from the HW:

if {W_i}_{i in I} is a collection of linear sub.'s of V,
    with I possibly infinite,
then their sum is defined to be

sum_{i in I} W_i = {sum_{i in J} w_i | J sub I finite,
    w_i in W_i for all i}

it is a direct sum iff every elt has a unique expr.
    sum_i w_i with w_i in W_i for all i

Prop   sum_{i in I} W_i is the minimal lin. sub.
    containing W_i for all i

(Axler §2A)   {v_i}_{i in I} any set of vectors in V

Df   the span of {v_i}_i is (simultaneously)

1) {sum_{i in J} a_iv_i | J sub I finite,
    a_i in F for all i}
2) sum_{i \in I} Fv_i, where Fv_i = {av_i | a in F}
3) the minimal linear subspace of V containing v_i for all i

i.e. 1), 2), 3) are all the same
    and {v_i}_i is said to span [verb] it

the vector sum_{i in J} a_iv_i is said to be
a linear combination of the v_i's with coeff's a_i

Ex in $F[x]$ = {set of polynomials in x over F}:

$\{x^k \mid k \geq 0\} = \{1, x, x^2, x^3, \ldots\}$ spans $F[x]$

[why? every polynomial is a sum of monomials]

Ex let $\mathbf{N} = \{1, 2, 3, \ldots\}$
in $F^{\mathbf{N}}$ = {functions from N into F}:
let $e_i : \mathbf{N}$ to F be the function
$e_i(i) = 1$,
$e_i(j) = 0$ for $j \neq i$

$\{e_i \mid i \text{ in } \mathbf{N}\}$ does not span $F^{\mathbf{N}}$

[why?] consider the function f s.t. $f(i) = 1$ for all i

Df $\{v_i\}_i$ is said to be
a linearly independent set of vectors iff
either:

I) for any finite set $\{a_i\}_{i \text{ in } J}$ of elements of F,

$\sum_{i \text{ in } J} a_i v_i = \mathbf{0}$ implies ($a_i = 0$ for all i)

II) $\sum_i F v_i$ is a direct sum

Lem I) and II) are indeed equivalent

Pf

suppose II)
then $\sum_i 0 v_i = \mathbf{0}$ is the unique expr for $\mathbf{0}$

suppose I)
suppose v in sum_i Fv_i has two expr.'s

sum_{i in J} a_iv_i = v = sum_{i in J'} a'_iv_i

let J'' = J cup J'

let a'_i = 0 for i in J – J'

let a_i = 0 for i in J' – J

then sum_{i in J''} (a_i - a'_i)v_i = **0**

so a_i – a'_i = 0 for all i

we say {v_i}_i is a linearly dependent set iff
it is not linearly indepndent

we say the eqn sum_{i in J} a_iv_i = 0 is a linear dependence for the v_i iff some a_i is nonzero

Lem {v_i}_{i in I} is linearly dependent
iff
there exist J sub I finite and
i notin J

s.t. v_i is a linear combo of the v_j with j in J

Pf exercise

[most striking thm thus far:]

Thm (Steinitz Exchange) if
{v_1, …, v_k} is a lin. independent set in V,
{e_1, …, e_n} spans V

then k ≤ n

[crucially, both sets of vectors are finite]

Cor if V is spanned by n vectors,
then any set with $> n$ vectors
has some linear dependence

Cor if there is a linearly independent set
of k vectors in V,
then any set with $< k$ vectors
cannot span V

Pf of Thm let $S_0 = \{e_1, \ldots, e_n\}$

will prove that for $\ell = 1, \ldots, k$,
we can construct S_ℓ from $S_{\ell - 1}$ s.t.

1) S_ℓ still spans V
2) S_ℓ has one more v_i and one fewer e_j than $S_{\ell - 1}$

thus $\ell \leq n$ at each step [and $k \leq n$ at the last step]

WLOG reindex the v_i's and e_j's s.t.
$S_{\ell - 1} = \{v_1, \ldots, v_{\ell - 1}, e_\ell, \ldots, e_n\}$
since $S_{\ell - 1}$ spans V,
$$v_\ell = \sum_{i = 1}^{\ell - 1} a_i v_i + \sum_{j = \ell}^{n} b_j e_j$$
with some coeff nonzero
if $b_j = 0$ for all j, then $\{v_i\}_i$ lin. dep.
so we can pick j s.t. $b_j \neq 0$
so $e_j = (1/b_j)(v_\ell - \text{other stuff})$

build S_ℓ by appending v_ℓ and removing e_j □

Df a basis for V is a set of vectors $\{v_i\}_i$

s.t. 1) $\{v_i\}_i$ spans V
2) $\{v_i\}_i$ is a linearly independent set

Cor if V has a finite basis of size r,
then any basis for V has size r

Pf if $\{e_1, \ldots, e_r\}$ is a basis,
and $\{f_1, \ldots, f_s\}$ is another:

$r \leq s$ because
$\{e_i\}_i$ is lin. indep. and $\{f_j\}_j$ is spanning
$s \geq r$ because
$\{f_i\}_i$ is lin. indep. and $\{e_j\}_j$ is spanning

Df if V has a finite basis,
then we define the dimension of V to be

$\dim(V)$ = size of any basis for V

else we say V is infinite-dimensional

"the Good, the Bad, and the Ugly"

V has finite dimension
V has infinite dimension, yet has an (infinite) basis
e.g., $F[x]$
V has infinite dimension and no basis
e.g., $F^{\mathbf{N}}$