Fock Spaces, Braid Varieties, and Block Equivalences

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1 Charged Partitions

Fix an integer $l > 0$.

An integer partition $\lambda \in \Pi$ is called an *l-core* iff it has no hook lengths divisible by l.

- 1-cores: $\varnothing$.
- 2-cores: staircase partitions.
- l-cores for $l \geq 3$: complicated.

An analogue of long division for partitions:

$$l\text{-core} \times l\text{-quotient} : \Pi \xrightarrow{\sim} \Pi_{l\text{-}cor} \times \Pi^l.$$

Our starting point is its application to *quantum groups* and *finite reductive groups*.

First, repackage it as a bijection

$$\Upsilon_l : \Pi \times \mathbf{Z} \xrightarrow{\sim} \Pi^l \times \mathbf{Z}^l.$$

Elements of $\Pi^l \times \mathbf{Z}^l$ are called *charged l-partitions*.

We'll need $\mathbf{B} = \{\beta \mid \mathbf{Z}_{<x} \subseteq \beta \subseteq \mathbf{Z}_{<y} \text{ for some } x, y\}$.

Elements of $\mathbf{B}^l$ are *l-abacus configurations*.

Step 1. $\Pi \times \mathbf{Z} \simeq \mathbf{B}$ via

$$|\pi, s\rangle \leftrightarrow \{s + \pi_i - i + 1 \mid i = 1, 2, 3, \ldots\}.$$

Step 2. $\vec{v}_l : \mathbf{B} \xrightarrow{\sim} \mathbf{B}^l$ given by

$$v_l^{(r)}(\beta) = \{q \in \mathbf{Z} \mid lq + r \in \beta\} \quad \text{for all } r \bmod l.$$

$$\Upsilon_l(\pi, 0) = |\vec{\pi}, \vec{s}\rangle \iff \begin{cases} \Upsilon_l(l\text{-core}(\pi), 0) = |\vec{\varnothing}, \vec{s}\rangle, \\ l\text{-quotient}(\pi) = \vec{\pi}. \end{cases}$$

Ex Take $|\pi, s\rangle = |(2, 2, 1), 4\rangle$ and $l = 3$.

$$\begin{array}{cccccccccccc} \cdots & \bullet & \bullet & \bullet & & \bullet & & \bullet & \bullet & & \cdots \\ & -1 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & \end{array}$$

$$\Downarrow$$

$$\begin{array}{cccccccccc} 2 & \cdots & \bullet & \bullet & & \bullet & & & \cdots \\ 1 & \cdots & \bullet & \bullet & \bullet & & & & \cdots \\ 0 & \cdots & \bullet & \bullet & \bullet & \bullet & \bullet & & \cdots \\ & & -2 & -1 & 0 & 1 & 2 & 3 & \end{array}$$

The charged 3-partition: $|((\varnothing, \varnothing, (1)), (2, 0, 0)\rangle$.

We do have $\Upsilon_3(3\text{-core}(\pi), s) = |\vec{\varnothing}, (2, 0, 0)\rangle$.

2 Cyclotomic Hecke Algebras

Let $\mathfrak{S}_{N,l} = \mathfrak{S}_N \ltimes \mathbf{Z}_l^N$.

From partitions to representations:

- $\mathrm{Irr}(\mathfrak{S}_n) \simeq \{\pi \in \Pi \mid \pi \vdash n\}$.
- $\mathrm{Irr}(\mathfrak{S}_{N,l}) \simeq \{\vec{\pi} \in \Pi^l \mid |\vec{\pi} \vdash_l N\}$.

Actually, we'll use the *Ariki–Koike algebra*

$$H_{N,l}(u,\vec{v}) = \frac{\mathbf{C}[u^{\pm 1}, v_1^{\pm 1}, \dots, v_{\ell-1}^{\pm 1}]\mathfrak{B}_{N,l}}{\left\langle \begin{matrix} (\sigma_i - 1)(\sigma_i + u) \text{ for all } i, \\ (\tau - 1)(\tau - v_1)\cdots(\tau - v_{l-1}) \end{matrix} \right\rangle},$$

By Tits deformation, $\mathrm{Irr}(\mathfrak{S}_{N,l}) \simeq \mathrm{Irr}(H_{N,l}(u,\vec{v}))$.

For general m and $\vec{s}$, nontrivial *decomposition map*

$$\mathrm{K}_0(H_{N,l}(u,\vec{v})) \to \mathrm{K}_0(H_{N,l}(\zeta_m, \zeta_m^{\vec{s}})).$$

(Ariki) For l, m, and $\vec{s} \in \mathbf{Z}^l$: Description of

$$\mathbf{Q}\mathrm{K}_0(H_{N,l}(\zeta_m, \zeta_m^{\vec{s}})) \quad \text{for } N \geq 0$$

via a $\mathrm{U}'_v(\widehat{\mathfrak{sl}}_m)_{\mathbf{Q}}$-module

$$\Lambda_{\vec{s}} := \bigoplus\nolimits_{\vec{\lambda} \in \Pi^l} \mathbf{Q}(v)|\vec{\lambda}, \vec{s}\rangle$$

called the *Fock space of level l and charge $\vec{s}$*.

(Uglov) For $(\vec{s}, \vec{r}) \in \mathbf{Z}^l \times \mathbf{Z}^m$ such that $|\vec{s}| = s = |\vec{r}|$:

Commuting $\mathrm{U}'_v(\widehat{\mathfrak{sl}}_m)_{\mathbf{Q}}$- and $\mathrm{U}'_v(\widehat{\mathfrak{sl}}_l)_{\mathbf{Q}}$-actions on

$$\Lambda_{\vec{s}} \overset{\Upsilon_l}{=} \Lambda_s \overset{\bar{\Upsilon}_m}{=} \Lambda_{\vec{r}}$$

where $\bar{\Upsilon}_m$ is a modified version of Υ_m.

(Uglov) Bijections of the form below, matching decomposition numbers on the two sides:

$$\begin{array}{ccc} \mathrm{Irr}(\mathfrak{S}_{N,l})_{\mathsf{b}} & \simeq & \mathrm{Irr}(\mathfrak{S}_{N',m})_{\mathsf{c}} \\ \Uparrow & & \Uparrow \\ \mathsf{b} \trianglelefteq \mathrm{K}_0(H_{N,l}(\zeta_m, \zeta_m^{\vec{s}})) & & \mathrm{K}_0(H_{N',m}(\zeta_l, \zeta_l^{\vec{r}})) \trianglerighteq \mathsf{c} \end{array}$$

(Losev, Rouquier–Shan–Varagnolo–Vasserot, Webster)

$$\boxed{\mathsf{Rep}_{\mathsf{b}}(H^{\mathrm{rat}}_{N,l}(\vec{\nu}_l)) \simeq \mathsf{Rep}_{\mathsf{c}}(H^{\mathrm{rat}}_{N',m}(\vec{\nu}_m))}$$

for *cyclotomic rational DAHAs* $H^{\mathrm{rat}}_{N,l}, H^{\mathrm{rat}}_{N',m}$.

- $\zeta_m^{\vec{s}} = e^{2\pi i \vec{\nu}_l}$ and $\zeta_l^{\vec{r}} = e^{2\pi i \vec{\nu}_m}$.
- $\mathsf{Rep}_{\mathsf{b}}, \mathsf{Rep}_{\mathsf{c}}$ lift b, c.

These equivalences are called *level-rank dualities*.

3 Φ-Harish-Chandra Theories Ting Xue and I propose a generalization to *relative Weyl groups*.

Fix a prime power q. A reductive group $\mathbf{G}$ with Frobenius $F : \mathbf{G} \to \mathbf{G}$ over $\bar{\mathbf{F}}_q$ defines a

$$\textit{finite reductive group} \quad G = \mathbf{G}^F.$$

Let $\mathrm{Uch}(G)$ index its *unipotent irreducible characters*.

(Harish–Chandra) $\mathrm{Uch}(G) = \coprod_{(L,\lambda)} \mathrm{Uch}(G)_{L,\lambda}$.

- $L \subseteq G$ is an F-maximally split Levi.
- $\lambda \in \mathrm{Uch}(L)$ is *cuspidal*.
- $\mathrm{Uch}(G)_{L,\lambda} = \{\rho \in \mathrm{Uch}(G) \mid (\rho, \mathrm{Ind}_L^G(\lambda)) \neq 0\}$.

Relative Weyl groups: $W_{G,L,\lambda} = C_{N_G(L)/L}(\lambda)$.

How to introduce l, m?

(Broué–Malle–Michel) A Levi L is *Φ_l-split* iff

$$L = Z_G(T)^\circ \quad \text{for some torus } \mathbf{T} \subseteq \mathbf{G} \text{ such that } |T| \text{ is } \textit{generically} \text{ a power of } \Phi_l(q).$$

($\mathbf{T}$ need not be maximal!)

$\lambda \in \mathrm{Uch}(L)$ is *Φ_l-cuspidal* iff it does not occur in the *Lusztig induction* $\mathbf{R}_M^L$ from smaller Φ_l-split Levis M.

Φ_l-cuspidal pairs and *Φ_l-Harish-Chandra series*:

- $\mathrm{Uch}(G) = \coprod_{\Phi_l\text{-cuspidal } (L,\,\lambda)} \mathrm{Uch}(G)_{L,\lambda}$.
- Bijections $\mathrm{Uch}(G)_{L,\lambda} \xrightarrow{\chi_{L,\lambda}} \mathrm{Irr}(W_{G,L,\lambda})$.
- Signs $\mathrm{Uch}(G)_{L,\lambda} \xrightarrow{\varepsilon_{L,\lambda}} \{\pm 1\} \implies$ isometries $\varepsilon\chi$.

Ex Take $G = \mathrm{GL}_n(\mathbf{F}_q)$, so that $\mathrm{Uch}(G) \simeq \{\pi \vdash n\}$.

Φ_l-split Levi L	$\mathrm{GL}_N(\mathbf{F}_q) \times (\mathbf{F}_{q^l})^{\frac{n-N}{l}}$
Φ_l-cuspidal $\lambda \in \mathrm{Uch}(L)$	l-core $\lambda \vdash N$
$\mathrm{Uch}(G)_{L,\lambda}$	$\{\pi \vdash n \mid l\text{-core}(\pi) = \lambda\}$

Above, $W_{G,L,\lambda} \simeq S_N \ltimes \mathbf{Z}_l^N$. The map

$$\chi_{L,\lambda} : \mathrm{Uch}(G)_{L,\lambda} \to \mathrm{Irr}(W_{G,L,\lambda})$$

comes from the Π^l part of $\Upsilon_l(-, \mathsf{len}(\lambda))$.

$\mathbf{R}_L^G := \mathrm{H}_c^*(Y_L^G)$ for some *Deligne–Lusztig variety* Y_L^G.

Conj (BMM) $\boxed{\mathrm{End}_G(\mathrm{H}_c^*(Y_L^G)[\lambda]) \simeq H_{N,l}(q^l, q^{\vec{a}(\lambda)}).}$

Above, $\vec{a}(\lambda) = l\vec{a}'(\lambda) + (0, 1, \ldots, l-1)$, where $\vec{a}'$ is the $\mathbf{Z}^l$ part of $\Upsilon_l(\lambda, \mathsf{len}(\lambda))$.

Conj (BMM) For *general* G and Φ_l-cuspidal (L, λ),

$$\boxed{\mathrm{End}_G(\mathrm{H}_c^*(Y_L^G)[\lambda]) \simeq H_{W_{G,L,\lambda}}(q)}$$

for an *explicit* 1-parameter algebra $H_{W_{G,L,\lambda}}(x)$. And the commuting actions induce

$$\chi_{L,\lambda} : \mathrm{Uch}(G)_{L,\lambda} \to \mathrm{Irr}(H_{W_{G,L,\lambda}}(q)) = \mathrm{Irr}(W_{G,L,\lambda}).$$

(Lusztig) True for $l = 1$ cases and "Coxeter tori".

(Digne–Michel–Rouquier) Progress in types A, B, D_4.

Our generalization of level-rank duality will involve

$$H_{W_{G,L,\lambda}}(\zeta_m) \quad \text{versus} \quad H_{W_{G,M,\mu}}(\zeta_l),$$

for Φ_l-cuspidal (L, λ) and Φ_m-cuspidal (M, μ).

Let $\mathrm{Uch}^G_{L,\lambda,M,\mu} = \mathrm{Uch}(G)_{L,\lambda} \cap \mathrm{Uch}(G)_{M,\mu}$. Form

$$\boxed{\mathrm{Irr}(W_{G,L,\lambda}) \xleftarrow{\chi_{L,\lambda}} \mathrm{Uch}^G_{L,\lambda,M,\mu} \xrightarrow{\chi_{M,\mu}} \mathrm{Irr}(W_{G,M,\mu}).}$$

Conj (T–Xue)

1 The left / right image is a union of preimages of blocks of $H_{G,L,\lambda}(\zeta_m)$ / $H_{G,M,\mu}(\zeta_l)$.

2 The maps descend to a bijection

$$\{H_{G,L,\lambda}(\zeta_m)\text{-blocks}\} \simeq \{H_{G,M,\mu}(\zeta_l)\text{-blocks}\}.$$

3 For matching blocks, an equivalence of their highest-weight covers ($=$ blocks of rational DAHAs).

Thm (T–Xue) Take $G = \mathrm{GL}_n$ and l, m coprime.

Then (1)–(3) hold when $H_{G,L,\lambda}(x) = H_{N,l}(x^l, x^{\vec{a}(\lambda)})$. In (1), the images are single blocks.

4 Steinberg Varieties for $\mathbf{G}F$ (Recall $G = \mathbf{G}^F$.)

A more explicit version of the BMM conjecture:

$$R_L^G(\lambda) := \sum_i (-1)^i \mathrm{H}_c^i(Y_L^G)[\lambda]$$
$$= \sum_{\rho \in \mathrm{Uch}(G)_{L,\lambda}} \rho \otimes \varepsilon_{L,\lambda}(\rho) \chi_{L,\lambda}(\rho)_q$$

as a virtual $(G, H_{W_{G,L,\lambda}}(q))$-bimodule.

Suggests looking at

$$R_L^G(\lambda) \otimes_{\mathbf{C}G} R_M^G(\mu)$$

as a virtual $(H_{W_{G,L,\lambda}}(q), H_{W_{G,M,\mu}}(q))$-bimodule.

Let $\mathcal{B}$ be the flag variety of $\mathbf{G}$. For $w \in W$, set

$$\mathcal{Y}_w = \{(g, B) \in \mathbf{G} \times \mathcal{B} \mid B \xrightarrow{w} gFB(gF)^{-1}\}.$$

Action $\mathbf{G} \curvearrowright \mathcal{Y}_w$ via $x \cdot (g, B) = xgF(x)^{-1}, xBx^{-1})$.

If L is a maximal torus of type $[w]$, then

$$\mathbf{R}_L^G(1_L) = \mathrm{H}_c^*(Y_L^G)[1_L] \simeq \mathrm{H}_{c,\mathbf{G}}^*(\mathcal{Y}_w).$$

For L, M maximal tori of types $[w], [v]$, we are led to consider the *generalized Steinberg variety*

$$\mathcal{Y}_w \times_{\mathbf{G}}^{\mathrm{L}} \mathcal{Y}_v.$$

Some (derived) Künneth-type formula should show

$$R_L^G(\lambda) \otimes_{\mathbf{C}G} R_M^G(\mu) = \sum_i (-1)^i \mathrm{H}_{c,\mathbf{G}}^i(\mathcal{Y}_w \times_{\mathbf{G}}^{\mathrm{L}} \mathcal{Y}_v).$$

5 Steinberg Varieties for $\mathbf{G}$

Earlier, I introduced similar varieties in a totally independent context.

Let $\mathcal{U} \subseteq \mathbf{G}$ be the *unipotent locus*. Let

$$\mathcal{U}_w = \{(u, B) \in \mathcal{U} \times \mathcal{B} \mid B \xrightarrow{w} uBu^{-1}\}.$$

For example, $\mathcal{U}_e$ is the (groupy) Springer resolution.

I studied an action of $\mathbf{H}_W := \mathrm{gr}^{\mathsf{W}}_* \mathrm{H}^*_{c,\mathbf{G}}(\mathcal{U}_e \times^{\mathsf{L}}_{\mathcal{U}} \mathcal{U}_e)$ on

$$\mathrm{gr}^{\mathsf{W}}_* \mathrm{H}^*_{c,\mathbf{G}}(\mathcal{U}_e \times^{\mathsf{L}}_{\mathcal{U}} \mathcal{U}_w).$$

(Actually a braid version motivated by link homology.)

Above, $\mathbf{H}_W \simeq \mathbf{C}W \ltimes \mathrm{Sym}(X_*(\mathbf{A}))$, where $\mathbf{A} \subseteq \mathbf{G}$ is a maximally split maximal torus. ($W = W_{G,A,1}$.)

Via the W-action, the $\mathbf{G}$-equivariant virtual weight polynomial defines a $\mathbf{Z}W[\mathsf{x}]$-valued *virtual character*.

Thm (T) Suppose that $w \in W$ is *regular* of order m. Then the virtual character of $\mathcal{U}_e \times^{\mathsf{L}}_{\mathcal{U}} \mathcal{U}_w$ is given by

$$\sum_{\rho \in \mathrm{Uch}(G)_{A,1,M,1}} D_m(\rho)[\Delta_{1/m}(\chi_{A,1}(\rho))],$$

where:

- M is any Φ_m-split maximal torus of G.
- $D_m(\rho) = \varepsilon_{M,1}(\rho) \deg \chi_{M,1}(\rho)$.
- For any χ, we write $[\Delta_{1/m}(\chi)]$ for the character of the associated Verma of the rational DAHA $H^{\mathrm{rat}}_W(\frac{1}{m})$.

$H^{\mathrm{rat}}_W(\frac{1}{m})$ is a highest-weight cover of $H_W(\zeta_m)$!

A very strange analogy emerges.

Below M is a Φ_m-split maximal torus of G.

$\mathbf{G}F$	$\mathbf{G}$
$\mathcal{Y}_e \times^{\mathrm{L}}_{\mathbf{G}} \mathcal{Y}_w$	$\mathcal{U}_e \times^{\mathrm{L}}_{\mathcal{U}} \mathcal{U}_w$
(q, q)	$(\zeta_m, 1)$
$H_W(q), H_{W_{G,M,1}}(q)$	$H_W^{\mathrm{rat}}(\frac{1}{m}), H_{W_{G,M,1}}(1)$

Where else do we expect the formula on the $\mathbf{G}$ side?

- Work of Oblomkov–Yun, Losev–Boixeda-Alvarez, *et al.* on *affine Springer fibers*.
- Work of Lusztig and Abreu–Nigro on analogues of $\mathcal{U}_w$ replacing $\mathcal{U}$ with a *regular semisimple* class of G.

Thank you for listening.