(Munkres §54) wrap-up of:

<u>Thm</u> let $p : E \to X$ be a covering, e in E

1) for any path $\gamma : [0, 1] \to X$ s.t.

$p(e) = \gamma(0)$,

a <u>unique</u> $\Gamma : [0, 1] \to E$ s.t.

$\Gamma(0) = e$ and $\gamma = p \circ \Gamma$,

which we call the <u>lift</u> of γ to E

2) for any homotopy $h : [0, 1]^2 \to X$ s.t.

$p(e) = h(0, 0)$,

a <u>unique</u> homotopy $H : [0, 1]^2 \to E$ s.t.

$H(0, 0) = e$ and $h = p \circ H$

3) in 2), if h is a path homotopy, then so is H

last time, proved 1) via [what lemma?]
the Lebesgue number lemma

<u>Pf of 2)</u> as last time:
pick an open cover $\{U_\alpha\}_\alpha$ s.t.
each U_α is evenly covered by p

by double application of Lebesgue, can find

$$0 = s_0 < s_1 < \ldots < s_n = 1$$
$$0 = t_0 < t_1 < \ldots < t_m = 1$$

s.t. $h : [0, 1]^2 \to X$ maps each rectangle
$[s_i, s_{i+1}] \times [t_j, t_{j+1}]$ into a single U_α at a time

again, build $H : [0, 1]^2 \to E$ inductively:

set $H(0, 0) = e$

order the rectangles lexicographically

for a given (i, j),

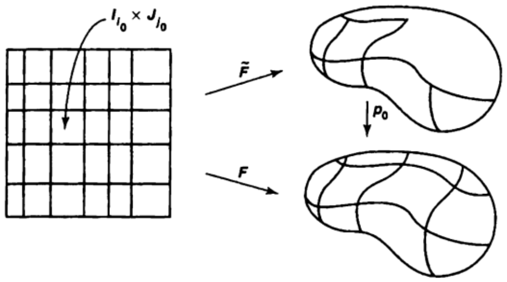
let $R_{i, j} = [s_i, s_{i+1}] \times [t_j, t_{j+1}]$

let $A_{i, j}$ be the union of $[0, 1] \times \{0\}$,

$\{0\} \times [0, 1]$, all rectangles prior to R

note that $A_{i, j}$ cap $R_{i, j}$ is a subset of $\partial A_{i, j}$

want to extend $H|_{A_{\{i, j\}}}$ to $H|_{A_{\{i, j\}} \text{ cup } R_{\{i, j\}}}$



similarly to last time:

h maps $R_{i, j}$ into a single U_α,

U_α is evenly covered by p

the key difference from last time:

we seek a lift of $h|_{R_{\{i, j\}}}$ that extends

the subspace $H(A_{i, j}$ cap $R_{i, j})$ of $p^{-1}(U_\alpha)$

not just a single point $\Gamma(s_i)$ in $p^{-1}(U_\alpha)$

[what saves us?]

$H(A_{i, j}$ cap $R_{i, j})$ is connected

so inside $p^{-1}(U_\alpha)$, it must be contained in

a single homeomorphic copy of U_α

now the rest of the proof is analogous to

the proof of 1) □

Pf of 3) suppose h is a path homotopy between two paths from x to y in X

then $H(1, t)$ in $p^{-1}(y)$ for all t in $[0, 1]$
but $p^{-1}(y)$ is discrete, since p is a covering
so $H(1, t)$ is constant for all t
so H is a path homotopy in E

Cor if $\gamma_0 \sim_p \gamma_1$ in X and e in E s.t. $p(e) = \gamma_0(0) = \gamma_1(0)$,

then γ_0, γ_1 have unique lifts Γ_0, Γ_1 in E starting at e
and $\Gamma_0 \sim_p \Gamma_1$

Applications

Cor 1 if $p : E \to X$ is a covering and $p(e) = x$, then $p_* : \pi_1(E, e) \to \pi_1(X, x)$ is injective

Pf let Γ_0, Γ_1 be loops in E based at e s.t. $p_*([\Gamma_0]) = p_*([\Gamma_1])$

by construction, $p_*(\Gamma_i) = [p \circ \Gamma_i]$
and Γ_0, Γ_1 are lifts of $p \circ \Gamma_0$, $p \circ \Gamma_1$
so $[\Gamma_0] = [\Gamma_1]$

Ex recall that for any integer $n > 0$ there is an n-fold covering $p_n : S^1 \to S^1$

under $\pi_1(S^1) = Z$, we have $\text{im}(p_{n,*}) = nZ$ [draw]

each $[\gamma]$ in $\pi_1(X, x)$ defines a permutation of $p^{-1}(x)$ as follows:

each e in $p^{-1}(x)$ is the <u>start</u> of a unique lift of γ
the permutation sends $e \mapsto e \cdot [\gamma]$, where
$e \cdot [\gamma]$ is the <u>end</u> of that lift
[draw solenoid]

<u>Cor 2</u> (Lifting Correspondence) given e in $p^{-1}(x)$:

a) $e \cdot [\gamma] = e$ if and only if $[\gamma]$ in $p_*(\pi_1(E, e))$
thus, an injective map

$$\varphi_e : p_*(\pi_1(E, e))\backslash\pi_1(X, x) \to p^{-1}(x)$$

defined by $\varphi_e(p_*(\pi_1(E, e)) * [\gamma]) = e \cdot [\gamma]$

b) if E is path-connected, then φ_e is bijective

c) if E is path-connected and simply-connected, then φ_e is a bijection $\pi_1(X, x) \to p^{-1}(x)$

<u>Pf of a)</u> $[\gamma] = p_*([\Gamma])$ for some $[\Gamma]$ in $\pi_1(E, e))$
iff the lift of γ to e is a loop (namely, Γ)
iff $e \cdot [\gamma] = e$

<u>Pf of b)</u> for any e' in $p^{-1}(x)$,
we can pick a path Γ from e to e' in E
let $\gamma = p \circ \Gamma$
then Γ is the lift of γ to e, so $e \cdot [\gamma] = e'$

<u>Pf of c)</u> immediate from b)

Ex recall that $RP^2 = S^2/\sim$
where ~ is antipodal identification

S^2 is simply-conn & $S^2 \to RP^2$ is a 2-fold covering
so $|\pi_1(RP^2)| = 2$
so new proof that $\pi_1(RP^2) = Z/2Z$

Df a pointed covering is a pair (p, e) s.t.
$p : E \to X$ is a covering
e in E

if $(p : E \to X, e)$ and $(p' : E' \to X, e')$ are
pointed coverings of the same X,
then a pointed equivalence from p to p' is
a homeo $f : (E, e) \to (E', e')$ s.t. $p = p' \circ f$

for such f, we write $(E, e) \sim (E', e')$
we also see that f_* is an isomorphism
$\pi_1(E, e) \to \pi_1(E', e')$
thus $p_*(\pi_1(E, e)) = p'_*(f_*(\pi_1(E, e))) = p'_*(E', e')$
[we’ve proven injectivity, but not bijectivity, in:]

Cor 3 (Galois Correspondence)

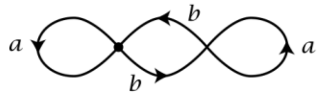
the map sending $(p : E \to X, e)$ to $p_*(\pi_1(E, e))$
defines a bijection

$\{\text{pointed coverings of } X\}/\sim$
$\to \{\text{subgroups of } \pi_1(X, x)\}$

Rem if $E = E'$ but $e \neq e'$, then $(E, e), (E', e')$
might not be equivalent

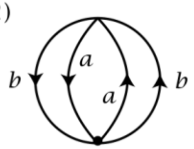
Ex coverings of the figure-eight
and corresponding subgroups of its π_1:

(1)



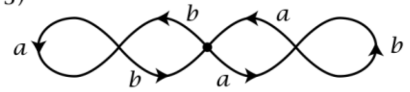
$\langle a, b^2, bab^{-1} \rangle$

(2)



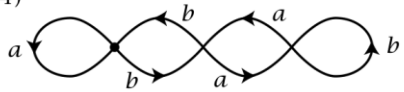
$\langle a^2, b^2, ab \rangle$

(3)



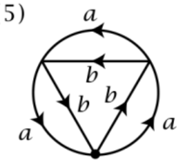
$\langle a^2, b^2, aba^{-1}, bab^{-1} \rangle$

(4)



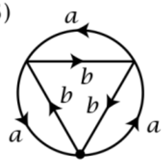
$\langle a, b^2, ba^2b^{-1}, baba^{-1}b^{-1} \rangle$

(5)



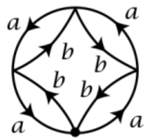
$\langle a^3, b^3, ab^{-1}, b^{-1}a \rangle$

(6)



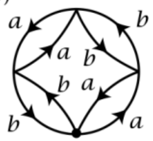
$\langle a^3, b^3, ab, ba \rangle$

(7)



$\langle a^4, b^4, ab, ba, a^2b^2 \rangle$

(8)



$\langle a^2, b^2, (ab)^2, (ba)^2, ab^2a \rangle$