



HOMFLYPT Twisting as a Homotopy Equivalence

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$G = \mathrm{GL}_n$

$\mathcal{U} = \{\text{unipotent } n \times n \text{ matrices}\} \subseteq G$



Ex For $n = 2$,

$$\mathcal{U} = \left\{ \begin{pmatrix} 1+a & b \\ c & 1-a \end{pmatrix} \middle| a^2 + bc = 0 \right\}$$

because u is unipotent $\iff u - I$ is nilpotent
 $\iff a + d = ad - bc = 0.$

B upper-triangular subgroup of G

Bruhat, Cartan, Chevalley

$$G = \bigsqcup_{w \in S_n} B\dot{w}B$$

where \dot{w} is the permutation matrix lifting w .

Lets us study G in terms of the *Borel subgroup* B and the *Weyl group* S_n .

What can we say about \mathcal{U} in terms of

$$\mathcal{U} = B \cap \mathcal{U}?$$

B lower-triangular subgroup of G

Fine–Herstein '58, Steinberg '65

$$\begin{aligned} |\mathcal{U}(\mathbb{F}_q)| &= q^{n(n-1)} \\ &= |U(\mathbb{F}_q)|^2 \\ &= |UU_-(\mathbb{F}_q)|. \end{aligned}$$

where $U_- = B_- \cap \mathcal{U}$.

\mathcal{U}_w = $\mathcal{U} \cap BwB$, \mathcal{V}_w = $UU_- \cap BwB$

Kawanaka '75 For any $w \in S_n$,

$$|\mathcal{U}_w(\mathbb{F}_q)| = |\mathcal{V}_w(\mathbb{F}_q)|.$$

Ex For any n , we have $\mathcal{U}_{\text{id}} = U = \mathcal{V}_{\text{id}}$.

Ex For $n = 3$ and $w = \begin{pmatrix} & 1 & 1 \\ & 1 & \end{pmatrix}$,

$$\mathcal{U}_w \simeq U \times \{(a, b, c, d) \mid a, b \neq 0, (1+ab)^3 = abcd\}$$

$$\mathcal{V}_w \simeq U \times \{(a, b, c, d) \mid a, b \neq 0, 1+ab = abcd\}$$

These are not even homeomorphic over \mathbb{C} .

T diagonal subgroup

$T \curvearrowright \mathcal{U}_w, \mathcal{V}_w$ by conjugation.

Thm (T) For any $w \in S_n$,

$$\mathrm{gr}_*^W H_{c,T}^*(\mathcal{U}_w(\mathbb{C})) \simeq \mathrm{gr}_*^W H_{c,T}^*(\mathcal{V}_w(\mathbb{C})).$$

$H_{c,T}^*$ is *equivariant cohomology with compact support*.

$W_{\leq *}$ is its *weight filtration*.

This implies Kawanaka's identity

$$|\mathcal{U}_w(\mathbb{F}_q)| = |\mathcal{V}_w(\mathbb{F}_q)|$$

via results of Katz.

The map

$$UU_- \rightarrow \mathcal{U} \text{ given by } xy \mapsto xyx^{-1}$$

is T -equivariant.

Conj (T) It restricts to a homotopy equivalence

$$\mathcal{V}_w(\mathbb{C}) \rightarrow \mathcal{U}_w(\mathbb{C}).$$

This would imply the theorem about cohomology.

(The original map is itself a homotopy equivalence:
 UU_- and \mathcal{U} are both contractible.)

Each $w \in S_n$ lifts to some *braid* $\sigma_w \in Br_n^+$.

For a positive braid $\beta = \sigma_{w_1} \cdots \sigma_{w_k}$, let

$$X_\beta = B\dot{w}_1 B \times^B B\dot{w}_2 B \times^B \cdots \times^B B\dot{w}_k B$$

where \times^B means “ \times mod antidiagonal B .”

For any conjugation-stable set $C \subseteq G$, form

$$\begin{array}{ccc} X_\beta^C & \rightarrow & X_\beta \\ \downarrow & & \downarrow \\ C & \rightarrow & G \end{array}$$

$\mathcal{U}_w, \mathcal{V}_w$ are special cases of the X_β^C .

The *full twist* π :



Turns out: $\mathcal{U}_w = X_{\sigma_w}^{\mathcal{U}}$ and $\mathcal{V}_w = X_{\sigma_w \pi}^1$.

Thm (T) For any $\beta \in Br_n^+$,

$$\begin{aligned} |X_\beta^{\mathcal{U}}(\mathbb{F}_q)| &= |X_{\beta \pi}^1(\mathbb{F}_q)|, \\ \text{gr}_*^W H_{c,T}^*(X_\beta^{\mathcal{U}}(\mathbb{C})) &\simeq \text{gr}_*^W H_{c,T}^*(X_{\beta \pi}^1(\mathbb{C})). \end{aligned}$$

Surprisingly, the proof uses knot theory.

Previous theorem is the case $\beta = \sigma_w$.

HOMFLYPT poly P : $\{\text{links}\} \rightarrow \mathbf{Z}[[q][a, q^{-1/2}],$
 KhR superpoly \mathbf{P} : $\{\text{links}\} \rightarrow \mathbf{Z}[[q][a, q^{-1/2}, t]$

Kálmán '09 Writing $\widehat{\beta}$ for the closure of β ,

$$P(\widehat{\beta})[a^{|\beta|-n+1}] = P(\widehat{\beta\pi})[a^{|\beta|+n-1}]$$

where $P[a^i]$ means “ q -coefficient of a^i .”

Gorsky–Hogancamp–Mellit–Nakagane '19

True with \mathbf{P} in place of P .

We will relate $X_\beta^{\mathcal{U}}, X_\beta^1$ to these pieces of \mathbf{P} .

$X_\beta^{\mathcal{U}}, X_\beta^1$ are pieces of a larger variety $\tilde{X}_\beta^{\mathcal{U}}$.

Springer resolution of \mathcal{U} :

$$\tilde{\mathcal{U}} = \{(u, gB) \in \mathcal{U} \times G/B \mid ugB = gB\}.$$

Pullback squares:

$$\begin{array}{ccccccc} X_\beta^1 \times G/B & \rightarrow & \tilde{X}_\beta^{\mathcal{U}} & \rightarrow & X_\beta^{\mathcal{U}} & \rightarrow & X_\beta \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \{1\} \times G/B & \rightarrow & \tilde{\mathcal{U}} & \rightarrow & \mathcal{U} & \rightarrow & G \end{array}$$

To relate $X_\beta^{\mathcal{U}}, X_\beta^1$ to pieces of \mathbf{P} , relate $\tilde{X}_\beta^{\mathcal{U}}$ to \mathbf{P} .

Thm (T) For any $\beta \in Br_n^+$,

$$S_n \curvearrowright \text{gr}_*^W H_{c,T}^*(\tilde{X}_\beta^{\mathcal{U}}(\mathbb{C})).$$

Moreover,

$$\mathbf{P}(\widehat{\beta}) \propto (\Lambda^*(\mathbb{V}), \text{gr}_*^W H_{c,T}^*(\tilde{X}_\beta^{\mathcal{U}}(\mathbb{C})))_{S_n}$$

where $\mathbb{V} = \mathbb{C}^{n-1}$ is the standard irrep of S_n .

The variable a corresponds to the Λ^* -grading.

$$\implies \mathbf{P}(\widehat{\beta})[a^{\text{lo}}] = \dim \text{gr}_*^W H_{c,T}^*(X_\beta^{\mathcal{U}}(\mathbb{C})),$$

$$\implies \mathbf{P}(\widehat{\beta})[a^{\text{hi}}] = \dim \text{gr}_*^W H_{c,T}^*(X_\beta^1(\mathbb{C})).$$

For P , we can use *symmetric functions* to be more concrete.

Hotta–Springer '77 For $u \in \mathcal{U}$ of Jordan type $\mu \vdash n$,

$$S_n \curvearrowright H^*(\tilde{\mathcal{U}}_u(\mathbb{C}))$$

is given by the *Hall–Littlewood polynomial* $\tilde{H}_\mu(q)$.

Consequently:

$$P(\widehat{\beta})[a^{\text{lo}+2k}] \propto \sum_{\mu \vdash n} \langle s_{\Lambda^k}, \tilde{H}_\mu(q) \rangle \cdot |X_\beta^\mu(\mathbb{F}_q)|.$$

s_{Λ^k} is the *Schur function* of type $(n-k, 1^k)$.

$X_\beta^\mu \subseteq X_\beta^{\mathcal{U}}$ is the preimage of $\mathcal{U}_\mu \subseteq \mathcal{U}$.

Why should \mathcal{U} be related to knot invariants?

P arises from traces on *Hecke algebras*.

\mathbf{P} arises from traces on *Hecke categories*.

$$\mathrm{D}_{mix,G}^b \mathrm{Perv}(\mathcal{U})$$

$$\simeq \mathrm{D}^b \mathrm{Mod}(\mathbb{C} S_n \ltimes \mathrm{Sym}(\mathbb{V})) \quad (\text{Rider})$$

$$\simeq \textcolor{red}{\mathrm{hTr}}(\mathrm{Hecke}(S_n)) \quad (\text{Gorsky--Wedrich})$$

Thank you for listening.

Passing from X_β to $\tilde{X}_\beta^{\mathcal{U}}$ is like passing from β to its
annular closure.