6.

Notes about formulas for the rational (q,t)-Catalan numbers and related polynomials.

6.1.

6.1. Fix coprime integers $d, n > 0$. Let $\delta = \frac{1}{2}(d-1)(n-1)$. The following non-standard normalization of the (q,t)-Catalan numbers will be more convenient:

$$\mathsf{P}_{d/n}(q,t) = q^{\delta}\mathsf{C}_{d/n}(\tfrac{1}{q},t) = \sum_{(d\times n)\text{-Dyck paths } D} q^{\delta-\mathrm{dinv}(D)}t^{\mathrm{area}(D)}.$$

For example, we have

$$\begin{aligned}\mathsf{P}_{d/2} &= 1 + qt + \cdots + (qt)^{\frac{d-1}{2}},\\ \mathsf{P}_{4/3} &= 1 + qt + q^2t + q^2t^2 + q^3t^3.\end{aligned}$$

We are interested in formulas for $\mathsf{P}_{d/n}(q,t)$, and counterparts for Kreweras numbers and parking spaces, which are inspired by algebraic geometry.

6.2. Our first inspiration is work of Gorsky–Mazin and parallel work of Hikita.

Gorsky–Mazin study a projective variety known as the local compactified Jacobian of $Y^n = X^d$, which we will denote $\mathcal{M}_{d/n}$. It is stratified by affine spaces, and these strata are indexed by $d\times n$ Dyck paths. For each Dyck path D, let $\mathcal{M}_D$ be the corresponding stratum. There is an increasing filtration of $\mathcal{M}_{d/n}$ by closed subvarieties $\mathcal{M}_{d/n,\leq i}$, in which each subvariety is a union of strata. Gorsky–Mazin essentially prove that

$$\begin{aligned}\mathrm{area}(D) &= \min\{i \mid \mathcal{M}_D \subseteq \mathcal{M}_{d/n,i} \text{ and } \mathcal{M}_D \not\subseteq \mathcal{M}_{d/n,j} \text{ for } j < i\},\\ \delta - \mathrm{dinv}(D) &= \dim(\mathcal{M}_D).\end{aligned}$$

6.3. At the same time, the variety $\mathcal{M}_{d/n}$ is an example of an affine Springer fiber for SL_n. Hikita studied $\mathcal{M}_{d/n}$ in this Lie-theoretic setting. Below, we summarize the part of his work that corresponds to Gorsky–Mazin's.

Let $\mathbf{Z}^n_0 \subseteq \mathbf{Z}^n$ be the set of integral vectors ξ for which $\mathrm{sum}(\xi) := \sum_i \xi_i = 0$. Let

$$\mathfrak{D}_{d/n} = \left\{\xi \in \mathbf{Z}^n \;\middle|\; \begin{array}{l}\xi_1 \geq \xi_2 \geq \cdots \geq \xi_n,\\ \xi_1 - \xi_n \leq d\end{array}\right\}.$$

There is a known bijection between $d\times n$ Dyck paths and points of $\mathfrak{D}_{d/n} \cap \mathbf{Z}^n_0$. It turns out that if the Dyck path D corresponds to the point $\xi \in \mathfrak{D}_{d/n} \cap \mathbf{Z}^n_0$, then we can recover the statistics for D from ξ as follows. First, for $1 \leq i \leq n$, set

$$a_i(\xi) = n\xi_i + d(i-1).$$

Hikita essentially showed

$$\mathrm{area}(D) = \delta - \min\{a_1(\xi), \ldots, a_n(\xi)\}.$$

For all $i, j, k \in \mathbf{Z}$ with $1 \leq i, j \leq n$ and $i \neq j$, let $\alpha_{i,j,k} : \mathbf{Z}^n \to \mathbf{Z}$ be the affine root $\alpha_{i,j,k}(\xi) = \xi_i - \xi_j - k$. Let

$$\mathfrak{A}_{d/n} = \{\alpha_{i,j,k} \mid 0 \leq \alpha_{i,j,k}(\tfrac{d}{n}\rho^\vee) < \tfrac{d}{n}\},$$

where $\rho^\vee = (\frac{n-1}{2}, \frac{n-3}{2}, \ldots, \frac{1-n}{2})$, and let

$$\mathfrak{A}_{d/n}(\xi) = \{\alpha_{i,j,k} \in \mathfrak{A}_{d/n} \mid \alpha_{i,j,k}(\xi) < 0\}.$$

The set $\mathfrak{A}_{d/n}(\xi)$ was implicitly introduced by Goresky–Kottwitz–MacPherson in "Purity of Equivalued Affine Springer Fibers". They showed that

$$\dim(\mathcal{M}_D) = |\mathfrak{A}_{d/n}(\xi)|.$$

6.4. By work of Mellit, later extended by Elias and Hogancamp, $\mathsf{P}_{d/n}(q,t)$ is the (a,q,t)-superpolynomial of the (d,n)-torus knot, up to normalization. A conjecture of Oblomkov–Rasmussen–Shende about plane curve singularities, specialized to the case of $Y^n = X^d$, matches the superpolynomial with a generating function for the virtual weight polynomials of the components of the Hilbert scheme of the singularity. We now recast their conjecture in a combinatorial form à la Hikita.

Let $\mathcal{H}_{d/n}$ be the Hilbert scheme in question. We may view $\mathcal{H}_{d/n}$ as the "positive part" of an affine Springer fiber for GL_n. It is stratified by affine spaces, just like $\mathcal{M}_{d/n}$, except now, the strata are indexed by the points of $\mathfrak{D}_{d/n} \cap \mathbf{Z}^n_{\geq 0}$.

For each $\xi \in \mathfrak{D}_{d/n} \cap \mathbf{Z}^n_{\geq 0}$, let $\mathcal{H}_\xi$ be the corresponding stratum. It belongs to the ℓth connected component of $\mathcal{H}_{d/n}$, meaning the component that parametrizes ideals of colength ℓ, precisely when $\mathrm{sum}(\xi) = \ell$. Let

$$\mathfrak{A}^+_{d/n}(\xi) = \{\alpha_{i,j,k} \in \mathfrak{A}_{d/n}(\xi) \mid k \leq \xi_i\}.$$

Either by applying Goresky–Kottwitz–MacPherson's argument, or by imitating an argument of Piontkowski, one can show:

Lemma 6.1. $\dim(\mathcal{H}_\xi) = |\mathfrak{A}^+_{d/n}(\xi)|$.

One can then deduce that:

Proposition 6.2. *The ORS conjecture for $Y^n = X^d$ is equivalent to the identity*

$$\frac{1}{1-x}\mathsf{P}_{d/n}(xy, x) = \sum_{\xi \in \mathfrak{D}_{d/n} \cap \mathbf{Z}^n_{\geq 0}} x^{\mathrm{sum}(\xi)} y^{|\mathfrak{A}^+_{d/n}(\xi)|}. \tag{6.1}$$

Here we have avoided using the letters q and t, because the (q,t) in ORS corresponds to the (t,q) in Haglund's book.

6.2.

6.5. Surprisingly, we do know how to express the left-hand side of (6.1) in a form very similar to the right-hand side, but using a different region in $\mathbf{Z}^n_{\geq 0}$, and using yet another variant of $\mathfrak{A}_{d/n}(-)$.

The combinatorial motivation for what follows is the paper "Recursions for Rational (q,t)-Catalan Numbers", by Gorsky–Mazin–Vazirani. The geometric motivation is a family of generalizations of the Hilbert scheme of $Y^n = X^d$: namely, the Quot schemes of various $R_{d/n}$-modules, where $R_{d/n}$ is the ring of formal germs of functions on the singularity. Each of these Quot schemes can be interpreted as the positive part of some affine Springer fiber for GL_n.

6.6. Fix $\mu \in \mathbf{Z}^n$. Let

$$\mathfrak{D}_{d/n}(\mu) = \left\{ \xi \in \mathbf{Z}^n \;\middle|\; \begin{array}{l} \alpha_{1,2}(\xi + \mu), \ldots, \alpha_{n-1,n}(\xi + \mu) \geq 0, \\ \alpha_{1,n}(\xi + \mu) \leq d \end{array} \right\},$$

and let

$$\begin{aligned} \mathfrak{A}^{\mu}_{d/n} &= \{\alpha_{i,j,k} \mid 0 \leq \alpha_{i,j,k}(\tfrac{d}{n}\rho^\vee - \mu) < \tfrac{d}{n}\}, \\ \mathfrak{A}^{\mu}_{d/n}(\psi) &= \{\alpha_{i,j,k} \in \mathfrak{A}^{\mu}_{d/n} \mid \alpha_{i,j,k}(\psi - \mu) < 0\}, \\ \mathfrak{A}^{\mu,+}_{d/n}(\psi) &= \{\alpha_{i,j,k} \in \mathfrak{A}^{\mu}_{d/n}(\psi) \mid k \leq \psi_i - \mu_i\}. \end{aligned}$$

Then the generating function

$$\mathsf{F}^{\mu}_{d/n}(x,y) = \sum_{\psi \in \mathfrak{D}_{d/n}(\mu) \cap \mathbf{Z}^n_{\geq 0}} x^{\mathrm{sum}(\psi)} y^{|\mathfrak{A}^{\mu,+}_{d/n}(\psi)|}.$$

recovers the right-hand side of (6.1) when μ is the zero vector.

Theorem 6.3. *Suppose that* $\{a_i(\mu)\}_i = \{0, 1, \ldots, n-1\}$. *Then*

$$\frac{1}{1-x}\mathsf{P}_{d/n}(y,x) = \mathsf{F}^{\mu}_{d/n}(x,y).$$

Moreover, in this case, $\mathfrak{A}^{\mu,+}_{d/n}(\psi) = \mathfrak{A}^{\mu}_{d/n}(\psi)$ *for all* $\psi \in \mathfrak{D}^{\mu}_{d/n} \cap \mathbf{Z}^n_{\geq 0}$.

Corollary 6.4. *Suppose that* $\{a_i(\mu)\}_i = \{0, 1, \ldots, n-1\}$. *Then the ORS conjecture for* $Y^n = X^d$ *is equivalent to the identity*

$$\mathsf{F}^{0}_{d/n}(x,y) = \mathsf{F}^{\mu}_{d/n}(xy,y). \tag{6.2}$$

Remark 6.5. In the case where $d = kn + 1$ for some integer k, the condition $\{a_i(\mu)\}_i = \{0, 1, \ldots, n-1\}$ is equivalent to taking $\mu_i = -(i-1)k$ for all i.

Example 6.6. Take $(d, n) = (5, 2)$ and $\mu = (0, -2)$. We compute that

$$\begin{aligned} \mathfrak{D}_{5/2} = \mathfrak{D}_{5/2}(0) &= \{\xi \in \mathbf{Z}^2 \mid 0 \leq \xi_1 - \xi_2 \leq 5\}, \\ \mathfrak{D}_{5/2}(\mu) &= \{\psi \in \mathbf{Z}^2 \mid -2 \leq \psi_1 - \psi_2 \leq 3\}. \end{aligned}$$

In the following picture of $\mathfrak{D}_{5/2} \cap \mathbf{Z}^2_{\geq 0}$, the bottom left point is the origin, and we have marked each point ξ in the region with the value of $|\mathfrak{A}^{+}_{5/2}(\xi)|$.

$$\begin{array}{cccccccccc} & & & & 2 & 1 & 0 & 0 & 1 & 2 \\ & & & 2 & 1 & 0 & 0 & 1 & 2 & \\ & & 2 & 1 & 0 & 0 & 1 & 2 & & \\ & 1 & 1 & 0 & 0 & 1 & 2 & & & \\ 0 & 0 & 0 & 0 & 1 & 2 & & & & \end{array}$$

In the following picture of $\mathfrak{D}_{5/2}(\mu) \cap \mathbf{Z}^2_{\geq 0}$, we have marked each point ψ in the region with the value of $|\mathfrak{A}^{\mu,+}_{5/2}(\psi)| = |\mathfrak{A}^{\mu}_{5/2}(\psi)|$.

$$\begin{array}{cccccccc} & & 2 & 1 & 0 & 0 & 1 & 2 \\ & 2 & 1 & 0 & 0 & 1 & 2 & \\ 2 & 1 & 0 & 0 & 1 & 2 & & \\ 1 & 0 & 0 & 1 & 2 & & & \\ 0 & 0 & 1 & 2 & & & & \end{array}$$

Now (6.2) amounts to a bijection between these sets that preserves the numbers assigned, and translates any point assigned the number ℓ upwards to the anti-diagonal "ℓ steps above" its current anti-diagonal.

In fact, Oscar Kivinen and I proved an identity for an arbitrary plane curve singularity that specializes to Theorem 6.3 in the case of $Y^n = X^d$.