# MATH 250: Topology I

[intros]

[recap:   website
          LaTeX
          textbook
          late hw policy
          exam dates]

[initial reading]

I) algebraic topology:

  how to tell shapes apart?

Euler: Königsberg (1736), "V – E + F = 2" (1750)
Poincaré: Analysis situs (1890s–1900s)

II) point-set topology:

  what IS shape?

Weyl: rigorous defn of "surface" (1913)
Hausdorff: topological spaces (1914)

[we first do (II), then do (I)]

(Munkres §12)        fix a set X

<u>Def</u>        a topology on X is
        a collection T of subsets of X s.t.


1)    Ø, X in T
2)    if {U_i}_i is a subcollection of T, then the
union of the U_i in X is in T
3)    if {U_i}_i is a <u>finite</u> subcollection of T, then
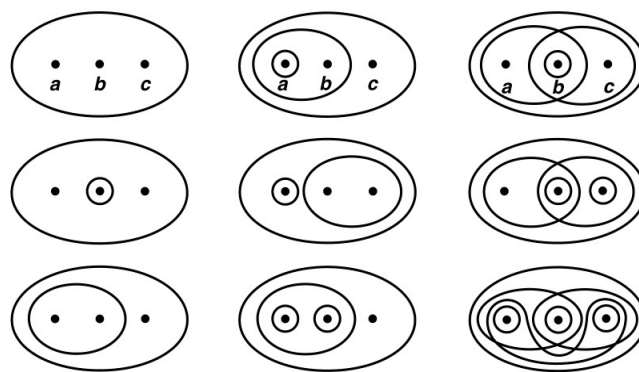the intersection of the U_i in X is in T

we say that
        (X, T) is a <u>topological space</u>
        the elements of T are its <u>open sets</u>

<u>Ex</u>        X = {a, b, c}



[in each case, Ø is not depicted]

[example collection of subsets that isn't a
topology?]

Ex          X = R^n with |x| = sqrt(sum_i x_i^2)

write B(x, δ) = {y in R^n | |y − x| < δ}

we say U sub R^n is _analytically open_ iff
    for all x in U,
    there is some δ > 0 s.t. B(x, δ) sub U

Thm         {analytically open sets} is a topology
            on R^n

1)   easy
2)   suppose U_i anlytc opens, U = bigcup_i U_i
     pick x in U
         x belongs to x in U_j for some j
         have δ > 0 s.t. B(x, δ) sub U_j sub U

3)    suppose finitely many i,
          U_i analytic opens, V = bigcap_i U_i
      pick x in V
      for all i, pick δ_i > 0 s.t. B(x, δ_i) sub U_i

      [what next?] let δ = min_i δ_i
      then B(x, δ) sub B(x, δ_i) sub U_i for all i
      therefore B(x, δ) sub V

[observe: 3) wouldn't work for infinite δ_i → 0]

we call this the _analytic topology_ T_{an} on R^n

Rem      not the only topology possible:
         also _discrete_ and _indiscrete_ topologies

Ex        X arbitrary

T_f = {Ø} cup {U sub X s.t. X – U is finite}

Thm       T_f is a topology on X

1)    easy
2)    suppose {U_i}_i is a subcollection of T_f,
          U = bigcup_i U_i
      [what happens if U_i = Ø for all i?]
      [what if U_i ≠ Ø for some i?]
3)    suppose finitely many i,
          U_i subcollection of T_f
          V = bigcap_i U_i
      [what happens if U_i = Ø for some i?]
      [what if U_i ≠ Ø for all i?]

we call this the finite complement topology

Df        given topologies T, T′ on the same X
          s.t. T is a subcollection of T′,

we say that
      T is coarser than T′
      T′ is finer than T

[T′ is more refined: it sees more open sets]

Ex    [how do the topologies on R^n compare?
       analytic, discrete, indiscrete, finite-comp]

      T_{indisc} sub T_f sub T_{an} sub T_{disc}

Rem    topologies can be incomparable:
       think about $X = \{a, b, c\}$

(Munkres §13)    $\{B_i\}_{i \in I}$ any collection of
                 subsets of X

Df     $\{B_i\}_{i \in I}$ is a <u>basis</u> for a top on X iff

1)  $X = \bigcup_{i \in I} B_i$
2)  for all i, j in I, and x in $B_i \cap B_j$,
        have k in I s.t. x in $B_k \subseteq B_i \cap B_j$
        ["$B_i \cap B_j$ is covered by $B_k$'s"]

Rem    Munkres also discusses "subbases"

[idea: reduce q's abt topologies to q's abt bases]

any basis <u>generates</u> a topology:

Thm    if $\{B_i\}_{i \in I}$ is a basis on X, then

       $T = \{\bigcup_{i \in J} B_i \mid$ any $J \subseteq I\}$

is a topology on X [note! formula includes $J = \varnothing$]

Pf      1) $\varnothing$, X in T [why?]
        2) T is closed under unions [why?]

to show 3) T is closed under finite intersections:

by induction, just need to show: if U, V in T,
       then $U \cap V$ in T

U = bigcup_{j in J} B_i for some J sub I
V = bigcup_{k in K} B_i for some K sub I
U cap V = bigcup_{j in J, k in K} (B_j cap B_k)

for all j in J and k in K and x in B_j cap B_k,
    have ℓ in I s.t. x in B_ ℓ sub B_j cap B_k
so B_j cap B_k is union of B_ℓ's
so U_1 cap U_2 is too

<u>Ex</u>    {B(x, δ) | x in R and δ > 0} is a basis for
      the analytic top on R [why?]

same as {(a, b) | a < b} [why?]

<u>Rem</u>   different bases can generate
      the same topology

[criterion to check when a subcoll is a basis]

<u>Thm</u>   suppose T is a topology on X,
         {B_i}_{i in I} a subcollection of T
s.t.    for all U in T, and x in U,
      there is some k in I s.t. x in B_k sub U

then {B_i}_{i in I} is a basis, and it generates T

<u>Pf</u>    1) X = bigcup_{i in I} B_i [why?]
      2) pick i, j, and x in B_i cap B_j
         B_i cap B_j in T since B_i, B_j in T
         get k s.t. x in B_k sub B_i cap B_k
      so {B_i}_i is a basis

finally, T = {bigcup_{i in J} B_i | J sub I} [why?]

<u>Ex</u>     recall: {(a, b) |  a < b} is a basis for
     the analytic top on R

is {(a, b) | a < b and a, b in Q} a basis? [yes]
is {(a, b) | a < b and a, b in Z} a basis? [no]