

naive idea behind Seifert–van Kampen:

if X is covered by open U_i 's [draw]
then can compute $\pi_1(X)$ from $\pi_1(U_i)$'s

but only works when all intersections of U_i 's are path-connected

Thm (Seifert–van Kampen)

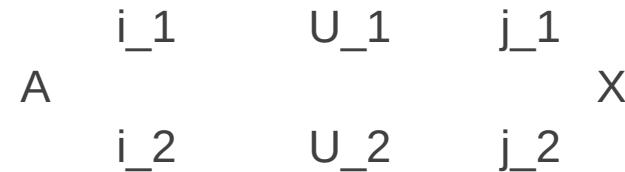
suppose A, U_1, U_2 sub X are open and x in A

s.t. $X = U_1 \cup U_2$,

$A = U_1 \cap U_2$

A, U_1, U_2 are all path-connected

with inclusion maps



then:

1) $j_{\{1, *}, j_{\{2, *}\}}$ induce a surjective hom

$\pi_1(U_1, x) * \pi_1(U_2, x)$ to $\pi_1(X, x)$

2) via this hom, $\pi_1(X, x)$ is the largest quotient of $\pi_1(U_1, x) * \pi_1(U_2, x)$ in which

$i_{\{1, *}\}([y]) \sim i_{\{2, *}\}([y])$ for all $[y]$ in $\pi_1(A, x)$

Ex

X the figure-eight,
 U_1, U_2 open thickenings of the S^1 's,
x the intersection point
[draw]

since $\pi_1(A, x)$ is trivial:

$$\begin{aligned}\pi_1(X, x) &\simeq \pi_1(U_1, x) * \pi_1(U_2, x) \\ &\simeq \pi_1(S^1, x) * \pi_1(S^1, x) \\ &\simeq \mathbb{Z} * \mathbb{Z}\end{aligned}$$

Ex

$X = S^2$,
 U_1, U_2 open thickenings of opposed
hemispheres,
[draw]

here, $\pi_1(A, x) \simeq \pi_1(S^1, x) \simeq \mathbb{Z}$
but $\pi_1(U_1, x), \pi_1(U_2, x)$ trivial
so $\pi_1(X, x)$ also trivial

Rem

Seifert–van Kampen alone
cannot compute $\pi_1(S^1)$

(Munkres §53–54) let $o = (1, 0)$ in S^1

recall: $\omega_n(s) = (\cos(2\pi ns), \sin(2\pi ns))$

Thm

$\Phi : \mathbb{Z}$ to $\pi_1(S^1, o)$ def by $\Phi(n) = [\omega_n]$
is an isomorphism

Pf

Step 1. Φ is a homomorphism
Step 2. Φ is bijective

Step 1 must show that $\Phi(m + n) = \Phi(m) + \Phi(n)$,
meaning $[\omega_{m+n}] = [\omega_m * \omega_n]$

key idea: let $p : R$ to S^1 be

$$p(x) = (\cos(2\pi x), \sin(2\pi x))$$

Lem for any a, b in Z s.t. $b - a = n$, we have

$$\omega_n = p \circ \omega_{a, b}$$

where $\omega_{a, b} : [0, 1]$ to R is defined by

$$\omega_{a, b}(s) = (1 - s)a + sb$$

[draw]

Lem for any a, b, c in Z , we have

$$[\omega_{a, b} * \omega_{b, c}] = [\omega_{a, c}]$$

[draw]

therefore,

$$\begin{aligned} & [(p \circ \omega_{a, b}) * (p \circ \omega_{b, c})] \\ &= [p \circ (\omega_{a, b} * \omega_{b, c})] \text{ [Munkres 327]} \\ &= [p \circ \omega_{a, c}] \text{ [PS5, #1]} \end{aligned}$$

$$\text{giving } [\omega_m * \omega_n] = [\omega_{m+n}]$$

before Step 2, we introduce the notion of covering map inspired by $p : R$ to S^1