



# Zeta Functions as Knot Invariants

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- 1 The Riemann Hypothesis
- 2 Weil's Rosetta Stone
- 3 From Curves to Knots
- 4 Cherednik's New Hypothesis

[Rational noncrossing Coxeter–Catalan combinatorics.](#)

*Proc. London Math. Soc.* (2024), 50 pages.

+ P. Galashin, T. Lam, N. Williams

[The Hilb-vs-Quot conjecture.](#)

*J. reine angew. Math. (Crelle)* (2025), 44 pages.

+ O. Kivinen

[From the Hecke category to the unipotent locus.](#) 88

pages. [arXiv:2106.07444](#)

## 1 The Riemann Hypothesis

(Euler ~1730s) Proof of the infinitude of primes:

If there were finitely many, then we'd have

$$\prod_{\text{prime } p > 0} \left(1 + \frac{1}{p} + \frac{1}{p^2} + \cdots\right) = \sum_{n=1}^{\infty} \frac{1}{n}.$$

But the series diverges—a contradiction.

He also implicitly studied the [zeta function](#)

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

For  $s > 1$ , we have  $\zeta(s) = \prod_p \left(1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \cdots\right).$

What if we allow  $s$  to be complex?

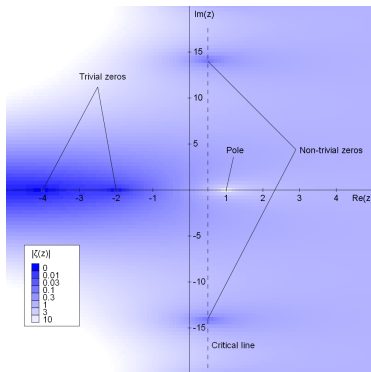
(Riemann 1859) A unique  $\mathbf{C}$ -valued function  $\zeta$  that is

- *holomorphic* (complex-differentiable) when  $s \neq 1$ .
- given by  $\sum_{n=1}^{\infty} \frac{1}{n^s}$  when  $\operatorname{Re}(s) > 1$ .

He checked that  $\zeta(n) = 0$  for  $n = -2, -4, -6, \dots$  by relating these zeros to poles of the *gamma function*.

He speculated from examples that all other zeros of  $\zeta$  live on the *critical line*  $\operatorname{Re}(s) = \frac{1}{2}$ .

Location of zeros  $\leftrightarrow$  distribution of prime numbers.



Wikipedia

(Hardy 1914) Infinitely many zeros on the line.

(Pratt–Robles–Zaharescu–Zeindler 2020) Among *nontrivial zeros*, over five-twelfths on the line.

(Dedekind ~1860s) Generalize the formula

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

by replacing  $\mathbf{Z}$  with other *rings*  $R$ .

Thus  $R$  is a set with operations  $+$  and  $\cdot$  resembling addition and multiplication.

$$\zeta_R(s) = \sum_{\substack{\text{ideals } I \subseteq R \\ |R/I| < \infty}} \frac{1}{|R/I|^s}$$

An *ideal*  $I$  is the collection of all finite linear combos  $c_1 \cdot x_1 + \cdots + c_k \cdot x_k$  for some fixed  $x_1, x_2, \dots \in R$ .

The *quotient*  $R/I$  is the set of translates  $y + I \subseteq R$ .

**Note** For  $\zeta_R$  to make sense, the number of  $I$  such that  $|R/I| = n$  must be finite for each  $n > 0$ .

**Ex** Every ideal of  $\mathbf{Z}$  takes the form

$$n\mathbf{Z} = \{\text{multiples of } n\} \quad \text{for some integer } n \geq 0.$$

For instance, the ideal generated by 30 and 2025 is

$$\{c_1 \cdot 30 + c_2 \cdot 2025 \mid c_1, c_2 \in \mathbf{Z}\} = 15\mathbf{Z}.$$

We see that

$$\zeta_{\mathbf{Z}}(s) = \sum_{\substack{n\mathbf{Z} \subseteq \mathbf{Z} \\ n > 0}} \frac{1}{|\mathbf{Z}/n\mathbf{Z}|^s} = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

In other rings, ideals may not simplify so nicely.

Why care?

(Hilbert–Polyá ~1910s) To prove RH, prove that

$$\{e^{it} \mid \tfrac{1}{2} + it \text{ is a nontrivial zero of } \zeta\}$$

is the set of eigenvalues of an infinite *unitary* matrix.

( $\implies e^{it}$  on the unit circle of  $\mathbf{C} \implies t$  real.)

(Weil ~1940s) Fix a particular prime  $p$ .

Can we prove an analogue for  $\zeta_R$ , for certain rings  $R$  appearing in *algebraic geometry* modulo  $p$ ?

(Grothendieck–Deligne ~1960s–70s) Yes.

2 Weil's Rosetta Stone Algebraic geometry studies *varieties*: shapes cut out by polynomial equations.

For simplicity, we'll stick to (*affine*) *hypersurfaces*

$$V_f = \{\vec{a} = (a_0, a_1, \dots, a_d) \mid f(\vec{a}) = 0\},$$

cut out by a single polynomial  $f \in \mathbf{Z}[x_0, x_1, \dots, x_d]$ .

$V_f$  is *smooth* at  $\vec{a} \bmod p$  when  $\frac{\partial f}{\partial x_i}(\vec{a}) \not\equiv 0 \pmod{p}$  for some  $i$ . Else, *singular*.

Ex For  $d = 1$ , hypersurfaces are plane curves, like

$$f(x, y) = y^2 - x^3 - c \quad \text{for constant } c.$$

For which  $c$  is  $V_f$  smooth everywhere mod  $p$ ?

The *ring of polynomial functions* on  $V_f \bmod p$  is

$$R_{f,p} := \frac{\mathbf{F}_p[x_0, \dots, x_d]}{\mathbf{F}_p[x_0, \dots, x_d] \cdot f}, \quad \text{where } \mathbf{F}_p := \mathbf{Z}/p\mathbf{Z}.$$

In a letter to his sister, Weil described a dictionary:

$\mathbf{Z}$	$R_{f,p}$	$V_f \bmod p$
$n\mathbf{Z}$	ideals	subvarieties
$p\mathbf{Z}$	maximal ideals	points

The first and last columns: a Rosetta stone between number theory and algebraic geometry.

Conj (Weil 1948) Assume  $V_f$  is smooth everywhere.

Then zeros of  $\zeta_{R_{f,p}}(s)$  have  $\operatorname{Re}(s) \in \{\frac{1}{2}, \frac{3}{2}, \dots, \frac{2d-1}{2}\}$ .

Weil proved it for many cases.

Set  $\zeta_{f,p}(s) := \zeta_{R_{f,p}}(s)$  for convenience.

(Grothendieck ~1964)  $\zeta_{f,p}(s)$  is a rational function in

$$q := p^{-s}.$$

In fact: polynomials  $\phi_0, \phi_1, \dots, \phi_{2d-1}$  such that

$$\zeta_{f,p}(s) = \frac{\phi_1(q) \cdot \phi_3(q) \cdots \phi_{2d-1}(q)}{\phi_0(q) \cdot \phi_2(q) \cdots \phi_{2d-2}(q)}.$$

$\phi_k$  is the charpoly of a certain operator on a certain vector space, describing the *étale topology* of  $V_f$ .

Conj For all  $k$ , the roots of  $\phi_k(q)$  live on the circle

$$|q| = p^{-k/2}.$$

$\implies$  Weil's Riemann Hypothesis.

(Deligne 1974) True for all  $f$  (smooth mod  $p$ ).

*In fact, Weil conjectured—and Deligne proved—results for all varieties, not just hypersurfaces.*

Ex Taking  $d = 1$  and  $f(x, y) = y^2 - x^3 - c$ :

$$\phi_0(t) = 1 - p\mathbf{q}$$

$$\phi_1(t) = 1 - \textcolor{red}{a}_p \mathbf{q} + p\mathbf{q}^2 \quad \text{for some integer } \textcolor{red}{a}_p,$$

giving  $\zeta_{f,p}(s) = \frac{1 - \textcolor{red}{a}_p p^{-s} + p^{1-2s}}{1 - p^{1-s}}$ . It turns out:

- $|a_p| \leq 2p^{1/2}$ .
- So the two roots of  $\phi_1(\mathbf{q})$  satisfy  $|\mathbf{q}| = p^{-1/2}$ .
- So the zeros of  $\zeta_{f,p}(s)$  satisfy  $\text{Re}(s) = \frac{1}{2}$ .

What if  $V_f$  has singularities?

Simplest case:  $f(x, y)$  with unique singularity at  $(0, 0)$ .

It turns out that here,

$$\zeta_{f,p}(s) = \zeta_{f,p}^\times(s) \cdot \zeta_{f,p}^0(s),$$

where:

- $\zeta_{f,p}^\times$  satisfies Weil's Riemann Hypothesis.
- $\zeta_{f,p}^0$  is analogous to  $\zeta_{f,p}$ , with the power-series ring

$$R_{f,p}^0 := \frac{\mathbf{F}_p[[x, y]]}{\mathbf{F}_p[[x, y]] \cdot f}$$

in place of  $R_{f,p}$ .

Does  $\zeta_{f,p}^0(s) = \sum_{\substack{I \subseteq R_{f,p}^0 \\ |R_{f,p}^0/I| < \infty}} \frac{1}{|R_{f,p}^0/I|^s}$  satisfy a RH?

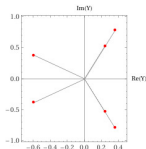
Ex For  $f = y^2 - x^3$ ,

$$\zeta_{f,p}^0(s) = \frac{1 - p^{1-2s}}{1 - p^{-s}} = \frac{1 + pq^2}{1 - q}.$$

Ex For  $f = y^3 - x^4$ ,

$$\zeta_{f,p}^0(s) = \frac{1 + pq^2 + p^2q^3 + p^2q^4 + p^3q^6}{1 - q}.$$

Here, not all roots satisfy  $|q| = p^{-1/2}$ .



WolframAlpha

3 From Curves to Knots For general  $f(x, y)$ ,

it turns out there's  $\Phi_f(t, q) \in \mathbf{Z}\left[t, q, \frac{1}{1-q}\right]$  such that

$$\zeta_{f,p}^0(s) = \frac{\Phi_f(p, p^{-s})}{1 - p^{-s}} \quad \text{for almost all } p.$$

These polynomials have many surprising features.

(Piontkowski 2007) Take  $f = y^n - x^{n+1}$ .

Then  $\Phi_f(1, 1) = \frac{(2n)!}{(n+1)!n!}$ , the  $n$ th Catalan number.

Ex If  $f = y^3 - x^4$ , then

$$\Phi_f(t, q) = 1 + tq^2 + t^2q^3 + t^2q^4 + t^3q^6,$$

$$\Phi_f(1, 1) = 5.$$



The  $\Phi_f$  also arise from *knot/link invariants*.

A *knot* is an embedding of a circle into  $\mathbf{R}^3$  or  $S^3$ .



A *link* is a generalization allowing multiple circles.



Two links are *isotopic* when we can deform one into the other without self-intersections.



Chmutov–Duzhin–Mostovoy

Let  $S_\epsilon^3 = \{(x, y) \in \mathbf{C}^2 \mid |x|^2 + |y|^2 = \epsilon\}$ . The subset

$$L_f = \{(x, y) \in S_\epsilon^3 \mid f(x, y) = 0\}$$

is a link in  $S_\epsilon^3$  when  $\epsilon > 0$  is small enough.

Ex If  $f = y^n - x^m$ , then  $L_f$  is the  $(m, n)$  torus link. It's a knot when  $m$  and  $n$  are coprime.



Wolfram Language

Ex If  $f = y^4 - 2x^3y^2 - 4x^5y + x^6 - x^7$ , then  $L_f$  is the closure of the braid



Cherednik–Danilenko

Conj (Oblomkov–Shende ~2010)

$$\Phi_f(1, q^2) = \lim_{a \rightarrow 0} \left[ (q/a)^\mu \mathbb{P}_{L_f}(a, q) \right],$$

where  $\mu \in \mathbf{Z}$  and  $\mathbb{P}$  is the *HOMFLYPT invariant*, discovered in 1986 and defined by the *skein rules*:

$$(1) \quad a \mathbb{P}_{\nearrow \searrow} - a^{-1} \mathbb{P}_{\nwarrow \swarrow} = (q - q^{-1}) \mathbb{P}_{\searrow \nearrow}$$

$$(2) \quad \mathbb{P}_{\bigcirc} = 1$$

Full statement incorporates  $a$ , by upgrading  $\Phi_f$ .

(Maulik 2012) True for all plane curves.

*Proof sketch* Blow up the singularity repeatedly.

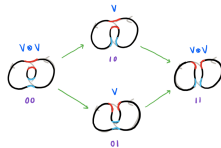
Control  $\Phi_f$  via *wall crossing* and  $L_f$  via skein algebra.

Conj (Oblomkov–Rasmussen–Shende ~2013)

$$\Phi_f(t^2, q^2) = \lim_{a \rightarrow 0} \left[ (q/a)^\mu \mathbf{P}_{L_f}(a, t, q) \right],$$

where  $\mathbf{P}$  is a refinement of  $\mathbb{P}$ , discovered in the 2000s by Khovanov–Rozansky.

$\mathbf{P}$  is defined by *categorifying* (1)–(2). Unknown how to categorify Maulik’s proof.



Melissa Zhang

(Kivinen–T 2025) True for  $f = y^3 - x^m$  with  $3 \nmid m$ .

Cor (Kivinen–T) New closed formula for  $\mathbf{P}_{\text{torus}(m,3)}$ .

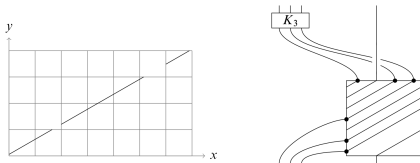
## Proof Sketch

1 Recursions that compute  $\mathbf{P}_{\text{torus}(m,n)}(\mathbf{a}, \mathbf{t}, \mathbf{q})$ , due to Elias–Hogancamp–Mellit.

$$\text{Diagram with } K_{n-1} \text{ box and strands} \simeq \left( (TQ^{-1})^{1-n} \text{Diagram with } K_n \text{ box and strands} \rightarrow Q^2 \text{Diagram with } K_{n-1} \text{ box and strands} \right)$$

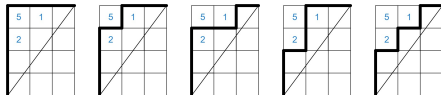
Elias–Hogancamp

Encoding of braids into grid diagrams, due to Mellit.



Mellit

2 For  $m, n$  coprime, yields a sum over  $\frac{(m+n-1)!}{m!n!}$  many *rational Dyck paths*.



At the same time,  $R_{f,p}^0 \simeq \mathbf{F}_p[[u^m, u^n]]$ .

We relate Dyck paths to  $R_{f,p}^0$ -submodules  $M \subseteq \mathbf{F}_p[[u]]$ .

3 We then relate

$$\sum_M \frac{1}{|\mathbf{F}_p[[u]]/M|^s} \quad \text{and} \quad \sum_I \frac{1}{|R_{f,p}^0/I|^s}$$

Uses *Serre duality*. For now, requires  $\min(m, n) \leq 3$ .

Big Picture I study special functions that appear in

- *algebraic geometry*
- *knot theory*
- *combinatorics*

Using *representation theory*, we can decompose them into simpler functions, like in Fourier analysis.

(Hikita 2016) The Dyck-path decomposition of  $\Phi_f$  arises from the rep theory of *symmetric groups*.

Another case:

(T 2021) New formulas for  $\mathbb{P}$  and  $\mathbf{P}$ , from the rep theory of finite Coxeter groups.

(Galashin–Lam–T–Williams 2024) Ideas from (T) solve conjectures in Coxeter combinatorics from 2012.

## 4 Cherednik's New Hypothesis

Recall: For  $f = y^3 - x^4$  and prime  $p$ , the roots of

$$\Phi_f(p, q) = 1 + pq^2 + p^2q^3 + p^2q^4 + p^3q^6$$

do not all satisfy  $|q| = p^{-1/2}$ .

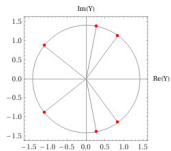
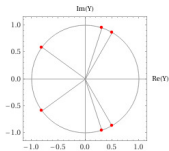
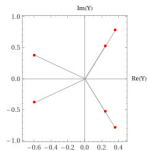
Conj (Cherednik 2018) For any plane curve  $f$ :

$$0 < t \leq \frac{1}{2} \implies \begin{array}{l} \text{all roots of } \Phi_f(t, q) \text{ satisfy} \\ |q| = t^{-1/2}. \end{array}$$

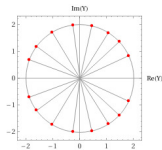
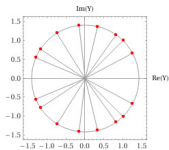
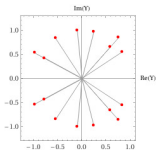
Would imply *arithmetic* constraints on  $\mathbf{P}_{L_f}(a, t, q)$ .

Dream (Cherednik) Related to the Lee–Yang theorem on zeros of *partition functions* in statistical mechanics.

$$f = y^3 - x^4, \quad t = 2, 1, \frac{1}{2}$$



$$f = y^4 - 2x^3y^2 - 4x^5y + x^6 - x^7, \quad t = 1, \frac{1}{2}, \frac{1}{4}$$



*Thank you for listening.*