

(Axler §7A, cont.) last time, we saw:

- given inner prod. spaces $(V, \langle \cdot, \cdot \rangle)$, $(W, \langle \cdot, \cdot \rangle)$,
 $T : V \rightarrow W$ defines an adjoint $T^* : W \rightarrow V$
characterized by $\langle Tv, w \rangle = \langle v, T^*w \rangle$
- if V, W are finite-dim'l
then get orthonormal bases for V and W
i.e., the basis vectors e_i satisfy
$$\langle e_i, e_i \rangle = 1$$
$$\langle e_j, e_i \rangle = 0 \text{ if } j \neq i$$

let M be the matrix of T wrt the orthonormal bases

Q what is the matrix of T^* ?
[let them cook a bit]

A WLOG can take $V = F^n$ and $W = F^m$
under their dot / skew-dot products
and choose the bases to be the std bases

$F = R$: for all v and w , we require
$$\langle Tv, w \rangle = (Tv)^t w = v^t T^t w$$
$$\langle v, T^*w \rangle = v^t T^*w$$
taking $v = e_j$ and $w = e_i$ shows
$$T^t_{\{j, i\}} = T^*_{\{j, i\}} \text{ for all } j, i$$
so $T^* = T^t$

$F = C$: for all v and w ,
$$\langle Tv, w \rangle = (Tv)^t w^- = v^t T^t w^-$$
$$\langle v, T^*w \rangle = v^t (T^*w)^- = v^t (T^*)^- w^-$$
so $T^t_{\{j, i\}} = (T^*)^-_{\{j, i\}} \text{ for all } j, i$ so $T^* = (T^t)^-$ [that is:]

Thm wrt orthonormal bases for both V and W ,
 $T : V$ to W and $T^* : W$ to V have
 matrices that are mutual conjugate
 transposes [we may write $M^* = (M^t)^{-1}$]

[works for both \mathbb{R} and \mathbb{C} : conjugation does nothing
 in the case of \mathbb{R}]

[we deduce:]

Properties of Adjoints

- $(aS + T)^* = a^*S^* + T^*$ [for all a in F and S, T]
- $\text{Id}^* = \text{Id}$
- $(T^*)^* = T$
- $(S \circ T)^* = T^* \circ S^*$
- T^* is invertible iff T is, and in this case,
 $(T^*)^{-1} = (T^{-1})^*$

[backing up a bit to §6B–6C, we revisit]
Orthogonal Complements

Df given linear $U \subseteq V$, its orthogonal
 complement (wrt $\langle \cdot, \cdot \rangle$) is

$$U^\perp = \{v \in V \mid \langle v, u \rangle = 0 \text{ for all } u \in U\}$$

- by definiteness of $\langle \cdot, \cdot \rangle$, $U^\perp \cap U = \{0\}$
- [we saw last time:] by Gram–Schmidt,
 if V is finite-dim'l, then $V = U + U^\perp$

if V is finite-dim'l, then we also have:

- $(U_1 + U_2)^\perp = [\text{what?}] U_1^\perp \cap U_2^\perp$
 [and why?]
- $(U^\perp)^\perp = U$

[notice similarity to properties of adjoints...]

Q how do adjoints interact with complements?

Thm 1) $\text{im}(T)^\perp = \ker(T^*)$ and $\ker(T^*)^\perp = \text{im}(T)$
2) $\ker(T)^\perp = \text{im}(T^*)$ and $\text{im}(T^*)^\perp = \ker(T)$

Pf 2) follows from 1) by swapping T and T^*

to show $\text{im}(T)^\perp = \ker(T^*)$:

$w \in \ker(T^*)$ iff $T^*w = \mathbf{0}_V$
iff $\langle v, T^*w \rangle = 0$ for all $v \in V$
iff $\langle Tv, w \rangle = 0$ for all $v \in V$
iff $w \in \text{im}(T)^\perp$

taking $(\)^\perp$ of both sides, we get $\ker(T^*)^\perp = \text{im}(T)$

Cor if V, W are finite-dim'l, then direct sums:

$$V = \ker(T) + \text{im}(T^*)$$

$$W = \text{im}(T) + \ker(T^*)$$

now consider a linear operator $T : V$ to V

Df [a linear op] T is self-adjoint iff $T^* = T$,
i.e., $\langle Tv', v \rangle = \langle v', Tv \rangle$ for all v, v'

if M is the matrix of T wrt an orthonormal basis,
then $T^* = T$ iff $M^* = M$

[where M^* denotes the conjugate transpose]

Prop if T is self-adjoint (over either \mathbb{R} or \mathbb{C})
then every eigenval of T is real
[is the converse true? no]

Pf let v be an eigenvector with eigenvalue λ
 [what is $\langle Tv, v \rangle$? pause]

$$\langle Tv, v \rangle = \langle \lambda v, v \rangle = \lambda \langle v, v \rangle$$

but also $\langle v, Tv \rangle = \langle v, \lambda v \rangle = \lambda^{-1} \langle v, v \rangle$

since $v \neq \mathbf{0}$, we know $\langle v, v \rangle \neq 0$ by definiteness
 so $\lambda = \lambda^{-1}$

[here is a slightly weaker notion:]

Df a linear operator T is normal iff
 $T^* \circ T = T \circ T^*$, i.e., they commute

[thus, any self-adjoint operator is normal]

Ex let $M : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be

$$M = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \text{ so that } M^* = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

then $M^* \neq M$, yet $M^*M = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = MM^*$

Prop T is normal iff $\|Tv\| = \|T^*v\|$ for all v in V

Pf T is normal
 iff $T^* \circ T - T \circ T^*$ is zero
 iff $\langle (T^* \circ T - T \circ T^*)v, v \rangle = 0$ for all v
 iff $\langle T^*Tv, v \rangle = \langle TT^*v, v \rangle$ for all v
 iff $\langle Tv, Tv \rangle = \langle T^*v, T^*v \rangle$ for all v

Thm if $T : V$ to V is normal, then:

- 1) $\ker(T^*) = \ker(T)$
- 2) $\text{im}(T^*) = \text{im}(T)$
- 3) $T - \lambda$ is normal for all λ in F

Pf 1) follows from the prop

2) from $\text{im}(T^*) = \ker(T)^\perp = \ker(T^*)^\perp = \text{im}(T)$

3) from $(T - \lambda) \circ (T - \lambda)^*$
 $= (T - \lambda) \circ (T^* - \lambda^-)$
 $= T \circ T^* - \lambda^- T - \lambda T^* + |\lambda|^2$
 $= T^* \circ T - \lambda^- T - \lambda T^* + |\lambda|^2$
 $= (T^* - \lambda^-) \circ (T - \lambda)$
 $= (T - \lambda)^* \circ (T - \lambda)$

Cor if $T : V$ to V is normal
then V is a direct sum $\ker(T) + \text{im}(T)$

Pf V is a direct sum $\ker(T) + \text{im}(T^*)$
but $\text{im}(T^*) = \text{im}(T)$

Cor if $T : V$ to V is normal
then for all v in V and λ in F , we have
 $Tv = \lambda v$ if and only if $T^*v = \lambda^-v$

Pf $\ker(T^* - \lambda^-) = \ker((T - \lambda)^*) = \ker(T - \lambda)$

[our goal next time: the] Spectral Thm

if V is finite-dim'l and $T : V$ to V is normal
then T is diagonalizable