Warmup how many topologies on $\varnothing$? [one]

Q is $\varnothing$ connected? [yes!]

Q is R^ω connected in the box topology? [no]

say that (x_i) in R^ω is bounded iff there is a fixed $C > 0$ s.t. $|x_i| < C$ for all i

$U = \{(x_i)_i \text{ is bounded}\}$
$V = \{(x_i)_i \text{ is not bounded}\}$

disjoint, nonempty, and satisfy $R^\omega = U \cup V$

to prove that U is open:

Lem if $(x_i)_i$ in U, then for any fixed $\delta > 0$, we have $\prod_i (x_i - \delta, x_i + \delta) \subset U$

Pf pick $(y_i)_i$ in $\prod_i (x_i - \delta, x_i + \delta)$
pick C s.t. $|x_i| < C$ for all i

then $|y_i| < |x_i + \delta| \leq |x_i| + \delta < C + \delta$ for all i
so $(y_i)_i$ in U

a similar proof shows that V is open, so:

Thm R^ω is disconnected in the box top

Q for integer n, is R^n connected in the box = product = analytic topology? [yes!]

Facts

1) top on R^n = product top arising from analytic R^{n-1} and R
   [more general: Munkres 118, #4]
2) R is connected
3) if X, Y are connected, then $X \times Y$ is too

Thm R^n is connected

Pf if $n = 0$, then $R^n = \{0\}$, so true
suspose $n > 0$ and induct:

by 1), enough to show $R^{n-1} \times R$ connected in product top
by 2), enough to show analytic R^{n-1} and R are connected
by 3), R is connected
by inductive hypothesis, R^{n-1} is connected

Q examples of connected subspaces of R?
[$\varnothing$, singleton sets, intervals]
how about connected subspaces of Q?

Thm the only connected subsets of Q are $\varnothing$ and singleton sets

Pf suppose A sub Q contains distinct a, b
pick irrational α s.t. $a < \alpha < b$

then take $U = A \cap (-\infty, \alpha)$ and $V = A \cap (\alpha, \infty)$
then U, V are disjoint, nonempty, open sets of A s.t. $A = U \cup V$

Df a top space is totally disconnected iff its only connected subspaces are $\varnothing$ and singletons

so Q is totally disconnected, but not discrete, in the subspace top it inherits from analytic R

[we'll use the same proof strategy to show:]

Thm (Intermediate Value Thm) suppose
X connected,
f : X to R cts,
x, y in X

if f(x) ≤ α ≤ f(y), then there is z in X s.t. f(z) = α

Pf if no such z, then
U = f^{-1}((–∞, α)) and V = f^{-1}((α, ∞))
would be a separation of X

Moral to study general top spaces, helpful to compare them to R via cts functions

Df a path in X from x to y is a cts map γ : [0, 1] to X s.t. γ(0) = x and γ(1) = y, where we give [0, 1] the analytic top

[sometimes replace [0, 1] with [a, b], where a < b]

Df X is path-connected iff, for all x, y in X, there is a path from x to y

Thm	if X is path-connected,
then X is connected

Pf	suppose X is not connected
pick a separation U, V
pick x in U and y in V
then no path from x to y is possible

Q	if X is connected, is it path-connected?
[no]

counterex known as the "topologist's sine curve"