recall: V defines $V^\vee = \mathrm{Hom}(V, F)$

<u>Q</u> let U sub V be a linear subspace
do we then have $U^\vee$ sub $V^\vee$?

<u>A</u> no: dual to inclusion $i : U$ to V is $i^\vee : V^\vee$ to $U^\vee$
[instead:]

<u>Df</u> the annihilator $\mathrm{Ann}_{V^\vee}(U)$ sub $V^\vee$
is def by $\{\theta$ in $V^\vee \mid \theta(u) = 0$ for all u in $U\}$

<u>Lem</u>

1) $\mathrm{Ann}_{V^\vee}(U)$ is a linear subspace of $V^\vee$
2) U sub U' iff
$\mathrm{Ann}_{V^\vee}(U')$ sub $\mathrm{Ann}_{V^\vee}(U)$

<u>Pf</u> 1) if θ, θ' in $\mathrm{Ann}(U)$ and α in F, then
$(\theta + \theta')(u) = \theta(u) + \theta'(u) = 0$
$(\alpha \bullet \theta)(u) = \alpha \bullet \theta(u) = 0$

2) suppose U sub U'
pick φ in $\mathrm{Ann}_{V^\vee}(U')$
for all u in U, have u in U'
so $\varphi(u) = 0$
so φ in $\mathrm{Ann}_{V^\vee}(U)$

other direction is also boring

recall: if V is finite-dim'l, then $\dim V^\vee = \dim V$

<u>Q</u> what is $\dim \mathrm{Ann}_{V^\vee}(U)$ in terms of
$\dim V$, $\dim U$?

[note: U larger means Ann(U) smaller]

<u>A</u> [guess] dim $\mathrm{Ann}_{V^\vee}(U) = \dim V - \dim U$

[note: $\dim U = \dim U^\vee$]

[relate Ann, $V^\vee$, $U^\vee$?]

<u>Lem</u> $\mathrm{Ann}_{V^\vee}(U) = \ker(i^\vee : V^\vee \text{ to } U^\vee)$

<u>Pf</u> what is $i^\vee$?

$i^\vee(\theta) = \theta|_U$ [restrict the domain]

now, θ in $\mathrm{Ann}_{V^\vee}(U)$

iff $\theta|_U$ is zero

iff θ in $\ker(i^\vee)$ □

[so it remains to show $\mathrm{im}(i^\vee) = U^\vee$]

<u>Lem</u> if V is finite-dim'l and U sub V

then $V^\vee$ to $U^\vee$ is surjective

<u>Pf</u> pick ψ in $U^\vee$

want to exhibit θ in $V^\vee$ s.t. $\theta|_U = \psi$

lem from a long time ago:

any basis of U can be extended to one of V

so pick $u_1, \ldots, u_\ell, v_1, \ldots, v_m$ s.t.

u's form a basis of U

u's and v's form a basis of V

let θ be def by

$\theta(u_i) = \psi(u_i)$

$\theta(v_j)$ = anything!

Thm 1 dim $\mathrm{Ann}_{V^\vee}(U) = \dim V - \dim U$

Pf $\dim \mathrm{Ann} = \dim V^\vee - \mathrm{im}(i^\vee)$
$= \dim V^\vee - \dim U^\vee$
$= \dim V - \dim U$

[putting the last lemma in context:]

Thm 2 let $T : U$ to V be any lin map
with dual $T^\vee : V^\vee$ to $U^\vee$
1) if V is fin. dim. and T inj., then $T^\vee$ is surj.
2) if T is surj., then $T^\vee$ is inj.
[with no hypotheses of fin-dim'lity]

Thm 2 in turn follows from Thm 3 below:

Df let $T : U$ to V be any lin map
let Ω be any vector space

pullback $T^\vee : \mathrm{Hom}(V, \Omega)$ to $\mathrm{Hom}(U, \Omega)$ is def by

$$T^\vee(\theta) = \theta \circ T$$

Thm 3 above:
1) if V is fin. dim. and T inj., then $T^\vee$ is surj.
2) if T is surj., then $T^\vee$ is inj.

Pf pf of 1) very similar to pf of earlier lem

to prove 2): suppose T surj.
pick $\theta, \theta' : V$ to Ω s.t. $T^\vee(\theta) = T^\vee(\theta')$
want to show $\theta = \theta'$

$T^\vee(\theta) = T^\vee(\theta')$ says $\theta(T(u)) = \theta'(T(u))$ for all u in U
while $\theta = \theta'$ says $\theta(v) = \theta'(v)$ for all v in V

but for all v in V, can pick u in U s.t. $T(u) = v$
because T surj.
then $\theta(v) = \theta(T(u)) = \theta'(T(u)) = \theta'(v)$ □

<u>Rem</u> to show $T^\vee$ inj., would have been easier
to show $\ker(T^\vee) = \{\mathbf{0}\}$
but this proof generalizes:

(<u>Thm</u>) for any sets X, Y, Z and map f : X to Y
if f is surj., then

Maps(Y, Z) to Maps(X, Z) is inj.
θ mapsto $\theta \circ f$

(Axler §3E) have seen:

giving a lin. subsp. of V is equiv to
giving a vector space U and inj. lin. map i : U to V
key: i injective iff U is iso to im(i)

[what about <u>surj.</u> lin. maps <u>out of</u> V?]

<u>Df</u> suppose U is a lin. subsp. of V
set $v + U = \{v + u \mid u \text{ in } U\}$ for all v in V

warning: $v + U = v' + U$ whenever $v - v'$ in U

the linear quotient of V by U is
the vec. sp. formed as follows from
$V/U = \{v + U \mid v \text{ in } V\}$ [“V mod U”]:

for all $v + U$ and $w + U$ and a in F:

$(v + U) + (w + U) = (v + w) + U$

$a \cdot (v + U) = a \cdot v + U$

<u>Lem</u> above, + and • are well-defined
so V/U is indeed a vector space

<u>Pf</u> if $v + U = v' + U$ and $w + U = w' + U$

then $(v + w) - (v' + w')$
$= (v - v') + (w - w')$
in $U + U$
in U

so $(v + w) + U = (v' + w') + U$

pf that • is well-defined is similar

now see:

giving a lin. quotient of V is equiv to
giving a vector sp. W and surj. lin. map $q : V$ to W
key: q surjective iff W is iso to $V/\ker(q)$

more parallels:

lin. subsp. U to V gives lin. quotient $V^\vee$ to $U^\vee$
when V is fin. dim'l

lin. quot. V to V/U gives lin. subsp. $(V/U)^\vee$ to $V^\vee$
[naturally suggests:]

<u>Lem</u> $\mathrm{Ann}_{V^\vee}(U) = \mathrm{im}((V/U)^\vee \text{ to } V^\vee)$

Pf let $q : V$ to V/U be the quotient map
$q^\vee : (V/U)^\vee$ to $V^\vee$ is given by pullback:
$q^\vee(\psi) = \psi \circ q$

now θ in $\mathrm{im}(q^\vee)$
iff $\theta = \psi \circ q$ for some ψ in $(V/U)^\vee$

key: latter occurs iff $\ker(\theta)$ supset $\ker(q) = U$
"only if": PS4, #3
"if": if $\ker(\theta)$ supset U
then $\psi(v + U) = \theta(v)$ well-def, lin.

so $\theta = \psi \circ q$ for some ψ iff $\theta|_U$ is zero

Summary for any V:

U	inj	V	surj	V/U	dualizes to
$(V/U)^\vee$	inj	$V^\vee$	to	$U^\vee$	

and:
- $(V/U)^\vee$ iso to $\mathrm{Ann}_{V^\vee}(U)$
- for V fin. dim'l, the last map is surj