

## 2.

Today we discuss Borel subgroups of reductive groups and the corresponding quotients.

### 2.1.

We have discussed how an algebraic group  $G$  over  $\bar{\mathbf{F}}_q$ , equipped with a Frobenius map  $F : G \rightarrow G$ , gives rise to a finite group  $G^F$ . When  $G$  is reductive, the structure of  $G$ , *resp.*  $G^F$ , closely resembles that of  $\mathrm{GL}_n$ , *resp.*  $\mathrm{GL}_n(\mathbf{F}_q)$ .

Even earlier, we discussed the [Bruhat decomposition](#). For now, let  $k$  be an arbitrary algebraically closed field. Let  $B \subseteq \mathrm{GL}_n$  be the subgroup of upper-triangular matrices, and for all  $w \in S_n$ , let  $\dot{w} \in \mathrm{GL}_n$  be the permutation matrix of  $w$ . The Bruhat decomposition on  $k$ -points is

$$\mathrm{GL}_n(k) = \coprod_{w \in S_n} B(k)\dot{w}B(k).$$

Its proof is similar to how we used row reduction to establish the Schubert cell decomposition of any Grassmannian. Namely, it suffices to show:

**Theorem 2.1.** *The coset space  $B(k) \backslash \mathrm{GL}_n(k)$  is the disjoint union of the subsets  $B(k) \backslash (B(k)\dot{w}B(k))$  for  $w \in S_n$ .*

*Proof.* We can identify cosets of  $B(k)$  with (complete) flags in  $k^n$  via the map  $B(k)g \mapsto \vec{V} \cdot g$ , where  $\vec{V} = (V_i)_i$  is the standard flag in row notation. The rows of  $g$  define an ordered basis  $(v_i)_i$  such that  $V_i \cdot g = \langle v_1, \dots, v_i \rangle$  for all  $i$ .

So apply row reduction to  $g$ . The result is an upper-triangular matrix  $b \in B(k)$ . From the algorithm, we also get a permutation  $w^{-1} \in S_n$  that only depends on the flag  $\vec{V} \cdot g$ : the composition of the row swaps (from the left) used to reduce  $g$  to  $b$ . We have  $\vec{V} \cdot g = \vec{V} \cdot \dot{w}b$ .  $\square$

Note that the expression  $B\dot{w}B$  can be reduced even further. For instance, we can always take  $b$  unipotent in the proof above. Another way to see this: Recall that  $B = TU$ , where  $T$ , *resp.*  $U$  is the subgroup of diagonal, *resp.* unipotent matrices, and observe that permutation matrices normalize  $T$ , meaning  $\dot{w}T = T\dot{w}$  for all  $w$ .

Something stronger is true. For any algebraic groups  $H \subseteq G$ , Milne Prop. 1.83 exhibits an algebraic group  $N_G(H)$  such that  $N_G(H)(R) = N_{G(R)}(H(R))$  for any  $k$ -algebra  $R$ . It turns out that the connected components  $N_{\mathrm{GL}_n}(T)$  are precisely the cosets  $\dot{w}T$  for  $w \in S_n$ . This suggests how Bruhat decomposition ought to generalize beyond  $\mathrm{GL}_n$ .

Define the [Weyl group](#) of a maximal torus  $T \subseteq G$  to be the normalizer  $W = W(G, T) := N_G(T)/T$ . Note that for any  $w \in W$ , and any algebraic subgroup  $B \subseteq G$  containing  $T$ , the notation  $wB$  is unambiguous.

**Theorem 2.2.** *Suppose that  $G$  is a reductive algebraic group. Let  $B = T \ltimes U \subseteq G$  be a Borel subgroup, where  $U = [B, B]$ . Then  $B(k) \backslash G(k)$  is the disjoint union of the subsets  $B(k) \backslash (B(k)wB(k))$  for  $w \in W(G, T)$ .*

## 2.2.

As in the first lecture, we switch notation from left-hand quotients back to right-hand quotients. The set  $G(k)/B(k)$  is precisely the set of  $k$ -points of the fppf sheaf quotient  $G/B$ , essentially because  $\text{Spec } k$  has no nontrivial fppf covers. However, we can be much more concrete about spaces like this. The key idea is a representation-theoretic characterization of algebraic subgroups:

**Theorem 2.3** (Chevalley). *If  $G$  is any affine algebraic group with algebraic subgroup  $G'$ , then there exist a (finite-dimensional) representation  $V$  of  $G$  and a subspace  $V' \subseteq V$  such that  $G'(k) = \{g \in G(k) \mid gV' \subseteq V'\}$ . We can even choose  $V, V'$  so that  $V'$  is a line.*

*Proof.* Let  $I$  be the kernel of the quotient map  $k[G] \rightarrow k[G']$ . We can pick a finite generating set for  $I$  as an ideal. Then we can pick a finite-dimensional  $k[G]$ -comodule  $V \subseteq k[G]$  containing these generators. (This is similar to the proof of the linearity of affine algebraic groups, except that here, our representation of  $G$  is the comodule itself, not its dual.) To get  $V' \subseteq V$ , we take  $V' = V \cap I$ .

If  $g \in G'(k)$ , then  $gI \subseteq I$ , so  $gV' \subseteq V'$ . Conversely, if  $gV' \subseteq V'$ , then  $g$  sends every generator of  $I$  to another element of  $I$ , but  $g$  acts on  $k[G]$  by algebra automorphisms, so  $gI \subseteq I$  and hence  $I$  is also the kernel of the quotient map  $k[G] \rightarrow k[G'g]$ , which forces  $g \in G'$ .

Finally, once we have such  $V, V'$ , we see that the same characterization of  $G'$  holds when we replace  $V, V'$  by  $\bigwedge^d V, \bigwedge^d V'$ , respectively, where  $d = \dim(V')$ .  $\square$

**Corollary 2.4** (Chevalley–Plücker). *If  $G$  is a smooth affine algebraic group with algebraic subgroup  $G'$ , then there is a locally closed,  $G$ -equivariant embedding  $G/G' \rightarrow \mathbf{P}V$  for some representation  $V$  of  $G$ . In particular,  $G/G'$  is a quasiprojective variety.*

*Proof.* Take  $V, V'$  as in the theorem, with  $V'$  a line. Let  $G/G' \rightarrow \mathbf{P}V$  be induced by the map from  $G$  onto the orbit of  $[V']$ . The smoothness of  $G$  ensures that the latter is faithfully flat, allowing us to identify the orbit with  $G/G'$ .  $\square$

## 2.3.

An algebraic subgroup  $P \subseteq G$  is *parabolic* if and only if  $G/P$  is projective, not merely quasiprojective. As it turns out, there is a nice characterization of parabolic subgroups. For the proof of the following fixed-point theorem, see Milne Chapter 17.

**Theorem 2.5** (Borel). *If  $B$  is a connected, smooth, solvable algebraic group acting on a nonempty proper variety  $X$ , then  $X^B$  is nonempty.*

**Corollary 2.6.** *Suppose that  $G$  is a smooth affine algebraic group.*

- (1) *If  $B$  is a Borel subgroup and  $P$  a parabolic subgroup of  $G$ , then some  $G(k)$ -conjugate of  $B$  is contained in  $P$ .*
- (2) *Conversely, any algebraic subgroup of  $G$  that contains a Borel is parabolic.*

**Remark 2.7.** In the setup above,  $G$  must have *some* Borel subgroup, because  $G$  has finite dimension and  $\{1\}$  is connected, smooth, and solvable.

*Proof.* (1): By Borel's theorem, the action of  $B$  by left multiplication on  $G/B$  must have a fixed point  $gP$ , in which case  $g^{-1}Bg$  is a Borel contained in  $P$ .

(2): Since the image of any proper variety is proper, it suffices to show that if  $B \subseteq G$  is a Borel, then  $G/B$  is proper. We induct on the dimension of  $G$ . Pick a faithful representation  $V$  of  $G$ . The action of  $G$  on  $\mathbf{P}V$  must have a closed orbit. The stabilizer of any  $k$ -point of this orbit is a parabolic subgroup  $P \subseteq G$ . By (1), it contains some conjugate of  $B$ , and without loss of generality, we may replace  $B$  with this conjugate. Two cases: Either  $P$  is smaller than  $G$ , in which case  $P/B$  is proper by the inductive hypothesis, and hence  $G/B$  is proper, or else  $P = G$ , in which case  $V^G$  contains a line, and we can replace  $V$  with  $V/(V^G)$  until we either reach  $\{0\}$  or reduce to the previous case.  $\square$

**Corollary 2.8.** *Any two Borels in a smooth affine algebraic group are conjugate.*

**Example 2.9.** Any Borel of  $\mathrm{GL}_n$  is conjugate to the subgroup of upper-triangular matrices. Similarly, any parabolic of  $\mathrm{GL}_n$  is conjugate to some subgroup of *block* upper-triangular matrices.

Recall that by Milne Thm. 16.27, any two maximal tori in a connected, solvable, affine algebraic group are conjugate. This fact combined with Corollary 2.8 proves the Cartan–Lie–Kolchin theorem stated last time.

Another way to state Corollary 2.8 is: The conjugation action of  $G(k)$  on the set of Borel subgroups of  $G$  is transitive. Milne Thm. 17.48 shows that the stabilizer of any Borel is itself:

**Theorem 2.10.** *If  $B$  is any Borel subgroup of a connected, smooth, affine algebraic group  $G$ , then  $N_G(B) = B$ . Thus the map  $gB \mapsto gBg^{-1}$  is a bijection from  $(G/B)(k)$  to the set of Borel subgroups of  $G$ .*

When we regard  $G/B$  as the variety of Borel subgroups of  $G$ , we will call it the *flag variety* and denote it by  $\mathcal{B}$ . Indeed, for  $G = \mathrm{GL}_n$ , the above theorem follows from Corollary 2.8 together with the fact that if  $B$  is the stabilizer of a flag  $\vec{V}$ , then  $gBg^{-1}$  is the stabilizer of  $g \cdot \vec{V}$ .

## 2.4.

The orbit decomposition of  $G/B$  under the left action of  $B$  gives rise, on  $k$ -points, to the Bruhat decomposition that we discussed at the start. Note that we have not yet proven the precise decomposition for general  $G$ . We can sketch the gist modulo the following result. For  $G = \mathrm{GL}_n$ , it is a byproduct of the argument that proves Jordan–Hölder, via the flag interpretation of  $\mathcal{B}(k)$ .<sup>1</sup>

**Theorem 2.11.** *If  $G$  is reductive, then any two Borel subgroups of  $G$  contain a common maximal torus of  $G$ .*

*Sketch of Bruhat decomposition.* We exhibit a map from  $B(k) \backslash G(k) / B(k)$  to the Weyl group  $W = W(G, T)$ . For any  $g \in G(k)$ , pick a maximal torus  $S \subseteq B \cap gBg^{-1}$ . By Cartan–Lie–Kolchin, we can write

$$S = bTb^{-1} = (gb'g^{-1})(gTg^{-1})(gb'g^{-1})^{-1}$$

for some  $b, b' \in B(k)$ . But then  $b^{-1}gb'$  normalizes  $T$ , so we obtain an element  $[b^{-1}gb'] \in W$ . One has to check that this element only depends on  $BgB$ .  $\square$

## 2.5.

Henceforth,  $k = \bar{\mathbf{F}}_q$ . In what follows, recall that we often write  $X^F$  in place of  $X^F(k)$  when  $F$  is a (relative) Frobenius map on  $X$ .

Any Frobenius map  $F : G \rightarrow G$  that respects the group law and stabilizes an algebraic subgroup  $H \subseteq G$  induces an analogous map  $F : G/H \rightarrow G/H$ . The identification  $(G/H)(k) = G(k)/H(k)$  induces an identification

$$(G/H)^F \{F\text{-stable orbits of } H(k) \text{ on } G(k)\}.$$

The action of  $G(k)$  on  $(G/H)(k)$  restricts to an action of  $G^F$  on  $(G/H)^F$ .

Consider the *standard Frobenius map*  $F : \mathrm{GL}_n \rightarrow \mathrm{GL}_n$  given by raising each matrix coordinate to the  $q$ th power, so that  $\mathrm{GL}_n^F$  is the group classically denoted  $\mathrm{GL}_n(\mathbf{F}_q)$ . Then  $F$  stabilizes  $B$  and fixes  $\dot{w}$  for all  $w$ . Hence the Bruhat decomposition of  $\mathrm{GL}_n(k)$  into double cosets of  $B(k)$  implies an analogous decomposition of  $\mathrm{GL}_n^F$  into double cosets of  $B^F$ . With more work,<sup>2</sup> one can further show that  $((B\dot{w}B)/B)^F = (B^F\dot{w}B^F)/B^F$ , and hence,  $(\mathrm{GL}_n/B)^F = \mathrm{GL}_n^F/B^F$ . What happens for general  $G, B, F$ ?

It turns out that the connectedness of  $B$  ensures that  $(G/B)^F = G^F/B^F$  holds for any  $F$ -stable Borel  $B$ . On Problem Set 1, you will use the theorem below to deduce (a version of) the first corollary following it:

<sup>1</sup>Here I borrow from the answers to <https://mathoverflow.net/q/15438>.

<sup>2</sup>... than I originally thought was necessary, during the lecture...

**Theorem 2.12** (Lang). *Let  $H$  be a connected, smooth algebraic group over  $k$  and  $F : H \rightarrow H$  the Frobenius map for some  $\mathbf{F}_q$ -form. Then the Lang map*

$$h \mapsto h^{-1}F(h) : H \rightarrow H$$

*is surjective.*

The proof of Lang's theorem is given on Wikipedia. The key idea is to calculate the induced map on Lie algebras, using the fact that the differential of  $F$  vanishes to show bijectivity.

*Remark 2.13.* Note that the Lang map is finite étale, and its fiber over the identity is precisely  $H^F$ . For this reason, one can think of the theorem as presenting  $H$  as an  $H^F$ -principal bundle over itself in the étale topology. This leads to bizarre topological conclusions for, say,  $H = \mathbf{G}_a$  and  $F(x) = x^q$ .

*Remark 2.14.* In the affine case, Steinberg generalized Lang's theorem from Frobenius maps to any surjective map  $F$  with finitely many fixed points. So I sometimes speak of the Lang–Steinberg theorem even where it is overkill.

**Corollary 2.15.** *Let  $G$  be a connected, smooth algebraic group over  $k$  with a Frobenius map  $F : G \rightarrow G$ . Let  $\mathcal{X}$  be a set with a  $G(k)$ -action and a map  $f : \mathcal{X} \rightarrow \mathcal{X}$  such that  $f(g \cdot x) = F(g) \cdot f(x)$  for all  $g \in G(k)$  and  $x \in \mathcal{X}$ .<sup>3</sup> Then:*

- (1) *Every  $f$ -stable  $G(k)$ -orbit on  $\mathcal{X}$  contains an  $f$ -fixed point.*
- (2) *If  $\mathcal{X} = G(k)/H(k)$  for some  $F$ -stable  $H \subseteq G$ , and  $f$  is induced by  $F$ , then  $\mathcal{X}^f = G^F/H^F$ .*

**Corollary 2.16.** *A connected, smooth affine algebraic group with Frobenius map  $F$  always contains an  $F$ -stable Borel pair. In particular, any  $F$ -stable Borel contains an  $F$ -stable maximal torus.*

*Proof.* Take  $\mathcal{X}$  to be the set of all Borel pairs, and  $f : \mathcal{X} \rightarrow \mathcal{X}$  to be defined by  $f(B, T) = (F(B), F(T))$ . Now Cartan–Lie–Kolchin implies the first statement. Replacing the ambient group with a given  $F$ -stable Borel, we deduce the second statement.  $\square$

However, a given  $F$ -stable maximal torus need not be contained in an  $F$ -stable Borel. If it is contained in such a Borel, then it is called  *$F$ -maximally split*.

**Example 2.17.** Let  $F$  be the standard Frobenius map on  $\mathrm{GL}_2$ . Then the diagonal torus of  $\mathrm{GL}_2$  is  $F$ -maximally split. By contrast, the  $\mathbf{F}_q$ -form of  $\mathbf{G}_m$  that we denoted  $\mathrm{U}(1)$  last time produces, under base change, a different  $F$ -stable maximal torus  $T \subseteq \mathrm{GL}_2$ . If  $T$  were contained in an  $F$ -stable Borel, then that Borel would have an  $\mathbf{F}_q$ -form containing  $\mathrm{U}(1)$ , which we can rule out by computation.

<sup>3</sup>This setup generalizes the notion of compatible Frobenius maps defined earlier.

**Corollary 2.18.** *Any two  $F$ -stable Borel subgroups of a connected, smooth affine algebraic group  $G$  are conjugate under  $G^F$ , not just under  $G(k)$ .*

*Proof.* Pick an  $F$ -stable Borel  $B$ . The isomorphism  $gB \mapsto gBg^{-1} : G/B \rightarrow B$  is  $F$ -equivariant. Now use  $(G/B)^F = G^F/B^F$ .  $\square$

*Remark 2.19.* In the setting of a more general field  $k = \bar{K}$ , a  $K$ -form of an algebraic group  $G$  is called *quasi-split* if and only if there is a Borel subgroup of  $G$  that descends to the  $K$ -form. In this language, the last result essentially says that  $\mathbf{F}_q$ -forms are always quasi-split.

2.6.

We now focus on the setting where  $G$  is a reductive algebraic group over  $\bar{\mathbf{F}}_q$  with Frobenius map  $F$ , and  $(B, T)$  is an  $F$ -stable Borel pair.

Observe that if  $H \subseteq G$  is any  $F$ -stable algebraic subgroup and  $F$  is surjective on  $H(k)$ , then  $N_G(H)$  is  $F$ -stable. In particular, the  $F$ -action on  $N_G(T)$  descends to  $W$ , and the Bruhat decomposition of  $G(k)$  restricts to

$$G^F = \coprod_{w \in W^F} B^F w B^F.$$

By Corollary 2.15,  $W^F = N_{G^F}(T^F)/T^F$ . We say that  $(G, T)$  is *split* under  $F$  if and only if  $F$  acts trivially on  $W$ .

As usual, write  $U = [B, B]$ . Since  $B, T$  are  $F$ -stable, so is  $U$ . The  $G^F$ -action on  $G^F/U^F$  defines a representation of  $G^F$  on

$$I = \text{Ind}_{U^F}^{G^F}(1) := \{\mathbf{C}\text{-valued functions on the finite set } G^F/U^F\}.$$

The  $G^F$ -stable summands of  $I$  are called the *principal series representations* of  $G^F$ . Some standard theory shows that  $I \simeq \bigoplus_{\chi} I_{\chi}$  as a representation, where the sum runs over characters  $\chi : B^F \rightarrow \mathbf{C}^{\times}$  that factor through  $T^F \simeq B^F/U^F$ , and

$$I_{\chi} = \text{Ind}_{B^F}^{G^F}(\chi)$$

for all  $\chi$ . To determine the irreducible summands of  $I$  and  $I_{\chi}$ , we should analyze  $\text{End}_{G^F}(I)$  and  $\text{End}_{G^F}(I_{\chi})$ .

Here is a very general principle. Suppose that  $\Gamma$  is a finite group and  $\Xi$  a finite set with a  $\Gamma$ -action. Let  $\mathbf{C}\Xi$  be the representation of  $\Gamma$  formed by the  $\mathbf{C}$ -valued functions on  $\Xi$  under  $[g \cdot f](-) = f(g^{-1} \cdot -)$ . Let  $\Gamma$  act on  $\Xi \times \Xi$  diagonally, and endow  $\mathbf{C}(\Xi \times \Xi)$  with the *convolution* product

$$(f_1 * f_2)(x, y) = \sum_{z \in \Xi} f_1(x, z) f_2(z, y).$$

Note that  $\mathbf{C}(\Xi \times \Xi)^{\Gamma}$  forms a subalgebra of  $\mathbf{C}(\Xi \times \Xi)$ .

**Proposition 2.20.** *There is an isomorphism of  $\mathbf{C}$ -algebras*

$$\mathbf{C}(\Xi \times \Xi)^\Gamma \xrightarrow{\sim} \text{End}_\Gamma(\mathbf{C}\Xi),$$

$$1_O \mapsto \left( 1_x \mapsto \sum_{\substack{y \in \Xi \\ (x,y) \in O}} 1_y \right),$$

where  $O$  denotes any  $\Gamma$ -orbit of  $\Xi \times \Xi$ , and  $1_O, 1_x$  refer to indicator functions on  $O, \{x\}$ .

Above, the image of  $1_O$  in  $\text{End}_\Gamma(\mathbf{C}\Xi)$  is called the *Hecke operator* for  $O$ . In the case where  $\Xi = \Gamma/H$  for some subgroup  $H \subseteq \Gamma$ , we have a further bijection

$$\Gamma \backslash (\Gamma/H \times \Gamma/H) \xrightarrow{\sim} H \backslash \Gamma/H,$$

$$(yH, xH) \mapsto Hy^{-1}xH,$$

which induces an isomorphism of vector spaces  $\mathbf{C}(\Xi \times \Xi)^\Gamma \simeq (\mathbf{C}\Gamma)^{H \times H}$ . Taking  $\Gamma = G^F$  and  $H = U^F, B^F$ , we deduce:

**Corollary 2.21.** *As a vector space,  $\text{End}_{G^F}(I)$ , resp.  $\text{End}_{G^F}(I(1))$ , has a basis indexed by  $U^F \backslash G^F / U^F$ , resp.  $B^F \backslash G^F / B^F$ . In particular, the latter is also indexed by  $W^F$ .*

The arguments above can be pushed further to analyze  $\text{Hom}_{G^F}(I_\chi, I_\psi)$  for any  $\chi, \psi$ . Returning to the abstract setup, let  $A, B$  be subgroups of  $\Gamma$ , and let  $\alpha$ , resp.  $\beta$ , be an arbitrary  $\mathbf{C}$ -valued character of  $A$ , resp.  $B$ . For all  $g \in \Gamma$ , we set  $A^g = g^{-1}Ag$ , so that  $\alpha^g(-) := \alpha(g(-)g^{-1})$  is a  $\mathbf{C}$ -valued character of  $A^g$ .

The following is proved via Frobenius reciprocity in most texts on character theory, such as Serre's book:

**Theorem 2.22** (Mackey). *Above, there is an isomorphism of vector spaces*

$$\text{Hom}_\Gamma(\text{Ind}_A^\Gamma(\alpha), \text{Ind}_B^\Gamma(\beta)) \simeq \bigoplus_{g \in A \backslash \Gamma / B} \text{Hom}_{A^g \cap B}(\alpha^g, \beta).$$

**Corollary 2.23.** *We have*

$$\text{Hom}_{G^F}(I_\chi, I_\psi) \simeq \bigoplus_{w \in W^F} \text{Hom}_{(B^F)^w \cap B^F}(\chi^w, \psi).$$

*In particular,  $I_\chi$  is irreducible if and only if  $\chi^w \neq \chi$  as characters of  $T^F$  for all  $w \neq e$ .*