Last Time $x = (1, 0)$ in S^1

$\omega_n : [0, 1]$ to S^1 def by
$\omega_n(s) = (\cos(2\pi ns), \sin(2\pi ns))$ for all n in Z

Thm $\Phi : Z$ to $\pi_1(S^1, x)$ def by $\Phi(n) = [\omega_n]$
is an isomorphism

we proved Φ is a homomorphism using some auxiliary maps [which ones?]:

- $p : R$ to S^1 def by
  $p(x) = (\cos(2\pi x), \sin(2\pi x))$
- $\acute{\omega}_{\{a, b\}} : [0, 1]$ to R
  $\acute{\omega}_{\{a, b\}}(s) = (1 - s)a + sb$
  for any a, b in Z

if $b - a = n$, then $\omega_n = p \circ \acute{\omega}_{\{a, b\}}$

Claim Φ is bijective

Assumptions (to be postponed till later)

1) for any path $\gamma : [0, 1]$ to S^1 and a in R s.t.
   $p(a) = \gamma(0)$,
   there is a unique $\Gamma : [0, 1]$ to R s.t.
   $\Gamma(0) = a$ and $\gamma = p \circ \Gamma$,
   which we will call a lift of γ to R

2) for any path homotopy $h : [0, 1]^2$ to S^1 s.t.
   $h(-, 0) = \gamma$, and lift Γ of γ,
   there is a path homotopy $H : [0, 1]^2$ to R s.t.
   $H(-, 0) = \Gamma$ and $h = p \circ H$

Conditional Pf of Claim

Φ is surjective:

pick an elt of $\pi_1(S^1, x)$, where $x = (1, 0)$
it must be $[\gamma]$ for some loop γ based at x
note that $p^{-1}(x) = \mathbb{Z} \subseteq \mathbb{R}$

pick a in $\mathbb{Z}$
by 1), get a lift Γ of γ to $\mathbb{R}$ starting at a
let $b = \Gamma(1)$
then we also need $p(b) = \gamma(1) = x$
so we also need b in $p^{-1}(x) = \mathbb{Z}$
so Γ is a path in $\mathbb{R}$ from a to b
so get a path-homotopy from Γ to $\acute{\omega}_{a, b}$
so get a path-homotopy from γ to $\omega_{b - a}$

Φ is injective:

enough to show that $[\omega_m] = [\omega_n]$ implies
$m = n$
pick path homotopy h from ω_m to ω_n
recall that $\acute{\omega}_{0, m}$ lifts ω_m
by 2), get path homotopy $H : [0, 1]^2 \to \mathbb{R}$ s.t.
$H(-, 0) = \acute{\omega}_{0, m}$ and $h = p \circ H$

let $\Gamma = H(-, 1)$
as H is a path homotopy, must have
$\Gamma(0) = \acute{\omega}_{0, m}(0) = 0$,
$\Gamma(1) = \acute{\omega}_{0, m}(1) = m$
but Γ is also a lift of ω_n to $\mathbb{R}$ starting at 0
and $\acute{\omega}_{0, n}$ is another such lift
so by the uniqueness in 1), need $\Gamma = \omega_n$
so $m = \Gamma(1) = n$ □

[what can we do with the iso $\pi_1(S^1, x) = Z$?]

recall that if G, K are groups, so is $G \times K$
under a coordinate-wise law

Lem for any x in X and y in Y, there is an iso

$\pi_1(X \times Y, (x, y))$ to $\pi_1(X, x) \times \pi_1(Y, y)$
[what is the map?]

Pf send $[\gamma]$ mapsto $([p \circ \gamma], [q \circ \gamma])$
where $p : X \times Y$ to X and $q : X \times Y$ to Y
are the projections

we define the (surface) torus to be $T = S^1 \times S^1$

Cor $\pi_1(T, (x, y)) = Z^2$ for any x, y in S^1

Cor the torus and the 2-sphere are
not homeomorphic

now consider $Y = R^2 - \{(0, 0)\}$ and $x = (1, 0)$
what is $\pi_1(Y, x)$?

π_1 still iso to (Z, +)
but Y is not homeomorphic to S^1 [why?]
so it is possible for non-homeomorphic spaces
to have isomorphic fundamental groups

however, Y is still similar in shape to S^1
[to quantify that:]

(Munkres §55, 58) let X, Y be top spaces

recall: cts maps $f_0, f_1 : X$ to Y are homotopic iff there is a cts $h : X \times [0, 1]$ to Y s.t.

$h(–, 0) = f_0$,
$h(–, 1) = f_1$

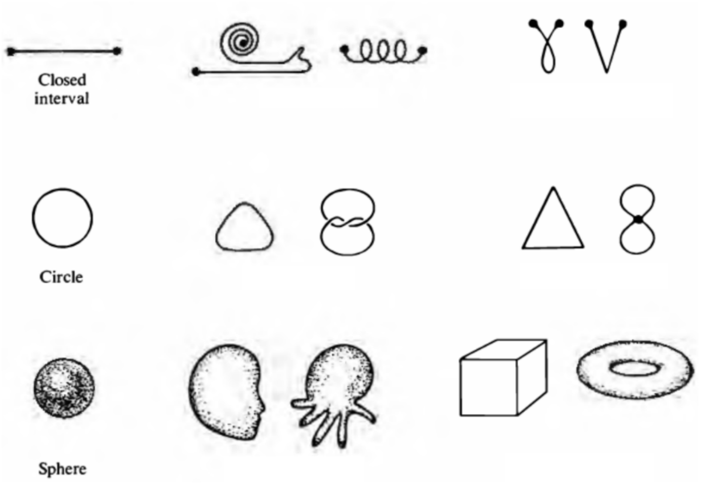
<u>Df</u> a <u>homotopy equivalence</u> btw X and Y consists of maps $f : X$ to Y and $g : Y$ to X

s.t. $g \circ f$ is homotopic to id_X
$f \circ g$ is homotopic to id_Y

often, we abuse language by saying f or g alone is the homotopy equivalence

we write $X \sim Y$ iff there is a homotopy equivalence btw them

then $\sim$ is an equivalence relation on top spaces so when $X \sim Y$, we also say X and Y are <u>homotopy equivalent</u>



<u>Ex</u> let $Y = R^2 - \{(0, 0)\}$
we claim that the inclusion $f : S^1 \to Y$
is (half of) a homotopy equivalence

first we need to give $g : Y \to S^1$: [what works?]
take the radial projection

$g(v) = v/|v|$ where $|(x, y)| = \sqrt{x^2 + y^2}$

now $g \circ f = id_{S^1}$ on the nose, while $f \circ g = g$
so just need a homotopy btw g and id_Y: eas(ier)

<u>Df</u> in general, if $s : X \to Y$ and $r : Y \to X$
satisfy $r \circ s = id_X$
then s is called a <u>section</u> to r
r is called a <u>retract</u> [!] to s

in this case, s_* is inj and r_* is surj [PS6, #7]
more generally:

<u>Lem</u> if $\varphi : G \to G'$ and $\psi : G' \to G''$ are maps
s.t. $\psi \circ \varphi$ is bijective
then φ is inj and ψ is surj

<u>Thm</u> if $f : X \to Y$ is a homotopy equivalence
then $f_* : \pi_1(X, x) \to \pi_1(Y, f(x))$ is
an iso for any x in X

<u>Next Time</u> proof of this thm