Intuitive Infinitesimal Calculus

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Contents

1 Differentiation

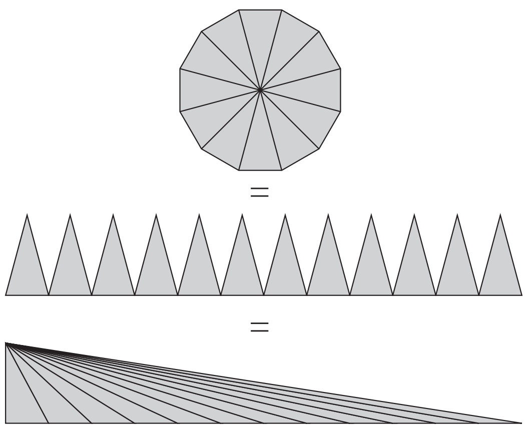
§ 1.1. Infinitesimals

§ 1.1.1. Lecture worksheet

The basic idea of the calculus is to analyse functions by means of their behaviour on a "micro" level. Curves can be very complicated when taken as a whole but if you zoom in far enough they all look straight, and if you slice a complicated area thin enough the slices will pretty much be rectangles. Lines and rectangles are very basic to work with, so at the micro level everything is easy. Here is an example:

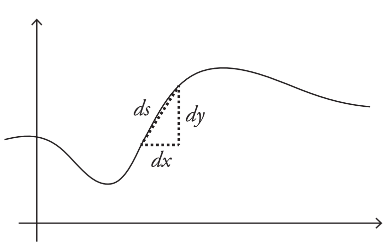
1.1.1. The area of a circle is equal to that of a triangle with its radius as height and circumference as base.

  (a) Explain how this follows from the figure below.



  (b) Explain why this is equivalent to the school formula $A = \pi r^2$.

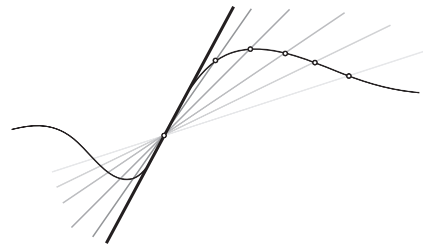
In the context of the calculus we utilise this idea by systematically dividing the x-axis into "infinitely small" or "infinitesimal" pieces, which we call dx ("d" for "difference"). Here I have drawn such a dx and the associated change dy in the value of the function:



Since dx is so small, the curve may be considered to coincide with the hypotenuse ds on this interval. Of course in the figure this is not quite so, but the figure is only schematic: in reality dx is infinitely small, so the hypotenuse ds really does coincide with the curve exactly, we must imagine.

In fact, if we extend the hypotenuse segment ds we get the tangent line to the curve.

1.1.2. Explain why. Hint: a tangent line may be considered a line that cuts a curve in "two successive points," i.e., as the limit of a secant line as the two points of intersection are brought closer and closer together:



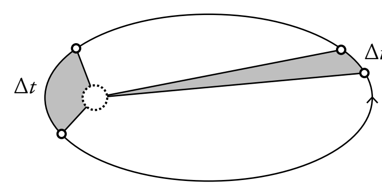
1.1.3. Why is it called a "tangent" line? Hint: the dance "tango" and the adjective "tangible" share the same Latin root.

1.1.4. Argue that $\frac{dy}{dx}$ is the slope of the graph.

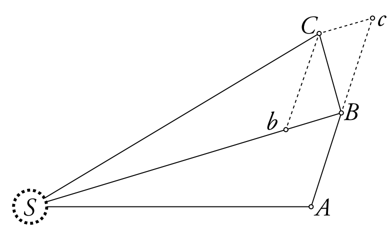
§ 1.1.2. Problems

1.1.5. Explain how the result of problem 1.1.1 can also be obtained by considering the area as made up of infinitesimally thin concentric rings instead of "pizza slices."

1.1.6. † Isaac Newton's *Philosophiae Naturalis Principia Mathematica* (1687) is arguably the most important scientific work of all time. The very first proposition in this work is Kepler's law of equal areas. The law says that planets sweep out equal areas in equal times:



Newton's proof uses nothing but very simple infinitesimal geometry.



In an infinitely small period of time the planet has moved from A to B. If we let an equal amount of time pass again then the planet would continue to c if it was not for the gravity of the sun, which intervenes and deflects the planet to C. Since the time it takes for the planet to move from B to C is infinitely small, the gravitational pull has no time to change direction from its initial direction BS, thus causing cC to be parallel to BS.

  (a) Conclude the proof of the law.

§ 1.2. The derivative

§ 1.2.1. *Lecture worksheet*

The idea that $\frac{dy}{dx}$ is the slope of the graph of $y(x)$ is very useful. It has many faces besides the geometrical one:

- *Geometrically,* $\frac{dy}{dx}$ is the slope of the graph of y.
- *Verbally,* $\frac{dy}{dx}$ is the rate of change of y.
- *Algebraically,*

$$\frac{dy}{dx} = \frac{y(x+dx) - y(x)}{dx}.$$

- *Physically,* the rate of change of distance is velocity; the rate of change of velocity is acceleration.

1.2.1. Sketch the graphs of the distance covered, the speed, and the acceleration of a sprinter during a 100 meter race, and explain how your graphs agree with the above characterisations of these quantities.

Of course the rate of change of $y(x)$ is generally different for different values of x. We use $y'(x)$ to denote the function whose value is the rate of change of $y(x)$. We call $y'(x)$ the *derivative* of $y(x)$. Thus $y(x)$ is the *primitive* function, meaning the starting point, while $y'(x)$ is merely *derived* from it. For example, $y'(0) = 3$ and $y'(1) = -1$ means that the function is at first rising quite steeply but is later coming back down, albeit at a less rapid rate.

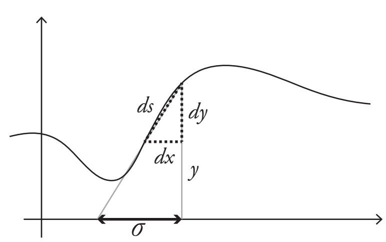
1.2.2. Express symbolically in terms of derivatives:

(a) The volume of the arctic ice $V(t)$ is shrinking.

(b) The population $P(t)$ grows at a rate of 10% per unit time.

§ 1.2.2. *Problems*

1.2.3. The length σ is called the subtangent:



(a) Express σ in terms of y and y'.

(b) A famous curve has "constant subtangent"—indeed this is how it was usually referred to in the 17th century. Which curve is it?

1.2.4. The derivative $y'(x)$ represents the "instantaneous" rate of change. In the case of a moving object the derivative of its distance is the velocity "at a given instant." Nevertheless it is useful to read the fractional interpretation $\frac{dy}{dx}$ of the derivative as "so many steps in y for so many steps in x." This is what we do for example when we speak of so-and-so many "miles per hour." It can be confusing that concepts like "per hour" or "per step in the x-direction" occurs in the description of something that is supposedly instantaneous and not at all ongoing for hours.

The confusion can be illustrated with the following scenario. A police officer stops a car.

OFFICER: The speed limit here is 60 km/h and you were going 80.

DRIVER: 80 km/h? That's impossible. I have only been driving for ten minutes.

OFFICER: No, it doesn't mean that you have been driving for an hour. It means that if you kept going at that speed for an hour you would cover 80 km.

DRIVER: Certainly not. If I kept going like that I would soon smash right into that building there at the end of the street.

(a) How can the officer better explain what a speed of 80 km/h really means?

Imagine an electric train travelling frictionlessly on an infinite, straight railroad. The train is running its engine at various rates, speeding up and slowing down accordingly. Then at a certain point it turns off the engine. Of course the train keeps moving inertially.

(b) Explain how this image captures both the "instantaneous" and the "per hour" aspect of a derivative in a concrete way.

1.2.5. Sometimes we consider the derivative of the derivative, or the "second derivative," y''. The second derivative is also denoted $\frac{d^2y}{dx^2}$.

(a) Argue that this notation makes algebraic sense by considering

$$\frac{d}{dx}\left(\frac{dy}{dx}\right)$$

(b) Also write out the meaning of

$$y''(x) = \frac{y'(x+dx) - y'(x)}{dx}$$

and show that it leads to the same result.

1.2.6. "In the fall of 1972, President Nixon announced that the rate of increase of inflation was decreasing. This was the first time a sitting president had used the third derivative to advance his case for re-election." (*Notices of the AMS*, Oct. 1996, 43(10), p. 1108.)

If Nixon was speaking of f''', what is f?

☐ rate of change of quantity of goods purchasable by one dollar

☐ quantity of goods purchasable by one dollar

☐ quantity of goods accumulated from t=0 until now by someone spending one dollar per unit time

☐ net change in quantity of goods purchasable by one dollar from t=0 until now

§ 1.3. Derivatives of polynomials

§ 1.3.1. Lecture worksheet

To find the derivative of $y(x)$ we should:

- let x increase by an infinitesimal amount dx;
- calculate the corresponding change in y, which is denoted dy;
- divide the two to obtain the rate of change $\frac{dy}{dx}$.

At the final stage we typically discard any remaining terms involving dx on the right hand side, since dx is infinitely small. We cannot, however, discard all dx's from the outset. This is because even though it is infinitely small, it still has an impact when considered in relation to dy. This is the way any small numbers work. If we have $5+0.00001$ then the second term can pretty much be discarded since it is so insignificant compared to the first. However, if we have $5+\frac{0.00001}{0.000001}$ then in fact those tiny numbers become very significant indeed, in this case even outweighing the "big" number and making the end result 15. Thus 0.00001 and 0.000001, though tiny on their own, become big when taken in ratio. In the same way, infinitesimal terms can be discarded in expressions like $5+dx$ but not in expressions like $\frac{dy}{dx}$. It is therefore safest to do our discarding only at the final step of our three-step plan for finding derivatives, since that is when we are done dividing.

In the case of $y(x)=x$ this goes as follows. Suppose x increases by dx. What is dy, the corresponding change in y? Quite clearly $dy=dx$ in this case, since the function "doesn't do anything" to the variable, but rather merely passes it along, whence the change in output equals the change in input. The derivative, therefore, is

$$\frac{dy}{dx}=\frac{dx}{dx}=1$$

That is to say, the rate of change of $y=x$ is 1; its slope is 1; it's always heading one step up for each step over.

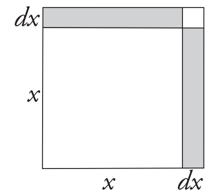
For the derivative of $y=x^2$ we follow the same plan. Suppose x increases by dx. What is the corresponding dy? It is

$$dy=(x+dx)^2-x^2=2x\,dx+(dx)^2$$

so

$$\frac{dy}{dx}=\frac{2x\,dx+(dx)^2}{dx}=2x+dx$$

Since dx is so small we can throw it away. Thus the derivative is $\frac{dy}{dx}=2x$. Note that the calculations correspond to this picture:



1.3.1. Find the derivative of x^3 and draw the corresponding picture. Thus, as its side length grows, the volume of a cube grows at a rate of $\boxed{}$ times its [side length, diagonal, surface area, volume, number of sides].

The pattern continues, giving the general differentiation rule $(x^n)'=nx^{n-1}$. This rule works also for non-integer exponents, as we shall see in problems 1.3.3, 1.3.4, 1.3.5, 1.5.3, 1.5.4.

1.3.2. (a) What is the formula for the volume of a sphere? (Just state it for now. We shall prove it in problem 4.1.1.)

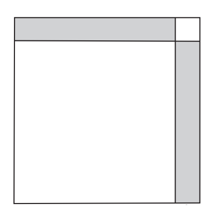
(b) Take its derivative with respect to the radius. What is the geometrical meaning of the result? The derivative of the volume of a sphere with respect to its radius is: [the radius, the radius squared, the surface area of the sphere, the area of the equatorial circle, the volume of the sphere, the volume of the circumscribed cube, none of the above].

(c) Draw the corresponding picture and compare with problem 1.3.1.

(d) The derivative of the area of a circle with respect to its radius is: [the radius, the radius squared, the area, the diameter, the area of the circumscribed square, the circumference, none of the above].

§ 1.3.2. Problems

1.3.3. Consider the function $f(x)=\sqrt{x}$. Geometrically, we can interpret $f(x)$ as the [area, side, diagonal, perimeter] of a square whose [area, side, diagonal, perimeter] is x. To investigate the derivative of $f(x)$, we let x increase by dx and look for the change df in $f(x)$. We can visualise it like this:



where (with everything expressed in terms of x and dx):

(a) area of big white square = $\boxed{}$

(b) total shaded area = $\boxed{}$

(c) short side of each shaded rectangle = $\boxed{}$

(d) Therefore, $df/dx=\boxed{}$

1.3.4. Find the derivative of $1/x$ algebraically and confirm that it agrees with the rule $(x^n)' = nx^{n-1}$. Hint: after writing out dy, combine the terms on a common denominator.

1.3.5. A pizza is to be shared among x friends. They cut it into so many equal pieces. Then one more friend shows up. The pizza now has to be divided into $x+1$ pieces, but the cutting had already taken place. Therefore each person cuts off an x^{th} piece of their slice and give it to the newcomer.

(a) How much smaller did each piece of pizza become?

(b) This illustrates the fact that the derivative of $f(x) = \boxed{}$ is $f'(x) = \boxed{}$.

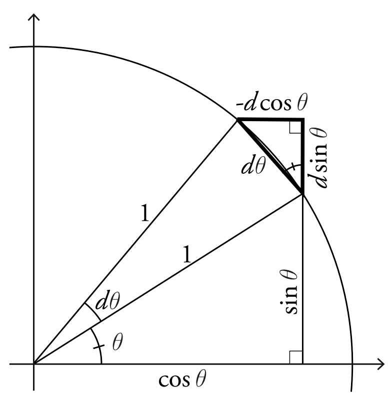
(c) Does everyone have the same amount of pizza in the end?

1.3.6. Imagine a string wrapped around the earth's equator. How much longer would the string need to be for it to be able to be raised one meter above the earth's surface at all points? What if you used a beach ball in place of the earth? By considering the derivative of the circumference of a circle with respect to its radius we see that the results are [equal, proportional to r, proportional to r^2, proportional to r^3].

§ 1.4. Derivatives of elementary functions

§ 1.4.1. Lecture worksheet

1.4.1. Explain why investigating the derivatives of sine and cosine leads to the following figure.



(a) To prove that the angles marked as equal really are equal, which of the following are useful?

☐ addition formula for cosine

☐ addition formula for sine

☐ angle sum of triangle

☐ radius of circle meets circumference at right angle

☐ product rule of differentiation

☐ Pythagorean Theorem

(b) If degrees were used instead of radians, which aspect(s) of the figure, if any, would become invalid?

☐ parts marked 1

☐ parts marked $d\theta$

☐ parts marked $\sin(\theta)$ and $\cos(\theta)$

☐ parts marked $d\sin(\theta)$ and $-d\cos(\theta)$

(c) Complete the ratio based on similar triangles: $\sin(\theta)\ /\ 1 = \boxed{} / \boxed{}$

In §A.5 we saw that exponential functions have the property that they grow in proportion to their size.

1.4.2. Formulate this in terms of derivatives.

In fact, the number e we mentioned there can be defined as the number such that $(e^x)' = e^x$. In other words, e is the base for the exponential function that is its own derivative.

1.4.3. To find the derivative of $\ln(x)$, write $y = \ln(x)$. Then the derivative we seek is $\frac{dy}{dx}$.

(a) Rewrite $y = \ln(x)$ in exponential form and find $\frac{dx}{dy}$.

(b) Invert the fraction to find $\frac{dy}{dx}$.

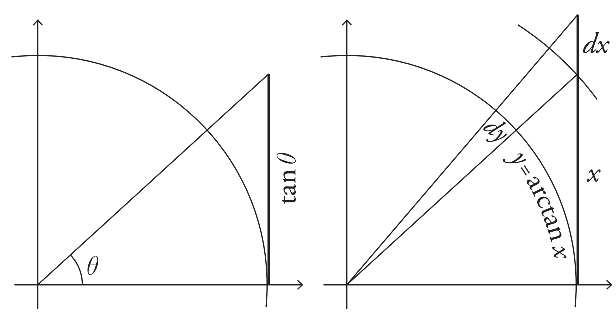
(c) What is the derivative of $\ln(x)$? Of course the answer should be expressed purely in terms of x.

§ 1.4.2. Problems

1.4.4. Find the derivative of $y = \arcsin(x)$ by differentiating $x = \sin(y)$ and inverting the fraction in the manner of problem 1.4.1.

Later we shall see a more geometrical way of arriving at this result in problem 4.2.4.

1.4.5. *Derivative of arctangent.* Recall the geometrical definitions of the tangent and arctangent functions from §A.3:



Let us find the derivative of the arctangent. In other words we are looking for dy/dx. In the figure I made x increase by an infinitesimal amount dx and marked the corresponding change in y. We need to find how the two are related. To do this I drew a second circle, concentric with the first but larger, which cuts off an infinitesimal triangle with dx as its hypotenuse.

(a) Show that this infinitesimal triangle is similar to the large one that has x as one of its sides.

(b) By what factor is the second circle larger than the first? (Hint: Find the hypotenuse of the triangle with x in it. Remember that the first circle was a unit circle.)

(c) Use this to express the "arc" leg of the infinitesimal triangle as a multiple of dy.

(d) Find dy/dx by similar triangles.

1.4.6. If $f(x) = \arctan(x) + \arctan(1/x)$, compute $f(1)$, $f(-1)$ and $f'(x)$, and explain the "paradox."

§ 1.5. Basic differentiation rules

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§ 1.5.1. Lecture worksheet

Above we found the derivatives of various standard functions. We must now consider how to find derivatives of functions that are built up by combining these functions in various ways.

Let us start with these simple rules:

$$\begin{aligned} (f+g)' &= f' + g' \qquad \text{(sum rule)} \\ (cf)' &= cf' \qquad \text{(coefficient rule)} \end{aligned}$$

These rules are quite evident already without any calculations.

1.5.1. Justify these rules in purely verbal, "common-sense" terms.

A less obvious rule is:

$$(fg)' = f'g + fg' \qquad \text{(product rule)}$$

1.5.2. Prove the product rule by viewing fg as the area of a rectangle with sides f and g and considering how the area changes as x (the implied variable) grows by dx.

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§ 1.5.2. Problems

1.5.3. Find the derivative of $1/x$ by letting $y = 1/x$ and differentiating xy.

1.5.4. Come up with a similar proof for $1/\sqrt{x}$.

1.5.5. To find the derivative of a quotient f/g of two functions we can simply write it as $f \cdot \frac{1}{g}$ and apply the product rule. So there is really no need for a separate rule. However, we can save ourselves some time by working out this product rule calculation once and for all, so that we have a rule for quotients "ready to go" for future reference. Do so.

1.5.6. What is the product rule for a product of three functions?

§ 1.6. The chain rule

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§ 1.6.1. Lecture worksheet

The final differentiation rule we need tells us what happens when one function is "trapped" inside another (like the links of a chain):

$$(f(g(x)))' = f'(g(x)) \cdot g'(x) \qquad \text{(chain rule)}$$

1.6.1. Argue that the chain rule can be written

$$\frac{d}{dx} f(g(x)) = \frac{df}{dg}\frac{dg}{dx}$$

and use this to give an algebraic justification for its truth.

The chain rule is typically quite evident when its meaning is spelled out for a real-world example; in fact you would probably often use it intuitively without even thinking about it as a formal rule. Consider for example the following scenario. A scientist observes that the boundary of a polar ice cap is receding by 3 km/year. The boundary is currently 2000 km from the pole. The ice cap may be considered a circle centered at the pole.

1.6.2. How fast is the area of the ice cap shrinking? What does this have to do with the chain rule?

1.6.3. Explain how the composite function of problem A.2.3 can be used to illustrate the chain rule. Hint: it is probably best to use specific numbers for illustration purposes.

§ 1.7. Limits

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§ 1.7.1. Lecture worksheet

Some people find the notions of infinitesimals and the infinitely small disagreeable. Mathematics should not be built on such mysterious and quasi-metaphysical notions, they say. No, give me good old real numbers, they say; otherwise it's not mathematics.

This kind of conservatism can be accommodated using the notion of limits. The limit of $f(x)$ as x goes to a, or in symbols $\lim_{x\to a} f(x)$, means the number that $f(x)$ approaches as x is taken closer and closer to the given number a. Thus for example $\lim_{x\to\infty} 1/x = 0$ because 1 divided by something very big is virtually zero. Also $\lim_{x\to 0} 1/x = \infty$ because as x becomes closer and closer to 0 (think of $x = 0.0001$ and such numbers) the function $1/x$ becomes very big and grows beyond bounds.

1.7.1. Actually we should be a bit more careful and write $\lim_{x\to 0^+} 1/x = \infty$ and $\lim_{x\to 0^-} 1/x = -\infty$. How so? Illustrate with a figure.

1.7.2. What is the visual meaning of $\lim_{x\to a} f(x) = f(a)$ in terms of the graph of f? In such cases we say that $f(x)$ is *continuous*.

The principles of the calculus can be formulated in terms of limits. Then instead of speaking of an infinitesimal increment dx we can speak of the limit of a finite increment Δx as $\Delta x \to 0$. Thus for example the derivative may be more formally defined as

$$\lim_{\Delta x \to 0} \frac{y(x+\Delta x) - y(x)}{\Delta x} \quad \text{instead of} \quad \frac{y(x+dx) - y(x)}{dx}$$

1.7.3. Write out the proof that $(x^2)' = 2x$ in both manners of expression side by side.

The limit approach has some advantages over freewheeling use of infinitesimals in certain technical contexts. However, as the above example shows, this comes at the cost of often needless pedantry. The infinitesimal manner of speaking is simpler and more suggestive, and we shall therefore stick to it throughout this book. But it will do us well to know that we can fall back on a more formal explication in the language of limits if we should run into tricky problems where the meaning of infinitesimals becomes unclear.

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§ 1.7.2. Problems

1.7.4. *L'Hôpital's rule.* Suppose you want to compute the limit

$$\lim_{x \to a} \frac{f(x)}{g(x)}$$

(a) Explain why this would be easy if f and g were continuous functions and $f(a) = 8$ and $g(a) = 2$.

If $f(a)$ and $g(a)$ are both zero, however, the situation is trickier. But consider this analogy. Suppose $f(x)$ and $g(x)$ represent the distance from the finish line of two sprinters in a race x seconds after they took off.

(b) What is the meaning of $f(a) = g(a) = 0$ in this context?

Suppose now that one of the runners was running twice as fast as the other during the last second of the race. Imagine watching a slow-motion replay of the finish.

(c) Argue that this makes it clear that

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

Thus in these kinds of "0/0" situations we can differentiate top and bottom of the fraction without altering the value of the limit. This often makes it possible to compute the limit.

1.7.5. The number e can be defined as a limit. This idea arises most naturally in the context of economics. Imagine that we have a bank account with a 100% interest rate. (This is a fantasy, no doubt, but mathematically speaking it is the simplest possible interest rate, so it is the natural starting point for a "pure" mathematical theory of interest.) In its simplest form this would mean that, once a year, on December 31, the bank looked at the balance of our account (or maybe the average balance across the year) and then gave us 100% of that amount. So if we started with \$1 we would have \$2 on January 1, and \$4 the year after that. However, why should interest calculations be based on the notion of a calendar year? There is no intrinsic reason for this. The bank could just as well give us interest payments every month or every week or whatever.

If we simply deposit an initial amount B_0 and leave it to grow, the formula for our balance after t years will be

$$B(t) = B_0\left(1 + \frac{100}{100}\right)^t$$

if the interest is added yearly,

$$B(t) = B_0\left(1 + \frac{100/12}{100}\right)^{12t}$$

if the interest is added monthly, and

$$B(t) = B_0\left(1 + \frac{100/52}{100}\right)^{52t}$$

if the interest is added weekly.

(a) What is the balance after one year in each of these scenarios?

(b) ⋆ Explain how these differences arise. Hint: consider the phenomena of "interest upon interest."

Thus the account holder should insist on more frequent compounding. And so should the mathematician, because mathematically there is no reason why the theory of interest should be based on any particular chunks of time that are nothing but arbitrary social conventions. Better then to denote the number of compounding occasions in a year by n and let $n \to \infty$ instead of fixing it at 1, 12, 52, or any other arbitrary number.

(c) Write down the balance formula for n compounding occasions per year.

(d) What is the balance after one year in this case?

(e) Let $n \to \infty$ in this expression. This limit is the number e.

(f) In §1.4 we said that e could be defined as the number such that $(e^x)' = e^x$. Explain how this is related to the economic definition.

§ 1.8. Reference summary

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§ 1.8.1. Meaning of derivative

$$\begin{aligned} y' &= \text{slope of graph of } y \\ &= \text{rate of change of } y \\ &= \text{number of units by which } y \text{ increases} \\ &\qquad \text{per unit increase in } x \text{ (at current rate)} \end{aligned}$$

$$y' \text{ positive} \iff \text{graph of } y \text{ heading upwards}$$

$$y' \text{ negative} \iff \text{graph of } y \text{ heading downwards}$$

$$y' = 0 \iff \text{graph of } y \text{ horizontal}$$

$$\text{magnitude of } y' = \text{ steepness of graph of } y$$

Δx	change in x; difference between two x-values (Δ = delta = difference)
dx	"infinitesimal" or "infinitely small" change in x (d = difference)
Δy, dy	change in y resulting from the change in x

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§ 1.8.2. Physical meaning of derivative

$$\frac{d}{dt}\text{distance} = \text{velocity}$$

$$\frac{d}{dt}\text{velocity} = \text{acceleration}$$

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§ 1.8.3. Derivatives of elementary functions

function	derivative	function	derivative
x^n	nx^{n-1}	$\arctan x$	$1/(1+x^2)$
$\sin x$	$\cos x$	$\arcsin x$	$1/\sqrt{1-x^2}$
$\cos x$	$-\sin x$	$\arccos x$	$-1/\sqrt{1-x^2}$
$\ln x$	$1/x$	$\log_a(a)$	$1/(x\ln(a))$
e^x	e^x	a^x	$a^x \ln(a)$

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§ 1.8.4. Derivatives of composite functions

$(f(g(x)))'$	$= f'(g(x)) \cdot g'(x)$	chain rule
$(fg)'$	$= f'g + fg'$	product rule
$(f/g)'$	$= (f'g - fg')/g^2$	quotient rule
$(f+g)'$	$= f' + g'$	sum rule
$(cf)'$	$= cf'$	coefficient rule

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§ 1.8.5. Problem guide

- Differentiate a given function when the function consists of …
  - … a number.

    Derivative is zero. (Derivative means rate of change and numbers don't change.)
  - … a number times something.

    Coefficient rule. Keep the number in front and differentiate the something.

    $(5x^2)' = 10x$
  - … an x is inside a root or in the denominator of a fraction.

    Rewrite in form x^n and apply differentiation rule for this form.

function	equivalent form	derivative
$\sqrt{x}$	$x^{1/2}$	$\frac{1}{2\sqrt{x}}$
$1/x$	x^{-1}	$-\frac{1}{x^2}$
$1/x^2$	x^{-2}	$-\frac{2}{x^3}$

  - … function times function.

    Product rule. Differentiate one and keep the other as it is, then vice versa, and add the results.

function	derivative
$x^3\sin(x)$	$3x^2\sin(x) + x^3\cos(x)$
xe^x	$e^x + xe^x$

  - … function divided by function.

    Quotient rule.

    $\left(\frac{2x}{1+x^2}\right)' = \frac{2(1+x^2)-2x(2x)}{(1+x^2)^2} = \frac{2-2x^2}{(1+x^2)^2}$
  - … function plus or minus function.

    Sum rule. Differentiate each separately and keep the sign in between.

    $(5x+3x^3)' = 5+9x^2$
  - … one expression contained inside another.

    Chain rule. Differentiate the outer function, leaving the inside as it is, then multiply by the inner derivative.

    In other words: Replace inside function by a single letter g, and differentiate as if g was the variable. Then substitute back the full expression for the inside function in place of g in the result. Then multiply the result by the derivative of g with respect to the true variable.

function to differentiate	outer	inner	derivative
$f(g(x))$	f	g	$f'(g(x)) \cdot g'(x)$
$\sqrt{1-x}$	$\sqrt{g}$	$1-x$	$\frac{1}{2\sqrt{1-x}} \cdot (-1)$
$\sin(x^2)$	$\sin(g)$	x^2	$\cos(x^2) \cdot 2x$
$\sin^3(x)$	g^3	$\sin(x)$	$3\sin^2(x) \cdot \cos(x)$
$(1+2x)^8$	g^8	$1+2x$	$8(1+2x)^7 \cdot 2$
$\frac{1}{x+x^3}$	g^{-1}	$x+x^3$	$-\frac{1}{(x+x^3)^2} \cdot (1+3x^2)$

– … an expression inside another inside another inside another …

Apply chain rule repeatedly. The inside derivatives will spit out inside derivatives of their own. Just keep multiplying.

> Differentiate $f(x) = \left(\sin(\sqrt{x})\right)^2$.
>
> ---
>
> $f' = \left(\left(\sin(\sqrt{x})\right)^2\right)' = 2\left(\sin(\sqrt{x})\right)\left(\sin(\sqrt{x})\right)' = 2\left(\sin(\sqrt{x})\right)\cos(\sqrt{x})(\sqrt{x})' = 2\left(\sin(\sqrt{x})\right)\cos(\sqrt{x})(\frac{1}{2}x^{-\frac{1}{2}}) = \sin(\sqrt{x})\cos(\sqrt{x})/\sqrt{x}$.

– … a very complicated algebraic expression involving roots, exponents and/or fractions.

If using the usual rules seems to daunting, try taking the logarithm of both sides of the equation, simplify with logarithm laws, and then use implicit differentiation.

- Estimate how much y changes (Δy) when x goes from some value x_0 to some other value x_1, given derivative of y.

  Since $\frac{\Delta y}{\Delta x} \approx \frac{dy}{dx}$, we get $\Delta y \approx \frac{dy}{dx} \cdot \Delta x$. Fill in $\Delta x = x_1 - x_0$ and $\frac{dy}{dx} = y'(x_0)$.

- Estimate $y'(a)$ given the graph or a table of values for $y(x)$.

  Focus on two x-values not too far apart, both in the vicinity of $x = a$. Find the difference between them. This is Δx. Determine the corresponding change Δy in y (i.e., how much y changes when you go from one of your x's to the other). Divide to obtain $\frac{\Delta y}{\Delta x} \approx \frac{dy}{dx}$.

- Find the derivative of $y(x)$ from first principles.

  Let x increase by an infinitesimal amount dx; calculate the corresponding change $dy = y(x + dx) - y(x)$ in y; divide the two to obtain the rate of change $\frac{dy}{dx}$.

---

§ 1.8.6. Limits

$$\lim_{x\to a} f(x) = \text{the number that } f(x) \text{ approaches as } x \text{ gets closer and closer to } a$$

L'Hôpital's rule: If, when trying to compute the limit

$$\lim_{x\to a} \frac{f(x)}{g(x)},$$

you get $\frac{0}{0}$ or $\frac{\infty}{\infty}$ when plugging in $x = a$, you can take the derivative of both numerator and denominator without altering the value of the limit:

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

- Find $\lim_{x\to a} f(x)$.

  Plug a into $f(x)$. If the result is a number: this is the answer. If the result is ∞: determine whether $f(x)$ is positive or negative as x approaches a; the answer is $+\infty$ or $-\infty$ accordingly. If the result is of the form $0/0$ or ∞/∞: rewrite $f(x)$ and try again. Rewriting could involve factoring, cancelling, algebraic/trigonometric/logarithmic identities, and L'Hôpital's rule. If $f(x)$ is oscillating (e.g. $\sin(x)$) without approaching any one value: limit does not exist.

---

§ 1.8.7. Examples

Differentiate:

> $\cos(x^2)$
>
> ---
>
> $-2x\sin(x)$

> $x^2 + \sin(x)$
>
> ---
>
> $2x + \cos(x)$

> $5\ln(x)$
>
> ---
>
> $5 \cdot \frac{1}{x}$

> xe^{-x^2}
>
> ---
>
> $e^{-x^2} - 2x^2 e^{-x^2}$

> $xe^{1/x}$
>
> ---
>
> $e^{1/x} + xe^{1/x}(-\frac{1}{x^2}) = e^{1/x}(1 - \frac{1}{x})$

> $(\frac{x+1}{x-1})^4$
>
> ---
>
> $4(\frac{x+1}{x-1})^3 \frac{1\cdot(x-1)-(x+1)\cdot 1}{(x-1)^2} = 4(\frac{x+1}{x-1})^3 \frac{-2}{(x-1)^2} = -8\frac{(x+1)^3}{(x-1)^5}$

> xe^{x^2+3x-2}
>
> ---
>
> $e^{x^2+3x-2} + xe^{x^2+3x-2}(2x+3) = e^{x^2+3x-2}(1 + x(2x+3)) = e^{x^2+3x-2}(2x^2+3x+1)$

> 2^x
>
> ---
>
> $2^x \cdot \ln 2$

> $\arctan\frac{1}{x}$
>
> ---
>
> $\frac{1}{1+\frac{1}{x^2}}(-\frac{1}{x^2}) = -\frac{1}{x^2+1}$

2 Applications of differentiation

§ 2.1. Maxima and minima

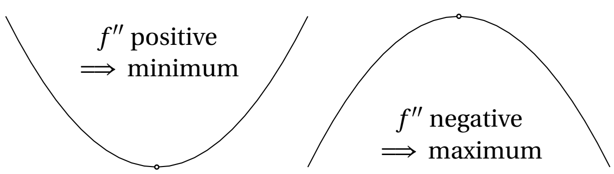
§ 2.1.1. Lecture worksheet

To find the maximum or minimum values of a function of one variable we look for x's such that $f'(x) = 0$.

2.1.1. Explain why. Hint: It may be easiest to argue this point in contrapositive form: if $f' \neq 0$ then it can't be a max or min.

2.1.2. To get a more hands-on feel for this, draw a closed curve with various squiggles in it on a piece of paper. Hold the paper up against the wall. Consider the tangent lines at the lowest and highest points. Rotate the paper in various ways an repeat.

Once found, the points where $f' = 0$ can be classified using second derivatives:



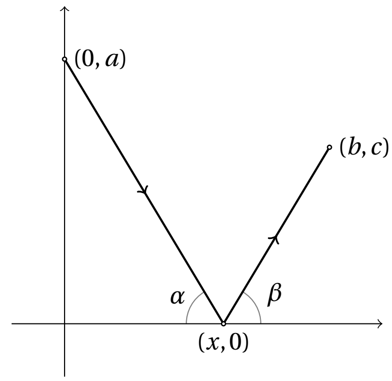
It is easy to remember which is which since a *positive* second derivative corresponds to a "happy mouth" shape and vice versa.

2.1.3. Explain why the classification works. Hint: What does f'' say about the slopes?

2.1.4. Prove that a square has greater area than any rectangle of the same perimeter.

According to ancient sources this principle used to be important for the purposes of division of arable land. Those ignorant of mathematics could be fooled into accepting a smaller plot by being led to believe that the value of a plot is determined by the number of paces around it. Then as now, it pays to be educated in mathematics.

2.1.5. The law of reflection says that when light reflects off a mirror, the angle of incidence α is equal to the angle of reflection β.



This follows from the principle that "nature does nothing in vain." To prove this, suppose light has to travel from $(0, a)$ to (b, c) via some point $(x, 0)$ on the x-axis.

(a) Express the distance travelled as a function of x.

(b) Find the derivative of this function and set it equal to zero.

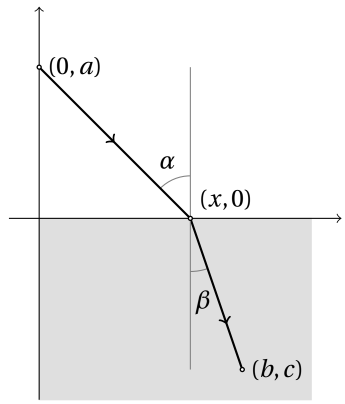
(c) Show that the law of reflection follows. Hint: interpret the fractions in your equation as ratios of sides of the triangles in the figure.

§ 2.1.2. Problems

2.1.6. Newton writes in his *Treatise of the Method of Fluxions and Infinite Series* (1671): "When a quantity is the greatest or the least that it can be, at that moment it neither flows backwards nor forwards: for if it flows forwards or increases it was less, and will presently be greater than it is; and on the contrary if it flows backwards or decreases, then it was greater, and will presently be less than it is. Wherefore find its Fluxion … and suppose it to be nothing." What result is Newton explaining? Translate his reasoning into modern calculus language.

2.1.7. The law of refraction says that when light passes from a medium where its velocity is c_1 to a medium where its velocity is c_2, the angles of incidence α and refraction β are related by

$$\frac{\sin(\alpha)}{\sin(\beta)} = \frac{c_1}{c_2}$$



We can prove this in essentially the same way as the law of reflection (problem 2.1.5). Suppose light has to travel from $(0,a)$ to (b,c) via some point $(x,0)$ on the x-axis.

(a) Express the time it takes the light to travel this path as a function of x.

(b) Find the derivative of this function and set it equal to zero.

(c) Show that the law of reflection follows. Hint: interpret the fractions in your equation as ratios of sides of the triangles in the figure.

(d) ⋆ Place a coin at the bottom of a bowl. Place your eye so that your view of the coin is just obstructed by the edge of the bowl. Pour water into the bowl. You can now see the coin. Explain the result in terms of the law of refraction.

§ 2.2. Concavity

§ 2.2.1. Lecture worksheet

Geometrically, $f(x)$ tells us how high up we are, and $f'(x)$ whether we are heading up or down. What is the geometrical meaning of $f''(x)$? It is the derivative of $f'(x)$, so it say whether $f'(x)$ is increasing or decreasing. In terms of $f(x)$ this has to do with how the graph "bends." We say that when $f''(x)$ is positive the graph is "concave up" and when it is negative the graph is "concave down." A point where we switch from one to the other is called an "inflection point."

2.2.1. What can you say about the concavity of a quadratic function?

2.2.2. Are the following true or false? Illustrate with figures. An inflection point could be:

(a) A maximum or a minimum.

(b) A point where $f' = 0$ yet no maximum or minimum occurs.

(c) A point where $f' \neq 0$.

2.2.3. (a) Argue that landing an airplane smoothly calls for a curve with an inflection point, since the flight path needs to be horizontal at the beginning and end of descent, yet sloping downwards in between.

In light of the above (and perhaps problem A.2.8a), we are inclined to try to model the descent path by a cubic polynomial $y(x) = ax^3 + bx^2 + cx + d$. Suppose descent starts at $(-L, H)$ and ends at $(0,0)$.

(b) What is the meaning of L and H?

(c) What can you infer about the values of the coefficients a, b, c, d based on what is known about the path and its endpoint slopes?

Assume that the plane needs to maintain a constant horizontal speed V throughout the descent. This is not entirely realistic; in reality planes do slow down some during descent. But it is not too far from the truth because the plane needs to keep a good speed so as to maintain flight aerodynamics and not go into free-fall mode, just like a waterskier starts to sink if he's not being pulled fast enough.

(d) Express the vertical acceleration in terms of L, H and V. Hint: find dy/dt using the chain rule.

(e) This equation can be used to compute any one of the variables in terms of the other three. What would be the most realistic practical application of this?

§ 2.3. Tangent lines

§ 2.3.1. Lecture worksheet

Derivatives can be used to find the equation for a tangent line of a curve since they give the slope m needed for the equation for a line $y = mx + b$. The y-intercept can then be determined by plugging in a known point.

2.3.1. Consider the parabola $y = x^2$.

(a) Find the equation for its tangent line in the point $(1,1)$.

(b) Find the equation for its tangent line in a general point (X, X^2).

(c) Do the same for the more general parabola $y = ax^2$.

(d) How can you characterise the y-intercept of these tangent lines in general, verbal terms?

(e) Use this information to give a calculus-free recipe for constructing tangent lines to parabolas.

This is in fact how tangents to parabolas were characterised over two thousand years ago by Apollonius in his classic Κωνικά, Prop. I.33.

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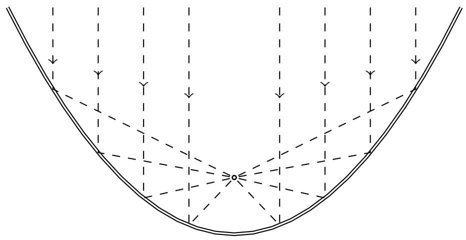
§ 2.3.2. Problems

2.3.2. *Focal property of the parabola.* Parabolic reflectors concentrate all incoming rays parallel to its axis in a single point:



This is useful for picking up as much as possible of a signal that has been weakened by travelling a long distance, such as communications from a satellite. It can also be used to focus the rays of the sun; legend has it that Archimedes set fire to enemy ships this way. The mirror also works in the converse direction: light originating at the focal point will be reflected into parallel beams. This is useful for making a limited amount of light reach as far as possible in a focussed direction, such as in the headlights of a car.

Let us prove that the focal property in fact holds. Suppose the parabola $y = x^2$ is a mirror. Consider a light ray parallel to the y-axis that comes in from above and strikes the parabola in the point where its slope is 1.

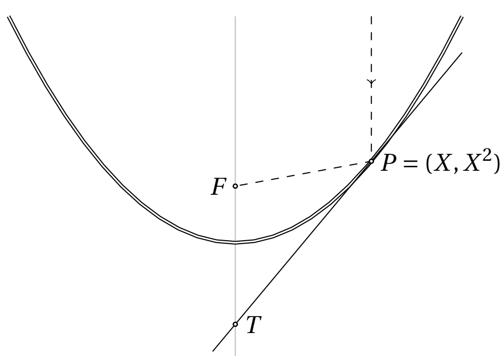
(a) What point is this?

(b) Use the law of reflection (problem 2.1.5) to find the slope of the ray after it has been reflected in the tangent line.

(c) What is the equation for the reflected ray?

(d) What is the y-intercept of the reflected ray?

Call this point F. The focal property says that all other rays are also reflected toward this point. To check that this is so, let $P = (X, X^2)$ be a general point on the parabola. Draw its tangent and let T be its intersection with the y-axis.

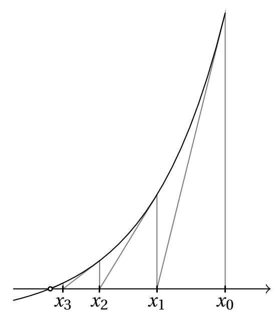


(e) What are the coordinates of T? Hint: this was determined in problem 2.3.1.

(f) Find the lengths FT and FP.

(g) Conclude the proof of the focal property of the parabola.

2.3.3. *Newton's method for finding roots of equations numerically.* Tangent lines can be used to find roots of equations numerically. Suppose we are looking for the points where a certain function $y(x)$ is zero. Often we cannot determine these values directly by setting the function equal to zero, since this equation may be much too difficult to solve by hand. In such a case we can proceed as follows. Let x_0 be out best guess as to where the root might be. Evaluating $y(x_0)$ will probably show us that it was not quite zero as we had aimed for. However, we then compute the tangent line to the function at this point and find its x-intercept. If our guess was reasonably close to a root this will give us a new x-value, x_1, which is closer to the root, as the figure shows. Then we can repeat the process, getting closer and closer.



(a) ⋆ This process only makes sense if we cannot solve $y(x) = 0$ but we can compute the tangent line in various points. Is it realistic for this to happen in practice?

(b) Given one of the x_n, find a formula for the next one, x_{n+1}. Hint: this is closely related to problem 1.2.3a.

(c) Apply this method to the equation $1/x - a = 0$. What expression for x_{n+1} does this give? What is the limit of x_n as $n \to \infty$?

(d) Explain briefly how this can be used to compute any quotient A/B without actually dividing (i.e., without using the division algorithm you learned in school). Some computer systems use this method for division since it requires fewer steps than the usual division algorithm, and thus saves computing time.

§ 2.4. Conservation laws

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§ 2.4.1. Lecture worksheet

Many useful physical laws are conservation laws, i.e., laws that say that some particular quantity is preserved. Mathematically speaking, proving a conservation law means proving that something has derivative zero with respect to time. In this section we shall derive a conservation law for motion in this way. As we shall see, proving that the derivative is zero follows at once from the basic laws of physics we summarised in §A.7.12. Thus this conservation law—and many others—are in a way really nothing but other basic laws in disguise (and a disguise, furthermore, that the calculus readily unmasks). Nevertheless conservation laws are so useful that physicists have made up names for the things that are preserved, such as "energy."

Suppose we fire a projectile straight up into the air. If its velocity is great enough if will never fall back down. The minimum velocity for which this happens is called the escape velocity. Calculating it is an example of a situation where using a conservation law is very convenient. Let m = mass of the projectile, v = velocity of the projectile, G = gravitational constant, M = mass of the earth, y = distance from the projectile to the center of the earth. I say that this conservation law holds:

$$\frac{mv^2}{2} - G\frac{Mm}{y} = \text{constant}$$

2.4.1. Check this by calculating the derivative with respect to time. Hint: recall from §A.7.12 that:

$$ma = F = \text{gravity} = -G\frac{Mm}{y^2}$$

(Later we shall reason our way to this conservation law in a more intuitive way, as opposed to merely checking it "after the fact," as it were. See §4.4.)

How does this help us calculate the escape velocity? If we have just enough initial velocity to escape to $y = \infty$ we will get there with zero velocity $v = 0$.

2.4.2. Plug in these conditions to determine the value of the constant in the conservation law corresponding to this scenario.

2.4.3. Use the conservation law with this value for the constant to determine the escape velocity. Hint: when the projectile is fired, y = radius of the earth.

2.4.4. ⋆ What are the physical names for the quantities in our conservation law?

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§ 2.4.2. Problems

2.4.5. Commercial aircrafts fly at an altitude of about 10,000 meters. With what velocity does a projectile need to be fired from the surface of the earth to reach this altitude?

2.4.6. The escape velocity for a black hole exceeds the speed of light. To what radius would the mass of the earth need to be compressed for it to become a black hole?

§ 2.5. Differential equations

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§ 2.5.1. Lecture worksheet

In many real-world scenarios we want to know the value of a certain function but we know, initially, only its derivative. If I put some money in the bank I may want to know how much I will have some years from now, for example when I retire. But this is not what the bank tells me. Instead they tell me the interest rate; that is to say, the rate at which the money is growing, or the derivative. In physics it's even worse. We may want to know for example the position of a satellite. But, like the bank, nature doesn't tell us. We have to start with Newton's law

$$\text{Force} = \text{mass} \times \text{acceleration},$$

and figure out the position from there. So nature tells us the second derivative (acceleration) of what we really want to know (position).

What we are given in these kinds of situations is a differential equation, i.e., not the function itself but some condition that its derivatives must fulfil. Differential equations are equations involving derivatives, such as $y' = y$. In words, this equation says: the rate of growth is equal to the current amount. So the more you have the faster it grows. Things that have this property include money and rabbits, as noted in §A.5.

A solution to a differential equation is a function that satisfies it. In our example e^x is a solution, since if $y = e^x$ then $y' = e^x = y$. But there are also other solutions. You can multiply e^x by any constant and it will still solve the equation: if $y = ce^x$ then $y' = ce^x = y$.

2.5.1. What is the real-world meaning of the constant c in the case of money? In general, constants that occur in solutions to differential equations are determined by plugging in known *initial conditions*.

2.5.2. Match each differential equation with the real-world scenario it models, and explain the meaning of the variables and constants involved.

   A. Growth of population with unlimited resources.

   B. Growth of population with limited resources.

   C. Motion of pendulum.

   D. Body in free fall.

   E. Predator-prey system (foxes/rabbits, shark/fish, etc.).

   F. Predator-prey system with harvesting (hunting, fishing, etc.).

   G. Growth of population with limited resources and harvesting.

   H. Conventional two-army warfare.

   I. Conventional army versus guerrilla. (Imagine the battle taking place in a jungle, with the guerrillas hiding in the trees and bushes. The guerrillas can target enemies as in any battle, but the conventional army, since they cannot see the guerrillas, can only fire their machine guns into the jungle somewhat randomly, hoping to hit a guerrilla.)

i. $y'' = -ky$

ii. $y' = ky$

iii. $y'' = -k$

iv. $y' = ky(a-y) - b$

v. $y' = ky(a-y)$

vi. $\begin{cases} x' = -by \\ y' = -ax \end{cases}$

vii. $\begin{cases} x' = ax - bxy - ex \\ y' = -cy + dxy - ey \end{cases}$

viii. $\begin{cases} x' = -by \\ y' = -axy \end{cases}$

ix. $\begin{cases} x' = ax - bxy \\ y' = -cy + dxy \end{cases}$

2.5.3. The case of pendulum motion is more important than it might seem. We shall come back to it later, but for now we can use it to introduce an important distinction between two types of equilibria. What are the two positions in which a pendulum (with a rigid rod) can be in equilibrium? What is the qualitative difference between them? This is a useful metaphor for many less concrete situations.

This is similar to which of the following?

- ☐ A ball rolling in a landscape of hills and valleys.
- ☐ A ball thrown vertically into the air.
- ☐ An elastic beam that is bent and then released.
- ☐ A planet orbiting the sun.

2.5.4. A disease spreads in proportion to the number of encounters between infected and healthy individuals. Argue that this can be captured by a differential equation of the same form as the population growth model in problem 2.5.2, but with a, y, k now corresponding to:

- ☐ number of infected
- ☐ infectiousness of disease
- ☐ recovery rate
- ☐ total population
- ☐ number of uninfected
- ☐ incubation time

Hence “the growth of a population is the spread of the disease of life,” so to speak. Differential equations often reveal analogies like this.

2.5.5. If we ignore air resistance the only force acting on a falling object is the constant gravitational acceleration ($g \approx 10$; see §A.7.12), which gives the differential equation $v' = -10$. Suppose you fire a gun straight up into the air. The initial velocity of the bullet is 1000 meters/second.

(a) Solve the differential equation for v as a function of t with the given initial condition.

(b) Find the height of the bullet as a function of time.

(c) With what velocity does the bullet strike the ground when it lands?

Of course this is unrealistic. The resistance of the air is considerable. Experience shows that it is roughly proportional to the square of the velocity. Then the differential equation is something like $v' = -10 - 0.004v|v|$.

(d) Why did I write $v|v|$ instead of v^2? Hint: Consider the difference between going up and coming down.

(e) With what velocity does the bullet strike the ground according to this model? Hint: Instead of solving the differential equation to find this out, use the reasonable assumption that the descending bullet will reach terminal velocity (i.e., a velocity at which it is no longer accelerating) before hitting the ground.

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§ 2.5.2. Problems

2.5.6. It makes sense that Newton’s law $F = ma$ has acceleration in it, because to stand still and to move with constant velocity is physically equivalent. That is, no physical experiment can tell one state from the other. This was known to Galileo, who explained it as follows in his *Dialogue Concerning the Two Chief World Systems* (1632).

> Shut yourself up with some friend in the main cabin below decks on some large ship, and have with you there some flies, butterflies, and other small flying animals. Have a large bowl of water with some fish in it; hang up a bottle that empties drop by drop into a wide vessel beneath it. With the ship standing still, observe carefully how the little animals fly with equal speed to all sides of the cabin. The fish swim indifferently in all directions; the drops fall into the vessel beneath; and, in throwing something to your friend, you need throw it no more strongly in one direction than another, the distances being equal; jumping with your feet together, you pass equal spaces in every direction. When you have observed all these things carefully (though doubtless when the ship is standing still everything must happen in this way), have the ship proceed with any speed you like, so long as the motion is uniform and not fluctuating this way and that. You will discover not the least change in all the effects named, nor could you tell from any of them whether the ship was moving or standing still. In jumping, you will pass on the floor the same spaces as before, nor will you make larger jumps toward the stern than toward the prow even though the ship is moving quite rapidly, despite the fact that during the time that you are in the air the floor under you will be going in a direction opposite to your jump. In throwing something to your companion, you will need no more force to get it to him whether he is in the direction of the bow or the stern, with yourself situated opposite. The droplets will fall as before into the vessel beneath without dropping toward the stern, although while the drops are

in the air the ship runs many spans. The fish in their water will swim toward the front of their bowl with no more effort than toward the back, and will go with equal ease to bait placed anywhere around the edges of the bowl. Finally the butterflies and flies will continue their flights indifferently toward every side, nor will it ever happen that they are concentrated toward the stern, as if tired out from keeping up with the course of the ship, from which they will have been separated during long intervals by keeping themselves in the air. And if smoke is made by burning some incense, it will be seen going up in the form of a little cloud, remaining still and moving no more toward one side than the other.

This means that physical laws cannot speak directly about velocity. An observer on the shore thinks the guy in the ship is moving; but the guy in the ship could claim that he is in fact standing still and that it is the guy on the shore that is moving. As we just saw, no physical experiment can settle their dispute, so they must both be considered to be equally right so far as physics is concerned. Nature does not distinguish between them, so her laws must be equally true for them both.

To illustrate this more formally, let the person on the shore be the origin of a coordinate system, and let the ship be traveling in the positive x-direction with constant velocity v. Now imagine releasing a butterfly inside the ship, in the manner described by Galileo. Suppose the butterfly moves in the x-direction only, and let $X(t)$ be its position in the coordinate system of an observer on the ship (i.e., taking a point inside the ship as the origin).

(a) Find the general formula for the position $x(t)$ of the butterfly in the coordinate system of the observer on the shore.

(b) Express the position, velocity, and acceleration of the butterfly in terms of both coordinate systems.

(c) What is the conclusion?

2.5.7. Rashevsky (*Looking At History Through Mathematics*, M.I.T. Press, 1968) proposed the following model of the increase of agnosticism on a historical timescale.

Assume that most people are receptive to common faiths while a small fraction pN of the total population N is naturally agnostic. Let's say that the birth rate is b and the death rate is d, so that $N' = (b-d)N$. The agnostic population A will grow because the agnostics bring up their children to be agnostic, while a fraction p of other births are naturally agnostic.

(a) Thus $A' = \boxed{}$.

The agnostics constitute a growing fraction $y(t)$ of the population, so that $A = yN$.

(b) Take the derivative of both sides in this equation with respect to time t. (Note that both N and y are functions of t.)

(c) From this we see that

☐ $y' = pb(1-y)$

☐ $y' = pb(y-1)$

☐ $y' = p(b-d)(1-y)$

☐ $y' = p(b-d)(y-1)$

(d) Therefore, with the initial condition $y(0) = 1/1000$, $y(t) = \boxed{}$.

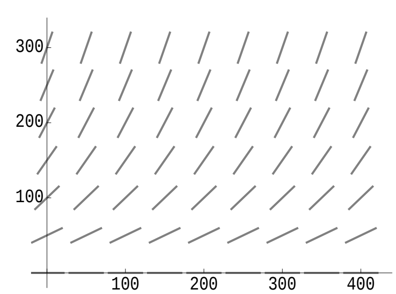
According to Rashevsky, "If we roughly assume that for religious beliefs … only about one person in a thousand in early antiquity was a natural agnostic" (i.e., $y(0) = p = 1/1000$) and that "the order of magnitude of b is about 10^{-2} individuals/year," then "in about the last 10,000 years, the time that has elapsed since the emergence of mankind from a primitive state, we find an increase of y from 1/1000 to only about 1/100," and indeed "we actually find that all the major religions … still share between them practically all of humanity." As for the future, "in 100,000 years the fraction y … will have increased to only about 2/3."

(e) Do we have all the information we need to verify these calculations with our formula for $y(t)$?

§ 2.6. Direction fields

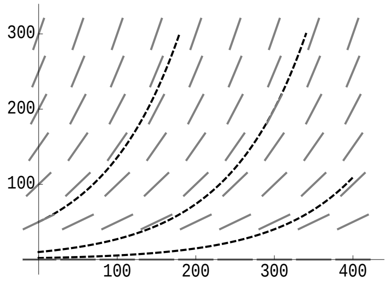
§ 2.6.1. Lecture worksheet

A useful tool for understanding a differential equation is its direction field. You construct it as follows: pick a point (x, y), plug these values for x and y into the differential equation, solve for y', and draw a little line segment with this slope at the point in question. Then you repeat this for many points until you see the pattern. Let us take the population growth with unlimited resources as an example. This is the direction field for $y' = 0.01y$:



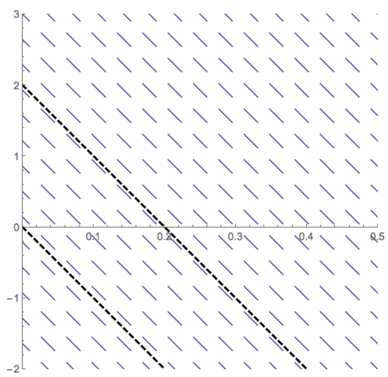
For example, the slope at the point $(0, 100)$ is 1, since plugging $x = 0$ and $y = 100$ into $y' = 0.01y$ gives $y' = 1$.

A solution to the differential equation must follow the direction field lines at every point. Thus once we have drawn the direction field we can easily see what the solutions of the equation will look like. Here I have drawn the solution curves corresponding to initial populations of 2, 10, and 50:



We see that the "biblical" case is slow to get off the ground.

2.6.1. Below is the direction field for the differential equation $y' = \boxed{}$. This corresponds to the differential equation of a freely falling object from problem 2.5.5 (with time on the x-axis and velocity on the y-axis).



The dashed solution curves correspond to falling objects with different [air resistance/initial height/initial velocity/weight].

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§ 2.6.2. Problems

2.6.2. One of the differential equations in problem 2.5.2 describes the growth of a population with limited resources and harvesting, such as the population of fish in a lake that is being fished by humans.

(a) Which one? Explain briefly the real-world meaning of the terms in the equation.

(b) Consider the case where $k = 0.01$, $a = 1000$, $b = 900$. Find the equilibrium values of y. What do they represent in real-world terms?

(c) Sketch a direction field for this differential equation.

(d) There is a qualitative difference between the two equilibria—what is it? Hint: What happens if you increase harvesting temporarily for one week?

(e) ⋆ One of the equilibrium values is lower than b. Explain why this might at first seem paradoxical in view of the real-world meaning of these values. Nevertheless the model makes sense even with these values for the constants. How can this be?

(f) Other constants remaining as above, what is the highest harvesting rate that can be maintained without eventually depleting the fish population (assuming that the initial population is large enough)?

(g) Draw the direction fields for this rate of fishing, and for a higher rate of fishing. Explain what these pictures show.

2.6.3. This problem is based on the differential equations for warfare in problem 2.5.2.

(a) In these equations, the derivatives are taken with respect to time. So y' means $\frac{dy}{dt}$. But by dividing one equation by the other we can obtain a new equation involving only $\frac{dy}{dx}$ and no t. Do this.

(b) Draw the corresponding direction field and explain how the course of a battle is reflected in this picture. Treat both the case where the armies have equal fighting efficiency and a case where one army is stronger.

(c) Use direction fields to illustrate what happens if one army receives troop reinforcements mid-battle.

(d) What if they receive better weaponry mid-battle instead, increasing their killing efficiency? Discuss what happens in terms of direction fields.

(e) Draw the direction field for conventional-versus-guerilla warfare, and use it to say something about the real-world differences between this and the conventional case.

§ 2.7. Reference summary

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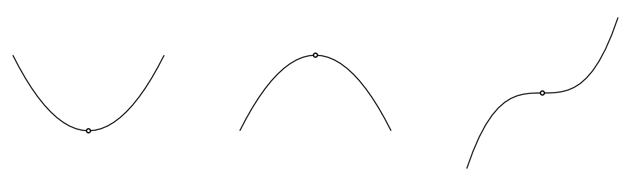
§ 2.7.1. Maxima and minima

Types of points with derivative zero:



minimum maximum inflection point

Second derivative test:

$$f'(a) = 0 \text{ and } f''(a) \text{ positive} \implies \text{(local) minimum}$$

$$f'(a) = 0 \text{ and } f''(a) \text{ negative} \implies \text{(local) maximum}$$

- Find (local) max. or min. of $f(x)$.

  Set $f'(x) = 0$ and solve for x.

- Classify values where $f'(a) = 0$ as max., min., or neither.

  $f''(a)$ positive $\implies$ min.; $f''(a)$ negative $\implies$ max. If $f''(a) = 0$, it could be max., min., or inflection point. To find out which, determine the value of $f(x)$ for values of x slightly greater than and less than a.

Find and classify the critical points of $f(x) = 2x^2 - \ln|x|$.

$f' = 4x - 1/x$, $f'' = 4 + 1/x^2$. $f' = 0 \implies 4x^2 - 1 = 0 \implies x = \pm\frac{1}{2}$. $f''(\pm\frac{1}{2}) = 8 > 0$, so $x = \frac{1}{2}$ and $x = -\frac{1}{2}$ are both local minima.

- Find global max. or min. of $f(x)$.

  Find local max. and min. as above. Check the value of $f(x)$ at each of these points. The biggest of these values is the global max. and the smallest the global min., if such exist. Investigate the values of $f(x)$ as x approaches $\pm\infty$ or any point at which $f(x)$ is not defined (such as a point corresponding to division by zero). If $f(x) \to \infty$ in any of these cases, no global maximum exists. If $f(x) \to -\infty$ in any of these cases, no global minimum exists.

Find any local and global extrema of $f(z) = 10 + 4z + z^2 - \frac{2}{3}z^3$.

Stationary points occur where $f'(z) = 4 + 2z - 2z^2 = 0$, which means $z_1 = -1$ or $z_2 = 2$. The second derivative is $f''(z) = 2 - 4z$. Since $f''(-1) = 6 > 0$ we see that $z_1 = -1$ is a local minimum (with value $f(-1) = \frac{23}{3}$). Since $f''(2) = -6 < 0$ we see that $z_2 = 2$ is a local maximum (with value $f(2) = -\frac{50}{3}$). Since $f(z) = 10 + 4z + z^2 - \frac{2}{3}z^3 = z^3(\frac{10}{z^3} + \frac{4}{z^2} + \frac{1}{z} - \frac{2}{3})$, we see that $\lim_{z\to-\infty} f(z) = \infty$ and $\lim_{z\to\infty} f(z) = -\infty$. Hence the function has no global maximum or global minimum.

- Find max. or min. of $f(x)$ when x is limited to a specific interval.

  Set $f'(x) = 0$ and solve for x; list the x-values that are in the given interval. Also include the endpoints of your interval in your list. Any max. or min. occurs at one of x-values in this list.

  To classify whether max., min., or neither, evaluate $f(x)$ at the given points and determine which points give the greatest and smallest values. Non-endpoints may also still be classified with the second derivative test.

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§ 2.7.2. Shape of graphs

f'' positive $\implies$ graph of f concave up
(slopes increasing, turning upwards)

f'' negative $\implies$ graph of f concave down
(slopes decreasing, turning downwards)

$f'' = 0$ and changing sign $\implies$ inflection point of f

- Sketch the shape of the graph of a function based on knowledge of its derivative.

  Find where the derivative is zero; at these points the function "goes flat," i.e., has a horizontal tangent.

  For each interval between these points, determine the sign of the derivative. Positive or negative derivative means that the graph goes up or down respectively. Assuming that the derivative is continuous, its sign will be the same at any point within one such interval; therefore it is enough to evaluate the derivative at any one point in the interval. The sign of the derivative on these intervals can also be inferred from the second derivative, if known (y'' positive $\implies$ y' increasing, and so on).

  Also determine what happens to the function as x goes to plus or minus ∞, either by examining the function or by determining whether the derivative is positive or negative for big values of x.

  A sketch agreeing with these three types of information will be a good approximation of the shape of the graph.

  Note that from information about the derivative alone it is not possible to know the vertical position of the graph, i.e., whether it needs to be shifted up or down. However, knowing the value of the function at any one point is enough to determine the vertical position of the graph.

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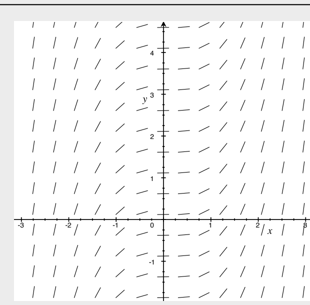
§ 2.7.3. Direction fields

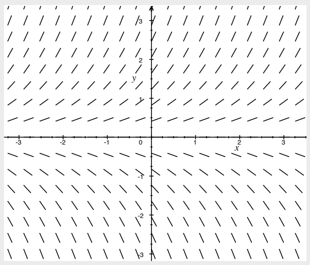
- Sketch the direction field of a differential equation.

  Pick a point (x, y), plug these values for x and y into the differential equation, solve for y', and draw a little line segment with this slope at the point in question. Repeat for many points until you see the pattern.

Draw the direction field for $y' = x^2$.



Draw the direction field for $y' = y$.



- Infer properties of a solution of a differential equation given its direction field.

  A solution curve will follow the direction field at all points. If a point on the solution curve is specified, the solution curve can be drawn by tracing along the direction field lines from that point.

In each of the cases above, if $y(x)$ is a solution to the given differential equation, what are the possible values $y(x)$ can approach as $x \to \infty$?

For $y' = x^2$: ∞. For $y' = y$: ∞, 0, or $-\infty$.

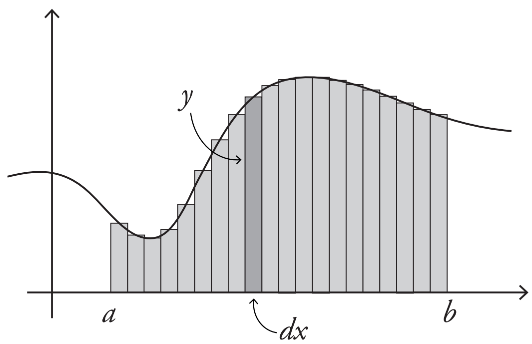
3 Integration

§ 3.1. Integrals

§ 3.1.1. *Lecture worksheet*

The integral $\int_a^b y\,dx$ means:

- *Algebraically,* the sum (hence the $\int$, which is a kind of "s") of infinitesimal rectangles with height y and base dx:



And thus:

- *Geometrically,* the integral $\int_a^b y\,dx$ is the area under the graph of $y(x)$ from $x = a$ to $x = b$. (Technically the "signed area": area below the x-axis is negative since the height of the rectangle, y, is a negative number.)

I now wish to convince you that also:

- *Physically,* the integral of velocity is distance; the integral of acceleration is velocity.

3.1.1. If I drive 50 km/h for one hour and then 100 km/h for three hours, how far did I go? Show that this is the area under the graph of the speed function.

3.1.2. Objects in free fall have a constant acceleration, according to Galileo. Draw the graphs of the acceleration, velocity, and distance fallen functions of a dropped stone, and explain how your graphs agree with the above characterisations of these quantities in terms of integrals.

Finally:

- *Verbally,* integrals often represent some kind of net change or net effect, though this viewpoint is not always applicable.

This shall become clearer in §3.2 but we can already feel it in the above examples.

3.1.3. (a) Argue that the verbal description too applies well to problem 3.1.1.

(b) How do these descriptions play out if I then drive backwards at 50 km/h for two hours? In which two senses can "distance travelled" be interpreted now? How can you express each as an integral?

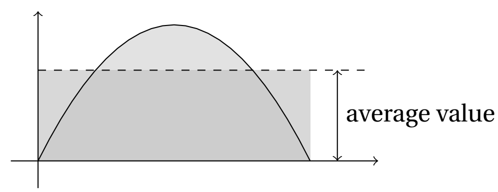
3.1.4. If $\sigma(h)$ is the density of water at depth h, what does the integral of $\sigma(h)$ from the surface to the bottom of the sea represent?

3.1.5. Which of the following have a common etymological root meaning with the mathematical term integral?

- ☐ We need to integrate immigrants into society.
- ☐ Common name for $1, 2, 3, \ldots$
- ☐ Spaghetti integrale.
- ☐ None of the above

§ 3.1.2. *Problems*

3.1.6. Argue that $\frac{1}{b-a}\int_a^b f(x)\,dx$ represents the average value of $f(x)$ on the interval $[a, b]$.



Hint: this can be done very concretely by thinking of the area under the graph as so much sand, for example.

3.1.7. Argue that $\int_a^b f(x) - g(x)\,dx$ represents the area between the graphs of these two functions. Illustrate with a figure.

3.1.8. *Social science: index of inequality.* Class inequalities are often quantified by saying that the richest so-and-so percent have so-and-so much of all the world's wealth. To put this in analytic form, let $F(x)$ be the fraction of a particular resource, such as money, owned by the poorest fraction x of the population. Thus $F(0.3) = 0.1$ means that the poorest 30% of the population owns 10% of the resource. Gini's index of inequality is one way to measure how evenly the resource is distributed. It is defined as the integral

$$2\int_0^1 x - F(x)\,dx.$$

(a) Show graphically what this integral represents in terms of the graph of $F(x)$.

(b) Which of the following must always be true in any society? (Assume that no one can have a negative amount of the resource, and that $F(x)$ is twice differentiable.)

- ☐ $F(0) = 0$
- ☐ $F(1) = 1$
- ☐ $F'(x) \geq 0$
- ☐ $F''(x) \geq 0$

☐ $F(x) < x$

☐ None of the above

(c) The maximum possible value of Gini's index of inequality is $\boxed{\quad}$ and the minimum is $\boxed{\quad}$. In the latter case $F(x) = \boxed{\quad}$.

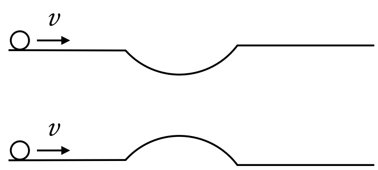
(d) Which of the following represents the most unequal society?

☐ $F(x) = x$

☐ $F(x) = x^2$

☐ $F(x) = x^3$

3.1.9. Which ball reaches the endpoint first? Disregard friction. (Hint: By energy conservation, both balls reach the endpoint with velocity v. Consider the graph of their velocity as a function of time.)



§ 3.2. Relation between differentiation and integration

§ 3.2.1. Lecture worksheet

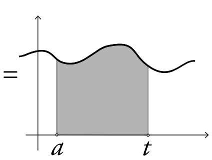
The fundamental theorem of calculus says that derivatives and integrals are each other's inverses in the following ways:

$$\frac{d}{dt}\int_a^t y(x)\,dx = y(t) \qquad \text{(FTC1)}$$

$$\int_a^b y'(x)\,dx = y(b) - y(a) \qquad \text{(FTC2)}$$

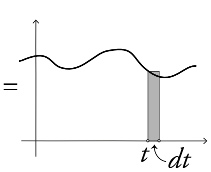
To prove FTC1 we proceed as with any derivative. In this case the variable is t and the function is $\int_a^t y(x)\,dx$.

$\int_a^t y(x)\,dx =$ area under $y(x)$ from a to $t =$



so if t increases by dt then $\int_a^t y(x)\,dx$ increases by

area under $y(x)$ from t to $t + dt =$



$= y(t)\,dt$,

so

$$\frac{d\int_a^t y(x)\,dx}{dt} = \frac{y(t)\,dt}{dt} = y(t),$$

which proves FTC1.

3.2.1. What happens if we take the derivative with respect to the lower bound instead? If $y(x)$ is a positive function, $\frac{d}{dt}\int_t^a y(x)dx = \boxed{\quad}$ because if the $\boxed{\quad}$ endpoint of integration is moved to the $\boxed{\quad}$ the area decreases.

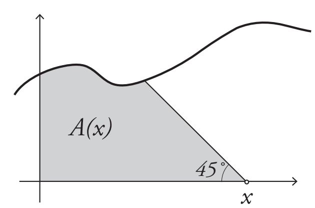
FTC2 is even easier to prove:

$$\begin{aligned}\int_a^b y'\,dx = \int_a^b \frac{dy}{dx}\,dx = \int_a^b dy\\ &= \text{sum of little changes in } y \text{ from } a \text{ to } b\\ &= \text{net change in } y \text{ from } a \text{ to } b\\ &= y(b) - y(a)\end{aligned}$$

§ 3.2.2. Problems

3.2.2. *Discrete analog of the fundamental theorem of calculus.* Write down a list of eight arbitrary numbers, leaving generous spaces around them. Above the gap between each pair of numbers, write down the sum of all numbers up to this point. Below the gap between each pair of numbers, write down the difference between those two numbers. Above and below the new lists, write down the sums of the difference list, and the differences of the sum list. Explain how this is related to the fundamental theorem of calculus.

3.2.3. † What happens if, in FTC1, we use a slanted line instead of a perpendicular one? In other words, what is $\frac{dA}{dx}$, with $A(x)$ defined like this:



3.2.4. Argue that FTC1 can be obtained by differentiating FTC2.

3.2.5. Explain why

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

in two ways:

(a) Geometrically in terms of the sign of dx.

(b) Algebraically in terms of FTC2.

(c) Which explanation do you prefer? Why?

3.2.6. (a) If I save all the money I earn, argue that my savings balance is the integral of my salary.

(b) Explain the meaning of FTC1 and FTC2 in the context of this example.

§ 3.3. Evaluating integrals

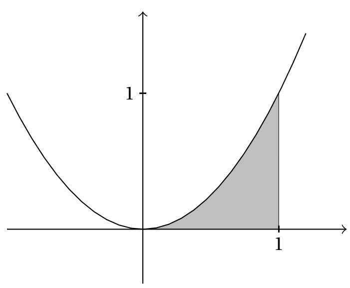
§ 3.3.1. Lecture worksheet

FTC2 says that in order to integrate some function $f(x)$ one has only to find an antiderivative $F(x)$, that is, a function such that $F' = f$, because then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

so to evaluate the integral we just have to plug in the bounds into $F(x)$ and take the difference between them.

To see that this is a very powerful result, consider the problem of finding the area under the parabola $y = x^2$ between $x = 0$ and $x = 1$.



Without the FTC alphabet soup you wouldn't know where to start, would you? But with this machinery the whole thing becomes reduced to a straightforward matter of manipulating symbols in a predictable way:

$$\int_0^1 x^2\,dx = \left[\frac{x^3}{3}\right]_0^1 = \frac{1^3}{3} - \frac{0^3}{3} = \frac{1}{3}$$

So in this case the antiderivative $F(x)$ was $\frac{x^3}{3}$. Figuring this out amounts to doing a differentiation problem "backwards": We are used to problems like:

The derivative of x^2 is $\boxed{}$.

But now we have to "anti-differentiate," i.e., solve problems like:

The derivative of $\boxed{}$ is x^2.

It follows that we can integrate or anti-differentiate most standard functions by reading our tables of derivatives backwards.

As in the case of differentiation, we also need rules for how to deal with functions that are built up from standard functions combined in various ways. Two simple rules are:

$$\int c f(x)\,dx = c \int f(x)\,dx$$

$$\int f(x) + g(x)\,dx = \int f(x)\,dx + \int g(x)\,dx$$

3.3.1. Explain why these rules are quite obvious.

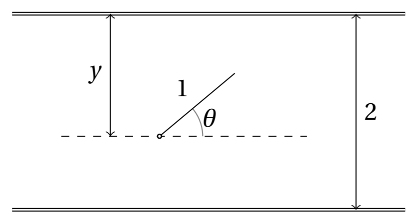
3.3.2. Evaluate $\int_1^2 1 + 2x^4\,dx$.

§ 3.3.2. Problems

3.3.3. In pre-revolutionary France, Georges Louis Leclerc, Comte de Buffon, spent his bourgeois leisure time tossing needles on the floor.

> I suppose that in a room in which the parquet floor is simply divided by parallel joints, one throws a stick in the air and one of the players bets that the stick will not cross any of the parallels of the parquet floor, and that the other bets on the contrary that the stick will cross some of these parallels; the chances of these two players are asked for. [Buffon, *Essai d'Arithmetique Morale*, 1777]

Let's say that we toss a 1-inch needle on a floor with 2-inch floor boards going east-west. The position of the needle is determined by two parameters: the distance y from the southern end of the needle to the joint to the north of it, and the angle θ the needle makes with the floorboards. Thus the possible values of y are $0 \le y < 2$ and the possible values of θ are $0 \le \theta < \pi$. Call this the "possibility space."



(a) Draw a coordinate system with θ and y as the x and y coordinates respectively, and indicate the possibility space in this picture.

(b) The needle will cross a joint if and only if $y <$ $\boxed{}$. (Use trigonometry.)

(c) Shade the subset of points of the possibility space satisfying this condition. The area of this shaded region is $\boxed{}$.

(d) The probability of the needle hitting a joint is the area of this shaded region divided by the total area of the possibility space. So the probability is $\boxed{}$.

3.3.4. The population $P(t)$ of a country is growing continuously at a rate of $(1 + t)$% per year, where t is the number of years from today.

(a) This means that $P'(t)/P(t) =$ $\boxed{}$.

(b) Find $\int_a^b \frac{P'(t)}{P(t)}\,dt$ in terms of $P(a)$ and $P(b)$. Hint: The integrand is a "logarithmic derivative." The integral evaluates to $\boxed{}$.

(c) How many percent bigger will the population be in 6 years compared to today? Hint: Work out the integral $\int_0^6 \frac{P'(t)}{P(t)}\,dt$ in two ways, once using (a) and once using (b). The population will have grown by $\boxed{\quad}$%.

§ 3.4. Change of variables

§ 3.4.1. Lecture worksheet

What is the antiderivative of $2x\cos(x^2)$? By "guess and check" you can find the answer $\sin(x^2)$, which can be verified using the chain rule. But it is not always so easy to do this kind of "backwards chain rule" problem in your head. A systematic technique for such integrals and more is substitution, or change of variables. To find $\int 2x\cos(x^2)\,dx$ we could introduce the new variable $u = x^2$. Then $\frac{du}{dx} = 2x$, so $du = 2x\,dx$. So the integral rewritten in terms of u becomes $\int 2x\cos(x^2)\,dx = \int \cos(x^2)(2x\,dx) = \int \cos(u)\,du$. This is easy to integrate: it's $\sin(u)+C$, or, putting the answer back in terms of x, $\sin(x^2)+C$.

This technique always works when we need to integrate something where a function is "trapped" inside another function, and the derivative of the trapped function appears on the outside (give or take a constant). In such a situation we should choose the trapped function as our new variable u. For example:

$\int x^2 e^{x^3}\,dx$	$u = x^3$
$\int 5x\sqrt{1-x^2}\,dx$	$u = 1-x^2$
$\int \cos(8x+1)\,dx$	$u = 8x+1$
$\int (\cos x)^7 \sin x\,dx$	$u = \cos x$

3.4.1. Solve the second example. If we drop the $5x$ the integral becomes much harder but also much more interesting—why? See problem 1.1.1.

3.4.2. Select all that are true:

- ☐ $\int \frac{1}{2x}dx = \frac{1}{2}\int \frac{1}{x}dx = \frac{1}{2}\ln(x)+C$
- ☐ Using the substitution $u = 2x$, $\int \frac{1}{2x}dx = \frac{1}{2}\int \frac{1}{u}du = \frac{1}{2}\ln(2x)+C$
- ☐ Using the substitution $u = 2x$, $\int \frac{1}{2x}dx = \int \frac{1}{u}du = \ln(2x)+C$

Explain the "paradox."

Let's solve another example: $\int_0^{\pi/2}\sin^3 x\cos^3 x\,dx$. Here it is not so clear which function is the "trapped" one. It was easier when we had all sines and one cosine, or the other way around, as in the last example in the table. But this can easily be arranged, for the identity $\sin^2 x+\cos^2 x = 1$ enable us to essentially "trade two cosines for two sines," or conversely. If we "trade in" two of the cosines in our integral we get $\int_0^{\pi/2}\sin^3 x(1-\sin^2 x)\cos x\,dx = \int_0^{\pi/2}\sin^3 x\cos x\,dx - \int_0^{\pi/2}\sin^5 x\cos x\,dx$. In each of these integrals it is clear that the substitution should be $u = \sin x$. As always when we make a substitution we immediately take its derivative, because we are going to need it to replace the dx in the original integral. In this case $\frac{du}{dx} = \cos x$, so $du = \cos x\,dx$. We must also remember the bounds of integration. These are of course x-values, so when we rewrite the integral in terms of u we must replace them with the corresponding u-values. For this we use the basic relation between u and x, namely $u = \sin x$, to find that the lower bound is $u = \sin 0 = 0$, and the upper bound is $u = \sin\pi/2 = 1$. Altogether we get

$$\begin{aligned}\int_0^{\pi/2}\sin^3 x\cos^3 x\,dx &= \int_0^{\pi/2}\sin^3 x\cos x\,dx - \int_0^{\pi/2}\sin^5 x\cos x\,dx\\ &= \int_0^1 u^3\,du - \int_0^1 u^5\,du\\ &= \left[\frac{u^4}{4}\right]_0^1 - \left[\frac{u^6}{6}\right]_0^1 = \frac{1}{4}-\frac{1}{6} = \frac{1}{12}\end{aligned}$$

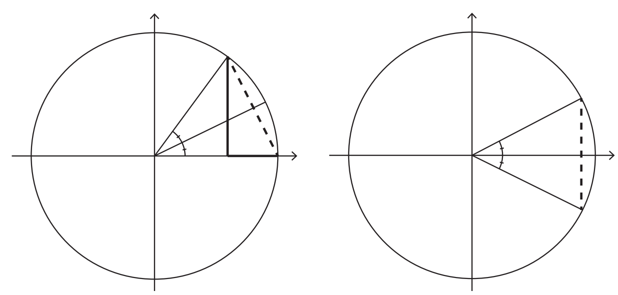
For $\int \sin^2 x\,dx$ and $\int \cos^2 x\,dx$ our trading trick doesn't work. The following example illustrates how we can deal with such cases.

3.4.3. Geometrically, $\int_0^1 \sqrt{1-x^2}\,dx$ is the area of:

(a) $\boxed{\quad}$

(b) Using the substitution $x = \cos(u)$, the integral becomes $\int_0^{\pi/2} = \boxed{\quad}\,du$

We can find a clever way of rewriting this integrand by expressing the dashed length in the figure below in two different ways (by the Pythagorean theorem in the left figure and in terms of sines in the right one).



(c) This gives $2\sin(u) = \boxed{\quad}$ and hence $\sin^2(u) = \boxed{\quad}$

(d) Hence the integral becomes $[\boxed{\quad}]_0^{\pi/2} = \boxed{\quad}$

§ 3.5. Integration by parts

§ 3.5.1. Lecture worksheet

Like the chain rule, the product rule also has an integral counterpart:

$$\int fg' = fg - \int f'g$$

3.5.1. Show that this follows immediately from the product rule. (So there is really no need to memorise it as a separate formula.)

This is called integration by parts. Basically it allows us to trade one integral for another, namely $\int fg'$ for $\int f'g$. For this to be a profitable trade the integral we bought should be simpler than the one we sold, i.e., the integral should simplify if we take the derivative of one factor (f) and the antiderivative of the other (g'). So good choices for f would be polynomials and logarithms, because they become simpler when differentiating, and good choices for g' would be sines, cosines, and exponential functions, because they do not become worse when integrating. For example:

$\int e^{x/2} x\,dx$	$f = x$	$g' = e^{x/2}$
$\int x^3 \sin x\,dx$	$f = x^3$	$g' = \sin x$
$\int x^2 \ln x\,dx$	$f = \ln x$	$g' = x^2$

In the second case we have to integrate by parts three times to "run down" the polynomial. In the last case we are forced to anti-differentiate x^2, which in itself makes the integral worse. But this is a small price to pay for getting a clear shot at the logarithm, which becomes vastly simpler when differentiating.

The rest is simple fill-in-the-blanks. The first example above for instance goes like this:

$$\int \underbrace{x}_{f}\underbrace{e^{x/2}}_{g'}\,dx = \underbrace{x}_{f}\underbrace{2e^{x/2}}_{g} - \int \underbrace{1}_{f'}\underbrace{2e^{x/2}}_{g}\,dx$$
$$= 2xe^{x/2} - 4e^{x/2} + C$$

3.5.2. Work out the last example in the table above.

---

§ 3.5.2. Problems

3.5.3. † *Geometrical interpretation of integration by parts.*

(a) Sketch a schematic representation of the parametric curve $(f(t), g(t))$. Assume that it is always heading upwards and to the right.

(b) Pick two t-values a, b and express the area under the curve between these two points as an integral.

(c) Draw and express the areas of the axis-parallel rectangles with lower-left point at the origin and upper-right point at $(f(a), g(a))$ and $(f(b), g(b))$ respectively.

(d) Explain how the integration by parts formula is geometrically evident from your figure.

§ 3.6. Partial fractions

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§ 3.6.1. Lecture worksheet

Consider the problem of integrating a function with several factors in the denominator, such as

$$\int \frac{1}{x(1-x)}\,dx.$$

The trick here is to split the integrand into *partial fractions*:

$$\frac{1}{x(1-x)} = \frac{A}{x} + \frac{B}{1-x}.$$

So I gave each factor of the denominator its own fraction, leaving the numerators as unknown constants. I say that we can in fact find numbers A and B that make this equation true. To find these numbers, multiply both sides by $x(1-x)$ to clear the denominators. This gives us $1 = A(1-x) + Bx$. Now, for the partial fraction decomposition to be valid this equation must be true for any value of x. So in particular it must be true if we plug in $x = 1$, for instance. I chose this value of x because it simplifies the equation so nicely; in fact, the equation now says $1 = B$, so we have figured out one of our constants. Another clever choice of x will be the other root, $x = 0$, which gives $1 = A$. So actually both constants are 1. Therefore

$$\int \frac{1}{x(1-x)}\,dx = \int \frac{1}{x} + \frac{1}{1-x}\,dx = \ln|x| - \ln|1-x| + C.$$

3.6.1. Find $\int \frac{3+x}{x^2-4}\,dx$.

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§ 3.6.2. Problems

3.6.2. (a) One of the differential equations in problem 2.5.2 describes the growth of a population with limited resources. Which one? Explain briefly the real-world meaning of the terms in the equation.

(b) For the case of the human species inhabiting the earth, make a rough estimate as to the values of the constants in the equation. Give brief justifications.

(c) Solve the differential equation. Hint: First find the derivative of time with respect to population, integrate this expression, and then solve for population as a function of time.

(d) Use the "biblical" case of an initial population of 2 to determine the constant of integration and sketch (possibly with computer assistance) the graph for this case.

(e) Mark the present-day population on the graph. How many years after $t = 0$ is it?

(f) At what population size does the inflection point occur? Hint: Use the differential equation instead of the formula for the population.

(g) Complete the sentence: "When population growth stops accelerating, the population has reached $\boxed{}$."

3.6.3. *Chemistry: rate of reaction.* Consider a chemical reaction in which one molecule of reagent A combines with one molecule of reagent B to produce one molecule of a compound X. The rate at which molecules of X are produced is proportional to the concentration of the reagents:

$$\frac{dx}{dt} = k(a-x)(b-x),$$

where x is the concentration of X, a, b are the initial concentrations of each reagent, in mols per unit volume, and k is a constant. Let us assume that $a < b$.

(a) Rewrite the equation in the form $\frac{dt}{dx} = \cdots$.

(b) Find t as a function of x. Determine the constant of integration using the fact that no molecules of the compound X are present at the beginning of the reaction.

(c) Plot the solution curve for $a = 1$, $b = 2$, $k = 1$. (Note that the same graph, when rotated, can be read as a graph of x as a function of t.)

(d) Usually one can speed up chemical reactions by increasing the temperature. Suppose this increases k to 2, but reduces each of the concentrations a and b by 10% due to heat expansion of the solution. Plot this new situation. Is the reaction faster than before?

§ 3.7. Reference summary

§ 3.7.1. Meaning and properties of integrals

$$\int_a^b y\,dx = \text{(signed) area under } y(x) \text{ from } x = a \text{ to } x = b$$

$$\int y\,dx = \text{indefinite integral of } y(x)$$
$$= \text{the general anti-derivative of } y(x)$$
(always includes constant of integration "$+C$")

(Terminology: $F' = f \iff F =$ anti-derivative of f)

$$\int cf(x)\,dx = c\int f(x)\,dx$$

$$\int f(x) + g(x)\,dx = \int f(x)\,dx + \int g(x)\,dx$$

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty}\int_a^b f(x)\,dx$$

§ 3.7.2. Applied meaning of integrals

$$\int_a^b \text{rate of change of something } dt = \text{ net change in that thing}$$

$$\int_a^b \text{velocity } dt = \text{ net distance travelled}$$

$$\int_a^b \text{acceleration } dt = \text{ net increase in velocity}$$

§ 3.7.3. Fundamental theorem of calculus

$$\frac{d}{dt}\int_a^t y(x)\,dx = y(t) \qquad \text{(FTC1)}$$

$$\int_a^b y'(x)\,dx = y(b) - y(a) \qquad \text{(FTC2)}$$

$$\int_a^b f(x)\,dx = F(b) - F(a) \qquad \text{(FTC2)}$$

§ 3.7.4. Integrals of elementary functions

function	anti-derivative	function	anti-derivative
x^n	$\frac{x^{n+1}}{n+1}$	$\frac{1}{x}$	$\ln\lvert x\rvert$
$\sin x$	$-\cos x$	$\cos x$	$\sin x$
$\frac{1}{1+x^2}$	$\arctan x$	$\frac{1}{\sqrt{1-x^2}}$	$\arcsin x$
e^x	e^x	a^x	$a^x / \ln(a)$

§ 3.7.5. Rules of integration

Substitution, simplest case:

$$\int g'(x) f(g(x))\,dx = F(g(x))$$

Integration by parts:

$$\int fg' = fg - \int f'g$$

Partial fractions, simplest case:

$$\int \frac{f(x)}{(x-a)(x-b)} = \int \frac{A}{x-a} + \int \frac{B}{x-b}$$

§ 3.7.6. Problem guide

- Integrate: a number times a function.

  Move the number out in front of the integral and integrate the function. The number remains a coefficient of the answer.

  $\int 5x dx = 5\int x dx = 5\frac{x^2}{2} + C$

- Integrate: a power of x.

  Integrate with power rule, i.e., increase exponent by 1 and divide by the new exponent.

  $\int x^3 dx = \frac{1}{4}x^4 + C$

  $\int \frac{dx}{x^3} = \int x^{-3} dx = \frac{x^{-3+1}}{-3+1} + C = -\frac{1}{2}x^{-2} + C = -\frac{1}{2x^2} + C$

- Integrate: an x is inside a root or in the denominator of a fraction.

  Rewrite in form x^n and apply integration rule for this form.

function	equivalent form	anti-derivative
$\sqrt{x}$	$x^{1/2}$	$\frac{2x^{3/2}}{3}$
$1/\sqrt{x}$	$x^{-1/2}$	$2\sqrt{x}$
$1/x^2$	x^{-2}	$-\frac{1}{x}$

$\int_1^4 \frac{1}{\sqrt{x}} dx$

$= \int_1^4 x^{-1/2} dx = 2x^{1/2}\big]_1^4 = 2(\sqrt{4}-\sqrt{1}) = 2$

- Integrate: function plus or minus function.

  Integrate each separately and keep the sign in between.

$\int (1-\frac{1}{\sqrt{x}}) dx$

$= \int (1-x^{-\frac{1}{2}}) dx = x - 2x^{\frac{1}{2}} + C = x - 2\sqrt{x} + C$

$\int \left(e^{-x} + \frac{1}{x^2}\right) dx$

$= \int \left(e^{-x} + x^{-2}\right) dx = -e^{-x} - \frac{1}{x} + C$

- Integrate: one expression contained inside another (including contained in denominator).

  If (a constant times) the derivative of the inside function occurs on the outside: Let u = (the inside function) and solve by substitution.

- Perform a substitution (change of variables) in an integral.

  Express the new variable u in terms of the old variable x or vice versa. Find $\frac{du}{dx}$ or $\frac{dx}{du}$ and use this to solve for dx. Use the resulting expression to replace the dx in the integral. Rewrite any remaining expressions involving x in the integrand in terms of u. Also translate the bounds from x-values into u-values by plugging them in for x in the formula defining u. Solve the resulting integral. If indefinite integral, rewrite the answer in terms of x by substituting back using the formula defining u.

$\int x \sin(x^2) dx$

Substitute $u = x^2$. Then $du/dx = 2x$, so $dx = du/2x$. Thus $\int x\sin(x^2)dx = \frac{1}{2}\int \sin(u)du = -\frac{1}{2}\cos(u) + C = -\frac{1}{2}\cos(x^2) + C$.

$\int_0^1 (x-1)^9 dx$

Substitute $u = x-1$. Then $\frac{du}{dx} = 1 \Rightarrow dx = du$, so the integral becomes $\int_{-1}^0 u^9\, du = \frac{u^{10}}{10}\big]_{-1}^0 = -\frac{1}{10}$.

$\int \frac{(\ln t)^{10}}{t} dt$

The substitution $u = \ln t$, which implies $du = \frac{1}{t}dt$, gives $\int \frac{(\ln t)^{10}}{t} dt = \int u^{10} du = \frac{1}{11}u^{11} + C = \frac{1}{11}(\ln t)^{11} + C$.

$\int \frac{1}{x \ln x} dx$

Let $u = \ln x$. Then $du = dx/x$, so $\int \frac{1}{x\ln x} dx = \int \frac{1}{\ln x}\frac{1}{x} dx = \int \frac{1}{u} du = \ln|u| + C = \ln|\ln x| + C$.

$\int \tan x\, dx$

Let $u = \cos x$. Then $du = -\sin x\, dx$. Thus $\int \tan x\, dx = \int \frac{\sin x}{\cos x} dx = \int \frac{1}{\cos x}\sin x\, dx = \int \frac{-1}{u} du = -\ln|u| + C = -\ln|\cos x| + C$.

- Integrate: function times function.

  Unless of the above form, try integration by parts:

  First determine which of the two functions you would gain most by differentiating: this will be your f. The sooner a function occurs in the following list, the more inclined you should be to make it your f: logarithmic functions, inverse trigonometric functions, algebraic functions, trigonometric functions, exponential functions. The remaining function is your g'.

  Now differentiate your f-function to find f' and anti-differentiate your g' to find g. Fill this into the integration by parts formula:

$$\int f g'\, dx = \underbrace{\quad}_{f}\underbrace{\quad}_{g} - \int \underbrace{\quad}_{f'}\underbrace{\quad}_{g} dx$$

  If bounds are involved, put them everywhere:

$$\int_a^b f g'\, dx = [\underbrace{\quad}_{f}\underbrace{\quad}_{g}]_a^b - \int_a^b \underbrace{\quad}_{f'}\underbrace{\quad}_{g} dx$$

  If a power of x is involved, repeated integration by parts is generally needed to bring it down one degree at a time.

  If the integrand is a sine or cosine times an exponential function, integrate by part twice. Denote the sought integral by I. You will find that I occurs also in your final expression. Replace that occurrence also by I, then solve for I in the resulting equation.

$\int e^{x/2} x\, dx$

$$\int \underbrace{x}_{f}\underbrace{e^{x/2}}_{g'} dx = \underbrace{x}_{f}\underbrace{2e^{x/2}}_{g} - \int \underbrace{1}_{f'}\underbrace{2e^{x/2}}_{g} dx$$
$$= 2xe^{x/2} - 4e^{x/2} + C$$

$\int x \cos x \, dx$

$= x \sin x - \int \sin x \, dx = x \sin x + \cos x + C.$

$\int x e^{2x} dx$

Integrate by parts with $f = x$ and $g' = e^{2x}$, which gives $f' = 1$ and $g = \frac{e^{2x}}{2}$. Hence $\int x e^{2x} dx = x \frac{e^{2x}}{2} - \int \frac{e^{2x}}{2} dx = x \frac{e^{2x}}{2} - \frac{e^{2x}}{4} + C = (\frac{x}{2} - \frac{1}{4}) e^{2x} + C.$

$\int \frac{\ln x}{x^2} dx$

$= -\frac{1}{x} \ln x + \int \frac{1}{x^2} dx = -\frac{1}{x} \ln x - \frac{1}{x} + C$

$\int \sin(x) e^x \, dx$

If you try to solve this by integration by parts you will feel that you are going in circles: when you have done integration by parts twice you are back to the same integral that you started with:

$$\int \sin(x) e^x \, dx = \sin(x) e^x - \cos(x) e^x - \int \sin(x) e^x \, dx.$$

But this is not as pointless as it might look. Denote the integral you are looking for by I. Then:

$$I = \sin(x) e^x - \cos(x) e^x - I.$$

Now you can solve for I to get

$$I = \frac{\sin(x) e^x - \cos(x) e^x}{2},$$

plus a constant of course.

$\int_0^2 t e^{-2t} dt$

$= [t \cdot \frac{e^{-2t}}{-2}]_0^2 - \int_0^2 \frac{e^{-2t}}{-2} dt = [t \cdot \frac{e^{-2t}}{-2}]_0^2 - [\frac{e^{-2t}}{(-2)^2}]_0^2 = 2 \cdot \frac{e^{-4}}{-2} - 0 - (\frac{e^{-4}}{4} - \frac{e^0}{4}) = -e^{-4} - \frac{e^{-4}}{4} + \frac{1}{4} = \frac{1 - 5e^{-4}}{4}$

- Integrate: $\ln(x)$.

  View as $1 \cdot \ln(x)$ and integrate by parts.

  $\int \ln(x) \, dx$

  $= x \ln x - \int x \frac{1}{x} dx = x \ln x - x + C.$

- Integrate: an inverse (arc) trigonometric function.

  View as 1 times the function and integrate by parts.

  $\int \arctan x \, dx$

  $= x \arctan x - \int \frac{x}{1+x^2} dx = x \arctan x - \frac{1}{2} \ln(1 + x^2) + C.$

- Integrate: a product of powers of sines and/or cosines.

  Only even powers occur: Rewrite using double angle formula

  $$\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$$

  The power of one of them is 1: Make that function u and solve by substitution.

  Other cases: Rewrite using $\sin^2 x + \cos^2 x = 1$ to the forms above.

  $\int_0^{\pi/2} \sin x \cos x \, dx$

  Let $u = \cos x$. Then $du = -\sin x \, dx$. Thus $\int_0^{\pi/2} \sin x \cos x \, dx = \int_1^0 -u \, du = \int_0^1 u \, du = \frac{1}{2} u^2 \Big]_0^1 = 1/2.$

  $\int \cos^2 x \, dx$

  $\int \cos^2 x \, dx = \int \frac{1+\cos(2x)}{2} dx = \int \left(\frac{1}{2} + \frac{1}{2}\cos(2x)\right) dx = \frac{1}{2} x + \frac{1}{2} \frac{\sin(2x)}{2} + C = \frac{1}{2} x + \frac{\sin(2x)}{4} + C.$

  $\int 3 \sin^3 x dx$

  $= \int 3 \sin x (1 - \cos^2 x) dx$. Substitute $u = \cos x$. Then $\frac{du}{dx} = -\sin x \Rightarrow du = -\sin x dx$. Hence the integral becomes $\int -3(1 - u^2) du = \int -3 + 3u^2 du = -3u + u^3 + C = -3\cos x + \cos^3 x + C.$

- Integrate: a product of sine and/or cosine factors with different coefficients of x.

  Rewrite using addition formulas for sine or cosine.

- Integrate: a ratio of two polynomials.

  (The method below requires that the degree of the numerator is lower than the degree of the denominator. If it is not, first bring it into such a form for example by factoring something out or performing a division of polynomials (§A.7.6).)

  Factor the denominator. The original fraction is equal to a sum of "partial" fractions found as follows:

  – A simple linear factor $(ax - b)$ in the denominator contributes a term $\frac{A}{ax-b}$.

  – A double linear factor $(ax-b)^2$ in the denominator contributes the terms $\frac{A}{ax-b} + \frac{B}{(ax-b)^2}$.

  – A triple linear factor $(ax - b)^3$ in the denominator contributes the terms $\frac{A}{ax-b} + \frac{B}{(ax-b)^2} + \frac{C}{(ax-b)^3}$.

  – And so on for higher powers.

  – An irreducible quadratic factor $(ax^2 + bx + c)$ in the denominator contributes a term $\frac{Ax+B}{ax^2+bx+c}$.

  – A repeated irreducible quadratic factor: analogous to repeated linear factor.

  Use a separate letter for the constants A, B, etc., in each term. These are some numbers yet to be determined. We find these numbers as follows. Set the original fraction equal to the sum of the partial fractions. Proceed in one of two ways:

- (Best for simpler cases.) Multiply both sides by the denominator of the original fraction. Many things cancel and we are left with an equation without fractions. Plug in the values for x that make one of the factors zero (i.e., the roots of the original denominator). Each time you plug in one of these values many terms will become zero and you will get some quite simple equations from which you can figure out the values of the constants A, B, etc.

- (Better in advanced cases.) Multiply out all parentheses and identify coefficients of like terms on left and right sides (coefficient of x^n on left hand side = coefficient of x^n on right hand side).

We have now reduced the problem of integrating the original fraction to the problem of integrating the sum of the partial fractions. This is done term by term.

- $\frac{A}{ax-b}$ integrates to a logarithm (substitute $u = ax - b$).
- $\frac{B}{(ax-b)^2}$, $\frac{C}{(ax-b)^3}$, etc., integrates using power rule (substitute $u = ax - b$).
- $\frac{Ax+B}{ax^2+bx+c}$ integrates to an arctangent (first complete the square in the denominator).

$\int \frac{x+1}{(x-1)(x-3)^2}\,dx$

$\frac{x+1}{(x-1)(x-3)^2} = \frac{A}{(x-1)} + \frac{B}{(x-3)} + \frac{C}{(x-3)^2}$
$\Rightarrow x+1 = A(x-3)^2 + B(x-1)(x-3) + C(x-1)$

$$\Rightarrow \left\{ \begin{array}{c} 0 = A+B \\ 1 = -6A-4B+C \\ 1 = 9A+3B-C \end{array} \right\} \Rightarrow \left\{ \begin{array}{c} A = \frac{1}{2} \\ B = -\frac{1}{2} \\ C = 2 \end{array} \right\}$$

So $\int \frac{x+1}{(x-1)(x-3)^2}\,dx = \int \frac{\frac{1}{2}}{(x-1)} + \frac{-\frac{1}{2}}{(x-3)} + \frac{2}{(x-3)^2}\,dx = \frac{\ln(x-1)}{2} - \frac{\ln(x-3)}{2} - \frac{2}{x-3} + C.$

$\int \frac{(x+2)}{(x+3)(x+4)}\,dx$

$= \int \frac{2}{x+4} - \frac{1}{x+3}\,dx = [2\ln(x+4) - \ln(x+3)] + C = \ln\frac{(x+4)^2}{x+3} + C$

$\int_1^2 \frac{dx}{x^2-3x-4}$

$= \int_1^2 \frac{1/5}{x-4} - \frac{1/5}{x+1}\,dx = \frac{1}{5}[\ln|x-4| - \ln|x+1|]_1^2 = \frac{2}{5}\ln\frac{2}{3}$

$\int \frac{dx}{x^2+4x+5}$

$= \int \frac{dx}{(x+2)^2+1} = \arctan(x+2) + C$

- Integrate: an expression involving $\sqrt{\pm a^2 \pm x^2}$.

  Use a trigonometric substitution that enables you to rewrite the expression under the root as a perfect square using the Pythagorean identity $\sin^2 x + \cos^2 x = 1$ or some variant of it (e.g. divided by $\cos^2 x$). In this way the root can be eliminated which should make the function easier to integrate. For the purposes of substituting back a trigonometric answer to the original variable, it is useful to interpret the original root expression as one of the sides of a right triangle; the trigonometric answer can then be interpreted as a ratio in this figure and translated accordingly.

$\int_{-3}^{3} \sqrt{9-x^2}\,dx$

Let $x = 3\sin\theta$. Then $dx = 3\cos\theta\ d\theta$. Thus $\int_{-3}^{3} \sqrt{9-x^2}\,dx = \int_{-\pi/2}^{\pi/2} \sqrt{9-9\sin^2\theta}(3\cos\theta)\,d\theta = \int_{-\pi/2}^{\pi/2} 3\sqrt{9\cos^2\theta}\cos\theta\,d\theta = \int_{-\pi/2}^{\pi/2} 3|3\cos\theta|\cos\theta\,d\theta = \int_{-\pi/2}^{\pi/2} 9\cos^2\theta\,d\theta = \int_{-\pi/2}^{\pi/2} \frac{9}{2}\big(1 + \cos(2\theta)\big)\,d\theta = \frac{9}{2}\big(\theta + \frac{1}{2}\sin(2\theta)\big)\Big]_{-\pi/2}^{\pi/2} = \frac{9}{2}\pi.$

- Integrate: an expression involving $(\pm a^2 \pm x^2)^n$.

  If other rules are inapplicable, a substitution strategy analogous to that of the previous case may help to simplify the integrand.

- Integrate: a more complicated case not covered above which involves …

  - … a rational function of $\sin(x)$ and $\cos(x)$.

    Unless some simplification using trigonometric identities suggests itself, substitute $u = \tan(x/2)$. This should turn the integrand into a rational function of u, which can then be integrated. (For geometric interpretation see problem 7.3.5.)

  - … a root expression.

    Try substituting u = this root expression. If this does not help, try substituting u^2 = the interior of the root expression

  - … a fraction of exponential expressions.

    Try to find a substitution that will rationalise the expression (i.e., turn it into a ratio of polynomials).

- Evaluate an integral where one of the bounds is infinite, or on an interval where the integrand becomes infinite (i.e., has vertical asymptote).

  Evaluate the integral with a generic bound, say a, in place of the exceptional one, and take the limit of the answer as a goes to the exceptional point. Split into two integrals if the exceptional point is in the interior of the interval. The integral is convergent if the limit exists and is finite, otherwise divergent.

- Find the derivative of an integral $\int f(x)\,dx$ with respect to t where t …

  - … is the upper bound of integration.

    $f(t)$.

  - … is the lower bound of integration.

    $-f(t)$.

  - … occurs in an expression $g(t)$ for the upper bound of integration.

Combine the above with chain rule: $f(g(t))g'(t)$.

– … occurs in an expression $g(t)$ for the lower bound of integration.

Combine the above with chain rule: $-f(g(t))g'(t)$.

– … occurs in both bounds.

Treat each bound separately according to the above; add the results.

Differentiate $F(x) = \int_0^x \frac{e^t}{t+2}dt$.

$F'(x) = e^x x + 2$

Differentiate $G(x) = \int_0^{x^2} \frac{e^t}{t+2}dt$.

$G'(x) = \frac{e^{x^2}}{x^2+2} \cdot 2x$

---

§ 3.7.7. Examples

$\int (\frac{1}{2} + x)^2 \ln(1+2x)dx$

The substitution $1+2x = t$ gives $2dx = dt$ and $\int(\frac{1}{2} + x)^2 \ln(1+2x)dx = \int(\frac{1+2x}{2})^2 \ln(1+2x)dx = \int \frac{t^2}{4} \ln t \frac{1}{2}dt = \frac{1}{8}\int t^2 \ln t dt$. Integrating by parts, $\int t^2 \ln t dt = \frac{1}{3}t^3 \ln t - \int \frac{1}{3}t^3 \frac{1}{t}dt = \frac{1}{3}t^3 \ln t - \frac{1}{3}\int t^2 dt = \frac{1}{3}t^3 \ln t - \frac{1}{9}t^3 + C = \frac{1}{3}(1+2x)^3 \ln(1+2x) - \frac{1}{9}(1+2x)^3 + C$. The final answer is thus $\frac{1}{24}(1+2x)^3 \ln(1+2x) - \frac{1}{72}(1+2x)^3 + C_1$.

$\int_0^2 x^2(9-x^3)^{10}dx$

The substitution $9 - x^3 = t$ gives $-3x^2dx = dt$, and hence $x^2dx = -\frac{1}{3}dt$. Moreover, t varies between 9 and 1 as x varies between 0 and 2. Therefore $\int_0^2 x^2(9-x^3)^{10}dx = \int_9^1 t^{10}(-\frac{1}{3})dt = -\frac{1}{3}\int_9^1 t^{10}dt = \frac{1}{3}\int_1^9 t^{10}dt = \frac{1}{33}t^{11}]_1^9 = \frac{9^{11}-1}{33}$.

$\int_0^1 x\sqrt{x+3}dx$

$\int_0^1 x\sqrt{x+3}dx = [x\frac{2}{3}(x+3)^{3/2}]_{x=0}^{x=1} - \int_0^1 \frac{2}{3}(x+3)^{3/2}dx = \frac{2}{3}4^{3/2} - \frac{2}{3}\frac{2}{5}(x+3)^{5/2}]_{x=0}^{x=1} = \frac{2}{3}2^3 - \frac{4}{15}[(4)^{5/2} - (3)^{5/2}] = \frac{16}{3} - \frac{4}{15}[32 - 9\sqrt{3}] = \frac{80}{15} - \frac{128}{15} + \frac{36}{15}\sqrt{3} = \frac{-16+12\sqrt{3}}{5}$.

$\int x^2 \cos x dx$

Integrate by parts: $= x^2 \sin x - \int 2x \sin x = x^2 \sin x + 2x\cos x - \int 2\cos x = x^2 \sin x + 2x\cos x - 2\sin x + C$.

$\int_1^5 \frac{dt}{\sqrt{3t+1}}$

Make the substitution $3t+1 = u$. Then $\frac{du}{dt} = 3$ which gives $dt = \frac{1}{3}du$. The upper bound of integration is $3\cdot4+1 = 16$ and the lower bound $3\cdot1+1 = 4$. Thus $\int_1^5 \frac{1}{\sqrt{3t+1}}dt = \int_4^{16} \frac{1}{\sqrt{u}}\frac{1}{3}du = \frac{1}{3}\int_4^{16} u^{-\frac{1}{2}}du = \frac{1}{3}[\frac{u^{\frac{1}{2}}}{\frac{1}{2}}]_4^{16} = \frac{2}{3}(\sqrt{16} - \sqrt{4}) = \frac{4}{3}$.

$\int_0^{\ln 3} \frac{e^x}{1+e^x}dx$

Substitute $u = 1+e^x$, so that $e^x dx = du$. Then the integral becomes $\int_2^4 \frac{1}{u}du = [\ln u]_2^4 = \ln 2$.

$\int_0^{1/2} \frac{1}{2+8x^2}dx$

$= \frac{1}{2}\int_0^{1/2} \frac{1}{1+(2x)^2}dx = \frac{1}{4}[\arctan(2x)]_0^{1/2} = \frac{\pi}{16}$

$\int_1^4 \sqrt{x}\ln x dx$

By parts: $= [\frac{x^{3/2}\ln x}{3/2}]_1^4 - \int_1^4 \frac{\sqrt{x}}{3/2}dx = \frac{16\ln 4}{3} - \frac{28}{9}$

$\int_0^2 \frac{x}{(x^2+4)^{1/3}}dx$

$= \frac{1}{2}\int_4^8 \frac{du}{u^{1/3}} = [\frac{3u^{2/3}}{4}]_4^8 = 3 - \frac{3}{2^{2/3}}$

$\int (2r-1)e^{-2r}dr$

By parts: $\int(2r-1)e^{-2r}dr = (2r-1)\frac{e^{-2r}}{-2} - \int 2\frac{e^{-2r}}{-2}dr = (2r-1)\frac{e^{-2r}}{-2} + \int e^{-2r}dr = (2r-1)\frac{e^{-2r}}{-2} + \frac{e^{-2r}}{-2} + C = (2r-1+1)\frac{e^{-2r}}{-2} + C = -re^{-2r} + C$

$\int \frac{x+\frac{1}{2}}{e^{1+x+x^2}}dx$

The substitution $t = 1+x+x^2$ implies $\frac{dt}{dx} = 1+2x$ which gives $dt = (1+2x)dx = 2(x+\frac{1}{2})dx$. Hence $\int \frac{x+\frac{1}{2}}{e^{1+x+x^2}}dx = \frac{1}{2}\int \frac{1}{e^t}dt = \frac{1}{2}\int e^{-t}dt = \frac{1}{2}\cdot\frac{e^{-t}}{-1} + C = -\frac{1}{2e^t} + C = -\frac{1}{2e^{1+x+x^2}} + C$.

$\int_0^{\sqrt{3}} \arctan x dx$

By parts: $= [x\arctan x]_0^{\sqrt{3}} - \int_0^{\sqrt{3}} \frac{x}{1+x^2}dx = \frac{\sqrt{3}\pi}{3} - [(1/2)\ln(1+x^2)]_0^{\sqrt{3}} = \frac{\sqrt{3}\pi}{3} - \ln 2$

$\int_0^1 \arcsin x dx$

By parts: $= [x\arcsin x]_0^1 - \int_0^1 \frac{x}{\sqrt{1-x^2}}dx = \frac{\pi}{2} + [\sqrt{1-x^2}]_0^1 = \frac{\pi}{2} - 1$.

$\int_1^\infty \frac{\pi}{x^3}dx$

$= \pi\int_1^\infty x^{-3}dx = \lim_{t\to\infty}\pi[-\frac{1}{2x^2}]_1^t = \lim_{t\to\infty}\pi(-\frac{1}{2t^2} + \frac{1}{2}) = \frac{\pi}{2}$.

$\int_{-1}^{1} \frac{1}{(1+x)^2}\,dx$

This is an improper integral because $\frac{1}{(1+x)^2}$ has an asymptote at $x = -1$. Hence we must find it using limits: $\int_{-1}^{1} \frac{1}{(1+x)^2}\,dx = \lim_{t\to -1^+} -\frac{1}{1+x}\Big]_t^1 = \lim_{t\to -1^+} -\frac{1}{2} + \frac{1}{1+t} = \infty$. In other words, the integral is divergent.

$\int_1^{\infty} \frac{dx}{x^2+x}$

$= \int_1^{\infty}(\frac{1}{x} - \frac{1}{x+1})\,dx = \lim_{R\to\infty}\int_1^R(\frac{1}{x} - \frac{1}{x+1})\,dx = \lim_{R\to\infty}[\ln x - \ln(x+1)]_1^R = \lim_{R\to\infty}[\ln\frac{x}{x+1}]_1^R = \ln 2$

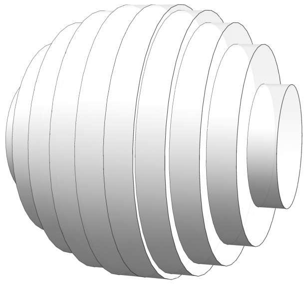
Determine the area enclosed by the curves $y = x^2$ and $y = -x+1$.

Setting the y-values equal to find the points intersection of the curves gives $-x+1 = x^2$, which has the solutions $x = \frac{1}{2}(-1-\sqrt{5})$ and $x = \frac{1}{2}(-1+\sqrt{5})$. Between these points, the upper curve is $y = -x+2$. Hence the area is $\int_{\frac{1}{2}(-1-\sqrt{5})}^{\frac{1}{2}(-1+\sqrt{5})}(-x+1) - x^2\,dx = \left[-\frac{x^2}{2} + x - \frac{x^3}{3}\right]_{\frac{1}{2}(-1-\sqrt{5})}^{\frac{1}{2}(-1+\sqrt{5})} = 5\frac{\sqrt{5}}{6}$.

§ 4.1. Volume

§ 4.1.1. Lecture worksheet

4.1.1. As the area under $y(x)$ is made up of rectangles with hight y and base dx, so is the volume of a sphere made up of cylindrical slices with thickness dx.



(a) Find a formula for the volume of such a slice, i.e., base area times height expressed in terms of r and x. We assume that the x-axis skewers the cylinders right through the middle.

(b) Sum up the pieces (i.e., integrate your expression) to find the famous formula for the volume of a sphere.

4.1.2. Generalise your argument to find an integral expression for the volume of any solid of revolution, i.e., volume obtained when the area under a graph $y(x)$ is rotated about the x-axis.

4.1.3. *Volume of pyramid.* In a manner similar to problem 4.1.1, the integral $\int x^2\,dx$ can be interpreted as the volume of a pyramid.

(a) Explain how, by interpreting each x^2 as an actual geometrical square.

(b) Show how the well-known formula for the volume of a pyramid agrees with a well-known integration rule.

§ 4.1.2. Problems

4.1.4. Find an integral formula for the volume of the solid of revolution generated when area under $y(x)$ is revolved about the y-axis. Hint: Consider the volume to be made up of thin cylindrical shells with height y and thickness dy. What is the circumference and volume of such a shell?

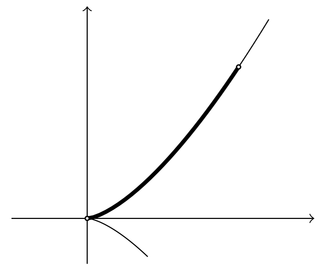
§ 4.2. Arc length

§ 4.2.1. Lecture worksheet

4.2.1. Explain what $\int_a^b ds$ represents. (The meaning of ds is shown in a figure in §1.1.)

4.2.2. Show that $\int_a^b \sqrt{1+\left(y'(x)\right)^2}\,dx$ expresses the arc length of the curve $y(x)$ from $x=a$ to $x=b$. Hint: This is really nothing but the Pythagorean Theorem applied to infinitesimal triangles. Express ds in terms of dx and dy and then factor out dx.

4.2.3. Find the arc length of the semi-cubical parabola $y^2=x^3$ from $(0,0)$ to $(1,1)$.



4.2.4. (a) Use the geometrical definition of the inverse trigonometric functions from §A.3 to express $\arccos(x)$ as an integral.

(b) What is the derivative of $\arccos(x)$? Explain how this follows from the above.

(c) Do the same for $\arcsin(x)$.

§ 4.2.2. Problems

4.2.5. Argue that the surface area of a surface of revolution (curve $y(x)$ revolved about the x-axis) is $\int 2\pi y\,ds$.

4.2.6. Consider the curve $y=1/x$ from $x=1$ to $x=\infty$.

(a) Using the techniques of §4.3, find the volume generated when the area under this curve is rotated about the x-axis.

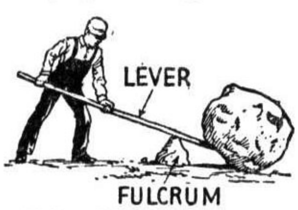
(b) Using problem 4.2.5, find the surface area of the same shape.

(c) Is the result "paradoxical"?

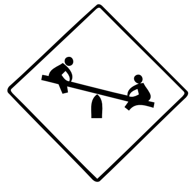
§ 4.3. Center of mass

§ 4.3.1. Lecture worksheet

The "law of the lever" says that a lever multiplies the effect of a force by the length of the lever arm from the fulcrum to the point where the force is applied. Thus we can lift a stone with, say, a three times smaller force than that required to lift it directly by using a lever with a three times longer arm on our side than on the stone's side.



Man lifting a stone with a lever



We can see the same principle in action on a children's playground seesaw: one child can balance two on the other side if he sits twice as far out on the beam.

Algebraically, an equilibrium evidently corresponds to $\sum mx = 0$, where x is the position along the axis (with the fulcrum as origin) and m is the mass applied at that point. (We are assuming that the lever bar itself is weightless.)

We can also reverse the problem: given some masses m_i and their positions x_i along the axis (measured from any reference point, such as for instance the left endpoint), find where the fulcrum needs to be placed to achieve equilibrium; that is, find the point where you can balance the whole thing on the tip of your finger. This point is called he center of mass and is denoted $\bar{x}$ ("x bar"). Another way of putting it is that the original distribution of masses is equivalent to all masses being stacked at this one point $\bar{x}$: pooling your masses in this way would not alter the behaviour of the system as a whole as far as balances, levers, seesaws, etc. are concerned.

4.3.1. Find the center of mass $\bar{x}$ in terms of the x_i and m_i using the equilibrium equation above.

Hint: That equation was set up with the fulcrum as the origin of the coordinate system. How does it change if the origin of the coordinate system is, e.g., the left endpoint of teh bar and the fulcrum is at x-coordinate $\bar{x}$?

4.3.2. Argue that the result can be interpreted in a natural way as a kind of average.

4.3.3. Why does it not matter where the origin of the coordinate system is located?

Suppose now that the masses are located at various points (x_i, y_i) in a plane (say placed on a thin metal tray) instead of along a single axis.

4.3.4. What is the physical meaning of your expression for $\bar{x}$ in this context?

4.3.5. Find an expression for the center of mass in this case (i.e., the point at which you could balance the whole tray on the tip of your finger).

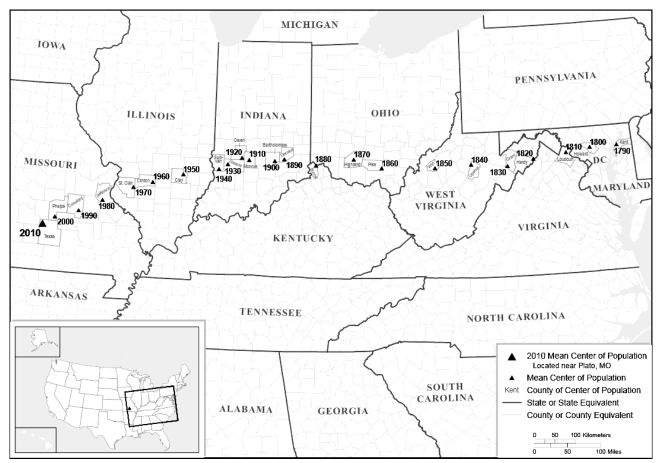
Suppose we want to find the center of mass of a figure, such as for example the area between the parabola $y = x^2 - 1$ and the x-axis. We can imagine this area being cut out of a sheet of metal and we want to know on which point this piece of metal could be balanced on the tip of a needle. Above we dealt only with the center of mass of a system of point-masses, but the idea is easily extended by thinking of the figure as made up of many little point-masses. In our example, we see already without calculations that $\bar{x}$ for symmetry reasons, meaning that our piece of metal would balance on the edge of a knife placed along the y-axis. But what is $\bar{y}$? Along which horizontal line does the shape balance?

4.3.6. Find $\bar{y}$ by slicing the area into dy-thin horizontal strips and considering each strip a separate point mass.

We can also calculate the center of mass of a curve in an analogous way. We assume then that mass is proportional to arc. Physically we may think of a piece of metal wire, for example.

4.3.7. Express $\bar{x}$ and $\bar{y}$ as integrals with respect to arc length. Hint: each segment ds of the the curve (cf. §4.2) may be considered a point mass.

The idea of the center of mass $(\bar{x}, \bar{y})$ is useful and suggestive beyond its physical context. As we saw, it can be interpreted as a kind of "average position." This is a useful notion in geometry, where $(\bar{x}, \bar{y})$ is also called the centroid to de-emphasise its physical connotations. The center of mass can also be seen as a "nutshell" summary of a complex system. For example, this map of the center of mass of the population of the United States at various points in time vividly captures a complex historical development:



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§ 4.3.2. Problems

4.3.8. Above we sliced the area into horizontal strips when calculating $\bar{y}$ of a figure.

(a) Explain why this is in a way more natural than using vertical slicing.

(b) Explain how you could nevertheless compute $\bar{y}$ on the basis of vertical strips. Hint: first pool the mass of each strip at its center of mass.

(c) Give an example where vertical slice are more convenient than horizontal ones.

4.3.9. *Theorem of Pappus.* Show that if some plane area is rotated about the y-axis then the volume generated is equal to the area of the region times the distance travelled by its center of mass. Hint: compare the integral expression for $\bar{x}$ with that for rotational volume (§4.6.1).

4.3.10. Find a similar theorem for the surface area of a solid of revolution.

4.3.11. Galileo thought that the shape of a necklace held up by its endpoints is a parabola. We shall prove that it is not. The physical principle we need for this is: nature strives to arrange the necklace in such a way that its center of gravity is as low as possible.

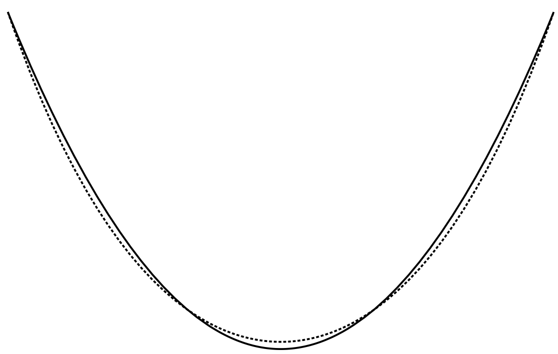
Consider the parabola $y = x^2 - 1$ and view the portion below the x-axis as the shape of a necklace suspended from the points $(-1,0)$ and $(1,0)$.

(a) Find the center of mass of this part of the parabola.

I say that a necklace placed along this curve and released would instead attain the shape

$$y = 0.314\left(e^{x/0.628} + e^{-x/0.628}\right) - 1.607$$

These two curves look like this, with the parabola drawn solid:



(b) Verify that the curves have the same arc length.

(c) Compute the center of mass of the second curve.

We shall see how to arrive at the true shape of the necklace in problem 6.2.2.

§ 4.4. Energy and work

§ 4.4.1. Lecture worksheet

In §2.4 we studied conservation laws in physics. We made the general point that proving such laws corresponds to showing that the derivative of some quantity is zero. But the inquisitive reader may have objected: it's all fine and well to prove that some given quantity is constant by differentiating it, but how do we know what quantity to differentiate in the first place? In other words, how do we discover what the conservation law is, as opposed to proving it once it has been proposed?

Since differentiation and integration are inverse processes, you will perhaps not be surprised to learn that, just as differentiation can bring a conservation law "back down" to basic laws like those of §A.7.12, so integration can be used to go the other way and "build up" to the quantities occurring in conservation laws starting from what is known. This is done by means of the concept of work, which is another way of looking at energy. We may define work as force times distance, or, more generally, the integral of force with respect to distance, $\int F\,ds$.

4.4.1. Looking back at §2.4, explain how the two energy terms can be obtained as work integrals given known expressions for force (§A.7.12):

(a) $-G\frac{Mm}{y}$.

(b) $\frac{mv^2}{2}$. Hint: rewrite $\int ma\,ds$ as an integral with respect to v.

But why should work, as defined above, be the same thing as energy? How can we arrive at this definition of work in the first place? The rest of this section is devoted to developing our physical intuition about this.

An object of mass m at a not-too-great height h above the surface of the earth has a potential energy of mgh. This means that we could, potentially, have it do so much work for us. You can think of for example a water wheel driven by a water fall: this device takes advantage of the potential energy stored in the water by virtue of its altitude, and harnesses it for some other purpose. Thinking in terms of water wheels, it is easy to understand why potential energy is proportional to mass and height. For if the height is double, you can have the water run through twice as many wheels on its way down, so you get twice as much work out of it. And if the mass is double you can split it in half and run each part through the water wheels separately, which makes it clear that you get twice the work in this case also. By the same argument we obtain the general relation

$$\text{work} = \text{force} \times \text{distance}$$

which may be taken as the formal definition of work, as above.

Potential energy is energy by virtue of position; kinetic energy is energy by virtue of velocity. Water can drive a water wheel not only by falling from a certain height (potential energy) but also by rushing ahead in a stream at a certain velocity (kinetic energy). I shall now prove to you that just as potential energy is measured by mgh, so kinetic energy is measured by $\frac{1}{2}mv^2$. First I want make it clear that kinetic energy is "stored work." Imagine yourself pushing a wagon along a railway track. When you are done pushing and let the wagon go, all the work you put into it is now "stored" in the wagon in the form of kinetic energy. We can get it back out again for example by our prototype method of water wheels, which we could have the wagon set spinning as it hits them along its path. Experience shows that it takes the same amount of effort to stop the wagon as it did to get it moving, so it is clear that the amount of work stored in the wagon is the same as that you put into it.

When you push the wagon to get it moving you are applying a certain force across a certain distance. The product of the two is the work you do, we saw above. This, then, is a measure of the kinetic energy, but not a very nice one. Kinetic energy is quite clearly intrinsic to the moving wagon, so it is awkward to characterise it in terms of the action of the worker who set it moving and however long of a run-up he used. We should much prefer to express it in terms of the mass and velocity of the wagon. But this is easily done, for we know that

$$\text{force} = \text{mass} \times \text{acceleration}$$

and

$$\text{distance} = \text{average velocity} \times \text{time}$$

"Distance" here means the length of your run-up before you released the wagon, and "time" how long you took to complete it. Let's say that you push equally hard throughout, so that the force, and thus acceleration, is constant.

4.4.2. Conclude from this that the kinetic energy is $\frac{1}{2}mv^2$.

The two forms of energy that we have studied are clearly interchangeable: when an object falls it "trades in" potential energy for kinetic, and conversely when its velocity is directed upwards. By means of some ramps we could turn a water fall into a stream and conversely, so we would quite like to know which is better for driving water wheels. But it turns out to be all the same. The economy of nature is such that the exchange rate in these kinds of transactions is one to one. Energy is conserved. This agrees with experience but we can also prove it formally.

4.4.3. Prove, by taking its time-derivative, that the total energy $mgh + \frac{1}{2}mv^2$ is constant for a freely falling object.

Another useful way of establishing this sort of result is to prove that if it didn't hold one could exploit the discrepancy to build a perpetual-motion machine which could create energy out of nothing, which is known to be impossible or at least a point on which we would be very pleasantly surprised to be proven wrong.

4.4.4. Argue on such grounds that $mgh + \frac{1}{2}mv^2$ is constant.

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§ 4.4.2. Problems

4.4.5. When you are pushing the wagon to get it moving, if you push it for twice as long, while maintaining the same constant force, then you double its final speed. But the kinetic energy doesn't double but quadruple since it is proportional to v^2. So by doubling the input effort you got four times the output energy stored in the system: a violation of energy conservation. Resolve the paradox.

4.4.6. Consider a lever with one lever arm just over twice as long as the other. Attached to the shorter arm is a weight of mass 2. You lift a weight of mass 1 and attach it to the longer lever arm. Then this weight will sink and the other one will rise. Doesn't this prove that lifting a unit weight is equivalent to lifting two unit weights? By connecting several levers you could even lift any weight of mass 2^n with no more effort required to lift the original unit weight. This surely violates energy conservation. Resolve the paradox.

4.4.7. The Great Pyramid of Giza, Egypt, has a square base wide side length 230 meters, and its original height was 146 meters. Its interior is basically solid stone throughout, except for a few small chambers. The stones used weigh about 2700 kg/m^3.

(a) Calculate the total work done in erecting the pyramid. Hint: Slice the pyramid into horizontal layers and express the work required to lift the stones for each layer. Then integrate to get the total work needed for all layers.

According to the ancient Greek historian Herodotus, the pyramid was built in 20 years by 100 000 workers. Let's check whether this seems plausible.

(b) Estimate how much work a man can do in one hour. Hint: Picture the man lifting weights onto a ledge of height 1 meter. What weight can he lift and how many times can he repeat this in one hour?

(c) Based on your estimate, how many man hours would have been needed to lift the stones into place for the pyramid?

(d) Does Herodotus's claim seem plausible?

§ 4.5. Logarithms redux

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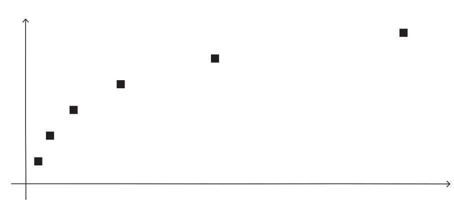
§ 4.5.1. Lecture worksheet

Integrals give us a new way of looking at logarithms, which is more illuminating in certain respects. In particular, this perspective enables us to understand the derivative of the logarithm in a more direct way than the approach we used in problem 1.4.1.

In §A.4 we saw that the essence of logarithms is that they turn multiplication into addition:

$$\log(ab) = \log(a) + \log(b) \tag{L1}$$

and that a table of powers of some integer has this property when "read backwards." If we plot the values of such a table in a coordinate system we get a picture like this:



We are looking to extend this table to include all intermediate values as well. In §A.4 we did this in an algebraic fashion. Now we wish to do it geometrically. Thus we look at the plot and try to characterise the function that runs through these points.

4.5.1. (a) Argue that it seems plausible that a function running through these points has derivative $1/x$.

(b) Explain why the function $f(x) = \int_c^x \frac{1}{t}\,dt$, where c is any constant, has this derivative.

We now wish to check whether the function so defined in fact has the property (L1), as desired.

(c) For the formula $f(ab) = f(a) + f(b)$ to hold for all positive numbers a and b, it is necessary that $f(p) = 0$ for a certain number p. Explain why. (Hint: Consider simple values for a and b.) What is p?

(d) Therefore, what is c?

(e) How is the derivative of $f(ax)$ related to the derivative of $f(x)$?

(f) What does this imply about how $f(ax)$ is related to $f(x)$? Use the value that you found in (b) to further specify this relation.

(g) Show that this implies $f(ab) = f(a) + f(b)$, as desired.

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§ 4.5.2. Problems

4.5.2. † Discuss what you consider to be the advantages and disadvantages of the two ways of defining logarithms presented in sections §A.4 and §4.5. Hint: aspects to consider might include how these definitions incorporate non-fractional exponents, and what is "natural" about the natural logarithm.

4.5.3. With the logarithm defined as in §4.5.1, we can define the exponential function e^x as its inverse. Prove $(e^x)' = e^x$ and $e^{xy} = e^x e^y$ starting with this definition.

4.5.4. We write $\ln|x|$ rather than $\ln x$ for the antiderivative of $1/x$. This may seem like a hassle. Of course, when the logarithm is defined as above it only exists for positive numbers, but what's stopping us from simply extending the definition to include negative numbers as well, so that $\ln x = \ln|x|$, which would spare us the trouble of writing the absolute value bars all the time? Hint: What are some other important properties that we want the logarithm function to have?

4.5.5. Argue that $\ln|x| + C$ is not quite the most general antiderivative of $1/x$. Hint: replace C with a locally constant function.

§ 4.6. Reference summary

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§ 4.6.1. Geometrical applications of integration

Volume of solid of revolution generated when area under $y(x)$ is revolved about x-axis (seen as made up of thin disks):

$$\int_a^b \pi y^2 \, dx$$

Find the rotational volume generated when $f(x) = \sqrt{\sin(2x)}$, $0 \le x \le \frac{\pi}{2}$, is rotated about the x-axis.

$= \pi \int_0^{\pi/2} (\sqrt{\sin(2x)})^2 dx = \pi \int_0^{\pi/2} \sin(2x) dx = \frac{\pi}{2}[-\cos(2x)]_0^{\pi/2} = \pi.$

Determine the rotational volume generated when the curve $y(x) = (1+a)e^{-ax}$, $x \ge 0$ is rotated around the positive x-axis.

$V(a) = \pi \int_0^{\infty} (y(x))^2 dx = \pi(1+a)^2 \int_0^{\infty} e^{-2ax} dx = \pi(1+a)^2 [\frac{-1}{2a} e^{-2ax}]_0^{\infty} = \frac{\pi(1+a)^2}{2a}.$

Find the rotational volume generated when $y = \sqrt{ax}e^{-ax^2}$, $x \ge 0$, is rotated about the x-axis.

$= \pi \int_0^{\infty} axe^{-2ax^2} dx = [2ax^2 = t,\ 4axdx = dt] = \frac{\pi}{4} \int_0^{\infty} e^{-t} dt = \frac{\pi}{4}[-e^{-t}]_0^{\infty} = \frac{\pi}{4} \lim_{T\to\infty}(1 - e^{-T}) = \frac{\pi}{4}.$

Volume of solid of revolution generated when area under $y(x)$ is revolved about y-axis (seen as made up of thin cylindrical shells):

$$\int_a^b 2\pi x y \, dx$$

Find the volume generated when the area under the curve $y = x^2$, $1 < x < 2$, is rotated about the line $x = 2$.

$= \int_1^2 2\pi(2-x)x^2 dx = 2\pi \int_1^2 2x^2 - x^3 dx = 2\pi\left[\frac{2x^3}{3} - \frac{x^4}{4}\right]_1^2 = 2\pi\left(\frac{16}{3} - \frac{16}{4} - \left(\frac{2}{3} - \frac{1}{4}\right)\right) = \frac{11\pi}{6}.$

Arc length of curve $y(x)$ from $x = a$ to $x = b$:

$$\int_a^b ds = \int_a^b \sqrt{1 + (y')^2} \, dx$$

Surface area of a surface of revolution (curve $y(x)$ revolved about the x-axis):

$$\int_a^b 2\pi y \, ds = \int 2\pi y \sqrt{1 + (y')^2} \, dx$$

Average value of $f(x)$ on given interval:

$$\frac{1}{b-a} \int_a^b f(x) \, dx$$

Area between $f(x)$ and $g(x)$ where $f(x)$ is the upper function (if no interval $[a, b]$ is given, intersections are the implied endpoints: find them by setting $f(x) = g(x)$):

$$\int_a^b f(x) - g(x) \, dx$$

§ 4.6.2. Center of mass

$(\bar{x}, \bar{y})$ = centroid = center of mass (assuming uniform density).

Center of mass of region under graph of $y(x)$ from $x = a$ to $x = b$:

$$\bar{x} = \frac{\int_a^b xy\,dx}{\int_a^b y\,dx} \qquad \bar{y} = \frac{\int_a^b \frac{1}{2} y^2\,dx}{\int_a^b y\,dx}$$

Center of mass of region between graphs of $f(x)$, $g(x)$ from $x = a$ to $x = b$:

$$\bar{x} = \frac{\int_a^b x(f-g)\,dx}{\int_a^b (f-g)\,dx} \qquad \bar{y} = \frac{\int_a^b \frac{1}{2}(f^2-g^2)\,dx}{\int_a^b (f-g)\,dx}$$

Center of mass of curve $y(x)$ from $x = a$ to $x = b$:

$$\bar{x} = \frac{\int_a^b x\,ds}{\int_a^b ds} = \frac{\int_a^b x\sqrt{1+\left(y'\right)^2}\,dx}{\int_a^b \sqrt{1+\left(y'\right)^2}\,dx}$$

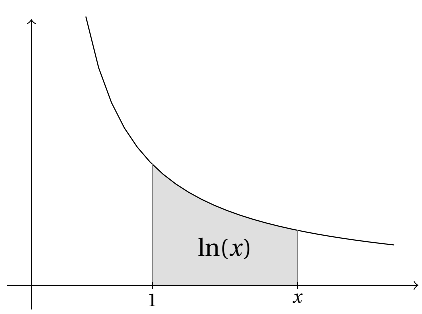
$$\bar{y} = \frac{\int_a^b y\,ds}{\int_a^b ds} = \frac{\int_a^b y\sqrt{1+\left(y'\right)^2}\,dx}{\int_a^b \sqrt{1+\left(y'\right)^2}\,dx}$$

§ 4.6.3. Work

$$\text{work} = \text{force} \times \text{distance} = \int F\,ds$$

§ 4.6.4. Integral definition of the logarithm

$$\ln(x) = \int_1^x \frac{1}{t}\,dt$$



5 Power series

§ 5.1. The idea of power series

§ 5.1.1. Lecture worksheet

Functions can be expressed as power series:

$$f(x) = A + Bx + Cx^2 + Dx^3 + \cdots$$

We can think of the coefficients as so many "degrees of freedom," i.e., free choices we can make when picking the coefficients.

5.1.1. These "degrees of freedom" have a direct visual meaning.

(a) Argue visually that by suitable choices of the constants a, b, c, you can make a parabola of the form $y = ax^2 + bx + c = A(x-B)^2 + C$ pass through any $\boxed{\quad\quad}$ predetermined points.

(b) For $y = bx + c$ the number of points I can make it pass through is $\boxed{\quad\quad}$

(c) For $y = c$ the number of points I can make it pass through is $\boxed{\quad\quad}$

(d) Conclude that it makes sense that any function can be represented by an "infinite polynomial."

The power series for a given function $f(x)$ can be found by plugging zero into f, f', f'', etc., which gives the values of A, B, C, etc., respectively.

5.1.2. Show that this gives

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$
$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$
$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

5.1.3. Suppose you know the power series for sine and cosine but have no calculator at your disposal. In which of the following situations could you use the power series to resolve your problem?

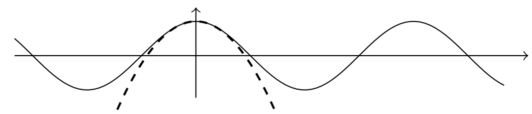
☐ I remember the wavy shape of the graph of the sine function, but I forget how to plot it and tell it apart from the cosine graph.

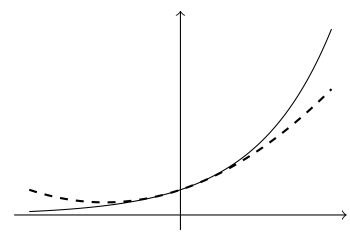
☐ I remember that the sine and cosine functions are basically each other's derivative, except there is a minus sign somewhere, and I forget where it goes.

☐ I remember that $\sin(60°)$ is something quite simple, but I forget the exact value.

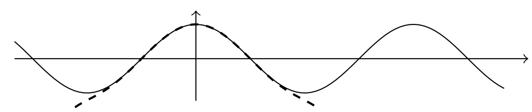
This method of repeated differentiation for finding power series is in principle always applicable. But in practice we rarely derive power series by the method of repeated differentiation. Instead we build them up from standard series such as the above, by algebraic manipulations like substituting, multiplying, and so on, as we are used to doing for ordinary polynomials. Furthermore we shall see below that many important series arise more naturally in other ways altogether.

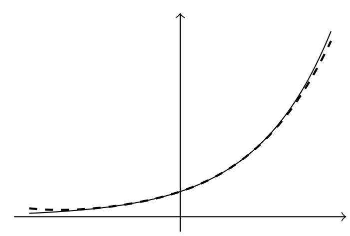
The above series give us a way of estimating these functions by polynomials. If we cut the series off after for instance the second-degree term we get the best possible parabolic approximation. Here I have illustrated this for the cosine and exponential functions:





If we include more terms of the series we will see the polynomial "hugging" the function more and more closely, like this:





5.1.4. By picturing the graph of $\ln(x)$, I feel that the power series for $\ln(1+x)$ starts with a [positive/negative/zero] constant term, a [positive/negative/zero] linear term, and a [positive/negative/zero] quadratic term.

5.1.5. Argue that the power series for the sine implies that $\sin x \approx x$ when x is small. (This is a useful approximation in many situations. We mentioned it already in §A.3, and we also effectively used this approximation in §6.4 when deriving the differential equation for pendulum motion.)

5.1.6. Show that applied to a general function $f(x)$ the method gives

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots$$

One use of the idea of polynomial approximation is to tackle difficult integrals. We quite often face integrals for which none of our usual integration tricks work, such as the integral of e^{-x^2} (the normal distribution function at the heart of statistical theory) or $\sqrt{\frac{a^4+(b^2-a^2)x^2}{a^4-a^2x^2}}$ (the arc-length of the ellipse

$x^2/a^2 + y^2/b^2 = 1$, an important problem in astronomy since planets move in elliptical orbits). In such cases the best we can do is often to expand the function as a power series and integrate term by term, which gives us the desired integral in series form.

5.1.7. Let us evaluate $\int e^{-x^2}\,dx$ in this way.

(a) If I include only the first four non-zero terms, the integral is $\boxed{\quad\quad}$

(b) Though not an exact solution in closed form, this is still very useful. For example, I could use it to find a good approximation to $\int_0^1 e^{-x^2}dx$. Suppose I use only the first three non-zero terms for this. This must already be quite good because I see from the above that the next term would only affect the $\boxed{\quad\quad}$ decimal and subsequent terms are even smaller.

5.1.8. Suppose I use the first five terms of the power series for e^x to approximate $e^{0.1}$, then use this result to find an approximation for e by [raising the result to the power 10, taking 1 divided by the result, multiplying the result by 10, taking the ln of the result and multiplying by 10]. Alternatively, I could find an approximation for e directly from the series by [plugging in $x = 0$, plugging in $x = 1$, using a geometric series, using a binomial series]. Which of the two methods will be more accurate? [the first, the second, both equal]

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§ 5.1.2. Problems

5.1.9. (a) Estimate the sine of 1° using nothing but a simple calculator that only has the operations $+$, $-$, $\times$, $/$.

(b) Check your answer using a more advanced calculator that has a sine button.

(c) Is the "more advanced" calculator really more advanced, or does it just have the algorithm of (a) on "speed dial"?

5.1.10. (a) By considering the roots of $\sin(x)/x$, argue that its power series

$$\sin(x)/x = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots$$

can be factored as

$$\left(1-\frac{x^2}{\pi^2}\right)\left(1-\frac{x^2}{4\pi^2}\right)\left(1-\frac{x^2}{9\pi^2}\right)\cdots$$

by analogy with the way one factors ordinary polynomials, such as $x^2 - x - 2 = (x+1)(x-2)$.

(b) What is the coefficient of x^2 when the product is expanded?

(c) Equate this with the coefficient of x^2 in the ordinary power series and use the result to find a formula for the sum of the reciprocals of the squares, $\sum 1/n^2$.

§ 5.2. The geometric series

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§ 5.2.1. Lecture worksheet

5.2.1. (a) What is the greatest number smaller than 1? One is inclined to suggest $a = 0.99999\ldots$, but argue against this by considering $10a - a$.

(b) Generalise your argument to find a closed formula for $1 + x + x^2 + x^3 + \cdots$

5.2.2. Explain how power series are related to the paradox of motion mentioned by Aristotle, *Physics*, 239b11: "[Zeno] asserts the non-existence of motion on the ground that that which is in locomotion must arrive at the half-way stage before it arrives at the goal," and then the half-way stage of what is left, etc., ad infinitum.

5.2.3. Derive the series

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

by first noting that

$$\ln(1+x) = \int_1^{x+1} \frac{1}{t}\,dt = \int_0^x \frac{1}{1+u}\,du,$$

then applying the geometric series, then integrating term by term.

---

§ 5.2.2. Problems

5.2.4. What is $0.888\ldots$? Does it "spill over" like we saw $0.999\ldots$ do in problem 5.2.1a?

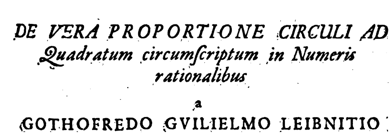
5.2.5. By the fundamental theorem of calculus, the arctangent is the integral of its derivative.

(a) Use this to find a power series for the arctangent. (To make sure that you take the constant of integration into account, check that your constant term is correct using the geometrical definition of the arctangent.)

(b) Find the value of arctan(1) in two ways: by the geometrical definition, and from the power series.

(c) Equate these two expressions for arctan(1) to find an infinite series representation for π.

When Leibniz found this series he concluded that "God loves the odd integers," as you can see in the figure below (taken from his 1682 paper).


DE VERA PROPORTIONE CIRCULI AD
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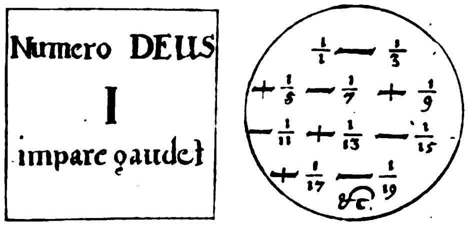
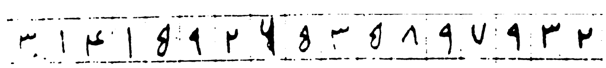


(d) ⋆ What does Leibniz's series have to do with a square of area 1, which is what Leibniz has drawn on the left? Hint: The use of the Greek letter π to denote the famous circle constant is a relatively recent invention. It was never used in Leibniz's time, and certainly not by the ancient Greeks. This is because they preferred to formulate mathematical truths geometrically rather than by "formulas." So to understand Leibniz's mode of expression you should consider how to express your formula for π in purely geometrical terms.

Leibniz's series is beautiful but it is not really very efficient for computing π. Already in 1424 al-Kashi had computed π with 16-decimal accuracy using different methods.

(e) ⋆ Estimate how many terms of Leibniz's series must be added together to achieve al-Kashi's accuracy.

Here is al-Kashi's result in his own notation:



That's 3.1415926535897932. Note the interesting way in which our symbols for 2 and 3 are derived from their Arabic counterparts. The Arabic symbols are perhaps a more natural way of denoting "one and then some."

5.2.6. In this problem we shall show that e is not a rational number, i.e., not a ratio of two integers. We shall do this by assuming on the contrary that $e = p/q$ for some integers p and q, and showing that this leads to a contradiction.

(a) Find a series representation of e using the power series for e^x, and set it equal to p/q.

(b) Multiply both sides by $q!$. You will find that the infinite series now starts with a series of integers and then from a certain point onwards becomes a series of fractions. The first non-integer term is $\boxed{}$

(c) By writing down this and the next few terms, we see that, from this point onwards, the series is [</=/>] the geometric series

$$\frac{1}{q+1} + \frac{1}{(q+1)^2} + \frac{1}{(q+1)^3} + \cdots$$

which in closed form = $\boxed{}$

(d) Consequently, we have shown that [e, p, q, $p!$, $q!$, $pq!/q$] can be written as [an integer, a negative number, a non-integer, 0] which is absurd. Therefore our initial assumption $e = p/q$ must have been false.

§ 5.3. The binomial series

§ 5.3.1. *Lecture worksheet*

The binomial series

$$(1+x)^q = 1 + qx + \frac{q(q-1)}{2!}x^2 + \frac{q(q-1)(q-2)}{3!}x^3 + \cdots$$

is perhaps best thought of by analogy with the integer-exponent case. When q is an integer the series dies after the $(q+1)^{\text{th}}$ term, and one has the trivial results

$$(1+x)^2 = 1 + 2x + x^2, \quad (1+x)^3 = 1 + 3x + 3x^2 + x^3, \quad \text{etc.}$$

Here we can think of, say, the coefficient of x^2 in the last expression as follows. To get an x^2-term when expanding $(1+x)^3 = (1+x)(1+x)(1+x)$ we need to choose xs from two of the parentheses and 1 from the third. For the first x we have three choices, and for the second x we have two choices, giving $2 \cdot 3 = 6$ choices in total, except that we must divide by the number of ways in which these two things we choose can be ordered internally (choosing first the third parenthesis and then the first is the same as choosing first the first and then the third), which is 2!, thus explaining the coefficient 3.

The general binomial theorem can be thought of in the same way, even though q is no longer an integer. Let's try the x^2 coefficient again. To get an x^2-term when expanding $(1+x)^q$ we need to choose xs from "two of the q parentheses" so to speak (whatever that is supposed to mean when q is something like $-1/2$). For the first x we have q choices, and for the second x we have $q-1$ choices, and correcting for internal ordering the coefficient for x^2 should be $\frac{q(q-1)}{2!}$.

5.3.1. The binomial series for $q = 2$, $q = -1$, $q = \frac{1}{2}$ have very different standing. Match them with a suitable description:

- ☐ becomes another famous series
- ☐ becomes finite
- ☐ amazingly works
- ☐ equals a logarithm function

We typically use the binomial series for functions involving:

- ☐ powers
- ☐ fractions
- ☐ roots
- ☐ logarithms other than ln

An important application of the binomial series concerns the inverse trigonometric functions.

5.3.2. (a) Recall from problem 4.2.4 how to express $\arcsin(x)$ as an integral.

(b) Find a power series representation of $\arcsin(x)$ by expanding the integrand as a binomial series and integrating term by term.

---

§ 5.3.2. Problems

5.3.3. The general binomial series can be bypassed in specific cases in the following way. Let's say we are looking for the power series of $\sqrt{1+x^2}$. This means that we want a series such that

$$\sqrt{1+x^2} = A + Bx + Cx^2 + Dx^3 + \cdots,$$

or, in other words,

$$1+x^2 = (A+Bx+Cx^2+Dx^3+\cdots)(A+Bx+Cx^2+Dx^3+\cdots).$$

Find the series for $\sqrt{1+x^2}$ by multiplying out the right hand side and identifying coefficients with the left hand side (constant term equal on both sides, coefficient of x equal on both sides, coefficient of x^2 equal on both sides, etc.).

5.3.4. (a) What is the largest r for which the circle with center $(0,r)$ and radius r fits inside the parabola $y = x^2$? Use the power series expansion of the circle at the origin to find out. Explain how you can see from the power series that a smaller or larger r would not work.

(b) Answer the same question with $y = 1-\cos(x)$ in place of the parabola.

Leibniz called this the osculating circle, or "kissing circle" if translated literally from the Latin.

§ 5.4. Divergence of series

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§ 5.4.1. Lecture worksheet

5.4.1. Argue that the geometric series

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$$

is clearly nonsense for certain values of x.

How can this be when we "proved" this formula in problem 5.2.1b?

5.4.2. (a) To diagnose the problem, try your argument from problem 5.2.1b on the finite series $1+x+x^2+\cdots+x^n$.

(b) If we let $n\to\infty$, what does this tell us about when the geometric series works?

This case illustrates two points that apply generally:

- We cannot naively assume that we can always manipulate infinite series according to the same rules as finite expressions (although this works more often than not).
- We can often avoid pitfalls and be more careful in our reasoning by considering the infinite series as the limit case of a finite expression.

For the purposes of finitistic analysis, we can cut the series off and add up all the terms we have up to that point. This is called a partial sum. Often the terms of the series shrink very quickly. Then the partial sums will soon be basically equal to the whole series since the cut-off terms are so small. In such a case we say that the series *converges*, i.e., approaches a particular value. Convergent series are "almost like a finite expression" in this sense, so it is not surprising that they can generally be treated as such without any need to worry about falling into absurdities like those of problem 5.4.1.

When the partial sums of a series do not approach a particular value—i.e., when the cut-off part never becomes negligible no matter how far out you go—we say that the series *diverges*.

5.4.3. Argue that the geometric series exhibits two different "kinds" of divergence. Hint: consider $x = 2$ and $x = -1$.

Divergent series are the dangerous ones that can lead us into absurdities if we carelessly assume that they obey all ordinary rules of algebra.

5.4.4. Show for example that, given this assumption, the series for $x=-1$ that you obtained in problem 5.4.3 can easily be used to "prove" that $1 = 0$.

In the geometric series examples, the divergent cases were easy to spot in that their terms did not shrink to zero. No wonder then that we can never discard the cut-off part, since its terms are still large. Unfortunately, however, telling convergent from divergent series is not always as easy as this. It is in fact possible for the terms of a series to shrink to zero and for the series to nevertheless diverge at the same time. The following is such an example.

5.4.5. Consider the following purported proofs that

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots = \infty$$

First proof. The second term is $\geq \frac{1}{2}$, the sum of the next two terms is $\geq \frac{1}{2}$, the sum of the next four terms is $\geq \frac{1}{2}$, the sum of the next eight terms is $\geq \frac{1}{2}$, and so on. I see this by estimating each term by the [minimum/maximum/average] of the terms in that group. The result follows.

Second proof. I can picture the sum as the areas of rectangles, where the value of the term is the height of a rectangle of base 1. By placing these rectangles in a coordinate system starting from $[(0,0),(-1,0),(1,0)]$ I see that $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\cdots\ [</=/>]\ \int_a^\infty \frac{1}{x}\,dx$, where the a corresponding to this sum is $\boxed{\quad}$. I can therefore use integration to prove the result.

Which proof is valid? [the first/the second/both/neither]. The result shows that a series can [be computed/converge/diverge/vanish/oscillate] even though [its terms go to 0/all its terms are positive/it has only integer denominators/it has the same derivative as ln]. Can the same thing happen with a geometric series? [yes/no]

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§ 5.4.2. Problems

5.4.6. Another proof that the harmonic series diverges can be given on the basis of the inequality

$$\frac{1}{n-1}+\frac{1}{n}+\frac{1}{n+1}>\frac{3}{n}$$

(a) Prove this inequality. Hint: geometrically, it reflects the fact that the function $1/x$ "flattens out" as x increases.

(b) Apply this inequality to the harmonic series

$$\begin{aligned}&1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots\\ =&1+\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}\right)+\cdots\end{aligned}$$

and show how the divergence of the series follows from this.

5.4.7. I claimed that it is important to distinguish divergent series from convergent ones on the ground that the former can lead to absurdities if ordinary algebra is assumed to apply to them. However, I showed only that divergence of the types in problems 5.4.3 lead to absurdities, not that divergence of the type of problem 5.4.5 does so also. Derive an absurdity by careless reasoning with the latter series to establish that my point was well taken.

5.4.8. In fact, even convergent series are not entirely free of "paradoxes."

(a) Using the power series for the logarithmic function, write down a series expression for ln(2).

(b) Rearrange the order of these terms by moving some negative terms toward the beginning of the series in such a way that after every positive term comes the next two negative terms. Now subtract from every positive term the negative term that follows, and sum the resulting series.

(c) Discuss.

5.4.9. Consider the product

$$\prod_{p\text{ prime}}\frac{1}{1-\frac{1}{p}}=\left(\frac{1}{1-\frac{1}{2}}\right)\left(\frac{1}{1-\frac{1}{3}}\right)\left(\frac{1}{1-\frac{1}{5}}\right)\left(\frac{1}{1-\frac{1}{7}}\right)\cdots$$

(a) Expand each term as a geometric series and argue that the result of multiplying everything out must be

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\cdots$$

since every integer has a unique prime factorisation.

(b) Deduce from this and problem 5.4.5 that there must be infinitely many primes.

§ 5.5. Reference summary

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§ 5.5.1. Standard series

General power series of $f(x)$:

$$f(x)=f(0)+\frac{f'(0)}{1!}x+\frac{f''(0)}{2!}x^2+\frac{f'''(0)}{3!}x^3+\cdots$$

General power series of $f(x)$ centered at $x=a$:

$$f(x)=f(a)+\frac{f'(a)}{1!}(x-a)+\frac{f''(a)}{2!}(x-a)^2+\frac{f'''(a)}{3!}(x-a)^3+\cdots$$

Power series of elementary functions:

$$\begin{aligned}e^x&=1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots\\ \sin x&=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots\\ \cos x&=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots\\ \ln(1+x)&=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots\end{aligned}$$

Geometric series ($|x|<1$):

$$\frac{1}{1-x}=1+x+x^2+x^3+\cdots$$

Binomial series ($|x|<1$):

$$(1+x)^q=1+qx+\frac{q(q-1)}{2!}x^2+\frac{q(q-1)(q-2)}{3!}x^3+\cdots$$

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§ 5.5.2. Terminology

Taylor series	power series of a function centered at some point $x=a$
Maclaurin series	power series of of a function centered at the origin; special case $a=0$ of Taylor series
partial sum of a series	sum of cut off series; the terms of the series added together up to a certain point
convergent series	the partial sums of the series approach a specific value as more and more terms are added

divergent series	the partial sums of the series do no approach a specific value as more and more terms are added (instead go to $\pm\infty$ or oscillate)
alternating series	every other term positive, every other negative
$n!$	Multiply n by every integer below it.

$$4! = 4\cdot 3\cdot 2\cdot 1$$

$\sum_{n=a}^{b} f(n)$	For every integer n starting with a and going to b, compute $f(n)$, and add all of the results together. (Σ = sigma = sum.)

$$\sum_{n=0}^{\infty}(-1)^n x^n = 1 - x + x^2 - x^3 + x^4 - x^5 + \cdots$$

---

§ 5.5.3. Problem guide

- Find the power series for a given function.

  This is typically best done using the standard series (§5.5.1) in conjunction with the below techniques. Alternatively, the general power series formula can be used, though this is usually more work.

- Find power series for $f(g(x))$ (such as $f(3x)$, $f(x^2)$, etc.) when the series for $f(x)$ is known.

  Insert $g(x)$, enclosed in brackets, in place of x in the series for $f(x)$.

  $\sin(x^2)$

  $= (x^2) - \frac{(x^2)^3}{3!} + \frac{(x^2)^5}{5!} - \frac{(x^2)^7}{7!} + \cdots$
  $= x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \cdots$

- Find the power series for a given function from first principles.

  (Hardly ever the easiest way to find a series in practice.) Set the function equal to $A + Bx + Cx^2 + Dx^3 + \cdots$. Plug in zero to determine A. Take the derivative of both sides, and plug in zero again to determine B. Repeat.

- Multiply two series.

  Multiply as ordinary polynomials. When cutting the answer off, take care that the terms you have not yet multiplied would not affect the terms before the cut-off point in your answer.

  $e^x \sin x$

  $= \left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right)$
  $= x + x^2 + \left(\frac{1}{2!} - \frac{1}{3!}\right)x^3 + \left(\frac{1}{3!} - \frac{1}{3!}\right)x^4 + \cdots$
  $= x + x^2 + \frac{x^3}{3} + \cdots$

  (Note that we could not multiply the 1 by the $\frac{x^5}{5!}$ to put a $\frac{x^5}{5!}$ in the answer, since there is a $\frac{x^4}{4!}$ hiding in the dots of the e^x-series, which when multiplied by x would affect the fifth-power term.)

- Estimate the magnitude of the error when an infinite series is replaced with one of its partial sums.

  By looking at the magnitude of the next term or two, as well as considering how fast the terms are shrinking, you can make an educated guess. In an alternating series with shrinking terms, the error is always less than the magnitude of the next term.

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§ 5.5.4. Examples

Write as a power series:

e^{-3x}

$= 1 + (-3x) + \frac{(-3x)^2}{2!} + \frac{(-3x)^3}{3!} + \frac{(-3x)^4}{4!} + \cdots = 1 - 3x + \frac{9}{2}x^2 - \frac{9}{2}x^3 + \frac{27}{8}x^4 + \cdots$

$\frac{1}{1+x^2}$

$= 1 - x^2 + x^4 - x^6 + x^8 + \cdots$

$(1+x)^{1/3}$

$= 1 + \frac{1}{3}x + \frac{(\frac{1}{3})(-\frac{2}{3})}{2}x^2 + \frac{(\frac{1}{3})(-\frac{2}{3})(-\frac{5}{3})}{3!}x^3 + \frac{(\frac{1}{3})(-\frac{2}{3})(-\frac{5}{3})(-\frac{8}{3})}{4!}x^4 + \cdots = 1 + \frac{1}{3}x - \frac{1}{9}x^2 + \frac{5}{81}x^3 - \frac{10}{243}x^4 + \cdots$

Evaluate $\int_0^1 e^{-x^2}\,dx$ using a power series.

$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \implies e^{-x^2} = 1 - \frac{x^2}{1!} + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots = 1 - x^2 + \frac{x^4}{2} - \frac{x^6}{6} + \cdots$. Thus $\int_0^1 e^{-x^2}\,dx = \int_0^1 1 - x^2 + \frac{x^4}{2} - \frac{x^6}{6} + \cdots dx = [x - \frac{x^3}{3} + \frac{x^5}{10} - \frac{x^7}{42} + \cdots]_0^1 = 1 - \frac{1}{3} + \frac{1}{10} - \frac{1}{42} + \cdots$.

Approximate $f(x) = x^2 + \int_0^x \sin^2 t\,dt$ by a power series of degree 2.

Using the chain rule and the FTC we get $f'(x) = 2x + \sin^2 x$ and $f''(x) = 2 + 2\sin x\cos x$. Hence $f(0) = 0, f'(0) = 0, f''(0) = 2$. Thus the power series approximation is $f(x) \approx x^2$.

§ 6.1. Separation of variables

§ 6.1.1. Lecture worksheet

The simplest strategy for solving differential equations is *separation of variables*: move all x's to one side and all y's to the other, then integrate both sides. Thus if we have the equation $\frac{dy}{dx} = x^2 y$ we rewrite it as $dy/y = x^2 dx$ and then integrate both sides to get $\ln|y| = x^3/3 + C$, or $y = \pm e^{x^3/3+C}$.

This example alerts us to some technical points that often come up in this context. First of all you may be upset that I included the constant of integration only on the right hand side of the equation, even though I integrated both sides. But this comes to the same thing, for if I had included constants on both sides, say C_1 on the left and C_2 on the right, then I could just move them to the same side to get $C_2 - C_1$ on the right, which we might as well denote by a single letter C since a constant minus a constant is just another constant. Also, we dislike having constants in the exponents; it's impractical. Therefore the standard trick in these kinds of situations is to rewrite the solution as $y = e^{x^3/3+C} = e^{x^3/3}e^C = Ae^{x^3/3}$. Again, e^C is just another constant so there is no point in writing it this way. It is neater to just give it its own letter, A.

6.1.1. Solve the differential equation for population growth with unlimited resources (from problem 2.5.2). Note that our manner of rewriting constants makes the final constant easy to interpret in real-world terms.

6.1.2. Find the equations for warfare in problem 2.5.2. For the conventional warfare case, we shall now prove the famous military-strategic maxim "never divide your forces," or, if you prefer, "divide and conquer" (divide the enemy, that is).

(a) In these equations, the derivatives are taken with respect to time. So y' means $\frac{dy}{dt}$. But by dividing one equation by the other we can obtain a new equation involving only $\frac{dy}{dx}$ and no t. Do this.

(b) Solve this differential equation.

(c) What is the real-world meaning of the constant of integration? Hint: Consider the cases where one army has been depleted.

Suppose you and the enemy have equal fighting efficiency, $a = b = 1$. You have 5000 soldiers and the enemy has 7000.

(d) If you took the enemy head on, how many of their soldiers would survive the battle and march on toward your capital?

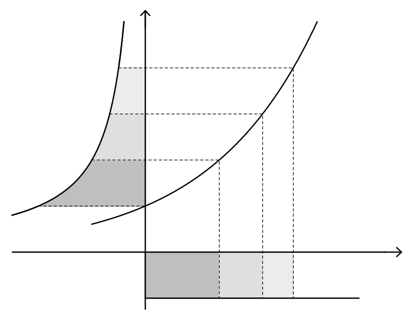
(e) Suppose you managed to split the enemy into two groups of 4000 and 3000 soldiers, for example by blowing up a bridge. If you take them on one at a time, how many enemies will survive to march on your capital in this case?

(f) Explain how, more generally, the conclusion "never divide your forces" can be deduced from 6.1.2b directly.

(g) Solve the equations for guerrilla warfare. Does the same maxim apply in this case?

§ 6.1.2. Problems

6.1.3. *Geometrical interpretation of separation of variables.* Explain how solving $y' = y$ by separation of variables corresponds to the figure below. Areas in the same shade are equal. The point generalises to any separable differential equation.



6.1.4. Consider a differential equation with separated variables, $f(x)\,dx = g(y)\,dy$, to be solved for the initial condition $y(x_0) = y_0$. Instead of taking the indefinite integral of both sides and including a constant of integration we can take the definite integrals $\int_{x_0}^{x} f(x)\,dx = \int_{y_0}^{y} g(y)\,dy$ which gives us the solution directly, bypassing the need for the constant of integration. Explain why this works. Hint: this can be done using problem 6.1.3.

6.1.5. *Forensic medicine.* Newton's law of cooling says that the temperature, H, of a hot object decreases at a rate proportional to the difference between its temperature and that of its surroundings, S:

$$\frac{dH}{dt} = -k(H - S)$$

(a) $\star$ If you stir your coffee, does it cool faster or slower? How is this reflected in Newton's law?

The body of a murder victim is found at noon in a room with a constant temperature of 20°C. At noon the temperature of the body is 35°C; two hours later the temperature of the body is 33°C.

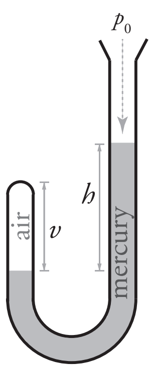
(b) Find the temperature of the body as a function of t, the time in hours since it was found.

(c) Explain how you can check your work by considering the cases $t = 0$ and $t \to \infty$.

(d) When did the murder occur? Assume that the victim had the normal body temperature 37°C at the time of the murder. Provide the answer in both exact and decimal form.

6.1.6. Since atmospheric pressure decreases when you climb a mountain, it ought to be possible to determine one's altitude simply by measuring the atmospheric pressure. In this problem we shall derive a formula that does precisely this.

For this purpose we need Boyle's law of gases, which states that pressure p is proportional to density σ, i.e., $p = a\sigma$, for some constant a. (Background: Boyle discovered this law in 1662 using "a long glass-tube, which, by a dexterous hand and the help of a lamp, was in such a manner crooked at the bottom, that the part turned up was almost parallel to the rest of the tube." The pressure exerted on the enclosed air is the combined effect of the atmospheric pressure p_0 and the weight of the excess mercury (measured by h). The density of the enclosed air is of course readily measured by v. Thus, by pouring in more mercury, we can test the effect of an increase in pressure on density, which reveals Boyle's law: every unit increase in pressure causes an increase in density of a units.)



The atmospheric pressure $p(h)$ at any given altitude h is determined by the weight of the "column of air" weighing down upon it, i.e.,

$$p(h) = \int_h^\infty \sigma(\lambda)\, d\lambda$$

where $\sigma(\lambda)$ is the density of the air at height λ. (We know that this weight is considerable because trees are made out of air.) In terms of the pressure p_0 at the earth's surface, this can be rewritten as

$$p(h) = p_0 - \int_A^B \sigma(\lambda)\, d\lambda$$

(a) where $A = \boxed{}$ and $B = \boxed{}$.

(b) Use Boyle's law to eliminate density from this equation (and replace it with pressure).

The resulting formula relates pressure to altitude, which is what we wanted. However, the formula is quite useless as a means of determining altitude since evaluating the integral experimentally would require measuring pressure at many different intervals of height.

(c) Remarkably, this problem can be alleviated by differentiating both sides. Do so! This leads to the differential equation $dp/dh = \boxed{}$.

(d) Solve the resulting differential equation for p as a function of h. Consider $p(0)$ in order to determine the constant of integration.

(e) Explain how you can check your work by considering the physical meaning of $p(\infty)$.

(f) Solve for h as a function of p.

This formula gives an easy way of finding the altitude from the pressure, as sought. Note that the constants in the final formula are easily determined once and for all, so that pressure is indeed the only input that needs to be measured in the field.

(g) ⋆ Does the formula also work below sea level?

The "column of air" part of the argument may have bothered you. Robert Hooke (*Micrographia*, 1665) explains it as follows: "I say Cylinder, not a piece of a cone, because, as I may elsewhere shew in the Explication of Gravity, that triplicate proportion of the shels of a Sphere, to their respective diameters, I suppose to be removed by the decrease of the power of Gravity." In other words, while the base area of a cone with its vertex at the surface of the earth is as the height squared, gravity is as the inverse height squared, meaning that the weight is equivalent to that of a cylinder with constant gravity.

6.1.7. We shall consider a model for the spread of a disease in an isolated population, such as the students at a boarding school. There are three variables: S = the number of susceptibles, the people who are not yet sick but who could become sick; I = the number of infected, the people who are currently sick; R = the number of recovered, or removed, the people who have been sick and can no longer infect others or be reinfected.

(a) Explain why the following differential equations are a reasonable model for the spread of the disease:

$$\frac{dS}{dt} = -aSI \qquad \frac{dI}{dt} = aSI - bI \qquad \frac{dR}{dt} = bI$$

Consider a school with 1000 students. Let's say that one student develops the flu, and that one day later two more students are infected.

(a) Use the first equation above to estimate a on the basis of this information.

(b) Let's say that $b = 0.5$. What is the real-world meaning of this?

(c) Sketch the direction field for this system with S on the x-axis and I on the y-axis.

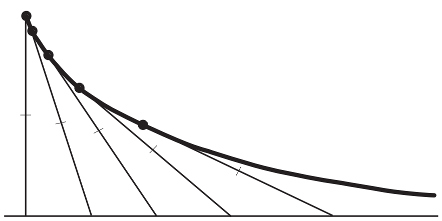
(We really only care about S and I since these are the variables that determine the course of the epidemic. Once people fall into R they might as well not exist as far as the spread of the disease is concerned, so we do not need R in our analysis.)

(d) Find the equation for I as a function of S and draw its graph for the case of one initial sick student and the others all susceptible.

(e) How many students remain uninfected at the end of the epidemic?

(f) Suppose half the students were vaccinated against the flu (and thus not part of the susceptible population). With a and b as above, and $I = 1$ at $t = 0$, how many students remain uninfected in this case? Indicate how this relates to your direction field.

6.1.8. The tractrix is the curve traced by a weight dragged along a horizontal surface by a string whose other end moves along a straight line:



In the *physique de salon* of 17th-century Paris, a pocket watch on a chain was the preferred way for gentlemen to trace this curve.

(a) The same curve can also be interpreted as the "pursuit path" of a ship [trying to catch up with-/shadowing at a safe distance] an enemy ship that [doesn't know it is being pursued/always steers straight away from its pursuer].

Let's say that the length of the string is 1. Consider it as the hypotenuse of a triangle with its other sides parallel to the axes. Draw a figure of this triangle and write in the lengths of its sides (1 for the hypotenuse, y for the height, and the last side by the Pythagorean Theorem).

(b) Find a differential equation for the tractrix by expressing the slope of the curve in terms of this triangle. $dy/dx = \boxed{}$

(c) Separate the variables and integrate. Hint: a possible method is: substitute $u^2 = 1 - y^2$; factor out a u and then split into partial fractions (since partial fractions are not applicable when numerator and denominator have the same degree); bring the u back in the numerators and integrate each fraction by a substitution that simplifies its denominator.

Express x as a function of y, and choose the constant of integration so that the asymptote (along which the free end of the string is pulled) is the x-axis and the point $(0, 1)$ corresponds to the vertical position. Then: $x = \boxed{}$

§ 6.2. Statics

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§ 6.2.1. Lecture worksheet

Statics is the study of physical systems in equilibrium. Things that don't move, in other words. In the problems of this section we shall see how two interesting statics problems reduce to differential equations. This comes about because they involve tangential forces, and tangents are related to derivatives.

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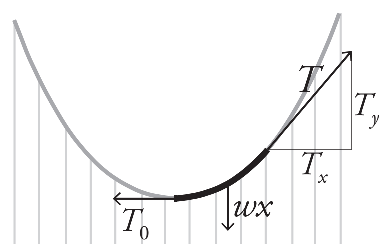
§ 6.2.2. Problems

6.2.1. In this problem we shall find the shape of a suspension bridge cable. We assume that the weight of the roadway and cars is uniformly distributed with density w, so that a segment of the bridge with length x has weight wx. This weight translates into tension in the cable. In the figure we have indicated the tension forces T_0 and T. If the cable was cut at one of these points, then T_0 or T indicates the force we would need to apply at that point to keep the shape of the cable intact.



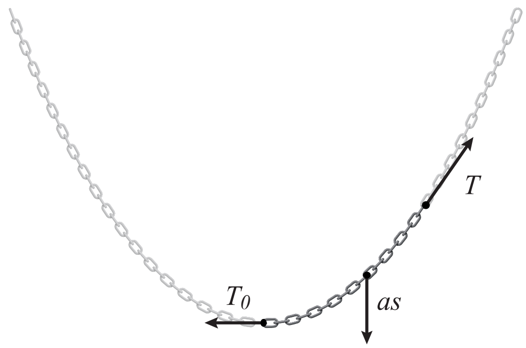
Thus there are three forces acting on the cable segment: the tensions T_0 and T, and the weight wx. Since the cable is in equilibrium these forces must cancel out. Therefore the horizontal component T_x of T must equal T_0, and the vertical component T_y of T must equal wx.

(a) Argue that $T_y/T_x = dy/dx$.

(b) Use this to express the condition of equilibrium as a differential equation.

(c) Solve it.

6.2.2. The shape of a freely hanging chain suspended from two points is called the "catenary," from the Latin word for chain. In principle any piece of string would do, but one speaks of a chain since a chain with fine links embodies in beautifully concrete form the ideal physical assumptions that the string is non-stretchable and that its elements have complete flexibility independently of each other.



We can find a differential equation for the catenary by considering the forces acting on a segment of it. These forces are: the tension forces at the endpoints, which act tangentially, and the gravitational force, which is proportional to the arc s measured from the lowest point of the catenary.

(a) Deduce by an equilibrium of forces argument that the differential equation for the catenary is

$$\frac{dy}{dx} = s$$

for some appropriate choice of units. Hint: consider the vertical and horizontal components of T.

(b) Explain how this equation seems plausible visually, quite apart from the physical argument regarding forces.

Ultimately, we seek an expression for the catenary in terms of x and y only, but the differential equation involves also the variable s. Work around this problem as follows.

(c) Use the basic property of arc lengths $ds^2 = dx^2 + dy^2$ (this is the Pythagorean theorem applied to an infinitesimal triangle; see §1.1) to eliminate dx in the differential equation.

(d) Separate the variables in the resulting expression.

(e) Solve the resulting differential equation. Take the constant of integration to be zero (this corresponds to a convenient choice of coordinate system).

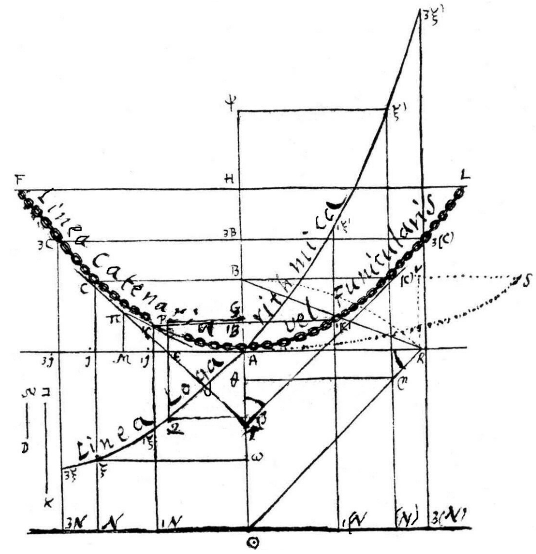
(f) Solve for s in the resulting expression.

(g) Substitute this expression for s into the original differential equation for the catenary.

(h) Verify by differentiation that $y = (e^x + e^{-x})/2$ is a solution of the resulting differential equation.

(i) Sketch the graphs of e^x, e^{-x}, and the catenary, as well as the coordinate system axes in the same figure.

The link between the catenary and logarithms led Leibniz to suggest that measurements on an actual hanging chain could be used in place of logarithm tables for calculations. For your amusement I have included below the figure from Leibniz's 1692 paper on the "linea catenaria," as he calls it. Having solved the problem, you can easily understand what Leibniz means by "linea logarithmica."



§ 6.3. Dynamics

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§ 6.3.1. Lecture worksheet

Dynamics is the study of physical systems involving motion. We have already noted that such systems give rise to differential equations since they are governed by Newton's law $F = ma$, in which a is the second derivative of position. In principle this is the law at bottom of all problems of dynamics, but in practice it is sometimes better to circumvent a direct "brute force" attack using this law in favour of instead employing a suitable conservation principle (cf. §2.4). In the problems of this section we shall see some examples of such approaches.

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§ 6.3.2. Problems

6.3.1. Problems about balls rolling frictionlessly down curved ramps are reduced to differential equations by the fact that the speed acquired is equal to the speed of an object in free fall having covered the same vertical distance. This is a simple consequence of energy conservation: no matter how the ball descends, the speed it acquires must be precisely sufficient to take it back up to its starting point, whether by the same or any other path. The speed of an object falling under constant gravitational acceleration is of course proportional to time, but to characterise the curve geometrically we do not want time to figure in our equations. Therefore we note that, since distance fallen is proportional to time squared, time is proportional to the square root of the distance fallen.

(a) Prove this by integrating the equation acceleration

$= g$ twice (using the initial conditions corresponding to an object dropped from rest).

Thus we have speed in geometrical terms as proportional to the square root of the vertical distance covered. Stated as a differential equation, this becomes

$$\frac{\sqrt{dx^2 + dy^2}}{dt} = a\sqrt{y}.$$

The appearance of time in the above equation is an obstacle to finding a solution in purely geometrical terms as an equation in x and y. However, consider the special problem of finding a curve along which a ball descends at uniform vertical speed, so that $dt = dy$.

(b) Sketch a rough guess of what the solution curve will look like based on your physical intuition.

(c) Find an equation for the solution curve by solving the differential equation. Graph it and check your guess.

6.3.2. The following problem establishes a general result about beads on ramps which will be of use on multiple occasions in later problems. Consider a bead sliding down a ramp of shape $y(x)$.

(a) Using the same reasoning as in problem 6.3.1, express the velocity ds/dt of a particle released from the top of the ramp as a function of y.

(b) The time it takes the particle to reach the bottom is $\int dt$. Using the above, rewrite this integral in terms of x and y.

6.3.3. Many second-order differential equations arise from Newton's law $F = ma$. However, the actual Newton's law is not $F = ma$ but $F = \frac{d}{dt}(mv)$.

(a) Explain in terms of rules of differentiation why these two forms of the law are often equivalent in practice.

The saying that something relatively easy is "not rocket science" was perhaps coined by someone familiar with this distinction, because in rocket science it is in fact necessary to use the more complicated form $F = \frac{d}{dt}(mv)$. Consider a rocket in outer space with no external forces acting on it. Then mv is constant ("conservation of momentum") since $\frac{d}{dt}(mv) = F = 0$. But the rocket can still move forward by throwing out parts of its mass in the form of exhaust products, say with velocity $-b$ relative to the ship. Thus for any infinitesimal time period dt we have the following "before" and "after" scenarios:

	time	mass	velocity
ship	t	m	v
ship	$t+dt$	$m+dm$	$v+dv$
exhaust	$t+dt$	$-dm$	$v-b$

(b) Using conservation of momentum, show that this leads to the differential equation

$$\frac{dv}{dm} = -\frac{b}{m}$$

Hint: Recall the principles for discarding negligible infinitesimal expressions from §1.3.

(c) Solve the differential equation. (Find v as a function of m, i.e., your answer should have the form $v = \cdots$)

You have a choice between two space ships. Ship A has a mass of $m = 1$ and an exhaust velocity of $b = 1$. Ship B also has a mass of $m = 1$ but has an exhaust velocity of $b = 2$. Ship A is cheaper so if you buy it you can afford more fuel: a mass of 2 instead of just a mass of 1 if you buy Ship B. Thus when you start your journey (from rest, $v = 0$) the masses of the ships would be $m = 3$ and $m = 2$ respectively for Ship A and Ship B.

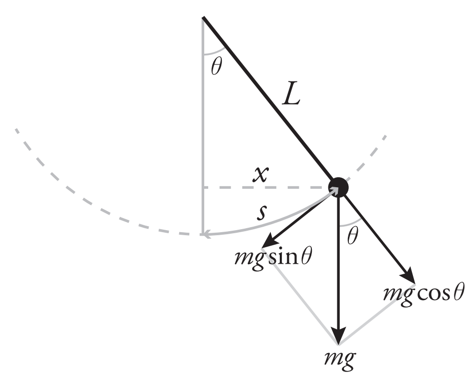
(d) Use these initial conditions to determine the value of the constant in your expression for v for each ship.

(e) Which ship has a higher terminal velocity (i.e., velocity when fuel is exhausted, i.e., when $m =$ ship's mass $= 1$)?

§ 6.4. Second-order differential equations

§ 6.4.1. *Lecture worksheet*

Pendulum motion is the prototype for all periodic phenomena. Indeed, we shall see in this chapter that the many variants of the pendulum motion problem exhaust the better part of the theory of second-order differential equations. But our first order of business is to derive the equation for pendulum motion in its very simplest case.



We wish to find $s(t)$, the elevation of the pendulum measured along its arc. As always we must start with Newton's law $F = ma$. The force involved is the component of gravity that pulls in the direction of the tangent; this is $-gm\sin\theta$ (negative because it acts to decrease s), so $F = ma$ says $m\ddot{s} = -gm\sin\theta$. Since we are looking for a differential equation for $s(t)$ we want s and t to be the only variables. But θ is also variable, so we must get rid of it. We could make it $\sin\theta = x/L$, which doesn't seem much better since x is also variable. But here is the trick: horizontal displacement is almost equal to displacement along the arc, i.e., $x \approx s$, at least for small θ. With this approximation we can get rid of all unwanted variables and obtain $m\ddot{s} = -\frac{gm}{L}s$. In other words, $s(t)$ is a function such that when you differentiate it twice you get back the function itself times $-g/L$. Which functions behave like this? Well, $\sin(\sqrt{g/L}t)$ does, and you could also put a constant A in front and it would still work. And $B\cos(\sqrt{g/L}t)$ does too. So $s(t)$ must have the form

$$s(t) = A\sin(\sqrt{g/L}t) + B\cos(\sqrt{g/L}t)$$

for some constants A and B.

6.4.1. Select all that are true:

- ☐ The higher the starting point, the greater the swing time, according to the solution formula.
- ☐ The smaller the swings, the more accurate the solution.
- ☐ This gives an easy way to estimate g.
- ☐ The fact that the solution has precisely two undetermined constants in it corresponds to the fact that the highest derivative in the differential equation was of order two.
- ☐ None of the above.

Pendulum motion is the archetype of periodic or oscillatory phenomena resulting from a kind of "delayed feedback" mechanism. Gravity is always trying to pull the pendulum down to its lowest position. One might say that gravity "wants" the pendulum to reach this position. But if this is what gravity is trying to achieve, gravity acts a bit stupidly, because it always overshoots its target. The problem is that gravity controls the acceleration rather than the velocity of the pendulum: When the pendulum reaches the lowest position, gravity makes the acceleration zero, but the pendulum still has velocity, so it keeps going anyway.

An analogous situation is that of a thermostat trying to keep the temperature of a room at a fixed level by means of a warm-water radiator. The thermostat "wants" the temperature in a room to be at a desired level, say $20°C$. But it doesn't control the rate of change of temperature, but rather the rate of change of the rate of change of temperature. This is because it controls the valve of the radiator, not the radiator temperature itself. If the thermostat wants it to be warmer, it opens the valve and lets warm water into the radiator. When the thermostat doesn't want to increase the temperature anymore, it closes the valve. But when the valve closes there is still warm water left in the radiator, which will keep warming up the room for a while longer. So the thermostat missed its target.

Another example is what economists call the "pork cycle," which arises due to the tension between the immediate current demand for pork and the delay of raising a pigs to a suitable age for slaughter.

Using the pendulum case as a prototype, we might say:

$$\begin{aligned} y &= \text{height of pendulum} \\ &= \text{the thing you want to control} \\ \dot{y} &= \text{speed of pendulum} \\ &= \text{change in the thing you want to control} \\ \ddot{y} &= \text{gravity on pendulum} \\ &= \text{change in the change in the thing you want to control} \end{aligned}$$

6.4.2. What corresponds to y, $\dot{y}$, and $\ddot{y}$ in the other two scenarios?

- ☐ hot water valve / pig births
- ☐ temperature of room / pigs to sell
- ☐ radiator temp. / adolescent pigs

To repeat the above equation schematically, Newton's law $F = ma$ in the case of pendulum motion is of the form

$$-cx = \ddot{x}$$

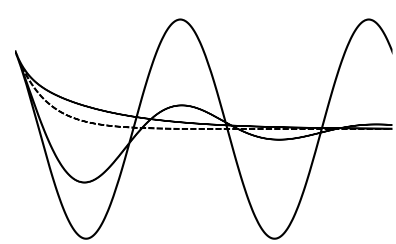
Now we wish introduce air resistance. This is another force, so it will go on the left (F) side of the equation. Like gravity, it too is working against the motion of the pendulum, so it has a minus sign on front of it. But unlike gravity it does not depend on position but on velocity: the faster you go the more the air "pushes back." So the new schematic equation is

$$-cx - b\dot{x} = \ddot{x}$$

Four qualitatively different scenarios are possible:

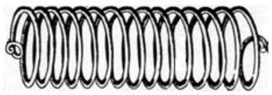
- The "undamped" ($b = 0$) pendulum keeps on swinging forever. This is the idealised case where there is no air resistance.
- The "damped" (b small) pendulum gradually swings shorter and shorter arcs. This is the realistic case of an ordinary pendulum with air resistance.
- The "overdamped" (b big) pendulum goes slowly to its lowest point without oscillating. This could be for example a pendulum submerged in thick syrup.
- The "critically damped" pendulum is right on the boundary between damped and overdamped.

Here are graphs of these four cases:



I have varied the resistance b while keeping other things the same. The critically damped case (dashed) has "just the right amount of resistance." Therefore it is often desirable for applications such as shock-absorbing suspensions.

It is not for nothing that we ended up with an example involving springs rather than pendulums. Strictly speaking, the spring rather than the pendulum is a truer archetype for the theory of second-order differential equations. In fact, the differential equation we derived above holds exactly for springs, whereas it holds only approximately for pendulums (because of our approximation horizontal displacement ≈ arc).



6.4.3. Derive the differential equation for a spring from Hooke's law "ut tensio, sic vis" ("as the extension, so the force," i.e., force is proportional to the displacement of the weight from its equilibrium position).

Nevertheless I prefer the pendulum as my archetype. Its benefits include: the physics behind it is of much greater generality ($F = ma$ and gravity versus Hooke's law); it exemplifies both stable and unstable equilibria in their purest forms; we have greater intuition about it and encounter it more frequently in everyday life. This last point makes a difference especially when "forcing" is introduced, i.e., when the pendulum is being pushed by an external force, like a child on a swing being pushed by a parent. Our schematic equation for a pendulum left to its own devices can be rewritten as

$$\ddot{x} + b\dot{x} + cx = 0$$

With forcing this becomes

$$\ddot{x} + b\dot{x} + cx = f(t)$$

So the forcing term "doesn't care about x"; it imposes itself following its own formula regardless of what the pendulum is doing at that time. It is in fact often true for differential equations more generally that the terms involving the function and its derivatives represent the internal dynamics of the situation while other terms represent external forces artificially imposed upon it from without.

6.4.4. Go back to problem 2.5.2 and see if you can find some other such examples.

There is a systematic way of solving all second-oder differential equations with constant coefficients. For the "homogenous" (i.e., non-forced) case $\ddot{x} + b\dot{x} + cx = 0$, the strategy is to first solve a related quadratic equation, the "characteristic equation" $m^2 + bm + c = 0$. So the power of m in this quadratic equation equals the order of the derivative in the differential equation. If this equation has two distinct real roots m_1, m_2, then the solution is $x(t) = Ae^{m_1 t} + Be^{m_2 t}$. If the characteristic equation has a double root $m_1 = m_2$ then the solution is $x(t) = Ae^{m_1 t} + Bte^{m_1 t}$. These are the overdamped and critically damped cases. The oscillating cases will come from characteristic equations that have no real roots at all, such as $x^2 + 1 = 0$. Such cases will be treated in the next section.

We know that we have found all solutions in this way since a general solution for a second-order problem should have two constants in it (as in problem 6.4.1). It doesn't really matter how we found the solutions because once we have them we can verify them by simply plugging them into the differential equation. Nevertheless one may wonder where the idea of the characteristic equation came from. The heuristic behind it is that one guesses that there will be a solution of the form e^{mt} and then plugs this in, giving

$$(e^{mt})'' + b(e^{mt})' + ce^{mt} = (m^2 + bm + c)e^{mt} = 0$$

Since e^{mt} is always positive, $m^2 + bm + c$ must be zero, and thus we are led to the characteristic equation.

So much for the homogenous case. When a forcing term is present one must split the problem into two parts. First throw away the forcing term and solve the corresponding homogenous case as above. Call the solution x_h. Then find a particular solution to the forced equation, call it x_p. Adding these two solutions together solves the general equation.

6.4.5. Show that $x_h + x_p$ is a solution, and argue that there are no others.

We find the particular solution by educated guess-and-check: We hypothesise a solution of the same form as the forcing term, but with undetermined constants in front of every term, then we plug this guess into the equation to determine the constants.

6.4.6. Solve $\ddot{x} + 2\dot{x} + x = t$.

A special case not covered by this form of educated guessing is the equation $\ddot{x} + c^2 x = A\cos ct$, which has the particular solution $x(t) = \frac{A}{2c} t \sin ct$. Similarly $\ddot{x} + c^2 x = A\sin ct$ has the particular solution $x(t) = -\frac{A}{2c} t \cos ct$.

6.4.7. Interpret these special cases in terms of a child on a swing.

6.4.8. I have solved a second-order differential equation and found the homogenous solution x_h and the particular solution x_p. Which of the following must be true?

- ☐ $x_h + x_p$ is a solution to the differential equation.
- ☐ x_h is a solution to the differential equation.
- ☐ x_p is a solution to the differential equation.
- ☐ $x_p + 5$ is another particular solution.
- ☐ $5x_p$ is another particular solution.
- ☐ x_h contains two undetermined constants.
- ☐ The choice of particular solution is not unique.
- ☐ Once I have an initial condition such as $x(0) = 2$ I can give one concrete answer for the general solution.

§ 6.5. Second-order differential equations: complex case

§ 6.5.1. Lecture worksheet

The method for solving second-order differential equations introduced above needs some tweaking to apply to cases where the roots of the characteristic equation are complex numbers (§A.6). Though this is mathematically the most complicated case, it includes the physically simplest situation of all: the simple pendulum equation introduced already in problem 2.5.2.

Suppose we want to solve $\ddot{x} + b\dot{x} + cx = 0$ and find that the characteristic equation $m^2 + bm + c = 0$ has the complex roots $m = a \pm bi$ (complex roots always come in pairs like this). If we proceed as above we would then have solutions of the form $x(t) = Ae^{(a+bi)t} + Be^{(a-bi)t}$. But of course we do not want complex numbers in our solution since we are interested in real things like pendulums. We therefore break apart the complex exponentials using problem A.6.7,

$$e^{(a+bi)t} = e^{at}e^{bti} = e^{at}(\cos bt + i \sin bt),$$

and then just stick constants in front of every term to get the final answer

$$x(t) = Ae^{at}\cos bt + Be^{at}\sin bt.$$

As always, we can verify our answer by plugging it back into the original equation, and we know that it is the most general solution since it has as many constants as the order of the equation. Note that we only need to break apart one of the complex exponentials to get all real solutions. The other one, $e^{(a-bi)t}$, would not add anything new since the constants will "eat up" the minus sign.

6.5.1. Solve $\ddot{x} - 2\dot{x} + 2x = 0$ given the initial conditions $x(0) = 1$ and $x'(0) = 0$.

6.5.2. What values of a and b correspond to undamped and damped pendulum motion respectively? Are there any other possibilities?

6.5.3. The standard form for a second-order differential equation is $\ddot{x} + b\dot{x} + cx = f(t)$. In terms of a pendulum, b is air resistance, c is gravity, and $f(t)$ is forcing (i.e., an external force pushing the pendulum). By picturing this prototype example we can get a good feeling for the behaviour of such a differential equation. Use this way of thinking to associate the following differential equations with their corresponding scenario and type of solution.

Equations:

☐ $\ddot{x} + x = 0$

☐ $\ddot{x} + x = t$

☐ $\ddot{x} + 2x = \sin(t)$

☐ $\ddot{x} + x = \sin(t)$

☐ $\ddot{x} + \dot{x} - x = 0$

☐ $\ddot{x} + 0.1\dot{x} + 2x = 0$

☐ $\ddot{x} - 0.1\dot{x} + 2x = 0$

Scenarios: A = pendulum with no air resistance; B = child on swing pushed well by parent; C = child on swing given out-of-synch pushes; D = pendulum in syrup; E = air "encouragement" instead of air resistance; F = "negative gravity"; G = pendulum with slight air resistance; H = pendulum pulled in one direction.

Types of solution: a = perpetual oscillations; b = dying oscillations; c = growing oscillations; d = slow approach to equilibrium without oscillations; e = running off to infinity; f = jerky motion.

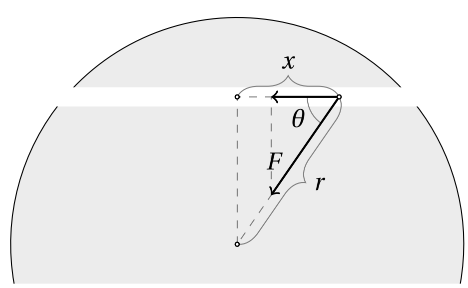
§ 6.5.2. Problems

6.5.4. † Suppose we dig a straight, frictionless tunnel between any two points of the earth's surface, such as New York and Paris. Then, if we jump into the tunnel at one end, gravity will transport us to the other in less than 45 minutes. Let's prove this.

The motion is of course governed by $F = ma$. The force in question is gravity. The gravitational pull exerted on an object with mass m inside the earth at a distance r from the center of the earth is $F = \frac{GM}{R^3}mr$, where G is the usual gravitational constant, M is the mass of the earth, R is the radius of the earth. This is so for the following reasons. First of all the mass of the earth further than r from the center will have no influence on the object since the net gravitational effect of this outer shell is zero.

(a) Argue that this is so. Hint: Consider the object as the vertex of a double cone, and argue that the two pieces of any thin outer shell that the cones cut out cancel each other in terms of their gravitational effect.

(b) Show that therefore the force of gravity acting on the body at a distance r from center of earth is $G\frac{M}{R^3}mr$.



This force tries to pull the object towards the center of the earth, but since the object can only move along the tunnel it is only the part of the force that pulls in the direction of the tunnel that has any effect. Let the tunnel be the x-axis of a coordinate system with the tunnel's midpoint as the origin.

(c) Use trigonometry to find a formula for the effective part of the force. (Note that the force is acting in the negative x-direction.)

We have thus found the part of gravity that acts in the direction of the tunnel, i.e., the actual force F in the law $F = ma$ governing the motion of the object along the tunnel.

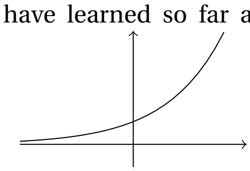
(d) Set this force equal to ma and solve the resulting differential equation. Assume that we enter the tunnel with zero velocity.

(e) Is $x(t)$ periodic? What is its period? What determines the period?

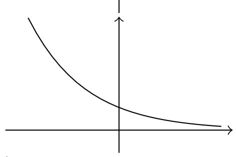
§ 6.6. Phase plane analysis

§ 6.6.1. Lecture worksheet

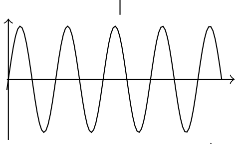
The classification of exponential expressions that we have carried out in the previous two sections is useful beyond its application to pendulum-type differential equations. In this section we shall see that it also helps us understand the nature of equilibrium points of systems of differential equations. But let us first summarise what we have learned so far about expressions of the form $e^{(a+bi)t}$.



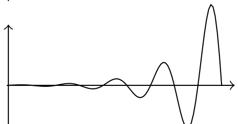
Exponent real and positive ($a > 0$, $b = 0$): exponential growth.



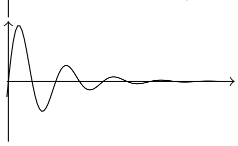
Exponent real and negative ($a < 0$, $b = 0$): exponential decay.



Exponent pure imaginary ($a = 0$, $b \neq 0$): periodic oscillations.



Exponent complex with positive real part ($a > 0$, $b \neq 0$): growing oscillations.

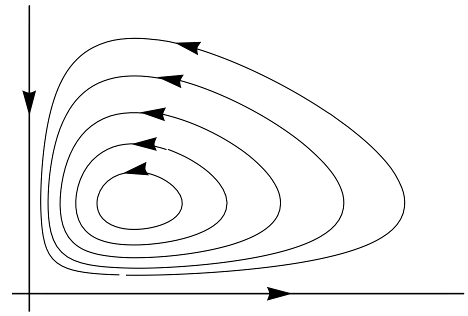


Exponent complex with negative real part ($a < 0$, $b \neq 0$): shrinking oscillations.

These things will come up again when we study the equilibria of systems of two equations, such as those we have seen for warfare and two-species interaction. In the predator-prey system

$$\begin{cases} \dot{x} = ax - bxy \\ \dot{y} = -cy + dxy \end{cases}$$

there are two equilibrium points which are of very different character, as one can see from the phase plane diagram:



6.6.1. (a) Explain briefly the real-world meaning of this diagram.

(b) Calculate the equilibria.

(c) Find the equilibria of the system when harvesting is introduced:

$$\begin{cases} \dot{x} = ax - bxy - ex \\ \dot{y} = -cy + dxy - ey \end{cases}$$

(d) During World War I, when overall fishing was reduced, the catches of Italian fishermen contained a larger percentage of sharks. Explain why by equations as well as commonsensically.

How does one draw the phase plane diagram for a given system of differential equations equations? In principle one can divide the two time-derivatives by each other; since

$$\frac{\dot{y}}{\dot{x}} = \frac{dy/dt}{dx/dt} = \frac{dy}{dx}$$

this gives an expression for the slopes, so that we can plot a direction field as in §2.6. However, the resulting formula can be very complicated to work with, as one may well expect from such a brute-force approach that, in dividing away the time variable, disregards the underlying nature of the problem. We shall consider instead a second approach, which is more sensitive to the internal dynamics of the system.

Let us begin with the simple system

$$\begin{cases} \dot{x} = ax + by \\ \dot{y} = cx + dy \end{cases}$$

6.6.2. Show that in general this system has only one equilibrium point, at $(0,0)$.

This system has solutions of the form

$$\begin{cases} x = Ae^{m_1 t} + Be^{m_2 t} \\ y = Ce^{m_1 t} + De^{m_2 t} \end{cases}$$

as is not hard to imagine. Checking this would be tedious but straightforward. If you carried out the computations you would find that

$$m = \frac{1}{2}\left(a + d \pm \sqrt{(a+d)^2 - 4(ad - bc)}\right)$$

or, if we use the shorthand notation $p = a + d$, $q = ad - bc$, $\Delta = p^2 - 4q$,

$$m = \frac{1}{2}\left(p \pm \sqrt{\Delta}\right)$$

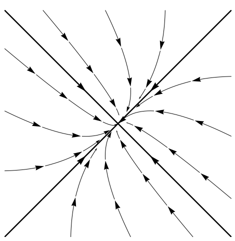
The point is that we do not need to worry about all these horrible calculations, only classify which types of exponential expressions we are dealing with. This will give us a clear enough picture of what is going on without having to worry about the numerical details.

If for example the exponents m are purely imaginary (which obviously happens if $p = 0$ and $q > 0$) then both $x(t)$ and $y(t)$ are periodically oscillating functions, as seen in our table above. In the phase plane this gives closed loops like those shown in the predator-prey figure above. As time ticks away, we go around and around in the same loop over and over again.

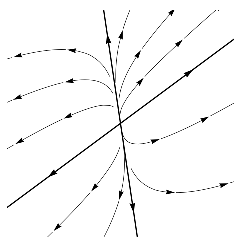
If the exponents are complex with a negative real part then x and y are oscillating towards zero, so we get an inward spiral.

And so on. Altogether we get the following classification.

- $q > 0$, $\Delta > 0$: node.

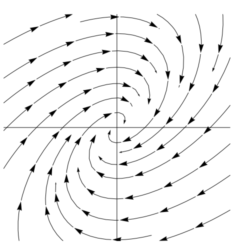


$p < 0$: stable node.

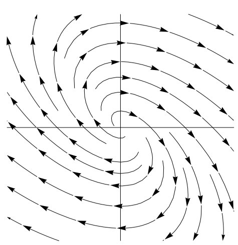


$p > 0$: unstable node

- $q > 0$, $\Delta < 0$: spiral.

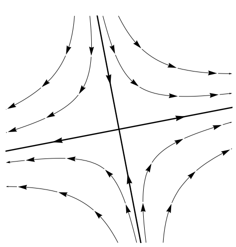


$p < 0$: stable spiral.

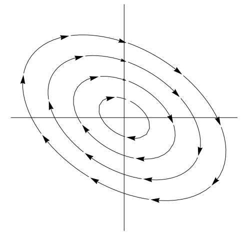


$p > 0$: unstable spiral

- $q < 0$: saddle.



- $p = 0$, $q > 0$: centre.



As we see, nodes and saddles involve exceptional lines. So to be able to draw a good picture in those cases we must first know the slopes m of these lines. The characteristic property of these lines is: if we are on the line, we stay on the line. Being on the line means that $y = mx$ (no constant term since the line passes through the equilibrium, i.e., the origin), and staying on the line means that we keep moving in the direction of the line, i.e., $dy/dx = m$, so we get the equation

$$\frac{\dot{y}}{\dot{x}} = \frac{cx + dy}{ax + by} = \frac{cx + dmx}{ax + bmx} = m$$

which can be solved for m (the rightmost equality yields a quadratic equation, corresponding to the two exceptional lines).

The directional arrows are found by plugging in particular points near the equilibrium. If for example $\dot{x}$ and $\dot{y}$ are both positive at a particular point then they are both growing so we must be heading "northeast," so we place an arrow in our diagram to that effect.

This classification was for a simple linear system with a single equilibrium at the origin, but in fact any equilibrium can be approximately reduced to this situation. This is done as follows. Take for example the predator-prey system

$$\begin{cases} \dot{x} = 3x - xy \\ \dot{y} = -4y + 2xy \end{cases}$$

which we know has an equilibrium at $(c/d, a/b) = (2,3)$. First we want to make a change of variables that brings this point to the origin. We do this by making $x = 2 + X$ and $y = 3 + Y$, which means that the equilibrium $(2,3)$ is the origin $(0,0)$ in the new coordinate system (X, Y). With this change of variables the system becomes

$$\begin{cases} \dot{x} = 2x - xy = 3(2+X) - (2+X)(3+Y) = -2Y + XY \\ \dot{y} = -3y + 2xy = -4(3+Y) + 2(2+X)(3+Y) = 6X + XY \end{cases}$$

(We can check our work by the fact that the constant terms must disappear since the derivatives must vanish at the equilibrium $(X, Y) = (0,0)$.) But we can still not use our classification since there are nonlinear terms present (XY). However, since we are interested in the behaviour close to the equilibrium point we can discard any higher-order terms: when X and Y are very small, XY is very, very small, so it is negligible in comparison with X and Y. With this linear approximation we get

$$\begin{cases} \dot{x} = -2Y \\ \dot{y} = 6X \end{cases}$$

Now we can classify the equilibrium using the table above. In this case, $p = 0$ and $q = 12$, so $(2,3)$ is a centre, i.e., the equilibrium is encircled by closed loops. To find the direction of the loops we can consider for example a point just to the right of the origin, $(X,Y) = (1,0)$. There $\dot{x} = 0$ and $\dot{y} > 0$, so we are heading straight upwards from the "three o'clock" position, meaning that the loops go counterclockwise.

The other equilibrium is at $(0,0)$ already so we do not need to change the variables. We can simply linearise directly to get

$$\begin{cases} \dot{x} = 3X \\ \dot{y} = -4Y \end{cases}$$

Here $q = -12$ so we have a saddle point. As we saw above, normally when we have a saddle point we would look for the exceptional lines by solving

$$\frac{\dot{y}}{\dot{x}} = \frac{-4Y}{3X} = \frac{-4mX}{3X} = m$$

for m. In this case, however, the method breaks down since the solutions are the x-axis (which is not found since $Y = 0$ is impossible in the above equation) and the y-axis (which is not found since it has $m = \infty$). But it is clear enough that the axes are the exceptional lines and that along the y-axis we are crashing to zero and along the x-axis we are running off to infinity. This is also necessary given the counterclockwise orientation of the loops around the other equilibrium.

In general, once we have analysed the equilibria and drawn a local picture for each we can fill in the rest of the phase plane by making one local picture transition smoothly into the other (this is of course permitted whenever $\dot{x}$ and $\dot{y}$ are continuous).

6.6.3. Draw the phase plane diagram of

$$\begin{cases} \dot{x} = y + xy \\ \dot{y} = 3x + xy \end{cases}$$

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§ 6.6.2. Problems

6.6.4. (a) Argue that the system

$$\begin{cases} \dot{x} = x - 2yx \\ \dot{y} = 2y - 4xy \end{cases}$$

represents two species competing for the same resources.

(b) Discuss the difference between the two species. In particular, what could be the biological reason for the difference in the coefficients for the nonlinear terms?

(c) Sketch the phase plane diagram using the approximate linearisation method.

(d) Interpret the diagram in real-world terms.

6.6.5. Consider the system $dr/dt = -j$, $dj/dt = r$, where $r(t)$ represents Romeo's love (positive values) or hate (negative values) for Juliet at time t, and $j(t)$ similarly represents Juliet's feelings toward Romeo.

(a) $\boxed{}$ "loves to be loved," while $\boxed{}$ is intrigued by rejection.

(b) Romeo's and Juliet's families are enemies. This can be expressed in the initial condition $(r, j) =$ $\boxed{}$ at time $t = 0$.

(c) What happens in the long run?

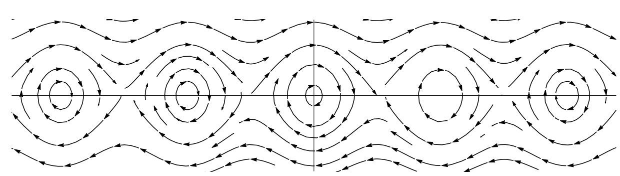
(d) "In the Spring a young man's fancy lightly turns to thoughts of love," says Tennyson. What differential equation concept is best invoked to capture this idea?

6.6.6. The method we have studied in this section for drawing the phase portrait of a system of two first-order differential equations can also be applied to second-order differential equations. The latter is reduced to the former by taking the derivative of the function as a new variable. Let us carry this out in the case of the pendulum equation. Recall from 6.4 that the actual pendulum equation is $\dot{x}' = -k\sin(x)$. We simplified this to $\dot{x}' = -kx$, which is accurate for small oscillations since $\sin(x) \approx x$ when x is small. But now we wish to use the actual equation $\dot{x}' = -k\sin(x)$, which holds for oscillations of any size.

(a) Let $y = \dot{x}$ and note how the pendulum equation becomes a system of two first-order differential equations.

(b) Carry out the phase plane analysis for this system. (You will need to use the approximation $\sin(x) \approx x$ when linearising near equilibria.)

You should obtain a picture like this:



(c) Explain the two types of equilibria in physical terms.

(d) Draw a few phase curves and explain their physical meaning.

6.6.7. In problem 6.6.1d you probably assumed that the equilibrium values for x and y give a good indication of the average numbers of x and y also for orbits other than the equilibrium. This looks about right from the phase plane diagram, but let us confirm it by calculation.

Since the equilibrium is a centre, we know that x and y are periodic functions, say with period T.

(a) Show that $\dot{x}/x = a - by$ and integrate both sides of this equation from 0 to T. Hint: The left hand side is a "logarithmic derivative."

(b) Use the resulting equation to find the average of y. (See problem 3.1.6.)

(c) Find the average of x similarly.

§ 6.7. Reference summary

§ 6.7.1. Separation of variables

To solve a differential equation by separation of variables: Write $y' = \frac{dy}{dx}$. Move all x's (including dx) to one side and all y's (including dy) to the other. Integrate both sides. Include a constant of integration ("$+C$") on one side. If given, plug in initial condition (specific values of y and x) to determine C. Solve for y to find the function that solves the differential equation.

$y' = -2xy^2$

The equation can be written as $\frac{dy}{dx} = -2xy^2$. An evident solution is $y = 0$. When $y \neq 0$, separation of variables gives $\frac{dy}{y^2} = -2x dx$. Integration of both sides gives $\frac{1}{y} = -x^2 + C$, and hence $y = \frac{1}{x^2 - C}$.

$1 + x = 2xyy'$, $y(1) = 2$

Write $y' = dy/dx$ and use separation of variables. $1 + x = 2xyy' \implies (\frac{1}{x}+1)dx = 2y\,dy \implies \ln|x| + x = y^2 + C$. Plugging in the condition $x = 1$ when $y = 2$ gives $\ln|1| + 1 = 2^2 + C \implies C = -3$. Thus $y(x) = \pm\sqrt{3 + x + \ln|x|}$.

$y' e^{x+y} = 1$, $y(0) = 1$

$\frac{dy}{dx} e^y = e^{-x} \Rightarrow e^y dy = e^{-x} dx \Rightarrow \int e^y dy = \int e^{-x} dx \Rightarrow e^y = -e^{-x} + C$. Since $y(0) = 1$, we get $C = e + 1$. Thus the solution is $e^y = -e^{-x} + e + 1 \Leftrightarrow y = \ln(-e^{-x} + e + 1)$.

$y' = y^2$, $y(0) = 1$

An evident solution is $y = 0$, but it does not satisfy the initial condition $y(0) = 1$, so it can be disregarded. When $y \neq 0$ we can divide by y^2, which gives $\frac{y'}{y^2} = 1$ and $\frac{dy}{y^2} = dx$. Integrating gives $\frac{1}{y} = x + C$ and plugging in $y(0) = 1$ gives $C = -1$. Thus $y = \frac{1}{1-x}$.

$(1 + \cos x)y' = y \sin x$, $y(0) = 1$

$(1 + \cos x)y' = y \sin x \Leftrightarrow \frac{dy}{y} = \frac{\sin x dx}{1+\cos x} \Leftrightarrow \int \frac{dy}{y} = \int \frac{\sin x dx}{1+\cos x} \Leftrightarrow \ln y = -\ln(1 + \cos x) + C$. The condition $y(0) = 1$ gives $\ln 1 = -\ln 2 + C \Leftrightarrow 0 = -\ln 2 + C \Leftrightarrow C = \ln 2$. Hence $\ln y = -\ln(1 + \cos x) + \ln 2 \Leftrightarrow y = e^{-\ln(1+\cos x)+\ln 2} \Leftrightarrow y = \frac{2}{1+\cos x}$.

$(y + 1)y' + \cos x = 0$, $y(0) = 0$

The equation separates to $(y+1)dy = -\cos x dx \Leftrightarrow \frac{1}{2}y^2 + y = -\sin x + C$. Plugging in $y(0) = 0$ gives $\frac{1}{2}\cdot 0 + 0 = 0 + C \Rightarrow C = 0$. We multiply by 2 to get $y^2 + y = -2\sin x \Leftrightarrow (y + 1)^2 = 1 - 2\sin x$. The initial condition means that only the positive roots are relevant, so $y = \sqrt{1 - 2\sin x} - 1$.

§ 6.7.2. Other methods for first-order differential equations

Linear differential equation $y' + P(x)y = Q(x)$. Multiply both sides by integrating factor $e^{\int P(x)\,dx}$ and interpret left hand side as outcome of product rule to integrate.

$y' - y = x$

Since, $\int(-1)dx = -x$ (with $C = 0$), we see that e^{-x} is an integrating factor. This gives $e^{-x}y' - e^{-x}y = xe^{-x}$ and hence $(e^{-x}y)' = xe^{-x}$ and $e^{-x}y = \int xe^{-x}dx = -xe^{-x} + \int e^{-x}dx = -xe^{-x} - e^{-x} + C$. Thus the general solution is $y = -x - 1 + Ce^x$.

$y' + y = x$, $y(0) = 0$

Use integrating factor e^x: $y'e^x + ye^x = xe^x$, $ye^x = \int xe^x dx = xe^x - e^x + C$, $y = x - 1 + Ce^{-x}$. The condition $y(0) = 0$ gives $0 = -1 + C$, so $C = 1$ and $y = x - 1 + e^{-x}$.

$y' + 2xy = x$

Since $\int 2x dx = x^2$ (where $C = 0$), we see that e^{x^2} is an integrating factor. This gives $e^{x^2}y' + 2xe^{x^2}y = xe^{x^2}$, and hence $(e^{x^2}y)' = xe^{x^2}$ and $e^{x^2}y = \int xe^{x^2}dx = [x^2 = t,\ 2x dx = dt] = \frac{1}{2}\int e^t dt = \frac{1}{2}e^t + C = \frac{1}{2}e^{x^2} + C$. The general solution is thus $y = \frac{1}{2} + Ce^{-x^2}$.

$x^2 y' + y = e^{1/x}$, $y(1) = e$.

$x^2 y' + y = e^{1/x} \Leftrightarrow y' + \frac{1}{x^2}y = \frac{1}{x^2}e^{1/x}$. Multiplying by the integrating factor $e^{-1/x}$ gives $e^{-1/x}y' + \frac{1}{x^2}e^{-1/x}y = \frac{1}{x^2} \Leftrightarrow D(e^{-1/x}y) = \frac{1}{x^2}$, which upon integrating yields $e^{-1/x}y = C - \frac{1}{x} \Leftrightarrow y = (C - \frac{1}{x})e^{1/x}$. The boundary condition $y(1) = e$ implies $e = y(1) = (C - 1)e \Rightarrow C = 2$. Hence the solution is $y = (2 - \frac{1}{x})e^{1/x}$.

When above methods not applicable: A substitution may make it so. Standard substitutions:

- If the differential equation is of the form $y' = f(x, y)$ where f is invariant under scaling ($f(\lambda x, \lambda y) = f(x, y)$), substitute $u = y/x$.
- If the differential equation involves terms $ay \pm bx$, try making this u.
- If the differential equation is of the form $y' + P(x)y = Q(x)y^n$, substitute $u = y^{1-n}$.

To carry out the substitution: Differentiate the equation defining u to find $\frac{du}{dx}$. Using this and the defining equation for u, eliminate all y's and dy's from the differential equation, so as to produce a differential equation for u as a function of x. Solve this equation. Substitute back to express the answer in terms of the original variables.

§ 6.7.3. *Second-order differential equations*

- Solve $\ddot{x}+b\dot{x}+cx=0$.

  Form the corresponding characteristic equation $m^2+bm+c=0$. The roots of this equation tells you the solution:

roots of $m^2+bm+c=0$:	solution of $\ddot{x}+b\dot{x}+cx=0$:
distinct real	$x(t)=Ae^{m_1 t}+Be^{m_2 t}$
double root	$x(t)=(A+Bt)e^{mt}$
complex $a\pm bi$	$x(t)=Ae^{at}\cos bt+Be^{at}\sin bt$

$y''+3y'-4y=0$

The characteristic equation $m^2+3m-4=0$ has two real roots: $m_1=1$ and $m_2=-4$. Thus $y(x)=Ae^x+Be^{-4x}$.

$y''+2y'+y=0$

The characteristic equation $m^2+2m+1=0$ has a double root: $m_{1,2}=-1$. Thus $y(x)=(Ax+B)e^{-x}$.

$y''+2y'+2y=\sin 2x$

The characteristic equation $m^2+2m+2=0$ has the complex roots $m_1=-1+i$ and $m_2=-1-i$. Thus $y(x)=e^{-x}(A\sin x+B\cos x)$.

- Solve $\ddot{x}+b\dot{x}+cx=f(t)$.

  First consider the corresponding homogenous equation $\ddot{x}+b\dot{x}+cx=0$. Solve it as above. Call the solution x_h.

  Next find a particular solution x_p, meaning any one function that satisfies the differential equation. Do this by first guessing that x_p has the same form as $f(t)$, except with undetermined coefficients for each term. For instance:

If $f(t)$ has this form:	Try this as x_p :
$3e^{5t}$	Ce^{5t}
$\sin t$	$C\sin t+D\cos t$
$2\cos 3t$	$C\sin 3t+D\cos 3t$
$4x+2$	$Cx+D$
$8x^2$	Cx^2+Dx+E

  Special cases not covered by this method:

Differential equation	Particular solution x_p :
$\ddot{x}+c^2x=A\cos ct$	$\frac{A}{2c}t\sin ct$
$\ddot{x}+c^2x=A\sin ct$	$-\frac{A}{2c}t\cos ct$

  Then determine the coefficients needed to fulfil the equation by plugging x_p, x_p', x_p'' into the differential equation.

The full solution of the differential equation is the sum of the homogenous and particular solutions: $x(t)=x_h+x_p$.

$y''+4y'+4y=x+1$

The characteristic equation $m^2+4m+4=0$ has a double root $m_{1,2}=-2$. Hence $y_h=(C_1x+C_2)e^{-2x}$. We seek a particular solution of the form $y_p=Ax+B$. We see that $y_p''+4y_p'+4y_p=4Ax+(4A+4B)=x+1$. Thus $4A=1$ and $4A+4B=1$, which gives $A=\frac{1}{4}$, $B=0$. Thus $y_p=\frac{1}{4}x$ and the general solution is $y=(C_1x+C_2)e^{-2x}+\frac{1}{4}x$.

$y''+y=x^2$, $y(0)=0$, $y'(0)=0$

The characteristic equation $m^2+1=0$ has the solution $m=\pm i$, hence the homogenous solution is $y_h=C_1\sin x+C_2\cos x$. For the particular solution, assume $y_p=ax^2+bx+c$. Then $y''=2a$. The requirement $y_p''+y_p=x^2$ gives $2a+ax^2+bx+c=x^2$, which has the solution $a=1$, $b=0$, $c=-2$. So the particular solution is $y_p=x^2-2$, and the full solution is $y=y_p+y_h=x^2-2+C_1\sin x+C_2\cos x$. The condition $y(0)=0$ gives $0=-2+C_2$, $C_2=2$. Differentiation gives $y'=2x+C_1\cos x-C_2\sin x$ and the condition $y'(0)=0$ gives $C_1=0$. Hence $y=x^2-2+2\cos x$.

$y''-2y'-15y=6e^{-x}$

The characteristic equation $m^2-2m-15=0$ has the solutions $m_1=-3$ and $m_2=5$. Hence $y_h=C_1e^{-3x}+C_2e^{5x}$. We seek a particular solution of the form $y_p=Ae^{-x}$. Differentiation gives $y_p'=-Ae^{-x}$, $y_p''=Ae^{-x}$. Plugging this into the equation, we get $Ae^{-x}+2Ae^{-x}-15Ae^{-x}=6e^{-x}\Leftrightarrow -12A=6\Leftrightarrow A=-\frac{1}{2}$. The general solution is thus $y=y_h+y_p=C_1e^{-3x}+C_2e^{5x}-\frac{1}{2}e^{-x}$.

$y''-4y'+13y=e^{-2x}+1$

The characteristic equation $m^2-4m+13=0$ has the roots $m_{1,2}=2\pm 3i$. Thus $y_h=e^{2x}(D_1\cos 3x+D_2\sin 3x)$. We seek a particular solution of the form $y_p=Ae^{-2x}+B$. This gives $y_p''-4y_p'+13y_p=4Ae^{-2x}-4(-2Ae^{-2x})+13(Ae^{-2x}+B)=25Ae^{-2x}+13B=e^{-2x}+1$. Hence $A=1/25$, $B=1/13$ and this $y_p=\frac{1}{25}e^{-2x}+\frac{1}{13}$. The general solution is thus $y=e^{2x}(D_1\cos 3x+D_2\sin 3x)+\frac{1}{25}e^{-2x}+\frac{1}{13}$.

§ 6.7.4. *Phase plane analysis*

Classification of equilibrium $(0,0)$ of

$$\begin{cases}\dot{x}=ax+by\\ \dot{y}=cx+dy\end{cases}$$

in terms of $p = a + d$, $q = ad - bc$, $\Delta = p^2 - 4q$:

p	q	Δ	equilibrium
$+$	$+$	$+$	unstable node
$-$	$+$	$+$	stable node
$+$	$+$	$-$	unstable spiral
$-$	$+$	$-$	stable spiral
	$-$		saddle
0	$+$		centre

The system $\dot{x} = x - 5y$, $\dot{y} = x - y$ has an equilibrium at $(0,0)$. Classify it.

$a = 1, b = -5, c = 1, d = -1$, so $p = a + d = 0, q = ad - bc = 4, \Delta = p^2 - 4q = -16$. Since $p = 0$ and $q > 0$, the equilibrium is a centre.

The system $\dot{x} = 2x + 3y$, $\dot{y} = -3x - 3y$ has an equilibrium at $(0,0)$. Classify it.

$a = 2, b = 3, c = -3, d = -3 \Rightarrow p = -1, q = 3, \Delta = -11 \Rightarrow$ stable spiral.

The system $\dot{x} = x + y$, $\dot{y} = x - 2y$ has an equilibrium at $(0,0)$. Classify it.

$a = 1, b = 1, c = 1, d = -2 \Rightarrow p = -1, q = -3, \Delta = 13$. Since $q < 0$, the equilibrium point is a saddle. The slopes of the exceptional lines or axes of the saddle are given by $m = \dot{y}/\dot{x} = \frac{x-2mx}{x+mx} = \frac{1-2m}{1+m}$, which gives $m^2 + 3m - 1 = 0$, or $m = \frac{1}{2}(-3 \pm \sqrt{13})$.

- Find and classify the equilibria of a system $\dot{x} = f(x,y)$, $\dot{y} = g(x,y)$.

  Find all points (x,y) where $\dot{x} = 0$ and $\dot{y} = 0$ at the same time. These are the equilibria. Classify each equilibrium in turn.

  If $(0,0)$ is an equilibrium, classify it as above. If the expressions for $\dot{x}$ and $\dot{y}$ contain nonlinear terms (such as x^2, xy, etc.), discard these before classifying.

  If (A,B) is an equilibrium other than $(0,0)$, first make the change of variables $x = A + X$, $y = B + Y$. The equilibrium is now at $(0,0)$ in the new XY coordinate system. Classify it as above.

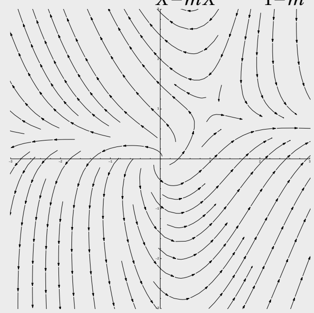
  For nodes and saddles, find the slope of the exceptional lines by substituting $Y = mX$ into $\dot{Y}/\dot{X} = m$.

- Sketch the phase plane diagram of a system $\dot{x} = f(x,y)$, $\dot{y} = g(x,y)$.

  Find and classify the equilibria as above. Mark the equilibria in a coordinate system. Draw phase curves in the vicinity of each equilibrium according to its classification. Extend the picture in such a way that it transitions smoothly between the local pictures.

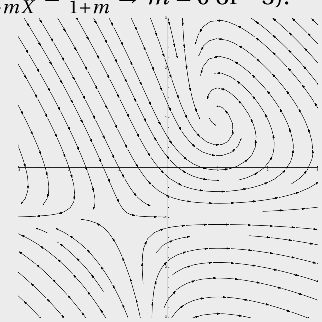
Find and classify the equilibria of the system $\dot{x} = x - y, \dot{y} = x + y - 2xy$. Draw the phase plane.

Equilibria occur where $x - y = 0, x + y - 2xy = 0$. Eliminating y gives $x(1-x) = 0$, so equilibrium points occur at $(0,0)$ and $(1,1)$. First look at $(0,0)$. Here $\dot{x} = x - y, \dot{y} \approx x + y$. Since $p = 2, q = 2$ and $\Delta = -4$, it's an unstable spiral. Next consider $(1,1)$. Introduce new variables by $x = 1 + X, y = 1 + Y$. Then $\dot{X} = X - Y, \dot{Y} = 2 + X + Y - 2(1+X)(1+Y) \approx -X - Y$. Hence $a = 1, b = 1, c = -1, d = -1$, so that $q = -2 < 0$, so this point is a saddle (with $m = \dot{Y}/\dot{X} = \frac{-X-mX}{X-mX} = \frac{-1-m}{1-m} \Rightarrow m = 1 \pm \sqrt{2}$).



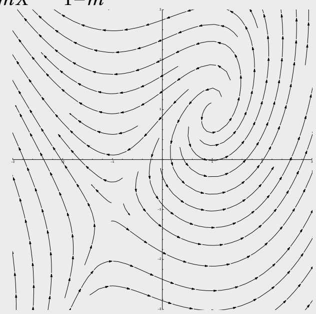
Find and classify the equilibria of the system $\dot{x} = 1 - xy, \dot{y} = (x-1)(y+1)$. Draw the phase plane.

Equilibria occur where $1 - xy = 0, (x-1)(y+1) = 0$, which is at $(1,1)$ and $(-1,-1)$. Consider $(1,1)$. Let $x = 1 + X, y = 1 + Y$. Then $\dot{X} \approx -X - Y, \dot{Y} = X$. Hence $a = -1, b = -1, c = 1, d = 0 \Rightarrow p = -1, q = 1, \Delta = -3 \Rightarrow$ stable spiral. Consider $(-1,-1)$. Let $x = -1 + X, y = -1 + Y$. Then $\dot{X} \approx X + Y, \dot{Y} \approx -2Y$. Hence $a = 1, b = 1, c = 0, d = -2 \Rightarrow q = -2 < 0 \Rightarrow$ saddle (with $m = \dot{Y}/\dot{X} = \frac{-2mX}{X+mX} = \frac{-2m}{1+m} \Rightarrow m = 0$ or -3).



Find and classify the equilibria of the system $\dot{x} = x - y, \dot{y} = x^2 - 1$. Draw the phase plane.

Equilibria occur where $x - y = 0, x^2 - 1 = 0$ which is at $(1,1)$ and $(-1,-1)$. Consider $(1,1)$. Let $x = 1 + X, y = 1 + Y$. Then $\dot{X} = X - Y, \dot{Y} \approx 2X$. Hence $a = 1, b = -1, c = 2, d = 0 \Rightarrow p = 1, q = 2, \Delta = -7 \Rightarrow$ unstable spiral. Consider $(-1,-1)$. Let $x = -1 + X, y = -1 + Y$. Then $\dot{X} = X - Y, \dot{Y} \approx -2X$. Hence $a = 1, b = -1, c = -2, d = 0 \Rightarrow p = 1, q = -2 \Rightarrow$ saddle (with $m = \dot{Y}/\dot{X} = \frac{-2X}{X - mX} = \frac{-2}{1-m} \Rightarrow m = 2$ or -1).

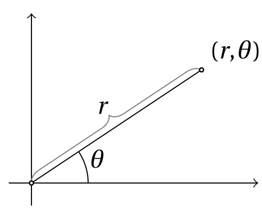


§ 7.1. Polar coordinates

§ 7.1.1. Lecture worksheet

We saw in problem A.1.2 that $y = \sqrt{1-x^2}$ is the equation for a unit circle, or rather the top half of it: to get a full circle of radius R we must write $y = \pm\sqrt{R^2 - x^2}$. Isn't this formula disturbingly complicated? One has the feeling that algebra almost fails to convey the simplicity of the circle in some sense. Think about how lines and curves are drawn in geometry: ruler and compass—the simplest of tools. The compass really captures what it means to be a circle; the equation $y = \sqrt{R^2 - x^2}$ almost seems to miss the point by comparison.

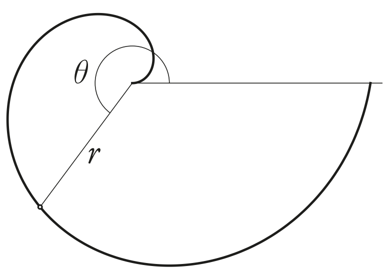
However, the real problem here is that we are trying to put a round peg in a square hole. Our coordinate system of x's and y's is intrinsically "rectangular" in nature: we describe the positions of points by saying "so far this way and so far in the perpendicular direction." But this is not the only way to describe the positions of points. Instead we could specify the position of a point by saying that it's so-and-so *far away* in such-and-such a *direction*. This is called polar coordinates:



7.1.1. Find the equation of a circle of radius R in polar coordinates.

So it's not that "circles hate algebra"; it's just that, like people, curves become difficult when you force them to abide by some system that clashes with their nature. Curves that "orbit" around a center point like polar coordinates better and will "play nice" if you let them have their way about this.

Another such example is the Archimedean spiral $r = \theta$:



In fact, in rectangular coordinates, spirals cannot be described by a polynomial equation of any kind.

7.1.2. $\star$ Can you think of a simple way of seeing this at a glance? Hint: cf. problems A.2.5–A.2.7.

§ 7.1.2. Problems

7.1.3. † Another important spiral is the logarithmic spiral $r = e^{k\theta}$.

(a) Prove that for this spiral a magnification is the same thing as a rotation. Hint: what do these operations mean in algebraic terms?

(b) Show that this means that the logarithmic spiral cuts all radial lines at the same angle. (This spiral is therefore central in the theory of navigation, where it corresponds to sailing at a constant compass course.)

Consider the portion of the spiral between $r = 0$ and some upper bound $r = R$. Divide it into triangles with central angle $d\theta$.

(c) Show that by flipping over every other of these triangles (so that the pointy ends of the triangles are pointing in alternating directions) they can be made to fit into one large triangle.

(d) Use this to infer the area and arc length of this portion of the spiral.

§ 7.2. Calculus in polar coordinates

§ 7.2.1. Lecture worksheet

In a figure in §1.1 we saw the differential triangle with sides dx, dy, ds.

7.2.1. Draw the analogous figure in the polar coordinate case.

Hint: The figure in §1.1 is showing the step ds from a point (x, y) on the curve to the "subsequent" point $(x + dx, y + dy)$ on the curve, as well as the decomposition of this step into the variable components dx and dy. The polar case in quite analogous, except of course the two coordinate components are now radial and circular rather than horizontal and vertical. Also, note that an increase in θ corresponds to a greater change in position the greater r is.

7.2.2. (a) Use the figure from problem 7.2.1 to express ds in terms of r and θ (and their differentials).

(b) Use this to express the arc length of a polar curve $r(\theta)$ as an integral in θ. Hint: this is in many ways analogous to problem 4.2.2.

7.2.3. Use the figure from problem 7.2.1 to express the angle the curve $r(\theta)$ makes with the radial line in terms of r and θ (and their differentials).

In §3.1 we saw how the area under the curve $y(x)$ is made up of rectangles with area $y\,dx$.

7.2.4. Draw the analogous picture for the polar coordinate case, and use it to express by means of an integral the area "swept out" by a polar curve.

Hint: Instead of rectangles the area will be made up of "pizza slices," which for the purposes of area calculation may be considered triangular.

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§ 7.2.2. Problems

7.2.5. In this problem we shall prove computationally what we saw geometrically in problem 7.1.3 regarding the logarithmic spiral $r = e^{k\theta}$.

(a) Use problem 7.2.3 to show that this curve makes equal angles with all radial lines.

Consider the portion of the spiral between $r = 0$ and some upper bound $r = R$.

(b) Use problem 7.2.2b to express the arc length of this portion.

(c) Use problem 7.2.4 to find the area of this portion.

(d) ⋆ Reconcile your results with the geometrical result of problem 7.1.3.

7.2.6. (a) Use problem 7.2.4 to find the area of a circle.

(b) ⋆ Discuss how this proof relates to the ones we have seen in problems 1.1.1 and 3.4.3.

§ 7.3. Parametrisation

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§ 7.3.1. Lecture worksheet

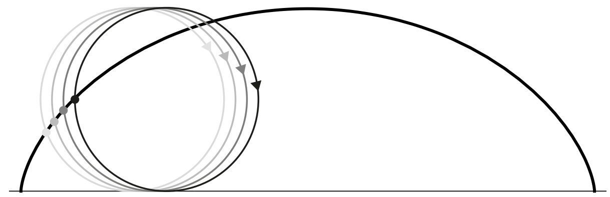
A *parametrisation* is a way of characterising a curve "one point at a time." For example, $(\cos t, \sin t)$ is a parametric representation of the unit circle, because as the parameter t runs through all numbers, $(\cos t, \sin t)$ runs around and around the unit circle. In fact no more than a restricted interval such as $0 \le t < 2\pi$ is enough to generate the circle.

Parametric representations of curves are often useful when the x and y coordinates of the curve are governed by separate laws. An important physical example is that of projectile motion. The initial firing velocity v_0 of a projectile can be decomposed into horizontal and vertical components $v_0 \cos\alpha$ and $v_0 \sin\alpha$. So if the projectile simply kept going in inertial motion its equations would be $x = tv_0 \cos\alpha$, $y = tv_0 \sin\alpha$. But these are not yet the correct equations. We must also take into account the constant gravitational acceleration affecting the horizontal position, giving the parametric representation of the trajectory $x = tv_0 \cos\alpha$, $y = tv_0 \sin\alpha - gt^2/2$.

7.3.1. Combine these equations so as to eliminate t and confirm that the result is the kind of curve you expected.

7.3.2. A bomb is dropped from a point directly above a target. We are located at a fortification some distance away. At the moment the bomb is dropped we fire a projectile aimed at the point from where the bomb is dropped. Show that we will hit the bomb mid-air as long as the firing velocity of our canon is greater than a certain threshold.

The curve traced by a point on a rolling circle is called a cycloid:



This is another example of a curve for which the essence of the generating motion is easier to capture in parametric form than through a standard Cartesian equation in x and y.

7.3.3. Show that the cycloid is parametrised by

$$x = t - \sin t$$
$$y = 1 - \cos t$$

Hint: begin by parametrising the motion of the midpoint of the circle.

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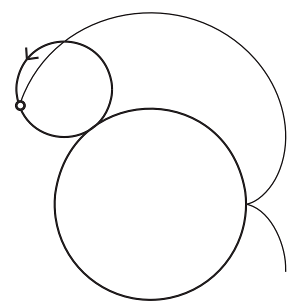
§ 7.3.2. Problems

7.3.4. As a variant of the cycloid we can let a circle roll on another circle:



(a) Sketch the resulting curve in the case where the two circles are of the same size.

(b) This curve is called the cardioid—why? Sound people speak of "cardioid microphones"—can you see what they mean?

(c) Show that the cardioid is parametrised by:

$$x = 2\cos t - \cos 2t$$
$$y = 2\sin t - \sin 2t$$

7.3.5. Consider a line through $(-1, 0)$ with slope t. This line cuts the unit circle $x^2 + y^2 = 1$ in one more point besides $(-1, 0)$. Find the coordinates of this point in terms of t.

7.3.6. Use problem 7.3.5 to explain why the substitution $u = \tan(\theta/2)$ turns a rational function of $\sin(\theta)$ and $\cos(\theta)$ into a rational function of u. (This is useful for integration purposes, as noted in §3.7.6.)

7.3.7. † One of the oldest recorded mathematical documents is a Babylonian clay tablet from almost four thousand years ago, way back in the Bronze Age. It consists in nothing but a long list of Pythagorean triples, i.e., integers a, b, c such that $a^2 + b^2 = c^2$. Explain how problem 7.3.5 can be used to generate Pythagorean triples.

7.3.8. A thread is wound around a circular spool. You grab the end of the thread and start unwinding it while keeping it as taunt as possible.

(a) Argue that the unwound piece of the string at any stage in this process is tangent to the circle at the point of contact.

(b) Sketch roughly the curve traced out by the free end of the thread as you unwind it.

(c) Find a parametric representation of this curve.

Hint: Let the spool be the unit circle and let the initial position of the free end of the thread be $(1,0)$. As parameter, use the angle θ between the x-axis and the point where the unwound portion of the string touches the spool.

§ 7.4. Calculus of parametric curves

§ 7.4.1. *Lecture worksheet*

In this section we shall see how to deal with tangents, areas, and arc lengths of curves given parametrically. The idea is simple: start with the usual expressions, and the rewrite them in terms of the parametrisation, as outlined in §7.5.4. Let us keep using the cycloid of problem 7.3.3 as our guiding example in this investigation.

7.4.1. (a) Using the parametric representation of the cycloid, find an expression for the slope of its tangent at a given point.

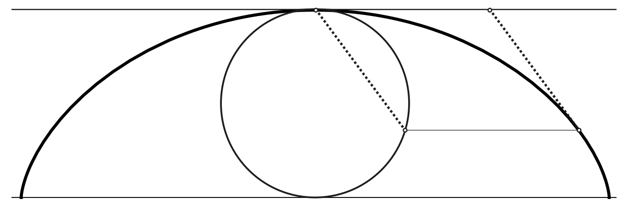
(b) How can you check whether the answer seems reasonable?

7.4.2. Show that the area under one arch of the cycloid is three times the area of the rolling circle.

7.4.3. Show that the length of one arch of the cycloid is four times the diameter of the rolling circle.

§ 7.4.2. *Problems*

7.4.4. Show that the result of problem 7.4.1a can be interpreted geometrically as saying that the tangent is parallel to the other dotted line in the figure below.



7.4.5. In this problem we shall prove that the bob of a perfect pendulum clock swings along a cycloidal path. This means that even as the pendulum's swings are dampened it still takes the same time to complete one full swing. In other words, a particle sliding down a cycloidal ramp (a cycloid turned upside-down compared to our previous pictures) will take the same time to reach the bottom no matter where it started. The fact that we are now considering an upside-down cycloid can easily be accounted for by simply letting the y-axis be directed downwards. Then y represents the vertical distance fallen from the highest point of the cycloid (the start of the ramp), and our previous parametric equations carry over without change.

(a) Using the parametric representation of the cycloid, rewrite the integral of problem 6.3.2 in terms of θ.

(b) What are the bounds of integration?

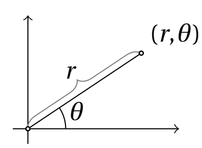
(c) Evaluate the integral.

(d) Repeat all of the above steps for the case where the particle is released not from $y = 0$ but some lower point $y = y_0$.

(e) Explain how this establishes what we wanted to show.

§ 7.5. Reference summary

§ 7.5.1. *Polar coordinates*



- Given a point (r,θ) in polar coordinates, find its cartesian coordinates (x, y).

  $x = r\cos\theta$, $y = r\sin\theta$.

- Given a point (x, y) in cartesian coordinates, find its polar coordinates (r,θ).

  $r = \sqrt{x^2 + y^2}$. If x is positive, $\theta = \arctan(y/x)$. If x is negative, $\arctan(y/x)$ is the negative of the angle with the negative x-axis, whence $\theta = \arctan(y/x) + 180°$.

§ 7.5.2. *Calculus in polar coordinates*

Arc length of polar curve $r(\theta)$ from $\theta = a$ to $\theta = b$:

$$\int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

Angle $r(\theta)$ makes with radial line:

$$\arctan\left(\frac{r\,d\theta}{dr}\right)$$

Area of polar curve $r(\theta)$ from $\theta = a$ to $\theta = b$:

$$\int_a^b \tfrac{1}{2} r^2\, d\theta$$

§ 7.5.3. *Parametrisation*

A parametrisation is an expression $(x(t), y(t))$ that gives one point for each t-value one plugs into it. As the parameter t runs through all possible values, a curve is generated.

Motion of projectile fired at angle α with velocity v_0: $x = t v_0 \cos\alpha$, $y = t v_0 \sin\alpha - g t^2/2$.

- Understand the curve determined by a given parametrisation.

  Fill in various values for t and plot the resulting points. These are all points on the curve.

  If you can solve for t in one of the equations $x = x(t)$ and $y = y(t)$ then you can plug this expression for t into the other equation to obtain the ordinary cartesian equation for the curve.

- Find the parametrisation of a given curve.

  Some simple cases: If the curve is of the form $y = f(x)$, you can let x be the parametrising variable; then $x = t$ and $y = f(t)$. The circle with midpoint (a, b) and radius r is parametrised by $x = a + r\cos t$, $y = b + r\sin t$.

  In general: Try to choose the parameter t to correspond to a natural characteristic of the curve, such as a certain angle, time, arc length, etc. We are free to do this since the only formal requirement for the parameter t is that plugging in numbers for it generates all points on the curve.

  Next we need to look at the x and y coordinates of a point on the curve in isolation from each other. Forget about y and try to express x purely as a function of t. Then do the same for y. This is can often be done directly when the nature of the curve is based on inherently rectilinear principles, such as gravity and inertia.

  If the curve consists of multiple interacting or compounding motions, first try to express each separately. The parametrised point can often be described as the sum of such component effects, for example as a sum of vectors.

§ 7.5.4. *Calculus of parametric curves*

- Find the slope of the tangent to a parametric curve $(x(t), y(t))$.

  $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$

- Find the area enclosed by a parametric curve $(x(t), y(t))$.

  Start with the usual area integral $\int y\,dx$ and rewrite it as an integral purely in terms of the parameter t.

- Find the arc length of a parametric curve $(x(t), y(t))$.

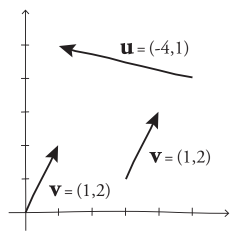
  Start with the usual arc length integral $\int ds$, where $ds^2 = dx^2 + dy^2$, and rewrite it as an integral purely in terms of the parameter t.

8 Vectors

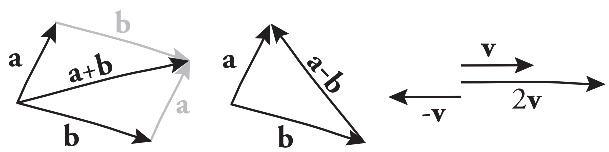
§ 8.1. Vectors

§ 8.1.1. Lecture worksheet

In this section we shall see that geometrical problems regarding projections and perpendicularity can be reduced to algebra in a remarkably simple way. This is done using the language of vectors. A vector is a directed line segment: it is so-and-so long and it goes one way rather than the other. We draw it as an arrow and denote it **v**. We can also express it in coordinate form by putting its foot end at the origin and recording the coordinates of its endpoint (this is why the **v** is fat: it is "stuffed" with more than one number). But the vector is the same no matter where it starts, like these two **v**'s:



The arithmetic of vectors goes like this:



So to add vectors we put them "head to tail" to make a "vector train." It follows that $\mathbf{a}-\mathbf{b}$ points from **b** to **a**, because $\mathbf{a}-\mathbf{b}$ must be something such that when you add **b** to it you get **a**.

8.1.1. What is the sum of the twelve vectors pointing from the centre of a clock to the hours?

8.1.2. Which of the following expressions correspond to the midpoint of the diagonal of the parallelogram spanned by **a** and **b**?

- ☐ $\frac{1}{2}(\mathbf{a}+\mathbf{b})$
- ☐ $\mathbf{a}+\frac{1}{2}(\mathbf{a}-\mathbf{b})$
- ☐ $\mathbf{b}+\frac{1}{2}(\mathbf{a}-\mathbf{b})$
- ☐ $\mathbf{a}-\frac{1}{2}(\mathbf{a}-\mathbf{b})$
- ☐ $\mathbf{b}+\frac{1}{2}(\mathbf{b}-\mathbf{a})$

In physics we often encounter quantities that have both magnitude and direction, such as force or velocity. Vectors are the natural language for describing such phenomena. This is in contrast with quantities that have magnitude only, such as temperature.

§ 8.1.2. Problems

8.1.3. You want to paddle across a river in a canoe. You can paddle twice as fast as the speed of the flow of the river. At what angle should you aim the nose of your canoe in order to go straight across the river?

8.1.4. Prove that if you join the midpoints of the sides in any quadrilateral you always get a parallelogram. Hint: Think of quadrilateral as $\mathbf{a}+\mathbf{b}=\mathbf{c}+\mathbf{d}$, divide by 2, interpret geometrically.

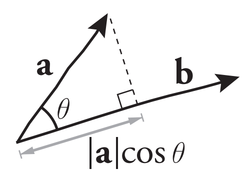
8.1.5. (a) Prove both algebraically and visually that $|\mathbf{a}+\mathbf{b}| = |\mathbf{a}-\mathbf{b}|$ if and only if **a** and **b** are perpendicular.

(b) Under what conditions is $|\mathbf{a}+\mathbf{b}|^2 = |\mathbf{a}|^2+|\mathbf{b}|^2$?

§ 8.2. Scalar product

§ 8.2.1. Lecture worksheet

Vectors can be used to express projections in a very convenient way.

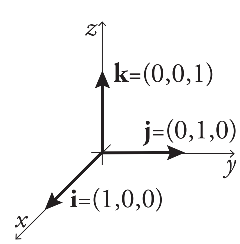


The length of **a**'s projection onto **b** is easily expressed trigonometrically. It is $|\mathbf{a}|\cos\theta$, where $|\mathbf{a}|$ means the length of the vector **a** (just as absolute value always means distance to the origin, or simply magnitude). The remarkable thing is that the length of the projection can also be expressed in a very simple way in terms of the coordinates of the vectors. This is codified in the *scalar product*

$$\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta = a_1b_1+a_2b_2+a_3b_3$$

When **b** is a unit vector (i.e., has length 1) the middle expression is precisely the length of the projection, so the formula tells us that we can find it simply by multiplying the vectors component-wise and adding the results. Nothing could be easier.

What is the reason behind this magical harmony of geometry and algebra? To see this it is useful to introduce the unit vectors **i**, **j**, **k** pointing in the direction of the axes:



Then by breaking up **a** and **b** into their coordinate components we get

$$\mathbf{a}\cdot\mathbf{b} = (a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k})\cdot(b_1\mathbf{i}+b_2\mathbf{j}+b_3\mathbf{k})$$

Now, the projection properties of **i**, **j**, **k** are particularly simple: any one of them projected onto itself gives 1, and projected onto each of the other two gives zero. Therefore when we multiply out the parenthesis all the cross terms go away and only the "like with like" terms survive. So the result is $a_1b_1 + a_2b_2 + a_3b_3$, as claimed.

8.2.1. Which of the following are assumptions made in the above proof?

- ☐ $\mathbf{u}\cdot\mathbf{v} = \mathbf{v}\cdot\mathbf{u}$ for any vectors $\mathbf{u}, \mathbf{v}$.
- ☐ $(\mathbf{u}+\mathbf{v})\cdot\mathbf{w} = \mathbf{u}\cdot\mathbf{w} + \mathbf{v}\cdot\mathbf{w}$ for any vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$
- ☐ $(a\mathbf{u})\cdot b\mathbf{v} = ab(\mathbf{u}\cdot\mathbf{v})$ for any vectors $\mathbf{u}, \mathbf{v}$, and constants a, b.
- ☐ $\mathbf{u}\cdot\mathbf{v} = u_1v_1 + u_2v_2 + u_3v_3$ for any vectors $\mathbf{u}, \mathbf{v}$

To complete the proof any such assumptions would have to be proved

- ☐ geometrically (using cosine form)
- ☐ algebraically (using coordinate form)

Our argument about **i**, **j**, **k** also highlighted two useful special cases of the scalar product: projection onto itself, and perpendicularity. The scalar product of a vector with itself gives the length squared, $\mathbf{a}\cdot\mathbf{a} = a_1^2 + a_2^2 + a_3^2 = |\mathbf{a}|^2$, and the scalar product of two vectors is zero if and only if they are perpendicular. Indeed, if a problem has right angles in it, chances are that you can solve it by scalar products. Here is an example:

8.2.2. Consider two lines, $y = ax + b$ and $y = cx + d$, that intersect in some point. The vectors ($\boxed{}$, a) and ($\boxed{}$, c) point in the direction of these lines respectively. If the lines are perpendicular, the product of their slopes is $\boxed{}$

Another example illustrating the same point is problem 8.2.10.

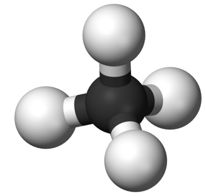
8.2.3. $|\mathbf{a}+\mathbf{b}| = |\mathbf{a}-\mathbf{b}|$ if and only if $\mathbf{a}$ and $\mathbf{b}$ are perpendicular. One can see this geometrically by means of [the Pythagorean Theorem, Theorem of Thales, diagonals of parallelograms, trigonometry]. To prove it algebraically, it is useful to consider:

- ☐ $(\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}-\mathbf{b})$
- ☐ $(\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}+\mathbf{b})$
- ☐ $(\mathbf{a}-\mathbf{b})\cdot(\mathbf{a}-\mathbf{b})$

---

§ 8.2.2. Problems

8.2.4. A molecule of methane, CH_4, forms a regular tetrahedron with the four hydrogen atoms at the vertices and the carbon atom at the centroid. Let the vertices of the first three hydrogen atoms be $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$.



(a) Argue that any point of the form (a, a, a) is equidistant from each of these three points.

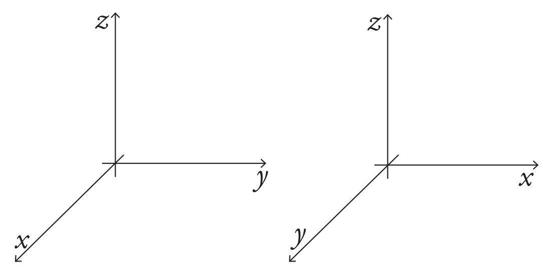
(b) Determine the coordinates of the fourth hydrogen atom using the condition that all hydrogen atoms must be equidistant.

(c) Determine the coordinates of the carbon atom by computing the "average" position of the four hydrogen atoms (i.e., add the hydrogen position vectors and divide by 4).

(d) Compute the bond angle, i.e., the angle between the lines that join the carbon atom to two of the hydrogen atoms.

8.2.5. Show by an example that $\mathbf{a}\cdot\mathbf{b} = \mathbf{a}\cdot\mathbf{c}$ does not necessarily imply $\mathbf{b} = \mathbf{c}$. In other words, you cannot "cancel" the $\mathbf{a}$.

8.2.6. Consider these two ways of setting up a coordinate system of three variables:



Is there a difference? Is one more natural than the other?

8.2.7. In Proclus's commentary on Euclid's *Elements* one reads:

> The Epicureans are wont to ridicule this theorem, saying it is evident even to an ass and needs no proof. ... That the present theorem is known to an ass they make out from the observation that, if straw is placed at one extremity of the sides [of a triangle], an ass in quest of provender will make his way along the one side and not by way of the two others.

Benno Artmann, in his book *Euclid: The Creation of Mathematics*, adds:

> The Epicureans of today might as well add that one could see the proof on every campus where people completely ignorant of mathematics traverse the lawn in the manner of the ass.

What is the theorem in question? Prove it using vector methods. Hint: First prove that $|\mathbf{a}\cdot\mathbf{b}| \le |\mathbf{a}||\mathbf{b}|$, and then consider $(\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}+\mathbf{b})$ and estimate it upwards.

8.2.8. (a) Prove the Pythagorean theorem by vector methods.

(b) Is this proof circular? I.e., was the Pythagorean theorem needed to establish the properties of vectors that you used in your proof?

(c) Generalise to the case where the angle between the two "legs" is no longer a right angle. This is the so-called "law of cosines."

8.2.9. For this problem, use *only* scalar products and properties of vectors. Do *not* use coordinates, x's and y's, trigonometry, functions, theorems of Euclidean geometry, etc.

(a) Let A and B be two diametrically opposite points on a circle, and let C be an arbitrary third point on the circle. Prove that $\angle ACB$ is a right angle.

(b) Let ABC be a triangle with a right angle at C, and let O be the midpoint of AB. Prove that $|OA| = |OB| = |OC|$. (Note that this result is obvious when the triangle is inscribed in a circle as in the previous problem, but here you are asked to prove it independently without making any reference to circles.)

8.2.10. Shortest (i.e., perpendicular) distance from a line to a point.

(a) Explain why $\mathbf{a} + t\mathbf{b}$ generates a line as t runs through all real numbers.

(b) Let $\mathbf{a} = (0,0)$ and $\mathbf{b} = (2,1)$. Express the vector pointing from $\mathbf{a} + t\mathbf{b}$ to the point $(3,3)$.

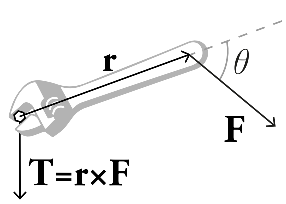
(c) Find the value of t for which this vector is perpendicular to the line.

(d) Find the shortest distance from the line to the point.

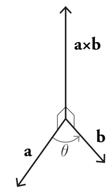
§ 8.3. Vector product

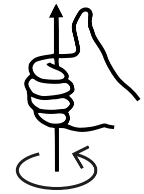
§ 8.3.1. Lecture worksheet

There is another way of "multiplying" vectors, which is written with a cross instead of a dot: $\mathbf{a} \times \mathbf{b}$. This is called the vector product because the result is a vector. The meaning of the product $\mathbf{a} \times \mathbf{b}$ is most vividly seen in physical terms. If $\mathbf{r}$ is a wrench onto which I apply the force $\mathbf{F}$, then $\mathbf{r} \times \mathbf{F}$ is the torque $\mathbf{T}$, i.e., the force with which the bolt moves.

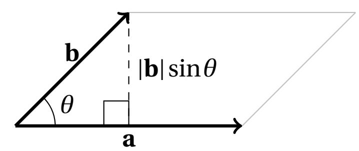


This interpretation makes the main properties of $\mathbf{a} \times \mathbf{b}$ obvious. First of all $\mathbf{a} \times \mathbf{b}$ is clearly perpendicular to both $\mathbf{a}$ and $\mathbf{b}$. Also its direction can be determined from our everyday experience with how bolts and screws move:





Lastly, it is evident from the law of the lever that the magnitude $|\mathbf{a} \times \mathbf{b}|$ is the area of the parallelogram spanned by $\mathbf{a}$ and $\mathbf{b}$, i.e., $|\mathbf{a}||\mathbf{b}|\sin\theta$. In the wrench interpretation, $|\mathbf{b}|\sin\theta$ is the part of the applied force that is perpendicular to the wrench shaft, i.e., the part of the force that actually has an effect in rotating the wrench.



To find out how to compute the vector product algebraically, we split $\mathbf{a}$ and $\mathbf{b}$ into components and consider

$$\mathbf{a} \times \mathbf{b} = (a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}) \times (b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k})$$

Let us assume for the moment that the vector product behaves much like ordinary multiplication, so that we can "multiply out" the parenthesis in the usual way. Then each of the terms are easy to find since it is evident from the wrench interpretation that for example $\mathbf{i} \times \mathbf{j} = \mathbf{k}$ and $\mathbf{i} \times \mathbf{i} = \mathbf{0}$.

8.3.1. Explain these equalities.

In this way we see that the product becomes

$$\mathbf{a} \times \mathbf{b} = (a_2b_3 - a_3b_2)\mathbf{i} + (a_3b_1 - a_1b_3)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}$$

8.3.2. Check this.

This result can be written more elegantly in terms of determinants as

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

This is the formula to use for computing vector products in practice. Once we have arrived at this formula through this heuristic line of reasoning, we can reverse our path and take this formula as our definition of the vector product and then derive its various properties by direct computation. In particular we could then verify our assumption about the algebra of vector multiplication. It would be easier to be formally complete this way but in terms of insight it would add little. In fact we could also justify our assumptions physically.

8.3.3. Explain the following in terms of the wrench interpretation.

(a) $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$

(b) $(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c}$

(c) $(\lambda\mathbf{a}) \times \mathbf{b} = \mathbf{a} \times (\lambda\mathbf{b}) = \lambda(\mathbf{a} \times \mathbf{b})$

(d) $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$

§ 8.3.2. Problems

8.3.4. Find $(1,2,3) \times (2,1,2)$ and use it to determine the area of the parallelogram spanned by these two vectors. Note that the result is very easily obtained in this way from a formula that we found using physical reasoning, whereas it would be much harder to compute directly by brute-force analytic geometry.

§ 8.4. Geometry of vector curves

§ 8.4.1. Lecture worksheet

Many geometrical properties of curves are more naturally and elegantly expressed in vector language than in terms of explicit formulae in x and y. If $\mathbf{x}(t) = (x(t), y(t))$ is the position of a moving particle, then its velocity is $\dot{\mathbf{x}}(t) = \mathbf{v}(t) = (\dot{x}(t), \dot{y}(t))$. Geometrically, $\dot{\mathbf{x}}$ is a tangent vector since it indicates the "direction of instantaneous change."

8.4.1. Consider a particle moving counterclockwise along the unit circle at unit speed. Explain how you could determine $\dot{\mathbf{x}}$ geometrically without actually differentiating $\mathbf{x}$. Then, by considering the derivative of $\mathbf{x}$, explain how this leads to a new proof of the differentiation rules for the sine and the cosine.

8.4.2. (a) Find the acceleration vector $\mathbf{a} = \ddot{\mathbf{x}}$ for this uniform circular motion and illustrate with a figure.

(b) Interpret this result is physical terms, using Newton's law $\mathbf{F} = m\mathbf{a}$, in the case where $\mathbf{x}$ is the motion of a planet about the sun.

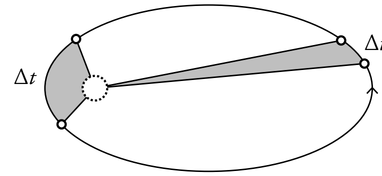
8.4.3. By definition, $\mathbf{a} = \frac{\mathbf{v}(t+dt)-\mathbf{v}(t)}{dt}$ is proportional to the difference between two successive velocity vectors. Argue on this basis that $\mathbf{a}$ is always perpendicular to $\mathbf{v}$ for any fixed-speed motion.

When studying the geometry of curves we prefer to use unit-speed parametrisations of our curves, $|\dot{\mathbf{x}}| = 1$, since this gives the curve in its purest form, uncontaminated by physical considerations. For unit-speed parametrisations, as problem 8.4.3 suggests, the geometrical meaning of $|\ddot{\mathbf{x}}|$ is "how much the curve is turning," or *curvature*. This is the same curvature studied in §13.1. We see that vector language is more naturally suited to the problem and simplifies a mess of a formula into the simple and intuitive $|\ddot{\mathbf{x}}|$. And this simplification is no mere game with symbols, as the following problem shows.

8.4.4. Compute the curvature of a circle using the vector method, and compare with the non-vector way of doing this (problem 13.1.3).

§ 8.4.2. Problems

8.4.5. Kepler's area law says that planets sweep out equal areas in equal times.



This is easily proved by vector methods. Let $\mathbf{r}$ be the position vector of the planet with the sun as the origin.

(a) The gravitational force is directed towards the sun. Explain what this means in terms of $\ddot{\mathbf{r}}$. Hint: cf. problem 8.4.2.

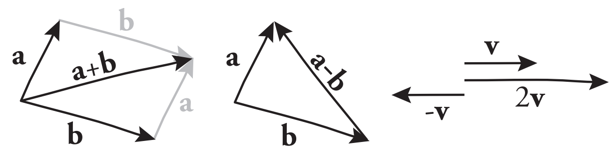
(b) Explain how the area covered by the planet is measured by $|\mathbf{r} \times \dot{\mathbf{r}}|/2$.

(c) Prove the product rule for vector products: $\frac{d}{dt}(\mathbf{u} \times \mathbf{v}) = \dot{\mathbf{u}} \times \mathbf{v} + \mathbf{u} \times \dot{\mathbf{v}}$.

(d) Use this to prove that the planet covers equal areas in equal times.

§ 8.5. Reference summary

§ 8.5.1. Vectors



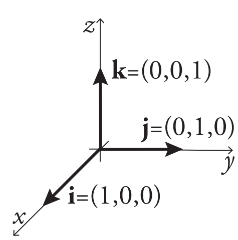
Vector addition. Algebraically: $(a,b)+(c,d) = (a+c, b+d)$. Geometrically: arrows head to tail.

$k\mathbf{a} = k(a,b) = (ka, kb)$ = magnification of $\mathbf{a}$ by factor k.

$|\mathbf{x}| = \sqrt{x_1^2 + x_2^2 + x_3^2}$ = length of $\mathbf{x}$.

$$\mathbf{a} \text{ parallel to } \mathbf{b} \iff \mathbf{a} = k\mathbf{b}$$

position vector of point P	$\overrightarrow{OP}$; vector pointing from the origin to P
unit vector	vector of length 1
orthogonal	perpendicular; making a right angle; scalar product 0
orthonormal	orthogonal and of unit length



$$(2,5,1) = 2\hat{\mathbf{i}} + 5\hat{\mathbf{j}} + \hat{\mathbf{k}}$$

§ 8.5.2. Scalar product

$$\mathbf{a}\cdot\mathbf{b} = \underbrace{|\mathbf{a}||\mathbf{b}|\cos\theta}_{\text{cosine form}} = \underbrace{a_1b_1 + a_2b_2 + a_3b_3}_{\text{coordinate form}}$$
$$= \text{(length of) projection of } \mathbf{a} \text{ onto } \mathbf{b} \text{ if } |\mathbf{b}| = 1$$
$$\mathbf{a}\cdot\mathbf{b} = 0 \Rightarrow \mathbf{a} \text{ perpendicular to } \mathbf{b}$$
$$\mathbf{a}\cdot\mathbf{a} = |\mathbf{a}|^2$$

$$(1,2,2)\cdot(3,0,5) = 1\cdot 3 + 2\cdot 0 + 2\cdot 5 = 13$$

§ 8.5.3. Vector product

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$
$$= (a_2b_3 - a_3b_2)\mathbf{i} + (a_3b_1 - a_1b_3)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}$$

$$\mathbf{a}\times\mathbf{b} \text{ is perpendicular to } \mathbf{a} \text{ and } \mathbf{b}$$
$$|\mathbf{a}\times\mathbf{b}| = \text{ area of parallelogram spanned by } \mathbf{a} \text{ and } \mathbf{b}$$

§ 8.5.4. Geometry of vector curves

For a curve given parametrically by $\mathbf{x}(t)$:

$\dot{\mathbf{x}}$ = tangent vector.

$|\ddot{\mathbf{x}}| = |\kappa|$ = curvature.

§ 8.5.5. Problem guide

- Find a vector pointing from **a** to **b**.

  $\mathbf{b} - \mathbf{a}$.

- Find unit vector pointing in same direction as **a**.

  $\mathbf{a}/|\mathbf{a}|$.

- Express a line in vector form.

  Find a position vector **a** for a point on the line. Using a second point on the line, or its slope or direction, find a vector **b** that points in the direction of (is parallel to) the line, i.e., so that $\mathbf{a}+\mathbf{b}$ is a point on the line. Now $\mathbf{a}+\lambda\mathbf{b}$ is a point on the line for any number λ, so this expression gives the line in vector form.

- Find centroid (midpoint, average position) of a set of position vectors.

  Add the vectors and divide by how many there are.

- Find center of mass (weighted average position) of a set of weighted position vectors.

  Multiply each position vector by its mass, add all together, and divide by total mass.

- Ensure that two vectors are perpendicular.

  Set scalar product (coordinate form) equal to zero.

  Find a value of x such that $(x,-1,2)$ is perpendicular to $(1,2,2)$.

  $(1,2,2)\cdot(x,-1,2) = x-2+4 = 0 \implies x = -2$.

  Find a value of a such that the vectors $(6,5,-\frac{1}{2})$ and $(\frac{1}{3},a,16)$ are perpendicular.

  The vectors are perpendicular when $(6,5,-\frac{1}{2})\cdot(\frac{1}{3},a,16) = 0$. Hence $6\cdot\frac{1}{3} + 5a - 16\cdot\frac{1}{2} = 0$ and thus $a = \frac{6}{5}$.

- Determine angle between two vectors.

  Solve for θ in equality between coordinate and cosine forms of the scalar product.

  Determine the angle θ between the vectors $(1,5)$ and $(3,2)$.

  From the scalar product identity $(1,5)\cdot(3,2) = |(1,5)|\cdot|(3,2)|\cdot\cos\theta$, we obtain $\cos\theta = \frac{1\cdot 3+5\cdot 2}{\sqrt{1^2+5^2}\cdot\sqrt{3^2+2^2}} = \frac{13}{\sqrt{26}\sqrt{13}} = \frac{1}{\sqrt{2}}$. The angle is thus $\pi/4$.

  Find the angle between $\mathbf{v} = (1,0,1)$ and $\mathbf{w} = (1,2,2)$.

  On the one hand, $\mathbf{v}\cdot\mathbf{w} = (1,0,1)\cdot(1,2,2) = 3$. On the other hand, $\mathbf{v}\cdot\mathbf{w} = |\mathbf{v}||\mathbf{w}|\cos\theta$, and $|\mathbf{v}| = \sqrt{1^2+0^2+1^2} = \sqrt{2}$, and $|\mathbf{w}| = \sqrt{1^2+2^2+2^2} = \sqrt{9} = 3$. Thus $3 = 3\sqrt{2}\cos\theta \implies \cos\theta = 1/\sqrt{2} \implies \theta = \frac{\pi}{4} = 45°$.

- Find area of parallelogram.

  Find two vectors **a**, **b** that span it (i.e., point from one vertex to the two adjacent ones). Area is $|\mathbf{a}\times\mathbf{b}|$.

- Find area of triangle.

  Consider as half a parallelogram and proceed as above.

  Find the area of the triangle with vertices $P = (1,1,1)$, $Q = (1,1,0)$, $R = (0,0,1)$.

  $\frac{1}{2}|QP \times RP| = \frac{1}{2}|(-1,1,0)| = \frac{1}{2}\sqrt{2}$.

- Determine the direction of $\mathbf{a}\times\mathbf{b}$.

  Point the index finger of your right hand in the direction of **a**, with your palm facing towards **b**. The vector product $\mathbf{a}\times\mathbf{b}$ points perpendicularly upwards in the direction of your thumb. (See figure and screwdriver analogy in §8.3.1.)

Given that $|\mathbf{u}| = 2$, $|\mathbf{v}| = 3$, and $\mathbf{u} \cdot \mathbf{v} = -1$, determine the length of the vector $2\mathbf{u} - 3\mathbf{v}$.

$|2\mathbf{u}-3\mathbf{v}|^2 = (2\mathbf{u}-3\mathbf{v})\cdot(2\mathbf{u}-3\mathbf{v}) = 4\mathbf{u}\cdot\mathbf{u}-6\mathbf{u}\cdot\mathbf{v}-6\mathbf{v}\cdot\mathbf{u}+9\mathbf{v}\cdot\mathbf{v} = 4|\mathbf{u}|^2-12\mathbf{u}\cdot\mathbf{v}+9|\mathbf{v}|^2 = 4\cdot 2^2-12(-1)+9\cdot 3^2 = 109$. The length of $2\mathbf{u}-3\mathbf{v}$ is hence $\sqrt{109}$.

Find the distance from the point $(2,3,-1)$ to the plane $2x - y + 2z = 2$.

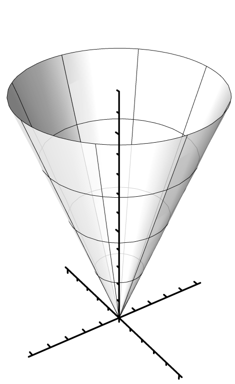
$(x,y,z) = (2,3,-1)+t(2,-1,2) = (2+2t, 3-t, -1+2t)$ is a parametric representation of the line in the direction of the normal through the given point. Plugging this into the equation for the plane gives $2 = 2(2+2t)-(3-t)+2(-1+2t) = 9t-1$, and hence $t = \frac{1}{3}$. The distance is thus $d = |t|\cdot|(2,-1,2)| = \frac{1}{3}\cdot\sqrt{2^2+(-1)^2+2^2} = 1$.

Find the equation for a plane through the points $P = (1,1,1)$, $Q = (1,1,0)$, $R = (0,0,1)$.

The vectors $QP = (0,0,1)$ and $RP = (1,1,0)$ are parallel to the plane. A normal vector to the plane is $QP \times RP = (0,0,1) \times (1,1,0) = (-1,1,0)$. The equation for the plane thus has the form $-x+y = D$. Plugging in one of the points, for example $P = (1,1,1)$, into this equation, shows that $D = 0$. Thus the equation for the plane is $-x+y = 0$.

§ 9.1. Functions of several variables

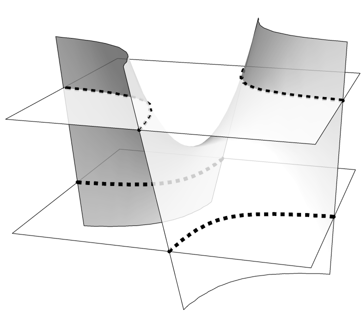
§ 9.1.1. Lecture worksheet

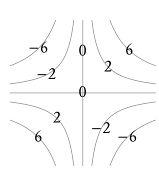


The calculus can be extended to functions of more than one variable. Then instead of $y(x)$ one has $z(x, y)$. So both x and y are "input" variables now, and z is the "output." We visualise the xy-plane as a horizontal "ground level," and z as the height. So the function gives a specific elevation for each point on the ground. Thus the graph of $z(x, y)$ is a surface. An ice cream cone, depicted here, is a simple example of a surface. What is the function $z(x, y)$ corresponding to this surface? Three variables is a lot to keep in one's mind, so to deal with this kind of question we often have to break the problem into more manageable parts. In this case we notice for example that the horizontal cross sections are circles. The equation for a circle is $x^2 + y^2 = r^2$. We want one circle for each z, and bigger ones as we go higher, so we might guess $z = x^2 + y^2$, which clearly satisfies both of these requirements. We can check our guess by taking another cross section. If we cut an ice cream cone in half right down the middle we of course get a V-shaped cross section. This corresponds to cutting for example along the zx-plane, i.e., along the plane $y = 0$. But if we slice our guess $z = x^2 + y^2$ in this way we see that we get the cross section $z = x^2$. This is a U rather than a V, so altogether $z = x^2 + y^2$ looks more like a bowl or a wine glass than an ice cream cone. We would rather prefer $z = |x|$ as our intersection, and this can be arranged by taking $z = \sqrt{x^2 + y^2}$.

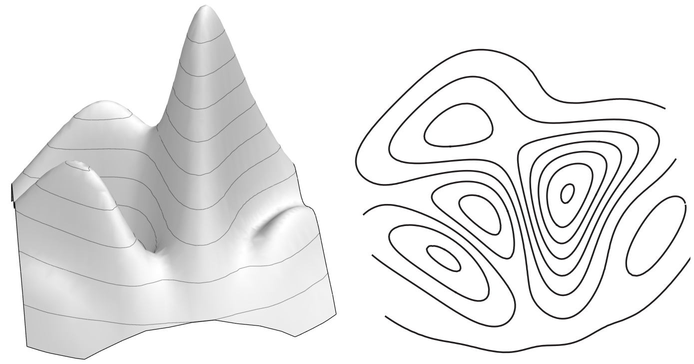
9.1.1. This is almost an ice cream cone, but one more adjustment needs to be made. Explain what it is and how to amend the equation.

The idea of understanding a surface by its cross sections is systematised in contour plots. To produce a contour plot we slice the surface by horizontal planes. Here for example I have sliced the saddle-shaped surface $z = xy$ at one positive and one negative z-value:





Algebraically, this corresponds to plugging $z = c$ into the equation for the surface. For the saddle this gives the hyperbolas $y = c/x$. These curves we then plot in the ordinary xy-plane and label them with their corresponding value for $z = c$, as shown on the right. You are already familiar with contour plots, no doubt, from their use in topographical maps. Here, for example, is a mountain and its corresponding contour plot:



9.1.2. (a) If you needed to scale the highest peak, how could you use the contour plot to determine the path of easiest ascent?

(b) Conversely, if you are a daredevil skier, which way should you go down?

9.1.3. (a) Draw contour plots of the parts of the cone $z = 5 - \sqrt{x^2 + y^2}$ and the unit sphere that lie above the xy-plane.

(b) If these were mountains, which would be hardest to climb?

9.1.4. Match the functions with the real-world object their graphs resemble.

- ☐ $x^2 + y^2 = z$
- ☐ $3(x^2 + y^2) = z$
- ☐ $x^2 + y^2 = z^2$
- ☐ $5(x^2 + y^2) = z^2$
- ☐ wine glass
- ☐ ice cream cone
- ☐ beach ball
- ☐ Asian-style farmer's hat
- ☐ pyramid
- ☐ hour glass
- ☐ champagne glass

A function of two variables has two *partial derivatives*, f_x and f_y. They answer the questions: How fast is the height f changing when I take a small step in the x direction? And how much for a step in the y direction? When moving in one of these directions, the other variable doesn't change. Computationally, this means that, to find the derivative of $f(x, y)$ with respect to x you treat y as a constant and vice versa. So when differentiating with respect to x you can secretly think of any y's in the formulas as if they were 5's or some such innocuous number. For example, if $f(x,y)=x^2y$ then $f_x=2xy$ and $f_y=x^2$. To get a feeling for the meaning of these derivatives, go back to the ice cream cone.

9.1.5. For the ice cream cone of the previous problem, compute the following partial derivatives and interpret geometrically.

(a) $z_x(1,0)$

(b) $z_y(1,0)$

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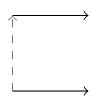
§ 9.1.2. Problems

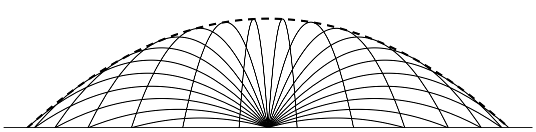
9.1.6. † *The mixed partial derivatives f_{xy} and f_{yx} are equal.* The derivative f_x means: if I take a step in the x-direction, how much does the value of f change? So $(f_x)_y$ means: if I take a step in the y-direction, how much does the change in f per step in the x-direction change? In other words, $(f_x)_y$ is the change in f along the top arrow minus the change in f along the bottom arrow:



Interpret $(f_y)_x$ similarly and explain why $(f_x)_y-(f_y)_x=0$.

9.1.7. † This problem illustrates the context in which partial differentiation was first conceived in the late 17$^{\text{th}}$ century. The creators of the calculus were not interested in multivariable calculus and partial derivatives as it is taught today, since most physical phenomena can be understood in two-dimensional form. However, examples like the one below led them to consider partial derivatives nevertheless, and shows their use even for two-dimensional problems.

Consider the trajectories of projectiles fired from a canon at varying angles:



Note that the figure is two-dimensional: the projectiles are not "coming towards you"; they are all within the same plane.

Ignoring air resistance, the trajectories are of course parabolas, as Galileo discovered. We want to calculate the dashed "safety curve." Beyond this curve we are always safe, whereas anywhere inside this curve we can be hit. This curve can be computed using the fact that the trajectories of projectiles fired at two almost identical angles, say α and $\alpha+d\alpha$, intersect at the safety curve.

(a) ⋆ Explain briefly how this fact is evident from the figure.

Now to put this into equations. Let the trajectory for firing angle α be $f(x,y,\alpha)=0$. That is, the equation for the curve in x and y coordinates will be some formula involving x, y and α set equal to zero (i.e., we have moved all terms to the left hand side). What we just proved above is that a point (x,y) on the safety curve satisfies both $f(x,y,\alpha)=0$ and $f(x,y,\alpha+d\alpha)=0$.

(b) Show that this implies that $\frac{d}{d\alpha}f(x,y,\alpha)=0$.

Therefore we can find the safety curve by combining the two equations $f(x,y,\alpha)=0$ and $\frac{d}{d\alpha}f(x,y,\alpha)=0$ so as to eliminate α. This will give us all the points with the required property in terms of x and y only, which is what we want.

(c) Express the trajectory in parametric form in the manner of §7.3.

(d) Obtain one equation for the trajectory involving x, y, and α, but not t. (I.e., combine the equations so as to eliminate t.) This essentially gives us $f(x,y,\alpha)=0$.

(e) Find the equation for the safety curve using the method given above.

(Hint: You may want to use $\frac{1}{\cos^2\alpha}=\frac{\sin^2\alpha+\cos^2\alpha}{\cos^2\alpha}=1+\tan^2\alpha$ or some similar trick to relate different trigonometric expression to each other.)

(f) Explain how you can see from the equation that the curve you found has roughly the right shape and position.

9.1.8. A function of one variable is called continuous if its graph can be drawn without lifting the pen, i.e., if it has no "gaps" or "jumps." More technically, $f(x)$ is continuous at a point x_0 if $f(x)$ approaches $f(x_0)$ as x approaches x_0. For a function of two variables to be continuous, it is required that $f(x,y)$ approaches $f(x_0,y_0)$ no matter how (i.e., in which direction or along which curve) (x,y) approaches (x_0,y_0).

The famous French mathematician Cauchy claimed in the early 19$^{\text{th}}$ century that if a function of two variables is continuous at a point in each variable separately then it is continuous at that point. He was mistaken, however. Consider the function $z(x,y)=\frac{xy}{x^2+y^2}$. This function is defined everywhere except at $(0,0)$, since division by zero is undefined. We extend it to a function defined everywhere by defining $z(0,0)$ to be 0.

(a) Show that this function is continuous in each variable separately (i.e., $z(x,0)$ and $z(0,y)$ are continuous as one-variable functions).

(b) Show, however, that the function is discontinuous at the origin by finding another way of approaching the origin so that the z-values do not approach the same value as above.

A trickier example still is $z(x,y)=2xy^2/(x^2+y^4)$.

(c) Show that if we approach the origin on any straight line, z approaches zero, but z has different limits when the origin is being approached along the two parabolas $x = \pm y^2$.

(d) Illustrate both examples with plots of the functions.

§ 9.2. Tangent planes

§ 9.2.1. Lecture worksheet

The simplest surfaces are planes. They have equations of the form $Ax + By + Cz = D$, as is easy to imagine since they are the three-dimensional analog of lines in two dimensions, which can be written $Ax + By = C$.

9.2.1. How many points does it take to determine a plane? How does this square with the argument of problem 5.1.1?

The vector (A, B, C) is perpendicular to the plane—it is a normal vector, as it is called.

9.2.2. Is the following proof correct? If not, indicate its first erroneous step.

I want to investigate whether the vector (A, B, C) is always a normal vector (i.e., perpendicular) to the plane $Ax + By + Cz = D$. To do this I reason as follows.

[1] If (A, B, C) is perpendicular to the plane, then its scalar product with any vector in the plane should be zero.

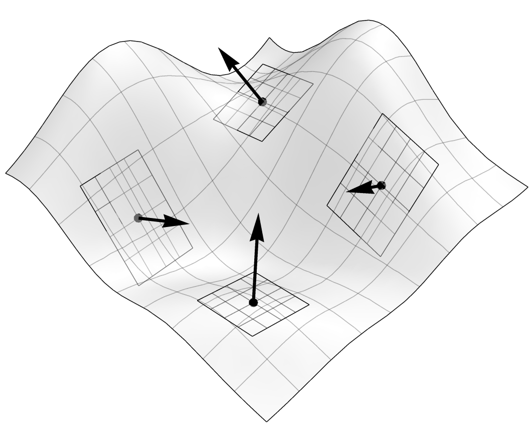
[2] Let (X, Y, Z) be a vector in this plane. In other words, $AX + BY + CZ = D$.

[3] Then the scalar product is $(A, B, C) \cdot (X, Y, Z) = AX + BY + CZ = D$.

[4] Since this is not zero the vector is not perpendicular to the plane.

9.2.3. Find the shortest distance from the point $(1, 1, 1)$ to the plane $2x - y + 3z = 1$. (Hint: Walk in the direction of the normal until you hit the plane.) The shortest distance is $\boxed{}$ times the length of the normal vector.

Tangents are important in calculus, and in three dimensions that means tangent planes. Here I have drawn a surface and some of its tangent planes and normals:



The tangent plane to the surface $f(x, y)$ above the point (x_0, y_0) in the xy-plane is

$$z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

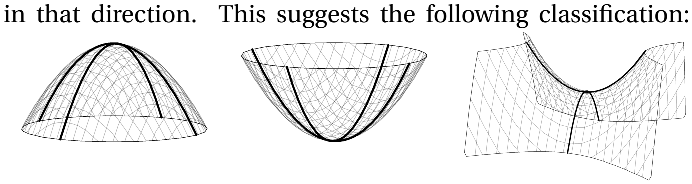
9.2.4. Explain why.

9.2.5. Give an expression for the normal vector in terms of derivatives.

§ 9.3. Unconstrained optimisation

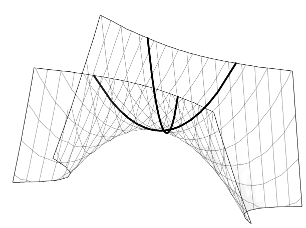
§ 9.3.1. Lecture worksheet

Finding maxima and minima is much the same for functions of two variables as for the one-variable case (§2.1). At a maximum or minimum point both partial derivatives must be zero, for the same reason as in the one-variable case. And again the second derivatives help us classify these extremum points. As before, the second derivatives f_{xx} and f_{yy} tell us whether the function makes a "happy" or "sad" shape in that direction. This suggests the following classification:



both negative: max	both positive: min	one of each: saddle

This is almost right, except for one thing. The fact that a surface is curving upwards in both axes directions does not mean that it its turning upwards in *every* direction: if it dips along the diagonal it is a saddle after all.



both positive
but still a saddle

Such a "tricky saddle" would throw us off if we looked only at f_{xx} and f_{yy}. To catch it we must study the mixed second derivative f_{xy}, because this basically measures "how much different the function is along the diagonal than along the axes" (as is quite clear from the reasoning in problem 9.1.6).

So the final version of our classification is this. If $f_{xx}f_{yy} - (f_{xy})^2 > 0$ then the "diagonal effect" is not strong enough to throw off the simple classification based on the axes directions, so we get either a max or a min according to the signs of f_{xx} and f_{yy}. If $f_{xx}f_{yy} - (f_{xy})^2 < 0$ we have a saddle one way or the other (either a simple saddle if $f_{xx}f_{yy}$ alone is negative, or a tricky one if it becomes negative only after the diagonal effect has been subtracted). Just as in the one-variable case it can happen that none of these classification rules apply. In such cases we are on our own and must seek another way of understanding what is going on.

9.3.1. Confirm that the function $z = xy$ that I showed you in §9.1 has a saddle at the origin.

9.3.2. What type of extremum does the function have at the origin?

- ☐ $f(x, y) = (x + y)^2$
- ☐ $f(x, y) = \sqrt{x^2 + y^2}$
- ☐ $f(x, y) = -x^2$
- ☐ $f(x, y) = \ln(1/e^{x^2+y^2})$
- ☐ $f(x, y) = x^2 - y^2$
- ☐ $f(x, y) = x^2 + y^2 + 3xy$
- ☐ monkey saddle
- ☐ tricky saddle
- ☐ simple saddle
- ☐ unique maximum
- ☐ unique minimum
- ☐ non-unique minimum
- ☐ non-unique maximum

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§ 9.3.2. Problems

9.3.3. Museums and theatres often offer discounts to senior citizens, typically out of profit interest rather than kindness. The reason is that seniors are more sensitive to price, so discounts have a greater impact on their purchasing decisions. Suppose a theatre can sell q senior tickets and Q full tickets at prices €p and €P respectively, according to the demand functions $q(p) = 100p^{-4}$ and $Q(P) = 1000P^{-2}$.

(a) Sketch the graphs of the demand functions and explain how they reflect the differences in sensitivity to price between the two consumer groups.

(b) The theatre has operating costs of €1 per visitor. How should the theatre set the prices for full and senior tickets so as to maximise their overall profit? (Express profit as a function of p and P.)

(c) Confirm that your answer is a maximum using second-order derivatives. Explain, however, why this method is rather overkill in this instance.

9.3.4. A product to be sold in the Dutch market can be manufactured in either the Netherlands or China. The cost of producing q items is $20q^2 - 60q + 100$ in one country and $10q^2 - 40q + 90$ in the other.

(a) Plot these two functions in the same coordinate system. Which function do you think corresponds to which country? Explain.

(b) The price at which each item can be sold is a function of the total quantity produced, $p = 520 - 10q$. What is the real-world reason for this?

(c) Find the production levels that yield maximum profit.

9.3.5. A manufacturer produces a quantity q of a product to be sold on two markets. The prices on each market depends on the quantity sold there according to the formulas

$$p_a = 57 - 5q_a \qquad p_b = 40 - 7q_b$$

(a) Interpret the difference between the two markets in real-world terms.

(b) How should production be divided over the two markets in order to maximise profit? Does this make sense in terms of your description of the markets?

§ 9.4. Gradients

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§ 9.4.1. Lecture worksheet

The derivative f_x says: if you go one step over in the x-direction, then the height will change by this much. And f_y does the same for a step in the y-direction. What about a step in any other direction? How much would that change the height? Easy: just break the step into its x and y components and use the derivatives for each. This gives us the *directional derivative* $f_x \cos\theta + f_y \sin\theta$, where θ is the direction in question given as the angle it makes with the positive x-axis. This can be more elegantly expressed in vector form. If the direction is given as a unit vector $\mathbf{s} = (\cos\theta, \sin\theta)$ then the directional derivative is simply $\mathbf{s} \cdot \nabla f$, where ∇f, the *gradient* of $f(x, y)$, is the vector made up of its partial derivatives:

$$\nabla f = (f_x, f_y)$$

Besides its use for finding directional derivatives, the gradient vector also has an interesting meaning in itself: it points in the direction of steepest ascent of f.

9.4.1. Prove this in vector terms. Hint: which direction $\mathbf{s}$ makes the directional derivative $\mathbf{s} \cdot \nabla f$ the biggest?

We can get an intuitive feeling for why the gradient is the direction of steepest ascent by the following experiment. Grab something flat, such as the paper on which this text is printed, and hold it up horizontally in front of you. Now tilt its right end upwards just a little, and then its far end upwards quite a bit. If you stood in the middle of this plane, which way should you go if you want to go up as fast as possible? Surely you want to go mostly straight ahead because you gave it the most tilt that way. But you can do even better if you deviate a bit to the right, in order to utilise that slope as well.

9.4.2. Explain how this agrees with the gradient pointing in the direction of steepest ascent.

9.4.3. Go back and try out the gradient on the ice cream cone from §9.1. Interpret visually.

9.4.4. If $z = f(x, y)$ is the roof of a building, in what direction will rain water flow?

Since ∇f tells us the direction of steepest ascent, it follows that $-\nabla f$ is the direction of steepest descent, and that the directions perpendicular to ∇f correspond to no ascent at all, i.e., to going sideways while staying at the same height. (Visualise this on your tilted plane.) Another way of saying this is that the gradient is perpendicular to the contour curves (since the contour curves correspond to fixed height).

A clever application of these ideas is to the problem of finding the normal to a given curve. Let's say I want to find the normal to for example the curve $x^2y + 4y = 5$ at the point $(1, 1)$. Then my first step is to think of this as a level curve of the function $f(x, y) = x^2y + 4y$. At first this may seem like a very circumspect way of going about things—after all, the original problem was a nice and simple two-dimensional problem and here I am making rather a mess of it by imagining it to be a cross section of a surface situated in three-dimensional space. But sometimes generality is simplicity, and certainly so in this case. For now I know by the very simple arguments above that $\nabla f = (f_x, f_y) = (2xy, x^2 + 4)$ is perpendicular to the curve, so I immediately see that the normal at $(1, 1)$ is $(2, 5)$.

9.4.5. Find normal vectors for a few points on the curve $y = 3x^2 + x + 2$. Illustrate with a sketch.

All of these things also generalise to higher dimensions. In particular, we get for free the rather powerful result that the normal to a surface $f(x, y, z) = c$ is given by the gradient (f_x, f_y, f_z).

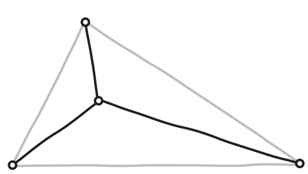
9.4.6. Go back to §9.2.1 and re-explain in this new light what was said about normal vectors there.

9.4.7. Show by an example that the gradient method is sometimes a more convenient way of finding a normal vector than the formula for normals that follows from the tangent plane equation (problem 9.2.5).

§ 9.4.2. *Problems*

9.4.8. What is the geometrical meaning of $|\nabla f|$? Hint: what happens if I take a unit step in this direction?

9.4.9. † Three cities are to be connected by roads. To minimise cost and environmental footprint, we want to minimise the total length of the roads. This is done by finding a point (x, y) between the three cities such that the sum of the distances to the cities, $D = D_1 + D_2 + D_3$, is a small as possible.



Once the coordinates of the cities are given the total distance is a function of x and y; for example, if the first city is at (a, b) then $D_1 = \sqrt{(x-a)^2 + (y-b)^2}$ and so on for the other cities. So the total function to be minimised is a sum of three such root expressions. We could minimise this formula the brute force way by the method of §9.3, but the calculations would not be pretty. Here is a more clever method.

(a) Find the gradient of D_1 as a function of x and y. Hint: This can be done without calculations (if you have solved problem 9.4.8).

(b) Do the same for D_2 and D_3 and draw the gradients in the figure.

(c) Explain why the sum of the gradients must be 0 at the optimum point.

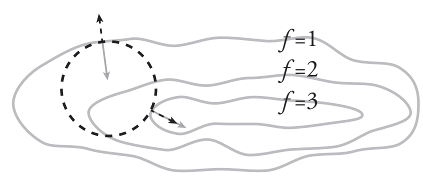
(d) Argue that this implies that the angles between the gradients must be 120°, and that this fact is enough to find the optimum.

(e) ⋆ The above method breaks down in certain exceptional cases. Explain.

§ 9.5. Constrained optimisation

§ 9.5.1. *Lecture worksheet*

The geometry of gradients also leads immediately to a simple way of solving constrained optimisation problems (the so-called *Lagrange multiplier* method). Suppose for example that we want to make f as big or as small as possible while being constrained to this dashed circle:

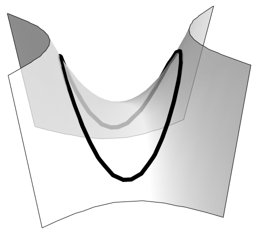


Clearly the extrema will occur at the points where the constraint curve precisely touches one of f's contour curves, as it does for $f = 1$ and $f = 3$. When the constraint curve cuts right through a contour, as it does for $f = 2$, there is one side with bigger values and one with smaller, so it can't be a minimum nor a maximum. This idea is captured analytically as follows. The extremum points of $f(x, y)$ subject to the constraint $g(x, y) = c$ are found by solving the system of equations

$$f_x = \lambda g_x$$
$$f_y = \lambda g_y$$
$$g = c$$

The first two equations say that the gradient vectors of g and f differ only by a multiple, λ. Geometrically, this means that the normals of the constraint curve and a contour of f are parallel. This happens precisely where the curves touch each other, as shown in the picture.

For example, let's find the maximum values of the saddle $f(x,y) = xy$ subject to the constraint $x^2 + y^2 = 1$. So the constraint curve is the unit circle. When I round it in the xy-plane, the corresponding points on the graph of f look like this:



So it's a roller-coaster with two maxima and two minima. In the reference summary they are computed analytically.

9.5.1. Draw the contour picture for this example.

9.5.2. Solving the above system of equations gives us a list of candidate points that must contain all maxima and minima. But how to tell which is which? Explain why the second-derivative tests of §9.3 are of no use here. What to do instead?

9.5.3. If $f(x,y)$ is the revenue as a function of money spent on television (x) and print (y) advertisements, what is the meaning of $g(x,y) = c$?

(a) A realistic expression for $f(x,y)$ is [xy, $\sin(xy)$, $1/xy$, $3x - y$] and a realistic expression for $g(x,y)$ is [xy, $(x+y)^2$, $\sqrt{x+y}$, $2x+y$].

(b) If $c = 20$, the maximum profit is $\boxed{\quad\quad}$

(c) Solving for λ in the Lagrange equations suggests that λ has the real-world interpretation:

- ☐ profit as fraction of spending
- ☐ spending as fraction of profit
- ☐ extra dollars earned per extra dollar spent
- ☐ extra dollars spent per extra dollar earned
- ☐ payoff of TV ads relative to print ads
- ☐ payoff of print ads relative to TV ads
- ☐ increase in cost of ads with increasing spending

(d) In our case, $\lambda = \boxed{\quad\quad}$

(e) Ad spending should be [increased, decreased].

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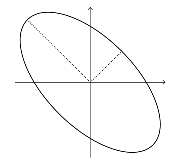
§ 9.5.2. Problems

9.5.4. A rectangular building is to be designed to minimise heat loss. The walls lose heat at a rate of 2 units/m^2 per day, except the south wall, which, since it receives more sun, loses only half this much heat. The floor and the roof are better insulated and lose heat at a rate of 1 unit/m^2 per day. The volume of the building must be exactly 1000 m^3. The dimensions that minimise heat loss are: height $= \boxed{\quad\quad}$; length of south-facing wall $= \boxed{\quad\quad}$; length of east-facing wall $= \boxed{\quad\quad}$.

9.5.5. $5x^2 + 6xy + 5y^2 = 8$ is a tilted ellipse:



Find the lengths of its semi-axes (= greatest and least distance to the origin) using Lagrange multipliers. Hint: Use $g(x,y) = 5x^2+6xy+5y^2 = 8$ as the constraint in a suitable optimisation problem.

9.5.6. *Economics: Cobb–Douglas model.* Suppose the production P of a company depends on the available labour L and the capital investment K according to the formula $P = L^\alpha K^\beta$. Assume further that the production is scalable, i.e., if L and K both grow by a certain factor then P also grows by that same factor.

(a) What does this imply about the relation between α and β? This is best seen by means of the [chain rule, product rule, laws of exponents, laws of logarithms].

(b) If $\alpha = 0.75$, then $\beta = \boxed{\quad\quad}$.

(c) Sketch a few contours of P in a coordinate system with L and K on the axes.

(d) Explain how the equation $2L + K = 1000$ can be interpreted as a budget constraint.

(e) Include the constraint line in your figure. Visually, how can you find the maximum production?

(f) Find the maximum production using Lagrange multipliers and explain how this corresponds to the visual method.

§ 9.6. Multivariable chain rule

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§ 9.6.1. Lecture worksheet

Imagine standing at a point (x_0, y_0) in the xy-plane underneath a surface $z(x,y)$. Suppose you take an infinitesimal vectorial step in some direction, (dx, dy). How much does the height z change? The answer is given by the so-called total differential:

$$dz = \frac{\partial z}{\partial x}\,dx + \frac{\partial z}{\partial y}\,dy$$

9.6.1. Explain why. Is it important that (dx, dy) is infinitesimal? Explain the precise meaning of the terms in the formula. Explain how the notation of this formula can be confusing and show how it can be expressed differently to avoid this problem.

The same reasoning leads to the chain rule for partial derivatives, which says: Given $z(x, y)$ and $x(t)$ and $y(t)$ as functions of t:

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

9.6.2. Explain why.

A more general version of the chain rule is: Given $z(x, y)$ and $x(u, v)$ and $y(u, v)$ as functions of u and v:

$$\frac{dz}{du} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}$$

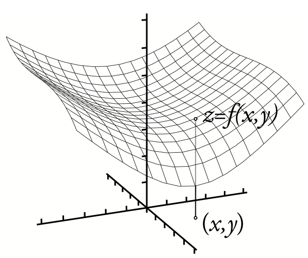
9.6.3. Explain why one cannot "cancel the ∂x's" in this formula. Express the formula in a different notation, which avoids this temptation. Is this better?

9.6.4. Consider the parabolic "bowl" $z = x^2 + y^2$. Starting at the point $(1, 1)$ in the xy-plane, how fast is the height z changing if we move radially (straight away from the origin) or circularly (remaining at the same radius from the origin)? Use polar coordinates and the chain rule to find out.

§ 9.7. Reference summary

§ 9.7.1. *Functions of two variables*

A function $f(x, y)$ of two variables specifies a height $z = f(x, y)$ above each point (x, y) in the plane. Geometrically, it therefore defines a surface.



§ 9.7.2. *Partial derivatives*

$$f_x = \frac{\partial f}{\partial x} = \text{ partial derivative of } f \text{ with respect to } x$$
$$= \text{ rate of change of } f \text{ as one moves in the } x \text{ direction}$$

- Find the partial derivative of a function with respect to a given variable.

  Differentiate as usual with respect to this variable, while treating all other variables as constants.

Find the partial derivatives of $f(x, y) = xy^2 + \cos(x)$.

$$f_x = y^2 - \sin(x) \qquad f_y = 2xy$$

Find the partial derivatives of $f(x, y) = xe^{xy}$.

$$f_x = e^{xy} + xye^{xy} \qquad f_y = x^2e^{xy}$$

§ 9.7.3. *Planes*

Equation for plane:

$$Ax + By + Cz = D \quad \text{normal vector} = (A, B, C)$$

The point $(10, 18, 3)$ is in a plane with normal vector $(3, -2, 4)$. Find the equation of the plane.

Since $(3, -2, 4)$ is a normal vector, the equation for the plane is $3x-2y+4z = D$ for some constant D. Plugging in the point $(10, 18, 3)$, we find that $D = 3\cdot10-2\cdot18+4\cdot3 = 30-36+12 = 6$. The equation of the plane is hence $3x - 2y + 4z = 6$.

Tangent plane to $f(x, y)$ above the point (x_0, y_0):

$$z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

Find the tangent plane to $f(x, y) = x^2+y^2-1$ above the point $(1, 3)$.

$f_x = 2x$, $f_y = 2y$, so the tangent plane is

$$\begin{aligned} z &= f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) \\ &= f(1, 3) + f_x(1, 3)(x - 1) + f_y(1, 3)(y - 3) \\ &= 9 + 2(x - 1) + 6(y - 3) \\ &= 2x + 6y - 11 \end{aligned}$$

§ 9.7.4. *Unconstrained optimisation*

Stationary points of $f(x, y)$ occur where the partial derivatives f_x and f_y are both zero. Classification of stationary points:

$f_{xx}f_{yy} - (f_{xy})^2$	f_{xx} and/or f_{yy}	type of equilibrium
+	+	minimum
+	−	maximum
−		saddle

Find and classify the stationary points of $f(x, y) = x^3 + y^2 - 3x - 4y + 2$.

$f_x = 3x^2 - 3 = 0 \Rightarrow x = \pm1$ and $f_y = 2y - 4 = 0 \Rightarrow y = 2$. There are thus two stationary points:$(1, 2)$ and $(-1, 2)$. We calculate: $f_{xx} = 6x, f_{yy} = 2, f_{xy} = 0$. At $(1, 2)$: $f_{xx} = 6 > 0, f_{yy} = 2 > 0, f_{xx}f_{yy} - f_{xy}^2 = 12 > 0 \Rightarrow$ minimum. At $(-1, 2)$: $f_{xx} = -6 < 0, f_{yy} = 2 > 0, f_{xx}f_{yy} - f_{xy}^2 = -12 < 0 \Rightarrow$ saddle.

Find and classify the stationary points of $g(x,y)=\frac{x^2+y^2}{2}-\frac{1}{xy}$.

Stationary points occur where the partial derivatives are zero. This gives $g'_x(x,y)=x+\frac{1}{x^2y}=0$, which simplifies to $x^3y=-1$, and $g'_y(x,y)=y+\frac{1}{xy^2}=0$, which simplifies to $3xy^3=-1$. Solving for y in the first equation gives $y=-\frac{1}{x^3}$, which plugged into the second gives $x(-\frac{1}{x^3})^3=-1 \Leftrightarrow \frac{1}{x^8}=1 \Leftrightarrow x^8=1 \Leftrightarrow x=\pm 1$. Combining this with $y=-\frac{1}{x^3}$, we see that the critical points are $(1,-1)$ and $(-1,1)$. To classify these points we need the second derivatives: $g''_{xx}=1-\frac{2}{x^3y}$, $g''_{xy}=-\frac{1}{x^2y^2}$, $g''_{yy}=1-\frac{2}{xy^3}$. For $(1,-1)$ we get $g''_{xx}=1-(-2)=3$, $g''_{xy}=-1$, $g''_{yy}=1-(-2)=3$. Thus $g''_{xx}>0$ and $g''_{xx}g''_{yy}>g''^2_{xy}$ so the point is a minimum. For $(-1,1)$ we get $g''_{xx}=1-(-2)=3$, $g''_{xy}=-1$, $g''_{yy}=1-(-2)=3$. These are the same values as before, so this point is also a minimum.

Find and classify the stationary points of $f(x,y)=xye^{x-y}$.

For ease of writing, let $E=e^{x-y}$. Note that $E>0$.
$f_x=yE+xyE=0 \Rightarrow y+xy=0 \Rightarrow y=0$ or $x=-1$.
$f_y=xE-xyE=0 \Rightarrow x-xy=0 \Rightarrow x=0$ or $y=1$.
So the stationary points are $(-1,1)$ and $(0,0)$.
The second derivatives are $f_{xx}=2yE+xyE$, $f_{yy}=-2xE+xyE$, and $f_{xy}=E-yE+xE-xyE$.
$(x,y)=(-1,1) \Rightarrow f_{xx}=E>0, f_{yy}=E>0, f_{xy}=0$, so minimum.
$(x,y)=(0,0) \Rightarrow f_{xx}=0, f_{yy}=0, f_{xy}=E \Rightarrow f_{xx}f_{yy}-(f_{xy})^2=-E^2<0$, so saddle.

Find and classify the stationary points of $f(x,y)=x^3+y^3+6xy+2$.

Stationary points occur where $\frac{\partial f(x,y)}{\partial x}=\frac{\partial f(x,y)}{\partial y}=0$, which in our case becomes $3x^2+6y=0$ and $3y^2+6x=0$, or, simplifying, $x^2+2y=0$ and $y^2+2x=0$. Solving for y in the first equation gives $y=-\frac{1}{2}x^2$, which inserted in the second gives $(-\frac{1}{2}x^2)^2+2x=0$, or $x^4+8x=0$. This factors into $x(x^3+8)=0$. Thus $x=0$ or $x^3=-8$, which means $x=-2$. When $x=0$ we get $y=0$ and when $x=-2$ we get $y=-2$. We thus have two stationary points: $P_1=(0,0)$ and $P_2=(-2,-2)$. To classify them we calculate the second derivatives: $A=\frac{\partial^2 f(x,y)}{\partial x^2}=6x$, $C=\frac{\partial^2 f(x,y)}{\partial y^2}=6y$, $B=\frac{\partial^2 f(x,y)}{\partial x \partial y}=6$. For P_1 we get $AC-B^2=-36<0$, so it's a saddle. For P_2 we get $AC-B^2=144-36>0$, while $A=-12<0$, so this is a maximum.

§ 9.7.5. *Gradients*

$$\begin{aligned}\nabla f &= (f_x, f_y)\\ &= \text{direction of steepest ascent of } f\\ &= \text{normal to level curve of } f\end{aligned}$$

$$|\nabla f| = \text{ rate of steepest ascent of } f$$

$f(x,y)=\frac{1}{x^2+y^2}$. You are standing at $(1,1)$ in the xy plane. Which way should you go to make f grow the fastest?

$f_x=\frac{-2x}{(x^2+y^2)^2} \Rightarrow f_x(1,1)=-\frac{1}{2}$ and $f_y=\frac{-2y}{(x^2+y^2)^2} \Rightarrow f_y(1,1)=-\frac{1}{2}$. The direction of fastest increase in f is $\nabla f=(f_x,f_y)=(-\frac{1}{2},-\frac{1}{2})$ (in words: toward the origin).

Directional derivative in direction of unit vector $\hat{\mathbf{s}}$:

$$\hat{\mathbf{s}}\cdot\nabla f$$

Directional derivative in direction θ:

$$f_x\cos\theta+f_y\sin\theta$$

Find the rate of change of $f(x,y)=xy^3$ when moving away from the origin from the point $(1,1)$.

$f_x=y^3 \Rightarrow f_x(1,1)=1$ and $f_y=3xy^2 \Rightarrow f_y(1,1)=3$. When standing at $(1,1)$, going away from the origin means going in the direction $\theta=\frac{\pi}{4}$. Therefore the directional derivative is $f_x\cos\theta+f_y\sin\theta=\cos\frac{\pi}{4}+3\sin\frac{\pi}{4}=2\sqrt{2}$.

Find the rate of change of $f(x,y)=xy^3$ when moving in the direction $\mathbf{w}=(-2,1)$ from the point $(1,1)$.

We know from the above example that $\nabla f=(1,3)$ at this point. Moreover, $\mathbf{w}$ converted to a unit vectors is $\hat{\mathbf{w}}=\mathbf{w}/|\mathbf{w}|=(-\frac{2}{\sqrt{5}},\frac{1}{\sqrt{5}})$. Thus the directional derivative is $\hat{\mathbf{w}}\cdot\nabla f=(-\frac{2}{\sqrt{5}},\frac{1}{\sqrt{5}})\cdot(1,3)=1/\sqrt{5}$.

- Find the normal to a given curve or surface at a given point.

  Write in form $f(x,y)=0$ (curve) or $f(x,y,z)=0$ (surface). Compute the gradient of f and evaluate it at the given point. This is the normal.

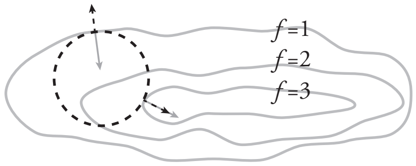
Find a normal vector to the surface $z=x^2+y^2$ at the point $(1,1,2)$.

The surface is the level surface $f=0$ of the function $f(x,y,z)=x^2+y^2-z$. A normal vector is therefore $\nabla f=(2x,2y,-1)$, which evaluated at the point in question is $(2,2,-1)$.

§ 9.7.6. *Constrained optimisation*

(Lagrange multipliers.) To extremise $f(x,y)$ subject to the constraint $g(x,y)=c$, solve

$$\begin{aligned}f_x&=\lambda g_x\\ f_y&=\lambda g_y\\ g&=c\end{aligned}$$



Any local maxima or minima will be among the pairs of points (x, y) that satisfy these equations. To determine which are max., min., or neither, plug them into $f(x, y)$ and compare their values. If the constraint curve has endpoints, these are also potential max. or min. If the constraint curve is infinite, also investigate the limit behaviour of the function there (e.g., there is no global max. or min. if the function grows or shrinks without bounds in such a direction).

Find the maximum value of $f(x, y) = xy$ subject to the constraint $x^2 + y^2 = 1$.

We need to solve the system

$$\begin{aligned} y &= 2\lambda x \\ x &= 2\lambda y \\ x^2 + y^2 &= 1 \end{aligned}$$

Putting the first equation into the second gives $x = 2\lambda(2\lambda x)$. If we divide by x we get $4\lambda^2 = 1$, or $\lambda = \pm\frac{1}{2}$. Of course, whenever we divide by x we must take care that we are not dividing by zero. But in this case x cannot be zero, since if it is then so is y, which is impossible by the last equation. Putting $\lambda = \pm\frac{1}{2}$ back into the first two equations we find that $y = \pm x$. Combining this with the last equation we get $x^2 + (\pm x)^2 = 1$, or $x = \pm\frac{1}{\sqrt{2}}$. Since $y = \pm x$, all possible solutions are $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$, $(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$, $(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$, $(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$. To see which are maxima and which are minima, we plug these back into the original function $f(x, y) = xy$. The largest value of f is thus $\frac{1}{2}$ and the smallest $-\frac{1}{2}$.

Find the maximum value of $f(x, y) = 2xy$ subject to the constraint $5x + 4y = 100$.

$$g = c \Leftrightarrow 5x + 4y = 100$$

$$f_x = \lambda g_x \Leftrightarrow 2y = 5\lambda \Rightarrow \lambda = 2y/5$$

$$f_y = \lambda g_y \Leftrightarrow 2x = 4\lambda \Rightarrow \lambda = x/2$$

Hence $2y/5 = x/2$, which substituted into the constraint gives $x = 10$ and $y = 12.5$. Our candidate for a maximum point is thus $(10, 12.5)$, where $f = 250$. But since the constraint curve is an infinite line we must also investigate the limit values of f as we go toward infinity along the line in either direction. The constraint conditions shows that if $x \to \infty$ then $y \to -\infty$ and if $x \to -\infty$ then $y \to \infty$. In either case, $f \to -\infty$. Hence our candidate maximum is indeed the biggest f ever gets.

Determine the maxima and minima of $f(x, y) = 5x^2 - 2y^2 + 10$ on the curve $x^2 + y^2 = 1$.

$$g = c \Leftrightarrow x^2 + y^2 = 1$$

$$f_x = \lambda g_x \Leftrightarrow 10x = 2\lambda x \Rightarrow \lambda = 5 \text{ or } x = 0$$

$$f_y = \lambda g_y \Leftrightarrow -4y = 2\lambda y \Rightarrow \lambda = -2 \text{ or } y = 0$$

So the stationary points are $(\pm 1, 0)$ and $(0, \pm 1)$. $f(\pm 1, 0) = 15$ and $f(0, \pm 1) = 8$, so $(\pm 1, 0)$ are maxima and $(0, \pm 1)$ minima.

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§ 9.7.7. Multivariable chain rule

Total differential:

$$dz = \frac{\partial z}{\partial x}\,dx + \frac{\partial z}{\partial y}\,dy$$

Chain rule for partial derivatives when x and y are functions of t:

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

Chain rule for partial derivatives when x and y are functions of u and v:

$$\frac{dz}{du} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}$$

10 MULTIVARIABLE INTEGRAL CALCULUS

§ 10.1. Multiple integrals

§ 10.1.1. *Lecture worksheet*

We know that $\int y(x)\,dx$ is an area made up of thin rectangles with base dx and height $y(x)$. Likewise, $\iint f(x,y)\,dx\,dy$ is a volume made up of think rectangular blocks with base area $dx\,dy$ and height $f(x,y)$. This is called a double integral. To evaluate it we integrate twice: once with respect to x and once with respect to y. Geometrically, the first integration gives an expression for the cross-sectional areas of the shape, and the second integration finds the volume by endowing each of these areas with an infinitesimal thickness.

10.1.1. Consider for example the tetrahedron with vertices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, $(0,0,1)$.

(a) Argue that the "roof" of this tetrahedron is given by $z = 1 - x - y$.

Suppose I intersect the tetrahedron with a plane perpendicular to the y-axis.

(b) Argue that the cross-sectional area is $\int_0^{1-y} 1 - x - y\,dx$.

(c) Argue that the volume of the tetrahedron is $\int_0^1 \left(\int_0^{1-y} 1 - x - y\,dx\right) dy$.

(d) Find the volume by evaluating first the inner then the outer of these integrals.

In general terms, to compute the double integral $\iint_R f(x,y)\,dx\,dy$ over some region R, we write it as an iterated integral

$$\int_a^b \int_{x_0(y)}^{x_1(y)} f(x,y)\,dx\,dy$$

where a and b are the y-values between which the region R is contained, and $x_0(y)$ and $x_1(y)$ are the x-values between which any given cross-section of R perpendicular to the y-axis is contained. Alternatively we can invert the roles of x and y to instead express the integral as

$$\int_c^d \int_{y_0(x)}^{y_1(x)} f(x,y)\,dy\,dx$$

10.1.2. Find, in both ways, $\iint_R y\,dx\,dy$ where R is the region between $y = x$ and $y = x^2$.

Depending on the shape of R, it may be much easier to specify the bounds $x_0(y)$, $x_1(y)$ than $y_0(x)$, $y_1(x)$, or vice versa.

10.1.3. Think of an example to illustrate this.

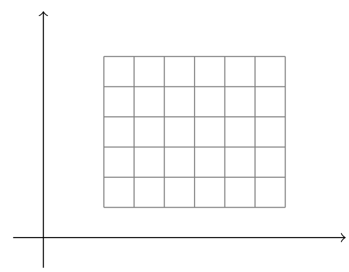
§ 10.2. Polar coordinates

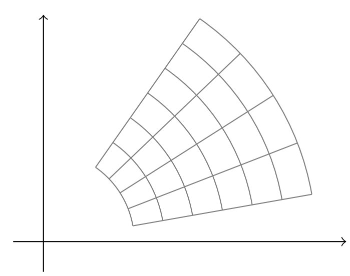
§ 10.2.1. *Lecture worksheet*

Polar coordinates (§7.1) are useful when evaluating double integrals since many regions of a circular or radial nature are much more naturally and easily described in polar than rectangular coordinates. When an integral in ordinary rectangular coordinates x, y is rewritten in polar coordinates r, θ it becomes

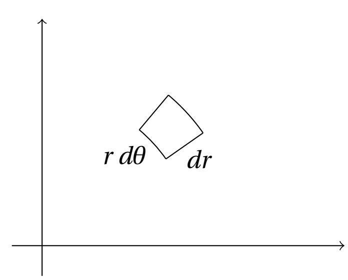
$$\iint_R f(x,y)\,dx\,dy = \int_\alpha^\beta \int_{r_0(\theta)}^{r_1(\theta)} f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$

where α, β are the angular (θ) bounds of R, and $r_0(\theta)$, $r_1(\theta)$ are the radial (r) bounds for R for any given θ. The manner in which x and y have been translated we recognise from before (§7.5.1). It remains to explain where the extra r comes from in the area element expression $r\,dr\,d\theta$. We can understand this as follows. In an integral in rectangular coordinates, if we let x increase by increments dx and y by increments dy, this generates a grid of identical rectangles. But if we let r increase by increments dr and θ by increments $d\theta$, this generates a different kind of grid in which the cells have different sizes.





10.2.1. Argue that each cell can be considered a rectangle with sides dr and $r\,d\theta$.



10.2.2. Find the area generated by one full revolution of the Archimedean spiral $r = \theta$ (shown in §7.1.1).

§ 10.2.2. Problems

10.2.3. Consider the integral $I = \int_0^\infty e^{-x^2}\,dx$. The function e^{-x^2} cannot be integrated in closed form using any of our previous integration techniques, so we cannot evaluate I by direct integration. Nevertheless we can evaluate this integral by an ingenious use of multiple integration, as we shall now see.

(a) Argue that
$$I^2 = \left(\int_0^\infty e^{-x^2}\,dx\right)\left(\int_0^\infty e^{-y^2}\,dy\right)$$

(b) Argue that this can be rewritten as
$$\int_0^\infty \int_0^\infty e^{-(x^2 y^2)}\,dx\,dy$$

(c) Evaluate this integral using polar coordinates.

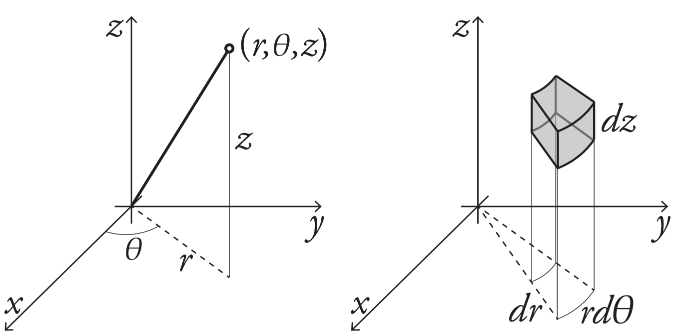
(d) What is I?

§ 10.3. Cylindrical coordinates

§ 10.3.1. Lecture worksheet

If we are integrating over a cylindrical region the bounds are complicated to express in rectangular coordinates x, y, z. It is better to use *cylindrical coordinates*, which means polar coordinates in the base xy-plane and the ordinary z-coordinate for the height. Similar to the polar coordinate case, the integral then becomes

$$\iiint_R f(x,y,z)\,dx\,dy\,dz = \iiint_R f(r\cos\theta, r\sin\theta, z)\,r\,dr\,d\theta\,dz$$



10.3.1. Explain this on the basis of the figure.

10.3.2. Something is stored in a cylindrical silo. It compresses under weight in such a way that the density of each layer is proportional to the height above it to the top of the stack. Find an expression for the net mass as a function of the height of the stack.

Cylindrical coordinates are useful not only for cylinders but often for any shape with an axis of symmetry.

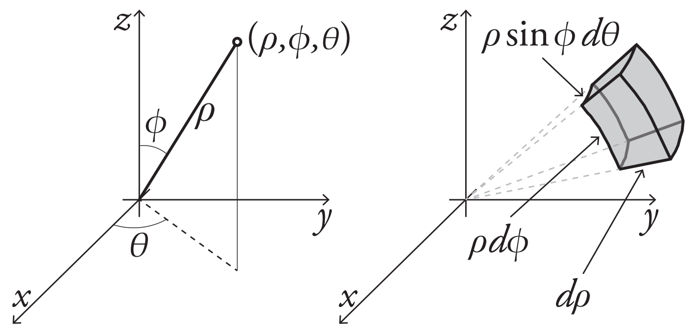
10.3.3. Solve problem 10.3.2 when the silo has the shape of the cone $z = x^2 + y^2$.

§ 10.4. Spherical coordinates

§ 10.4.1. Lecture worksheet

Spherical coordinates characterise points in 3-dimensional space by means of one radial coordinate and two angular coordinates (analogous to longitude and latitude). Translating an integral into spherical coordinates gives:

$$\iiint_R f(x,y,z)\,dx\,dy\,dz =$$
$$\iiint_R f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$



10.4.1. Explain this on the basis of the figure. Hint: To verify the lengths of the arcs, ask yourself what the radius is of the circle of which it is part.

10.4.2. Find the volume of a sphere as a triple integral in spherical coordinates.

An important application of spherical coordinates is to the gravitational theory of spherical bodies.

10.4.3. Suppose a mass M is uniformly distributed in the form of a thin spherical shell of radius R and thickness dr centered at the origin. We shall show that the gravitational force exerted by this shell on a particle of mass m located some distance $a > R$ away at the point $(a,0,0)$ is the same as that of a point-mass M located at the origin.

(a) What are the implications of this result for computing the gravitational influences of celestial bodies?

(b) What is the volume of the shell? Hint: surface area × thickness.

(c) Therefore, what is its density?

Consider an infinitesimal piece of the shell (in the manner of the above figure).

(d) What is the mass of this piece?

(e) Find the gravitational force it exerts on the mass m in terms of the distance s between them and the angle α the line connecting them makes with the x-axis.

(f) Write down a triple integral expression for the net gravitational force of the whole shell.

(g) Argue that two of the integrals can be evaluated without knowing s. Do so.

Hint: Use the law of cosines to find the relation between s and the current variable. Also use trigonometry to rewrite the integrand purely in terms of s.

(h) For the remaining integral, make a change of variables to rewrite the integral as an integral in s.

(i) Evaluate the integral.

(j) Conclude.

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§ 10.4.2. Problems

10.4.4. Solve problem 10.4.3 in the case where the mass m is located inside the shell.

(This result was used in problem 6.5.4.)

§ 10.5. Surface area

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§ 10.5.1. Lecture worksheet

10.5.1. Show that the area of the surface $z = f(x, y)$ above an infinitesimal rectangle of sides dx, dy is

$$\sqrt{1 + f_x^2 + f_y^2}\, dx\, dy$$

Hint: one way of doing this is to consider the area as the parallelogram spanned by two vectors and computing it using a vector product.

10.5.2. Find the surface area of a sphere using this method.

§ 10.6. Reference summary

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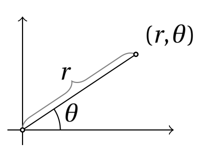
§ 10.6.1. Polar coordinates



Coordinate transformation formulas:

$$x = r\cos\theta$$
$$y = r\sin\theta$$

Area scaling factor:

$$dx\, dy = r\, dr\, d\theta$$

Integral transformation formula:

$$\iint_R f(x, y)\, dx\, dy = \int_\alpha^\beta \int_{r_0(\theta)}^{r_1(\theta)} f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

where α, β are the angular (θ) bounds of R, and $r_0(\theta)$, $r_1(\theta)$ are the radial (r) bounds for R for any given θ.

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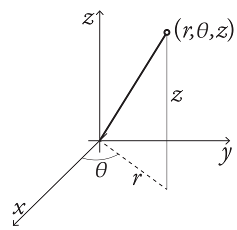
§ 10.6.2. Cylindrical coordinates



Coordinate transformation formulas:

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$z = z$$

Volume scaling factor:

$$dx\, dy\, dz = r\, dr\, d\theta\, dz$$

Integral transformation formula:

$$\iiint_R f(x, y, z)\, dx\, dy\, dz = \iiint_R f(r\cos\theta, r\sin\theta, z)\, r\, dr\, d\theta\, dz$$

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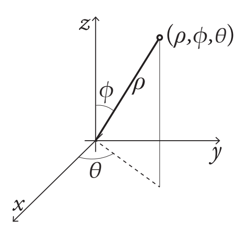
§ 10.6.3. Spherical coordinates



Coordinate transformation formulas:

$$x = \rho\sin\phi\cos\theta$$
$$y = \rho\sin\phi\sin\theta$$
$$z = \rho\cos\phi$$

Volume scaling factor:

$$dx\, dy\, dz = \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$$

Integral transformation formula:

$$\iiint_R f(x, y, z)\, dx\, dy\, dz =$$

$$\iiint_R f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$$

Find the cartesian coordinates (x,y,z) of the points $(r,\theta,\phi) = (1,0,0)$ and $(r,\theta,\phi) = (2,\pi/2,\pi/2)$.

$(0,0,1)$, $(0,2,0)$

Find the spherical polar coordinates (r,θ,ϕ) of the points $(x,y,z) = (0,1,0)$ and $(x,y,z) = (1,2,2)$.

$(r,\theta,\phi) = (1,\pi/2,\pi/2)$, $(r,\theta,\phi) = (3,\arccos(2/3),\arctan(2))$

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§ 10.6.4. Surface area

Area of the surface $z = f(x,y)$ above the region R:

$$\iint_R \sqrt{1+f_x^2+f_y^2}\,dx\,dy$$

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§ 10.6.5. Problem guide

- Evaluate a double integral $\iint_R f(x,y)\,dx\,dy$.

  First decide whether the region of integration R is most easily described in terms of rectangular coordinates x, y or polar coordinates r,θ. If you choose polar coordinates, translate the given function $f(x,y)$ into r,θ and multiply it with the area scaling factor (r), as shown above.

  Next write the integral as an iterated integral. We seek to choose the order of integration in such a way that the bounds are most easily expressed. We will evaluate the integrals from the inside out, so the inner differential and the inner bounds of integration correspond to the first integration. But when writing down the bounds it is easier to work from the outside in. For the outer integral, the bounds should be numbers (constants) expressing the bounds between which the region is contained as far as the outer variable is concerned. When specifying the inner bounds of integration you may use the outer variable in your expressions for these bounds; this is necessary whenever the bounds of the region with respect to the inner variable are different for different values of the outer variable. If expressing the inner bounds becomes very complicated, this suggests that we should try the other ordering of the variables.

  When the integrals have been written down completely, evaluate them one at a time, going from the inside out.

  $\iint_D \frac{1}{1+x^2}dxdy$ where D is given by $0 \le y \le x \le 1$.

  $= \int_0^1 (\int_0^x \frac{1}{1+x^2} dy) dx = \int_0^1 \frac{x}{1+x^2} dx = [\dfrac{1}{2}\ln(1+x^2)]_0^1 = \dfrac{1}{2}(\ln 2 - \ln 1) = \dfrac{\ln 2}{2}$

  $\iint_D e^x dxdy$, where D is the triangular region with vertices $(0,0)$, $(1,1)$, $(1,2)$.

  The sides of the triangles are on the lines $y = x$, $y = 2x$, $x = 1$. Hence D corresponds to $x \le y \le 2x, 0 \le x \le 1$. Thus $\iint_D e^x dxdy = \int_0^1 (\int_x^{2x} e^x dy) dx = \int_0^1 [e^x y]_{y=x}^{y=2x} = \int_0^1 xe^x dx = [xe^x]_0^1 - \int_0^1 e^x dx = e - [e^x]_0^1 = e - e + 1 = 1$.

  $\iint_D x^2 y dxdy$ where $D = \{(x,\ y)\ :\ x^2+y^2 \le 4,\ x,\ y \ge 0\}$.

  In polar coordinates D is given by $0 \le \theta \le \frac{\pi}{2}$ and $0 \le r \le 2$. Thus $\iint_D x^2 y dxdy = \int_0^{\frac{\pi}{2}} \int_0^2 (r\cos\theta)^2 (r\sin\theta) r drd\theta = \int_0^{\frac{\pi}{2}} \cos^2\theta \sin\theta d\theta \int_0^2 r^4 dr = [-\frac{1}{3}\cos^3\theta]_0^{\frac{\pi}{2}} [\frac{1}{5}r^5]_0^2 = (-\frac{1}{3})(0-1)\cdot \frac{1}{5}(32-0) = \frac{32}{15}$.

  $\iint_D e^{-(x^2+y^2)} dxdy$, where $D = \{(x,\ y)\ :\ 0 \le y \le x,\ x^2+y^2 \le 3\}$.

  In polar coordinates D corresponds to $E = \{(r,\theta) : 0 \le r \le \sqrt{3},\ 0 \le \theta \le \pi/4\}$. Hence $\iint_D e^{-(x^2+y^2)} dxdy = \iint_E re^{-r^2} drd\theta = \int_0^{\sqrt{3}} (\int_0^{\pi/4} re^{-r^2} d\theta) dr = \frac{\pi}{4} \int_0^{\sqrt{3}} re^{-r^2} dr = [r^2 = t,\ 2rdr = dt] = \frac{\pi}{8} \int_0^3 e^{-t} dt = \frac{\pi}{8}[-e^{-t}]_0^3 = \frac{\pi}{8}(1-e^{-3})$.

  $\iint_D xy^2 dxdy$, where $D = \{(x,y) : x^2+y^2 \le 4,\ x \ge 0\}$.

  In polar coordinates the region corresponds to $\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ and $0 \le r \le 2$. Thus: $\iint_D xy^2 dxdy = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^2 (r\cos\theta)(r\sin\theta)^2 r drd\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2\theta \cos\theta d\theta \int_0^2 r^4 dr = [\frac{1}{3}\sin^3\theta]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} [\frac{1}{5}r^5]_0^2 = \frac{1}{3}(1-(-1))\cdot\frac{1}{5}(32-0) = \frac{64}{15}$.

  $\iint_D \frac{x}{(x^2+y^2)^2} dxdy$, where $D = \{(x,\ y)\ :\ 0 \le y \le x,\ 1 \le x^2+y^2 \le 2\}$.

  In polar coordinates D corresponds to $E = \{(r,\ \theta)\ :\ 1 \le r \le \sqrt{2}, 0 \le \theta \le \pi/4\}$. Thus $\iint_D \frac{x}{(x^2+y^2)^2} dxdy = \iint_E \frac{r^2\cos\theta}{r^4} drd\theta = \int_1^{\sqrt{2}} r^{-2} dr \int_0^{\pi/4} \cos\theta d\theta = [1/r]_1^{\sqrt{2}} [\sin\theta]_0^{\pi/4} = (-\frac{1}{\sqrt{2}}+1)\cdot\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} - \frac{1}{2}$.

- Evaluate a triple integral $\iiint_R f(x,y,z)\,dx\,dy\,dz$.

  First decide whether the region of integration R is most easily described in terms of rectangular coordinates x,y,z, cylindrical coordinates r,θ,z (for regions with a symmetry axis, which we make the z-axis), or spherical coordinates ρ,ϕ,θ. If you choose non-rectangular coordinates, translate the given function $f(x,y,z)$ into your chosen coordinates (using standard formulas $x = \cdots, y = \cdots, z = \cdots$), and multiply it with the volume scaling factor, as shown above.

  Next write the integral as an iterated integral. We seek to choose the order of integration in such a way that the bounds are most easily expressed. We will evaluate the three integrals from the inside out, so the innermost differential and the innermost bounds of integration correspond to the first integration. But when writing down the bounds it is easier to work from the outside in. For the outermost integral, the bounds should be numbers (constants) expressing the bounds between which the region is contained as far as the outermost variable is concerned. When specifying the next bounds of integration you may use the outermost variable in your expressions for these bounds; this is necessary whenever the bounds of the region with respect to the current variable are different for different values of the outermost variable. Similarly, expression for the last bounds may contain both of the two outer variables. If expressing the inner bounds becomes very complicated, this suggests that we should try another ordering of the variables.

When the integrals have been written down completely, evaluate them one at a time, going from the inside out.

$\iiint_D f(x,y,z) = y^2z^3\,dxdydz$ where D is a cylinder symmetric about the z-axis, with radius a and cut off at $z=0$ and $z=1$.

$= \int_{z=0}^{1}\int_{\phi=0}^{2\pi}\int_{\rho=0}^{a}(\rho^2\sin^2\phi z^3)\rho\,d\rho d\phi dz = \int_{z=0}^{1} z^3 dz \cdot \int_{\phi=0}^{2\pi}\sin^2\phi d\phi \cdot \int_{\rho=0}^{a}\rho^3 d\rho = \frac{1}{4}\cdot\pi\cdot\frac{a^4}{4} = \frac{\pi a^4}{16}$

$\iiint_D \sqrt{x^2+y^2+z^2}\,dxdydz$, where D is the region $x^2+y^2+(z-1)^2 \le 1$.

In spherical coordinates, $x^2+y^2+(z-1)^2 \le 1 \Leftrightarrow x^2+y^2+z^2 \le 2z \Leftrightarrow \rho^2 \le 2\rho\cos\theta \Leftrightarrow \rho \le 2\cos\theta$, so D corresponds to $0\le\theta\le\pi/2$, $0\le\phi\le 2\pi$, $0\le\rho\le 2\cos\theta$. Over this region, $\iiint \rho\rho^2\sin\theta\,d\rho d\theta d\phi = 2\pi\int_0^{\pi/2}(\int_0^{2\cos\theta}\rho^3 d\rho)\sin\theta d\theta = 2\pi\int_0^{\pi/2}([\frac{1}{4}\rho^4]_{\rho=0}^{\rho=2\cos\theta})\sin\theta d\theta = 8\pi\int_0^{\pi/2}\cos^4\theta\sin\theta d\theta = 8\pi[-\frac{1}{5}\cos^5\theta]_0^{\pi/2} = \frac{8}{5}\pi$.

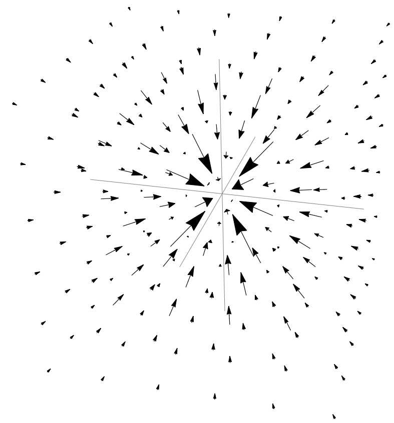
$\iiint_K (1+z^2)\,dxdydz$, where $K = \{(x,y,z) : x^2+y^2 \le \cos z, -\frac{\pi}{2} \le z \le \frac{\pi}{2}\}$.

$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left((1+z^2)\iint_{x^2+y^2\le\cos z} dxdy\right)dz = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(1+z^2)\pi\cos z dz = \pi[(1+z^2)\sin z]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} - \pi\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2z\sin z dz = \pi(2+\frac{1}{2}\pi^2) - \pi[2z(-\cos z)]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + \pi\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2(-\cos z)dz = \pi(2+\frac{1}{2}\pi^2) - 2\pi[\sin z]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{1}{2}\pi^3 - 2\pi$.

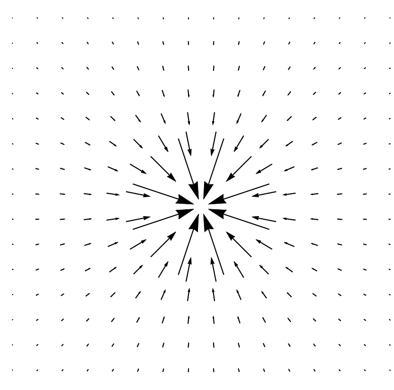
§ 11.1. Vector fields

§ 11.1.1. Lecture worksheet

The gravitational vector field of a heavy object, such as the sun, located at the origin looks like this:



A two-dimensional cross-section might be clearer on paper:



At each point a vector shows how an object placed there would be pulled by gravity. The gravitational pull is stronger closer to the origin. Electrostatic forces work much the same way: the same picture shows how a negative charge at the origin would attract a positively charged particle.

It is useful to think of situations like these in terms of imaginary fluid flows, even though no actual fluids are involved. We can imagine a large pool of water, into which someone has stuck a pipe that ends right in the middle of the pool. If water is sucked out through the pipe then this will cause the water in the pool to flow in the manner of the picture above. (This is very much an idealisation of course; in particular we are ignoring gravity and consider only the internal pressure of the water.)

11.1.1. One very nice aspect of this analogy is that it makes it obvious "why" the force of gravity (and electrostatic attraction) diminishes as the inverse square of the distance. Explain how. (Hint: Imagine how the water would need to move to fill a hole pumped out in the middle. It may help to consider a conical section of the water.)

11.1.2. Find an explicit formula for the vector field as follows.

(a) Argue that $-(x, y, z)$ is a vector that points from a given point (x, y, z) to the origin.

(b) Find a unit vector that points from (x, y, z) to the origin.

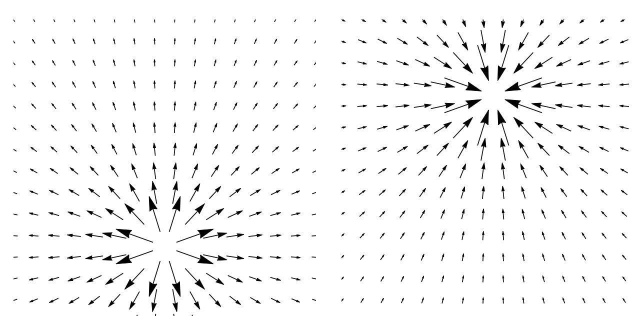
(c) Find a formula for the gravitational vector field **F**.

The origin in our example is called a *sink*. If water was being pumped in instead that would be a *source*. If we stick several pipes in our pool to make several sinks and sources then a more complicated fluid flow will arise. In fact it will be the superposition of the individual flows: that is, the net vector field is obtained by vector addition of the individual vector fields of each source and sink considered separately. Or at least an idealised fluid with this property is easy to imagine, and it is evidently what is required to correctly represent for example the combined gravitational field of the sun and Jupiter. Actual water, as it happens, is not quite so simple (because it is compressible and has a kind of internal friction), but that need not concern us since this whole business is hypothetical anyway; all this talk of fluids is meant only as a conceptual tool and a useful aid to our imagination.

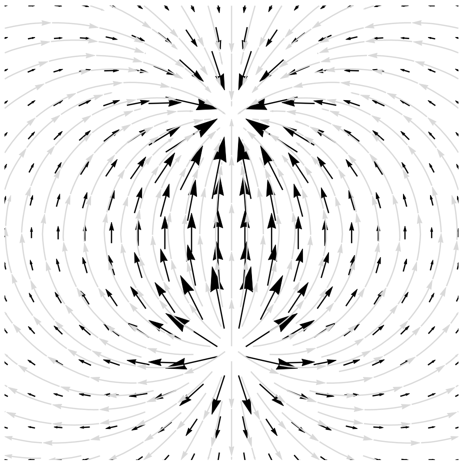
The same kind of reasoning also works if we restrict ourselves to a plane. We can then imagine a thin layer of fluid, say trapped between two glass planes.

11.1.3. Adapt the argument of problem 11.1.1 to show that forces diminish linearly in this case.

The vector fields of a source and a sink in this case look like this:



If I pump water in at one point and suck it out at another I can find the net effect by superimposing these two pictures. It looks like this:



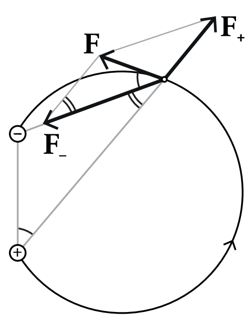
11.1.4. Sketch visually how the net field is obtained by adding up of the previous two.

11.1.5. Give an explicit formula for **F** in this case.

This fluid flow scenario models the way electric current behaves if we connect the two poles of a battery to two distinct points of a conducting sheet of metal. The analogy between electricity and fluid flow is so powerful that thinking in terms of an "electric fluid" is often the most intuitive way of understanding electric phenomena.

In the combined field I also traced (in light gray) the "flow curves" that follow the arrows.

11.1.6. Prove that the flow curves are in fact (parts of) circles, as follows. (We assume, of course, that the source and the sink are of equal magnitude.)



(a) Prove that the double-striped angles are equal and that the their triangles are similar. Hint: the "$1/r$" force law is the key to similarity.

(b) Infer that the single-striped angles are equal.

(c) Infer that the flow curve is a circle. Hint: draw the midpoint of the circle and the relevant radii, and consider the relations among the base angles of the isosceles triangles that arise to prove that **F** is tangent to the circle.

This last problem gives us occasion to reflect further on the relationship between a force field and its fluid flow analog. The problem shows that a small piece of paper dropped into our imaginary fluid will flow along such a circular arc, not that a particle in the corresponding force field will move in this way: the particle's momentum will cause it to deviate from the flow lines, as planets and projectiles do in the gravitational force field. The *velocity* of the imaginary fluid is the *acceleration* (or force, which in effect comes to the same things since $F = ma$) of the particle in the force field. So the fluid analogy is just a way of conceptualising forces, not an actual flow in which real-world objects can ride along like boats.

Note also that, as again highlighted by this last problem, the flow of our imaginary fluid is determined solely by its moving in the direction of lowest pressure; it does not accumulate momentum, which would interfere with this defining property. We can imagine our fluid as composed of a myriad little particles moving about chaotically and bouncing into each other all over the place. There will then be a net tendency for the fluid to move towards areas of lower pressure (since there are fewer particles to bump into in that direction), but at the same time there will never be a single, coordinated mass moving in any one direction, so the issue of momentum does not arise.

§ 11.2. Divergence

§ 11.2.1. Lecture worksheet

When thinking of a vector field **F** as a fluid flow, a fundamental question is how much fluid is being generated at a given point. This is called the *divergence* of the field.

The divergence can easily be found in terms of the derivatives of $\mathbf{F} = (P, Q, R)$ as follows. Consider an infinitesimal cube, and consider first the two walls of the cube that are pierced by the x-direction. The divergence in this direction is the difference between how much is flowing in through the left wall and how much is flowing out through the right wall. The difference in flow intensity is how much P changes in between, so $\frac{\partial P}{\partial x}\,dx$. This flow intensity acts across the area $dy\,dz$ of the wall, so the total excess flux generated inside the cube in this direction is $\frac{\partial P}{\partial x}\,dx\,dy\,dz$. And the same in the other directions. So the flux generated per unit volume is

$$\operatorname{div}\mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

Another notation for this is $\nabla\cdot\mathbf{F}$, where ∇ ("nabla") is the formal vector $(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})$.

11.2.1. Sketch the fields $\mathbf{A} = (x, 0, 0)$ and $\mathbf{B} = (0, x, 0)$. Compute their divergence. Interpret in terms of fluid flow.

For any region is space, the net flow out of it is the amount of fluid generated inside it. This evident fact is expressed in the so-called Divergence Theorem or Gauss's Theorem:

$$\iiint \operatorname{div}\mathbf{F}\; dV = \iint (\mathbf{F}\cdot\mathbf{n})\, dS$$

The left hand side is the integral over a region in space with volume element dV, so it expresses the amount of fluid generated inside it. The right hand side is the integral across the boundary of this region with normal $\mathbf{n}$ and surface element dS, so it measures how much of the flow is going out of the region (i.e., in the direction of the normal).

§ 11.3. Line integrals

§ 11.3.1. Lecture worksheet

Given a force field **F**, we recall from §4.4 that we can find the work it performs on an object being made to traverse a path in it by the integral $\int \mathbf{F} \cdot d\mathbf{s}$ taken along the path (although we are integrating along a curve the integral is nevertheless called a *line integral*—a very stupid name). Here $d\mathbf{s}$ is an infinitesimal piece of the curve, and by taking the scalar product we are projecting **F** onto it, i.e., we are counting only the component of the force that acts in the direction of the curve. In particular, if the force is perpendicular to the curve it has no effect at all and might as well be absent altogether. Thus we must not think that any effort is required to keep the object from deviating from the path; rather the object must be understood to be constrained to the path in a natural manner. A prototype example would be a hockey puck sliding on the surface of a frozen lake. Friction aside, gravity has no effect on the puck's free motion along this surface, and no effort is required to keep the puck from "turning downwards." Generalising from this example, we can imagine any line integral as a puck following a frictionless ice channel, such as a groove in the ice. The idea is that the ice channel ensures that the puck's inertial velocity is directed along the path of motion in a "lossless" fashion, as if this had been its natural inertial motion. Under these conditions the work done by the field amounts to how much it speeds up or slows down the motion along the path.

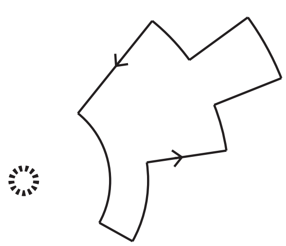
11.3.1. Discuss some line integrals in the gravitational force field near the earth, $\mathbf{F} = (0, -mg)$. Explain how this relates to potential energy. Also explain the meaning of the sign of the integral.

On a more astronomical scale, the gravitational pull of an object of great mass is $-GMm/r^2$, directed radially towards it. The net work done when moving a body along any closed path in this field is zero.

11.3.2. Show that this follows from energy conservation. (Hence such fields are called *conservative*.)

11.3.3. Also prove the same result more directly as follows.

(a) First prove the result for a path made up of pieces that are either radial or circular with respect to the center of force, such as this:



(b) Now consider an infinitesimal right triangle whose two legs are such radial and circular lines, and prove that the work along these legs is the same as the work along the hypothenuse.

(c) Conclude the proof of the desired result.

11.3.4. Infer that, in this force field, the work done in bringing an object from one point to another is independent of the path taken. Note that this holds for any conservative field.

Since distance travelled can also be expressed as velocity times time, $d\mathbf{s} = \dot{\mathbf{x}}\,dt$, another way of writing $\int \mathbf{F} \cdot d\mathbf{s}$ is $\int \mathbf{F} \cdot \dot{\mathbf{x}}\,dt$, which is a more practical form of the work integral for cases where the path **x** is given as a parametrised curve.

11.3.5. (a) Find an explicit formula for **F** for the radial gravitational force field. Hint: The vector (x, y, z) points from the origin to this point, so the vector $(-x, -y, -z)$ points from this point to the origin. It remains only to scale it so that it has the right magnitude.

(b) Confirm by explicit calculation that the work done by the field on an object traversing the circle $(a + \cos t, \sin t, 0)$ is zero. Note that the special case $a = 0$ is easy to deal with both physically and computationally.

Another way of seeing that a field is conservative is to think of its integrand as the derivative of something. For if $\int_0^T \mathbf{F} \cdot \dot{\mathbf{x}}\,dt = \int_0^T y'\,dt$ then by the fundamental theorem of calculus the integral evaluates to $y(T) - y(0)$, which is zero if the start and end points are the same. This happens precisely when the field is a gradient field, i.e., $\mathbf{F} = \nabla f$ for some function $f(x, y, x)$.

11.3.6. Explain why in this case

$$\frac{d}{dt} f(\mathbf{x}(t)) = \nabla f(\mathbf{x}(t)) \cdot \dot{\mathbf{x}}(t)$$

and use this to show that

$$\int_0^T \mathbf{F} \cdot \dot{\mathbf{x}}\,dt = f(\mathbf{x}(T)) - f(\mathbf{x}(0))$$

This function f has an important physical meaning as the following problem shows.

11.3.7. (a) Find an f such that $\mathbf{F} = \nabla f$ for the radial gravitational field **F**.

(b) Show that f is the potential energy. By definition the potential energy is the same thing as the work obtained by letting the object fall to the center of force.

(c) An application of this: For a body orbiting the sun, if at some point in its orbit it is twice as far from the sun as at another point, what is the difference in velocity between these two points? Hint: Use energy conservation.

This meaning of f as the potential energy generalises to any conservative force field, as we can see by the following reasoning.

11.3.8. Let **F** be any conservative field and define $f(\mathbf{x})$ as $\int_{\mathbf{o}}^{\mathbf{x}} \mathbf{F} \cdot d\mathbf{s}$, where the integral is taken along any path from the arbitrarily fixed origin **o** to the general point **x**.

(a) Explain why it necessary for $\mathbf{F}$ to be conservative for this construction to make sense.

(b) Show that $\nabla f = \mathbf{F}$.

(c) Conclude that f is a potential energy function. What does the arbitrariness of $\mathbf{o}$ mean in physical terms?

Restated in purely mathematical terms, this shows that any conservative field is a gradient field.

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§ 11.3.2. Problems

11.3.9. Show that the field $\mathbf{F}(x, y) = (P(x, y), Q(x, y))$ is conservative if and only if $P_y = Q_x$ at every point in the plane.

§ 11.4. Circulation

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§ 11.4.1. Lecture worksheet

We have seen that if $\mathbf{F}$ is a force field then $\int \mathbf{F}\cdot d\mathbf{s}$ is the work done by the field in traversing a certain path, that is, the "boost" that field is giving us, like a wind in our backs, as we traverse the path.

A problem with this image is that we must disregard the forces that are trying to push us off the path. This is what we accomplished above with our "ice channel" idea. The analogous idea for fluid flows would be to imagine that all of the fluid except for a narrow channel is instantaneously frozen. Then the fluid in the channel will, in general, continue flowing around one way or the other depending on which direction had the greater momentum in the original flow, and this net balance of momenta is precisely what the integral $\oint \mathbf{F}\cdot d\mathbf{s}$ computes.

For this reason, when the work integral $\oint \mathbf{F}\cdot d\mathbf{s}$ is taken around a closed path (indicated by the little circle on the integral sign) it is called the *circulation*. To fix the sign, the convention is that we traverse the path so that the inside is on our left. The figure thus shows a negative circulation.

The suggestive idea of the ice channel is all we really need, but if you wish to think more about what actually happens to the fluid you should be able to convince yourself that, owing to its incompressibility, the fluid will settle into a circulation of uniform speed (as long as there are no sources or sinks inside the channel). So, when we freeze the rest of the fluid, the flow in the channel also alters, and therefore ceases to represent the forces in the force field interpretation of $\mathbf{F}$. Nevertheless, it remains a viable image for the net work done along the channel as a whole, which was its purpose in the first place.

You may also wish to think about how to reconcile the momentum-based account of circulation with the particle-kinetic fluid model that we used at the end of §11.1 to argue *against* momentum effects. Hint: the narrow channel now means that the momenta are coordinated after all.

We shall now show that the circulation along a loop can be computed as the sum of the circulations around its interior pieces. To this end it is useful to consider first the circulation around an infinitesimal square, and then taking general shapes to be made up of them. So consider an infinitesimal square with its sides parallel to the axes. What is the circulation around this square? Consider first the two vertical sides. The force $\mathbf{F} = (P, Q)$ will be almost the same along both of these sides, namely Q evaluated there. As we walk around the square, one of these Q-forces go with the circulation and the other against it, so their net effect is zero except for the fact that they are not quite equal: since the two sides are dx apart their Q-values differ by $\frac{\partial Q}{\partial x}\,dx$. This, then, is the net force contributing to the circulation, and since it acts across a distance of dy its net contribution is $\frac{\partial Q}{\partial x}\,dx\,dy$.

11.4.1. Continue this line of reasoning to show that the circulation around the infinitesimal square is

$$\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

Hint: The signs are easily understood by recalling that our convention is to round the square counter-clockwise and considering whether an increase in P or Q helps or hinders this motion.

We now wish to consider any region as an aggregate of infinitesimal squares, which will give us Green's Theorem:

$$\oint P\,dx + Q\,dy = \iint \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

The left hand side is just the usual work or circulation integral $\oint \mathbf{F}\cdot d\mathbf{s}$ written out in terms of the components of $\mathbf{F} = (P, Q)$. The right hand side is the sum of the circulations about infinitesimal squares.

11.4.2. Complete the proof of Green's Theorem as follows.

(a) Suppose two infinitesimal squares are joined along one side. Argue that the circulation around the new region is the sum of the circulations of the each constituent square considered separately. (The picture suggests the idea that the shared edge "cancels," but make sure that your explanation makes sense in terms of the fluid flow interpretation of circulation.)

(b) We can approximate any region very closely by infinitesimal squares, but the boundary will be "jagged." Prove that the flow remains the same if we cut across diagonally instead of following two edges of the boundary squares.

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§ 11.4.2. Problems

11.4.3. Show that the area of a region in the plane is given by the integral $\oint x\,dy$ or $-\oint y\,dx$ taken around its boundary. Hint: Cut the boundary of the figure into infinitesimal pieces and draw the rectangles $x\,dy$ for each piece.

These results are often presented as a corollary of Green's Theorem even though it is much more illuminating to understand them directly from first principles.

11.4.4. By averaging the two area expressions in the previous problem we can also write the area as $\frac{1}{2}\oint x\,dy - y\,dx$. This formula can also be interpreted in terms of determinants: Show how it is obtained from a computation of the areas of triangles with corner points $(0,0)$, (x,y), $(x+dx, y+dy)$ by determinant methods.

11.4.5. Find the area of the ellipse $x^2/a^2 + y^2/b^2 = 1$.

11.4.6. Show that the Divergence Theorem reduced to two dimensions gives essentially Green's Theorem (only with trivial modifications in signs).

§ 11.5. Curl

§ 11.5.1. Lecture worksheet

We would now like to extend our investigation of circulation into three dimensions. We do this by means of the *curl* of a vector field $\mathbf{F} = (P,Q,R)$:

$$\operatorname{curl}\mathbf{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}$$

This expression is more conveniently expressed in terms of the formal nabla vector as $\operatorname{curl}\mathbf{F} = \nabla\times\mathbf{F}$.

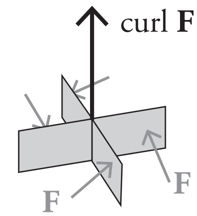
The z-component of the curl vector is the circulation around an infinitesimal square parallel to the xy-plane, just as in the previous section, and the other components are the analogous expressions for the other directions. The meaning of the curl vector, therefore, is that $\operatorname{curl}\mathbf{F}\cdot\mathbf{n}$ measures how much a wheel with axis $\mathbf{n}$ would be made to rotate by the fluid. For example, if $\mathbf{n}$ points in the z-direction we are asking for the rotation of a wheel with this axis, which is just $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$, or the circulation parallel to the xy-plane, just as before. If $\mathbf{n}$ points in some oblique direction then the rotation of a wheel with this axis will be a combination of the various coordinate-axis-plane rotations taken in proportions depending on what coordinate axis $\mathbf{n}$ agrees more with, and this is precisely what the scalar product accomplishes.

Green's Theorem about circulation extended to three dimensions, where is is called Stokes' Theorem, therefore becomes

$$\oint \mathbf{F}\cdot d\mathbf{s} = \iint \operatorname{curl}\mathbf{F}\cdot\mathbf{n}\,dS$$

The curl vector can also be interpreted directly, without projecting it onto a specific direction vector, in a manner very similar to how the direction and magnitude of ∇f were found to have interesting intrinsic meaning once this vector had been most naturally introduced in terms of its scalar products (cf. §9.4).

11.5.1. Convince yourself that the direction of $\operatorname{curl}\mathbf{F}$ is the axis of maximal circulation (rather like the vortical axis in a pitcher of lemonade being stirred) and that its magnitude is the intensity of rotation.



§ 11.5.2. Problems

11.5.2. Prove computationally that $\operatorname{curl}\nabla f = 0$ and argue physically that the curl of a field is zero if and only if the field is conservative.

11.5.3. (a) Prove computationally that $\operatorname{div}\operatorname{curl}\mathbf{F} = 0$.

(b) † Is there a physical interpretation of this result?

11.5.4. Prove that $\operatorname{div}\operatorname{curl}\mathbf{F} = 0$ by considering the limit of Stokes' Theorem as the loop shrinks to a point and then applying the Divergence Theorem to the resulting integral.

11.5.5. Find the curl of $\mathbf{F} = z\mathbf{i}$ and interpret the result physically.

11.5.6. The fact that "you can't go uphill both ways to school" is an instance of what vector calculus theorem?

§ 11.6. Electrostatics and magnetostatics

§ 11.6.1. Lecture worksheet

Space is permeated by two fields: the electric field $\mathbf{E}$, which describes how a positively charged (static) particle would move, and the magnetic field $\mathbf{B}$, which describes in which direction the north pole of a compass needle points. All electromagnetic phenomena can be characterised in terms of these fields. In particular, all information transmitted through all forms of wireless communication is encoded in these fields.

The complete theory of electromagnetism is contained in a few simple laws, analogous to Newton's law of classical mechanics. These are the equations of Maxwell. Before stating these laws in full generality we wish to study the electrostatic and magnetostatic special cases. For electrostatics Maxwell's equations reduce to

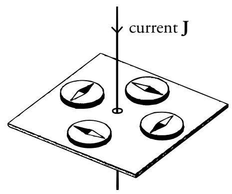
$$\nabla\cdot\mathbf{E} = \rho/\epsilon_0 \quad\text{and}\quad \nabla\times\mathbf{E} = 0$$

where ρ is electric charge and ϵ_0 is a constant. Thus the first law is saying that electric charges produce divergence—i.e., act as sources or sinks—in the electric field, as we have already discussed above. Note that Coulomb's inverse-square law is automatically incorporated in this more elegant divergence law (in the manner of problem 11.1.1). The second equation says that the curl is always zero, as of course we would expect since an electrostatic field is analogous to a gravitational one and therefore conservative.

For magnetostatics Maxwell's equations reduce to

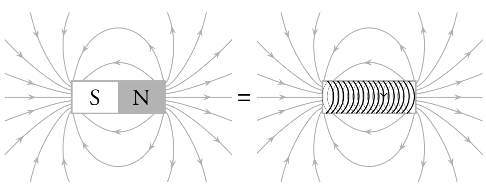
$$\nabla\times\mathbf{B} = \mu_0\mathbf{J} \quad\text{and}\quad \nabla\cdot\mathbf{B} = 0$$

where **J** is the (constant) flow of electric current and μ_0 is a constant. So the "magnetic fluid" spins around and around, in a direction perpendicular to the current:



The magnetic field is like a pitcher of lemonade that someone is stirring, and the direction of the current is the axis of its rotation. The divergence is zero: no magnetic fluid is generated or destroyed. This absence of divergence corresponds to the fact that there are no magnetic monopoles, i.e., magnets always come in north-south pairs, never a piece of "north only," in contrast to the charged particles that generate divergence in the electric field. (As Humphry Davy once wrote to a woman to whom he was attracted: "You are my magnet, though you differ from a magnet in having no repulsive points.")

When we speak of the current **J** in this law we must understand that we do not mean only currents from a battery or a wall socket. An ordinary bar magnet also contains a current: in fact, at an atomic level, for a material to be magnetised means that the rotations of its electrons are synchronised in such a way that they amount to a microscopic current along the surface of the material. This is why a magnet can always be replaced by a coiled wire with a current in it: the two are fundamentally the same thing. So while we have all played with ordinary magnets and might have expected them to be the most primitive objects of magnetic theory, they are in fact from a theoretical point of view a bit of an exotic curiosity; it is rather the definition of magnetic fields in terms of current that is the basic one.



In addition to Maxwell's four equations describing the fields **E** and **B** one also needs a "0^{th} equation" to understand the effect of the fields on a charged particle. This law says that the force on a particle of charge q moving at velocity **v** is

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

So the electric field is pushing things on directly as we have already seen, whereas the effect of the magnetic field is a bit more subtle. When a charge is moving we can think of its velocity vector as a wrench and the magnetic field as a force pushing the wrench. The resulting torque is the force that the particle experiences.

11.6.1. Explain how the following experimental facts are accounted for by these equations.

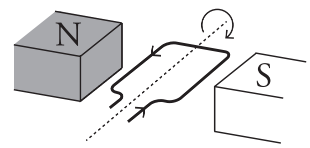
(a) On the table in front of me I have a bar magnet standing with its north pole pointing upwards. In the air above the magnet runs a straight wire, fixed in position. When I turn on the current in the wire, the magnet tips over. (In which direction?)

(b) If in the same arrangement the magnet is glued fixed in its position, white the wire is hanging freely, the wire is pushed to the side instead. (In which direction?)

(c) Two parallel wires with currents going the same way are attracted towards each other.

11.6.2. Electric motors harness the above principles to convert electric current to mechanical work.

(a) Explain how this is achieved by means of this arrangement:



(b) Electric motors switch the direction of the current once every half turn. Explain why.

§ 11.6.2. Problems

11.6.3. Fields with curl are non-conservative, i.e., do work along closed paths. The magnetic field has curl. So why can we not solve the world's energy crisis with nothing but a bunch of magnets?

§ 11.7. Electrodynamics

§ 11.7.1. Lecture worksheet

The full versions of Maxwell's equations are:

$$\nabla \cdot \mathbf{E} = \rho / \epsilon_0$$

$$\nabla \times \mathbf{E} = -\dot{\mathbf{B}}$$

$$\nabla \times \mathbf{B} = \mu_0 \left(\mathbf{J} + \epsilon_0 \dot{\mathbf{E}}\right)$$

$$\nabla \cdot \mathbf{B} = 0$$

In the previous section we studied the "static" cases where the time-derivatives were zero. Of the two new terms, the one in the second equation is the most interesting. This equation says that if the magnetic field is moving then it causes a "stir" in the electric fluid with the direction of motion being the axis of stirring. If a conducting wire is placed so that its particles are caught in the stirred-up vortex then a current is generated.

11.7.1. Using the setup of problem 11.6.1b, explain how a current can be created in the wire by moving it back and forth. Which way does the current go?

11.7.2. Draw the field $\dot{\mathbf{B}}$ and explain how the motor in problem 11.6.2 can be "run backwards" to generate electric current from mechanical work. This is how power plants generate electricity.

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§ 11.7.2. Problems

11.7.3. An alternating current is a current that oscillates from one direction to the other like a sine wave. This is the kind of current that comes out of our wall sockets.

(a) Consider the magnetic field $\mathbf{B}$ generated by a current in a coiled wire, and draw the field $\dot{\mathbf{B}}$ resulting if the current is alternating.

(b) Consider a second coiled wire placed next to the first, with its axis aligned with it. Show that the alternating current in the first induces, without a direct connection, a current in a second wire.

This is used for example in electric toothbrush chargers, where an exposed electrical connector would be hazardous. Induction cooking is also based on the same principle, in conjunction with the fact that certain metals produce a lot of resistive heat when a current is induced in them.

11.7.4. Discuss the following passage from Maxwell's paper *On Faraday's lines of force* (1855).

> The student [of electrical science] must make himself familiar with a considerable body of most intricate mathematics, the mere retention of which in the memory materially interferes with further progress. The first process therefore in the effectual study of the science, must be one of simplification and reduction of the results of previous investigation to a form in which the mind can grasp them. The results of this simplification may take the form of a purely mathematical formula or of a physical hypothesis. In the first case we entirely lose sight of the phenomena to be explained ; and though we may trace out the consequences of given laws, we can never obtain more extended views of the connexions of the subject. If, on the other hand, we adopt a physical hypothesis, we see the phenomena only through a medium, and are liable to that blindness to facts and rashness in assumption which a partial explanation encourages. We must therefore discover some method of investigation which allows the mind at every step to lay hold of a clear physical conception, without being committed to any theory founded on the physical science from which that conception is borrowed, so that it is neither drawn aside from the subject in pursuit of analytical subtleties, nor carried beyond the truth by a favourite hypothesis.

§ 11.8. Reference summary

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§ 11.8.1. Line integrals

Work done by a field along a path:

$$\int \mathbf{F}\cdot d\mathbf{s} = \int \mathbf{F}\cdot\dot{\mathbf{x}}\,dt$$

Reversal of direction:

$$\int_{-\gamma} \mathbf{F}\cdot d\mathbf{x} = -\int_{\gamma} \mathbf{F}\cdot d\mathbf{x}$$

where $-\gamma$ is the curve γ traversed backwards.

Fundamental theorem for line integrals:

$$\int_{\mathbf{a}}^{\mathbf{b}} \nabla f\cdot d\mathbf{x} = f(\mathbf{b}) - f(\mathbf{a})$$

where $\mathbf{a}$ and $\mathbf{b}$ are the start and end points of any curve.

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§ 11.8.2. Vector field concepts

$$\begin{aligned}
&\mathbf{F} \text{ is conservative}\\
\iff &\mathbf{F} \text{ is a gradient field } (\mathbf{F} = \nabla f)\\
\iff &\text{ line integrals independent of path}\\
\iff &\text{ line integrals around closed paths} = 0\\
\iff &\text{curl } \mathbf{F} = 0\\
\iff &\mathbf{F} \text{ irrotational}
\end{aligned}$$

In this case $-f$ is called the *potential* function. A special case is the potential energy of a gravitational field.

$$\text{div } \mathbf{F} = \nabla\cdot\mathbf{F} = \text{ generated flux}$$

$$\text{curl } \mathbf{F} = \nabla\times\mathbf{F} = \text{ axis of maximal circulation}$$

$$\text{curl } \mathbf{F}\cdot\mathbf{n} = \text{ circulation around axis } \mathbf{n}$$

$$\text{direction of curl } \mathbf{F} = \text{ axis of maximal circulation}$$

$$|\text{curl } \mathbf{F}| = \text{ intensity of maximal circulation}$$

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§ 11.8.3. Properties of curves

closed	returns to where it started; final point = initial point
simple	does not intersect itself
positively oriented	interior on left as traversed

§ 11.8.4. Vector calculus theorems

Divergence Theorem (Gauss's Theorem):

$$\iiint \operatorname{div} \mathbf{F}\, dV = \iint (\mathbf{F}\cdot\mathbf{n})\, dS$$

Green's Theorem:

$$\oint P\,dx + Q\,dy = \iint \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

Stokes' Theorem:

$$\oint \mathbf{F}\cdot d\mathbf{s} = \iint \operatorname{curl} \mathbf{F}\cdot \mathbf{n}\, dS$$

These theorems are generally not applicable if the functions involved have discontinuities in the region in question.

§ 11.8.5. Electromagnetism

Maxwell's equations:

$$\nabla\cdot\mathbf{E} = \rho/\epsilon_0$$

$$\nabla\times\mathbf{E} = -\dot{\mathbf{B}}$$

$$\nabla\times\mathbf{B} = \mu_0\left(\mathbf{J} + \epsilon_0\dot{\mathbf{E}}\right)$$

$$\nabla\cdot\mathbf{B} = 0$$

Effect of fields on moving particle:

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v}\times\mathbf{B})$$

§ 11.8.6. Problem guide

- Evaluate 2D line integral $\int_\gamma \mathbf{F}\cdot d\mathbf{x} = \int_\gamma P(x,y)\,dx + Q(x,y)\,dy$, where γ is a given curve.

  Is $\mathbf{F} = (P, Q)$ the gradient field (f_x, f_y) of some function $f(x,y)$? If so, use the fundamental theorem for line integrals.

  Are P_y, Q_x, and the interior of γ relatively simple? If so, consider using Green's Theorem.

  Parametrise γ (see §7.5.3), or, if easier, pieces of it one at a time.

  Rewrite the integral purely in terms of t. The integrand can be rewritten directly by plugging in corresponding expression for t from the parametrisation equations. We must also differentiate the parametrisation equations to find dx, dy in terms of dt.

  The integral can now be evaluated as an ordinary single-variable integral.

Evaluate $\int_C xy\,dx + 2y\,dy$ on the curve $y = x^2$ from $x = 0$ to $x = 2$.

$\int_C xy\,dx + 2y\,dy = \int_0^2 x^3 + 4x^3\,dx = [\frac{5x^4}{4}]_0^2 = 20.$

$\int_C (1-y)\,dx + x\,dy$, where C is a curve consisting of three parts: C_1, the arc of $y = 1 - x^2$ from $(1,0)$ to $(0,1)$; C_2, the line segment from $(0,1)$ to $(-1,0)$; C_3 the line segment from $(-1,0)$ to $(1,0)$.

Along C_1 we have $y = 1 - x^2 \Rightarrow dy = -2x\,dx$, so $\int_{C_1}(1-y)\,dx + x\,dy = \int_{C_1} x^2\,dx + x(-2x\,dx) = \int_1^0 -x^2\,dx = [-x^3/3]_1^0 = 1/3$. Along C_2 we have $y = 1 + x \Rightarrow dy = dx$, so $\int_{C_2}(1-y)\,dx + x\,dy = \int_{C_2} -x\,dx + x\,dx = 0$. Along C_3 we have $y = 0$ and $dy = 0$, so $\int_{C_1}(1-y)\,dx + x\,dy = \int_{C_1} dx = [x]_{-1}^1 = 2$. Thus $\int_C (1-y)\,dx + x\,dy = 1/3 + 0 + 2 = 7/3$.

$\int_\gamma \mathbf{F}\cdot d\mathbf{r}$, where $\mathbf{F} = (\sin(y-x), 2xy + \sin(x-y))$ and γ is the curve $y = \sqrt{x}$, $0 \le x \le 1$.

Let γ_1 be the line segment $y = x, 0 \le x \le 1$, and let D be the region enclosed by γ and γ_1. Note that $\gamma_1 - \gamma$ is positively oriented with respect to D. Since $\partial Q/\partial x - \partial P/\partial y = 2y$, Green's Theorem gives $\int_{\gamma_1-\gamma}\mathbf{F}\cdot d\mathbf{r} = \iint_D 2y\,dx\,dy = \int_0^1(\int_x^{\sqrt{x}} 2y\,dy)\,dx = \int_0^1 [y^2]_{y=x}^{y=\sqrt{x}} = \int_0^1 (x - x^2)\,dx = \frac{1}{6}$. A parametrisation of γ_1 is $r(t) = (t,t)$, $0 \le t \le 1$, so $\int_{\gamma_1}\mathbf{F}\cdot d\mathbf{r} = \int_0^1 (0, 2t^2)\cdot(1,1)\,dt = \int_0^1 2t^2\,dt = \frac{2}{3}$. Hence $\int_\gamma \mathbf{F}\cdot d\mathbf{r} = \frac{2}{3} - \frac{1}{6} = \frac{1}{2}$.

$\int_{\gamma_a} yze^{xyz}\,dx + zxe^{xyz}\,dy + xye^{xyz}\,dz$ where γ_a is the curve $x = \cos t$, $y = \sin t$, $z = t$, $0 \le t \le a$.

This is a gradient field with potential function e^{xyz}. Hence the integral is $[e^{x(t)y(t)z(t)}]_{t=0}^{t=a} = e^{a\cos a \sin a} - 1$.

$\int_\gamma \frac{2x}{2x^2+3y^2}\,dx + \frac{3y}{2x^2+3y^2}\,dy$, where γ is the curve $(x(t), y(t)) = (\cos t + e^{\sin t}, e^{\cos t})$, $0 \le t \le \pi$.

Note that $Q'_x = P'_y$ which means that (P, Q) is a gradient field. A potential function is $U(x,y) = \frac{1}{2}\ln(2x^2 + 3y^2)$. Hence $\int_\gamma \frac{2x}{2x^2+3y^2}\,dx + \frac{3y}{2x^2+3y^2}\,dy = U(b) - U(a) = U(0, e^{-1}) - U(2, e) = \frac{1}{2}\ln(\frac{3}{e^2(8+3e^2)})$.

$\int_\gamma x\sin(y^2)\,dx + (x^2 y\cos(y^2) + 2x)\,dy$, where γ is the ellipse $x^2 + 4y^2 = 1$ traversed counter-clockwise.

Let $D = \{(x,y) : x^2 + 4y^2 \le 1\}$. Green's Theorem gives $\int_\gamma x\sin(y^2)\,dx + (x^2 y\cos(y^2) + 2x)\,dy = \iint_D (\frac{\partial}{\partial x}(x^2 y\cos(y^2) + 2x) - \frac{\partial}{\partial y}(x\sin(y^2)))\,dx\,dy = 2\int\int_D dx\,dy = 2\cdot\pi\cdot 1\cdot\frac{1}{2} = \pi$. (Using the formula $A = \pi ab$ for the area of an ellipse.)

$\int_\gamma \frac{2xydx-x^2dy}{x^4+y^2}$, where γ is the curve $y = x^2+2x-4$ from $(1,-1)$ to $(2,4)$.

Since $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y} = \frac{2x^5-2xy^2}{(x^4+y^2)^2}$, we can change the path of integration to e.g. the three line segments from $(1,-1)$ to $(1,0)$ (γ_1), $(1,0)$ to $(2,0)$ (γ_2), and $(2,0)$ to $(2,4)$ (γ_3). Then $\int_\gamma \frac{2xydx-x^2dy}{x^4+y^2} = \int_{\gamma_1}+\int_{\gamma_2}+\int_{\gamma_3} = I_1+I_2+I_3$. We compute each integral separately. $I_1 = \int_{\gamma_1}\frac{2xydx-x^2dy}{x^4+y^2} = \int_{-1}^0 \frac{-dy}{1+y^2} = [-\arctan y]_{-1}^0 = -\arctan 0+\arctan(-1) = -\frac{\pi}{4}$. $I_2 = \int_{\gamma_2}\frac{2xydx-x^2dy}{x^4+y^2} = \int_1^2 0dx = 0$. $I_3 = \int_{\gamma_3}\frac{2xydx-x^2dy}{x^4+y^2} = \int_0^4 \frac{-4dy}{16+y^2} = \left\{\begin{array}{ll} y= & 4t \\ dy= & 4dt\end{array}\right\} = \int_0^1 \frac{-16dt}{16+16t^2} = \int_0^1\frac{-dt}{1+t^2} = [-\arctan t]_0^1 = -\arctan 1+\arctan(0) = -\frac{\pi}{4}$. Altogether: $\int_\gamma = I_1+I_2+I_3 = -\frac{\pi}{4}+0-\frac{\pi}{4} = -\frac{\pi}{2}$.

- Decide whether a given vector field $\mathbf{F} = (P,Q)$ is the gradient (f_x, f_y) of some scalar function $f(x,y)$.

  If such an f exists you may find it as follows. Integrate P with respect to x. The constant of integration could be any function of y, say $C(y)$. The resulting expression is our candidate f. Take its derivative with respect to y. Compare the result with Q. If $C(y)$ can be chosen so that they match then $\mathbf{F} = (P,Q)$ is the gradient field of the resulting f.

  If such an f does not exists, the quickest way to show this may be to check whether $P_y = Q_x$. This would have to be the case for f to exist (equality of mixed partial derivatives).

- Compute surface integral $\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iint_S f\,dS$, where S is a given surface.

  If div $\mathbf{F}$ is relatively simple, consider using the Divergence Theorem. Otherwise:

  Express the surface S (or pieces of it) as the result of letting two variables range between certain bounds (polar, cylindrical, or spherical coordinates may be useful). Express the integrand f and the surface area element dS in terms of these variables. Evaluate as a double integral.

Find the flow of the field $\mathbf{F} = \frac{\mathbf{r}}{r^2}$, where $\mathbf{r} = (x,y,z)$ and $r = |\mathbf{r}|$, out of the region $D = \{(x,y,z) : 2 \le x^2+y^2+z^2 \le 3\}$.

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div $\mathbf{F} = 1/r^2$. Hence the Divergence Theorem gives $\Phi = \iiint_D \frac{1}{r^2}dV = \int_{\sqrt{2}}^{\sqrt{3}}(\iint_{x^2+y^2+z^2=r^2}\frac{1}{r^2}dS)dr = \int_{\sqrt{2}}^{\sqrt{3}} 4\pi dr = 4\pi(\sqrt{3}-\sqrt{2})$.

Find the flow of the field $\mathbf{F} = (-xy^2, x\sin z - y, zy^2)$ into the region $K = \{(x,y,z)\in\mathbb{R}^3 : 1\le x\le 3, 1\le y\le 4, 2\le z\le 4\}$.

With ∂K oriented with outward-pointing normal, the flow into the region is $-\iint_{\partial K}\mathbf{F}dS = -\iiint_K \text{div}\,\mathbf{F}dxdydz = -\iiint_K -1dxdydz =$ Volume$(K) = 12$.

$\int\int_Y \mathbf{F}\cdot\mathbf{N}dS$ where $\mathbf{F} = (x\sin y, x+\cos y, z-1)$ and Y is the part of the ellipsoid $\{x^2+2y^2+4z^2 = 1\}$ with $z \ge 0$ (normal oriented upwards).

Complete Y with the “floor” $Y_0 = \{x^2+2y^2 \le 1, z = 0\}$ (normal pointing downwards). We can then apply the Divergence Theorem to the resulting closed surface: $\iint_Y \mathbf{F}\cdot\mathbf{N}dS + \iint_{Y_0}\mathbf{F}\cdot\mathbf{N}dS = \iiint_K \text{div}\,\mathbf{F}dxdydz$. Two terms of this equation are readily evaluated, namely: $I_0 = \int\int_{Y_0}\mathbf{F}\cdot\mathbf{N}dS = \iint_{x^2+2y^2\le 1}(x\sin y, x+\cos y, -1))\cdot(0,0,-1)dxdy = \int\int_{x^2+2y^2\le 1} 1dxdy = \frac{\pi}{\sqrt{2}}$ and $I_1 = \iiint_K \text{div}\,\mathbf{F} = \iiint_K(\sin y - \sin y+1)dxdydz = \iiint_K 1dxdydz = \frac{1}{2}\frac{4\pi}{3}\cdot 1\cdot\frac{1}{\sqrt{2}}\cdot\frac{1}{2} = \frac{\sqrt{2}\pi}{6}$ (using the formula $V = \frac{4\pi}{3}abc$ for the volume of an ellipsoid with semi-axes a, b, c). Hence the sought integral is $I_1 - I_0 = -\frac{\sqrt{2}\pi}{3}$.

- Evaluate 3D line integral $\int_\gamma \mathbf{F}\cdot d\mathbf{x}$, where γ is a given curve.

  If curl $\mathbf{F}$ is relatively simple, consider using Stokes' Theorem.

  Is $\mathbf{F}$ a gradient field? If so, use the fundamental theorem for line integrals.

  For direct computation, use parametrisation as in 2D case.

Determine the work done by $\mathbf{F} = (3x^2+yz, \cos y+xz, 4e^z+xy)$ along a curve γ from $(0,\frac{\pi}{2},1)$ to $(1,0,0)$.

The work done is independent of the path since curl $\mathbf{F} = (x-x, y-y, z-z) = (0,0,0)$. There is thus a potential function U such that $(\frac{\partial U}{\partial x},\frac{\partial U}{\partial y},\frac{\partial U}{\partial z}) = (3x^2+yz, \cos y+xz, 4e^z+xy)$. Hence $\frac{\partial U}{\partial x} = 3x^2+yz \Rightarrow U = x^3+xyz+G(y,z)$ which gives $\cos y + xz = \frac{\partial U}{\partial y} = xz+\frac{\partial G}{\partial y} \Rightarrow \frac{\partial G}{\partial y} = \cos y$, and hence $G(y,z) = \sin y + H(z)$. Thus we can write $U = x^3+xyz+\sin y+H(z)$, which gives $4e^z+xy = \frac{\partial U}{\partial z} = xy+\frac{\partial H}{\partial z} \Rightarrow H(z) = 4e^z+C$. We are free to choose $C = 0$, so that $U = x^3+xyz+\sin y+4e^z$. The work done is thus $U(1,0,0)-U(0,\frac{\pi}{2},1) = 5-(1+4e) = 4-4e$.

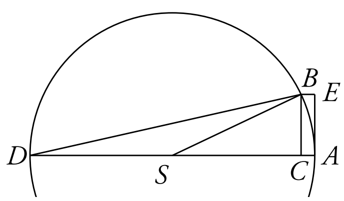
§ 12.1. Newton's moon test

Prerequisites: §1.1.

The moon is kept in its orbit by the earth's gravitational pull, or so your high school textbook told you. How do you know that it is really so? How do you know that the moon is not towed about by a bunch of angels? This question doesn't seem to arise in today's authoritarian classrooms, but Newton gave an excellent answer if anyone is interested.

"That force by which the moon is held back in its orbit is that very force which we usually call 'gravity'," says Newton (*Principia*, Book III, Prop. IV). And his proof goes like this. Consider the hypothetical scenario that "the moon be supposed to be deprived of all motion and dropped, so as to descend towards the earth." If we knew how far the moon would fall in, say, one second, then we could compare its fall to that of an ordinary object such as an apple. Ignoring air resistance, the two should fall equally far if dropped from the same height.

Of course we cannot actually drop the moon, but with the power of infinitesimals we can deduce what would happen if we did. Here is a picture of the moon's orbit, with the earth in the center:



Suppose the moon moves from A to B along a circle with center S in an infinitely small interval of time. If there were no gravity the moon would have moved along the tangent to the circle to some point E instead of to B (BE is parallel to ASD because the time interval is infinitely small so gravity has no time to change direction).

12.1.1. (a) Prove that ABC is similar to ABD.

Thus $AC/AB = AB/AD$, i.e., (diameter of the orbit)/(arc)=(arc)/(distance fallen).

(b) Explain how one can use this relation to find how far the moon falls in one second.

(c) Carry out the calculation. (You will need to look up some parameters of the moon's motion.)

(d) Compute how far the moon would fall if dropped at the surface of the earth, where gravity is 60^2 times stronger since the moon is 60 earth radii away.

(e) Is the result the same as for a falling apple? Hint: consult §A.7.12 if your physics is rusty.

§ 12.2. The rainbow

Prerequisites: §2.1.

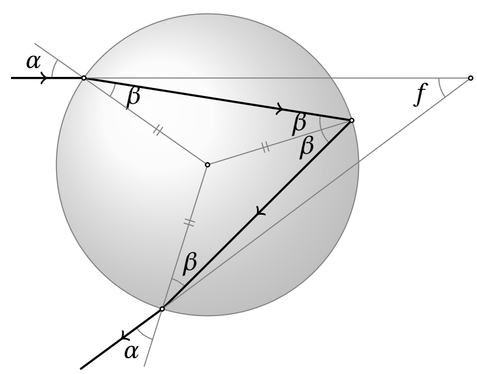
Rainbows are the result of light from the sun "bouncing" through raindrops. In this problem we shall show that raindrops tend to concentrate the rays of the sun—almost like a magnifying glass—at one particular outgoing angle, namely about 42°.

From this we can infer the shape and position of the rainbow as follows. Imagine that you have the sun in your back and that there is a wall of raindrops some distance ahead of you. Since the sun is so far away its rays may be considered parallel. For any given raindrop, picture the ray from the sun that hits it, and picture the line of sight from your eye to the raindrop. Consider the angle at which these lines meet.

12.2.1. Characterise the set of all raindrops for which this angle is 42°. Hint: consider the ray from the sun that passes through your head for reference purposes.

12.2.2. You never see rainbows at mid-day in mid-summer, no matter how much it rains. Why not?

Now we must understand where the number 42° comes from. At any given contact surface between air and water, some light is reflected and some light is refracted, in the manner described in problems 2.1.5 and 2.1.7 respectively. The light that constitutes the rainbow is the light that refracted into the raindrop, reflected at the back of it, and refracted back out gain through the front. In the plane containing the rays from the sun, the midpoint of the drop, and the observer, this looks as follows:



We are assuming that raindrops are spherical.

12.2.3. (a) Explain why all β's in the figure are equal, and why the two α's are equal.

(b) Express the ray's net reflection f as a function of α and β. Hint: it may be easier to first consider $\pi - f$, which is the angular difference between the incoming and outgoing angle, i.e., the "total amount the ray has turned."

(c) Express β in terms of α using the law of refraction. The ratio of the speed of light c_1 in air and c_2 in water is about $c_1/c_2 = 4/3$.

(d) Express f purely in terms of α and find its derivative.

(e) Set the derivative equal to zero and solve for α.

(f) Explain why we should expect a concentration of light at this angle.

(g) $\star$ From an airplane it is possible to see a full circle rainbow. Explain.

You may be disappointed that we did all this work about rainbows, yet said nothing about its colours. Indeed, rainbows are arguably a light-concentration phenomenon more than a colour phenomenon, scientifically speaking. We are often so distracted by the beautiful colours that we fail to notice that the rainbow is also significantly brighter than the surrounding sky. Black-and-white photos can help us see this more clearly. The colours of the rainbow result from the fact that light rays of different colours have slightly different refraction angles.

§ 12.3. Addiction modelling

Prerequisites: §2.5.

McPhee (*Formal Theories of Mass Behavior*, 1963) offers the following mathematical model of what he calls "the logic of addiction." Let C stand for consumption, scaled so that $C = 1$ corresponds to "normal maximum" (e.g., in the case of alcohol consumption, a few glasses of wine or so). The change in C depends on the stimulation s to consume, which is proportional to the remaining room $1 - C$ to consume up to normal satisfaction, and the resistance r, which is proportional to C. Thus $C' = s(1-C) - rC$. But drinking also has an intrinsic effect i, which increases stimulation and weakens resistance ("one drink leads to another", p. 187). So $C' = (s+i)(1-C) - (r-i)C$.

12.3.1. Extremum occurs when $C' = 0$, i.e. when $C = \boxed{\quad\quad\quad}$.

12.3.2. A normal drinker has s=moderate, i=moderate, and r=moderate, giving a $\boxed{\quad\quad\quad}$ extremum consumption. A typical alcoholic may have s=big, i=big, r=–small, giving a $\boxed{\quad\quad\quad}$ extremum consumption.

The model also explains a different type of behaviour among some alcoholics, illustrated by this quotation from a sociology study:

> I don't drink every day and I'll go weeks without drinking. Then when I'm on top of the world and everything is going swell, I flop like a dope. What causes it I don't know! When I really should get drunk is after I've sobered up and I've got all kinds of problems. When I start drinking is, when I don't have any. Everything looks fine and rosy and everything. . . . I went in with him for no reason—not planned—and had a couple. The next thing I knew I woke up in a hotel room. (p. 210)

12.3.3. Argue that this type of alcoholic seems to have i=big and r=–small like a regular alcoholic, but has s=–small.

12.3.4. Explain his behaviour in terms of the differential equation.

McPhee concludes: "If other models confirm that this is a quite general consequence, then it means not to keep on looking endlessly for behavioral 'reasons' for the alcoholic's loss-of-control phenomenon. Rather, we might have to face the awful truth that what the alcoholic has been saying for years is the truth: no 'reason' is really necessary." (p. 213)

§ 12.4. Estimating $n!$

Prerequisites: §5.

The factorial function $n!$ is intractable to compute and grasp for large n since its definition involves compounded multiplications that quickly grow beyond bounds. Therefore it is useful to have a closed formula approximating $n!$. We can find such a formula as follows.

12.4.1. (a) Decompose $\ln(n!)$ into a sum.

(b) Interpret this sum geometrically as a sum of the areas of rectangles with base 1 along the x-axis.

(c) Estimate the area from below by an integral.

(d) Estimate the area of the pieces left out by the integral approximation. Hint: align these pieces by sliding them horizontally to the y-axis.

(e) Deduce an estimate for $n!$.

(f) Check the accuracy of this estimation for a few large values of n.

A slightly better estimate may be obtained as follows. First we note that $n!$ can be expressed as an integral.

12.4.2. Show that

$$\int_0^\infty x^n e^{-x}\,dx = n!$$

for any integer n.

We now apply a standard trick for estimating integrals: the "full width at half maximum" rule, which says that the area under a graph with a peak in it may be approximated by multiplying the height of the peak with the width of the graph at the y-value corresponding to half the maximum value. This guesstimation trick can be of use as a last resort when one cannot evaluate an integral analytically, or even when the graph is only known visually rather than as a formula, which can happen for instance if its is the output of a physical measuring device.

12.4.3. To estimate $n!$ using the "full width at half maximum" heuristic, let $f(t)$ denote the integrand of the integral in problem 12.4.2.

(a) Sketch a rough graph of $f(t)$ from $t = 0$ to $t = \infty$. Hint: first consider the graphs of the factors separately.

(b) Find the maximum, $f_{\max}$, of $f(t)$ on this interval.

We now need to find the t-values corresponding to half maximum, $f(t) = \frac{1}{2}f_{\max}$. We cannot directly solve this equation analytically. But we can approximate the solutions as follows.

(c) Apply logarithms to both sides of this equation.

(d) Replace the left hand side by the first two non-zero terms of its power series expansion about its maximum.

(e) Find the desired t-values.

(f) Conclude the estimation of $n!$.

(g) Check the accuracy of this estimation for a few large values of n.

§ 12.5. Wallis's product for π

Prerequisites: §5.

Wallis's product expression for π says:

$$\frac{\pi}{2} = \frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdot\frac{8}{7}\cdot$$

12.5.1. Argue that Wallis's product follows from the expression in problem 5.1.10a. Hint: plug in a specific value for x.

12.5.2. The following is an alternative proof of Wallis's product expression for π.

(a) Evaluate $\int_0^{\pi/2}(\sin x)^{2m}\,dx$.

(b) Evaluate $\int_0^{\pi/2}(\sin x)^{2m+1}\,dx$.

(c) Divide the two results to find an expression for $\frac{\pi}{2}$.

The expression contains the ratio of the two integrals.

(d) What needs to be the limit of this ratio as $m\to\infty$ for Wallis's expression to follow?

To establish this limit we need to estimate the ratio from above and from below.

(e) Find one of these estimates by considering what multiplying by $\sin x$ does to the values of a function on the interval $(0,\pi/2)$.

(f) Find the other estimate by comparing $\int_0^{\pi/2}(\sin x)^{2m+1}\,dx$ and $\int_0^{\pi/2}(\sin x)^{2m-1}\,dx$.

(g) Conclude the proof.

(h) ⋆ † Can you see why some people prefer this proof to that of problem 12.5.1?

§ 12.6. Power series by interpolation

Prerequisites: §5.

12.6.1. In problem 5.1.1 we argued that a first-degree polynomial can be made to go through two points, a second-degree polynomial to go through three points, and so on. Indeed, Newton constructed such a polynomial, namely a polynomial $p(x)$ which takes the same values as a given function $y(x)$ at the x-values $0, b, 2b, 3b, \ldots$. Here is the construction. First, our polynomial $p(x)$ is supposed to have the same value as the given function $y(x)$ when $x = 0$. Therefore we should start by setting $p(x) = y(0)$. Next, we want $p(x)$ to take the same value as $y(x)$ when $x = b$. This is easily done by setting

$$p(x) = y(0) + \frac{x}{b}\big(y(b) - y(0)\big).$$

This polynomial obviously agrees with $y(x)$ when x is 0 or b. Now we need to add a quadratic term to make it agree when x is $2b$ as well. We want the new term to contain the factor $(x)(x-b)$ because then it will vanish when x is 0 or b, so our previous work will be preserved. If we set $x = 2b$ in the piece of $p(x)$ that we have so far we get

$$p(2b) = y(0) + 2y(b) - 2y(0) = 2y(b) - y(0).$$

So we want the quadratic term to have the value $y(2b) - 2y(b) + y(0)$ at $x = 2b$.

(a) Use this reasoning to write down a second-degree polynomial $p(x)$ that agrees with $y(x)$ when x is 0, b or $2b$. (Keep the factor $(x)(x-b)$ as it is, i.e., do *not* reduce the expression to the form $p(x) = A+Bx+Cx^2$.)

In the same manner we could add a cubic term to make $p(x)$ agree with $y(x)$ at $x = 3b$, and so on.

The formula becomes more transparent if we introduce the notation $\Delta y(x)$ for the "forward difference" $y(x+b) - y(x)$, and $\Delta^2 y(x)$ for the forward difference of forward differences $\Delta y(x+b) - \Delta y(x)$, etc., so that

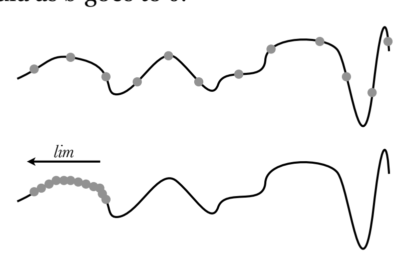
$$\begin{aligned}
\Delta y(0) &= y(b) - y(0)\\
\Delta^2 y(0) &= \Delta y(b) - \Delta y(0) = y(2b) - 2y(b) + y(0)\\
\Delta^3 y(0) &= \Delta^2 y(b) - \Delta^2 y(0) = y(3b) - 3y(2b) + 3y(b) - y(0)\\
&\vdots
\end{aligned}$$

(b) Rewrite your formula for $p(x)$ using this notation, and then extend it to the third power and beyond "at pleasure by observing the analogy of the series," as Newton puts it.

(c) Show that Taylor's series

$$y(x) = y(0) + y'(0)x + \frac{y''(x)}{2!}x^2 + \frac{y'''(x)}{3!}x^3 + \cdots$$

is the limiting case of Newton's forward-difference formula as b goes to 0.



This is indeed how Taylor himself proved his theorem in 1715. The nowadays more common method of finding the series by repeated differentiation (as in problem 5.1.2) was used by Maclaurin in 1742.

§ 12.7. Path of quickest descent

Prerequisites: §2.1, §6.3, §7.4.

Consider a children's playground slide, and suppose it is covered with ice so that there is no friction. Which shape of the slide will take you as quickly as possible from the starting point to the endpoint? This is the brachistochrone problem, or the problem of quickest descent. Johann Bernoulli solved this problem in 1697 by means of the following ingenious optical analogy.

Recall first from problem 6.3.1 that the speed of a particle sliding down a ramp under the influence of gravity is proportional to the square root of the vertical distance covered. Let us therefore imagine slicing the space to be covered into thin horizontal strips. In each strip the speed may be considered constant, but from strip to strip it varies in the manner just outlined. But this reminds us of problem 2.1.7, where we found how to choose the quickest path in cases involving two mediums of different speed.

12.7.1. Use this information to obtain a differential equation for the desired curve as follows.

(a) What does problem 2.1.7 imply about the multi-layer case? Hint: apply the result once for every layer-crossing, and combine the results.

(b) Use a limit argument to express the result in terms of the angle β and velocity v considered as continuous functions, rather than a step-by-step process.

(c) Express the result in terms of x, y, y', and use this to form the differential equation.

12.7.2. Check that an appropriately oriented cycloid satisfies the differential equation. (Cf. §7.3.3.)

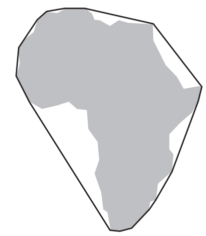
§ 12.8. Isoperimetric problem

Prerequisites: §10, §11.3.

The isoperimetric problem asks: Among all figures with a given perimeter L, which encloses the greatest area A? The solution is the circle, which can also be expressed as an inequality: $L^2 - 4\pi A \geq 0$.

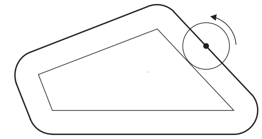
We are going to give a proof of this fact inspired by problem 3.3.3: to wit, we could get information about how much area a figure covers by considering how good it is at being hit by needles.

But before we get to our proof we must explain two preliminary details. First we note that a solution to the isoperimetric problem must be "convex," meaning that is has no cavities, so to speak. Africa is not convex and its "convex hull," the least convex figure containing it, is obtained by snapping a rubber band around it and taking that as the new figure:



12.8.1. Explain why we can rule out non-convex figures as possible solutions to the isoperimetric problem.

Second, we need the concept of a "parallel curve." Take a convex figure and roll a circle on its boundary. The curve that is being traced out by the midpoint of the circle is what we call a parallel curve.

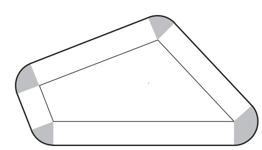


Or, if you prefer, dip the circle in paint and have it bounce around with its midpoint trapped inside the figure. It will then paint the parallel figure.

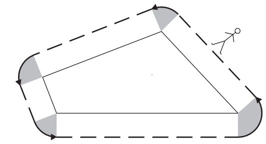
12.8.2. Prove that the area enclosed by the parallel curve is $A + Lr + r^2\pi$, where A is the area of the original figure and r is the radius of the circle used to construct the parallel curve.

Hint 1: First prove the result for polygons, and then infer the general case.

Hint 2: The new area is the old area plus the area of the strips plus the area of the shaded pieces.

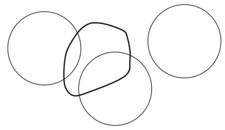


This is all easy to calculate once we realise that the shaded pieces fit together to form a disc, because the sum of the angles of the shaded pieces is how much we turn when we walk around the figure once.



Now we are ready for our proof of the isoperimetric inequality.

12.8.3. Consider a figure of perimeter L. Now, instead of throwing needles at it, throw circles. And use circles of the same perimeter as the figure, i.e., with radius $r = \frac{L}{2\pi}$.



We hope that this will give us information about the area of our figure—the more circles intersecting it, the greater the area—and we capture this intuition by considering the weighted area of the plane

$$\iint_{\mathbb{R}^2} \#\text{intersections } dx\,dy$$

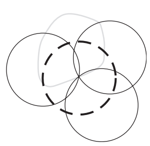
By this we mean that, to determine the weight of an infinitesimal square, we should put the midpoint of one of our circles there and count how many times it intersects the boundary of our figure.

Only the area at most r away from the figure will be given nonzero weight, that is, the area inside the r outer parallel figure. Also, if we drop one of our circles with its midpoint anywhere inside this parallel figure, then it will intersect the boundary of our figure at least twice (there will be no zero-area in the middle, for the circle cannot fit inside the figure).

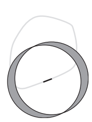
(a) Therefore

$$\iint_{\mathbb{R}^2} \#\text{intersections } dx\,dy \geq \iint_{\text{the parallel figure}} 2\,dx\,dy = 2\left(\boxed{}\right)$$

Now we will calculate the weighted area by brute force. Consider a point on the boundary of our figure. What circles intersect this point? It is of course the circles with midpoints r away from it.



So consider an infinitesimal ds-segment of the boundary of our figure. What circles intersect this segment? This must then be circles with midpoints that lie in two strips like this:



The strips are $2r$ high and have constant width ds, so the segment contributes a total amount of weighted area $4r\,ds$.

(b) Summing these up,

$$\iint_{\mathbb{R}^2} \#\text{intersections } dx\,dy = \oint_{\text{the boundary}} 4r\,ds = \boxed{}$$

So now we have two expressions for the weighted area that we can combine.

(c) Deduce from this the isoperimetric inequality.

§ 12.9. Isoperimetric problem II

Prerequisites: §11.

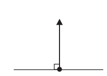
This section gives an alternative approach to the isoperimetric problem of §12.8. It is based on the following physical intuition. Where the earth is perfectly spherical, we can balance a stick by putting it down perpendicular to the ground. Where the earth is not spherical, on the side of a hill or so, putting down a stick perpendicular to the ground will cause it tip over. Let's agree that this experience convinces us that if we were stranded on a non-spherical planet we could always find places where putting a stick perpendicular to the ground would cause it to tip over. So it is only for the sphere that gravity always acts in the direction of the normal of the surface. Let's agree also that this still holds when the universe is flat—when planets are plane figures. To capture the mathematics of gravity, we should think of this in terms of vector fields, and to make it easier we consider the negative of the gravitational field—just take the ordinary gravitational field and multiply the vectors by -1, pretending that we are in a dual universe where gravity pushes instead of pulls. Now, take all figures of a given area and fill them with cement. Then they all produce as much negative gravity. This negative gravity flows out from the figure, but only for the circle does it always flow out along the normal. For any other figure, the negative gravity flows out askew, which we feel is an inefficient use of perimeter. So perhaps, then, this will force the perimeter to be greater than that of the circle of the same area.

So how do we capture these ideas to make a proof? Well, the amount of negative gravity being produced can be calculated by summing what is flowing out

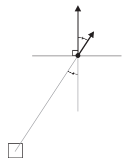
$$\int_{\text{the boundary}} \text{flow along the normal } ds$$

We wish to show that a non-circular figure spreads the outflow over greater perimeter. This would follow at once if we could show that the flow along the normal from a point on a circle is greater than the flow along the normal from any point on the boundary of any figure of the same area. That would mean that not only does a non-circular figure dilute the outflow at some places but also that it cannot concentrate the flow somewhere else. Indeed this is so, as we shall now calculate.

Fix a point and its normal:

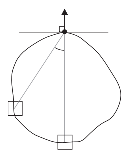


We now have a given area to distribute under this point to make the flow of negative gravity in the direction of the normal as large as possible. How will an arbitrary infinitesimal square contribute if we choose to include it?



The force it causes to act on the point is proportional to the area divided by the distance (in the manner of problem 11.1.3). Then the angle θ the force vector makes with the normal determines what part of the force that acts in the direction of the normal: $\cos\theta$.

Suppose we have found the optimal shape of the area. Walk straight ahead to the last infinitesimal square we included in that direction, and do the same thing for an angle θ.



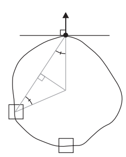
Say that the straight-ahead square is at a distance of r_0, and the other square r_θ.

12.9.1. (a) Argue that these two squares contributes as much negative gravity flow in the direction of the normal.

(b) Express this as an equation.

(c) Infer that the figure is a circle.

Hint:



§ 13.1. Curvature

§ 13.1.1. Lecture worksheet

In this section we shall investigate a way of measuring how curved a given curve is at a given point. We quantify this by means of the *curvature* κ, which we define as the rate at which the tangent to the curve is turning as we move along the curve. If we let ϕ denote the angle the tangent makes with the x-axis, we thus define $\kappa = d\phi/ds$.

13.1.1. Express the curvature κ in terms of x and y.

Hint: Recall the geometrically immediate facts that $\tan\phi = \frac{dy}{dx}$ and $ds^2 = dx^2 + dy^2$.

13.1.2. What curves have curvature zero?

13.1.3. Compute the curvature of a circle of radius R.

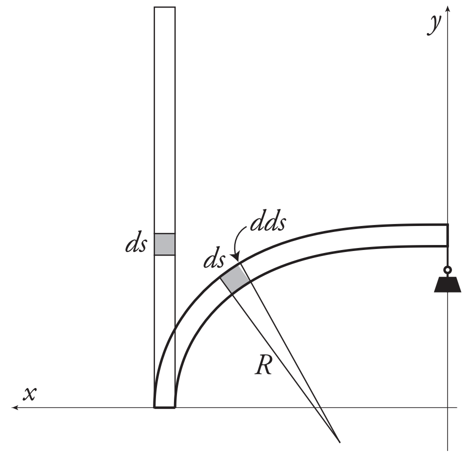
13.1.4. *Radius of curvature.* Another way of characterising curvature is this. Imagine a curve and select a point on it. Now draw the normal to the curve at this point. Then pick a second point infinitesimally close to the first, and draw the normal at this point as well. Let R be the distance from the original point to the point where these two normals intersect; this is called the radius of curvature.

(a) Using problem problem 13.1.3, what is the relation between κ and R in the case of a circle?

(b) † Argue that this relation holds generally for any curve. Hint: approximate the curve locally by a circle.

§ 13.1.2. Problems

13.1.5. An example showing how the idea of radius of curvature occurs naturally in physical problems is the *elastica*: the shape of a bent elastic beam. The simplest case is that of a beam fixed vertically at its foot and weighed down with such a weight that the tangent at its endpoint is horizontal.



We can obtain a differential equation for the elastica as follows. The outer side of the elastic beam is thought of as consisting of springs, while the inner side is taken to maintain fixed length. When the beam is bent by the weight, the spring in a given position extends by *dds*. The extension is proportional to the force acting on the spring (this is the so-called Hooke's law of springs; cf. problem 6.4.3). This force is found by thinking of the remainder of the beam as the arm of a lever, through which the weight acts. Since the force of the weight is vertical, the horizontal component of the beam is the effective lever arm. Thus the extension *dds* is proportional to the horizontal position x. On the other hand it is evident that the extension is inversely proportional to the radius of curvature R defined by the two normals drawn.

(a) Obtain a differential equation (expressed in terms of x, y and their derivatives) for the elastica by equating these two expressions for the extension.

§ 13.1.3. Reference summary

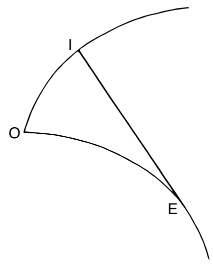
Curvature: $\kappa = \frac{y''}{(1+(y')^2)^{3/2}}$ = rate at which angle of tangent is turning.

Radius of curvature: $R = 1/|\kappa|$ = radius of circle best approximating curve at given point = distance to point of intersection of two successive normals to the curve.

§ 13.2. Evolutes and involutes

§ 13.2.1. Lecture worksheet

In this figure, OI is the path traced by the end of a string unwound from OE:



We say that OE is the *evolute* of the *involute* OI.

13.2.1. Argue that the length IE can be characterised in terms of curvature (§13.1).

This gives us a way of determining the evolute given the involute. Suppose the involute OI is given parametrically as $(x(t), y(t))$.

13.2.2. Find expressions for the coordinates (X, Y) of the point E in terms of x, y, $\dot{x}$, $\dot{y}$, κ, R.

To make concrete use of this we need parametric expressions for curvature.

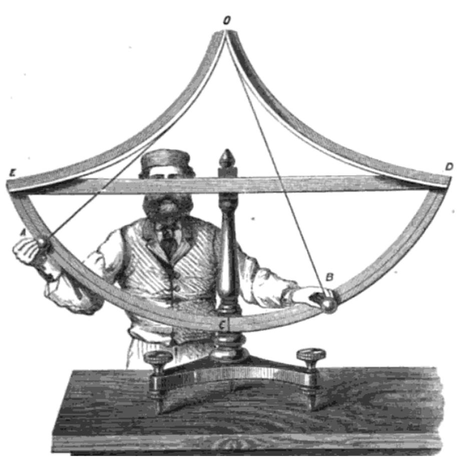
13.2.3. Express the radius of curvature R in terms of $\dot{x}$, $\ddot{x}$, $\dot{y}$, $\ddot{y}$ in one of the following two ways.

(a) From geometrical first principles, as in §13.1.

(b) By expressing y' and y'' parametrically and substituting into the formula found in §13.1. Hint: for y'', use chain and quotient rules.

13.2.4. Show that the evolute of a cycloid is another cycloid congruent to the first.

13.2.5. Argue that if you wish to make a pendulum bob follow a cycloidal path you should have it swing between cycloidal "cheeks." Problem 7.4.5 shows why this is important.



§ 13.2.2. Problems

13.2.6. Evolutes and involutes can also be used to find the arc lengths of curves.

(a) Argue that, in the above definition of evolute and involute, the arc OE = the line segment EI.

(b) Use this to find the length of a cycloidal arc.

§ 13.3. Fourier series

§ 13.3.1. Lecture worksheet

Fourier series can be seen as a generalisation of the idea of the scalar product. Consider on the one hand the ordinary vectors $\mathbf{v}_1 = (1,1)$ and $\mathbf{v}_2 = (1,-1)$, and on the other hand the functions $\mathbf{w}_1 = \sin(x)$, $\mathbf{w}_2 = \sin(2x)$, $\mathbf{w}_3 = \sin(3x)$, ..., which we shall think of as a kind of "vectors" as well. The vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ form a basis for $\mathbb{R}^2$, as we say, i.e., any vector in the plane can be written in the form $a\mathbf{v}_1 + b\mathbf{v}_2$ for some real numbers a and b. To express some vector $\mathbf{u}$ in this form we go through the following steps:

- "Normalise" the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ by dividing each vector by its length. Call the resulting unit vectors $\mathbf{v}_1^*$ and $\mathbf{v}_2^*$.
- Check that $\mathbf{v}_1^*$ and $\mathbf{v}_2^*$ are orthogonal (i.e., perpendicular) using scalar products.
- "Project" $\mathbf{u}$ onto $\mathbf{v}_1^*$ and $\mathbf{v}_2^*$ using scalar products.
- The fact that $\mathbf{v}_1^*$ and $\mathbf{v}_2^*$ are orthogonal ensures that $\mathbf{u}$ is decomposed into independent components, so adding up the two projections gives $\mathbf{u} = a\mathbf{v}_1^* + b\mathbf{v}_2^*$, as sought.

13.3.1. Carry out these steps for the vector $\mathbf{u} = (4,1)$. Include a picture showing $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_1^*$, $\mathbf{v}_2^*$, $\mathbf{u}$, $a\mathbf{v}_1^*$ and $b\mathbf{v}_2^*$.

We are now going to generalise this idea to "spaces" where vectors are functions. To do so we essentially define the scalar product of two functions $f(x)$ and $g(x)$ as $\int f(x)g(x)\,dx$ by analogy with the usual scalar product $\sum f_i g_i$. More precisely, when dealing with sine functions we are going to focus on only one of their periods, so we define the scalar product to be $\int_{-\pi}^{\pi} f(x)g(x)\,dx$.

13.3.2. Now let us try to carry out the same steps as above in this new setting.

(a) Find the normalised vectors $\mathbf{w}_1^*$, $\mathbf{w}_2^*$, $\mathbf{w}_3^*$, ... (above you may have used the Pythagorean theorem for this step, but now you will need to formulate the normalisation purely in terms of scalar products, using the usual relation between scalar products and lengths).

(b) Check that these vectors are "orthogonal" (in the sense of scalar products).

(c) Use the scalar product method to "project" $\mathbf{u}_1 = x$ onto $\mathbf{w}_1^*$, $\mathbf{w}_2^*$, $\mathbf{w}_3^*$, $\mathbf{w}_4^*$, $\mathbf{w}_5^*$, $\mathbf{w}_6^*$.

This gives you real numbers a_1, a_2, a_3, a_4, a_5, a_6, such that

$$\mathbf{u}_1 \approx a_1\mathbf{w}_1^* + a_2\mathbf{w}_2^* + a_3\mathbf{w}_3^* + a_4\mathbf{w}_4^* + a_5\mathbf{w}_5^* + a_6\mathbf{w}_6^*$$

In other words, you have approximated the function x by a sum of sine functions with various coefficients.

(d) Plot the function $a_1\mathbf{w}_1^* + a_2\mathbf{w}_2^* + a_3\mathbf{w}_3^* + a_4\mathbf{w}_4^* + a_5\mathbf{w}_5^* + a_6\mathbf{w}_6^*$ and note that it somewhat approximates x on the interval $(-\pi, \pi)$.

(e) Use the same method to obtain an approximation for $\mathbf{u}_2 = x/|x|$.

(f) Plot $x/|x|$ and its approximating function in the same graph.

(g) ⋆ Why can we not obtain power series approximations in a similar way (i.e., by "projecting" a function onto x, x^2, x^3, ... with the integral scalar product to find its Taylor coefficients)? Explain in terms of the "vectors" x, x^2, x^3, ..., and give an analogy with vectors in $\mathbb{R}^2$ that illustrates the problem.

This method of approximating functions by trigonometric series has an interesting physical meaning in terms of sounds. Functions of the form $\sin(nx)$ are "pure notes"—they describe the vibrations of tuning forks. The fact that any function can be approximated by such trigonometric functions thus corresponds to the fact that any sound—say, for example, Beethoven's ninth symphony (including the chorus!)—can be produced by nothing but tuning forks. The numbers a_n are telling us how hard to strike each tuning fork. There is also a converse physical meaning: a tuning fork will start vibrating spontaneously whenever its tone is being played ("sympathetic resonance"). The human ear is based on this principle. It consists of many hairs that are in effect tuning forks sensitive to a particular frequency. When a sound arrives which includes this frequency as a component, the hair will vibrate with a strength a_n determined by the strength of that tone in the sound heard. Thus the information sent to the brain is the coefficient a_n, so you have been computing scalar products all your life, as it were, whether you were aware of it or not.

§ 13.4. Hypercomplex numbers

§ 13.4.1. Lecture worksheet

Generalising complex numbers into higher dimensions is problematic. Three-dimensional numbers are in a sense impossible, as the following argument shows.

13.4.1. Consider the set of all points, seen as "hypercomplex numbers," in three-dimensional space with the usual (vector) notions of magnitude and addition. Suppose there is some way of multiplying such numbers which satisfies the usual laws of algebra. We shall now show that these assumptions are contradictory.

For the usual laws of algebra to hold there must be a multiplicative identity, call it $\mathbf{1}$, and since we are in three dimensions there must be two other numbers of unit length, $\mathbf{i}$ and $\mathbf{j}$, such that these three numbers are all mutually perpendicular.

(a) Show that $|\mathbf{1} - \mathbf{i}^2| = 2$.

(b) Therefore what is $\mathbf{i}^2$? And $\mathbf{j}^2$? Hint: What is its distance to $\mathbf{0}$? To $\mathbf{0}$?

(c) Show that if $|\mathbf{u}| = 1$ then the angle between $\mathbf{v}$ and $\mathbf{w}$ is the same as that between $\mathbf{uv}$ and $\mathbf{uw}$. Hint: first show that the mapping $\mathbf{z} \mapsto \mathbf{uz}$ is distance-preserving.

(d) Infer from this that $\mathbf{ij}$ is perpendicular to $\mathbf{1}$.

(e) Prove that $(\mathbf{ij})^2$ must be both $\mathbf{1}$ and $-\mathbf{1}$, a contradiction.

(f) Note that the contradiction is avoided if we sacrifice commutativity and make $\mathbf{ij} = -\mathbf{ji}$.

So if we continue this line of thought and try to salvage what we can from the wreckage, then we must try to figure out what number this mysterious $\mathbf{ij}$ can be. So far we know only that it is perpendicular to $\mathbf{1}$ and equal to $-\mathbf{ji}$.

13.4.2. (a) Show that it is also perpendicular to $\mathbf{i}$ and $\mathbf{j}$.

(b) Conclude that $\mathbf{ij}$ must go off in a "fourth dimension."

So let us write $\mathbf{ij} = \mathbf{k}$ and consider the set of all four-dimensional numbers $a\mathbf{1} + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$. The set of all such numbers may be called quaternions (owing to their fourness) and denoted $\mathbb{H}$ (after their discoverer, Hamilton, since $\mathbb{Q}$ is already taken).

13.4.3. (a) Show that $\mathbb{H}$ is spatially closed, i.e., that any product involving $\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k}$ can be expressed as a linear combination of $\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k}$ without the need for any fifth dimension.

(b) Show that the "multiplication table"

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -\mathbf{1}$$

contains all the information needed to reduce any product of numbers in $\mathbb{H}$ down to standard form.

Now that the arithmetic of $\mathbb{H}$ is well defined it would be straightforward to go back and check that it satisfies all the various properties we desired of hypercomplex numbers except for the commutativity of multiplication. So by "following our nose" from our failure with three-dimensional numbers we arrived at the next best thing.

§ 13.4.2. Problems

13.4.4. Quaternions once rivalled vectors, and as the following problem shows they are in some ways almost equivalent.

(a) Show that if you take the quaternion product $(a\mathbf{i} + b\mathbf{j} + c\mathbf{k})(d\mathbf{i} + e\mathbf{j} + f\mathbf{k})$ and discard all its real terms then the result is the same as that of the corresponding vector product $(a\mathbf{i} + b\mathbf{j} + c\mathbf{k}) \times (d\mathbf{i} + e\mathbf{j} + f\mathbf{k})$. Hint: brute-force calculation is not the only way of seeing this.

(b) Show that the scalar part discarded above is the negative of the corresponding scalar product.

13.4.5. The following argument gives a simpler and independent proof that there can in any case not be any very simple formula for three-dimensional multiplication.

Surely any multiplication worthy of its name should at the very least satisfy the multiplicative property of the magnitude: $|z_1 z_2| = |z_1||z_2|$. Written out in terms of the components of the numbers this says

$$\sqrt{(a^2+b^2+c^2)}\sqrt{(d^2+e^2+f^2)} = \sqrt{(\alpha^2+\beta^2+\gamma^2)}$$

or by squaring

$$(a^2+b^2+c^2)(d^2+e^2+f^2) = \alpha^2+\beta^2+\gamma^2$$

where α, β, γ are some functions of a, b, c, d, e, f.

(a) In the case of ordinary complex numbers (i.e., $c = f = \gamma = 0$), what is α and β?

These functions produce integer output for integer input. Assume that this holds for α, β, γ in three dimensions as well.

(b) Show that then 15 must be a sum of three squares (of integers).

(c) Show that it is not.

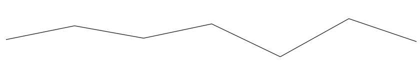
§ 13.5. Calculus of variations

§ 13.5.1. Lecture worksheet

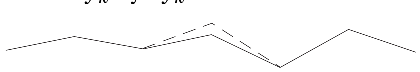
Consider the problem of finding a function $y(t)$ that extremises an integral that depends on it,

$$\int f(y, \dot{y}, t)\, dt.$$

$y(t)$ is an extremum if wiggling it a little causes no change in the value of the integral, just as, in ordinary calculus, x is an extremum of $f(x)$ if wiggling it a little causes no change in f. For this purpose, split the t-axis into infinitesimal segments $dt/2$, and assume that $y(t)$ is linear on these intervals.



Then $y(t)$ and $\dot{y}(t)$ are determined by the value of $y(t)$ at the break points, let's call them $y_1, y_2, y_3, \ldots$. Let the point y_k vary, so that we increase y_k by dy_k.



What happens to the integral?

13.5.1. First there is the direct change caused by the change in y_k. This change causes the integrand to increase by

$$dy_k \left(\frac{\partial f}{\partial y_k}\right)$$

on this interval dt, and thus the integral by

(a) $\boxed{}$

Then there are the changes caused by the changes in the derivatives on the sides of y_k, call them $\dot{y}_k$ and $\dot{y}_{k+1}$. When y_k changes by dy_k, $\dot{y}_k$ changes by dy_k/dt, so it causes the change in the integrand of

(b) $\boxed{}$

which translates into a change in the integral of

(c) $\boxed{}$

since this change only applies for half the interval dt. Similarly, $\dot{y}_{k+1}$ changes by $-dy_k/dt$ and causes the change

(d) $\boxed{}$

So the equation for the change being zero altogether is

(e) $\boxed{}$

This is the equation when a single value y_k is altered. In general y may be altered in any manner, meaning that any number of y_i's may be altered. To obtain the criterion for y being stationary in this general case we must therefore sum the previous equation over all k. In doing so one finds that each term $\frac{\partial f}{\partial \dot{y}_i}$ is counted twice, which cancels the 2 in the denominator and leaves simply

$$\frac{\partial f}{\partial y} - \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{y}}\right) = 0.$$

This is the Euler–Lagrange equation of the calculus of variations.

§ 13.5.2. Problems

13.5.2. In problem 6.3.2 we found an integral expression for the time taken to slide down a ramp as a function of its shape $y(x)$.

(a) Use the Euler–Lagrange equation to find a differential equation for the path of quickest descent.

(b) As in problem 12.7.2, check that a cycloid solves this differential equation.

13.5.3. If we can write Newton's equation $F = ma$ in the form of the Euler–Lagrange equation we can infer an integral-variational formulation of the basic equation of motion. Indeed, this is easily done: just let $f = T + U$, where $T = mv^2/2$ is the kinetic energy and U is the potential function. In other words, U is a function whose derivatives give the forces, as in $U = -mgy$ for gravity, giving constant gravitational acceleration $m\ddot{y} = \frac{\partial U}{\partial y} = -mg$.

(a) Show that in this case the Euler–Lagrange equation reduces to $F = ma$.

(b) In other words, to find the trajectory of a particle we used to solve the differential equation $F = ma$, but now we can instead determine it by the equivalent problem: $\boxed{}$. This is the so-called principle of least action.

(c) Instead of $\int T + U\, dt$, Euler uses the integral $\int m v\, ds$. Argue that this is equivalent.

(d) $\star$ Discuss Euler's interpretation of this result: "Because of their inertia, bodies are reluctant to move, and obey applied forces as though unwillingly; hence, external forces generate the smallest possible motion consistent with the endpoints."

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§ 13.5.3. Reference summary

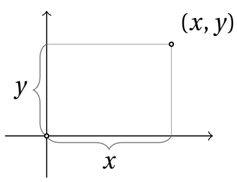
Find y that extremises $\int f(y, \dot{y}, t)\, dt$	$\Leftrightarrow$	Solve the differential equation $\frac{\partial f}{\partial y} - \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{y}}\right) = 0$

A PRECALCULUS REVIEW

§ A.1. Coordinates

§ A.1.1. *Lecture worksheet*

The position of a point is characterised analytically by its coordinates (x, y), meaning its vertical and horizontal distance from some designated origin point $(0,0)$:



By means of this device geometry is turned into algebra, so to speak. For example, a line can be described as all the points (x, y) that satisfy the relation $y = mx + b$ for some fixed numbers m and b.

A.1.1. Explain why. Hint: a line is characterised by the property that any horizontal step always corresponds to the same number of vertical steps.

A.1.2. What kind of figure is $y = \sqrt{1-x^2}$? Hint: consider x and y as legs of a right triangle.

A.1.3. What kind of figure is $xy = 0$?

A.1.4. ⋆ Rotate the figure in the previous problem by 45°. What is its equation now? Hint: What is a formula for combining x and y that will give you zero if you're at $(5,0)$ or $(0,5)$? What is a formula for combining x and y that will give you zero if you're at $(5,5)$? At $(5,-5)$?

§ A.1.2. *Problems*

A.1.5. (a) The enemy has one cannon movable along a shoreline $y = -\frac{1}{4}$ and one cannon located at a fortress off the shore at the point $\left(0, \frac{1}{4}\right)$. You must sail between them. What path should you choose to minimise the danger of being hit?

(b) † The positions of the canons in fact correspond to (and the problem is meant to illustrate) certain theoretical notions that always apply to this kind of curve. Explain.

§ A.2. Functions

§ A.2.1. *Lecture worksheet*

A function $y(x)$ ("y of x") is a rule assigning a specific output y to a specific input x. In other words, to say that y is a function of x is to say that y depends on x, or is determined by x. We can picture a function as a kind of "machine" where you stick some input in one end, turn some cranks, and receive a processed version of the input out at the other end. Often this takes the form of a formula with x's in it, into which one can "plug" whatever value for x to find the associated output value $y(x)$.

For instance, $f(x) = 2x - 1$ is a function that doubles the input and subtracts 1. So $f(3) = 5$, for example. It is often useful to put this kind of information in a table for overview:

x	1	2	3	4	5	6	7	8
$f(x)$	1	3	5	7	9	11	13	15

This table can help us see for example a second way of characterising $f(x)$ verbally, namely as an "odd-number machine," so to speak:

A.2.1. When x is a positive integer, $\boxed{}$ is the $\boxed{}$ odd number.

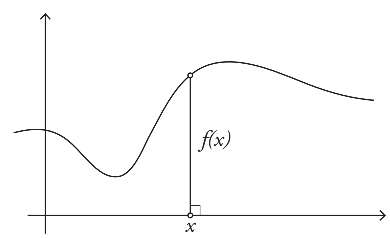
A.2.2. What is the 127th odd number?

The notion and notation of a function $f(x)$ is powerful in its flexibility and scope. For one thing, functions can be composed, meaning that the output of one function is the input of another, which is often a useful way of describing composite operations.

A.2.3. If $d(x)$ is the number of dollars you get for x euros, and $h(x)$ is the number of hot dogs you can buy for x dollars, what is $h(d(x))$?

A.2.4. If $f(x) = 2x$ and $g(x) = x/2$, explain both in formulas and purely verbally why $f(g(x)) = x$ and $g(f(x)) = x$.

Visually speaking, functions are curves. The value of the function for a particular x-value corresponds to the height of the function above the x-axis at that point:



This is called the graph of the function.

A.2.5. Draw the graph of $f(x) = x^2$ and show visually that a line can cut the curve in at most two points.

A.2.6. Draw the graph of $f(x) = x^3$ and show visually that a line can cut the curve in at most three points.

A.2.7. Find a fourth-degree graph which a line can cut in four points.

A.2.8. (a) "Quadratic functions are U's and cubic functions are S's." Discuss.

(b) Do fourth-degree curves correspond to some letter of the alphabet?

A function is called "even" if $f(-x) = f(x)$, and "odd" if $f(-x) = -f(x)$. Most functions are neither one nor the other, but for those that are it is often useful to be aware of these simple rules for how minuses behave, just as in ordinary arithmetic you wouldn't start all over again to compute $53 \times (-74)$ if you had just computed 53×74, or keep minding the sign at each step when evaluating $(-2)^5$.

A.2.9. Is $f(x) = x^n$ an even or odd function?

A.2.10. What is the visual meaning of a function being even or odd?

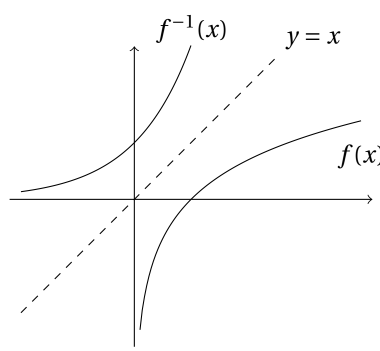
A.2.11. Illustrate with the graphs of x, x^2, x^3, x^4.

A.2.12. Make up some other even and odd functions by drawing graphs with two hands using two pens, both starting at the origin, one going right and one going left.

A.2.13. Speaking of the graphs of x, x^2, x^3, x^4, explain what happens graphically as the exponent increases ($x^5, x^6, x^7, \ldots$). Hint: What happens to 0.9^n and n becomes bigger and bigger? To 1.1^n?

$f^{-1}(x)$ means the inverse function of $f(x)$. It's "f backwards," or the function that "undoes" f. In symbols this means $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$: if you do one and then the other you get back what you started with. Thus, for example, if $f(x) = 2x$ then $f^{-1}(x) = x/2$. Another way of putting it is that, for f^{-1}, the output of f becomes the input and the input becomes the output.

Visually, the graph of $f^{-1}(x)$ is the graph of $f(x)$ mirrored in the line $y = x$, since this transformation interchanges the roles of x and y, i.e., input and output.



A.2.14. Consider the function $f(x) = x^2$.

(a) What is $f^{-1}(x)$?

(b) Find the graph of $f^{-1}(x)$ using the rule for graphs of inverse functions. What kind of curve is it?

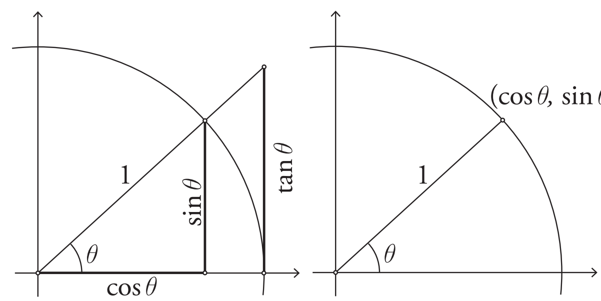
(c) How can you see the same thing directly from the equation for $f^{-1}(x)$?

§ A.3. Trigonometric functions

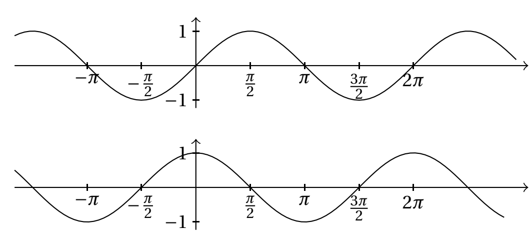
§ A.3.1. *Lecture worksheet*

Trigonometric functions are so called for their applications to triangles (§A.7.8), but in the context of the calculus they take on a much broader significance. In fact, they are the language in which all kinds of periodic phenomena are best described. Nature exhibits obvious periodicity in phenomena such as day and night, summer and winter, and the ebb and flow of the sea. But more important examples still are the many kinds of invisible waves that constitute a big part of our lives, including light waves, sound waves, and virtually all man-made forms of wireless communication.

The periodic nature of the trigonometric functions is obvious from their unit-circle definition, which generalises their definition in terms of triangles:



On the left we see, in effect, the geometrical definitions of the trigonometric functions embedded in a coordinate system. On the right we see a further step of abstraction where the cosine and sine are simply defined as the x and y coordinate respectively of a point moving along a unit circle. In this way the functions are liberated from their trigonometric origins, as it were. In particular, in this way we can define their values for any angle, including angles that are too great to ever occur in a triangle. Defined in this manner, the sine and cosine become beautiful periodic functions that look like this:



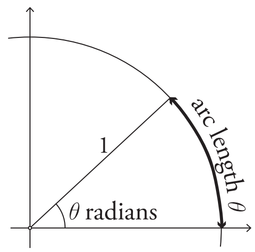
The iconic wave shape is unmistakeable, but to draw one of these graphs correctly we must also know how to start it off correctly, say when x is 0. We can do this by recalling a very useful fact—a quick and dirty trick has been the lazy mathematician's friend since time immemorial—namely that $\sin x \approx x$ when x is small.

A.3.1. Explain why this is so, using the unit circle definition of the sine.

A.3.2. Therefore, which graph is which of the two above?

A.3.3. What are some other features of the graphs that you can confirm using the unit circle definition?

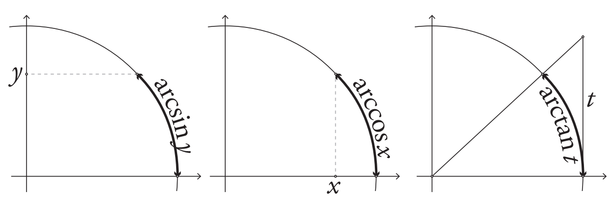
In analytical trigonometry we measure angles not in degrees but in radians. This means that an angle is measured by the corresponding arc length of a unit circle. In short, when using radians, angle is the same thing as (unit-circle) arc:



In calculus we always use radians rather than angles. The precise reason for this is seen in problem 1.4.1, but we can already appreciate that radians is the superior angle measure. After all, the notion that a full revolution corresponds to 360° is an arbitrary social construction. Basing a theory on such arbitrary starting points leads to arbitrary repercussions later, as is hardly surprising. Radian angle measure, by contrast, doesn't introduce any artificial conventions, but rather characterises angles by means intrinsic to geometry itself.

What are the inverses of the trigonometric functions? In the manner of §A.2 we can define them abstractly and denote them $\sin^{-1}(x)$ etc., as is sometimes done. However, it is also rewarding to think about their meaning more concretely. By definition $\sin^{-1}(x)$ inverts $\sin(x)$. What does this mean in terms of the geometrical definition of $\sin(x)$? The sine takes an angle—or rather, as we have now learned to say, arc—as its input and gives a corresponding coordinate as its output. The inverse sine, $\sin^{-1}(x)$, does the opposite: it takes the coordinate as its input and tells you what the corresponding arc is. For this reason $\sin^{-1}(x)$ is also denoted $\arcsin(x)$.

Here, then, are the geometrical meanings of the inverse trigonometric functions:



A.3.4. Find the value of each of the following and illustrate with a figure.

(a) $\sin(\pi/4)$

(b) $\arccos(-1)$

(c) $\arctan(1)$

(d) $\arctan(\infty)$

A.3.5. Argue that degree and radian angle measures can be interpreted as "observers's viewpoint" and "mover's viewpoint" respectively.

§ A.4. Logarithms

§ A.4.1. *Lecture worksheet*

Logarithms were first developed in the early 17th century as a means of simplifying long calculations. Long calculations were involved for example in navigation at sea, which was of increasing importance in this era. Indeed, the first ship of slaves from Africa to America set sail only four years after the publication of the first book on logarithms.

The essence of logarithms is that they *turn multiplication into addition*:

$$\log(ab) = \log(a) + \log(b) \tag{L1}$$

This simplifies calculations because if you have to compute by hand it is much easier to add than to multiply. In this way logarithms "doubled the lifetime of the astronomer," it was said at the time. Not so long ago, before the advent of pocket calculators, people still learned logarithms for this purpose in school. You can still see the traces of this today when you go to a used bookstore and look at the mathematics section: usually you will find there tables of logarithms published in the first half of the 20th century.

We can rediscover logarithms for ourselves in the following way. Consider a table of powers of some integer, such as 2:

n	1	2	3	4	5	6	$\cdots$
2^n	2	4	8	16	32	64	$\cdots$

A.4.1. Explain how for example 4×8 can be found using this table without actually performing any multiplication.

That's a neat trick, but it only works for numbers that happen to occur in the bottom row. We need to be able to multiply any numbers. Fortunately it is not hard to extend the idea to produce a table without such big gaps.

A.4.2. Explain how.

Thus, to produce a table of a function that has the property (L1), all we have to do, it turns out, is to make a table of values for some exponential function $f(x) = a^x$ and then read it backwards. Logarithms are simply the inverse of exponentiation.

In our table we used the base 2, but any number would have worked. We get a different logarithm for each base, but all of them have the crucial property (L1). The logarithm associated with our table would be denoted $\log_2(x)$. It shall emerge later that a certain number $e = 2.71828\ldots$ is the mathematician's favourite base, and that the associated logarithm is the most "natural" of all logarithms and therefore denoted $\ln(x)$.

A.4.3. (L1) is the defining property of logarithms, and the mother of all logarithm laws. Show how:

(a) The logarithm of 1 follows from (L1) by restricting one of its values to an identity element.

(b) The logarithm of an exponential expression follows from (L1) by regarding multiplication as repeated addition.

(c) The logarithm of a quotient follows from (L1) by considering that / cancels × and − cancels +.

§ A.5. Exponential functions

§ A.5.1. Lecture worksheet

The essence of exponential functions, such as $f(x) = 2^x$, is that they describe *things that grow in proportion to their size.* The more you have the faster it grows.

A.5.1. Argue that rabbit populations and money both have this property.

A.5.2. Verify that $f(x) = 2^x$ has this property by considering how $f(x+1)$ is related to $f(x)$.

A.5.3. I put \$1000 in a savings account earning 10% interest annually.

(a) How much money do I have after 1 year? After 2 years? After x years?

I want to know: How long will it take for my money to double?

(b) Write down an equation involving the required time T.

There are two ways of tackling this equation. Nowadays we can simply:

(c) Have a computer or calculator solve the equation.

However, we can also make some progress by hand:

(d) Use logarithms to solve for T, i.e., write the equation as $T = \ldots$.

Did we accomplish anything this way? You may say no, because we still need a calculator to find the numerical value of T, and we could already do that without logarithms anyway.

(e) ⋆ Explain why this method used to make more sense when there were no calculators.

Nevertheless solving for T in this way does serve a purpose in other contexts, where solving for T may be merely a substep in a more complex investigation.

Exponential functions also describe things that *shrink* in proportion to their size, of which radioactive decay is an important example.

A.5.4. *Radiocarbon dating.* Carbon is an essential atom in plants and animals. Plants absorb it through carbon dioxide in the atmosphere and animals absorb it through their food. A small portion of this carbon is in the form of the isotope ^{14}C ("carbon-14"). This isotope is radioactive, meaning that it is in an unstable state and will eventually revert into another form without external influence. This unnatural state is created by cosmic radiation, which converts nitrogen into ^{14}C. Left to its own devises, the ^{14}C will eventually decay back to nitrogen. However, this may not happen until many years later. ^{14}C decays in proportion to the amount present at such a rate that in 5730 years only half of the isotopes originally present have decayed back into nitrogen.

(a) Use this information to express the amount y of ^{14}C as a function of time t measured in years.

Living organisms continually replace their carbon, so their ^{14}C levels are kept at a constant level. However, when a plant or animal dies, it stops replacing its carbon and the amount of ^{14}C begins to decrease through radioactive decay. Thus any dead organic material can be dated on the basis of its ^{14}C content. For example, in antiquity precious treatises were written on parchment (dried animal skin).

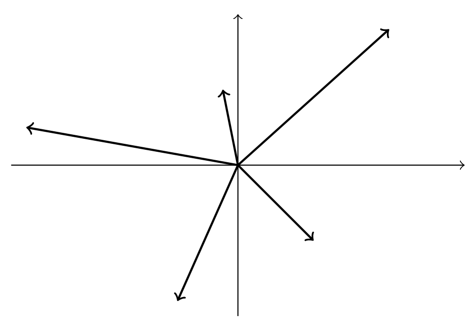
(b) A parchment manuscript has 80% as much ^{14}C as living material. Find an equation expressing the age of the parchment.

(c) Find the age of the parchment.

§ A.6. Complex numbers

§ A.6.1. Lecture worksheet

"Complex numbers" are an expanded universe of numbers in which any polynomial equation has as many roots as its degree. This is achieved by generalising ordinary real numbers into one more dimension. So in this universe numbers are no longer confined to one line or axis, but rather live in a whole plane. These numbers are points in the plane, or, if you prefer, they are arrows pointing from the origin to that point:



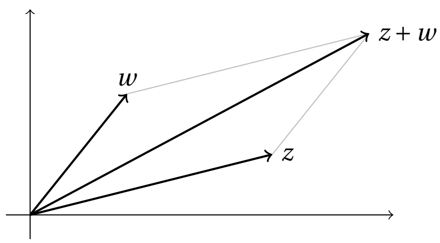
Now we ask ourselves: how does one add and multiply such numbers? The goal of generalisation is to retain old things but to think of them in new ways that give them wider applicability. Indeed, the following is a strange way of looking at ordinary multiplication, which, however, has the advantage that it generalises readily to two-dimensional numbers.

A.6.1. Let $|z|$ denote the magnitude, or distance to the origin, of a number z, and let $\arg(z)$, the "argument" of the number, denote the angle it makes with the positive x-axis (measured in radians). Then:

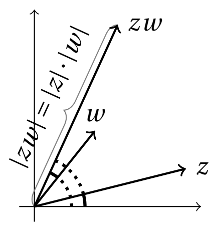
z	$\|z\|$	$\arg(z)$
2		
-3		
$2\cdot(-3)$		

z	$\|z\|$	$\arg(z)$
-1		
-4		
$(-1)\cdot(-4)$		

Generalising from this to two-dimensional complex numbers, we get the following rules. We add by "concatenation":



And we multiply by multiplying the lengths and adding the angles:



A.6.2. A multiplication by a complex number is sometimes called an "amplitwist." Explain the rationale behind this name.

A.6.3. Use geometric reasoning to find the following in a simple way.

(a) $(1-i)^{10}$.

(b) For what value of a are there multiple solutions to $(\frac{1-i}{a})^n = (\frac{1-i}{a})^{10}$?

(c) For this value of a, what is the next n beyond 10 for which the equation holds?

A.6.4. If $z = w^2$ we say that w is a square root of z. To answer the following questions it will be useful to picture geometrically the possible square roots of some complex numbers, such as -1 and $1+i$.

(a) If z and w are both square roots of the same complex number, which of the following must be true?

☐ $\arg(w) = -\arg(z)$

☐ $w = \bar{z}$

☐ $|w| = |z|$

☐ $w = -z$

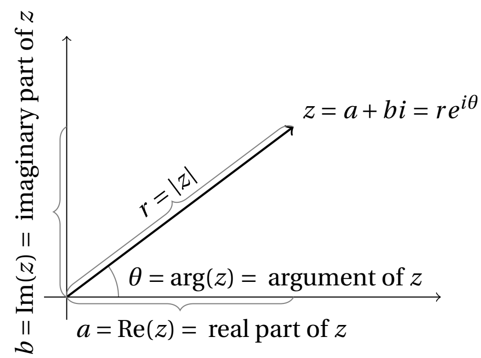
☐ $wz = 1$

(b) Does any complex number always have precisely two distinct square roots?

Already we are realising that complex numbers have two faces. On the one hand they are points in the plane and can be characterised by their x and y coordinates. When we think of complex numbers in this way we write them as $a+bi$. The i is for "imaginary." It is a number such that $i^2 = -1$, which certainly tests the imagination.

A.6.5. Explain why i corresponds to the point $(0,1)$.

On the other hand complex numbers can also be usefully characterised in terms of their length r and the angle θ that they make with the x-axis. In this case we write them in the "polar form" $re^{i\theta}$.



These two notations make it easy to do algebra with complex numbers. In the $a+bi$ notation we just do ordinary algebra with the added rule $i^2 = -1$. And in the polar form $re^{i\theta}$ we are free to use the usual laws of exponents. Indeed, note that the algebraic identity $r_1e^{i\theta_1}r_2e^{i\theta_2} = r_1r_2e^{i(\theta_1+\theta_2)}$ reflects precisely our geometrical definition of multiplication of complex numbers. This convenient fit between algebra and geometry is why this notation is chosen.

A.6.6. Evaluate $e^{i\pi}$. Write your answer in the form of a formula relating "the five most famous numbers."

A.6.7. Show that $e^{i\theta} = \cos\theta + i\sin\theta$. This is an important bridge between the two notations.

A.6.8. (a) Solve the equation $x^2-2x+2=0$ in complex numbers using the usual solution formula for quadratic equations

$$x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

and the fact that $\sqrt{-1} = i$.

(b) Express the roots in both notations.

(c) For one of the roots x, draw x^2, $-2x$, and 2 in the complex plane, and note what happens when you add them together.

§ A.6.2. Problems

A.6.9. (Requires knowledge of derivatives and preferably power series.) The appearance of the number e in the polar form of complex numbers may seem mysterious.

(a) Study again the justification for this notation given in the lecture and argue that e could just as well be replaced by, say, the number 3 so far as this justification is concerned.

(b) However, make a case for e on the basis of its distinctive property $(e^x)' = e^x$, not shared by other exponential functions.

(c) Also make a case for e by expanding both sides of the identity in problem A.6.7 using the series in problem 5.1.2.

(d) ⋆ Are these two arguments for e different or is it the same reason in different guises?

A.6.10. Explain what is wrong in the following argument:

$$-1 = \sqrt{-1}\sqrt{-1} = \sqrt{(-1)(-1)} = \sqrt{1} = 1$$

A.6.11. (a) Prove from the geometric definition of complex numbers that $a(b+c) = ab+ac$.

(b) ⋆ Are there any other important algebraic rules we need to derive in order to be justified in treating the algebraic and geometric conceptions of complex numbers as equivalent?

A.6.12. When we know complex numbers we no longer need to memorise the trigonometric addition formulas

$$\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \sin(\beta)\cos(\alpha)$$

$$\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$$

because we can easily re-derive them by simply multiplying $e^{i\alpha}$ by $e^{i\beta}$.

(a) Show how. (Express the product in two different ways and identify real and imaginary parts.)

(b) ⋆ Is this proof circular? That is, does the algebraic machinery it involves already rest on the sum formulas for trigonometric functions in some implicit way?

A.6.13. Complete the mathematical pun: Why did my stomach hurt after Christmas? $\sqrt{\frac{-1}{64}}$, $\sqrt{-49}$, $\frac{1+i}{1-i}$, or $\frac{i}{1+i}$?

A.6.14. With complex numbers we can solve any quadratic equation, or so the textbooks tell us. But what kind of "solutions" are these weird things with i's in them anyway? Indeed, the first person to publish on complex numbers, Cardano in his 1545 treatise *Ars magna*, called them "as subtle as they are useless," a sentiment perhaps shared by students today. But despite this lack of parental love from their father, these underdog numbers gradually triumphed over adversity by proving themselves useful again and again in field after field.

Their first triumph, however, was not the quadratic equations found in textbooks today but rather cubic ones, i.e., equations of degree 3. For cubic equations there is a formula analogous to the common quadratic formula, namely the solution of $y^3 = py+q$ is

$$y = \sqrt[3]{\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 - \left(\frac{p}{3}\right)^3}} + \sqrt[3]{\frac{q}{2} - \sqrt{\left(\frac{q}{2}\right)^2 - \left(\frac{p}{3}\right)^3}}.$$

(a) Apply the formula to $x^3 = 15x+4$. The two cube roots that arise are in fact equal to $2+i$ and $2-i$. Check this!

(b) So what solution does the formula give? Is it correct?

The conclusion is that even if you think answers with i's in them are hocus-pocus you still have to admit that complex numbers are useful for answering questions about ordinary real numbers as well.

§ A.7. Reference summary

§ A.7.1. Basic algebra

Exponents:

$$\cdots \quad x^{-2} = \frac{1}{x^2} \quad x^{-1} = \frac{1}{x} \quad x^0 = 1 \quad x^1 = x \quad x^2 = x\cdot x \quad \cdots$$

$$x^{\frac{1}{2}} = \sqrt{x} \quad x^{\frac{1}{3}} = \sqrt[3]{x} \quad \cdots$$

$$a^x a^y = a^{x+y} \quad \frac{a^x}{a^y} = a^{x-y} \quad (a^x)^y = a^{xy} \quad (ab)^x = a^x b^x$$

Fractions:

$$\frac{A}{B}\times\frac{C}{D} = \frac{AC}{BD} \qquad \frac{A}{B}+\frac{C}{D} = \frac{AD+BC}{BD}$$

$$\frac{A\cancel{C}}{B\cancel{C}} = \frac{A}{B} \qquad \frac{A}{B} = \frac{A}{B}\times\frac{C}{C}$$

$$\frac{X}{\frac{A}{B}} = X\times\frac{B}{A} \qquad \frac{A}{B} = A\times\frac{1}{B}$$

Roots:

$$\sqrt{a^2} = a \qquad \sqrt[n]{a^n} = a \qquad (a \text{ positive})$$

$$\sqrt{ab} = \sqrt{a}\sqrt{b} \qquad \sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$$

Polynomials:

$$a^2+2ab+b^2 = (a+b)^2$$

$$a^2-2ab+b^2 = (a-b)^2$$

$$a^2-b^2 = (a+b)(a-b)$$

$$a^3 - b^3 = (a-b)\left(a^2 + ab + b^2\right)$$
$$a^3 + b^3 = (a+b)\left(a^2 - ab + b^2\right)$$

Quadratic formula:

$$ax^2 + bx + c = 0 \quad \Longrightarrow \quad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

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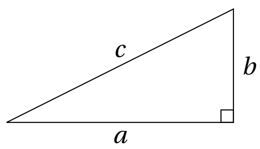
§ A.7.2. Pythagorean Theorem

$$a^2 + b^2 = c^2$$



Distance between two points (x_1, y_1) and (x_2, y_2):

$$\sqrt{(x_1 - x_2)^2 + \left(y_1 - y_2\right)^2}$$

Distance between two points (x_1, y_1, z_1) and (x_2, y_2, z_2):

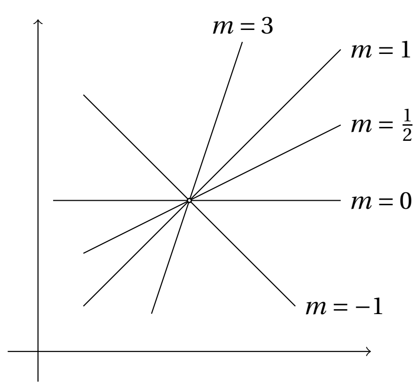
$$\sqrt{(x_1 - x_2)^2 + \left(y_1 - y_2\right)^2 + (z_1 - z_2)^2}$$

---

§ A.7.3. Lines

Equation for a line:

$$\begin{aligned} y &= mx + b \\ m &= \text{ slope } = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \text{ rise over run''} \\ b &= \ y\text{-intercept } = y(0) \end{aligned}$$



$$\text{lines with slopes } m_1, m_2 \text{ parallel} \iff m_1 = m_2$$
$$\text{lines with slopes } m_1, m_2 \text{ perpendicular} \iff m_1 = -1/m_2$$

- Find the equation for a line with a given slope m passing through a given point (x_1, y_1).

  Fill in what you know in $y - y_1 = m(x - x_1)$ and then rewrite it in the form $y = mx + b$.

- Find the equation for a line passing through two given points (x_1, y_1), (x_2, y_2).

  Find the slope $m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$, and then proceed as in the previous problem.

---

§ A.7.4. Circles

Circle with center at origin:

$$x^2 + y^2 = r^2 \quad \Longrightarrow \quad y = \pm\sqrt{r^2 - x^2}$$

Circle with center at (a, b):

$$(x-a)^2 + (y-b)^2 = r^2$$

$$\text{area } = \pi r^2 \qquad \text{circumference } = 2\pi r$$

---

§ A.7.5. Parabolas

Parabola with vertical axis:

$$y = ax^2 + bx + c = A(x-B)^2 + C$$

$$\begin{aligned} A &= \text{ ``steepness''} \\ A \text{ positive } &\Longrightarrow \text{ upward or ``happy'' parabola} \\ A \text{ negative } &\Longrightarrow \text{ downward or ``sad'' parabola} \\ B &= \ x\text{-value of axis of symmetry} \\ C &= \text{ vertical shift } = \text{ distance of vertex from } x\text{-axis} \end{aligned}$$

- Convert a quadratic function given in the form $y = x^2 + bx + c$ into the form $y = A(x-B)^2 + C$.

  Rewrite $x^2 + bx$ as $x^2 + bx + \left(\frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 = (x + \frac{b}{2})^2 - \left(\frac{b}{2}\right)^2$.

- Convert a quadratic function given in the form $y = ax^2 + bx + c$ into the form $y = A(x-B)^2 + C$.

  Divide by a, proceed as above to obtain $y/a = (x-B)^2 + C$, then multiply by a.

---

§ A.7.6. Factoring

Fundamental theorem of algebra: a polynomial of degree n can be factored into n linear factors: $a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n = k(x - r_1)(x - r_2)\cdots(x - r_n)$, where the roots r may be complex numbers or repeat occurrences of the same number.

- Factor a second-degree polynomial, $x^2 + Bx + C$.

  If the coefficients and roots are simple numbers: determine a and b such that $a + b = B$ and $ab = C$. The factorisation is $(x+a)(x+b)$.

In general: Find the roots r_1, r_2 of $x^2 + Bx + C = 0$, e.g. using the quadratic formula. The factorisation is $(x - r_1)(x - r_2)$.

- Factor a third-degree polynomial (or higher).

  Find one root r by trial-and-error or educated guessing. Factor out $(x-r)$. This can be done systematically by polynomial long division (see below), or, in many simpler cases, simply by writing $(x-r)(\boxed{}x^2+\boxed{}x+\boxed{})$ and determining by inspection what numbers need to go in the blanks to make this expression equal to the original cubic expression.

- Divide one polynomial, $p(x)$, by another, $q(x)$.

  To find "how many times $q(x)$ goes into $p(x)$," first determine the numbers a_1 and n_1 such that $a_1 x^{n_1}$ times the highest-degree term of $q(x)$ equals the highest-degree term of $p(x)$. Write down $a_1 x^{n_1}$ as the first term of the answer. Next compute $a_1 x^{n_1} q(x)$, and subtract the result from $p(x)$. This leaves the remainder of the division, $r_1(x)$.

  Now repeat the process with $r_1(x)$ in place of $p(x)$. Keep repeating this process until it can't go any further, i.e., until the remainder is 0 or of lower degree than $q(x)$.

  If the remainder is 0, then the answer gives the result of the division, i.e., $\frac{p(x)}{q(x)} = a_1 x^{n_1} + a_2 x^{n_2} + \cdots$.

  If the remainder is $r_k(x) \neq 0$, then the remaining division that could not be carried out must be added to the answer, i.e., $\frac{p(x)}{q(x)} = a_1 x^{n_1} + a_2 x^{n_2} + \cdots + \frac{r_k(x)}{q(x)}$.

$$
\begin{array}{r|lll}
 & a_1 x^{n_1} & +a_2 x^{n_2} & +\cdots \\ \hline
q(x) & p(x) & & \\
 & & & \\
 & -a_1 x^{n_1} q(x) & & \\ \cline{2-4}
 & & r_1(x) & \\
 & & & \\
 & & -a_2 x^{n_2} q(x) & \\ \cline{3-4}
 & & & r_2(x) \\
 & & & \ddots
\end{array}
$$

---

§ A.7.7. Functions and graphs

A *polynomial* is an expression of the form $a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$. The numbers a_i are the *coefficients* of the various x-terms.

A *rational* function is one polynomial divided by another.

An *algebraic* function is a function defined by an equation built up in any manner from the operations $+$, $-$, $\times$, div , $\sqrt{}$, $\sqrt[3]{}$, …, the variables x, y, and numbers.

A *transcendental* function is a function that is not algebraic.

$$
\begin{aligned}
f(x) \text{ even} &\iff f(-x) = f(x) \\
&\iff \text{graph symmetric in } y\text{-axis} \\
f(x) \text{ odd} &\iff f(-x) = -f(x) \\
&\iff \text{graph symmetric in } y\text{-axis except upside-down}
\end{aligned}
$$

- Given $f(x)$ as a formula, find f(whatever given expression).

  Replace all occurrences of x in the formula for $f(x)$ with the given expression enclosed in brackets.

- Evaluate a composite function $f(g(x))$.

  Work "inside out": first find $g(x)$, then plug the result into $f(x)$.

- Find the graph of a function $f(x)$ given as a formula.

  Pick some value for x, compute $f(x)$, mark the point $(x, f(x))$. Repeat for various values of x. When the pattern is clear, connect the dots.

- Infer the properties of the graph of a polynomial function.

  If the polynomial has a factor $(x - a)$, the graph intersects or touches the x-axis at $x = a$.

  If the degree of the polynomial is n, the graph has no more than $n - 1$ turning points, and no more than n intersections with any line.

  If the highest-degree term is x raised to an odd power, the function goes to $+\infty$ for big x's and $-\infty$ for big negative x's.

  If the highest-degree term is x raised to an even power, the function goes to $+\infty$ for big x's and for big negative x's.

- Recognise how the graph of a function closely related to $f(x)$ differs from that of $f(x)$.

  $f(x) + c$ moves the graph c steps up.

  $-f(x)$ flips the graph upside-down (i.e., mirrors it in the x-axis).

  $f(-x)$ flips the graph the other way around (i.e., mirrors it in the y-axis).

  $kf(x)$ stretches the graph by a factor k in the y-direction; the bigger the k, the "steeper" or "more accentuated" the graph becomes.

  $f(kx)$ stretches the graph by a factor k in the x direction; the bigger the k, the more "flattened out" the graph becomes.

  $f^{-1}(x)$ interchanges x and y, i.e., reflects the graph in the line $y = x$.

- Find the inverse of a function given as a formula $y = f(x)$.

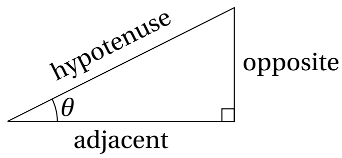
  Solve for the output variable, i.e., rewrite the given equation in the form $x =$ (something with y's). The right hand side is then the desired inverse function $f^{-1}(y)$. (Its variable is now called y. If you prefer to forget what x denoted when you started the problem and simply consider f^{-1} in its own right, then you can simply replace all the y by x's and you have $f^{-1}(x)$.)

Find the inverse of $f(x) = \left(\sin(\sqrt{x})\right)^2$.

$y = \left(\sin(\sqrt{x})\right)^2 \implies \pm\sqrt{y} = \sin(\sqrt{x}) \implies \arcsin(\pm\sqrt{y}) = \sqrt{x} \implies x = (\arcsin(\pm\sqrt{y}))^2 = (\arcsin(\sqrt{y}))^2$. So $f^{-1}(x) = (\arcsin(\sqrt{x}))^2$.

§ A.7.8. *Trigonometry*

Geometrical definitions of trigonometric functions:



$$\sin\theta = \frac{\text{opposite side}}{\text{hypotenuse}} \qquad \cos\theta = \frac{\text{adjacent side}}{\text{hypotenuse}}$$

$$\tan\theta = \frac{\text{opposite side}}{\text{adjacent side}} = \frac{\sin\theta}{\cos\theta}$$

Reciprocal trigonometric functions:

$$\sec x = \frac{1}{\cos x} \quad \csc x = \frac{1}{\sin x} \quad \cot x = \frac{1}{\tan x}$$

Pythagorean property:

$$\sin^2\theta + \cos^2\theta = 1$$

Symmetry properties:

$$\sin(-\theta) = -\sin(\theta)$$

$$\cos(-\theta) = \cos(\theta)$$

$$\tan(-\theta) = -\tan(\theta)$$

Area of triangle (C = angle opposite side c):

$$\frac{\text{base} \times \text{height}}{2} = \frac{1}{2} ab \sin C$$

Law of sines:

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

Law of cosines:

$$a^2 = b^2 + c^2 - 2bc\cos A$$

Compound angle formulas:

$$\sin(x+y) = \sin x\cos y + \cos x \sin y$$

$$\sin(x-y) = \sin x\cos y - \cos x \sin y$$

$$\cos(x+y) = \cos x\cos y - \sin x \sin y$$

$$\cos(x-y) = \cos x\cos y + \sin x \sin y$$

Double angle formulas:

$$\sin 2x = 2\sin x\cos x$$

$$\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$$

Half angle formulas:

$$\sin\frac{x}{2} = \sqrt{\frac{1-\cos x}{2}}$$

$$\cos\frac{x}{2} = \sqrt{\frac{1+\cos x}{2}}$$

Radian angle measure: measuring angles by the length of the corresponding arc of a unit circle. In particular, a full revolution = circumference of unit circle = 2π.

- Convert $\theta°$ into radians.

  $\theta \cdot \frac{2\pi}{360}$

- Convert θ radians into degrees.

  $\theta \cdot \frac{360}{2\pi}$

Trigonometric table:

degrees	radians	sin	cos	tan
0°	0	0	1	0
30°	$\pi/6$	$1/2$	$\sqrt{3}/2$	$1/\sqrt{3}$
45°	$\pi/4$	$1/\sqrt{2}$	$1/\sqrt{2}$	1
60°	$\pi/3$	$\sqrt{3}/2$	$1/2$	$\sqrt{3}$
90°	$\pi/2$	1	0	

§ A.7.9. *Exponential functions*

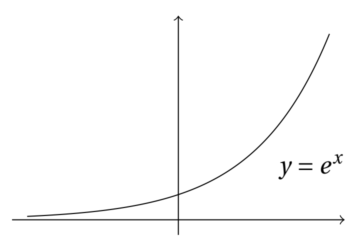
$y(t) = y_0 e^{kt}$: exponential growth/decay function.

$y_0 = y(0) =$ initial amount

$k =$ growth/decay rate constant

(positive $k \iff$ growth, negative $k \iff$ decay)

"After one unit of time, λ times the initial amount remains" $\iff k = \ln(\lambda)$.



- Given an exponential growth/decay function, determine the time at which λ times the initial amount is present. (E.g., "half life," $\lambda = \frac{1}{2}$.)

  Set $y(T) = \lambda y(0)$ and solve for T.

- Given the initial value y_0 and one data point $y(t_1) = y_1$ of an exponential growth/decay function $y(t)$, find the expression for $y(t)$.

  We need y_0 and k in the formula $y(t) = y_0 e^{kt}$. Fill in y_0, which is given, right away. Using the resulting expression, write out $y(t_1) = y_1$ and solve for k.

- Given two data points, $y(t_1) = y_1$ and $y(t_2) = y_2$, of an exponential growth/decay function $y(t)$, find the expression for $y(t)$.

  We need y_0 and k in the formula $y(t) = y_0 e^{kt}$. Use this formula to write out $y(t_1) = y_1$, and solve the resulting equation for y_0. Also write out $y(t_2) = y_2$, and substitute the found expression for y_0, and then solve for k. Plug back into the expression for $y(t)$.

---

§ A.7.10. Logarithms

Logarithm as inverse of exponentiation:

$$\begin{aligned}\log_a x &= \text{ the inverse of } a^x \\ &= \text{ the number to which } a \text{ needs to be raised to give } x\end{aligned}$$

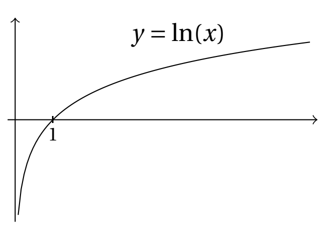
$$\ln = \log_e$$

$$y = \ln(x) \iff x = e^y \qquad y = \log_b(x) \iff x = b^y$$

Laws of logarithms (log can be any logarithm):

$$\begin{aligned}\log(ab) &= \log a + \log b \\ \log(a/b) &= \log a - \log b \\ \log(a^b) &= b \log a \\ \log 1 &= 0\end{aligned}$$

$$\ln(e^x) = x \qquad e^{\ln x} = x \qquad \ln e = 1$$



- Solve for x in an equation in which x occurs in an exponent.

  Isolate the exponential expression on one side of the equation (move other terms to the other side and divide away the coefficient of the exponential term, if it has one), and take logarithms of both sides.

- Solve for x in an equation in which x occurs inside a logarithm.

  Isolate the logarithmic expression on one side of the equation (move other terms to the other side; if there are several logarithmic terms, combine them using laws of logarithms; divide away the coefficient of the logarithmic term, if it has one), and write $e^{\text{left hand side}} = e^{\text{right hand side}}$.

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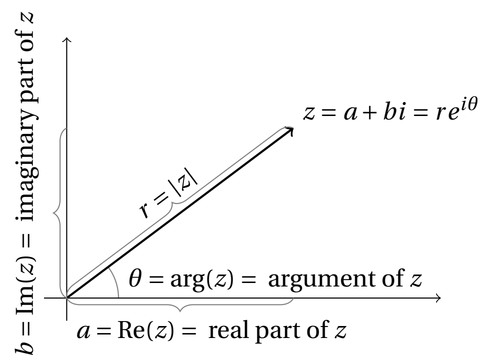
§ A.7.11. Complex numbers

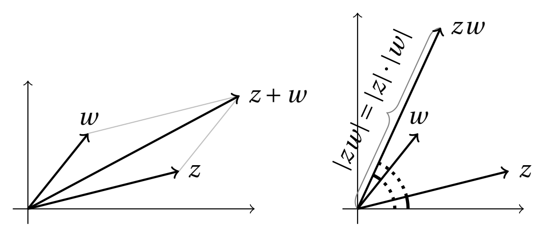


$$i^2 = -1 \qquad e^{i\theta} = \cos\theta + i\sin\theta$$

$$(\cos x + i \sin x)^n = \cos nx + i \sin nx$$

Arithmetic of complex numbers, algebraically: as ordinary algebra with added rule $i^2 = -1$.

Arithmetic of complex numbers, geometrically:



$$\overline{a + bi} = \text{ complex conjugate of } a + bi = a - bi$$

- Solve a quadratic equation with complex roots.

  Solve with usual quadratic formula and when the root comes out as $\sqrt{-A}$, rewrite it as $\sqrt{A}\sqrt{-1} = \sqrt{A}i$.

> Solve $z^2 - 2z + 4 = 0$.
>
> $z = \frac{2 \pm \sqrt{4-16}}{2} = \frac{2 \pm \sqrt{-12}}{2} = \frac{2 \pm 2\sqrt{-3}}{2} = 1 \pm i\sqrt{3}$.

- Simplify a fraction with a complex number $a + bi$ as its denominator.

  Multiply top and bottom of the fraction by the conjugate $a - bi$ of the denominator. After simplifying, there will be no i's left in the denominator.

> Express $\frac{3}{1+5i}$ in the standard form $a + bi$.
>
> $\frac{3}{1+5i} = \frac{3(1-5i)}{(1+5i)(1-5i)} = \frac{3-15i}{1+25} = \frac{3}{26} - \frac{15}{26}i$.

> Simplify $\frac{1+i}{2} + \frac{1}{1+i}$.
>
> $\frac{1+i}{2} + \frac{1}{1+i} = \frac{1+i}{2} + \frac{1-i}{(1+i)(1-i)} = \frac{1+i}{2} + \frac{1-i}{2} = 1$.

Simplify $\frac{3+2i}{3-2i}$.

$= \frac{(3+2i)^2}{(3-2i)(3+2i)} = \frac{5+12i}{9+4} = \frac{5}{13} + \frac{12}{13}i$.

- Convert from rectangular form $a+bi$ to polar form $re^{i\theta}$ or vice versa.

  Direct visualisation often suggests how to find r and θ using basic geometry. For explicit formulas, see §7.5.1.

Write in polar form ($re^{i\theta}$): $1-i$, $\sqrt{3}+i$, $2i$.

$\sqrt{2}e^{-i\pi/4}$, $2e^{i\pi/6}$, $2e^{i\pi/2}$.

Write $\frac{1+i}{\sqrt{3}+i}$ in polar form.

$= \frac{\sqrt{2}e^{i\pi/4}}{2e^{i\pi/6}} = \frac{1}{\sqrt{2}}e^{i(\pi/4-\pi/6)} = \frac{1}{\sqrt{2}}e^{i\pi/12}$.

Write in the form $a+bi$: $e^{3\pi i/2}$, $2e^{\pi i/6}$, $3e^{i\pi/4}$.

$-i$, $\sqrt{3}+i$, $\frac{3}{\sqrt{2}} + \frac{3}{\sqrt{2}}i$.

Write $\frac{e^{\pi i}}{1-i}$ in the standard form $a+bi$.

$e^{\pi i}$ means a complex number with $r=1$ and $\theta=\pi$, so $e^{\pi i}=-1$. Thus

$$\frac{e^{\pi i}}{1-i} = \frac{-1}{1-i} = \frac{-1}{1-i}\cdot\frac{1+i}{1+i} = \frac{-1-i}{1+2} = -\frac{1}{2} - \frac{1}{2}i$$

- Simplify expression involving powers or roots of complex numbers.

  Often useful in trickier cases: convert to polar form and use laws of exponents or geometric properties.

Simplify $(1-i)^{12}$.

For the polar form of $1-i$ we have $r=\sqrt{1^2+(-1)^2}=\sqrt{2}$, $\theta=-\frac{\pi}{4}$, so for $(1-i)^{12}$ we have $r=(\sqrt{2})^{12}=2^6=64$, $\theta=-\frac{\pi}{4}\cdot 12=-3\pi$, which is equivalent to $-\pi$. Thus $(1-i)^{12}=-64$.

Write $(1-\sqrt{3}i)^{11}$ in the form $a+bi$.

Let $z=1-\sqrt{3}i$. Observe that $|z|=2$, and $\arg z=\theta=\frac{5\pi}{3}$, so $z=2e^{\frac{5\pi}{3}i}$. Thus $z^{11}=(2e^{\frac{5\pi}{3}i})^{11}=2^{11}e^{\frac{55\pi}{3}i}=2^{11}e^{(18\pi+\frac{\pi}{3})i}=2^{11}e^{\frac{\pi}{3}i}=2^{11}(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3})=2^{11}(\frac{1}{2}+i\frac{\sqrt{3}}{2})=2^{10}(1+i\sqrt{3})=1024+1024\sqrt{3}i$.

Simplify $(\frac{1+\sqrt{3}i}{1+i})^8$.

$= \frac{2^8(\cos(8\cdot 60°)+i\sin(8\cdot 60°))}{(\sqrt{2})^8(\cos(8\cdot 45°)+i\sin 8(\cdot 45°))} = \frac{2^8}{2^4}\cdot\frac{\cos 120°+i\sin 120°}{\cos 0°+i\sin 0°} = 2^4(-\frac{1}{2}+i\frac{\sqrt{3}}{2}) = -8+8\sqrt{3}i$.

Simplify $(\frac{1}{2}+\frac{\sqrt{3}}{2}i)^{666}$.

$=(1(\cos 60°+i\sin 60°))^{666} = 1^{666}(\cos(666\cdot 60°)+i\sin(666\cdot 60°)) = \cos(111\cdot 360°)+\sin(111\cdot 360°) = \cos 0° + i\sin 0° = 1$.

§ A.7.12. *Physics*

Basic arithmetic of motion:

$$\text{distance} = \text{velocity} \times \text{time}$$

Newton's law:

$$F = ma$$

$$\text{Force} = \text{mass} \times \text{acceleration}$$

Law of gravity:

$$F = \frac{GMm}{r^2}$$

$$\begin{aligned} F &= \text{gravitational force (N)} \\ M, m &= \text{masses of the two objects (kg)} \\ r &= \text{distance between the objects (m)} \\ G &= \text{gravitational constant} \approx 6.67\cdot 10^{11}\ \frac{\text{m}^3}{\text{kg}\cdot\text{s}} \end{aligned}$$

Law of gravity close to the earth's surface:

$$a = g$$

$$\text{acceleration} = \text{constant}$$

$$g \approx 9.8\ \frac{\text{m}}{\text{s}^2}$$

§ B.1. Introduction

§ B.1.1. *Lecture worksheet*

$7x+3y$ is a "linear" expression, as opposed to anything involving x^2, xy, or such things of higher order. $7x+3y$ is a "linear combination" of x and y: it's so much of the one plus so much of the other. Linear relationships between quantities is the simplest kind of relation, and they occur everywhere. Innumerable scenarios are modelled by linear systems of equations such as

$$\begin{aligned} 5x+2y&=3 \\ x-4y&=1 \end{aligned}$$

or linear transformations such as

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x-y \\ 2x+y \end{pmatrix}$$

where some "state" (x, y) of some system is being transformed into a new state that is a linear transformation of the original state.

Linear algebra gives us concepts and techniques to deal with relationships of this type. At first sight these concepts may not seem very impressive or well-motivated, but the more you encounter these types of relationships "in the wild," the more you will come to agree that linear algebra really hit the nail on the head and singled out some structurally fundamental concepts and ideas. This makes linear algebra a bit hard to teach and learn in terms of motivation. The ideas of linear algebra are not well motivated by any one particular problem. Rather, they prove their worth in that the capture patterns that recur in a vast number of diverse situations. At first sight, the concepts of linear algebra might seem like complicated ways of saying simple things. I will show you various applications, but you may well object that any one of them you could have handled with simpler means instead of using the fancy words of linear algebra with their intricate definitions. This is true. But the point is not that the notions of linear algebra helps you solve any one particular problem, but that they help us highlight patterns that occur in a vast range of problems across a multitude of contexts. The "start-up cost" of learning these concepts might seem like a high price to pay for the applications you get, but it's a long-term investment if there ever was one. Linear algebra is distilled experience. Mathematicians spent centuries working with linear relationships the hard way, and linear algebra is the box of insiders' tricks that emerged as the most common and fundamental structural patterns they encountered.

§ B.2. Matrices

§ B.2.1. *Lecture worksheet*

A matrix is an array of numbers such as $\left[\begin{smallmatrix} 3 & 7 \\ 4 & 1 \end{smallmatrix}\right]$. There is a certain rule for multiplying such things that turns out to be very fundamental. Let's learn the rule algebraically first, and then we shall see what it all means. The rule is given in the reference summary. We see that matrices are multiplied by multiplying rows of the left matrix by columns of the right matrix. To me, multiplying matrices is a tactile experience. I run my left index finger across the row and my right one down the column, tapping the numbers that are to be multiplied together.

B.2.1. Find

$$\begin{bmatrix} 5 & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ -2 & 0 \end{bmatrix}$$

and make sure you tap along with your fingers so that matrix multiplication becomes ingrained in your muscle memory.

We shall see many examples in which matrix multiplication represents the transition from one state to the next. For example, the population dynamics of a city can be characterised by how many people move between the city center and the suburbs. Perhaps mostly young people leave the suburbs to go live in the city, while the older generation tend to remain in the suburbs. This could mean for example that 80% of the suburban population stay and 20% move in a given decade. Those who move to the city perhaps do so on a more temporary basis, for example until they have children of their own. Thus we can imagine that 50% of the city population move and 50% stay in any given decade. This information is encoded in the matrix $\left[\begin{smallmatrix} .8 & .5 \\ .2 & .5 \end{smallmatrix}\right]$, because if x_n and y_n represent the two populations after n decades, then

$$\begin{bmatrix} .8 & .5 \\ .2 & .5 \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$$

B.2.2. (a) We also have

$$\begin{bmatrix} .8 & .5 \\ .2 & .5 \end{bmatrix}^n \begin{bmatrix} x_k \\ y_k \end{bmatrix} = \begin{bmatrix} x_m \\ y_m \end{bmatrix}$$

where $m = \boxed{}$

(b) Which is the city population? x or y?

(c) The population distribution is in equilibrium when the city population is $\boxed{}$% of the suburban population.

So figuring out what happens with the populations over time is just a matter of multiplying by the transition matrix so many times. Below we shall return to this example and see how its long-term behaviour can be analysed.

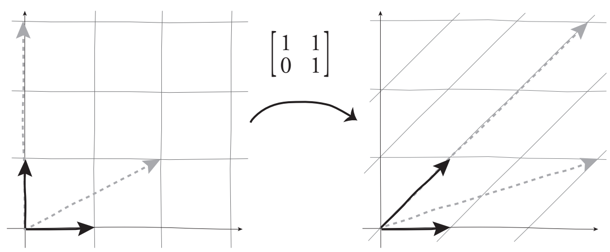
§ B.3. Linear transformations

§ B.3.1. *Lecture worksheet*

Matrix multiplication also has a geometrical meaning. We can think of a matrix as a function that takes as its input a point in the plane written as a column vector $\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]$. Then for example $\left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]$ leaves $\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$ alone but tilts $\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]$ since

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

So the transformation looks like this:



I have drawn the grid and dashed vectors here to emphasise that the matrix is determined by its effect on the two unit-basis vectors $\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]$. Its effect on for example $\left[\begin{smallmatrix} 2 \\ 1 \end{smallmatrix}\right]$ is just twice its effect on $\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$ plus once its effect on $\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]$ since

$$\begin{aligned} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} &= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \left(2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) \\ &= 2 \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &= 2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \end{aligned}$$

To summarise in general terms: the columns of a matrix represent its effect on the unit-basis vectors, and its general effect can be extrapolated from there.

B.3.1. Match each of the matrices

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

with its geometrical description: 90° clockwise rotation, 180° rotation, reflection in x-axis, magnification, projection onto y-axis, reflection in the line $y = x$.

B.3.2. Use this to find matrices A and B such that $AB \neq BA$.

We say that matrix multiplication is "not commutative." Can you see the etymology of this term, and its connection with everyday phrases such as "I have a long commute to work"?

B.3.3. Use the above to find non-zero matrices A, B, C such that $AB = AC$ but $B \neq C$. Explain why this shows another way in which matrix algebra is unlike ordinary algebra.

The matrix $I = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$ is called the identity matrix since it changes nothing.

B.3.4. Which of the above matrices come back to I when multiplied by itself a certain number of times? How many multiplications does it take? Confirm by both calculation and geometrical interpretation.

§ B.3.2. *Problems*

B.3.5. Find the matrix representation of a reflection in the line $y = 2x$.

§ B.4. Gaussian elimination

§ B.4.1. *Lecture worksheet*

When faced with a system of linear equations such as

$$\begin{aligned} x + 2y &= 0 \\ 3x - y &= 0 \end{aligned}$$

students are often tempted to "set them equal": $x + 2y = 3x - y$.

B.4.1. Explain why this is a bad idea in that it "destroys information."

A better strategy, which works for any system of linear equations, is to subtract a multiple of the first equation from the second so as to "kill" all the x's in the second equation.

B.4.2. Solve the same system by this strategy.

B.4.3. Use the same strategy to solve the system

$$\begin{aligned} x - 2y &= 1 \\ 3x + 4y &= 8 \end{aligned}$$

Note that when solving systems of linear equations in this way one really only plays around with the coefficients; the x and the y are merely placeholders. Matrices make for a convenient bookkeeping device in such situations. Thus the last system above can be encoded by the coefficient matrix

$$\left[\begin{array}{rr|r} 1 & -2 & 1 \\ 3 & 4 & 8 \end{array}\right]$$

and solved as follows. First subtract 3 times the first row from the second:

$$\left[\begin{array}{rr|r} 1 & -2 & 1 \\ 0 & 10 & 5 \end{array}\right]$$

Divide the last row by 10:

$$\left[\begin{array}{rr|r} 1 & -2 & 1 \\ 0 & 1 & \frac{1}{2} \end{array}\right]$$

Add 2 times the last row to the first:

$$\left[\begin{array}{rr|r} 1 & 0 & 2 \\ 0 & 1 & \frac{1}{2} \end{array}\right]$$

These steps are nothing but shorthand versions of standard ways of manipulating algebraic equations—most likely the same steps you took in problem B.4.3. Translating the final matrix back into equations again, this last matrix says $x = 2$ and $y = \frac{1}{2}$, so we have our solution.

This strategy for solving linear equations is two thousand years old: it was used in ancient China, where the coefficient matrix was represented and manipulated on a counting board. One is reminded of those Mancala board games where marbles are placed in a grid of pits on a wooden board—a delightfully concrete version of a matrix, and one very well suited for this kind of calculation. Let me give you a sample problem from an ancient Chinese text for practice. See if you can feel the marbles moving as you solve it in coefficient matrix (or "counting board") form.

B.4.4. "[We are to ascend a mountain carrying a weight of 40 dan] given one superior horse, two common horses, and three inferior horses. ... The superior horse together with one common horse, the [group of two] common horses together with one inferior horse, and the [group of three] inferior horses together with one superior horse, are all able to ascend. Problem: How much weight do the superior horse, common horse, and inferior horse each have the strength to pull?"

§ B.5. Inverse matrices

§ B.5.1. *Lecture worksheet*

The inverse A^{-1} of a matrix A is a matrix that "undoes" A, i.e., $A^{-1}A = AA^{-1} = I$. Computationally, we can find the inverse of a matrix A in the following way. First form the double matrix $[A|I]$ consisting of the given matrix A and the identity matrix of the same dimensions written to the right of it. Now perform row manipulations just as in §B.4 to transform A into I. When we perform these operations we are focussing on the left, or A, part of our double matrix. However, we are also perform the same operations on the right half of the double matrix. Thus the I we started with there will be turned into some other matrix B. I say that this B is in fact the sought inverse A^{-1}.

B.5.1. Prove this by arguing that solving for A^{-1} in $AA^{-1} = I$ amounts to three separate system-of-equations problems like the ones studied in §B.4, and that the method just described is simply the Gaussian elimination way of solving them all at the same time.

For the special case of a 2×2 matrix the inverse is:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

B.5.2. Name two ways in which you could prove this.

§ B.5.2. *Problems*

B.5.3. *Cryptology.* A spy is encoding four-letter text messages by first translating the letters into numbers according to the table below, then forming a column vector $\mathbf{v}$ from these numbers, then calculating $A\mathbf{v}$, where

$$A = \begin{bmatrix} 1 & -1 & 1 & -1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

and then translating the resulting column vector back into a four letter word. The spy sends the message CRAR. Decode the message.

a	b	c	d	e	f	g	h	i	j	k	l	m	n
1	2	3	4	5	6	7	8	9	10	11	12	13	14

o	p	q	r	s	t	u	v	w	x	y	z
15	16	17	18	19	20	21	22	23	24	25	26

§ B.6. Determinants

§ B.6.1. *Lecture worksheet*

Above we saw that for a 2×2 matrix,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

The expression in the denominator is called the determinant,

$$\begin{aligned} \det A = \begin{vmatrix} a & b \\ c & d \end{vmatrix} &= ad - bc \\ &= \pm \text{ area of parallelogram spanned by } \left[\begin{smallmatrix} a \\ c \end{smallmatrix}\right] \text{ and } \left[\begin{smallmatrix} b \\ d \end{smallmatrix}\right] \\ &= \text{ area magnification factor of the transformation } A \end{aligned}$$

No wonder, then, that it appears in the denominator of the inverse, since the inverse must shrink any area magnification that occurred back down again.

We can see why determinants correspond to areas by observing (from direct computation with the algebraic definition) that

$$\begin{vmatrix} k\cdot a & b \\ k\cdot c & d \end{vmatrix} = k\begin{vmatrix} a & b \\ c & d \end{vmatrix} \quad\text{and}\quad \begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & b-a \\ c & d-c \end{vmatrix}$$

B.6.1. Thinking of determinants as areas of parallelograms, in what terms are these rules best interpreted?

- ☐ reshaping a parallelogram like a stack of books
- ☐ reshaping a parallelogram like four sticks
- ☐ stacking parallelograms side to side
- ☐ introducing a third dimension/thickness
- ☐ a similar but scaled parallelogram
- ☐ the perimeter of the parallelogram

Hint: visualise for example how the transformations

$$\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} \to \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \text{ and } \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} \to \begin{vmatrix} 1 & 3 \\ 0 & 3 \end{vmatrix}$$

affect area.

B.6.2. Argue that it follows that determinants are areas since $\left|\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right| = 1$ is the area of the unit square and all other parallelograms can be built up from there by the above operations (or, conversely, can be brought back down to a unit square by these operations).

B.6.3. Technically, determinants are "signed areas" since they are sometimes negative, although their magnitude always corresponds to the area. This is reflected in determinant algebra by the fact that if we switch two columns the determinant changes sign. Compute the determinants of the transformations in problem B.3.1 and argue that a negative determinant corresponds to areas being "flipped upside down."

B.6.4. † ⋆ 3×3 determinants are volumes of parallelepipeds, the natural generalisation of the 2×2 case. They are computed by breaking them into 2×2 determinants as shown in the reference summary. Generalise the argument of problem B.6.2 to the 3×3 case to justify the interpretation of a 3×3 determinant as a volume.

The rules for manipulating determinants that we found above can be used to simplify calculations. For example, if we are looking for the determinant

$$\begin{vmatrix}1 & 1 & 5\\ 0 & 0 & 1\\ 2 & 2 & 11\end{vmatrix}$$

we simply subtract twice the first row and once the middle row from the last row, which then becomes a row of all zeroes. Therefore the entire determinant is zero.

B.6.5. Explain why the last sentence is clear both computationally and geometrically.

The case of a determinant being zero is often of special interest. It means that the column/row vectors are "linearly dependent," i.e., one of them can be obtained by combining the others with certain coefficients, like the last row was a combination of the previous two in our example. So in such cases there is a kind of redundancy: the last vector "doesn't add anything new." This idea will be important later.

B.6.6. Select all that are true.

- ☐ If $\mathbf{u},\mathbf{v},\mathbf{w}$ are vectors such that $\{\mathbf{u},\mathbf{v}\}$, $\{\mathbf{u},\mathbf{w}\}$, and $\{\mathbf{v},\mathbf{w}\}$ are each linearly independent sets, then $\{\mathbf{u},\mathbf{v},\mathbf{w}\}$ is a linearly independent set.
- ☐ If the columns of a matrix are linearly dependent, then its determinant is zero.
- ☐ If the rows of a matrix are linearly dependent, then its determinant is zero.
- ☐ Just as every real number except 0 has a multiplicative inverse, so every square matrix that has no 0 entries has an inverse.
- ☐ A diagonal matrix commutes with anything: that is, if D is a diagonal square matrix then $AD = DA$ for all matrices A of the same dimensions.

§ B.7. Eigenvectors and eigenvalues

§ B.7.1. Lecture worksheet

Above we discussed a population dynamics example where the movements each decade were described by a matrix,

$$\begin{bmatrix}.8 & .5\\ .2 & .5\end{bmatrix}\begin{bmatrix}x_n\\ y_n\end{bmatrix} = \begin{bmatrix}x_{n+1}\\ y_{n+1}\end{bmatrix}$$

What will happen in the long run in this situation? It seems likely that the population distribution will eventually settle at an equilibrium, so that the number of people moving one way is equal to the number of people moving in the other direction. In equations this means

$$\begin{bmatrix}.8 & .5\\ .2 & .5\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix} = \begin{bmatrix}x\\ y\end{bmatrix}$$

or in other words

$$\begin{aligned}.8x + .5y &= x\\ .2x + .5y &= y\end{aligned}$$

or

$$\begin{aligned}-.2x + .5y &= 0\\ .2x - .5y &= 0\end{aligned}$$

The second equation is just minus one times the first, so there is really only one equation and two unknowns. Therefore there are infinitely many solutions. In such a situation we can pick any value for one of the variables, say $x = t$, and there will always be a corresponding value for y that solves the equation, in this case $y = \frac{2}{5}t$. We can then express all solutions in vector form as $\left[\begin{smallmatrix}t\\ \frac{2}{5}t\end{smallmatrix}\right]$, or $t\left[\begin{smallmatrix}1\\ 2/5\end{smallmatrix}\right]$. Or, if we prefer to write it without fractions, $t\left[\begin{smallmatrix}5\\ 2\end{smallmatrix}\right]$, since any constant multiple is absorbed by the parameter t, which runs through all numbers. So we see that an equilibrium is reached when the populations are in the proportions 5 to 2, i.e., when the city population is 40% of the suburban population. The parameter t reflects the fact that we did not know the total number of people to begin with, so we know only the proportions but not the absolute numbers.

In more general terms, a matrix identity $A\mathbf{x} = \mathbf{0}$, or $\left[\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right]\left[\begin{smallmatrix}x\\ y\end{smallmatrix}\right] = \left[\begin{smallmatrix}0\\ 0\end{smallmatrix}\right]$, is really a system of linear equations,

$$\begin{aligned}ax + by &= 0\\ cx + dy &= 0\end{aligned}$$

Geometrically, each equation is a line. These lines are either the same (as in the above example), or they intersect in one point. So there is either one solution or infinitely many.

B.7.1. Why are parallel lines not a possibility?

The difference between these two possibilities is reflected in the determinant of A:

$$\det A = 0 \quad\Leftrightarrow\quad \text{number of solutions} = \infty$$

$$\det A \neq 0 \quad\Leftrightarrow\quad \text{number of solutions} = 1$$

B.7.2. Explain why this is clear in terms of both the area and linear independence interpretations of the determinant.

I would like to generalise from the population example and consider any vector that, when multiplied by a matrix A, is sent to a multiple of itself,

$$A\begin{bmatrix}x\\y\end{bmatrix}=\lambda\begin{bmatrix}x\\y\end{bmatrix}$$

Such a vector is called an eigenvector of A, and the number λ is the corresponding eigenvalue. In the population example we were interested in the special case $\lambda = 1$.

B.7.3. For each of the matrices in problem B.3.1, find all eigenvectors and eigenvalues both algebraically and by geometrical reasoning.

The eigenvector equation $A\mathbf{x} = \lambda\mathbf{x}$ can also be rewritten as $A\mathbf{x} = \lambda I\mathbf{x}$ and then $(A-\lambda I)\mathbf{x} = \mathbf{0}$. This is precisely the kind of system we studied above. So we know that there is either one solution or infinitely many, and we can find out which by computing $\det(A-\lambda I)$. Obviously there is always the trivial solution $\mathbf{x} = \mathbf{0}$, but we do not count $\mathbf{0}$ as an eigenvector. So eigenvectors occur precisely when there are infinitely many solutions, i.e., when $\det(A-\lambda I) = 0$.

So if we are looking for the eigenvectors and eigenvalues of our population matrix $A = \left[\begin{smallmatrix}.8 & .5\\ .2 & .5\end{smallmatrix}\right]$ we begin by solving the equation

$$\det(A-\lambda I)=\begin{vmatrix}.8-\lambda & .5\\ .2 & .5-\lambda\end{vmatrix}=(.8-\lambda)(.5-\lambda)-0.1=0$$

The two roots are $\lambda = 1$ and $\lambda = .3$. To find the corresponding eigenvectors we plug each of these values into $(A-\lambda I)\mathbf{x} = \mathbf{0}$. For $\lambda = 1$ this gives $\left[\begin{smallmatrix}-.2 & .5\\ .2 & -.5\end{smallmatrix}\right]\left[\begin{smallmatrix}x\\ y\end{smallmatrix}\right]=\left[\begin{smallmatrix}0\\ 0\end{smallmatrix}\right]$, so $y=\frac{2}{5}x$, so the eigenvector is $t\left[\begin{smallmatrix}1\\ 2/5\end{smallmatrix}\right]$ or $t\left[\begin{smallmatrix}5\\ 2\end{smallmatrix}\right]$, as we already knew. For $\lambda = .3$ we get $\left[\begin{smallmatrix}.5 & .5\\ .2 & .2\end{smallmatrix}\right]\left[\begin{smallmatrix}x\\ y\end{smallmatrix}\right]=\left[\begin{smallmatrix}0\\ 0\end{smallmatrix}\right]$, so $y=-x$, so the eigenvector is $t\left[\begin{smallmatrix}1\\ -1\end{smallmatrix}\right]$.

B.7.4. How can you check these answers? Do so.

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§ B.7.2. Problems

B.7.5. (a) Prove that if each column of a matrix A sums to 1 then A must have 1 as an eigenvalue. Hint: Show that you can create a row of zeroes in the matrix $A-I$ by applying row operations, and then consider what this means for the system of equations $(A-I)\mathbf{x} = 0$. For ease of writing you may assume that A is a 3×3 matrix.

(b) What does this result mean in terms of systems where the entries of the matrix represent mutually exclusive probabilities (as in the city population example)?

B.7.6. The population of a species is divided into three age groups: child, adolescent, adult. Let the number of individuals in each group be encoded as a column vector

$$\begin{bmatrix}x_n\\y_n\\z_n\end{bmatrix}$$

where n is the generation. To obtain the population distribution in the next generation one multiplies by a transition matrix such as

$$A=\begin{bmatrix}0 & 2 & 4\\ 1/16 & 0 & 0\\ 0 & 1/4 & 0\end{bmatrix}$$

$$B=\begin{bmatrix}0 & 2 & 4\\ 1/4 & 0 & 0\\ 0 & 1/2 & 0\end{bmatrix}$$

$$C=\begin{bmatrix}0 & 1/2 & 1/2\\ 1/4 & 0 & 0\\ 0 & 1/2 & 0\end{bmatrix}$$

Match each matrix with its corresponding real-world scenario among those listed below. Also, without calculations, by reasoning about the real-world meaning of eigenvectors and eigenvalues, deduce which eigenvalues and eigenvectors from the options provided should go with each scenario. Confirm your inferences computationally (perhaps using a computer). Possible eigenvectors:

$$\mathbf{v}_1=\begin{bmatrix}2\\1\\1\end{bmatrix}\quad \mathbf{v}_2=\begin{bmatrix}8\\2\\1\end{bmatrix}\quad \mathbf{v}_3=\begin{bmatrix}16\\2\\1\end{bmatrix}$$

Possible eigenvalues: 0.5, 1.

(a) Good environmental conditions: matrix $\boxed{}$ eigenvalue $\boxed{}$ eigenvector $\boxed{}$

(b) Presence of deadly toxins: matrix $\boxed{}$ eigenvalue $\boxed{}$ eigenvector $\boxed{}$

(c) Presence of toxins that diminish fertility: matrix $\boxed{}$ eigenvalue $\boxed{}$ eigenvector $\boxed{}$

B.7.7. Consider the following model of an expanding economy. There are three variables: steel, food and labour. The production of each good consumes a part of what was produced the year before: a new unit of steel requires .4 units of existing steel and .5 units of labour; a new unit of food requires .1 units of existing food and .7 units of labour; producing (or maintaining) a unit of labour costs .8 units of food and .1 units of steel and labour.

(a) Represent this situation by a 3×3 matrix A such that

$$A\begin{bmatrix}s_1\\f_1\\l_1\end{bmatrix}=\begin{bmatrix}s_0\\f_0\\l_0\end{bmatrix}$$

and explain why this matrix equation correctly represents the information given above.

(Note that this equation is "backwards" in the sense that it expresses the "input as a function of the output," so to speak.)

(b) It seems people spend [more/less] time on agriculture than childcare.

(c) Compute the eigenvalues and eigenvectors of the matrix (perhaps by computer). If the proportions of s:f:l are 1 : $\boxed{}$: $\boxed{}$ then the economy is [growing/shrinking] by $\boxed{}$% per year.

B.7.8. Consider an economy based on oil and steel. Extracting oil costs both steel and oil: steel for drills and pipes, and oil to run the pumps. Similarly, mining for steel requires steel drills and rails and oil-driven machinery. Extracting one unit of oil costs .04 units of oil and .08 units of steel. Extracting one unit of steel costs .04 units of oil and .01 units of steel. This is encoded in the matrix

$$A = \begin{bmatrix} .04 & .04 \\ .08 & .01 \end{bmatrix}$$

There is a yearly market demand for 100 units of oil and 20 units of steel, which we may express by the matrix

$$D = \begin{bmatrix} 100 \\ 20 \end{bmatrix}$$

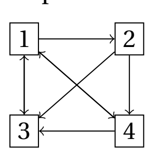
Finally, $X = \left[\begin{smallmatrix} \text{oil} \\ \text{steel} \end{smallmatrix}\right]$ is a column vector expressing the yearly production quantities.

(a) What is the real-world meaning of AX? Hint: $A\left[\begin{smallmatrix} o_n \\ s_n \end{smallmatrix}\right] = \left[\begin{smallmatrix} o_{n-1} \\ s_{n-1} \end{smallmatrix}\right]$.

(b) What is the economic sense of the equation $(I - A)X = D$?

(c) Solve this equation for X and interpret your answer in real-world terms.

B.7.9. † When you perform a Google search, the order of the results is determined using matrix algebra. The basic idea is that if a web page contains n links to other pages then it "passes on" $1/n$ times its own importance to each of those sites. We can think of the web as the board of a board game, on which stacks of coins placed on each site is being moved around in this manner. One "turn" of all websites passing on their importance according to this principle can be encoded in a matrix A such that if $\mathbf{x}$ is the column vector of the importance of the websites then $A\mathbf{x}$ is the column vector of importances after the passing on has taken place. An example is shown below.



$$A = \begin{bmatrix} 0 & 0 & 1 & 1/2 \\ 1/3 & 0 & 0 & 0 \\ 1/3 & 1/2 & 0 & 1/2 \\ 1/3 & 1/2 & 0 & 0 \end{bmatrix}$$

We can now rank the pages by supposing that each website starts with equal importance, i.e.,

$$\mathbf{x}_0 = \begin{bmatrix} 1/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{bmatrix},$$

and applying the matrix A many times. The process can be thought of as modelling the behaviour of a "random surfer" who clicks the links of the page he is on with equal probability. The pages with the highest rankings are those which the random surfer ends up hitting most often.

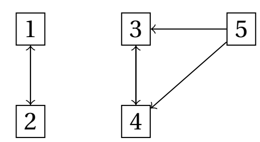
(a) Compute $A^5\mathbf{x}_0$, $A^{20}\mathbf{x}_0$ and $A^{100}\mathbf{x}_0$, and notice that the results are stabilising at particular values. These are the relative importances of these pages according to Google's algorithm.

(b) Explain how this is related to eigenvectors and eigenvalues. Hint: Compare $A^{101}\mathbf{x}_0 \approx A^{100}\mathbf{x}_0$ with $A\mathbf{x} = \lambda\mathbf{x}$.

(c) Show how the same ranking (and relative importances) can found by an eigenvector calculation instead of computing powers of A. (Recall that if $\mathbf{v}$ is an eigenvector then so is any multiple of $\mathbf{v}$.)

(d) Suppose the owner of page 4 tries to boost his ranking by creating a page 5 which links to page 4; page 4 also links to page 5. Does the new page 5 help page 4's ranking?

Consider this disconnected web:



(e) Compute the eigenvalues and eigenvectors of this web. Why does the ranking strategy used above not work here?

(f) ⋆ Does this problem occur for any disconnected web? Does it ever occur for a connected web? Explain using the idea of a stable state and its meaning in terms of eigenvectors and eigenvalues.

(g) To fix this and other problems the actual Google matrix is not A but $0.85A + 0.15B$, where B is a matrix with all entries $1/N$ (where N is the number of pages). Explain how this can be interpreted in terms of the "random surfer" mentioned above.

(h) With this modification, rank the pages of the disconnected web using the eigenvector method.

§ B.8. Diagonalisation

§ B.8.1. Lecture worksheet

In this section we shall find a clever way of figuring out the power A^n of a matrix without actually having to multiply it out so many times. This is done by finding the "diagonalisation" of A. A diagonal matrix is a matrix with all zeros except along the diagonal. Diagonal matrices are very convenient because

$$\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}^n = \begin{bmatrix} a^n & 0 \\ 0 & b^n \end{bmatrix}$$

B.8.1. Interpret this result geometrically.

Therefore we seek a diagonal representation of A, so that we can take its powers in a convenient way. I claim that in fact

$$M = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix}^{-1} A \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

where $\mathbf{v}_1$ and $\mathbf{v}_2$ are the eigenvectors of A written as columns. This is a splendid fact, because if we solve for A in this equation we obtain

$$A = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix}^{-1} \qquad (*)$$

and therefore

$$A^n = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}^n \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix}^{-1}$$

In this way we need only three matrix multiplications instead of a hundred to compute A^{100}.

To prove my claim I only need to compute:

$$\begin{aligned} M \begin{bmatrix} 1 \\ 0 \end{bmatrix} &= \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix}^{-1} A \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix}^{-1} A\mathbf{v}_1 = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix}^{-1} \lambda_1 \mathbf{v}_1 \\ &= \lambda_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \lambda_1 \\ 0 \end{bmatrix} \end{aligned}$$

B.8.2. Justify each step in this calculation by matching the first, second, third, and fourth equalities with a corresponding justification:

- ☐ definition of A
- ☐ rule for scalar product of vector with itself
- ☐ definition of M
- ☐ simplifying by column operations
- ☐ matrix multiplication computation
- ☐ reasoning backwards: what input gives this input?
- ☐ definition of eigenvector

In the same way one finds that $M\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 0 \\ \lambda_2 \end{smallmatrix}\right]$, so M must be $\left[\begin{smallmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{smallmatrix}\right]$, as claimed.

If we apply this to the population example we find that

$$\begin{aligned} A^n &= \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}^n \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix}^{-1} \\ &= \begin{bmatrix} 5 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1^n & 0 \\ 0 & 0.3^n \end{bmatrix} \begin{bmatrix} 5 & 1 \\ 2 & -1 \end{bmatrix}^{-1} \\ &= \begin{bmatrix} 5 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0.3^n \end{bmatrix} \frac{1}{-7} \begin{bmatrix} -1 & -1 \\ -2 & 5 \end{bmatrix} \\ &= \frac{1}{7} \begin{bmatrix} 5+2\cdot 0.3^n & 5-5\cdot 0.3^n \\ 2-2\cdot 0.3^n & 2+5\cdot 0.3^n \end{bmatrix} \end{aligned}$$

In particular,

$$\lim_{n\to\infty} A^n \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \frac{1}{7} \begin{bmatrix} 5 & 5 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \frac{1}{7} \begin{bmatrix} 5(x_0+y_0) \\ 2(x_0+y_0) \end{bmatrix}$$

So no matter what the initial numbers x_0 and y_0 are, in the long run the ratio will stabilise at five sevenths in the suburbs and two sevenths in the city. We already knew that this was the equilibrium, but now we have confirmed explicitly that we always approach this equilibrium regardless of initial condition.

B.8.3. Diagonalise one or two of the matrices from problem B.7.3. Then use the diagonalisation to find a simple expression for an arbitrary power of the matrix A^n and note that the answer could easily have been predicted geometrically.

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§ B.8.2. Problems

B.8.4. Explain how formula $(*)$ from the text can be used to generate a matrix with given eigenvectors and eigenvalues. (This is useful for teachers designing exam problems. If you make up some matrices with simple numbers in them and compute their eigenvalues and eigenvectors you will find that these are typically not simple at all, so this is not a good way of designing manageable exam problems.)

B.8.5. Re-solve problem B.3.5 by finding the eigenvectors and eigenvalues through geometric reasoning and applying formula $(*)$.

B.8.6. Argue that $(*)$ can be interpreted geometrically as follows. To apply the transformation A is the same thing as to: (1) perform a change of variables so that the eigenvectors are the new basis vectors, (2) apply the transformation in this new coordinate system, where it is simply a dilation of each of the basis vectors, (3) revert back to the original variables.

B.8.7. The Fibonacci sequence is a sequence of numbers in which every number is the sum of the two previous numbers: $1, 1, 2, 3, 5, 8, 13, 21, \ldots$, or in symbols $F_n = F_{n-1} + F_{n-2}$. These numbers are found in many places in nature. For example, if you turn a pine cone or a pineapple upside-down and count the number of spirals going clockwise and counter-clockwise you will find that these are two consecutive Fibonacci numbers.

Fibonacci was a 13th century Italian mathematician who used this sequence and many other computational problems to show the superiority of the Arabic numerals that we use today over the Roman numerals still used in Europe at that time. He introduced his sequence by means of the following rabbit population scenario. Suppose each pair of adult rabbits produces one pair of baby rabbits per season. Next season these baby rabbits become adults and start producing offspring of their own. Thus the number of adult rabbit pairs F_n in generation n equals all the rabbit pairs from last generation F_{n-1} plus the new rabbits produced by the rabbits who are reproductively active, i.e., the rabbits who were born at least two seasons ago, F_{n-2}. Thus $F_n = F_{n-1} + F_{n-2}$, as above.

We shall now obtain a formula for F_n that will enable us to find for example F_{1000} in one step instead of the thousand steps required to write out the entire Fibonacci sequence up to this point.

(a) Find a matrix A such that

$$\begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix} = A \begin{bmatrix} F_n \\ F_{n-1} \end{bmatrix}.$$

The eigenvalues of this matrix are $a = \frac{1+\sqrt{5}}{2}$ and $b = \frac{1-\sqrt{5}}{2}$, and the corresponding eigenvectors are $\begin{bmatrix} a \\ 1 \end{bmatrix}$ and $\begin{bmatrix} b \\ 1 \end{bmatrix}$. Diagonalise A and find a formula for F_n by computing a suitable power of A multiplied with $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

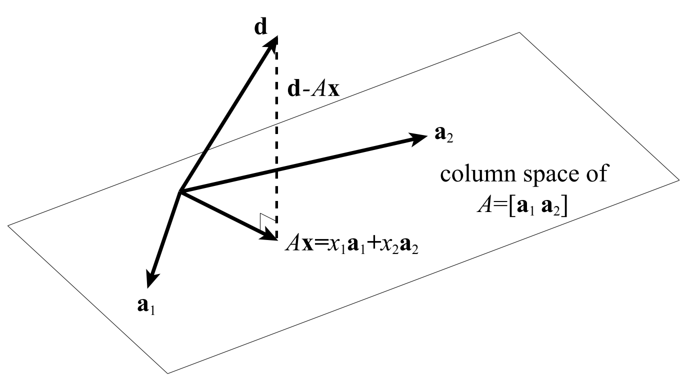
(b) This leads to $F_n = \boxed{}$ (expressed as a formula in terms of a, b and n).

§ B.9. Data

§ B.9.1. Lecture worksheet

Linear algebra is very often used to process big data. Remarkably, many useful numerical techniques for dealing with big sets of data correspond to readily visualisable geometrical ideas concerning vectors. The following is an example of this.

I'm given a vector $\mathbf{d}$ and I am looking for its projection onto a particular plane, namely the plane spanned by the vectors $\mathbf{a}_1$ and $\mathbf{a}_2$. In other words, I am trying to find some combination of these vectors, say $x_1\mathbf{a}_1 + x_2\mathbf{a}_2$, that is as close as possible to $\mathbf{d}$. I can formulate this in terms of matrices by forming a matrix A that has the vectors $\mathbf{a}_1$ and $\mathbf{a}_2$ as columns. Then my problem becomes: choose $\mathbf{x} = (x_1, x_2)$ so that $A\mathbf{x}$ is a close as possible to $\mathbf{d}$.



The best choice of $\mathbf{x}$ is the one that makes the error vector $\mathbf{d} - A\mathbf{x}$ perpendicular to $\mathbf{a}_1$ and $\mathbf{a}_2$. In other words, $\mathbf{a}_1 \cdot (\mathbf{d} - A\mathbf{x}) = 0$ and $\mathbf{a}_2 \cdot (\mathbf{d} - A\mathbf{x}) = 0$, or in matrix terms: $A^T(\mathbf{d} - A\mathbf{x}) = \mathbf{0}$, which can also be written $A^T A\mathbf{x} = A^T\mathbf{d}$. The only unknown in this equation is $\mathbf{x}$. We have thus reduced the geometrical problem of finding $\mathbf{x}$ to a straightforward matter of solving a system of equations numerically.

Now consider a data problem that at first sight appears unrelated to the above but in fact turn out to be the same thing. Let's say I want to investigate the relation between the ages of male and female actors portraying couples in Hollywood movies. Here for example is a small data set:

movie	Brad Pitt's age (t)	age of actress playing his wife (w)
Se7en	32	22
Mr. & Mrs. Smith	41	30
World War Z	49	37

Does this data fit a linear relationship $w = mt + b$? Not exactly, but not far from it. The equations this data would satisfy if the relationship was linear would be:

$$b + mt_1 = w_1$$
$$b + mt_2 = w_2$$
$$b + mt_3 = w_3$$

or in matrix terms:

$$\begin{bmatrix} 1 & t_1 \\ 1 & t_2 \\ 1 & t_3 \end{bmatrix} \begin{bmatrix} b \\ m \end{bmatrix} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$$

In terms of our geometric example, this corresponds to $A\mathbf{x} = \mathbf{d}$. The b and the m are the unknowns $\mathbf{x}$ that we are looking for, and the wife's ages are the data $\mathbf{d}$ that we are trying to capture. But just as in the geometric example, it is not possible to solve this equation $A\mathbf{x} = \mathbf{d}$. But we can solve the problem to the closest possible approximation by the same trick as above, that is, solve $A^T A\mathbf{x} = A^T\mathbf{d}$ instead. In the movie case this becomes:

$$\begin{bmatrix} 1 & 1 & 1 \\ t_1 & t_2 & t_3 \end{bmatrix} \begin{bmatrix} 1 & t_1 \\ 1 & t_2 \\ 1 & t_3 \end{bmatrix} \begin{bmatrix} b \\ m \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ t_1 & t_2 & t_3 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$$

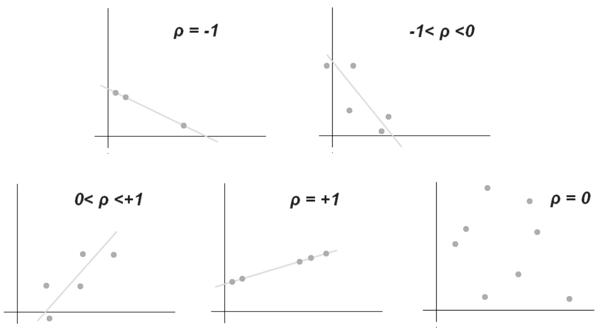
It is straightforward to write this out as a system of equations and solve for the unknowns b and m. This gives $b = -1350/217 \approx -6.22$ and $m = 383/434 \approx 0.88$. Thus $w = 0.88t - 6.22$ is the best linear fit for the given data. So Brad Pitt's movie wives were, so to speak, born a bit over six years after him and furthermore age only 0.88 years for every one of his.

I used only three age pairs in this example, but the method works for a data set of any size. The three-dimensional visualisation enabled us to see the method in an intuitive way, but once we translated it into matrix language we could just as well extend it to any number of dimensions.

§ B.9.2. Problems

B.9.1. Find an interesting data set and perform a linear fit analysis using the matrix method as above.

B.9.2. Another example of "data geometry." The correlation coefficient ρ captures "how correlated" two variables are:



Each data point (x_k, y_k) is an empirical observation such as, for example, a person's salary (y_k) and the number of years of education that person has (x_k).

(a) Which plot is the likeliest depiction of this case?

(b) Come up with plausible real-world scenarios corresponding to the other plots.

The correlation coefficient can be formally defined and calculated using the idea that the scalar product measures the "amount of agreement" between two vectors. This is done as follows. Think of the data as two vectors $(x_1, x_2, x_3, \ldots, x_n)$ and $(y_1, y_2, y_3, \ldots, y_n)$. First we have to "center the data," meaning shift it so that the mean of each variable is zero. To this end, compute the mean values $\mu_x = (x_1+x_2+x_3+\ldots+x_n)/n$ and $\mu_y = (y_1+y_2+y_3+\ldots+y_n)/n$, and now work with the vectors $(x_1-\mu_x, x_2-\mu_x, x_3-\mu_x, \ldots, x_n-\mu_x)$ and $(y_1-\mu_y, y_2-\mu_y, y_3-\mu_y, \ldots, y_n-\mu_y)$ instead. Now compute $\cos\theta$ for the angle between these vectors. This is the correlation coefficient.

(c) Carry out these steps for the first plot above. (Introduce a scale on the axes by taking the x-coordinate of the first point to be 1.)

(d) Visualise the vectors in question in three-dimensional space, and interpret the result visually.

§ B.10. Reference summary

§ B.10.1. Basic matrix algebra

$$ABC = (AB)C = A(BC) \qquad A(B+C) = AB + AC$$

- Add a matrix to a matrix, $A+B$.

Add entry-by-entry:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a+e & b+f \\ c+g & d+h \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 \\ 2 & 0 \end{bmatrix} + \begin{bmatrix} 4 & 2 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 1+4 & 3+2 \\ 2+2 & 0-1 \end{bmatrix} = \begin{bmatrix} 5 & 5 \\ 4 & -1 \end{bmatrix}$$

- Multiply a matrix by a number, kA.

The number multiplies onto each entry:

$$k \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix}$$

$$3 \begin{bmatrix} 1 & 3 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 3\cdot 1 & 3\cdot 3 \\ 3\cdot 2 & 3\cdot 0 \end{bmatrix} = \begin{bmatrix} 3 & 9 \\ 6 & 0 \end{bmatrix}$$

- Multiply a matrix by a matrix, AB.

Highlight the first row of A and the first column of B. Multiply the first entry in the row by the first entry in the column, the second entry in the row by the second entry in the column, end so on, and then add the results. Write down the result as the entry in the first row and first column of the answer.

Next highlight the second column of B instead and repeat the process. Keep going until you have exhausted all possible combinations of rows of A with columns of B. Each such combination gives another entry of the answer (namely that in that row and that column).

Note that in general $AB \neq BA$: the order matters when multiplying matrices.

$$\begin{bmatrix} 2 & 1 \\ 3 & -5 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ 1 & -1 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} \mathbf{2} & \mathbf{1} \\ 3 & -5 \end{bmatrix} \begin{bmatrix} \mathbf{0} & 2 \\ \mathbf{1} & -1 \end{bmatrix}}_{2\cdot 0 + 1\cdot 1 = \mathbf{1}} = \begin{bmatrix} \mathbf{1} & \\ & \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} \mathbf{2} & \mathbf{1} \\ 3 & -5 \end{bmatrix} \begin{bmatrix} 0 & \mathbf{2} \\ 1 & \mathbf{-1} \end{bmatrix}}_{2\cdot 2 + 1\cdot -1 = \mathbf{3}} = \begin{bmatrix} 1 & \mathbf{3} \\ & \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} 2 & 1 \\ \mathbf{3} & \mathbf{-5} \end{bmatrix} \begin{bmatrix} \mathbf{0} & 2 \\ \mathbf{1} & -1 \end{bmatrix}}_{3\cdot 0 + -5\cdot 1 = \mathbf{-5}} = \begin{bmatrix} 1 & 3 \\ \mathbf{-5} & \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} 2 & 1 \\ \mathbf{3} & \mathbf{-5} \end{bmatrix} \begin{bmatrix} 0 & \mathbf{2} \\ 1 & \mathbf{-1} \end{bmatrix}}_{3\cdot 2 + -5\cdot -1 = \mathbf{11}} = \begin{bmatrix} 1 & 3 \\ -5 & \mathbf{11} \end{bmatrix}$$

So altogether the result is that

$$\begin{bmatrix} 2 & 1 \\ 3 & -5 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ -5 & 11 \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{bmatrix}$$

§ B.10.2. Linear transformations

Geometrically, matrices represent linear transformations, meaning transformations that preserve lines. Such transformations are rotations, reflections, dilations (i.e., magnifications, or scalings), linear projections, and combinations of these. Matrix multiplication always leaves the origin intact so translations (i.e., vertical or horizontal displacements) are not included.

Algebraically, this corresponds to the multiplication $A\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] = \left[\begin{smallmatrix} X \\ Y \end{smallmatrix}\right]$ A transforms the input point (x, y) into some output point (X, Y).

The first column of A is its effect on the unit vector $\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$, the second on the unit vector $\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]$.

Matrix multiplication AB represents the composition of the linear transformations A and B in the order "first apply B then apply A" (since input vectors are "plugged in on the right").

- Find the matrix representing a given linear transformation specified in geometrical language (rotation, reflection, etc.).

  Determine the effect of the transformation of the unit basis vectors. Write the results as the columns of a matrix. This is the sought transformation matrix.

Find the 2×2 matrix representing a reflection in the line $y = -x$.

$$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

Find the 3×3 matrix representing a reflection of three-dimensional space in the plane $y = z$.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

- Characterise geometrically the effect of a given matrix A.

  Calculate A's effect on the unit basis vectors ($A\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$ and so on), and picture the resulting vectors along with the original unit basis vectors. Determine by visualisation what linear transformation sends the latter vectors onto the former. This is the answer.

What geometric transformation is $\begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$?

45° counterclockwise rotation.

What geometric transformation is $\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$?

90° rotation about the z-axis, clockwise as seen from "above" (positive z position).

§ B.10.3. Gaussian elimination

Correspondence between matrices and systems of linear equations:

$$\begin{aligned} ax + by &= c \\ dx + ey &= f \end{aligned} \quad \leftrightarrow \quad \left[\begin{array}{cc|c} a & b & c \\ d & e & f \end{array}\right]$$

Gaussian elimination rules:

– You may add or subtract any row from any other row.

– You may multiply or divide any row by any number.

– You may switch places of any two rows.

The goal of Gaussian elimination is generally to turn the matrix into identity-matrix form, or at least upper-triangular form (i.e., having all 0's in the bottom-left half below its principal diagonal).

- Solve a system of linear equations.

  Translate the system into matrix form as above. Apply Gaussian elimination to turn the matrix (the part of it before the bar) into upper-triangular form. Translate the last row back into an ordinary equation. This gives you the value for the last variable. Translate the next-to-last row back into an ordinary equation, and plug in the value for the last variable. This gives you the value for the next-to-last variable. And so on.

  If you reach an equation of the form $0 = a$ where $a \neq 0$: the system has no solutions.

  If you reach an equation of the form $0 = 0$: the system has infinitely many solutions. Instead of entering a specific value for the variable corresponding to this equation, set it equal to a parameter, such as t, and proceed as usual (this means that the other variables will become expressed in terms of t also). Your formulas for the values of the variables give a solution for the system of equation for any value of t you plug into it.

- Determine whether a system of linear equations has 0, 1, or infinitely many solutions.

  Attempt to solve the system as above and see the rules for the number of solutions there.

$$\begin{array}{ccccc} x & + & y & = & 2 \\ 2x & + & 2y & = & 4 \end{array}$$

$$\sim \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 2 & 2 & 4 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right]$$

Hence

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2-t \\ t \end{bmatrix}$$

is a solution for any value of t.

$$\begin{array}{rcrcl} x & + & y & = & 2 \\ 2x & + & 2y & = & 5 \end{array}$$

$$\sim \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 2 & 2 & 5 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 0 & 1 \end{array}\right]$$

There are no values for x and y that make $0 = 1$. Hence the system of equations has no solutions.

For what value(s) of a does the following system of equations have no solutions?

$$\begin{array}{rcrcrcl} x & - & 2y & + & 3z & = & 2 \\ 2x & - & y & + & 2z & = & 3 \\ x & + & y & + & az & = & a \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & -2 & 3 & 2 \\ 2 & -1 & 2 & 3 \\ 1 & 1 & a & a \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -2 & 3 & 2 \\ 0 & 3 & -4 & -1 \\ 0 & 3 & a-3 & a-2 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & -2 & 3 & 2 \\ 0 & 3 & -4 & -1 \\ 0 & 0 & a+1 & a-1 \end{array}\right]$$

There are no solutions when $a - 1 \neq 0$ and at the same time $a + 1 = 0$, thus when $a = -1$.

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§ B.10.4. Special matrices

I	Identity matrix. A matrix with 1's on its principal diagonal and 0's elsewhere, as in $\left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$. Multiplying by I has no effect: $AI = IA = A$.
A^T	Transpose of A. A with rows and columns interchanged: the rows become the columns and the columns become the rows.

$$\begin{bmatrix} 2 & 5 & 6 \\ 1 & 3 & 4 \end{bmatrix}^T = \begin{bmatrix} 2 & 1 \\ 5 & 3 \\ 6 & 4 \end{bmatrix}$$

A^{-1}	Inverse of A. A matrix such that $AA^{-1} = A^{-1}A = I$.
orthogonal matrix	A matrix whose columns (and rows) form a system of orthogonal unit vectors, or equivalently: a matrix A such that $A^{-1} = A^T$.
singular matrix	A matrix with determinant zero. A singular matrix is non-invertible.
symmetric matrix	A matrix that equals its own reflection in its principal diagonal; matrix A such that $A = A^T$.
antisymmetric matrix	Matrix A such that $A = -A^T$.

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§ B.10.5. Determinants

$$\begin{aligned} \det A = \begin{vmatrix} a & b \\ c & d \end{vmatrix} &= ad - bc \\ &= \pm \text{ area of parallelogram spanned by } \left[\begin{smallmatrix} a \\ c \end{smallmatrix}\right] \text{ and } \left[\begin{smallmatrix} b \\ d \end{smallmatrix}\right] \\ &= \pm \text{ area scaling factor of linear transformation } A \end{aligned}$$

Similarly for 3×3 determinants which represent the volume of the parallelepiped spanned by its column vectors, and so on.

$$\det AB = (\det A)(\det B) \qquad \det A^{-1} = \frac{1}{\det A} \qquad \det A = \det A^T$$

- Compute the determinant of a matrix.

  For a 2×2 matrix, $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$.

  For a bigger matrix, select any one row or column of the matrix (preferably one with many 0's in it). For each entry in this row or column:

  - Block out the row and column containing this entry, and write down the determinant of the matrix that remains.
  - Write the entry itself as a coefficient in front of this determinant.
  - Decide whether the entry is associated with a plus or minus sign. The top left entry (of the original matrix) is positive and every time you go one step over or one step down the sign changes. Once the sign is determined, write it in front of the entry coefficient.

  The original matrix is equal to the resulting sum of smaller matrices (with their appropriate signs and coefficients). Apply the same process to these smaller matrices until you have broken them down to 2×2 determinants, which can be evaluated as above.

Find the determinant: $\begin{vmatrix} 2 & 1 & 4 \\ 1 & 5 & 1 \\ 2 & 3 & 1 \end{vmatrix}$

We can do this by expanding by any one row or column. Let's pick the first column. Then each entry of this column needs to be multiplied by the 2×2 determinant that remains when its row and column is blocked out:

$$2\begin{vmatrix} 2 & 1 & 4 \\ 1 & 5 & 1 \\ 2 & 3 & 1 \end{vmatrix} = 2\begin{vmatrix} 5 & 1 \\ 3 & 1 \end{vmatrix} = 4$$

$$1\begin{vmatrix} 2 & 1 & 4 \\ 1 & 5 & 1 \\ 2 & 3 & 1 \end{vmatrix} = 1\begin{vmatrix} 1 & 4 \\ 3 & 1 \end{vmatrix} = -11$$

$$2\begin{vmatrix} 2 & 1 & 4 \\ 1 & 5 & 1 \\ 2 & 3 & 1 \end{vmatrix} = 2\begin{vmatrix} 1 & 4 \\ 5 & 1 \end{vmatrix} = -38$$

These results are to be added together except first some terms must be given a minus sign based on "where it came from" according to the pattern

$$\begin{matrix} + & - & + \\ - & + & - \\ + & - & + \end{matrix}$$

In other words, to find the sign of a given position we "count it off" in alternating plusses and minuses starting from a plus in the top left corner. In our case, therefore, the middle term must be negated. So the final answer is $4 + 11 - 38 = -23$.

Find the determinant: $\begin{vmatrix} 3 & 4 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 4 & 2 \end{vmatrix}$

Expanding by first row: $= 3\begin{vmatrix} 2 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 4 & 2 \end{vmatrix} - 4\begin{vmatrix} 1 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 4 & 2 \end{vmatrix} =$

$3 \cdot 2\begin{vmatrix} 3 & 1 \\ 4 & 2 \end{vmatrix} - 4\begin{vmatrix} 3 & 1 \\ 4 & 2 \end{vmatrix} = 6 \cdot 2 - 4 \cdot 2 = 4$

- Simplify a determinant before computing it.

  If you add or subtract any multiple of one row to another row, the value of the determinant does not change. If you add or subtract any multiple of one column to another column, the value of the determinant does not change. The best use of this is usually to create as many 0's as possible.

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = \begin{vmatrix} a & b & c \\ d & e & f \\ g-a & h-b & i-c \end{vmatrix}$$

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = \begin{vmatrix} a & b-a & c \\ d & e-d & f \\ g & h-g & i \end{vmatrix}$$

If all entries in a row (or column) are multiples of the same number k, then you can factor this number out. Place the k as a coefficient of the determinant and divide away this factor in all the entries of the row (or column) in question, leaving other rows (or columns) unchanged.

$$\begin{vmatrix} k \cdot a & b & c \\ k \cdot d & e & f \\ k \cdot g & h & i \end{vmatrix} = k\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$$

You can switch places of two of the rows (or two of the columns), but then you must also switch the sign of the determinant (multiply by -1 in front). In this way the value of the determinant remains the same.

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = -\begin{vmatrix} d & e & f \\ a & b & c \\ g & h & i \end{vmatrix}$$

Evaluate the determinant

$$\begin{vmatrix} 1 & 2 & 4 & 4 \\ 0 & 2 & 9 & 1 \\ 0 & 0 & 1 & 8 \\ 2 & 4 & 8 & 9 \end{vmatrix}$$

To make more 0s, subtract twice the first row from the last row. Then expand by the first column repeatedly.

$$\begin{vmatrix} 1 & 2 & 4 & 4 \\ 0 & 2 & 9 & 1 \\ 0 & 0 & 1 & 8 \\ 2 & 4 & 8 & 9 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 4 & 4 \\ 0 & 2 & 9 & 1 \\ 0 & 0 & 1 & 8 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

$$= 1 \cdot \begin{vmatrix} 2 & 9 & 1 \\ 0 & 1 & 8 \\ 0 & 0 & 1 \end{vmatrix} = 1 \cdot 2 \cdot \begin{vmatrix} 1 & 8 \\ 0 & 1 \end{vmatrix} = 1 \cdot 2 \cdot 1 \cdot 1 = 2$$

Find the determinant of the matrix

$$A = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & a^3 & a^2 & a \\ 1 & a & 2 & 1 \end{pmatrix}$$

Subtract the first row from the last, and then expand by the first column:

$$\det A = \begin{vmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & a^3 & a^2 & a \\ 0 & a & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ a^3 & a^2 & a \\ a & 1 & 1 \end{vmatrix}$$

Subtract the first row from the third:

$$\det A = \begin{vmatrix} 1 & 1 & 1 \\ a^3 & a^2 & a \\ a-1 & 0 & 0 \end{vmatrix}$$

Expand by the third row:

$$\det A = (a-1)\begin{vmatrix} 1 & 1 \\ a^2 & a \end{vmatrix} = (a-1)(a-a^2) = -a(a-1)^2$$

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§ B.10.6. Matrix inverses

- Find the inverse of a given matrix A.

  For a 2×2 matrix,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

  For a bigger matrix, transform $[A|I]$ into $[I|B]$ by row manipulations; then B is the inverse of A. In greater detail this means the following. Form the augmented matrix $[A|I]$ consisting of A and an identity matrix I of the same dimensions written next to it. Focussing on the left half the augmented matrix, but carrying out all operations also on the right half, rewrite A into I using the Gaussian elimination rules (§B.10.3). As you turn the left half of the matrix into I, the right half will keep changing with every operation. When the left side has become I, what remains on the right half is the inverse of A, as sought.

$$\begin{pmatrix} 1 & -1 \\ 3 & 2 \end{pmatrix}^{-1} = \frac{1}{5}\begin{pmatrix} 2 & 1 \\ -3 & 1 \end{pmatrix} = \begin{pmatrix} \frac{2}{5} & \frac{1}{5} \\ -\frac{3}{5} & \frac{1}{5} \end{pmatrix}$$

Invert $A = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$\left[\begin{array}{ccc|ccc} 0 & 1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

Thus

$$A^{-1} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Find the inverse of $A = \begin{bmatrix} 0 & 1 & 1 \\ 2 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$\left[\begin{array}{ccc|ccc} 0 & 1 & 1 & 1 & 0 & 0 \\ 2 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 2 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

So

$$A^{-1} = \begin{bmatrix} 0 & \frac{1}{2} & 0 \\ 1 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

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§ B.10.7. Eigenvectors and eigenvalues

Definition: If $A\mathbf{v} = \lambda\mathbf{v}$ for some number λ then $\mathbf{v}$ ($\neq \mathbf{0}$) is called an eigenvector of A and λ is the eigenvalue associated with it.

- Find the eigenvalues λ of a given matrix A.

  Solve $\det(A - \lambda I) = 0$.

Find the eigenvalues of $A = \begin{pmatrix} 7 & 1 \\ 2 & 6 \end{pmatrix}$.

$0 = \begin{vmatrix} 7-\lambda & 1 \\ 2 & 6-\lambda \end{vmatrix} = (7-\lambda)(6-\lambda) - 2 = \lambda^2 - 13\lambda + 40 = (\lambda-5)(\lambda-8)$, so the eigenvalues are $\lambda_1 = 5$ and $\lambda_2 = 8$.

- Find the eigenvectors $\mathbf{v}$ of a given matrix A.

  Having determined the eigenvalues λ as above, plug each of them into $(A - \lambda I)\mathbf{v} = \mathbf{0}$ and solve for the corresponding eigenvector.

Find the eigenvectors and eigenvalues of $A = \left[\begin{smallmatrix} 3 & 1 \\ 0 & 2 \end{smallmatrix}\right]$.

$$\det(A - \lambda I) = \begin{vmatrix} 3-\lambda & 1 \\ 0 & 2-\lambda \end{vmatrix} = (3-\lambda)(2-\lambda) = 0$$

Thus the two eigenvalues are $\lambda = 2$ and $\lambda = 3$. To find the corresponding eigenvectors we plug each of these values into $(A - \lambda I)\mathbf{x} = \mathbf{0}$. For $\lambda = 2$ this gives $\left[\begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix}\right]\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] = \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]$, so $x = -y$, so the eigenvector is $t\left[\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right]$. For $\lambda = 3$ we get $\left[\begin{smallmatrix} 0 & 1 \\ 0 & -1 \end{smallmatrix}\right]\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] = \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]$, so $y = 0$ and x can be anything, so the eigenvector is $t\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$.

Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}$.

$\det(A - \lambda I) = 0 \Rightarrow (3-\lambda)(2-\lambda) = 0 \Rightarrow \lambda_1 = 2$ and $\lambda_2 = 3$.

For λ_1: $\begin{bmatrix} 3-2 & 1 \\ 0 & 2-2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{cases} x + y = 0 \\ 0\cdot x + 0\cdot y = 0 \end{cases}$

$\Rightarrow y = -x \Rightarrow \mathbf{v}_1 = t\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

For λ_2: $\begin{bmatrix} 3-3 & 1 \\ 0 & 2-3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{cases} 0\cdot x + 0\cdot y = 0 \\ 0\cdot x + 1\cdot y = 0 \end{cases}$

$\Rightarrow y = 0 \Rightarrow \mathbf{v}_2 = t\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

§ B.10.8. Diagonalisation

$$A = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix}\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix}^{-1}$$

where $\mathbf{v}_1$ and $\mathbf{v}_2$ are the eigenvectors of A written as columns.

- Diagonalise a given matrix A.

  Find its eigenvectors and eigenvalues and enter them into the above formula.

Diagonalise $A = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}$, that is, find a matrix C, C^{-1}, and a diagonal matrix D such that $A = CDC^{-1}$.

Above we found $\lambda_1 = 2$, $\lambda_2 = 3$, $\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Therefore

$$A = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}$$

- Compute a power A^n of a given matrix A.

  Diagonalise A. Multiplying its diagonalised expression with itself repeatedly yields

$$A^n = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix}\begin{bmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{bmatrix}\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix}^{-1}$$

Given $A = \begin{bmatrix} -1 & 4 \\ 0 & 1 \end{bmatrix}$ find A^{139} using diagonalisation.

$\det(A - \lambda I) = \lambda^2 - 1 = 0 \Rightarrow \lambda_1 = -1, \lambda_2 = 1$. To find the eigenvector for $\lambda_1 = -1$ we form the equation $\det(A - \lambda_1 I)\vec{v} = 0 \Rightarrow \begin{bmatrix} 0 & 4 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Thus $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. By a similar calculation, $\vec{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. By diagonalisation we know that $A = SDS^{-1}$ where $S = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$. To find the required power we multiply $A^{139} = (SDS^{-1})^{139} = SDS^{-1}SDS^{-1}\cdots SDS^{-1} = SD^{139}S^{-1}$ which becomes

$$A^{139} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}^{139}\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 4 \\ 0 & 1 \end{bmatrix}$$

§ B.10.9. Examples

Give an example of a 2×2 matrix B such that $B^8 = I$ but $B \neq I$, $B^2 \neq I$, and $B^4 \neq I$.

Reasoning geometrically, we see that a rotation 45° either clockwise or counterclockwise has the desired properties. This corresponds to the matrices

$$B = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \text{ or } B = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

Given $A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 7 \\ 2 & -4 \\ 0 & 2 \end{bmatrix}$ find a matrix X such that $XA = B$.

Multiply both sides of the equation by $A^{-1} = \frac{1}{5}\begin{bmatrix} 4 & -3 \\ -1 & 2 \end{bmatrix}$ on the right. Since $XAA^{-1} = BA^{-1}$ simplifies to $X = BA^{-1}$ we get $X = \frac{1}{5}\begin{bmatrix} 1 & 7 \\ 2 & -4 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 4 & -3 \\ -1 & 2 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} -3 & 11 \\ 12 & -14 \\ -2 & 4 \end{bmatrix}$

Find all eigenvalues and eigenvectors of $A = \begin{bmatrix} 3 & -4 & 8 \\ 2 & -3 & 8 \\ 0 & 0 & 1 \end{bmatrix}$

Then find $A^{11}\vec{v}$ where $\vec{v} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$

$\det(A - \lambda I) = 0 \Rightarrow \begin{vmatrix} (3-\lambda) & -4 & 8 \\ 2 & (-3-\lambda) & 8 \\ 0 & 0 & (1-\lambda) \end{vmatrix} = 0$

$\Rightarrow (1-\lambda)(\lambda^2 - 1) = 0 \Rightarrow (1-\lambda)(\lambda-1)(\lambda+1) = 0.$

The eigenvectors for $\lambda_1 = 1$ are found by

$$\begin{bmatrix} 2 & -4 & 8 \\ 2 & -4 & 8 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

which reduces to $x - 2y + 4z = 0$. The solutions can be written as

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = s \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix}$$

so these are the two eigenvectors.

For $\lambda_2 = -1$ we get

$$\begin{bmatrix} 4 & -4 & 8 \\ 2 & -2 & 8 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

or in other words $x - y + 2z = 0$ and $z = 0$, which means that the solutions are

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

This is our eigenvector for this eigenvalue.

For the second part we diagonalise the matrix: our eigencalulations show that $A = PDP^{-1}$ where

$$P = \begin{bmatrix} 2 & -4 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

It follows that $A^{11} = PD^{11}P^{-1}$. But

$$D^{11} = \begin{bmatrix} 1^{11} & 0 & 0 \\ 0 & 1^{11} & 0 \\ 0 & 0 & (-1)^{11} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = D$$

so $A^{11} = PD^{11}P^{-1} = PDP^{-1} = A$. Hence

$$A^{11}\vec{v} = A\vec{v} = \begin{bmatrix} 3 & -4 & 8 \\ 2 & -3 & 8 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 9 \\ 8 \\ 1 \end{bmatrix}$$

C NOTATION REFERENCE TABLE

§ C.1. Logic

$\implies$	implies
$\iff$	is equivalent to; if and only if

§ C.2. Calculus

f'	derivative of f
$\frac{df}{dx}$	derivative of f
$\frac{d}{dx} f(x)$	derivative of f
$\dot{x}$	derivative of x when x is a function of time
Δx	change in x; difference between two x-values (Δ = delta = difference)
dx	"infinitesimal" or "infinitely small" change in x (d = difference)
$f^{-1}(x)$	inverse function of $f(x)$
$[F(x)]_a^b$	$F(b) - F(a)$
$\to$	goes to; approaches
∞	infinity

§ C.3. Algebra

$\lvert x \rvert$	absolute value of x; "size," distance to origin Example: $\lvert -5 \rvert = 5$.
$\pm x$	$+x$ or $-x$.
$\pm \cdots \mp$	$+$ in the first position goes with $-$ in the second, and vice versa.
$n!$	Multiply n by every integer below it.

$$4! = 4 \cdot 3 \cdot 2 \cdot 1$$

$\sum_{n=a}^{b} f(n)$	For every integer n starting with a and going to b, compute $f(n)$, and add all of the results together. (Σ = sigma = sum.)

$$\sum_{n=0}^{\infty} (-1)^n x^n = 1 - x + x^2 - x^3 + x^4 - x^5 + \cdots$$

$\prod_{n=a}^{b} f(n)$	For every integer n starting with a and going to b, compute $f(n)$, and multiply all of the results together. (Π = pi = product.)

$$\prod_{n=1}^{3} (1 - x^n) = (1 - x)(1 - x^2)(1 - x^3)$$

§ C.4. Vectors

$\mathbf{v}$ or $\vec{v}$	vector
$\hat{\mathbf{v}}$	unit vector (length 1)
$\lvert \mathbf{v} \rvert$	length of vector $\mathbf{v}$
$\overrightarrow{AB}$	vector pointing from point A to point B

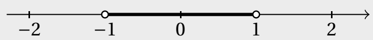
§ C.5. Multivariable and vector calculus

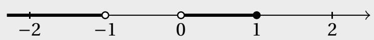
∂x	infinitesimal change in x in a multivariable context
$f_x, \frac{\partial f}{\partial x}$	partial derivative of f with respect to x
$\left(\frac{\partial f}{\partial x}\right)_y$	Partial derivative of f with respect to x, emphasising the fact that y is considered fixed; not different in meaning from f_x, but useful to avoid confusion in certain contexts.
∇f	gradient of f; (f_x, f_y, f_z)
∇	formal vector $\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$

§ C.6. Sets and intervals

(a,b)	the interval from a to b; all numbers between a to b
(or)	endpoint not included
[or]	endpoint included
$\cup$	union (the aggregate, "everything combined," "pool your resources")
$\cap$	intersection (what is common to both)
$\setminus$ or $-$	"set minus"; difference
$A \setminus B$ or $A - B$	A with everything from B taken out
$\in$	is an element of
$A \subset B$	A is contained in B, is a subset of B
$A \subseteq B$	A is contained in B, and is possibly equal to it
$\mathbb{N}$	the natural numbers $(1, 2, 3, \ldots)$
$\mathbb{Z}$	the whole numbers, the integers $(\ldots, -2, -1, 0, 1, 2, 3, \ldots)$

$\mathbb{Q}$	the rational numbers; numbers that are the ratio of two integers
$\mathbb{R}$	the real numbers; a “whole axis”
$\mathbb{C}$	the complex numbers
$\emptyset$ or $\varnothing$	the empty set; nothing

$(-1,1)$ = 

$(-\infty,-1)\cup(0,1]$ = 

$$1\in[0,1]\quad 1\notin(0,1)$$

$$[-2,2]\cap(0,4)=(0,2]$$

$$[0,2]-(1,\infty)=[0,1]$$

$$\varnothing\subset\mathbb{N}\subset\mathbb{Z}\subset\mathbb{Q}\subset\mathbb{R}\subset\mathbb{C}$$

$$2\in\mathbb{Z}\quad \tfrac{3}{2}\in\mathbb{Q}\quad \pi\in\mathbb{R}\quad \pi\notin\mathbb{Q}$$

§ C.7. Book-organisational

	Calculator or computer to be used for this problem.
⋆	“Bonus” material; asides that can be skipped.
†	Harder problem.