

Warmup

recall 2×2 inversion formula:

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad M^{-1} = (1/\det(M)) \operatorname{adj}(M)$$

$$\text{where } \det(M) = ad - bc$$

$$\operatorname{adj}(M) = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

what are the inverses of:

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad ? \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad [\text{reflection}]$$

$$\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \quad ? \quad \begin{pmatrix} 1 & -b \\ 0 & 1 \end{pmatrix} \quad [\text{shear}]$$

$$\text{in general } \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} \quad \text{are shears}$$

but there are others:

$$M = \begin{pmatrix} 5/2 & -3/2 \\ 3/2 & -1/2 \end{pmatrix} \quad \text{is a shear too: why?}$$

$$M \text{ sends } \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ to } \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} -1 \\ 1 \end{pmatrix} \text{ to } \begin{pmatrix} -4 \\ -2 \end{pmatrix} = \begin{pmatrix} -3 \\ -3 \end{pmatrix} + \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

[draw]

Idea have: qualitative defs of geometric ops
 and examples given by matrices
 want: matrix-independent defs

Q how to formalize “matrix-independent”?

fix linear map $T : V$ to W

(e_1, \dots, e_n) ordered basis of V ,

(f_1, \dots, f_m) ordered basis of W

M matrix of T wrt $(v_i)_i, (w_j)_j$

(e_1, \dots, e_n) another ordered basis of V ,

(f_1, \dots, f_m) another ordered basis of W

matrix of T wrt $(e_i)_i, (f_i)_i$? in terms of M ?

	id_V		T		id_W		
V	to	V	to	W	to	W	
e_i		v_i		w_j		f_j	

let	A be the matrix of	id_V	“from e_i to v_i ”
	B	id_W	“from w_j to f_j ”

A, B are invertible [since they represent iso's]

the matrix of T wrt $(e_i)_i, (f_j)_j$ is $B \cdot M \cdot A$

possibly confusing:

ith col of A expresses e_i as $\sum_j A_{\{k, i\}} v_k$

jth col of B w_j as sum $\ell \in B \setminus \{i, j\}$ of ℓ

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[id_V sends e_i to e_i; id_W sends w_j to w_j]
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suppose $W = V$,
 $(w_j)_j = (v_i)_i$,
 $(f_j)_j = (e_i)_i$

here T is a linear op on V ,
 M is its matrix “from $(v_i)_i$ to $(v_i)_i$ ”

A is the matrix of the id map “from e_i to v_i ”

B is the matrix of the id map “from v_i to e_i ”

so $A = B^{-1}$

Thm if M is the matrix of a lin op $T : V$ to V
wrt a basis $(v_i)_i$ for V ,
then BMB^{-1} is its matrix
wrt another basis $(e_i)_i$
where $v_j = \sum_i B_{\{j, i\}} e_i$

Rem general case $T : V$ to W also useful,
but the statement is cumbersome –
easier to rederive from picture of maps

Ex $V = \{p \text{ in } F[x] \mid p = 0 \text{ or } \deg p \leq 3\}$
 $T(p) = dp/dx$
 $(v_1, v_2, v_3, v_4) = (1, x, x^2, x^3)$
 $(e_1, e_2, e_3, e_4) = (1, x, x^2/2, x^3/6)$

$M =$	0	1	0	0	$B =$	1	0	0	0
	0	0	2	0		0	1	0	0
	0	0	0	3		0	0	2	0
	0	0	0	0		0	0	0	6

[compute B^{-1}]

[compute BMB^{-1}]

Df matrices M, N are conjugate* iff,
for some matrix P ,
 $P \cdot M \cdot P^{-1}$ is well-defined,
equals N

* also say “ M and N are conjugates of each other”

Rem almost always, we only use this notion
in the context where M, N are square

Df a property of M is conjugation-invariant
iff it is the same for all conjugates of M

Cor if M is the matrix of a lin op T ,
then any property of T is
a conjugation-invariant property of M

and conversely!
any conjugation-invariant property of M
only depends on T

Ex entries of M are not conj-invariant

but $\text{tr}(M)$ = sum of diagonal entries
and $\det(M)$ = ??? [to be discussed later]
are

for a 2×2 matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$\text{tr}(M) = a + d$, $\det(M) = ad - bc$

[can check by brute force, if you want]

Ex fix an integer $k > 0$

the property $(M^k = 0_n)$ is conj-invariant
because

$$(BMB^{-1})^k = BM^kB^{-1} = 0_n$$

corresponds to having $T \circ \dots \circ T = \text{zero}$
where T is iterated k times

Df a lin op, resp. matrix, is nilpotent iff
some iterate, resp. power, is zero

it is unipotent iff it takes the form
 $\text{id} + T$, resp. $I_n + M$, where T , resp. M ,
is nilpotent

Ex since nilpotence is conj-invariant,
unipotence is conj-invariant:

$$\begin{aligned} B(I_n + M)B^{-1} &= BI_nB^{-1} + BMB^{-1} \\ &= I_n + BMB^{-1} \end{aligned}$$

above, used the left & right distributive properties
for matrix multiplication

Ex for fixed $k > 0$,
the property $M^k = I_n$ is conj-invariant

M is called an involution iff $M^2 = I_n$

Rem 2×2 shears are unipotent
[hyperplane] reflections are involutions