1 Introduction

Let X be a smooth manifold. Doing local geometry means that what you're interested in depends only on a neighborhood of a point, or perhaps that you're interested in something like cohomology that has interesting local-to-global behavior. Doing microlocal geometry means that instead of studying X we study its cotangent bundle $M = T^*(X)$, so that the quantities we discuss are associated not only to a point but to a point and a cotangent vector.

$M = T^*(X)$ is naturally a symplectic manifold, and the natural setting for many of the things you'd like to do to $T^*(X)$ can be done on any symplectic manifold. So microlocal geometry can be thought of as polarized symplectic geometry (a polarization is an identification of a symplectic manifold with a cotangent bundle).

1.1 Reasonable spaces

What is a reasonable space? (What Grothendieck called tame topology.) Roughly speaking we want a space which is locally described by a finite amount of data. What is an example of an unreasonable space? Take, for example $0 \cup \left\{ \frac{1}{2^n} : n \in \mathbb{N} \right\} \subset \mathbb{R}$. This space is not locally contractible; that's a reasonable requirement.

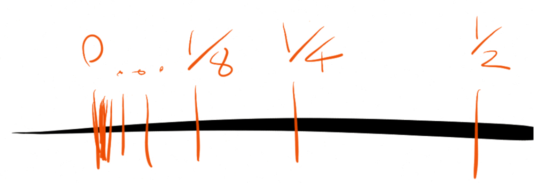

Figure 1: Infinitely many points and a limit point.

An example of an unreasonable subspace of a space is a spiral in $\mathbb{R}^2$. This is unreasonable because its intersection with the y-axis, say, is the space above, which is unreasonable.

Yet another unreasonable space is infinitely many distinct lines passing through the origin in $\mathbb{R}^2$ (in particular this is also not locally contractible).

So what is a reasonable space?

Definition An *n-step stratified space* X is a space with a filtration $X_0 \subseteq X_1 \subseteq ... \subseteq X_n = X$ by closed subspaces such that

1. $S_i = X_i \setminus X_{i-1}$ is an i-manifold (the i^{th} *stratum*), and

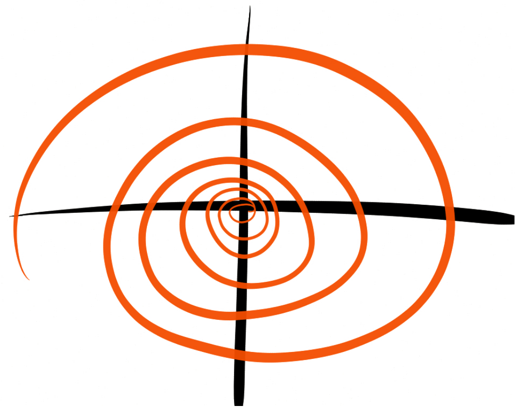

Figure 2: An infinite spiral.

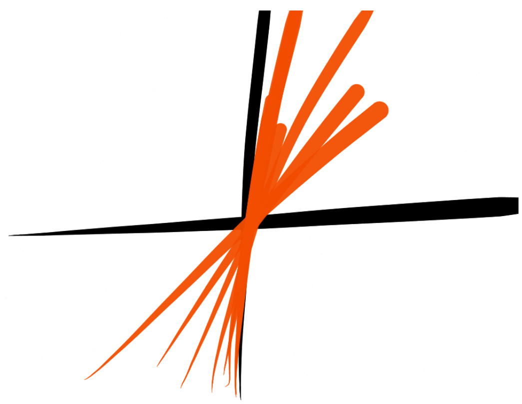

Figure 3: Infinitely many lines through the origin.

2. for all $x \in S_i$, there exists a neighborhood $x \in U \subseteq X$ and a filtration-preserving homeomorphism

$$U \cong \mathbb{R}^i \times \mathrm{Cone}(L_x) \tag{1}$$

where L_x (the *link* of x) is a compact $(n-i-1)$-step stratified space.

Here $\mathrm{Cone}(Y)$ is the space obtained by gluing $Y \times [0, \infty)$ to a point along $Y \times \{0\}$. The cone of a filtered space acquires a filtration.

Exercise 1.1. *Condition 2 implies condition 1.*

Example A 0-step stratified space is a collection of points.

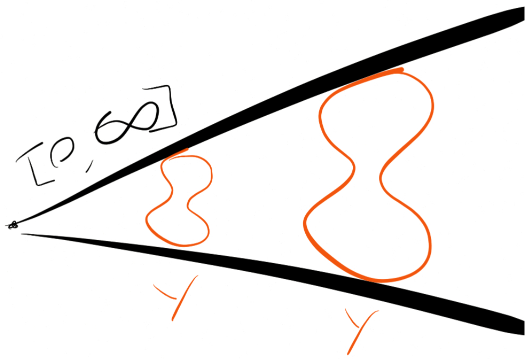

Figure 4: A picture of a cone.

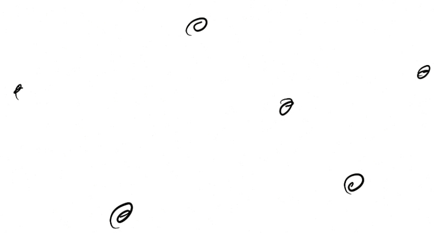

Figure 5: A collection of points.

Example A 1-step stratified space is in particular a space containing a collection of points such that their complement is a 1-manifold. Each point in S_1 must have a neighborhood homeomorphic to $\mathbb{R}$, and each point in S_0 must have a neighborhood homeomorphic to the cone on a finite collection of points. Hence we get locally finite graphs, possibly with isolated loops not attached to a vertex.

Example A famous example when $n = 2$ is called Whitney's umbrella. It consists of the union of a line and a 2-manifold, but the naive stratification one attempts to build this way fails to be a stratification because one point has unusual neighborhoods.

Example Another famous example when $n = 2$ is called Whitney's cusp. Again, the naive stratification fails to be a stratification because one point has unusual neighborhoods.

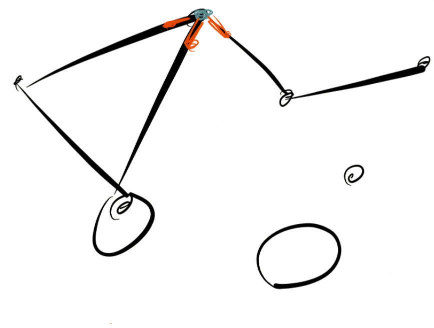

Figure 6: A locally finite graph.

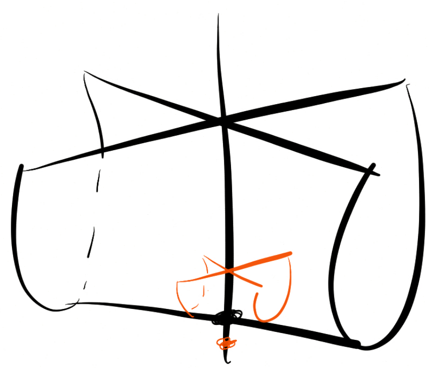

Figure 7: The Whitney umbrella and one of its unusual neighborhoods.

Exercise 1.2. *Let $x \in S_i$ and suppose we regard it as its own stratum. Then its link L'_x, when regarded as its own stratum, is the join $S^{i-1} * L_x$. In particular, if $i = 1$ then the join with S^0 (two points) is the suspension.*

Example A (locally finite) simplicial complex with at most n-simplices is an n-step stratified space where the i^{th} part of the filtration is the i-skeleton. This is because for any point x, which lies in the interior of some simplex σ_i, we can take the neighborhood U to be $\mathrm{Star}(\sigma_i)$ and we can take the link to be the link in the simplicial sense.

We can think of stratified spaces as simplicial complexes where the simplices are allowed to be manifolds.

Exercise 1.3. *Let X be a stratified space. Then the group of stratified homeomorphisms of X acts transitively on every path component of every stratum.*

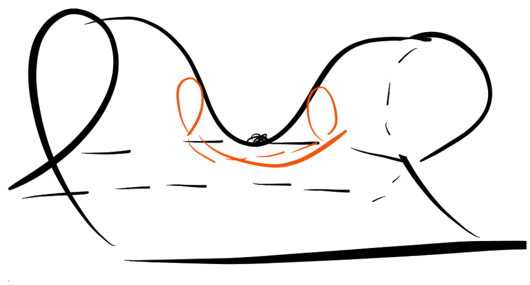

Figure 8: The Whitney cusp and one of its unusual neighborhoods.

Non-Example By contrast, most CW-complexes are not stratified spaces with their natural filtration. One problem is that we can write down CW-complexes with strata not satisfying the above condition.

We could call an n-step stratified space an n-dimensional stratified space, but then we should also require that $X = \overline{S_n}$.

2 Whitney stratifications

Yesterday we defined an n-step stratified space. Various exercises could have been but weren't assigned, e.g. that the link of a point has some uniqueness property.

Today we will talk about Whitney stratifications, which in particular provide stratified spaces. Let M be a smooth manifold; it is usually enough to take $M = \mathbb{R}^n$. Let $X \subseteq M$ be a closed subspace of M. Whitney described infinitesimal conditions under which M admits a stratification. First, it needs a decomposition as a disjoint union of strata

$$X = \bigsqcup_{\alpha} S_\alpha \tag{1}$$

which we will assume are connected manifolds. This decomposition should satisfy two conditions: it should be locally finite (every point has a neighborhood which intersects finitely many strata), and the closure of each stratum should be a disjoint union of strata. Finally, the induced decomposition into strata in each open neighborhood should also have the same property.

Non-Example Consider two line segments perpendicular to each other and intersecting at the endpoint of one of the line segments. Then we cannot write down a stratification consisting of one line segment and its complement because the closure of a stratum won't be a union of strata (it will be a union of a stratum and a point).

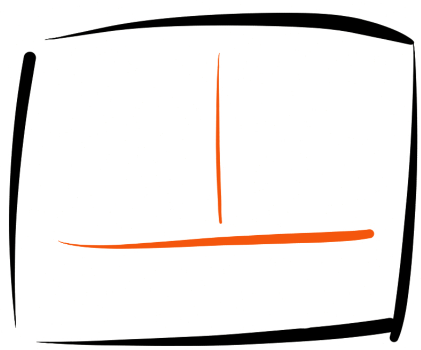

Figure 1: A non-example of the above condition.

Whitney's Condition A can be motivated by looking at the Whitney umbrella again. The unusual point has the property that, as we approach the unusual point, the tangent space of the unusual point is not contained in the tangent spaces of the points by which we approach it (loosely speaking). More precisely, we want the following condition: if S_β, S_α are two strata and $y \in S_\beta, x_i \in S_\alpha$ are points such that $x_i \to y$, then for all subsequences such that the limit $T_{x_i}(S_\alpha)$ exists (in some Grassmannian bundle), it should contain $T_y(S_\beta)$.

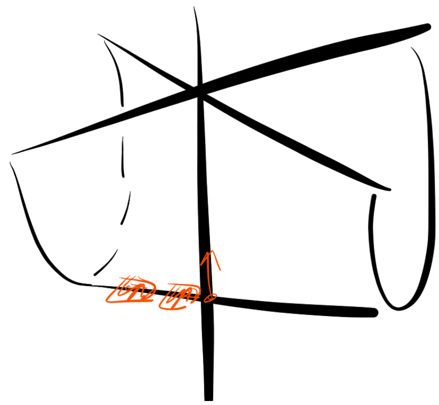

Figure 2: A poorly-behaved sequence of tangent spaces.

Whitney's Condition B can be motivated by looking at the Whitney cusp again. The unusual point has the property that we can find two sequences of points approaching it such that the secant lines between them (we need some extra structure on the ambient manifold to define this) are not contained in the tangent space of the point. More precisely, we want the following condition: if S_β, S_α are two strata, $y_i \in S_\beta, x_i \in S_\alpha$ are two sequences such that the limit of the tangent spaces $T_{x_i}(S_\alpha)$ and the limit of secant lines $\overline{x_i y_i}$ both exist in $T_y(M)$ (again, we need extra structure for this, e.g. a Riemannian metric), then the second limit should be contained in the first limit.

Figure 3: A poorly-behaved sequence of secant lines.

Exercise 2.1. *Condition B does not depend on the choice of extra structure, e.g. a Riemannian metric.*

Definition A *Whitney stratification* of X is a decomposition of X into a disjoint union $\bigsqcup_\alpha S_\alpha$ as above satisfying Whitney's Conditions A and B.

Exercise 2.2. *Condition B implies Condition A.*

The above discussion admits a microlocal interpretation. Recall that thinking microlocally means replacing the study of a manifold M with the study of its cotangent bundle $T^*(M)$. To a submanifold S_α we can associate the conormal bundle $T^*_{S_\alpha}(M)$, which is the bundle of cotangent vectors vanishing on the tangent vectors to S_α. Hence we can associate to X the disjoint union

$$\Lambda = \bigsqcup_\alpha T^*_{S_\alpha}(M). \tag{2}$$

Unfortunately Λ may not even be a closed subset of $T^*(M)$.

Exercise 2.3. *Whitney's Condition A holds iff Λ is closed.*

Please tell me if you know a microlocal interpretation of Condition B!

We want to turn the infinitesimal data above into non-infinitesimal data. Generically the way to do this is ODEs. For example, vector fields have flows for some time ϵ, which give diffeomorphisms.

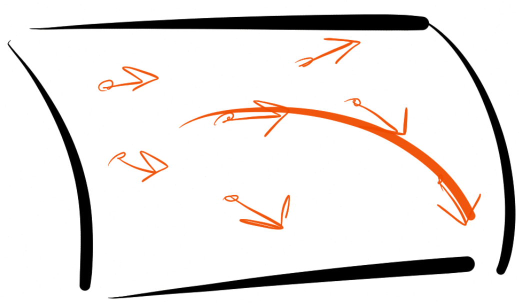

Figure 4: Flowing along a vector field.

To write down diffeomorphisms sending one part of a manifold to another, it suffices to construct appropriate vector fields. For a more concrete example:

Theorem 2.4. *(Ehresmann) Let $f : M \to P$ be a proper (inverse image of compact subspaces is compact) submersion ($df_x : T_x(M) \to T_{f(x)}(P)$ is surjective) of manifolds. Then f is a fibration.*

Proof. (Sketch). Since f is a submersion, we can lift tangent vectors from P to M. Locally this implies that we can lift vector fields, and now we can take the flows of these vector fields. These flows will identify fibers. $\square$

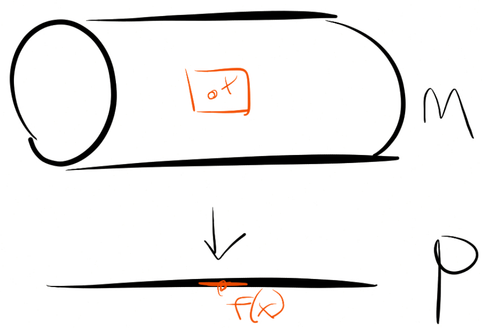

Figure 5: A proper submersion.

Thom generalized this to the case where M is not just a manifold.

Theorem 2.5. *(Thom's first isotopy theorem) Let $f : M \to P$ be a proper smooth map between smooth manifolds. Let X_α be a Whitney stratification of M. Assume that $f|X_\alpha$ is a submersion for all α. Then f is a stratified fibration: each fiber has a stratification, and locally $M \cong P \times F$ (P has a trivial fibration).*

The proof will start out like the previous proof, but we need to make sure vector fields glue nicely, and they might not. Consider the following example involving the Whitney cusp:

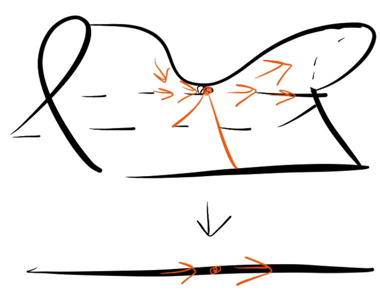

Figure 6: A non-fibration.

This is a proper map which is a submersion with respect to the naive stratification, but the Whitney conditions fail and the map is not a fibration. This is because vector fields do not lift nicely: the lift of the vector field $\frac{d}{dt}$ on the base does not have a flow on the complement of the central line for any positive time ϵ because it cannot intersect the central line.

In the theorem above, the identification $M \cong P \times F$ is a homeomorphism, and this cannot be made any stronger.

Example Consider four planes intersecting at a common line in $\mathbb{R}^3$, projecting down to a line, such that one of the planes is wiggling. After compactifying suitably, this is proper.

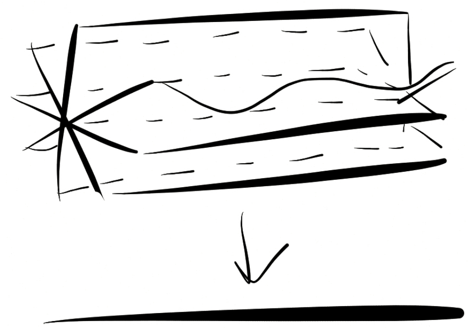

Figure 7: A picture of the space above and its projection to a line.

There is an obvious stratification with respect to which the projection is a stratified submersion, so Thom's theorem asserts that the map is a fibration. The identification $M \cong P \times F$ cannot be made into a diffeomorphism: the fibers cannot be identified smoothly because, with three of the planes fixed, the map on tangent spaces must be a scalar, so cannot correctly deal with the wiggling plane.

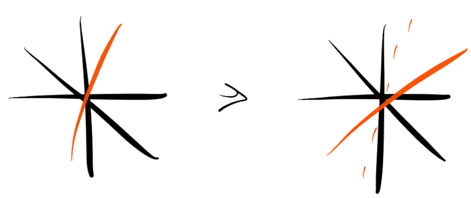

Figure 8: The fibers cannot be smoothly identified.

3 Thom's first isotopy theorem

Lemma 3.1. *Let X, Y be Whitney stratified subsets of a manifold M whose strata always intersect transversally. Then the intersections $X_\alpha \cap Y_\beta$ are a Whitney stratification.*

(There may be a problem with local finiteness...?)

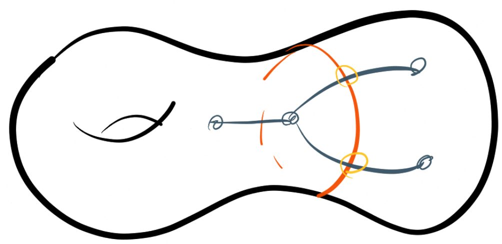

Figure 1: A transverse intersection of stratifications.

In the previous lecture we stated Thom's first isotopy theorem. We can reformulate its hypotheses microlocally. If $f : M \to P$ is a smooth map, microlocally we get a manifold $T^*(P) \times M$ with a projection map to $T^*(P)$ and a derivative map to $T^*(M)$, giving a Lagrangian correspondence between these two symplectic manifolds. It can also be thought of as the conormal bundle $T^*_{\Gamma_f}(M \times P)$ of the graph of f. This is our replacement for f. Recall also that a stratification S_α of $X \subseteq M$ gives a collection of conormal bundles $T^*_{X_\alpha}(M)$ in $T^*(M)$.

We want to rephrase the condition that f is a submersion microlocally. If $P = \mathbb{R}$, then f is a stratified submersion iff

$$f^* \, dt \cap \left(\bigsqcup_\alpha T^*_{X_\alpha}(M) \right) = \emptyset. \tag{1}$$

In general we have the following.

Exercise 3.2. *f is a stratified submersion iff*

$$df^*(p^{-1}(T^*(P) \setminus P)) \cap \left(\bigsqcup_\alpha T^*_{X_\alpha}(M) \right) = \emptyset. \tag{2}$$

The key idea in the proof of Thom's theorem is the notion of a tube system.

Definition A *tube* around a stratum S_α consists of the following data:

1. $E_\alpha \xrightarrow{\pi_\alpha} S_\alpha$, a vector bundle on S_α (of dimension the codimension of S_α, usually the normal bundle);

2. ρ_α, a metric on this vector bundle;

3. η_α, a function $S_\alpha \to \mathbb{R}$;

4. U_α, an open neighborhood of S_α; and

5. a diffeomorphism $U_\alpha \cong \{\rho_\alpha < \eta_\alpha\} \subseteq E_\alpha$.

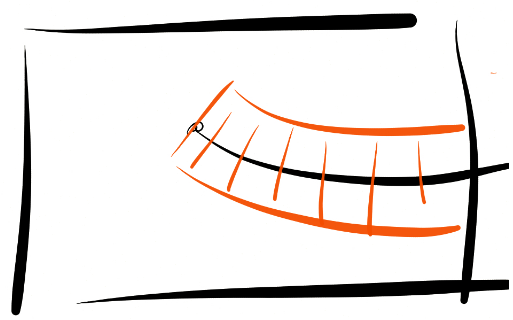

Figure 2: A tube.

We want a collection of tubes around each strata which interact well with each other.

Definition A *tube system* is a collection of tubes around each stratum S_α such that $\pi_\beta = \pi_\beta \pi_\alpha$ (whenever both of these are defined on some U_α) and $\rho_\beta = \rho_\beta \pi_\alpha$ (same condition).

The rest of the proof of Thom's theorem is complicated but has no new ideas: one just applies the proof of Ehresmann's theorem carefully.

Let $f : M \to P$ satisfy the hypotheses of Thom's theorem.

Definition f is *controlled by* a tube system T if $f|_{U_\alpha} = f \circ \pi_\alpha$.

Definition A collection of vector fields v_α, one on each S_α, is *controlled by* a tube system T if $d\pi_\alpha v_\beta = v_\alpha$ and $v_\beta(\rho_\alpha) = 0$ (whenever these make sense).

From here a rough sketch of the proof of Thom's theorem is as follows. Given f, we construct a tube system T controlling f, and we show that vector fields on P lift to controlled vector fields on M. This lets us run the proof of Ehresmann's theorem.

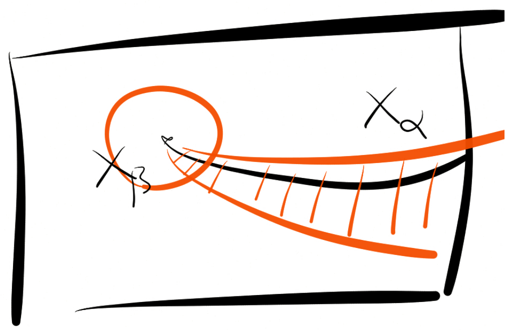

Figure 3: A tube system.

4 Tame topology

Earlier we described three kinds of reasonable spaces: stratified spaces, tube systems, and Whitney stratifications. The implication between these is that Whitney stratifications give a tube system, and the proof of Thom's first isotopy theorem shows that a tube system gives a stratification.

Theorem 4.1. *A Whitney stratification gives a stratification.*

Proof. (Sketch) Let M be an ambient manifold and M_α a Whitney stratification of it. Pick local coordinates on some M_α so that it looks locally like points of the form $(x_1, ...x_k, 0, ...0)$. Let f be the function $(x_1, ...x_k)$ to $\mathbb{R}^k$. This map turns out to be a submersion on all nearby strata by Whitney's Condition A, so it is a fibration by Thom's theorem. This tells us that small neighborhoods of a point on M_α look like $\mathbb{R}^k$ times the fiber of this map, which we now want to show is a cone. We can do this by taking a suitable neighborhood of a point, removing the point, and applying Thom's theorem to a suitable radial function. $\square$

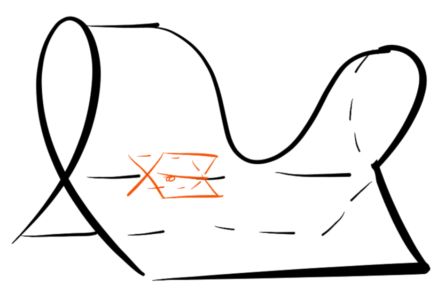

Figure 1: Local coordinates on a stratum.

When do we get Whitney stratifications? One place is when doing real algebraic geometry. Here we study *semialgebraic sets* in $\mathbb{R}^n$, which are described by finite unions and intersections of polynomials equalities and inequalities. The reason we care about inequalities is that sets cut out by polynomial equalities are not closed under projection; for example, the unit circle projects in $\mathbb{R}^2$ projects to a closed interval in $\mathbb{R}$.

Theorem 4.2. *(Tarski-Seidenberg) Semialgebraic sets are closed under projections $\mathbb{R}^{n+1} \to \mathbb{R}^n$.*

This is equivalent to a statement about quantifier elimination, the point being that projecting a set corresponds to tacking on a quantifier.

Theorem 4.3. *All semialgebraic sets admit a Whitney stratification.*

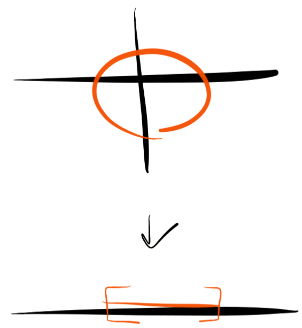

Figure 2: A circle projecting to an interval.

There is a similar theory involving *semianalytic sets*, where we (locally) use real analytic functions rather than polynomials. But the Tarski-Seidenberg theorem fails here even for proper projections.

Example Consider the map

$$f : \mathbb{R}^2 \ni (x_1, x_2) \mapsto (x_1, x_1x_2, x_1x_2e^{x_2}) \in \mathbb{R}^3. \tag{1}$$

Let (y_1, y_2, y_3) be the coordinates on $\mathbb{R}^3$. Ignoring the last coordinate, the first two coordinates almost supply a diffeomorphism, but there is a problem when $x_1 = 0$. When $x_1 = 0$, the y_2 coordinate is identically zero; otherwise everything is fine. So ignoring the third coordinate the image is the y_1, y_2-plane minus the line $y_2 = 0$ together with the point $(y_1, y_2) = (0, 0)$. The actual image is a graph living over this set.

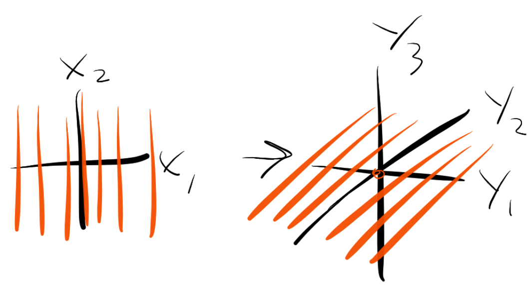

Figure 3: The first two coordinates.

We want to show that no real analytic function can vanish on the actual image without vanishing identically. Suppose $G(x_1, x_1x_2, x_1x_2e^{x_2}) = 0$. Write G as the sum of its homogeneous components

$$G(y_1, y_2, y_3) = \sum_{k=0}^{\infty} G_k(y_1, y_2, y_3). \tag{2}$$

Then

$$G(x_1, x_1x_2, x_1x_2e^{x_2}) = \sum_{k=0}^{\infty} x_1^k G_k(1, x_2, x_2e^{x_2}). \tag{3}$$

If this vanishes identically, then each component $G_k(x_1, x_2, x_2e^{x_2})$ must also vanish identically. But G_k is a polynomial, and e^{x_2} is not an algebraic function.

To fix this, we consider *subanalytic sets*, which are the closure of semianalytic sets under proper images. These also have Whitney stratifications. This all falls under the general heading of analytic-geometric categories: the goal here is to find some reasonable collection of subsets of $\mathbb{R}^n$.

Definition An *analytic-geometric category* is an assignment, to each real analytic manifold M, of a collection $\mathcal{C}(M)$ of subsets of M. This is required to satisfy the following axioms:

1. $\mathcal{C}(M)$ is closed under finite unions, intersections, and complements, and $M \in \mathcal{C}(M)$.

2. The map $A \to A \times \mathbb{R}$ sends subsets in $\mathcal{C}(M)$ to subsets in $\mathcal{C}(M \times \mathbb{R})$.

3. The projection from subsets of M to subsets of N under a real analytic map $f : M \to N$ should send $\mathcal{C}(M)$ to $\mathcal{C}(N)$ provided that the restriction of f to the subset of $\mathcal{C}(M)$ in question is proper.

4. Membership in $\mathcal{C}(M)$ is locally defined: a subset A is in $\mathcal{C}(M)$ iff, in some open cover $M = \bigcup_i U_i$ of M by real analytic submanifolds, $A \cap U_i \in \mathcal{C}(U_i)$.

5. Every bounded set in $\mathcal{C}(\mathbb{R})$ has finite boundary.

The fifth axiom is the one with real bite. From this data, we can describe a category whose objects are the subsets $A \in \mathcal{C}(M)$ and whose morphisms are functions $f : A \to B$ whose graphs $\Gamma(F) \subseteq A \times B \subseteq M \times N$ are in $\mathcal{C}(M \times N)$.

There is a smallest analytic-geometric category, namely the subanalytic sets. The axioms above are enough to give us some standard tools:

1. Elements of $\mathcal{C}(M)$ are always Whitney stratifiable.

2. Closed sets are the zero loci of functions in the category.

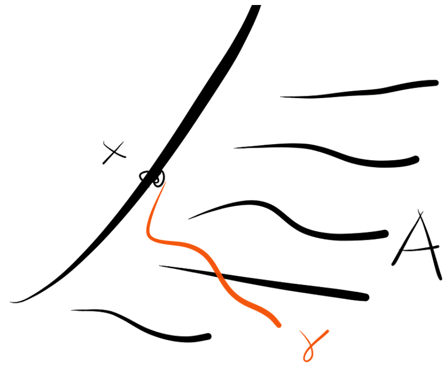

Figure 4: Curve selection.

3. The *curve selection lemma* holds: let $A \in \mathcal{C}(M)$ and let $x \in \mathrm{cl}(A) \setminus A$. Then there exists a curve $\gamma : [0, 1) \to M$ in the category such that $\gamma((0, 1)) \subset A$ and $\gamma(0) = x$.

The curve selection lemma is used to run arguments such as showing that a sequence of points exhibiting some bad behavior can be lifted to a curve of points exhibiting some bad behavior.

Analytic-geometric categories are the same thing as *o-minimal structures* containing analytic sets, except that we work in $\mathbb{R}^n$ rather than real analytic manifolds.

5 Homology and cohomology of reasonable spaces

Today we'll only consider subanalytic sets.

Let $X = \mathbb{R}$. Recall that its homology and cohomology are both concentrated in degree 0. But these are very different. We can think of the homology as being generated by a point somewhere on $\mathbb{R}$ with a 1 attached to it. Any two such points are homologous by a chain, which is a line with a number 1 attached to it and an orientation.

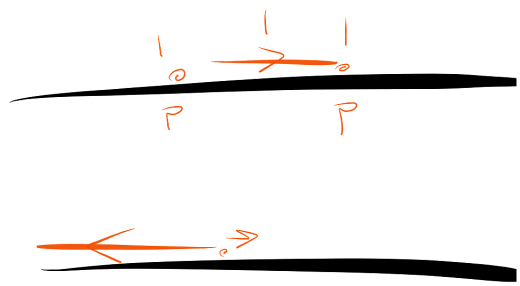

Figure 1: Homology and cohomology.

We can think of the cohomology as being generated by a copy of the entire real line $\mathbb{R}$ with a 1 attached to it (with respect to the singular definition of cohomology, we are implicitly using some kind of Poincarë duality here). H^1 vanishes because any point is the boundary of a cochain given by the line to the left of it.

In general, homology and cohomology linearize a space. We can think of cohomology as being like functions on a space, while homology is like distributions on a space. But in topology everything has to be locally constant.

We'll describe both homology and cohomology via complexes of subanalytic chains and cochains. Here are some guiding principles:

Homology	Cohomology
Compact support	Not necessarily compact
Oriented	Cooriented
Bad intersections with singularities	Strongly transverse to singularities
Indexed by dimension	Indexed by codimension
Covariant	Contravariant
Love relative sets	Hate relative sets

Definition A stratified space has *pure dimension n* if it is the closure of its top stratum, which is n-dimensional.

Definition Let X be a subanalytic set of pure dimension n (closed in some ambient real analytic manifold M). A *subanalytic chain* consists of the following data:

1. a compact subanalytic subset $\sigma_k \subseteq X$ of pure dimension k,
2. an orientation of an open dense smooth locus $\sigma_k^\circ \subseteq \sigma_k$,
3. a number attached to each component of σ_k°.

We can take linear combinations of chains (which has something to do with the numbers and orientations; there's an equivalence relation we need to write down).

Example Let X be a circle decorated with some extra lines. It's homotopy equivalent to a circle, so its homology is generated by an element in degree 0 and an element in degree 1. H_0 can be generated by any point with a 1 attached. Some of these points are singular.

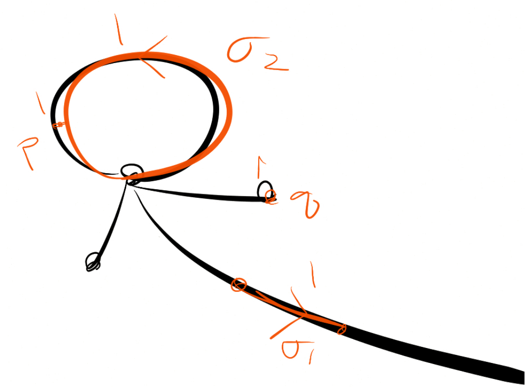

Figure 2: Chains on X.

1-chains are given by unions of line segments (and their endpoints) together with numbers and orientations attached to them. There is a boundary map from 1-chains to 0-chains given by taking the closure of the codimension-1 strata, together with some convention on orientations. H_1 can be generated by the obvious loop, which is stratified by taking a point.

Recall that

1. a *coorientation* of a submanifold in a (Riemannian?) manifold is a nonvanishing section of the top exterior power of the normal bundle,
2. if x is a point in a stratified space with a neighborhood $U_x \cong \mathbb{R}^k \times \text{Cone}(L_x)$, then the *slice* to x is $\text{Cone}(L_x)$,
3. a *singular point* in a stratified space of pure dimension is a point not in the top-dimensional stratum.

Definition Let X be as above. A *subanalytic cochain* consists of the following data:

1. a closed subanalytic subset $\sigma^k \subseteq X$ of pure codimension k which is *strongly transverse* (locally contains the slice to any singular point),
2. a coorientation of an open dense smooth locus $(\sigma^k)^\circ \subseteq \sigma^k$,
3. a number attached to each component of $(\sigma^k)^\circ$.

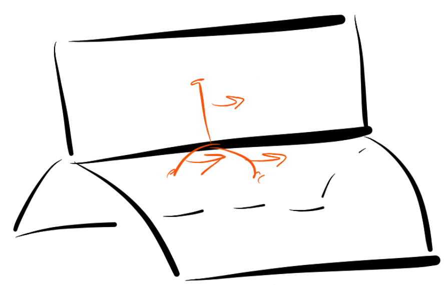

Figure 3: Strong transversality.

Example Let's return to the space X we looked at earlier. A 0-cochain is a line segment with a number attached to it, but coorientations are trivial since line segments have codimension 0. At a singular point, we need to contain a slice. We also have a boundary map from 0-cochains to 1-cochains which is given by intersection with the top-dimensional stratum (this throws away singular points) and then taking the closure of the topological boundary there. H^0 is generated by a cochain covering the entire space.

1-cochains are points, and coorientations are arrows pointing in one direction or another. H^1 is generated by a cochain in the loop; cochains not on the loop are boundaries, so are zero in H^1.

The intuition here is that cochains are like functions in that they should be determined by their behavior on an open dense subset (here the top-dimensional stratum).

Some more words about boundaries. In the following example, the topological boundary of the 0-cochain contains an extra line (the intersection with the line passing through the middle) which we want to get rid of because it is singular in the ambient space. The definition above removes it.

Example Let X be the kissing banana (S^2 with the north and south pole identified). H_0 is generated by a point. For H_1 there are two candidate loops (the space resembles a torus)

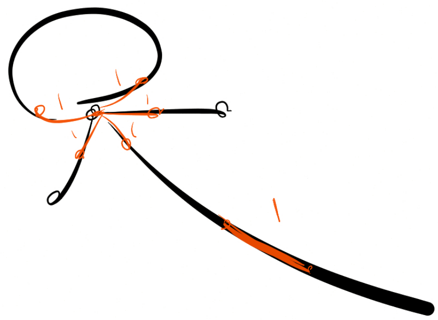

Figure 4: Cochains on X.

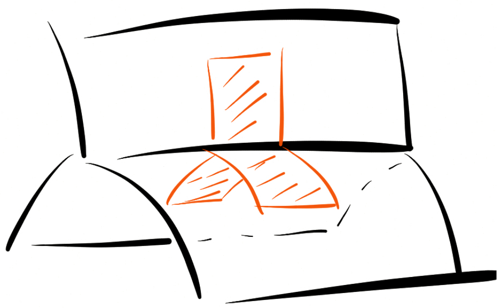

Figure 5: The boundary operator on cochains.

but one of them is zero because it is the boundary of a horn. And H^2 is generated by the entire space; they are all generated by one nonzero element.

Poincarë duality doesn't work here, but it will turn out that H^0, H^1, H^2 are also all generated by one nonzero element. H^0 and H^2 are straightforward, but H^1 is generated by the loop that was zero above; we can't use the horn because it is not transverse to the singular point. The loop that generates H_1 is also not allowed in cohomology because it is not transverse to the singular point.

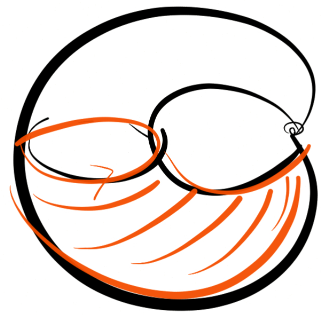

Figure 6: The boundary of the horn is a loop.

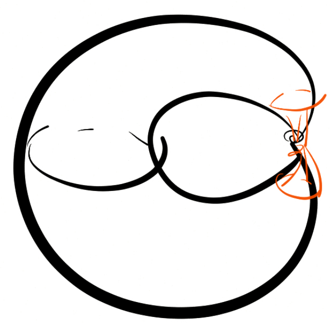

Figure 7: Transversality to the singular point.

6 More about homology and cohomology

Having talked about homology and cohomology for subanalytic spaces, we'd like to talk about long exact sequences, cup and cap products, and Poincaré duality. Eventually we'd like to make our way to intersection cohomology.

We should say some more about strong transversality. With a tube system in place, we can be more specific about what a slice is: namely, it is the inverse image of π_α (part of the data $(\pi_\alpha, E_\alpha, \rho_\alpha, \eta_\alpha)$ associated to a stratum in a tube system).

Why should the construction we've given describe ordinary cohomology? It turns out that we can construct a deformation retract collapsing a neighborhood of the singular stratum to the singular stratum.

Someone wanted to do the Klein bottle as an example.

Example Consider the Klein bottle. The homology computation resembles singular homology and is nothing new. The cohomology computation is more interesting because it does not resemble usual computations with singular or simplicial cohomology. H^2 is $\mathbb{Z}/2\mathbb{Z}$ generated by a point because there is a 1-cochain whose boundary is twice it. H^1 is $\mathbb{Z}$ generated by one of the obvious 1-cochains. H^0 is $\mathbb{Z}$ enerated by the whole thing.

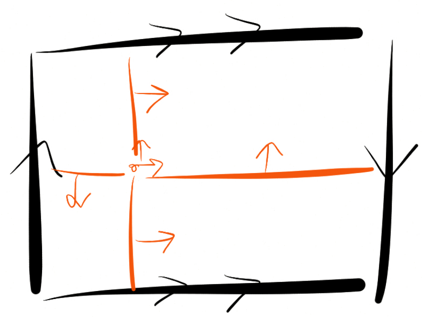

Figure 1: 1-cochains on a Klein bottle.

Example Consider the suspension ΣT^2 of a torus. Here there are two singular points unlike the above. We will compute with $\mathbb{C}$ coefficients. $H_0 = \mathbb{C}$ represented by any point. $H_1 = 0$ because any 1-chain not touching a singular point is the boundary of its cone with respect to the other singular point, and we can always move a 1-chain so that it doesn't touch a singular point. $H_2 = \mathbb{C}^2$ generated by the suspension of the obvious 1-chains in the torus, and $H_3 = \mathbb{C}$ generated by the whole thing.

For cohomology, $H^0 = \mathbb{C}$ generated by the whole thing and $H^1 = 0$ because a codimension-1 cochain can't touch a singular point, so can be coned over. $H^2 = \mathbb{C}^2$ generated by the obvious 1-cochains on the torus because anything they might be the boundary of can't touch the singular points. Similarly $H^3 = \mathbb{C}$ generated by any point.

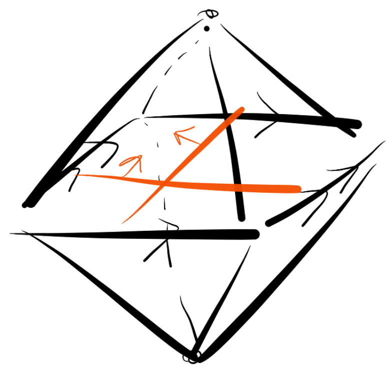

Figure 2: Chains on the suspension of a torus.

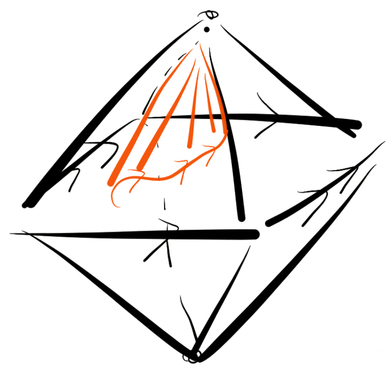

Figure 3: Cochains on the suspension of a torus.

6.1 Functoriality

Let $f : X \to Y$ be a subanalytic map of subanalytic sets. If $\sigma \subseteq X$ is a chain we'd like to map it to $f(\sigma) \subseteq Y$, but the dimension of σ might change. We will fix this by removing any lower-dimensional pieces of $f(\sigma)$.

The same idea doesn't work for pushing forward cochains because we might introduce singularities intersecting the image. Instead, if $\sigma \subseteq Y$ is a cochain we'd like to map it to $f^{-1}(\sigma) \subseteq X$. The problem we might run into is a lack of transversality, which we can fix by moving σ so that it is transverse to f (in the sense that f restricted to each stratum has an image transverse to σ).

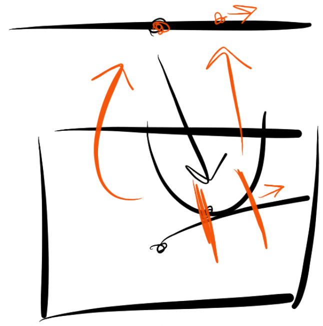

Figure 4: Possible failure of transversality.

6.2 Poincaré duality

Let X be oriented and compact (but possibly with singular points). There is a special element $[x] \in H_n$ called the fundamental class, and taking the cap product with $[x]$ gives an operation $H^k \to H_{n-k}$ given by using the orientation on X to turn a coorientation into an orientation. Because chains, unlike cochains, don't have to worry about singular points, this includes cochains into chains. If X also has no singular points, then cochains can be identified with chains in a way consistent with boundary maps, which reproduces Poincaré duality.

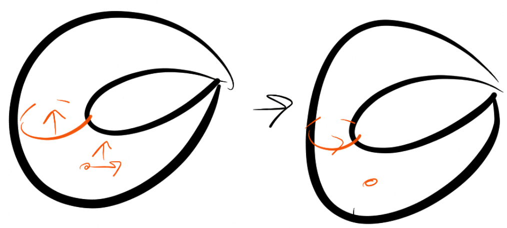

Figure 5: Capping with the fundamental class.

With singularities, the difference between chains and cochains in the above picture is interesting, and there will be a microlocal picture of this.

6.3 Cups and caps

In this picture, the cup product on cohomology is given by taking transverse intersections of cochains. The cap product between homology and cohomology is given in a similar way. The corresponding thing for chains is problematic because it is not always possible to make a chain transverse to itself due to the presence of singularities.

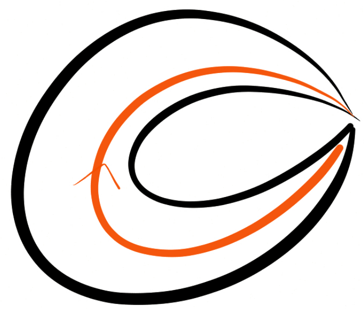

Figure 6: A chain that cannot be made transverse to itself.

6.4 Relative stuff

Let $Y \subseteq X$ be a union of strata in X. We want to define relative chains and cochains to define relative homology and cohomology. Relative chains are chains on X modulo chains on Y. Relative cochains are cochains not intersecting Y. In the following example, relative H_0 vanishes because every point is homologous to a point in Y. Relative H^0 vanishes because no codimension-0 cochain can fail to intersect Y. Relative H^1, H_1 are both $\mathbb{C}^2$.

6.5 Lefschetz duality

Let X be oriented and compact and let $Y \subseteq X$ be closed. Then pairing with the fundamental class gives identifications $H^k(X,Y) \to H_{n-k}(X \setminus Y)$ and $H^k(X \setminus Y) \cong H_{n-k}(X,Y)$. The model we use above makes this straightforward to see.

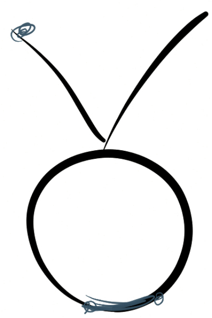

Figure 7: A relative example.

7 Intersection cohomology

Let X be an n-dimensional Whitney stratified space. Assume that X is oriented (by which we mean that the smooth locus is oriented). Because capping with the fundamental class exchanges orientations and coorientations, we may behave as if everything is oriented. Recall also that capping with the fundamental class gave an inclusion $C^k(X) \to C_{n-k}(X)$ of cochains into chains. The difference between these has something to do with the singularities of X. Intersection cohomology tries to find something in between cochains and chains which is self-dual. Roughly speaking, we are trying to find the center term in the sequence of inclusions $L^\infty \to L^2 \to L^1$, or perhaps the sequence of inclusions

$$\text{functions} \to L^2 \to \text{distributions.} \tag{1}$$

(It was conjectured for decades that there is a de Rham presentation of intersection cohomology formalizing this analogy; this has been recently proven?)

An important point of view for us will be that a cochain is completely determined by what it does on the smooth locus. This suggests the possibility of discussing the cohomology of X via the de Rham theory of its smooth locus, with some restrictions.

Example Let X be the kissing banana again.

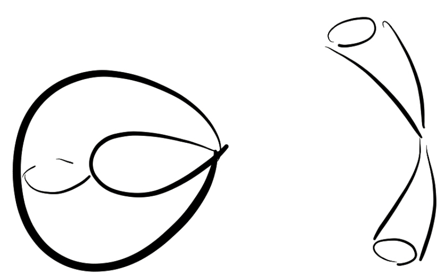

Figure 1: A neighborhood of the singular point.

In a neighborhood of the singular point, the only interesting cochain is the whole thing in degree 0 (it is the only strongly transverse cochain). There are two interesting chains in degree 2 given by the top and bottom half of the neighborhood and an interesting chain in degree 1 given by a line through the singular point.

Intersection cochains are as follows: we keep the top and bottom half of the neighborhood but don't keep the line through the singular point. The corresponding intersection cohomology of X is 1-dimensional in degrees 0 and 2 and 0-dimensional in degree 1; in particular it agrees with the cohomology of the sphere.

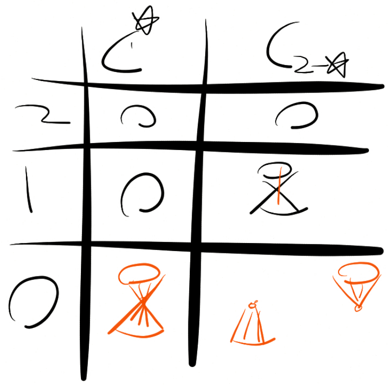

Figure 2: Interesting chains and cochains.

Proposition 7.1. *Let $f : \tilde{X} \to X$ a finite resolution (a finite, in particular proper, map where $\tilde{X}$ is smooth, which is an isomorphism on an open dense subspace). Then $IH^*(X)$, the* intersection cohomology *of X, is the ordinary cohomology of $H^*(\tilde{X})$.*

We can take this as the definition of intersection cohomology. In any case, this tells us what intersection cochains should be.

Note that finiteness is a very strong condition. The kissing banana can be resolved by a sphere, which gives a finite resolution, or by a torus, which does not.

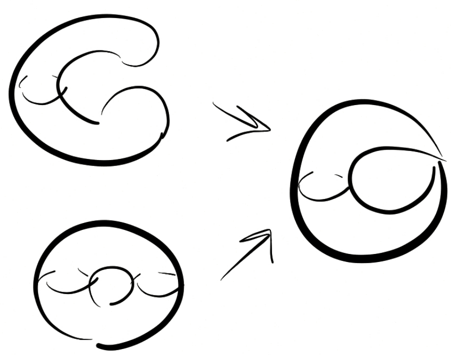

Figure 3: Two resolutions of the kissing banana.

Something funny happens when X has odd-dimensional strata. For example, a wedge of two circles has two finite resolutions which have different cohomology. We will need to restrict to spaces with only even-dimensional strata for uniqueness of the cohomology of resolutions to hold: in particular, any space admitting a stratification by complex manifolds has this property.

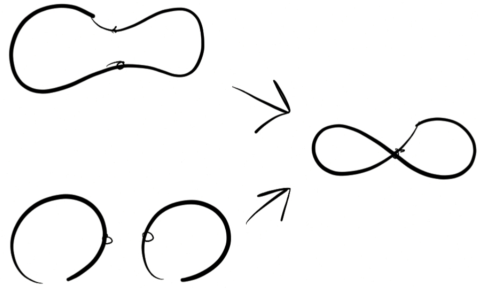

Figure 4: Two finite resolutions of the wedge of two circles.

The local picture of intersection cohomology is as follows. Consider a space X of dimension $2n$ which is the cone over some space Y of odd dimension. Poincaré duality on Y means reflecting over a line between H_{n-1} and H_n. We will allow the cone over anything in codimension $n-1$ and below (dimension n and above) to be a cochain on X.

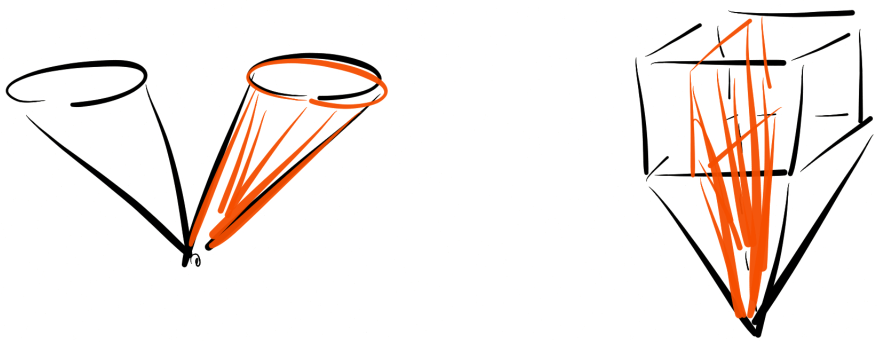

Figure 5: Intersection cochains on cones.

The motivation for the above definition is as follows. For a cone $X = \mathrm{Cone}(Y)$ as above, homology and cohomology are both just $\mathbb{C}$ in degree 0 The failure of Poincaré duality reflects itself in the fact that the relative cohomology of X with respect to Y (which is the reduced cohomology of Y, shifted) should be but is not dual to the cohomology of X; in other words, Lefschetz duality does not hold.

The definition above restores this lack of duality. For a cone $X = \mathrm{Cone}(Y)$, the intersection cohomology $IH^*(X)$ is the cohomology of Y in degrees 0 to $n-1$ and zero otherwise (by taking cones on the low-codimension cochains on Y). The relative intersection cohomology is zero in degrees 0 to n and a shift of the cohomology of Y in degrees $n+1$ to $2n$

(by keeping the high-codimension cochains on Y). Now these two have a Poincaré duality pairing inherited from that of Y.

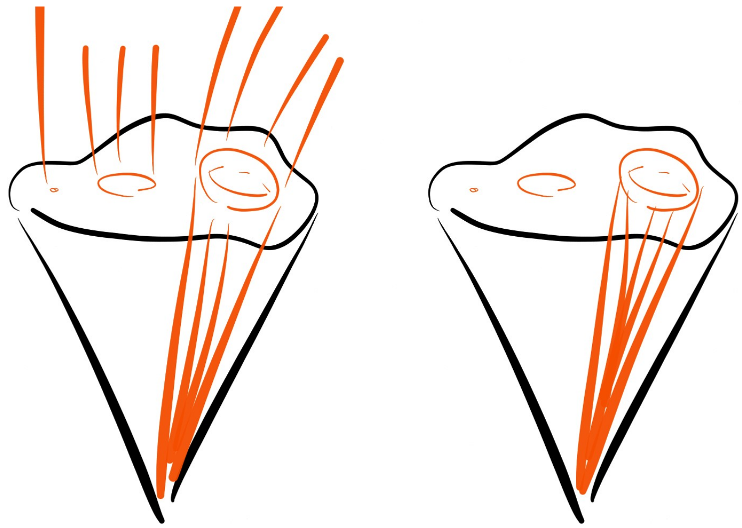

Figure 6: Intersection cochains and relative intersection cochains.

Why did we need X to be even-dimensional? If X is the cone over an even-dimensional Y (say $\dim Y = 2n$), then the middle cohomology $H^n(Y)$ is self-dual, and it's unclear how to divide it up (as we did above). We might want to split $H^n(Y)$ into two dual subspaces, but this is impossible, for example, if its dimension is odd. Even when we can do this, we may not be able to do it canonically.

From the perspective of sheaf theory, this story is complicated. The microlocal viewpoint will make intersection cohomology seem more obvious.

8 More about intersection cohomology

Theorem 8.1. *Let X^{2n} be a (reasonable) $2n$-dimensional stratified space with only even-dimensional strata (e.g. any complex algebraic variety). Write S_{2k} for its $2k$-dimensional strata and assume that S_{2n} is oriented. Then there is a unique cochain theory (sheaf) IC_X such that*

1. *When restricted to S_{2n}, we get cochains on S_{2n} in the usual sense,*
2. *IC_X is self-dual in the sense that $IC_X^\bullet(U_x)$ is quasi-isomorphic to $IC_X^{2n-\bullet}(U_x, \partial U_x)^\vee$ in sufficiently small neighborhoods U_x of points (those looking like cones on the link times a ball),*
3. *IC_X satisfies local vanishing in the sense that if $x \in S_{2k}$ and U_x is a neighborhood as above, then the cohomology of $IC_X^\bullet(U_x)$ vanishes for $\bullet \geq n-k$.*

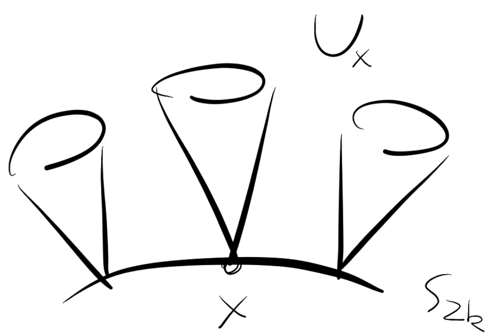

Figure 1: A neighborhood U_x.

Our earlier construction in the special case that $X = \text{Cone}(L^{2n-1})$, where L is $2n-1$-dimensional, satisfied these conditions. The idea is to inductively perform this construction to construct intersection cochains in general.

Example Let $L = S^1$, so X is the cone over a circle. The ordinary cohomology of the link is $\mathbb{C}$ in degree 0 (generated by the whole thing) and 1 in degree 1 (generated by a point). The intersection cohomology is $\mathbb{C}$ in degree 0 (generated by the cone over the link) and zero otherwise, and the relative intersection cohomology is $\mathbb{C}$ in degree 2 (generated by a point in the link).

We now inductively define intersection cochains as follows by inducting on codimension. Let $x \in S_{2k}$. Consider a neighborhood $U_x \ni x$ of the form $\text{Cone}(L_x) \times B^{2k}$. Moving along

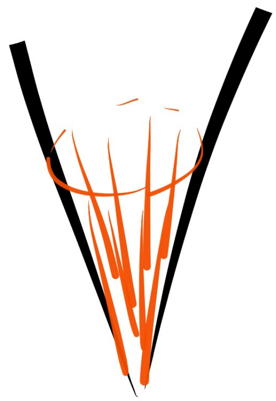

Figure 2: The cone over a circle.

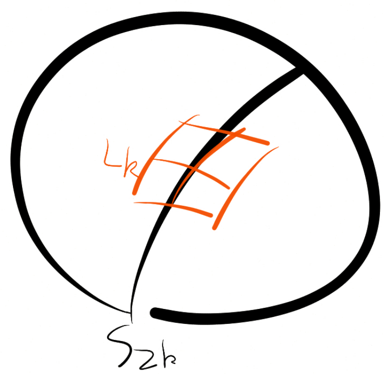

Figure 3: A neighborhood.

S_{2k} won't change anything, so we'll take a slice and reduce to the case that $x \in S_0$ and $U_x \cong \text{Cone}(L_x)$.

(Previously we only allowed L_x to be a manifold, but now L_x is itself a stratified space all of whose strata are odd-dimensional). Since L_x was obtained by taking a slice, by induction we have a self-dual cochain theory on $L_x \times (-\epsilon, \epsilon)$ which restricts (e.g. by taking transverse intersections) to a self-dual cochain theory on L_x. Now we cone off cochains of degree less than or equal to $\frac{\dim L_x - 1}{2}$ as before and ignore the others.

Example Let X be the cone over a pair L of 3-dimensional tori meeting along a circle (so all strata are odd-dimensional, or equivalently have even codimension). L has a finite resolution given by a pair of 3-dimensional tori, so the self-dual cohomology that X sees above is just

$\mathbb{C}^2, \mathbb{C}^6, \mathbb{C}^6, \mathbb{C}^2$. Alternatively, we know what happens on the smooth locus, and the singular locus looks like two tori meeting at a point times an interval.

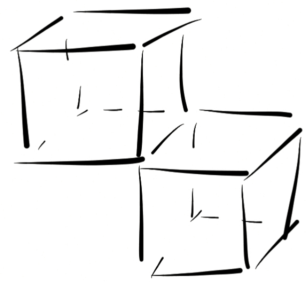

Figure 4: Two 3-tori meeting along a circle.

9 Calculating intersection cohomology

Let X^{2n} be a ($2n$-dimensional) stratified space with even-dimensional strata. Let $\mathrm{or}_{S^{2n}}$ be the orientation local system on the smooth locus (over $\mathbb{C}$), and let $\mathcal{L}$ be a finite-dimensional local system on S^{2n}.

Theorem 9.1. *There is a unique cochain theory $IC^\bullet(X, \mathcal{L})$ satisfying the following conditions:*

1. *When restricted to the smooth locus, we get cochains $C^\bullet(S^{2n}, \mathcal{L})$ on S^{2n} with values in $\mathcal{L}$;*

2. *Local duality holds in the sense that, for neighborhoods $U_x \ni x$ of a point x with compact closure, we have $IC^\bullet(U_x, \mathcal{L})$ quasi-isomorphic to*

$$IC^{2n-\bullet}\left((U_x, \partial U_x); \mathcal{L}^\vee \otimes or_{S^{2n}}\right)^\vee . \tag{1}$$

3. *Local vanishing holds in the sense that if $x \in S^{2k}$ then the cohomology of $IC^\bullet(U_x, \mathcal{L})$ vanishes for $\bullet \geq n-k$.*

Definition A resolution $\tilde{X} \xrightarrow{\pi} X$ is *small* if, for all $x \in S^{2k} \subseteq X$, where $k < n$, the dimension of the fiber $F_x = \pi^{-1}(x)$ over x is less than $n-k$ (half the codimension).

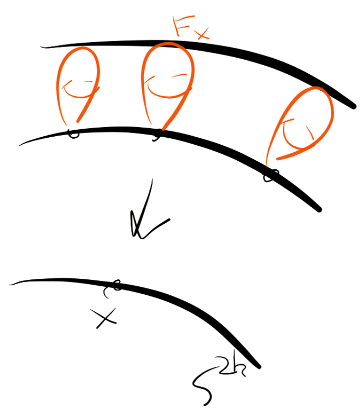

Figure 1: Fibers of a resolution.

Proposition 9.2. *Let $\tilde{X} \to X$ be a small resolution (to be defined later). Then $IC^\bullet(X)$ is quasi-isomorphic to $C^\bullet(\tilde{X})$.*

The definition of small is cooked up more or less to satisfy local vanishing.

Example Let Q^3 be the (complex) 3-dimensional quadric in 2×2 complex matrices such that $\det A = 0$. There is a toric picture of Q as a cone over a square (which is a toric picture of $\mathbb{P}^1 \times \mathbb{P}^1$). There is a small resolution $\tilde{Q}$ consisting of pairs (A, ℓ) where A is a matrix and $\ell \in \ker(A)$ is a line, and a toric picture of $\tilde{Q}$ where the singular point has been resolved to a line. But $\tilde{Q}$ retracts onto $\mathbb{P}^1$ since we can scale the matrix to zero, so the ordinary cohomology of $\tilde{Q}$ is the cohomology of $\mathbb{P}^1$.

Figure 2: A toric picture.

$\mathbb{P}^1 \times \mathbb{P}^1$ arises as follows. If we remove the singular point, we get nonsingular matrices of determinant 0, and these are determined by their column span and their row span up to complex scaling; hence up to complex scaling we get $\mathbb{P}^1 \times \mathbb{P}^1$. If we only work up to real scaling we have a circle action, so the link of the singular point in Q is a circle bundle over $\mathbb{P}^1 \times \mathbb{P}^1$. We can then compute the cohomology of the link using the Serre spectral sequence, and from here compute the intersection cohomology of Q. This was harder than finding a resolution.

As a simpler example of the same phenomenon, the quadric

$$Q^2 = \{(x, y, z) \in \mathbb{C}^3 : xz - y^2 = 0\} \tag{2}$$

has singular point $(0, 0, 0)$, and the link of its singular point is a circle bundle over $\mathbb{P}^1$. In fact it is the degree-2 circle bundle, so the link is $\mathbb{RP}^3$. One way to see this is to consider the map

$$\mathbb{C}^2 \ni (a, b) \mapsto (a^2, ab, b^2) \in Q^2 \tag{3}$$

and notice that the link of the singular point in $\mathbb{C}^2$ is S^3 and that the map is a double cover. From here we can compute the intersection cohomology: the cohomology of the link over $\mathbb{C}$ is the cohomology of a 3-sphere, so the intersection cohomology is $\mathbb{C}$ in degree 0 and vanishes otherwise.

(As it turns out, Q^2 does not admit a small resolution, so we cannot use a resolution to compute its intersection cohomology.)

The following theorem is a deep generalization of the hard Lefschetz theorem.

Theorem 9.3. *(Decomposition) Let $\tilde{X} \xrightarrow{\pi} X$ be a proper complex algebraic map. Then there exists a unique collection of closed subvarieties Y of X and local systems $\mathcal{L}_Y$ on their smooth parts such that*

$$IC^*(\tilde{X}) = \bigoplus_{(Y,\mathcal{L}_Y)} IC^*(Y, \mathcal{L}_Y)[shifts]. \tag{4}$$

In a small resolution, the only subvariety that appears is X with the trivial local system.

Question: why might we expect something like this to be true?

Answer: in general, to compute the cohomology of a fibration $\tilde{X} \xrightarrow{\pi} X$ we need a spectral sequence involving the cohomology of X and a fiber. When the spectral sequence is particularly nice the cohomology of $\tilde{X}$ is just the tensor product of the cohomology of X and of a fiber. The decomposition theorem is something like this.

Example We return to the quadric Q^2 and a resolution $\tilde{Q}^2$ of it, with toric pictures as follows.

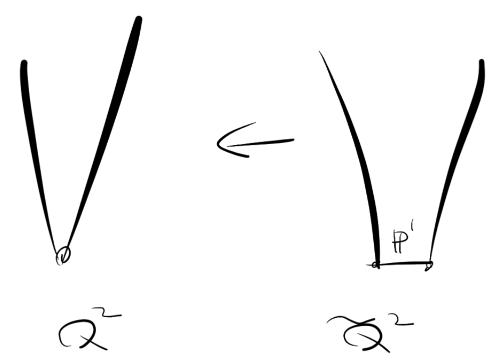

Figure 3: Another toric picture.

Here it turns out that

$$C^\bullet(\tilde{Q}^2) = IC^\bullet(Q^2) \oplus IC^\bullet(\mathrm{pt})[-2]. \tag{5}$$

Exercise 9.4. *Consider the map obtained by collapsing the curve $\mathbb{P}^1 \times \{0\}$ on $\mathbb{P}^1 \times \mathbb{C}$ to a point. This map is not algebraic. Show that the decomposition theorem fails here.*

10 Local systems

Everything is with coefficients in $\mathbb{C}$ as usual.

Definition The category Sh(pt) of (complexes of) *constructible sheaves* on a point is the dg-category of chain complexes of $\mathbb{C}$-vector spaces with finite-dimensional cohomology localized at quasi-isomorphisms.

(Finite-dimensional cohomology means bounded and finite-dimensional in each degree.)

Our chain complexes will be cohomological, so with differentials

$$... \xrightarrow{d} F^{-1} \xrightarrow{d} F^0 \xrightarrow{d} F^1 \xrightarrow{d} ... \tag{1}$$

of degree 1. This category is enriched in chain complexes, hence a dg-category, because we can take the Hom complex. A quasi-isomorphism is a morphism of chain complexes which induces isomorphisms on homology. Localizing at quasi-isomorphisms means that we want to add formal inverses to all quasi-isomorphisms. Generally we only care about chain complexes up to quasi-isomorphism (this is stronger than passing to cohomology). This story is a more modern way of talking about triangulated categories.

(Keller has an ICM address on dg-categories which is a nice reference.)

Definition Let M be a manifold. The category Sh(M) of (complexes of) *constructible sheaves* on M is the dg-category of (derived) $\mathbb{C}$-local systems on M.

The above definition requires elaboration. The objects are presheaves $U \mapsto F(U)$ of chain complexes on M. We want the sheaf axioms to be satisfied, but in a more sophisticated sense; we need to consider all intersections rather than just intersections of pairs of open subsets.

Why do we need to do this? Associate to M the simplicial set Simp(M) of simplices in M. The idea is that we want to assign to every 0-simplex a chain complex F_{σ_0}, to every 1-simplex a quasi-isomorphism of chain complex, to every 2-simplex a chain homotopy between quasi-isomorphisms, etc. Because chain complexes have all higher morphisms we cannot stop here as we could with sheaves of abelian groups.

(Abstractly we are writing down an ∞-functor from some model of the fundamental ∞-groupoid of M to chain complexes.)

Constructibility means two things. First, whatever we assign to balls (contractible neighborhoods) is something with finite-dimensional cohomology. Second, the restriction map from a ball to a smaller ball is a quasi-isomorphism. This is enough to get the simplicial picture above.

Example Let $M = S^1$. We'll pick the CW-decomposition with a 0-cell and a 1-cell. A (derived? ∞?) local system assigns a chain complex F to the 0-cell and a quasi-isomorphism $m : F \to F$ to the 1-cell. Since we're working over $\mathbb{C}$ a complex with finite-dimensional cohomology is quasi-isomorphic to its cohomology, so we just have an automorphism of a finite-dimensional graded vector space.

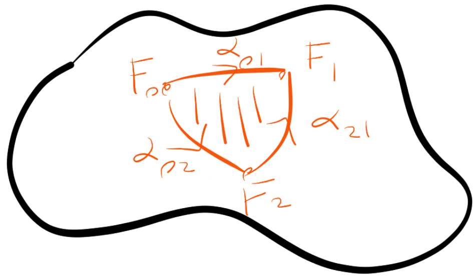

Figure 1: Data assigned to simplices.

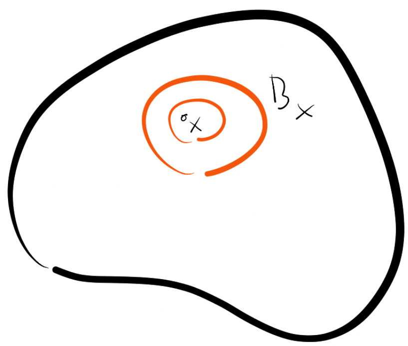

Figure 2: Restriction from balls to small balls.

Example Let $M = S^2$. Note that local systems on S^2 in the ordinary sense are just vector spaces, since $\pi_1(S^2)$ is trivial. But local systems in our sense are more interesting.

We'll pick the CW-decomposition with a 0-cell and a 2-cell. A local system assigns a chain complex F to the 0-cell and a map $m_{-1} : F \to F$ of degree -1 to the 2-cell (higher monodromy). How should we think about this map?

Let F be $\mathbb{C}$ in degrees 0 and 1 and zero otherwise. We can take $m_{-1} = 0$, which is some kind of trivial monodromy, or we can take m_{-1} to be an isomorphism from degree 0 to degree -1, which is interesting monodromy.

We can construct these things geometrically as follows. There is a projection map $\pi : S^1 \times S^2 \to S^2$, and we can take $F(U)$ to be cochains on $\pi^{-1}(U)$. This gives us trivial monodromy. We can also take the Hopf fibration $\pi : S^3 \to S^2$ and again take cochains on $\pi^{-1}(U)$. This gives us interesting monodromy.

One way to see this monodromy is to attempt to trivialize the Hopf fibration. Conflating

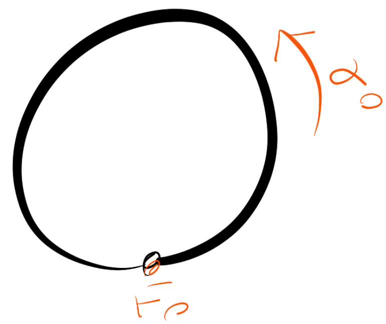

Figure 3: Local systems on a circle.

the Hopf fibration with the fibration $\mathbb{RP}^3 \to S^2$, which is the unit tangent bundle of S^2, we can attempt to write down a unit vector field that does not vanish away from some point. When we try to do this the unit vectors we assign to small neighborhoods of the point wind around the point twice.

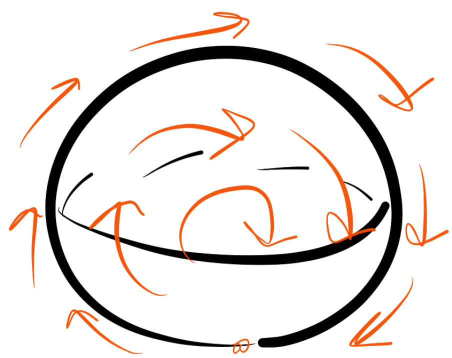

Figure 4: Attempting to write down a nonvanishing unit vector field.

11 Constructible sheaves

Last time we introduced two perspectives on local systems on a smooth manifold M. On the one hand, we can think of it as a full subcategory of the category of sheaves of chain complexes on M. On the other hand, we can think of it as a functor from the fundamental ∞-groupoid (e.g. the simplicial set) of M to chain complexes. Roughly speaking these are analogous to vector bundles with flat connections.

Figure 1: Two views of local systems.

Example Singular cochains $C_M^\bullet$ is a local system. On a smooth manifold this is quasi-isomorphic to the de Rham complex $\Omega_M^\bullet$. The Poincaré lemma tells us that a small ball gets assigned the constant sheaf, so these local systems are in turn quasi-isomorphic to the sheaf of locally constant functions. We can think of singular cochains and the de Rham complex as resolutions of the constant sheaf.

Example Borel-Moore chains $C_{M,-\bullet}^{BM}$ assigns to an open set singular chains, not necessarily of compact support. A small ball gets assigned the orientation sheaf shifted by the dimension of M. This has something to do with Poincaré duality.

Definition Let X be a reasonable space and S a fixed stratification. *Constructible sheaves* $\mathrm{Sh}_S(X)$ on X with respect to the stratification S is the dg-category of sheaves of $\mathbb{C}$-cochain complexes with S-constructible cohomology. This means that

1. Small balls get assigned cochain complexes with finite-dimensional cohomology, and

2. If U is a neighborhood of a point x and $V \subseteq U$ is a neighborhood of another point y in the same stratum as x, then the restriction map is a quasi-isomorphism.

This definition can also be reformulated simplicially using the notion of an exit path ∞-category. Here we only allow 1-simplices that can move to less singular but not to more singular strata, and the condition on higher simplices is analogous but somewhat complicated.

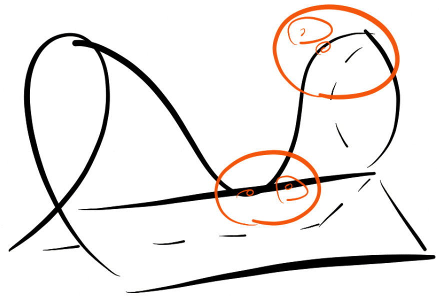

Figure 2: Two restriction maps, one of which must be a quasi-isomorphism and one of which needn't be.

Example Consider $\mathbb{R}$ with a stratification consisting of a single singular point. A constructible sheaf assigns to the singular point a chain complex F_0, assigns to the negative part another chain complex F_-, and assigns to the positive part another chain complex F_+. The exit paths out of the singular point give maps $F_- \leftarrow F_0 \rightarrow F_+$.

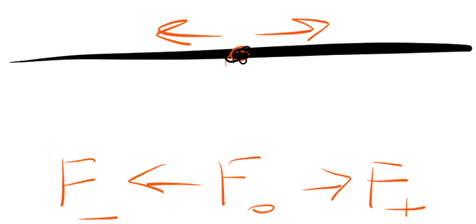

Figure 3: Constructible sheaves on $\mathbb{R}$ with a singular point.

Example Consider $\mathbb{R}^2$ with a stratification consisting of a single singular point. A constructible assigns to the singular point a chain complex F_0 and a ball in the complement another chain complex F_1. The exit path out of the singular point gives a map $r : F_0 \rightarrow F_1$. We can also consider monodromy of the ball around the singular point, which gives a map $m : F_1 \rightarrow F_1$. Finally, there is a map $h : F_1 \rightarrow F_1$ of degree -1 whose boundary exhibits a homotopy between r and $m \circ r$.

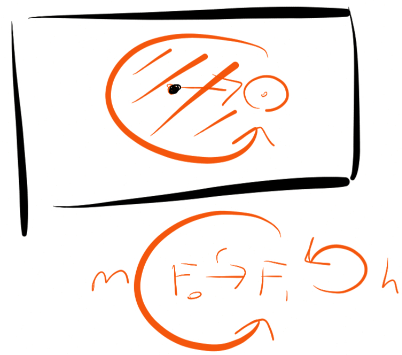

Figure 4: Constructible sheaves on $\mathbb{R}^2$ with a singular point.

Suppose a stratification S *refines* another stratification T in the sense that every stratum of T is a union of strata of S. Then there is a natural inclusion $\mathrm{Sh}_S(X) \supseteq \mathrm{Sh}_T(X)$.

Definition *Constructible sheaves* $\mathrm{Sh}(X)$ are the union of $\mathrm{Sh}_S(X)$ over all (reasonable) stratifications of X.

In other words, a sheaf is constructible if it is constructible with respect to some stratification. In particular, any local system on X is constructible.

If F is a constructible sheaf, then restricted to each stratum S_α it is a local system $F|_{S_\alpha}$. Can we reconstitute F from this data?

Let $i : Y \to X$ be a reasonable closed subset of X and let $j : U \to X$ be the inclusion of its complement. Then there is a distinguished triangle of sheaves

$$i_! i^! F \to F \to j_* j^* F \xrightarrow{[1]} . \tag{1}$$

(We may need to refine the strata and we will discuss the operations above.) This is what Grothendieck calls recollement. We can think of this as saying that F is (quasi-isomorphic to) the cone over a certain map $f : j_* j^* F[-1] \to i_! i^! F$.

To explain the symbols, j^* is the restriction map $\mathrm{Sh}(X) \to \mathrm{Sh}(U)$. This map has a right adjoint (in a suitable homotopical sense) $j_* : \mathrm{Sh}(U) \to \mathrm{Sh}(X)$ such that

$$(j_* G)(B) = G(B \cap U). \tag{2}$$

The composition $j_* j^*$ ignores things that are not happening on U. The unit of the adjunction is a natural map $F \to j_* j^* F$ given by the restriction map $F(B) \to F(B \cap U)$.

$i_! : \mathrm{Sh}(Y) \to \mathrm{Sh}(X)$ is defined in the same way as j_* in this case. This map has a right adjoint (in a suitable homotopical sense) $i^! : \mathrm{Sh}(X) \to \mathrm{Sh}(Y)$ defined as follows. Think of Y as being surrounded by a tube. Given an open set V in Y we can consider the tube T living around this open set, and $(i^* F)(V)$ is sections on T relative to its boundary ∂T (by which we mean the boundary of the long part of the tube rather than its ends). We can think of this as the cone of the restriction map $F(T) \to F(\partial T)$ (where if desired we can replace ∂T with a thickening of it). This is a derived version of sections living on Y.

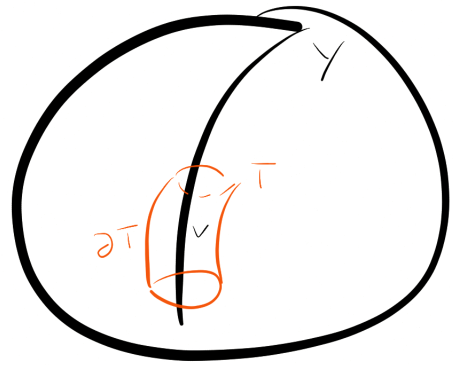

Figure 5: A tube.

The counit of this adjunction is a natural map $i_! i^! F \to F$ sending sections supported along Y into all sections. Together with the unit of the adjunction above we get our distinguished triangle.

Example Let $Y \subseteq X$ be a submanifold of a manifold and let $F = C^\bullet_X$ be cochains. Then the distinguished triangle is

$$C^\bullet_Y \otimes \mathrm{or}_{Y/X}[\mathrm{codim}_X(Y)] \to C^\bullet_X \to C^\bullet_U \xrightarrow{[1]} . \tag{3}$$

This is because, when interpreting a cochain on Y as a cochain on X, we need to keep track of codimension and coorientation.

Example Let $X = S^2$, let Y be a point, and let U be its complement. Let L be a local system on X. Then by the above, the distinguished triangle is

$$L|_Y \otimes \mathrm{or}[-2] \to L \to L|_U \xrightarrow{[1]} . \tag{4}$$

Call Y the point 0 and call another point the point 1. Then $L|_U$ is just $L|_1$. Then the boundary map in our distinguished triangle is a map $L|_1 \to L|_0 \otimes \mathrm{or}[-2]$ which is the homotopy we were looking for earlier.

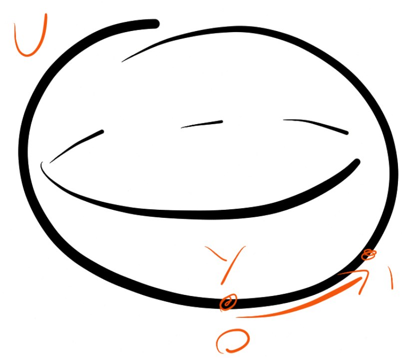

Figure 6: Constructible sheaves on S^2.

12 More about constructible sheaves

Let X be a reasonable space, $i : Y \to X$ be the inclusion of a closed subspace, and $j : U \to X$ be the inclusion of its open complement. Last time we wrote down a distinguished triangle $i_! i^! F \to F \to j_* j^* F \xrightarrow{[1]}$. The operations $j_* j^*$ ere naive operations but $i^!$ was a somewhat more sophisticated operation. With $F = C_X^\bullet$ and $Y \subseteq X$ a submanifold of a manifold X this took the form

$$C_Y^\bullet \otimes \mathrm{or}_{Y/X}[-\mathrm{codim}_{Y/X}] \to C_X^\bullet \to C_U^\bullet \xrightarrow{[1]} . \tag{1}$$

Taking cohomology, the first term can be reinterpreted as the relative cohomology $H^\bullet(X; U)$, and this gives us a long exact sequence in cohomology.

The importance of this sequence is that it tells us we can build constructible sheaves on a stratified space inductively in terms of pushforwards of local systems on the strata. In particular we will be able to give Deligne's definition of intersection cohomology in the language of sheaf theory.

Let X be a space stratified by even-codimensional strata. We will be indexing strata by codimension. Let $j_0 : S^0 \to X$ be the inclusion of the 0-codimensional stratum. Let $F^0 \in \mathrm{Loc}(S^0)$ be a local system concentrated in degree 0. Consider the (derived) pushforward $(j_0)_* L^0 \in \mathrm{Sh}(X)$. This computes the cohomology of a small ball intersect the open stratum (with coefficients in F^0). Consider the restriction $\tilde{F}^{\leq 2}$ of this pushforward to $S^0 \cup S^2$. We will cohomologically truncate this sheaf; this is the throwing out of certains chains and cochains that we did earlier. More precisely we take $F^{\leq 2} = \tau^{\leq 0} \tilde{F}^{\leq 2}$ where $\tau^{\leq 0}$ replaces a complex

$$\cdots \to A^{-1} \to A^0 \xrightarrow{d^0} A^1 \to \cdots \tag{2}$$

with

$$\cdots \to A^{-1} \to \ker(d^0) \to 0 \to \cdots . \tag{3}$$

(This is adjoint to the inclusion of complexes which are zero in positive degree into all complexes.) We now repeat this construction: we'll push $F^{\leq 2}$ forward along the inclusion $j_{\leq 2} : S^0 \cup S^2 \to X$ to get $\tilde{F}^{\leq 4}$, then take a truncation $F^{\leq 4} = \tau^{\leq 1} \tilde{F}^{\leq 4}$ in degrees less than or equal to 1, etc. Then intersection cohomology with coefficients in F^0 is the final output $F^{\leq \dim X}$ of this construction. In particular it is a constructible sheaf. Geometrically $F^{\leq 4}$ corresponds to only allowing codimension 0 and 1 cochains to intersect S^4.

If we try to apply the distinguished triangle we did earlier to intersection cohomology, $j^* F$ is intersection cohomology again, but $i^! F$ contains complicated information. It is an interesting question to try to write intersection cohomology in terms of local systems on the strata.

There is a distinguished triangle dual to the one we wrote above. It takes the form

$$j_! j^! F \to F \to i_* i^* F \xrightarrow{[1]} \tag{4}$$

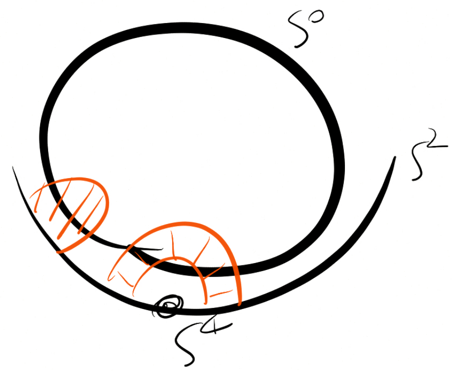

Figure 1: Sections.

where now we've switched the roles of i and j. Here $i_* = i_!$ because i is the inclusion of a closed subspace and $j^! = i^*$ because j is the inclusion of an open subspace. $j_! : \mathrm{Sh}(U) \to \mathrm{Sh}(X)$ is universal with the property that sections on any small ball around a point in Y are zero.

Example Let $Y \subseteq X$ be the inclusion of a submanifold into a manifold and let $F = C_X^\bullet$ again. Then the distinguished triangle takes the form

$$C_{X;Y}^\bullet \to C_X^\bullet \to C_Y^\bullet \xrightarrow{[1]} \tag{5}$$

where $C_{X;Y}^\bullet$ is cochains away from Y, and passing to cohomology gives another long exact sequence in cohomology.

So far we've encountered four of Grothendieck's six operations. Let $f : X \to Y$ be a reasonable map between reasonable spaces. This induces two (derived) adjunctions (f^*, f_*) and $(f_!, f^!)$ between categories of constructible sheaves. $f^* : \mathrm{Sh}(Y) \to \mathrm{Sh}(X)$ is the (derived) pullback. $f_* : \mathrm{Sh}(X) \to \mathrm{Sh}(Y)$ is the (derived) pushforward. $f_! : \mathrm{Sh}(X) \to \mathrm{Sh}(Y)$ is the (derived) pushforward with proper support: here, if $U \subseteq Y$, then $f_! G(U)$ would be sections of $G(f^{-1}(U))$ such that f restricted to their support is a proper map.

$f^!$ is the right adjoint to (derived) pushforward with proper support. For a closed inclusion it's the same as f^*, and we also described it for an open inclusion. We will also describe $f^!$ when $f : X \to \mathrm{pt}$; in general $f^!$ is defined fiberwise. $f^!$ of the constant sheaf on the point is shifted Borel-Moore chains (closed support, not compact support) $C_{-\bullet}^{BM}[\dim X]$. It is sometimes called the Verdier dualizing complex on X.

The claim that this is the right adjoint of $f_!$ implies in particular that

$$\mathrm{Hom}(f_! C_X^\bullet, \mathbb{C}_{\mathrm{pt}}) \cong \mathrm{Hom}(C_X^\bullet, f^! \mathbb{C}_{\mathrm{pt}}). \tag{6}$$

Morphisms from a constant sheaf give global sections, so the RHS is just Borel-Moore homology $H_{-\bullet}^{BM}(X)$. The LHS is the dual of cohomology with compact support $H_c^\bullet(X)^\vee$.

This is a version of the standard duality between homology and cohomology. (This has nothing to do with Poincaré duality. To get Poincaré duality we need to know that on a compact manifold we have $f^! \cong f^* \otimes \mathrm{or}[\dim X]$.)

(The other two of Grothendieck's operations are sheaf tensor and sheaf hom.)

Some identities between the operations that are used in practice:

1. If f is proper, then $f_! = f_*$.

2. If f is a fibration with smooth fibers, then $f^! = f^* \otimes \mathrm{or}[\dim F]$ where F is the fiber.

3. In particular, if f is an open inclusion, then $f^* = f^!$.

What also gets used in practice is base change. Consider a pullback square

$$\begin{tikzcd} X \times_Y Y' \arrow[r, "s'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "s"] & Y \end{tikzcd} . \tag{7}$$

If F is a sheaf on X, we can try to turn it into a sheaf on Y' by passing through Y or by passing through $X \times_Y Y'$. These two possibilities are compatible in the sense that $s^! f_* F \cong (f')_* (s')^! F$ and $s^* f_! F \cong (f')_! (s')^* F$. This is not true if all of the shrieks are replaced with stars.

Example Let $X = \mathbb{R}^2 \setminus \{(0,0)\}$ and let $f : X \to \mathbb{R}$ be the projection onto the x-axis. Let $s : \mathrm{pt} \to \mathbb{R}$ be the inclusion of the origin, so the pullback is the fiber X' over the origin of f.

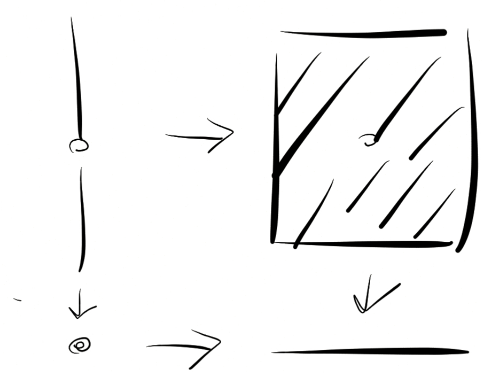

Figure 2: The spaces above.

Let $F = C^\bullet_X$. Then

1. $f_* C^\bullet_X$ is $C^\bullet_{\mathbb{R}} \oplus C^\bullet_{\{0\}}[-1]$,

2. $(s')^* C^\bullet_X$ is $C^\bullet_{X'}$,
3. $(f')_* C^\bullet_X = \mathbb{C} \oplus \mathbb{C}$, but
4. $s^* \left(C^\bullet_{\mathbb{R}} \oplus C^\bullet_{\{0\}}[-1] \right)$ is $\mathbb{C} \oplus \mathbb{C}[-1]$

so base change with all stars fails. Morally speaking, without shrieks we are ignoring what the neighborhood of the fiber looks like.

13 Microlocal support

Last time we associated to every reasonable space X the dg-category $\mathrm{Sh}(X)$ of derived constructible sheaves, and we associated to every morphism $f : X \to Y$ a pair of adjunctions (f^*, f_*) and $(f_!, f^!)$. The last two of Grothendieck's six operations are sheaf tensor and sheaf hom. We also wrote down some identities between these operations, including base change.

One thing we get for free here is an equivalence $D_X : \mathrm{Sh}(X) \to \mathrm{Sh}(X)^{op}$ called Verdier duality with D_X^2 naturally isomorphic to the identity. It is given by $D_X F \cong \mathrm{Hom}(F, \mathbb{D}_X)$ where $\mathbb{D}_X = \pi^! \mathbb{C}_{\mathrm{pt}}$ is the Verdier dualizing sheaf (where $\pi : X \to \mathrm{pt}$ is the unique map), or Borel-Moore chains shifted by $\dim X$. We have $D_X C_X \cong \mathbb{D}_X$ and $D_X \mathbb{D}_X \cong C_X$. Verdier duality conjugates the above two adjunctions in the sense that

$$D_Y f_* D_X = f_! \tag{1}$$

$$D_X f^* D_Y = f^! \tag{2}$$

and in fact Verdier duality intertwines the adjunctions as well. So understanding (f^*, f_*) and understanding Verdier duality means we understand $(f_!, f^!)$ as well.

Exercise 13.1. *Show that $D_X IC_X \cong IC_X$ (where X has even-dimensional strata and its smooth locus is oriented).*

(This is a manifestation of the idea that IC_X sits between C_X and $\mathbb{D}_X$.)

Let $F \in \mathrm{Sh}(X)$. The *support* $\mathrm{supp}(F)$ of F is the complement of the set of points such that, on an open ball around such a point, F vanishes (is quasi-isomorphic to zero). There is also a function $\chi(F)$ given by taking the Euler characteristic of the stalk at a point. This is a constructible (integer-valued) function (locally constant on strata).

Exercise 13.2. *Show that χ induces an isomorphism between the Grothendieck group $K(Sh(X))$ and constructible (integer-valued) functions $Fun(T^*(X))$ on X.*

In microlocal geometry we will consider more refined versions of these invariants. For X an ambient smooth manifold, the refined version of support is the *microsupport* or *singular support* $\mu\mathrm{supp}(F)$. This will be a closed, $\mathbb{R}_+$-conic, and Lagrangian subspace of $T^*(X)$. The refined version of χ is the *characteristic cycle* $\mathrm{cc}(F) \in \mathrm{Fun}(T^*(X))$. These refine the above in the sense that

$$\mathrm{supp}(F) = X \cap \mu\mathrm{supp}(F) \tag{3}$$

$$\chi(F) = \mathrm{cc}(F)|_X. \tag{4}$$

Consider the example $X = \mathbb{R}$, so $T^*(X) \cong \mathbb{R} \times \mathbb{R}^\vee$. A constructible sheaf can be described by describing some set of points and some restriction maps from those points to the intervals bordering them. This is a local but not a microlocal description.

Now instead of describing sheaves at a point x we want to describe sheaves at (x, ξ) where $\xi \neq 0$ is a covector. We want to measure the change in the topology in the positive direction (where ξ tells us which direction is positive). Pick a ball $B_x \ni x$ and a function $f : B_x \to \mathbb{R}$ such that $df_x = \xi$, and define

$$N_{x,\xi} = \{f(y) = -\epsilon\} \subset B_x, \epsilon > 0. \tag{5}$$

We want to consider sections $F(B_x, N_{x,\xi})$ relative to $N_{x,\xi}$ (in a derived sense). In terms of the six operations, let $i : B_x \to X$ be the inclusion. We first restrict F to the interval $\{f > -\epsilon\}$ in B_x, getting a sheaf i^*F. Letting $j : \{f > -\epsilon\} \to \{f \geq -\epsilon\}$, we then take $j_!$, getting a sheaf $j_! i^* F$. Said another way, we are looking at the cone of the map $\Gamma(B_x, F) \to \Gamma(N_{x,\xi}, F)$.

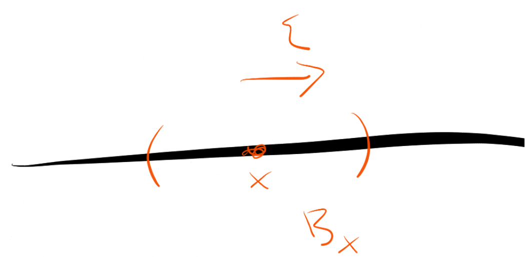

Figure 1: Change in the positive direction.

Example Let $X = \mathbb{R}$ and let F be a local system. Taking sections on a small ball and sections on a smaller ball in the negative half-space, we are looking at the cone of a quasi-isomorphism, which is quasi-isomorphic to zero. Hence $F_{x,\xi}$ vanishes for all $\xi \neq 0$. The singular support or microsupport is X, and the characteristic cycle of L is $\chi(L)$.

Example Let $X = \mathbb{R}$ and let $F = \mathbb{C}_{\text{pt}}$ be a skyscraper sheaf at a point pt. Because F is concentrated at pt, the restriction to any open subset not containing pt is zero, so we have $F_{x,\xi} \cong \mathbb{C}$ if ξ is based at pt and it vanishes otherwise. The microsupport consists of pairs (pt, ξ).

Example Let $j : (a, b) \to \mathbb{R}$ be the inclusion of an open interval and let $F = j_* \mathbb{C}^\bullet_{(a,b)}$. The support of F is the closed interval $[a, b]$ and the corresponding constructible function takes value 1 on $[a, b]$ and value 0 otherwise.

Now for the microlocal picture. We want to compare how interesting the sheaf is in the past to the sheaf in the future. Inside (a, b) the future is exactly as interesting as the past (restriction maps are quasi-isomorphisms). At b, the restriction maps are interesting if positive points into the interval, and similarly at a. The microsupport is a jagged line and $\text{cc}(F)$ is equal to 1 on it.

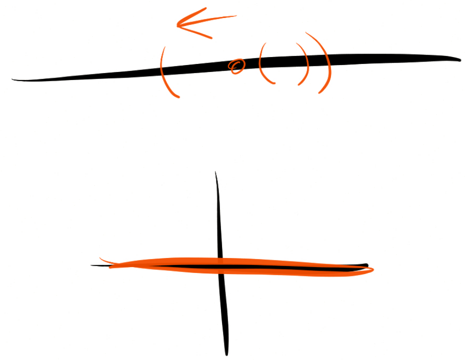

Figure 2: The singular support of a local system.

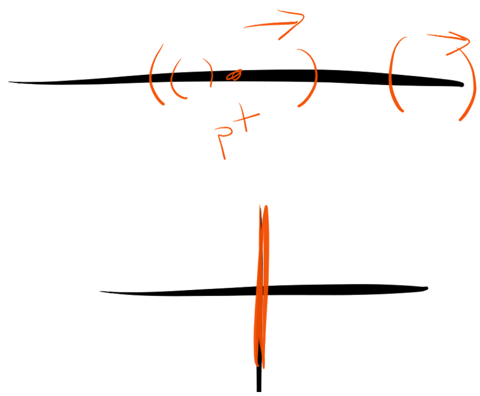

Figure 3: The singular support of a skyscraper sheaf.

Example Let $j : (a, b) \to \mathbb{R}$ be the inclusion of an open interval and let $F = j_! \mathbb{C}^{\bullet}_{(a,b)}$. The shriek extension from an open set is extension by zero, or sections with proper support. The support of F is again $[a, b]$, but now a section of F has to stay away from the boundary, so chains can be moved out of open neighborhoods of a and b and $\chi(F) = 0$ at a, b. The microsupport should be another jagged line.

Let $F \in \mathrm{Sh}_S(X)$, where $S = \{X_\alpha\}$ is a stratification. The singular support of F will be a closed, $\mathbb{R}_+$-conic, Lagrangian subset of

$$T^*_S(X) = \bigsqcup_{\alpha} T^*_{X_\alpha}(X) \subseteq T^*(X). \tag{6}$$

For example, if $X = \mathbb{R}$ and S is a stratification with a finite number of singular points, then $T^*_S(X)$ consists of cotangent vectors at the singular points.

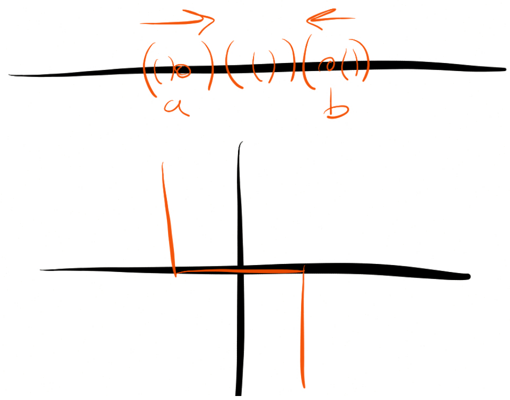

Figure 4: The singular support of the pushforward.

Let (x, ξ) be in the smooth locus of $T^*_S(X)$. Choose a small ball $B_x \ni x$ and a function $f : B_x \to \mathbb{R}$ such that $f(x) = 0, df_x = \xi$, and such that the graph of df is transverse to $T^*_S(X)$. (When $\xi = 0$ this means that f is locally a Morse function.) Define $F_f \cong F(B_x, N_f)$ where $N_f = \{f < 0\} \subset B_x$.

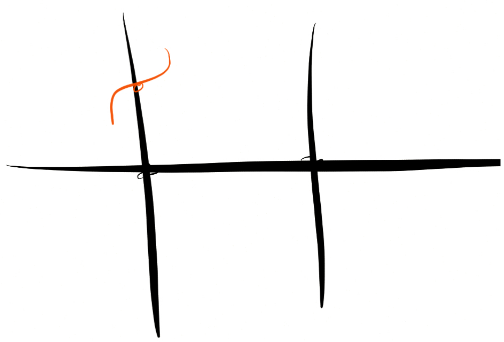

Figure 5: A transverse choice of f.

What is invariant about this construction (if we change f)? First, whether it is quasi-isomorphic or not is invariant. More can be invariant if we fix a convention about the quadratic part of f, e.g. ensure that f has a local minimum at x. If we do this then F_f is independent of the choice of f. This construction will give us both the microsupport and the characteristic cycle.

14 More about microlocal support

Let $Y \subseteq X$ be a submanifold, 0 a point on Y, and ξ a covector in the conormal bundle. Write y for (multi-)coordinates in the Y direction and x for (multi-)-coordinates normal to Y. Last time we considered functions $f : X \to \mathbb{R}$ such that $f(0) = 0, df_0 = \xi$, and such that the graph Γ_{df} of df is transverse to $T^*_Y(X)$. The first condition means that the Taylor series expansion of f has no constant term. The second condition means the linear term is ξx. The third term means that the quadratic term is nondegenerate in y, e.g. $f(x, y) = \xi x + Ay^2 + ...$ (where A is a matrix, which we may later want to require to be positive-definite), so f behaves like a Morse function along y.

Let $F \in \mathrm{Sh}_S(X)$. To each (x, ξ) in the smooth locus of $T^*_S(X)$ we will assign a complex $F_{(x,\xi)}$ as follows. If B_x is a small ball around x, we choose a function $f_{(x,\xi)} : B_x \to \mathbb{R}$ satisfying the above conditions such that along the stratum containing x it has a local minimum at f. Letting $N_{(x,\xi)} = \{f = -\epsilon\} \subset B_x$, we want to consider relative sections $F(B_x, N_{(x,\xi)})$. This fits into a long exact sequence of the form

$$F(N_{(x,\xi)}) \leftarrow F(B_x) \leftarrow F_{(x,\xi)}. \tag{1}$$

Exercise 14.1. *$F_{(x,\xi)}$ is independent of f up to quasi-isomorphism.*

We can always ask whether $F_{(x,\xi)}$ is quasi-isomorphic to zero or what its Euler characteristic is. The former gives us singular support $\mathrm{ss}(F)$ and the latter is the value of the characteristic cycle $\mathrm{cc}(F)$.

Example Let $S = X = \mathbb{R}^2$ and let $F = C^\bullet_X$ be cochains. If $\xi \neq 0$ then the restriction map induced by the inclusion $N_{(x,\xi)} \to B_x$ is always a quasi-isomorphism independent of the choice of f, so nothing depends on f. If $\xi = 0$ then we have a choice of three possible quadratic terms $f = x_1^2 + x_2^2, f = x_1^2 - x_2^2, f = -x_1^2 - x_2^2$. In the first case $N_{(x,\xi)}$ is empty and $F_{(x,\xi)}$ is $\mathbb{C}$ in degree zero. in the second case $N_{(x,\xi)}$ has two components and $F_{(x,\xi)}$ is $\mathbb{C}$ in degree 1, twisted by coorientation (so the Euler characteristic is different). In the third case $N_{(x,\xi)}$ is an annulus and $F_{(x,\xi)}$ is $\mathbb{C}$ in degree 2, again twisted by coorientation.

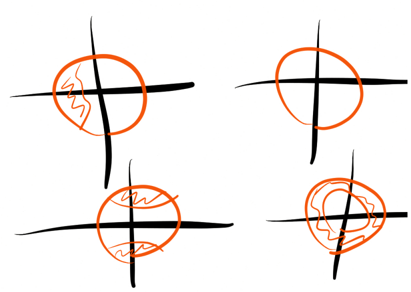

Figure 1: Dependence on the quadratic part.

Example Let $X = \mathbb{R}^2$ stratified by a cuspidal curve γ. Let $F = C^\bullet_\gamma$ be the constant sheaf on the curve. This sheaf is constructible. With respect to this stratification, $T^*_S(X)$ is the zero section on the smooth locus, consists of normal covectors on the smooth locus of γ, and is the whole cotangent space on the cusp 0. The smooth covectors are the covectors on the smooth locus, the covectors (y, ξ) on the smooth locus of γ with $\xi \neq 0$, and the covectors $(0, \xi)$ where ξ is not a limit of covectors on the smooth locus. (This happens generally.)

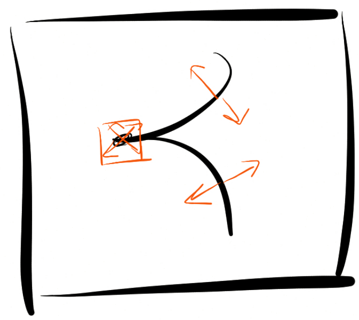

Figure 2: Cotangent vectors.

The computation on the smooth locus is uninteresting. On the smooth locus of γ, let t, s be local coordinates where t points along the curve. We want to pick an f of the form $\xi s + t^2$. Then $N_{(x,\xi)}$ doesn't intersect the curve, so $F_{(x,\xi)}$ is $\mathbb{C}$ concentrated in degree zero.

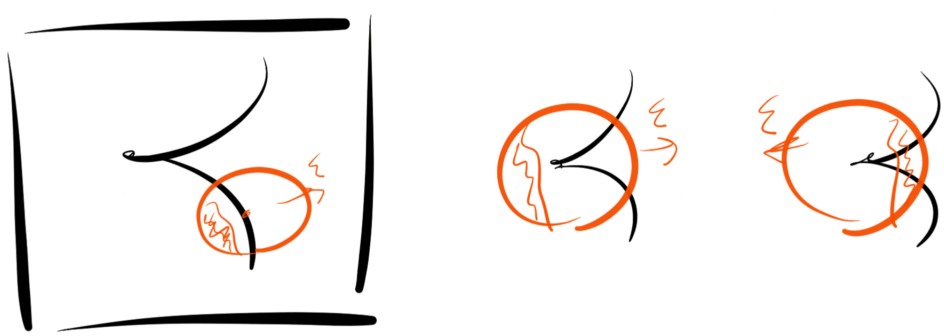

Figure 3: On the smooth locus of γ and at the cusp.

At the cusp two things can happen. Either $N_{(x,\xi)}$ does not intersect the curve, so again $F_{(x,\xi)}$ is $\mathbb{C}$ in degree 0, or $N_{(x,\xi)}$ intersects the curve in two components, so $F_{(x,\xi)}$ is $\mathbb{C}$ in degree 1, again twisted by coorientation.

Altogether the singular support consists of all covectors in $T^*_S(X)$ on γ.

Example Let X be as above, but consider the sheaf F obtained from the constant sheaf on half of the complement of the curve, pushed forward to the bottom half of the curve by $*$ and pushed forward to the top half by !. These are cochains which can touch the bottom half of the curve but not the top half. What happens on the smooth locus and the smooth locus of the curve is similar to the above, except that the choice of ! rather than $*$ pushforward switches the behavior.

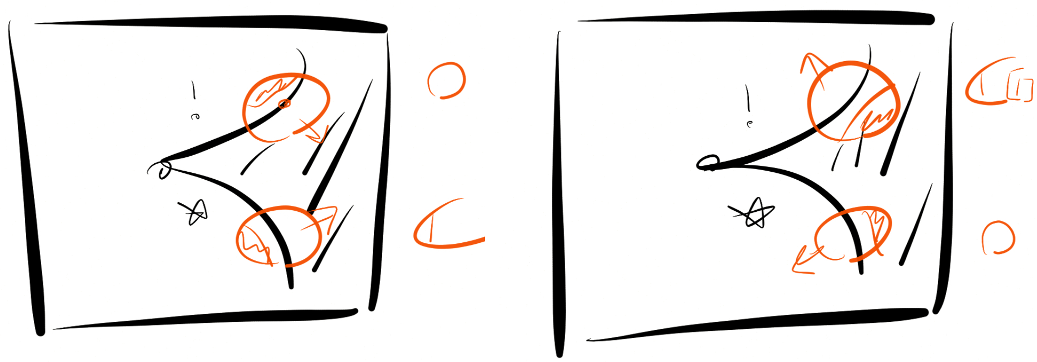

Figure 4: On the smooth locus of the curve.

Something interesting happens at the cusp. The singular support contains the conormal vector pointing upward by closure. But in fact it contains no other conormal vectors; in all other cases the conditions on boundary behavior make $F_{(x,\xi)}$ quasi-isomorphic to zero.

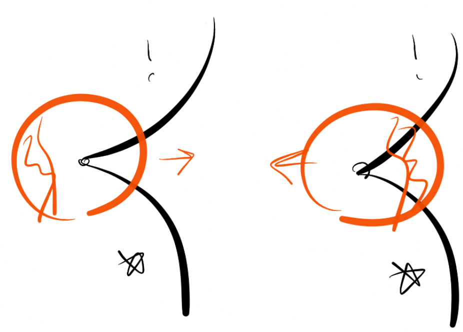

Figure 5: At the cusp.

15 Microlocal Morse theory

Let M be a manifold with a stratification S and F an S-constructible sheaf on M. Last time we associated, to every point $(x, \xi) \in T^*_S(M)$, a complex $F_{(x,\xi)}$ using a function $f : B_x \to \mathbb{R}$, where $B_x \ni x$ is a small ball.

When $S = \{M\}$ this is more or less Morse theory. In this setting F is a local system. Suppose M is compact. We might want to calculate the (derived) global sections of F. We can do this using open covers but this is messy. Instead we will choose a Morse function $f : M \to \mathbb{R}$ and look at its critical points, which are the intersection $\Gamma_{df} \cap M$ in $T^*(M)$. A Morse function equips $\Gamma(M, F)$ with a filtration by $\Gamma(M_{f \le c}, F)$, and

1. if $c \le c'$ and there are no critical points between them, then the comparison map from $\Gamma(M_{f \le c'}, F) \to \Gamma(M_{f \le c}, F)$ is a quasi-isomorphism as long as there are no critical points between c and c', and
2. if there is a critical point $c \le f(p) \le c'$ between them, then relative sections $\Gamma((M_{\le c'}, M_{\le c}), F)$ is the stalk F_p of F at the critical point together with a shift and a twist.

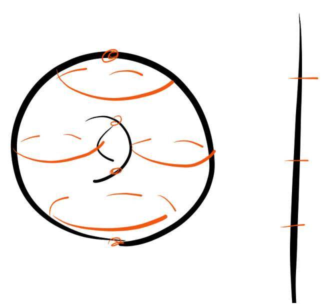

Figure 1: An example of a Morse decomposition.

Example Consider the sphere S^2 equipped with a Morse function giving it two horns. Going up a few critical points gives us a long exact sequence relating the cohomology of a ball, the cohomology of a circle, and the cohomology of a circle relative to a ball. We can calculate the first and third thing and this lets us calculate the second thing.

We haven't really calculated $\Gamma(M, F)$ because we would need to know some boundary maps, but at least we can calculate the Euler characteristic as a sum over critical points.

We would like a microlocal analogue of this story when M comes with a stratification S. Ordinarily $f : M \to \mathbb{R}$ being a Morse function is equivalent to Γ_{df} being transverse to the zero section in $T^*(M)$. With the stratification, let us now say that f is Morse if Γ_{df} is transverse to $T^*_S(M)$; their intersection is our new version of critical points. If F is an S-constructible sheaf, $\Gamma(M, F)$ again admits a filtration by $\Gamma(M_{\le c}, F)$ and we would like the analogues of the standard statements in Morse theory, namely that

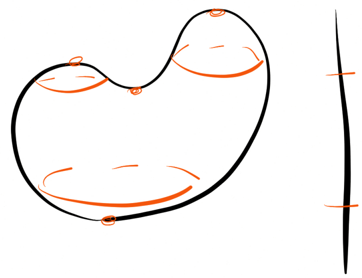

Figure 2: A Morse decomposition of the sphere.

1. if $c \leq c'$ and there is no critical point between them then $\Gamma(M_{\leq c'}, F) \to \Gamma(M_{\leq c}, F)$ is a quasi-isomorphism, and
2. if $c \leq f(p) \leq c'$ with one critical point (x, ξ) in between then $\Gamma((M_{\leq c'}, M_{\leq c}), F)$, suitably defined, should be $F_{(x,\xi)}$.

Once we know this, we get an index theorem, the Dobson-Kashiwara index theorem, giving $\chi(M, F)$ as an alternating sum of Euler characteristics $\chi(F_{(x,\xi)})$. All of this is proven using the usual Morse theory proofs via the Thom isotopy lemma.

Example Consider the kissing banana with a vertical Morse function.

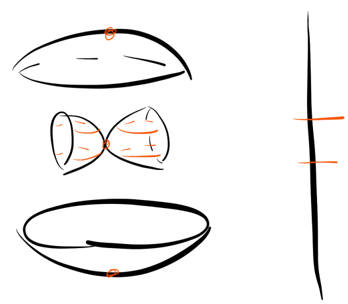

Figure 3: X-ray vision of a Morse decomposition of the kissing banana.

We will think of it as a sphere S^2 with two points squashed together on the inside. This Morse function has three critical points. Let M be intersection cohomology (recall that this is the same as the cohomology of S^2). At the bottom, $M_{(x,\xi)} \cong \mathbb{C}$ in degree zero. At the top, $M_{(x,\xi)} \cong \mathbb{C}$ in degree two with a twist. In the middle, $M_{(x,\xi)} \cong 0$; intersection cohomology is not sensitive to this critical point.

Now let F be the constant sheaf. At the bottom and the top it's the same as above. In the middle there is a 1-cochain stuck around the singular point, so $F_{(x,\xi)}$ is $\mathbb{C}$ in degree 1.

16 Characteristic cycles

Let M be an oriented manifold with stratification S, let F be an S-constructible sheaf on M, and let $(x, \xi) \in T^*_S(M)$ be a smooth point. What can we say if we are only interested in the Euler characteristic $\chi(F_{x,\xi})$?

Recall that we claimed there is an isomorphism from the Grothendieck group $K(\mathrm{Sh}(M))$ to constructible functions on M given by sending a sheaf F to the constructible function which assigns to $x \in M$ the Euler characteristic of the stalk F_x.

Question: consider the fibration $f : Kl \to S^1$ of the Klein bottle over S^1 and consider the sheaf $f_* C^\bullet_{Kl}$. It looks like the constructible function $\chi(f_* C^\bullet_{Kl})$ is zero, but is this sheaf really zero in the Grothendieck group?

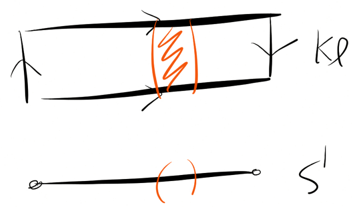

Figure 1: The Klein bottle.

Answer: yes, it is really zero in the Grothendieck group. To see this we can first write it as a direct sum $C^\bullet_{S^1} \oplus C^\bullet_{S^1}(\text{fiber or})[-1]$. We can cut up these sheaves by considering a point $\mathrm{pt} \in S^1$ with complement U and looking at the distinguished triangles associated to the inclusions $U \xrightarrow{j} S^1 \xleftarrow{i} \mathrm{pt}$ which settles the issue (and this is how the theorem is proven in general).

We would like a microlocal version of this story telling us something about characteristic cycles $\chi(F_{x,\xi})$. First, observe that $F \mapsto F_{(x,\xi)}$ and $F_{(x,\xi)} \to \chi(F_{x,\xi})$ both behave nicely with respect to distinguished triangles. Hence $\chi(F_{x,\xi})$ factors through the Grothendieck group, so equivalently it must factor through constructible functions.

Example Consider the real cusp in $\mathbb{R}^2$ and let F be the constant sheaf on the curve. The constructible function assigns 1 to every point on the curve and 0 otherwise. Now let x be the cusp and let ξ be a non-vertical cotangent vector at x. Then $F_{x,\xi} = \mathbb{C}[-1]$ twisted by coorientation, and $\chi(F_{x,\xi}) = -1$. The claim is that we can get this number from the constructible function above.

We can do this as follows. $\chi(F_{x,\xi})$ is the relative Euler characteristic of F on a small ball B_x with respect to the region $N_{(x,\xi)} = \{f = -\epsilon\}$ for small ϵ. So we can compute it by subtracting these two. On B_x this is the integral of the constructible function we get with respect to the Euler characteristic (we split up into parts and add up each part weighted by

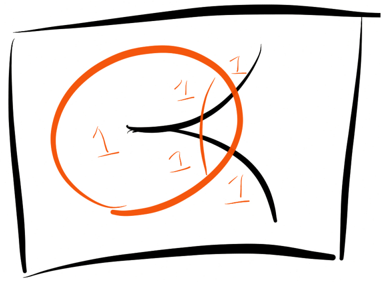

Figure 2: The real cusp in the plane.

the Euler characteristic). We get 3 points and 2 line segments, so the answer is $3-2=1$. On $N_{(x,\xi)}$ we get 2 points, so the answer is 2. Overall the answer is $1-2=-1$.

This gives us a map which assigns, to any sheaf, a function $(x,\xi)\mapsto\chi(F_{x,\xi})$ (the characteristic cycle $\mu\chi$ of F) on the smooth locus of $T^*_S(M)$. This map factors through the Grothendieck group, and we claim that as a map on the Grothendieck group it is injective. This is essentially the content of the following lemma.

Lemma 16.1. *Let f be a nonzero constructible function and let S_α be a stratum which is open in the support of f. Then for any $x\in S_\alpha$ and $(x,\xi)\in T^*_{S_\alpha}(M)$ we will have*

$$(f)_{x,\xi}=\int_{B_x}f-\int_{N_(x,\xi)}f\neq 0. \tag{1}$$

(Since we observed that $F\mapsto\chi(F_{x,\xi})$ factors through constructible functions we can regard it as acting on constructible functions.)

What is the image of the characteristic cycle map $\mu\chi$? It turns out to be conical Lagrangian cycles. This is the union of top Borel-Moore cycles $Z^{BM}_{\text{top}}(T^*_S(M))$ over all stratifications S.

One way to interpret the computation of $\chi(F)$ (the ordinary Euler characteristic of a sheaf F) that we did last time is that it is the intersection $\Gamma_{df}\cap\mu\chi(F)$ of cycles.

17 Computations with characteristic cycles

Last time we discussed isomorphisms

$$K(\mathrm{Sh}_S(M)) \cong \mathrm{Fun}_S(M) \cong H^{BM}_{\mathrm{top}}(T^*_S(M)). \tag{1}$$

The first isomorphism was given by taking local Euler characteristics χ, while the second was given by taking microlocal Euler characteristics $\mu\chi$ (and twisting by orientation). In particular, $\mu\chi(F)$ only depends on $\chi(F)$. We can see this by constructing an inverse. Let $x \in M, B_x \ni x$, and $j : B_x \to M$ be the inclusion. Then $\Gamma(B_x, F) \cong \mathrm{Hom}(j_!\mathbb{C}_{B_x}, F)$. We can compute its Euler characteristic by intersecting the characteristic cycle $\mu\chi$ of $j_!\mathbb{C}_{B_x}$ (possibly after applying some dualization) and the characteristic cycle of F.

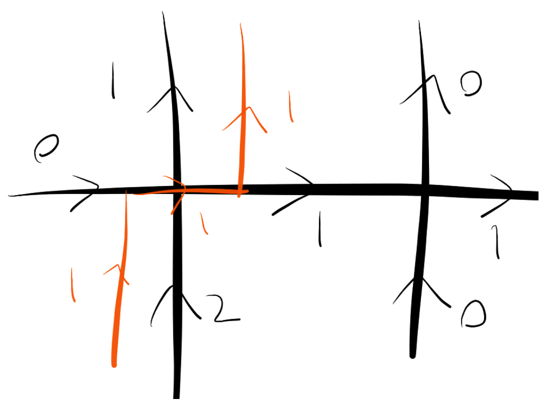

Figure 1: Intersecting cycles.

One curious fact about this story is that constructible functions have an obvious basis given by the indicator functions on each stratum. Characteristic cycles don't have such an obvious basis. However, characteristic cycles are better in that they make some symmetries more visible.

Now let's do some calculations.

Example Let $M = \mathbb{C}^2$ and consider the complex cusp $C = \{y^2 = x^3\}$ in it. Let F be intersection homology. C is homeomorphic to its normalization, which is just $\mathbb{A}^1$. However, it is embedded in an interesting way. The intersection of a small sphere around the cusp with C turns out to be the trefoil knot. In particular, around the cusp it is far from being a submanifold.

The only interesting thing to calculate is at the cusp. At the cusp there is a (complex) 2-dimensional conormal space. The smooth locus here is the complement of a (complex) 1-dimensional subspace, and in particular it is connected. Hence our computation will not depend on the choice of covector. However, there will potentially be interesting monodromy as we move our covector around. If $B_x \ni x$ is a small ball, the intersection of B_x with the

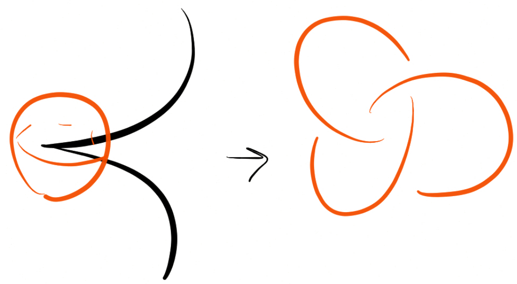

Figure 2: A real picture of the complex cusp and the trefoil knot.

curve will be a disc, and the intersection of N_x with the curve will be two arcs. Hence there is one interesting cochain in degree 1, and $F_{(x,\xi)} \cong \mathbb{C}[-1]$.

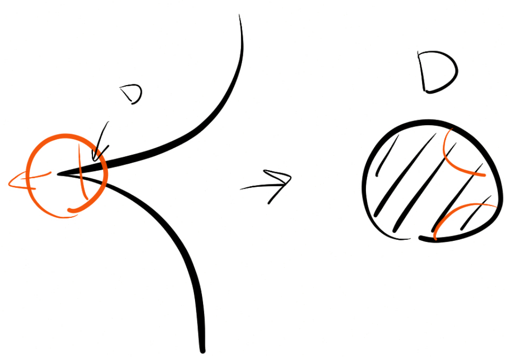

Figure 3: Microlocal calculations at the cusp.

What happens as we move ξ around? The intersection of N_x with the curve twists around, and this induces a monodromy of -1.

Example Let $M = \mathbb{C}^3$ and consider the quadratic cone $Q^2 = \{xz = y^2\}$. The intersection of Q^2 with a small sphere S^5 around the cone point is $\mathbb{RP}^3$ based on looking at the action of SO(3). In particular $\mathbb{RP}^3$ is close to S^3, so Q^2 is close to smooth - it is rationally smooth.

We'll compute microlocal Euler characteristic for two sheaves which are equal in the Grothendieck group but which have different microlocal stalks. First, let j be the inclusion $Q^2_{\text{sm}} \to Q^2$ of smooth points, and let $F = j_* C^\bullet_{Q^2_{\text{sm}}}$. The only interesting thing happens at the cone point. Here the smooth conormal vectors are $\mathbb{C}^3$ minus a dual light cone. The

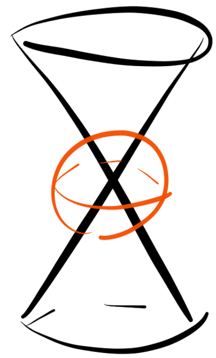

Figure 4: A real picture of the quadratic cone.

sheaf F is not supported at the cone point, so we can push things onto the link $\mathbb{RP}^3$ to do calculations.

Let (x, ξ) be a smooth point, where x is the cone point. $B_x \cap Q^2_{\mathrm{sm}}$ should be $\mathbb{RP}^3 \times (0, 1)$. What is $N_{(x,\xi)}$? It should be a complex hyperbola. After being pushed into $\mathbb{RP}^3$ it should be a copy of $\mathbb{RP}^1 \cong S^1$. So we want to compute the cohomology of $\mathbb{RP}^3$ relative to $\mathbb{RP}^1$. Over $\mathbb{C}$ this is $\mathbb{C}$ in dimensions 2 and 3, and it vanishes otherwise. In particular, the Euler characteristic is zero, so the characteristic cycle is not supported at (x, ξ) even though (x, ξ) is part of the singular support. The smooth conormal vectors have some interesting π_1 so there is some interesting monodromy which we will not calculate.

Now, Q^2_{sm} is homotopy equivalent to $\mathbb{RP}^3$, so its π_1 is $\mathbb{Z}_2$. Hence we can consider a nontrivial local system L on Q^2_{sm} and repeat the above story. This gives $F_{(x,\xi)}$ as the relative cohomology of $\mathbb{RP}^3$ with respect to $\mathbb{RP}^1$ with coefficients in L. The cohomology of both $\mathbb{RP}^1$ and $\mathbb{RP}^3$ with coefficients in L vanishes, hence so does this relative cohomology. In particular (x, ξ) is not part of the singular support.

18 Sheaves on the affine line

Let $M = \mathbb{C}$ with stratification the origin and its complement. We want to study $\mathrm{Sh}_S(M)$. Recall that a constructible sheaf is described by giving its stalk F_0 at 0, its stalk F_1 at 1, the restriction map $r : F_0 \to F_1$, and the monodromy map $m : F_1 \to F_1$. Moreover, mr is homotopic to r via a homotopy given by a map $h : F_1 \to F_1$ of degree -1. This description is somewhat unsatisfactory in that it does not see some symmetries of the situation. We will give a better microlocal description.

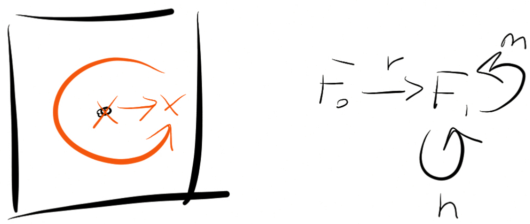

Figure 1: Previous description of sheaves on the affine line.

The cotangent bundle $T^*(M)$ looks like $\mathbb{C}^2$. There are two coordinate axes which are in some sense on equal footing. Microlocally, we will still measure the stalk F_ϵ at a nonzero point, but we also want to measure the microlocal stalk $F_{\epsilon^\vee}$ at some smooth cotangent vector $(0, \epsilon^\vee dx)$.

Example Let $F = C_M^\bullet$. Then $F_{\epsilon^\vee} = 0$ but $F_\epsilon = \mathbb{C}$.

Example Let $F = \mathbb{C}_{\{0\}}$. Then $F_{\epsilon^\vee} = \mathbb{C}$ but $F_\epsilon = 0$.

Microlocally these look very similar, whereas in the usual picture they might not. Moreover, $F_\epsilon, F_{\epsilon^\vee}$ together are faithful in the sense that if they both vanish then $F = 0$.

In addition to these two stalks there are also two monodromies $m : F_\epsilon \to F_\epsilon$ and $m^\vee : F_{\epsilon^\vee} \to F_{\epsilon^\vee}$.

Example Let $j : \mathbb{C}^\times \to \mathbb{C}$ be the inclusion and let $F = j_* L_\alpha$ where L_α is the local system with monodromy given by multiplication by some $\alpha \in \mathbb{C}^\times$ not equal to 1. Then $F_0 = 0$ because the nontrivial monodromy means there are no sections on a small ball. $F_\epsilon = \mathbb{C}$, and $F_{\epsilon^\vee} = \mathbb{C}[-1]$ because we are computing the relative cohomology of a circle relative to a point. The monodromy m is multiplication by α, and so is the monodromy $m^\vee$.

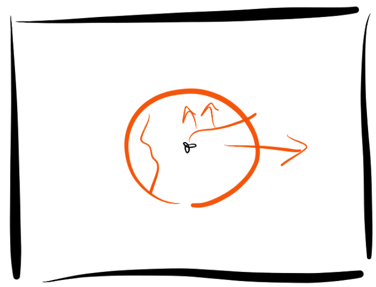

Figure 2: A microlocal stalk.

Note that there is a natural triangle

$$F_{\epsilon^\vee} \to F_0 \to F_{-\epsilon}. \tag{1}$$

Example Here is an example where m and $m^\vee$ disagree. Let F be the sheaf which is constant away from the origin and which, at the origin, is cochains which stay away from half of (small balls around) the origin. Then $F_\epsilon = \mathbb{C}$ with identity monodromy because it's away from the origin. $F_0 = \mathbb{C}[-1]$ because it is again the relative cohomology of a circle relative to a point. And $F_{\epsilon^\vee} = \mathbb{C}^2[-1]$ because it is the relative cohomology of a circle relative to two points.

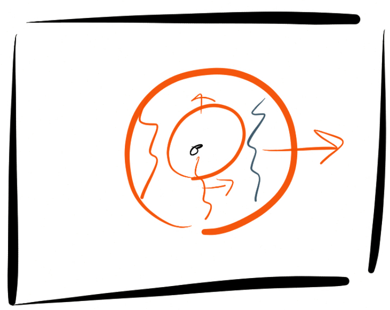

Figure 3: An interesting sheaf.

The monodromy $m^\vee$ is interesting. With an appropriate choice of basis, it turns out to be $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$; in particular, it is not the identity.

The natural triangle tells us that there is a coboundary map $\delta : F_\epsilon \to F_{\epsilon^\vee}$ of degree 1. There is also a map $\sigma : F_{\epsilon^\vee} \to F_\epsilon$. In homology it looks like the following. With B a disk and N half a disk, there is a boundary map from relative chains $C_\bullet(B, N)$ (which like N) to chains $C_\bullet(N)$ of degree -1. There is also a map in the other direction of degree 1 which, given a homology class on N, sweeps it out along a path through B back to N to get a relative homology class on B of one higher degree.

The dual maps in cohomology look like the following. The coboundary map looks like a kind of thickening or Gysin map. The dual to the sweeping map is a relative restriction.

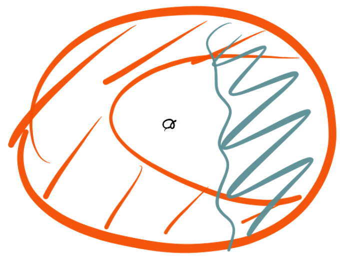

Figure 4: The cosweeping map.

Example Consider again $F = j_* L_\alpha$. Whether or not $\alpha = 1$ we always have $F_\epsilon = \mathbb{C}$ and $F_{\epsilon^\vee} = \mathbb{C}[-1]$, and moreover both monodromies m are equal to α.

Now we want to calculate δ and σ. When $\alpha = 1$, the coboundary map δ is zero (a thickened cochain can be sent away), but the cosweeping map σ is an isomorphism. When $\alpha \neq 1$, δ is now an isomorphism because the nontrivial monodromy prevents us from sending a thickened cochain away, and so is σ. With a suitable choice of generators, $\delta = 1 - \alpha$ and $\sigma = 1$.

We can now ask about relations between the maps we wrote down. Let's rename δ to p and rename σ to q. Then it turns out that $1 - pq = m^\vee$, but homotopically: there is an $h^\vee$ such that $\delta h^\vee = (1 - pq) - m^\vee$. Similarly, $1 - qp = m$, but homotopically: there is an h such that $\delta h = (1 - qp) - m$. This turns out to be all of the data in a constructible sheaf on M.

Note that the description we've given of a constructible sheaf on M is now completely symmetric: we can exchange the roles of F_ϵ and $F_{\epsilon^\vee}$. There is a kind of Fourier transform here that would be hard to see in the usual description of constructible sheaves on M.

Exercise 18.1. *Assume $F_{\epsilon^\vee}$ is concentrated in degree 0 and F_ϵ is concentrated in degree -1. Classify the possible sheaves.*

Exercise 18.2. *Assume in addition that m is the identity. Show that there exist five indecomposable sheaves.*

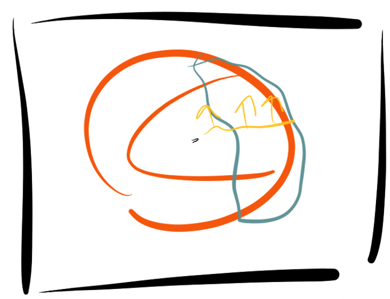

Figure 5: The cosweeping map.

19 Sheaves on the projective line, the Fourier transform

Last time we described constructible sheaves on $\mathbb{C}$ with a singular point in terms of modules over the quiver

$$\begin{tikzcd} \varepsilon^\vee \arrow[r, bend left, "p[1]"] & \varepsilon \arrow[l, bend left, "q[-1]"] \end{tikzcd} \tag{1}$$

where $1-qp$ and $1-pq$ are invertible (these give the monodromy up to homotopy, which we can forget since they don't arise in any other conditions). The left vertex is the microlocal stalk $F_{(0,\varepsilon^\vee dx)}$ and the right vertex is the usual stalk $F_\epsilon[-1]$, which we'll shift for convenience so p, q can have degree 0.

Now let's consider sheaves on $M = \mathbb{P}^1$ with a singular point.

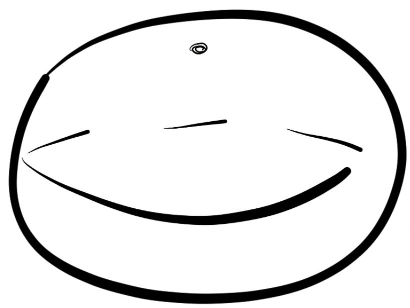

Figure 1: The sphere with a singular point.

This has the effect of adding a 2-cell which kills the monodromy, hence $pq = \delta h$ for some h of degree -1. Letting M_0, M_1 be the stalks above, we want to find indecomposable modules over this quiver with M_0, M_1 in degree 0 ($h = 0$ in this case). There are exactly 5 of these. They can be thought of as perverse sheaves and have something to do with $\mathfrak{sl}_2$-modules of highest weight with central character 0. Here they are:

$$\mathbb{C} \underset{0}{\overset{0}{\rightleftarrows}} 0\,,\ \mathbb{C} \underset{0}{\overset{\sim}{\rightleftarrows}} \mathbb{C}\,,\ 0 \underset{0}{\overset{0}{\rightleftarrows}} \mathbb{C}\,,\ \mathbb{C} \underset{\sim}{\overset{0}{\rightleftarrows}} \mathbb{C}\,,\ \mathbb{C} \oplus \mathbb{C} \underset{i_2}{\overset{\pi_1}{\rightleftarrows}} \mathbb{C}\,. \tag{2}$$

Exercise 19.1. *These are all the indecomposable modules.*

The sheaf corresponding to the first module is $\mathbb{C}_{\{0\}}$ and the sheaf corresponding to the third module is $\mathbb{C}_{\mathbb{P}^1}[1]$. The sheaf corresponding to the second module is $j_* \mathbb{C}_U[1]$ where

$j : U \to \mathbb{P}^1$ is the inclusion of $\mathbb{C}$ into $\mathbb{P}^1$. The sheaf corresponding to the fourth module is $j_!\mathbb{C}_U[1]$. Finally, the sheaf corresponding to the fifth module is Harold's pushforward.

Now let's talk about the Fourier transform. Let V be a real vector space and $V^\vee$ its dual. Then $V \times V^\vee$ can be thought of as both $T^*(V)$ and as $T^*(V^\vee)$. In $V \times V^\vee$ consider the subset S of pairs (v, λ) with $\lambda(v) \geq 0$.

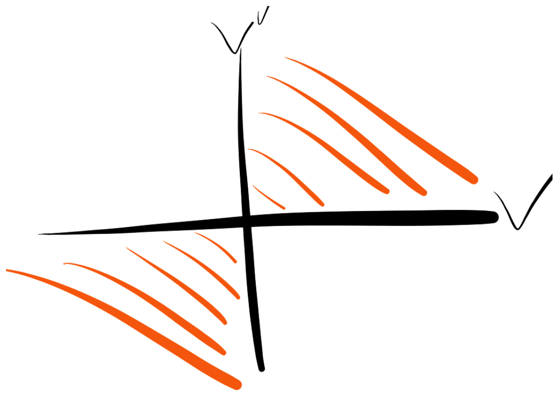

Figure 2: The subset S.

If $C_S^\bullet \in D(V \times V^\vee)$, we can consider the functor

$$D(V) \ni F \mapsto E(F) = (\pi_{V^\vee})_!(C_S^\bullet \otimes \pi_V^* F) \in D(V^\vee). \tag{3}$$

This is an equivalence on sheaves which are $\mathbb{R}_+$-conical. The construction we are performing here should be thought of as analogous to integration against a kernel

$$\hat{f}(\xi) = \int f(x) K(x, \xi)\, dx. \tag{4}$$

Example Let's compute a few examples, all of which may be off by a shift, for $V = \mathbb{R}$. Here $\mathbb{C}_{\{0\}}$ gets sent to $\mathbb{C}_{V^\vee}$. $\mathbb{C}_V$ gets sent to $\mathbb{C}_{\{0\}}$. The pushforward $j_*\mathbb{C}_{(-\infty,0)}$ gets sent to $j_!\mathbb{C}_{(0,\infty)}$.

20 Nearby and vanishing cycles

Up to now we've been playing the following game. Given a manifold M with a stratification S, we constructed $T^*_S(M) \subseteq T^*(M)$. Given $(x,\xi) \in T^*_S(M)$ in the smooth locus, we chose nice functions $f_{(x,\xi)} : M \to \mathbb{R}$ and used them to talk about microlocal support and characteristic cycles. Now we're going to play this game with an arbitrary function in place of $f_{(x,\xi)}$. Also, we'll mostly be interested in the holomorphic or complex setting.

Let f be a complex variety and let $f : X \to \mathbb{C}$ be regular. We want to study how the fibers of f change.

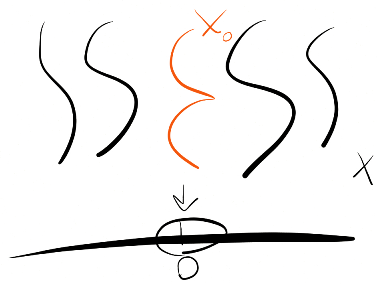

Figure 1: The fibers of a map.

The end result of this study will be a triangle of functors $\mathrm{Sh}(X) \to \mathrm{Sh}(X_0)$. Let $X_0 = f^{-1}(0)$ and let $i_0 : X_0 \to X$ be the inclusion. One functor is restriction $i_0^* : \mathrm{Sh}(X) \to \mathrm{Sh}(X_0)$. Another functor is a functor ψ_f called nearby cycles together with a map $i_0^* \to \psi_f$; roughly speaking this tells us the cohomology of a nearby fiber. Finally, there will be a functor φ_f together with a map $\varphi_f \to i_0^*$ called vanishing cycles. These fit together into a triangle

$$\varphi_f \to i_0^* \to \psi_f \xrightarrow{[1]} . \tag{1}$$

The first is some analogue of (B_0, F_ε), the second is some analogue of B_0, and the third is some analogue of F_ε.

Example Let

$$f : \mathbb{C} \ni z \mapsto z^n \in \mathbb{C} \tag{2}$$

and let $F = C^\bullet_X$. The i_0^* term is $\mathbb{C}^\bullet_{\mathrm{pt}}$ because 0 has one preimage. The nearby term is $\mathbb{C}^\bullet_{F_\varepsilon}$, or the direct sum of n copies of $\mathbb{C}^\bullet_{\mathrm{pt}}$, because a nearby fiber has n points. The vanishing term is the relative cohomology $C^\bullet_{(B,F_\varepsilon)}$, which is (up to some choices) $n-1$ copies of $\mathbb{C}^\bullet_{\mathrm{pt}}[-1]$.

This story depends on the choice of ε: as we choose a nearby point to take the fiber of and vary that point, we get monodromies m_φ and m_ψ of the vanishing and nearby cycles functors.

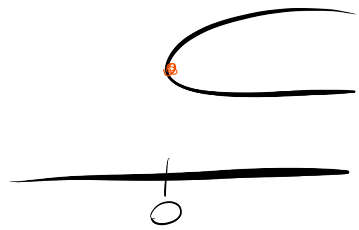

Figure 2: A real picture of $z \mapsto z^2$.

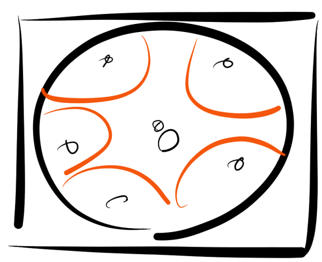

Figure 3: A complex picture of the nearby fiber.

Example Let

$$f : \mathbb{C}^2 \ni (x, y) \mapsto xy \in \mathbb{C} \tag{3}$$

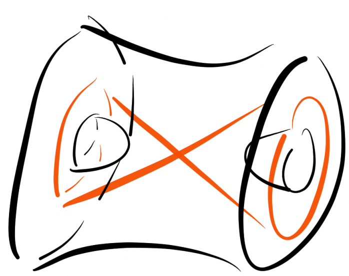

Figure 4: A picture of the fibers of f.

and let $F = C^{\bullet}_X$. The fiber X_0 over 0 looks like a cone and the nearby fibers look like cylinders. We need to describe three sheaves on X_0. The middle term is the restriction, which is just $C^\bullet_{X_0}$. To describe the others, first let $B_\kappa(x)$ be a small ball around x and let ε be very small. Take

$$\psi_f(F)(B_\kappa(x) \cap X_0) = F(B_k(x) \cap \{f^{-1}(\varepsilon)\}). \tag{4}$$

With $F = C^\bullet_X$, away from the singular point, this sheaf looks like the constant sheaf again. At the singular point we instead assign the cohomology of a cylinder (since that's what a nearby fiber looks like in a ball). Now φ_f is the cone of the map $C^\bullet_{X_0} \to \psi_f C^\bullet_X$. Away from the singular point this vanishes, and at the singular point we have a cochain in degree 2. So this is $\mathbb{C}^\bullet_{\text{singpt}}[-2]$.

(φ_f is dual to some homology that vanishes at X_0, hence the name.)

As ε varies, the monodromy m_ψ vanishes on global sections, hence so does the monodromy m_φ. But m_ψ is not the identity as a sheaf endomorphism; cycles get acted on by a Dehn twist.

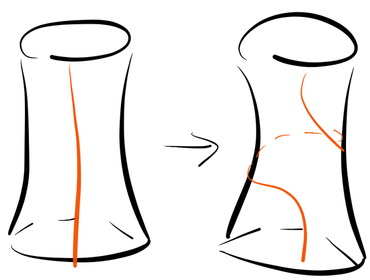

Figure 5: A Dehn twist.

It makes sense to say this because the monodromy can be trivialized at infinity; equivalently, m_ψ is the identity on sheaves away from the singular point. Write $v_\psi = 1 - m_\psi$ for the variation of the monodromy of the identity. The action of v_ψ sends the twisted cycle to a cycle not supported at infinity, and regarded as a map from the stalk at the singular point to the relative stalk, it is nontrivial.

Example Let

$$f : \mathbb{C}^2 \ni (x, y) \mapsto x^2 + y^3 \in \mathbb{C}. \tag{5}$$

The generic fiber is an elliptic curve while the special fiber is a cusp: the extra handle on the elliptic curve degenerates to the cusp.

Here $\psi C^\bullet_X$ is $C^\bullet_{X_0}$ away from the singular point and $C^\bullet$(torus) at the singular point, and $\varphi C^\bullet_X$ is $\mathbb{C}^2_{\text{singpt}}[-2]$.

Question: how do we compute things here?

Answer: one way is to use the Thom-Sebastiani theorem. For $f : \mathbb{C}^{n+1} \to \mathbb{C}$ with 0 the only critical point, the nearby fiber F_ε over a small ε is homotopy equivalent to a wedge

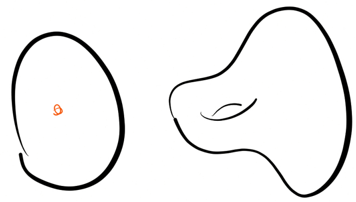

Figure 6: The generic and special fibers.

of spheres. Moreover, if $f(x,y) = g(x) + h(y)$, then F^f_ε is homotopy equivalent to the join $F^g_\varepsilon * F^h_\varepsilon$.

21 More about nearby and vanishing cycles

Consider again the map $f(x,y) = xy$ with singular fiber X_0 and F the constant sheaf. We wrote down a triangle

$$\varphi C_X^\bullet \to C_{X_0}^\bullet \to \psi C_X^\bullet \tag{1}$$

informally, and now we should do it more formally. First, $\psi \mathbb{C}_X^\bullet$ is equipped with a filtration, and we know its associated graded. The first step is $\mathbb{C}_{\text{singpt}}[-1]$. We should think of this as being generated by the cochain surrounding the singular point in a nearby fiber.

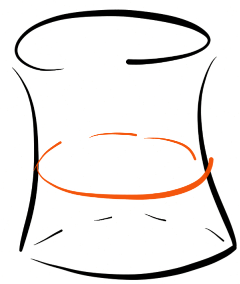

Figure 1: The cochain.

The next step is a sheaf I of cochains which either don't intersect the singular cochain or which contain the entire thing. In particular, the quotient of I by the previous step is IC_{X_0}. The last step in the filtration is $\psi \mathbb{C}_X^\bullet$, and the quotient by I is $\mathbb{C}_{\text{singpt}}[-1]$.

This filtration is compatible with monodromy (it is almost the filtration obtained from the nilpotent part $1 - m_\psi$ of the monodromy). In particular, $I = \ker(1 - m_\psi)$ and $\mathbb{C}_{\text{singpt}}[-1] = \text{im}(1 - m_\psi)$.

We can think about this sheaf using microlocal calculations, in particular computing the microlocal stalk at the singular point. The filtration becomes

$$\mathbb{C}[-1] \xrightarrow{0} \mathbb{C}[-1] \to \mathbb{C}[-1]^{\oplus 2} \tag{2}$$

with the monodromy acting by $\left[\begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array} \right]$ on $\mathbb{C}[-1]^{\oplus 2}$.

Here is a somewhat more formal definition. If $f : X \to \mathbb{C}$ is a nice map, consider a tube X_B around the singular fiber X_0.

Stratify it with a stratification S. There is a map

$$\pi : X_B \to X_0 \tag{3}$$

which is almost a retraction, and we will define $\psi F = \pi_*(F|_{X_\varepsilon})$, where X_ε is a nearby fiber inside X_B. (Assume that $X \setminus X_0$, as a stratified space, looks like $X_\varepsilon \times \mathbb{C}^\times$.) This

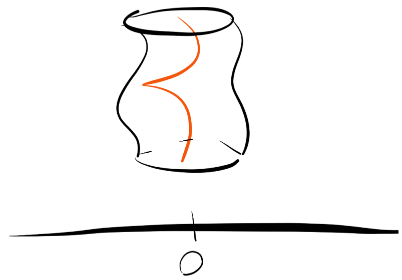

Figure 2: A tube.

construction is done using tube systems. (Almost means that $\pi|_{X_0}$ is not id_{X_0}, but induces the identity on $\mathrm{Sh}_S(X_0)$. This construction is independent of the choice of π.)

Here is an even more formal definition. Nearby cycles does not care about the singular fiber, so we are free to restrict away from it to get a map $X^\times \to \mathbb{C}^\times$. If F is a sheaf on $X^\times$, we can pull back along the universal cover $\exp : \mathbb{C} \to \mathbb{C}^\times$ to get a new sheaf on the pullback $\tilde{X}^\times$. (This is the algebraic geometer's way of taking a nearby fiber without mentioning a specific ε.) We now push this sheaf forward to $X^\times$ again, then to X, then pull it back to X_0. Writing $i : X_0 \to X, j : X^\times \to X$, and $\exp : \tilde{X}^\times \to X^\times$, this gives

$$\psi F = i^* j_* \exp_* \exp^* F. \tag{4}$$

$\exp_* \exp^*$ locally has the effect of replacing fibers by a $\mathbb{Z}$ worth of the same fiber, but globally it unwraps the monodromy.

Here is a standard way to reduce the complexity of the situation. If $f : X \to \mathbb{C}$ is a map, we can factor it into a composite

$$X \xrightarrow{\Gamma_f} X \times \mathbb{C} \xrightarrow{p} \mathbb{C} \tag{5}$$

where the first map is the inclusion of the graph and the second map is a coordinate projection. Starting with a sheaf F on X we can work with the pushforward $(\Gamma_f)_* F$ in relation to the coordinate projection, so we've made the map we want to study simpler at the cost of making the sheaf more complicated. This lets us split up the construction of nearby and vanishing cycles into two steps, the point being that ψ, φ, i_0^* factor through deformation to the normal bundle (we will define this later). In the case of the map $z \mapsto z^2$, the real picture looks like stretching out a parabola to a double line.

Figure 3: Stretching out a parabola.

22 Deformation to the normal cone

Last time we gave two definitions of nearby cycles. One was in terms of an almost retract $\pi : X \to X_0$ to the singular fiber. The other was in terms of pulling back and pushing forward from the universal cover.

Question: in general, let $f : \tilde{B} \to B$ be, say, a fibration. How should I think about $f_* f^* F$?

Answer: there's a projection formula which in this case shows that

$$f_* f^* F \cong f_* \mathbb{C}_{\tilde{B}} \otimes F. \tag{1}$$

As another example, if f is a covering map and F is a local system then it is just a representation of $\pi_1(B)$, the pullback is the restriction to $\pi_1(\tilde{B})$, and the pushforward is the induction.

Now let's talk about deformation to the normal cone. Recall that by considering the graph map $\Gamma_f : X \to X \times \mathbb{C}$ we reduced the study of nearby and vanishing cycles to the case of a projection map $\pi : X \times \mathbb{C} \to \mathbb{C}$ at the cost of considering arbitrary sheaves. We can try to make our lives easier by simplifying the sheaf by degenerating to the normal cone of $X \times \{0\} \subset X \times \mathbb{C}$.

In general, let $Y \subset M$ be a submanifold of a manifold. Recall that the normal bundle $N_{Y/M}$ (the tangent bundle of M, restricted to Y, quotiented by the tangent bundle of Y) is diffeomorphic to a small neighborhood of Y.

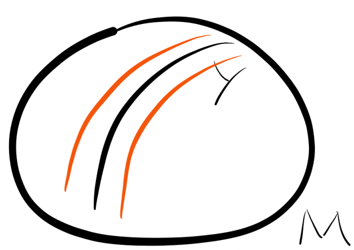

Figure 1: The normal bundle.

We want to think about this picture more dynamically. We will construct a family $F \to \mathbb{C}$ with the following properties. First, notice that there is a $\mathbb{C}^\times$ action here. We want a special point $0 \in \mathbb{C}$ such that the fiber over 0 is $N_{Y/M}$ and such that the fiber over $\mathbb{C}^\times$ is $M \times \mathbb{C}^\times$. So there is a diagram of the form

$$\begin{tikzcd} N_{Y/M} \arrow[r] \arrow[d] & F \arrow[d] & M \times \mathbb{C}^\times \arrow[l] \arrow[d] \\ \{0\} \arrow[r] & \mathbb{C} & \mathbb{C}^\times \arrow[l] \end{tikzcd} \tag{2}$$

Example Think about circles fibered over $\mathbb{R}$ getting stretched out larger and larger until, at 0, they become two straight lines.

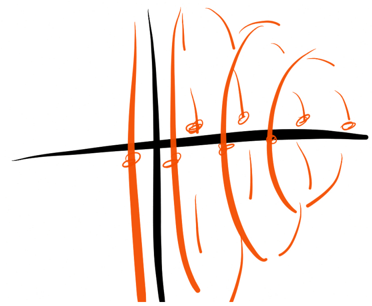

Figure 2: Stretched out circles.

In particular, everything not attached to Y (which is two points here, while the circle is M) is going away to infinity as we get to 0.

More formally, begin with the projection $M \times \mathbb{C} \to \mathbb{C}$. The first step is to blow up $M \times \mathbb{C}$ along $Y \times \{0\}$. The second step is to remove the closure of $(M \setminus Y) \times \{0\}$. The fiber over a nonzero point is M, but the fiber at 0 is the projective space $\mathbb{P}(N_{Y/M} \oplus \text{triv})$ minus the "divisor at infinity," and this is in fact $N_{Y/M}$ again.

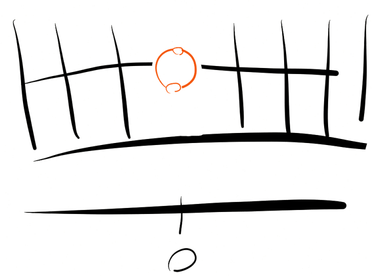

Figure 3: Blowup and removal.

Now we return to the setting of some sheaf F on $M = X \times \mathbb{C}$. We perform the above construction with respect to the submanifold $X \times \{0\}$. This gives us a family over $\mathbb{C}$ which has generic fiber M and special fiber $N_{X/M}$, and we want to send F to $F \boxtimes C^{\bullet}_{\mathbb{C}^{\times}}$, a sheaf on our family. Finally, we take nearby cycles ψ with respect to this family, giving us a new sheaf F_{con} on $N_{X/M}$. This sheaf is constructible with respect to a $\mathbb{C}^{\times}$-invariant stratification on the normal bundle, so in particular it is simpler than an arbitrary sheaf.

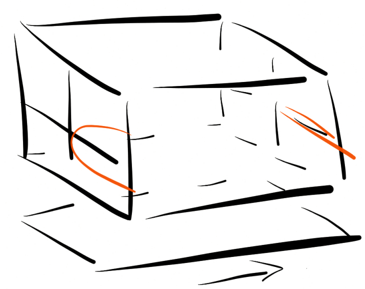

Figure 4: Degeneration to the normal cone.

23 More about deformation to the normal cone

Example Consider the map

$$f : X \ni z \mapsto z^n \in Y \tag{1}$$

where $X = Y = \mathbb{C}$, and the constant sheaf $F = \mathbb{C}_X$ on X. We already know that the nearby cycles are $\mathbb{C}^n$ and the vanishing cycles are $\mathbb{C}^{n-1}$. But we want to try deformation to the normal cone here. So now we consider the sheaf $\tilde{F} = \mathbb{C}_{\Gamma_f}$ supported on $\Gamma_f \subseteq X \times Y$ and the projection $\pi_Y : X \times Y \to Y$.

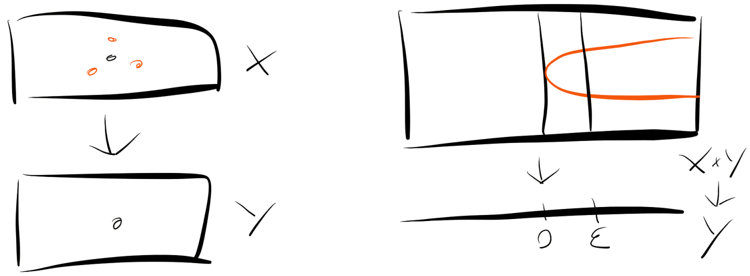

Figure 1: A complex picture and a real picture.

Here the normal bundle $N_{X/(X\times Y)}$ is trivial. When we degenerate to the normal cone the graph $y - x^n = 0$ starts to look $y - (\text{big})x^n = 0$ and eventually becomes $x^n = 0$. The new sheaf $\psi\tilde{F}$ on $N_{X/(X\times Y)}$ is $\mathbb{C}^n$ on the degenerate graph, $\mathbb{C}$ on the singular point, with monodromy around the singular point, and the restriction map $\mathbb{C}^n \to \mathbb{C}$ is sum. In particular, it has nicer support than before. This reproduces the nearby and vanishing calculations we did earlier.

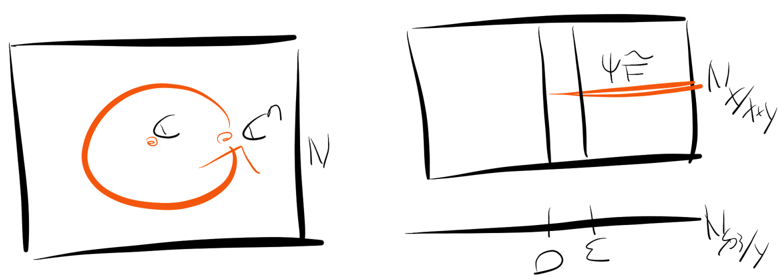

Figure 2: Monodromy and support.

The goal of doing this is not to make calculations easier; rather it is to talk about Fourier transforms, which we need to linearize to do first.

Example Consider the map

$$f : X \cong \mathbb{C}^2 \ni (x_1, x_2) \mapsto x_1 x_2 \in \mathbb{C} \cong Y \tag{2}$$

and F the constant sheaf again. Once again we can consider the sheaf $\tilde{F}$ supported on the graph $\Gamma_f \subset X \times Y \cong \mathbb{C}^3$ and the projection $\pi_Y : X \times Y \to Y$.

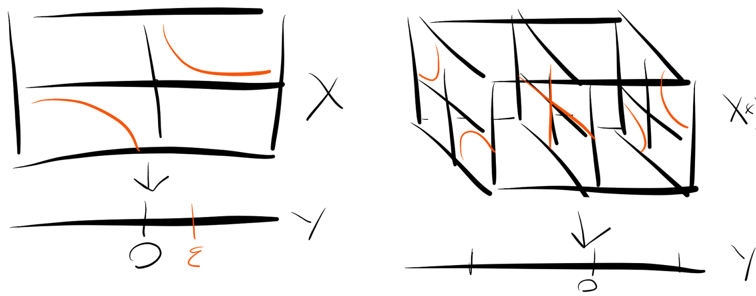

Figure 3: A real picture of f and the corresponding projection.

The degeneration to the normal cone sends the graph $y = x_1 x_2$ to the family of graphs $ty = x_1 x_2$ as $t \to \infty$. Before we pictured a parabola getting thin; now we should picture a saddle getting steep and thin. On each slice, the corresponding hyperbolas are getting thin. The support of $\psi \tilde{F}$ is now $N_{X/(X\times Y)}$ restricted to X_0. We again reproduce the nearby and vanishing cycles computations from earlier.

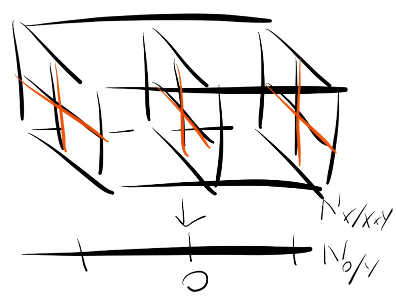

Figure 4: Thin hyperbolas.

One way to describe what we're doing is that we're computing nearby cycles for all ε simultaneously.

Example For a new example, consider

$$f : X \cong \mathbb{C}^3 \ni (x, y, z) \mapsto xy - z^2 \in \mathbb{C} \cong Y. \tag{3}$$

We can think of this as the determinant map $\mathfrak{sl}_2(\mathbb{C}) \to \mathbb{C}$. This is a 2-dimensional quadric; we already computed the 0-dimensional and 1-dimensional quadrics. The 1-dimensional case was somewhat complicated but the 0-dimensional and 2-dimensional case are complicated; the complication oscillates based on parity. For k-dimensional quadrics the nearby slice will be diffeomorphic to $T(S^k)$, which reflects that S^k is the vanishing cycle. (When $k = 0$ this is two points and when $k = 1$ this is a cylinder.)

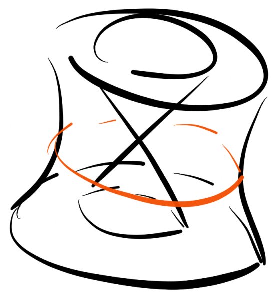

Figure 5: The vanishing cycle.

For $k = 2$ we claim that ψC_X is $\mathbb{C}_{X_0} \oplus \mathbb{C}_{\{0\}}[-2]$, which is simpler than the $k = 1$ case.

To write down this computation let's recall that we resolved $\mathfrak{sl}_2(\mathbb{C})$ to

$$\widetilde{\mathfrak{sl}}_2(\mathbb{C}) = \{(A, \ell) \in \mathfrak{sl}_2(\mathbb{C}) \times \mathbb{P}^1 | A\ell = \ell\} \tag{4}$$

which is equipped with a map to $\mathfrak{sl}_2(\mathbb{C})$. The generic fiber has 2 points since a generic matrix has 2 eigenvectors, but the special fiber has been blown up. This is called the Grothendieck-Springer resolution.

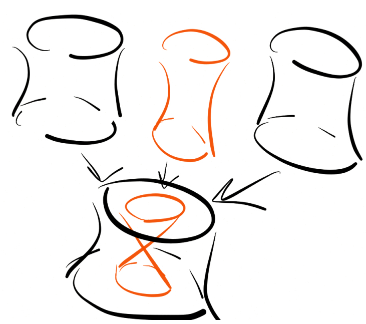

Figure 6: The resolution.

Nearby cycles commutes with proper pushforward, so we can compute nearby cycles on $\widetilde{\mathfrak{sl}}_2(\mathbb{C})$, where nothing happens because the family is topologically trivial, and then compute the pushforward by the decomposition theorem.

24 Microlocal homology

Recall that we started with a map $f : X \to Y \cong \mathbb{C}$ and a sheaf $F \in \mathrm{Sh}(X)$, replaced it with the projection map $\pi_Y : X \times Y \to Y$ and a sheaf $\tilde{F} = (\Gamma_f)_* F \in \mathrm{Sh}(X \times Y)$, then deformed to the normal cone to get the projection map $\pi : N_{X/(X\times Y)} \to N_{0/Y}$ and a sheaf $\psi\tilde{F} \in \mathrm{Sh}(N_{X/(X\times Y)})$. This is a kind of universal nearby cycles.

We would like to add one more step to this procedure, namely a Fourier transform, which will give us the projection map $\pi : N^*_{X/(X\times Y)} \to N^*_{0/Y}$ and a sheaf $\hat{F} \in \mathrm{Sh}(N^*_{X/(X\times Y)})$. The Fourier transform $\hat{F}$ has the property that its restriction to the zero fiber will be the horizontally compactly supported sections of $\psi\tilde{F}$ and its restriction to the ε-fiber will be vanishing cycles $\varphi_\varepsilon F$ (all possibly up to shifts).

In particular, suppose $F = C^\bullet_X$. Then $\hat{F}$ is supported on $X_0 \times \mathbb{C}$. Its sections over $X_0 \times \{0\}$ are $C^\bullet_{X_0}$ (cohomology), while its sections over $X_0 \times \mathbb{C}$ are Borel-Moore chains $C^{X_0}_{\mathrm{BM}}$ (homology). There is a natural map between these given by capping with the fundamental class.

Here is a thesis problem: $\hat{F}$ can be restricted to various subvarieties. These restrictions give chain theories living between homology and cohomology. For example, in Hamiltonian reduction we could take f to be a moment map. Then the sections over $X_0 \times \mathfrak{g}^*$ would be homology, the sections over $X_0 \times \{0\}$ would be cohomology, but we could also consider, say, the sections over $X_0 \times N$ where N is the nilpotent cone. What do these sections look like?

In the one-dimensional case, for example, we can consider sections over $X_0 \times \mathbb{C}^\times$, which gives the cone of $1 - m$ where m is the monodromy on φ.

Example Consider again

$$f : \mathbb{C} \cong X \ni z \mapsto z^2 \in Y \cong \mathbb{C}. \tag{1}$$

After deformation to the normal cone, $\psi\tilde{F} \in \mathrm{Sh}(N)$, where $N = N_{0/Y}$, is a direct sum $C^\bullet_N \oplus j_* L_{-1}$ where j is the inclusion $j : N \setminus \{0\} \to N$ and L_{-1} is the local system with monodromy -1.

Now we perform a Fourier transform. Recall that this exchanges skyscraper sheaves and constant sheaves. We get the sheaf $C^\bullet_{\{0\}} \oplus j_* L_{-1}$ (the second direct summand is Fourier self-dual). Here capping with the fundamental class is an isomorphism, so Poincaré duality is satisfied (rationally; integrally there will be problems).

Recall that the Fourier transform is a functor $\mathrm{Sh}(E) \to \mathrm{Sh}(E^*)$ where E is a vector bundle on some space X. The set of pairs $k = \{(v, \lambda) \in E \times E^* : \lambda(v) \geq 0\}$ gives rise to a sheaf $K = C^\bullet_k$ on the bundle product $E \times E^*$, and the Fourier transform is

$$\mathrm{FT}(F) = (\pi_{E^*})_! \left(K \otimes \pi_E^* F\right). \tag{2}$$

This is an equivalence from $\mathbb{R}_+$-conical sheaves on E to $\mathbb{R}_+$-conical sheaves on E^*.

Now let's do another example.

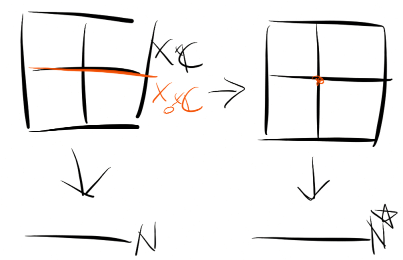

Figure 1: Fourier transform.

Example Again consider

$$f : \mathbb{C}^2 \cong X \ni (x, y) \mapsto xy \in Y \cong \mathbb{C} \tag{3}$$

and the constant sheaf. After deforming to the normal cone and computing universal nearby cycles, the resulting sheaf is supported on $X_0 \times N$, equipped with the projection to N. The Fourier transform is supported on axes; it vanishes most of the time because there aren't vanishing cycles most of the time.

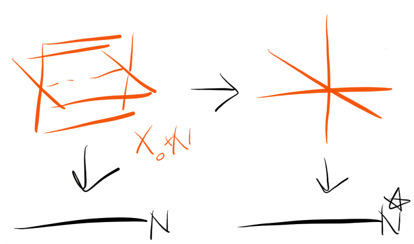

Figure 2: Another Fourier transform.

We get a sheaf whose global sections are Borel-Moore chains on the special fiber X_0 (a cone) and whose sections supported on X_0 is the constant sheaf. The former is $\mathbb{C}^2$ in degree 0 and $\mathbb{C}$ in degree 1 while the latter is $\mathbb{C}$ in degree 0, so Poincaré duality is not satisfied. The cone of the map from the latter to the former, which is the diagonal map $\mathbb{C} \to \mathbb{C}^2$ in degree 0, is $\mathbb{C}$ in degrees 0 and 1. This is the cone of $1 - m$ where m is the monodromy on φ. By contrast, in the previous example the cone of $1 - m$ was trivial.

Some philosophical points.

One lesson here is that homology of a space (say a local complete intersection, realized as a singular fiber as above) can be spread out in cotangent directions, and also that the difference between homology and cohomology on such a fiber has something to do with vanishing cycles.

The sheaf $\hat{F}$ we produced is a perverse sheaf. The $\mathfrak{G}_m$-action gives a natural filtration on it that has something to do with intersection cohomology.

25 Hamiltonian reduction

Let (M^{2n}, ω) be a symplectic manifold (so ω is a closed 2-form and $\omega^n \neq 0$). Darboux's theorem assures us that locally M^{2n} is $(\mathbb{R}^{2n}, \omega)$ with

$$\omega = \sum_{i=1}^{n} dx_i \wedge dy_i. \tag{1}$$

Hence symplectic manifolds in some sense have no local invariants.

Example Let $M = T^*(M)$ be a cotangent bundle. The corresponding symplectic form has the form $\omega = d\theta$ where $\theta \in \Omega^1(T^*(M))$ has the form

$$\theta = \sum y_i \, dx_i. \tag{2}$$

What should a morphism of symplectic manifolds $(M, \omega_M) \to (N, \omega_N)$ be? We can talk about symplectomorphisms, but we really should be talking about Lagrangian correspondences (Lagrangian subvarieties of $(M \times N, -\omega_M \times \omega_N)$; we allow singularities), which are more general (the graph of a symplectomorphism is a Lagrangian correspondence). In particular, a morphism $\text{pt} \to N$ is a Lagrangian submanifold of N and a morphism $M \to \text{pt}$ is a Lagrangian submanifold of M^{op} (M with the opposite symplectic form).

At some point we might want to quotient by a group action $X \to X/G$. Usually the quotient of a symplectic manifold by a group action will fail to be symplectic.

Example Let $M = S^2$ with symplectic form the volume form. (Here every curve is a Lagrangian submanifold.) $G = S^1$ acts by rotation. The quotient map in topological spaces is a line segment; in particular it fails to be even-dimensional and fails to be a manifold.

The singularity is not important; we could take $M = T^2$.

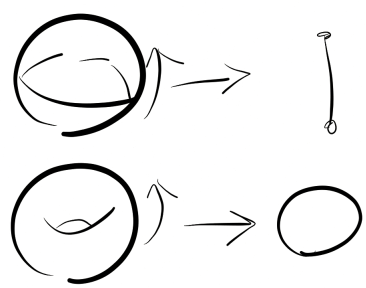

Figure 1: Quotients.

Question: is the problem that S^1 is not symplectic?

Answer: yes and no. In some sense we can replace S^1 with the cotangent bundle $T^*(S^1)$.

The solution to our problems is Hamiltonian reduction. For starters let's think about the quotient by an $\mathbb{R}$-action, so a vector field. The idea is that we won't take a quotient but a subquotient; we'll first pass to a subspace before quotienting. (Correspondences are a good way to talk about subquotients.) Recall that a symplectic form ω gives an identification $\eta : T(M) \to T^*(M)$. If $H : M \to \mathbb{R}$ is a function then it determines a Hamiltonian vector field $\eta^{-1}(dH) = v_H$. (If H is a Hamiltonian of a classical mechanical system then this vector field determines dynamics.) Every Hamiltonian vector field generates a symplectomorphism.

Example Let $M = T^*(\mathbb{R})$ and consider the Hamiltonian

$$H(x, y) = y^2. \tag{3}$$

This Hamiltonian has only a kinetic term. We have $dH = 2y\,dy$ and $v_H = \pm 2y\,dx$ depending on sign conventions. This vector field vanishes when $y = 0$ and looks like a shear in general. This reflects the fact that y is the momentum.

There is an exact sequence

$$0 \to \mathcal{O}_{\text{loc}}(M) \to \mathcal{O}(M) \xrightarrow{d} \Omega^1_{\text{ex}}(M) \tag{4}$$

where $\mathcal{O}_{\text{loc}}(M)$ is the locally constant functions, and a second exact sequence

$$0 \to \Omega^1_{\text{ex}}(M) \xrightarrow{\eta^{-1}} \mathfrak{symp}(M) \xrightarrow{\eta} H^1(M, \mathbb{R}) \to 0 \tag{5}$$

where $\mathfrak{symp}(M)$ is the Lie algebra of vector fields preserving the symplectic form, which can be identified with $\Omega^1_{\text{cl}}(M)$. In particular, if $H^1(M, \mathbb{R})$ vanishes then we can identify $\text{symp}(M)$ with exact 1-forms dH, so an infinitesimal symplectomorphism is the same thing as a function $H : M \to \mathbb{R}$ up to locally constant functions. This information can be thought of as the information of a function $H : M \to \mathfrak{g}^*$ where $\mathfrak{g} = \mathbb{R}$ is the Lie algebra of $\mathbb{R}$. This is a moment map, and we will perform Hamiltonian reduction by first restricting to a fixed value of the moment map and then quotienting:

$$(M//\mathbb{R})_\lambda = H^{-1}(\lambda)/\mathbb{R}. \tag{6}$$

Example When $M = S^2$ we can take H to be a coordinate, and then the Hamiltonian reduction is either a point or empty.

Exercise 25.1. *If λ is a regular value (and maybe the action needs to be reasonable) then the Hamiltonian reduction is symplectic.*

This gives a correspondence $M \leftarrow H^{-1}(\lambda) \to (M//\mathbb{R})_\lambda$.

In general, a moment map for a group action of a Lie group G on a symplectic manifold M is a map

$$H : M \to \mathfrak{g}^* \tag{7}$$

which is G-equivariant and which has the property that, for $v \in \mathfrak{g}$, the image of $v \in \mathfrak{symp}(M)$ is $\eta^{-1}(d(H(v)))$.

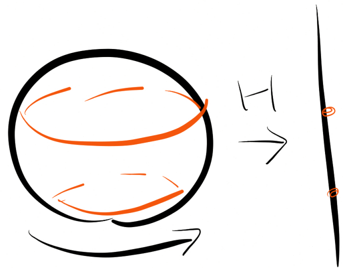

Figure 2: Hamiltonian reduction.

Example Let $G = \mathrm{SU}(2)$ acting on S^2. Here the moment map is a map from S^2 to a 3-dimensional vector space which in good cases is an inclusion.

Exercise 25.2. *Suppose G acts on a manifold X. Find the moment map for G acting on $T^*(X)$. In particular do this in the case that $X = G/B$.*

Moment maps are a good source of interesting maps with interesting fibers which we can apply the machinery we've developed to.

26 More about Hamiltonian reduction

Meta-theorem: in practice, every symplectic manifold arises as Hamiltonian reduction of a cotangent bundle (in physics, often an infinite-dimensional vector space).

Last time we considered an action of a Lie group G on a smooth manifold X, giving an induced action by symplectomorphisms on $T^*(X)$. Up to some H^1 business we should get a moment map $H : T^*(X) \to \mathfrak{g}^*$. We can describe this by giving a map $\mathfrak{g} \to \mathcal{O}(T^*(X))$. For starters, the action of G on X gives a map from $\mathfrak{g}$ to vector fields on X. But vector fields are sections of $T(X)$, hence can be evaluated against covectors to get functions on $T^*(X)$ as desired.

When we have a natural moment map H like this it's natural to consider $H^{-1}(0)$ when taking Hamiltonian reduction. This is a union

$$H^{-1}(0) = \bigcup_{G\text{-orbit } Y} T^*_Y(X) \tag{1}$$

of the conormals to the G-orbits, and is in particular singular. Let's call this $T^*_G(X)$. Then the Hamiltonian reduction is $T^*_G(X)/G$. Although they are singular it is possible to describe in what sense they're symplectic using derived geometry.

Example Let $X = \mathbb{CP}^1$ and let $G = \mathbb{C}^\times$ act on it by multiplication on one coordinate. In local coordinates the vector field generating the action above can be written $v = x_1 \, \partial_{x_1}$, so the moment map can be written

$$T^*(X) \ni (x_1, \xi_1) \mapsto x_1 \xi_1 \in \mathfrak{g}^*. \tag{2}$$

The zero level $T^*_G(X)$ is the union of conormals as above; it is a $\mathbb{CP}^1$ with two copies of $\mathbb{C}$ at the two fixed points. The cohomology is $\mathbb{C}, 0, \mathbb{C}$ (the cohomology of the sphere), but the Borel-Moore homology is $0, \mathbb{C}, \mathbb{C}^3$; all 0-cycles can be sent to infinity, there is a 1-cycle connecting the fixed points, and there are three 2-cycles coming from the $\mathbb{CP}^1$ and each of the two $\mathbb{C}$s.

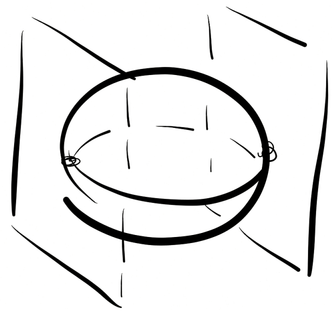

Figure 1: A tie fighter.

Every cochain is in particular a chain, so there is a map from cohomology to Borel-Moore homology. The map in degree 2 is the diagonal $\mathbb{C} \to \mathbb{C}^3$ since the generator of H^2 is the sum of all three 2-cycles we identified above. In particular this map is not an isomorphism, so Poincaré duality is not satisfied. The cone is $0, \mathbb{C}^2, \mathbb{C}^2$. (Recall that way back when we wrote down a sheaf on a space containing a singular fiber X_0 whose sections over X_0 were cohomology, whose global sections were homology, and whose sections away from X_0 were vanishing cycles. We can interpret this cone construction in terms of this sheaf.)

Example Now consider $G = \mathbb{C}^1$ acting on $X = \mathbb{CP}^1$ $(a(x_0 : x_1) = (x_0 : x_1 + ax_0))$. In the coordinate patch where $x_0 = 1$, this is translation. The corresponding vector field is $v = \partial_x$ and the moment map is

$$H(x_1, \xi_1) = \xi_1. \tag{3}$$

In the other coordinate patch the corresponding vector field is $v = \pm x_0^2 \partial_{x_0}$ and the moment map is

$$H(x_0, \xi_0) = x_0^2 \xi_0. \tag{4}$$

The zero level has a double line at infinity. The cohomology is again just $\mathbb{C}, 0, \mathbb{C}$, the cohomology of $\mathbb{CP}^1$, since it is a homotopy invariant. The Borel-Moore homology is $0, 0, \mathbb{C}^2$; we've lost a 2-cycle and we don't see the doubling. The cone is $0, \mathbb{C}, \mathbb{C}$.

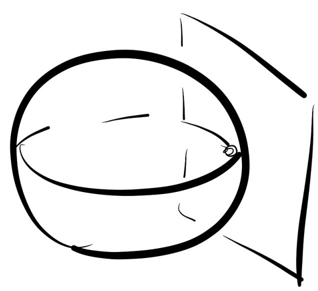

Figure 2: A tie fighter missing one wing.

So, why is all this interesting? There are many natural moduli spaces that are singular and given by equations (e.g. some kind of Hamiltonian reduction), and chains with support conditions on these spaces are important (e.g. in physics). For example, let X be a Riemann surface and consider G-local systems on X. It would be interesting to calculate e.g. microlocal chains with nilpotent support (support on the nilpotent cone in $\mathfrak{g}^*$).