Knots, Plethysms, and the Riordan Group

Minh-Tâm Quang Trinh

Howard University

1 Fruit

"You can't add together apples and oranges."

Well, not in real life.

But in mathematics, you can make up a new world where this is possible.

The *free vector space* on $X = \{\text{apple}, \text{orange}, \text{pear}\}$:

$\mathbf{C}\langle X\rangle = \{a \cdot \text{apple} + b \cdot \text{orange} + c \cdot \text{pear} \mid a, b, c \in \mathbf{C}\}$.

Natural ways to do addition and scalar multiplication on $\mathbf{C}\langle X\rangle$.

Too dumb? The vectors "apple" and "orange" just sum to "apple + orange".

But there's a vector space where it simplifies further.

(1) Start with some relations like

$$\text{pear} \sim \text{apple} + \text{orange}, \quad \text{orange} \sim 2 \cdot \text{apple}.$$

(2) Let *Rel* be the span of "pear − apple − orange" and "orange − 2 · apple".

(3) Extend $\sim$ to an equivalence relation on $\mathbf{C}\langle X\rangle$:

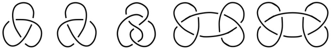
$$v \sim v' \iff v - v' \in \mathit{Rel}.$$

The set of equivalence classes is a new vector space $\mathbf{C}\langle X\rangle/\mathit{Rel}$, in which $\sim$ defines equality.

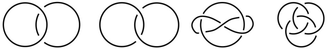
2 Knots and Links

I'm interested in *knots* and *links*.

Knot diagrams:



Links allow multiple circles:



An *oriented link diagram* is a link diagram where we give each circle a direction/orientation.

Also interested in diagrams that are strictly inside a given region $\Omega \subseteq \mathbf{R}^2$.

Will mainly focus on $\Omega = \mathbf{R}^2$ and $\Omega = \mathbf{R}^2 \setminus \mathbf{0}$.

We will treat two diagrams in Ω as equal as long as they are *isotopic*:

That is, we can deform one into the other within Ω, without tearing any circles.

Let $\mathcal{L}_\Omega$ be the set of all oriented link diagrams in Ω, including the empty diagram.

$$\mathbf{C}\langle\mathcal{L}_\Omega\rangle = \{\text{finite linear combos of elements of } \mathcal{L}_\Omega\}$$

is usually huge!

To study it, try to find equivalence relations on it that give us smaller, more tractable vector spaces.

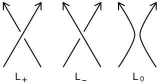
This is called *skein theory*.

The interesting parts of links are the crossings.

One thing that we can do with links, which we could not do with fruit:

We can specify *local relations*, that relate diagrams whenever they differ at a single crossing.

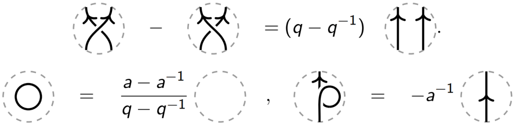
E.g., we might have three diagrams that only differ at one crossing:



We will interpret a relation on these crossings as a relation on *every* such triple of oriented link diagrams.

Fix constants $a \neq 0$ and $q \neq 0, 1$.

It turns out that the following local *skein relations* are especially interesting.



When $\Omega \neq \mathbf{R}^2$, we will be a bit more restrictive:

We will only apply the relations when the drawings take place inside an open disk inside Ω*.*

E.g., we will not simplify a circle using the bottom left relation when Ω has a hole inside the circle.

The relations give us a linear subspace $Rel_\Omega \subseteq \mathbf{C}\langle \mathcal{L}_\Omega \rangle$.

The *HOMFLYPT skein module* of Ω is

$$\mathrm{Sk}_\Omega = \mathbf{C}\langle \mathcal{L}_\Omega \rangle / Rel_\Omega.$$

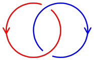
Thm (≈ HOMFLYPT 1986)

$$\boxed{\mathrm{Sk}_{\mathbf{R}^2} = \mathbf{C}.}$$

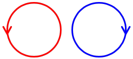
That is, any diagram in $\mathbf{R}^2$ is a scalar multiple of the empty diagram modulo the skein relations.

The *(unreduced) HOMFLYPT invariant* of a link is the scalar that we get from any diagram of it.

Example Consider the following element in $\mathbf{C}\langle \mathcal{L}_{\mathbf{R}^2} \rangle$:

$L =$ 

Modulo the "crossing" rule,

$L =$  $+ (q - q^{-1})$ 

Modulo $\bigcirc = \frac{a-a^{-1}}{q-q^{-1}} \cdot \emptyset$,

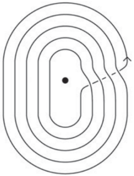
$$L = \left(\frac{a - a^{-1}}{q - q^{-1}} \right)^2 \cdot \emptyset + (a - a^{-1}) \cdot \emptyset.$$

So the scalar is $\left(\frac{a-a^{-1}}{q-q^{-1}} \right)^2 + a - a^{-1}$.

For $\Omega = \mathbf{R}^2 \setminus \mathbf{0}$, what happens?

Cannot simplify circles in $\mathbf{R}^2 \setminus \mathbf{0}$ that go around $\mathbf{0}$.

In fact: pairwise distinct diagrams p_n for all $n \in \mathbf{Z}$.

$p_5 =$ 

($n > 0$ is counterclockwise, $n < 0$ clockwise.)

We set $p_0 = \emptyset$ as a matter of convention.

There are even more diagrams in $\mathbf{R}^2 \setminus \mathbf{0}$ that we cannot simplify.

If we have two diagrams L and L', then we can put L *around* L' to get a new diagram

$$L \cdot L'.$$

Note that $L \cdot L'$ and $L' \cdot L$ are isotopic.

Extend this to a binary operation on $\mathrm{Sk}_{\mathbf{R}^2 \setminus \mathbf{0}}$, by making it distribute over addition.

Think of this as a multiplication law, which turns $\mathrm{Sk}_{\mathbf{R}^2 \setminus \mathbf{0}}$ into a *ring*.

Monomials in the p_n's, like $p_1 p_2 p_3$ or p_{-1}^2, do not simplify further.

Thm (Turaev 1988)

The collection of all monomials in the p_n's is a basis for $\mathrm{Sk}_{\mathbf{R}^2\setminus\mathbf{0}}$ as a vector space.

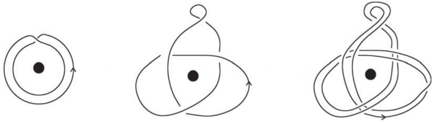
Corollary As a *ring*,

$$\boxed{\mathrm{Sk}_{\mathbf{R}^2\setminus\mathbf{0}} = \mathbf{C}[p_0, p_{\pm 1}, p_{\pm 2}, \ldots].}$$

Remark

The subring generated by $p_0, p_1, p_2, \ldots$ is isomorphic to a very famous ring in combinatorics, called the *ring of symmetric functions*.

3 Plethysm Another operation on $\mathrm{Sk}_{\mathbf{R}^2\setminus\mathbf{0}}$:



The first diagram above is p_2. Call the middle one L.

The last diagram is the *plethysm* $L \circ p_2$.

If L had multiple knot components, then we would form $L \circ p_2$ by inserting p_2 into each component, following their orientations.

For any n, we define $L \circ p_n$ analogously.

It can be fun to check that:

(1) $p_m \circ p_n = p_{mn}$ for any m, n.

How to define $L \circ K$ for any K and L?

Every element of $\mathrm{Sk}_{\mathbf{R}^2 \setminus \mathbf{0}}$ is a polynomial in the p_n's, so it is enough to declare:

(2) $- \circ K$ distributes over $+$ and $\cdot$, for all K.

(3) $p_n \circ -$ distributes over $+$ and $\cdot$, for all n.

Thm (1)–(3) define a binary operation on $\mathrm{Sk}_{\mathbf{R}^2 \setminus \mathbf{0}}$.

This operation is associative, non-commutative, and has identity element p_1:

$L \circ p_1 = L = p_1 \circ L$ for any L.

Let $\mathbf{C}[t]$ be the ring of polynomials in t.

$\mathrm{Sk}_{\mathbf{R}^2 \setminus \mathbf{0}}$	p_1	plethysm
$\mathbf{C}[t]$	t	composition of polynomials

By comparison, the composition operation

$$(g \circ f)(t) = g(f(t))$$

on $\mathbf{C}[t]$ is characterized by:

(1) $f(t) \circ t = f(t) = t \circ f(t)$ for any f.

(2) $- \circ f(t)$ distributes over $+$ and $\cdot$, for any f.

Remark t^n is analogous to p_1^n, not to p_n:

E.g., $t^n \circ (f_1(t) + f_2(t)) \neq t^n \circ f_1(t) + t^n \circ f_2(t)$.

4 Riordan Revisited

In combinatorics, we like to study number sequences $c_0, c_1, c_2, \ldots$ through *generating functions*

$$c(t) = c_0 + c_1 t + c_2 t^2 + \ldots,$$

a.k.a. *formal power series*. They form a ring $\mathbf{C}[\![t]\!]$.

The word "formal" means we don't worry about whether $c(t)$ converges at any given value of t.

Any polynomial is a power series: $\mathbf{C}[t] \subseteq \mathbf{C}[\![t]\!]$.

But $\circ$ does not extend to a binary operation on $\mathbf{C}[\![t]\!]$.

Example If $c(t) = 1 + t + t^2 + \ldots$, then $c(1)$ diverges.

Similarly, $c(1 + \mathrm{blah}(t))$ will never work. By contrast:

$$c(t + t^2) = 1 + (t + t^2) + (t + t^2)^2 + (t + t^2)^3 + \ldots$$

$$= \left\{ \begin{array}{lllll} 1 & & & & \\ + t + t^2 & & & & \\ & + t^2 + 2t^3 + t^4 & & & \\ & & + t^3 & + 3t^4 + \cdots & \\ & & & + t^4 & + \cdots \end{array} \right.$$

$$= 1 + t + 2t^2 + 3t^3 + 5t^4 + \ldots$$

In general, we can form $g \circ f$ as long as f has zero constant term.

Let $\mathbf{C}[\![t]\!]^\circ$ be the further subset of power series with zero constant term and nonzero linear term.

Thm Any element of $\mathbf{C}[\![t]\!]^\circ$ has an inverse under $\circ$.

In other words:

$\mathbf{C}[\![t]\!]^\circ$ forms a *group* under $\circ$, with identity t.

If you think about what I've covered, you'll realize:

There is an analogous group where we replace

$$\mathbf{C}[\![t]\!] \supseteq \mathbf{C}[t]$$

with a certain containment

$$\widehat{\mathrm{Sk}}_{\mathbf{R}^2\setminus\mathbf{0}} \supseteq \mathrm{Sk}_{\mathbf{R}^2\setminus\mathbf{0}}$$

and replace composition with plethysm.

Maybe interesting for symmetric functions.

Thm Any element of $\mathbf{C}[\![t]\!]^\circ$ has an inverse under $\circ$.

Proof sketch For any $f \in \mathbf{C}[\![t]\!]^\circ$, let M_f be the infinite matrix whose columns record the powers of f:

$$M_f = \begin{pmatrix} 1 & & & \\ 0 & c_{1,1} & & \\ 0 & c_{2,1} & c_{2,2} & \\ \vdots & \vdots & & \ddots \end{pmatrix},$$

where the $c_{i,j}$ are given by $f(t)^j = \sum_{i\geq 0} c_{i,j} t^i$.

For example, $M_t = I$, the identity matrix.

In general, we can recover f from M_f by looking at the second column.

Since M_f is lower-triangular with nonzero diagonal entries, it is *invertible*.

Now check the following facts:

(1) M_f^{-1} takes the form M_g for some $g \in \mathbf{C}[\![t]\!]^\circ$.

(2) $M_{f_1 \circ f_2} = M_{f_1} \cdot M_{f_2}$.

We deduce that for any f, there's some g s.t.

$$M_{g \circ f} = M_g \cdot M_f = M_f^{-1} \cdot M_f = I = M_t.$$

Thus $g(t) \circ f(t) = t$. $\square$

This proof shows that the group $\mathbf{C}[\![t]\!]^\circ$ *embeds* into the group of invertible infinite matrices GL_∞.

Recall that the set $\mathbf{C}[\![t]\!]^\times$ of power series with *nonzero* constant term forms a group under $\times$.

The map $f \mapsto M_f$ can be extended to an embedding

$$\begin{array}{rcl} \mathbf{C}[\![t]\!]^\times \rtimes \mathbf{C}[\![t]\!]^\circ & \hookrightarrow & \mathrm{GL}_\infty, \\ (u, f) & \mapsto & M_{u,f}. \end{array}$$

Shapiro's *Riordan group* is the image.

This also has an analogue in the world of plethysm.

Thank you for listening.