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$$G = \mathrm{GL}_n$$

$$\mathcal{U} = \{\text{unipotent } n \times n \text{ matrices}\} \subseteq G$$

$$\underline{\text{Ex}} \quad \text{for } n = 2, \text{ have } \mathcal{U} \simeq \left\{ \begin{pmatrix} 1+x & y \\ z & 1-x \end{pmatrix} \middle| x^2 + yz = 0 \right\}$$

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$$B \subseteq G \text{ upper-triangular}$$

$$\underline{\text{Bruhat, Chevalley}} \quad \text{understand } G \text{ via } B:$$

$$G = \bigsqcup_{w \in S_n} B \dot{w} B$$

$$\text{can we understand } \mathcal{U} \text{ via } U := B \cap \mathcal{U}?$$

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$$U_- \subseteq B_- \subseteq G \text{ lower-triangular}$$

$$\underline{\text{Fine-Herstein '58, Steinberg '65}}$$

$$|\mathcal{U}(\mathbb{F}_q)| = q^{n(n-1)} = |U(\mathbb{F}_q)|^2 = |UU_-(\mathbb{F}_q)|$$

$$\underline{\text{Kawanaka '75}} \quad \text{for any } w \in S_n,$$

$$|\overbrace{(\mathcal{U} \cap B \dot{w} B)}^{\mathcal{U}_w}(\mathbb{F}_q)| = |\overbrace{(UU_- \cap B \dot{w} B)}^{\mathcal{V}_w}(\mathbb{F}_q)|$$

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$$\underline{\text{Ex}} \quad \text{for any } n, \text{ have } \mathcal{U}_{\mathrm{id}} = U = \mathcal{V}_{\mathrm{id}}$$

$$\underline{\text{Ex}} \quad \text{for } n = 3 \text{ and } \dot{w} = \begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix}, \text{ not even homeo over } \mathbb{C}$$

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diagonal $T \curvearrowright \mathcal{U}_w, \mathcal{V}_w$ by conjugation

Thm (T) $\mathrm{gr}_*^{\mathbf{W}} \mathrm{H}_{c,T}^*(\mathcal{U}_w(\mathbb{C})) \simeq \mathrm{gr}_*^{\mathbf{W}} \mathrm{H}_{c,T}^*(\mathcal{V}_w(\mathbb{C}))$

where \mathbf{W} is the weight filtration on $\mathrm{H}_{c,T}^*$

almost implies (Kawanaka) via results of Katz

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Conj (T) the T -equivariant map

$$UU_- \rightarrow \mathcal{U} \quad \text{given by } xy \mapsto xyx^{-1}$$

restricts to a homotopy equivalence $\mathcal{V}_w(\mathbb{C}) \rightarrow \mathcal{U}_w(\mathbb{C})$

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for $\beta = \sigma_{w_1} \cdots \sigma_{w_k} \in Br_n^+$,

$$\begin{array}{ccc} X_\beta^C & \rightarrow & X_\beta := B\dot{w}_1 B \times^B B\dot{w}_2 B \times^B \cdots \times^B B\dot{w}_k B \\ \downarrow & & \downarrow \\ C & \rightarrow & G \end{array}$$

$\mathcal{U}_w = X_{\sigma_w}^{\mathcal{U}}$ and $\mathcal{V}_w = X_{\sigma_w \pi}^1$, where π is the full twist

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Thm (T) for any $\beta \in Br_n^+$,

$$\begin{aligned} |X_\beta^{\mathcal{U}}(\mathbb{F}_q)| &= |X_{\beta\pi}^1(\mathbb{F}_q)|, \\ \mathrm{gr}_*^{\mathbf{W}} \mathrm{H}_{c,T}^*(X_\beta^{\mathcal{U}}(\mathbb{C})) &\simeq \mathrm{gr}_*^{\mathbf{W}} \mathrm{H}_{c,T}^*(X_{\beta\pi}^1(\mathbb{C})) \end{aligned}$$

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Kálmán '09 writing P for HOMFLYPT,

$$P(\widehat{\beta})[a^{|\beta|-n+1}] = P(\widehat{\beta\pi})[a^{|\beta|+n-1}]$$

Gorsky–Hogancamp–Mellit–Nakagane '19

true with KhR superpolynomial \mathbb{P} in place of P

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Springer resolution $\tilde{\mathcal{U}} = \{(u, gB) \in \mathcal{U} \times G/B \mid ugB = gB\}$

$$\begin{array}{ccccccc} X_{\beta}^1 \times G/B & \rightarrow & \tilde{X}_{\beta}^{\mathcal{U}} & \rightarrow & X_{\beta}^{\mathcal{U}} & \rightarrow & X_{\beta} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \{1\} \times G/B & \rightarrow & \tilde{\mathcal{U}} & \rightarrow & \mathcal{U} & \rightarrow & G \end{array}$$

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Thm (T) if $\beta \in Br_n^+$, then $S_n \curvearrowright \mathrm{gr}_*^W \mathrm{H}_{c,T}^*(\tilde{X}_{\beta}^{\mathcal{U}})$ and

$$\mathbb{P}(\hat{\beta}) \propto (\Lambda^*(\mathbb{V}), \mathrm{gr}_*^W \mathrm{H}_{c,T}^*(\tilde{X}_{\beta}^{\mathcal{U}}(\mathbb{C})))_{S_n}, \quad \text{where } \mathbb{V} = \mathbb{C}^{n-1}$$

$$a \text{ tracks } \Lambda^*\text{-grading} \implies \mathbb{P}(\hat{\beta})[a^{\mathrm{lo}}] \propto \mathrm{gr}_*^W \mathrm{H}_{c,T}^*(X_{\beta}^{\mathcal{U}}(\mathbb{C}))$$

$$\implies \mathbb{P}(\hat{\beta})[a^{\mathrm{hi}}] \propto \mathrm{gr}_*^W \mathrm{H}_{c,T}^*(X_{\beta}^1(\mathbb{C}))$$

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Jordan type: $\mathcal{U} = \bigcup_{\lambda \vdash n} \mathcal{U}_{\lambda}$

$$\begin{aligned} \tilde{X}_{\beta}^{\mathcal{U}} &= \bigcup_{\lambda} (X_{\beta}^{\mathcal{U}_{\lambda}} \times \tilde{\mathcal{U}}_{u_{\lambda}}) \quad \text{for fixed } u_{\lambda} \in \mathcal{U}_{\lambda} \\ P(\hat{\beta})[a^{\mathrm{lo}+2k}] &= \sum_{\lambda} \underbrace{(\Lambda^k(\mathbb{V}), [\mathrm{H}^*(\tilde{\mathcal{U}}_{u_{\lambda}}(\mathbb{C}))]_q)_{S_n}}_{\text{known}} |X_{\beta}^{\mathcal{U}_{\lambda}}(\mathbb{F}_q)| \end{aligned}$$

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just as P arises from traces on Hecke algebras,

$$\begin{aligned} D_{mix,G}^b \text{Perv}(\mathcal{U}) &\simeq D^b \text{Mod}(\mathbb{C}S_n \ltimes \text{Sym}) && \text{(Rider)} \\ &\simeq \text{hTr}(\text{Hecke}(S_n)) && \text{(Gorsky–Wedrich)} \end{aligned}$$