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$$G = \mathrm{GL}_n$$

$$\mathcal{U} = \{\text{unipotent } n \times n \text{ matrices}\} \subseteq G$$

$$\underline{\text{Ex}} \quad \text{for } u \in \mathrm{GL}_2, \text{ write } u - I = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\begin{aligned} u \text{ is unipotent} &\iff u - I \text{ is nilpotent} \\ &\iff a + d = ad - bc = 0 \end{aligned}$$

$$\text{so for } n = 2, \text{ have } \mathcal{U} = \left\{ \begin{pmatrix} 1+a & b \\ c & 1-a \end{pmatrix} \middle| a^2 + bc = 0 \right\}$$

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B upper-triangular subgroup of G

Bruhat, Cartan, Chevalley understand G via

$$G = \bigsqcup_{w \in S_n} B\dot{w}B, \quad \text{where } \dot{w}\text{'s are permutation matrices}$$

how to understand \mathcal{U} via $U := B \cap \mathcal{U}$?

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Thm (Fine–Herstein '58, Steinberg '65)

$$|\mathcal{U}(\mathbb{F}_q)| = |U(\mathbb{F}_q)|^2 \quad (= q^{n(n-1)})$$

Thm (Kawanaka '75)

$$|\overbrace{(\mathcal{U} \cap B\dot{w}B)}^{\mathcal{U}_w}(\mathbb{F}_q)| = |\overbrace{(UU_- \cap B\dot{w}B)}^{\mathcal{V}_w}(\mathbb{F}_q)| \quad \text{for any } w$$

where $U_- \subseteq B_-$ are lower-triangular

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Ex for any n , have $\mathcal{U}_{\text{id}} = U = \mathcal{V}_{\text{id}}$

Ex for $n = 3$ and $w = \begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix}$,

$$\mathcal{U}_w \simeq U \times \{(a, b, c, d) \mid a, b \neq 0, (1 + ab)^3 = abcd\}$$

$$\mathcal{V}_w \simeq U \times \{(a, b, c, d) \mid a, b \neq 0, 1 + ab = abcd\}$$

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$B = T \ltimes U$ and $B_- = T \ltimes U_-$, where T is diagonal

$$T \curvearrowright \mathcal{U}_w, \quad t \cdot u = tut^{-1}$$

$$T \curvearrowright \mathcal{V}_w, \quad t \cdot xy = (txt^{-1})(tyt^{-1})$$

Thm (T)

$$\text{gr}_*^W H_{c,T}^*(\mathcal{U}_w(\mathbb{C})) \simeq \text{gr}_*^W H_{c,T}^*(\mathcal{V}_w(\mathbb{C})) \quad \text{for all } w$$

where $W_{\leq *}$ is the weight filtration on $H_{c,T}^*$

implies (Kawanaka) via results of Katz

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a T -equivariant map

$$\Phi: UU_- \rightarrow \mathcal{U} \quad \text{def by } \Phi(xy) = xyx^{-1}$$

Conj (T) Φ restricts to a homotopy equivalence

$$\mathcal{V}_w(\mathbb{C}) \rightarrow \mathcal{U}_w(\mathbb{C}) \quad \text{for any } w$$

would imply thm about $\mathcal{U}_w, \mathcal{V}_w$

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recall: $w \in S_n$ lifts to $\sigma_w \in Br_n^+$

for $\beta = \sigma_{w_1} \cdots \sigma_{w_k} \in Br_n^+$,

$$\begin{array}{ccc} X_\beta^C & \rightarrow & X_\beta := B\dot{w}_1 B \times^B B\dot{w}_2 B \times^B \cdots \times^B B\dot{w}_k B \\ \downarrow & & \downarrow \\ C & \rightarrow & G \end{array}$$

$\mathcal{U}_w = X_{\sigma_w}^{\mathcal{U}}$ and $\mathcal{V}_w = X_{\sigma_w \pi}^1$, where $\pi := \sigma_{w_\circ}^2$

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HOMFLYPT poly $P: \{\text{links in } \mathbb{R}^3\} \rightarrow \mathbf{Z}[[q]][a, q^{-1/2}]$

KhR superpoly $\mathbb{P}: \{\text{links in } \mathbb{R}^3\} \rightarrow \mathbf{Z}[[q]][a, q^{-1/2}, t]$

satisfying $\mathbb{P}|_{t \rightarrow -1} = P$

Thm (Kálmán '09) writing $\widehat{\beta}$ for the closure of β ,

$$P(\widehat{\beta})[a^{|\beta|-n+1}] = P(\widehat{\beta\pi})[a^{|\beta|+n-1}]$$

Thm (Gorsky–Hogancamp–Mellit–Nakagane '19)

true with \mathbb{P} in place of P

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Thm (T) if $\beta \in Br_n^+$, then

$$\begin{aligned} |X_\beta^{\mathcal{U}}(\mathbb{F}_q)| &= |X_{\beta\pi}^1(\mathbb{F}_q)|, \\ \mathrm{gr}_*^{\mathrm{W}} \mathrm{H}_{c,T}^*(X_\beta^{\mathcal{U}}(\mathbb{C})) &\simeq \mathrm{gr}_*^{\mathrm{W}} \mathrm{H}_{c,T}^*(X_{\beta\pi}^1(\mathbb{C})) \end{aligned}$$

will deduce from Kálmán and GHMN respectively

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$X_\beta^\mathcal{U}, X_\beta^1$ are pieces of a larger variety

recall the Springer resolution

$$\tilde{\mathcal{U}} = \{(u, gB) \in \mathcal{U} \times G/B \mid u \in gBg^{-1}\}$$

pullback squares:

$$\begin{array}{ccccc} \tilde{X}_\beta^\mathcal{U} & \rightarrow & X_\beta^\mathcal{U} & \rightarrow & X_\beta \\ \downarrow & & \downarrow & & \downarrow \\ \tilde{\mathcal{U}} & \rightarrow & \mathcal{U} & \rightarrow & G \end{array}$$

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Thm (T) if $\beta \in Br_n^+$, then $S_n \curvearrowright \mathrm{gr}_*^W H_{c,T}^*(\tilde{X}_\beta^\mathcal{U})$ and

$$\mathbb{P}(\hat{\beta}) \propto (\mathrm{gr}_*^W H_{c,T}^*(\tilde{X}_\beta^\mathcal{U}(\mathbb{C})))[\Lambda^*], \quad \text{where } \Lambda^i = \Lambda^i(\mathrm{std})$$

Hotta–Springer: if $\mathrm{Jordan}(u) = \lambda$, then $H^*(\tilde{\mathcal{U}}_u) \simeq \mathrm{Ind}_{S_\lambda}^{S_n}(1)$

Cor (T)

- 1) $\mathbb{P}(\hat{\beta})[a^{|\beta|-n+1}] = \mathrm{gr}_*^W H_{c,T}^*(\tilde{X}_\beta^\mathcal{U}(\mathbb{C}))[\Lambda^0] = \mathrm{gr}_*^W H_{c,T}^*(X_\beta^\mathcal{U}(\mathbb{C}))$
- 2) $\mathbb{P}(\hat{\beta})[a^{|\beta|+n-1}] = \mathrm{gr}_*^W H_{c,T}^*(\tilde{X}_\beta^\mathcal{U}(\mathbb{C}))[\Lambda^{n-1}] = \mathrm{gr}_*^W H_{c,T}^*(X_\beta^1(\mathbb{C}))$

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why?

recall: P arises from Markov traces on Hecke algebras

$$\begin{aligned} D_{mix,G}^b \mathrm{Perv}(\mathcal{U}) &\simeq D^b \mathrm{Mod}(\mathbb{C}S_n \ltimes \mathrm{Sym}) && (\text{Rider}) \\ &\simeq \mathrm{hTr}(\mathrm{Hecke}(S_n)) && (\text{Gorsky–Wedrich}) \end{aligned}$$