recall Heine–Borel: for A sub R,

A is compact iff A is closed and bounded

last time:

1) [a, b] is always compact
2) closed subspaces of cpct spaces are cpct

proves the “if” direction:

suppose A closed and bounded
A sub [a, b] for some a < b
R – A open, so [a, b] – A open in [a, b]
so A closed in [a, b]
so A compact

[what about “only if”?]

[eas(ier) to show that cpct in R implies bounded]

<u>Thm</u> if X is Hausdorff and A sub X compact,
then A is closed

<u>Lem</u> if X is Hausdorff, A sub X is compact,
and x in X – A,
then there exist disjoint open U, V sub X
s.t. A sub U and x in V

<u>Pf</u> for all a in A, pick disjoint open U_a, V_a
s.t. a in U_a and x in V_a
{U_a}_{a in A} is a cover in X of A
pick a finite subcover {U_a}_{a in I}
U = bigcup_{a in I} U_a
V = bigcap_{a in I} V_a

Pf of Thm for all x in $X - A$,
pick disj open U_x, V_x
s.t. $A \subset U_x$ and $x \in V_x$

then $X - A = \bigcup_{x \in X - A} V_x$
so $X - A$ is open

[Cor if X is compact, Y is Hausdorff,
and $f : X \to Y$ is a cts bijection,
then f is a homeomorphism

Ex the cts bijection $f : [0, 1) \to S^1$ def by
$f(t) = (\cos(2\pi t), \sin(2\pi t))$
is not a homeo: but $[0, 1)$ is not compact

Pf want to show f^{-1} cts
know $(f^{-1})^{-1}(U) = f(U)$

so enough to show:
if $U \subset X$ is open, then $f(U)$ is open
[that is, f is an open map]

since f is bijective, $f(X - A) = Y - f(A)$ for any A
so enough to show:
if $A \subset X$ is closed, then $f(A)$ is closed

indeed:
- X compact + A closed implies A compact
- A compact implies $f(A)$ compact
- Y Hausdorff + $f(A)$ compact implies A closed]

(Munkres §51) let X be a top space

new goal: study "holes" in X

Df a loop in X is
a path $\gamma : [0, 1]$ to X s.t. $\gamma(0) = \gamma(1)$

in this case, we say $\gamma(0)$ is the basepoint of γ

idea: fix x in X
compose loops based at x using "pasting"

$\beta, \gamma : [0, 1]$ to X yield $\beta * \gamma : [0, 1]$ to X def by

$$(\beta * \gamma)(s) = \begin{cases} \beta(2s) & \text{if } s \leq 1/2 \\ \gamma(2s - 1) & \text{if } s \geq 1/2 \end{cases}$$

[left-to-right composition!] [draw picture]

problem: all of the group axioms fail
- no associativity
- no id elt
- no inverses

idea: id elt should be the constant loop at x
[so, consider paths only up to some equiv relation]

Df let f, f': S to X be cts

a homotopy from f to f' is cts $h : S \times [0, 1]$ to X

s.t. for all s in S, $h(s, 0) = f(s)$
$h(s, 1) = f'(s)$

for paths, want a more restrictive notion:

<u>Df</u> let $\gamma, \gamma' : [0, 1] \to X$ be paths
s.t. $\gamma(0) = \gamma'(0)$ and $\gamma(1) = \gamma'(1)$

a <u>path homotopy</u> from γ to γ' is a homotopy
$h : [0, 1] \times [0, 1] \to X$ from γ to γ'

s.t. for all t in $[0, 1]$,
$$h(0, t) = \gamma(0) = \gamma'(0)$$
$$h(1, t) = \gamma(1) = \gamma'(1)$$

[draw picture]