Last time (f : R to R^2, g : R^2 to R) def by $f(x) = (x, 0)$ and $g(x, y) = x$ is a homotopy equivalence

note: $g \circ f = \mathrm{id}_R$, stronger than $g \circ f \sim \mathrm{id}_R$

Rem a homotopy equivalence (f, g) where f is an inclusion and $g \circ f = \mathrm{id}$ is called a deformation retraction

Rem in general, if $g \circ f = \mathrm{id}$, then f is injective and g is surjective

Prob give a deformation retraction from $R^2 - \{(0, 0)\}$ onto S^1 [draw]

Prob if $X \sim Y$ via (f, g) and $Y \sim Z$ via (j, k) then $X \sim Z$ via $(j \circ f, g \circ k)$ [diagram]

need to prove: $j \circ f \sim \mathrm{id}_X$ and $g \circ k \sim \mathrm{id}_Z$

Cor $\sim$ is an equiv relation on spaces

[Pf $X \sim X$ via $(\mathrm{id}_X, \mathrm{id}_X)$
$X \sim Y$ via (f, g) iff $Y \sim X$ via (g, f)
$X \sim Y$ and $Y \sim Z$ imply $X \sim Z$ by prob]

Rem if $X \sim Y$, then
1) X is connected iff Y is
2) X is path connected iff Y is

but X compact need not imply Y compact

(Munkres §51) a path in X is a cts map $\gamma : [0, 1]$ to X

$\gamma(0)$ is the starting pt and $\gamma(1)$ is the ending pt

Df given paths $\gamma, \gamma' : [0, 1]$ to X s.t. $\gamma(0) = \gamma'(0)$ and $\gamma(1) = \gamma'(1)$

a path homotopy from γ to γ' is a homotopy

$$\varphi : [0, 1] \times [0, 1] \text{ to } X \text{ from } \gamma \text{ to } \gamma'$$

s.t. $\varphi(0, t) = \gamma(0) = \gamma'(0)$ and $\varphi(1, t) = \gamma(1) = \gamma'(1)$ for all times t

in this case, we write $\gamma \sim_p \gamma'$

Prob for any space X, points x, y in X, paths $\gamma_1, \gamma_2, \gamma_3$ from x to y,

given path homotopies φ from γ_1 to γ_2, ψ from γ_2 to γ_3,

find path homotopy from γ_1 to γ_3 using φ, ψ

Cor $\sim_p$ is an equiv rel on paths from x to y

[why interesting?]

Df for paths β, γ s.t. $\beta(1) = \gamma(0)$ [draw], set

$$(\beta * \gamma)(s) = \begin{cases} \beta(2s) & \text{for } s \leq 1/2 \\ \gamma(2s - 1) & \text{for } s \geq 1/2 \end{cases}$$

$\beta * \gamma$ is a path called the <u>concatenation</u> of β and γ

<u>Thm</u> write $[\gamma]$ for "path homotopy class of γ"

if $\beta \sim_p \beta'$ and $\gamma \sim_p \gamma'$, then $\beta * \gamma \sim_p \beta' * \gamma'$

that is: $[\beta * \gamma]$ only depends on $[\beta]$ and $[\gamma]$

(Munkres §52) a loop based at x is a path γ
s.t. $\gamma(0) = x = \gamma(1)$

<u>Ex</u> write e_x for the <u>constant loop</u> at x

<u>Prob</u> show that if X is convex,
or even <u>star convex</u> [draw],
then any loop based at x is $\sim_p e_x$

(Munkres §52)

<u>Df</u> the <u>fundamental group</u> of X at x is

$\pi_1(X, x) = \{\text{loops in } X \text{ based at } x\}/\sim_p$

under the operation $[\beta] * [\gamma] := [\beta * \gamma]$

the id element is $[e_x]$

[<u>Rem</u> in general, there is a group $\pi_n(X, x)$
called the <u>nth homotopy group</u>]

<u>Cor</u> $\pi_1(R, x)$ and $\pi_1(R^2, x)$ are trivial
[by previous prob]