Q heat, T temperature

[heat is kinetic energy transferred btw systems]
[temperature is avg kinetic energy of a system]

(1) ΔT propto $Q/\Delta m$

thin cylinder with length coord z

[draw]

(2) Δm propto Δz

temperature [is a] function $T(z, t)$

[look at cross-section from z to $z + \Delta z$:]
[heat the cylinder and study T:]
[heat to cross-section prop to change in $\partial T/\partial z$ from z to $z + \Delta z$, not change in T itself]

(3) Q propto $[(\partial T/\partial z)(z + \Delta z, t) - (\partial T/\partial z)(z, t)]\, \Delta t$

altogether

$$\Delta T/\Delta t = c\, (1/\Delta z)\, [(\partial T/\partial z)(z + \Delta z, t) - (\partial T/\partial z)(z, t)]$$

taking limits,

$$\partial T/\partial t = c\, \partial^2 T/\partial z^2$$ [“heat eq for cylinder”]

now suppose cylinder is from $z = 0$ to $z = 1$
and $T(0, 0) = T(1, 0) = 0$ [abs zero]

to solve heat eq: start with separable case

$$T(z, t) = A(z)B(t)$$

then $B'(t)/B(t) = c\, A''(z)/A(z)$

so there is λ s.t. $\quad A'' = \lambda A$
$\quad B' = c\lambda B$

$\lambda > 0$: $\quad A = c1\, e^{\sqrt{\lambda} z} + c2\, e^{-\sqrt{\lambda} z}$
boundary cond's force $c1 = c2 = 0$

$\lambda = 0$: [what happens?] again $A = 0$

only $\lambda < 0$ interesting:

$$A = c1 \cos(\sqrt{|\lambda|} z) + c2 \sin(\sqrt{|\lambda|} z)$$
$$B = e^{c\lambda t}$$

boundary cond's force $c1 = 0$ and $\sqrt{|\lambda|} = \pi n$

<u>Thm</u> (Fourier) separable solns look like

$$T(z, t) = \sin(\pi n z)\, e^{-c(\pi n)^2 t}$$

gen'l solution is a superposition:

$$T(z, t) = \sum_{n \geq 1} a_n \sin(\ )\, e^{\ }$$

[does anyone know the name for these coeffs?]

<u>Thm</u> (Fourier) inversion formula for "nice" T:

$$a_n = 2 \int_0^1 T(z, 0) \sin(\pi n z)\, dz$$

now suppose at $t = 0$: Dirac delta at $z = 1/2$, abs zero elsewhere

$$\int_0^1 T(z, 0)\varphi(z)\, dz = \varphi(1/2)$$

then

$$a_n = 2 \sin(\pi n/2) = \begin{cases} 0 & n \equiv 0 \bmod 4 \\ 2 & n \equiv 1 \bmod 4 \\ 0 & n \equiv 2 \bmod 4 \\ -2 & n \equiv 3 \bmod 4 \end{cases}$$

$$\begin{aligned} T(z, t) &= 2\,[\ \sin(\pi z)e^{-c\pi^2 t} \\ &\qquad - \sin(3\pi z)e^{-c(3\pi)^2 t} \\ &\qquad + \sin(5\pi z)e^{-c(5\pi)^2 t} \\ &\qquad - \dots\ ] \\ &= \sum_{n \in \mathbb{Z}} e^{2\pi i(n + 1/2)(z - 1/2)} e^{-c(4\pi^2)(n + 1/2)^2 t} \end{aligned}$$

by de Moivre

[renormalize to:]

<u>Df</u> the Jacobi theta function

$$\Theta(z, \tau) = \sum_{n \in \mathbb{Z}} e^{2\pi i n z} e^{\pi i n^2 \tau}$$

for z, τ s.t. $\mathrm{Im}(\tau) > 0$ ["upper half-plane"]

[c = 1/(4π):
T = e^{πi(z – 1/2) – t/4} ϴ(z + it/2 – 1/2, it)]

ϴ(z, τ) highly symmetric: focus on θ(τ) = ϴ(0, τ)

θ(τ + 1) = θ(τ)
θ(–1/τ) = sqrt{–iτ}θ(τ), where Im sqrt{ } > 0

#2 deeper: uses “Poisson summation”

also related to number theory via q = e^{πiτ}:

θ = sum_n q^{n^2},
θ^2 = sum_m r_2(m)q^m

where r_2(m) = |{(a, b) in Z^2 | m = a^2 + b^2}|

Brahmagupta:

r_2(m), r_2(n) > 0 implies r_2(mn) > 0
[why?]
(a^2 + b^2)(c^2 + d^2)
= (ac + bd)^2 + (ad – bc)^2

Fermat: studied r_2(m) for prime m

2 = 1^2 + 1^2
3
5 = 1^2 + 2^2
7
11
13 = 2^2 + 3^2 [pattern?]

let $d_1(m) = |\{\text{divisors} \equiv 1 \bmod 4 \text{ of } m\}|$
$d_3(m) = |\{\text{divisors} \equiv 3 \bmod 4 \text{ of } m\}|$

$\xi(\tau) = 1 + 4 \sum_{m \geq 1} (d_1(m) - d_3(m))q^m$

<u>Thm</u> (Jacobi) $\theta^2 = \xi$
$r_2(m) = 4(d_1(m) - d_3(m))$

<u>Sketch</u> $\xi(\tau) = 1 + 4 \sum_m q^m/(1 + q^{2m})$
$= \sum_{n \in \mathbb{Z}} 2/(q^n + q^{-n})$

can then show that θ^2, ξ are functions f s.t.

- $f(\tau + 2) = f(\tau)$ and $f(-1/\tau) = -i\tau f(\tau)$
- f analytic and nonvanishing on $\{\mathrm{Im}(\tau) > 0\}$
- as $\mathrm{Im}(\tau)$ to infty,
  $f(\tau)$ to 1 and $f(1 - 1/\tau) \sim -4i\tau e^{\pi i\tau/2}$

the ratio $F = \xi/\theta^2$ satisfies

- $F(\tau + 2) = F(\tau)$ and $F(-1/\tau) = F(\tau)$
- F is analytic and bounded on $\{\mathrm{Im}(\tau) > 0\}$

[F is a "holomorphic modular form of weight 0
for the theta group"]

analogue of Liouville's thm: all such F constant

[F descends to a bdd analytic fn on $\{\mathrm{Im}(\tau) > 0\}/\sim$
and this surface is compact]

but constant terms of θ^2 and ξ are both 1