MATH 250: TOPOLOGY I FINAL

FALL 2025

You have till 3:00 pm to do these problems in any order. You do not need to write in complete sentences, but please show your work. Throughout the exam, $\mathbf{R}$ has the analytic topology unless otherwise specified.

You are allowed to look at the textbook (Munkres, *Topology*, 2nd Ed.) and any notes on paper that you wrote prior to the exam. You may omit proofs for statements proved in the textbook, unless otherwise specified. You are not allowed to use electronic devices of any kind, such as phones, computers, tablets, or audio devices. If you need to use the bathroom, please give any phones or other electronics to the proctor first.

Problem 1 (4 points). Let $A = \{0\} \cup \{\frac{1}{n} \mid n = 1, 2, 3, \ldots\}$ as a subspace of $\mathbf{R}$. Show that A is compact directly from the definition, without using Heine–Borel or similar theorems.

Solution. Let $\{U_\alpha\}_\alpha$ be an open cover of A. For each index α, we know that $U_\alpha = A \cap V_\alpha$ for some open $V_\alpha \subseteq \mathbf{R}$. Moreover, there is some index α_0 such that $0 \in U_{\alpha_0} \subseteq V_\alpha$. Pick $\delta > 0$ such that $(-\delta, \delta) \subseteq V_\alpha$.

The set $\mathcal{S}$ of integers $n > 0$ such that $\frac{1}{n} \geq \delta$ is finite. For each $n \in \mathcal{S}$, pick an index α_n such that $\frac{1}{n} \in U_{\alpha_n}$. Then $A \subseteq U_{\alpha_0} \cup \bigcup_{n \in \mathcal{S}} U_{\alpha_n}$, so the collection of U_{α_n} for $n \in \{0\} \cup \mathcal{S}$ is a finite subcover of the original open cover of A. $\square$

Problem 2 (4 points). Let $X = \mathbf{R}/\sim$, where $p \sim q$ for distinct $p, q \in \mathbf{R}$ if and only if p, q are both rational. Show that X is not Hausdorff as a quotient space of $\mathbf{R}$. You may assume that any nonempty open set in $\mathbf{R}$ contains some rational number.

Solution. Let $x, y \in X$ be distinct points. Suppose that $U \ni x$ and $V \ni y$ are open. Then their respective preimages $\tilde{U}$ and $\tilde{V}$ must be nonempty and open in $\mathbb{R}$. So they each contain rational numbers: say, $\alpha \in \tilde{U} \cap \mathbf{Q}$ and $\beta \in \tilde{V} \cap \mathbf{Q}$. But $\alpha \sim \beta$, so they have a common image in X, contained in both U and V. So U and V intersect. So X is not Hausdorff. $\square$

Problem 3 (5 points). (1) Show that for any space X, the map

$$\Delta\colon X \to X^\omega \quad \text{defined by } \Delta(x) = (x, x, x, \ldots)$$

is continuous with respect to the product topology on X^ω.

(2) Show that for any collection $(g_i)_{i\geq 1}$ of continuous maps $g_i\colon Y_i \to Z_i$, the map

$$g\colon \prod_{i\geq 1} Y_i \to \prod_{i\geq 1} Z_i \quad \text{defined by } g((y_i)_{i\geq 1}) = (g_i(y_i))_{i\geq 1}$$

is continuous with respect to the product topologies on the domain and range.

Solution. (1) We see that $\mathrm{pr}_i \circ \Delta = \mathrm{id}_X$, a continuous map, for all integers $i \geq 1$, so by Munkres, Δ is continuous for the product topology on X^ω.

(2) It suffices to check on a basis for the product topology on $\prod_i Z_i$. A basis element looks like $V = \prod_i V_i \subseteq \prod_i Z_i$, where V_i is open in Z_i for all i, and equal to Z_i for all but finitely many i. Since g_i is continuous, $g_i^{-1}(V_i)$ is open in Y_i for all i. Moreover, if $V_i = Z_i$, then $g_i^{-1}(V_i) = Y_i$. So $g^{-1}(V) = \prod_i g_i^{-1}(V_i)$ is open in $\prod_i Y_i$. $\square$

Problem 4 (5 points). Let $n \geq 2$, and let $\vec{0}$ be the origin in $\mathbf{R}^n$. Let

$$M = \mathbf{R}^n \setminus \{\vec{0}\}, \qquad H = \{(1, x_2, \ldots, x_n) \mid x_2, \ldots, x_n \in \mathbf{R}\}.$$

Show that, for any point $p = (a_1, a_2, \ldots, a_n) \in M$, there is an <u>explicit</u> path from p to some point of $H \subseteq M$. You can leave path concatenations unsimplified. *Hint:* Handle the case where $a_2 = \cdots = a_n = 0$ separately.

Solution. If $a_2 = \cdots = a_n = 0$, then $\beta\colon [0,1] \to \mathbf{R}^n$ defined by $\beta(s) = (a_1, s, 0, \ldots, 0)$ is a path from p to $q = (a_1, 1, 0, \ldots, 0)$ in $\mathbf{R}^n$, which stays in M because $p \neq \vec{0}$ and $\beta(s)$ has nonzero second coordinate for all $s > 0$.

This reduces us to the case where one of the coordinates $a_2, \ldots, a_n$ is nonzero, because if we have a solution in that case, then we can apply it with q in place p, then pre-concatenate with β to get a solution for the $a_2 = \cdots = a_n = 0$ case.

Now assume that one of $a_2, \ldots, a_n$ is nonzero, say a_i. Then $\gamma\colon [0,1] \to \mathbf{R}^n$ defined by $\gamma(s) = ((1-s)a_1 + s, a_2, \ldots, a_n)$ is a path from p to M in $\mathbf{R}^n$, which stays in M because a_i is nonzero. $\square$

Problem 5 (5 points)**.** Let $S^1 \subseteq \mathbf{R}^2$ be the unit circle, and let $p = (1,0) \in S^1$.

(1) Show that the paths $\alpha, \beta \colon [0,1] \to S^1$ defined by

$$\alpha(s) = (\cos(\pi s), \sin(\pi s)) \quad \text{and} \quad \beta(s) = (\cos(\pi s), -\sin(\pi s))$$

are <u>not</u> path-homotopic in S^1. *Hint:* Think about $\alpha * \overline{\beta}$ and what you know about $\pi_1(S^1, p)$.

(2) Are α and β still <u>homotopic</u> in S^1? Just state yes or no.

Solution. (1) If α, β were path-homotopic in S^1, then by Munkres, $\alpha * \overline{\beta}$ and $\beta * \overline{\beta}$ would be as well. But $\alpha * \overline{\beta}$ performs one full counterclockwise revolution around the circle, so it corresponds under our standard isomorphism $\pi_1(S^1, p) \simeq \mathbf{Z}$ to 1. Meanwhile, $\beta * \overline{\beta}$ is nulhomotopic, so it corresponds to 0. Contradiction.

(2) Yes. □

Problem 6 (5 points)**.** Suppose that X is a union of path-connected open sets U and V, whose intersection A is also path-connected, and that $x \in A$.

State whether each claim below is true or false. If it is false, state (or draw) a counterexample. You do not need to justify the counterexample.

(1) If $\pi_1(X, x)$ is trivial, then $\pi_1(U, x)$ and $\pi_1(V, x)$ are trivial.
(2) If $\pi_1(U, x)$ and $\pi_1(V, x)$ are trivial, then $\pi_1(X, x)$ is trivial.
(3) If $\pi_1(U, x)$ and $\pi_1(V, x)$ are trivial, then $\pi_1(A, x)$ is trivial.
(4) X is path-connected.

Solution. (1) False. Take X to be an open disk, fix distinct points $p, q \in X$, and take $U = X \setminus \{p\}$ and $V = X \setminus \{q\}$. (2) True. (3) False. Take $X = S^2$. Fix $\delta > 0$. Take U to consist of all points less than distance δ south of the equator, and V to consist of all points less than distance δ north of the equator. (4) True. □

Problem 7 (3 points)**.** Give, for any space X, an <u>explicit</u> homotopy equivalence between $\mathbf{R} \times X$ and X, and justify that it is a homotopy equivalence. If you wish, you may identify X with $\{0\} \times X$.

Solution. First we build a homotopy equivalence between $\mathbf{R}$ and $\{0\}$: Take $r\colon \mathbf{R} \to \{0\}$ to be the only possible map, and take $i\colon \{0\} \to \mathbf{R}$ to be the inclusion map. Then $i \circ r \sim \mathrm{id}_{\mathbf{R}}$ via the homotopy $h(s,t) = st$, while $r \circ i = \mathrm{id}_{\{0\}}$.

Now, $\tilde{r} = r \times \mathrm{id}_X$ and $\tilde{i} = i \times \mathrm{id}_X$ define a homotopy equivalence between $\mathbf{R} \times X$ and $\{0\} \times X$, because $\tilde{i} \circ \tilde{r} \sim \mathrm{id}_{\mathbf{R}\times X}$ via the homotopy $\tilde{h}((s,x),t) = (h(s,t),x) = (st,x)$ and $\tilde{r} \circ \tilde{i} = \mathrm{id}_{\{0\}\times X}$.

[Writing p for the homeomorphism from $\{0\} \times X$ to X given by projection, we deduce that $p \circ \tilde{r}$ and $\tilde{i} \circ p^{-1}$ define a homotopy equivalence between $\mathbf{R} \times X$ and X.] $\square$

Problem 8 (3 points)**.** Suppose that $f\colon X \to Y$ and $g\colon Y \to X$ are continuous maps such that $g \circ f\colon X \to X$ is a self-homeomorphism. Show that

$$f_*\colon \pi_1(X,x) \to \pi_1(Y, f(x))$$

is <u>injective</u> for any $x \in X$.

Solution. We have $(g \circ f)_* = g_* \circ f_*$ as maps from $\pi_1(X,x)$ to $\pi_1(X, g(f(x)))$. Since $g \circ f$ is a homeomorphism, $(g \circ f)_*$ must be an isomorphism. Therefore, g_* is a left inverse to f_*. In particular, if $f_*(a) = f_*(b)$, then $a = g_*(f_*(a)) = g_*(f_*(b)) = b$, so f_* is injective. $\square$

Problem 9 (2 points)**.** Let $\Delta\colon X \to X \times X$ be defined by $\Delta(x) = (x, x)$. Show that

$$\Delta_*\colon \pi_1(X, x_0) \to \pi_1(X \times X, (x_0, x_0))$$

is injective for any $x_0 \in X$. *Hint:* The previous problem.

Solution. Observe that $\mathrm{pr}_i \circ \Delta = \mathrm{id}_X$ for $i = 1, 2$. Since identity maps are homeomorphisms, we can apply the previous problem to deduce that Δ_* is injective. □

Problem 10 (2 points)**.** State (or draw) a subset of $\mathbf{R}^2$ that is not star-convex (so not convex, either), but is still contractible.

Solution. Pick any point $p \in S^1 \subseteq \mathbf{R}^2$. Then $S^1 \setminus \{p\}$ is a contractible but not star-convex subset of $\mathbf{R}^2$. □

Problem 11 (2 points)**.** Draw two examples of 3-fold covering spaces of the figure-eight $S^1 \vee S^1$, such that one is connected and the other is disconnected.

Solution. [For the connected covering space: The graph with vertices a, b, c, edges between each pair of distinct vertices, and a loop on each vertex.]

[For the disconnected covering space: The union of three disjoint copies of the figure-eight.] □