(Axler §7B) last time:

<u>Thm</u> if $T : V$ to V is normal, then:

1–2) $\ker(T^*) = \ker(T)$, $\mathrm{im}(T^*) = \mathrm{im}(T)$

3) $T - \lambda$ is normal for all λ in F

4) $\ker(T - \lambda) = \ker(T^* - \bar{\lambda})$

today, we work over $F = C$

<u>Spectral Thm</u>

if V is finite-dim'l over C and $T : V$ to V is normal then T is diagonalizable

in fact:
V has a basis of <u>orthonormal eigenvectors</u> for T

<u>Restatement in Matrices</u>

let $(e_1, \ldots, e_n)$ be any orthonormal basis for V
A the matrix of T wrt $(e_i)_i$

let $(u_1, \ldots, u_n)$ be the basis of orthonormal eigenvectors for T
λ_i defined by $Tu_i = \lambda_i u_i$
P the $n \times n$ matrix defined by $Pe_i = u_i$
D the $n \times n$ diagonal matrix with diagonal $\lambda_1, \ldots, \lambda_n$

[what's A in terms of P, D?] then $A = P^{-1}DP$

<u>Note 1</u> we proved last time: if T is self-adjoint, not just normal, then the λ_i's are all real

Note 2 — the cols of P expand the u_i's into e_i's
but the u_i's are orthonormal, so

$$PP^* = I$$

that is: $Pu \cdot \overline{(Pv)} = u^t PP^* \overline{v} = u \cdot \overline{v}$ for all u, v

Df — recall:
an operator Q is orthogonal wrt $\langle\ ,\ \rangle$
iff $\langle Qu, Qv\rangle = \langle u, v\rangle$ for all u, v

if $F = \mathbb{C}$, then we often say "unitary" rather than "orthogonal"

we say a matrix P is unitary iff $Pu \cdot \overline{(Pv)} = u \cdot \overline{v}$
[which occurs] iff $PP^* = I$

Pf of Thm — induct on $n := \dim V$
if $n = 0$, then done

suppose $n > 0$
then [recall:] T has an eigenvector v, say,
with eigenvalue λ

the line $\mathbb{C}v$ is T-stable
let $W = (\mathbb{C}v)^\perp = \{w \text{ in } V \mid \langle w, v\rangle = 0\}$
recall that the Gram–Schmidt process shows

$$V = \mathbb{C}v + W \quad \text{and this sum is direct}$$

so it remains to show:

Claim — W is T-stable

Claim Finishes Pf   note dim $W = n - 1$

by inductive hypothesis, W has a basis of
  orthonormal eigenvectors $u_1, \ldots, u_{n-1}$
all are orthogonal to v
now set $u_n = v/||v||$ □

Pf of Claim   pick w in W
  want Tw in W: that is, $\langle Tw, v\rangle = 0$

know $\langle Tw, v\rangle = \langle w, T^*v\rangle$
but [recall!] v in $\ker(T - \lambda) = \ker(T^* - \bar{\lambda})$
now, $\langle w, T^*v\rangle = \langle w, \bar{\lambda}v\rangle$
$= \bar{\lambda}\langle w, v\rangle$
$= 0$

Rem   claim + its proof generalize:

if $T : V$ to V is normal and U sub V is T^*-stable
then $U^\perp$ is T-stable

Applications

Cor   if $TT^* = T^*T$ and all eigenvalues of T
  are real and positive,
then $T = S^*S$ for some $S : V$ to V

in particular, $S^* = S$
[because $(S^*S)^* = S^*S^{**} = S^*S$]

Pf   pick a basis of orthonormal eigenvectors
the matrix of T in this basis is diagonal

call it D

let C be diagonal s.t. $C^2 = D$

let $S : V$ to V be the op with matrix C in that basis

<u>Cor</u> TFAE for an $n \times n$ matrix A:

1) the pairing $\langle u, v\rangle := u^tA\bar{v}$ is an inner product

2) A is Hermitian and positive-definite

[pos-def: $v^t A\bar{v} > 0$ for $v \neq \mathbf{0}$]

3) $A = BB^*$ for some $B : V$ to V

<u>Pf</u> the direction 1) implies 2) implies 3) are PS8, #8, part (1)

[use the previous corollary]

conversely, if $A = B^*B$, then:

A is Hermitian, via the argument earlier

[$(B^*B)^* = B^*B^{**} = B^*B$]

$v^tA\bar{v} = v^tB^*B\bar{v} = (\bar{B}v)^t\overline{(\bar{B}v)} > 0$

since the skew-dot product is pos-def

so $\langle\ ,\ \rangle$ is pos-def and conj-symmetric

<u>Df</u> B^*B is called the Gram matrix of B

its eigenvals are all real and positive

their sq roots are the <u>singular vals</u> of B

similar lingo for linear operators

[why are singular values useful?]

[just as vec's in an inner product space get norms, so too do operators on it]

Df the L^2 operator norm of $S : V \to V$ is

$$\|S\| = \max_{v \text{ s.t. } \|v\| = 1} \|Sv\|$$

i.e., the largest factor by which S rescales the norm of a vector

Cor $\|S\| = \max \{\text{singular values of } S\}$

Pf 1 pick a basis of orthonormal eigenvectors for S^*S

now, e.g., Lagrange multipliers show:
$\|Sv\|^2 = \langle Sv, Sv\rangle = \langle v, S^*Sv\rangle$ is maximized on $\{\|v\| = 1\}$ when v is an eigenvec for the largest eigenval of S^*S

Pf 2 in some orthonormal basis, the matrix of S^*S looks like $P^{-1}DP$ with P orthogonal and D diagonal
[so, enough to show:]

Lem if P, Q are unitary and D is anything then $\|QDP\| = \|DP\| = \|D\|$

Pf the set $\{v \mid \|v\| = 1\}$ is stable under unitary ops

more general result [has a "Min-Max" version]:

Thm (Max-Min) $\max_{\dim U = i} \min_{v \in U \mid \|v\| = 1} \|Sv\|$ = ith largest singular value of S [since $\|v\| = 1$ iff $\|\bar{v}\| = 1$ and $\dim U = \dim \bar{U}$:]

Cor S and S* have the same singular vals
i.e.,
S*S and SS* have the same eigenvals