MATH 430: INTRODUCTION TO TOPOLOGY
PROBLEM SET #2

SPRING 2025

Due Wednesday, January 29. You may consult books, papers, and websites as long as you cite all sources and write up your solutions in your own words.

Problem 1. Endow $\mathbf{R}^2$ with the analytic topology. How is the subspace topology on $\mathbf{R}$, viewed as the x-axis of $\mathbf{R}^2$, related to the analytic topology on $\mathbf{R}$?

Problem 2. Endow $\mathbf{R}$ with the analytic topology, and

$$X = \{\tfrac{1}{n} \mid n = 1, 2, 3, \ldots\} \cup \{0\}$$

with the subspace topology.

(1) Show that for all integers $n > 0$, the singleton set $\{\frac{1}{n}\}$ is *clopen*: both closed and open.

(2) Show that $\{0\}$ is closed but not open.

Problem 3 (Munkres 128, #9(c)–(d))**.** Recall that the Euclidean norm on $\mathbf{R}^n$ is given by $\|u\| = \sqrt{u \cdot u}$, where

$$u \cdot v := u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$$

for general $u, v \in \mathbf{R}^n$.

(1) Use the Cauchy–Schwarz inequality $|u \cdot v| \leq \|u\|\|v\|$ to show that $\|u + v\| \leq \|u\| + \|v\|$ for all $u, v \in \mathbf{R}^n$.

(2) Conclude that the *Euclidean metric* $d(x, y) = \|x - y\|$ really is a metric $d : \mathbf{R}^n \times \mathbf{R}^n \to [0, \infty)$.

Problem 4 (Munkres 129, #11)**.** Let X be arbitrary, and let $d : X \times X \to [0, \infty)$ be an arbitrary metric. Let $e : X \times X \to [0, \infty)$ be defined by

$$e(x, y) = \frac{d(x, y)}{1 + d(x, y)}.$$

Show that:

(1) e is a (bounded) metric. *Hint:* The mean value theorem shows that if $f(x) = \frac{x}{1+x}$, then $f(a + b) - f(b) \leq f(a)$ for all $a, b \geq 0$.

(2) d and e induce the same topology on X.

Problem 5. Recall that metrics $d, d' : X \times X \to [0, \infty)$ are *equivalent* if and only if there are constants $A, B > 0$ such that

$$d(x, y) \leq Ad'(x, y) \text{ and } d'(x, y) \leq Bd(x, y) \text{ for all } x, y \in X.$$

In the setting of Problem 4, show that e is not equivalent to d when $X = \mathbf{R}$ and d is the Euclidean metric.

Problem 6 (Munkres 127, #6)**.** The *uniform topology* on

$$\mathbf{R}^\omega := \{\text{sequences } (a_1, a_2, a_3, \ldots) \text{ with } a_i \in \mathbf{R} \text{ for all } i\}$$

is induced by the *uniform metric* $\bar{\rho}(x, y) = \sup_{i>0} \min\{1, |x_i - y_i|\}$. For all $x \in \mathbf{R}^\omega$ and $\delta > 0$, show that:

(1) The set $U(x, \delta) := (x_1 - \delta, x_1 + \delta) \times (x_2 - \delta, x_2 + \delta) \times \cdots$ is not open in the uniform topology.

(2) Nonetheless, $B_{\bar{\rho}}(x, \epsilon) = \bigcup_{\delta<\epsilon} U(x, \delta)$.

Problem 7. The *box topology* on $\mathbf{R}^\omega$ is defined as follows: U is open if and only if, for all $x \in U$, there is some set of the form $V = (a_1, b_1) \times (a_2, b_2) \times \cdots$ such that $x \in V \subseteq U$. In what follows, assume the Axiom of Choice.

(1) Show that the box topology really is a topology on $\mathbf{R}^\omega$.

(2) Use Problem 6 to verify that the box topology is strictly finer than the uniform topology.

Problem 8 (Munkres 127, #5)**.** Let $\mathbf{R}^\infty \subseteq \mathbf{R}^\omega$ be the subset of sequences $(a_i)_{i>0}$ such that $a_i \neq 0$ for only finitely many i. Determine the closure of $\mathbf{R}^\infty$ in the uniform topology on $\mathbf{R}^\omega$.