PROBLEMS ON SYMPLECTIC REFLECTION ALGEBRAS

7. Hochschild cohomology and deformations

Exercise 7.1. *Let A_0 be an algebra and $A_1 = A_0 \oplus P \otimes A_0$ its first order deformation with product defined by $\mu(a,b) = ab + \mu_1(a,b)$ for $a, b \in A_0$, where $\mu_1 \in P \otimes C^2(A,A)$. Show that the product μ is associative iff $d\mu_1 = 0$.*

Exercise 7.2. *Let $A_1, A_1' = A_0 \oplus P \otimes A_0$ be two 1st order deformations of A_0 with products μ, μ'. Let $\sigma : A_1 \to A_1'$ be an $S(P)$-module isomorphism such that $\sigma(a) = a + \sigma_1(a)$ for $a \in A_0$, where $\sigma_1 \in P \otimes C^1(A,A)$. Show that σ is an algebra homomorphism iff $\mu' = \mu + d\sigma$.*

Exercise 7.3. *Let A_k be a deformation of A_0 over $S(P)/(P^{k+1})$, $A_k = \bigoplus_{i=0}^k S^i(P) \otimes A_0$. Let $\mu(a,b) = ab + \sum_{i=1}^k \mu_k(a,b)$ be the product. Consider the equation*

$$(1)\quad \mu_k(a,b)c + \mu_k(ab,c) - \mu_k(a,bc) - a\mu_k(b,c) = \sum_{i=1}^{k-1} \left(\mu_i(a, \mu_{k-i}(b,c)) - \mu_i(\mu_{k-i}(a,b),c)\right).$$

Show that the r.h.s. is a cocycle if $k = 2$.

Problem 7.1. *Show that the r.h.s. of (1) is a cocycle for any k.*

Below A_0 is a $\mathbb{Z}_{\geqslant 0}$-graded algebra.

Exercise 7.4. *Consider the inverse system of graded deformations A_k of A_0 over $S(P)/(P^{k+1})$ and set $A^i := \varprojlim_k A_k^i$ and $A := \bigoplus_{i=0}^\infty A^i$. Equip A with a graded $S(P)$-algebra structure so that $A \twoheadrightarrow A_k$ for all k. Further, check that A is a free graded $S(P)$-module.*

Exercise 7.5. *Show that if $\mathrm{HH}^1(A_0)^i = 0$ for $i \leqslant -2$, then an equivalence in the proposition on the universal property of A_{un} is unique.*

Problem 7.2. *Let A be a graded deformation of A_0 over $S(P)$. Describe the set of all auto-equivalences of A in terms of the groups $\mathrm{HH}^1(A_0)^i$ with $i \leqslant -2$.*

Exercise 7.6. *Use the Koszul resolution to check that $\mathrm{HH}^\bullet(\mathbb{C}[x], \mathbb{C}[x]\gamma)$ is the cohomology of the complex $\mathbb{C}[x]\gamma \to \mathbb{C}[x]\gamma$, where the map is a left $\mathbb{C}[x]$-module homomorphism given by $1\gamma \mapsto (x - \gamma(x))\gamma$, where we shift the grading on the target module by 1 (so that 1 there has degree -1).*

Exercise 7.7. *Let Γ be a finite subgroup of $\mathrm{Sp}(V)$. Show that $\gamma \in \Gamma$ has even number of eigenvalues different from 1. Deduce that $\mathrm{HH}^2(S(V), S(V)\gamma)^i = 0$ for $i < -2$ and $\mathrm{HH}^3(S(V), S(V)\gamma)^i = 0$ for $i < -3$.*

Exercise 7.8.
1. *Suppose that γ is not a symplectic reflection. Prove that any element in $\mathrm{HH}^\bullet(S(V), S(V)\gamma)$ has homological degree at least 4.*
2. *Let γ be a symplectic reflection. Show that $\mathrm{HH}^2(S(V), S(V)\gamma)^{-2}$ is one dimensional.*
3. *Let $S = S_1 \sqcup S_2 \sqcup \ldots \sqcup S_m$. Show that $\dim \left(\bigoplus_{\gamma \in S_i} \mathrm{HH}^2(S(V), S(V)\gamma)^{-2}\right)^\Gamma = 1$.*
4. *Show that as a Γ-module, $\mathrm{HH}^2(S(V), S(V))^{-2}$ is $\bigwedge^2 V$. Deduce that*
$$\dim \left(\mathrm{HH}^2(S(V), S(V))^{-2}\right)^\Gamma = 1.$$