MATH 251: TOPOLOGY II

SPRING 2026 PRACTICE PROBLEMS

Contents

NOTE: All citations are to Munkres's textbook, *Topology*, 2nd Edition. When a problem statement has a proof in Munkres, try your best to find your own proof, before comparing with his.

For any integer $n > 0$, the topology on $\mathbf{R}^n$ is the analytic topology unless otherwise specified.

0. Review of Topology I

Problem 1. Show that no two of the spaces

$$(0,1), \quad (0,1], \quad [0,1]$$

are homeomorphic. *Hint:* What happens if you remove any point from $(0,1)$?

Problem 2. Suppose that $A \subseteq X$. Recall that the *interior* of A in X is

$$\mathrm{Int}_X(A) := \{x \in X \mid x \in U \text{ for some } U \text{ open in } X \text{ such that } U \subseteq A\}$$

and the *closure* of A in X is

$$\mathrm{Cl}_X(A) := \{x \in X \mid x \in K \text{ for all } K \text{ closed in } X \text{ such that } K \supseteq A\}.$$

Show the identity $\mathrm{Cl}_X(A) = X \setminus \mathrm{Int}_X(X \setminus A)$.

Problem 3. Suppose that $A \subseteq Y \subseteq X$, where Y is a subspace of X.

(1) Show that $\mathrm{Int}_Y(A) \supseteq \mathrm{Int}_X(A)$.
(2) Give an example where $\mathrm{Int}_Y(A) \neq \mathrm{Int}_X(A)$. *Hint:* You can assume $Y = A$.

Problem 4. Let X be the real line $\mathbf{R}$ in its finite complement, or *cofinite*, topology. Show that every sequence of points in X converges to every point of X simultaneously. Deduce that X is not Hausdorff.

Problem 5. Show that any metric space is Hausdorff.

Problem 6. Show that for any integer $n \geq 1$, the analytic topology on $\mathbf{R}^n$ matches the product topology on $\mathbf{R} \times \cdots \times \mathbf{R}$, where there are n factors.

Problem 7. Let $p\colon \mathbf{R} \to S^1$ be the map

$$p(t) = (\cos(2\pi t), \sin(2\pi t)).$$

For any $a, b \in \mathbf{R}$, we define the *(open) arc* $J_{a,b} \subseteq S^1$ to be

$$J_{a,b} = \{p(t) \mid a < t < b\}.$$

Show that the collection of arcs $\{J_{a,b} \mid a, b \in \mathbf{R}\}$ satisfies the definition of a basis (Munkres page 78).

Problem 8. Show that the following topologies on S^1 are all the same:

- The topology generated by the basis $\{J_{a,b} \mid a, b \in \mathbf{R}\}$ in Problem 7.
- The subspace topology that S^1 inherits from its inclusion into $\mathbf{R}^2$.
- The quotient topology that S^1 inherits from the surjective map $p\colon \mathbf{R} \to S^1$.

Problem 9. Let $f_1, f_2, f_3\colon A \to X$ be continuous maps. Suppose that φ is a homotopy from f_1 to f_2 and ψ is a homotopy from f_2 to f_3. Construct a homotopy from f_1 to f_3 explicitly in terms of φ and ψ.

Problem 10. In the setup of Problem 9, suppose that

$$\begin{aligned} A &= [0,1], \\ f_1(0) &= f_2(0) = f_3(0), \\ f_1(1) &= f_2(1) = f_3(1). \end{aligned}$$

Show that in this case, if φ and ψ are path homotopies, then we can choose the solution of (1) to be a path homotopy as well.

Problem 11. Let $f, g\colon X \to Y$ and $F, G\colon Y \to Z$ be continuous. Show that

$$\text{if } f \sim g \text{ and } F \sim G, \quad \text{then } F \circ f \sim G \circ g.$$

Problem 12. Use the $f = g$ case of Problem 11 to show: If X, Y are nonempty and Y is contractible, then any continuous map from X into Y is *nulhomotopic*: that is, homotopic to a constant map.

Problem 13. Give an explicit homotopy equivalence between S^1 and $\mathbf{R}^2 \setminus \{(0,0)\}$, and show that it is indeed a homotopy equivalence.

Problem 14. Suppose that:

- $f\colon X \to Y$ and $g\colon Y \to X$ define a homotopy equivalence between X and Y.
- $j\colon Y \to Z$ and $k\colon Z \to Y$ define a homotopy equivalence between Y and Z.

Show that $j \circ f\colon X \to Z$ and $g \circ k\colon Z \to X$ define a homotopy equivalence between X and Z.

Hint: Among other things, you will need to show that $g \circ k \circ j \circ f \sim \mathrm{id}_X$. But we are given that $k \circ j \sim \mathrm{id}_Y$. Use Problem 11 to show that $k \circ j \sim \mathrm{id}_Y$ implies $g \circ k \circ j \circ f \sim \mathrm{id}_X$.

Problem 15. Show that if the identity map of a space X is homotopic to a constant map with value $c \in X$, then X is homotopy equivalent to $\{c\}$. Deduce that all *contractible* spaces are homotopy equivalent to each other.

Problem 16. A subset $A \subseteq \mathbf{R}^n$ is *star convex* if and only if there is some point $c \in A$ such that the line segment between c and any other point of A is contained within A. That is, for any $a \in A$ and $t \in [0,1]$, we have $(1-t)c + ta \in A$.

(1) Show that if A is star convex, then A is contractible.
(2) Show that if A is star convex, then any loop in A based at a_0 is path homotopic to the constant loop.
(3) Give a star convex subset of $\mathbf{R}^2$ that is not convex.

Exam 1 Topics

Exam 1 will be a **closed-book** exam. It will be held in-class, 12:40–2:00 pm, on ~~January 26~~ ~~February 2~~ February 4.

You should know by heart the definitions of the following terms. You should be able to state multiple examples of each (with justification), and in some cases, non-examples.

- Munkres §13. Bases
- §18. Continuous maps
- §15. The product topology (only on finite direct products)
- §16. Subspaces
- §22. Quotient spaces
- §17. Interiors and closures
- §23–24. Separations, connectedness
- §23. Paths, path connectedness
- §26–27 Compactness
- §51. Homotopies, path homotopies
- §58. Homotopy equivalences

1. Fundamental Groups and Covering Spaces

Problem 17. Show that if X is path connected, then X is connected.

Problem 18. We say that X is *totally disconnected* if and only if its only nonempty connected subspaces are one-point sets.

(1) Show that if X is discrete, then X is totally disconnected.
(2) Show that the set of rational numbers $\mathbf{Q}$, as a subspace of (analytic) $\mathbf{R}$, is totally disconnected, but not discrete.
(3) Show that if X is totally disconnected, then $\pi_1(X, x)$ is trivial for all $x \in X$. *Hint:* Use Problem 17.

Problem 19. Consider the points $p = (1, 0)$ and $q = (-1, 0)$ in $\mathbf{R}^2$.

(1) Give two paths in S^1 from p to q that are not path homotopic.
(2) Explain why the paths in (1) become path homotopic when viewed as paths in $\mathbf{R}^2$.

Problem 20. Let $x_0, x_1 \in X$. Let $\alpha\colon [0, 1] \to X$ be a path from x_0 to x_1, and let $\bar{\alpha}$ be the reverse path defined by $\bar{\alpha}(s) = \alpha(1 - s)$.

(1) Show that $\alpha * \bar{\alpha}$ is path homotopic to e_{x_0}.
(2) Using part (1), show that the map

$$\hat{\alpha} : \pi_1(X, x_0) \to \pi_1(X, x_1) \quad \text{given by } \hat{\alpha}([\gamma]) = [\bar{\alpha} * \gamma * \alpha]$$

is a group homomorphism.
(3) Show that $\hat{\bar{\alpha}}$ is a two-sided inverse of $\hat{\alpha}$. Deduce that $\hat{\alpha}$ is an isomorphism.

Problem 21. Let $f\colon X \to Y$ be a continuous map.

(1) Show that if $\beta, \gamma\colon [0, 1] \to X$ are paths such that $\beta(1) = \gamma(0)$, then

$$f \circ (\beta * \gamma) = (f \circ \beta) * (f \circ \gamma).$$

(2) Let $x \in X$, Using part (1), show that the map

$$f_*\colon \pi_1(X, x) \to \pi_1(Y, f(x)) \quad \text{given by } f_*([\gamma]) = [f \circ \gamma]$$

is a group homomorphism.
(3) Let $g\colon Y \to Z$ be another continuous map. Show that $(g \circ f)_* = g_* \circ f_*$ as homomorphisms from $\pi_1(X, x)$ into $\pi_1(Z, g(f(x)))$.

Problem 22. Let X be a subspace of some space X', and let $f\colon X \to Y$ be a continuous map. Suppose that

$$f = f'|_X \quad \text{for some continuous map } f'\colon X' \to Y,$$

and that X' is *simply connected*, meaning X' is path connected and its fundamental group is trivial. Show that in this case, the homomorphism $f_*\colon \pi_1(X, x) \to \pi_1(Y, f(x))$ is *trivial* for all $x \in X$, meaning it sends every element of $\pi_1(X, x)$ to the identity element of $\pi_1(Y, f(x))$. *Hint:* Use part (3) of Problem 21.

Problem 23. Let X_1, X_2 be spaces with points $x_1 \in X_1$ and $x_2 \in X_2$. Let $X = X_1 \times X_2$ and $x = (x_1, x_2) \in X$. For $i = 1, 2$, let $\mathrm{pr}_i \colon X \to X_i$ be the appropriate *projection map* (Munkres page 87).

(1) Show that the map

$$\varphi \colon \pi_1(X, x) \to \pi_1(X_1, x_1) \times \pi_1(X_2, x_2) \quad \text{given by } \varphi([\gamma]) = ([\mathrm{pr}_1 \circ \gamma], [\mathrm{pr}_2 \circ \gamma])$$

is a homomorphism.

(2) Show that the map

$$\psi \colon \pi_1(X_1, x_1) \times \pi_1(X_2, x_2) \to \pi_1(X, x) \quad \text{given by } \psi([\gamma_1], [\gamma_2]) = [\gamma_1 \times \gamma_2]$$

is a two-sided inverse to φ. Deduce that

$$\pi_1(X, x) \simeq \pi_1(X_1, x_1) \times \pi_1(X_2, x_2).$$

(3) Using the conclusion to part (2), show that $\mathbf{Z}^n$ is the fundamental group of an explicit (path-connected) topological space, for any integer $n > 0$.

Problem 24. By considering what happens on fundamental groups when a point is removed, show that $\mathbf{R}^2$ and $\mathbf{R}^n$ are not homeomorphic if $n > 2$.

Problem 25. Learn the definitions of a *free group* and of a *presentation of a group (by generators and relations)*. (They are covered in Munkres §69, but you may find other sources online helpful as well.)

Let F_2 denote a free group on generators a and b. Show that the homomorphism

$$\varphi \colon F_2 \to \mathbf{Z}^2 \quad \text{given by } \varphi(a) = (1, 0) \text{ and } \varphi(b) = (0, 1)$$

is surjective and that its kernel contains the *commutator* of a and b: that is, the element $aba^{-1}b^{-1}$.

(In fact, the kernel is exactly the smallest normal subgroup of F_2 containing the commutator. This shows that $\mathbf{Z}^2$ has a presentation with generators a, b and a single relation $aba^{-1}b^{-1}$.)

Problem 26. Suppose that A, U_1, U_2 are path-connected open subsets of X such that $X = U_1 \cup U_2$ and $A = U_1 \cap U_2$. For $k = 1, 2$, let $i_k \colon A \to U_k$ and $j_k \colon U_k \to X$ be the appropriate inclusion maps. Fix a point $x \in A$.

(1) Describe a homomorphism

$$\Psi \colon \pi_1(U_1, x) \star \pi_1(U_2, x) \to \pi_1(X, x)$$

such that $\Psi|_{\pi_1(U_k, x)} = (j_k)_*$ for $k = 1, 2$. (It is necessarily unique.)

(2) Show that for any $[\gamma] \in \pi_1(A, x)$, the element

$$(i_1)_*([\gamma]) * (i_2)_*([\gamma])^{-1} \in \pi_1(U_1, x) \star \pi_1(U_2, x)$$

belongs to the kernel of Ψ. *Hint:* Compare $j_1 \circ i_1$ and $j_2 \circ i_2$.

Problem 27. Recall that $X \sqcup Y$ denotes the disjoint union of X and Y. The *wedge sum* of X and Y at points $x \in X$ and $y \in Y$ is a quotient space

$$X \vee Y = (X \sqcup Y)/{\sim},$$

where $\sim$ is an equivalence relation that only identifies two distinct points when those points are x and y. Use Seifert-van Kampen to show that $\pi_1(S^1 \vee S^1, p) \simeq \mathbf{Z} \star \mathbf{Z}$ for any basepoint p.

Problem 28. Keep the hypotheses of Problem 26. Thus, the Seifert–van Kampen theorem applies to the homomorphism Ψ in part (1) of that problem. Give examples where:

(1) A is simply-connected, but X, U_1, U_2 are not. *Hint:* Problem 27.
(2) X, U_1, U_2 are simply-connected, but A is not. *Hint:* Munkres §59.
(3) X and U_1 are simply-connected, but U_2 is not.
(4) X is simply-connected, but U_1 and U_2 are not.

Problem 29. For each of the following spaces, the fundamental group is either trivial, $\mathbf{Z}$, or the free group $F_2 \simeq \mathbf{Z} \star \mathbf{Z}$. Determine which option is the case. You do not need to give explicit homeomorphisms or homotopy equivalences, but give informal descriptions (or pictures) to support your reasoning.

(1) $D \times S^1$, where $D = \{(x, y) \in \mathbf{R}^2 \mid x^2 + y^2 < 1\}$.
(2) The *torus* $S^1 \times S^1$.
(3) The bounded cylinder $S^1 \times [0, 1]$.
(4) The unbounded cylinder $S^1 \times \mathbf{R}$.
(5) The punctured plane $\mathbf{R}^2 - \{p\}$, where p is any point.
(6) *Harder:* The punctured torus $S^1 \times S^1 - \{q\}$, where q is any point.

Problem 30. Let $p \colon E \to B$ be a covering map. Show that if B is connected and $p^{-1}(b_0)$ has k elements for some $b_0 \in B$, then $p^{-1}(b)$ has k elements for all $b \in B$. In this case, we say that p is a *k-fold covering*.

Problem 31. Draw every 2-fold covering space of $S^1 \vee S^1$ up to homeomorphism. You do not need to prove that your list is exhaustive.

(Note that $S^1 \vee S^1$ has a symmetry of order two. If two covering spaces differ by a lift of this symmetry, you do not need to draw both.)

Problem 32. Let $\mathbf{R}_+$ be the set of positive numbers. Let $p \colon \mathbf{R} \times \mathbf{R}_+ \to \mathbf{R}^2 - \{(0, 0)\}$ be defined by $p(u, r) = (r\cos(2\pi u), r\sin(2\pi u))$. This is a covering map. Find liftings along p of the following paths in $\mathbf{R}^2 - \{(0, 0)\}$:

$$\begin{aligned} f(t) &= (2 - t, 0), \\ g(t) &= ((1 + t)\cos(2\pi t), (1 + t)\sin(2\pi t)), \\ h &= f * g. \end{aligned}$$

Sketch (the images of) these paths and their liftings.

Problem 33. Let $p\colon \mathbf{R} \to S^1$ be defined by

$$p(t) = (\cos(2\pi u), \sin(2\pi u)).$$

Consider the path in $S^1 \times S^1$ given by

$$f(t) = ((\cos(2\pi t), \sin(2\pi t)), (\cos(4\pi t), \sin(4\pi t))).$$

Find an explicit lifting $\tilde{f}$ of f along $(p, p) : \mathbf{R} \times \mathbf{R} \to S^1 \times S^1$, and sketch it.

Exam 2 Topics

Exam 2 will be an **open-book** exam. It will be held in-class, 12:40–2:00 pm, on March 2.

Topics Included.

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Topics Not Included.

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2. Separation Theorems in the Plane

3. Simplicial Complexes and Surfaces

Exam 3 Topics

4. Homology