



Fock Spaces, Braid Varieties, and Block Equivalences

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1 Charged Partitions Fix an integer $l > 0$.

An integer partition $\lambda \in \Pi$ is called an l -core iff it has no hook lengths divisible by l .

- 1-cores: \emptyset .
- 2-cores: staircase partitions.
- l -cores for $l \geq 3$: complicated.

An analogue of long division for partitions:

$$l\text{-core} \times l\text{-quotient} : \Pi \xrightarrow{\sim} \Pi_{l\text{-cor}} \times \Pi^l.$$

Our starting point is its application to *quantum groups* and *finite reductive groups*.

First, repackage it as a bijection

$$\Upsilon_l : \Pi \times \mathbf{Z} \xrightarrow{\sim} \Pi^l \times \mathbf{Z}^l.$$

Elements of $\Pi^l \times \mathbf{Z}^l$ are called *charged l -partitions*.

We'll need $\mathbf{B} = \{\beta \mid \mathbf{Z}_{<x} \subseteq \beta \subseteq \mathbf{Z}_{<y} \text{ for some } x, y\}$.

Elements of \mathbf{B}^l are *l -abacus configurations*.

Step 1. $\Pi \times \mathbf{Z} \simeq \mathbf{B}$ via

$$|\pi, s\rangle \leftrightarrow \{s + \pi_i - i + 1 \mid i = 1, 2, 3, \dots\}.$$

Step 2. $\vec{v}_l : \mathbf{B} \xrightarrow{\sim} \mathbf{B}^l$ given by

$$v_l^{(r)}(\beta) = \{q \in \mathbf{Z} \mid lq + r \in \beta\} \quad \text{for all } r \bmod l.$$

$$\Upsilon_l(\pi, 0) = |\vec{\pi}, \vec{s}\rangle \iff \begin{cases} \Upsilon_l(l\text{-core}(\pi), 0) = |\vec{\emptyset}, \vec{s}\rangle, \\ l\text{-quotient}(\pi) = \vec{\pi}. \end{cases}$$

Ex Take $|\pi, s\rangle = |(2, 2, 1), 4\rangle$ and $l = 3$.

$$\begin{array}{ccccccccccc} \cdots & \bullet & \bullet & \bullet & & \bullet & & \bullet & \bullet & & \cdots \\ & -1 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & \\ & & & & & \Downarrow & & & & & \\ 2 & \cdots & \bullet & \bullet & & \bullet & & & & \cdots \\ 1 & \cdots & \bullet & \bullet & \bullet & & & & & \cdots \\ 0 & \cdots & \bullet & \bullet & \bullet & \bullet & \bullet & & & \cdots \\ & & -2 & -1 & 0 & 1 & 2 & 3 & & \end{array}$$

The charged 3-partition: $|((\emptyset, \emptyset, (1)), (2, 0, 0))\rangle$.

We do have $\Upsilon_3(3\text{-core}(\pi), s) = |\vec{\emptyset}, (2, 0, 0)\rangle$.

2 Cyclotomic Hecke Algebras Let $\mathfrak{S}_{N,l} = \mathfrak{S}_N \ltimes \mathbf{Z}_l^N$.

From partitions to representations:

- $\text{Irr}(\mathfrak{S}_n) \simeq \{\pi \in \Pi \mid \pi \vdash n\}$.
- $\text{Irr}(\mathfrak{S}_{N,l}) \simeq \{\vec{\pi} \in \Pi^l \mid |\vec{\pi}| \vdash_l N\}$.

Actually, we'll use the *Ariki-Koike algebra*

$$H_{N,l}(u, \vec{v}) = \frac{\mathbf{C}[u^{\pm 1}, v_1^{\pm 1}, \dots, v_{\ell-1}^{\pm 1}] \mathfrak{B}_{N,l}}{\left\langle \begin{array}{l} (\sigma_i - 1)(\sigma_i + u) \text{ for all } i, \\ (\tau - 1)(\tau - v_1) \cdots (\tau - v_{l-1}) \end{array} \right\rangle},$$

By Tits deformation, $\text{Irr}(\mathfrak{S}_{N,l}) \simeq \text{Irr}(H_{N,l}(u, \vec{v}))$.

For general m and \vec{s} , nontrivial *decomposition map*

$$K_0(H_{N,l}(u, \vec{v})) \rightarrow K_0(H_{N,l}(\zeta_m, \vec{\zeta}_m)).$$

(Ariki) For l, m , and $\vec{s} \in \mathbf{Z}^l$: Description of

$$\mathbf{Q}K_0(H_{N,l}(\zeta_m, \vec{\zeta}_m)) \quad \text{for } N \geq 0$$

via a $U'_v(\widehat{\mathfrak{sl}}_m)_{\mathbf{Q}}$ -module

$$\Lambda_{\vec{s}} := \bigoplus_{\vec{\lambda} \in \Pi^l} \mathbf{Q}(v) |\vec{\lambda}, \vec{s}\rangle$$

called the *Fock space of level l and charge \vec{s}* .

(Uglov) For $(\vec{s}, \vec{r}) \in \mathbf{Z}^l \times \mathbf{Z}^m$ such that $|\vec{s}| = s = |\vec{r}|$:

Commuting $U'_v(\widehat{\mathfrak{sl}}_m)_{\mathbf{Q}}$ - and $U'_v(\widehat{\mathfrak{sl}}_l)_{\mathbf{Q}}$ -actions on

$$\Lambda_{\vec{s}} \stackrel{\Upsilon_l}{=} \Lambda_s \stackrel{\bar{\Upsilon}_m}{=} \Lambda_{\vec{r}}$$

where $\bar{\Upsilon}_m$ is a modified version of Υ_m .

(Uglov) Bijections of the form below, matching decomposition numbers on the two sides:

$$\begin{array}{ccc} \mathrm{Irr}(\mathfrak{S}_{N,l})_{\mathbf{b}} & \simeq & \mathrm{Irr}(\mathfrak{S}_{N',m})_{\mathbf{c}} \\ \uparrow & & \uparrow \\ \mathbf{b} \trianglelefteq K_0(H_{N,l}(\zeta_m, \zeta_m^{\vec{s}})) & & K_0(H_{N',m}(\zeta_l, \zeta_l^{\vec{r}})) \trianglerighteq \mathbf{c} \end{array}$$

(Losev, Rouquier–Shan–Varagnolo–Vasserot, Webster)

$$\mathrm{Rep}_{\mathbf{b}}(H_{N,l}^{\mathrm{rat}}(\vec{\nu}_l)) \simeq \mathrm{Rep}_{\mathbf{c}}(H_{N',m}^{\mathrm{rat}}(\vec{\nu}_m))$$

for *cyclotomic rational DAHAs* $H_{N,l}^{\mathrm{rat}}, H_{N',m}^{\mathrm{rat}}$.

- $\zeta_m^{\vec{s}} = e^{2\pi i \vec{\nu}_l}$ and $\zeta_l^{\vec{r}} = e^{2\pi i \vec{\nu}_m}$.
- $\mathrm{Rep}_{\mathbf{b}}, \mathrm{Rep}_{\mathbf{c}}$ lift \mathbf{b}, \mathbf{c} .

These equivalences are called *level-rank dualities*.

3 Φ -Harish-Chandra Theories Ting Xue and I propose a generalization to *relative Weyl groups*.

Fix a prime power q . A reductive group \mathbf{G} with Frobenius $F : \mathbf{G} \rightarrow \mathbf{G}$ over $\bar{\mathbf{F}}_q$ defines a

$$\text{finite reductive group } G = \mathbf{G}^F.$$

Let $\mathrm{Uch}(G)$ index its *unipotent irreducible characters*.

(Harish–Chandra) $\mathrm{Uch}(G) = \coprod_{(L,\lambda)} \mathrm{Uch}(G)_{L,\lambda}$.

- $L \subseteq G$ is an F -maximally split Levi.
- $\lambda \in \mathrm{Uch}(L)$ is *cuspidal*.
- $\mathrm{Uch}(G)_{L,\lambda} = \{\rho \in \mathrm{Uch}(G) \mid (\rho, \mathrm{Ind}_L^G(\lambda)) \neq 0\}$.

Relative Weyl groups: $W_{G,L,\lambda} = C_{N_G(L)/L}(\lambda)$.

How to introduce l, m ?

(Broué–Malle–Michel) A Levi L is Φ_l -split iff

$$L = Z_G(T)^\circ \quad \text{for some torus } \mathbf{T} \subseteq \mathbf{G} \text{ such that} \\ |T| \text{ is generically a power of } \Phi_l(q).$$

(\mathbf{T} need not be maximal!)

$\lambda \in \text{Uch}(L)$ is Φ_l -cuspidal iff it does not occur in the Lusztig induction \mathbf{R}_M^L from smaller Φ_l -split Levis M .

Φ_l -cuspidal pairs and Φ_l -Harish-Chandra series:

- $\text{Uch}(G) = \coprod_{\Phi_l\text{-cuspidal } (L, \lambda)} \text{Uch}(G)_{L, \lambda}.$
- Bijections $\text{Uch}(G)_{L, \lambda} \xrightarrow{\chi_{L, \lambda}} \text{Irr}(W_{G, L, \lambda}).$
- Signs $\text{Uch}(G)_{L, \lambda} \xrightarrow{\varepsilon_{L, \lambda}} \{\pm 1\} \implies \text{isometries } \varepsilon \chi.$

Ex Take $G = \text{GL}_n(\mathbf{F}_q)$, so that $\text{Uch}(G) \simeq \{\pi \vdash n\}.$

Φ_l -split Levi L	$\text{GL}_{\mathbf{N}}(\mathbf{F}_q) \times (\mathbf{F}_{q^l})^{\frac{n-N}{l}}$
Φ_l -cuspidal $\lambda \in \text{Uch}(L)$	l -core $\lambda \vdash N$
$\text{Uch}(G)_{L, \lambda}$	$\{\pi \vdash n \mid l\text{-core}(\pi) = \lambda\}$

Above, $W_{G, L, \lambda} \simeq S_N \ltimes \mathbf{Z}_l^N$. The map

$$\chi_{L, \lambda} : \text{Uch}(G)_{L, \lambda} \rightarrow \text{Irr}(W_{G, L, \lambda})$$

comes from the Π^l part of $\Upsilon_l(-, \text{len}(\lambda)).$

$\mathbf{R}_L^G := \text{H}_c^*(Y_L^G)$ for some Deligne–Lusztig variety Y_L^G .

Conj (BMM) $\text{End}_G(\text{H}_c^*(Y_L^G)[\lambda]) \simeq H_{N, l}(q^l, q^{\vec{a}(\lambda)}).$

Above, $\vec{a}(\lambda) = l\vec{a}'(\lambda) + (0, 1, \dots, l-1)$, where \vec{a}' is the \mathbf{Z}^l part of $\Upsilon_l(\lambda, \text{len}(\lambda)).$

Conj (BMM) For *general* G and Φ_l -cuspidal (L, λ) ,

$$\text{End}_G(H_c^*(Y_L^G)[\lambda]) \simeq H_{W_{G,L,\lambda}}(q)$$

for an *explicit* 1-parameter algebra $H_{W_{G,L,\lambda}}(x)$. And the commuting actions induce

$$\chi_{L,\lambda} : \text{Uch}(G)_{L,\lambda} \rightarrow \text{Irr}(H_{W_{G,L,\lambda}}(q)) = \text{Irr}(W_{G,L,\lambda}).$$

(Lusztig) True for $l = 1$ cases and “Coxeter tori”.

(Digne–Michel–Rouquier) Progress in types A, B, D_4 .

Our generalization of level-rank duality will involve

$$H_{W_{G,L,\lambda}}(\zeta_m) \text{ versus } H_{W_{G,M,\mu}}(\zeta_l),$$

for Φ_l -cuspidal (L, λ) and Φ_m -cuspidal (M, μ) .

Let $\text{Uch}_{L,\lambda,M,\mu}^G = \text{Uch}(G)_{L,\lambda} \cap \text{Uch}(G)_{M,\mu}$. Form

$$\text{Irr}(W_{G,L,\lambda}) \xleftarrow{\chi_{L,\lambda}} \text{Uch}_{L,\lambda,M,\mu}^G \xrightarrow{\chi_{M,\mu}} \text{Irr}(W_{G,M,\mu}).$$

Conj (T–Xue)

- 1 The left / right image is a union of preimages of blocks of $H_{G,L,\lambda}(\zeta_m)$ / $H_{G,M,\mu}(\zeta_l)$.
- 2 The maps descend to a bijection

$$\{H_{G,L,\lambda}(\zeta_m)\text{-blocks}\} \simeq \{H_{G,M,\mu}(\zeta_l)\text{-blocks}\}.$$

- 3 For matching blocks, an equivalence of their highest-weight covers (= blocks of rational DAHAs).

Thm (T–Xue) Take $G = \text{GL}_n$ and l, m coprime.

Then (1)–(3) hold when $H_{G,L,\lambda}(x) = H_{N,l}(x^l, x^{\vec{a}(\lambda)})$. In (1), the images are single blocks.

4 Steinberg Varieties for \mathbf{G}^F (Recall $G = \mathbf{G}^F$.)

A more explicit version of the BMM conjecture:

$$\begin{aligned} R_L^G(\lambda) &:= \sum_i (-1)^i H_c^i(Y_L^G)[\lambda] \\ &= \sum_{\rho \in \text{Uch}(G)_{L,\lambda}} \rho \otimes \varepsilon_{L,\lambda}(\rho) \chi_{L,\lambda}(\rho)_q \end{aligned}$$

as a virtual $(G, H_{W_{G,L,\lambda}}(q))$ -bimodule.

Suggests looking at

$$R_L^G(\lambda) \otimes_{\mathbf{C}G} R_M^G(\mu)$$

as a virtual $(H_{W_{G,L,\lambda}}(q), H_{W_{G,M,\mu}}(q))$ -bimodule.

Let \mathcal{B} be the flag variety of \mathbf{G} . For $w \in W$, set

$$\mathcal{Y}_w = \{(g, B) \in \mathbf{G} \times \mathcal{B} \mid B \xrightarrow{w} gFB(gF)^{-1}\}.$$

Action $\mathbf{G} \curvearrowright \mathcal{Y}_w$ via $x \cdot (g, B) = xgF(x)^{-1}, xBx^{-1}$.

If L is a maximal torus of type $[w]$, then

$$\mathbf{R}_L^G(1_L) = H_c^*(Y_L^G)[1_L] \simeq H_{c,\mathbf{G}}^*(\mathcal{Y}_w).$$

For L, M maximal tori of types $[w], [v]$, we are led to consider the *generalized Steinberg variety*

$$\mathcal{Y}_w \times_{\mathbf{G}}^L \mathcal{Y}_v.$$

Some (derived) Künneth-type formula should show

$$R_L^G(\lambda) \otimes_{\mathbf{C}G} R_M^G(\mu) = \sum_i (-1)^i H_{c,\mathbf{G}}^i(\mathcal{Y}_w \times_{\mathbf{G}}^L \mathcal{Y}_v).$$

5 Steinberg Varieties for \mathbf{G}

Earlier, I introduced similar varieties in a totally independent context.

Let $\mathcal{U} \subseteq \mathbf{G}$ be the *unipotent locus*. Let

$$\mathcal{U}_w = \{(u, B) \in \mathcal{U} \times \mathcal{B} \mid B \xrightarrow{w} gBg^{-1}\}.$$

For example, \mathcal{U}_e is the (groupy) Springer resolution.

I studied an action of $\mathbf{H}_W := \mathrm{gr}_*^W \mathrm{H}_{c, \mathbf{G}}^*(\mathcal{U}_e \times_{\mathcal{U}}^L \mathcal{U}_e)$ on

$$\mathrm{gr}_*^W \mathrm{H}_{c, \mathbf{G}}^*(\mathcal{U}_e \times_{\mathcal{U}}^L \mathcal{U}_w).$$

(Actually a braid version motivated by link homology.)

Above, $\mathbf{H}_W \simeq \mathbf{C}W \ltimes \mathrm{Sym}(X_*(\mathbf{A}))$, where $\mathbf{A} \subseteq \mathbf{G}$ is a maximally split maximal torus. ($W = W_{G, \mathbf{A}, 1}$.)

Via the W -action, the \mathbf{G} -equivariant virtual weight polynomial defines a $\mathbf{Z}W[x]$ -valued *virtual character*.

Thm (T) Suppose that $w \in W$ is *regular* of order m .

Then the virtual character of $\mathcal{U}_e \times_{\mathcal{U}}^L \mathcal{U}_w$ is given by

$$\sum_{\rho \in \mathrm{Uch}(G)_{A, 1, M, 1}} D_m(\rho) \Delta_{1/m}(\chi_{A, 1}(\rho)),$$

where:

- M is any Φ_m -split maximal torus of G .
- $D_m(\rho) = \varepsilon_{M, 1}(\rho) \deg \chi_{M, 1}(\rho)$.
- For any χ , we write $[\Delta_{1/m}(\chi)]$ for the character of the associated Verma of the rational DAHA $H_W^{\mathrm{rat}}(\frac{1}{m})$.

$H_W^{\mathrm{rat}}(\frac{1}{m})$ is a highest-weight cover of $H_W(\zeta_m)$!

A very strange analogy emerges.

Below M is a Φ_m -split maximal torus of G .

\mathbf{GF}

\mathbf{G}

$$\mathcal{Y}_e \times_{\mathbf{G}}^{\mathbf{L}} \mathcal{Y}_w$$

$$\mathcal{U}_e \times_{\mathcal{U}}^{\mathbf{L}} \mathcal{U}_w$$

$$(q, q)$$

$$(\zeta_m, 1)$$

$$H_W(q), H_{W_{G,M,1}}(q) \quad H_W^{\text{rat}}(\frac{1}{m}), H_{W_{G,M,1}}(1)$$

Thank you for listening.

Where else do we expect the formula on the \mathbf{G} side?

- Work of Oblomkov–Yun, Losev–Boixeda–Alvarez, *et al.* on *affine Springer fibers*.

- Work of Lusztig and Abreu–Nigro on analogues of \mathcal{U}_w replacing \mathcal{U} with a *regular semisimple* class of G .