1.

Notes on LLT polynomials, using the definition in [HHLRU, §5] and [HHL, §3].

1.1. We write $\mathbf{N}$ for the set of positive integers.

For our purposes, an integer partition is a weakly decreasing sequence of integers $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots)$ that stabilizes at zero after finitely many entries. Recall that λ defines a subset of $\mathbf{N} \times \mathbf{N}$ called its Young diagram: A point $\square = (j, i)$ belongs to the subset if and only if $j \leq \lambda_i$. The *content* of $\square$ is

$$\kappa(\square) = j - i.$$

Abusing notation, we will conflate λ with its Young diagram where convenient.

1.2. Consider sets of the form $\nu = \lambda \setminus \mu$, where λ, μ are partitions with $\mu \subseteq \lambda$. We declare two such sets to be equivalent whenever they differ by translation by a vector (k, k) with $k \in \mathbf{Z}$: that is, whenever they differ by translation but their elements have the same contents. We define a *skew shape* to be an equivalence class of such sets. Abusing notation, we again write ν to denote the skew shape it represents.

If ν is nonempty, then its *head* is its bottommost rightmost element. In this case, the *content* of ν, denoted $\kappa(\nu)$, is the content of its head. This is also the maximum content among the elements of ν.

1.3. We will work with d-tuples of skew shapes: say, $\vec{\nu} = (\nu^{(0)}, \ldots, \nu^{(d-1)})$. We define the *size* of $\vec{\nu}$ to be $|\vec{\nu}| = \sum_i |\nu^{(i)}|$.

A *semistandard Young tableau of (skew) shape* $\vec{\nu}$ is a function $T : \coprod_i \nu^{(i)} \to \mathbf{N}$ that, within each skew shape $\nu^{(i)}$, is weakly increasing along rows and strictly increasing along columns. In symbols,

$$\left.\begin{array}{rl} \text{if } \square' \text{ lies left of } \square, & \text{then } T(\square') \leq T(\square), \\ \text{and if } \square' \text{ lies above } \square, & \text{then } T(\square') > T(\square), \end{array}\right\} \quad \text{for all } i \text{ and } \square, \square' \in \nu^{(i)}.$$

We write $\mathrm{SSYT}(\vec{\nu})$ for the set of all semistandard Young tableaux of shape $\vec{\nu}$.

The *weight* of T is the sequence $\mu(T) = (\mu_1(T), \mu_2(T), \ldots)$ in which $\mu_k(T) = |T^{-1}(k)|$. For any integer partition μ, we write $\mathrm{SSYT}(\vec{\nu}, \mu) \subseteq \mathrm{SSYT}(\vec{\nu})$ for the subset of semistandard Young tableaux T such that $\mu(T) = \mu$. Note that it is nonempty only if $|\mu| = |\vec{\nu}|$.

For $0 \leq i < d$ and $\square \in \nu^{(i)}$, we define the *adjusted content* of $\square$ by

$$c(\square) = d\kappa(\square) + i.$$

Given a semistandard Young tableau T of shape $\vec{\nu}$, the *inversion set* $\mathrm{Inv}(T)$ is the set of tuples $(i, i', \square, \square')$, with $\square \in \nu^{(i)}$ and $\square' \in \nu^{(i')}$, such that:

$$T(\square) < T(\square') \quad \text{and} \quad 0 < c(\square) - c(\square') < d.$$

Compare to [HHLRU, 216].

1.4. Let Λ_t be the ring of symmetric functions in variables $x_1, x_2, \ldots$ with coefficients in $\mathbf{Z}[t^{\pm 1}]$. We write $\langle -, - \rangle$ for the $\mathbf{Z}[t^{\pm 1}]$-linear Hall pairing on Λ_t. We write $(h_\mu)_\mu$ for the basis of complete homogeneous symmetric functions, where μ runs over integer partitions. For any sequence of nonnegative integers $\mu = (\mu_1, \mu_2, \ldots)$, we set

$$x^\mu = x_1^{\mu_1} x_2^{\mu_2} \cdots$$

For our purposes, the *(modified) Lascoux–Leclerc–Thibon (LLT) polynomial* of $\vec{\nu}$ is the quasisymmetric function

$$\mathrm{LLT}_{\vec{\nu}}(t) = \sum_{T \in \mathrm{SSYT}(\vec{\nu})} t^{|\mathrm{Inv}(T)|} x^{\mu(T)},$$

which turns out to be symmetric in $x_1, x_2, \ldots$, and hence, defines an element of Λ_t. Indeed, $\mathrm{LLT}_{\vec{\nu}}$ is the unique symmetric function such that

$$\langle h_\mu, \mathrm{LLT}_{\vec{\nu}} \rangle = \sum_{T \in \mathrm{SSYT}(\vec{\nu}, \mu)} t^{\mathrm{Inv}(T)} \quad \text{for all } \mu \text{ with } |\mu| = |\vec{\nu}|.$$

1.5. We say that a skew shape ν is a *ribbon* if and only if the (unadjusted) contents of its elements form a single (possibly empty) interval of consecutive integers, without repetition. In informal terms, ν is connected and contains no square. In the literature, ribbons are also known as *border strips* or *rim hooks*.

We order the elements of a nonempty ribbon by decreasing content, so that the head is the first element. We say that a non-head element is a *step up* if and only if it lies above its predecessor: that is, it marks a place where the ribbon moves up, not left.

Henceforth, we assume that $\vec{\nu}$ is a d-tuple of ribbons. (Their individual sizes may vary.) Let

$$\begin{aligned} C(\vec{\nu}) &= \{c(\square) \in \mathbf{Z} \mid \square \in \nu^{(i)} \text{ for some } i\}, \\ D(\vec{\nu}) &= \{c(\square) \in \mathbf{Z}^2 \mid \square \in \nu^{(i)} \text{ for some } i \text{ and } \square \text{ is a step up}\}. \end{aligned}$$

Note that $D(\vec{\nu})$ would be what [HHL, §3] would call the descent set of $\vec{\nu}$, except that we use the adjusted content c while they use κ.

Let us say that an arbitrary subset $C \subseteq \mathbf{Z}$ is *d-contiguous* if and only if, for all $0 \leq i < d$, the set $\{(c - i)/d \mid c \in C_{(i)}\}$ is a (possibly empty) interval of consecutive integers, where

$$C_{(i)} = \{c \in C \mid c \equiv i \pmod{d}\}.$$

Let $\mathbf{A}(d)$ be the set of pairs (C, D), with $D \subseteq C \subseteq \mathbf{Z}$, such that:

(1) C is bounded and d-contiguous.
(2) If $c \in D$, then $c + d \in D$.

Then, by construction:

Lemma 1.1. *The map $\vec{\nu} \mapsto (C(\vec{\nu}), D(\vec{\nu}))$ restricts to a bijection from the set of d-tuples of ribbons to the set $\mathbf{A}(d)$.*

1.6. Let $(C, D) \in \mathbf{A}(d)$. We define a *coloring* of (C, D) to be a function $T : C \to \mathbf{N}$ such that:

$$\left.\begin{array}{rl} \text{if } c \notin D, & \text{then } T(c) \leq T(c+d), \\ \text{and if } c \in D, & \text{then } T(c) > T(c+d), \end{array}\right\} \quad \text{for all } c \in C \text{ such that } c+d \in C.$$

We write $\mathrm{Col}(C, D)$ for the set of colorings of (C, D).

The *weight* $\mu(T)$ is defined as before: $\mu(T) = (\mu_k(T))_{k \geq 1}$ with $\mu_k(T) = |T^{-1}(k)|$. For any integer partition μ, we write $\mathrm{Col}(C, D, \mu) \subseteq \mathrm{Col}(C, D)$ for the subset of colorings T such that $\mu(T) = \mu$.

The *inversion set* $\mathrm{Inv}(T)$ is the set of pairs (c, c') such that:

$$T(c) < T(c') \quad \text{and} \quad 0 < c - c' < d.$$

We conclude:

Lemma 1.2. *If $(C, D) = (C(\vec{\nu}), D(\vec{\nu}))$ for some d-tuple of ribbons $\vec{\nu}$, then pullback along the function $c : \coprod_i \nu^{(i)} \to \mathbf{Z}$ defines a bijection*

$$\mathrm{Col}(C, D) \xrightarrow{\sim} \mathrm{SSYT}(\vec{\nu})$$

that preserves weights and inversion sets. In particular, $\mathrm{LLT}_{\vec{\nu}} = \mathrm{LLT}_{C,D}$, where

$$\mathrm{LLT}_{C,D}(t) = \sum_{T \in \mathrm{Col}(C,D)} t^{|\mathrm{Inv}(T)|} x^{\mu(T)}.$$

We will restate the theorems of Haglund–Haiman–Loehr and Hikita using the formalism of pairs (C, D) introduced above.

1.7. *Macdonald Polynomials.* Fix a partition λ with at most d parts, meaning $\lambda_{i+1} = 0$ for $i \geq d$. Let

$$C(\lambda) = \{i - dj \in \mathbf{Z} \mid 0 \leq i < d,\ 0 \leq j < \lambda_{i+1}\}.$$

Then $C(\lambda)$ forms a bounded, d-contiguous subset of $\mathbf{Z}$, For any $(C, D) \in \mathbf{A}(d)$, let

$$\begin{aligned} \mathrm{maj}(D) &= \sum_{0 \leq i < d} \sum_{\substack{0 \leq j < \lambda_{i+1} \\ i - dj \in D}} (\lambda_{i+1} - j) \\ &= |\{(c, c') \in D \times C \mid c - c' \in d\mathbf{N}\}|. \end{aligned}$$

Let λ' denote the transpose of λ, and let

$$\begin{aligned} \mathrm{hhl}(D) &= \sum_{0 \leq i < d} \sum_{\substack{0 \leq j < \lambda_{i+1} \\ i - dj \in D}} (\lambda'_{j+1} - i - 1) \\ &= |\{(c, c') \in C \times D \mid d\lfloor c/d \rfloor \leq c' < c\}|. \end{aligned}$$

Let $\tilde{H}_\lambda(q,t) \in \Lambda_t \otimes \mathbf{Z}[q^{\pm 1}]$ be the modified Macdonald polynomial of λ. Recall the identity

$$\tilde{H}_{\lambda'}(t,q) = \tilde{H}_\lambda(q,t),$$

Taking the μ in [HHL] to be our λ', and applying the symmetry identity, we can rewrite the main result of *ibid.* as:

Theorem 1.3 (Haglund–Haiman–Loehr)**.** *For any partition λ, we have*

$$\tilde{H}_\lambda(q,t) = \sum_{\substack{D \text{ such that} \\ (C(\lambda),D)\in \mathbf{A}(d)}} q^{\mathrm{maj}(D)} t^{-\mathrm{hhl}(D)} \mathrm{LLT}_{C(\lambda),D}(t).$$

Below, our notation for λ will omit its zero entries. We also write $(m_\mu)_\mu$ for the basis of Λ_t of monomial symmetric functions, which is dual to the basis $(h_\mu)_\mu$ under the Hall pairing, and $(s_\mu)_\mu$ for the basis of Schur functions, which is orthonormal.

Example 1.4. We work out the Schur expansion of $\tilde{H}_\lambda$ for the partitions $\lambda \vdash 3$. We will use the fact that $\mathrm{hhl}(D) = 0$ for all D encountered below, along with the following table of pairings $\langle s_\pi, m_\mu\rangle$, also known as inverse Kostka numbers:

	$s_{(3)}$	$s_{(2,1)}$	$s_{(1^3)}$
$m_{(3)}$	1	-1	1
$m_{(2,1)}$		1	-2
$m_{(1^3)}$			1

- $\lambda = (3)$. We can take $d = 1$, giving $C(\lambda) = \{0,-1,-2\}$. There are 4 choices for D, which give the following values for $\mathrm{maj}(D)$ and the coefficients in the monomial and Schur expansions of $\mathrm{LLT}_{C(\lambda),D}$:

maj	$m_{(3)}$	$m_{(2,1)}$	$m_{(1^3)}$	$s_{(3)}$	$s_{(2,1)}$	$s_{(1^3)}$
0	1	1	1	1		
1		1	2		1	
2		1	2		1	
3			1			1

  Altogether, $\tilde{H}_{(3)}(q,t) = s_{(3)} + (q+q^2)s_{(2,1)} + q^3 s_{(1^3)}$.
- $\lambda = (2,1)$. We can take $d = 2$, giving $C(\lambda) = \{0,1,-2\}$. There are 2 choices for D, which give:

maj	$m_{(3)}$	$m_{(2,1)}$	$m_{(1^3)}$	$s_{(3)}$	$s_{(2,1)}$	$s_{(1^3)}$
0	1	$1+t$	$1+2t$	1	t	
1		1	$2+t$		1	t

  Altogether, $\tilde{H}_{(2,1)}(q,t) = s_{(3)} + (q+t)s_{(2,1)} + qts_{(1^3)}$.

- $\lambda = (1^3)$. We can take $d = 3$, giving $C(\lambda) = \{0, 1, 2\}$. The only choice for D is the empty set, which gives:

maj	$m_{(3)}$	$m_{(2,1)}$	$m_{(1^3)}$
0	1	$1 + t + t^2$	$1 + 2t + 2t^2 + t^3$

Altogether, $\tilde{H}_{(1^3)}(q, t) = s_{(3)} + (t + t^2)s_{(2,1)} + t^3 s_{(1^3)}$.

Example 1.5. The first cases where $\mathrm{hhl}(D) \neq 0$ occur when $\lambda = (2, 2) \vdash 4$. Here we need the following inverse Kostka numbers:

	$s_{(4)}$	$s_{(3,1)}$	$s_{(2^2)}$	$s_{(2,1^2)}$	$s_{(1^4)}$
$m_{(4)}$	1	-1		1	-1
$m_{(3,1)}$		1	-1	-1	2
$m_{(2^2)}$			1	-1	1
$m_{(2,1^2)}$				1	-3
$m_{(1^4)}$					1

Taking $d = 2$ gives $C(\lambda) = \{0, 1, -2, -1\}$. There are 4 choices for D, which give:

maj	hhl	$m_{(4)}$	$m_{(3,1)}$	$m_{(2^2)}$	$m_{(2,1^2)}$	$m_{(1^4)}$
0	0	1	$1 + t$	$1 + t + t^2$	$1 + 2t + t^2$	$1 + 3t + 2t^2$
1	0		t	t	$2t + t^2$	$3t + 3t^2$
1	1		t	t	$2t + t^2$	$3t + 3t^2$
2	1			t	$t + t^2$	$2t + 3t^2 + t^3$

Altogether, the Schur expansion of $\tilde{H}_{(2^2)}(q, t)$ is

$$s_{(4)} + (q + qt + t)s_{(3,1)} + (q^2 + t^2)s_{(2^2)} + (q^2 t + qt + qt^2)s_{(3,1)} + q^2 t^2 s_{(1^4)}.$$

1.8. *Hikita Polynomials.* Fix a positive integer n coprime to d. Write $d = mn + b$, where m, b are integers and $0 < b < n$. Let $\mathbf{Z}_0^n \subseteq \mathbf{Z}^n$ be the subset of tuples ξ such that $\xi_1 + \cdots + \xi_n = 0$, and let

$$\mathbf{Z}_0^n(d) = \left\{ \xi \in \mathbf{Z}_0^n \;\middle|\; \begin{array}{lll} \text{if} & j = i + b, & \text{then} \quad \xi_j \leq \xi_i + m; \\ \text{if} & j = i + b - n, & \text{then} \quad \xi_j \leq \xi_i + m + 1 \end{array} \right\}.$$

For all $\xi \in \mathbf{Z}_0^n$, let

$$C(\xi) = \{n\xi_j + j - 1 \in \mathbf{Z} \mid 1 \leq j \leq n\}.$$

We see that $C(\xi)$ is a full and irredundant set of representatives for residue classes modulo n. It turns out that if $\xi \in \mathbf{Z}_0^n(d)$, then $C(\xi)$ is also d-contiguous: This fact is implicit in the work of Gorsky–Mazin [GM13, GM16]. For all such ξ, we set

$$\begin{aligned} &\mathrm{area}(\xi) = -\min C(\xi), \\ (1.1) \qquad &\mathrm{Inv}(\xi) = \{(c, c', l) \in C(\xi)^2 \times \{1, \ldots, d-1\} \mid c - c' \in l + n\mathbf{N}\}. \end{aligned}$$

These definitions essentially come from [GM13]. The statistic denoted $\mathrm{dinv}(\xi)$ in the literature corresponds to $\delta - |\mathrm{Inv}(\xi)|$, where

$$\delta := \frac{1}{2}(d-1)(n-1),$$

the maximum value of $|\mathrm{Inv}|$. See also [GMV], where codinv corresponds to $|\mathrm{Inv}|$.

Let $S'_{d/n}(q,t) \in \Lambda_t \otimes \mathbf{Z}[q^{\pm 1}]$ be the image, under $s_\lambda \mapsto s_{\lambda'}$, of the q,t-symmetric function in the rational shuffle theorem. The following result is essentially [H, Thm. 4.15], except that Hikita does not use area as we have defined it: He instead uses $\mathrm{area}((-)^\tau)$, where τ is the involution on $\mathbf{Z}_0^n$ defined by

$$(\xi_1, \ldots, \xi_n)^\tau = (-\xi_n, \ldots, -\xi_1),$$

which preserves $\mathbf{Z}_0^n(d)$. For all $\xi \in \mathbf{Z}_0^n(d)$, it can be shown that $|\mathrm{Inv}(\xi^\tau)| = |\mathrm{Inv}(\xi)|$ and $\mathrm{LLT}_{C(\xi^\tau),\emptyset} = \mathrm{LLT}_{C(\xi),\emptyset}$: for instance, through the reinterpretations of these invariants that we will give later.

Theorem 1.6 (Hikita). *For any $n > 0$ coprime to d, we have*

$$t^\delta S'_{d/n}(q,t^{-1}) = \sum_{\xi \in \mathbf{Z}_0^n(d)} q^{\mathrm{area}(\xi)} t^{|\mathrm{Inv}(\xi)|} \mathrm{LLT}_{C(\xi),\emptyset}(t).$$

Example 1.7. Take $d = 4$ and $n = 3$, so that $\delta = 3$. We compute:

$$\mathbf{Z}_0^3(4) = \{(0,0,0), (1,0,-1), (0,1,-1), (1,-1,0), (-1,0,1)\}.$$

Below. we list the corresponding sets $C(\xi)$, the values for $\mathrm{area}(\xi)$ and $|\mathrm{Inv}(\xi)|$, and the coefficients in the monomial and Schur expansions of $\mathrm{LLT}_{C(\xi),\emptyset}$.

$C(x)$	area	$\lvert\mathrm{Inv}\rvert$	$m_{(3)}$	$m_{(2,1)}$	$m_{(1^3)}$	$s_{(3)}$	$s_{(2,1)}$	$s_{(1^3)}$
$\{0,1,2\}$	0	0	1	$1+t+t^2$	$1+2t+2t^2+t^3$	1	$t+t^2$	t^3
$\{3,1,-1\}$	1	1	1	$1+t$		1	t	
$\{0,4,-1\}$	1	2	1	$1+t$		1	t	
$\{3,-2,2\}$	2	2	1	$1+t$		1	t	
$\{-3,1,5\}$	3	3	1			1		

Thus the Hikita polynomial is

$$(1 + qt + qt^2 + q^2t^2 + q^3t^3)s_{(3)} + (t + t^2 + qt^2 + qt^3 + q^2t^3)s_{(2,1)} + t^3 s_{(1^3)}.$$

Sending $t \mapsto t^{-1}$, then multiplying by t^3, we get $S'_{4/3}(q,t)$, which has symmetry between q and t:

$$(t^3 + qt^2 + qt + q^2t + q^3)s_{(3)} + (t^2 + t + qt + q + q^2)s_{(2,1)} + s_{(1^3)}.$$

Note that here, the LLT polynomials happen to be Hall–Littlewood polynomials. For other pairs (d,n), like $(7,3)$ or $(5,4)$, this is not the case.