fix top spaces X, Y and x in X

<u>Thm</u> if $f : X$ to Y is a homotopy equiv., then
$f_* : \pi_1(X, x)$ to $\pi_1(Y, f(x))$ is an iso

<u>Lem 1</u> if $\varphi : G$ to G' and $\psi : G'$ to G'' are maps
s.t. $\psi \circ \varphi$ is bijective
then φ is injective and ψ is surjective

<u>Lem 2</u> let $h : X \times [0, 1]$ to Y be a homotopy
set $f_0(s) = h(s, 0)$,
$f_1(s) = h(s, 1)$,
$\alpha(t) = h(0, t)$ [starting pt at time t]

then we have
$f_{1, *} = \breve{\alpha} \circ f_{0, *} : \pi_1(X, x)$ to $\pi_1(Y, f_1(x))$

here, α is a path in Y and $\breve{\alpha}([\gamma]) = [\alpha'] * [\gamma] * [\alpha]$
thus $\breve{\alpha}$ is an <u>automorphism</u> of $\pi_1(Y, f_1(x))$

<u>Pf of Thm from Lem's</u>

have $g : Y$ to X s.t. $g \circ f$ is homotopic to id_X
$f \circ g$ is homotopic to id_Y

by lem 2,
$$g_* \circ f_* = (g \circ f)_* = \breve{\alpha}_X \circ id_{X, *} = \breve{\alpha}_X$$
$$f_* \circ g_* = (f \circ g)_* = \breve{\alpha}_Y \circ id_{Y, *} = \breve{\alpha}_Y$$

for some paths α_X, α_Y in X, Y, respectively

so, by lem 1, f_* is both injective and surjective
i.e., f_* is bijective □

(Munkres §68–69) next: Seifert–van Kampen
first: more group theory

Review on Quotient Groups

Df if H is a subgroup of G, then a left coset of H in G is a subset of G with the form

$$gH = \{gh \mid h \text{ in } H\} \qquad \text{for some } g$$

the set of left cosets is written $G/H = \{gH \mid g \text{ in } G\}$

similarly, a right coset is a set $Hg = \{hg \mid h \text{ in } H\}$

Ex if $G = \mathbb{Z}$ and $H = m\mathbb{Z}$, then

$$G/H = \{0 + m\mathbb{Z}, 1 + m\mathbb{Z}, \ldots\}, \text{ which has } m \text{ elts}$$

e.g., $\mathbb{Z}/3\mathbb{Z} = \{0 + 3\mathbb{Z}, 1 + 3\mathbb{Z}, 2 + 3\mathbb{Z}\}$

- instead of using $g = 0, 1, 2$, we could have used other rep's: e.g., $g = 0, -1, -2$ [draw]
- the + law on $\mathbb{Z}$ descends to a + law on $\mathbb{Z}/3\mathbb{Z}$: $(a + 3\mathbb{Z}) + (b + 3\mathbb{Z}) = (a + b) + 3\mathbb{Z}$ well-def'd

Q when does the law on G descend to a well-defined law on G/H in which

$$g_1H \bullet g_2H = (g_1g_2)H?$$

observe: if $Hg_2 = g_2H$, then
$(g_1H)(g_2H) = g_1(g_2H)H = (g_1g_2)H$

<u>Df</u> we say H is a normal subgroup iff
$gH = Hg$ for all g in G

here, G/H forms a group: the <u>quotient</u> of G by H

recall: the <u>kernel</u> of a homomorphism $\varphi: G \to Q$ is

$$\ker(\varphi) = \{g \in G \mid \varphi(g) = e_Q\}$$

<u>Thm</u> if $\varphi: G \to Q$ is surjective and $H = \ker(\varphi)$
then there is a well-def'd isomorphism
$[\varphi]: G/H \to Q$, namely, $[\varphi](gH) = \varphi(g)$
["first isomorphism thm"]

[will use thm to define presentations of groups]

<u>Free Groups</u> fix an arbitrary set X

<u>Df</u> let $X^\pm = X \cup \{x^{-1} \mid x \in X\}$, where
x^{-1} is a formal symbol indexed by x

- a <u>(signed) word</u> in X is a finite sequence of elts of $X^\pm$
- a word is <u>reduced</u> iff no consecutive elts look like "x, x^{-1}" or "x^{-1}, x"

an <u>elementary reduction</u> in a word w is the operation of deleting such consec elts from w

a <u>reduction</u> of w is a reduced[!] word obtained from w by successive elementary reductions

Ex let $X = \{g, h, k\}$ and [from Terry Tao]

$w = g^{-1}k^{-1}gk\,k^{-1}\,g^{-1}h^{-1}gh\,k$

elementary reductions give $g^{-1}k^{-1}h^{-1}ghk$

Thm every word in X has a unique reduction

Pf existence: words have finite length
uniqueness: induct on the length $|w|$

if w is empty, then done
else let w to u_1 to u_2 to … to u_m
w to v_1 to v_2 to … to v_n
be two chains of elementary reductions
with u_m and v_n reduced

Claim either $u_1 = v_1$
or there is a word w' obtained from both
by a single elementary reduction

Pf of Claim either the elementary reductions
from w to u_1, v_1 overlap
or they do not:
check each case separately

Claim Implies Thm

if $u_1 = v_1$, then $u_m = v_n$ by the inductive hyp.
because $|u_1| = |v_1| < |w|$
otherwise, let w'' be a reduction of w'
then w'' is also a reduction of both u_1 and v_1
so $u_m = w'' = v_n$, again by the inductive hyp.

Df let $v|w$ denote concatenation of v and w

the free group generated by X is

F_X = {reduced words in X}

under the group law $v \bullet w$ = reduction($v|w$)
we call this concatenation as well, and drop the •
associativity:

reduct(reduct(u|v)|w)
= reduct(u|v|w)
= reduct(u|reduct(v|w))

[what is the id elt?] id_{F_X} = empty word
[inverses should be clear]

Universal Property of Free Groups

for any group G, there is a bijection

{set-theoretic maps X to G} = {hom.'s F_X to G}

$f : X$ to G goes to $\varphi_f : F_X$ to G def by

$\varphi_f(x_1^{e_1}, x_2^{e_2}, \ldots) = f(x_1)^{e_1} * f(x_2)^{e_2} * \ldots)$

[F_X is the "freest", or "most universal", way to build a group from an arbitrary set X]

if X is finite, and we only care about $n = |X|$, then we write F_n in place of F_X

Ex F_1 is a copy of Z

Groups via Generators and Relations

Df a subset S sub G is a *generating set* for G iff either of the following hold:
[they are equivalent]

1) no proper subgroup of G contains S
2) the homomorphism F_S to G induced by the inclusion S to G is surjective

here, G is iso to F_S/H, where $H = \ker(F_S \text{ to } G)$
[but usually, F_S is much, much larger than G]

Df for any R sub G and generating set S:

R is a set of relations for G wrt S iff $\ker(F_S \text{ to } G)$ is the minimal normal subgroup of G containing R

in this case, we write $G = \langle S \mid R \rangle$
and say G is *gen'd by S modulo the relations R*

Ex up to iso, a unique group of size 2:

$G = \{e, s\}$ s.t. $e*e = s*s = e$
$s*e = e*s = s$

$S = \{s\}$ generates G

[F_S to G sends powers of s to powers of s]
$\ker(F_S \text{ to } G) = \{\text{even powers of } s\}$

altogether, $G = \langle s \mid s^2 \rangle$
[technically, $S = \{s\}$ and $R = \{s^2\}$, but we omit the brackets]