



# Knots, Plethysms, and the Riordan Group

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## 1 Fruit

“You can’t add together apples and oranges.”

Well, not in real life.

But in mathematics, you can make up a new world where this is possible.

The *free vector space* on  $X = \{\text{apple}, \text{orange}, \text{pear}\}$ :

$$\mathbf{C}\langle X \rangle = \{a \cdot \text{apple} + b \cdot \text{orange} + c \cdot \text{pear} \mid a, b, c \in \mathbf{C}\}.$$

Natural ways to do addition and scalar multiplication on  $\mathbf{C}\langle X \rangle$ .

Too dumb? The vectors “apple” and “orange” just sum to “apple + orange”.

But there’s a vector space where it simplifies further.

(1) Start with some relations like

$$\text{pear} \sim \text{apple} + \text{orange}, \quad \text{orange} \sim 2 \cdot \text{apple}.$$

(2) Let  $Rel$  be the span of “pear – apple – orange” and “orange – 2 · apple”.

(3) Extend  $\sim$  to an equivalence relation on  $\mathbf{C}\langle X \rangle$ :

$$v \sim v' \iff v - v' \in Rel.$$

The set of equivalence classes is a new vector space  $\mathbf{C}\langle X \rangle / Rel$ , in which  $\sim$  defines equality.

## 2 Knots and Links I'm interested in *knots* and *links*.

Knot diagrams:



Links allow multiple circles:



An *oriented link diagram* is a link diagram where we give each circle a direction/orientation.

Also interested in diagrams that are strictly inside a given region  $\Omega \subseteq \mathbf{R}^2$ .

Will mainly focus on  $\Omega = \mathbf{R}^2$  and  $\Omega = \mathbf{R}^2 \setminus \mathbf{0}$ .

We will treat two diagrams in  $\Omega$  as equal as long as they are *isotopic*:

That is, we can deform one into the other within  $\Omega$ , without tearing any circles.

Let  $\mathcal{L}_\Omega$  be the set of all oriented link diagrams in  $\Omega$ , including the empty diagram.

$$\mathbf{C}\langle \mathcal{L}_\Omega \rangle = \{\text{finite linear combos of elements of } \mathcal{L}_\Omega\}$$

is usually huge!

To study it, try to find equivalence relations on it that give us smaller, more tractable vector spaces.

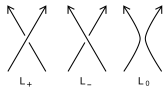
This is called *skein theory*.

The interesting parts of links are the crossings.

One thing that we can do with links, which we could not do with fruit:

We can specify *local relations*, that relate diagrams whenever they differ at a single crossing.

E.g., we might have three diagrams that only differ at one crossing:



We will interpret a relation on these crossings as a relation on *every* such triple of oriented link diagrams.

Fix constants  $a \neq 0$  and  $q \neq 0, 1$ .

It turns out that the following local *skein relations* are especially interesting.

$$\begin{aligned} \text{(crossing with top over)} - \text{(crossing with bottom over)} &= (q - q^{-1}) \text{(two parallel strands)} \\ \text{(circle)} &= \frac{a - a^{-1}}{q - q^{-1}} \text{(empty disk)} , \quad \text{(strand with loop)} = -a^{-1} \text{(strand)} \end{aligned}$$

When  $\Omega \neq \mathbf{R}^2$ , we will be a bit more restrictive:

*We will only apply the relations when the drawings take place inside an open disk inside  $\Omega$ .*

E.g., we will not simplify a circle using the bottom left relation when  $\Omega$  has a hole inside the circle.

The relations give us a linear subspace  $Rel_{\Omega} \subseteq \mathbf{C}\langle \mathcal{L}_{\Omega} \rangle$ .

The *HOMFLYPT skein module* of  $\Omega$  is

$$Sk_{\Omega} = \mathbf{C}\langle \mathcal{L}_{\Omega} \rangle / Rel_{\Omega}.$$

Thm ( $\approx$  HOMFLYPT 1986)

$$Sk_{\mathbf{R}^2} = \mathbf{C}.$$

That is, any diagram in  $\mathbf{R}^2$  is a scalar multiple of the empty diagram modulo the skein relations.

The *HOMFLYPT invariant* of a link is the scalar that we get from any diagram of the link.

**Example** Consider the following element in  $\mathbf{C}\langle \mathcal{L}_{\mathbf{R}^2} \rangle$ :



Modulo the “crossing” rule,

$$L = \text{red circle} + (q - q^{-1}) \text{black circle}$$

$$\text{Modulo } \bigcirc = \frac{a - a^{-1}}{q - q^{-1}} \cdot \emptyset,$$

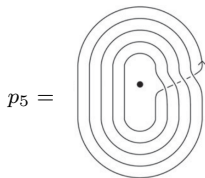
$$L = \left( \frac{a - a^{-1}}{q - q^{-1}} \right)^2 \cdot \emptyset + (a - a^{-1}) \cdot \emptyset.$$

$$\text{So the scalar is } \left( \frac{a - a^{-1}}{q - q^{-1}} \right)^2 + a - a^{-1}.$$

For  $\Omega = \mathbf{R}^2 \setminus \mathbf{0}$ , what happens?

Cannot simplify circles in  $\mathbf{R}^2 \setminus \mathbf{0}$  that go around  $\mathbf{0}$ .

In fact: pairwise distinct diagrams  $p_n$  for all  $n \in \mathbf{Z}$ .



( $n > 0$  is counterclockwise,  $n < 0$  clockwise.)

We set  $p_0 = \emptyset$  as a matter of convention.

There are even more diagrams in  $\mathbf{R}^2 \setminus \mathbf{0}$  that we cannot simplify.

If we have two diagrams  $L$  and  $L'$ , then we can put  $L$  around  $L'$  to get a new diagram

$$L \cdot L'.$$

Note that  $L \cdot L'$  and  $L' \cdot L$  are isotopic.

Extend this to a binary operation on  $\text{Sk}_{\mathbf{R}^2 \setminus \mathbf{0}}$ , by making it distribute over addition.

Think of this as a multiplication law, which turns  $\text{Sk}_{\mathbf{R}^2 \setminus \mathbf{0}}$  into a *ring*.

Monomials in the  $p_n$ 's, like  $p_1 p_2 p_3$  or  $p_{-1}^2$ , do not simplify further.

Thm (Turaev 1988)

The collection of all monomials in the  $p_n$ 's is a basis for  $\text{Sk}_{\mathbf{R}^2 \setminus \mathbf{0}}$  as a vector space.

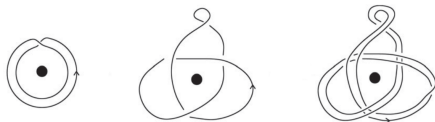
Corollary As a *ring*,

$$\text{Sk}_{\mathbf{R}^2 \setminus \mathbf{0}} = \mathbf{C}[p_0, p_{\pm 1}, p_{\pm 2}, \dots].$$

Remark

The subring generated by  $p_0, p_1, p_2, \dots$  is isomorphic to a very famous ring in combinatorics, called the *ring of symmetric functions*.

3 Plethysm Another operation on  $\text{Sk}_{\mathbf{R}^2 \setminus \mathbf{0}}$ :



The first diagram above is  $p_2$ . Call the middle one  $L$ .

The last diagram is the *plethysm*  $L \circ p_2$ .

If  $L$  had multiple knot components, then we would form  $L \circ p_2$  by inserting  $p_2$  into each component, following their orientations.

For any  $n$ , we define  $L \circ p_n$  analogously.

It can be fun to check that:

$$(1) \quad p_m \circ p_n = p_{mn} \text{ for any } m, n.$$

How to define  $L \circ K$  for any  $K$  and  $L$ ?

Every element of  $\text{Sk}_{\mathbf{R}^2 \setminus \mathbf{0}}$  is a polynomial in the  $p_n$ 's, so it is enough to declare:

$$(2) \quad - \circ K \text{ distributes over } + \text{ and } \cdot, \text{ for all } K.$$

$$(3) \quad p_n \circ - \text{ distributes over } + \text{ and } \cdot, \text{ for all } n.$$

**Thm** (1)–(3) define a binary operation on  $\text{Sk}_{\mathbf{R}^2 \setminus \mathbf{0}}$ .

This operation is associative, non-commutative, and has identity element  $p_1$ :

$$L \circ p_1 = L = p_1 \circ L \text{ for any } L.$$

Let  $\mathbf{C}[t]$  be the ring of polynomials in  $t$ .

$$\begin{array}{c|c|c} \text{Sk}_{\mathbf{R}^2 \setminus \mathbf{0}} & p_1 & \text{plethysm} \\ \mathbf{C}[t] & t & \text{composition of polynomials} \end{array}$$

By comparison, the composition operation

$$(g \circ f)(t) = g(f(t))$$

on  $\mathbf{C}[t]$  is characterized by:

$$(1) \quad f(t) \circ t = f(t) = t \circ f(t) \text{ for any } f.$$

$$(2) \quad - \circ f(t) \text{ distributes over } + \text{ and } \cdot, \text{ for any } f.$$

**Remark**  $t^n$  is analogous to  $p_1^n$ , not to  $p_n$ :

$$\text{E.g., } t^n \circ (f_1(t) + f_2(t)) \neq t^n \circ f_1(t) + t^n \circ f_2(t).$$



## 4 Riordan Revisited

In combinatorics, we like to study number sequences  $c_0, c_1, c_2, \dots$  through *generating functions*

$$c(t) = c_0 + c_1 t + c_2 t^2 + \dots,$$

a.k.a. *formal power series*. They form a ring  $\mathbf{C}[[t]]$ .

The word “formal” means we don’t worry about whether  $c(t)$  converges at any given value of  $t$ .

Any polynomial is a power series:  $\mathbf{C}[t] \subseteq \mathbf{C}[[t]]$ .

But  $\circ$  does not extend to a binary operation on  $\mathbf{C}[[t]]$ .

**Example** If  $c(t) = 1 + t + t^2 + \dots$ , then  $c(1)$  diverges.

Similarly,  $c(1 + \text{blah}(t))$  will never work. By contrast:

$$\begin{aligned} c(t + t^2) &= 1 + (t + t^2) + (t + t^2)^2 + (t + t^2)^3 + \dots \\ &= \begin{cases} 1 \\ + t + t^2 \\ + t^2 + 2t^3 + t^4 \\ + t^3 + 3t^4 + \dots \\ + t^4 + \dots \end{cases} \\ &= 1 + t + 2t^2 + 3t^3 + 5t^4 + \dots \end{aligned}$$

In general, we can form  $g \circ f$  as long as  $f$  has zero constant term.

Let  $\mathbf{C}[[t]]^\circ$  be the further subset of power series with zero constant term and nonzero linear term.

**Thm** Any element of  $\mathbf{C}[[t]]^\circ$  has an inverse under  $\circ$ .

In other words:

$\mathbf{C}[[t]]^\circ$  forms a *group* under  $\circ$ , with identity  $t$ .

If you think about what I've covered, you'll realize:

There is an analogous group where we replace

$$\mathbf{C}[[t]] \supseteq \mathbf{C}[t]$$

with a certain containment

$$\widehat{\mathbf{Sk}}_{\mathbf{R}^2 \setminus \mathbf{0}} \supseteq \mathbf{Sk}_{\mathbf{R}^2 \setminus \mathbf{0}}$$

and replace composition with plethysm.

Maybe interesting for symmetric functions.

**Thm** Any element of  $\mathbf{C}[[t]]^\circ$  has an inverse under  $\circ$ .

**Proof sketch** For any  $f \in \mathbf{C}[[t]]^\circ$ , let  $M_f$  be the infinite matrix whose columns record the powers of  $f$ :

$$M_f = \begin{pmatrix} 1 & & & \\ 0 & c_{1,1} & & \\ 0 & c_{2,1} & c_{2,2} & \\ \vdots & \vdots & & \ddots \end{pmatrix},$$

where the  $c_{i,j}$  are given by  $f(t)^j = \sum_{i \geq 0} c_{i,j} t^i$ .

For example,  $M_t = I$ , the identity matrix.

In general, we can recover  $f$  from  $M_f$  by looking at the second column.

Since  $M_f$  is lower-triangular with nonzero diagonal entries, it is *invertible*.

Now check the following facts:

- (1)  $M_f^{-1}$  takes the form  $M_g$  for some  $g \in \mathbf{C}[[t]]^\circ$ .
- (2)  $M_{f_1 \circ f_2} = M_{f_1} \cdot M_{f_2}$ .

We deduce that for any  $f$ , there's some  $g$  s.t.

$$M_{g \circ f} = M_g \cdot M_f = M_f^{-1} \cdot M_f = I = M_t.$$

Thus  $g(t) \circ f(t) = t$ .  $\square$

This proof shows that the group  $\mathbf{C}[[t]]^\circ$  *embeds* into the group of invertible infinite matrices  $\mathbf{GL}_\infty$ .

Recall that the set  $\mathbf{C}[[t]]^\times$  of power series with *nonzero* constant term forms a group under  $\times$ .

The map  $f \mapsto M_f$  can be extended to an embedding

$$\begin{aligned} \mathbf{C}[[t]]^\times \rtimes \mathbf{C}[[t]]^\circ &\hookrightarrow \mathbf{GL}_\infty, \\ (u, f) &\mapsto M_{u,f}. \end{aligned}$$

Shapiro's *Riordan group* is the image.

This also has an analogue in the world of plethysm.

Thank you for listening.