5.

Notes on discussions with Oscar during his 4/28 visit to Yale.

Fix $f \in \mathbf{C}[x, y]$, unibranch through $(0, 0)$. Fix a coordinate t on the normalization of $f(x, y) = 0$. Let $R = \mathbf{C}[[x, y]]/(f)$ and $S = \mathbf{C}[[t]]$, so that the normalization map defines an embedding $R \hookrightarrow S$. Abusing notation, let $x(t), y(t) \in S$ be the images of $x, y \in R$.

5.1.

Fix $k > 0$. Form

$$R_k = R[[x, y]]/(f^k) \quad \text{and} \quad S_r = \mathbf{C}[[t]][\epsilon]/(\epsilon^k).$$

We want to lift $R \hookrightarrow S$ to an embedding $R_k \hookrightarrow S_k$. By construction, such a lift must send $x \mapsto x(t) + \epsilon p$ and $y \mapsto y(t) + \epsilon q$ for some $p, q \in S_k$.

Example 5.1. Take $f = y^2 - x^3$ and $k = 2$. Then we may assume that $x(t) = t^2$ and $y(t) = t^3$ If the lift $R_2 \hookrightarrow S_2$ exists, then it must send $x \mapsto t^2 + \epsilon p$ and $y \mapsto t^3 + \epsilon q$, where we may assume that p, q only involve t. Then it also sends f to

$$\begin{aligned} f(t^2 + \epsilon p, t^3 + \epsilon q) &= (t^3 + \epsilon q)^2 - (t^2 + \epsilon p)^3 \\ &= (t^6 + 2t^3 \epsilon q) - (t^6 + 2t^2 \epsilon p) \\ &= 2t^2 \epsilon (tq - p). \end{aligned}$$

So many lifts are possible: for example, $p = t - 1$ and $q = 1$.

5.2.

Recall that the conductor ideal of the inclusion $R \subseteq S$ is precisely

$$\{p \in S \mid \mathrm{val}_t(p) \geq 2\delta\} = t^{2\delta} S.$$

(In fact, this result holds for any Gorenstein integral domain embedded in $\mathbf{C}[[t]]$, according to Serre as cited by Pfister–Steenbrink in the introduction of their article.)

Following Carlsson–Oblomkov, let $\bar{R} = R/t^{2\delta} S$. For any R-module M, let $\mathscr{Q}(M)$ be the Quot scheme parametrizing (finite-colength) R-submodules of M. For any (finite-colength) R-submodule $N \subseteq M$, let $\mathscr{G}(M, N)$ be the closed subscheme parametrizing the R-submodules of M that contain N. Then Carlsson–Oblomkov claim that there is an isomorphism

$$\Psi : \mathscr{G}(S, R) \xrightarrow{\sim} \mathscr{Q}(\bar{R}) \quad \text{given by } \Psi(M) = \mathrm{Ext}^1_R(S/M, R).$$

To clarify how the map is defined: Observe that applying $\mathrm{Hom}_R(-, R)$ to the s.e.s. of R-modules $0 \to R \to S \to S/R \to 0$, then simplifying, we get

$$0 = \mathrm{Hom}(S/R, R) \to \mathrm{Hom}(S, R) \to \mathrm{Hom}(R, R) \to \mathrm{Ext}^1(S/R, R) \to \mathrm{Ext}^1(S, R),$$

which further simplifies to

$$0 \to t^{2\delta} S \to R \to \mathrm{Ext}^1(S/R, R) \to 0.$$

Thus $\mathrm{Ext}^1(S/R, R) = \bar{R}$. But applying $\mathrm{Hom}_R(-, R)$ to the s.e.s. of R-modules $0 \to M/R \to S/R \to S/M \to 0$, then simplifying, we get

$$0 = \mathrm{Hom}(M/R, R) \to \mathrm{Ext}^1(S/M, R) \to \mathrm{Ext}^1(S/R, R).$$

Thus $\mathrm{Ext}^1(S/M, R)$ forms an R-submodule of $\bar{R}$.

Note that we can further identify $\mathscr{Q}(\bar{R})$ with $\mathscr{G}(R, t^{2\delta}S)$ via pullback along the quotient map $R \to \bar{R}$.

Also note that, by `quot` Lem. 2.4, the fundamental domain for the semilattice action on the Quot scheme $\mathscr{Q}(S)$ is given on points by

$$\mathscr{D} = \{M \in \mathscr{G}(S) \mid M \cap S^\times \neq \emptyset\},$$

whereas we can check that

$$\mathscr{G}(S, R) = \{M \in \mathscr{G}(S) \mid M \ni 1\}.$$

So we always have $\mathscr{G}(S, R) \subseteq \mathscr{D}$, but this inclusion is usually not an equality, since $S^\times$ is usually larger than $R^\times$. (In fact, $S^\times/R^\times$ is the local Picard group.)

Example 5.2. Take $f = y^2 - x^3$, so that $\delta = 1$ and $R = \mathbf{C}[[t^2, t^3]]$ under the embedding into S. For all integers $i \geq 0$ and $\lambda \in \mathbf{C}$, let

$$\begin{aligned} M_{i,\lambda} &= \langle t^i + \lambda t^{i+1} \rangle, \\ N_i &= \langle t^i, t^{i+1} \rangle. \end{aligned}$$

By `quot` Ex. 2.8, the Quot schemes $\mathscr{Q}^\ell(S)$ parametrizing R-submodules of S of specific colengths ℓ look like

$$\begin{aligned} \mathscr{Q}^0(S) &= \{N_0 = S\}, \\ \mathscr{Q}^\ell(S) &= \{M_{\ell-1,\lambda} \mid \lambda \in \mathbf{C}\} \cup \{N_\ell\} \quad \text{for } \ell \geq 1. \end{aligned}$$

Moreover, $\mathscr{D} = \{N_0\} \cup \{M_{0,\lambda} \mid \lambda \in \mathbf{C}\}$. By comparison, $\mathscr{G}(S, R) = \{N_0, M_{0,0}\}$ and $\mathscr{G}(R, t^{2\delta}S) = \{M_0, N_2\}$.

5.3.

There is a silver lining. For any $[u] \in S^\times/R^\times$, we may replace the s.e.s.'s in the arguments above with

$$0 \to uR \to S \to S/uR \to 0 \quad \text{and} \quad 0 \to M/uR \to S/uR \to S/M \to 0$$

to deduce that Carlsson–Oblomkov's map generalizes to an isomorphism

$$\Psi_u : \mathscr{G}(S, uR) \xrightarrow{\sim} \mathscr{G}(R, t^{2\delta}S).$$

However, the $\mathscr{G}(S, uR)$ intersect: for example, because they all contain $N_0 = S$. It seems that the intersections can be complicated when f is complicated.