# Symmetries of Homogeneous Affine Springer Fibers

## 1

$G$ complex reductive alg group, $\quad A \subseteq B \subseteq G$ Borel pair,
$X$ complex alg curve

$$\begin{array}{ccccccc} \mathcal{M}_{G,\mathrm{B}}(X) & \approx & \overset{\text{nonabelian Hodge}}{\mathcal{M}_{G,\mathrm{dR}}(X)} & \approx & \mathcal{M}_{G,\mathrm{Dol}}(X) & & \\ & & & & \updownarrow & & \text{Langlands} \\ & & & & \mathcal{M}_{G^\vee,\mathrm{Dol}}(X) & & \end{array}$$

HMS: $\quad \mathrm{Coh}_S(\mathcal{M}_{G,\mathrm{B}}) \xrightarrow{?} \mathrm{Fuk}(\mathcal{M}_{G^\vee,\mathrm{Dol}}) \simeq \mathcal{D}(\mathcal{M}_{G^\vee,\mathrm{Dol}})$

## 2

Ex 1 $\quad G = \mathrm{GL}_n$

| | |
|---|---|
| $\mathcal{M}_{\mathrm{B}}$ | local systems $\varrho : \pi_1(X) \to G$ |
| $\mathcal{M}_{\mathrm{dR}}$ | flat connections $(E, \nabla : E \to E \otimes \Omega^1)$ |
| $\mathcal{M}_{\mathrm{Dol}}$ | Higgs bundles $(E, \theta : E \to E \otimes \Omega^1)$ |

$X$ of genus $g$, $\quad G = \mathrm{GL}_1$

$$\mathcal{M}_{\mathrm{B}} = (\mathbf{C}^\times)^{2g}, \quad \mathcal{M}_{\mathrm{dR}} = \mathcal{M}_{\mathrm{Dol}} = T^*\mathrm{Jac}(X) \approx \mathbf{C}^g \times (S^1)^{2g}$$

## 3

Ex 2 (BBMY) $\quad X = \mathbf{P}^1 - \{0, \infty\}, \quad \gamma \in \mathfrak{g}[z]$ homogeneous

| | |
|---|---|
| $\mathcal{M}_{\mathrm{B}}$ | "braid variety" |
| $\mathcal{M}_{\mathrm{Dol}}$ | $\{$wild Higgs bundles with flag at 0, tail $\gamma \frac{dz}{z}$ at $\infty\}$ |

BBMY–Feng–Le Hung: for $\gamma^\vee$ of "integral slope", a map

$$\mathrm{K}_0(\mathrm{Coh}(\mathcal{M}_{G,\mathrm{B}})) \to \mathrm{K}_0(\mathrm{Fuk}(\mathcal{M}_{G^\vee,\mathrm{Dol}}))$$

$\approx$ Breuil–Mézard $\quad \mathrm{K}_0(\mathrm{Rep}_{\bar{\mathbf{F}}_p}(G(\mathbf{F}_p))) \to \mathrm{Ch}_{\mathrm{mid}}(\mathcal{X}^{\mathrm{EG}})$

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geometry of $\mathcal{M}_{\mathrm{Dol}}^{\mathrm{BBMY}}$:

- $\mathbf{C}^{\times}$-action contracting to Lagrangian central fiber $\mathcal{F}l_\gamma$
- $\mathcal{F}l_\gamma$ is an "Iwahori affine Springer fiber"
- $\mathrm{H}^*_{\mathbf{C}^\times}(\mathcal{F}l_\gamma)$ is a $(\widetilde{W}, \widetilde{W})$-bimodule (for integral slope)

BBMY expect mirror symmetry to be biequivariant

F–L use biequivariance to make their analogy precise

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Affine Springer Fibers (fpqc) affine flag variety

$$\mathcal{F}l := G(\!(z)\!)/I, \quad \text{where } I \subseteq G(\!(z)\!) \text{ lifts } B \subseteq G$$

$\gamma \in \mathfrak{g}[\![z]\!]$ defines a vector field with fixed-point set

$$\mathcal{F}l_\gamma := \{gI \in \mathcal{F}l \mid \gamma \in \mathrm{Lie}(gIg^{-1})\}$$

$\gamma$ is regular semisimple iff $T := Z^\circ_{G(\!(z)\!)}(\gamma)$ is a max torus

Kazhdan–Lusztig: if $\gamma$ is reg ss, then $\mathcal{F}l_\gamma$ is finite-dim'l

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as moduli of parabolic Higgs bundles over $D = \operatorname{Spec} \mathbf{C}[\![z]\!]$:

$$\mathcal{F}l_\gamma \simeq \left\{ (E, \theta, \tilde{E}_0, \iota) \;\middle|\; \begin{array}{l} (E,\theta) \in \mathcal{M}_{\mathrm{Dol}}(D), \\ \tilde{E}_0 \text{ is a } \theta_0\text{-stable flag in } E_0, \\ \iota : (E,\theta)|_{D^\circ} \xrightarrow{\sim} (E^{\mathrm{triv}}, \gamma)|_{D^\circ} \end{array} \right\}$$

$\mathcal{F}l_\gamma \hookrightarrow \mathcal{M}_{\mathrm{Dol}}^{\mathrm{BBMY}}$ defined by gluing bundles

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for $\frac{d}{m} \in \mathbf{Q}_+$ in <u>lowest terms</u>, let $\mathbf{C}^\times \curvearrowright G(\!(z)\!), \mathfrak{g}(\!(z)\!)$ by

$$c \cdot g(z) = \mathrm{Ad}(c^{d\rho^\vee})g(c^m z), \quad \text{where } \rho^\vee = \textstyle\sum_i \omega_i^\vee$$

$\gamma$ is homogeneous of slope $d/m$ iff $c^m \cdot \gamma(z) = c^d \gamma(z)$

<u>Ex</u> take $G = \mathrm{SL}_2$ and $B$ upper-triangular

$\begin{pmatrix} 1 & \\ & -1 \end{pmatrix}, \begin{pmatrix} & 1 \\ z & \end{pmatrix}, \begin{pmatrix} z & \\ & -z \end{pmatrix}$ are reg ss: slopes $0, \frac{1}{2}, 1$

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<u>Symmetries</u> $\mathfrak{c}^{\mathrm{rs}}_{d/m} = \{\text{homog reg ss } \gamma \text{ of slope } \frac{d}{m}\} /\!\!/ G(\!(z)\!)_0$

$$\mathcal{F}l_{\begin{pmatrix} z & \\ & -z \end{pmatrix}} = \mathbf{P}^1 \sqcup_{\mathrm{pt}} \mathbf{P}^1 \sqcup_{\mathrm{pt}} \cdots \sqcup_{\mathrm{pt}} \mathbf{P}^1 \curvearrowleft \langle s_1, z^{\rho^\vee} \rangle = \widetilde{W}$$

— action of centralizer lattice $\pi_0(T)$ $\quad z^{\rho^\vee}$

— action of monodromy group $\pi_1(\mathfrak{c}^{\mathrm{rs}}_{d/m})$ on $\mathrm{H}^*_{\mathbf{C}^\times}$ $\quad s_1$

<u>Conj</u> (T–Xue) formula for monodromy using $\mathrm{Irr}(G(\mathbf{F}_q))$

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$W := N_G(A)/A$ is a rat'l refl group

$C := N_{G(\!(z)\!)}(T)/T$ is a comp'x refl grp; $\mathrm{Br}_C = \pi_1(\mathfrak{c}^{\mathrm{rs}}_{d/m})$

<u>Ex</u> if $G = \mathrm{SL}_n$, then $m \mid n$ and $C \simeq S_{n/m} \wr \mathbf{Z}/m\mathbf{Z}$

$$G_{\mathbf{C}(\!(z)\!)}, A_{\mathbf{C}(\!(z)\!)}, T_{\mathbf{C}(\!(z)\!)} \quad \leftrightsquigarrow \quad G_{\mathbf{F}_q}, A_{\mathbf{F}_q}, T_{\mathbf{F}_q}$$

Lusztig: induction $R_T^G : \mathrm{K}_0(T(\mathbf{F}_q)) \to \mathrm{K}_0(G(\mathbf{F}_q))$

$$\mathrm{HC}_T = \{\rho \in \mathrm{Irr}(G(\mathbf{F}_q)) \mid (\rho, R_T^G(1)) \neq 0\}$$

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$$\begin{aligned} \text{Iwahori:} \quad & \chi : \mathrm{HC}_A \xrightarrow{\sim} \mathrm{Irr}(W) \\ \text{Broué–Malle–Michel:} \quad & \psi : \mathrm{HC}_T \xrightarrow{\sim} \mathrm{Irr}(C) \end{aligned}$$

BMM define a ring $\mathcal{H}_T(x) = \mathbf{C}[x^{\pm 1/m}][\mathrm{Br}_C]/{\sim}$ s.t.

(1) $\mathcal{H}_T(e^{2\pi i/m}) \simeq \mathbf{C}C$

(2) conjecturally, via $\psi_q : \mathrm{HC}_T \xrightarrow{\sim} \mathrm{Irr}(\mathcal{H}_T(q))$,

$$R_T^G(1) = \sum_{\rho \in \mathrm{HC}_T} \varepsilon(\rho)\rho \otimes \psi_q(\rho) \quad \text{for some } \varepsilon(\rho) \in \{\pm 1\}$$

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take $G$ ss and $V_\gamma^* = \mathrm{H}^*_{\mathbf{C}^\times}(\mathcal{F}l_\gamma)^{\pi_0(T)}|_{\epsilon \to 1}$ $\quad (\epsilon \in \mathrm{H}^2_{\mathbf{C}^\times}(\mathrm{pt}))$

<u>Conj 1</u> (T–Xue) $\mathrm{Br}_C \curvearrowright V_\gamma^*$ factors through $H_T(1)$

expect commutant of $\mathrm{Br}_C$ to be generated by:

— action of $\widetilde{W}$ via Springer

— action of $\mathrm{H}^*_B(\mathrm{pt}) = \mathbf{C}[X^*(A)]$ via Chern classes

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rational DAHA: $\mathcal{D}_A(\frac{d}{m}) = (\mathbf{C}[\mathfrak{a}^*] \otimes \mathbf{C}W \otimes \mathbf{C}[\mathfrak{a}])/{\sim}$

Oblomkov–Yun: for <u>elliptic</u> $\gamma$, <u>perverse filtration</u> $\mathrm{P}_{\leq *}$,

$$\mathbf{C}\widetilde{W} \otimes \mathbf{C}[X^*(A)] \curvearrowright V_\gamma \quad \rightsquigarrow \quad \mathcal{D}_A(\tfrac{d}{m}) \curvearrowright \mathrm{gr}^{\mathrm{P}}_* V_\gamma$$

<u>Conj 2</u> (T–Xue) as virtual $(\mathcal{D}_A(\frac{d}{m}), \mathcal{H}_T(1))$-bimodules,

$$\sum_i (-1)^i \mathrm{gr}^{\mathrm{P}}_* V_\gamma^i = \sum_{\rho \in \mathrm{HC}_A \cap \mathrm{HC}_T} \varepsilon(\rho)\Delta_{d/m}(\chi(\rho)) \otimes \psi_1(\rho)$$

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$$\Delta_{d/m}(\chi) = \mathrm{Ind}_{\mathbf{C}W \ltimes \mathbf{C}[\mathfrak{a}]}^{\mathcal{D}_A(d/m)}(\chi) \qquad \text{(“Verma modules”)}$$

<u>Thm</u> (T–Xue) Conj 2 is true for:

(1) $m$ the Coxeter number of $W$ ($C$ cyclic)

(2) (twisted) $G$ of rank 2

compare to virtual $(\mathcal{H}_A(q), \mathcal{H}_T(q))$-bimodule

$$R_A^G(1) \otimes_{G(\mathbf{F}_q)} R_T^G(1) = \sum_{\rho \in \mathrm{HC}_A \cap \mathrm{HC}_T} \varepsilon(\rho)\chi_q(\rho) \otimes \psi_q(\rho)$$