Warmup X topological space, x in X

[draw picture]

Df Y sub X is a neighborhood of x iff
x in U sub Y for some open U sub X

Df the interior of Y sub X is the set

Int(Y) = bigcup_{U sub Y, open in X} U
= {x in X | Y is a nbd of x} [why?]

in the analytic topology on R, compute:

Int([0, 1)) = (0, 1)
Int({0, 1, 2, …}) = Ø
Int(R – {0, 1, 2, …}) = R – {0, 1, 2, …}
Int({1/n | n is a positive integer}) = Ø

Df Z sub X is closed iff X – Z is open in X

Df the closure of Y sub X is the set

Ȳ = Cl(Y) = bigcap_{Z supset Y, closed in X} Z
= X – bigcup_{V sub X – Y, open in X} Y
= X – Int(X – Y)

in analytic R, compute:

Cl([0, 1)) = [0, 1]
Cl({0, 1, 2, …}) = {0, 1, 2, …}
Cl(R – {0, 1, 2, …}) = R
Cl({1/n | n is a positive integer}) = {1/n}_n cup {0}

Review take $X = R$ and $A = [0, \infty)$ analytic

U open in A iff $U = A \cap V$ for some V open in X

Claim $[0, b)$ open in A [but not in X]

Pf $[0, b) = A \cap (-1, b)$

Claim if $a > 0$, then $[a, b)$ not open in A

Pf must show: no V open in X s.t.
$[a, b) = A \cap V$

if V exists, then there is $\delta > 0$ s.t. $B(a, \delta) \subset V$
after shrinking δ, can assume $B(a, \delta) \subset A$
but then $B(a, \delta) \subset A \cap V = [a, b)$

(Munkres §20–21) recall from real analysis:

Df a metric on a set X is a function
$d : X \times X$ to $[0, \infty)$

s.t., for all x, y, z in X,

1) $d(x, y) = 0$ implies $x = y$
2) $d(x, y) = d(y, x)$
3) $d(x, y) + d(y, z) \geq d(x, z)$

given $\delta > 0$, let $B_d(x, \delta) = \{y \text{ in } X \mid d(x, y) < \delta\}$

[note that x in $B_d(x, \delta)$, because $d(x, x) = 0 < \delta$]

Df the metric topology on X induced by d:

U is open in the metric topology iff
for all x in U, there is a $\delta > 0$ s.t. $B_d(x, \delta)$ sub U

Idea metric topology on X generalizes
analytic topology on R^n

Thm the metric topology really is a topology

Pf exactly like the proof that
the anlytic topology is a topology

[so how much weirder can it be?]

Ex in any X: the discrete metric defined by

$d(x, x) = 0, \quad d(x, y) = 1$ when $x \neq y$

1) and 2) easy
3) [how many cases to check? 5 but can combine]
if $x = z$:
$d(x, y) + d(y, z) \geq 0 = d(x, z)$
[because $d(-, -) \geq 0$]
if $x \neq z$:
either $y \neq x$ or $y \neq z$
so $d(x, y) + d(y, z) \geq 1 = d(x, z)$

observe $B_d(x, 1) = \{x\}$ for all x. thus:

Prop metric topology from the discrete metric is the discrete topology

Df we say a topology or topological space is metrizable iff the topology is induced by some metric

sometimes, different metrics induce the same topology

Lem suppose d induces T on X, d' induces T' on X

then T' is finer than T iff for all x in X and $\varepsilon > 0$, there is $\delta > 0$ s.t. $B_{d'}(x, \delta) \subseteq B_d(x, \varepsilon)$.

Pf exercise (Munkres Lem 20.2)

Ex [picture of $B_d(x, \delta)$ versus $B_\rho(x, \delta)$]

euclidean metric:
$d(x, y) = \sqrt{(x_1 - y_1)^2 + \dots + (x_n - y_n)^2}$

square metric:
$\rho(x, y) = \max(|x_1 - y_1|, \dots, |x_n - y_n|)$

observe:

$$\begin{aligned} d(x, y) &\leq \sqrt{n \max_i (x_i - y_i)^2} \\ &= \sqrt{n}\, \rho(x, y) \\ \rho(x, y) &= \sqrt{\max_i |x_i - y_i|^2} \\ &\leq d(x, y) \end{aligned}$$

shows:

$B_\rho(x, \varepsilon/\sqrt{n}) \subset B_d(x, \varepsilon)$

$B_d(x, \varepsilon) \subset B_\rho(x, \varepsilon)$

[in general:]

Df metrics d, d' are called equivalent iff there exist $A, B > 0$ s.t.

$d(x, y) \leq A\, d'(x, y)$ and $d'(x, y) \leq B\, d(x, y)$

uniformly in x and y

Lem if two metrics are equivalent, then their metric topologies coincide

Cor Euclidean and square metrics both induce the analytic topology on $\mathbb{R}^n$

Rem converse is false: given a metric d, let

$$d'(x, y) = d(x, y)/(1 + d(x, y))$$

then 1) d' is still a metric

2) metric topologies for d, d' coincide

3) d and d' need not be equivalent

[reason: equivalence involves uniformity in x, y]

Ex $\mathbb{R}^\omega = \{(x_1, x_2, \ldots) \mid x_i\text{'s in } \mathbb{R}\}$

$\mathbb{R}^\infty = \{(x_1, x_2, \ldots) \mid x_i \text{ eventually } 0\}$

Euclidean and square metrics still work on $\mathbb{R}^\infty$, but not on $\mathbb{R}^\omega$

but $u(x, y) = \min\{1, \sup_i |x_i - y_i|\}$ works…