(Axler §7B) last time:

<u>Thm</u> if $T : V$ to $V$ is normal, then:

1–2) $\ker(T^*) = \ker(T)$, $\mathrm{im}(T^*) = \mathrm{im}(T)$

3) $T - \lambda$ is normal for all $\lambda$ in $F$

4) $\ker(T - \lambda) = \ker(T^* - \bar{\lambda})$

today, we work over $F = C$

## <u>Spectral Thm</u>

if $V$ is finite-dim'l over $C$ and $T : V$ to $V$ is normal
then $T$ is diagonalizable

in fact:
$V$ has a basis of <u>orthonormal eigenvectors</u> for $T$

## <u>Restatement in Matrices</u>

let $(e_1, \ldots, e_n)$ be any orthonormal basis for $V$
$A$ the matrix of $T$ wrt $(e_i)_i$

let $(u_1, \ldots, u_n)$ be the basis of
orthonormal eigenvectors for $T$
$\lambda_i$ defined by $Tu_i = \lambda_i u_i$
$P$ the $n \times n$ matrix defined by $Pe_i = u_i$
$D$ the $n \times n$ diagonal matrix with diagonal
$\lambda_1, \ldots, \lambda_n$

[what's $A$ in terms of $P$, $D$?] then $A = P^{-1}DP$

<u>Note 1</u> we proved last time: if $T$ is self-adjoint,
not just normal, then the $\lambda_i$'s are all real

Note 2 the cols of $P$ expand the $u_i$'s into $e_i$'s
but the $u_i$'s are orthonormal, so

$$PP^* = I$$

that is: $Pu \bullet \overline{(Pv)} = u^t PP^* \bar{v} = u \bullet \bar{v}$ for all $u, v$

Df recall:
an operator $Q$ is orthogonal wrt $\langle\ ,\ \rangle$
iff $\langle Qu, Qv\rangle = \langle u, v\rangle$ for all $u, v$

if $F = \mathbb{C}$, then we often say "unitary" rather than "orthogonal"

we say a matrix $P$ is unitary iff $Pu \bullet \overline{(Pv)} = u \bullet \bar{v}$
[which occurs] iff $PP^* = I$

Pf of Thm induct on $n := \dim V$
if $n = 0$, then done

suppose $n > 0$
then [recall:] $T$ has an eigenvector $v$, say,
with eigenvalue $\lambda$

the line $\mathbb{C}v$ is $T$-stable
let $W = (\mathbb{C}v)^{\perp} = \{w \text{ in } V \mid \langle w, v\rangle = 0\}$
recall that the Gram–Schmidt process shows

$$V = \mathbb{C}v + W \quad \text{and this sum is direct}$$

so it remains to show:

Claim $W$ is $T$-stable

Claim Finishes Pf    note dim $W = n - 1$

by inductive hypothesis, $W$ has a basis of
orthonormal eigenvectors $u_1, \ldots, u_{n-1}$
all are orthogonal to $v$
now set $u_n = v/||v||$ $\square$

Pf of Claim    pick $w$ in $W$
want $Tw$ in $W$: that is, $\langle Tw, v\rangle = 0$

know $\langle Tw, v\rangle = \langle w, T^*v\rangle$
but [recall!] $v$ in $\ker(T - \lambda) = \ker(T^* - \bar{\lambda})$
now, $\langle w, T^*v\rangle = \langle w, \bar{\lambda}v\rangle$
$= \bar{\lambda}\langle w, v\rangle$
$= 0$

Rem    claim + its proof generalize:

if $T : V$ to $V$ is normal and $U$ sub $V$ is $T^*$-stable
then $U^{\perp}$ is $T$-stable

Applications

Cor    if $TT^* = T^*T$ and all eigenvalues of $T$
are real and nonnegative,
then $T = S^*S$ for some $S : V$ to $V$

in particular, $S^* = S$
[because $(S^*S)^* = S^*S^{**} = S^*S$]

Pf    pick a basis of orthonormal eigenvectors
the matrix of $T$ in this basis is diagonal

call it D
let C be diagonal s.t. $C^2 = D$
let $S : V$ to $V$ be the op with matrix C in that basis

<u>Cor</u> TFAE for an $n \times n$ matrix A:

1) the pairing $\langle u, v \rangle := u^tA\bar{v}$ is
an inner product
2) A is Hermitian and positive-definite
[pos-def: $v^t A\bar{v} > 0$ for $v \neq \mathbf{0}$]
3) $A = BB^*$ for some <u>invertible</u> $B : V$ to $V$

<u>Pf</u> the direction 1) implies 2) implies 3)
are PS8, #8, part (1)
[use the previous corollary]
[why do we need B invertible?]

conversely, if $A = B^*B$, then:
A is Hermitian, via the argument earlier
[$(B^*B)^* = B^*B^{**} = B^*B$]
$v^tA\bar{v} = v^tB^*B\bar{v} = (\bar{B}v)^t\overline{(\bar{B}v)} > 0$
since the skew-dot product is pos-def
so $\langle \, , \, \rangle$ is pos-def and conj-symmetric

<u>Df</u> for general square B,
$B^*B$ is called the <u>Gram matrix</u>
its eigenvals are all real and nonegative
their sq roots are the <u>singular vals</u> of B

similar lingo for linear operators

[why useful? just as vector in an inner product space get norms, so too do operators on it]

Df the L^2 operator norm of S : V to V is

||S|| = max_{v s.t. ||v|| = 1} ||Sv||

i.e., the largest factor by which S rescales
the norm of a vector

Cor ||S|| = max {singular values of S}

Pf 1 pick a basis of orthonormal eigenvectors
for S*S

now, e.g., Lagrange multipliers show:
||Sv||^2 = <Sv, Sv> = <v, S*Sv> is maximized
on {||v|| = 1} when v is an eigenvec
for the largest eigenval of S*S

Pf 2 in some orthonormal basis,
the matrix of S*S looks like P^{-1}DP
with P orthogonal and D diagonal
[so, enough to show:]

max_{v s.t. ||v|| = 1} ||P^{-1}DPv||
= max_{v s.t. ||v|| = 1} ||Dv||

indeed, the set {v | ||v|| = 1} is stable
under unitary ops like P, P^{-1}

more general result [has a “Min-Max” version]:

Thm (Max-Min) max_{dim U = i}
min_{v in U | ||v|| = 1} ||Sv||
= ith largest singular value of S

[since $\|v\| = 1$ iff $\|\bar{v}\| = 1$ and $\dim U = \dim \bar{U}$:]

<u>Cor</u> S and $S^*$ have the same singular vals
i.e.,
$S^*S$ and $SS^*$ have the same eigenvals