MATH 251: TOPOLOGY II EXAM 1

SPRING 2026

You have 80 minutes to do these problems in any order. There are 36 points total. Please show all work. You are not allowed to use electronic devices of any kind, including phones, computers, tablets, or audio devices. Throughout, $\mathbf{R}^N$ has the analytic topology, for any N. You may assume that intervals in $\mathbf{R}$ are connected, and that closed intervals are compact.

Problem 1 (8 points)**.** View the intervals $(-1, 1), [-1, 1]$ as subspaces of $\mathbf{R}$. Show that:

(a) $\mathbf{R}$ and $[-1, 1]$ are not homeomorphic.

(b) $\mathbf{R}$ and $(-1, 1)$ are homeomorphic.

Solution. (a) By Heine–Borel, $[-1, 1]$ is compact while $\mathbf{R}$ is not. [Alternatively: $\mathbf{R} - \{x\}$ is not path-connected for any $x \in \mathbf{R}$, but $[-1, 1] - \{1\} = [-1, 1)$ is path-connected.]

(b) Let $f\colon (-1, 1) \to \mathbf{R}$ and $g\colon \mathbf{R} \to (-1, 1)$ be defined by $f(t) = \tan(\pi t/2)$ and $g(x) = (2/\pi) \arctan x$. Then f and g are continuous and form two-sided inverses of each other, so they are homeomorphisms. □

Problem 2 (6 points)**.** Show that if A is a set of rational numbers, and A contains at least two elements, then A is disconnected as a subspace of $\mathbf{R}$. You may assume that any nonempty open interval in $\mathbf{R}$ contains an irrational number.

Solution. Let $a, b \in A$ be distinct points. Then the open interval (a, b) is nonempty, so it contains some irrational number α. Since α is irrational, $\alpha \notin A$. So A is the union of its subsets $U = A \cap (-\infty, \alpha)$ and $V = A \cap (\alpha, \infty)$.

Note that $a \in U$ and $b \in V$. Altogether, U, V are nonempty disjoint open subsets of A whose union is A. So they form a separation. □

Problem 3 (6 points)**.** Show that if X is compact, and $f\colon X \to Y$ is continuous and surjective, then Y is compact.

Solution. Let $\{V_i\}_i$ be an open cover of Y. Then $U_i := f^{-1}(V_i)$ is open for all i, by continuity of f. Moreover, $Y = \bigcup_i V_i$ implies $X = \bigcup_i U_i$. So the collection $\{U_i\}_i$ is an open cover of X. So it has a finite subcover $\{U_i\}_{i \in I}$, by compactness of X.

It remains to show that $\{V_i\}_{i \in I}$ is still an open cover of Y. Indeed, for all $y \in Y$, we can find $x \in X$ such that $f(x) = y$, since f is surjective. Then $x \in U_{i_0}$ for some $i_0 \in I$. Since $U_{i_0} = f^{-1}(V_{i_0})$, we conclude that $y \in f(U_{i_0}) \subseteq V_{i_0}$. □

Problem 4 (7 points)**.** Give, for any space Z, an explicit homotopy equivalence between $\mathbf{R} \times Z$ and its subspace $\{0\} \times Z$, and prove that it is indeed a homotopy equivalence.

Solution. Let $i\colon \{0\} \times Z \to \mathbf{R} \times Z$ be the inclusion $i(0, z) = (0, z)$. Let $r\colon \mathbf{R} \times Z \to \{0\} \times Z$ be the projection $r(x, z) = (0, z)$. Then $r \circ i = \mathrm{id}_{\{0\}\times Z}$. It remains to show that $i \circ r$ is homotopic to $\mathrm{id}_{\mathbf{R}\times Z}$.

Let $\varphi\colon (\mathbf{R} \times Z) \times [0, 1] \to \mathbf{R} \times Z$ be defined by $\varphi((x, z), t) = (xt, z)$. This is continuous because xt is a polynomial in x, t and products of continuous maps are continuous. Moreover, $\varphi((x, z), 0) = (0, z) = i(r(x, z))$ and $\varphi((x, z), 1) = (x, z)$. Therefore, φ is the desired homotopy. □

Problem 5 (5 points)**.** Let D be contractible, meaning the identity map of D is homotopic to a constant map (from D to itself). Show that if $g\colon D \to E$ is continuous, then g is homotopic to a constant map (from D to E).

Solution. By assumption, there is a point $c \in D$ such that id_D is homotopic to the constant map (from $D \to D$) with value c. Let $\varphi\colon D \times [0, 1] \to D$ be a homotopy from id_D to this constant map.

Now consider $\psi\colon D \times [0, 1] \to E$ defined by $\psi(d, t) = g(\varphi(d, t))$. This is continuous because compositions of continuous maps are continuous. Moreover, $\psi(d, 0) = g(\varphi(d, 0)) = g(d)$ and $\psi(d, 1) = g(\varphi(d, 1)) = g(c)$. Therefore, ψ is a homotopy from g to the constant map (from D to E) with value $g(c)$. □

Problem 6 (4 points)**.** Let $a, b \in S^1$ be the points $a = (1, 0)$ and $b = (-1, 0)$. Give two explicit paths in S^1 from a to b that are not path-homotopic. You do not need to prove the nonexistence of the path homotopy.

Solution. Take $\gamma_+, \gamma_-\colon [0, 1] \to S^1$ defined by

$$\gamma_+(t) = (\cos(\pi t), \sin(\pi t)),$$
$$\gamma_-(t) = (\cos(\pi t), -\sin(\pi t)).$$

(Note that $\gamma_+(0) = \gamma_-(0) = (1, 0)$ and $\gamma_+(1) = \gamma_-(1) = (-1, 0)$.) □