2.

Today we discuss Borel subgroups of reductive groups and the corresponding quotients.

2.1.

We have discussed how an algebraic group G over $\bar{\mathbf{F}}_q$, equipped with a Frobenius map $F : G \to G$, gives rise to a finite group G^F. When G is reductive, the structure of G, *resp.* G^F, closely resembles that of GL_n, *resp.* $\mathrm{GL}_n(\mathbf{F}_q)$.

Even earlier, we discussed the *Bruhat decomposition*. For now, let k be an arbitrary algebraically closed field. Let $B \subseteq \mathrm{GL}_n$ be the subgroup of upper-triangular matrices, and for all $w \in S_n$, let $\dot{w} \in \mathrm{GL}_n$ be the permutation matrix of w. The Bruhat decomposition on k-points is

$$\mathrm{GL}_n(k) = \coprod_{w \in S_n} B(k)\dot{w}B(k).$$

Its proof is similar to how we used row reduction to establish the Schubert cell decomposition of any Grassmannian. Namely, it suffices to show:

Theorem 2.1. *The coset space $B(k)\backslash\mathrm{GL}_n(k)$ is the disjoint union of the subsets $B(k)\backslash(B(k)\dot{w}B(k))$ for $w \in S_n$.*

Proof. We can identify cosets of $B(k)$ with (complete) flags in k^n via the map $B(k)g \mapsto \vec{V} \cdot g$, where $\vec{V} = (V_i)_i$ is the standard flag in row notation. The rows of g define an ordered basis $(v_i)_i$ such that $V_i \cdot g = \langle v_1, \ldots, v_i\rangle$ for all i.

So apply row reduction to g. The result is an upper-triangular matrix $b \in B(k)$. From the algorithm, we also get a permutation $w^{-1} \in S_n$ that only depends on the flag $\vec{V} \cdot g$: the composition of the row swaps (from the left) used to reduce g to b. We have $\vec{V} \cdot g = \vec{V} \cdot \dot{w}b$. □

Note that the expression $B\dot{w}B$ can be reduced even further. For instance, we can always take b unipotent in the proof above. Another way to see this: Recall that $B = TU$, where T, *resp.* U is the subgroup of diagonal, *resp.* unipotent matrices, and observe that permutation matrices normalize T, meaning $\dot{w}T = T\dot{w}$ for all w.

Something stronger is true. For any algebraic groups $H \subseteq G$, Milne Prop. 1.83 exhibits an algebraic group $N_G(H)$ such that $N_G(H)(R) = N_{G(R)}(H(R))$ for any k-algebra R. It turns out that the connected components $N_{\mathrm{GL}_n}(T)$ are precisely the cosets $\dot{w}T$ for $w \in S_n$. This suggests how Bruhat decomposition ought to generalize beyond GL_n.

Define the *Weyl group* of a maximal torus $T \subseteq G$ to be the normalizer $W = W(G,T) := N_G(T)/T$. Note that for any $w \in W$, and any algebraic subgroup $B \subseteq G$ containing T, the notation wB is unambiguous.

Theorem 2.2. *Suppose that G is a reductive algebraic group. Let $B = T \ltimes U \subseteq G$ be a Borel subgroup, where $U = [B, B]$. Then $B(k)\backslash G(k)$ is the disjoint union of the subsets $B(k)\backslash(B(k)wB(k))$ for $w \in W(G, T)$.*

2.2.

As in the first lecture, we switch notation from left-hand quotients back to right-hand quotients. The set $G(k)/B(k)$ is precisely the set of k-points of the fppf sheaf quotient G/B, essentially because $\operatorname{Spec} k$ has no nontrivial fppf covers. However, we can be much more concrete about spaces like this. The key idea is a representation-theoretic characterization of algebraic subgroups:

Theorem 2.3 (Chevalley)**.** *If G is any affine algebraic group with algebraic subgroup G', then there exist a (finite-dimensional) representation V of G and a subspace $V' \subseteq V$ such that $G'(k) = \{g \in G(k) \mid gV' \subseteq V'\}$. We can even choose V, V' so that V' is a line.*

Proof. Let I be the kernel of the quotient map $k[G] \to k[G']$. We can pick a finite generating set for I as an ideal. Then we can pick a finite-dimensional $k[G]$-comodule $V \subseteq k[G]$ containing these generators. (This is similar to the proof of the linearity of affine algebraic groups, except that here, our representation of G is the comodule itself, not its dual.) To get $V' \subseteq V$, we take $V' = V \cap I$.

If $g \in G'(k)$, then $gI \subseteq I$, so $gV' \subseteq V'$. Conversely, if $gV' \subseteq V'$, then g sends every generator of I to another element of I, but g acts on $k[G]$ by algebra automorphisms, so $gI \subseteq I$ and hence I is also the kernel of the quotient map $k[G] \to k[G'g]$, which forces $g \in G'$.

Finally, once we have such V, V', we see that the same characterization of G' holds when we replace V, V' by $\bigwedge^d V, \bigwedge^d V'$, respectively, where $d = \dim(V')$. $\square$

Corollary 2.4 (Chevalley–Plücker)**.** *If G is a* smooth *affine algebraic group with algebraic subgroup G', then there is a locally closed, G-equivariant embedding $G/G' \to \mathbf{P}V$ for some representation V of G. In particular, G/G' is a quasiprojective variety.*

Proof. Take V, V' as in the theorem, with V' a line. Let $G/G' \to \mathbf{P}V$ be induced by the map from G onto the orbit of $[V']$. The smoothness of G ensures that the latter is faithfully flat, allowing us to identify the orbit with G/G'. $\square$

2.3.

An algebraic subgroup $P \subseteq G$ is *parabolic* if and only if G/P is projective, not merely quasiprojective. As it turns out, there is a nice characterization of parabolic subgroups. For the proof of the following fixed-point theorem, see Milne Chapter 17.

Theorem 2.5 (Borel). *If B is a connected, smooth, solvable algebraic group acting on a nonempty proper variety X, then X^B is nonempty.*

Corollary 2.6. *Suppose that G is a smooth affine algebraic group.*

(1) If B is a Borel subgroup and P a parabolic subgroup of G, then some $G(k)$-conjugate of B is contained in P.

(2) Conversely, any algebraic subgroup of G that contains a Borel is parabolic.

Remark 2.7. In the setup above, G must have *some* Borel subgroup, because G has finite dimension and $\{1\}$ is connected, smooth, and solvable.

Proof. (1): By Borel's theorem, the action of B by left multiplication on G/P must have a fixed point gP, in which case $g^{-1}Bg$ is a Borel contained in P.

(2): Since the image of any proper variety is proper, it suffices to show that if $B \subseteq G$ is a Borel, then G/B is proper. We induct on the dimension of G. Pick a faithful representation V of G. The action of G on $\mathbf{P}V$ must have a closed orbit. The stabilizer of any k-point of this orbit is a parabolic subgroup $P \subseteq G$. By (1), it contains some conjugate of B, and without loss of generality, we may replace B with this conjugate. Two cases: Either P is smaller than G, in which case P/B is proper by the inductive hypothesis, and hence G/B is proper, or else $P = G$, in which case V^G contains a line, and we can replace V with $V/(V^G)$ until we either reach $\{0\}$ or reduce to the previous case. □

Corollary 2.8. *Any two Borels in a smooth affine algebraic group are conjugate.*

Example 2.9. Any Borel of GL_n is conjugate to the subgroup of upper-triangular matrices. Similarly, any parabolic of GL_n is conjugate to some subgroup of *block* upper-triangular matrices.

Recall that by Milne Thm. 16.27, any two maximal tori in a connected, solvable, affine algebraic group are conjugate. This fact combined with Corollary 2.8 proves the Cartan–Lie–Kolchin theorem stated last time.

Another way to state Corollary 2.8 is: The conjugation action of $G(k)$ on the set of Borel subgroups of G is transitive. Milne Thm. 17.48 shows that the stabilizer of any Borel is itself:

Theorem 2.10. *If B is any Borel subgroup of a connected, smooth, affine algebraic group G, then $N_G(B) = B$. Thus the map $gB \mapsto gBg^{-1}$ is a bijection from $(G/B)(k)$ to the set of Borel subgroups of G.*

When we regard G/B as the variety of Borel subgroups of G, we will call it the *flag variety* and denote it by $\mathcal{B}$. Indeed, for $G = \mathrm{GL}_n$, the above theorem follows from Corollary 2.8 together with the fact that if B is the stabilizer of a flag $\vec{V}$, then gBg^{-1} is the stabilizer of $g \cdot \vec{V}$.

2.4.

The orbit decomposition of G/B under the left action of B gives rise, on k-points, to the Bruhat decomposition that we discussed at the start. Note that we have not yet proven the precise decomposition for general G. We can sketch the gist modulo the following result. For $G = \mathrm{GL}_n$, it is a byproduct of the argument that proves Jordan–Hölder, via the flag interpretation of $\mathcal{B}(k)$.[1]

Theorem 2.11. *If G is reductive, then any two Borel subgroups of G contain a common maximal torus of G.*

Sketch of Bruhat decomposition. We exhibit a map from $B(k)\backslash G(k)/B(k)$ to the Weyl group $W = W(G,T)$. For any $g \in G(k)$, pick a maximal torus $S \subseteq B \cap gBg^{-1}$. By Cartan–Lie–Kolchin, we can write

$$S = bTb^{-1} = (gb'g^{-1})(gTg^{-1})(gb'g^{-1})^{-1}$$

for some $b, b' \in B(k)$. But then $b^{-1}gb'$ normalizes T, so we obtain an element $[b^{-1}gb'] \in W$. One has to check that this element only depends on BgB. □

2.5.

Henceforth, $k = \bar{\mathbf{F}}_q$. In what follows, recall that we often write X^F in place of $X^F(k)$ when F is a (relative) Frobenius map on X.

Any Frobenius map $F : G \to G$ that respects the group law and stabilizes an algebraic subgroup $H \subseteq G$ induces an analogous map $F : G/H \to G/H$. The identification $(G/H)(k) = G(k)/H(k)$ induces an identification

$$(G/H)^F \{F\text{-stable orbits of } H(k) \text{ on } G(k)\}.$$

The action of $G(k)$ on $(G/H)(k)$ restricts to an action of G^F on $(G/H)^F$.

Consider the *standard Frobenius map* $F : \mathrm{GL}_n \to \mathrm{GL}_n$ given by raising each matrix coordinate to the qth power, so that GL_n^F is the group classically denoted $\mathrm{GL}_n(\mathbf{F}_q)$. Then F stabilizes B and fixes $\dot{w}$ for all w. Hence the Bruhat decomposition of $\mathrm{GL}_n(k)$ into double cosets of $B(k)$ implies an analogous decomposition of GL_n^F into double cosets of B^F. With more work,[2] one can further show that $((B\dot{w}B)/B)^F = (B^F\dot{w}B^F)/B^F$, and hence, $(\mathrm{GL}_n/B)^F = \mathrm{GL}_n^F/B^F$. What happens for general G, B, F?

It turns out that the connectedness of B ensures that $(G/B)^F = G^F/B^F$ holds for any F-stable Borel B. On Problem Set 1, you will use the theorem below to deduce (a version of) the first corollary following it:

[1] Here I borrow from the answers to https://mathoverflow.net/q/15438.

[2] . . . than I originally thought was necessary, during the lecture. . .

Theorem 2.12 (Lang)**.** *Let H be a connected, smooth algebraic group over k and $F : H \to H$ the Frobenius map for some $\mathbf{F}_q$-form. Then the* Lang map

$$h \mapsto h^{-1} F(h) : H \to H$$

is surjective.

The proof of Lang's theorem is given on Wikipedia. The key idea is to calculate the induced map on Lie algebras, using the fact that the differential of F vanishes to show bijectivity.

Remark 2.13. Note that the Lang map is finite étale, and its fiber over the identity is precisely H^F. For this reason, one can think of the theorem as presenting H as an H^F-principal bundle over itself in the étale topology. This leads to bizarre topological conclusions for, say, $H = \mathbf{G}_a$ and $F(x) = x^q$.

Remark 2.14. In the affine case, Steinberg generalized Lang's theorem from Frobenius maps to any surjective map F with finitely many fixed points. So I sometimes speak of the Lang–Steinberg theorem even where it is overkill.

Corollary 2.15. *Let G be a connected, smooth algebraic group over k with a Frobenius map $F : G \to G$. Let $\mathcal{X}$ be a set with a $G(k)$-action and a map $f : \mathcal{X} \to \mathcal{X}$ such that $f(g \cdot x) = F(g) \cdot f(x)$ for all $g \in G(k)$ and $x \in \mathcal{X}$.*[3] *Then:*

1. *Every f-stable $G(k)$-orbit on $\mathcal{X}$ contains an f-fixed point.*
2. *If $\mathcal{X} = G(k)/H(k)$ for some F-stable $H \subseteq G$, and f is induced by F, then $\mathcal{X}^f = G^F/H^F$.*

Corollary 2.16. *A connected, smooth affine algebraic group with Frobenius map F always contains an F-stable Borel pair. In particular, any F-stable Borel contains an F-stable maximal torus.*

Proof. Take $\mathcal{X}$ to be the set of all Borel pairs, and $f : \mathcal{X} \to \mathcal{X}$ to be defined by $f(B, T) = (F(B), F(T))$. Now Cartan–Lie–Kolchin implies the first statement. Replacing the ambient group with a given F-stable Borel, we deduce the second statement. □

Remark 2.17. In the setting of a more general field $k = \bar{K}$, a K-form of an algebraic group G is called *quasi-split* if and only if there is a Borel subgroup of G that descends to the K-form. In this language, the last result essentially says that $\mathbf{F}_q$-forms are always quasi-split.

However, a given F-stable maximal torus need not be contained in an F-stable Borel. If it is contained in such a Borel, then it is called *F-maximally split*.

[3] This setup generalizes the notion of compatible Frobenius maps defined earlier.

Example 2.18. Let F be the standard Frobenius map on GL_2. Then the diagonal torus of GL_2 is F-maximally split. At the same time, there is a different F-stable maximal torus $T \subseteq \mathrm{GL}_2$ defined on k-points by

$$T(k) = \left\{ \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \middle| \, a^2 + b^2 \neq 0 \right\}.$$

It descends to an $\mathbf{F}_q$-form T_1 with the same matrix presentation. If T were contained in an F-stable Borel, then that Borel would have an $\mathbf{F}_q$-form containing T_1, which we can rule out by computation.

Note that the $\mathbf{F}_q$-form of $\mathbf{G}_m$ called $\mathrm{U}(1)$ embeds into T_1. However, the Frobenius map on $\mathbf{G}_m$ coming from $\mathrm{U}(1)$ is not related to the F above.

Corollary 2.19. *Any two F-stable Borel subgroups of a connected, smooth affine algebraic group G are conjugate under G^F, not just under $G(k)$.*

Proof. Pick an F-stable Borel B. The isomorphism $gB \mapsto gBg^{-1} : G/B \to \mathcal{B}$ is F-equivariant. Now use $(G/B)^F = G^F/B^F$. □

2.6.

We now focus on a connected, smooth affine algebraic group G over $\bar{\mathbf{F}}_q$ with a Frobenius map F corresponding to an $\mathbf{F}_q$-form and an F-stable Borel pair (B, T). Recall that $W = N_G(T)/T$. We claim that:

Proposition 2.20. *If $H \subseteq G$ is an F-stable, smooth algebraic subgroup and $N_G(H)/H$ is finite, then $N_G(H)$ is also F-stable. If H is morever connected, then $N_G(H)^F = N_{G^F}(H^F)$.*

Proof. First, the hypotheses imply that $N := N_G(H)$ is smooth, and hence reduced. So the kernel of the quotient map $k[G] \to k[N]$ equals its radical. So by the Nullstellensatz, it is determined by $N(k) = \mathrm{Hom}(k[N], k)$ as a subset of $G(k) = \mathrm{Hom}(k[G], k)$ (where these are Hom spaces in the category of k-algebras). We deduce that if $N(k)$ is F-stable, then N is too.

Next, F must induce a self-bijection of $H(k)$. So for all $n \in N(k)$ and $h \in H(k)$, we see that $F(n) \cdot h = F(n \cdot F^{-1}(h)) \in F(H(k)) = H(k)$, where $\cdot$ here denotes the conjugation action. Hence $F(n) \in N(k)$, as needed.

The inclusion $N_G(H)^F \subseteq N_{G^F}(H^F)$ follows from the definitions. For the converse, we must show that if an element of G^F normalizes H^F, then it normalizes H. Rewrite G^F, H^F in terms of the $\mathbf{F}_q$-points of the corresponding $\mathbf{F}_q$-forms, then use an argument similar to that in the first paragraph. □

Corollary 2.21. *In our setup, if G is reductive, then $N_G(T)$ is F-stable. In particular, the F-action on $N_G(T)$ descends to W.*

By Corollary 2.15, we have $W^F = N_G(T)^F/T^F = N_{G^F}(T^F)/T^F$. The Bruhat decomposition of $G(k)$ restricts to

$$G^F = \coprod_{w \in W^F} B^F w B^F.$$

We say that (G, T) is *split* under F if and only if F acts trivially on W.

Example 2.22. Let $(-)^\tau : \mathrm{GL}_n \to \mathrm{GL}_n$ be the "anti-transpose" given by $g^\tau = Jg^tJ$, where g^t is the usual transpose and $J \in \mathrm{GL}_n(k)$ is the matrix with 1's along the anti-diagonal and 0's everywhere else. Then

$$F'(g) := (g^\tau)^{-q} = (g^{-q})^\tau$$

is a Frobenius map that differs from the standard one when $n \geq 3$, or when q is odd and $n = 2$. It defines the $\mathbf{F}_q$-form of GL_n promised earlier, $\mathrm{GU}(n)$. For $n \geq 3$, we can check that F' acts on $W = S_n$ nontrivially.

2.7.

As usual, write $U = [B, B]$. Since B, T are F-stable, so is U. The G^F-action on G^F/U^F defines a representation of G^F on

$$I = \mathrm{Ind}_{U^F}^{G^F}(1) := \{\mathbf{C}\text{-valued functions on the finite set } G^F/U^F\}.$$

The G^F-stable summands of I are called the *principal series representations* of G^F. Some standard theory shows that $I \simeq \bigoplus_\theta I_\theta$ as a representation, where the sum runs over characters $\theta : B^F \to \mathbf{C}^\times$ that factor through $T^F \simeq B^F/U^F$, and

$$I_\theta = \mathrm{Ind}_{B^F}^{G^F}(\theta)$$

for all θ. To determine the summands of I_θ, we analyze $\mathrm{End}_{G^F}(I_\theta)$.

Here is a very general principle. Suppose that Γ is a finite group and Ξ a finite set with a Γ-action. Let $\mathbf{C}\Xi$ be the representation of Γ formed by the $\mathbf{C}$-valued functions on Ξ under $[g \cdot f](-) = f(g^{-1} \cdot -)$. Let Γ act on $\Xi \times \Xi$ diagonally, and endow $\mathbf{C}(\Xi \times \Xi)$ with the *convolution* product

$$(f_1 * f_2)(x, y) = \sum_{z \in \Xi} f_1(x, z) f_2(z, y).$$

Note that $\mathbf{C}(\Xi \times \Xi)^\Gamma$ forms a subalgebra of $\mathbf{C}(\Xi \times \Xi)$.

Proposition 2.23. *There is an isomorphism of* $\mathbf{C}$*-algebras*

$$\mathbf{C}(\Xi \times \Xi)^\Gamma \xrightarrow{\sim} \mathrm{End}_\Gamma(\mathbf{C}\Xi),$$

$$1_O \mapsto \left(1_x \mapsto \sum_{\substack{y \in \Xi \\ (x,y) \in O}} 1_y\right),$$

where O denotes any Γ-orbit of $\Xi \times \Xi$, and $1_O, 1_x$ refer to indicator functions on $O, \{x\}$.

Above, the image of 1_O in $\mathrm{End}_\Gamma(\mathbf{C}\Xi)$ is called the *Hecke operator* for O. In the case where $\Xi = \Gamma/\mathrm{H}$ for some subgroup $\mathrm{H} \subseteq \Gamma$, we have a further bijection

$$\Gamma\backslash(\Gamma/\mathrm{H} \times \Gamma/\mathrm{H}) \xrightarrow{\sim} \mathrm{H}\backslash\Gamma/\mathrm{H},$$
$$(y\mathrm{H}, x\mathrm{H}) \mapsto \mathrm{H}y^{-1}x\mathrm{H},$$

which induces an isomorphism of vector spaces $\mathbf{C}(\Xi \times \Xi)^\Gamma \simeq (\mathbf{C}\Gamma)^{\mathrm{H}\times\mathrm{H}}$. Taking $\Gamma = G^F$ and $\mathrm{H} = U^F, B^F$, we deduce:

Corollary 2.24. *As a vector space,* $\mathrm{End}_{G^F}(I)$, resp. $\mathrm{End}_{G^F}(I(1))$*, has a basis indexed by* $U^F\backslash G^F/U^F$*,* resp. $B^F\backslash G^F/B^F$*. In particular, the latter is also indexed by* W^F*.*

The arguments above can be pushed further to analyze $\mathrm{Hom}_{G^F}(I_\chi, I_\psi)$ for any χ, ψ. Returning to the abstract setup, let A, B be subgroups of Γ, and let α, *resp.* β, be an arbitrary $\mathbf{C}$-valued character of A, *resp.* B. For all $g \in \Gamma$, we set $\mathrm{A}^g = g^{-1}\mathrm{A}g$, so that $\alpha^g(-) := \alpha(g(-)g^{-1})$ is a $\mathbf{C}$-valued character of A^g.

The following is proved via Frobenius reciprocity in most texts on character theory, such as Serre's book:

Theorem 2.25 (Mackey)**.** *Above, there is an isomorphism of vector spaces*

$$\mathrm{Hom}_\Gamma(\mathrm{Ind}_\mathrm{A}^\Gamma(\alpha), \mathrm{Ind}_\mathrm{B}^\Gamma(\beta)) \simeq \bigoplus_{g \in \mathrm{A}\backslash\Gamma/\mathrm{B}} \mathrm{Hom}_{\mathrm{A}^g \cap \mathrm{B}}(\alpha^g, \beta).$$

Corollary 2.26. *We have*

$$\mathrm{Hom}_{G^F}(I_\theta, I_\eta) \simeq \bigoplus_{w \in W^F} \mathrm{Hom}_{(B^F)^w \cap B^F}(\theta^w, \eta).$$

In particular, I_θ *is irreducible if and only if* $\theta^w \neq \theta$ *as characters of* T^F *for all* $w \neq e$*.*