

# CENTRAL ELEMENTS, CELL DECOMPOSITIONS, AND PARTIAL SPRINGER RESOLUTIONS

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ABSTRACT. For any finite Weyl group  $W$  and parabolic subgroup  $W_J$ , arising from a finite reductive group  $G$  and parabolic subgroup  $P_J$ , we prove identities relating the partial Springer resolutions of type  $J$  to central elements in the Hecke algebra for  $G$ , given by sums over minimal- or maximal-length representatives of cosets of  $W_J$  in  $W$ . We deduce formulas for Hecke traces arising from these central elements, generalizing work of Wan–Wang beyond type  $A$  via Lusztig’s exotic Fourier transform, and cell decompositions of new braid varieties involving  $J$ , generalizing work of Shende–Treumann–Zaslow. From the latter, we construct noncrossing sets that interpolate between rational Catalan and parking objects, generalizing our work with Galashin–Lam, and new formulas for arbitrary  $a$ -degrees of the HOMFLYPT polynomials of positive braid closures.

## 1. INTRODUCTION

1.1. Fix a finite Coxeter system  $(W, S)$  and a subset  $J \subseteq S$  generating a subgroup  $W_J \subseteq W$ . Let  $H_W$  and  $H_{W_J}$  be the Hecke algebras over  $\mathbf{Z}[\mathbf{q}^{\pm 1}]$  corresponding to  $W$  and  $W_J$ . We take the convention where the Hecke operators  $T_s \in H_W$ , for  $s \in S$ , obey the relations  $T_s^2 = (\mathbf{q} - 1)T_s + \mathbf{q}$ . We identify  $H_{W_J}$  with the subalgebra of  $H_W$  generated by the elements  $T_s$  with  $s \in J$ .

Under this embedding, the center  $Z(H_{W_J})$  need not embed into the center  $Z(H_W)$ . Nonetheless, Hoefsmit–Scott constructed an injective, linear *relative norm* map

$$N_J^S : Z(H_{W_J}) \rightarrow Z(H_W),$$

that they, and L. K. Jones, used to study induction from  $H_{W_J}$  to  $H_W$  [Jon90]. To define  $N_J^S$ , recall that each right coset of  $W_J$  in  $W$  contains a unique representative of minimal Bruhat length. Let  $W^J$  be the set of such representatives. Then

$$N_J^S(\alpha) = \sum_{v \in W^J} \mathbf{q}^{-\ell(v)} T_{v^{-1}} \alpha T_v \quad \text{for all } \alpha \in Z(H_{W_J}),$$

where  $T_v$  and  $\ell(v)$  denote the Hecke operator for and Bruhat length of  $v$ .

When  $W$  is crystallographic, we can interpret it as the Weyl group of a split finite reductive group  $G$ . We can then interpret the above algebras geometrically, as convolution algebras of functions on the flag variety of  $G$  or its square. The main observation of this paper is that in the geometric framework, the relative norm  $N_J^S$  is related to the two partial Springer resolutions for  $J$ , defined in (1.1).

From this relationship, we obtain applications to traces on  $H_W$ , generalizing work of Lascoux [Las06] and Wan–Wang [WW15]; cell decompositions of *partial braid*

*Steinberg varieties*, which we expect to generalize work of Shende–Treumann–Zaslow [STZ17]; and the rational parking combinatorics of  $(W, S)$ , generalizing our prior work with Galashin–Lam [GLTW24].

**1.2. Partial Resolutions.** Let  $\mathbf{F}$  be a finite field of order  $q$ . Let  $\mathbf{G}$  be a connected reductive algebraic group over  $\bar{\mathbf{F}}$ , equipped with a Frobenius map  $F : \mathbf{G} \rightarrow \mathbf{G}$ . We assume that the characteristic of  $\mathbf{F}$  is a good prime for  $\mathbf{G}$  [Car93, 28].

Fix an  $F$ -stable maximal torus in an  $F$ -stable Borel:  $\mathbf{T} \subseteq \mathbf{B} \subseteq \mathbf{G}$ . Let  $\mathbf{W} = N_{\mathbf{G}}(\mathbf{T})/\mathbf{T}$ . We now take  $W$  to be the finite Coxeter group  $\mathbf{W}^F$ . Similarly, we write  $G, B$ , etc. for the groups formed by the  $F$ -fixed points of  $\mathbf{G}, \mathbf{B}$ , etc.

The  $G$ -invariant,  $\mathbf{Z}[q^{\pm 1}]$ -valued functions on  $(G/B)^2$  form a convolution algebra  $H_B^G$ . If  $G$  is *split*, meaning  $W = \mathbf{W}$ , then  $H_B^G$  is the specialization at  $q \rightarrow q$  of the algebra  $H_W$  presented earlier. Explicitly,  $T_w$  specializes to the indicator function on the set of pairs  $(hB, gB)$  such that  $Bh^{-1}gB = BwB$ . In Section 2, we review the presentation of  $H_B^G$  for general  $G$ . *In the rest of this introduction, we assume that  $G$  is split, for simplicity.*

We take  $S$  to be the system of simple reflections arising from  $\mathbf{B}$ . Let  $\mathbf{P}_J = \mathbf{B}\mathbf{W}_J\mathbf{B}$ , a parabolic subgroup of  $\mathbf{G}$ . Let  $\mathbf{U}_J$  be its unipotent radical and  $\mathbf{V}_J$  the variety of all unipotent elements in  $\mathbf{P}_J$ . If  $J = \emptyset$ , then  $\mathbf{P}_J = \mathbf{B}$  and  $\mathbf{U}_J = \mathbf{V}_J$ ; otherwise,  $\mathbf{V}_J$  is larger than  $\mathbf{U}_J$ . At the level of points, the two *partial Springer resolutions* of type  $J$  are defined by

$$(1.1) \quad \begin{aligned} \mathbf{Spr}_J^+ &= \{(u, y\mathbf{P}_J) \in \mathbf{G} \times \mathbf{G}/\mathbf{P}_J \mid u \in y\mathbf{V}_Jy^{-1}\}, \\ \mathbf{Spr}_J^- &= \{(u, y\mathbf{P}_J) \in \mathbf{G} \times \mathbf{G}/\mathbf{P}_J \mid u \in y\mathbf{U}_Jy^{-1}\}. \end{aligned}$$

The  $+$  case is a partial resolution of singularities of the unipotent variety  $\mathbf{V} \subseteq \mathbf{G}$ , while the  $-$  case is a resolution of the closure of the Richardson orbit for  $J$ .

It will be convenient to set  $\mathbf{E}_J^{\pm} := \mathbf{G}/\mathbf{B} \times \mathbf{Spr}_J^{\pm}$  and  $E_J^{\pm} = (\mathbf{E}_J^{\pm})^F$ . There is a natural left  $G$ -action on  $E_J^{\pm}$ , under which the map  $f : E_J^{\pm} \rightarrow (G/B)^2$  defined by

$$f(hB, u, yP_J) = (hB, uhB)$$

is equivariant. For any set  $E$  equipped with a  $G$ -action and an equivariant map  $f : E \rightarrow (G/B)^2$ , we write  $f_!\delta_E \in H_B^G$  to denote the function whose value at a point in  $(G/B)^2$  is the size of its preimage in  $E$ .

Let  $w_{\circ}$  and  $w_{J_{\circ}}$  respectively denote the longest elements of  $W$  and  $W_J$ . For convenience, we set  $\ell_S = \ell(w_{\circ})$  and  $\ell_J = \ell(w_{J_{\circ}})$ . Recall that  $w_{\circ}, w_{J_{\circ}}$  are involutions, and that  $T_{w_{J_{\circ}}}^2$  is central in  $H_{W_J}$  [BMR98]. We can now state the split case of our main result, proven for general  $G$  in Section 3.

**Theorem 1.1.** *For any  $J \subseteq S$ , we have*

$$\begin{aligned} f_!\delta_{E_J^-} &= q^{\ell_S - \ell_J} N_J^S(1)|_{q \rightarrow q}, \\ f_!\delta_{E_J^+} &= q^{\ell_S - \ell_J} N_J^S(T_{w_{J_{\circ}}}^2)|_{q \rightarrow q}. \end{aligned}$$

Let  $W^{J,-} = W^J$ , and by analogy, let  $W^{J,+}$  of *maximal-length* representatives for the right cosets of  $W_J$  in  $W$ , so that multiplication by  $w_{J_0}$  interchanges  $W^{J,-}$  with  $W^{J,+}$ . Then the identities above can be rewritten as:

$$\begin{aligned} f_! \delta_{E_J^-} &= q^{\ell_S - \ell_J} \sum_{w \in W^{J,-}} q^{-\ell(v)} T_{v^{-1}} T_v, \\ f_! \delta_{E_J^+} &= q^{\ell_S} \sum_{w \in W^{J,+}} q^{-\ell(v)} T_{v^{-1}} T_v. \end{aligned}$$

We emphasize that the  $+$  case is deeper than the  $-$  case. The  $-$  case only uses standard results about Bruhat decomposition. Under the assumption that  $G$  is split, we can refine it to an algebro-geometric statement about  $\mathbf{E}_J^-$ : See Proposition 3.3. By contrast, the  $+$  case relies on a difficult theorem of Kawanaka [Kaw75]. We expect that it can only be refined to a statement at the level of cohomology. This issue is related to the sheafification of Kawanaka's work discussed in [Tri22].

Theorem 1.1 suggests some compatibility between  $N_J^S$  and parabolic induction from the Levi quotient of  $P_J$  up to  $G$ , which we will study in future work.

**1.3. Traces.** Recall that a *trace* on an algebra is a linear map that vanishes on commutators. We write  $R(H_W)$  to denote the vector space of  $\mathbf{Q}(\mathbf{q})$ -valued traces on  $H_W$ . Our first application of Theorem 1.1 is to identify certain elements of  $R(H_W)$  arising from  $N_J^S(1)$  and  $N_J^S(T_{w_{J_0}}^2)$ .

Let  $e \in W$  be the identity. Let  $\tau : H_W \rightarrow \mathbf{Z}[\mathbf{q}^{\pm 1}]$  be the trace given by  $\tau(T_e) = 1$  and  $\tau(T_w) = 0$  for all  $w \neq e$ . Then any central element  $\zeta \in Z(H_W)$  gives rise to a trace  $\tau[\zeta] : H_W \rightarrow \mathbf{Z}[\mathbf{q}^{\pm 1}] \subseteq \mathbf{Q}(\mathbf{q})$ : namely,

$$\tau[\zeta](\beta) = \tau(\beta\zeta).$$

These traces are best understood when  $W$  is a symmetric group.

Let  $S_n$ , the symmetric group on  $n$  letters, and let  $\Lambda_n$  be the vector space of symmetric functions over  $\mathbf{Q}(\mathbf{q})$  of degree  $n$  in variables  $X = (X_1, X_2, \dots, X_n)$ . Then  $R(H_{S_n})$  is isomorphic to  $\Lambda_n$ , as both of these vector spaces have bases indexed by the integer partitions of  $n$ . Let  $ch_{\mathbf{q}} : R(H_{S_n}) \xrightarrow{\sim} \Lambda_n$  be the  *$\mathbf{q}$ -deformed Frobenius characteristic* isomorphism that sends the irreducible character  $\chi_{\mathbf{q}}^\lambda$  to the Schur function  $s_\lambda(X)$ , for any partition  $\lambda \vdash n$ .

For  $W = S_n$ , we take  $S = \{s_1, \dots, s_{n-1}\}$ , where  $s_i = (i, i+1)$ . This choice sets up a bijection between subsets  $J \subseteq S$  and integer *compositions*  $\nu$  of  $n$ . Let  $e_\nu(X)$  and  $h_\nu(X)$  respectively denote the elementary and complete homogeneous symmetric functions in  $\Lambda_n$  indexed by  $\nu$ . Wan–Wang [WW15], recasting work of Lascoux [Las06], show that if  $J$  corresponds to  $\nu$ , then

$$\begin{aligned} (1.2) \quad ch_{\mathbf{q}}(\tau[N_J^S(1)]) &= (\mathbf{q} - 1)^n e_\nu \left( \frac{X}{\mathbf{q} - 1} \right), \\ ch_{\mathbf{q}}(\tau[N_J^S(T_{w_{J_0}}^2)]) &= \mathbf{q}^{\ell_J} (\mathbf{q} - 1)^n h_\nu \left( \frac{X}{\mathbf{q} - 1} \right). \end{aligned}$$

Using these formulas, they show that the maps  $N_J^S$  give rise to a ring structure on the direct sum of the centers  $Z(\mathbf{Q}(\mathbf{q}) \otimes H_{S_n})$ , isomorphic to the ring of symmetric functions over  $\mathbf{Q}(\mathbf{q})$ . We will generalize the formulas to any crystallographic  $W$ .

Recall that Springer constructed a  $W$ -action on the étale cohomologies of the fibers of his resolution of the unipotent variety, now called *Springer fibers*. In [Tri21], the first author used this action to construct a trace on  $H_W$  valued in  $\mathbf{Q}(\mathbf{q})$ -linear traces on  $W$ , or equivalently, a *bitrace*

$$\tau_G : \mathbf{Q}W \otimes H_W \rightarrow \mathbf{Q}(\mathbf{q}),$$

which refines the Markov traces on  $H_W$  studied by Gomi [Gom06] and Webster–Williamson [WW11]. These, in turn, were motivated by the traces used by Ocneanu to construct the HOMFLYPT link polynomial [Jon87].

In this paper, we use a normalization for  $\tau_G$  characterized by the formula

$$\tau_G(z \otimes T_w)|_{\mathbf{q} \rightarrow \mathbf{q}} = \frac{1}{|G|} \sum_{\substack{u \in G \\ \text{unipotent}}} |O(w)_u| \chi_u(z) \quad \text{for all } z, w \in W,$$

where  $\chi_u$  is the total Springer character for  $u$ , reviewed in §4.3, and  $O(w)_u$  is the set of pairs  $(hB, gB)$  such that  $h^{-1}gB = BwB$  and  $gB = uhB$ . Let  $e_{J,-}$ , *resp.*  $e_{J,+}$ , denote the antisymmetrizer, *resp.* symmetrizer, in  $\mathbf{Q}W_J$ , reviewed in §4.4. By combining Theorem 1.1 with work of Borho–MacPherson [BM83], we show:

**Theorem 1.2.** *For any  $J \subseteq S$ , we have*

$$\begin{aligned} \tau[N_J^S(1)] &= (\mathbf{q} - 1)^{\text{rk}(G)} \tau_G(e_{J,-} \otimes ( \quad )), \\ \tau[N_J^S(T_{w_{J_0}}^2)] &= \mathbf{q}^{\ell_J} (\mathbf{q} - 1)^{\text{rk}(G)} \tau_G(e_{J,+} \otimes ( \quad )) \end{aligned}$$

as traces on  $H_W$ , where  $\text{rk}(G)$  is the rank of the maximal torus  $T$ .

From [Tri21], there is a purely algebraic formula for  $\tau_G$  involving the *exotic Fourier transform*: a pairing introduced by Lusztig to relate the set  $\text{Irr}(W)$  of irreducible characters of  $W$  to the set of (unipotent) irreducible characters of  $G$ . Let

$$\{-, -\} : \text{Irr}(W) \times \text{Irr}(W) \rightarrow \mathbf{Q}$$

be its “truncation” to  $\text{Irr}(W)$ , and for all  $\chi \in \text{Irr}(W)$ , let  $\chi_{\mathbf{q}} \in R(H_W)$  be the Tits deformation of  $\chi$ . We deduce the following formulas. For  $G = \text{GL}_n(\mathbf{F})$ , the pairing is trivial and our formulas recover (1.2).

**Corollary 1.3.** *The multiplicity of  $\chi_{\mathbf{q}}$  in  $\tau[N_J^S(1)]$ , *resp.*  $\tau[N_J^S(T_{w_{J_0}}^2)]$ , is*

$$(\mathbf{q} - 1)^{\text{rk}(G)} \sum_{\psi \in \text{Irr}(W)} \{\chi, \psi\} \left[ \frac{\psi(e_{J,-})}{\det(\mathbf{q} - e_{J,-} \mid \mathbf{V}_G)}, \quad \text{resp.} \quad \frac{\mathbf{q}^{\ell_J} \psi(e_{J,+})}{\det(\mathbf{q} - e_{J,+} \mid \mathbf{V}_G)} \right],$$

where  $\mathbf{V}_G$  is the representation of  $W$  on the (rational) cocharacters of  $\mathbf{T}$ .

**1.4. Cell Decompositions.** Our second application of Theorem 1.1 is to provide point-counting formulas for new kinds of braid varieties through Deodhar-type decompositions. In what follows,  $h\mathbf{B} \xrightarrow{w} g\mathbf{B}$  will mean  $\mathbf{B}h^{-1}g\mathbf{B} = \mathbf{B}w\mathbf{B}$ .

Let  $\vec{s} = (s^{(1)}, s^{(2)}, \dots, s^{(\ell)})$  be a word in  $S$ . Recall that in [Deo85], Deodhar showed how to decompose a certain *Richardson variety* for  $\vec{s}$  into subvarieties of the form  $\mathbf{A}^{\mathbf{d}} \times \mathbf{G}_m^{\mathbf{e}}$ , now called *Deodhar cells*. As in [GLTW24], we will work with a variant definition depending on an element  $v \in W$ :

$$\mathbf{R}^{(v)}(\vec{s}) = \{\vec{g}\mathbf{B} = (g_i\mathbf{B})_i \in (\mathbf{G}/\mathbf{B})^\ell \mid vw_o\mathbf{B} \xrightarrow{s^{(1)}} g_1\mathbf{B} \xrightarrow{s^{(2)}} \dots \xrightarrow{s^{(\ell)}} g_\ell\mathbf{B} \xleftarrow{vw_o} \mathbf{B}\}.$$

To describe the cell decomposition, recall that a *subword* of  $\vec{s}$  is a sequence  $\vec{\omega}$  of the same length with  $\omega^{(i)} \in \{e, s^{(i)}\}$  for all  $i$ . We set  $\omega_{(i)} := \omega^{(1)} \dots \omega^{(i)}$ . For any  $v \in W$ , a *v-distinguished subword* of  $\vec{s}$  is a subword  $\vec{\omega}$  such that

$$v\omega_{(i)} \leq v\omega_{(i-1)}s^{(i)} \quad \text{for all } i.$$

Let  $\mathcal{D}^{(v)}(\vec{s})$  be the set of  $v$ -distinguished subwords  $\vec{\omega}$  for which  $\omega_{(\ell)} = e$ . Then the Deodhar cells of  $\mathbf{R}^{(v)}(\vec{s})$  are indexed by  $\mathcal{D}^{(v)}(\vec{s})$ . The cell for a given element  $\vec{\omega}$  is isomorphic to  $\mathbf{A}^{\mathbf{d}_{\vec{\omega}}} \times \mathbf{G}_m^{\mathbf{e}_{\vec{\omega}}}$  for certain disjoint subsets  $\mathbf{d}_{\vec{\omega}}, \mathbf{e}_{\vec{\omega}} \subseteq \{1, 2, \dots, \ell\}$ , allowing us to count  $R^{(v)}(\vec{s}) := \mathbf{R}^{(v)}(\vec{s})^F$ :

$$(1.3) \quad |R^{(v)}(\vec{s})| = \sum_{\vec{\omega} \in \mathcal{D}^{(v)}(\vec{s})} q^{|\mathbf{d}_{\vec{\omega}}|} (q-1)^{|\mathbf{e}_{\vec{\omega}}|}.$$

We give further detail in Section 5.

Using Theorem 1.1, we relate the disjoint union of the sets  $R^{(v)}(\vec{s})$  for  $v \in W^{J,\mp}$  to the set  $Z_J^\pm(\vec{s}) := \mathbf{Z}_J^\pm(\vec{s})^F$  for a certain variety

$$\mathbf{Z}_J^\pm(\vec{s}) = \{(\vec{g}\mathbf{B}, u, y\mathbf{P}_J) \in (\mathbf{G}/\mathbf{B})^\ell \times \mathbf{Spr}_J^\pm \mid u^{-1}g_\ell\mathbf{B} \xrightarrow{s^{(1)}} g_1\mathbf{B} \xrightarrow{s^{(2)}} \dots \xrightarrow{s^{(\ell)}} g_\ell\mathbf{B}\}.$$

Note the sign flip above, which arises because the element  $w_o$  in the formula for  $\mathbf{R}^{(v)}(\vec{s})$  interchanges  $W^{J,-}$  with  $W^{J,+}$ . We obtain identities of point counts:

**Theorem 1.4.** *For any word  $\vec{s}$ , we have*

$$\begin{aligned} \frac{|Z_J^-(\vec{s})|}{|G|} &= \frac{1}{q^{\ell_J}(q-1)^{\text{rk}(G)}} \sum_{v \in W^{J,+}} \sum_{\vec{\omega} \in \mathcal{D}^{(v)}(\vec{s})} q^{|\mathbf{d}_{\vec{\omega}}|} (q-1)^{|\mathbf{e}_{\vec{\omega}}|}, \\ \frac{|Z_J^+(\vec{s})|}{|G|} &= \frac{1}{(q-1)^{\text{rk}(G)}} \sum_{v \in W^{J,-}} \sum_{\vec{\omega} \in \mathcal{D}^{(v)}(\vec{s})} q^{|\mathbf{d}_{\vec{\omega}}|} (q-1)^{|\mathbf{e}_{\vec{\omega}}|}. \end{aligned}$$

(Note the sign flip between the left and right sides of each identity.)

Note that  $\mathbf{Z}_\emptyset^+(\vec{s})$  and  $\mathbf{Z}_\emptyset^-(\vec{s})$  coincide: They match the *braid Steinberg variety* introduced in [Tri21]. At the other extreme,  $\mathbf{Z}_S^+(\vec{s})$  and  $\mathbf{Z}_S^-(\vec{s})$  are the varieties respectively denoted  $\mathcal{U}(\vec{s})$  and  $\mathcal{X}(\vec{s})$  in *ibid*.

For  $G = \text{PGL}_n(\mathbf{F})$ , the variety  $\mathcal{X}(\vec{s})$  was studied earlier by Shende–Treumann–Zaslow [STZ17], who used contact geometry to construct a decomposition of  $\mathcal{X}(\vec{s})$

resembling Deodhar's. For general  $J$ , we exhibit a decomposition of  $\mathbf{Z}_J^-(\vec{s})$  into varieties equivariantly cohomologous to Deodhar cells: See Corollary 5.9. It appears to specialize to the decomposition in [STZ17], as we explain in Remark 5.10.

**1.5. Combinatorics.** Our third application of Theorem 1.1, by way of Theorem 1.2, is to construct noncrossing sets of interest in the Catalan combinatorics of  $(W, S)$ . *In the rest of this introduction,  $W$  is irreducible with Coxeter number  $h$ .*

Let  $d_1, \dots, d_{|S|}$  be the fundamental degrees of the action of  $W$  on its (irreducible) reflection representation. For each  $i$ , let  $e_i = d_i - 1$ . For any positive integer  $p$  coprime to  $h$ , the *rational Catalan number* of  $(W, p)$  is

$$\text{Cat}_{W,p} := \prod_i \frac{p + e_i}{d_i},$$

while the *rational parking number* of  $(W, p)$  is  $p^{|S|}$ . These numbers enumerate disparate families of combinatorial objects. Most are constructed from root-theoretic data generalizing nonnesting partitions and parking functions, respectively. The collective study of these families and the bijections between them is the “nonnesting” side of rational Catalan/parking combinatorics. In [GLTW24], we instead sought, and constructed, “noncrossing” families: those depending on a chosen ordering of  $S$ , or *Coxeter word*.

For any word  $\vec{s}$  in  $S$ , let  $\mathcal{M}^{(v)}(\vec{s}) \subseteq \mathcal{D}^{(v)}(\vec{s})$  be the subset of elements  $\vec{w}$  such that  $|\mathbf{e}_{\vec{w}}| = |S|$ , the minimum possible value [GLTW24, Cor. 4.9]. Let  $\vec{c}$  be a Coxeter word for  $(W, S)$ , and  $\vec{c}^p$  its  $p$ -fold concatenation. The main results of [GLTW24] are the identities

$$\text{Cat}_{W,p} = |\mathcal{M}^{(e)}(\vec{c}^p)| \quad \text{and} \quad p^{|S|} = \sum_{v \in W} |\mathcal{M}^{(v)}(\vec{c}^p)|,$$

proved by way of  $\mathbf{q}$ -deformed identities involving  $\mathcal{D}^{(v)}(\vec{c}^p)$  and taking  $\mathbf{q} \rightarrow 1$ .

In Section 6, we prove an identity that interpolates between the two above. Let  $d_1^J, \dots, d_{|J|}^J$  be the fundamental degrees of  $W_J$ . Let  $e_1^J, \dots, e_{|J|}^J$  be the exponents of the  $W_J$ -action on the reflection representation of  $W$ . We define the *rational parabolic parking numbers* of  $(W, p, J)$  to be

$$\text{Park}_{W,p}^{J,\pm} = \prod_i \frac{p \pm e_i^J}{d_i^J}.$$

Then  $\text{Park}_{W,p}^{S,+} = \text{Cat}_{W,p}$  and  $\text{Park}_{W,p}^{\emptyset,+} = \text{Park}_{W,p}^{\emptyset,-} = p^{|S|}$ . We relate these numbers to  $\tau_G$  via a result from [Tri21], which describes  $\tau_G((\ ) \otimes T_{\vec{c}}^p)$  for a certain  $T_{\vec{c}} \in H_W$  as the graded character of a *rational parking space* for  $(W, p)$ , in the sense of [ARR15] and [ALW16]. Ultimately, we obtain:

**Corollary 1.5.** *For any Coxeter word  $\vec{c}$ , integer  $p > 0$  coprime to  $h$ , and subset  $J \subseteq S$ , we have*

$$\text{Park}_{W,p}^{J,\pm} = \sum_{v \in W^{J,\mp}} |\mathcal{M}^{(v)}(\vec{c}^p)|.$$

(Note the sign flip.) That is,  $\coprod_{v \in W^{J, \pm}} \mathcal{M}^{(v)}(\vec{\mathcal{P}})$  is a  $\vec{c}$ -noncrossing set enumerated by the  $\mp$ -rational parabolic parking number of  $(W, p, J)$ .

**1.6.  $a$ -Degrees in Markov Traces.** In Section 7, we prove results about Markov traces and rational Kirkman numbers that are respectively parallel to Theorem 1.2 and Corollary 1.5.

First, for any  $v \in W$ , recall the *left ascent set*  $\text{Asc}(v) = \{s \in S \mid \ell(sv) > \ell(v)\}$  and *descent set*  $\text{Des}(v) = \{s \in S \mid \ell(sv) < \ell(v)\}$ . Observe that  $W^{J, -}$ , *resp.*  $W^{J, +}$ , consists of those  $v$  such that  $\text{Asc}(v) \supseteq J$ , *resp.*  $\text{Des}(v) \supseteq J$ . Hence,  $N_J^S(1)$  and  $\mathbf{q}^{-\ell_J} N_J^S(T_{w_{J_0}}^2)$  decompose as sums, over supersets  $I \supseteq J$ , of elements

$$\zeta_I^+ := \sum_{\text{Asc}(v)=I} \mathbf{q}^{-\ell(v)} T_{v^{-1}} T_v \quad \text{and} \quad \zeta_I^- := \sum_{\text{Des}(v)=I} \mathbf{q}^{-\ell(v)} T_{v^{-1}} T_v.$$

Note that  $\zeta_S^+ = \zeta_\emptyset^- = 1$  and  $\zeta_\emptyset^+ = \zeta_S^- = \mathbf{q}^{-\ell_S} T_{w_\circ}^2$ . By inclusion-exclusion, the elements  $\zeta_I^\pm$  are again central in  $H_W$ .

**Question 1.6.** For general  $W$  and  $I$ , is there a more familiar description of the traces on  $H_W$  of the form  $\tau[\zeta_I^\pm]$ ?

We now take  $W = S_n$  and  $S = \{s_1, \dots, s_{n-1}\}$ . The HOMFLYPT Markov trace on  $H_{S_n}$  can be written as a  $\mathbf{Q}(\mathbf{q}^{1/2})[a^{\pm 1}]$ -valued trace  $\mu_n$ . For  $0 \leq k \leq n-1$ , let  $\mu_n^{(k)} : H_W \rightarrow \mathbf{Q}(\mathbf{q}^{1/2})$  be the coefficient of the  $k$ th highest power of  $a$  in  $\mu_n$ , and let

$$I_k = \{s_1, s_2, \dots, s_{n-1-k}\}.$$

Using work of Bezrukavnikov–Tolmachov [BT22, Cor. 6.1.2], we show:

**Theorem 1.7.** *For any integer  $k$ , we have*

$$(1.4) \quad \tau[\zeta_{I_k}^-] = (\mathbf{q} - 1)^{n-1} \mu_n^{(k)}$$

as traces on  $H_{S_n}$ .

Let  $e_{\wedge^k} \in \mathbf{Q}W$  be the Young symmetrizer of the hook partition  $(n-k, 1, \dots, 1) \vdash n$ , which indexes the  $k$ th exterior power of the reflection representation of  $S_n$ . By combining (1.4) with the result in [Tri21] relating the Markov trace to  $\tau_G$ , we deduce this analogue of Theorem 1.2:

**Corollary 1.8.** *For  $G$  split semisimple of type  $A_{n-1}$ , and any integer  $k$ , we have*

$$\tau[\zeta_{I_k}^-] = (\mathbf{q} - 1)^{n-1} \tau_G(e_{\wedge^k} \otimes ( \quad ))$$

as traces on  $H_{S_n}$ .

For general  $W$  and  $0 \leq k \leq |S| - 1$ , we can use the rational parking space for  $(W, p)$  mentioned earlier to define rational generalizations  $\text{Kirk}_{W,p}^{(k)}$  of the Kirkman numbers studied in [ARR15]. For  $W = S_n$ , the preceding result implies this analogue of Corollary 1.5:

**Corollary 1.9.** *Take  $W = S_n$  and  $S = \{s_1, \dots, s_{n-1}\}$ . Then for any Coxeter word  $\vec{c}$ , any integer  $p > 0$  coprime to  $n$  and integer  $k$ , we have*

$$\text{Kirk}_{W,p}^{(k)} = \sum_{\text{Asc}(v)=I_k} |\mathcal{M}^{(v)}(\vec{c}^p)|.$$

*That is,  $\coprod_{\text{Asc}(v)=I_k} \mathcal{M}^{(v)}(\vec{c}^p)$  is a  $\vec{c}$ -noncrossing set enumerated by the  $k$ th rational Kirkman number of  $(W, p)$ .*

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## 2. THE GEOMETRIC HECKE ALGEBRA

**2.1.** In this section, we review the general definition of the convolution algebra  $H_B^G$  without assuming  $G$  to be split, following [Car95, §3.3]. At the end, we explain how to adapt  $N_J^S$  to this generality. Along the way, we review Bruhat decomposition and related facts about Borel subgroups. Our geometric setup is essentially Kawanaka's in [Kaw75]. See also [Car93, Car95].

We keep  $\mathbf{F}$ ,  $q$ ,  $\mathbf{G}$ ,  $\mathbf{B}$ ,  $\mathbf{T}$ ,  $\mathbf{W}$  as in §1.2. Let  $S_{\mathbf{B}}$  be the system of simple reflections of  $\mathbf{W}$  arising from  $\mathbf{B}$ , and let  $\ell_{\mathbf{B}}$  be the Bruhat length function on  $\mathbf{W}$  defined by  $S_{\mathbf{B}}$ .

**2.2. Bruhat Decomposition.** Note that  $w\mathbf{B}$  and  $\mathbf{B}w$  are well-defined for any  $w \in \mathbf{W}$ . Bruhat decomposition says that as we run over all  $w$ , the double cosets  $\mathbf{B}w\mathbf{B}$  are pairwise disjoint and partition  $\mathbf{G}$ .

Let  $\mathbf{U}$  be the unipotent radical of  $\mathbf{B}$ , so that  $\mathbf{B} = \mathbf{T} \ltimes \mathbf{U}$ . Let  $\mathbf{U}_-$  be the unipotent radical of the opposed Borel  $\mathbf{B}_-$ . Note that  $w\mathbf{U}w^{-1}$  and  $w\mathbf{U}_-w^{-1}$  are well-defined for all  $w \in \mathbf{W}$ . Let

$$\begin{aligned} \mathbf{U}_w &= \mathbf{U} \cap w\mathbf{U}w^{-1}, \\ \mathbf{U}_w^- &= \mathbf{U} \cap w\mathbf{U}_-w^{-1}. \end{aligned}$$

Then  $\mathbf{U}_w, \mathbf{U}_w^-$  are stable under the conjugation action of  $\mathbf{T}$  on  $\mathbf{U}$ . The following results are proved in [Car93, §2.5]:

**Lemma 2.1.** *For all  $w \in \mathbf{W}$ :*

- (1) *If  $\ell_{\mathbf{B}}(wv) = \ell_{\mathbf{B}}(w) + \ell_{\mathbf{B}}(v)$ , then  $\mathbf{U}_{wv}^- = \mathbf{U}_w^- \mathbf{U}_v^-$ , and  $\mathbf{U}_w^- \cap \mathbf{U}_v^- = \{1\}$ .*
- (2)  *$\mathbf{U} = \mathbf{U}_w \mathbf{U}_w^- = \mathbf{U}_w^- \mathbf{U}_w$ , and  $\mathbf{U}_w \cap \mathbf{U}_w^- = \{1\}$ .*
- (3)  *$\mathbf{B}w\mathbf{B} = \mathbf{U}_w^- w \mathbf{B}$ , and the map  $\mathbf{U}_w^- \rightarrow \mathbf{U}_w^- w \mathbf{B} / \mathbf{B}$  is an isomorphism.*
- (4) *As an algebraic variety (but not group),  $\mathbf{U}_w^-$  is the product of the root subgroups inverted by  $w$ , hence an affine space of dimension  $\ell_{\mathbf{B}}(w)$ .*



**2.3. Bott–Samelson Varieties.** The double cosets of  $\mathbf{B}$  in  $\mathbf{G}$  are in bijection with the set of diagonal  $\mathbf{G}$ -orbits on  $(\mathbf{G}/\mathbf{B})^2$ . As in the introduction, we write  $h\mathbf{B} \xrightarrow{w} g\mathbf{B}$  to mean  $\mathbf{B}h^{-1}g\mathbf{B} = \mathbf{B}w\mathbf{B}$ . Such pairs  $(h\mathbf{B}, g\mathbf{B})$  form the points of the  $\mathbf{G}$ -orbit of  $(\mathbf{G}/\mathbf{B})^2$  corresponding to  $w$ , which we will denote by  $\mathbf{O}(w)$ .

More generally, for any sequence of elements  $\vec{w} = (w^{(1)}, w^{(2)}, \dots, w^{(k)})$  in  $\mathbf{W}$ , let  $\mathbf{O}(\vec{w})$  be the subvariety of  $(\mathbf{G}/\mathbf{B})^{1+k}$  defined on points by

$$\mathbf{O}(\vec{w}) = \{\vec{g}\mathbf{B} = (g_0\mathbf{B}, g_1\mathbf{B}, \dots, g_k\mathbf{B}) \mid g_0\mathbf{B} \xrightarrow{w^{(1)}} g_1\mathbf{B} \xrightarrow{w^{(2)}} \dots \xrightarrow{w^{(k)}} g_k\mathbf{B}\}.$$

The Zariski closure of  $\mathbf{O}(\vec{w})$  is called the *Bott–Samelson variety* of  $\vec{w}$ . For this reason,  $\mathbf{O}(\vec{w})$  may be called the *open Bott–Samelson variety*.

For any subset  $I \subseteq \{1, \dots, k\}$ , we write  $pr_I : \mathbf{O}(\vec{w}) \rightarrow (\mathbf{G}/\mathbf{B})^I$  to denote the map that sends  $pr_I(\vec{g}\mathbf{B}) = (g_i\mathbf{B})_{i \in I}$ . When writing out  $\vec{w}$ , *resp.*  $I$ , explicitly, we will omit the parentheses, *resp.* brackets, where convenient.

Lemma 2.1 implies that if  $\ell_{\mathbf{B}}(wv) = \ell_{\mathbf{B}}(w) + \ell_{\mathbf{B}}(v)$ , then  $pr_{0,2}$  induces an explicit isomorphism  $\mathbf{O}(w, v) \xrightarrow{\sim} \mathbf{O}(wv)$ . By induction, any variety of the form  $\mathbf{O}(\vec{w})$  is explicitly isomorphic to one of the form  $\mathbf{O}(\vec{s})$ , where  $\vec{s}$  is a word in  $S_{\mathbf{B}}$ .

**2.4. Frobenius Maps.** For algebraic varieties over  $\bar{\mathbf{F}}$  equipped with Frobenius maps, we use italics to denote the corresponding sets of Frobenius-fixed points.

As in §1.2, we fix a Frobenius map  $F : \mathbf{G} \rightarrow \mathbf{G}$  arising from an  $\mathbf{F}$ -form, such that  $\mathbf{B}$  and  $\mathbf{T}$  are  $F$ -stable. Then  $\mathbf{W}$  and  $S_{\mathbf{B}}$  are also  $F$ -stable. The group  $W := \mathbf{W}^F$  is again a Coxeter group, which can be identified with  $N_G(T)/T$ .

*Remark 2.2.* When  $\mathbf{G}$  is almost-simple, the options for  $G$  and  $W$  are listed in [Car95, §1.5–1.6]. Notably,  $W$  is crystallographic except when it has factors of type  ${}^2F_4$ .

There is a system of simple reflections for  $W$ , which we will denote  $S$ , indexed by the  $F$ -orbits on  $S_{\mathbf{B}}$ : Each element  $s \in S$  is the product of all the elements in the given  $F$ -orbit, which pairwise commute and form a reduced word in  $S_{\mathbf{B}}$  in any order. Let  $\ell$  be the Bruhat length function on  $W$  defined by  $S$ .

By Lang’s theorem,  $g\mathbf{B}$  is  $F$ -stable if and only if  $g \in G$ , and in this case,  $g\mathbf{B} = (x\mathbf{B})^F$ . Similarly,  $\mathbf{B}w\mathbf{B}$  is  $F$ -stable if and only if  $w \in W$ , and in this case,  $BwB = (\mathbf{B}w\mathbf{B})^F$ . Thus, the double cosets  $BwB$  for  $w \in W$  partition  $G$ , while the  $G$ -orbits on  $(G/B)^2$  are the sets  $O(w)$  for  $w \in W$ . As explained in [Car93], parts (1)–(3) of Lemma 2.1 have exact analogues with  $\mathbf{W}$  replaced by  $W$ . See also [Kaw75, §1].

**Lemma 2.3.** *For all  $w \in W$ :*

- (1) *If  $\ell(wv) = \ell(w) + \ell(v)$ , then  $U_{wv}^- = U_w^- U_v^-$ , and  $U_w^- \cap U_v^- = \{1\}$ .*
- (2)  *$U = U_w U_w^- = U_w^- U_w$ , and  $U_w \cap U_w^- = \{1\}$ .*
- (3)  *$BwB = U_w^- wB$ , and the map  $U_w^- \rightarrow U_w^- wB/B$  is a bijection.*

The one point where caution is needed concerns the sizes of  $U_w$  and  $U_w^-$ , as they involve  $\ell_{\mathbf{B}}(w)$ , not  $\ell(w)$  [Car93, 74]:

**Lemma 2.4.** *For all  $w \in W$ , we have  $|U_w^-| = q^{\dim(\mathbf{U}_w^-)} = q^{\ell_{\mathbf{B}}(w)}$ .*

**2.5. Operations on Functions.** For any finite set  $X$  equipped with the action of a finite group  $G$ , we write  $\mathcal{C}_G(X)$  to denote the free module of  $\mathbf{Z}$ -valued,  $G$ -invariant functions on  $X$ . For any  $G$ -stable subset  $Y \subseteq X$ , we write  $\delta_Y \in \mathcal{C}_G(X)$  to denote the indicator function on  $Y$ .

For a  $G$ -equivariant map  $f : Y \rightarrow X$ , the *pullback* of functions along  $f$  is the linear map  $f^* : \mathcal{C}_G(X) \rightarrow \mathcal{C}_G(Y)$  given by  $f^*(\varphi)(y) = \varphi(f(y))$ . The *pushforward*, or *integral*, of functions along  $f$  is the linear map  $f_! : \mathcal{C}_G(Y) \rightarrow \mathcal{C}_G(X)$  given by

$$f_!(\psi)(x) = \sum_{y \in f^{-1}(x)} \psi(y).$$

When  $f$  can be understood from context, we omit  $f_!$  from our notation.

Let  $*$  denote the *convolution product* on  $\mathcal{C}(X \times X)$  defined in terms of the three projection maps  $pr_{i,j} : X^3 \rightarrow X^2$  by

$$\varphi_1 * \varphi_2 = pr_{1,3,!}(pr_{1,2}^*(\varphi_1) \cdot pr_{2,3}^*(\varphi_2)),$$

where  $\cdot$  denotes pointwise multiplication. Explicitly,

$$(\varphi_1 * \varphi_2)(y, x) = \sum_{z \in X} \varphi_1(y, z) \varphi_2(z, x).$$

The indicator function of the diagonal  $X \subseteq X^2$  is the identity element for this operation. If  $X$  is equipped with a  $G$ -action, and  $G$  acts on  $X^2$  diagonally, then  $*$  restricts to an operation on  $\mathcal{C}_G(X \times X)$  with the same identity element.

Iwahori proved that the ring formed by  $\mathcal{C}_G(G/B \times G/B)$  under convolution is freely generated by the elements  $\delta_w := \delta_{O(w)}$  for  $w \in W$  modulo the following relations for all  $w \in W$  and  $s \in S$ :

$$\delta_s * \delta_w = \begin{cases} \delta_{sw} & \ell(sw) > \ell(w), \\ |U_s^-| \delta_{sw} + (|U_s^-| - 1) \delta_w & \ell(sw) < \ell(w). \end{cases}$$

See [Car95, §3.3] or [Kaw75, Thm. 2.6]. We define  $H_B^G$  to be the  $\mathbf{Z}[\frac{1}{q}]$ -algebra

$$H_B^G = \mathcal{C}_G(G/B \times G/B)[\frac{1}{q}].$$

If  $G$  is *split*, meaning  $W = \mathbf{W}$ , then  $\ell_{\mathbf{B}}(s) = \ell(s) = 1$  and  $|U_s^-| = q$  for all  $s \in S$ . This is the case on which the introduction focused. Here,  $W$  is crystallographic, and  $H_B^G$  is a specialization of the  $\mathbf{Z}[\mathbf{q}^{\pm 1}]$ -algebra  $H_W$  freely generated by elements  $T_w$  for  $w \in W$  modulo the following relations for all  $w \in W$  and  $s \in S$ :

$$T_s T_w = \begin{cases} T_{sw} & \ell(sw) = \ell(w) + 1, \\ \mathbf{q} T_{sw} + (\mathbf{q} - 1) T_w & \ell(sw) = \ell(w) - 1. \end{cases}$$

**2.6. Parabolic Subgroups.** Fix an  $F$ -stable subset  $J_{\mathbf{B}} \subseteq S_{\mathbf{B}}$ , corresponding to a subset  $J \subseteq S$ . Let  $\mathbf{W}_J \subseteq \mathbf{W}$ , *resp.*  $W_J \subseteq W$ , be the subgroup generated by  $J_{\mathbf{B}}$ , *resp.*  $J$ . Then  $\mathbf{W}_J$  is  $F$ -stable and  $W_J = \mathbf{W}_J^F$ .

Let  $\mathbf{P}_J = \mathbf{B}\mathbf{W}_J\mathbf{B} \subseteq \mathbf{G}$ . We can write  $\mathbf{P}_J = \mathbf{L}_J \ltimes \mathbf{U}_J$ , where  $\mathbf{L}_J$  is reductive with Weyl group  $\mathbf{W}_J$  and  $\mathbf{U}_J$  the unipotent radical of  $\mathbf{P}_J$ . These subgroups are  $F$ -stable, and on  $F$ -fixed points, we have  $P_J = L_J \ltimes U_J$ .

By construction,  $\mathbf{B}_J = \mathbf{L}_J \cap \mathbf{B}$  is a Borel subgroup of  $\mathbf{L}_J$ . The inclusion  $L_J \subseteq P_J$  descends to an  $L_J$ -equivariant bijection  $L_J/B_J \simeq P_J/B$ , which in turn yields an isomorphism of algebras

$$\mathcal{C}_{L_J}(L_J/B_J \times L_J/B_J) \simeq \mathcal{C}_{L_J}(P_J/B \times P_J/B).$$

Once we adjoin  $\frac{1}{q}$ , the left-hand side becomes  $H_{B_J}^{L_J}$ , and the right-hand side becomes the subalgebra of  $H_B^G$  generated by the elements  $\delta_w$  with  $w \in W_J$ . Henceforth, we identify these  $\mathbf{Z}[\frac{1}{q}]$ -algebras with each other.

As in the introduction, let  $W^{J,-} \subseteq W$  be the set of minimal-length right coset representatives for  $W_J$ . By Lemma 2.1(4) and Lemma 2.4, the split case of the definition below recovers the  $\mathbf{q} \rightarrow q$  specialization of the relative norm map in §1.1.

**Definition 2.5.** The *relative norm* map  $N_J^S : H_{B_J}^{L_J} \rightarrow H_B^G$  is defined by

$$N_J^S(\alpha) = \sum_{v \in W^{J,-}} \frac{1}{|U_v^-|} \delta_{v^{-1}} * \alpha * \delta_v.$$

We have implicitly used Lemma 2.4 to ensure that  $|U_v^-|$  is a power of  $q$ .

2.7. Let  $w_\circ$  and  $w_{J_\circ}$  respectively denote the longest elements of  $W$  and  $W_J$  with respect to  $S$ . Then  $U = U_{w_\circ}$  and  $U_J = U_{w_{J_\circ}}$ . The following fact will be useful:

**Lemma 2.6.** For any  $J \subseteq S$  and  $v \in W^{J,-}$ , we have

$$U_J \cap U_v = U_{w_{J_\circ}v} \quad \text{and} \quad U_J \cap U_v^- = U_v^-.$$

In particular,  $U_J = U_{w_{J_\circ}v}U_v^- = U_v^-U_{w_{J_\circ}v}$  and  $U_{w_{J_\circ}v} \cap U_v^- = \{1\}$ . In the split case, the analogous identities hold with  $U_J$ ,  $U_v$ , etc. in place of  $U_J$ ,  $U_v$ , etc..

*Proof.* To show  $U_J \cap U_v = U_{w_{J_\circ}v}$ : In general, if  $w, v \in W$  satisfy  $\ell(wv) = \ell(w) + \ell(v)$ , then  $U_{wv}^- = U_w^-U_v^-$  and  $U_w^- \cap U_v^- = \{1\}$  by Lemma 2.3(1), which implies that  $U_{wv} = U_w \cap U_v$  by Lemma 2.3(2).

To show  $U_J \cap U_v^- = U_v^-$ , meaning  $U_v^- \subseteq U_J$ : In general, if  $w \in W_J$  and  $v \in W^{J,-}$ , then the  $F$ -orbits of root subgroups of  $\mathbf{U}_J$  inverted by  $wv$  are precisely those inverted by  $w$ . Taking  $w = e$  gives the result.

In the split case,  $\ell_{\mathbf{B}} = \ell$ , and thus,  $v$  minimizes  $\ell_{\mathbf{B}}$  in  $\mathbf{W}_J v$ . So we can repeat all the arguments above with the varieties in place of the sets.  $\square$

### 3. PARTIAL SPRINGER RESOLUTIONS

3.1. Recall the partial Springer resolutions  $\mathbf{Spr}_J^\pm \subseteq \mathbf{G} \times \mathbf{G}/\mathbf{P}_J$  and the varieties  $\mathbf{E}_J^\pm = \mathbf{G}/\mathbf{B} \times \mathbf{Spr}_J^\pm$  from §1.2. The latter are stable under the left  $\mathbf{G}$ -action on  $\mathbf{G}/\mathbf{B} \times \mathbf{G} \times \mathbf{G}/\mathbf{P}_J$  defined by

$$(3.1) \quad g \cdot (h\mathbf{B}, u, y\mathbf{P}_J) = (gh\mathbf{B}, gug^{-1}, gy\mathbf{P}_J).$$

Let  $f : \mathbf{G}/\mathbf{B} \times \mathbf{G} \times \mathbf{G}/\mathbf{P}_J \rightarrow (\mathbf{G}/\mathbf{B})^2$  be the  $\mathbf{G}$ -equivariant map defined by

$$f(h\mathbf{B}, u, y\mathbf{P}_J) = (h\mathbf{B}, uh\mathbf{B}).$$

On  $F$ -fixed points, it restricts to  $G$ -equivariant maps  $f : E_J^\pm \rightarrow (G/B)^2$ . These recover the maps  $f$  in §1.2. The goal of this section is to prove the identities

$$(3.2) \quad \begin{aligned} f_! \delta_{E_J^-} &= |U_J| N_J^S(1), \\ f_! \delta_{E_J^+} &= |U_J| N_J^S(\delta_{w_{J\circ}}^2), \end{aligned}$$

where  $N_J^S$  is now given by Definition 2.5. They recover Theorem 1.1 in the split case.

**3.2. Reduction to Strata.** Observe that  $\mathbf{E}_J^\pm$  is a union of  $\mathbf{G}$ -stable subvarieties  $\mathbf{E}_{J,v}^\pm$  for  $\mathbf{W}_J v \in \mathbf{W}_J \backslash \mathbf{W}$ , where on points,

$$\mathbf{E}_{J,v}^\pm = \{(h\mathbf{B}, u, y\mathbf{P}_J) \in \mathbf{G}/\mathbf{B} \times \mathbf{Spr}_J^\pm \mid \mathbf{P}_J y^{-1} h\mathbf{B} = \mathbf{P}_J v \mathbf{B}\}.$$

From §2.4, we see that  $\mathbf{P}_J v \mathbf{B}$  is  $F$ -stable if and only if  $v \in W$ , and in this case,  $\mathbf{P}_J v \mathbf{B} = (\mathbf{P}_J v \mathbf{B})^F$ . Therefore,  $E_J^\pm$  is the union of its  $G$ -stable subsets  $E_{J,v}^\pm$  as  $v$  runs over a full set of right coset representatives for  $W_J$ : for instance,  $W^{J,-}$ . As Lemma 2.6 shows that  $U_J \simeq U_{w_{J\circ}v} \times U_v^-$ , we reduce (3.2) to:

**Theorem 3.1.** *If  $v \in W^{J,-}$ , then:*

- (1)  $f_! \delta_{E_{J,v}^-} = |U_{w_{J\circ}v}| \delta_{v^{-1}} * \delta_v$ .
- (2)  $f_! \delta_{E_{J,v}^+} = |U_{w_{J\circ}v}| \delta_{v^{-1}} * \delta_{w_{J\circ}}^2 * \delta_v$ .

**3.3. Reduction to the Borel.** Let  $\check{\mathbf{E}}_{J,v}^\pm \subseteq \mathbf{G}/\mathbf{B} \times \mathbf{G} \times \mathbf{G}/\mathbf{B}$  be the subvariety defined on points by

$$\check{\mathbf{E}}_{J,v}^\pm = \{(h\mathbf{B}, u, y\mathbf{B}) \mid (u, y\mathbf{B}) \in \mathbf{Spr}_J^\pm \text{ and } y\mathbf{B} \xrightarrow{v} h\mathbf{B}\}$$

The forgetful map  $\mathbf{G}/\mathbf{B} \rightarrow \mathbf{G}/\mathbf{P}_J$  induces a map  $\check{\mathbf{E}}_{J,v}^\pm \rightarrow \mathbf{E}_{J,v}^\pm$ .

**Lemma 3.2.** *If  $v \in W^{J,-}$ , then  $\check{E}_{J,v}^\pm \rightarrow E_{J,v}^\pm$  is a bijection. In the split case, this bijection arises from an isomorphism  $\check{\mathbf{E}}_{J,v}^\pm \rightarrow \mathbf{E}_{J,v}^\pm$ .*

*Proof.* The first claim is just the fact that if  $v$  minimizes  $\ell$  in  $W_J v$ , then there are compatible bijections from  $U_v^-$  to the Schubert cells  $BvB/B$  and  $BvP_J/P_J$ .

For the second claim: As in the proof of Lemma 2.6,  $v$  minimizes  $\ell_{\mathbf{B}}$  in  $\mathbf{W}_J v$ . So we can repeat the argument above, but with the varieties  $\mathbf{U}_v^-, \mathbf{B}, \mathbf{P}_J$  in place of the sets  $U_v^-, B, P_J$ , and isomorphisms in place of bijections.  $\square$

The varieties  $\check{\mathbf{E}}_J^\pm$  are stable under the  $\mathbf{G}$ -action on  $\mathbf{G}/\mathbf{B} \times \mathbf{G} \times \mathbf{G}/\mathbf{B}$  analogous to (3.1). Let  $\check{f} : \mathbf{G}/\mathbf{B} \times \mathbf{G} \times \mathbf{G}/\mathbf{B} \rightarrow (\mathbf{G}/\mathbf{B})^3$  be the equivariant map defined by

$$\check{f}(h\mathbf{B}, u, y\mathbf{B}) = (h\mathbf{B}, y\mathbf{B}, uh\mathbf{B}).$$

The proofs of the two parts of Theorem 3.1 will use  $\check{f}$  in different ways.

3.4. **Proof of (1).** In the notation of Section 2,

$$pr_{0,2,!}\delta_{O(v^{-1},v)} = \delta_{v^{-1}} * \delta_v.$$

This suggests comparing  $\mathbf{E}_{J,v}^-$  to a bundle over  $\mathbf{O}(v^{-1}, v)$ . It turns out that  $\check{\mathbf{E}}_{J,v}^-$  is the bundle we seek.

Observe that if  $(h\mathbf{B}, u, y\mathbf{B})$  is a point of  $\check{\mathbf{E}}_{J,v}^-$ , then  $\mathbf{B}y^{-1}uh\mathbf{B} = \mathbf{B}y^{-1}h\mathbf{B} = \mathbf{B}v\mathbf{B}$ . Therefore,  $\check{f}$  restricts to a map from  $\check{\mathbf{E}}_{J,v}^-$  into  $\mathbf{O}(v^{-1}, v)$ , giving an equivariant commutative diagram:

$$\begin{array}{ccc} \check{\mathbf{E}}_{J,v}^- & \longrightarrow & \mathbf{E}_{J,v}^- \\ \check{f} \downarrow & & \searrow f \\ \mathbf{O}(v^{-1}, v) & & \\ pr_{0,2} \downarrow & \swarrow & \\ (\mathbf{G}/\mathbf{B})^2 & & \end{array}$$

By Lemma 3.2 and this diagram, we reduce case (1) of Theorem 3.1 to:

**Proposition 3.3.** *If  $v \in W^{J,-}$ , then*

$$\check{f}_! \delta_{\check{\mathbf{E}}_{J,v}^-} = |U_{w_{J_0}v}| \delta_{O(v^{-1},v)}$$

in  $\mathcal{C}_G(O(v^{-1}, v))$ . In the split case, this identity arises from  $\check{f} : \check{\mathbf{E}}_{J,v}^- \rightarrow \mathbf{O}(v^{-1}, v)$  being a smooth fiber bundle that restricts to a  $\mathbf{U}_{w_{J_0}v}$ -torsor over the subvariety of  $\mathbf{O}(v^{-1}, v)$  where  $(g_0\mathbf{B}, g_1\mathbf{B}) = (v\mathbf{B}, \mathbf{B})$ .

*Proof.* For the first claim: Recall that the  $G$ -action on pairs  $(g_0B, g_1B) \in O(v^{-1})$  is transitive. So by equivariance of  $\check{f}$  and homogeneity, it suffices to compute  $\check{f}$  over a subset of  $O(v^{-1}, v)$  where these coordinates are fixed.

We take  $(g_0B, g_1B) = (vB, B)$ . Over this pair, the fiber of  $\check{\mathbf{E}}_{J,v}^-$  consists of  $(vB, u, B)$  with  $u \in U_J$ , the fiber of  $O(v^{-1}, v)$  consists of  $(vB, B, gB)$  with  $gB \in BvB/B$ , and  $\check{f}$  is given by  $u \mapsto uvB$ . Therefore, under the bijections  $U_J \simeq U_{w_{J_0}v} \times U_v^-$  of Lemma 2.6 and  $BvB/B \simeq U_v^-$  of Lemma 2.3(3),  $\check{f}$  corresponds to the projection  $U_{w_{J_0}v} \times U_v^- \rightarrow U_v^-$ . This proves the claim.

For the second claim: As in the proof of Lemma 2.6, we observe that  $v$  minimizes  $\ell_{\mathbf{B}}$  in  $\mathbf{W}_Jv$ . So we can repeat the arguments above with the varieties  $\mathbf{G}$ ,  $\mathbf{O}(v)$ , etc. in place of the sets  $G$ ,  $O(v)$ , etc., and Lemma 2.1 in place of Lemma 2.3.  $\square$

3.5. **Proof of (2).** In the notation of Section 2 (*nota bene* §2.7),

$$pr_{0,4,!}\delta_{O(v^{-1}, w_{J_0}, w_{J_0}, v)} = \delta_{v^{-1}} * \delta_{w_{J_0}}^2 * \delta_v.$$

This suggests comparing  $\mathbf{E}_{J,v}^+$  to a bundle over  $\mathbf{O}(v^{-1}, w_{J_0}, w_{J_0}, v)$ . But unlike the situation in case (1), there is no obvious map from  $\check{\mathbf{E}}_{J,v}^+$  into the latter variety.

We do know that  $\check{f}$  restricts to a map from  $\check{\mathbf{E}}_{J,v}^+$  into  $\mathbf{O}(v^{-1}) \times \mathbf{G}/\mathbf{B}$ , giving an equivariant commutative diagram:

$$\begin{array}{ccc}
 \check{\mathbf{E}}_{J,v}^+ & \xrightarrow{\quad} & \mathbf{E}_{J,v}^+ \\
 \check{f} \downarrow & & \uparrow f \\
 \mathbf{O}(v^{-1}) \times \mathbf{G}/\mathbf{B} & & \\
 pr_0 \times \text{id} \downarrow & \swarrow & \\
 (\mathbf{G}/\mathbf{B})^2 & & 
 \end{array}$$

At the same time, we have a map

$$\mathbf{O}(v^{-1}, w_{J_0}, w_{J_0}, v) \xrightarrow{pr_{0,1,4}} \mathbf{O}(v^{-1}) \times \mathbf{G}/\mathbf{B}.$$

So by Lemma 3.2 and this discussion, we reduce case (2) of Theorem 3.1 to:

**Proposition 3.4.** *If  $v \in W^{J,-}$ , then*

$$\check{f}_! \delta_{\check{E}_{J,v}^+} = |U_{w_{J_0}v}| pr_{0,1,4,!} \delta_{O(v^{-1}, w_{J_0}, w_{J_0}, v)}$$

in  $\mathcal{C}_G(O(v^{-1}) \times G/B)$ .

*Proof.* Since the  $O(w)$  partition  $(G/B)^2$ , it suffices to fix  $w \in W$  and restrict to

$$O(v^{-1}) \times_w G/B = \{(hB, yB, gB) \in O(v^{-1}) \times G/B \mid Bh^{-1}gB = BwB\},$$

the preimage of  $O(w)$  along  $pr_0 \times \text{id}$ . Recall that the  $G$ -action on  $O(w)$  is transitive. So by equivariance and homogeneity, the fibers of  $\check{E}_{J,v}^+$  and  $O(v^{-1}, w_{J_0}, w_{J_0}, v)$  have constant size over  $O(v^{-1}) \times_w G/B$ . So it suffices to compare them over a subvariety of  $O(v^{-1}) \times_w G/B$  where the coordinates  $(hB, gB)$  are fixed. Moreover, to do this, it suffices to fix  $hB$  and average over  $gB \in hBwB/B$ .

We take  $hB = B$ . Then we must compare the preimages of

$$(3.3) \quad \{(B, yB, gB) \in O(v^{-1}) \times_w G/B\}$$

in  $\check{E}_J^+$  and  $O(v^{-1}, w_{J_0}, w_{J_0}, v)$ . Since  $v \in W^{J,-}$ , we can trade the latter set and the map  $pr_{0,1,4}$  for the set  $O(v^{-1}, w_{J_0}, w_{J_0}v)$  and the map  $pr_{0,1,3}$ .

The preimage of (3.3) in  $\check{E}_J^+$  consists of  $(B, u, yB)$  such that  $u \in yV_J y^{-1}$  and  $u \in BwB$ . Hence it has size

$$(3.4) \quad |yV_J y^{-1} \cap BwB|.$$

The preimage of (3.3) in  $O(v^{-1}, w_{J_0}, w_{J_0}v)$  consists of  $(B, yB, zB, gB)$  such that

$$yB \xleftarrow{w_{J_0}} zB \xrightarrow{w_{J_0}v} gB$$

and  $gB \in BwB/B$ . Observe that  $yB \in Bv^{-1}B/B$ , so homogeneity under left multiplication by  $B$  lets us count the preimage for a given  $yB$  by averaging over the

preimages for all  $yB \in Bv^{-1}B/B$ . Since  $v \in W^{J,-}$ , Lemma 2.3(1) shows that the union of these preimages is parametrized by  $(zB, gB)$  such that

$$(3.5) \quad B \xleftarrow{w_{J\circ}v} zB \xrightarrow{w_{J\circ}v} gB$$

and  $gB \in BwB/B$ . It also shows that there is a bijection from  $U_{(w_{J\circ}v)^{-1}}^- \times U_{w_{J\circ}v}^-$  to the set of pairs  $(zB, gB)$  satisfying (3.5), given by

$$(u, u') \mapsto (u(w_{J\circ}v)^{-1}B, u(w_{J\circ}v)^{-1}u'w_{J\circ}vB).$$

So the set of  $(zB, gB)$  satisfying (3.5) and  $gB \in BwB/B$  is parametrized by

$$(U_{(w_{J\circ}v)^{-1}}^-(w_{J\circ}v)^{-1}U_{w_{J\circ}v}^-w_{J\circ}v) \cap BwB.$$

Since  $U_{(w_{J\circ}v)^{-1}}^- \subseteq B$ , this last set can be identified with

$$U_{(w_{J\circ}v)^{-1}}^- \times ((w_{J\circ}v)^{-1}U_{w_{J\circ}v}^-w_{J\circ}v \cap BwB).$$

By Lemma 2.3(3), we have  $|U_{v^{-1}}^-|$  many choices for  $yB \in Bv^{-1}B/B$ , and since  $v \in W^{J,-}$ , we also have  $|U_{(w_{J\circ}v)^{-1}}^-| = |U_{w_{J\circ}}^-||U_{v^{-1}}^-|$ . Altogether, we conclude that the size of the preimage of (3.3) in  $O(v^{-1}, w_{J\circ}, w_{J\circ}v)$  is

$$(3.6) \quad |U_{w_{J\circ}}^-||U_{(w_{J\circ}v)^{-1}}^-U_{w_{J\circ}v}^-w_{J\circ}v \cap BwB|.$$

Finally, we compare (3.4) and (3.6). Theorem 4.1 of [Kaw75] says

$$|yV_Jy^{-1} \cap BwB| = |U_{v^{-1}}^-||U_{(w_{J\circ}v)^{-1}}^-U_{w_{J\circ}v}^-w_{J\circ}v \cap BwB|.$$

Again using  $|U_{(w_{J\circ}v)^{-1}}^-| = |U_{w_{J\circ}}^-||U_{v^{-1}}^-|$ , we see that  $|U_{v^{-1}}^-| = |U_{(w_{J\circ}v)^{-1}}^-||U_{w_{J\circ}}^-| = |U_{w_{J\circ}v}^-||U_{w_{J\circ}}^-|$ , giving the desired identity.  $\square$

*Remark 3.5.* The asymmetry of the variety  $\mathbf{O}(v^{-1}) \times \mathbf{G}/\mathbf{B}$  may seem defective. To make the geometry more symmetrical, one might try to replace the diagram

$$\mathbf{E}_J^+ \xrightarrow{\check{f}} \mathbf{O}(v^{-1}) \times \mathbf{G}/\mathbf{B} \xleftarrow{pr_{0,1,4}} \mathbf{O}(v^{-1}, w_{J\circ}, w_{J\circ}, v)$$

with the diagram

$$\mathbf{E}_J^+ \xrightarrow{\check{f}'} \mathbf{O}(v^{-1}) \times \mathbf{O}(v) \xleftarrow{pr_{0,1,3,4}} \mathbf{O}(v^{-1}, w_{J\circ}, w_{J\circ}, v)$$

in which  $\check{f}'(h\mathbf{B}, u, x\mathbf{B}) = (h\mathbf{B}, x\mathbf{B}, ux\mathbf{B}, uh\mathbf{B})$ . Then one would hope that

$$\check{f}'_! \delta_{E_{J,v}^+} = |U_J| pr_{0,1,3,4,!} \delta_{O(v^{-1}, w_{J\circ}, w_{J\circ}, v)}$$

in  $\mathcal{C}_G(O(v^{-1}) \times O(v))$ . However, Kawanaka's work does not seem to establish this stronger identity.

#### 4. TRACES ON THE HECKE ALGEBRA

4.1. The goal of this section is to prove a version of Theorem 1.2 for general  $G$ , and deduce Corollary 1.3 for split  $G$ . We keep the general setup of Section 2.

**4.2. Traces from Relative Norms.** As in §1.3, let  $\tau : H_B^G \rightarrow \mathbf{Z}[\frac{1}{q}]$  be the trace given by  $\tau(\delta_e) = 1$  and  $\tau(\delta_w) = 0$  for all  $w \neq e$ , and for any central element  $\zeta \in Z(H_B^G)$ , let  $\tau[\zeta] : H_B^G \rightarrow \mathbf{Z}[\frac{1}{q}]$  be the trace given by  $\tau[\zeta](\beta) = \tau(\beta * \zeta)$ .

**Lemma 4.1.** *For all  $J \subseteq S$  and  $w \in W$  and  $\alpha \in Z(H_{B_J}^{L_J})$ , we have*

$$\frac{1}{|B|} \tau[N_J^S(\alpha)](\delta_w) = \frac{1}{|G|} \sum_{(hB, gB) \in O(w)} N_J^S(\iota(\alpha))(hB, gB),$$

where  $\iota$  is the additive anti-involution of  $H_{B_J}^{L_J}$  given by  $\iota(\delta_w) = \delta_{w^{-1}}$ .

*Proof.* For any  $\beta \in H_B^G$  and  $xB \in G/B$ , we have  $\tau(\beta) = \beta(xB, xB)$ . Moreover,  $|G/B| = |G|/|B|$ . So for any  $\zeta \in Z(H_B^G)$ , we have

$$\frac{|G|}{|B|} \tau[\zeta](\beta) = \sum_{xB \in G/B} (\beta * \zeta)(xB, xB).$$

Next, for any  $w, v, z \in W$ , observe that there is a bijection

$$\begin{aligned} & \{(x_0B, x_1B, x_2B, x_3B, x_4B) \in O(w, v^{-1}, z, v) \mid x_0B = x_4B\} \\ & \xrightarrow{\sim} \{(g_0B, g_1B, g_2B, g_3B) \in O(v^{-1}, z^{-1}, v) \mid g_0B \xrightarrow{w} g_3B\} \end{aligned}$$

given by  $(g_0B, g_1B, g_2B, g_3B) = (x_4B, x_3B, x_2B, x_1B)$ . This shows the identity

$$\sum_{gB \in G/B} (\delta_w * \delta_{v^{-1}} * \delta_z * \delta_v)(gB, gB) = \sum_{(hB, gB) \in O(w)} (\delta_{v^{-1}} * \delta_{z^{-1}} * \delta_v)(hB, gB).$$

By expanding  $\alpha$  in the basis  $(\delta_z)_{z \in W_J}$  for  $H_{B_J}^{L_J}$ , and summing over all  $v \in W^{J,-}$ , we deduce that

$$\sum_{xB \in G/B} (\beta * N_J^S(\alpha))(xB, xB) = \sum_{(hB, gB) \in O(w)} N_J^S(\iota(\alpha))(hB, gB),$$

concluding the proof.  $\square$

**4.3. Springer Fibers.** A reference for this subsection is [Sho88].

In order to work with étale cohomology, we fix a prime  $\ell$  invertible in  $\mathbf{F}$ . The notation  $H^*(-, \bar{\mathbf{Q}}_\ell)$  will always mean étale cohomology with coefficients in the constant  $\bar{\mathbf{Q}}_\ell$ -sheaf. Henceforth, let  $\mathbf{V} = \mathbf{V}_\emptyset$  and

$$\mathbf{Spr} = \mathbf{Spr}_\emptyset^+ = \mathbf{Spr}_\emptyset^- \subseteq \mathbf{V} \times \mathbf{G}/\mathbf{B}.$$

By the *Springer resolution*, we mean either  $\mathbf{Spr}$  or the projection map from  $\mathbf{Spr}$  onto  $\mathbf{V}$ . For any  $u \in \mathbf{V}$ , the *Springer fiber* over  $u$  is the (reduced) fiber of this map over  $u$ , viewed as a subvariety  $\mathbf{Spr}_u$  of  $\mathbf{G}/\mathbf{B}$ . On points,

$$\mathbf{Spr}_u = \{y\mathbf{B} \in \mathbf{G}/\mathbf{B} \mid u \in y\mathbf{U}y^{-1}\}.$$

Springer showed that this is a projective variety with no odd cohomology. For  $u \in V := \mathbf{V}^F$ , he constructed an action of  $W$  on  $H^*(\mathbf{Spr}_u)$  through a type of Fourier



transform. Later, other authors gave independent constructions, generalizing to other base fields like the complex numbers.

In this paper, we use the  $W$ -action on  $H^*(\mathbf{Spr}_u)$  constructed through perverse sheaf theory, which differs from Springer's original action by a sign twist. Let  $\chi_u : \mathbf{Q}W \rightarrow \bar{\mathbf{Q}}_\ell$  be the trace defined by

$$\chi_u(w) = \mathrm{tr}(Fw \mid H^*(\mathbf{Spr}_u)).$$

For our choice of action, the sign character of  $W$  only occurs in  $\chi_1$ .

As reviewed in [Sho88, §15], it is now known  $\chi_u$  arises from the specialization at  $q \rightarrow q$  of a  $\mathbf{Z}[q]$ -valued trace on  $\mathbf{Z}W$ . In particular,  $\chi_u(w) \in \mathbf{Z}$  for all  $w \in W$ .

**4.4. Partial Springer Fibers.** For all  $J \subseteq S$ , the *symmetrizer* and *antisymmetrizer* in  $\mathbf{Q}W_J$  are respectively defined by

$$e_{J,+} = \frac{1}{|W_J|} \sum_{w \in W_J} w \quad \text{and} \quad e_{J,-} = \frac{1}{|W_J|} \sum_{w \in W_J} (-1)^{\ell(w)} w.$$

These are central elements of  $\mathbf{Q}W_J$ , such that  $\mathbf{Q}W_J e_{J,+}$  and  $\mathbf{Q}W_J e_{J,-}$  respectively afford the trivial and sign representations of  $W_J$ .

Borho–MacPherson related  $e_{J,-}$  and  $e_{J,+}$  to the *partial Springer fibers*

$$\begin{aligned} \mathbf{Spr}_{J,u}^- &= \{y\mathbf{P}_J \in \mathbf{G}/\mathbf{P}_J \mid u \in y\mathbf{U}_J y^{-1}\}, \\ \mathbf{Spr}_{J,u}^+ &= \{y\mathbf{P}_J \in \mathbf{G}/\mathbf{P}_J \mid u \in y\mathbf{V}_J y^{-1}\}. \end{aligned}$$

By §2.4, the set of  $F$ -fixed points  $\mathbf{Spr}_{J,u}^-$ , *resp.*  $\mathbf{Spr}_{J,u}^+$ , is the set of  $y\mathbf{P}_J \in G/P_J$  such that  $u \in y\mathbf{U}_J y^{-1}$ , *resp.*  $u \in y\mathbf{V}_J y^{-1}$ . For our choice of Springer action, the main result of [BM83] implies that for all  $J \subseteq S$  and  $u \in V$ , we have

$$(4.1) \quad \begin{aligned} \frac{1}{|U_{w_{J\circ}}^-|} \chi_u(e_{J,-}) &= |\mathbf{Spr}_{J,u}^-|, \\ \chi_u(e_{J,+}) &= |\mathbf{Spr}_{J,u}^+|. \end{aligned}$$

More precisely, these results come from transferring Borho–MacPherson's arguments from sheaves in the analytic topology over  $\mathbf{C}$  to sheaves in the étale topology over  $\bar{\mathbf{F}}$ , and keeping track of Tate twists arising from the  $\mathbf{F}$ -structure. The factor of  $|U_{w_{J\circ}}^-| = q^{\dim(\mathbf{L}_J/\mathbf{B}_J)}$  in the  $-$  case arises from a Tate twist of order  $2 \dim(\mathbf{L}_J/\mathbf{B}_J)$  that accompanies the cohomological shift in case (b) of [BM83, §3.4].

**4.5. The Bitrace.** As in §1.3, let  $O(w)_u$  be the subset of  $O(w)$  of pairs taking the form  $(hB, uhB)$ . Let  $\tau_G : \mathbf{Q}W \otimes H_B^G \rightarrow \mathbf{Q}$  be defined by

$$\tau_G(z \otimes \delta_w) = \frac{1}{|G|} \sum_{u \in V} |O(w)_u| \chi_u(z).$$

The framework of [Tri21] shows that this is, indeed, a bitrace, meaning  $\tau_G(z \otimes ( \ ))$  and  $\tau_G(( \ ) \otimes \delta_w)$  are traces for all  $z, w \in W$ . In the split case, it recovers the  $q \rightarrow q$  specialization of the trace denoted  $\tau_G$  in the introduction.

**Lemma 4.2.** *For all  $J \subseteq S$  and  $w \in W$ , we have*

$$\begin{aligned} \frac{1}{|U_{w_{J_0}}^-|} \tau_G(e_{J,-} \otimes \delta_w) &= \frac{1}{|G|} \sum_{(hB, gB) \in O(w)} f_! \delta_{E_J^-}(hB, gB), \\ \tau_G(e_{J,+} \otimes \delta_w) &= \frac{1}{|G|} \sum_{(hB, gB) \in O(w)} f_! \delta_{E_J^+}(hB, gB), \end{aligned}$$

where  $E_J^\pm$  and  $f$  are defined as in Section 3.

*Proof.* Apply (4.1) to the formula for  $\tau_G$ . Then observe that

$$\begin{aligned} \coprod_{u \in V} O(w)_u \times \text{Spr}_{J,u}^\pm &= \{(hB, u, yP_J) \in E_J^\pm \mid (hB, uhB) \in O(w)\} \\ &= \coprod_{(hB, gB) \in O(w)} f^{-1}(hB, gB). \end{aligned} \quad \square$$

The split case of the following result is the  $\mathbf{q} \rightarrow q$  specialization of Theorem 1.2. Since it amounts to a family of identities of Laurent polynomials in  $q$ , which hold for infinitely many  $q$ , we can lift it from  $q$  to  $\mathbf{q}$ .

**Theorem 4.3.** *For any  $J \subseteq S$ , we have*

$$\begin{aligned} \tau[N_J^S(1)] &= |T| \tau_G(e_{J,-} \otimes ( \quad )), \\ \tau[N_J^S(\delta_{w_{J_0}}^2)] &= |B_J| \tau_G(e_{J,+} \otimes ( \quad )) \end{aligned}$$

as traces on  $H_W$ .

*Proof.* Combine Lemmas 4.1–4.2 with (3.2), noting that 1 and  $\delta_{w_{J_0}}^2$  are invariant under  $\iota$ . Doing so gives

$$\begin{aligned} \frac{1}{|B|} \tau[N_J^S(1)] &= \frac{1}{|U_J| |U_{w_{J_0}}^-|} \tau_G(e_{J,-} \otimes ( \quad )) = \frac{1}{|U|} \tau_G(e_{J,-} \otimes ( \quad )), \\ \frac{1}{|B|} \tau[N_J^S(\delta_{w_{J_0}}^2)] &= \frac{1}{|U_J|} \tau_G(e_{J,+} \otimes ( \quad )). \end{aligned}$$

Then recall that  $B = T \times U = B_J \times U_J$ .  $\square$

**4.6. The Multiplicity Formula.** *Throughout this subsection, we assume that  $G$  is split. As in §1.3, we write:*

- $V_G$  for the representation of  $W$  on the  $\mathbf{Q}$ -span of the cocharacter lattice of  $\mathbf{T}$ .
- $\text{Irr}(W)$  for the set of irreducible characters of  $W$ .
- $\{-, -\}$  for the truncation of Lusztig's exotic Fourier transform to a  $\mathbf{Q}$ -valued pairing on  $\text{Irr}(W)$ .

In the notation of [Lus84], our pairing is the pullback of Lusztig's pairing  $\{-, -\}$  along his embedding (4.21.3).

By [Lus81],  $\mathbf{Q}(q^{1/2})$  is a splitting field for  $H_W$ . Hence, by Tits deformation [GP00, Ch. 7], each character  $\chi : W \rightarrow \mathbf{Q}$  defines a trace  $\chi_q : H_W \rightarrow \mathbf{Q}(q)$ . The set of traces  $\chi_q$  with  $\chi \in \text{Irr}(W)$  forms a basis for  $\mathbf{Q}(q^{1/2}) \otimes R(H_W)$  as a vector space.

The character formula in [Tri21] translates to an expansion of  $\tau_G(z \otimes ( ))$  in this basis for any  $z \in \mathbf{Q}W$ :

$$\tau_G(z \otimes ( )) = \sum_{\chi, \psi \in \text{Irr}(W)} \{\chi, \psi\} \frac{\psi(z)}{\det(\mathbf{q} - z \mid \mathbf{V}_G)} \otimes \chi \mathbf{q}.$$

Combining this with Theorem 1.2 gives Corollary 1.3.

**4.7. Recovering Lascoux–Wan–Wang.** In this subsection, we take  $\mathbf{G} = \mathbf{GL}_n$ , and  $F$  to be the standard Frobenius that raises each matrix coordinate to its  $q$ th power. Then  $G = \mathbf{GL}_n(\mathbf{F})$  and  $W = \mathbf{W} = S_n$ . For each integer partition  $\lambda \vdash n$ , let  $\chi^\lambda \in \text{Irr}(S_n)$  be the corresponding irreducible character.

As in §1.3, we take  $S = \{s_1, \dots, s_{n-1}\}$ , where  $s_i \in S_n$  is the transposition swapping  $i$  and  $i+1$ . We will use the bijection between integer compositions of  $n$  and subsets of  $S$  that matches  $\nu = (\nu_1, \nu_2, \dots) \vdash n$  with

$$J = S \setminus \{s_{\nu_1}, s_{\nu_1+\nu_2}, \dots\}$$

For this  $J$ , we find that  $W_J \subseteq W$  is the *Young subgroup*  $S_\nu \simeq S_{\nu_1} \times S_{\nu_2} \times \dots$ .

For  $G = \mathbf{GL}_n(\mathbf{F})$ , the pairing  $\{-, -\}$  in §4.6 is given by  $\{\chi, \chi\} = 1$  and  $\{\chi, \psi\} = 0$  whenever  $\chi \neq \psi$ . So to prove that Corollary 1.3 recovers Wan–Wang’s formulas (1.2), it remains to prove:

**Proposition 4.4.** *If the subset  $J$  corresponds to the integer composition  $\nu$ , then*

$$\begin{aligned} \frac{\chi^\lambda(e_{J,-})}{\det(\mathbf{q} - e_{J,-} \mid \mathbf{V}_G)} &= \left\langle s_\lambda(X), e_\nu \left( \frac{X}{\mathbf{q}-1} \right) \right\rangle, \\ \frac{\chi^\lambda(e_{J,+})}{\det(\mathbf{q} - e_{J,+} \mid \mathbf{V}_G)} &= \left\langle s_\lambda(X), h_\nu \left( \frac{X}{\mathbf{q}-1} \right) \right\rangle \end{aligned}$$

for any  $\lambda \vdash n$ , where  $\langle -, - \rangle$  is the Hall pairing on  $\Lambda_n$  in which the  $s_\lambda(X)$  are orthonormal.

As preparation, let  $R(S_n)$  be the vector space of  $\mathbf{Q}(\mathbf{q})$ -valued traces on  $\mathbf{Q}S_n$ . Let  $ch : R(S_n) \xrightarrow{\sim} \Lambda_n$  be the *(undeformed) Frobenius characteristic* isomorphism that sends  $\chi^\lambda$  to the Schur function  $s_\lambda(X)$ . It sends the multiplicity pairing on  $R(S_n)$  to the Hall pairing.

*Proof.* Recall that  $ch$  sends  $\chi^\lambda / \det(\mathbf{q} - ( ) \mid \mathbf{V}_G)$  to the plethystically transformed Schur  $s_\lambda(X/(\mathbf{q}-1))$ . At the same time, since  $W_J = S_\lambda$ , it sends the induced character of  $W = S_n$  arising from the trivial, *resp.* sign, character of  $W_J$  to the symmetric function  $h_\nu(X)$ , *resp.*  $e_\nu(X)$ . So by Frobenius reciprocity,

$$\frac{\chi^\lambda(e_{J,+})}{\det(\mathbf{q} - e_{J,+} \mid \mathbf{V}_G)} = \left\langle h_\nu(X), s_\lambda \left( \frac{X}{\mathbf{q}-1} \right) \right\rangle = \left\langle h_\nu \left( \frac{X}{\mathbf{q}-1} \right), s_\lambda(X) \right\rangle,$$

and similarly with  $e_{J,-}$ ,  $e_\nu$  in place of  $e_{J,+}$ ,  $h_\nu$ . □

*Remark 4.5.* The pairing  $\{-, -\}$  remains fairly mysterious. Notably, its definition in [Lus84] involves some case-by-case constructions. In this sense, Theorem 4.3 seems to be the closest we can get to a perfectly *uniform* generalization of Proposition 4.4 to Weyl groups.

## 5. BRAID VARIETIES AND CELL DECOMPOSITIONS

5.1. *For the rest of the paper, we assume that  $G$  is split.* In this section, we prove Theorem 1.4, relating partial braid Steinberg varieties to the cell decompositions of open braid Richardson varieties. In fact, we prove a refinement that respects individual cells.

We will freely use the terminology from Coxeter combinatorics that we reviewed in §1.4. Throughout, we fix a word  $\vec{s} = (s^{(1)}, \dots, s^{(\ell)})$  in  $S$ .

5.2. **Richardson Varieties.** Recall that for any  $v \in W$ , we defined the  *$v$ -twisted open Richardson variety* of  $\vec{s}$  on points by

$$\mathbf{R}^{(v)}(\vec{s}) = \{\vec{g}\mathbf{B} \in \mathbf{O}(\vec{s}) \mid g_0\mathbf{B} = vw_0\mathbf{B} \text{ and } \mathbf{B} \xrightarrow{vw_0} g_\ell\mathbf{B}\}.$$

Below, we give further detail about the cell decomposition mentioned in §1.4. For any  $v$ -distinguished subword  $\vec{\omega}$  of  $\vec{s}$ , let  $\mathbf{R}^{(v)}(\vec{s})_{\vec{\omega}} \subseteq \mathbf{R}^{(v)}(\vec{s})$  be the subvariety

$$\mathbf{R}^{(v)}(\vec{s})_{\vec{\omega}} = \{\vec{g}\mathbf{B} \in \mathbf{R}^{(v)}(\vec{s}) \mid \mathbf{B} \xrightarrow{v\omega_{(i)}w_0} g_i\mathbf{B}\}.$$

As before, let  $\mathcal{D}^{(v)}(\vec{s})$  be the set of  $v$ -distinguished subwords  $\vec{\omega}$  of  $\vec{s}$  such that  $\omega_{(\ell)} = e$ . For any  $\vec{\omega} \in \mathcal{D}^{(v)}(\vec{s})$ , let

$$\begin{aligned} \mathbf{d}_{\vec{\omega}} &= \{i \mid v\omega_{(i)} < v\omega_{(i-1)}\}, \\ \mathbf{e}_{\vec{\omega}} &= \{i \mid \omega^{(i)} = e\}, \end{aligned}$$

The main results of [Deo85] show that for any word  $\vec{s}$  in  $S$ :

- (1)  $\mathbf{R}^{(v)}(\vec{s})_{\vec{\omega}}$  is nonempty if and only if  $\omega \in \mathcal{D}^{(v)}(\vec{s})$ . In this case,

$$(5.1) \quad \mathbf{R}^{(v)}(\vec{s})_{\vec{\omega}} \simeq \left\{ \vec{t} \in \mathbf{A}^\ell \mid \begin{array}{ll} t_i \neq 0 & \text{for } i \in \mathbf{e}_{\vec{\omega}}, \\ t_i = 0 & \text{for } i \notin \mathbf{d}_{\vec{\omega}} \cup \mathbf{e}_{\vec{\omega}} \end{array} \right\}$$

from which  $R^{(v)}(\vec{s})_{\vec{\omega}} := \mathbf{R}^{(v)}(\vec{s})_{\vec{\omega}}^F$  satisfies

$$(5.2) \quad |R^{(v)}(\vec{s})_{\vec{\omega}}| = q^{|\mathbf{d}_{\vec{\omega}}|} (q-1)^{|\mathbf{e}_{\vec{\omega}}|}.$$

- (2) The subvarieties  $\mathbf{R}^{(v)}(\vec{s})_{\vec{\omega}}$  are pairwise disjoint and partition  $\mathbf{R}^{(v)}(\vec{s})$  as we run over  $\vec{\omega} \in \mathcal{D}^{(v)}(\vec{s})$ .

In light of (5.1), the varieties  $\mathbf{R}^{(v)}(\vec{s})_{\vec{\omega}}$  are called *Deodhar cells*.

**5.3. Change of Structure Group.** To compare them to the geometry in previous sections, we need a more symmetrical version of the open Richardson varieties. Let  $\mathbf{X}^{(v)}$ ,  $\mathbf{X}_{\square}^{(v)}$ ,  $\mathbf{R}^{(v)}$  be the varieties defined on points by

$$\begin{aligned}\mathbf{X}^{(v)} &= \{(h\mathbf{B}, x\mathbf{B}, g\mathbf{B}) \in (\mathbf{G}/\mathbf{B})^3 \mid h\mathbf{B} \xleftarrow{vw_{\circ}} x\mathbf{B} \xrightarrow{vw_{\circ}} g\mathbf{B}\} \\ &\simeq \mathbf{O}((vw_{\circ})^{-1}, vw_{\circ}), \\ \mathbf{X}_{\square}^{(v)} &= \{(h\mathbf{B}, g\mathbf{B}) \in (\mathbf{G}/\mathbf{B})^2 \mid h\mathbf{B} \xleftarrow{vw_{\circ}} \mathbf{B} \xrightarrow{vw_{\circ}} g\mathbf{B}\}, \\ \mathbf{R}^{(v)} &= \{vw_{\circ}\mathbf{B}\} \times \mathbf{B}vw_{\circ}\mathbf{B}/\mathbf{B}.\end{aligned}$$

By construction,  $\mathbf{R}^{(v)}(\vec{s})$  is the preimage of  $\mathbf{R}^{(v)}$  along  $\mathbf{O}(\vec{s}) \xrightarrow{pr_{0,\ell}} (\mathbf{G}/\mathbf{B})^2$ . We will relate the varieties above to one another, thereby relating  $\mathbf{R}^{(v)}(\vec{s})$  and its Deodhar cells to analogous varieties built from  $\mathbf{X}^{(v)}$ ,  $\mathbf{X}_{\square}^{(v)}$ .

Observe that  $\mathbf{X}^{(v)}$  is stable under the  $\mathbf{G}$ -action on  $(\mathbf{G}/\mathbf{B})^3$ . The action of  $\mathbf{G}$  on  $\mathbf{X}^{(v)}$  restricts to an action of  $\mathbf{B}$  on  $\mathbf{X}_{\square}^{(v)}$ , which in turn restricts to an action of

$$\mathbf{B}_v^- := \mathbf{B} \cap v\mathbf{B}_-v^{-1} = \mathbf{B} \cap (vw_{\circ})\mathbf{B}(vw_{\circ})^{-1}$$

on  $\mathbf{R}^{(v)}$ . By Lemma 2.1(2),  $\mathbf{B} = \mathbf{B}_v^- \mathbf{U}_v = \mathbf{U}_v \mathbf{B}_v^-$  and  $\mathbf{B}_v^- \cap \mathbf{U}_v = \{1\}$ .

**Lemma 5.1.** *For any  $v \in W$ , let  $\mathbf{B}$  act on  $\mathbf{G} \times \mathbf{X}_{\square}^{(v)}$  from the left by*

$$b \cdot (x, h\mathbf{B}, g\mathbf{B}) = (xb^{-1}, bh\mathbf{B}, bg\mathbf{B}).$$

*Then:*

- (1) *The map  $(\mathbf{G} \times \mathbf{X}_{\square}^{(v)})/\mathbf{B} \rightarrow \mathbf{X}^{(v)}$  that sends  $[x, h\mathbf{B}, g\mathbf{B}] \mapsto (xh\mathbf{B}, x\mathbf{B}, xg\mathbf{B})$  is an isomorphism.*
- (2) *The quotient  $\mathbf{X}_{\square}^{(v)}/\mathbf{U}_v$  forms an algebraic variety. The composition of maps*

$$\mathbf{R}^{(v)} \rightarrow \mathbf{X}_{\square}^{(v)} \rightarrow \mathbf{X}_{\square}^{(v)}/\mathbf{U}_v$$

*is an isomorphism.*

*Proof.* (1):  $\mathbf{X}_{\square}^{(v)}$  is the closed subvariety of  $\mathbf{X}^{(v)}$  cut out by the condition  $x\mathbf{B} = \mathbf{B}$ . The  $\mathbf{G}$ -action on  $\mathbf{X}^{(v)}$  is transitive on the coordinate  $x\mathbf{B}$ , and the stabilizer of the point  $\mathbf{B}$  is itself.

(2):  $\mathbf{R}^{(v)}$  is the closed subvariety of  $\mathbf{X}_{\square}^{(v)}$  cut out by the condition  $h\mathbf{B} = vw_{\circ}\mathbf{B}$ . By Lemma 2.1(3), the  $\mathbf{B}$ -action on  $\mathbf{X}_{\square}^{(v)}$  restricts to an action of  $\mathbf{U}_{vw_{\circ}}^- = \mathbf{U}_v$  that is simply transitive on the coordinate  $h\mathbf{B}$ .  $\square$

**Corollary 5.2.** *The maps  $(\mathbf{G} \times \mathbf{X}_{\square}^{(v)})/\mathbf{B} \rightarrow \mathbf{X}^{(v)}$  and  $\mathbf{R}^{(v)} \rightarrow \mathbf{X}_{\square}^{(v)}/\mathbf{U}_v$  on  $F$ -fixed points induced by the isomorphisms above are bijections.*

*Proof.* Immediate from Lang's theorem, since  $\mathbf{B}$ , resp.  $\mathbf{U}_v$ , is connected and acts freely on  $\mathbf{G} \times \mathbf{X}_{\square}^{(v)}$ , resp.  $\mathbf{X}_{\square}^{(v)}$ .  $\square$

Let  $\mathbf{X}^{(v)}(\vec{s}) = \mathbf{O}(\vec{s}) \times_{(\mathbf{G}/\mathbf{B})^2} \mathbf{X}^{(v)}$  and  $\mathbf{X}_{\square}^{(v)}(\vec{s}) = \mathbf{O}(\vec{s}) \times_{(\mathbf{G}/\mathbf{B})^2} \mathbf{X}_{\square}^{(v)}$ , where the fiber products are formed with respect to the maps  $pr_{0,\ell}$  on the left factors and the

coordinate pairs  $(h\mathbf{B}, g\mathbf{B})$  on the right factors. On points,

$$\mathbf{X}^{(v)}(\vec{s}) = \{(\vec{g}\mathbf{B}, x\mathbf{B}) \in \mathbf{O}(\vec{s}) \times \mathbf{G}/\mathbf{B} \mid g_0\mathbf{B} \xleftarrow{vw_\circ} x\mathbf{B} \xrightarrow{vw_\circ} g_\ell\mathbf{B}\},$$

$$\mathbf{X}_\square^{(v)}(\vec{s}) = \{\vec{g}\mathbf{B} \in \mathbf{O}(\vec{s}) \mid g_0\mathbf{B} \xleftarrow{vw_\circ} \mathbf{B} \xrightarrow{vw_\circ} g_\ell\mathbf{B}\}.$$

These varieties can respectively be partitioned into subvarieties

$$\mathbf{X}^{(v)}(\vec{s})_{\vec{\omega}} = \{(\vec{g}\mathbf{B}, x\mathbf{B}) \in \mathbf{O}(\vec{s}) \times \mathbf{G}/\mathbf{B} \mid x\mathbf{B} \xrightarrow{v\omega_{(i)}w_\circ} g_i\mathbf{B}\},$$

$$\mathbf{X}_\square^{(v)}(\vec{s})_{\vec{\omega}} = \{\vec{g}\mathbf{B} \in \mathbf{X}^{(v)}(\vec{s}) \mid \mathbf{B} \xrightarrow{v\omega_{(i)}w_\circ} g_i\mathbf{B}\}$$

as  $\vec{\omega}$  runs over  $\mathcal{D}^{(v)}(\vec{s})$ . Note that  $\mathbf{X}^{(v)}(\vec{s})_{\vec{\omega}}$  is stable under the  $\mathbf{G}$ -action on  $\mathbf{X}^{(v)}(\vec{s})$ , as are  $\mathbf{X}_\square^{(v)}(\vec{s})_{\vec{\omega}}$ , *resp.*  $\mathbf{R}^{(v)}(\vec{s})_{\vec{\omega}}$ , under  $\mathbf{B}$ , *resp.*  $\mathbf{B}_{vw_\circ}$ . Pulling back Lemma 5.1 along  $pr_{0,\ell} : \mathbf{O}(\vec{s})_{\vec{\omega}} \rightarrow (\mathbf{G}/\mathbf{B})^2$ , we see:

**Corollary 5.3.** *For any  $\vec{s}$  and  $\vec{\omega} \in \mathcal{D}^{(v)}(\vec{s})$ , the analogues of Lemma 5.1 and Corollary 5.2 hold with  $\diamond(\vec{s})_{\vec{\omega}}$  replacing  $\diamond$  for each  $\diamond \in \{\mathbf{X}^{(v)}, \mathbf{X}_\square^{(v)}, \mathbf{R}^{(v)}\}$ . Thus,*

$$|X^{(v)}(\vec{s})_{\vec{\omega}}| = \frac{|G||X_\square^{(v)}(\vec{s})_{\vec{\omega}}|}{|B|},$$

$$|X_\square^{(v)}(\vec{s})_{\vec{\omega}}| = |U_v||R^{(v)}(\vec{s})_{\vec{\omega}}|.$$

**5.4. Steinberg Varieties.** Fix  $J \subseteq S$ . As in §1.4, we define the *partial Steinberg varieties* of  $\vec{s}$  of type  $J$  on points by

$$\mathbf{Z}_J^\pm(\vec{s}) = \{(\vec{g}\mathbf{B}, u, y\mathbf{P}_J) \in \mathbf{O}(\vec{s}) \times \mathbf{Spr}_J^\pm \mid g_\ell\mathbf{B} = ug_0\mathbf{B}\}.$$

We let  $\mathbf{G}$  act on  $\mathbf{Z}_J^\pm(\vec{s})$  via its actions on  $\mathbf{Spr}_J^\pm$  and  $\mathbf{O}(\vec{s})$ . The coordinate triple  $(g_\ell\mathbf{B}, u, y\mathbf{P}_J)$  defines an equivariant map  $\mathbf{Z}_J^\pm(\vec{s}) \rightarrow \mathbf{E}_J^\pm$ . Pulling back the partition of  $\mathbf{E}_J^\pm$  by subvarieties  $\mathbf{E}_{J,v}^\pm$  in Section 3, we get a partition of  $\mathbf{Z}_J^\pm(\vec{s})$  into subvarieties

$$\mathbf{Z}_{J,v}^\pm(\vec{s}) = \{(\vec{g}\mathbf{B}, u, y\mathbf{P}_J) \in \mathbf{Z}_J^\pm(\vec{s}) \mid \mathbf{P}_J y^{-1} g_\ell\mathbf{B} = \mathbf{P}_J v\mathbf{B}\}$$

as  $W_J v$  runs over  $W_J \setminus W$ . Note that the points of  $\mathbf{Z}_{J,v}^\pm(\vec{s})$  also satisfy the condition  $\mathbf{P}_J y^{-1} g_0\mathbf{B} = \mathbf{P}_J v\mathbf{B}$ .

**Proposition 5.4.** *If  $v \in W^{J,-}$ , then:*

- (1)  $|Z_{J,v}^-(\vec{s})| = |U_{w_{J \circ v}}||X^{(vw_\circ)}(\vec{s})|.$
- (2)  $|Z_{J,v}^+(\vec{s})| = |U_{w_{J \circ v}}||X^{(w_{J \circ v}w_\circ)}(\vec{s})|.$

*Proof.* For any  $v \in W$ , we have

$$(5.3) \quad |Z_{J,v}^\pm(\vec{s})| = \sum_{\vec{g}\mathbf{B} \in \mathbf{O}(\vec{s})} f_i \delta_{E_{J,v}^\pm}(g_0\mathbf{B}, g_\ell\mathbf{B}),$$

$$(5.4) \quad |X^{(vw_\circ)}(\vec{s})| = \sum_{\vec{g}\mathbf{B} \in \mathbf{O}(\vec{s})} (\delta_{v^{-1}} * \delta_v)(g_0\mathbf{B}, g_\ell\mathbf{B}).$$

(The second identity used the involutivity of  $w_\circ$ .) Now apply Theorem 3.1.  $\square$

Since multiplication by  $w_\circ$  or  $w_{J_\circ}$  swaps  $W^{J,-}$  with  $W^{J,+}$ , the following result implies Theorem 1.4.

**Corollary 5.5.** *If  $v \in W^{J,-}$ , then*

$$\begin{aligned} \frac{|Z_{J,v}^-(\vec{s})|}{|G|} &= \frac{1}{q^{\ell_J}(q-1)^{\text{rk}(G)}} \sum_{\vec{\omega} \in \mathcal{D}^{(v)}(\vec{s})} q^{|\mathbf{d}_{\vec{\omega}}|} (q-1)^{|\mathbf{e}_{\vec{\omega}}|}, \\ \frac{|Z_{J,v}^+(\vec{s})|}{|G|} &= \frac{1}{(q-1)^{\text{rk}(G)}} \sum_{\vec{\omega} \in \mathcal{D}^{(v)}(\vec{s})} q^{|\mathbf{d}_{\vec{\omega}}|} (q-1)^{|\mathbf{e}_{\vec{\omega}}|}. \end{aligned}$$

*Proof.* We only do the  $-$  case, as the  $+$  case is similar. Observe that

$$\frac{|U_{w_{J_\circ}v}||X^{(vw_\circ)}(\vec{s})|}{|G|} = \frac{|U_{w_{J_\circ}v}||U_v||R^{(vw_\circ)}(\vec{s})|}{|B|} = \frac{|U_J||R^{(vw_\circ)}(\vec{s})|}{|B|} = \frac{|R^{(vw_\circ)}(\vec{s})|}{|B_J|}$$

by Corollary 5.3 and Lemma 2.6. Then apply Proposition 5.4 on the left and (5.2) on the right.  $\square$

*Remark 5.6.* It is not always the case that  $|Z_{J,v}^-(\vec{s})/G| = |Z_{J,v}^-(\vec{s})|/|G|$ . Indeed, the  $\mathbf{G}$ -action on  $\mathbf{Z}_{J,v}^\pm(\vec{s})$  need not be free, so we cannot apply Lang's theorem.

**5.5. Traces as Point Counts.** Here, we collect point-counting formulas for specific traces. Let  $\delta_{\vec{s}} = \delta_{s(1)} * \cdots * \delta_{s(\ell)}$ . Summing (5.3) over  $W_J v$  yields

$$\tau_G(e_{J,\pm} \otimes \delta_{\vec{s}}) = \frac{|Z_J^\pm(\vec{s})|}{|G|}.$$

Similarly, for any  $v \in W$ , (5.4) yields

$$(5.5) \quad \frac{1}{|B|} \tau(\delta_{\vec{s}} * \delta_{v^{-1}} * \delta_v) = \frac{|X^{(vw_\circ)}(\vec{s})|}{|G|}.$$

For the purpose of proving Theorem 1.4, we do not actually need these results. But in later sections, it will be useful to have a  $q$ -version of the formula

$$(5.6) \quad q^{-\ell(v)} \tau(\delta_{\vec{s}} * \delta_{v^{-1}} * \delta_v) = |R^{vw_\circ}(\vec{s})|$$

that follows from combining (5.5), Corollary 5.3, and Lemma 2.4. This formula is itself an easier version of Corollary 5.3 in [GLTW24].

Namely: Let  $T_{\vec{s}} = T_{s(1)} \cdots T_{s(\ell)}$ . Combining (1.3) and (5.6) gives an identity of Laurent polynomials in  $\delta_{\vec{s}}$  and  $q$  that holds for infinitely many  $q$ , hence lifts to

$$(5.7) \quad q^{-\ell(v)} \tau(T_{\vec{s}} T_{v^{-1}} T_v) = \sum_{\vec{\omega} \in \mathcal{D}^{(vw_\circ)}(\vec{s})} q^{|\mathbf{d}_{\vec{\omega}}|} (q-1)^{|\mathbf{e}_{\vec{\omega}}|},$$

an identity in  $T_{\vec{s}}$  and  $q$ .

**5.6. Decomposing Steinberg Varieties.** We can significantly refine case (1) of Proposition 5.4. For any  $\vec{\omega} \in \mathcal{D}^{(vw_\circ)}(\vec{s})$ , let  $\mathbf{Z}_{J,v}^\pm(\vec{s})_{\vec{\omega}}$  be the  $\mathbf{G}$ -stable subvariety of  $\mathbf{Z}_{J,v}^\pm(\vec{s})$  defined by

$$\mathbf{Z}_{J,v}^\pm(\vec{s})_{\vec{\omega}} = \{(\vec{g}\mathbf{B}, u, y\mathbf{P}_J) \in \mathbf{Z}_{J,v}^\pm(\vec{s}) \mid \mathbf{P}_J y^{-1} g_i \mathbf{B} = \mathbf{P}_J v w_\circ \omega_{(i)} w_\circ \mathbf{B}\}.$$

This subvariety only depends on  $W_J v w_\circ$ , even though  $\vec{\omega}$  depends on  $vw_\circ$  itself. Let  $\check{\mathbf{Z}}_{J,v}^-(\vec{s})_{\vec{\omega}}$  be the analogue of  $\mathbf{Z}_{J,v}^-(\vec{s})_{\vec{\omega}}$  with  $y\mathbf{B}$  in place of  $y\mathbf{P}_J$ . By pulling back Lemma 3.2 and Proposition 3.3 along  $pr_{0,\ell} : \mathbf{O}(\vec{s})_{\vec{\omega}} \rightarrow (\mathbf{G}/\mathbf{B})^2$ , we obtain:

**Proposition 5.7.** *If  $v \in W^{J,-}$  and  $\vec{\omega} \in \mathcal{D}^{(vw_\circ)}(\vec{s})$ , then the maps  $\check{\mathbf{E}}_{J,v}^- \xrightarrow{\sim} \mathbf{E}_{J,v}^-$  and  $\check{f} : \check{\mathbf{E}}_{J,v}^- \rightarrow \mathbf{O}(v^{-1}, v) = \mathbf{X}^{(vw_\circ)}$  of §3.3 fit into a cartesian diagram:*

$$(5.8) \quad \begin{array}{ccc} \mathbf{Z}_{J,v}^-(\vec{s})_{\vec{\omega}} & \longrightarrow & \mathbf{E}_{J,v}^- \\ \wr \uparrow & & \uparrow \wr \\ \check{\mathbf{Z}}_{J,v}^-(\vec{s})_{\vec{\omega}} & \longrightarrow & \check{\mathbf{E}}_{J,v}^- \\ \downarrow & & \downarrow \check{f} \\ \mathbf{X}^{(vw_\circ)}(\vec{s})_{\vec{\omega}} & \longrightarrow & \mathbf{X}^{(vw_\circ)} \end{array}$$

Hence,  $\check{\mathbf{Z}}_{J,v}^-(\vec{s})_{\vec{\omega}} \rightarrow \mathbf{X}^{(vw_\circ)}(\vec{s})_{\vec{\omega}}$  forms a smooth fiber bundle that restricts to a  $\mathbf{U}_{w_{J \circ v}}$ -torsor over the subvariety  $(h\mathbf{B}, x\mathbf{B}) = (v\mathbf{B}, \mathbf{B})$ .

**Corollary 5.8.** *If  $v \in W^{J,-}$ , then the  $\mathbf{Z}_{J,v}^\pm(\vec{s})_{\vec{\omega}}$  are pairwise disjoint and partition  $\mathbf{Z}_{J,v}^\pm(\vec{s})$  as  $\vec{\omega}$  runs over  $\mathcal{D}^{(vw_\circ)}(\vec{s})$ .*

*Proof.* Proposition 5.7 shows that if  $v \in W^{J,-}$ , then  $\mathbf{Z}_{J,v}^-(\vec{s})_{\vec{\omega}}$  arises from  $\mathbf{X}^{(vw_\circ)}(\vec{s})_{\vec{\omega}}$  by pullback. This establishes the statement for the  $-$  case. But the condition defining  $\mathbf{Z}_{J,v}^\pm(\vec{s})_{\vec{\omega}} \subseteq \mathbf{Z}_{J,v}^\pm(\vec{s})$  does not involve the coordinate  $u$  by which  $\mathbf{Z}_{J,v}^-(\vec{s})$  and  $\mathbf{Z}_{J,v}^+(\vec{s})$  differ. So we also get the statement for the  $+$  case.  $\square$

**Corollary 5.9.** *If  $v \in W^{J,-}$  and  $\vec{\omega} \in \mathcal{D}^{vw_\circ}(\vec{s})$ , then*

$$\begin{aligned} |Z_{J,v}^-(\vec{s})_{\vec{\omega}}| &= |U_{w_{J \circ v}}| |X^{(vw_\circ)}(\vec{s})|, \\ &= |G| q^{|\mathbf{d}_{\vec{\omega}}| - \ell_J} (q-1)^{|\mathbf{e}_{\vec{\omega}}| - \text{rk}(G)}, \end{aligned}$$

refining the  $-$  cases of Proposition 5.4 and Corollary 5.5.

Moreover, the  $\mathbf{G}$ -equivariant étale cohomology of  $\mathbf{Z}_{J,v}^-(\vec{s})_{\vec{\omega}}$  with  $\bar{\mathbf{Q}}_\ell$ -coefficients is isomorphic to the  $\mathbf{T}$ -equivariant étale cohomology of  $\mathbf{R}^{(vw_\circ)}(\vec{s})_{\vec{\omega}}$ . The analogous statement for compactly-supported cohomology holds up to a shift of degree  $\ell_J$ .

*Proof.* The first claim follows from Proposition 5.7 by taking  $F$ -fixed points. As for the second, let  $H_c^*$  denote compactly-supported étale cohomology. Then

$$\begin{aligned} H_{c,\mathbf{G}}^*(\mathbf{Z}_{J,v}^-(\vec{s})_{\vec{\omega}}) &\simeq H_{c,\mathbf{G}}^*(\mathbf{X}^{(vw_\circ)}(\vec{s})_{\vec{\omega}})[\dim \mathbf{U}_{w_{J \circ v}}] && \text{by Proposition 5.7} \\ &\simeq H_{c,\mathbf{B}}^*(\mathbf{R}^{(vw_\circ)}(\vec{s})_{\vec{\omega}})[\dim \mathbf{U}_{w_{J \circ v}} + \dim \mathbf{U}_{vw_\circ}] && \text{by Corollary 5.3} \\ &\simeq H_{c,\mathbf{T}}^*(\mathbf{R}^{(vw_\circ)}(\vec{s})_{\vec{\omega}})[\dim \mathbf{U}_{w_{J \circ v}} + \dim \mathbf{U}_{vw_\circ} - \dim \mathbf{U}] && \text{since } \mathbf{B} = \mathbf{T} \ltimes \mathbf{U}. \end{aligned}$$

Finally,  $\dim \mathbf{U}_{w_{J \circ v}} + \dim \mathbf{U}_{vw_\circ} - \dim \mathbf{U} = -\ell_J$  by Lemma 2.1. The statements for ordinary cohomology are the same, except there are no shifts.  $\square$



*Remark 5.10.* For  $J = S$ , we require  $v = e$  in (5.8) and the vertical arrows become trivial, giving isomorphisms  $\mathbf{E}_{\vec{S}} \simeq \check{\mathbf{E}}_{\vec{S}} \simeq \mathbf{G}/\mathbf{B}$  and  $\mathbf{Z}_{\vec{S}}(\vec{s}) \simeq \mathbf{X}^{(w_\circ)}(\vec{s})$ .

When  $G = \mathrm{PGL}_n(\mathbf{F})$ , so that  $W = S_n$ , and  $\beta$  is the positive braid on  $n$  strands defined by  $\vec{s}$ , the stack denoted  $\mathcal{M}(\beta^\circ)$  in [STZ17] is precisely  $[\mathbf{X}^{(w_\circ)}(\vec{s})/\mathbf{G}]$ . Their Proposition 6.31 gives a decomposition of another stack  $\mathcal{M}(\beta^\succ)$  into substacks indexed by rulings of a Legendrian link  $\beta^\succ$ . At the same time,

$$\mathcal{M}(\beta^\succ) \simeq \mathcal{M}((\Delta\beta\Delta)^\circ) \simeq \mathcal{M}((\beta\Delta^2)^\circ),$$

where  $\Delta$  is the *half-twist*: the minimal positive braid that lifts  $w_\circ \in S_n$ . (Note that  $\mathcal{M}((\Delta\beta\Delta)^\circ)$  is also isomorphic to  $[\mathbf{X}^{(e)}(\vec{s})/\mathbf{G}]$ .)

In this way, the varieties in our work generalize the stacks  $\mathcal{M}(\beta^\circ)$  and  $\mathcal{M}(\beta^\succ)$  in [STZ17]. The Deodhar-type decomposition of  $\mathbf{Z}_{J,v}^-(\vec{s})$  into subvarieties  $\mathbf{Z}_{J,v}^-(\vec{s})_{\vec{\omega}}$  seems to recover the ruling decomposition of  $\mathcal{M}(\beta^\succ)$  in [STZ17]. We leave the precise relationship to future work.

*Remark 5.11.* In §5.3, the passage from  $\mathbf{X}^{(v)}$  to  $\mathbf{X}_{\square}^{(v)}$  to  $\mathbf{R}^{(v)}$  encoded a passage from  $\mathbf{G}$ -symmetry to  $\mathbf{B}$ -symmetry to  $\mathbf{B}_v^-$ -symmetry. Instead of relating the Steinberg varieties and their strata to the  $\mathbf{B}$ -varieties  $\mathbf{X}^{(v)}$ , we could have used  $\mathbf{B}$ -varieties

$$\begin{aligned} \mathbf{Z}_{J,\square}^-(\vec{s}) &= \{(\vec{g}\mathbf{B}, u) \in \mathbf{O}(\vec{s}) \times \mathbf{U}_J \mid g_\ell \mathbf{B} = u g_0 \mathbf{B}\}, \\ \mathbf{Z}_{J,\square}^+(\vec{s}) &= \{(\vec{g}\mathbf{B}, u) \in \mathbf{O}(\vec{s}) \times \mathbf{V}_J \mid g_\ell \mathbf{B} = u g_0 \mathbf{B}\} \end{aligned}$$

and corresponding strata cut out by conditions of the form  $\mathbf{P}_J h \mathbf{B} = \mathbf{P}_J v \mathbf{B}$ . This is the approach in our FPSAC 2025 abstract.

Analogues of Proposition 5.7 and Corollary 5.9 hold for the  $\square$  versions. In fact, the  $\mathbf{G}$ -equivariant cohomology of  $\mathbf{Z}_{J,v}^-(\vec{s})_{\vec{\omega}}$  matches the  $\mathbf{B}$ -equivariant cohomology of its  $\square$  version, by construction.

## 6. PARKING NUMBERS

6.1. *In this subsection and the next,  $(W, S)$  denotes an arbitrary irreducible, finite Coxeter system with Coxeter number  $h$ . We write  $\mathbf{V}$  to denote the irreducible reflection representation of  $W$ , and  $\chi_{\mathbf{V}}$  to denote its character.*

For any integer  $p$ , let  $\mathbf{V}_p$  denote the *Galois conjugate* of  $\mathbf{V}$  that has the same underlying vector space but character given by  $\chi_{\mathbf{V}_p}(w) = \chi_{\mathbf{V}}(w^p)$ . If  $W$  is crystallographic and  $p$  is coprime to  $h$ , then  $\mathbf{V}_p \simeq \mathbf{V}$ .

For any integer  $k \geq 0$ , we set  $[k]_{\mathbf{q}} = 1 + \mathbf{q} + \cdots + \mathbf{q}^{k-1}$ . Generalizing the formula in §1.5 for the crystallographic case, we define the *rational parabolic  $\mathbf{q}$ -parking numbers* of  $(W, p, J)$  to be

$$(6.1) \quad \mathrm{Park}_{W,p}^{J,\pm}(\mathbf{q}) = \prod_{i=1}^{|J|} \frac{[p \pm e_i^{J,p}]_{\mathbf{q}}}{[d_i^J]_{\mathbf{q}}},$$

where  $d_1^J, \dots, d_{|J|}^J$  are the fundamental degrees of  $W_J$ , and  $e_1^{J,p}, \dots, e_{|J|}^{J,p}$  are the *exponents* or *fake degrees* of the  $W_J$ -action on  $\mathbf{V}_p^*$ , as defined in [BR11].

Recall that a Coxeter word in  $S$  is a word  $\vec{c}$  formed by placing the elements of  $S$  in any order. We write  $\vec{c}^p$  for the concatenation of  $p$  copies of  $\vec{c}$ . The goal of this section is the following identity, which implies Corollary 1.5 in the  $q \rightarrow 1$  limit.

**Theorem 6.1.** *If  $W$  is crystallographic, then for any Coxeter word  $\vec{c}$  in  $S$ , integer  $p > 0$  coprime to  $h$ , and subset  $J \subseteq S$ , we have*

$$\text{Park}_{W,p}^{J,\pm}(q) = \frac{1}{(q-1)^{|S|}} \sum_{v \in W^{J,\mp}} \sum_{\vec{\omega} \in \mathcal{D}^{(v)}(\vec{c}^p)} q^{|\mathbf{d}\vec{\omega}|} (q-1)^{|\mathbf{e}\vec{\omega}|}.$$

(Note the sign flip.)

**Conjecture 6.2.** *Theorem 6.1 generalizes to any irreducible finite Coxeter system when  $\text{Park}_{W,p}^{J,\pm}(q)$  is defined using (6.1).*

**6.2. From Products to Traces.** We continue to allow non-crystallographic  $W$ . Let  $\mathbf{K}$  be a splitting field for  $W$ , so that  $V_p$  is defined over  $\mathbf{K}$ . When  $W$  is crystallographic, we can take  $\mathbf{K} = \mathbf{Q}$ .

There is a graded representation  $L_{p/h} = \bigoplus_i L_{p/h}^i$  of  $W$  that may be called the *rational parking space* for  $(W, p)$ , in the spirit of [ARR15, ALW16], as its graded dimension is  $[p]_q^r$ . Explicitly,  $L_{p/h}$  is the representation of  $W$  underlying the simple spherical module of the rational Cherednik algebra of  $W$  at parameter  $p/h$ , equipped with a shift of the  $W$ -stable grading arising from the Euler element.

We view the graded character of  $L_{p/h}$  as a  $\mathbf{K}[q]$ -valued trace on  $\mathbf{K}W$ . To describe it explicitly, let  $S = \bigoplus_i S^i$  and  $\Lambda_p = \bigoplus_j \Lambda_p^j$ , where

$$S^i := \text{Sym}^i(V^*) \quad \text{and} \quad \Lambda_p^j := \bigwedge^j(V_p^*).$$

Then for all  $w \in W$ , we have

$$\begin{aligned} (6.2) \quad \sum_i q^i \text{tr}(w | L_{p/h}^i) &= \left[ \sum_{i,j} q^i t^j \text{tr}(w | S^i \otimes \Lambda_p^j) \right] \Big|_{t \rightarrow -q^p} \\ &= \frac{\det(1 - q^p w | V_p^*)}{\det(1 - qw | V^*)}. \end{aligned}$$

This formula arises from a so-called BGG-resolution of  $L_{p/h}$  by Verma modules for the rational Cherednik algebra, whose underlying  $W$ -representations take the form  $S \otimes \bigwedge^j$ .

**Proposition 6.3.** *For any integer  $p > 0$  coprime to  $h$  and subset  $J \subseteq S$ , we have*

$$\text{Park}_{W,p}^{J,\pm}(q) = \sum_i q^i \text{tr}(e_{J,\pm} | L_{p/h}^i).$$

*Proof.* We only do the  $+$  case, as the  $-$  case is similar.

Set  $U = V_p$ . Using the reflecting hyperplanes for  $S$ , we can decompose the  $W_J$ -action on  $V$  as a direct sum  $V \simeq V_J \oplus V_J^\Gamma$ , where  $V_J^\Gamma$  is a  $(|S| - |J|)$ -fold power of the trivial representation. Applying the Galois twist and grading shift that take  $V$  to

$U(-p)$ , we get a direct sum  $U(-p) \simeq U_J(-p) \oplus U_J^T(-p)$ , where  $U_J(-p) \simeq (V_J)_p(-p)$  and  $U_J^T(-p)$  remains a  $(|S| - |J|)$ -fold power of the trivial representation.

Therefore, the fake degrees for  $U(-p)$  as a representation of  $W_J$  are formed by taking the  $|J|$  fake degrees for  $(V_J)_p$ , appending  $|S| - |J|$  zeroes, and shifting everything up by  $p$ . In particular,  $U(-p)$  satisfies the hypothesis in Theorem 3.1 and Corollary 3.2 of [OS80] that the sum of the fake degrees is equal to the fake degree for its  $|S|$ th exterior power. We deduce that  $(S \otimes \wedge U(-p))^{W_J}$  remains isomorphic to an exterior algebra over  $S^{W_J}$ . So we arrive at the formula

$$\sum_{i,j} q^i t^j \dim (S^i \otimes \wedge_p^j)^{W_J} = \prod_i \frac{1 + t q^{p+e_i^J}}{1 - q^{d_i^J}},$$

which gives the desired product formula at  $t \rightarrow -1$ .  $\square$

**Example 6.4.** Taking  $J = \emptyset$  and  $J = S$  in Proposition 6.3, we recover the formulas

$$\text{Cat}_{W,p}(q) = \sum_i q^i \dim (\mathbb{L}_{p/h}^i)^W \quad \text{and} \quad [p]_q^r = \sum_i q^i \dim \mathbb{L}_{p/h}^i,$$

respectively.

**6.3. From Traces to Cells.** Recall the notation  $T_{\vec{c}} \in H_W$  from §5.5. In [Tri21], the first author showed that the value at  $T_{\vec{c}}$  of the trace on  $H_W$  corresponding to  $\tau_G$  is the graded character of  $\mathbb{L}_{p/h}$  up to a shift. In our notation, this is the identity

$$(6.3) \quad \tau_G(w \otimes T_{\vec{c}}^p) = \sum_i q^i \text{tr}(w \mid \mathbb{L}_{p/h}^i).$$

Now assume that  $W$  is crystallographic. Pick split semisimple  $G$  with Weyl group  $W$ . In this case,

$$\begin{aligned} \text{Park}_{W,p}^{J,\pm}(q) &= \sum_i q^i \text{tr}(e_{J,\pm} \mid \mathbb{L}_{p/h}^i). && \text{by Proposition 6.3} \\ &= \tau_G(e_{J,\pm} \otimes T_{\vec{c}}^p) && \text{by (6.3)} \\ &= \frac{1}{(q-1)^{|S|}} \sum_{v \in W^{J,\pm}} q^{-\ell(v)} \tau(T_{\vec{c}}^p T_{v^{-1}} T_v) && \text{by Theorem 1.2.} \end{aligned}$$

Applying (5.7) to the last expression, we get Theorem 6.1.

## 7. MARKOV TRACES AND KIRKMAN NUMBERS

**7.1.** In this section, we prove Theorem 1.7 and Corollary 1.9. Along the way, we review Markov traces, the HOMFLYPT polynomial, and rational Kirkman polynomials. Unless otherwise specified,  $W = S_n$  and  $S = \{s_1, \dots, s_{n-1}\}$ , as in §4.7.

**7.2. Markov Traces and HOMFLYPT.** As explained in [Jon87] (in a different normalization), there is a unique family of traces

$$\mu_n : H_{S_n} \rightarrow \mathbf{Q}(q^{1/2})[a^{\pm 1}]$$

satisfying these conditions:

- (1)  $\mu_1(1) = 1$ .
- (2) For all  $\beta \in H_{S_{n-1}}$ , we have

$$\mu_{n+1}(\beta T_{s_n}^{\pm 1}) = (-a^{-1} \mathbf{q}^{1/2})^{\pm 1} \mu_n(\beta).$$

In particular,  $\mu_{n+1}(\beta) = \frac{a - a^{-1}}{\mathbf{q}^{1/2} - \mathbf{q}^{-1/2}} \mu_n(\beta)$ , due to the quadratic relation on  $T_{s_n}$ .

These traces give rise to an isotopy invariant of (tame) topological links.

Namely: Any topological braid on  $n$  strands  $\beta$  defines an element of  $H_{S_n}$ , which we again denote by  $\beta$ , via the map from the braid group to  $H_{S_n}$  that sends the  $i$ th positive simple twist  $\sigma_i$  to the element  $\mathbf{q}^{-1/2} T_{s_i}$ . For instance, if  $\vec{s} = (s_{i_1}, \dots, s_{i_\ell})$ , then this map sends the positive braid  $\sigma_{i_1} \cdots \sigma_{i_\ell}$  to the element  $\mathbf{q}^{-\ell} T_{\vec{s}}$ . At the same time, closing up  $\beta$  by wrapping it around a solid torus, then embedding it into 3-space, defines a link  $\hat{\beta}$  up to isotopy, called the *closure* of  $\beta$ . Ocneanu showed that if  $e(\beta) \in \mathbf{Z}$  is the *writhe* of  $\beta$ , meaning its length with respect to positive simple twists, then

$$\mathbf{P}(\hat{\beta}) := (-a)^{e(\beta)} \mu_n(\beta) \in \mathbf{Q}(\mathbf{q}^{1/2})[a^{\pm 1}]$$

only depends on  $\hat{\beta}$ .

The Laurent polynomial  $\mathbf{P}(\hat{\beta})$  is now called its *reduced HOMFLYPT polynomial*, after its discoverers. (The “O” stands for Ocneanu; the adjective “reduced” means that the normalization satisfies  $\mathbf{P}(\text{unknot}) = 1$ .) The traces  $\mu_n$  are called *Markov traces*, as condition (2) in their definition corresponds to the so-called second Markov move on braids. For further details, see [Jon87].

In [Gom06], Y. Gomi introduced a uniform generalization of the traces  $\mu_n$  to finite Coxeter groups  $W$ . In [WW15], Webster–Williamson gave a construction of Gomi’s traces from weight filtrations on the cohomology of mixed sheaves. Building on their work, the main result of [Tri21] relates a categorification of Gomi’s traces to a Springer-type action of  $W$  on the weight-filtered,  $\mathbf{G}$ -equivariant cohomology of the Steinberg varieties  $\mathbf{Z}_\emptyset^-(\vec{s}) = \mathbf{Z}_\emptyset^+(\vec{s})$ .

**7.3. Individual  $a$ -Degrees.** Induction on  $|e(\beta)|$  shows that if  $\beta \in H_{S_n}$  arises from a topological braid, then the only exponents of  $a$  that can occur in  $\mu(\beta)$  are

$$-n + 1, \quad -n + 3, \quad \dots, \quad n - 1.$$

For  $0 \leq k \leq n - 1$ , we define  $\mu_n^{(k)} : H_{S_n} \rightarrow \mathbf{Q}(\mathbf{q}^{1/2})$  by

$$\mu_n^{(k)}(\beta) = \mathbf{Q}(\mathbf{q}^{1/2})\text{-coefficient of } a^{-n+1+2k} \text{ in } \mu_n(\beta).$$

By linearity, this is still a trace.

When  $G$  is (split) semisimple of type  $A_{n+1}$ , the formula for categorified traces in [Tri21] decategorifies to a formula relating  $\mu_n^{(k)}$  to  $\tau_G$ . To state it, let  $e_{\wedge^k} \in \mathbf{Q}S_n$  be the symmetrizer for the  $k$ th exterior power of the reflection representation  $\mathbf{V} \simeq \mathbf{V}^*$ .

For any finite, irreducible Coxeter group  $W$  of rank  $r = |S|$ , such elements  $e_{\wedge^k} \in \mathbf{Q}W$  may be defined for  $0 \leq k \leq r$  through the formal identity

$$(7.1) \quad \frac{1}{|W|} \sum_{w \in W} \det(1 + tw \mid \mathbf{V}) = \sum_{k=0}^r t^k e_{\wedge^k}.$$

Note that  $e_{\wedge^0} = e_{S,+}$  and  $e_{\wedge^{n-1}} = e_{S,-}$ , in the notation of §4.4. For  $G$  (split) semisimple of type  $A_{n+1}$ , we have:

$$(7.2) \quad \mu_n^{(k)} = (\mathbf{q} - 1)^{n-1} \tau_G(e_{\wedge^k} \otimes ( \quad )).$$

To summarize the proof: One starts from the analogue of (??) with  $G, \mathbf{V}$  in place of  $\mathrm{GL}_n, \mathbf{V}_n$ , then rearranges terms using (7.1) to arrive at the character-theoretic formula for  $\mu_n^{(k)}$  in [Gom06, §4.3].

Meanwhile, in [BT22], Bezrukavnikov–Tolmachov gave a formula that (in our normalization) relates  $\mu_n^{(k)}$  to  $\mu_n^{(n-1)}$ . To state it, we need the *multiplicative Jucys–Murphy elements*  $JM_k \in H_{S_n}$  defined by

$$JM_k = \mathbf{q}^{1-k} T_{s_{k-1} \cdots s_2 s_1} T_{s_1 s_2 \cdots s_{k-1}} \quad \text{for } 1 \leq k \leq n.$$

Let  $e_i(X_1, \dots, X_{n-1})$  be the elementary symmetric polynomial of degree  $i$  in variables  $X_1, \dots, X_{n-1}$ . Then [BT22, Cor. 6.1.1] is the identity

$$(7.3) \quad \mu_n^{(k)}(\beta) = \mu_n^{(n-1)}(\beta e_{n-1-k}(JM_1, \dots, JM_{n-1})).$$

It turns out that  $\mu_n^{(n-1)}$  is precisely the trace denoted  $\tau$  in §1.3, as one can also deduce from (7.2) and Theorem 1.2.

*Remark 7.1.* Jucys–Murphy elements were originally defined in the context of the group rings  $\mathbf{Z}S_n$ . One can show [IO05, (3)] that

$$\frac{JM_k - 1}{\mathbf{q} - 1} = \sum_{i=1}^{k-1} \mathbf{q}^{i-k} T_{s_{k-1} \cdots s_{i+1}} T_{s_i} T_{s_{i+1} \cdots s_{k-1}}.$$

At  $\mathbf{q} \rightarrow 1$ , the right-hand side specializes to the  $k$ th classical Jucys–Murphy element in  $\mathbf{Z}S_n$ . These elements generate a maximal commutative subalgebra of  $\mathbf{Z}S_n$ . Similarly, the  $JM_k$  generate a maximal commutative subalgebra of  $H_{S_n}$  [IO05, Prop. 1].

**7.4. Jucys–Murphy Products.** Recall that  $\mathrm{Asc}(v)$  and  $\mathrm{Des}(v)$  respectively denote the left ascent and descent sets of  $v$ . From (7.3), we reduce Theorem 1.7 to:

**Theorem 7.2.** *For all  $k$ , we have*

$$e_{n-1-k}(JM_1, \dots, JM_{n-1}) = \sum_{\mathrm{Des}(v)=I_k} \mathbf{q}^{-\ell(v)} T_{v^{-1}} T_v,$$

where  $I_k = \{s_1, \dots, s_{n-1-k}\} \subseteq S$ .

**Example 7.3.** Taking  $k = 0$  above, we get

$$JM_1 \cdots JM_{n-1} = \mathbf{q}^{-\ell_S} T_{w_0}^2.$$

Through this identity, (7.3) implies that the “lowest” and “highest”  $a$ -degrees of  $\mu_n$  are related by the *full twist*  $\Delta^2 := \mathbf{q}^{-\ell_S} T_{w_0}^2$ : explicitly,

$$\mu_n^{(0)}(\beta) = \mu_n^{(n-1)}(\beta \Delta^2),$$

an identity originally discovered by Kálmán [Kál09]. Compare to Remark 5.10.

The proof of Theorem 7.2 amounts to Lemmas 7.4–7.6 below. As preparation, for any subset  $I \subseteq \{1, \dots, n-1\}$ , let

$$JM(I) = \prod_{i \in I}^\downarrow JM_i \in H_{S_n} \quad \text{and} \quad c(I) = \prod_{i \in I}^\downarrow (s_1 \cdots s_i) \in S_n,$$

where the notation  $\prod_{i \in I}^\downarrow$  means the product over  $I$  in decreasing order.

**Lemma 7.4.** *For any subset  $I \subseteq \{1, \dots, n\}$ , we have*

$$JM(I) = \mathbf{q}^{-\ell(c(I))} T_{c(I)^{-1}} T_{c(I)}.$$

*Proof.* Let  $i_1 < i_2 < \cdots < i_j$  be the elements of  $I$ . For any  $i, k$  with  $1 \leq k < i \leq n-1$ , we have the relations

$$T_{s_k} T_{s_i \cdots s_2 s_1} = T_{s_i \cdots s_2 s_1} T_{s_{k+1}} \quad \text{and} \quad T_k T_{s_1 s_2 \cdots s_i} = T_{s_1 s_2 \cdots s_i} T_{s_{k-1}},$$

as one can check from braid diagrams. Using these relations, we can move the prefixes  $T_{s_1 s_2 \cdots s_i}$  in each factor  $JM_{i_k}$  of  $JM(I)$  from right to left, through each of  $JM_{i_{k+1}}, \dots, JM_{i_j}$ , giving the result.  $\square$

**Example 7.5.** In what follows, we omit brackets from  $I$  for clarity. When  $n = 4$  and  $|I| = 2$ , we have

$$\begin{aligned} JM(1, 2) &= \mathbf{q}^{-3} T_2 T_1^2 T_2 \cdot T_1^2, & c(1, 2) &= (s_1 s_2) \cdot s_1, \\ JM(1, 3) &= \mathbf{q}^{-4} T_3 T_2 T_1^2 T_2 T_3 \cdot T_1^2, & c(1, 2) &= (s_1 s_2 s_3) \cdot s_1, \\ JM(2, 3) &= \mathbf{q}^{-5} T_3 T_2 T_1^2 T_2 T_3 \cdot T_2 T_1^2 T_2, & c(2, 3) &= (s_1 s_2 s_3) \cdot (s_1 s_2), \end{aligned}$$

Lemma 7.4 says that

$$\begin{aligned} JM(1, 2) &= \mathbf{q}^{-3} (T_1 T_2 T_1) \cdot (T_1 T_2 T_1), \\ JM(1, 3) &= \mathbf{q}^{-4} T_1 \cdot (T_3 T_2 T_1) \cdot (T_1 T_2 T_3) \cdot T_1, \\ JM(2, 3) &= \mathbf{q}^{-5} (T_2 T_1) \cdot (T_3 T_2 T_1) \cdot (T_1 T_2 T_3) \cdot (T_1 T_2). \end{aligned}$$

**Lemma 7.6.** *For  $1 \leq j \leq n$ , we have*

$$\{c(I) \mid |I| = j\} = \{v \in S_n \mid \text{Des}(v) = \{s_1, \dots, s_j\}\}.$$

*Proof.* Let  $J = \{s_1, \dots, s_{j-1}\}$  and  $J' = J \cup \{s_{j+1}, \dots, s_{n-1}\}$  in what follows. We claim that any element  $v \in W$  with  $\text{Des}(v) = J$  must take the form  $w_{J_0} v'$ , where  $v'$  is a minimal-length right coset representative of  $W_{J'} \simeq S_j \times S_{n-j}$ . Indeed,  $\text{Des}(v) \supseteq J$  forces  $w_{J_0}$  to be a left factor of  $v$ , and if  $v = w_{J_0} v'$ , then the reverse inclusion  $\text{Des}(v) \subseteq J$  forces the condition on  $v'$ .

Note that there are exactly  $\binom{n}{j}$  elements of the form  $w_{J_0}v'$  with  $v' \in W^{J',-}$ . We claim that they are exactly the elements  $c(I)$  with  $|I| = j$ . Indeed, if we write  $i_1 < i_2 < \dots < i_j$  to denote the elements of  $I$ , then  $c(I)$  has the inversion set illustrated in Figure 1. This calculation shows that  $\prod_{i \in I}^\downarrow$  is reduced and that  $\text{Des}(c(I)) = J$ . As there are  $\binom{n}{j}$  choices for  $I$  such that  $|I| = j$ , the corresponding elements  $c(I)$  exhaust the elements  $v$  such that  $\text{Des}(v) = J$ .  $\square$

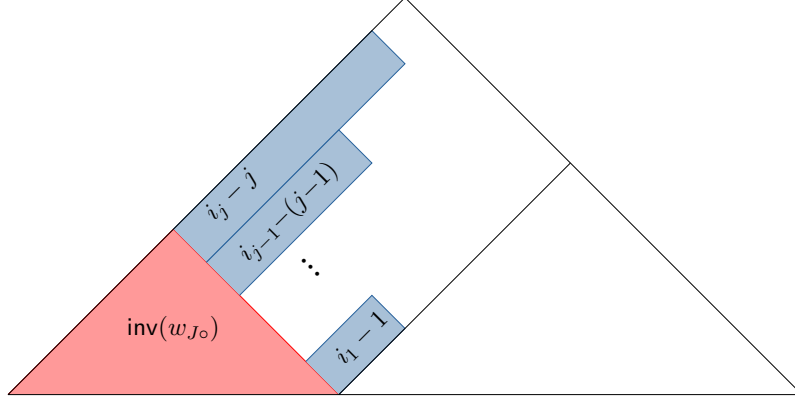


FIGURE 1. The inversions of  $c(I)$ , where  $I$  consists of  $i_1 < \dots < i_j$  and  $J = \{1, \dots, j\}$ . The  $i$ th diagonal from bottom left to top right consists of the transpositions  $(i, i+1), (i, i+2), \dots, (i, n+1)$ . The inversions in the bottom left triangle are the inversions of  $w_{J_0}$ . The remaining inversions take the form  $(k, j+2), (k, j+3), \dots, (k, i_k + j - k + 1)$ . ???

**Corollary 7.7.** *For any word  $\vec{s}$  in  $S = \{s_1, \dots, s_{n-1}\}$ , we have*

$$\mu_n^{(k)}(T_{\vec{s}}) = \frac{1}{(\mathbf{q} - 1)^{n-1}} \sum_{\text{Asc}(v) = I_k} \sum_{\vec{\omega} \in \mathcal{D}^{(v)}(\vec{s})} \mathbf{q}^{|\mathbf{d}_{\vec{\omega}}|} (\mathbf{q} - 1)^{|\mathbf{e}_{\vec{\omega}}|}.$$

*Proof.* Combine (7.3), Theorem 7.2, and (5.7) to arrive at a double sum over  $v$  such that  $\text{Asc}(v) = I_k$  and  $\vec{\omega}$  in  $\mathcal{D}^{(vw_0)}(\vec{s})$ . Then note that  $\ell(sv) < \ell(v)$  if and only if  $\ell(svw_0) > \ell(svw_0)$ .  $\square$

**7.5. Kirkman Numbers.** For any finite, irreducible Coxeter group  $W$  of rank  $r$  and Coxeter number  $h$ , and integer  $p > 0$  coprime to  $h$ , we define the *rational Kirkman polynomials* of  $(W, p)$  to be

$$\text{Kirk}_{W,p}^{(k)}(\mathbf{q}) = \frac{\det(1 - \mathbf{q}^p e_{\wedge^k} \mid \mathbf{V}_p^*)}{\det(1 - \mathbf{q} e_{\wedge^k} \mid \mathbf{V}^*)} \quad \text{for } 0 \leq k \leq r.$$

Equivalently, by (7.1),

$$\sum_{k=0}^r t^k \text{Kirk}_{W,p}^{(k)}(\mathbf{q}) = \frac{1}{|W|} \sum_{w \in W} \frac{\det(1 + tw \mid \mathbf{V}^*) \det(1 - \mathbf{q}^p w \mid \mathbf{V}_p^*)}{\det(1 - \mathbf{q} w \mid \mathbf{V}^*)}.$$

When  $p = h + 1$ , this definition recovers the *Kirkman polynomials* of  $W$  introduced in [ARR15, §9.2].

We define the *rational Kirkman numbers* of  $(W, p)$  by  $\text{Kirk}_{W,p}^{(k)} := \text{Kirk}_{W,p}^{(k)}(1)$ . One combinatorial motivation for these numbers is that at  $p = h + 1$ , they recover the  $f$ -vectors of the  $W$ -associahedron [ARW13].

In the  $q \rightarrow 1$  limit, the following identity implies Corollary 1.9 about the rational Kirkman numbers of  $S_n$ . Figure 2 at the end of the paper illustrates Corollary 1.5 and Corollary 1.9 simultaneously.

**Theorem 7.8.** *Take  $W = S_n$  and  $S = \{s_1, \dots, s_{n-1}\}$ . Then for any Coxeter word  $\vec{c}$ , integer  $p > 0$  coprime to  $n$ , and integer  $k$ , we have*

$$\text{Kirk}_{W,p}^{(k)}(q) = \frac{1}{(q-1)^{n-1}} \sum_{\text{Asc}(v)=I_k} \sum_{\vec{\omega} \in \mathcal{D}^{(v)}(\vec{c}^p)} q^{|\text{d}\vec{\omega}|} (q-1)^{|\text{e}\vec{\omega}|}.$$

*Proof.* Pick any split semisimple  $G$  of type  $A_{n+1}$ . Then

$$\begin{aligned} \text{Kirk}_{W,p}^{(k)}(q) &= \tau_G(e_{\wedge^k} \otimes T_{\vec{c}}^p) && \text{by (6.2) and (6.3)} \\ &= \frac{1}{(q-1)^{n-1}} \tau[\zeta_{I_k}^-](T_{\vec{c}}^p) && \text{by Theorem 1.7} \\ &= \frac{1}{(q-1)^{n-1}} \sum_{\text{Des}(v)=I_k} q^{-\ell(v)} \tau(T_{\vec{c}}^p T_{v^{-1}} T_v). \end{aligned}$$

Apply (5.7) to the sum over  $v$  such that  $\text{Des}(v) = I_k$ . Then conclude as in the proof of Corollary 7.7.  $\square$

**7.6. Other Types?** It is natural to seek generalizations of Corollary 7.7 and Theorem 7.8 to other  $W$ . We have been unable to find such a construction. This may be related to the absence of uniform formulas for Kirkman polynomials in general. Attractive formulas do exist for *coincidental types*, where the degrees of  $W$  form an arithmetic sequence [RSS21, §10].

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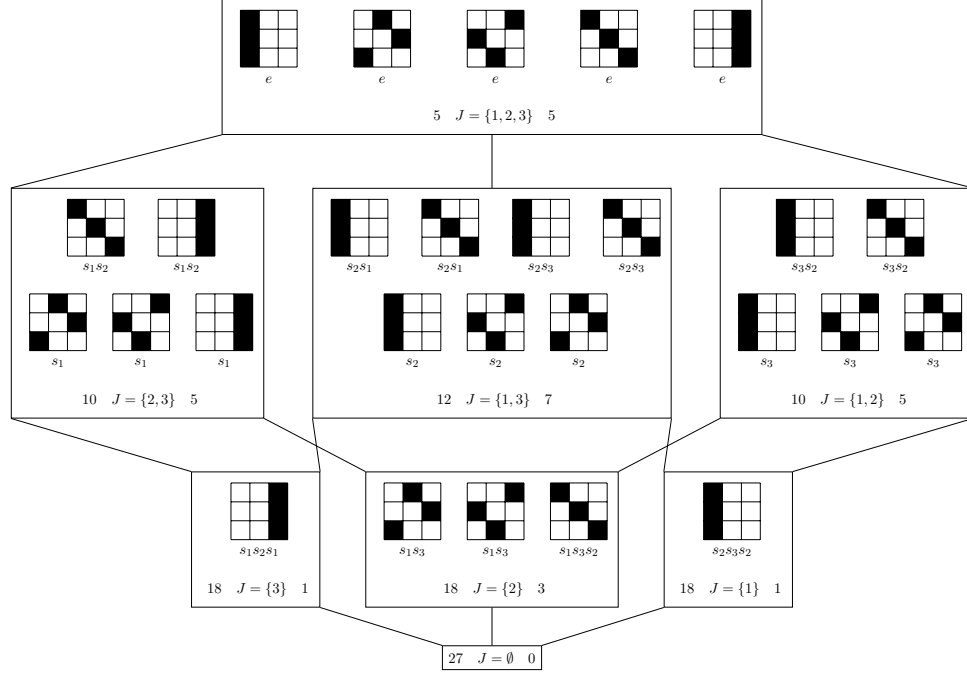


FIGURE 2. We take  $W = S_4$  and  $\vec{c} = (s_1, s_2, s_3)$  and  $p = 3$ . Each box is a set  $\mathcal{D}^J(\vec{c}^p) := \coprod_{v \in W^{J,+}} \mathcal{D}^{(v)}(\vec{c}^p)$  for some  $J$ . Edges between boxes are containments between  $J$ 's. Each  $\vec{w} \in \mathcal{D}^J(\vec{c}^p)$  is a  $3 \times 3$  box, with elements of  $\mathbf{e}_{\vec{w}}$  in black. For example,  $\begin{bmatrix} \blacksquare & & \\ & \blacksquare & \\ & & \blacksquare \end{bmatrix}$  represents  $\vec{w} = (e, s_2, s_3, s_1, e, s_3, s_1, s_2, e)$ . In each box, the number to the left, *resp.* right, of  $J$  is the number of  $w \in W$  with  $\text{Des}(w) \supseteq J$ , *resp.*  $\text{Des}(w) = J$ . The former is  $\text{Park}_{W,p}^{J,+}$ . The rightmost number in the  $(k+1)$ th row is  $\mu_4^k(L_{4,3})|_{q \rightarrow 1}$ . **change to LEFT descents**

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