MATH 430: INTRODUCTION TO TOPOLOGY
PROBLEM SET #9

SPRING 2025

Due Wednesday, April 23. You may consult books, papers, and websites as long as you cite all sources and write up your solutions in your own words.

Problem 1. Show that if X is path-connected and simply-connected, then for any two points $x, y \in X$, there is a unique path-homotopy class of paths from x to y. *Hint:* Given paths γ_0, γ_1 from x to y, consider $\gamma_0 * \bar{\gamma}_1 * \gamma_1$.

Problem 2. Recall the Möbius band $\mathcal{M} = [0,1]^2/{\sim}$ from Problem Set 7, #6, where $\sim$ consists of the identifications $(0, y) \sim (1, 1-y)$ for all $y \in [0, 1]$.

(1) What space do you get if you cut $\mathcal{M}$ along the image of the line segment in $[0,1]^2$ between $(0, \frac{1}{2})$ and $(1, \frac{1}{2})$?

(2) What space do you get if you glue two copies of $\mathcal{M}$ along their boundary? *Hint:* It's easier to work at the level of $[0,1]^2$. The way that the Möbius boundaries are oriented relative to each other will not matter.

Problem 3. A surface is a (path-connected) two-dimensional manifold. Given surfaces R and S, we form the *connect sum*

$$R\#S = (R \sqcup S)/{\sim}$$

by excising small open disks from R and S, then gluing them along the boundaries of the resulting holes. Let

$$X = P^2 \# P^2, \quad \text{where } P^2 \text{ is the real projective plane (Munkres 372),}$$

and let $x \in X$ be a point on the gluing boundary.

(1) Show that X is homeomorphic to a Klein bottle K, using the descriptions of P^2 and K as quotients of $[0,1]^2$.

(2) Compute $\pi_1(X, x)$ by applying Seifert–van Kampen to open neighborhoods of the two copies of P^2 used to form X. Do not simply cite the formula for the fundamental group of a Klein bottle.

Problem 4 (Munkres 453, #3(b))**.** (1) Find a 2-fold covering

$$p : T \to K,$$

where T is the torus and K the Klein bottle.

(2) Describe the homomorphism of fundamental groups induced by p.

Problem 5 (Munkres 341, #3)**.** Let $p : E \to B$ be a covering map. Show that if B is connected and $p^{-1}(b_0)$ has k elements for some $b_0 \in B$, then $p^{-1}(b)$ has k elements for all $b \in B$. In this case, we say that p is a *k-fold covering*.

Problem 6. Classify the 2-fold coverings of the figure-eight space $S^1 \vee S^1$, up to homeomorphisms of the covering space. Note that $S^1 \vee S^1$ has a symmetry of order two. As written, the problem statement is not distinguishing coverings that differ by a lift of this symmetry.

Problem 7 (Munkres 348, #4)**.** Let $\mathbf{R}_+ \subset \mathbf{R}$ be the subset of positive numbers, and let $\mathbf{0} = (0,0) \in \mathbf{R}^2$. Let

$$p : \mathbf{R} \times \mathbf{R}_+ \to \mathbf{R}^2 - \{\mathbf{0}\} \quad \text{be defined by } p(u,r) = (r\cos(2\pi u), r\sin(2\pi u)).$$

This is a covering map. Find liftings along p of the following paths in $\mathbf{R}^2 - \{\mathbf{0}\}$:

$$\begin{aligned} f(t) &= (2-t, 0), \\ g(t) &= ((1+t)\cos(2\pi t), (1+t)\sin(2\pi t)), \\ h &= f * g. \end{aligned}$$

Sketch (the images of) these paths and their liftings.

Problem 8 (Munkres 348, #5)**.** Let

$$p : \mathbf{R} \to S^1 \quad \text{be defined by } p(t) = (\cos(2\pi u), \sin(2\pi u)).$$

Consider the path in $S^1 \times S^1$ given by

$$f(t) = ((\cos(2\pi t), \sin(2\pi t)), (\cos(4\pi t), \sin(4\pi t))).$$

Sketch (the image of) f in $S^1 \times S^1$ when the latter is viewed as a doughnut surface. Find an (explicit) lifting $\tilde{f}$ of f along $(p,p) : \mathbf{R} \times \mathbf{R} \to S^1 \times S^1$, and sketch it.