4.

More notes on Mellit, "Poincaré Polynomials. . . ", refining `2302_06`.

4.1. Fix a field $\mathbf{F}$, which we will eventually take to be $\mathbf{F}_q$. Fix an integer $n > 0$. Let $G = \mathrm{GL}_n$ and $\mathfrak{g} = \mathfrak{gl}_n$ over $\mathbf{F}$. Let $\mathscr{N}$ be the nilpotent cone in $\mathfrak{g}$.

Let $F = \mathbf{F}((z))$ and $\mathscr{O} = \mathbf{F}[[z]]$. Let $\mathscr{V} \subseteq \mathscr{N}(F)^2$ be the set of pairs (η, θ) such that η and θ are conjugate under $G(F)$. Let $\mathscr{T} \subseteq \mathscr{N}(F)^2 \times G(F)$ be the set of triples (η, θ, g) such that $\eta = g\theta g^{-1}$. Forgetting g defines a map $\mathscr{T} \to \mathscr{V}$.

4.2. There is an action of $G(F)^2$ on $\mathscr{V}$ by conjugation in each entry. This lifts to an action on $\mathscr{T}$ defined by

$$(x, y) \cdot (\eta, \theta, g) = (x\eta x^{-1}, y\theta y^{-1}, xgy^{-1}).$$

For all η, let $\mathscr{V}_\eta \subseteq \mathscr{N}(F)$ be the fiber of $\mathscr{V}$ over η. For all θ, let $\mathscr{T}_{\eta,\theta} \subseteq G(F)$ be the fiber of $\mathscr{T}$ over (η, θ).

Let $G(F)_\eta$ and $G(F)_\theta$ be the respective stabilizers of η and θ under conjugation by $G(F)$. Then the action of $G(F)^2$ on $\mathscr{T}$ induces an action of $G(F)_\eta \times G(F)_\theta$ on $\mathscr{T}_{\eta,\theta}$. The resulting actions of $G(F)_\eta \times 1$ and $1 \times G(F)_\theta$ on $\mathscr{T}_{\eta,\theta}$ are simply transitive.

4.3. Let $\mathscr{V}^{\mathscr{O}} \subseteq \mathscr{V}$ and $\mathscr{T}^{\mathscr{O}} \subseteq \mathscr{T}$ be the subsets where $\eta, \theta \in \mathscr{N}(\mathscr{O})$.

Given $(\eta, \theta, g) \in \mathscr{T}^{\mathscr{O}}$, we say that g is *θ-kernel-strict* and g^{-1} is *η-kernel strict* if and only if any of the following equivalent conditions hold:

- $g : \ker(\theta) \xrightarrow{\sim} \ker(\eta)$ restricts to a bijection $\ker(\theta) \cap \mathscr{O}^n \xrightarrow{\sim} \ker(\eta) \cap \mathscr{O}^n$.
- $g^{-1} : \ker(\eta) \xrightarrow{\sim} \ker(\theta)$ restricts to a bijection $\ker(\eta) \cap \mathscr{O}^n \xrightarrow{\sim} \ker(\theta) \cap \mathscr{O}^n$.
- $\ker(\theta) \cap \mathscr{O}^n = \ker(\theta) \cap g^{-1}\mathscr{O}^n$ as subsets of F^n.
- $\ker(\eta) \cap \mathscr{O}^n = \ker(\theta) \cap g\mathscr{O}^n$ as subsets of F^n.

Let $\mathscr{T}^{\mathrm{ks}} \subseteq \mathscr{T}^{\mathscr{O}}$ be the subset of triples (η, θ, g) in which g is θ-kernel-strict. The action of $G(F)^2$ on $\mathscr{T}$ restricts to an action of $G(\mathscr{O})^2$ on $\mathscr{T}^{\mathscr{O}}$ that preserves $\mathscr{T}^{\mathrm{ks}}$. Let $\mathscr{T}^{\mathrm{ks}}_{\eta,\theta} = \mathscr{T}_{\eta,\theta} \cap \mathscr{T}^{\mathrm{ks}}$.

Let J_η and J_θ be the respective stabilizers of η and θ under conjugation by $G(\mathscr{O})$. Then the action of $G(F)_\eta \times G(F)_\theta$ on $\mathscr{T}_{\eta,\theta}$ restricts to an action of $J_\eta \times J_\theta$ on $\mathscr{T}^{\mathrm{ks}}_{\eta,\theta}$. Unlike before, the resulting actions of $J_\eta \times 1$ and $1 \times J_\theta$ on $\mathscr{T}^{\mathrm{ks}}_{\eta,\theta}$ are usually not simply transitive.

4.4. Let $G(F) = \coprod_{d \in \mathbf{Z}} G(F)^{(d)}$ be the decomposition in which $G(F)^{(d)}$ is the subset of elements whose determinant has z-valuation d. What follows is Lemma 3.7 in [M20]:

Lemma 4.1 (Mellit)**.** *Let $(\eta, \theta) \in \mathscr{V}^{\mathscr{O}}$. Suppose that $\eta \in \mathscr{N}(\mathbf{F})$. Then there is an integer $d_{\eta,\theta}$ such that $\mathscr{T}^{\mathrm{ks}}_{\eta,\theta} \subseteq G(F)^{(d_{\eta,\theta})}$. (By construction, it only depends on θ up to conjugation by $G(\mathscr{O})$.) Moreover, $d_{\eta,\theta}$ is nonnegative.*

4.5. Let $\mathscr{T}^{\mathrm{ks,max}}_{\eta,\theta} \subseteq \mathscr{T}^{\mathrm{ks}}_{\eta,\theta}$ be the subset of elements g such that the most negative z-valuation among the matrix entries of g is the most positive that it can be.

The groups J_η, J_θ are commeasurable in the following sense, adapted from Proposition 3.12 and Remark 3.14 of [M20].

Lemma 4.2 (Mellit). *Let $(\eta, \theta) \in \mathscr{V}^{\mathscr{O}}$. Suppose that $\eta \in \mathscr{N}(\mathbf{F})$. Then for any double coset $J_\eta g J_\theta \subseteq \mathscr{T}^{\mathrm{ks}}_{\eta,\theta}$ with $g \in \mathscr{T}^{\mathrm{ks,max}}_{\eta,\theta}$, the sets*

$$J_\eta \backslash J_\eta g J_\theta \subseteq G(\mathscr{O}) \backslash G(F)^{(d_{\eta,\theta})} \quad \textit{and} \quad J_\eta g J_\theta / J_\theta \subseteq G(F)^{(d_{\eta,\theta})}/G(\mathscr{O})$$

*are bounded (*e.g.*, in the Schubert stratification). If $\mathbf{F} = \mathbf{F}_q$, then the ratio of point counts*

$$\mathrm{wt}_{\eta,\theta}(q) := \frac{|J_\eta g J_\theta / J_\theta|}{|J_\eta \backslash J_\eta g J_\theta|}$$

is independent of g. (Again, this number only depends on θ up to conjugation by $G(\mathscr{O})$.)

4.6. For all $\mu \vdash n$, let $K_\mu \subseteq G(\mathscr{O})$ be the standard parahoric of type μ. Then $\mathscr{B}_\mu := G(\mathscr{O})/K_\mu$ is the usual partial flag variety of type μ.

For another $\lambda \vdash n$, let $\eta_\lambda \in \mathscr{N}(\mathbf{F}_q)$ be of Jordan type λ. Mellit introduces

$$C_{\lambda,\mu,q}(t) = \sum_{[\theta] \in \mathscr{V}^{\mathscr{O}}_{\eta_\lambda}/G(\mathscr{O})} t^{d_{\eta_\lambda,\theta}} \mathrm{wt}_{\eta_\lambda,\theta}(q) |\mathscr{B}^\theta_\mu(\mathbf{F}_q)|,$$

where $\mathscr{B}^\theta_\mu \subseteq \mathscr{B}_\mu$ is the Springer fiber for θ. After correction, his Theorem 5.15 is:

Theorem 4.3 (Mellit). *For all $\lambda, \mu \vdash n$, we have*

$$\begin{aligned} C_{\lambda,\mu,q}(t) &= \frac{\langle h_\mu, \tilde{H}_{\lambda^t}(q,t)\rangle}{\prod_{\substack{\square \in \lambda^t \\ l(\square) \neq 0}} (1 - t^{l(\square)} q^{-a(\square)-1})} \\ &= \frac{\langle h_\mu, \tilde{H}_{\lambda}(t,q)\rangle}{\prod_{\substack{\square \in \lambda \\ a(\square) \neq 0}} (1 - t^{a(\square)} q^{-l(\square)-1})}, \end{aligned}$$

where, on the right-hand side, we implicitly specialize q to the prime power $|\mathbf{F}_q|$.

Example 4.4. Taking $\lambda = (1^n)$, so that $\eta_\lambda = 0$, the identity becomes

$$C_{1^n,\mu,q}(t) = |\mathscr{B}_\mu(\mathbf{F}_q)| = \langle h_\mu, \tilde{H}_{1^n}(t,q)\rangle.$$

Note that above, each expression is a polynomial in q alone.

Example 4.5. Taking $t = 0$, we find that the Macdonald expression becomes

$$\langle h_\mu, \tilde{H}_\mu(0,q)\rangle = |\mathscr{B}^{\eta_\lambda}_\mu(q)|.$$

So in this limit, the identity becomes $C_{\lambda,\mu,q}(0) = |\mathscr{B}^{\eta_\lambda}_\mu(q)|$. This, in turn, is equivalent to the claim that any element $\theta \in \mathscr{V}^{\mathscr{O}}_{\eta_\lambda}$ with $d_{\eta_\lambda,\theta} = 0$ is already conjugate to η_λ under $G(\mathscr{O})$. These statements generalize the previous example.

Example 4.6. Take $n = 2$ and $\lambda = (2)$. For all integers $i \geq 0$, let

$$g_i = \begin{pmatrix} 1 & \\ & \varpi^i \end{pmatrix} \in G(F), \quad \theta_i = \begin{pmatrix} 0 & \varpi^i \\ & 0 \end{pmatrix} \in \mathscr{N}(\mathscr{O}).$$

Then $g_i^{-1}\eta g_i = \theta_i$, and in fact, the θ_i form a full set of representatives for the adjoint action of $G(\mathscr{O})$ on $\mathscr{V}^\eta$. Note that

$$G(F)_\eta = G(F)_{\theta_i} = \left\{\begin{pmatrix}1 & F \\ & 1\end{pmatrix}\right\},$$

from which $J_\eta = J_{\theta_i} = \left\{\begin{pmatrix}1 & \mathscr{O} \\ & 1\end{pmatrix}\right\}$. We compute:

$$\begin{aligned}
\mathscr{T}_{\eta,\theta_i} = \mathscr{T}^{\mathrm{ks}}_{\eta,\theta_i} &= G(F)_\eta g_i = g_i G(F)_{\theta_i} = \left\{\begin{pmatrix}1 & F \\ & \varpi^i\end{pmatrix}\right\} \subseteq G(F)_i^{\geq i}, \\
\mathscr{T}^{\mathrm{ks,max}}_{\eta,\theta_i} &= \left\{\begin{pmatrix}1 & \varpi^i\mathscr{O} \\ & \varpi^i\end{pmatrix}\right\}.
\end{aligned}$$

The latter display shows that for any $g \in \mathscr{T}^{\mathrm{ks,max}}_{\eta,\theta_i}$, we have

$$\begin{aligned}
J_\eta g J_{\theta_i} &= \left\{\begin{pmatrix}1 & \mathscr{O} \\ & \varpi^i\end{pmatrix}\right\}, \\
J_\eta g J_{\theta_i}/J_{\theta_i} &\simeq pt, \\
J_\eta \backslash J_\eta g J_{\theta_i} &\simeq \mathbf{F}^i.
\end{aligned}$$

We deduce that $d_{\eta,\theta_i} = i$ and $\mathrm{wt}_{\eta,\theta_i}(q) = q^{-i}$. Altogether,

$$\begin{aligned}
C_{\lambda,\mu,q}(t) = 1 + \left(\sum_{i\geq 1} t^i q^{-i}\right)|\mathscr{B}_\mu(\mathbf{F}_q)| &= 1 + \frac{tq^{-1}}{1-tq^{-1}} \cdot \begin{cases} 1 & \mu = (2), \\ 1+q & \mu = (1^2) \end{cases} \\
&= \frac{1}{1-tq^{-1}} \cdot \begin{cases} 1 & \mu = (2), \\ 1+t & \mu = (1^2). \end{cases}
\end{aligned}$$