8.

Much simpler:

8.1.

Fix a sequence of simple reflections $s_1, \dots, s_\ell \in W$ and set $\beta = \sigma_{s_1} \cdots \sigma_{s_\ell} \in Br_W$. Recall that

$$G(\beta) = \left\{(g, B_1, \dots, B_\ell) : B_\ell^g \xrightarrow{s_1} B_1 \xrightarrow{s_2} \cdots \xrightarrow{s_\ell} B_\ell\right\} \subseteq G \times \mathcal{B}^\ell.$$

Fix a pair of opposite Borel subgroups $B_+, B_- \in \mathcal{B}$. Let $G(\beta)_{B_\ell = B_+}$ be the fiber of $\mathrm{pr}_\ell : G(\beta) \to \mathcal{B}$ above B_+, so that

$$[G(\beta)_{B_\ell = B_+}/B_+] \simeq [G(\beta)/G].$$

We write $U_\pm \subseteq B_\pm$ for the unipotent radicals of the Borels and $T = B_+ \cap B_-$. We will study the above stack by means of the T-bundle

$$[G(\beta)_{B_\ell = B_+}/U_+] \to [G(\beta)_{B_\ell = B_+}/B_+].$$

As it turns out, the total space is more useful in some ways.

If H is an algebraic group, X is equipped with a right H-action, and Y is equipped with a left H^{op}-action, then we write

$$X \times^H Y = (X \times Y)/((x, y) \sim (xh, h^{-1}y)).$$

Let $w \mapsto \dot{w}$ be a section $W \to N_G(T)$, and let

$$\begin{aligned} G_+(\beta) &= B_+\dot{s}_1 U_+ \times^{U_+} \cdots \times^{U_+} U_+\dot{s}_\ell U_+ \\ &= B_+\dot{s}_1 B_+ \times^{B_+} \cdots \times^{B_+} B_+\dot{s}_\ell B_+. \end{aligned}$$

Then there is an isomorphism:

$$\begin{array}{ccc} G_+(\beta) & \xrightarrow{\sim} & G(\beta)_{B_\ell = B_+} \\ (g_1, g_2, \dots, g_\ell) & \mapsto & (g_1 g_2 \cdots g_\ell, B_+^{g_2 \cdots g_\ell}, \dots, B_+) \end{array}$$

The projection map $G(\beta)_{B_\ell = B_+} \to G$ corresponds to the multiplication map

$$\begin{array}{ccc} G_+(\beta) & \xrightarrow{m} & G \\ (g_1, g_2, \dots, g_\ell) & \mapsto & g_1 g_2 \cdots g_\ell. \end{array}$$

The fiber of m over $1 \in G$ is precisely the "braid variety" $X_0(\beta)$ studied by Mellit and Casals–Gorsky–Gorsky–Simental. This is explained in Appendix B of my preprint "From the Hecke Category…"

Moreover, m is equivariant with respect to the U_+-action on G by right conjugation and the U_+-action on $G_+(\beta)$ given by

$$(g_1, g_2 \ldots, g_\ell) \cdot x = (x^{-1} g_1, g_2, \ldots, g_\ell x).$$

We now set

$$\begin{aligned} \mathcal{T}(\beta) &= T\dot{s}_1 U_+ \times^{U_+} \cdots \times^{U_+} U_+ \dot{s}_k U_+ \\ &\simeq \mathbf{G}_m \times \mathbf{A}^\ell. \end{aligned}$$

Then the composition

$$\mathcal{T}(\beta) \to G_+(\beta) \to [G_+(\beta)/U_+]$$

is a homotopy equivalence. It is even an isomorphism of stacks as long as σ_{w_0} is a prefix of β. (Recall that by Garside's theorem, we can always write $\beta = \sigma_{w_0}^e \beta'$ for some integer e and positive braid β'.)

For any $c \in T /\!\!/ W$, we let G_c denote the preimage of c along the Chevalley map $G \to T /\!\!/ W$. Thus $G_{[1]} = \mathcal{U}$, the unipotent locus of G. We set

$$\begin{aligned} G_+(\beta, c) &= G_+(\beta) \times_G G_c, \\ \mathcal{T}(\beta, c) &= \mathcal{T}(\beta) \times_G G_c, \\ \mathcal{V}(\beta) &= \mathcal{T}(\beta, [1]). \end{aligned}$$

Then up to homotopy, there is a T-bundle

$$(\heartsuit) \qquad \mathcal{V}(\beta) \to [G_+(\beta, [1])/U_+] \to [G_+(\beta, [1])/B_+] \xrightarrow{\sim} [\mathcal{U}(\beta)/G],$$

where $\mathcal{U}(\beta)$ is the G-scheme over $\mathcal{U}$ that I introduced in my preprint.

Recall that the G-equivariant, weight-graded compactly-supported cohomology of $\mathcal{U}(\beta)$ records the lowest a-degree of the Khovanov–Rozansky homology of the conjugacy class $[\beta]$: Explicitly,

$$\mathrm{gr}^{\boldsymbol{w}}_{j+2r} \mathrm{H}^{j+k+2r}_{!,G}(\mathcal{U}(\beta)) \simeq \mathrm{HHH}^{0,j,k}(\beta),$$

where $r = \dim \mathfrak{t}$. Let $w \in W$ be the image of β, and let $r(w) = \dim \mathfrak{t}^w$. I expect the T-bundle ($\heartsuit$) to trivialize, via an argument involving some contracting action on $G_+(\beta, [1]) \to \mathcal{U}$. Then we would have

$$\mathrm{H}^*_{!,G}(\mathcal{V}(\beta)) \simeq \mathrm{H}^*_{!,G}(\mathbf{G}_m^{r-r(w)}) \otimes \mathrm{H}^*_{!,G}(\mathcal{U}(\beta)).$$

Above, $\mathbf{G}_m^{r-r(w)}$ stands for a subtorus of T complementary to $T^w \subseteq T$.

8.2.

Suppose that $G = \mathrm{SL}_2$, so that $W = \{1, s\}$. Then $s_1 = \cdots = s_\ell = s$. We fix a coordinate chart:

$$\begin{array}{rcl} \mathbf{G}_m \times \mathbf{A}^\ell & \xrightarrow{\sim} & \mathcal{T}(\beta) \\ (a, z_1, z_2, \ldots, z_\ell) & \mapsto & \left[\left(\begin{smallmatrix} a & \\ & 1/a \end{smallmatrix}\right)\left(\begin{smallmatrix} & 1 \\ 1 & z_1 \end{smallmatrix}\right), \left(\begin{smallmatrix} & 1 \\ 1 & z_2 \end{smallmatrix}\right), \ldots, \left(\begin{smallmatrix} & 1 \\ 1 & z_\ell \end{smallmatrix}\right)\right] \end{array}$$

Let $M_\ell = \left(\begin{smallmatrix} & 1 \\ 1 & z_1 \end{smallmatrix}\right) \cdots \left(\begin{smallmatrix} & 1 \\ 1 & z_\ell \end{smallmatrix}\right)$. We compute

$$M_2 = \begin{pmatrix} 1 & z_2 \\ z_1 & 1 + z_1 z_2 \end{pmatrix},$$

$$M_3 = \begin{pmatrix} z_2 & 1 + z_2 z_3 \\ 1 + z_1 z_2 & z_1 + z_3 + z_1 z_2 z_3 \end{pmatrix},$$

$$M_4 = \begin{pmatrix} 1 + z_2 z_3 & z_2 + z_4 + z_2 z_3 z_4 \\ z_1 + z_3 + z_1 z_2 z_3 & 1 + z_1 z_2 + z_1 z_4 + z_3 z_4 + z_1 z_2 z_3 z_4 \end{pmatrix}$$

A point $c \in T \mathbin{/\!\!/} W$ can be represented as an unordered pair of (inverse) eigenvalues $\{\lambda, \lambda^{-1}\}$ with $\lambda \in \mathbf{G}_m$. For instance, $[1]$ corresponds to $\lambda = \lambda^{-1} = 1$. The locus $\mathcal{T}(\beta, c) \subseteq \mathcal{T}(\beta)$ is precisely the closed subscheme where

$$\mathrm{tr}\left(\left(\begin{smallmatrix} a & \\ & 1/a \end{smallmatrix}\right) M_k\right) = \lambda + \lambda^{-1}.$$

In this way, we obtain the defining equation of $\mathcal{T}(\beta, c)$ for small values of ℓ:

ℓ	$\mathcal{T}(\beta, c)$
0	$a + a^{-1} = \lambda + \lambda^{-1}$
1	$a^{-1} z_1 = \lambda + \lambda^{-1}$
2	$a + a^{-1}(1 + z_1 z_2) = \lambda + \lambda^{-1}$
3	$a z_2 + a^{-1}(z_1 + z_3 + z_1 z_2 z_3) = \lambda + \lambda^{-1}$
4	$a(1 + z_2 z_3) + a^{-1}(1 + z_1 z_2 + z_1 z_4 + z_3 z_4 + z_1 z_2 z_3 z_4) = \lambda + \lambda^{-1}$

For instance:

(1) If $\ell = 0$, then $\mathcal{T}(\beta, c)$ parametrizes $a \in \mathbf{G}_m$ such that

$$a + a^{-1} = \lambda + \lambda^{-1}.$$

If $\lambda \neq \pm 1$, then there are precisely two solutions for a. Otherwise, there is only one solution: namely, $a = \lambda$.

(2) If $\ell = 1$, then $\mathcal{T}(\beta, c)$ parametrizes $(a, z_1) \in \mathbf{G}_m \times \mathbf{A}^1$ such that

$$z_1 = a(\lambda + \lambda^{-1}).$$

For any value of λ, this locus is a copy of $\mathbf{G}_m$ because a can be chosen freely.

(3) If $\ell = 2$, then $\mathcal{T}(\beta, c)$ parametrizes $(a, z_1, z_2) \in \mathbf{G}_m \times \mathbf{A}^2$ such that

$$z_1 z_2 = (\lambda + \lambda^{-1}) - (a + a^{-1}).$$

If $\lambda \neq \pm 1$, then there are precisely two values of $a \in \mathbf{G}_m$ where the right-hand side vanishes. Otherwise, $a = \lambda$ is the only such value. Thus, $\mathcal{T}(\beta, c)$ is a flat family of curves over $\mathbf{G}_m$, with generic fiber $\mathbf{G}_m$ and degenerate fiber(s) $\{z_1 z_2 = 0\} \subseteq \mathbf{A}^2$, such that if $\lambda \neq \pm 1$, then there are exactly two degenerate fibers, and else there is only one.

These examples and others lead us to a surprising possibility.

8.3.

Recall that the Steinberg scheme of β is the fiber product

$$\mathcal{Z}(\beta) = \mathcal{U}(\mathbf{1}) \times_{\mathcal{U}} \mathcal{U}(\beta),$$

where $\mathcal{U}(\mathbf{1}) \to \mathcal{U}$ is the Springer resolution. When β is elliptic, $[\mathcal{Z}(\beta)/G]$ is Deligne–Mumford. In this setting, I conjectured that its coarse space deformation retracts onto the Iwahori affine Springer fiber of any loop of braid class $[\beta]$. A discussion with Oscar suggested to me that this expectation still holds for general β, once we augment $[\mathcal{Z}(\beta)/G]$ by a T^w-bundle (where w is the image of β in W), and replace the affine Springer fiber with its full lattice quotient. At the same time, Michael and Roman conjecture that if $c \in T /\!\!/ W$ is *regular*, then the intersection

$$U_- T U_+ \cap G_c$$

retracts onto the lattice quotient of any Iwahori affine Springer fiber of braid class $[\pi]$, where π is the full twist.

The following observation reformulates part of the work of Jonathan Wang's PRIMES student, Andrew Gu: Namely, Theorem 3.11 in his draft from June 1.

Lemma 8.1. *For arbitrary $c \in T /\!\!/ W$, we have an isomorphism*

$$U_- T U_+ \cap G_c \simeq \mathcal{T}(\pi, c)$$

compatible with the natural maps to G_c on both sides.

Proof. Observe that the isomorphisms

$$\begin{aligned} U_- T U_+ &\simeq \dot{w}_0 U_+ \dot{w}_0^{-1} T U_+ \\ &\simeq T \dot{w}_0 U_+ \dot{w}_0 U_+ \\ &\simeq T \dot{w}_0 U_+ \times^{U_+} U_+ \dot{w}_0 U_+ \end{aligned}$$

respect the natural maps to G, hence can be restricted from G to G_c. $\square$

In conclusion: When $\beta = \pi$, we expect that $\mathcal{T}(\beta, c)$ and a certain T^w-bundle over $[\mathcal{Z}(\beta)/G]$ both retract onto the same variety. Note that in this case, $W = 1$, so $T^w = T$. For general β, there is an obvious T-bundle over $[\mathcal{Z}(\beta)/G]$: namely,

$$\begin{aligned}\mathcal{U}(\mathbf{1}) \times_{\mathcal{U}} \mathcal{V}(\beta) &\xrightarrow{\sim} [\mathcal{U}(\mathbf{1})/G] \times_{[\mathcal{U}/G]} \mathcal{V}(\beta) \\ &\to [\mathcal{U}(\mathbf{1})/G] \times_{[\mathcal{U}/G]} [\mathcal{U}(\beta)/G] \\ &\xrightarrow{\sim} [\mathcal{Z}(\beta)/G].\end{aligned}$$

We therefore arrive at:

Conjecture 8.2. *If $c \in T^{\mathrm{reg}} /\!\!/ W$, then there is a homotopy equivalence*

$$\mathcal{T}(\beta, c) \sim \mathcal{U}(\mathbf{1}) \times_{\mathcal{U}} \mathcal{V}(\beta)$$

(Recall that $\mathcal{V}(\beta) = \mathcal{T}(\beta, [1])$.)

In what follows, let $\mathfrak{c} = \mathfrak{t} /\!\!/ W$ and $\mathfrak{c}^{\mathrm{reg}} = \mathfrak{t}^{\mathrm{reg}} /\!\!/ W$.

Conjecture 8.3. *Let $a \in \mathfrak{c}^{\mathrm{reg}}(\mathbf{C}(\!(\epsilon)\!)) \cap \mathfrak{c}(\mathbf{C}[\![\epsilon]\!])$. Let $\mathfrak{X}_a$ and $\tilde{\mathfrak{X}}_a$ be the spherical and Iwahori affine Springer fibers for a, respectively, and let Λ_a be the usual (full) lattice that acts on these schemes.*

Let $[\beta] \subseteq Br_W$ be the braid conjugacy class associated with a, where β denotes a positive representative. Let $w \in W$ be the image of β, and let $r(w) = \dim \mathfrak{t}^w$. Then:

(1) $\mathcal{V}(\beta)$ retracts onto a $\mathbf{G}_m^{r-r(w)}$-bundle over $\mathfrak{X}_a/\Lambda_a$.

(2) $\mathcal{T}(\beta, c)$ retracts onto a $\mathbf{G}_m^{r-r(w)}$-bundle over $\tilde{\mathfrak{X}}_a/\Lambda_a$, for any $c \in T^{\mathrm{reg}} /\!\!/ W$.

(In particular, if $\beta = \pi$, then $w = 1$ and $r - r(w) = 0$.)