Warmup what's a basis for Mag(F)?

$$X = \begin{matrix} 1 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & -1 \end{matrix} \qquad Y = \begin{matrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{matrix}$$

$$Z = \begin{matrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{matrix}$$

Mag(F) = {aX + bY + cZ | a, b, c in F}
[works over any field F, by the way]

recall: we also view Mag(F) as a subspace of F^9
in this sense, (a, b, c) mapsto aX + bY + cZ
"embeds" F^3 into F^9

(Axler §3A) let V, W be vector spaces / F

Df an F-linear map from V to W is
a map/function T : V to W s.t.

1) T(v + v′) = T(v) + T(v′) for all v, v′ in V
2) T(a • v) = a • T(v) for all v in V and a in F

i.e., T "respects"/"preserves" the operations +, •

Lem if T : V to W linear, then T(**0_V**) = **0_W**

Pf T(**0_V**) = T(**0_V** + **0_V**)
= T(**0_V**) + T(**0_V**)

subtracting from both sides, **0_W** = T(**0_V**)

Ex if T : F^3 to W is a linear map, then

T((a, b, c))
= T((a, 0, 0) + (0, b, 0) + (0, 0, c)) [next?]
= T((a, 0, 0)) + T((0, b, 0)) + T((0, 0, c)) [next?]
= a • T((1, 0, 0)) + b • T((0, b, 0)) + c • T((0, 0, 1))
thus any linear map T : F^3 to W is determined by
the values T((1, 0, 0)), T((0, 1, 0)), T((0, 0, 1))
[note: {(1, 0, 0), (0, 1, 0), (0, 0, 1)} is a basis here]

“The Slogan”

if we know some basis for V,
then to give a linear map from V to W is to decide
where it sends the elements of the basis—
and any values will do

Ex any linear map F^2 to F^3 is det by
3 × 2 = 6 numbers:
T((1, 0)) = (x, y, z),
T((0, 1)) = (x′, y′, z′)
and any six numbers will work

Ex any linear map F^n to F^m is det by
mn numbers

Ex any linear map T : F[x] to W is det by
the set {T(1), T(x), T(x^2), …}

Ex take F = R
an R-linear map T : C to W is det by
the set {T(1), T(i)}
or, the set {T(1 + i), T(1 – i)} …

Ex let $V = R^R$ = {functions $f : R$ to R}
let U = {differentiable functions f}
let $D(f) = df/dx$

D is a well-defined map from U to V [why into V?]

also $D(f + g) = d(f + g)/dx = df/dx + dg/dx$
$= D(f) + D(g)$
$D(a \cdot f) = d(a \cdot f)/dx = a\, df/dx = a \cdot D(f)$

so D is R-linear

Review a map of sets $T : X$ to Y is:

1) injective iff ($T(x) = T(x')$ forces $x = x'$)
2) surjective iff (for all y in Y, some x s.t. $T(x) = y$)

know that T is bijective (both inj and surj) iff
it has a two-sided inverse $S : Y$ to X:
i.e., $S(T(x)) = x$ and $T(S(y)) = y$

is D injective? [no: D(any constant) = 0]
is D surjective? [no: D(f) must be locally integable]
[quantify lack of injectivity/surjectivity in general?]

(Axler §3B)

Df suppose $T : V$ to W is linear
1) the kernel or nullspace of T is
$\ker(T) = \{v \text{ in } V \mid T(v) = \mathbf{0\_W}\}$
2) the image or range of T is
$\mathrm{im}(T) = \{T(v) \text{ in } W \mid v \text{ in } V\}$

Rem some people say "range" to mean W,
not im(T) [why I prefer "image"]

Prop for a linear map T : V to W
1) T is injective iff ker(T) = {**0_V**}
2) T is surjective (onto W) iff im(T) = W

Pf 2) tautological

1) suppose ker(T) = {**0_V**}
suppose T(v) = T(v′)
then T(v – v′) = **0_W**
then v – v′ in ker(T), so v – v′ = **0_V**

conversely suppose T injective
if T(v) = **0_W**, then T(v) = T(**0_V**), so v = **0_V**

Prop 1) ker(T) is a linear subspace of V
2) im(T) is a linear subspace of W

Pf want: contain **0** and stable under +, •

observe that T(**0_V**) = **0_W**
so **0_V** in ker(T) and **0_W** in im(T)

ker(T) stable under +, • because
if v, v′ in ker(T), then
T(av + v′) = aT(v) + T(v′) = a**0_W** + **0_W**

im(T) stable under +, • because
if w, w′ in im(T), then
w = Tv and w′ = Tv′ for some v, v′ in V
aw + w′ = aT(v) + T(v′) = T(av + v′)

Ex let $T : F^3$ to F^3 be def by

$T((1, 0, 0)) = (1, 1, 1)$
$T((0, 1, 0)) = (0, 1, 2)$
$T((0, 0, 1)) = (-3, -2, -1)$

i.e., $T((a, b, c))$
$= (a, a, a) + (0, b, 2b) + (-3c, -2c, -c)$
$= (a - 3c, a + b - 2c, a + 2b - c)$

$$\ker(T) = \{(a, b, c) \mid a - 3c = 0,\ a + b - 2c = 0,\ a + 2b - c = 0\}$$

$$\operatorname{im}(T) = \{(a - 3c, a + b - 2c, a + 2b - c) \mid a, b, c\}$$

$$\ker(T) = \{(3c, b, c) \mid 3c + b - 2c = 0,\ 3c + 2b - c = 0\}$$
$$= \{(3c, b, c) \mid b + c = 0, 2c + 2b = 0\}$$
$$= \{(3c, -c, c)\} \text{ [dim 1]}$$

Moral solving a system of linear eq's is the same problem as describing a kernel explicitly

$$\operatorname{im}(T) = \operatorname{span}((1, 1, 1), (0, 1, 2), (-3, -2, -1))$$
$$= \operatorname{span}((1, 1, 1), (0, 1, 2))$$

[consider $-3(1, 1, 1) + (0, 1, 2)$]

as $\{(1, 1, 1), (0, 1, 2)\}$ is lin. indep., $\dim \operatorname{im}(T) = 2$

Thm if V is fin dim and $T : V$ to W is linear,
then $\dim V = \dim \ker(T) + \dim \operatorname{im}(T)$

Pf Sketch

pick basis $\{w_1, \ldots, w_r\}$ for $\operatorname{im}(T)$
pick $u_1, \ldots, u_r$ s.t. $T(v_i) = w_i$ for all i
let $U = \operatorname{span}(u_1, \ldots, u_r)$
pick basis $v_1, \ldots, v_k$ for $\ker(T)$
show that
- $\{u_1, \ldots, u_r\}$ is a basis for U
- $\ker(T) + U = V$
- $\ker(T) + U$ is a direct sum

by formula from last time,

$$\begin{aligned} \dim V &= \dim \ker(T) + \dim U \\ &= \dim \ker(T) + \dim \operatorname{im}(T) \end{aligned}$$