MATH 251: TOPOLOGY II

SPRING 2026 PRACTICE PROBLEMS

Contents

Note: All citations are to Munkres's textbook, *Topology*, 2nd Edition. When a problem statement has a proof in Munkres, try your best to find your own proof, before comparing with his.

0. Review of Topology I

Problem 1. Show that the following conditions on a space X are equivalent:

(1) For any pair of distinct points $x, y \in X$, we can find an open set of X containing x but not y.
(2) For any $x \in X$, the singleton set $\{x\}$ is closed.
(3) All finite subsets of X are closed.

In this situation, we say that X is a *T1 space*.

Problem 2. Let X be the real line $\mathbf{R}$ in its finite complement, or *cofinite*, topology. Show that every sequence of points in X converges to every point of X simultaneously. Deduce that X is not Hausdorff.

Problem 3. Show that any metric space is Hausdorff.

Problem 4. Show that for any integer $n \geq 1$, the analytic topology on $\mathbf{R}^n$ matches the product topology on $\mathbf{R} \times \cdots \times \mathbf{R}$ (where there are n factors).

Problem 5. Let $p \colon \mathbf{R} \to S^1$ be the map $p(t) = (\cos(2\pi t), \sin(2\pi t))$. For any $a, b \in \mathbf{R}$, we define the *(open) arc* $J_{a,b} \subseteq S^1$ to be

$$J_{a,b} = \{p(t) \mid a < t < b\}.$$

Show that the collection of arcs $\{J_{a,b} \mid a, b \in \mathbf{R}\}$ satisfies the definition of a basis (Munkres page 78).

Problem 6. Show that the following topologies on S^1 are all the same:

- The topology generated by the basis $\{J_{a,b} \mid a, b \in \mathbf{R}\}$ in Problem 5.
- The subspace topology that S^1 inherits from its inclusion into $\mathbf{R}^2$.
- The quotient topology that S^1 inherits from the surjective map $p \colon \mathbf{R} \to S^1$.

Problem 7. Suppose that $f_1, f_2, f_3 \colon X \to Y$ are all continuous maps. Given a homotopy φ from f_1 to f_2, and a homotopy ψ from f_2 to f_3, construct a homotopy from f_1 to f_3 explicitly in terms of φ and ψ.

Problem 8. Let $f, g \colon X \to Y$ and $F, G \colon Y \to Z$ be continuous. Show that

$$\text{if } f \sim g \text{ and } F \sim G, \quad \text{then } F \circ f \sim G \circ g.$$

Deduce that $F \circ f \sim F \circ g$ and $F \circ f \sim G \circ f$.

Problem 9. Use Problem 8 to show: If X, Y are nonempty and Y is contractible, then any two continuous maps from X into Y are homotopic.

1. Fundamental Groups and Covering Spaces

2. Separation Theorems in the Plane

3. Simplicial Complexes

4. Homology