GWU–Howard 2026

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$G = \mathrm{GL}_n$

$\mathcal{U} = \{\text{unipotent } n \times n \text{ matrices}\} \subseteq G$

Ex for $n = 2$, have $\mathcal{U} \simeq \left\{ \begin{pmatrix} 1+x & y \\ z & 1-x \end{pmatrix} \middle| \, x^2 + yz = 0 \right\}$

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$B \subseteq G$ upper-triangular

Bruhat, Chevalley understand G via B:

$$G = \bigsqcup_{w \in S_n} B\dot{w}B$$

can we understand $\mathcal{U}$ via $U := B \cap \mathcal{U}$?

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$U_- \subseteq B_- \subseteq G$ lower-triangular

Fine–Herstein '58, Steinberg '65

$$|\mathcal{U}(\mathbb{F}_q)| = q^{n(n-1)} = |U(\mathbb{F}_q)|^2 = |UU_-(\mathbb{F}_q)|$$

Kawanaka '75 for any $w \in S_n$,

$$|(\overbrace{\mathcal{U} \cap B\dot{w}B}^{\mathcal{U}_w})(\mathbb{F}_q)| = |(\overbrace{UU_- \cap B\dot{w}B}^{\mathcal{V}_w})(\mathbb{F}_q)|$$

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<u>Ex</u> for any n, have $\mathcal{U}_{\mathrm{id}} = U = \mathcal{V}_{\mathrm{id}}$

<u>Ex</u> for $n = 3$ and $\dot{w} = \begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix}$,

$$\mathcal{U}_w \simeq U \times \{(a, b, c, d) \mid a, b \neq 0,\ (1 + ab)^3 = abcd\},$$
$$\mathcal{V}_w \simeq U \times \{(a, b, c, d) \mid a, b \neq 0,\ 1 + ab = abcd\}$$

are not isomorphic (or even homeomorphic over $\mathbb{C}$)

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$\mathcal{U}_w$, $\mathcal{V}_w$ generalize to a construction for <u>positive braids</u>

$$\beta = \sigma_{w_1} \cdots \sigma_{w_k},$$

where $\sigma_w \in Br_n^+$ is the <u>simple</u> positive lift of $w \in S_n$

$$\begin{array}{ccc} X_\beta^C & \to & X_\beta := B\dot{w}_1 B \times^B B\dot{w}_2 B \times^B \cdots \times^B B\dot{w}_k B \\ \downarrow & & \downarrow \\ C & \to & G \end{array}$$

$\mathcal{U}_w = X_{\sigma_w}^{\mathcal{U}}$ and $\mathcal{V}_w = X_{\sigma_w \Delta^2}^1$, where $\Delta = \sigma_{w_\circ}$

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<u>Thm (T)</u> for any $\beta \in Br_n^+$,

$$(1) \qquad |X_\beta^{\mathcal{U}}(\mathbb{F}_q)| = |X_{\beta\Delta^2}^1(\mathbb{F}_q)|,$$

$$(2) \qquad \mathrm{gr}_*^{\mathsf{W}} \mathrm{H}_{c,T}^*(X_\beta^{\mathcal{U}}(\mathbb{C})) \simeq \mathrm{gr}_*^{\mathsf{W}} \mathrm{H}_{c,T}^*(X_{\beta\Delta^2}^1(\mathbb{C}))$$

(1) actually reduces to Kawanaka. (2) <u>categorifies</u> (1)

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$$\text{HOMFLYPT polynomial} \quad P\colon \{\text{links}\} \to \mathbf{Z}[\![q]\!][a, q^{-1/2}],$$
$$\text{KhR superpolynomial} \quad \mathbb{P}\colon \{\text{links}\} \to \mathbf{Z}[\![q]\!][a, q^{-1/2}, t]$$

P defined recursively by

$$P_{\bigcirc} = 1,$$
$$aP - a^{-1}P = (q^{1/2} - q^{-1/2})P$$

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Kálmán '09 writing P for HOMFLYPT,

$$P(\widehat{\beta})[a^{|\beta|-n+1}] = P(\widehat{\beta\Delta^2})[a^{|\beta|+n-1}]$$

Gorsky–Hogancamp–Mellit–Nakagane '19

true with KhR superpolynomial $\mathbb{P}$ in place of P

will show: (1), resp. (2), is equiv to Kálmán, resp. GHMN

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for all $u \in \mathcal{U}(\mathbb{C})$, Springer fiber

$$Spr_u = \{gB \in G/B \mid ugB = gB\}$$

Springer '76 $S_n \curvearrowright \mathrm{H}^*(Spr_u(\mathbb{C}))$, but not via the variety!

Springer resolution: $Spr = \{(u, gB) \mid ugB = gB\} \to \mathcal{U}$

Thm (T) $\mathbb{P}(\widehat{\beta})$ encoded in $\mathrm{H}^*_{c,T}(Spr \times_{\mathcal{U}} X^{\mathcal{U}}_{\beta})$

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Jordan type: $\mathcal{U} = \bigcup_{\lambda \vdash n} \mathcal{U}_\lambda$

Hall–Littlewood polynomial:

$$\tilde{H}_\lambda(q) = \sum_i q^i \,\mathrm{FrobChar}\left[\mathrm{H}^{2i}(Spr_{u_\lambda}(\mathbb{C}))\right] \quad \text{for } u_\lambda \in \mathcal{U}_\lambda(\mathbb{C})$$

<u>Cor (T)</u> for all $\beta \in Br_n^+$:

$$P(\widehat{\beta})[a^{\mathrm{lowest}+2k}] \propto \sum_{\lambda \vdash n} (s_{(n-k,1^k)}, \tilde{H}_\lambda(q)) \cdot |X_\beta^{\mathcal{U}_\lambda}(\mathbb{F}_q)|$$

where $(-,-)$ is the Hall pairing

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<u>Cor</u> for all $\beta \in Br_n^+$:

$$P(\widehat{\beta})[a^{|\beta|-n+1}] \propto |X_\beta^{\mathcal{U}}(\mathbb{F}_q)|, \tag{3}$$

$$P(\widehat{\beta})[a^{|\beta|+n-1}] \propto |X_\beta^{1}(\mathbb{F}_q)| \tag{4}$$

<u>Proof</u> $(s_{(n)}, \tilde{H}_\lambda(q)) = 1$ for all λ,

$(s_{(1^n)}, \tilde{H}_{(1^n)}(q)) = q^{n(n-1)/2}$,

$(s_{(1^n)}, \tilde{H}_\lambda(q)) = 0$ when $\lambda \neq (1^n)$

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why is $\mathcal{U}$ related to HOMFLYPT?

$$\begin{aligned} \mathsf{D}^b_{mix,G}\mathsf{Perv}(\mathcal{U}) &\simeq \mathsf{D}^b\mathsf{Mod}(\mathbb{C}S_n \ltimes \mathrm{Sym}) && \text{(Rider)} \\ &\simeq \mathsf{hTr}(\mathsf{Hecke}(S_n)) && \text{(Gorsky–Wedrich)} \end{aligned}$$

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Conj (T) the T-equivariant map

$$UU_- \to \mathcal{U} \quad \text{given by } xy \mapsto xyx^{-1}$$

restricts to a homotopy equivalence $\mathcal{V}_w(\mathbb{C}) \to \mathcal{U}_w(\mathbb{C})$

this would imply the theorem about cohomology