



Zeta Functions as Knot Invariants

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O. Kivinen, M. Trinh. [The Hilb-vs-Quot conjecture](#). *Crelle's Journal* (2025), 44 pp.

- 1 The Riemann Hypothesis
- 2 Weil's Rosetta Stone
- 3 From Curves to Knots
- 4 Cherednik's New Hypothesis

Also featuring:

M. Trinh. [From the Hecke category to the unipotent locus](#). 88 pp. [arXiv:2106.07444](#)

P. Galashin, T. Lam, M. Trinh, N. Williams. [Rational noncrossing Coxeter–Catalan combinatorics](#). *Proc. London Math. Soc.* (2024), 50 pp.

1 The Riemann Hypothesis

(Euler ~1730s) Proof of the infinitude of primes:

If there were finitely many, then we'd have

$$\prod_{\text{prime } p > 0} \left(1 + \frac{1}{p} + \frac{1}{p^2} + \cdots\right) = \sum_{n=1}^{\infty} \frac{1}{n}.$$

But the series diverges—a contradiction.

He also implicitly studied the [zeta function](#)

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

For $s > 1$, we have $\zeta(s) = \prod_p \left(1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \cdots\right).$

What if we allow s to be complex?

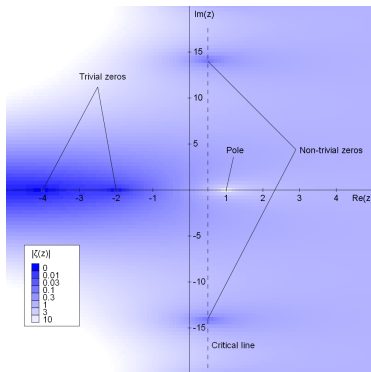
(Riemann 1859) A unique \mathbf{C} -valued function ζ that is

- *holomorphic* (complex-differentiable) when $s \neq 1$.
- given by $\sum_{n=1}^{\infty} \frac{1}{n^s}$ when $\operatorname{Re}(s) > 1$.

He checked that $\zeta(n) = 0$ for $n = -2, -4, -6, \dots$ by relating these zeros to poles of the *gamma function*.

He speculated from examples that all other zeros of ζ live on the *critical line* $\operatorname{Re}(s) = \frac{1}{2}$.

Location of zeros \leftrightarrow distribution of prime numbers.



Wikipedia

(Hardy 1914) Infinitely many zeros on the line.

(Pratt–Robles–Zaharescu–Zeindler 2020) Among *nontrivial zeros*, over five-twelfths on the line.

(Dedekind ~1860s) Generalize the formula

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

by replacing \mathbf{Z} with other *rings* R .

Thus R is a set with operations resembling addition and multiplication.

$$\zeta_R(s) = \sum_{\substack{\text{ideals } I \subseteq R \\ |R/I| < \infty}} \frac{1}{|R/I|^s}$$

An *ideal* $I \subseteq R$ is the set of all linear combinations $c_{\alpha_1} x_{\alpha_1} + \cdots + c_{\alpha_k} x_{\alpha_k}$ for some given $\{x_{\alpha}\}_{\alpha} \subseteq R$.

The *quotient* R/I is the set of translates $y + I \subseteq R$.

Note Requires that for each $n > 0$, there are finitely many I such that $|R/I| = n$.

Ex Every ideal of \mathbf{Z} takes the form

$$n\mathbf{Z} = \{\text{multiples of } n\} \quad \text{for some integer } n \geq 0.$$

For instance, the ideal generated by 30 and 2025 is

$$\{c_1 30 + c_2 2025 \mid c_1, c_2 \in \mathbf{Z}\} = 15\mathbf{Z}.$$

Check that $\mathbf{Z}/0\mathbf{Z} = \mathbf{Z}$, while $|\mathbf{Z}/n\mathbf{Z}| = n$ for $n > 0$.

$$\zeta_{\mathbf{Z}}(s) = \sum_{\substack{n\mathbf{Z} \subseteq \mathbf{Z} \\ n > 0}} \frac{1}{|\mathbf{Z}/n\mathbf{Z}|^s} = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

Why care?

(Hilbert–Polyá ~1910s) To prove RH, prove that

$$\{e^{i\gamma} \mid \tfrac{1}{2} + i\gamma \text{ is a nontrivial zero of } \zeta\}$$

is the set of eigenvalues of an infinite *unitary* matrix.

($\implies e^{i\gamma}$ on the unit circle of $\mathbf{C} \implies \gamma$ real.)

(Weil ~1940s) Fix a particular prime p .

Can we prove an analogue for ζ_R , for certain rings R appearing in *algebraic geometry* modulo p ?

(Grothendieck–Deligne ~1960s–70s) Yes.

2 Weil's Rosetta Stone Algebraic geometry studies *varieties*: shapes cut out by polynomial equations.

For simplicity, we'll stick to (*affine*) *hypersurfaces*

$$V_f = \{\vec{a} = (a_0, a_1, \dots, a_d) \mid f(\vec{a}) = 0\},$$

cut out by a single polynomial $f \in \mathbf{Z}[x_0, x_1, \dots, x_d]$.

V_f is *smooth* at $\vec{a} \bmod p$ when $\frac{\partial f}{\partial x_j}(\vec{a}) \not\equiv 0 \pmod{p}$ for some j . Else, *singular*.

Ex For $d = 1$, hypersurfaces are plane curves.

$$f(x, y) = y^2 - x^3 - c \implies V_f = \{y^2 = x^3 + c\}$$

For which c is V_f smooth everywhere mod p ?

The *ring of polynomial functions* on $V_f \bmod p$ is

$$R_{f,p} := \mathbf{F}_p[x_0, \dots, x_d] / f \mathbf{F}_p[x_0, \dots, x_d],$$

where $\mathbf{F}_p := \mathbf{Z}/p\mathbf{Z}$.

In a letter to his sister, Weil described a dictionary:

\mathbf{Z}	$R_{f,p}$	$V_f \bmod p$
$n\mathbf{Z}$	ideals	subvarieties
$p\mathbf{Z}$	maximal ideals	points

The first and last columns = Rosetta stone between number theory and algebraic geometry.

Conj (Weil 1948) Assume V_f is smooth everywhere.

Then zeros of $\zeta_{R_{f,p}}(s)$ have $\operatorname{Re}(s) \in \{\frac{1}{2}, \frac{3}{2}, \dots, \frac{2d-1}{2}\}$.

Weil proved it for many cases.

$$\text{Recall: } \zeta_{R_{f,p}}(s) = \sum_I \frac{1}{|R_{f,p}/I|^s}.$$

(Grothendieck ~1964) Introduce the variable

$$\mathbf{q} := p^{-s}.$$

There are polynomials $\phi_0, \phi_1, \dots, \phi_{2d-1}$ such that

$$\zeta_{R_{f,p}}(s) = \frac{\phi_1(\mathbf{q}) \cdot \phi_3(\mathbf{q}) \cdots \phi_{2d-1}(\mathbf{q})}{\phi_0(\mathbf{q}) \cdot \phi_2(\mathbf{q}) \cdots \phi_{2d-2}(\mathbf{q})}.$$

ϕ_k is the charpoly of a certain operator on a certain vector space, describing the *étale topology* of V_f .

Conj For all k , the roots of $\phi_k(\mathbf{q})$ live on the circle

$$|\mathbf{q}| = p^{-k/2}.$$

\implies Weil's Riemann Hypothesis.

(Deligne 1974) True for all f (smooth mod p).

Ex Taking $d = 1$ and $f(x, y) = y^2 - x^3 - c$:

$$\phi_0(t) = 1 - pq$$

$$\phi_1(t) = 1 - a_p q + pq^2 \quad \text{for some integer } a_p,$$

giving $\zeta_{R_{f,p}}(s) = \frac{1 - a_p p^{-s} + p^{1-2s}}{1 - p^{1-s}}$. It turns out:

- $|a_p| \leq 2p^{1/2}$.
- So the two roots of $\phi_1(q)$ satisfy $|q| = p^{-1/2}$.
- So the zeros of $\zeta_{R_{f,p}}(s)$ satisfy $\text{Re}(s) = \frac{1}{2}$.

In fact, Weil conjectured—and Deligne proved—results for all varieties, not just hypersurfaces.

What if V_f has singularities?

Simplest case: $f(x, y)$ with unique singularity at $(0, 0)$.

It turns out that here,

$$\zeta_{R_{f,p}}(s) = \zeta_{R_{f,p}}^*(s) \cdot \zeta_{R_{f,p}^0}(s),$$

where:

- $\zeta_{R_{f,p}}^*$ satisfies Weil's Riemann Hypothesis.
- $\zeta_{R_{f,p}^0}$ is analogous to $\zeta_{R_{f,p}}$, with

$$R_{f,p}^0 := \mathbf{F}_p[[x, y]]/f\mathbf{F}_p[[x, y]]$$

in place of $R_{f,p}$. Above, $[[\]]$ means power series.

Does $\zeta_{R_{f,p}^0}(s) = \sum_I \frac{1}{|R_{f,p}^0/I|^s}$ satisfy a RH?

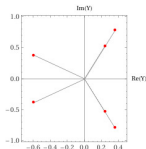
Ex For $f = y^2 - x^3$,

$$\zeta_{R_{f,p}}^0(s) = \frac{1 - p^{1-2s}}{1 - p^{-s}} = \frac{1 + pq^2}{1 - q}.$$

Ex For $f = y^3 - x^4$,

$$\zeta_{R_{f,p}}^0(s) = \frac{1 + pq^2 + p^2q^3 + p^2q^4 + p^3q^6}{1 - q}.$$

Here, not all roots satisfy $|q| = p^{-1/2}$.



WolframAlpha

3 From Curves to Knots For general $f(x, y)$,

it turns out there's $\Psi_f(t, q) \in \mathbb{Z}\left[t, q, \frac{1}{1-q}\right]$ such that

$$\zeta_{R_{f,p}}^0(s) = \frac{\Psi_f(p, p^{-s})}{1 - p^{-s}} \quad \text{for almost all } p.$$

These polynomials have many surprising features.

(Piontkowski 2007) Take $f = y^n - x^{n+1}$.

Then $\Psi_f(1, 1) = \frac{(2n)!}{(n+1)!n!}$, the n th Catalan number.

Ex If $f = y^3 - x^4$, then

$$\Psi_f(t, q) = 1 + tq^2 + t^2q^3 + t^2q^4 + t^3q^6,$$

$$\Psi_f(1, 1) = 5.$$

The Ψ_f also arise from *knot/link invariants*.

A *knot* is an embedding of a circle into \mathbf{R}^3 or S^3 .



A *link* is a generalization allowing multiple circles.



Two links are *isotopic* when we can deform one into the other without self-intersections.



Chmutov–Duzhin–Mostovoy

Let $S_\epsilon^3 = \{(x, y) \in \mathbf{C}^2 \mid |x|^2 + |y|^2 = \epsilon\}$. The subset

$$L_f = \{(x, y) \in S_\epsilon^3 \mid f(x, y) = 0\}$$

is a link in S_ϵ^3 when $\epsilon > 0$ is small enough.

Ex If $f = y^n - x^m$, then L_f is the (m, n) torus link. It's a knot when m and n are coprime.



Wolfram Language

Ex If $f = y^4 - 2x^3y^2 - 4x^5y + x^6 - x^7$, then L_f is the closure of the braid



Cherednik–Danilenko

Conj (Oblomkov–Shende ~2010)

$$\Psi_f(1, q^2) = \lim_{a \rightarrow 0} \left[(q/a)^\mu \mathbb{P}_{L_f}(a, q) \right],$$

where $\mu \in \mathbf{Z}$ and \mathbb{P} is the *HOMFLYPT invariant*, discovered in 1986 and defined by the *skein rules*:

$$(1) \quad \mathbb{P}_{\bigcirc} = 1$$

$$(2) \quad a \mathbb{P}_{\nearrow} - a^{-1} \mathbb{P}_{\nwarrow} = (q - q^{-1}) \mathbb{P}_{\frown}$$

Full statement incorporates a , by upgrading Ψ_f .

(Maulik 2012) True for all plane curves.

Proof sketch Blow up the singularity repeatedly.

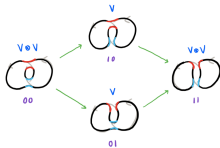
Control Ψ_f via *wall crossing* and \mathbb{P}_{L_f} via skein rules.

Conj (Oblomkov–Rasmussen–Shende ~2013)

$$\Psi_f(t^2, q^2) = \lim_{a \rightarrow 0} \left[(q/a)^\mu \mathbf{P}_{L_f}(a, t, q) \right],$$

where \mathbf{P} is the *Khovanov–Rozansky invariant*, a refinement of \mathbb{P} discovered in 2006.

\mathbf{P} is defined by *categorifying* (1)–(2). Unknown how to categorify Maulik’s proof.



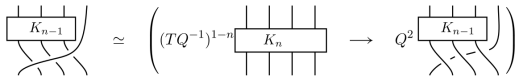
Melissa Zhang

(Kivinen–T 2025) True for $f = y^3 - x^m$ with $3 \nmid m$.

Cor (Kivinen–T) New closed formula for $\mathbf{P}_{\text{torus}(m,3)}$.

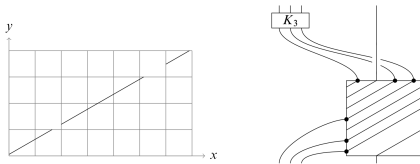
Proof Sketch $\mathbf{P}_{\text{torus}(m,n)} \rightsquigarrow \Psi_{y^n - x^m}$?

1 Recursions that compute $\mathbf{P}_{\text{torus}(m,n)}(\mathbf{a}, \mathbf{t}, \mathbf{q})$, due to Elias–Hogancamp–Mellit.



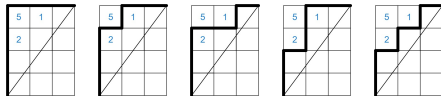
Elias–Hogancamp

Encoding of braids into grid diagrams, due to Mellit.



Mellit

2 For m, n coprime, yields a sum over *Dyck paths* in an $m \times n$ grid.



Meanwhile, $R_{f,p}^0 \simeq \mathbf{F}_p[[u^m, u^n]]$ when $f = y^n - x^m$.

We relate Dyck paths to $R_{f,p}^0$ -submodules $M \subseteq \mathbf{F}_p[[u]]$.

3 Recall $\frac{\Psi_f(p, p^{-s})}{1 - p^{-s}} = \zeta_{R_{f,p}^0}(s) = \sum_I \frac{1}{|R_{f,p}^0/I|^s}$.

We relate it to $\sum_M \frac{1}{|\mathbf{F}_p[[u]]/M|^s}$.

Uses *Serre duality*. For now, requires $\min(m, n) \leq 3$.

Big Picture I study special functions that appear in

- *algebraic geometry*
- *knot theory*
- *combinatorics*

We can decompose them into simpler functions via *representation theory*.

The Dyck-path decomposition of Ψ_f comes from the representation theory of *symmetric groups*.

Another case:

(T 2021) Generalizations of \mathbb{P} , \mathbf{P} arising from the representation theory of *Coxeter groups*.

(Galashin–Lam–T–Williams 2024) Ideas from (T) solve conjectures in Coxeter combinatorics from 2012.

4 Cherednik's New Hypothesis

Recall: For $f = y^3 - x^4$ and prime p , the roots of

$$\Psi_f(p, q) = 1 + pq^2 + p^2q^3 + p^2q^4 + p^3q^6$$

do not all satisfy $|q| = p^{-1/2}$.

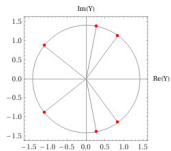
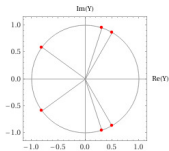
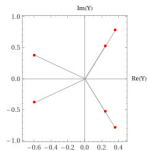
Conj (Cherednik 2018) For any plane curve f :

$$0 < t \leq \frac{1}{2} \implies \begin{array}{l} \text{all roots of } \Psi_f(t, q) \text{ satisfy} \\ |q| = t^{-1/2}. \end{array}$$

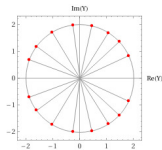
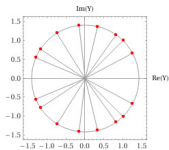
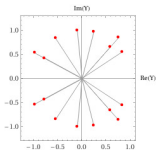
Would imply *arithmetic* constraints on $\mathbf{P}_{L_f}(a, t, q)$.

Dream (Cherednik) Related to the Lee–Yang theorem on zeros of *partition functions* in statistical mechanics.

$$f = y^3 - x^4, \quad t = 2, 1, \frac{1}{2}$$



$$f = y^4 - 2x^3y^2 - 4x^5y + x^6 - x^7, \quad t = 1, \frac{1}{2}, \frac{1}{4}$$



Thank you for listening.