(Munkres §24)

<u>Q</u> is R connected in the analytic topology?
why is this tricky?

last time, showed:
Q as a subspace of R is disconnected,
even though it is <u>dense</u> in R (i.e. Cl_R(Q) = R)

any proof that R is connected must use
some fact about R that fails for Q

<u>Df</u> given a set X with a total order <
it has the least upper bound property iff
every subset of X bounded above
has a least upper bound (= <u>sup</u>)

<u>Ex</u> in an intro analysis course,
we define R from scratch

the defn shows that R has the LUBP
but Q does not: e.g. {x in Q | x^2 < 2} has no LUB

the LUBP is not enough to prove R is connected:
why?

<u>Ex</u> if X sub R finite and |X| ≥ 2
then X disconnected but has the LUBP

<u>Df</u> a set X with a total order < is called
a linear continuum iff

1) it has the LUBP
2) for all x < y in X, there is z s.t. x < z < y

thus R is a linear continuum

Thm R is connected in the analytic topology

Pf suppose U, V form a separation of R
pick a in U and b in V
WLOG we can assume a < b

let A = [a, b] cap U and B = [a, b] cap V
then A, B form a separation of [a, b]
in its subspace top
but A is bounded above by b
so by the LUBP, sup A exists and is in [a, b]

remains to show 1) sup A notin B
2) sup A notin A

1) assume sup A in B
then sup A > a: else A = Ø
but B is open in [a, b], so (sup A – δ, sup A] sub B
for some δ > 0
but then, sup A – ε is a smaller upper bound for A
whenever 0 < ε ≤ δ

2) assume sup A in A
then sup A < b: else, B = Ø
but A is open in [a, b], so [sup A, sup + δ) sub A
for some δ > 0
but then sup A + ε is an elt of A above sup A
whenever 0 < ε < δ □

Rem thm generalizes to any linear continuum
in its order topology (we did not define)

for this course, just note:
the proof still works if we replace R with
(a, b), (a, b], [a, b), [a, b]
(–∞, b), (–∞, b], (a, ∞), [a, ∞)

Thm (Intermediate Value)

suppose X connected and f : X to R cts
wrt the analytic top on R
if a, b in X and f(a) ≤ α ≤ f(b)
then there is some c in X s.t. f(c) = α

Lem if X is connected, f : X to Y cts,
then f(X) sub Y is connected

Pf preimage of separation is a separation

Pf of Thm suppose no c in X s.t. f(c) = α

then f(X) has a separation: {f(x) < α | x in X},
{f(x) > α | x in X}

but X is connected, so contradiction with lemma

Moral the connectedness of intervals in R
is very well-understood

Idea study connectedness in other spaces
by comparing them to intervals in R

Df for any X and x, y in X
a path from x to y is a cts map
γ : [a, b] to X s.t. γ(a) = x and γ(b) = y

we say X is <u>path-connected</u> iff
there is a path between every pair of pts in X

<u>Rem</u> we require a ≤ b
but otherwise a, b can be any numbers

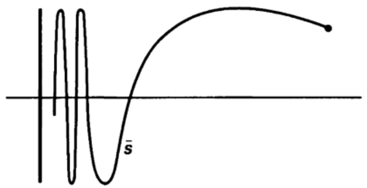
<u>Lem</u> if X is path-connected
then X is connected

<u>Pf</u> if X = Ø, then done
else, can fix x in X

for all y in X, pick a path γ_y from x to y
then X = bigcup_y γ_y([0, 1])
each γ_y([0, 1]) is connected and contains x
so X is connected

[famous non-example:]

<u>Ex</u> in analytic R^2: consider
S = {(x, sin(1/x)) in R^2 | 0 < x ≤ 1}
A = {(0, y) | –1 ≤ y ≤ 1}
Š = S cup A = the closure of S in R^2



the <u>topologist's sine curve</u>: Š in its subspace top.

not hard: S, A are path-connected
claim: Š is not

sketch: fix a in A and s in S

suppose $\gamma : [0, 1]$ to Š a path from a to s

claim there is a largest t_0 in $[0, 1]$ s.t. $\gamma(t_0)$ in A:

A is closed in R, hence in Š

so $\gamma^{-1}(A)$ is closed in $[0, 1]$

so $\gamma^{-1}(A)$ is its own closure

so it contains its sup

so for all t in $(t_0, 1]$, have $\gamma(t)$ in S

let $(t_n)_n$ be a decreasing seq converging to t_0

for all n: pick $0 < x_n <$ x-coord$(\gamma(t_n))$ s.t.

$\sin(1/x_n) = (-1)^n$

by IVT: x-coord$(\gamma(t_0)) < x_n <$ x-coord$(\gamma(t_n))$

means there is t'_n in $[t_0, t_n]$ s.t.

$\gamma(t'_n) = x_n$

(Munkres §25)

Lem for any X, the following are equiv. rel's:

1) $x \sim y$ iff there is a connected subspace of X that contains both x and y

2) $x \leftrightarrow y$ iff there is a path between x and y in X

Pf transitivity for 1): if x, y in A & y, z in B, then A cap B ni y

transitivity for 2): [Munkres 18.3:]

Pasting Lem let $Y = Y_1 \cup Y_2$

$f_i : Y_i$ to X for $i = 1, 2$

if f_1, f_2 cts and $f_1(x) = f_2(x)$ on $Y_1 \cap Y_2$

then $f : Y$ to X def by $f|_{Y_i} = f_i$ is cts

<u>Pf</u> boring

<u>Df</u> the connected components of X are
the equiv. classes under $\sim$
the path components of X are
the equiv. classes under $\leftrightarrow$

<u>Ex</u> with some work, we can show:

S and A are the conn. components of $\check{S}$,
and also, its path components