Warmup analytic topology T_{an} on R:

generated by the basis

$\{B(x, \delta) \mid x, \delta \text{ in } R\}$
$= \{(x - \delta, x + \delta) \mid x, \delta \text{ in } R\}$
$= \{(a, b) \mid a, b \text{ in } R\}$

[means: every open set is a union of (a, b)'s]

Df lower-limit topology T_ℓ on R:

generated by the basis

$\{[a, b) \mid a, b \text{ in } R\}$

Q1 is {[a, b) | a, b in R} really a basis? [yes]
Q2 how does T_ ℓ compare to T_{an}?

Prop T_ℓ is strictly finer than T_{an}

Pf T_ ℓ is finer than T_{an}:
suppose U anlytc open in R
suppose x in U
pick a < b s.t. x in (a, b) sub U
then [x, b) sub (a, b)

strict because [0, 1) in T_ℓ but notin T_{an}

T_indisc < T_f < T_{an} < T_ℓ < T_disc

will write R_ℓ to mean "R in the topology T_ℓ", etc.

(Munkres §18, 16) recall from real analysis:

$f : R^n$ to R^m is *Bolzano continuous* iff

for all x in R^n and $\varepsilon > 0$, there exists $\delta > 0$ s.t.

$|x - x'| < \delta$ implies $|f(x) - f(x')| < \varepsilon$

equivalently

for all x in R^n and $\varepsilon > 0$, there exists $\delta > 0$ s.t.

x' in $B(x, \delta)$ implies $f(x')$ in $B(f(x), \varepsilon)$

equivalently

for all x in R^n and $\varepsilon > 0$, there exists $\delta > 0$ s.t.

$B(x, \delta)$ sub $f^{-1}(B(f(x), \varepsilon))$

Goal generalize to topological spaces

Df a function $f : X$ to Y is *continuous* iff
V open in Y implies $f^{-1}(V)$ open in X

Thm $f : R^n$ to R^m is Bolzano cts
iff f is cts wrt the analytic topologies

Pf suppose f cts wrt analytic topologies:

for all x in R^n and $\varepsilon > 0$, want $\delta > 0$

s.t. $|x - x'| < \delta$ implies $|f(x) - f(x')| < \varepsilon$

know: $B(f(x), \varepsilon)$ is open in R^m

so $f^{-1}(B(f(x), \varepsilon))$ is open in R^n

so have $\delta > 0$ s.t. $B(x, \delta)$ sub $f^{-1}(B(f(x), \varepsilon))$

suppose f Bolzano cts:

for all V open in Y, want $f^{-1}(V)$ open in X

pick x in $f^{-1}(V)$

want $\delta > 0$ s.t. $B(x, \delta) \subseteq f^{-1}(V)$

pick $\varepsilon > 0$ s.t. $B(f(x), \varepsilon) \subseteq V$

pick $\delta > 0$ s.t. for all x' s.t. $|x - x'| < \delta$,

have $|f(x) - f(x')| < \varepsilon$

then $f(B(x, \delta)) \subseteq B(f(x), \varepsilon) \subseteq V$

so $B(x, \delta) \subseteq f^{-1}(V)$

<u>Thm</u> suppose that $\{C_i\}_{i \in I}$ is a basis for the topology on Y

then

$f : X \to Y$ is cts

iff $f^{-1}(C_i)$ is open in X for all i in I

<u>Pf</u> suppose f is cts

each C_i is open in Y

so each $f^{-1}(C_i)$ is open in X

now suppose $f^{-1}(C_i)$ open in X for all i in I

pick V open in Y

know: $V = \bigcup_{i \in I'} C_i$ for some $I' \subseteq I$

$$\begin{aligned} f^{-1}(V) &= f^{-1}\Big(\bigcup_{i \in I'} (C_i)\Big) \\ &= \bigcup_{i \in I'} f^{-1}(C_i) \end{aligned}$$

so $f^{-1}(V)$ is open in X

<u>Ex</u> which maps are continuous?

$f : \mathbb{R}_{an} \to \mathbb{R}_\ell$, $f(x) = x$ [no: $[0, 1)$]

$f : \mathbb{R}_\ell \to \mathbb{R}_{an}$, $f(x) = x$ [yes]

$f : \mathbb{R}_{an} \to \mathbb{R}_\ell$, $f(x) = 32$ [yes]

Df cts $f : X$ to Y is a homeomorphism
iff it has a two-sided inverse $g : Y$ to X
s.t. g is also cts

in this case, we say X and Y are homeomorphic

[“what is shape?” “X and Y have the same shape when there is a homeo between them”]

Rem any homeo is a cts bijection
but a cts bijection need not be a homeo:
[already have an example: which?]
$f : R_\ell$ to R_{an}, $f(x) = x$

Ex id : X to X is a homeo when we use the same topology for domain and range

Ex other homeo’s wrt analytic topologies:

$f : R$ to R, $f(x) = x^3$
$f : R^2$ to R^2 $f(x, y) = (x + y, (x - y)^3)$
[why?]

Q is there a homeo R to R^2? vice versa?

The Subspace Topology let A be a subset of X

Df the subspace topology on A induced by X is
$\{A \cap V \mid V \text{ is an open set in } X\}$

[is it really a topology?]

Ex take $X = R$ and $A = [0, \infty)$

open sets in subspace top on $[0, \infty)$
look like V cap $[0, \infty)$

what if $V = (a, b)$?

if $a \geq 0$, then (a, b) cap $[0, \infty) = (a, b)$
is open in both $[0, \infty)$ and R

if $a < 0$, then (a, b) cap $[0, \infty) = [0, b)$
is open in $[0, \infty)$, but not in R

Moral if U is open in the subspace top on A,
then U may or may not be open in X

Thm if {B_i}_{i in I} is a basis for the top on X,
then {A cap B_i}_{i in I} is a basis for
the subspace top on A

Pf by thm from last time, just need to show:
for all U open in A and x in U,
have i s.t. x in A cap B_i sub U

indeed, U = A cap V for some V open in X
then x in V, so have i s.t. x in B_i sub V
so x in A cap B_i, and also, A cap B_i sub U

[how do subspaces interact with cts maps?]

Observations

- if A sub X, then the inclusion map i : A to X
def by i(a) = a is cts
- compositions of cts maps are cts
- so if f : X to Y is cts, then f|_A : A to Y is cts

Bonus Material

recall that Z is the set of integers

Df the evenly spaced topology on Z is generated by the basis

$\{aZ + b \mid a, b \text{ in } Z \text{ and } a \neq 0\}$

[will show on PS1 that this really is a basis]

Thm there are infinitely many prime numbers

Proof (Furstenberg, 1955)

assume finitely many primes p

then Z – bigcup_p pZ = bigcap_p (Z – pZ) is open
because Z – pZ is open for all p

but Z – bigcup_p pZ = {±1}
because if |a| > 1, then some prime divides a

so {±1} is open
but {±1} is not an evenly spaced set in Z □