Thm (Jordan Canonical Form)

suppose V is fin. dim. over $\mathbb{C}$
$T : V$ to V is a linear op

I) there are scalars $\lambda_1, \dots, \lambda_\ell$ in $\mathbb{C}$ [pairwise distinct] s.t.

$$V = W_1 + \dots + W_\ell,$$

where $W_i = \bigcup_n \ker((T - \lambda_i)^n)$,
the generalized λ-eigenspace of T,
and the sum is a direct sum

II) each W_i has a basis given by a disjoint union of Jordan chains

(each chain looks like $e_d \to \dots \to e_1$, where
$Te_1 = \lambda e_1$,
$Te_i = \lambda e_i + e_{i-1}$ for all $i > 1$,
and d is some positive integer)

Cor every square matrix over $\mathbb{C}$ is conj to a Jordan canonical form matrix

i.e., a block-diagonal matrix with each block an upper-triangular Jordan block

Rem the JCF matrix is unique up to changing the blocks' order

e.g.

$$\begin{matrix} \lambda & & \\ & \mu & 1 \\ & & \mu \end{matrix} \quad \sim \quad \begin{matrix} \mu & 1 & \\ & \mu & \\ & & \lambda \end{matrix}$$

let JCF_n = {$n \times n$ JCF matrices over C}

Cor   if $f : Mat_n(C)$ to F is a conj-invariant fn
then f is determined by $f|_{JCF_n(C)}$

conversely:
if $g : JCF_n$ to C has the property that
$g(M) = g(M')$ whenever
M, M' differ by reordering of blocks
then g extends to a conj-inv. fn on Mat_n

Ex   a function with arg's $X_1, \ldots, X_n$
is called symmetric iff
it is unchanged by any permutation
of $(X_1, \ldots, X_n)$

e.g., if $g : JCF_n$ to F is a symmetric function of
the diagonal coords $x_{11}, x_{22}, \ldots, x_{nn}$
then g extends to a conj-invariant fn on Mat_n

Df   the trace of T is def by
$tr(T) = \sum_i (\dim W_i)\lambda_i$

Df   the determinant of T is def by
$det(T) = \prod_i \lambda_i^{(\dim W_i)}$:

Thm
1) tr = unique conj-invariant extension to $Mat_n(C)$ of
$x_{11} + \ldots + x_{nn} : JCF_n$ to C
2) det = unique conj-invariant extension to $Mat_n(C)$ of
$x_{11} \ldots x_{nn} : JCF_n$ to C

Pf these formulas hold on JCF_n

Rem tr is still given by x11 + … + xnn
on all of Mat_n(C)

but: det is not given by x11 … xnn
on non-triangular matrices!

Ex $M = \begin{matrix} a & b \\ c & d \end{matrix}$ det(T) $= ad - bc$
$\neq ad$ in gen'l

Df the characteristic polynomial of T is

charpoly_T(z)
$= \prod_i (z - \lambda_i)^{(\dim W_i)}$

Thm if T has a matrix M in some basis
then charpoly_T(z) = det(zI – M)

Pf this formula holds on JCF_n

Ex $M = \begin{matrix} a & b \\ c & d \end{matrix}$

charpoly_T(z) $= z^2 - (a + d)z + (ad - bc)$
$= z^2 - \mathrm{tr}(T)z + \det(T)$

is there a rship between charpoly and tr, det
beyond the 2 × 2 case?

Thm 1) tr = coeff of z^{n-1} in charpoly(z)
where n = dim V
2) det = constant term in charpoly(z)

Pf check these identities on JCF_n

More Observations

I) if V is a real, not complex, vector space then these results don't apply to T : V to V

but do apply to $T_C : V_C$ to V_C,
the complexification

PS2, #2: any basis for V produces one for V_C
thus: if V is iso to R^n, then V_C is iso to C^n

so for real T, we define
tr(T), det(T), charpoly_T
to be
$\text{tr}(T_C)$, $\det(T_C)$, $\text{charpoly}_{(T_C)}$

II) we can use det(T) to decide if T is invertible

Thm for V fin. dim.,
TFAE for any linear op T : V to V:

1) T is invertible
2) $\ker(T) = \{\mathbf{0}\}$
3) all eigenvals of T (or T_C) nonzero
4) $\det(T) \neq 0$

Pf 1), 2) equiv. by PS3, #1 + dim formula

2), 3) equiv. because nonzero elts of ker(T) are eigenvec's of T with eigenval 0

3), 4) equiv. by defn of det

III) recall:

minpoly_T(z) = (nonconst) monic poly p(z) in C[z] of lowest deg s.t. p(T) is zero

Thm if a(z) is any monic poly s.t. a(T) is zero then minpoly_T(z) divides a(z) as a poly

Pf a(z) = minpoly_T(z) q(z) + r(z) with deg r < deg minpoly_T

then 0 = a(T) = minpoly_T(T) q(T) + r(T) = r(T) forcing r(z) = 0 by minimality of minpoly_T

[but recall from last week:]

Thm if λ_1, …, λ_ℓ are the eigenvals of T then prod_i (T – λ_i)^(k_i) is zero

where k_i ≤ dim(gen'lized λ_i-eigensp. of T)

Cor for these k_i's,

minpoly_T(z) divides prod_i (z – λ_i)^(k_i)

Cor (Cayley–Hamilton)

minpoly_T(z) divides charpoly_T(z)