MATH 340: ADVANCED LINEAR ALGEBRA
PROBLEM SET #4

SPRING 2025

Due Wednesday, February 12. You may consult books, papers, and websites as long as you cite all sources and write up your solutions in your own words.

Problem 1. Consider the linear operators on $\mathbf{R}^2$ (in column notation) defined by the following matrices:

$$A = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}, \qquad B = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix},$$
$$C = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad D = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

(1) Which of the linear operators (represented by) A, B, C, D preserve length? There may be more than one.
(2) Which are invertible?
(3) Among those that are invertible, which flip the orientation of the triangle with vertices $\left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$? (Give your own definition—possibly informal—of what "orientation" means.)

Problem 2. Let V, W be vector spaces over $\mathbf{R}$, and let $T : V \to W$ be a linear map. Recall the complexifications $V_{\mathbf{C}}, W_{\mathbf{C}}$ defined in #1 on Problem Set 2. Let $T_{\mathbf{C}} : V_{\mathbf{C}} \to W_{\mathbf{C}}$ be defined by

$$T_{\mathbf{C}}(u + iv) = Tu + i(Tv), \qquad \text{where } u + iv \text{ is our new notation for } (u, v) \in V_{\mathbf{C}}.$$

Show that $T_{\mathbf{C}}$ is a $\mathbf{C}$-linear map. It is, of course, called the *complexification* of T.

Problem 3. Let $T : V \to W$ and $T' : W \to U$ be linear maps.

(1) How do $\ker(T)$ and $\ker(T' \circ T)$ compare, as linear subspaces of V? Relate the injectivity of $T' \circ T$ to the injectivity of T.
(2) How do $\operatorname{im}(T')$ and $\operatorname{im}(T' \circ T)$ compare, as linear subspaces of U? Relate the surjectivity of $T' \circ T$ to the surjectivity of T'.
(3) Give an example of $T : V \to W$ and $T' : W \to U$ where $T' \circ T$ is bijective, but neither T nor T' is bijective.

Problem 4. Let $T : V \to V$ be a linear operator such that $T \circ T = I$.

(1) Show that $V = V_+ + V_-$, where

$$V_\pm = \{v \in V \mid Tv = \pm v\},$$

and that this is a direct-sum decomposition of V.
(2) How is (1) related to #8 on Problem Set 1 and #7 on Problem Set 3?

Problem 5. A linear operator $T : V \to V$ is called a *projection* if and only if $T \circ T = T$.

(1) Show that if $T : V \to V$ is a projection, then $V = \ker(T) + \operatorname{im}(T)$.
(2) Give *distinct* projections $T, T' : \mathbf{R}^2 \to \mathbf{R}^2$ such that

$$\operatorname{im}(T) = \operatorname{im}(T') = \{(x, 0) \mid x \in F\}.$$

In words: There are actually many distinct ways to project the xy-plane onto the x-axis.
(3) Use (1) to show that if $T, T' : V \to V$ are projections such that

$$\operatorname{im}(T) = \operatorname{im}(T') \quad \text{and} \quad \ker(T) = \ker(T'),$$

then $T = T'$, in contrast to (2). *Hint:* Show that $Tv = v$ for all $v \in \operatorname{im}(T)$.

Problem 6. Let $D, S : \mathbf{R}[x] \to \mathbf{R}[x]$ be the linear operators defined by

$$D(p)(x) = p'(x) \quad \text{and} \quad S(p)(x) = xp(x),$$

where $'$ means the derivative with respect to x. Show that $D \circ S - S \circ D$ is the identity map.

Problem 7 (Axler §3D, #20)**.** Show that for all $q \in \mathbf{R}[x]$, there is a polynomial $p \in \mathbf{R}[x]$ such that

$$q(x) = (x^2 + x)p''(x) + 2xp'(x) + p(3).$$

Hint: The map $p(x) \mapsto (x^2 + x)p''(x) + 2xp'(x) + p(3)$ is linear. Study its effect on the monomial basis of $\mathbf{R}[x]$.

Problem 8. Look up the definition of a *ring*. (There are, in fact, conflicting definitions in the literature. We will require multiplication to have an identity element, but we will not require it to be commutative.) Verify that

$$\mathrm{Mat}_n(F) = \{n \times n \text{ matrices over } F\}$$

forms a ring under entrywise addition and matrix multiplication.

Hint: It is annoying to check the associativity of multiplication by hand. For a cleaner proof, recall that composition of maps between sets—hence, of linear maps between vector spaces—is associative.