Warmup idea of gluing:

$[0, 1]$ maps to $S^1 = \{z \text{ in } R^2 \mid |z| = 1\}$ via

$f(t) = (\cos(2\pi t), \sin(2\pi t))$

is it continuous [for the subspace topologies]?

method 1: f extends to a map $F : R$ to R^2

Lem if $F : X$ to A is cts and $F(Y) = B$
then $F|_Y : Y$ to B is cts wrt subsp. top's

F is cts with image S^1 [$X = R$, $A = R^2$]
so $F : R$ to S^1 is cts [$Y = R$, $B = S^1$]
so f is cts [now $Y = [0, 1]$, $A = B = S^1$]

method 2: check on basis for S^1

basis = {open arcs V}
if V does not pass through $\mathbf{p} := (1, 0)$
then $f^{-1}(V)$ is an open interval in $[0, 1]$
if it does pass through $\mathbf{p}$
then $f^{-1}(V)$ looks like $[0, a) \cup (b, 1]$

key idea:
[shape of S^1 close to shape of $[0, 1]$ because]
f injective everywhere except at the endpoints

Q describe topology of S^1 directly from topology of $[0, 1]$?

(Munkres §22) let $f : X \to A$ be surjective

<u>Df</u> the quotient topology on A wrt f is the finest topology s.t. $f : X \to A$ is cts

i.e., $V \subseteq A$ is open iff $f^{-1}(V)$ is open

[why is this a topology? preimages of cups/caps]

<u>Rem</u> sometimes A already has another top T

we say that $f : X \to A$ is a <u>quotient map</u> wrt T

iff

T is exactly the quotient topology wrt f

<u>Ex</u> the surjective map $[0, 1] \to S^1$ defines a quotient topology on S^1

claim: it is a quotient map wrt the subspace top

(subspace top) $\subseteq$ (quotient top) easy

conversely, if $V \subseteq S^1$ s.t. $f^{-1}(V)$ is open

either $f^{-1}(V)$ contains both 0 and 1

so $f^{-1}(V) \supseteq [0, a) \cup (b, 1]$

so $V \supseteq$ open arc around $\mathbf{p}$

or $f^{-1}(V)$ contains neither 0 nor 1

so V disjoint from arc around $\mathbf{p}$

[with more work]

can show V is union of open arcs in both cases

so (quotient top) $\subseteq$ (subspace top)

Ex the surjective map R to S^1 defines the same quotient topology on S^1 [exercise]

what partition of R corresponds to this map?
for all (cos(2πt), sin(2πt)) in S^1, preimage in R is

{…, t – 2, t – 1, t, t + 1, t + 2, …}
[draw picture]

Rem in language of group theory:
R is a group under +,
Z is a subgroup of R s.t. S^1 ≃ R/Z

Ex let Y = [0, 1) sub X = [0, 1]
f|_Y : Y to S^1 still surjective
same quotient topology? no [why?]

Moral if f : X to A is surjective and f(Y) = B then

the quot. top. on B induced by the sub. top. on Y could be strictly finer than
the sub. top. on B induced by the quot. top. on A

previous ex: X = [0, 1], Y = [0, 1), A = B = S^1

Ex let f : R to {–, 0, +} be the “sign” map
in quotient top wrt f, open sets are

Ø, {–}, {+}, {–, +}, {–, 0, +}

let g : {–1, 0, 1} to {–, 0, +} be restr of f
in quotient top wrt g, every set is open

Interlude

an <u>equivalence relation</u> on X is a rule $\sim$ s.t.

0) for all x, y in X, either $x \sim y$ or $x \nsim y$
1) $x \sim x$ for all x in X
2) $x \sim y$ if and only if $y \sim x$
3) ($x \sim y$ and $y \sim z$) together imply $x \sim z$

the following data are all "fungible"

I) a surjective map $f : X$ to A
II) a partition $X = \mathrm{coprod}_\alpha X_\alpha$
III) an equivalence relation $\sim$ on X

I) to II): set $X_\alpha = f^{-1}(\alpha)$ for all α in A

II) to I): $A = \{\text{indices } \alpha \text{ in partition}\}$
set $f(x) = \alpha$ for all x in X_α

II) to III): $x \sim y$ defined as:
x, y in X_α for the same α

III) to II): X_α defined to be maximal subsets s.t.
for all x, y in X_α, have $x \sim y$

(<u>Lem</u> these maximal subsets do partition X)
(<u>Df</u> these maximal subsets are called equivalence classes)

Ex if $f_i : X_i$ to A_i is surjective for $i = 1, 2$
then so is
$f = (f_1, f_2) : X_1 \times X_2$ to $A_1 \times A_2$

taking f_1, f_2 to be copies of $[0, 1]$ to S^1 gives

solid square $[0, 1] \times [0, 1]$ to torus $S^1 \times S^1$

two top's:
product top wrt quotient top's wrt f_1, f_2
quotient top wrt f

do they match? yes [but tricky]

(product top) sub (quotient top) by PS3, #2

suppose V sub $S^1 \times S^1$ s.t. $f^{-1}(V)$ open:
casework on "boxes" shows V open in prod top

[however, above example is misleading:]

Rem can find $f_i : X_i$ to A_i for $i = 1, 2$ s.t.

f_1, f_2 are surjective
but the prod top on $A_1 \times A_2$ wrt quot top's
is coarser than the quot top wrt $f_1 \times f_2$

see Munkres §22, Ex 7 and p. 145, #6

Comparison recall:

for any collection of top spaces $(X_i)_i$:
$(g_i)_i : Y$ to $\prod_i X_i$ is cts wrt product topology
iff $g_i : Y$ to X_i is cts for all i

similarly:

for any surjective $f : X$ to A:
$h : A$ to Z is cts wrt quotient topology
iff $h \circ f : X$ to Z is cts

note that $h \circ f$ will be constant along subsets of the form $f^{-1}(a)$ for a in A

Munkres calls these the saturated sets of X wrt f