MATH 340: ADVANCED LINEAR ALGEBRA
PROBLEM SET #7

SPRING 2025

Due Wednesday, April 9. You may consult books, papers, and websites as long as you cite all sources and write up your solutions in your own words.

Problem 1. Look up the definition of an integer partition. Let $p(n)$ be the number of partitions of an integer $n > 0$. Using Jordan canonical form, show that $p(n)$ is also the number of conjugacy classes of nilpotent $n \times n$ matrices over $\mathbf{C}$.

Problem 2. Let

$$\mathfrak{sl}(n, F) = \{n \times n \text{ matrices over } F \text{ of trace } 0\}.$$

The notation $\mathfrak{sl}$ stands for *special linear*.

(1) Verify that $\mathfrak{sl}(n, F)$ is a vector space over F of dimension $n^2 - 1$.
(2) Show that an element of $\mathfrak{sl}(2, F)$ is nilpotent if and only if its entries satisfy a certain polynomial equation.
(3) Using a suitable basis to identify $\mathfrak{sl}(2, \mathbf{R})$ with $\mathbf{R}^3$, sketch the subset of nilpotent matrices. (No need to prove the basis is a basis.)

In principle, the following two problems are solved in Axler's text. But it may be easier to think about them from scratch, than to start with Axler.

Problem 3. Let V, W be finite-dimensional vector spaces and $T : V \to W$ a linear map. Show that the kernel $\ker(T^\vee)$ and the annihilator $\mathrm{Ann}_{W^\vee}(\mathrm{im}(T))$ are the same subspace of $W^\vee$.

Problem 4. Keep the setup of Problem 3.

(1) Show that

$$\dim W - \dim \ker(T^\vee) = \dim V - \dim \ker(T).$$

Hint: You'll need Problem 3, a dimension formula relating U and $\mathrm{Ann}_{W^\vee}(U)$ for some $U \subseteq W$, and a dimension formula relating $\ker(T)$ and $\mathrm{im}(T)$.

(2) Deduce from (1) that

$$\dim \mathrm{im}(T^\vee) = \dim \mathrm{im}(T).$$

(3) Using (2), show that the column rank and row rank of any square matrix M agree.

You may use the fact (Axler §3.132) that if M represents a linear operator $T : V \to V$ in some basis for V, then the *transpose* matrix M^t defined by $(M^t)_{j,i} = M_{i,j}$ represents $T^\vee : V^\vee \to V^\vee$ in the dual basis.

Problem 5. Let V be a vector space over F, possibly infinite-dimensional. In each case below, show that T is a linear isomorphism without picking an explicit basis for V. You may still use the fact that a tensor product of vector spaces is spanned by pure tensors.

(1) $T : \{\vec{0}\} \to \{\vec{0}\} \otimes V$ defined in the only possible way.
(2) $T : V^{\oplus n} \to F^n \otimes V$, where $V^{\oplus n}$ is the n-fold direct sum of V, defined by

$$T(v^{(1)}, \ldots, v^{(n)}) = \sum_i e_i \otimes v^{(i)},$$

where $e_1, \ldots, e_n$ is an ordered basis for F^n. *Hint:* To show injectivity, use the definition of $e_i \otimes v^{(i)}$ as a bilinear functional and an ordered basis dual to $(e_i)_i$.

Problem 6. Let V, W, U be vector spaces. The set

$$\mathrm{Bil}(W, V \mid U) = \{\text{bilinear maps from } W \times V \text{ into } U\}$$

forms a vector space under $(\beta + \beta')(w, v) = \beta(w, v) + \beta'(w, v)$ and $(a \cdot \beta)(w, v) = a \cdot \beta(w, v)$. It recovers $\mathrm{Bil}(W, V)$ when $U = F$.

For all linear $T : W \otimes V \to U$, let $\beta_T : W \times V \to U$ be the bilinear map such that $\beta_T(w, v) = T(w \otimes v)$. Show that the map

$$\mathrm{B} : \mathrm{Hom}(W \otimes V, U) \to \mathrm{Bil}(W, V \mid U) \quad \text{defined by } \mathrm{B}(T) = \beta_T$$

is linear and injective, without picking explicit bases for the vector spaces involved. *Hint:* Again, pure tensors span $W \otimes V$.

It turns out that B *is an isomorphism, but starting from Axler's definition of* $W \otimes V$*, this is difficult to show without picking explicit bases for* V *and* W*.*

Problem 7. A bilinear form $\beta : V \times V \to F$ is *degenerate* if and only if there is some nonzero $v \in V$ such that either $\beta(v, -)$ or $\beta(-, v)$ is the zero functional on V. It is *nondegenerate* otherwise. Now set $V = F[x]$. Show that:

(1) If $\beta(p, q) = \displaystyle\int_0^1 p(x)q(x)\,dx$, then β is nondegenerate.
(2) If $\beta(p, q) = p(1)q(1)$, then β is degenerate.

Problem 8. Show that for all $n \geq 2$, there is a bilinear form β on F^n such that

$$\beta(w, v) \neq 0 \text{ for some } w, v \in F^n, \quad \text{but } \beta(v, v) = 0 \text{ for all } v.$$

Hint: Take $\beta(w, v) = w^t M v$ for some carefully chosen $n \times n$ matrix M.