Review recall F_X = free group on X

given a group G:

- S sub G is a generating set
  iff no smaller subgroup of G contains S
  iff the homomorphism F_S to G is surjective

in this case:

- R sub F_S is a set of relations wrt S
  iff ker(F_S to G) is the smallest kernel, i.e.,
  normal subgp of F_S, containing R

then we can speak of a presentation of G
by generators and relations: G = <S | R>
if R = Ø, then G = F_S and we write G = <S>

Rem any G has an "obvious" gen'ting set S:
[pause: what is it?]
take S = G itself
[usually we prefer to study smaller S]

Ex take G = Z
what is a one-elt gen'ting set? [pause]
S = {1} works [but also another:]
S = {–1} also works

what is a two-elt gen'ting set without ±1? [pause]
[e.g.] S = {2, 3}

Ex last time, saw that if G = {e, s}
then G = <s | s^2>
[abusing notation: s should be {s}, etc.]

Ex let $G = Z^2$ [under coordinate-wise +]
what is a generating set? [pause]
$S = \{(1, 0), (0, 1)\}$ works

write $a = (1, 0)$ and $b = (0, 1)$

what is ker(F_S to Z^2)? [pause]

elts of F_S are words in a, b, a^{-1}, b^{-1}
if such a word contains
- M a's,
- N b's,
- M' a^{-1}'s,
- N' b^{-1}'s

then it is mapped to $(M - M', N - N')$ in Z^2, so

ker(F_S to Z^2) = {words where the net exponent of a & the net exponent of b are both zero}

e.g., for any w, v in F_S, it contains the commutator $[w, v] := wvw^{-1}v^{-1}$
[here w^{-1} means the group inverse to w]

Fact ([follows from] Munkres 69.3–69.4)
- $\{[w, v] \mid w, v \text{ in } F_S\}$ is a generating set for ker(F_S to Z^2)
- the kernel is the smallest normal subgp containing $[a, b]$

[defer proof for now]

altogether, get the presentation

$\mathbb{Z}^2 = \langle a = (1, 0), b = (0, 1) \mid aba^{-1}b^{-1} \rangle$

Free Products [goal: Seifert–van Kampen:]
given groups
$G_1 = \langle S_1 \mid R_1 \rangle$,
$G_2 = \langle S_2 \mid R_2 \rangle$:

Df 1 the free product of G_1 and G_2 is

$$G_1 * G_2 = \langle S_1 \cup S_2 \mid R_1 \cup R_2 \rangle$$

Problem a priori, $G_1 * G_2$ could depend on how we present G_1 and G_2
[to solve this issue, new defn:]

Df 2 a free product of G_1, G_2 is a group G with maps $\iota_1 : G_1 \to G$, $\iota_2 : G_2 \to G$
s.t., for any group K, we have a bijection

$$\{\text{pairs of hom's } \varphi_1 : G_1 \to K,\ \varphi_2 : G_2 \to K\}$$
$$=$$
$$\{\text{hom's } \Phi : G \to K\}$$

given explicitly by $\varphi_1 = \Phi \circ \iota_1$ and $\varphi_2 = \Phi \circ \iota_2$

Thm the free product in definition #2 is unique up to iso [in fact, "unique iso"]

Pf suppose (G, ι_1, ι_2), (G', ι'_1, ι'_2) both work

taking $\varphi_k = \imath'_k$ above gives a hom $\Phi : G$ to G'
s.t. $\imath'_k = \Phi \circ \imath_k$
taking $\varphi_k = \imath_k$ above gives a hom $\Phi' : G'$ to G
s.t. $\imath_k = \Phi' \circ \imath'_k$

substituting, $\imath_k = \Phi' \circ \Phi \circ \imath_k$
so under the defining bijection for G,
id_G and $\Phi' \circ \Phi$ both correspond to $(\imath_1, \imath_2)$
[pause: what next?] so $id_G = \Phi' \circ \Phi$
similarly, $id_{G'} = \Phi \circ \Phi'$
so Φ and Φ' are each other's two-sided inverses □

[thm + proof illustrate "category-theoretic" ideas]

Lem $G_1 * G_2$ in defn #1 satisfies defn #2

Pf left as exercise

Ex the free group F_2 is isomorphic to $Z * Z$

more generally, $*$ is associative:
F_n is isomorphic to $Z * Z * \ldots * Z$ with n copies

Ex let $G = \{e, s\}$, the two-elt group
how to write down elts of $G * G$? [pause]

need to distinguish two copies of s: say, "s" and "t"

$G * G = \{e, s, t, st, ts, sts, tst, \ldots\}$

(Munkres §70) [but slightly changed notation]

<u>Thm</u> (Seifert–van Kampen) take open inclusions

$j_1 : U_1 \to X$,

$j_2 : U_2 \to X$

s.t. $X = U_1 \cup U_2$,

U_1 and U_2 are path connected,

$U := U_1 \cap U_2$ is path-connected

let $i_1 : U \to U_1$ and $i_2 : U \to U_2$ be inclusion

then for any x in U:

1) the homomorphism

$$\pi_1(U_1, x) * \pi_1(U_2, x) \to \pi_1(X, x)$$

arising from $(j_{1,*}, j_{2,*})$ via the defn of free product is <u>surjective</u>

2) the kernel of the homomorphism is the smallest normal subgp of the domain containing the elts of the form

$$i_{1,*}([\gamma])^{-1} i_{2,*}([\gamma])$$

as we run over elts $[\gamma]$ in $\pi_1(U, x)$

[above, $i_{k,*}([\gamma])$ in $\pi_1(U_k, x)$, but then we implicitly embed it into the free product]

<u>Cor</u> $\pi_1(X, x)$ is generated by the union of $\pi_1(U_1, x)$ and $\pi_1(U_2, x)$

<u>Cor</u> if there are open $U_1, U_2 \subset X$ s.t.

U_1, U_2 are simply-connected,

$X = U_1 \cap U_2$,

$U_1 \cap U_2$ is path-connected,

then X is simply-connected

[we stated the latter corollary in a previous class]

<u>Ex</u> take a figure-eight:

[draw]

take open U_1, U_2 s.t.
they deformation retract onto the two loops
$U_1 \cap U_2$ def. retracts onto the middle pt

[draw]

then $\pi_1(U_1, x) = \pi_1(U_2, x) = \pi_1(S^1) = \mathbb{Z}$
but $\pi_1(U_1 \cap U_2, x)$ is trivial

so $\pi_1(\text{figure-eight}, x) = \mathbb{Z} * \mathbb{Z} = F_2$