PROBLEMS ON SYMPLECTIC REFLECTION ALGEBRAS

6. SRA

Exercise 6.1. *Show that the Weyl algebra* $W(V)$ *is a filtered deformation of* $S(V)$ *(the case* $\dim V = 2$ *was considered before). Moreover, check that the Poisson bracket on* $S(V)$ *induced from* $W(V)$ *coincides with the initial bracket.*

Exercise 6.2. *Show that if* $\operatorname{gr} H_\kappa = S(V)\#\Gamma$*, then* $\kappa : \bigwedge^2 V^* \to \mathbb{C}\Gamma$ *is a* Γ*-equivariant map (where* Γ *acts on* $\mathbb{C}\Gamma$ *via the adjoint representation). Furthermore, show that if* $-1_V \in \Gamma$*, then the image of* κ *lies in* $\mathbb{C}\Gamma$*.*

Exercise 6.3. *For* $\kappa = \sum_{\gamma\in\Gamma} \kappa_\gamma \gamma$*, we have* $[\kappa(u, v), w] = \sum_{\gamma\in\Gamma} \kappa_\gamma(u, v)(\gamma(w) - w)\gamma$*.*

Exercise 6.4. *Show that* $\operatorname{im}(\gamma - 1_V) \oplus \ker(\gamma - 1_V) = V$ *for any* $\gamma \in \Gamma$*. Further, show that the summands are orthogonal with respect to* ω *and, in particular, the restrictions of* ω *to these subspaces are non-degenerate.*

Exercise 6.5. *Show that* $\omega_\gamma(u, v)(\gamma(w) - w) + \omega_\gamma(v, w)(\gamma(u) - u) + \omega_\gamma(w, u)(\gamma(v) - v) = 0$*.*

Exercise 6.6. *Show that a symplectically irreducible* Γ*-module* V *is either irreducible, or is the sum* $U \oplus U^*$*, where* U *is irreducible and not symplectic. Deduce that the space of* Γ*-invariant skew-symmetric forms on* V *is one-dimensional and so is generated by* ω*.*

Exercise 6.7. *Prove that* $S_0, \ldots, S_r$ *exhaust the conjugacy classes of symplectic reflections in* $\Gamma_n = \mathfrak{S}_n \ltimes \Gamma_1^n$*.*

Problem 6.1. *The goal of this problem is to construct a family of complex reflection groups, that includes the Weyl groups of types* B, D *(and, to some extent,* A*). Namely, fix* $n, \ell \geqslant 1$ *and a divisor* r *of* ℓ*. Consider all* $n \times n$*-matrices with the following properties: each column and each row contains a single non-zero element that is a root of* 1 *of order* ℓ*, and the product of these elements is a root of* 1 *of order* r*. Show that this is a complex reflection group. This group is denoted by* $G(\ell, r, n)$*. In particular,* $B_n = G(2, 1, n)$ *and* $D_n = G(2, 2, n)$*.*

Problem 6.2. *This problem concerns an exceptional complex reflection group,* G_4*. Take the Kleinian group* Γ *of type* E_6*. It has three two-dimensional irreducible representations,* $\mathbb{C}^2$ *and two other. Prove that the other two are dual to each other and* Γ *acts on them as a complex reflection group.*

Problem 6.3. *Show that the relations for a RCA can be written as follows. For a complex reflection* s *let* $\alpha_s \in \operatorname{im}(s - 1_{\mathfrak{h}^*}), \alpha_s^\vee \in \operatorname{im}(s - 1_{\mathfrak{h}})$ *be such that* $\langle \alpha_s, \alpha_s^\vee \rangle = 2$ *(this is motivated by the Weyl group case). Show that the relations for the RCA can be written as*

$$[x, x'] = 0, [y, y'] = 0, [y, x] = t\langle y, x\rangle - \sum_{s\in S} c_s \langle \alpha_s, y\rangle \langle \alpha_s^\vee, x\rangle s, \quad x, x' \in \mathfrak{h}^*, y, y' \in \mathfrak{h}.$$

Note that here the coefficients c_s *are not quite the same as in the presentation of an SRA. How are the coefficients related?*

Exercise 6.8. *Suppose B is a graded deformation of B_0 over A. Show that a basis in B can be obtained as follows. Let ι be any graded section of the projection $B \twoheadrightarrow B_0$. Then for a basis of B (viewed as an A-module) we can take $\iota(B_0)$.*

Exercise 6.9. *Check that $d^2 = 0$, where d is the Hochschild differential $C^\bullet(A, M) \to C^{\bullet+1}(A, M)$.*

Exercise 6.10. *Show that $\mathrm{HH}^0(A, M)$ coincides with the center of M, i.e., the space of all elements $m \in M$ such that $am = ma$. Show that Hochschild 1-cocycles are the derivations of M, i.e., the maps $A \to M$ that satisfy the Leibniz identity, while the Hochshild 1-coboundaries are inner derivations, i.e., maps $A \to M$ of the form $a \mapsto am - ma$ for $m \in M$. So $\mathrm{HH}^1(A, M)$ is the quotient of the two, the so called space of outer derivations.*