}$ over $F$. Note that $\Gamma$ is procyclic, being pro-generated by the field automorphism $\tau : \bar{F} \to \bar{F}$ defined by

$$\tau(t^{1/n}) = e^{2\pi i/n} t^{1/n}.$$

Since $a$ is defined over $F$, the $\Gamma$-action on $\bar{F}$ induces a $\Gamma$-action on $\mathfrak{t}^\circ_a$. Since $\mathfrak{t}^\circ_a$ is a torsor for $W$, we deduce that each $\xi \in \mathfrak{t}^\circ_a$ defines an element $w_\xi \in W$ such that

$$w_\xi \xi \tau = \xi.$$

(In [GKM], the $\Gamma$- and $W$-actions are both written on the left.)

For all $x \in W$, we can check that $(w_{x\xi}, \boldsymbol{r}_{x\xi}) = x \cdot (w_\xi, \boldsymbol{r}_\xi)$. Therefore, the $W$-orbit $[w_a, \boldsymbol{r}_a] := W \cdot (w_\xi, \boldsymbol{r}_\xi)$ only depends on $a$, not on $\xi$. This is the desired root valuation datum.

For a fixed datum $[w, \boldsymbol{r}]$, we write $\mathfrak{c}^\circ(F)_{w,\boldsymbol{r}}$ to denote the set of all $a \in \mathfrak{c}^\circ(F)$ such that $[w_a, \boldsymbol{r}_a] = [w, \boldsymbol{r}]$. These are the *root valuation strata* of $\mathfrak{c}^\circ(F)$.

**Example 6.2.** If $\mathfrak{g} = \mathfrak{sl}_2$, then the map $\mathfrak{t}^\circ(\bar{F}) \to \mathfrak{c}^\circ(\bar{F})$ simplifies to the squaring map $\bar{F}^\times \to \bar{F}^\times$. Moreover, for any $\xi \in \mathfrak{t}^\circ(\bar{F}) = \bar{F}^\times$,

- The function $\boldsymbol{r}_\xi$ is equivalent to the number $\mathrm{ord}_t\, \xi \in \frac{1}{2}\mathbf{Z}$.
- The element $w_\xi \in W = \{\pm 1\}$ equals 1 when $\xi \in F^\times$ and equals $-1$ otherwise.

In this setting, there are only two kinds of root valuation data: Those of the form $[1, m]$, and those of the form $[-1, m + \frac{1}{2}]$, where in both cases $m \in \mathbf{Z}$.

It is now known that when $G$ is a classical group, the $w$-part of a root valuation datum is redundant. More precisely, Sabitova showed that if $G$ is classical, then for all $w_1, w_2 \in W$ and functions $\boldsymbol{r} : \Phi \to \mathbf{Q}$ such that $\mathfrak{c}^\circ(F)_{w_1,\boldsymbol{r}} \cap \mathfrak{c}(\mathcal{O})$ and $\mathfrak{c}^\circ(F)_{w_2,\boldsymbol{r}} \cap \mathfrak{c}(\mathcal{O})$ are both nonempty, the elements $w_1, w_r$ are conjugate by an element of the $W$-stabilizer of $\boldsymbol{r}$, giving $[w_1, \boldsymbol{r}] = [w_2, \boldsymbol{r}]$.

*6.3.*

Let $\eta = \operatorname{Spec} F$ and $D = \operatorname{Spec} \mathcal{O}$. We view $D$ as an infinitesimal disk and $\eta$ as its generic point, i.e., as the punctured disk $D - 0$. Then the $F$-points of a variety become infinitesimal *loops* in the variety. These interpretations can be made rigorous using Grothendieck's theory of étale fundamental groups. Using the étale formalism, we will recast root valuation data in topological terms.

Our setup will work in the generality of arbitrary finite real reflection groups, not just the Weyl groups attached to semisimple Lie groups. In light of this, we redefine everything over from scratch. Let $\mathfrak{t}_{\mathbf{R}}$ be a finite-dimensional real vector space and $W$ a finite reflection group of $\mathfrak{t}_{\mathbf{R}}$. We set $\mathfrak{t} = \mathfrak{t}_{\mathbf{R}} \otimes \mathbf{C}$ and $\mathfrak{c} = \mathfrak{t} /\!\!/ W$. Again, let $\mathfrak{c}^\circ$ be the complement of the branch locus of $\mathfrak{t} \to \mathfrak{c}$, and let $\mathfrak{t}^\circ$ be its preimage in $\mathfrak{t}$.

We write $\Psi \subseteq W$ for the set of reflections. Each $w \in \Psi$ gives rise to a hyperplane $\mathfrak{t}^w \subseteq \mathfrak{t}$. By a lemma of Steinberg in "Endomorphisms of Linear Algebraic Groups,"

$$\mathfrak{t}^\circ = \mathfrak{t} - \bigcup_{w \in \Psi} \mathfrak{t}^w.$$

(That is, the $W$-stabilizer of a point in $\mathfrak{t}$ is trivial if and only if the point avoids every reflecting hyperplane.)

In what follows, let $F_m = \mathbf{C}(\!(t^{1/m})\!)$ and $\eta_m = \operatorname{Spec} F_m$, so that $F_1 = F$ and $\eta_1 = \eta$. We can fix identifications

$$\pi_1^{\text{ét}}(\eta_m, \bar{\eta}) \simeq \operatorname{Gal}(\bar{F} | F_m) = (\tau^m)^{\hat{\mathbf{Z}}},$$

where $\tau$ is the pro-generator of $\operatorname{Gal}(\bar{F}|F)$ described previously.

Fix a map $a : \eta \to \mathfrak{c}^\circ$. Let $\mathfrak{t}^\circ_a$ be the set of maps $\xi : \bar{\eta} \to \mathfrak{t}^\circ$ such that, for some $m$, we can factor $\xi$ through a map $\xi_m : \eta_m \to \mathfrak{t}^\circ$ that fits into a commutative diagram:

$$\begin{tikzcd} \eta_m \arrow[r, "\xi_m"] \arrow[d] & \mathfrak{t}^\circ \arrow[d] \\ \eta \arrow[r, "a"] & \mathfrak{c}^\circ \end{tikzcd}$$

We will re-explain in topological terms how $\xi$ gives rise to a root valuation datum:

For each $w \in \Psi$, fix a linear functional $\alpha_w : \mathfrak{t} \to \mathbf{C}$ such that $\mathfrak{t}^w = \ker \alpha_w$. (They need not be the roots of a root system.) Using the $\alpha_w$, we will show that $\xi \in \mathfrak{t}^\circ_a$ defines a function $r_\xi : \Psi \to \frac{1}{m}\mathbf{Z} \subseteq \mathbf{Q}$. Indeed, we have a morphism

$$\pi_1^{\text{ét}}(\eta_m, \bar{\eta}) \xrightarrow{\xi_{m,*}} \pi_1^{\text{ét}}(\mathfrak{t}^\circ, \xi(\bar{\eta})) \xrightarrow{\alpha_{w,*}} \pi_1^{\text{ét}}(\mathbf{G}_m, \alpha_w \xi(\bar{\eta})).$$

Abusing notation, we again write $\tau$ for the pro-generator of $\pi_1^{\text{ét}}(\mathbf{G}_m)$ analogous to our chosen pro-generator of $\pi_1^{\text{ét}}(\eta)$. Then the morphism above becomes a morphism $(\alpha_w \xi_m)_* : (\tau^m)^{\hat{\mathbf{Z}}} \to \tau^{\hat{\mathbf{Z}}}$, which induces a morphism $m\mathbf{Z} \to \mathbf{Z}$. We set

$$r_\xi(w) = \tfrac{1}{m}(\alpha_w \xi_m)_*(\tau^m) \in \tfrac{1}{m}\mathbf{Z}.$$

Then the function $r_\xi : \Psi \to \mathbf{Q}$ only depends on $\xi$ and $\Psi$, not on the choice of $m$ or the linear functionals $\alpha_w$. Moreover, the function $\boldsymbol{r}_\xi : \Phi \to \mathbf{Q}$ from the previous section factors through $r_\xi$.

We can identify $\mathrm{Gal}(F_m|F)$ with the group of deck transformations of $\eta_m \to \eta$. Let $\tau_m \in \mathrm{Gal}(F_m|F)$ be the image of $\tau \in \mathrm{Gal}(\bar{F}|F)$, viewed as a map $\tau_m : \eta_m \to \eta_m$. The $\mathrm{Gal}(F_m|F)$-action on $\mathfrak{t}_a^\circ$ is given by precomposition with $\tau_m$. At the same time, we can identify $W$ with the group of deck transformations of $\mathfrak{t}^\circ \to \mathfrak{c}^\circ$. Then there is a unique deck transformation $w_\xi \in W$ such that

$$w_\xi \circ \xi \circ \tau_m = \xi.$$

This is the same $w_\xi$ as in the previous section.

*6.4.*

Keeping the notation $a, \xi, \xi_m$ above, let

$$\begin{aligned} \beta_a^{\text{ét}} &= a_*(\tau) \in \pi_1^{\text{ét}}(\mathfrak{c}^\circ, a(\bar{\eta})), \\ \beta_{\xi,m}^{\text{ét}} &= \xi_*(\tau^m) \in \pi_1^{\text{ét}}(\mathfrak{t}^\circ, \xi(\bar{\eta})). \end{aligned}$$

In the sections to follow, we will relate the conjugacy classes $[\beta_{\xi,m}^{\text{ét}}] \subseteq \pi_1^{\text{ét}}(\mathfrak{t}^\circ)$ and $[\beta_a^{\text{ét}}] \subseteq \pi_1^{\text{ét}}(\mathfrak{c}^\circ)$ to the function $r_\xi$ and its orbit $r_a := W \cdot r_\xi$, respectively.

In doing so, we will lift these conjugacy classes to classes in the corresponding topological $\pi_1$'s. Recall that the *braid group* and *pure braid group* of the action of $W$ on $\mathfrak{t}$ are, respectively, the topological fundamental groups

$$\begin{aligned} Br_W &= \pi_1((\mathfrak{c}^\circ)^{an}), \\ PBr_W &= \pi_1((\mathfrak{t}^\circ)^{an}). \end{aligned}$$

We will use $r_\xi$ to construct a conjugacy class

$$[\beta_{\xi,m}] \subseteq PBr_W$$

that maps into $[\beta^{\text{ét}}_{\xi,m}]$ under the morphism $PBr_W \to \widehat{PBr_W} \simeq \pi_1^{\text{ét}}(\mathfrak{t}^\circ)$. This morphism fits into the following commutative diagram:

$$\begin{tikzcd}
1 \arrow[r] & PBr_W \arrow[r] \arrow[d] & Br_W \arrow[r] \arrow[d] & W \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & \pi_1^{\text{ét}}(\mathfrak{t}^\circ, \xi(\bar{\eta})) \arrow[r, "i"] & \pi_1^{\text{ét}}(\mathfrak{c}^\circ, a(\bar{\eta})) \arrow[r] & W \arrow[r] & 1
\end{tikzcd}$$

Above, the rows are short exact sequences. The morphism $i$ sends

$$\beta^{\text{ét}}_{\xi,m} \mapsto (\beta^{\text{ét}}_a)^m.$$

It should be a group-theoretic exercise(??) to show that all of this structure implies the existence of some conjugacy class $[\beta_a] \subseteq Br_W$ such that:

(1) $[\beta_a^m] = [\beta_{\xi,m}]$.
(2) $[\beta_a]$ maps into $[\beta^{\text{ét}}_a]$ under the morphism $Br_W \to \widehat{Br_W} \simeq \pi_1^{\text{ét}}(\mathfrak{c}^\circ)$.

For now, we leave this matter aside. Below, we explain how $r_\xi$ gives rise to, and is in fact equivalent to, $[\beta_{\xi,m}]$.

*6.5.*

The key will be a certain presentation of $PBr_W$ that only involves the lattice of flats of the hyperplane arrangement determined by $\Psi$. For any subset $F \subseteq \Psi$, we write

$$\mathfrak{t}^F := \bigcap_{w \in F} \mathfrak{t}^w.$$

Recal that $F$ is a *flat* of $\Psi$ iff, for all $w \in \Psi$, we have $\mathfrak{t}^w \supseteq \mathfrak{t}^F$ only if $w \in F$.

Let $t_1, \dots, t_N$ be an ordering of $\Psi$. For each flat $F = \{t_{i_1}, \dots, t_{i_\ell}\}_{i_1 < \cdots < i_\ell}$, we write $\mathfrak{R}_F(t)$ to denote the collection of formal relations

$$t_{i_1} t_{i_2} \dots t_{i_\ell} = t_{i_2} \dots t_{i_\ell} t_{i_1} = \cdots = t_{i_\ell} t_{i_1} \dots t_{i_{\ell-1}}.$$

(If $|F| \leq 1$, then $\mathfrak{R}_F(t)$ is vacuous.) What follows is essentially the main result of "The Fundamental Group of the Complement of a Union of Complex Hyperplanes."

**Theorem 6.3** (Randell 1982)**.** *We have*

$$PBr_W \simeq \langle \tau_1, \dots, \tau_N : \mathfrak{R}_F(\tau) \text{ for each flat } F \rangle.$$

*In particular, if we write $\pi_F = \tau_{i_1} \cdots \tau_{i_\ell}$ whenever $F = \{t_{i_1}, \dots, t_{i_\ell}\}_{i_1 < \cdots < i_\ell}$, then $\pi_F$ commutes with $\tau_i$ for each $t_i \in F$.*

**Example 6.4.** We have $PBr_{S_3} = \langle \tau_1, \tau_2, \tau_3 : \tau_1\tau_2\tau_3 = \tau_2\tau_3\tau_1 = \tau_3\tau_1\tau_2 \rangle$.

Cauchy's residue formula may be restated as an equality between an intersection multiplicity and a winding number. That is, there is a close relation between the intersection theory of divisors within, and the topology of formal loops within, an algebraic variety. We will assign an intersection multiplicity

$$e_{\xi,m}(F) \in \mathbf{Z}_{\geq 0}$$

to each flat $F$; then we will build up $[\beta^{\text{ét}}_{\xi,m}] \subseteq \pi_1^{\text{ét}}(\mathfrak{t}^\circ)$ from this collection of integers. The construction will show that it comes from a class $[\beta_{\xi,m}] \subseteq PBr_W$.

In what follows, let $\mathcal{O}_m = \mathbf{C}[\![t^{1/m}]\!]$ and $D_m = \operatorname{Spec} \mathcal{O}_m$. Henceforth, to simplify, we assume that $a : \eta \to \mathfrak{c}^\circ$ extends to a map $a : D \to \mathfrak{c}$ and that $\xi : \eta_m \to \mathfrak{t}^\circ$ extends to a map $\xi_m : D_m \to \mathfrak{t}$.

To make our plan work, we replace $\mathfrak{t}$ with the so-called *wonderful compactification* of $\mathfrak{t}^\circ$, introduced by de Concini–Procesi, who in turn built on Fulton–MacPherson. The usual definition of this space includes a boundary at infinity that, thanks to our new assumptions on $a, \xi, \xi_m$, we will not need. For our purpose, it is enough to use:

**Lemma 6.5.** *There is a birational map* $p : \mathfrak{t}_\sharp \to \mathfrak{t}$ *such that:*

*(1)* $p$ *restricts to an isomorphism over* $\mathfrak{t}^\circ \subseteq \mathfrak{t}$.
*(2)* $p^{-1}(\bigcup_{w \in \Psi} \mathfrak{t}^w)$ *is a simple normal crossings divisor. Its irreducible components are in bijection with the flats of the hyperplane arrangement.*

*Explicitly, for any flat* $F$*, the preimage*

$$\mathfrak{t}_\sharp^F = p^{-1}(\mathfrak{t}^F)$$

*is the corresponding irreducible component.*

Since $\xi_m(\eta_m) \subseteq \mathfrak{t}^\circ$, we can lift $\xi_m : D_m \to \mathfrak{t}$ to a unique map $\xi_{m,\sharp} : D_m \to \mathfrak{t}_\sharp$, essentially by the valuative criterion of properness. Now we set

$$e_{\xi,m}(F) = \operatorname{length}_{\mathbf{C}}(\xi_{m,\sharp}^{-1}(\mathfrak{t}_\sharp^F)).$$

For any reflection $w \in \Psi$, we have

$$r_\xi(w) = \frac{1}{m} \sum_{F \ni w} e_{\xi,m}(F). \tag{6.1}$$

So the function $r_\xi$ is determined by $m$ and the function $e_{\xi,m}$.

To see the converse: Let $\Lambda$ be the lattice formed by the flats, where the top element is $\Psi$ and the bottom element is $\emptyset$. We say that $F$ *covers* another flat $F'$ iff we have $F \supsetneq F'$ and there is no flat intermediate between $F$ and $F'$ in $\Lambda$.

**Lemma 6.6.** *Let* $\Lambda_\xi$ *be the sub-poset of flats* $F$ *such that* $e_{\xi,m}(F) \neq 0$.

*(1)* $\Lambda_\xi$ *has at most one maximal element.*

*(2) Any element of $\Lambda$ covers at most one element of $\Lambda_\xi$.*

*Thus, $\Lambda_\xi$ forms a descending chain.*

**Corollary 6.7.** *For any flat $F$, we have*

$$e_{\xi,m}(F) = m \cdot \min\{r_\xi(w) : w \in F\}. \tag{6.2}$$

*Thus, $e_{\xi,m}$ is determined by $m$ and $r_\xi$.*

Recall that each flat $F$ defines an element $\pi_F \in PBr_W$. If $F \supseteq F'$, then $\pi_F$ and $\pi_{F'}$ commute. Let $\Lambda_\xi$ be written in the form $F_1 \supseteq F_2 \supseteq \cdots \subseteq F_k$, and set

$$\beta_{\xi,m} = \pi_{F_1}^{e_{\xi,m}(F_1)} \cdots \pi_{F_k}^{e_{\xi,m}(F_k)} \in PBr_W. \tag{6.3}$$

Then $[\beta_{\xi,m}]$ is the desired conjugacy class:

**Proposition 6.8.** *The morphism $PBr_W \to \pi_1^{\text{ét}}(\mathfrak{t}^\circ)$ sends $[\beta_{\xi,m}]$ into $[\beta_{\xi,m}^{\text{ét}}]$. Altogether, $[\beta_a^{\text{ét}}] \subseteq \pi_1^{\text{ét}}(\mathfrak{c}^\circ)$ is an $m$th root of a conjugacy class of $\pi_1^{\text{ét}}(\mathfrak{t}^\circ)$ that can be lifted to a conjugacy class of $PBr_W$ that determines and is determined by $r_\xi$ by way of* (6.1)*,* (6.2)*,* (6.3)*.*

*6.6.*

It remains to show that:

**Conjecture 6.9.** *There exists a unique conjugacy class $[\beta_a] \subseteq Br_W$ such that $[\beta_a^m] = [\beta_{\xi,m}]$ and the morphism $Br_W \to \pi_1^{\text{ét}}(\mathfrak{c}^\circ)$ sends $[\beta_a]$ into $[\beta_a^{\text{ét}}]$.*

In type $A$, the uniqueness of $[\beta_a]$ is assured by the main result of González-Meneses, "The $n$th Root of a Braid is Unique up to Conjugacy." His proof relies on the Nielsen–Thurston classification of braids.