Warmup suppose $T : F^2$ to F^2 sends

$v1 = \begin{matrix} 1 \\ 1 \end{matrix}$ to $\begin{matrix} 2 \\ 2 \end{matrix}$ $v2 = \begin{matrix} -1 \\ 1 \end{matrix}$ to $\begin{matrix} 1 \\ -1 \end{matrix}$

matrix of T wrt std basis (e1, e2) of F^2?

1) matrix wrt (v1, v2): $M = \begin{matrix} 2 & 0 \\ 0 & -1 \end{matrix}$

2) $v1 = e1 + e2$ $B = \begin{matrix} 1 & -1 \\ 1 & 1 \end{matrix}$
$v2 = -e1 + e2$

3) $\det(B) = 2$ [$B^{(-1)} = \begin{matrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{matrix}$]
$\mathrm{adj}(B) = \begin{matrix} 1 & 1 \\ -1 & 1 \end{matrix}$

$$\begin{aligned} BMB^{(-1)} &= (1/2)\, B\, M\, \mathrm{adj}(B) \\ &= (1/2) \begin{matrix} 2 & 1 \\ 2 & -1 \end{matrix} \quad \begin{matrix} 1 & 1 \\ -1 & 1 \end{matrix} \\ &= (1/2) \begin{matrix} 1 & 3 \\ 3 & 1 \end{matrix} \\ &= \begin{matrix} 1/2 & 3/2 \\ 3/2 & 1/2 \end{matrix} \quad \text{[works]} \end{aligned}$$

(Axler §3A) given a linear op $T : V$ to V:

Df a subspace W sub V is T-stable iff
w in W implies Tw in W

Ex above: Fv1, Fv2 are T-stable lines
Fe1, Fe2 are not T-stable
matrix wrt (v1, v2) easier to understand

__Ex__ for any linear op T : V to V,
{**0**} and V are are [trivially] T-stable

__Ex__ suppose that wrt some basis,
T : F^3 to F^3 has the matrix

$$\begin{matrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & * \end{matrix}$$

nontrivial T-stable subspaces? {(x, y, 0) | x, y}
{(0, 0, z) | z}

when a matrix for T has nontrivial __diagonal blocks__,
the blocks indicate nontrivial T-stable subspaces
[why care?]

when V = sum_i W_i for T-stable W_i,
studying T reduces to studying T|_{W_i} for all i

best situation: each W_i is a line
[what means?]
in this case: there is a basis (w_i)_i s.t.
W_i = Fw_i,
Tw_i = a_iw_i for some i
matrix of T wrt w_i is __diagonal__

__Df__ for any linear op T : V to V

an __eigenline__ of T is a T-stable line [dim-1 sub.]
an __eigenvector__ of T is v in V s.t. Fv is an eigenline
(which forces v ≠ **0**)
if Tv = λv, then we call λ the __eigenvalue__ of the line

Ex $T : F^2 \to F^2$ given by the matrix wrt the std basis

$$\begin{pmatrix} 1/2 & 3/2 \\ 3/2 & 1/2 \end{pmatrix}$$

$\{(x, x)\}$ is an eigenline with eigenvalue 2

$\{(x, -x)\}$ is an eigenline with eigenvalue –1

Ex $P : F^2 \to F^2$ given by the matrix wrt the std basis

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

[does P have any eigenvectors?]

$$\begin{pmatrix} \lambda x \\ \lambda y \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ y \end{pmatrix}$$

[take λ = 0:] Fe_1 is an eigenline with eigenvalue 0
[take λ ≠ 0:] Fe_2 is an eigenline with eigenvalue 1

Ex $N : F^2 \to F^2$ given by the matrix wrt the std basis

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} \lambda x \\ \lambda y \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ x \end{pmatrix}$$

[regardless of λ:] $x = 0$
Fe_2 is the only eigenline with eigenvalue 0
[same regardless of whether F = R or F = C]

Ex   H : R^2 to R^2   given by the matrix

$$\begin{matrix} 1 & -1 \\ 1 & 1 \end{matrix}$$

wrt the std basis

$$\begin{matrix} \lambda x \\ \lambda y \end{matrix} = \begin{matrix} 1 & -1 \\ 1 & 1 \end{matrix} \quad \begin{matrix} x \\ y \end{matrix} = \begin{matrix} x - y \\ x + y \end{matrix}$$

messy to solve…

notice:

$$(1/\sqrt{2})\, H = \begin{matrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{matrix}$$

$$= \begin{matrix} \cos(\pi/4) & -\sin(\pi/4) \\ \sin(\pi/4) & \cos(\pi/4) \end{matrix}$$

so H is the composition of: rotate by $\pi/4$
scale by $\sqrt{2}$

no H-stable lines through **0**

compare to H′ : C^2 to C^2   given by

$$\begin{matrix} 1 & -1 \\ 1 & 1 \end{matrix}$$

claim: $1 + i$ and $1 - i$ are eigenvalues

$$\begin{matrix} (1 + i)x = x - y \\ (1 + i)y = x + y \end{matrix}$$

imply $ix = -y$ and $iy = x$

so $\left\{\begin{matrix} (x \\ ix) \end{matrix}\right\}$ is an eigenline with eigenvalue $1 + i$

similarly

$\left\{\begin{matrix} (x \\ -ix) \end{matrix}\right\}$ is an eigenline with eigenvalue $1 - i$

Rem $\mathbb{C}^2$ is iso to the complex'n of $\mathbb{R}^2$

$$\begin{array}{ccc} (\mathbb{R}^2)_{\mathbb{C}} & \xrightarrow{H_{\mathbb{C}}} & (\mathbb{R}^2)_{\mathbb{C}} \\ = & & \\ \mathbb{C}^2 & \xrightarrow[H']{} & \mathbb{C}^2 \end{array}$$

Moral choice of R vs C affects eigenstuff

even if $T : \mathbb{R}^n \to \mathbb{R}^n$ has no eigenvals,
$T_{\mathbb{C}} : \mathbb{C}^n \to \mathbb{C}^n$ may have eigenvals

[later: we will show it has at least one]

Summary

if W sub V is stable under $T : V \to V$
then we can study T in terms of smaller op's $T|_W$

nicer when W is a line
nicest when V is a sum of eigenlines

over R T may have no eigenlines
over C T will have some eigenline [later],
but V need not be sum(eigenlines)

Q for next time

let $D : F[x] \to F[x]$ be $D(p) = dp/dx$
what are the D-stable linear subspaces?