6.

Today, we discuss a big-picture overview of the consequences of Deligne–Lusztig theory and Lusztig's subsequent work on finite reductive groups. This is stolen from online notes by Chao Li.

6.1.

Suppose that G is a reductive algebraic group with Frobenius F and F-stable maximal torus T. If T is maximally split, *i.e.*, contained in an F-stable Borel $B \subseteq G$, then we can perform *parabolic induction* of representations from T^F to G^F in the spirit of Harish-Chandra: Pull back from T^F to B^F, then induct from B^F to G^F.

The key idea of Deligne–Lusztig—which they attribute to Macdonald—is to obtain the other irreducible characters of G^F by constructing, for all other F-stable maximal tori $S \subseteq G$, an analogous induction functor from S^F to G^F. What we have presented thus far is the construction of this functor when S^F is conjugate in $G(k)$ to T^{wF} for some $w \in W$. For such S, the virtual character that we denote by $R_{w,\theta}$ is, in other texts, constructed in terms of S and denoted by $R_{S,\theta}$, after replacing θ by the appropriate conjugate.

We are led to ask: Does every F-stable maximal torus $S \subseteq G$ have the property that S^F is conjugate to T^{wF} for some w? The answer is yes. Pick some $g \in G(k)$ such that $S = gTg^{-1}$. Then the F-stability of S and T implies that $g^{-1}F(g) \in N_G(T)(k)$, so we can take w to be the image of $g^{-1}F(g)$ in W. Suppose that we choose some other $g' \in G(k)$ such that $S = g'T(g')^{-1}$, and take w' to be the image of $(g')^{-1}F(g')$ in W. Then $g^{-1}g' \in N_G(T)(k)$, and its image $x \in W$ satisfies $xw'F(x)^{-1} = w$.

In general, we say that elements $w, w' \in W$ are *F-conjugate* if and only if $xw'F(x)^{-1} = w$ for some $x \in W$. We have constructed a map from the set of F-stable maximal tori of G to the set of F-conjugacy classes of W. The image of a torus under this map is sometimes called its *type*.

Proposition 6.1. *The map that sends an F-stable maximal torus to its type descends to a bijection*

$$\{F\text{-stable maximal tori of } G\}/G^F\text{-conjugacy} \xrightarrow{\sim} W/F\text{-conjugacy}.$$

Proof. We get surjectivity by choosing any section $W \to N_G(T)$. To show injectivity, observe that we can reduce to the $w = e$ case, where we must show that if $S = gTg^{-1}$ and $g^{-1}F(g)$ maps to the identity $e \in W$, then $S = hTh^{-1}$ for some $h \in G^F$. Indeed: $g^{-1}F(g) \in T(k)$, so by Lang's theorem, $g^{-1}F(g) = t^{-1}F(t)$ for some $t \in T(k)$. Now take $h = gt^{-1}$. $\square$

6.2.

So the virtual characters $R_{w,\theta}$ comprise all the virtual characters that we can get "geometrically" from F-stable maximal tori. The next problem is to determine which ones contribute the same irreducible summands as each other.

Last time, we stated a formula of Deligne–Lusztig implying that if $w, w' \in W$ belong to different F-conjugacy classes, then $R_{w,\theta}, R_{w'\theta'}$ are orthogonal for every pair of characers θ of T^{wF} and θ' of $T^{w'F}$. But we have also seen an example where w, w' are not F-conjugate and $R_{w,1}, R_{w',1}$ have the same *virtual* summands: For $G = \mathrm{SL}_2$ under the standard Frobenius, $R_{e,1} = 1 + \rho$ and $R_{s,1} = 1 - \rho$ for 1 the trivial character and ρ the Steinberg character.

Deligne–Lusztig found a stricter condition that rules out this sort of situation. As motivation, observe that since $G(k) = \bigcup_{m \geq 1} G^{F^m}$, every pair of maximal tori S, S' is conjugate under G^{F^m} for some $m \geq 1$. Since S is commutative, there is a group homomorphism

$$\mathbf{N}_m : S^{F^m} \to S^F,$$

called the *Galois norm*, that sends any element to the product of its conjugates under $F^0, F^1, \ldots, F^{m-1}$. Fix characters θ of S^F and θ' of $(S')^F$. We say that the pairs (S, θ) and (S', θ') are *geometrically conjugate* if and only if there exists some $m \geq 1$ and $g \in G^{F^m}$ such that $S' = {}^g S$ and $\theta' \circ \mathbf{N}_m = {}^g(\theta \circ \mathbf{N}_m)$. What follows is Corollary 6.3 in Deligne–Lusztig's paper.

Theorem 6.2 (Deligne–Lusztig)**.** *If $R_{S,\theta}, R_{S',\theta'}$ share an irreducible summand, then (S, θ) and (S', θ') are geometrically conjugate.*

6.3.

We also stated a formula about Lefschetz numbers, reducing the calculation of a Lefschetz function on G^F to those of other Lefschetz functions (for smaller schemes) on the subset of unipotent elements. It turns out that this formula reduces the calculation of $R_{S,\theta}$ to the case where $\theta = 1$, at the cost of replacing G with a collection of smaller reductive algebraic groups. For clarity in what follows, we will write $R_S^G(\theta)$ in place of $R_{S,\theta}$.

For any $g \in G^F$, let $g = g_s g_u = g_u g_s$ be its *Jordan decomposition*. This decomposition is uniquely determined by requiring g_s to be *semisimple*, meaning an element of some maximal torus of G, and g_u unipotent. It turns out that $g_s, g_u \in G^F$ as well, and that the centralizer $C(g_s) = C_G(g_s)$ is a reductive algebraic group. Let $C(g_s)^\circ \subseteq C(g_s)$ be the connected component at the identity. Observe that if g_s is contained in a torus, then the torus is contained in $C(g_s)^\circ$.

Theorem 6.3 (Deligne–Lusztig)**.** *For any F-stable maximal torus S and element $g \in G^F$, we have*

$$R_{S^F}^{G^F}(\theta)(g) = \frac{1}{|(C(g_s)^\circ)^F|} \sum_{\substack{x \in G^F \\ g_s \in {}^x S^F}} {}^x\theta(g_s) R_{{}^x S^F}^{(C(g_s)^\circ)^F}(1)(g_u).$$

This formula simplifies substantially when $g = g_s$. In particular, when $g = 1$, Deligne–Lusztig use the formula to express the dimension of any irreducible character ρ of G^F as a linear combination of the multiplicities $(\rho, R_{S^F}^{G^F}(\theta))_{G^F}$, running over all F-stable maximal tori S and characters θ of S^F. From this they deduce their Corollary 7.7:

Corollary 6.4 (Deligne–Lusztig)**.** *Every irreducible representation of G^F occurs as a virtual summand of $R_{S,\theta}$ for some S, θ: hence, of $R_{w,\theta}$ for some w, θ.*

6.4.

From these results, we begin to glimpse how the Deligne–Lusztig induction functors give structure to the set of irreducible characters of G^F, when G is a connected reductive algebraic group over k with Frobenius F.

(1) First, it is partitioned into subsets indexed by the geometric conjugacy classes of pairs (S, θ), where S is an F-stable maximal torus of G and θ is a character of S^F. Let $\mathcal{S}_{G,F}$ denote the set of such classes.
(2) Second, the values of the characters associated with a given (S, θ) can be computed from analogous values where G is replaced by a smaller reductive group, S is replaced by a conjugate torus, and θ is replaced by the trivial character.

Remarkably, we can repackage this structure in terms of the geometry of a different group. For any connected reductive G with Frobenius F, there is another connected reductive algebraic group $G^\vee$ over k, and a Frobenius on $G^\vee$ that we again denote by F, with the following properties.

(1) $\mathcal{S}_{G,F}$ is in bijection with the set of semisimple $G^\vee(k)$-conjugacy classes of $(G^\vee)^F$.
(2) In (1), the (single) geometric conjugacy class of pairs $(S, 1)$ corresponds to the conjugacy class of the identity element in $(G^\vee)^F$.
(3) The operation $(G, F) \mapsto (G^\vee, F)$ is involutive.

Granting its existence, we can state Lusztig's classification theorem. For any semisimple element $g \in (G^\vee)^F$, whose $G^\vee(k)$-conjugacy class (g) corresponds to a class $[(S, \theta)] \in \mathcal{S}_{G,F}$, let $\mathcal{E}(G^F, (g))$ denote the set of irreducible characters of G^F with nonzero multiplicity in $R_{S,\theta}$.

Theorem 6.5 (Deligne–Lusztig, Lusztig)**.** *We have a partition*

$$\mathrm{Irr}(G^F) = \coprod_{(g)} \mathcal{E}(G^F, (g)),$$

where (g) runs over the semisimple $G^\vee(k)$-conjugacy classes of $(G^\vee)^F$. Moreover, writing $H_g = C_{G^\vee}(g)^\vee$, so that $H_g^\vee = C_{G^\vee}(g)$, we have bijections

$$\begin{aligned}\mathcal{E}(G^F, (g)) &\xrightarrow{\sim} \mathcal{E}(H_g, (1)),\\ \rho &\mapsto \rho_u,\end{aligned}$$

such that the following property holds: Writing $[(S_g, \theta_g)] \in \mathcal{S}_{H_g,F}$ for the class corresponding to the $H_g^\vee(k)$-conjugacy class of g in $(H_g^\vee)^F$, we have

$$(\rho, R_{S,\theta})_{G^F} = \pm(\rho_u, R_{S_g,\theta_g})_{H_g^F},$$

where the sign can be made explicit and only depends on $G, S, (g)$.

We say that $\mathcal{E}(G^F, (g))$ is the *Lusztig series* indexed by (g). Hence, when (g) corresponds to the geometric conjugacy class of (T, θ) for a maximally split T, the Lusztig series for (g) is the principal series indexed by θ.

6.5.

The construction of $G^\vee$ requires some background from Lie theory: the classification of reductive algebraic groups in terms of root data, due to the work of Killing, É. Cartan, and Chevalley. Namely, we take the root datum of $G^\vee$ to be that dual to the root datum of G. It turns out that by the existence of F-stable Borel pairs, the Frobenius on G is determined by an automorphism of its root datum, and the Frobenius on $G^\vee$ can be defined in terms of an appropriate dual automorphism. A simpler statement is Deligne–Lusztig Proposition 5.7, which says:

Proposition 6.6 (Deligne–Lusztig)**.** *Let $T \subseteq G$ be a maximally split F-stable maximal torus. Then there is a bijection*

$$\mathcal{S}_{G,F} \xrightarrow{\sim} [(\mathrm{X}(T) \otimes \mathbf{Q}/\mathbf{Z})/W]^F,$$

where $\mathrm{X}(T)$ is the lattice of characters $T \to \mathbf{G}_m$, on which $W \rtimes \langle F \rangle$ acts by precomposition. If F corresponds to an $\mathbf{F}_q$-form of G, then for any torus $S = {}^gT$, the map sends $[(S, \theta)] \mapsto [{}^{g^{-1}}\theta]$ once we identify

$$\mathrm{X}(T) \otimes (\tfrac{1}{q^m-1}\mathbf{Z})/\mathbf{Z} \simeq \mathrm{Hom}(T^{F^m}, \bar{\mathbf{Q}}_\ell^\times[q^m - 1])$$

for all $m \geq 1$.