7.

On varieties studied by Bezrukavnikov–McBreen, Gu–Wang, Lusztig, and myself.

7.1.

Fix a braid $\beta = \sigma_1 \cdots \sigma_\ell$, where $\sigma_1, \ldots, \sigma_\ell$ is a sequence of simple twists with possible repetition. We have a variety

$$\begin{aligned} G(\beta) &= \left\{(g, B_1, \ldots, B_\ell) : B_\ell^g \xrightarrow{s_1} B_1 \xrightarrow{s_2} \cdots \xrightarrow{s_\ell} B_\ell\right\} \\ &\subseteq G \times \mathcal{B}^\ell, \end{aligned}$$

where $B^g = g^{-1}Bg$. The action of G on itself by right conjugation extends to an action on $G(\beta)$, namely,

$$(g, B_1, \ldots, B_\ell) \cdot x = (x^{-1}gx, B_1^x, \ldots, B_\ell^x).$$

Fix a pair of opposite Borel subgroups $B_+, B_- \subseteq G$. Let

$$G_+(\beta) = G(\beta)_{B_\ell = B_+},$$

the fiber of $\mathrm{pr}_\ell : G(\beta) \to \mathcal{B}$ above B_+. Then

$$[G_+(\beta)/B_{+,\mathrm{Ad}}] \simeq [G(\beta)/G_{\mathrm{Ad}}].$$

Let $U_\pm \subseteq B_\pm$ be the unipotent radicals, and let $T = B_+ \cap B_-$. We will be interested in the stacks $[G_+(\beta)/U_{+,\mathrm{Ad}}]$, viewed as T-bundles over $[G(\beta)/G_{\mathrm{Ad}}]$.

7.2.

Let $\pi = \sigma_{w_0}^2$, the full twist. By definition,

$$\begin{aligned} G_+(\pi) &= \left\{(g, B) : B_+^g \xrightarrow{w_0} B \xrightarrow{w_0} B_+\right\} \\ &\simeq \left\{(g, B_+h) : B_+^g \xrightarrow{w_0} B_+^h \xrightarrow{w_0} B_+\right\}. \end{aligned}$$

Pick a lift $\dot{w}_0 \in N_G(T)$ of w_0. Then the fiber of $G_+(\pi) \subseteq G \times \mathcal{B}$ above $g \in G$ is

$$G_+(\pi)_g = B_+ \backslash (B_+\dot{w}_0U_+g \cap B_+\dot{w}_0U_+).$$

In the document `2101_09`, we observed that the composition

$$B_+\dot{w}_0U_+ \cap \dot{w}_0U_+g \to B_+\dot{w}_0U_+ \cap B_+\dot{w}_0U_+g \to G_+(\pi)_g$$

was a bijection at the level of points. At the same time, there is another bijection:

$$\begin{array}{rcl} U_-U_+ \cap B_+g & \xrightarrow{\sim} & B_+\dot{w}_0U_+ \cap \dot{w}_0U_+g \\ vu = tzg & \mapsto & \dot{w}_0t^{-1}vu = \dot{w}_0zg \end{array}$$

(Note that $\dot{w}_0 t^{-1} v = (\dot{w}_0 t^{-1} \dot{w}_0^{-1})(\dot{w}_0 v \dot{w}_0^{-1})\dot{w}_0 \in B_+\dot{w}_0$.) Let

$$\begin{aligned} \mathcal{V} &= \{(g,u,v) : B_+ g = B_+ vu\} \\ &\subseteq G \times U_+ \times U_-, \\ \mathcal{V}_1 = \mathcal{V}_{u=1} &= \{(g,v) : B_+ g = B_+ v\} \\ &\subseteq G \times U_-. \end{aligned}$$

There is a U_+-action on $\mathcal{V}$ defined by

$$(g,u,v)\cdot x = (x^{-1}gx, ux, v).$$

The composition $\mathcal{V}_1 \to \mathcal{V} \to [\mathcal{V}/U_+]$ is an isomorphism. We conclude:

Lemma 7.1. *At the level of points, we have a U_+-equivariant bijection*

$$\begin{array}{rcl} \mathcal{V} & \to & G_+(\pi) \\ (g,u,v) & \mapsto & (g, B_+^{\dot{w}_0 z g}) \end{array}$$

where $z \in U_+$ is defined by $vu \in Tzg$. Thus, we have an equivalence of groupoids

$$\begin{array}{rcl} \mathcal{V}_1 & \to & [G_+(\pi)/U_{+,\mathrm{Ad}}] \\ (g,v) & \mapsto & [g, B_+^{\dot{w}_0 v}] \end{array}$$

(a posteriori, a bijection of sets).

Proof. To see the equivariance in the first statement: Note that $vu \in Tzg$ implies $vux \in Tzgx = Tzx(x^{-1}gx)$, and moreover, $B_+^{\dot{w}_0 zx(x^{-1}gx)} = (B_+^{\dot{w}_0 zg})^x$. To deduce the second statement from the first, set $u = 1$ and observe that in this situation, $Tzg = Tv$. $\square$

Next, we compare $\mathcal{V}_1$ to the varieties studied by Bezrukavnikov–McBreen. Since the U_--part of an element of U_-B_+ is uniquely determined, we have isomorphisms:

$$\begin{array}{rcl} U_-B_+ & \leftrightarrow & \mathcal{V}_1 \\ vb & \mapsto & (bv, v) \\ vgv^{-1} & \leftarrow & (g,v) \end{array}$$

Notably, these isomorphisms do *not* identify the inclusion $U_-B_+ \to G$ with the projection $\mathrm{pr}_1 : \mathcal{V}_1 \to G$. However, since $bv = b(vb)b^{-1}$, they do identify the further maps to $G/B_{+,\mathrm{Ad}}$. We conclude:

Proposition 7.2. *At the level of points, there is an equivalence of groupoids*

$$\begin{array}{rcl} U_-B_+ & \xrightarrow{\sim} & [G_+(\pi)/U_{+,\mathrm{Ad}}] \\ vb & \mapsto & [bv, B_+^{\dot{w}_0 v}] \end{array}$$

(a posteriori, a bijection of sets). It fits into a commutative diagram:

$$\begin{tikzcd} U_-B_+ \arrow[r, "\sim"] \arrow[d] & {[G_+(\pi)/U_{+,\mathrm{Ad}}]} \arrow[d] \\ G \arrow[r] & {[G/B_{+,\mathrm{Ad}}]} \end{tikzcd}$$

In particular, fix a point $\delta \in T /\!\!/ W$. Let $G_\delta \subseteq G$ and $[G/G_{\mathrm{Ad}}]_\delta \subseteq [G/G_{\mathrm{Ad}}]$ be its preimages along the maps

$$G \to [G/G_{\mathrm{Ad}}] \to T /\!\!/ W.$$

For any stack $\mathcal{X}$ over $[G/G_{\mathrm{Ad}}]$, we set

$$\mathcal{X}_\delta = \mathcal{X} \times_{[G/G_{\mathrm{Ad}}]} [G/G_{\mathrm{Ad}}]_\delta.$$

If $\mathcal{X} = [X/G]$ for some G-equivariant map of varieties $X \to G$, then $\mathcal{X}_\delta = [X_\delta/G]$. So the equivalence in the proposition restricts to an equivalence

$$(U_-B_+)_\delta \simeq [G_+(\pi)_\delta/U_{+,\mathrm{Ad}}].$$

Two special cases:

- When δ is regular, the left-hand side is the variety studied by Bezrukavnikov–McBreen that conjecturally retracts onto the full lattice quotient of an affine Springer fiber of split type.
- When $\delta = 1$, the scheme $G_+(\pi)_\delta$ is the scheme denoted $\mathcal{U}(\pi)_{B_+}$ in the notation of my preprint. The weight-graded Borel–Moore homology of the stack

  $$[\mathcal{U}(\pi)_{B_+}/B_{+,\mathrm{Ad}}] \simeq [\mathcal{U}(\pi)/G_{\mathrm{Ad}}]$$

  is what I called the $\mathbf{A}_W$-trace of the full twist π. The $\Lambda^*(\mathfrak{t})$-isotypic components of this bigraded representation of W recover the Khovanov–Rozansky homology of π, up to taking componentwise duals.

So for general β, we want to understand how the varieties $G_+(\beta)_\delta$ vary as δ varies in $T /\!\!/ W$.

7.3.

Let $\delta \in T /\!\!/ W$ be regular, and let $t \in T$ be a preimage of δ. Then G_δ is the adjoint orbit of t in G, which we can identify with the image of the embedding $T\backslash G \to G$ that sends $Tg \mapsto g^{-1}tg$. So we have

$$G_+(\pi)_\delta = \left\{(Tg, B) : B_+^{g^{-1}tg} \xrightarrow{w_0} B \xrightarrow{w_0} B_+\right\} \subseteq T\backslash G \times \mathcal{B}.$$