

# WHAT GAUSS KNEW ABOUT KNOTS AND BRAIDS

2021 MIT IAP MATHEMATICS LECTURE SERIES

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**Problem 1.** Show that the figure-8 knot is not tricolor.



**Problem 2.** Read about the [diagrammatic definition](#) of the linking number. Show that the link below, appropriately oriented, has linking number zero:



**Problem 3.** Show that for all  $n \geq 2$ , there is a braid on  $n$  strands whose closure is the unknot. It is *not* the identity.

**Problem 4.** Let  $\sigma_1$  denote the generator of  $Br_2$ .

- (1) Show that the HOMFLYPT invariants  $\mathbf{P}(\widehat{\sigma_1^n})$  obey a linear recurrence in  $n$ .
- (2) Deduce that the [\(2, n\)-torus links](#) are pairwise non-isotopic.

**Problem 5.** Show that in  $Br_4$ , the elements

$$(\sigma_2\sigma_1\sigma_3\sigma_2)^3\sigma_1^7 \quad \text{and} \quad (\sigma_1\sigma_2\sigma_3)^6\sigma_1$$

have the same link closure. How would you generalize this observation?

## REFERENCES

- [1] J. W. Alexander. A Lemma on Systems of Knotted Curves. *Proc. Nat. Acad. Sci. USA*, **9** (1923), 93-95.
- [2] J. W. Alexander. Topological Invariants of Knots and Links. *Trans. AMS*, **30**(2) (1928), 275-306.
- [3] E. Artin. Theory of Braids. *Ann. Math.*, Second Series, **48**(1) (1947), 101-126.
- [4] S. Chmutov, S. Duzhin, J. Mostovoy. *Introduction to Vassiliev Knot Invariants*. Cambridge University Press (2012). [arXiv:1103.5628](#)
- [5] D. DeTurck, H. Gluck, R. Komendarczyk, P. Melvin, H. Nuchi, C. Shonkwiler, D. S. Vela-Vick. Generalized Gauss Maps and Integrals for Three-Component Links: Toward Higher Helicities for Magnetic Fields and Fluid Flows, Part II. *Alg. Geom. Top.*, **13** (2013), 2897-2923.
- [6] M. Epple. Orbits of Asteroids, a Braid, and the First Link Invariant. *The Mathematical Intelligencer*, **20**(1) (1998), 45-52.
- [7] P. Freyd, D. Yetter, J. Hoste, W. B. R. Lickorish, K. Millett, A. Ocneanu. A New Polynomial Invariant of Knots and Links. *Bull. AMS*, **12**(2) (1985), 239-246.
- [8] V. F. R. Jones. Hecke Algebra Representations of Braid Groups and Link Polynomials. *Ann. Math.*, **126**(2) (Sep. 1987), 335-388.
- [9] N. Yu. Reshetikhin & V. G. Turaev. Ribbon Graphs and Their Invariants Derived from Quantum Groups. *Commun. Math. Phys.*, **127** (1990), 1-26.
- [10] E. Witten. Quantum Field Theory and the Jones Polynomial. *Commun. Math. Phys.*, **121** (1989), 351-399.