Warmup on $\mathbb{R}^\omega$, recall:

box topology:

basis opens $(a_1, b_1) \times (a_2, b_2) \times \ldots$

product topology:

basis opens $B_{J, \mathbf{a}, \mathbf{b}}$

for finite J and $(a_i)_{i \in J}$, $(b_i)_{i \in J}$

where

$B_{J, \mathbf{a}, \mathbf{b}} = \{x \mid a_i < x_i < b_i \text{ for } i \in J\}$

uniform (metric) topology:

basis opens $B_u(x, \delta)$

where, for $0 < \delta < 1$,

$B_u(x, \delta) = \{y \mid \sup_i |x_i - y_i| < \delta\}$

$T_{prod} \subseteq T_{unif} \subseteq T_{box}$

do any two coincide?

1) fix $0 < \delta < 1$

let $U = B_u(0, \delta) = \{x \mid |x_i| < \delta \text{ for all } i\}$

then U is not open in the product topology

because no $B_{J, \mathbf{a}, \mathbf{b}} \subseteq U$

2) let $V = (-1, 1) \times (-1, 1) \times (-1, 1) \times \ldots$

then V is not open in the uniform topology

because PS2, #6(1)

Rem on PS2, #8 show the closure of Y in X is

$\{x \in X \mid \text{every basis open at } x \text{ intersects } Y\}$

(Munkres §15, 19) [generalize T_{prod}, T_{box}]

let $\{X_i\}_i$ be any collection of topological spaces
their (set-theoretic) product is

$$\prod_i X_i = \{(x_i)_i \mid x_i \in X_i \text{ for all } i\}$$

the ith <u>projection</u> map is $pr_i : \prod_i X_i \to X_i$

<u>Df</u> the box topology on $\prod_i X_i$ is:

the topology generated by the basis
$$\{\prod_i U_i \mid U_i \text{ is open in } X_i \text{ for all } i\}$$

<u>Df</u> the product topology on $\prod_i X_i$ is:

I) the topology generated by the subbasis
$\{C_{i, U}$ for any i and open $U \subseteq X_i\}$
where $C_{i, U} = \{(x_j)_j \mid x_i \in U\}$

II) the coarsest topology s.t. pr_i is cts for all i

<u>Lem</u> I) and II) do define the same topology

<u>Pf</u> let T be any topology on $\prod_i X_i$. then

pr_i is cts wrt T
iff $C_{i, U}$ is open in T for all open $U \subseteq X_i$
iff the topology def by I) is a subcollection of T

Prop a) box top finer than product top
b) if there are finitely many X_i's,
then box top = product top

Pf a) proved similarly to analytic case
b) follows from observing
$\prod_i U_i = \bigcap_i C_{\{i, U_i\}}$

Ex suppose each X_i is discrete

the box top on $\prod_i X_i$ is also discrete
the product top on $\prod_i X_i$ need not be

e.g. take $X_i = \{0, 1\}$ for all i
then $\{0\} \times \{0\} \times \dots$ is not open in the product top

[why the product topology is nicer in general:]

Thm consider $f = \prod_i f_i : Y \to \prod_i X_i$

f is cts wrt the product topology
iff
$f_i = pr_i \circ f : Y \to X_i$ is cts for all i

Lem if $\{B_i\}_i$ is a basis for T on X, then
$f : Y \to X$ is cts wrt T
iff
$f^{-1}(B_i)$ is open in X for all i

Pf any open in X is a union of basis opens

Pf of Thm {finite intersections of $C_{i, U}$} is a basis for the product top on X so:

f is cts wrt product topology on X

iff f^{-1}(any fin intersxn of $C_{i, U}$'s) is open

iff $f^{-1}(C_{i, U})$ is open for all i, U

iff f_i is cts for all i □

Moral to give a cts function into (X, T_{prod}) is to give a cts function into X_i for each i

Another Moral to check continuity of f, check $f^{-1}(B)$ for basis elts B

Ex analogue of thm for box top is false, even when $X_i = Y$ = analytic R

let $f : R \to R^\omega$ be def by $f(x) = (x, x, x, \ldots)$

[what next?]

let $U = (-1, 1) \times (-1/2, 1/2) \times (-1/3, 1/3) \times \ldots$

then for all i,

$$f^{-1}(U) \subseteq f_i^{-1}((-1/i, 1/i)) = (-1/i, 1/i)$$

so $f^{-1}(U) = \{0\}$

<u>Digression</u> the Axiom of Choice says:

given <u>any</u> collection of nonempty sets $(X_i)_{i \in I}$
we can choose an elt from X_i for all i

equivalently
a product of nonempty sets is always nonempty

[is AoC true?]

sometimes the choice function is obvious
e.g., if $X_i = \mathbb{R}$ for all i,
then $\mathbb{R}^\omega = \prod_i X_i$ is nonempty too

the point is to deal with cases where it is not

<u>Ex</u> the collection of subsets of $\mathbb{R}^\omega$:

it contains
- $\varnothing$,
- $\{(0, 0, 0, \ldots)\}$,
- $\{(0, 1, 0, 1, \ldots)\}$,
- $\{(3, 1, 4, 1, 5, 9, \ldots)\}$,
- $\{(0, 0, 0, \ldots), (0, 1, 0, 1, \ldots)\}$,
- $\{x \text{ s.t. } x_{2025} = 0\}$,
- $\mathbb{R}^\infty$,
- …

can you describe a rule that,
given an arbitrary subset of $\mathbb{R}^\omega$,
exhibits an elt of that subset?