PROBLEMS ON SYMPLECTIC REFLECTION ALGEBRAS

13. Quantum CM systems and Rational Cherednik algebras

Exercise 13.1. *Prove that* $wD_aw^{-1} = D_{wa}$ *for all* $w \in W, a \in \mathfrak{h}$.

Exercise 13.2. *Prove an analog of Proposition on the properties of Dunkl operators for complex reflection groups.*

Exercise 13.3. *Let* W *be a complex reflection group.*

(1) *Show that* $\operatorname{ad} f$ *is a locally nilpotent operator on* $H_{\hbar,c}$ *for any* $f \in \mathbb{C}[\mathfrak{h}]^W$.
(2) *Deduce that the localization* $H_{\hbar,c}[\delta^{-1}]$ *exists.*
(3) *Show that the Dunkl homomorphism* $H_{\hbar,c} \to D_\hbar(\mathfrak{h}^{Reg})\#W$ *factors through a unique homomorphism* $H_{\hbar,c}[\delta^{-1}] \to D_\hbar(\mathfrak{h}^{Reg})\#W$.
(4) *Show that the homomorphism* $H_{\hbar,c}[\delta^{-1}] \to D_\hbar(\mathfrak{h}^{Reg})\#W$ *is an isomorphism.*

Exercise 13.4. *Prove that the Dunkl homomorphism* Θ *is injective modulo* $\hbar$ *and hence is injective.*

Exercise 13.5. *Define* φ *to be the identity on* $\mathbb{C}[\mathfrak{h}^{Reg}]\#W$ *and* $\varphi(a) = a + \sum_{s\in S} c_s \frac{\langle a,\alpha_s\rangle}{\alpha_s}$. *Show that* φ *extends to a* $\mathbb{C}[\hbar]$*-linear automorphism of* $D_\hbar(\mathfrak{h}^{Reg})\#W$.

Problem 13.1. *Let* $\Gamma = \mathfrak{S}_n$ *and* $\mathfrak{h} = \mathbb{C}^n$ *(and not the reflection representation, this is a minor technicality). The goal of this problem will be to relate the CM space* C *to the* $\operatorname{Spec}(eH_{0,c}e)$. *We are going to produce a morphism* $\operatorname{Rep}_\Gamma(H_{0,c}, \mathbb{C}\Gamma)/\!/ \operatorname{GL}(\mathbb{C}\Gamma)^\Gamma \to C$, *to show that this is an isomorphism. Then we prove that the natural morphism* $\operatorname{Rep}_\Gamma(H_{0,c}, \mathbb{C}\Gamma)/\!/ \operatorname{GL}(\mathbb{C}\Gamma)^\Gamma \to \operatorname{Spec}(eH_{0,c}e)$ *is an isomorphism.*

Let $y_1, \ldots, y_n$ *be the tautological basis in* $\mathbb{C}^n = \mathfrak{h}$ *and* $x_1, \ldots, x_n$ *be the dual basis in* $\mathfrak{h}^*$. *The elements* x_n, y_n *still act on* $N^{\mathfrak{S}_{n-1}} \cong \mathbb{C}^n$. *Show that* $[x_n, y_n] \in O = \{A |\operatorname{tr} A = 0, \operatorname{rk}(A+E) = 1\}$ *for a suitable choice of* c. *Deduce that we have a morphism* $\operatorname{Rep}_\Gamma(H_{0,c}, \mathbb{C}\Gamma) \to \mu^{-1}(O)$. *Show that it descends to a morphism* $\operatorname{Rep}_\Gamma(H_{0,c}, \mathbb{C}\Gamma)/\!/ \operatorname{GL}(\mathbb{C}\Gamma)^\Gamma \to C$. *Show that the latter is finite and birational. Deduce that it is an isomorphism.*

Show that a natural morphism $\operatorname{Rep}_\Gamma(H_{0,c}, \mathbb{C}\Gamma)/\!/ \operatorname{GL}(\mathbb{C}\Gamma)^\Gamma \to \operatorname{Spec}(eH_{0,c}e)$ *(how is it constructed, by the way?) is also finite and birational. Deduce that it is an isomorphism.*