(Munkres §16) let A be a subset of X

Df the subspace topology on A induced by X is $\{A \cap V \mid V \text{ is an open set in } X\}$

[is it really a topology?]

Ex take $X = \mathbf{R}$ and $A = [0, \infty)$

open sets in subspace top on $[0, \infty)$
look like $V \cap [0, \infty)$

what if $V = (a, b)$?

if $a \geq 0$, then $(a, b) \cap [0, \infty) = (a, b)$
is open in both $[0, \infty)$ and $\mathbf{R}$

if $a < 0$, then $(a, b) \cap [0, \infty) = [0, b)$
is open in $[0, \infty)$, but not in $\mathbf{R}$

Moral if U sub A is open in A,
then U may or may not be open in X

Thm if $\{B_i\}_{i \in I}$ is a basis for the top on X,
then $\{A \cap B_i\}_{i \in I}$ is a basis for
the subspace top on A

Pf by thm from last time, just need to show:
for all U open in A and x in U,
have i s.t. $x \in A \cap B_i \subseteq U$

indeed, $U = A \cap V$ for some V open in X
then x in V, so have i s.t. $x \in B_i \subseteq V$
so x in $A \cap B_i$, and also, $A \cap B_i \subseteq U$

[how do subspaces interact with cts maps?]

Observations

- if A sub X, then the inclusion map i : A to X def by i(a) = a is cts
- compositions of cts maps are cts
- so if f : X to Y is cts, then f|_A : A to Y is cts

(Munkres §20) recall from real analysis:

Df a metric on a set X is a function
d : X × X to [0, ∞)

s.t., for all x, y, z in X,

1) d(x, y) = 0 implies x = y
2) d(x, y) = d(y, x)
3) d(x, y) + d(y, z) ≥ d(x, z)

given δ > 0, let B_d(x, δ) = {y in X | d(x, y) < δ}

[note that x in B_d(x, δ), because d(x, x) = 0 < δ]

Df the metric topology on X induced by d:

U is open in the metric topology iff
for all x in U, there is a δ > 0 s.t. B_d(x, δ) sub U

Idea metric topology on X generalizes
analytic topology on R^n

Thm the metric topology really is a topology

Pf exactly like the proof that
the anlytic topology is a topology

[so how much weirder can it be?]

<u>Ex</u> in any X: the discrete metric defined by

$$d(x, x) = 0, \quad d(x, y) = 1 \text{ when } x \neq y$$

1) and 2) easy

3) [how many cases to check? 5 but can combine]

if $x = z$:

$d(x, y) + d(y, z) \geq 0 = d(x, z)$

[because $d(-, -) \geq 0$]

if $x \neq z$:

either $y \neq x$ or $y \neq z$

so $d(x, y) + d(y, z) \geq 1 = d(x, z)$

observe $B_d(x, 1) = \{x\}$ for all x. thus:
the discrete metric induces the discrete topology

<u>Df</u> we say a topology or topological space is metrizable iff the topology is induced by some metric

sometimes, different metrics induce the same topology

<u>Lem</u> suppose d induces T on X, d' induces T' on X

then T' is finer than T iff
for all x in X and $\varepsilon > 0$, there is $\delta > 0$ s.t.

$B_{d'}(x, \delta) \subseteq B_d(x, \varepsilon)$.

Pf exercise (Munkres Lem 20.2)

Ex [picture of $B_d(x, \delta)$ versus $B_\rho(x, \delta)$]

euclidean metric:
$d(x, y) = \sqrt{(x_1 - y_1)^2 + \ldots + (x_n - y_n)^2}$

square metric:
$\rho(x, y) = \max(|x_1 - y_1|, \ldots, |x_n - y_n|)$

observe:

$$\begin{aligned} d(x, y) &\leq \sqrt{n \max_i (x_i - y_i)^2} \\ &= \sqrt{n}\, \rho(x, y) \\ \rho(x, y) &= \sqrt{\max_i |x_i - y_i|^2} \\ &\leq d(x, y) \end{aligned}$$

shows:

$B_\rho(x, \varepsilon/\sqrt{n}) \subseteq B_d(x, \varepsilon)$
$B_d(x, \varepsilon) \subseteq B_\rho(x, \varepsilon)$

[in general:]

Df metrics d, d' are called equivalent iff there exist $A, B > 0$ s.t.
$d(x, y) \leq A\, d'(x, y)$ and $d'(x, y) \leq B\, d(x, y)$
uniformly in x and y

Lem if two metrics are equivalent, then their metric topologies coincide

Cor Euclidean and square metrics both induce the analytic topology on $\mathbb{R}^n$

Rem converse is false: given a metric d, let

$$d'(x, y) = d(x, y)/(1 + d(x, y))$$

then 1) d' is still a metric

2) metric topologies for d, d' coincide

3) d and d' need not be equivalent

[reason: equivalence involves uniformity in x, y]

Ex $\mathbb{R}^\omega = \{(x_1, x_2, \ldots) \mid x_i\text{'s in } \mathbb{R}\}$

$\mathbb{R}^\infty = \{(x_1, x_2, \ldots) \mid x_i \text{ eventually } 0\}$

Euclidean and square metrics still work on $\mathbb{R}^\infty$, but not on $\mathbb{R}^\omega$

but $u(x, y) = \sup_i \min\{1, |x_i - y_i|\}$ works…