(Munkres §53) main idea of covering spaces:
generalize the structure of

$p : R \to S^1 \qquad p(t) = (\cos(2\pi t), \sin(2\pi t))$

specifically, the <u>homotopy lifting property</u> for paths that we used to prove $\pi_1(S^1) = Z$

first attempt:

<u>Df</u> a continuous map $p : E \to X$ is <u>locally a homeomorphism</u> iff,

for all e in E,
there is an open neighborhood V of e s.t.
$p|_V$ is a homeomorphism from V onto $p(V)$

Ex $\quad p : \{0 < t < 1\} \to S^1$

again given by $p(t) = (\cos(2\pi t), \sin(2\pi t))$

is locally a homeomorphism

objection: p is not surjective

e.g., cannot lift loops in S^1 based at $(x, y) = (1, 0)$

Ex $\quad p : \{-1 < t < 1\} \to S^1$

again given by $p(t) = (\cos(2\pi t), \sin(2\pi t))$

is both surjective and locally a homeo

objection?

still cannot lift all loops...

Df suppose U is an open subset of X,

$V = p^{-1}(U)$ for some $p : E \to X$

we say that U is evenly covered by p iff both:

1) V is homeomorphic to a nonempty(!) disjoint union of copies of U
2) p restricts to a homeomorphism from each copy onto U

Df we say that $p : E \to X$ is a covering map iff,

for all x in X,

there's an open neighborhood of x evenly covered by p

Ex claim that $p : \{-1 < t < 1\} \to S^1$ is not a covering

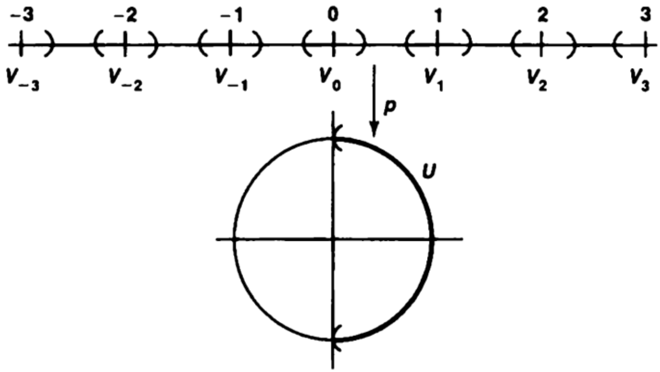
if $(x, y) \neq (1, 0)$
then (x, y) does have an open neighborhood
evenly covered by p

what goes wrong when $(x, y) = (1, 0)$?

Lem if p is a covering
then p is surjective and locally a homeo

(example shows that the converse can fail)

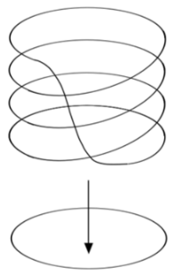
Ex p : $R \to S^1$ is indeed a covering



Ex paths in S^1 cannot be covering maps: why?

Ex identity maps Id : $X \to X$ are always covering maps

Ex more generally:



https://ahilado.wordpress.com/2017/04/14/covering-spaces/

for any $n > 0$, there is a covering $p : S^1 \to S^1$ s.t.
the fiber $p^{-1}(x)$ has cardinality n for all x in S^1

we say that the covering is of degree n, or n-fold

we say that R and S^1 form covering spaces, or covers, of S^1

will show later:

- $\mathbb{R}$ and S^1 are the <u>only</u> covers of S^1
- S^2 is the <u>only</u> cover of S^2

yet S^2 is a cover for another topological space X:
what is X?

<u>Ex</u> consider the relation ~ on S^2 identifying all pairs of antipodal points:

$$(x, y, z) \sim (-x, -y, -z)$$

the quotient map S^2 to $S^2/\sim$ is a 2-fold covering map

have we seen $S^2/\sim$ before?

General Properties of Covering Maps

if $p : E \to X$ is a covering, then:

- the fibers $p^{-1}(x)$ are discrete sets for all x in X

  (because for some open nbd U of x,
  $p^{-1}(U)$ is a bunch of copies of U mapped onto U
  homeomorphically by p)

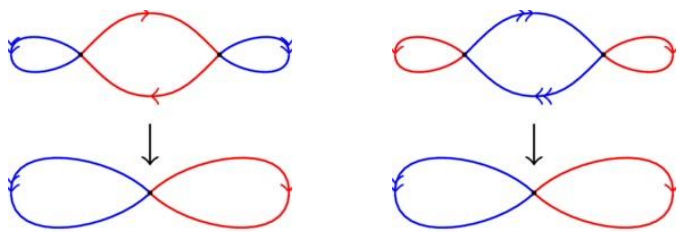
- p is a quotient map

  (true for any surjective map that's locally a homeo)

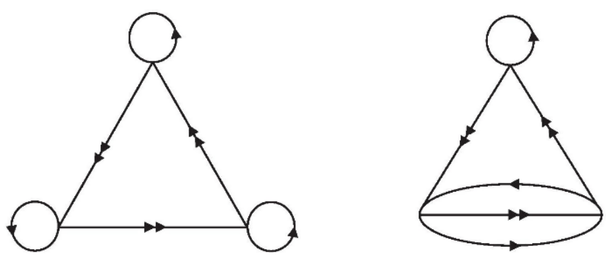
- if $p' : E' \to X'$ is another covering
  then $(p, p') : E \times E' \to X \times X'$ is a covering

Ex any pair of integers m, n > 0 defines a deg-(mn) covering $T \to T$, where $T = S^1 \times S^1$

Ex coverings of the figure-eight can be weird:

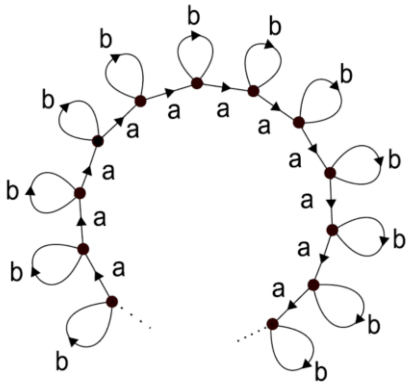


or weirder:

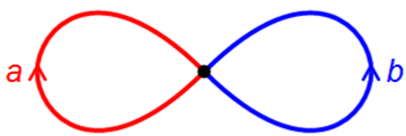


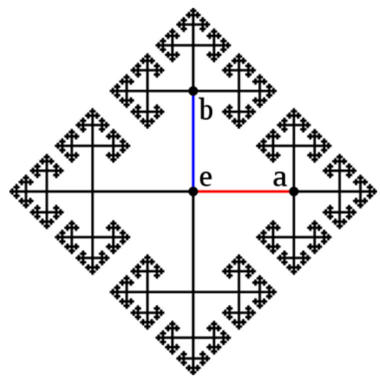
(1) https://www.homepages.ucl.ac.uk/~ucahjde/tg/html/cov-01.html (2) https://groupoids.org.uk/images/fig10-3.jpg

even weirder:



https://www.math.cmu.edu/~nkomarov/NK-NormalSubFreeGrp.pdf





https://math.stackexchange.com/a/3762676

(Munkres §54) the most important property:

Thm if $p : E \to X$ is a covering, then:

1) for any path $\gamma : [0, 1] \to X$ and e in E s.t.

$$p(e) = \gamma(0),$$

there's a unique $\Gamma : [0, 1] \to E$ s.t.

$$\Gamma(0) = e \text{ and } \gamma = p \circ \Gamma,$$

which we call the lift of γ to E

2) for any path homotopy $h : [0, 1]^2 \to X$ and e in E s.t.

$$p(e) = h(0, 0),$$

there's a unique(!) path homotopy $H : [0, 1]^2 \to E$ s.t.

$$H(0, 0) = e \text{ and } h = p \circ H$$

(slightly stronger than our version from 3/26) we'll do 2) on Mon, 4/21

<u>Pf of 1)</u> can pick an open cover $\{U_\alpha\}_\alpha$ of X s.t.
each U_α is evenly covered by p

recall from our proof that S^2 is simply-connected:
an argument showing that we can find

$$0 = s_0 < s_1 < \ldots < s_n = 1$$

s.t. $\gamma : [0, 1] \to X$ maps each segment $[s_i, s_{i+1}]$
into a single U_α at a time

(Munkres calls this the <u>Lebesgue number lemma</u>)

we build the lift $\Gamma : [0, 1] \to E$ inductively:

set $\Gamma(0) = e$

assume that for some i, we've defined Γ for $s \leq s_i$

to define Γ on $[s_i, s_{i+1}]$:

we know that γ maps $[s_i, s_{i+1}]$ into a single U_α,
that U_α is evenly covered by p
and $\Gamma(s_i)$ is in $p^{-1}(U_\alpha)$, which is a bunch of copies of U_α

let V_α be the copy containing $\Gamma(s_i)$
via the homeomorphism $V_\alpha \approx U_\alpha$,
$\gamma|_{[s\_i, s\_\{i + 1\}]}$ has a unique lift into V_α
define $\Gamma|_{[s\_i, s\_\{i + 1\}]}$ to be this lift

by the pasting lemma, we've now defined Γ for $s \leq s_{i+1}$ □