2.

On Macdonald polynomials, following Haglund–Haiman–Loehr and Guo–Ram.

2.1. For any positive integer m, we set $[m] = \{1, \dots, m\}$. Fix a positive integer n. For all $u = (i, j) \in [n] \times \mathbf{Z}$, we define the *cylindrical coordinate* of u to be

$$\mathbf{c}(u) = i + nj.$$

For all $k \in \mathbf{Z}$, we define $u + k \in [n] \times \mathbf{Z}$ by the identity $\mathbf{c}(u + k) = \mathbf{c}(u) + k$.

2.2. Suppose that $S \subset [n] \times \mathbf{Z}$ is a subset uniformly bounded above in the second coordinate. We picture S as a set of lattice points in the xy-plane, bounded in the positive y-direction. For all $u \in S$, let

$$\begin{aligned} \mathsf{attk}_S(u) &= S \cap (u - [n] + 1), & & \\ \mathsf{leg}_S(u) &= S \cap (u + n\mathbf{Z}_{>0}), & \ell_S(u) &= |\mathsf{leg}_S(u)|, \\ \mathsf{arm}_S(u) &= \{v \in \mathsf{attk}_S(u) \mid \ell_S(v) \leq \ell_S(u)\}, & a_S(u) &= |\mathsf{arm}_S(u)|. \end{aligned}$$

2.3. A *filling* of S is a function $f : S \to [m]$. We say that $u \in S$ is a *descent*, *resp.* *ascent*, of such f if and only if

$$u - n \in S \quad \text{and} \quad f(u) > f(u - n), \textit{resp.}, f(u) < f(u - n).$$

Let $\mathsf{Des}(f)$, *resp.* $\mathsf{Asc}(f)$, be the collection of descents, *resp.* ascents of f. Let

$$\begin{aligned} \mathrm{maj}(f) &= \sum_{u \in \mathsf{Des}(f)} (\ell_S(u) + 1), \\ a(f) &= \sum_{u \in \mathsf{Des}(f)} a_S(u). \end{aligned}$$

In the literature, pairs $(u, v) \in S^2$ with $v \in \mathsf{attk}_S(u)$ and $f(u) > f(v)$ are called *inversions* of f. For all $u \in S$, let

$$\mathsf{Inv}_f(u) = \{v \in \mathsf{attk}_S(u) \mid f(u) > f(v)\}.$$

2.4. Let $\mathbf{K} = \mathbf{C}(q, t)$. Fix a positive integer N, and let $\mathbf{K}[X] = \mathbf{K}[X_1, \dots, X_N]$. For any finite subset $S \subseteq [n] \times \mathbf{Z}$ and filling $f : S \to [N]$, we set

$$X^f = \prod_i X_i^{|f^{-1}(i)|} \in \mathbf{K}[X].$$

Fix a tuple $\mu = (\mu_1, \dots, \mu_n) \in \mathbf{Z}^n_{\geq 0}$. Let $|\mu| = \sum_i \mu_i$. We write $\tilde{H}^{(N)}_\mu \in \mathbf{K}[X]^{S_N}$ for the modified (symmetric) Macdonald polynomial of the underlying partition of μ, after truncation to N variables. In the case where $N = n$, we write $E_\mu \in \mathbf{K}[X]$ for the nonsymmetric Macdonald polynomial of μ. Moreover, for all $z \in S_n$, we write E^z_μ for the relative generalization of $E_\mu \in \mathbf{K}[X]$ depending on z, so that $E^{\mathrm{id}}_\mu = E_\mu$.

2.5. Let $\mathrm{dg}(\mu) \subset [n] \times \mathbf{Z}$ be defined by

$$\mathrm{dg}(\mu) = \{(i, j) \mid 1 \leq j \leq \mu_i\}.$$

Following Theorem 5.1.1 of [HHL08], the Haglund–Haiman–Loehr (HHL) formula for $\tilde{H}_\mu$ states:

Theorem 2.1 (HHL)**.** *For all μ, we have*

$$\tilde{H}_\mu^{(N)} = \sum_{\sigma:\mathrm{dg}(\mu)\to[N]} X^\sigma q^{\mathrm{maj}(\sigma)} t^{-a(\sigma)} \prod_{u\in\mathrm{dg}(u)} t^{|\mathsf{Inv}_\sigma(u)|}.$$

2.6. For a general filling $f : S \to [m]$, we see that $\mathrm{maj}(f), a(f)$ depend only on S and $\mathsf{Des}(f)$, not on f itself. So for a fixed subset $D \subseteq \mathrm{dg}(\mu)$, let

$$\begin{aligned}\mathrm{maj}_\mu(D) &= \sum_{u\in D} (\ell_{\mathrm{dg}(\mu)}(u) + 1),\\ a_\mu(D) &= \sum_{u\in D} a_{\mathrm{dg}(\mu)}(u).\end{aligned}$$

It turns out that the formula

$$F_{\mu,D}^{(N)} := \sum_{\substack{\sigma:\mathrm{dg}(\mu)\to[N]\\ \mathsf{Des}(\sigma)=D}} X^\sigma \prod_{u\in\mathrm{dg}(\mu)} t^{|\mathsf{Inv}_\sigma(u)|}$$

defines a Lascoux–Leclerc–Thibon (LLT) polynomial, hence an element of $\mathbf{K}[X]^{S_N}$. The HHL formula can be regrouped:

Corollary 2.2. *For all μ, we have*

$$\tilde{H}_\mu^{(N)} = \sum_{D\subseteq\mathrm{dg}(\mu)} q^{\mathrm{maj}_\mu(D)} t^{-a_\mu(D)} F_{\mu,D}^{(N)}.$$

Remark 2.3. Note that $F_{\mu,D}^{(N)} = 0$ whenever D contains elements of $\mathrm{dg}(\mu)$ of the form $(i, 1)$: *i.e.*, elements in the first row of $\mathrm{dg}(\mu)$.

2.7. Below we consider the examples where $n = 2$ and $N = |\mu| = 4$.

To describe the possible subsets $D \subseteq \mathrm{dg}(\mu)$, we will list the corresponding sets $\mathbf{c}(D) := \{\mathbf{c}(u) \mid u \in D\}$.

To compute the LLT polynomials, we will first compute their Hall pairings with the homogeneous symmetric functions h_ν: that is, their expansions into monomial symmetric functions $\{m_\nu\}_\nu$. To obtain their expansions into Schur functions, we will use the following table of inverse Kostka numbers:

	s_4	$s_{3,1}$	s_{2^2}	$s_{2,1^2}$	s_{1^4}
m_4	1	−1		1	−1
$m_{3,1}$		1	−1	−1	2
m_{2^2}			1	−1	1
$m_{2,1^2}$				1	−3
m_{1^4}					1

Example 2.4. Take $\mu = (2,2)$. There are four subsets $D \subseteq \mathrm{dg}(\mu)$ disjoint from the first row of $\mathrm{dg}(\mu)$. Below, we list them alongside the expansions of the corresponding LLT polynomials into monomial symmetric functions.

$\mathbf{c}(D)$	m_4	$m_{3,1}$	m_{2^2}	$m_{2,1^2}$	m_{1^4}
$\emptyset$	1	$1+t$	$1+t+t^2$	$1+2t+t^2$	$1+3t+2t^2$
$\{3\}$		t	t	$2t+t^2$	$3t+3t^2$
$\{4\}$		t	t	$2t+t^2$	$3t+3t^2$
$\{3,4\}$			t	$t+t^2$	$2t+3t^2+t^3$

Next, the values $\mathrm{maj}(D), a(D)$ alongside the Schur expansions of the LLT polynomials:

$\mathbf{c}(D)$	$\mathrm{maj}(D)$	$a(D)$	s_4	$s_{3,1}$	s_{2^2}	$s_{2,1^2}$	s_{1^4}
$\emptyset$	0	0	1	t	t^2		
$\{3\}$	1	0		t		t^2	
$\{4\}$	1	1		t		t^2	
$\{3,4\}$	2	1			t	t^2	t^3

Thus, $\tilde{H}^{(4)}_{2,2} = s_4 + (t+qt+q)s_{3,1} + (t^2+q^2)s_{2^2} + (qt^2+qt+q^2t)s_{2,1^2} + q^2t^2s_{1^4}$.

Example 2.5. If we instead take $\mu = (1,3)$, then the D's and m_ν-coefficients are:

$\mathbf{c}(D)$	m_4	$m_{3,1}$	m_{2^2}	$m_{2,1^2}$	m_{1^4}
$\emptyset$	1	$1+t$	$1+t$	$1+2t$	$1+3t$
$\{6\}$		1	$1+t$	$2+2t$	$3+5t$
$\{4\}$		1	2	$3+t$	$5+3t$
$\{4,6\}$				1	$3+t$

The values $\mathrm{maj}(D), a(D)$ and Schur coefficients are:

$\mathbf{c}(D)$	$\mathrm{maj}(D)$	$a(D)$	s_4	$s_{3,1}$	s_{2^2}	$s_{2,1^2}$	s_{1^4}
$\emptyset$	0	0	1	t			
$\{6\}$	1	0		1	t	t	
$\{4\}$	2	0		1	1	t	
$\{4,6\}$	3	0				1	t

Thus, $\tilde{H}^{(4)}_{1,3} = s_4 + (t+q+q^2)s_{3,1} + (qt+q^2)s_{2^2} + (qt+q^2t+q^3)s_{2,1^2} + q^3ts_{1^4}$.

Example 2.6. If we instead take $\mu = (3,1)$, then the data is almost the same as for $\mu = (1,3)$. We just list the final table:

$\mathbf{c}(D)$	$\mathrm{maj}(D)$	$a(D)$	s_4	$s_{3,1}$	s_{2^2}	$s_{2,1^2}$	s_{1^4}
$\emptyset$	0	0	1	t			
$\{5\}$	1	0		1	t	t	
$\{3\}$	2	1		t	t	t^2	
$\{3,5\}$	3	1				t	t^2

Thus, $\tilde{H}^{(4)}_{3,1} = \tilde{H}^{(4)}_{1,3}$, as expected.

2.8. For any filling $f : S \to [m]$, we say that f is *non-attacking* if and only if

$$f(u) \neq f(v) \quad \text{whenever } u \in S \text{ and } v \in \mathsf{attk}(u).$$

For all $u \in \mathsf{Des}(f) \cup \mathsf{Asc}(f)$, we set

$$\mathsf{Coinv}_f(u) = \left\{ v \in \mathsf{arm}_S(u) \;\middle|\; \begin{array}{ll} f(u) \leq f(v) \text{ or } f(u-n) \geq f(v) & \text{if } u \in \mathsf{Des}(f), \\ f(u) < f(v) < f(u-n) & \text{if } u \in \mathsf{Asc}(f). \end{array} \right\}.$$

Let $\mathrm{dg}_0(\mu) \subset [n] \times \mathbf{Z}$ be defined by

$$\mathrm{dg}_0(\mu) = \{(i,j) \mid 0 \leq j \leq \mu_i\}.$$

For any filling $\sigma : \mathrm{dg}(\mu) \to [m]$ and $z \in S_n$, let $\sigma^z : \mathrm{dg}_0(\mu) \to [m]$ be defined by

$$\sigma^z|_{\mathrm{dg}(\mu)} = \sigma \quad \text{and} \quad \sigma^z(i,0) = z(i) \text{ for all } i.$$

Suppose that $N = n$ in our preceding setup. Following Theorem 3.5.1 of [HHL08] and Theorem 1.1(b) of [GR], Guo–Ram's generalization of the HHL formula for E_μ states:

Theorem 2.7 (Guo–Ram–HHL)**.** *For all μ and z, we have*

$$E_\mu^z = \sum_{\substack{\sigma:\mathrm{dg}(\mu)\to[n] \\ \sigma^z \text{ non-attacking}}} X^\sigma q^{\mathrm{maj}(\sigma_z)} \prod_{u \in \mathsf{Des}(\sigma^z) \cup \mathsf{Asc}(\sigma^z)} \frac{(1-t)t^{\mathsf{Coinv}_{\sigma^z}(u)}}{1 - q^{\ell(u)+1} t^{a(u)+1}},$$

where $\ell(u) = \ell_{\mathrm{dg}_0(\mu)}(u) = \ell_{\mathrm{dg}(\mu)}(u)$ and $a(u) = a_{\mathrm{dg}_0(\mu)}(u)$ in the product.