Review fix a top space X with basept x

$$\pi_1(X, x) = \{[\gamma] \mid \text{loops } \gamma \text{ in } X \text{ based at } x\}$$

for any pointed cts map $f : (X, x)$ to (Y, y), have a

homomorphism $f_* : \pi_1(X, x)$ to $\pi_1(Y, y)$

defined by $f_*([\gamma]) = [f \circ \gamma]$

Thm for any $f : (X, x)$ to (Y, y),
$g : (Y, y)$ to (Z, z),

we have

$(g \circ f)_* = g_* \circ f_* : \pi_1(X, x)$ to $\pi_1(Z, z)$

Pf

$$\begin{aligned}(g_* \circ f_*)([\gamma]) &= [g \circ (f \circ \gamma)] \\ &= [(g \circ f) \circ \gamma] \\ &= (g \circ f)_*([\gamma]) \ \square\end{aligned}$$

[why "thm" if so easy? more useful than it seems]

Cor if f is a (pointed) homeomorphism
then f_* is a group isomorphism

Pf if $g = f^{-1}$, then $g_* = f_*^{-1}$

today: $\pi_1(R^n, x)$, $\pi_1(S^n, x)$
for any $n \geq 0$

hard case: S^1
[some details will be left till much later]

(Munkres ≈ §52, 54) [but I am skipping around]

<u>Df</u> we say X is simply connected iff
X is path connected and
$\pi_1(X, x)$ is trivial for some/any x

<u>Thm</u> for all $n \geq 0$, every <u>convex</u> subspace
A sub R^n is simply connected

in fact, generalizes to <u>star-convex</u> A

- convex: for all a, b in A, the segment between a and b stays in A
- star-convex: there is some a in A s.t., for all b in A, the segment between a and b stays in A

[note: line segments given by $(1 - t)a + tb$]
the claim for star-convex A will be PS7, #4 part (2)

<u>Idea of Pf</u> path-connectedness easy

use the "straight-line nulhomotopy" from any γ to the constant loop

formally, the n-sphere is

$$S^n = \{(x_0, x_1, \ldots, x_n) \mid \sum_i x_i^2 = 1\}$$

in the subspace top from analytic $R^{n + 1}$

note that S^n is path connected,
but not star-convex, let alone convex

Thm for any $n \geq 2$, the n-sphere is simply connected

Naïve Pf Idea [Munkres 370, #2]

pick a basepoint x
pick a loop $\gamma : [0, 1]$ to S^n based at x
pick a point p in S^n not in $\mathrm{im}(\gamma)$

there is a homeomorphism $S^n - \{p\}$ to R^n:
namely, stereographic projection
[how to finish?] R^n is simply-connected
so $[\gamma] = [e_x]$ [i.e., γ nulhomotopic] in $S^n - \{p\}$
hence $[\gamma] = [e_x]$ in S^n

[but where did we use $n \geq 2$?]

Problem how do we know p exists?
there are awful things called space-filling curves…

Claim if $n \geq 2$, then any loop based at x is path-homotopic to a non-surjective loop

fix the loop γ based at x
fix open hemispheres B and C s.t.

- $B \cup C = S^n$
- $B \cap C$ is homeo to an open annulus $S^{n-1} \times (-\varepsilon, \varepsilon)$
- x in $B \cap C$

note that $C - B$ is nonempty
[draw this]

we will path-homotope γ to a loop avoiding C – B
[idea: look at where γ crosses between B and C]

for all s in [0, 1], pick δ_s > 0 small enough that
if U_s = (s – δ_s, s + δ_s), then either
γ(U_s) sub B or γ(U_s) sub C
[possible bc B, C are open]
then {U_s}_s is an open cover of [0, 1]
so it has a finite subcover {U_{s_j}}_{0 ≤ j ≤ N}

after removing elts, can assume no U_{s_j}
contains another

after merging elts, can assume that
if γ(U_{s_j}) sub B, then γ(U_{s_{j + 1}}) sub C,
and vice versa

let t_j in U_{s_j} cap U_{s_{j + 1}} for 0 ≤ j ≤ N – 1
now we have [draw]

0 = t_0 < t_1 < t_2 < … < t_{N – 1} < t_N = 1

s.t. for all j, γ(t_j) in B cap C
and either γ([t_j, t_{j + 1}]) sub B
or γ([t_j, t_{j + 1}]) sub C

for j s.t. γ([t_j, t_{j + 1}]) sub C,
we path-homotope that restriction of γ
to a path between t_j and t_{j + 1} <u>inside</u> B cap C
[as B cap C = S^{n – 1} × (–ε, ε), this uses n ≥ 2]
after gluing, get the desired loop avoiding C – B □

first part of this argument adapts to a proof of:

Thm if i : B to X and j : C to X are inclusions of open subsets s.t.

X = B cup C,

B cap C is path-connected,

then for all x in B cap C, the images of

i_* : π_1(B, x) to π_1(X, x),
j_* : π_1(C, x) to π_1(X, x)

together generate π_1(X, x)

that is: every elt of π_1(X, x) is a (finite) iterated composition of elts of im(i_*) and im(j_*)

Cor if B and C are simply connected above, then X is simply connected

Rem PS7, #8 asks for a setup where X is simply connected but B, C are not

Rem turns out: if B, C sub S^1 are open s.t.

S^1 = B cup C,

B cap C is path-connected,

then either B = S^1 or C = S^1
so thm does not give insight for X = S^1

Thm π_1(S^1, x) is [what?] isomphc to (Z, +) for any x in S^1

say x = (1, 0) in S^1 sub R^2

[what map Z to π_1(S^1, x) gives the iso?]

for all n in Z, let $\omega_n : [0, 1] \to S^1$ be def by

$$\omega_n(s) = (\cos(2\pi ns), \sin(2\pi ns))$$

then take $\Phi(n) = [\omega_n]$

<u>Claim 1</u> Φ is a homomorphism Z to $\pi_1(S^1, x)$
<u>Claim 2</u> Φ is bijective

<u>Pf of Claim 1</u> want $[\omega_m] * [\omega_n] = [\omega_{m + n}]$

let $p : R \to S^1$ be $p(x) = (\cos(2\pi x), \sin(2\pi x))$
[draw]
for any a, b in Z, let $\acute{\omega}_{a, b} : [0, 1] \to R$ be

$$\acute{\omega}_{a, b}(s) = (1 - s)a + sb$$

if $b - a = n$, then $\omega_n = p \circ \acute{\omega}_{a, b}$
so it remains to show

$$[\acute{\omega}_{a, b}] * [\acute{\omega}_{b, c}] = [\acute{\omega}_{a, c}]$$

this is eas(ier)