



Hilb vs Quot vs HOMFLYPT

Minh-Tâm Quang Trinh

Howard University

- 1 Knots and Links
- 2 Plane Curve Singularities
- 3 Hilb vs Quot
- 4 Generic Singularities
- 5 Some Lie Theory

O. Kivinen, M. Trinh. [The Hilb-vs-Quot conjecture](#).
J. reine angew. Math., **828** (2025), 83–126.

image credits: Chmutov–Duzhin–Mostovoy, Bar-Natan,
 Penpa+, Cherednik–Danilenko

1 Knots and Links

Some *knots* in \mathbf{R}^3 (or S^3).



Links allow multiple circles.



Knot theory studies *isotopy* invariants of links.

Trade-off between being *strong* and being *practical*.

$\pi_1(S^3 \setminus L)$ is a strong, but impractical, invariant.

More practical: the *HOMFLYPT polynomial* $\mathbb{P}_L(a, q)$, defined via skein relations.

$$(1) \quad \mathbb{P}_{\bigcirc} = 1$$

$$(2) \quad a \mathbb{P}_{\nearrow \searrow} - a^{-1} \mathbb{P}_{\nwarrow \nearrow} = (q^{1/2} - q^{-1/2}) \mathbb{P}_{\searrow \nearrow}$$

The strange factor $q^{1/2} - q^{-1/2}$ will be useful later.

While $\pi_1(S^3 \setminus L)$ is intrinsic, \mathbb{P}_L is *diagrammatic*:

A priori, it depends on the planar diagram.

$$\pi_1(S^3 \setminus L) \simeq \mathbf{Z} \text{ implies } L = \bigcirc.$$

Unknown whether $\mathbb{P}_L = 1$ implies $L = \bigcirc$.

Khovanov–Rozansky '07 A further refinement

$$\mathbf{P}_L(a, q, t)$$

such that $\mathbf{P}_L(a, q, -1) = \mathbb{P}_L(a, q)$.

The dimension of a triply-graded vector space called the *HOMFLYPT homology* of L .

Defined via *categorified* skein relations.

Khovanov '08 Can be done in certain monoidal triangulated categories $\mathbf{K}^b(\mathbf{SBim}_n)$.

$$\text{Kronheimer–Mrowka '10 } \mathbf{P}_L = 1 \text{ implies } L = \bigcirc.$$

Proof related \mathbf{P}_L to gauge theory on $S^3 \setminus L$.

Mellit '16, Elias–Hogancamp–Mellit '15–19

Recursions in $K^b(\text{SBim}_n)$ computing \mathbf{P} for *torus links*.



\Rightarrow Mellit '16 A closed formula for any torus *knot*.

\Rightarrow Gorsky–Mazin–Vazirani '20 Another formula, valid for any torus *link*.

For torus knots, both formulas sum over *Dyck paths*.



Both formulas look like

$$\mathbf{P} \propto \sum_D q^{a(D)} t^{c(D)} \prod_{\square \in S_\bullet} f_{\bullet, D, \square}.$$

$a(D)$ counts shaded \square 's; $c(D)$ is messy.

$$S_{\text{Mellit}} = \{\square \mid \square \nearrow D\},$$

$$S_{\text{GMV}} = \{\square \mid D \swarrow \square \text{ with } \square \text{ shaded}\}.$$

Example For the $(3, 4)$ torus knot:

q^a	t^c	$\prod f_{\text{Mellit}}$	$\prod f_{\text{GMV}}$
q^3	t^3	1	$1 + aq^{-1}$
q^2	t^2	$1 + at$	$(1 + aq^{-1})(1 + aq^{-1}t)$
q	t^2	$1 + at$	$1 + aq^{-1}$
q	t	$1 + at$	$1 + aq^{-1}$
1	1	$(1 + at)(1 + at^2)$	1

2 Plane Curve Singularities Let

$$S_\epsilon^3 = \{(x, y) \in \mathbf{C}^2 \mid |x|^2 + |y|^2 = \epsilon\}.$$

Let $C \subseteq \mathbf{C}^2$ be an algebraic curve through $(0, 0)$.

The *link* of C at the origin is

$$L_C = \{(x, y) \in S_\epsilon^3 \mid f(x, y) = 0\},$$

independent of ϵ up to isotopy.

Example For $y^n = x^m$, it's the (m, n) torus link.

In general, *components* of L_C correspond to *branches* of C at the origin.

Example Take C parameterized by

$$(x(u), y(u)) = (u^4, u^6 + u^7).$$

Then L_C is the *closure* of



In general, the completed local ring of C looks like

$$R_C = R_{C_1} \times \cdots \times R_{C_b},$$

and the branches look like $R_{C_i} \simeq \mathbf{C}[[u^{n_i}, u^{m_i} + \cdots]]$ by Newton–Puiseux.

Puiseux exponents are *cabling* parameters of knots.

Oblomkov–Shende conjectured a formula for \mathbb{P}_{L_C} in terms of the *intrinsic* ring R_C .

Later, with Rasmussen, upgraded to \mathbf{P}_{L_C} .

Form the *Hilbert schemes*

$$\mathcal{H}_C^\ell = \{\text{ideals } I \subseteq R_C \mid \dim(R_C/I) = \ell\}.$$

Conj (ORS '12) The lowest a -degree $\mathbf{P}_{L_C}^{\text{lo}}$ satisfies

$$\boxed{\frac{\mathbf{P}_{L_C}^{\text{lo}}(q, qt)}{1 - q} \propto \sum_{\ell \geq 0} q^\ell \chi(t^{1/2}, \mathcal{H}_C^\ell),}$$

where χ denotes *virtual weight polynomials*.

Recall: $\chi(t^{1/2}, Z) = |Z(\mathbf{F}_t)|$ when t is a prime power and Z is especially nice.

Example $\mathbf{P}_{(2,3) \text{ torus}} \propto 1 + qt + at$, while

$$C = \{y^2 = x^3\} \implies \begin{cases} \mathcal{H}_C^0 = pt, \\ \mathcal{H}_C^1 = pt, \\ \mathcal{H}_C^\ell = \mathbf{CP}^1 \text{ for } \ell \geq 2, \end{cases}$$

giving $1 + q + \frac{q^2}{1-q}(1+t) = \frac{1}{1-q}(1 + q^2t)$.

Next, form *nested Hilbert schemes*

$$\mathcal{H}_C^{\ell,k} = \{(I, J) \in \mathcal{H}_C^\ell \times \mathcal{H}_C^{\ell+k} \mid I \supseteq J \supseteq \langle x, y \rangle J\}.$$

The full conjecture:

$$\boxed{\frac{\mathbf{P}_{L_C}(a, q, qt)}{1 - q} \propto \sum_{k, \ell \geq 0} a^k q^\ell t^{\binom{k}{2}} \chi(t^{1/2}, \mathcal{H}_C^{\ell,k}).}$$

Maulik '12 True at the level of $\mathbb{P}_L = \mathbf{P}_L|_{t \rightarrow -1}$.

Key idea is an analogue for \mathbb{P}_L of a *wall-crossing identity* from DT theory.

Unknown how to upgrade to \mathbf{P}_L .

Maulik–Yun, Migliorini–Shende '11

Why should the Hilbert schemes be hard?

They encode a *perverse filtration* on the *compactified Picard scheme* of C :

$$\sum_{\ell} q^{\ell} \chi(s, \mathcal{H}_C^{\ell}) = \frac{\sum_i q^i \chi(s, \mathrm{gr}_i^{\mathbf{P}} H^*(\bar{\mathcal{P}}_C / \mathbf{Z}^b))}{(1 - q)^b}$$

for $\bar{\mathcal{P}}_C = \{\text{full, fin. gen. } R_C\text{-submods of } \mathrm{Frac}(R_C)\}$.

3 Hilb vs Quot $\mathrm{Frac}(R_C) = \mathrm{Frac}(\tilde{R}_C)$, where

$$R_C \hookrightarrow \tilde{R}_C = \mathbf{C}[[u_1]] \times \cdots \times \mathbf{C}[[u_b]].$$

Form the *Quot schemes*

$$\mathcal{Q}_C^{\ell} = \{R_C\text{-submods } M \subseteq \tilde{R}_C \mid \dim(\tilde{R}_C/M) = \ell\}.$$

Initial motivation for these varieties:

Thm (Kivinen–T '23) We have

$$\sum_{\ell} q^{\ell} \chi(s, \mathcal{Q}_C^{\ell}) = \frac{\sum_i q^i \chi(s, \bar{\mathcal{P}}_C^{(i)} / \mathbf{Z}^b)}{(1 - q)^b}$$

for an explicit \mathbf{Z}^b -stable stratification $\bar{\mathcal{P}}_C = \coprod_i \bar{\mathcal{P}}_C^{(i)}$.

Recall ORS:

$$\frac{\mathbf{P}_{LC}(a, q, qt)}{1 - q} \propto \sum_{k, \ell \geq 0} a^k q^\ell t^{\binom{k}{2}} \chi(t^{1/2}, \mathcal{H}_C^{\ell, k}).$$

Form *nested Quot schemes*

$$\mathcal{Q}_C^{\ell, k} = \{(M, N) \in \mathcal{Q}_C^\ell \times \mathcal{Q}_C^{\ell+k} \mid M \supseteq N \supseteq \langle x, y \rangle M\}.$$

“Quot ORS” Conj (Kivinen–T ’23) For any C ,

$$\frac{\mathbf{P}_{LC}(a, q, t)}{1 - q} \propto \sum_{k, \ell \geq 0} a^k q^\ell t^{\binom{k}{2}} \chi(t^{1/2}, \mathcal{Q}_C^{\ell, k}).$$

Thm (Kivinen–T ’23) Quot ORS holds in full for:

- $y^n = x^m$ with m, n coprime.
- $y^n = x^{nk}$.

“Hilb-vs-Quot” Conj (Kivinen–T ’23) For any C ,

$$\sum_{\ell} q^\ell \chi(t^{1/2}, \mathcal{H}_C^\ell) = \sum_{\ell} q^\ell \chi((qt)^{1/2}, \mathcal{Q}_C^\ell).$$

Remarks on Hilb-vs-Quot

- $t \mapsto qt$ because \mathcal{Q}^ℓ is *larger* than \mathcal{H}^ℓ for fixed ℓ .
- Should really be an identity in $K_0(\mathbf{Var})$.
- Unibranch case proposed by Cherednik in another form, without the \mathcal{Q}^ℓ .

Example Take $C = \{y^3 = x^4\}$.

The \mathbf{C}^\times -action on C induces actions on the \mathcal{H}^ℓ , \mathcal{Q}^ℓ .

The \mathbf{C}^\times -orbits form affine pavings.

\mathcal{H}^0	\mathcal{H}^1	\mathcal{H}^2	\mathcal{H}^3	\mathcal{H}^4	\mathcal{H}^5	\mathcal{H}^6	\dots
pt			\mathbf{C}^2	\mathbf{C}^2		\mathbf{C}^3	\dots
		\mathbf{C}	\mathbf{C}		\mathbf{C}^2	\mathbf{C}^2	\dots
	pt			\mathbf{C}^2	\mathbf{C}^2	\mathbf{C}^2	\dots
		pt		\mathbf{C}	\mathbf{C}	\mathbf{C}	\dots
			pt	pt	pt	pt	\dots

The rows classify ideals as R_C -modules.

The colors are \mathcal{Q}^0 , \mathcal{Q}^1 , \mathcal{Q}^2 , \mathcal{Q}^3 , \dots

Similar picture for any $y^n = x^m$ with m, n coprime.

“Hilb ORS” is hard because Hilb-vs-Quot is hard.

Thm (Kivinen–T ’23) Hilb-vs-Quot holds for

$$y^n = x^m \quad \text{with } m, n \text{ coprime and } n \leq 3.$$

Key idea is that for fixed n , we can compute

$$\lim_{\substack{m \rightarrow \infty \\ \gcd(m, n) = 1}} \sum_{\ell} q^{\ell} \chi(t^{1/2}, \mathcal{H}_{y^n = x^m}^{\ell}),$$

$$\lim_{\substack{m \rightarrow \infty \\ \gcd(m, n) = 1}} \sum_{\ell} q^{\ell} \chi(t^{1/2}, \mathcal{Q}_{y^n = x^m}^{\ell})$$

by combinatorics.

If $n \leq 3$, then the limits determine the series at finite m , by a Serre duality trick.

4 Generic Singularities

Beyond torus links: Gorsky–Mazin–Oblomkov and Caprau–González–Hogancamp–Mazin.

\approx GMO '22 + CGHM '23

Quot ORS holds for the lowest a -degree $\mathbf{P}_{L_C}^{\text{lo}}$ when C has a *generic* unibranch singularity.

(Say the first pair of Puiseux exponents is md, nd for some m, n, d with m, n coprime. Then

$$R_C \simeq \mathbf{C}[[u^{nd}, u^{md} + u^{md+1} + \cdots]].$$

generically among unibranch C with that pair.)

L_C is the $(mnd + 1, n)$ *cable* of the (m, n) torus knot.

For such singularities, GMO give an affine paving of \mathcal{Q}^ℓ indexed by subsets $\Delta \subseteq \mathbf{Z}_{\geq 0}$ with ℓ gaps:

$$\mathcal{Q}_\Delta^\ell = \{M \in \mathcal{Q}^\ell \mid \Delta = \{\text{val}_u(f)\}_{f \in M}\}.$$

Gave subtle but elementary criterion for $\mathcal{Q}_\Delta^\ell \neq \emptyset$.

For such knots, CGHM generalize the GMV formula for \mathbf{P} to a sum over $md \times nd$ Dyck paths.

\approx Gorsky–Mazin–Vazirani '17 + GMO '23 Explicitly,

$$\{\Delta \mid \mathcal{Q}_\Delta^\ell \neq \emptyset\} \xrightarrow{\sim} \{\text{“shifted” Dyck paths}\}$$

such that $q^{a(D)}t^{c(D)} = q^\ell t^{\dim \mathcal{Q}_\Delta^\ell}$.

Recall GMV:

$$\sum_D q^{a(D)} t^{c(D)} \prod_{\square \in S_{\text{GMV}}(D)} f_{\text{GMV}, D, \square}(aq^{-1}, t).$$

$\prod_{\square} f_{\text{GMV}, D, \square}$ does not match nested Quot:

$$\sum_{k, \Delta'} a^k t^{\binom{k}{2}} \chi(t^{1/2}, \mathcal{Q}_{\Delta \supseteq \Delta'}^{\ell, k}).$$

Thm (T '25+) By contrast:

- 1 Mellit's formula for \mathbf{P} generalizes to the knots of generic unibranch singularities.
- 2 $\prod_{\square} f_{\text{Mellit}, D, \square}(a, t)$ does match nested Quot.
- 3 Quot ORS holds in full for such singularities.

5 Some Lie Theory What got me into this?

The *affine Grassmannian* of a group \mathbf{G} over \mathbf{C} is

$$X = \mathbf{G}((x))/\mathbf{G}[[x]].$$

For any $\gamma \in \mathbf{g}((x))$, the *affine Springer fiber* over γ is its fixed-point set in X :

$$X_{\gamma} = \{g\mathbf{G}[[x]] \mid \gamma \in \text{Ad}(g) \cdot \mathbf{g}[[x]]\},$$

Appears in number theory, with \mathbf{F}_p in place of \mathbf{C} .

Laumon '02 If C is a branched n -cover of a line, then

$$\bar{\mathcal{P}}_C = \{\text{full, fin. gen. submods of } \text{Frac}(R_C)\}$$

is an affine Springer fiber for $\mathbf{G} = \mathbf{GL}_n$.

Example Suppose $R_C = \mathbf{C}[[u^4, u^6 + u^7]]$.

Set $x = u^4$. Fix an isomorphism $\mathbf{C}((x))^4 \xrightarrow{\sim} \mathbf{C}((u))$.

$$u^6 + u^7 \curvearrowright \mathbf{C}((u)) \quad \rightsquigarrow \quad \gamma \curvearrowright \mathbf{C}((x))^4.$$

Two possibilities for γ :

$$\begin{pmatrix} & & x^6 - x^7 \\ 1 & & 4x^5 \\ & 1 & 2x^3 \\ & & 0 \end{pmatrix}, \begin{pmatrix} & x^2 & x^2 \\ & x^2 & x^2 \\ x & & x^2 \\ x & x & \end{pmatrix}.$$

Both give $\bar{\mathcal{P}}_C \simeq X_\gamma$, but different *positive truncations*

$$\{g\mathbf{G}[[x]] \in X_\gamma \mid g \text{ has no poles in } x\}.$$

Respectively, they are $\bigsqcup_\ell \mathcal{H}_C^\ell$ and $\bigsqcup_\ell \mathcal{Q}_C^\ell$.

This viewpoint also suggests:

- 1 Generalizing X to *partial affine flag varieties*.
- 2 Generalizing \mathbf{GL}_n to *reductive groups*.

(1) leads to flagged versions of \mathcal{H}^ℓ , \mathcal{Q}^ℓ that indirectly encode the nested versions and more.

(2) leads to conjectures relating affine Springer fibers to q, t -traces on *generalized braid groups*.

The braid group is $\mathbf{Br}_W = \pi_1(\mathbf{g}^{\text{rs}} // \mathbf{G})$. The map

$$\mathbf{g}((x)) \rightarrow (\mathbf{g}^{\text{rs}} // \mathbf{G})((x))$$

suggests how γ produces a conjugacy class in \mathbf{Br}_W .

Def (T '21) Suppose $\beta = \sigma_{s_1} \cdots \sigma_{s_\ell} \in \text{Br}_W^+$.

The *braid Steinberg variety* of β is

$$Z_\beta = \tilde{\mathcal{U}} \times_{\mathcal{U}} \mathcal{U}_\beta,$$

where $\tilde{\mathcal{U}} \rightarrow \mathcal{U}$ is the unipotent Springer resolution and

$$\mathcal{U}_\beta \simeq \{(ug_\ell \mathbf{B} \xrightarrow{s_1} g_1 \mathbf{B} \xrightarrow{s_2} \cdots \xrightarrow{s_\ell} g_\ell \mathbf{B}) \mid u \in \mathcal{U}\}$$

for a fixed Borel \mathbf{B} .

Thm (T '21) Suppose $\mathbf{G} = \text{PGL}_n$.

If L is the closure of β , then we can recover \mathbf{P}_L from

$$\text{the Springer action } W \curvearrowright H_{c,\mathbf{G}}^*(Z_\beta).$$

$$\begin{array}{ccccc} \text{Iwahori} & X_\gamma^{\mathbf{B}} & \xleftarrow{\approx} & & Z_\beta/\mathbf{G} \\ & \downarrow & & & \downarrow \\ \text{spherical} & X_\gamma & \xleftarrow{\approx} & & \mathcal{U}_\beta/\mathbf{G} \\ & & & & ? \end{array}$$

Thank you for listening.