

Warmup idea of gluing:

$[0, 1]$ maps to $S^1 = \{z \in \mathbb{R}^2 \mid |z| = 1\}$ via

$$f(t) = (\cos(2\pi t), \sin(2\pi t))$$

is it continuous [for the subspace topologies]?

method 1: f extends to a map $F : \mathbb{R} \rightarrow \mathbb{R}^2$

Lem if $F : X \rightarrow A$ is cts and $F(Y) = B$
then $F|_Y : Y \rightarrow B$ is cts wrt subsp. top's

F is cts with image S^1 [$X = \mathbb{R}$, $A = \mathbb{R}^2$]

so $F : \mathbb{R} \rightarrow S^1$ is cts [$Y = \mathbb{R}$, $B = S^1$]

so f is cts [now $Y = [0, 1]$, $A = B = S^1$]

method 2: check on basis for S^1

basis = {open arcs V }

if V does not pass through $\mathbf{p} := (1, 0)$

then $f^{-1}(V)$ is an open interval in $[0, 1]$

if it does pass through \mathbf{p}

then $f^{-1}(V)$ looks like $[0, a) \cup (b, 1]$

key idea:

[shape of S^1 close to shape of $[0, 1]$ because]

f injective everywhere except at the endpoints

Q describe topology of S^1 directly from
topology of $[0, 1]$?

(Munkres §22) let $f : X \rightarrow A$ be surjective

Df the quotient topology on A wrt f is
the finest topology s.t. $f : X \rightarrow A$ is cts

i.e., $V \subseteq A$ is open iff $f^{-1}(V)$ is open

[why is this a topology? preimages of cups/caps]

Rem sometimes A already has another top T

we say that $f : X \rightarrow A$ is a quotient map wrt T
iff

T is exactly the quotient topology wrt f

Ex the surjective map $[0, 1] \rightarrow S^1$ defines
a quotient topology on S^1

claim: it is a quotient map wrt the subspace top

(subspace top) \subseteq (quotient top) easy

conversely, if $V \subseteq S^1$ s.t. $f^{-1}(V)$ is open
either $f^{-1}(V)$ contains both 0 and 1
so $f^{-1}(V) \supseteq [0, a) \cup (b, 1]$
so $V \supseteq$ open arc around \mathbf{p}
or $f^{-1}(V)$ contains neither 0 nor 1
so V disjoint from arc around \mathbf{p}

[with more work]

can show V is union of open arcs in both cases
so (quotient top) \subseteq (subspace top)

Ex the surjective map R to S^1 defines
the same quotient topology on S^1
[exercise]

what partition of R corresponds to this map?
for all $(\cos(2\pi t), \sin(2\pi t))$ in S^1 , preimage in R is
 $\{\dots, t-2, t-1, t, t+1, t+2, \dots\}$
[draw picture]

Rem in language of group theory:
 R is a group under $+$,
 Z is a subgroup of R s.t. $S^1 \simeq R/Z$

Ex let $Y = [0, 1)$ sub $X = [0, 1]$
 $f|_Y : Y \rightarrow S^1$ still surjective
same quotient topology? no [why?]

Moral if $f : X \rightarrow A$ is surjective and $f(Y) = B$
then

the quot. top. on B induced by the sub. top. on Y
could be strictly finer than
the sub. top. on B induced by the quot. top. on A

(Prop in general, it is at least finer)

Ex let $f : R \rightarrow \{-, 0, +\}$ be the “sign” map
in quotient top wrt f , open sets are
 $\emptyset, \{-\}, \{+\}, \{-, +\}, \{-, 0, +\}$

let $g : \{-1, 0, 1\} \rightarrow \{-, 0, +\}$ be restr of f
in quotient top wrt g , every set is open

Interlude

an equivalence relation on X is a rule \sim s.t.

- 0) for all x, y in X , either $x \sim y$ or $x \not\sim y$
- 1) $x \sim x$ for all x in X
- 2) $x \sim y$ if and only if $y \sim x$
- 3) $(x \sim y \text{ and } y \sim z)$ together imply $x \sim z$

the following data are all “fungible”

- I) a surjective map $f : X$ to A
- II) a partition $X = \coprod_{\alpha} X_{\alpha}$
- III) an equivalence relation \sim on X

I) to II): set $X_{\alpha} = f^{-1}(\alpha)$ for all α in A

II) to I): $A = \{\text{indices } \alpha \text{ in partition}\}$
set $f(x) = \alpha$ for all x in X_{α}

II) to III): $x \sim y$ defined as:
 x, y in X_{α} for the same α

III) to II): X_{α} defined to be maximal subsets s.t.
for all x, y in X_{α} , have $x \sim y$

(Lem these maximal subsets do partition X)
(Df these maximal subsets are called
equivalence classes)

Ex if $f_i : X_i \rightarrow A_i$ is surjective for $i = 1, 2$
then so is

$$f = (f_1, f_2) : X_1 \times X_2 \rightarrow A_1 \times A_2$$

taking f_1, f_2 to be copies of $[0, 1] \rightarrow S^1$ gives

solid square $[0, 1] \times [0, 1] \rightarrow$ torus $S^1 \times S^1$

two top's:

product top wrt quotient top's wrt f_1, f_2

quotient top wrt f

do they match? yes [but tricky]

(product top) sub (quotient top) by PS3, #2

suppose $V \subset S^1 \times S^1$ s.t. $f^{-1}(V)$ open:
casework on “boxes” shows V open in prod top

[however, above example is misleading:]

Rem can find $f_i : X_i \rightarrow A_i$ for $i = 1, 2$ s.t.

f_1, f_2 are surjective

but the prod top on $A_1 \times A_2$ wrt quot top's
is coarser than the quot top wrt $f_1 \times f_2$

see Munkres §22, Ex 7 and p. 145, #6

Comparison recall:

for any collection of top spaces $(X_i)_i$:
 $(g_i)_i : Y \text{ to } \prod_i X_i$ is cts wrt product topology
iff $g_i : Y \text{ to } X_i$ is cts for all i

similarly:

for any surjective $f : X \text{ to } A$:
 $h : A \text{ to } Z$ is cts wrt quotient topology
iff $h \circ f : X \text{ to } Z$ is cts

note that $h \circ f$ will be constant along subsets of
the form $f^{-1}(a)$ for a in A

Munkres calls these the saturated sets of X wrt f