

MATH 251: TOPOLOGY II
SPRING 2026 PRACTICE PROBLEMS

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NOTE: All citations are to Munkres's textbook, *Topology*, 2nd Edition. When a problem statement has a proof in Munkres, try your best to find your own proof, before comparing with his.

For any integer $n > 0$, the topology on \mathbf{R}^n is the analytic topology unless otherwise specified.

0. REVIEW OF TOPOLOGY I

Problem 1. Show that no two of the spaces

$$(0, 1), \quad (0, 1], \quad [0, 1]$$

are homeomorphic. *Hint:* What happens if you remove any point from $(0, 1)$?

Problem 2. Suppose that $A \subseteq X$. Recall that the *interior* of A in X is

$$\text{Int}_X(A) := \{x \in X \mid x \in U \text{ for some } U \text{ open in } X \text{ such that } U \subseteq A\}$$

and the *closure* of A in X is

$$\text{Cl}_X(A) := \{x \in X \mid x \in K \text{ for all } K \text{ closed in } X \text{ such that } K \supseteq A\}.$$

Show the identity $\text{Cl}_X(A) = X \setminus \text{Int}_X(X \setminus A)$.

Problem 3. Suppose that $A \subseteq Y \subseteq X$, where Y is a subspace of X .

- (1) Show that $\text{Int}_Y(A) \supseteq \text{Int}_X(A)$.
- (2) Give an example where $\text{Int}_Y(A) \neq \text{Int}_X(A)$. *Hint:* You can assume $Y = A$.

Problem 4. Let X be the real line \mathbf{R} in its finite complement, or *cofinite*, topology. Show that every sequence of points in X converges to every point of X simultaneously. Deduce that X is not Hausdorff.

Problem 5. Show that any metric space is Hausdorff.

Problem 6. Show that for any integer $n \geq 1$, the analytic topology on \mathbf{R}^n matches the product topology on $\mathbf{R} \times \cdots \times \mathbf{R}$, where there are n factors.

Problem 7. Let $p: \mathbf{R} \rightarrow S^1$ be the map

$$p(t) = (\cos(2\pi t), \sin(2\pi t)).$$

For any $a, b \in \mathbf{R}$, we define the *(open) arc* $J_{a,b} \subseteq S^1$ to be

$$J_{a,b} = \{p(t) \mid a < t < b\}.$$

Show that the collection of arcs $\{J_{a,b} \mid a, b \in \mathbf{R}\}$ satisfies the definition of a basis (Munkres page 78).

Problem 8. Show that the following topologies on S^1 are all the same:

- The topology generated by the basis $\{J_{a,b} \mid a, b \in \mathbf{R}\}$ in Problem 7.
- The subspace topology that S^1 inherits from its inclusion into \mathbf{R}^2 .
- The quotient topology that S^1 inherits from the surjective map $p: \mathbf{R} \rightarrow S^1$.

Problem 9. Let $f_1, f_2, f_3: A \rightarrow X$ be continuous maps. Suppose that φ is a homotopy from f_1 to f_2 and ψ is a homotopy from f_2 to f_3 . Construct a homotopy from f_1 to f_3 explicitly in terms of φ and ψ .

Problem 10. In the setup of Problem 9, suppose that

$$\begin{aligned} A &= [0, 1], \\ f_1(0) &= f_2(0) = f_3(0), \\ f_1(1) &= f_2(1) = f_3(1). \end{aligned}$$

Show that in this case, if φ and ψ are path homotopies, then we can choose the solution of (1) to be a path homotopy as well.

Problem 11. Let $f, g: X \rightarrow Y$ and $F, G: Y \rightarrow Z$ be continuous. Show that

$$\text{if } f \sim g \text{ and } F \sim G, \quad \text{then } F \circ f \sim G \circ g.$$

Problem 12. Use the $f = g$ case of Problem 11 to show: If X, Y are nonempty and Y is contractible, then any continuous map from X into Y is *nulhomotopic*: that is, homotopic to a constant map.

Problem 13. Give an explicit homotopy equivalence between S^1 and $\mathbf{R}^2 \setminus \{(0, 0)\}$, and show that it is indeed a homotopy equivalence.

Problem 14. Suppose that:

- $f: X \rightarrow Y$ and $g: Y \rightarrow X$ define a homotopy equivalence between X and Y .
- $j: Y \rightarrow Z$ and $k: Z \rightarrow Y$ define a homotopy equivalence between Y and Z .

Show that $j \circ f: X \rightarrow Z$ and $g \circ k: Z \rightarrow X$ define a homotopy equivalence between X and Z .

Hint: Among other things, you will need to show that $g \circ k \circ j \circ f \sim \text{id}_X$. But we are given that $k \circ j \sim \text{id}_Y$. Use Problem 11 to show that $k \circ j \sim \text{id}_Y$ implies $g \circ k \circ j \circ f \sim \text{id}_X$.

Problem 15. Show that if the identity map of a space X is homotopic to a constant map with value $c \in X$, then X is homotopy equivalent to $\{c\}$. Deduce that all *contractible* spaces are homotopy equivalent to each other.

Problem 16. A subset $A \subseteq \mathbf{R}^n$ is *star convex* if and only if there is some point $c \in A$ such that the line segment between c and any other point of A is contained within A . That is, for any $a \in A$ and $t \in [0, 1]$, we have $(1 - t)c + ta \in A$.

- (1) Show that if A is star convex, then A is contractible.
- (2) Show that if A is star convex, then any loop in A based at a_0 is path homotopic to the constant loop.
- (3) Give a star convex subset of \mathbf{R}^2 that is not convex.

EXAM 1 TOPICS

Exam 1 will be a closed-book exam. It will be held in-class, 12:40–2:00 pm, on ~~January 26~~ ~~February 2~~ **February 4**.

You should know by heart the definitions of the following terms. You should be able to state multiple examples of each (with justification), and in some cases, non-examples.

- Munkres §13. Bases
- §18. Continuous maps
- §15. The product topology (only on finite direct products)
- §16. Subspaces
- §22. Quotient spaces
- §17. Interiors and closures
- §23–24. Separations, connectedness
- §23. Paths, path connectedness
- §26–27 Compactness
- §51. Homotopies, path homotopies
- §58. Homotopy equivalences

1. FUNDAMENTAL GROUPS AND COVERING SPACES

Problem 17. Show that if X is path connected, then X is connected.

Problem 18. We say that X is *totally disconnected* if and only if its only nonempty connected subspaces are one-point sets.

- (1) Show that if X is discrete, then X is totally disconnected.
- (2) Show that the set of rational numbers \mathbf{Q} , as a subspace of (analytic) \mathbf{R} , is totally disconnected, but not discrete.
- (3) Show that if X is totally disconnected, then $\pi_1(X, x)$ is trivial for all $x \in X$.

Hint: Use Problem 17.

Problem 19. Consider the points $p = (1, 0)$ and $q = (-1, 0)$ in \mathbf{R}^2 .

- (1) Give two paths in S^1 from p to q that are not path homotopic.
- (2) Explain why the paths in (1) become path homotopic when viewed as paths in \mathbf{R}^2 .

Problem 20. Let $x_0, x_1 \in X$. Let $\alpha: [0, 1] \rightarrow X$ be a path from x_0 to x_1 , and let $\bar{\alpha}$ be the reverse path defined by $\bar{\alpha}(s) = \alpha(1 - s)$.

- (1) Show that $\alpha * \bar{\alpha}$ is path homotopic to e_{x_0} .
- (2) Using part (1), show that the map

$$\hat{\alpha}: \pi_1(X, x_0) \rightarrow \pi_1(X, x_1) \quad \text{given by } \hat{\alpha}([\gamma]) = [\bar{\alpha} * \gamma * \alpha]$$

is a group homomorphism.

- (3) Show that $\hat{\bar{\alpha}}$ is a two-sided inverse of $\hat{\alpha}$. Deduce that $\hat{\alpha}$ is an isomorphism.

Problem 21. Let $f: X \rightarrow Y$ be a continuous map.

- (1) Show that if $\beta, \gamma: [0, 1] \rightarrow X$ are paths such that $\beta(1) = \gamma(0)$, then

$$f \circ (\beta * \gamma) = (f \circ \beta) * (f \circ \gamma).$$

- (2) Let $x \in X$. Using part (1), show that the map

$$f_*: \pi_1(X, x) \rightarrow \pi_1(Y, f(x)) \quad \text{given by } f_*([\gamma]) = [f \circ \gamma]$$

is a group homomorphism.

- (3) Let $g: Y \rightarrow Z$ be another continuous map. Show that $(g \circ f)_* = g_* \circ f_*$ as homomorphisms from $\pi_1(X, x)$ into $\pi_1(Z, g(f(x)))$.

Problem 22. Let X be a subspace of some space X' , and let $f: X \rightarrow Y$ be a continuous map. Suppose that

$$f = f'|_X \quad \text{for some continuous map } f': X' \rightarrow Y,$$

and that X' is *simply connected*, meaning X' is path connected and its fundamental group is trivial. Show that in this case, the homomorphism $f_*: \pi_1(X, x) \rightarrow \pi_1(Y, f(x))$ is *trivial* for all $x \in X$, meaning it sends every element of $\pi_1(X, x)$ to the identity element of $\pi_1(Y, f(x))$. *Hint:* Use part (3) of Problem 21.

Problem 23. Let X_1, X_2 be spaces with points $x_1 \in X_1$ and $x_2 \in X_2$. Let $X = X_1 \times X_2$ and $x = (x_1, x_2) \in X$. For $i = 1, 2$, let $\text{pr}_i: X \rightarrow X_i$ be the appropriate *projection map* (Munkres page 87).

(1) Show that the map

$$\varphi: \pi_1(X, x) \rightarrow \pi_1(X_1, x_1) \times \pi_1(X_2, x_2) \quad \text{given by } \varphi([\gamma]) = ([\text{pr}_1 \circ \gamma], [\text{pr}_2 \circ \gamma])$$

is a homomorphism.

(2) Show that the map

$$\psi: \pi_1(X_1, x_1) \times \pi_1(X_2, x_2) \rightarrow \pi_1(X, x) \quad \text{given by } \psi([\gamma_1], [\gamma_2]) = [\gamma_1 \times \gamma_2]$$

is a two-sided inverse to φ . Deduce that

$$\pi_1(X, x) \simeq \pi_1(X_1, x_1) \times \pi_1(X_2, x_2).$$

(3) Using the conclusion to part (2), show that \mathbf{Z}^n is the fundamental group of an explicit (path-connected) topological space, for any integer $n > 0$.

Problem 24. By considering what happens on fundamental groups when a point is removed, show that \mathbf{R}^2 and \mathbf{R}^n are not homeomorphic if $n > 2$.

Problem 25. Learn the definitions of a *free group* and of a *presentation of a group (by generators and relations)*. (They are covered in Munkres §69, but you may find other sources online helpful as well.)

Let F_2 denote a free group on generators a and b . Show that the homomorphism

$$\varphi: F_2 \rightarrow \mathbf{Z}^2 \quad \text{given by } \varphi(a) = (1, 0) \text{ and } \varphi(b) = (0, 1)$$

is surjective and that its kernel contains the *commutator* of a and b : that is, the element $aba^{-1}b^{-1}$.

(In fact, the kernel is exactly the smallest normal subgroup of F_2 containing the commutator. This shows that \mathbf{Z}^2 has a presentation with generators a, b and a single relation $aba^{-1}b^{-1}$.)

Problem 26. Suppose that A, U_1, U_2 are path-connected open subsets of X such that $X = U_1 \cup U_2$ and $A = U_1 \cap U_2$. For $k = 1, 2$, let $i_k: A \rightarrow U_k$ and $j_k: U_k \rightarrow X$ be the appropriate inclusion maps. Fix a point $x \in A$.

(1) Describe a homomorphism

$$\Psi: \pi_1(U_1, x) \star \pi_1(U_2, x) \rightarrow \pi_1(X, x)$$

such that $\Psi|_{\pi_1(U_k, x)} = (j_k)_*$ for $k = 1, 2$. (It is necessarily unique.)

(2) Show that for any $[\gamma] \in \pi_1(A, x)$, the element

$$(i_1)_*([\gamma]) * (i_2)_*([\gamma])^{-1} \in \pi_1(U_1, x) \star \pi_1(U_2, x)$$

belongs to the kernel of Ψ . *Hint:* Compare $j_1 \circ i_1$ and $j_2 \circ i_2$.

Problem 27. Recall that $X \sqcup Y$ denotes the disjoint union of X and Y . The *wedge sum* of X and Y at points $x \in X$ and $y \in Y$ is a quotient space

$$X \vee Y = (X \sqcup Y) / \sim,$$

where \sim is an equivalence relation that only identifies two distinct points when those points are x and y . Use Seifert-van Kampen to show that $\pi_1(S^1 \vee S^1, p) \simeq \mathbf{Z} \star \mathbf{Z}$ for any basepoint p .

Problem 28. Keep the hypotheses of Problem 26. Thus, the Seifert–van Kampen theorem applies to the homomorphism Ψ in part (1) of that problem. Give examples where:

- (1) A is simply-connected, but X, U_1, U_2 are not. *Hint:* Problem 27.
- (2) X, U_1, U_2 are simply-connected, but A is not. *Hint:* Munkres §59.
- (3) X and U_1 are simply-connected, but U_2 is not.
- (4) X is simply-connected, but U_1 and U_2 are not.

Problem 29. For each of the following spaces, the fundamental group is either trivial, \mathbf{Z} , or the free group $F_2 \simeq \mathbf{Z} \star \mathbf{Z}$. Determine which option is the case. You do not need to give explicit homeomorphisms or homotopy equivalences, but give informal descriptions (or pictures) to support your reasoning.

- (1) $D \times S^1$, where $D = \{(x, y) \in \mathbf{R}^2 \mid x^2 + y^2 < 1\}$.
- (2) The *torus* $S^1 \times S^1$.
- (3) The bounded cylinder $S^1 \times [0, 1]$.
- (4) The unbounded cylinder $S^1 \times \mathbf{R}$.
- (5) The punctured plane $\mathbf{R}^2 - \{p\}$, where p is any point.
- (6) *Harder:* The punctured torus $S^1 \times S^1 - \{q\}$, where q is any point.

2. SEPARATION THEOREMS IN THE PLANE

EXAM 2 TOPICS

3. SIMPLICIAL COMPLEXES AND SURFACES

EXAM 3 TOPICS

4. HOMOLOGY