The Gen and Cogen Formulas for Torus Knot Homology

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triply graded Khovanov–Rozansky homology

$$\mathrm{HHH} : \{\text{links}\}/\text{isotopy} \to \mathrm{Vect}_{\text{3-gr}}$$

KhR polynomial $\quad \mathbf{P}_L(a, q, t) = \sum_{i,j,k} a^i q^j t^k \, \mathrm{HHH}^{i,j,k}(L)$

$\mathbf{P}_{\text{unknot}}(a, q, t) = 1 \qquad$ (see [KT] for conventions)

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(Elias–Hogancamp–Mellit) computed $\mathbf{P}_{m,n} := \mathbf{P}_{\text{torus}(m,n)}$ for all $m, n > 0$

$$\text{torus}(m, n) = \widehat{\beta_n^m}, \quad \text{where } \beta_n = \sigma_1 \sigma_2 \cdots \sigma_{n-1} \in \mathrm{Br}_n.$$

<u>Ex</u> $\quad \mathbf{P}_{2,2} = \mathbf{P}_{\text{Hopf link}} = 1 + \frac{qt}{1-q} + \frac{at}{1-q}$

<u>Ex</u> $\quad \mathbf{P}_{2,3} = \mathbf{P}_{\text{trefoil}} = 1 + qt + at$

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<u>Thm</u> $\quad \sum_{\Delta \in D_{m,n}} q^{g_\Delta} t^{h_\Delta} f_\Delta^{\text{Gen}} = \mathbf{P}_{m,n} \overset{\text{GMV}}{=} \sum_{\Delta \in D_{m,n}} q^{g_\Delta} t^{h_\Delta} f_\Delta^{\text{Cogen}}$

where the LHS requires m, n coprime

<u>Ex</u> $\quad \mathbf{P}_{4,3}$ has 11 terms:

$q^{g_\Delta} t^{h_\Delta}$	Gen^+		Cogen	
$q^3 t^3$	$\emptyset$	1	$\{5\}$	$1 + aq^{-1}$
$q^2 t^2$	$\{5\}$	$1 + at$	$\{1, 2\}$	$(1 + aq^{-1})(1 + aq^{-1}t)$
qt^2	$\{1\}$	$1 + at$	$\{2\}$	$1 + aq^{-1}$
qt	$\{2\}$	$1 + at$	$\{1\}$	$1 + aq^{-1}$
1	$\{1, 2\}$	$(1 + at)(1 + at^2)$	$\emptyset$	1

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$\mathbf{N}_0 = \{0, 1, 2, \ldots\}$

$$D_{m,n} = \{\Delta \subseteq \mathbf{N}_0 \mid \Delta + m, \Delta + n \subseteq \Delta \text{ and } 0 \in \Delta\}.$$

$g_\Delta = |\mathbf{N}_0 - \Delta|$
$h_\Delta = \sum_{k \in \mathrm{Gen}_n(\Delta)} |\{k < j < k + n \mid j \notin \Delta\}|$, where

$$\mathrm{Gen}_n(\Delta) = \{k \in \Delta \mid k - n \notin \Delta\}.$$

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$\mathrm{Gen}(\Delta) = \{k \in \Delta \mid k - m, k - n \notin \Delta\}$
$\mathrm{Gen}^+(\Delta) = \mathrm{Gen}(\Delta) - \{0\}$
$\mathrm{Cogen}(\Delta) = \{k \in \mathbf{N}_0 - \Delta \mid k + m, k + n \in \Delta\}$

$$\begin{aligned} f_\Delta^{\mathrm{Gen}} &= \prod_{k \in \mathrm{Gen}^+} (1 + at^{|\{j \in \mathrm{Gen}_n \mid k-m<j<k\}|}), \\ f_\Delta^{\mathrm{Cogen}} &= \prod_{k \in \mathrm{Cogen}} (1 + aq^{-1}t^{|\{j \in \mathrm{Gen}_n \mid k+n<j<k+n+m\}|}) \end{aligned}$$

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<u>Ex</u> take $\Delta = \{0, 3, 4, 5, 6, \ldots\} = \mathbf{N}_0 - \{1, 2\} \qquad \in D_{4,3}$

$\mathrm{Gen}_3(\Delta) = \{0, 4, 5\}, \quad \mathrm{Gen}^+(\Delta) = \{5\}, \quad \mathrm{Cogen}(\Delta) = \{1, 2\},$

$$\begin{aligned} f_\Delta^{\mathrm{Gen}} &= 1 + at^{|\{j \in \mathrm{Gen}_3 \mid 5-4<j<5\}|} \\ &= 1 + at \\ f_\Delta^{\mathrm{Cogen}} &= (1 + aq^{-1}t^{|\{j \in \mathrm{Gen}_3 \mid 1+3<j<1+7\}|}) \\ &\qquad \cdot (1 + aq^{-1}t^{|\{j \in \mathrm{Gen}_3 \mid 2+3<j<2+7\}|}) \\ &= (1 + aq^{-1}t)(1 + aq^{-1}) \end{aligned}$$

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Gen formula from Mellit, <u>G&T</u> (2022)

Cogen formula from Hogancamp–Mellit, arXiv:1909.00418, via Gorsky–Mazin–Vazirani (2020)

Both rely on recursions from Elias–Hogancamp (2019)

In turn starts from Khovanov (2008):

$$\text{braid } \beta \quad \rightsquigarrow \quad \text{bimodule complex } F_\beta \quad \rightsquigarrow \quad \mathrm{HHH}(\hat{\beta})$$

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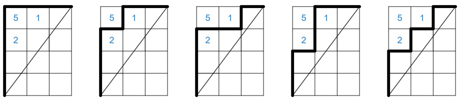
Mellit's recursion decomposes β_n^m into a sum indexed by m/n Dyck paths:



bijection $D_{m,n} \xrightarrow{\sim} \{m/n \text{ Dyck paths } \pi\}$

role of Dyck paths predicted by Gorsky–Neguţ (2016)

start with diagonal and "sweep up" applying local rules from Mellit, <u>Duke</u> (2021)

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$$\nu_*(\pi) = \{\text{bottom right corners of squares} \searrow \pi\}$$
$$\nu^*(\pi) = \{\text{top left corners of squares} \nwarrow \pi\}$$

<u>Lem</u> if $\Delta \mapsto \pi$, then:

(1) $\{j \in \mathrm{Gen}_n(\Delta) \mid k - m < j < k\} \xrightarrow{\sim} \nu_*(\pi)$

(2) $f_\Delta^{\mathrm{Gen}} = \prod_{p \in \nu_*(\pi)} (1 + at^{|\kappa_\pi(p)|}) = \frac{1}{1+a} \prod_{p \in \nu^*(\pi)} (1 + at^{|\kappa_\pi(p)|})$

where $\kappa_\pi(p) = \{\text{horizontal steps of } \pi \text{ meeting } l_{m/n}(p)\}$

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we used the Gen formula to show:

<u>Thm</u> for fixed $n > 0$,

$$\frac{1}{1-q} \lim_{\substack{m \to \infty \\ \gcd(m,n)=1}} \mathbf{P}_{m,n}(a, q, t) = \prod_{1 \le j \le n} \frac{1 + at^{j-1}}{1 - qt^{j-1}}$$

$\frac{1}{1-q}\,\mathbf{P}_{m,n}(a, q, t)$ is determined by its expansion up to q-degree $\frac{(m-1)(n-1)}{2}$, by by palindromicity of $\mathbf{P}$

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<u>Cor</u> formula for $\mathbf{P}_{m,n}$ when $n \le 3$ and $\gcd(m, n) = 1$

<u>Pf</u> $\frac{1}{1-q}\,\mathbf{P}_{m,n}$ matches $\frac{1}{1-q}\,\mathbf{P}_{\infty,n}$ up to q-degree m

<u>Cor</u> the ORS conjecture for $y^n = x^m$, for these (m, n):

$$\frac{\mathbf{P}_{m,n}(a, q, qt^2)}{1-q} = \sum_{k,\ell} a^k q^\ell t^{k^2-k} \chi(t, \underbrace{\mathrm{Hilb}^{\ell}_{k\text{-nest}}(y^n = x^m)}_{\subseteq \mathrm{Hilb}^{\ell} \times \mathrm{Hilb}^{\ell+k}})$$

<u>Pf</u> ORS established a matching formula on the Hilb side