<u>Warmup</u>   X topological space,   x in X

[draw picture]

<u>Df</u>   Y sub X is a neighborhood of x iff
x in U sub Y for some open U sub X

<u>Df</u>   Z sub X is closed iff X – Z is open in X

<u>Df</u>   the interior of Y sub X is the set

<u>Df</u>   the closure of Y sub X is the set

$Int(Y)$ $= bigcup\_\{U \text{ sub } Y, \text{ open in } X\} U$
$= \{x \text{ in } X \mid Y \text{ is a nbd of } x\}$ [why?]

$\bar{Y} = Cl(Y) = bigcap\_\{Z \text{ supset } Y, \text{ closed in } X\} Z$
$= X - bigcup\_\{V \text{ sub } X - Y, \text{ open in } X\} Y$
$= X - Int(X - Y)$

in the analytic topology on R, compute:

in analytic R, compute:

$Int([0, 1)) = (0, 1)$
$Int(\{0, 1, 2, \ldots\}) = \varnothing$
$Int(R - \{0, 1, 2, \ldots\}) = R - \{0, 1, 2, \ldots\}$
$Int(\{1/n \mid n \text{ is a positive integer}\}) = \varnothing$

$Cl([0, 1)) = [0, 1]$
$Cl(\{0, 1, 2, \ldots\}) = \{0, 1, 2, \ldots\}$
$Cl(R - \{0, 1, 2, \ldots\}) = R$
$Cl(\{1/n \mid n \text{ is a positive integer}\}) = \{1/n\}\_n \text{ cup } \{0\}$

<u>Review</u> take X = R and A = [0, ∞) analytic

U open in A iff U = A cap V for some V open in X

<u>Claim</u>   [0, b) open in A [but not in X]

<u>Pf</u>   [0, b) = A cap (−1, b)

<u>Claim</u>   if a > 0, then [a, b) not open in A

<u>Pf</u>   must show: no V open in X s.t.
                [a, b) = A cap V

if V exists, then there is δ > 0 s.t. B(a, δ) sub V
after shrinking δ, can assume B(a, δ) sub A
but then B(a, δ) sub A cap V = [a, b)

(Munkres §20)   recall from real analysis:

<u>Df</u>   a metric on a set X is a function
                d : X × X to [0, ∞)

s.t., for all x, y, z in X,
        1) $d(x, y) = 0$ implies $x = y$
        2) $d(x, y) = d(y, x)$
        3) $d(x, y) + d(y, z) \geq d(x, z)$

given δ > 0, let B_d(x, δ) = {y in X | d(x, y) < δ}

[note that x in B_d(x, δ), because d(x, x) = 0 < δ]

<u>Df</u>   the metric topology on X induced by d:

U is open in the metric topology iff
for all x in U, there is a $\delta > 0$ s.t. $B\_d(x, \delta)$ sub U

<u>Idea</u>   metric topology on X generalizes
         analytic topology on R^n

<u>Thm</u>   the metric topology really is a topology

<u>Pf</u>   exactly like the proof that
         the anlytic topology is a topology

[so how much weirder can it be?]

<u>Ex</u>   in any X: the discrete metric defined by

$$d(x, x) = 0, \quad d(x, y) = 1 \text{ when } x \neq y$$

1) and 2) easy
3) [how many cases to check? 5 but can combine]
    if x = z:
        $d(x, y) + d(y, z) \geq 0 = d(x, z)$
        [because $d(\text{-}, \text{-}) \geq 0$]
    if x ≠ z:
        either y ≠ x or y ≠ z
        so $d(x, y) + d(y, z) \geq 1 = d(x, z)$

observe $B\_d(x, 1) = \{x\}$ for all x. thus:

<u>Prop</u>   metric topology from the discrete metric
         is the discrete topology

<u>Df</u>   we say a topology or topological space
         is metrizable iff
         the topology is induced by some metric

sometimes, different metrics induce the same
topology

<u>Lem</u>   suppose   d induces T on X,
                    d′ induces T′ on X

         then T′ is finer than T iff
         for all x in X and $\varepsilon > 0$, there is $\delta > 0$ s.t.
             $B_{d'}(x, \delta)$ sub $B_d(x, \varepsilon)$.

<u>Pf</u>   exercise (Munkres Lem 20.2)

<u>Ex</u>   [picture of $B_d(x, \delta)$ versus $B_\rho(x, \delta)$]

euclidean metric:
$d(x, y) = \sqrt{(x_1 - y_1)^2 + \ldots + (x_n - y_n)^2}$

square metric:
$\rho(x, y) = \max(|x_1 - y_1|, \ldots, |x_n - y_n|)$

observe:
$d(x, y) \leq \sqrt{n \max_i (x_i - y_i)^2}$
$\quad\quad\quad = \sqrt{n}\, \rho(x, y)$
$\rho(x, y) = \sqrt{\max_i |x_i - y_i|^2}$
$\quad\quad\quad \leq d(x, y)$

shows:

$B_\rho(x, \varepsilon/\sqrt{n}) \subset B_d(x, \varepsilon)$

$B_d(x, \varepsilon) \subset B_\rho(x, \varepsilon)$

[in general:]

Df metrics d, d′ are called <u>equivalent</u> iff
  there exist A, B > 0 s.t.
  $d(x, y) \leq A\, d'(x, y)$ and $d'(x, y) \leq B\, d(x, y)$
   <u>uniformly</u> in x and y

Lem if two metrics are equivalent,
  then their metric topologies coincide

Cor Euclidean and square metrics
  both induce the analytic topology on R^n

Rem converse is false: given a metric d, let

$$d'(x, y) = d(x, y)/(1 + d(x, y))$$

then 1) d′ is still a metric
  2) metric topologies for d, d′ coincide
  3) d and d′ need not be equivalent
[reason: equivalence involves uniformity in x, y]

Ex $R^\omega = \{(x\_1, x\_2, \ldots) \mid x\_i\text{'s in } R\}$
  $R^\infty = \{(x\_1, x\_2, \ldots) \mid x\_i \text{ eventually } 0\}$

Euclidean and square metrics still work on R^∞,
  but not on R^ω

but $u(x, y) = \sup_i \min\{1, |x\_i - y\_i|\}$ works…