Last time $\pi_1(X, x) = \{[\gamma] \mid \text{loops } \gamma \text{ in } X \text{ at } x\}$
under the operation
$[\beta] * [\gamma] = [\beta * \gamma]$ [what's the id elt?]

Q how much does it depend on X, x?

$$[0, 1] \xrightarrow{\gamma} X \xrightarrow{f} Y$$

if γ is a path in X, then $f \circ \gamma$ is a path in Y

$$[0, 1] \times [0, 1] \xrightarrow{h} X \xrightarrow{f} Y$$

if h is a path homotopy from γ to γ'
then $f \circ h$ is a path homotopy from $f \circ \gamma$ to $f \circ \gamma'$

$$[0, 1] \xrightarrow{\beta * \gamma} X \xrightarrow{f} Y$$

$$[f \circ (\beta * \gamma)](s) = f((\beta * \gamma)(s))$$
$$=$$
$$[(f \circ \beta) * (f \circ \gamma)](s) = \begin{cases} f(\beta(2s)) & s \leq 1/2 \\ f(\gamma(2s)) & s \geq 1/2 \end{cases}$$

Thm suppose $f : X$ to Y is cts

1) if γ, γ' are paths in X s.t. $\gamma \sim_p \gamma'$
then $f \circ \gamma, f \circ \gamma'$ are paths in Y s.t. $f \circ \gamma \sim_p f \circ \gamma'$

2) if β, γ are paths in X s.t. $\beta(1) = \gamma(0)$,
then $f \circ (\beta * \gamma) = (f \circ \beta) * (f \circ \gamma)$

Cor suppose $f : X \to Y$ is cts and $f(x) = y$

1) well-def map $f_* : \pi_1(X, x) \to \pi_1(Y, y)$ s.t.

$$f_*([\gamma]) = [f \circ \gamma]$$

[if $[\gamma] = [\gamma']$, then $[f \circ \gamma] = [f \circ \gamma']$]

2) f_* is a group homomorphism:

$$f_*([\beta] * [\gamma]) = f_*([\beta]) * f_*([\gamma])$$

[LHS $= f_*([\beta * \gamma])$
$= [f \circ (\beta * \gamma)]$
$= [(f \circ \beta) * (f \circ \gamma)]$
$= [f \circ \beta] * [f \circ \gamma]$ $=$ RHS]

Cor if $f : X \to Y$ is a homeo
then f_* is an isomorphism

Ex if $f = id_X$, then $f_* = id_{\pi_1(X, x)}$

Q is the converse true?

Ex $\mathbb{R}$ and $\{0\}$ are not homeomorphic [why?]
but $\pi_1(\mathbb{R}, 0) = \pi_1(\{0\}, 0)$

[in fact:]

Thm if X is a convex subset of $\mathbb{R}^n$
then for any x and loop γ based at x,
have $\gamma \sim_p e_x$
thus $\pi_1(X, x) = \{[e_x]\}$

Pf recall the homotopy

$h(s, t) = (1 - t)*x + t*\gamma(s)$

is it a path homotopy?

$h(0, t) = (1 - t)*x + t*\gamma(0) = (1 - t)*x + t*x = x$

$h(1, t) = (1 - t)*x + t*\gamma(1) = (1 - t)*x + t*x = x$

Q given cts maps $f : X$ to Y and $g : Y$ to Z s.t. $f(x) = y$ and $g(y) = z$

how to relate $f_* : \pi_1(X, x)$ to $\pi_1(Y, y)$,

$g_* : \pi_1(Y, y)$ to $\pi_1(Z, z)$,

$(g \circ f)_* : \pi_1(X, x)$ to $\pi_1(Z, z)$?

Thm $(g \circ f)_* = g_* \circ f_*$

Pf

$$\begin{aligned}(g \circ f)_*([\gamma]) &= [g \circ f \circ \gamma] \\ &= [g \circ (f \circ \gamma)] \\ &= g_*([f \circ \gamma]) \\ &= g_*(f_*([\gamma]))\end{aligned}$$

Ex given spaces X, Y,

point x in X,

cts $f : X$ to Y and $g : Y$ to X

s.t. $g \circ f = id_X$

have f injective and g surjective by PS3, #8

also $g_* \circ f_* = (id_X)_* = id_{\pi_1(X, x)}$

so

f_* injective and g_* surjective

Ex $\quad x = (1, 0)$ in S^1

$i : S^1$ to $R^2 - \{(0, 0)\}$ $\quad i(x, y) = (x, y)$
$r : R^2 - \{(0, 0)\}$ to S^1 $\quad r(x, y) = (x, y)/|(x, y)|$

$r \circ i = id_{S^1}$
so i_* injective and r_* surjective

in fact: $\quad \pi_1(S^1, x) = \pi_1(R^2 - \{(0, 0)\}, x) = Z$
and i_*, r_* are isomorphisms

but $i \circ r \neq id_{R^2 - \{(0, 0)\}}$

but $i \circ r \sim id_{R^2 - \{(0, 0)\}}$
via $h(s, t) = ((1 - t) + t/|(x, y)|)*(x, y)$

(Munkres §58)

Df $\quad$ a homotopy equivalence btw X and Y
is a pair of cts maps

$f : X$ to Y and $g : Y$ to X

s.t. $\quad g \circ f \sim id_X$ and $f \circ g \sim id_Y$

if such maps exist, then we say that X and Y are homotopy equivalent

Thm $\quad$ if $f : X$ to Y and $g : Y$ to X form a homotopy equivalence

then f_* and g_* are isomorphisms of π_1's