today: 1) some history
2) group theory

<u>More History of Manifolds</u>

what does $x^4 - 4x^2 + y^2 + 1 = 0$ look like?

$$y^2 = -(x^2(x-2)(x+2) + 1)$$

very negative x	no real solns for y
$x = -2$	no real solns
$x = -1$	$y = \pm\sqrt{2}$
$x = 0$	no real solns
$x = 1$	$y = \pm\sqrt{2}$
$x = 2$	no real solns
very positive x	no real solns

[draw graph]
this is a manifold of dim 1

what does $x^4 - 4x^2 + y^2 = 0$ look like?

here $x = 0$ yields only one soln $y = 0$

[draw graph]
this is not a manifold: problem at $(0, 0)$

<u>Q</u> in gen'l, consider analytic $F(x, y)$
when does $F(x, y) = 0$ form a manifold?

<u>A</u> need to avoid singular points:
$F_x(a, b), F_y(a, b) \neq 0$ at all pts (a, b)

Ex if $F = x^4 - 4x^2 + y^2 + C$
then $(F_x, F_y) = (4x^3 - 8x, 2y)$

so $F(x, y) = 0$ forms a manifold
iff $F(0, 0) \neq 0$
iff $C \neq 0$

Implicit Function Thm let X sub $R^{n + m}$
consist of $(\mathbf{x}, \mathbf{y})$ s.t.

$F_1(x_1, \ldots, x_n, y_1, \ldots, y_m) = 0$
$F_2(x_1, \ldots, x_n, y_1, \ldots, y_m) = 0$
…
$F_m(x_1, \ldots, x_n, y_1, \ldots, y_m) = 0$

where $F_1, \ldots, F_m$ are cts'ly differentiable

let $(\mathbf{a}, \mathbf{b}) = (a_1, \ldots, a_n, b_1, \ldots, b_m)$ in X

if the Jacobian $(dF_i/dy_j)_{i, j}$ is invertible
then there exist
an analytic open U sub R^n around $\mathbf{a}$
cts'ly diff'ble fns $g_1, \ldots, g_n : U$ to R^n
s.t.
$g(\mathbf{a}) = b$ for $i = 1, \ldots, n$
$\{(\mathbf{x}, g(\mathbf{x}) \mid \mathbf{x}$ in $U\}$ sub X
i.e., X looks like U sub R^n near $(\mathbf{a}, \mathbf{b})$

Poincaré's First Defn of a Manifold

set of solns in R^N to $\mathbf{F}(x_1, \ldots, x_N) = 0, \ldots$
$\mathbf{\Phi}(x_1, \ldots, x_N) > 0, \ldots$

for some cts'ly diff'ble $F_1, \ldots, F_m, \Phi_1, \ldots, \Phi_\ell$ s.t.

for some choice of $x_{k_1}, \ldots, x_{k_m}$,
the Jacobian $(dF_i/d_{k_j})_{i, j}$ is invertible
at every soln

<u>Ex</u> $F(x1, x2) = x1^4 - 4x1^2 + x2^2 + 1$
$\Phi(x1, x2) = -x1$

$dF/dx2$ is invertible everywhere

$\Phi(x1, x2) > 0$ means $x1 < 0$
i.e., left connected component of $\{F(x1, x2) = 0\}$

too rigid: later, Poincaré gave a second defn
then Weyl + Hausdorff gen'lized

(Munkres 330–331) next goal:
fundamental groups
[of manifolds]

<u>Df</u> a group consists of
a set G
a function $- \bullet - : G \times G$ to G
called the group law/operation
obeying these rules:
1) (associativity) $(a \bullet b) \bullet c = a \bullet (b \bullet c)$
2) (identity) there is an elt e in G s.t.
$a \bullet e = a = e \bullet a$
3) (inversion) for all a in G, there is b in G s.t.
$a \bullet b = e = b \bullet a$

<u>Rem</u> we sometimes say G itself is "the group"

Ex $(G, \bullet) = (\mathbb{Z}, +)$ is a group:

1) $+$ is associative
2) 0 satisfies $a + 0 = a = 0 + a$
3) for all a in $\mathbb{Z}$, have $a + (-a) = 0 = (-a) + a$

Lem in any group $(G, \bullet)$,
1) the identity elt e is unique
2) the elt b s.t. $a \bullet b = e$ is unique to a

Ex $(G, \bullet) = (\mathbb{Z}, \times)$ is not a group:

1) $\times$ is associative
2) 1 satisfies $a \times 1 = a = 1 \times a$

but 3) fails
even if we replace $e = 1$ with a different e

[any set of numbers that forms a group under ×?]

Ex $(G, \bullet) = (\mathbb{R}, \times)$ is also not a group
$(G, \bullet) = (\mathbb{R}_+, \times)$ is a group

3) for all a in $\mathbb{R}_+$, have $a \times 1/a = 1 = 1/a \times a$
note: $1/a$ in $\mathbb{R}_+$ as well

Ex let X be any set

is $(G, \bullet) = (\{\text{maps from } X \text{ to itself}\}, \circ)$ a group?

1) $\circ$ is associative
2) id_X satisfies $f \circ id_X = f = id_X \circ f$

but 3) fails

$(G, \bullet) = (\{\text{bijections from } X \text{ to itself}\}, \circ)$ is a group!

3) for all f bijective, have two-sided inv f^{-1}

we denote it by Sym(X)

Rem Sym(X) not commutative, unlike Z, R_+
i.e. $g \circ f \neq f \circ g$ in general

commutative groups also called abelian groups

Ex let X be any topological space

$(G, \bullet) = (\{\text{homeo's from } X \text{ to itself}\}, \circ)$ is a group

we denote it by Homeo(X)

if X is not discrete
then most elts Homeo(X) may be hard to describe
[e.g., consider X = [0, 1] or S^1]

Ex pick $0 = a_0 < a_1 < \dots < a_k < 1$
$0 = b_0 < b_1 < \dots < b_k < 1$

there is a unique homeo f from [0, 1] to itself s.t.
$f(0) = 0$, $f(1) = 1$
$f(a_i) = b_i$ for all i
f is linear on $[a_{i-1}, a_i]$ for all i

exercise:
1) if f, g arise this way, then $g \circ f$ is too
2) $id_{[0, 1]}$ arises this way
3) if f arises this way, then f^{-1} does too

such f’s form a subset of Homeo([0, 1]),
stable under the group law ∘

<u>Df</u> a subgroup of (G, •) is
a group of the form (H, •), where
H is a subset of G stable under •
H contains e

(again writing • for the restricted map H × H to H)

<u>Ex</u> what subsets of Z are
stable under +
contain 0?

if H sub Z contains n
then it contains 2n, 3n, …
hence contains the inverses –n, –2n, –3n, …

<u>Lem</u> if H is a subgroup of (Z, +)
and H contains some elt n
then H supset nZ := {nk | k in Z}

<u>Thm</u> if H is a subgroup of (Z, +)
then H = mZ for some m