1.

Notes on truncated convolution. The goal is to prove a theorem that would:

(1) Imply a theorem of Bonnafé–Dudas–Rouquier [BDR]: roughly, by extending to the half-twist $\Delta(w_\circ)$ what they establish for the full twist $\Delta(w_\circ)^{*2}$.
(2) Prove, and in fact refine, a conjecture of Deligne–Lusztig about the structure of $\mathrm{H}^*_c(X_{w_\circ})$ as a representation of G^F [DL25].

1.1. Fix an algebraically closed field $\mathbf{k}$. Fix a connected reductive algebraic group G over $\mathbf{k}$. Let $\mathscr{B}$ be the flag variety of G. Let W be the Weyl group of G, and for each $w \in W$, let $j_w : \mathscr{O}_w \to \mathscr{B} \times \mathscr{B}$ be the inclusion of the corresponding G-orbit. We write e for the identity element of W, and $w_\circ$ for the longest element.

1.2. Suppose that either (I) $\mathbf{k} = \mathbf{C}$, or (II) $\mathbf{k} = \bar{\mathbf{F}}$ for a finite field $\mathbf{F}$ whose characteristic is a good prime for G. In the latter case, we fix a split $\mathbf{F}$-form of G in order to work with Frobenius weights.[1] Let $\mathsf{D} = \mathsf{D}_{G,m}(\mathscr{B} \times \mathscr{B})$ be the G-equivariant, bounded, mixed, constructible derived category of $\mathscr{B} \times \mathscr{B}$, defined in terms of either (I) mixed Hodge modules, or (II) mixed complexes of $\bar{\mathbf{Q}}_\ell$-sheaves, for a fixed prime ℓ invertible in $\mathbf{F}$. Recall that D is endowed with a convolution operation $*$. We write $\langle 1 \rangle$ for the shift-twist $[1](\frac{1}{2})$, where $(\frac{1}{2})$ is a formal half-Tate twist.

Let $\mathbf{C}_w$ be the constant sheaf over $\mathscr{O}_w$. For each $w \in W$, form the objects

$$L(w) = j_{w,!*}\mathbf{C}_w\langle \dim \mathscr{O}_w \rangle, \qquad \Delta(w) = j_{w,!}\mathbf{C}_w\langle \dim \mathscr{O}_w \rangle$$

in D. Observe that we have normalized $L(w), \Delta(w)$ to be perverse and pure of weight 0. The simple perverse sheaves in D are precisely the objects $L(w)$.

1.3. Let $\leq$ be Kazhdan–Lusztig's partial order on two-sided cells of W in which $\{e\}$ is maximal and $\{w_\circ\}$ is minimal. Fix a two-sided cell $\mathbf{c}$. Let $\mathsf{D}^{\leq \mathbf{c}}$, *resp.* $\mathsf{D}^{< \mathbf{c}}$, be the thick (additive) subcategory of D generated by objects K such that for all integers i, the composition factors of the perverse sheaf ${}^p\mathscr{H}^i(K)$ are objects $L(w)$ with $w \in \mathbf{c}'$ for some $\mathbf{c}' \leq \mathbf{c}$, *resp.* $\mathbf{c}' < \mathbf{c}$. Form the Serre quotient category

$$\mathsf{D}^{\mathbf{c}} := \mathsf{D}^{\leq \mathbf{c}}/\mathsf{D}^{< \mathbf{c}},$$

and let $E \mapsto \underline{E} : \mathsf{D}^{\leq \mathbf{c}} \to \mathsf{D}^{\mathbf{c}}$ denote the quotient functor.

Lusztig showed (*e.g.*, in [L15, Lemma 1.4(b)]) that $\mathsf{D}^{\leq \mathbf{c}}$ and $\mathsf{D}^{< \mathbf{c}}$ are stable under left and right convolution with any object of D. Thus, $\mathsf{D}^{\mathbf{c}}$ forms a bimodule category over D with respect to actions induced by convolution. For instance, the left action by an object $K \in \mathsf{D}$ sends $\underline{E} \mapsto \underline{K * E}$ for all $E \in \mathsf{D}^{\leq \mathbf{c}}$.

Let $a_{\mathbf{c}}$ be Lusztig's a-invariant for $\mathbf{c}$, a nonnegative integer. We are interested in the (invertible) endofunctor $\Xi_{\mathbf{c}} : \mathsf{D}^{\mathbf{c}} \to \mathsf{D}^{\mathbf{c}}$ defined by

$$\Xi_{\mathbf{c}}(\underline{E}) = (\underline{\Delta(w_\circ) * E)[a_{\mathbf{c}}]}.$$

[1] We hope to generalize to nonsplit forms later.

Mathas proved [M96] at the level of the triangulated, graded Grothendieck group—*i.e.*, the Hecke algebra—that there is a left-cell-preserving involution $w \mapsto w^!$ such that

$$[\Xi_{\mathbf{c}}(\underline{L(w)})] = [\underline{L(w^!)\langle a_{w_\circ \mathbf{c}}\rangle}] \quad \text{for all } w \in \mathbf{c}.$$

Moreover, as explained in [BDR], it follows from [BFO12, Remark 4.3] that $\Xi_{\mathbf{c}}^2$ is involutive: *i.e.*, isomorphic to the identity functor on $\mathsf{D}^{\mathbf{c}}$.

1.4. Let SBim_W be the category of Soergel bimodules for (W, V), where V is the representation of W on the cocharacter lattice in the root datum of G.

Via the weight realization functor from $\mathsf{K}^b\mathsf{SBim}_W$ into D, recent work of Elias–Hogancamp in [EH24] implies that $\Delta(w_\circ)$ lifts to a twisted Drinfeld center of D, in the following sense. First, [EH24] defines a monoidal involution of $\mathsf{K}^b\mathsf{SBim}_W$. An analogous construction yields an involution of D, which we again denote by Φ. We assume $\Phi \circ \Phi = \mathsf{id}$ from now on. The arguments of [EH24] show that there is an isomorphism of functors

$$\tau : \Delta(w_\circ) * (-) \xrightarrow{\sim} \Phi(-) * \Delta(w_\circ)$$

such that if τ_K is its value at $K \in \mathsf{D}$, then we have

$$(\mathsf{id}_{\Phi(K)} * \tau_L) \circ (\tau_K * \mathsf{id}_L) = \tau_{K*L} \quad \text{for all } K, L \in \mathsf{D}.$$

Loosely, we will refer to τ or similar data on related categories as *Φ-central structures*. Since Φ preserves the thick subcategories $\mathsf{D}^{\leq \mathbf{c}}$, $\mathsf{D}^{< \mathbf{c}}$ and commutes with the shift $[a_{\mathbf{c}}]$, we obtain Φ-central structures on the involutions $\Xi_{\mathbf{c}}$.

Note that if $w_\circ$ is central in W, then Φ is the identity map on objects. This occurs, for instance, in types B, C, D, E_7, E_8.

1.5. Let $\mathsf{C}^{\mathbf{c}}$ be the full subcategory of $\mathsf{D}^{\leq \mathbf{c}}$ whose objects are direct sums of the objects $L(w)$ for $w \in \mathbf{c}$. By construction, $\mathsf{C}^{\mathbf{c}}$ is a semisimple abelian category. Moreover, any morphism in $\mathsf{C}^{\mathbf{c}}$ that factors through $\mathsf{D}^{< \mathbf{c}}$ is already zero, so the composition of functors $\mathsf{C}^{\mathbf{c}} \to \mathsf{D}^{\leq \mathbf{c}} \to \mathsf{D}^{\mathbf{c}}$ is fully faithful. When convenient, we will identify $\mathsf{C}^{\mathbf{c}}$ with its essential image in $\mathsf{D}^{\mathbf{c}}$. We expect to prove:

Conjecture 1.1. *$\Xi_{\mathbf{c}}$ is exact in the perverse t-structure that $\mathsf{D}^{\mathbf{c}}$ inherits from $\mathsf{D}^{\leq \mathbf{c}}$. Equivalently, $\mathsf{C}^{\mathbf{c}}$ is stable under $\Xi_{\mathbf{c}}$.*

Let $\circledast$ be the *truncated convolution* operation on objects of $\mathsf{C}^{\mathbf{c}}$ defined by

$$\underline{E'} \circledast \underline{E} := \underline{{}^p\mathscr{H}^{a_{\mathbf{c}}}(E' * E)}.$$

In [L97], Lusztig showed that the associativity constraint on $*$ descends to one on $\circledast$. In this way, $\mathsf{C}^{\mathbf{c}}$ forms a tensor category. We see that if Conjecture 1.1 holds, then

$$\text{for all } E \in \mathsf{C}^{\mathbf{c}}, \quad \text{we have } \Xi_{\mathbf{c}}(\underline{E}) \simeq \underline{{}^p\mathscr{H}^0(\Delta(w_\circ) * E[a_{\mathbf{c}}])} \simeq \underline{{}^p\mathscr{H}^{a_{\mathbf{c}}}(\Delta(w_\circ) * E)}.$$

This leads us to speculate:

Conjecture 1.2. *There exist a $\circledast$-invertible object $J_{\mathbf{c}} \in \mathsf{C}^{\mathbf{c}}$ and an isomorphism*

$$\Xi_{\mathbf{c}}|_{\mathsf{C}^{\mathbf{c}}} \simeq (\underline{J_{\mathbf{c}}} \circledast -)\langle a_{w_\circ \mathbf{c}}\rangle$$

in the category of endofunctors of $\mathsf{C}^{\mathbf{c}}$. Moreover, this endofunctor categorifies Mathas's involution in the sense that $\underline{J_{\mathbf{c}}} \circledast \underline{L(w)} \simeq \underline{L(w^!)}$ for all $w \in \mathbf{c}$.

If Conjecture 1.2 holds, then the Φ-central structure on $\Xi_{\mathbf{c}}$ can be transported to a Φ-central structure on $J_{\mathbf{c}}$, where we again write Φ for the induced involution on $\mathsf{C}^{\mathbf{c}}$.

1.6. For any finite group $\mathscr{G}$, acting (from the left) on a finite set $\mathscr{X}$, we write $\mathsf{Coh}_{\mathscr{G}}(\mathscr{X})$ to denote the category of $\mathscr{G}$-equivariant coherent $\mathbf{K}$-sheaves on $\mathscr{X}$, where either (I) $\mathbf{K} = \mathbf{C}$, or (II) $\mathbf{K} = \bar{\mathbf{Q}}_\ell$. Recall that an object of this category is a $\mathbf{K}$-vector space V equipped with:

(1) A grading $V = \bigoplus_{x \in \mathscr{X}} V_x$.
(2) A (left) action $G \to \mathrm{GL}(V)$ such that $g \cdot V_x = V_{gx}$ for all $x \in \mathscr{X}$ and $g \in G$.

Assume for now that $\mathbf{c}$ is not an exceptional cell. Let $\mathscr{G}_{\mathbf{c}}$ be the finite group that Lusztig attaches to $\mathbf{c}$. Then by [BFO09, Theorem 4], there exist a finite $\mathscr{G}_{\mathbf{c}}$-set $\mathbf{X}_{\mathbf{c}}$ and an equivalence of tensor categories

$$(\mathsf{Coh}_{\mathscr{G}_{\mathbf{c}}}(\mathbf{X}_{\mathbf{c}} \times \mathbf{X}_{\mathbf{c}}), *) \xrightarrow{\sim} (\mathsf{C}^{\mathbf{c}}, \circledast).$$

In what follows, we write Φ for any endofunctor induced by Φ via an equivalence of tensor categories. We then get an isomorphism of twisted centers:

$$\mathsf{Z}_{\Phi}(\mathsf{Coh}_{\mathscr{G}_{\mathbf{c}}}(\mathbf{X}_{\mathbf{c}} \times \mathbf{X}_{\mathbf{c}})) \xrightarrow{\sim} \mathsf{Z}_{\Phi}(\mathsf{C}^{\mathbf{c}}). \tag{1.1}$$

If Conjecture 1.2 holds, then $J_{\mathbf{c}}$ lifts to an object of $\mathsf{Z}_{\Phi}(\mathsf{C}^{\mathbf{c}})$, hence defines an object of $\mathsf{Z}_{\Phi}(\mathsf{Coh}_{\mathscr{G}_{\mathbf{c}}}(\mathbf{X}_{\mathbf{c}} \times \mathbf{X}_{\mathbf{c}}))$. We expect that in many situations, we can simplify the above Z_{Φ}'s to Z's:

Conjecture 1.3. *If $w_\circ$ commutes with all of $\mathbf{c}$, then Φ is the identity functor on $\mathsf{C}^{\mathbf{c}}$.*

1.7. *Henceforth, we assume that $w_\circ$ commutes with all of $\mathbf{c}$.* By Morita equivalence for module categories over tensor categories, as explained in [EGNO, Example 7.12.19 and Corollary 7.16.2], we have a tensor equivalence

$$\mathsf{Coh}_{\mathscr{G}_{\mathbf{c}}}(\mathscr{G}_{\mathbf{c}}) \xrightarrow{\sim} \mathsf{Z}(\mathsf{Coh}_{\mathscr{G}_{\mathbf{c}}}(\mathbf{X}_{\mathbf{c}} \times \mathbf{X}_{\mathbf{c}})). \tag{1.2}$$

We can now make contact with the recent preprint [DL25] of Deligne–Lusztig.

Let $\{-,-\}$ be the exotic Fourier transform on the Grothendieck group $\mathsf{K}_{0,\mathscr{G}_{\mathbf{c}}}(\mathscr{G}_{\mathbf{c}})$. To describe it explicitly, recall that the isomorphism classes of simple objects in $\mathsf{Coh}_{\mathscr{G}_{\mathbf{c}}}(\mathscr{G}_{\mathbf{c}})$ are indexed by conjugacy classes of pairs (g, η), where $g \in \mathscr{G}_{\mathbf{c}}$ and η is a $\mathbf{K}$-valued irreducible character of the centralizer $Z(g) = Z_{\mathscr{G}_{\mathbf{c}}}(g)$. In this indexing,

$$\{[g,\eta],[g',\eta']\} = \frac{1}{|Z(g)||Z(g')|} \sum_{\substack{h \in \mathscr{G}_{\mathbf{c}} \\ h^{-1}ghg' = g'h^{-1}gh}} \eta(hg'h^{-1})\eta'(h^{-1}g^{-1}h).$$

Lusztig previously observed that

$$\{[g,\eta],[g',\eta']*[g'',\eta'']\} = \frac{|Z(g)|}{\eta(1)}\{[g,\eta],[g',\eta']\}\{[g,\eta],[g'',\eta'']\}.$$

That is, for fixed $[g,\eta]$, the linear map $\mathsf{K}_{0,\mathscr{G}_{\mathbf{c}}}(\mathscr{G}_{\mathbf{c}}) \to \mathbf{K}$ defined by

$$[g',\eta'] \mapsto \frac{|Z(g')|}{\eta'(1)}\{[g,\eta],[g',\eta']\}$$

is a ring homomorphism.

Theorem 1.4 (Deligne–Lusztig)**.** *For any W and two-sided cell $\mathbf{c} \subseteq W$, there is an invertible simple object $m_{\mathbf{c}}$ of $\mathsf{Coh}_{\mathscr{G}_{\mathbf{c}}}(\mathscr{G}_{\mathbf{c}})$ such that, for any $\chi \in \mathrm{Irr}^{\mathbf{c}}(W)$ corresponding to $[g,\eta] \in \mathsf{K}_{0,\mathscr{G}_{\mathbf{c}}}(\mathscr{G}_{\mathbf{c}})$, we have*

$$\{[g,\eta],[m_{\mathbf{c}}]\} = (-1)^{b_\chi - a_{\mathbf{c}}}\frac{\eta(1)}{|Z(g)|} \quad \textit{for all } [g,\eta],$$

where b_χ is the valuation of the fake degree of χ. If W is irreducible and $\mathbf{c}$ is not exceptional, then $j_{\mathbf{c}}$ is unique up to isomorphism.

One checks that tensoring with $m_{\mathbf{c}}$ is involutive on $\mathsf{Coh}_{\mathscr{G}_{\mathbf{c}}}(\mathscr{G}_{\mathbf{c}})$. Since $m_{\mathbf{c}}$ is simple and invertible, we get an involution of the set of isomorphism classes of simple objects of $\mathsf{Coh}_{\mathscr{G}_{\mathbf{c}}}(\mathscr{G}_{\mathbf{c}})$.

Let $j_{\mathbf{c}} \in \mathsf{Z}(\mathsf{Coh}_{\mathscr{G}_{\mathbf{c}}}(\mathbf{X}_{\mathbf{c}} \times \mathbf{X}_{\mathbf{c}}))$ be the image of $m_{\mathbf{c}}$. Then tensoring with $j_{\mathbf{c}}$ defines an involution of $\mathsf{Coh}_{\mathscr{G}_{\mathbf{c}}}(\mathbf{X}_{\mathbf{c}} \times \mathbf{X}_{\mathbf{c}})$. As before, we get an involution of the set of isomorphism classes of simple objects of $\mathsf{Coh}_{\mathscr{G}_{\mathbf{c}}}(\mathbf{X}_{\mathbf{c}} \times \mathbf{X}_{\mathbf{c}})$.

At the same time, recall that under Conjecture 1.2, $J_{\mathbf{c}}$ defines an object of $\mathsf{Z}(\mathsf{C}^{\mathbf{c}})$, and tensoring with $J_{\mathbf{c}}$ is an involution of $\mathsf{C}^{\mathbf{c}}$.

Conjecture 1.5. *Assume that $w_\circ$ commutes with all of $\mathbf{c}$. Then* (1.1) *takes $j_{\mathbf{c}}$ (with its central structure) to $J_{\mathbf{c}}$ (with its central structure), up to isomorphism.*

1.8. Finally, we explain the application to Deligne–Lusztig's conjecture at the end of [DL25]. *Henceforth, we assume that we are in setting (II), so that $\mathbf{k} = \bar{\mathbf{F}}$ for a finite field $\mathbf{F}$, and $\mathbf{K} = \bar{\mathbf{Q}}_\ell$, and D is defined in terms of mixed complexes of $\bar{\mathbf{Q}}_\ell$-sheaves relative to a split Frobenius $F : G \to G$.*

Let $\mathsf{Rep}_u^{\mathbf{c}}(G^F)$ be the full additive subcategory of $\mathsf{Rep}(G^F)$ generated by the unipotent representations in the family indexed by $\mathbf{c}$. The Harish-Chandra transform

$$\mathsf{HC}_F : \mathsf{Rep}(G^F) = \mathsf{D}^b_G(GF, \bar{\mathbf{Q}}_\ell) \to \mathsf{D}^b_{G,m}(\mathscr{B} \times \mathscr{B}, \bar{\mathbf{Q}}_\ell) =: \mathsf{D}$$

restricts to a functor $\mathsf{HC}_F : \mathsf{Rep}_u^{\mathbf{c}}(G^F) \to \mathsf{D}^{\leq \mathbf{c}}$. As explained in [BDR], the essential image of the latter is right-orthogonal to $\mathsf{D}^{<\mathbf{c}}$, in the sense that $\mathrm{Hom}_{\mathsf{D}}(K, \mathsf{HC}_F(\rho)) = 0$ for all $\rho \in \mathsf{Rep}_u^{\mathbf{c}}(G^F)$ and $K \in \mathsf{D}^{<\mathbf{c}}$. Moreover, Lusztig showed that the composition

$$\underline{\mathsf{HC}}_F : \mathsf{Rep}_u^{\mathbf{c}}(G^F) \xrightarrow{\mathsf{HC}_F} \mathsf{D}^{\leq \mathbf{c}} \to \mathsf{D}^{\mathbf{c}}$$

factors through a tensor equivalence

$$\mathsf{Rep}_u^{\mathbf{c}}(G^F) \xrightarrow{\sim} \mathsf{Z}(\mathsf{C}^{\mathbf{c}})$$

for a certain monoidal product on $\mathsf{Rep}_u^{\mathbf{c}}(G^F)$, introduced in [L15] by means of weight filtrations. Altogether, we get tensor equivalences

$$\mathsf{Rep}_u^{\mathbf{c}}(G^F) \xleftrightarrow{[L15]} \mathsf{Z}(\mathsf{C}^{\mathbf{c}}) \xleftrightarrow{(1.1)} \mathsf{Z}(\mathsf{Coh}_{\mathscr{G}_{\mathbf{c}}}(\mathbf{X}_{\mathbf{c}} \times \mathbf{X}_{\mathbf{c}})) \xleftrightarrow{(1.2)} \mathsf{Coh}_{\mathscr{G}_{\mathbf{c}}}(\mathscr{G}_{\mathbf{c}}).$$

Let $\mathrm{Uch}^{\mathbf{c}}(G^F)$ be the set of unipotent irreducible characters of G^F in the family indexed by $\mathbf{c}$. We then get a bijection between $\mathrm{Uch}^{\mathbf{c}}(G^F)$ and the set of $\mathscr{G}_{\mathbf{c}}$-conjugacy classes of pairs (g, η) with $g \in \mathscr{G}_{\mathbf{c}}$ and $\eta \in \operatorname{Irr} Z(g)$. This is precisely the bijection described by Lusztig in [L84].

Recall that tensoring with $m_{\mathbf{c}}$ induces an involution of the set of classes $[g, \eta]$. The corresponding involution on $\mathrm{Uch}^{\mathbf{c}}(G^F)$ is denoted $(-)^!$ in [DL25].

If all of our conjectures above hold, then for any $\rho \in \mathrm{Uch}^{\mathbf{c}}(G^F)$, we have

$$\Xi_{\mathbf{c}}(\underline{\mathsf{HC}}_F(\rho)) = (\underline{J_{\mathbf{c}}} \circledast \underline{\mathsf{HC}}_F(\rho))\langle a_{w_\circ \mathbf{c}}\rangle \simeq \underline{\mathsf{HC}}_F(\rho^!)\langle a_{w_\circ \mathbf{c}}\rangle = \underline{\mathsf{HC}_F(\rho^!)\langle a_{w_\circ \mathbf{c}}\rangle}. \tag{1.3}$$

To give the applications of this identity, let

$$\mathsf{CH}_F : \mathsf{D} \to \mathsf{Rep}(G^F)$$

denote the left adjoint to HC_F. For any sequence $\vec{w} = (w^{(1)}, w^{(2)}, \ldots, w^{(k)})$ of elements of W, let $X(\vec{w})$ be the associated Deligne–Lusztig variety over $\mathbf{k}$, and let

$$\Delta(\vec{w}) = \Delta(w^{(1)}) * \Delta(w^{(2)}) * \cdots * \Delta(w^{(k)})$$

in D. We get

$$\begin{aligned}
(\mathrm{H}_c^i(X(\vec{w})), \rho)_{G^F} &= \mathrm{Hom}_{G^F}(\mathsf{CH}_F(\Delta(\vec{w}))[i], \rho) \\
&\simeq \mathrm{Hom}_{\mathsf{D}}(\Delta(\vec{w})[i], \mathsf{HC}_F(\rho)) && \text{by adjunction} \\
&\simeq \mathrm{Hom}_{\mathsf{D}^{\mathbf{c}}}(\underline{\Delta(\vec{w})[i]}, \underline{\mathsf{HC}}_F(\rho)) && \text{by orthogonality of } \underline{\mathsf{HC}}_F \text{ to } \mathsf{D}^{<\mathbf{c}}.
\end{aligned}$$

If (1.3) holds, then

$$\begin{aligned}
\mathrm{Hom}_{\mathsf{D}^{\mathbf{c}}}(\underline{\Delta(\vec{w})[i]}, \underline{\mathsf{HC}}_F(\rho)) &\simeq \mathrm{Hom}_{\mathsf{D}^{\mathbf{c}}}(\underline{\Delta(w_\circ, \vec{w})[i]}, \underline{\Delta(w_\circ) * \mathsf{HC}_F(\rho)}) \\
&\simeq \mathrm{Hom}_{\mathsf{D}^{\mathbf{c}}}(\underline{\Delta(w_\circ, \vec{w})[i]}, \Xi_{\mathbf{c}}(\underline{\mathsf{HC}}_F(\rho))[-a_{\mathbf{c}}]) \\
&\simeq \mathrm{Hom}_{\mathsf{D}^{\mathbf{c}}}(\underline{\Delta(w_\circ, \vec{w})[i]}, \underline{\mathsf{HC}_F(\rho^!)\langle a_{w_\circ \mathbf{c}}\rangle[-a_{\mathbf{c}}]}) \\
&\simeq \mathrm{Hom}_{\mathsf{D}^{\mathbf{c}}}(\underline{\Delta(w_\circ, \vec{w})\langle -a_{w_\circ \mathbf{c}}\rangle[i + a_{\mathbf{c}}]}, \underline{\mathsf{HC}_F(\rho^!)}).
\end{aligned}$$

Altogether, writing W for weight filtrations on cohomology, we would get

$$(\mathrm{gr}_j^{\mathsf{W}} \mathrm{H}_c^i(X(\vec{w})), \rho)_{G^F} \simeq (\mathrm{gr}_{j - a_{w_\circ \mathbf{c}}}^{\mathsf{W}} \mathrm{H}_c^{i + a_{\mathbf{c}}}(X(w_\circ, \vec{w})), \rho^!)_{G^F}.$$

This would imply both Theorem B of [BDR] (by applying the identity twice, the second time with $(w_\circ, \vec{w})$ in place of $\vec{w}$), and the conjecture in [DL25] (by taking $\vec{w}$ empty).