17.

Having introduced the Hecke category and Rouquier complexes, we explain, following Khovanov, a categorification of Jones–Ocneanu's HOMFLYPT Markov trace.

17.1.

Let $k = \bar{\mathbf{F}}_q$ and $k_1 = \mathbf{F}_q$. Let $G, \mathcal{B}, O_w, j_w, B, T$ and their analogues G_1, *etc.* over k_1 be the same as last time.

We continue to assume that G_1 is a split form, so that Frobenius acts trivially on the Weyl group W and its system of simle reflections S. Recall the braid group attached to W: If

$$W = \langle S \mid s^2 = e \text{ and } \overbrace{sts\cdots}^{m_{s,t}} = \overbrace{tst\cdots}^{m_{s,t}} \text{ for all } s, t \in S\rangle,$$

then the associated *(Artin–Tits) braid group* is

$$Br_W = \langle (\sigma_s)_{s\in S} \mid \overbrace{\sigma_s\sigma_t\sigma_s\cdots}^{m_{s,t}} = \overbrace{\sigma_t\sigma_s\sigma_t\cdots}^{m_{s,t}} \text{ for all } s, t \in S\rangle.$$

If $w = s_{i_1}\cdots s_{i_\ell}$ is a word in S of minimal length, then $\sigma_w = \sigma_{s_{i_1}}\cdots\sigma_{s_{i_\ell}}$ is a well-defined element of Br_W depending only on w. Using this fact, one can show that Br_W admits the presentation

$$Br_W = \langle (\sigma_w)_{w\in W} \mid \sigma_w\sigma_{w'} = \sigma_{ww'} \text{ whenever } \ell(w) + \ell(w') = \ell(ww')\rangle,$$

which in turn gives rise to the identification

$$H_W(\mathsf{x}) = \frac{\mathbf{Z}[\mathsf{x}^{\pm 1}]Br_W}{\langle(\sigma_s - \mathsf{x})(\sigma_s + \mathsf{x}^{-1}) \mid s \in S\rangle}.$$

The composition $Br_W \to H_W(\mathsf{x})^\times \xrightarrow{\sim} [\mathsf{H}_W]^\times_\Delta$ sends $\sigma_w^{\pm 1} \mapsto [\mathcal{R}_w^{\pm}]$ for all w, and braid composition to convolution. We deduce that the classes $[\mathcal{R}_w^{\pm}]$ satisfy the braid relations.

Something stronger is true. The key is the following result, apparently first published by Deligne (though surely known earlier), which we may prove later in the course.

Lemma 17.1. *Whenever $\ell(w) + \ell(w') = \ell(ww')$, the forgetful map*

$$O_{w,1} \times_{\mathcal{B}_1} O_{w',1} \to O_{ww',1}$$

is a (G_1-equivariant) isomorphism of $\mathbf{F}_q$-schemes, where the fiber product uses the right-hand factor of $O_{w,1}$ and the left-hand factor of $O_{w',1}$.

This result implies that under the same hypothesis $\ell(w) + \ell(w') = \ell(ww')$, we have explicit isomorphisms in $\mathsf{D}_{G_1}(\mathcal{B}_1 \times \mathcal{B}_1)$ of the form

$$\Delta_{w,1} * \Delta_{w',1} \xrightarrow{\sim} \Delta_{ww',1} \quad \text{and} \quad \nabla_{w,1} * \nabla_{w',1} \xrightarrow{\sim} \nabla_{ww',1}.$$

Now recall the weight realization functor $\mathsf{real} : \mathsf{H}_W(\mathsf{x}) \to \mathsf{D}_{G_1}(\mathcal{B}_1 \times \mathcal{B}_1)$. Since real is fully faithful, monoidal, and sends $\mathcal{R}_w^+ \mapsto \Delta_{w,1}$ and $\mathcal{R}_w^- \mapsto \nabla_{w,1}$, we deduce that whenever $\ell(w) + \ell(w') = \ell(ww')$, we have explicit isomorphisms

$$\mathcal{R}_w^+ * \mathcal{R}_{w'}^+ \xrightarrow{\sim} \mathcal{R}_{ww'}^+ \quad \text{and} \quad \mathcal{R}_w^- * \mathcal{R}_{w'}^- \xrightarrow{\sim} \mathcal{R}_{ww'}^-.$$

In this sense, Rouquier complexes satisfy categorified braid relations under convolution. To be more accurate, we have only shown this statement for the $\mathcal{R}_w^+$ and the $\mathcal{R}_w^-$ separately. We should also check that:

Lemma 17.2. *For all $s \in S$, there are explicit isomorphisms*

$$\mathcal{R}_s^+ * \mathcal{R}_s^- \xleftarrow{\sim} \underline{E_{e,1}} \xrightarrow{\sim} \mathcal{R}_s^- * \mathcal{R}_s^+,$$

where $\underline{E_{e,1}}$ is the complex consisting of $E_{e,1}$ in degree zero.

These isomorphisms do not come from isomorphisms of varieties, effectively because they mix together ! and $*$. Instead, they are checked using Soergel's embedding of C_W into the category of finitely-generated graded R-bimodules for $R = \mathrm{H}_B^*(pt) = \mathrm{H}_T^*(pt)$. All in all:

Theorem 17.3. *For any sequences $\vec{w} \in W^m$ and $\vec{\epsilon} \in \{\pm\}^m$, let*

$$\begin{aligned}
\beta_{\vec{w}} &= \sigma_{w_1} \cdots \sigma_{w_m}, \\
\beta_{\vec{w},\vec{\epsilon}} &= \sigma_{w_1}^{\epsilon_1} \cdots \sigma_{w_m}^{\epsilon_m}, \\
O_{\vec{w},1} &= O_{w_1,1} \times_{\mathcal{B}} \cdots \times_{\mathcal{B}} O_{w_m,1}, \\
\mathcal{R}_{\vec{w},\vec{\epsilon}} &= \mathcal{R}_{w_1}^{\epsilon_1} * \cdots * \mathcal{R}_{w_m}^{\epsilon_m}.
\end{aligned}$$

Then:

1. *$O_{\vec{w}}, 1$ only depends on $\beta_{\vec{w}}$ up to isomorphism over $\mathcal{B} \times \mathcal{B}$, in the sense that if $\beta_{\vec{w}} = \beta_{\vec{w}'}$, then any explicit sequence of relations in the braid group producing the identity gives rise to an explicit isomorphism $O_{\vec{w},1} \simeq O_{\vec{w}',1}$ preserving the leftmost and rightmost projections to $\mathcal{B}$.*
2. *$\mathcal{R}_{\vec{w},\vec{\epsilon}}$ only depends on $\beta_{\vec{w},\vec{\epsilon}}$ up to isomorphism, in the sense that if $\beta_{\vec{w},\vec{\epsilon}} = \beta_{\vec{w}',\vec{\epsilon}'}$, then any explicit sequence of relations in the braid group producing the identity gives rise to an explicit isomorphism $\mathcal{R}_{\vec{w},\vec{\epsilon}} \xrightarrow{\sim} \mathcal{R}_{\vec{w}',\vec{\epsilon}'}$.*

When $\vec{\epsilon}$ consists solely of $+$'s, *resp.* solely of $-$'s, it will be convenient to write $\mathcal{R}_{\vec{w}}^+$, *resp.* $\mathcal{R}_{\vec{w}}^-$, in place of $\mathcal{R}_{\vec{w},\vec{\epsilon}}$. In this case, we also say that $\beta_{\vec{w},\vec{\epsilon}}$ is *positive*, *resp.* *negative*. (Thus $\beta_{\vec{w}}$ is positive.)

Remark 17.4. The original isomorphisms of varieties satisfy a strict form of associativity. Namely, the following square commutes on the nose:

$$\begin{tikzcd} O_{w,1} \times_{\mathcal{B}_1} O_{w',1} \times_{\mathcal{B}_1} O_{w'',1} \arrow[r] \arrow[d] & O_{w,1} \times_{\mathcal{B}_1} O_{w'w'',1} \arrow[d] \\ O_{ww',1} \times_{\mathcal{B}_1} O_{w'',1} \arrow[r] & O_{ww'w''} \end{tikzcd}$$

It implies similar associativity identities with $\Delta_{-,1}, \nabla_{-,1}, \mathcal{R}^+, \mathcal{R}^-$ in place of $O_{-,1}$. Using these identities, it is possible to describe more precisely the sense in which the isomorphisms of the form $O_{\vec{w},1} \xrightarrow{\sim} O_{\vec{w}',1}$ and $\mathcal{R}^{\pm}_{\vec{w}} \xrightarrow{\sim} \mathcal{R}^{\pm}_{\vec{w}'}$ are unique. Note that it suffices to handle the case where $\vec{w}, \vec{w}'$ are words in S. See Deligne, "Action du group des tresses sur un catégorie", for details.

17.2.

The discussion above leads us to regard Rouquier complexes as *categorified* braids. This in turn hints at categorifications of the braid and link invariants that we studied earlier, to be constructed from these complexes.

First observe that there is a naive analogue in algebraic geometry to forming the link closure of a (positive or negative) braid. For convenience below, we drop the subscript $_1$'s that indicate $\mathbf{F}_q$-structures. If

$$O_{\vec{w}} = \{(B_0 \xrightarrow{w_1} B_1 \cdots \xrightarrow{w_m} B_m) \in \mathcal{B}^{1+m}\}$$

represents such a braid itself, then

$$X_{\vec{w}} = \{(B_1, \ldots, B_m) \in \mathcal{B}^m \mid B_m \xrightarrow{w_1} B_1 \cdots \xrightarrow{w_m} B_m\}$$

the pullback of $O_{\vec{w}}$ along the diagonal $O_e = \mathcal{B} \to \mathcal{B} \times \mathcal{B}$, represents its cyclic closure. A more sophisticated construction: Form

$$G_{\vec{w}} = \{(g, B_1, \ldots, B_m) \in G \times \mathcal{B}^m \mid gB_mg^{-1} \xrightarrow{w_1} B_1 \cdots \xrightarrow{w_m} B_m\},$$

the pullback of $O_{\vec{w}}$ along the action map

$$\begin{aligned} G \times \mathcal{B} &\xrightarrow{act} \mathcal{B} \times \mathcal{B}, \\ (g, B) &\mapsto (gBg^{-1}, B). \end{aligned}$$

Everything here remains G-equivariant once we require that G acts on itself by left conjugation. Even though the definition of $G_{\vec{w}}$ does not have cyclic symmetry with respect to the coordinates $B_1, \ldots, B_m$, this symmetry is restored at the level of the quotient stack $[G \backslash G_{\vec{w}}]$.

17.3.

We now discuss how the passage from $O_{\vec{w}}$ to $G_{\vec{w}}$ looks at the level of Rouquier complexes. Recall from Soergel that the hypercohomology functor

$$\mathrm{H}^*_G(\mathcal{B} \times \mathcal{B}, -) : \mathsf{H}_W = \mathsf{K}^b(\mathsf{C}_W) \to \mathsf{K}^b(\mathsf{Mod}^{\mathsf{fg,gr}}_{R \otimes R^{\mathrm{op}}})$$

is fully faithful, and that the R-bimodule structure arises from the identification

$$[G\backslash(\mathcal{B} \times \mathcal{B})] \simeq [B\backslash G/B].$$

Pulling back along *act* on the left-hand side corresponds to pulling back along

$$[G/\mathrm{Ad}(B)] \to [B\backslash G/B]$$

on the right-hand side, where $\mathrm{Ad}(B)$ connotes B acting on G by left conjugation, not by multiplication.

Therefore, the effect of pulling back along *act before* G-equivariant hypercohomology corresponds to replacing the $*$-pushforward to $[B\backslash pt/B]$ with the $*$-pushforward to $[pt/\mathrm{Ad}(B)]$. If we were working at the underived level, then this would correspond to base change from $(R \otimes R^{\mathrm{op}})$-modules to R-modules along the map $R \otimes R^{\mathrm{op}} \to R$ that sends $f_1 \otimes f_2 \mapsto f_1 f_2$.

But since our sheaf operations happen at the derived level, we need to take the *derived* base change from $R \otimes R^{\mathrm{op}}$ to R. In more classical language, this is the *Hochschild homology* functor

$$\mathrm{HH}^* := \mathrm{Tor}_*^{R \otimes R^{\mathrm{op}}}(R, -).$$

To extend this operation from R-bimodules to complexes of R-bimodules up to homotopy, we just apply HH^* term by term to the complex. We will again write HH^* for the resulting functor $\mathsf{K}^b(\mathsf{Mod}^{\mathsf{fg,gr}}_{R \otimes R^{\mathrm{op}}}) \to \mathsf{K}^b(\mathsf{Vect}^{2\text{-}\mathsf{gr}}_{\bar{\mathbf{Q}}_\ell})$, where in general, $(-)^{d\text{-}\mathsf{gr}}$ will denote a $\mathbf{Z}^d$-grading.

To state the precise relation between Hochschild homology and G-quivariant hypercohomology over $\mathcal{B} \times \mathcal{B}$, we need to account somehow for the new, second *Hochschild grading*. Webster–Williamson showed that it has to do with weights. Recall that if $K_1 \in \mathsf{D}_{G_1}(X_1)$ is a mixed complex, then $\mathrm{H}^*_G(X, K)$ forms a $\bar{\mathbf{Q}}_\ell[F]$-module. We define a *weight filtration* $\mathsf{W}_{\leq *}$ on any such module V by setting $\mathsf{W}_{\leq \alpha} V$ to be the span of the F-eigenvectors in V with eigenvalue λ such that $|\lambda| \leq q^{\alpha/2}$ under any isomorphism $\bar{\mathbf{Q}}_\ell \simeq \mathbf{C}$.

Theorem 17.5 (Webster–Williamson)**.** *The functor* $\mathrm{H}^*_G(G \times \mathcal{B}, act^*(-))$*, where we first pull back along act before taking hypercohomology, factors as the composition*

$$\mathsf{H}_W \xrightarrow{\mathrm{H}^*_G(\mathcal{B}\times\mathcal{B},-)} \mathsf{K}^b(\mathsf{Mod}^{\mathsf{fg,gr}}_{R \otimes R^{\mathrm{op}}}) \xrightarrow{\mathrm{HH}^*} \mathsf{K}^b(\mathsf{Mod}^{\mathsf{fg,gr}}_{R})$$

followed by the regrading

$$\mathrm{gr}^R_{i+j}\,\mathrm{HH}^i(\mathbf{B}_w\langle m\rangle) = \mathrm{gr}^{\mathsf{W}}_{i+j}\,\mathrm{H}^j_G(G\times\mathcal{B}, act^* E_w\langle m\rangle),$$

where on the left, gr^R_* is the grading coming from the Soergel bimodule, and on the right, $\mathrm{gr}^{\mathsf{W}}_*$ is the weight grading.

We define *triply-graded Khovanov–Rozansky homology* to be the functor on H_W given by the composition

$$\text{(17.1)}\quad \mathsf{HHH} : \mathsf{H}_W \xrightarrow{\mathrm{H}^*_G(\mathcal{B}\times\mathcal{B},-)} \mathsf{K}^b(\mathsf{Mod}^{\mathsf{fg,gr}}_{R\otimes R^{\mathrm{op}}}) \xrightarrow{\mathrm{HH}^*} \mathsf{K}^b(\mathsf{Vect}^{2\text{-}\mathsf{gr}}_{\bar{\mathbf{Q}}_\ell}) \xrightarrow{\mathrm{H}_*} \mathsf{Vect}^{3\text{-}\mathsf{gr}}_{\bar{\mathbf{Q}}_\ell}.$$

Building on the original work of Khovanov–Rozansky, Khovanov observed:

Theorem 17.6 (Khovanov)**.** *Suppose that $W = S_n$. Then after renormalizing and shifting the triple grading, the function on $\mathsf{H}_{S_n}(\mathsf{x}) = [\mathsf{H}_{S_n}]_\triangle$ induced by the Euler characteristic of* HHH *is the Jones–Ocneanu trace. Here we take the Euler characteristic with respect to the "Rouquier" grading, coming from the degree in the functor* H_* *in the last step of* (17.1).

17.4.

We now calculate everything in the case of the identity object $E_{e,1}$. First, we have weight-preserving isomorphisms

$$\begin{aligned}\mathrm{H}^j_G(G\times\mathcal{B}, act^* E_e) &\simeq \mathrm{H}^j_{\mathrm{Ad}(B)}(B, \bar{\mathbf{Q}}_\ell)\\ &\simeq \mathrm{H}^j_T(T, \bar{\mathbf{Q}}_\ell)\\ &\simeq \bigoplus_i \mathrm{H}^i(T, \bar{\mathbf{Q}}_\ell)\otimes \mathrm{H}^{j-i}_T(pt, \bar{\mathbf{Q}}_\ell).\end{aligned}$$

Above, Frobenius acts by q^i on $\mathrm{H}^i(T)$ and by $q^{(j-i)/2}$ on $\mathrm{H}^{2d}_T(pt)$, so it acts by $q^{(i+j)/2}$ on the ith summand of the last expression. We deduce that

$$\mathrm{gr}^{\mathsf{W}}_{i+j}\,\mathrm{H}^j_G(G\times\mathcal{B}, act^* E_e) \simeq \mathrm{H}^i(T, \bar{\mathbf{Q}}_\ell)\otimes \mathrm{H}^{j-i}_T(pt, \bar{\mathbf{Q}}_\ell).$$

Compare this to

$$\mu_n(\sigma_e) = \mu_n(1) = \frac{\mathsf{a}-\mathsf{a}^{-1}}{\mathsf{x}-\mathsf{x}^{-1}}\cdot\mu_{n-1}(1) = \cdots = \left(\frac{\mathsf{a}-\mathsf{a}^{-1}}{\mathsf{x}-\mathsf{x}^{-1}}\right)^{n-1}.$$

(Recall that $\mu_1(1) = 1$.)