MATH 250: TOPOLOGY I PROBLEM SET #3

FALL 2025

Due Wednesday, October 1. Please attempt all of the problems. Six of them will be graded. You may consult books, papers, and websites as long as you cite all sources and write up your solutions in your own words. **Last update: 9/23**.

Problem 1 (Munkres 128, #4(1))**.** Consider the product, uniform, and box topologies on $\mathbf{R}^\omega = \mathbf{R} \times \mathbf{R} \times \mathbf{R} \times \dots$ (where each factor of $\mathbf{R}$ is analytic). In which topologies are the following functions continuous?

$$\begin{aligned} f(t) &= (t, 2t, 3t, \dots), \\ g(t) &= (t, t, t, \dots), \\ h(t) &= (t, \tfrac{1}{2}t, \tfrac{1}{3}t, \dots). \end{aligned}$$

Problem 2 (Munkres 128, #4(2))**.** Same setup as Problem 1. In which topologies do the following sequences converge?

$$\begin{aligned} (w_i)_i \text{ where } w_1 &= (1, 1, 1, 1, \dots), & (x_i)_i \text{ where } x_1 &= (1, 1, 1, 1, \dots), \\ w_2 &= (0, 2, 2, 2, \dots), & x_2 &= (0, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \dots) \\ w_3 &= (0, 0, 3, 3, \dots), & x_3 &= (0, 0, \tfrac{1}{3}, \tfrac{1}{3}, \dots), \\ &\dots & &\dots \end{aligned}$$

$$\begin{aligned} (y_i)_i \text{ where } y_1 &= (1, 0, 0, 0, \dots), & (z_i)_i \text{ where } z_1 &= (1, 1, 0, 0, \dots), \\ y_2 &= (\tfrac{1}{2}, \tfrac{1}{2}, 0, 0, \dots), & z_2 &= (\tfrac{1}{2}, \tfrac{1}{2}, 0, 0, \dots) \\ y_3 &= (\tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}, 0, \dots), & z_3 &= (\tfrac{1}{3}, \tfrac{1}{3}, 0, 0, \dots), \\ &\dots & &\dots \end{aligned}$$

Problem 3 (Munkres 118, #7)**.** Let $\mathbf{R}^\infty \subseteq \mathbf{R}^\omega$ be the subset of sequences $(a_i)_{i>0}$ such that $a_i \neq 0$ for only finitely many i. What is the closure of $\mathbf{R}^\infty$...

(1) ...in the box topology on $\mathbf{R}^\omega$?
(2) ...in the product topology on $\mathbf{R}^\omega$?

Problem 4 (Munkres 118, #8)**.** Fix sequences $(a_1, a_2, \dots), (b_1, b_2, \dots) \in \mathbf{R}^\omega$ such that $a_i > 0$ for all i. Let $h : \mathbf{R}^\omega \to \mathbf{R}^\omega$ be defined by

$$h(x_1, x_2, \dots) = (a_1 x_1 + b_1, a_2 x_2 + b_2, \dots)$$

(1) Show that in the product topology, h is a self-homeomorphism of $\mathbf{R}^\omega$.
(2) What happens in the box topology?

Problem 5 (Munkres 92, #3)**.** Endow $[-1, 1]$ with the analytic topology: *i.e.*, the subspace topology it inherits from analytic $\mathbf{R}$. Determine which of the following sets are open in $[-1, 1]$, and which are open in $\mathbf{R}$.

$$A = \{x \mid \tfrac{1}{2} < |x| < 1\}, \qquad B = \{x \mid \tfrac{1}{2} < |x| \leq 1\},$$
$$C = \{x \mid \tfrac{1}{2} \leq |x| < 1\}, \qquad D = \{x \mid \tfrac{1}{2} \leq |x| \leq 1\},$$
$$E = \{x \mid 0 < |x| < 1 \text{ and } \tfrac{1}{x} \notin \mathbf{Z}_+]\}.$$

Problem 6 (Munkres 92, #6)**.** Show that the countable collection

$$\{(a, b) \times (c, d) \mid a < b \text{ and } c < d \text{ and } a, b, c, d \text{ are rational}\}$$

is a basis for the analytic topology on $\mathbf{R}^2$. You may assume Problem 5 from Problem Set 1. *Hint:* How do the analytic and product topologies on $\mathbf{R}^2$ compare?

Problem 7 (Munkres 101, #11–13)**.** Show that:

(1) A product of two Hausdorff spaces is Hausdorff (in the product topology).
(2) A subspace of a Hausdorff space is Hausdorff (in the subspace topology).
(3) X is Hausdorff if and only if its *diagonal* $\Delta_X = \{(x, x) \mid x \in X\}$ is closed in (the product topology on) $X \times X$.

Problem 8. Let $s : A \to X$ and $r : X \to A$ be maps of sets such that $r \circ s$ is the identity map on A.

(1) Show that s must be injective and r must be surjective.
(2) Now suppose that X and A are endowed with topologies. Show that if s and r are both continuous, then the topology on A must be the quotient topology induced by the surjective map r.

In the situation above, we say that s is a continuous *section* of r, and that r is a continuous *retraction* of s.

Problem 9. Let $X = \mathbf{R}^2 - \{0\}$, endowed with the subspace topology it inherits from analytic $\mathbf{R}^2$. Let

$$S^1 = \{(x, y) \in \mathbf{R}^2 \mid x^2 + y^2 = 1\},$$

endowed with the subspace topology it inherits from X.

(1) Give a basis for the above topology on S^1. *Hint:* If $\mathcal{B}$ is a basis for the topology on X, then $\{S^1 \cap B \mid B \in \mathcal{B}\}$ is a basis for the topology on S^1.
(2) Give a retraction of the inclusion map from S^1 into X, in the sense of Problem 8. *Hint:* Polar coordinates.

Problem 10 (Munkres 152, #2)**.** Let $(A_n)_{n=1}^{\infty}$ be a sequence of connected subspaces of X, such that $A_n \cap A_{n+1} \neq \emptyset$ for all n. Show that $\bigcup_{n=1}^{\infty} A_n$ is connected.

Problem 11 (Munkres 152, #11)**.** Let $p : X \to Y$ be a quotient map. Show that if Y is connected and each subspace $p^{-1}(y) \subseteq X$ is connected, then X is connected.

Problem 12 (Munkres 152, #5)**.** We say that X is *totally disconnected* if and only if its only nonempty connected subspaces are one-point sets.

(1) Show that if X is discrete, then X is totally disconnected.
(2) Show that the set of rational numbers $\mathbf{Q}$, as a subspace of (analytic) $\mathbf{R}$, is totally disconnected, but not discrete.