MATH 340: ADVANCED LINEAR ALGEBRA
PROBLEM SET #8

SPRING 2025

Due Wednesday, April 16. You may consult books, papers, and websites as long as you cite all sources and write up your solutions in your own words.

Problem 1. In the notation of Problem Set 7, #6, let

$$\Phi : \mathrm{Bil}(W, V \mid U) \to \mathrm{Hom}(W, \mathrm{Hom}(V, U))$$

be the map such that, for all $\beta \in \mathrm{Bil}(W, V \mid U)$, we have $[\Phi(\beta)(w)](v) = \beta(w, v)$. Generalizing what we proved about $\mathrm{Bil}(W, V)$ in class, show that Φ is linear and bijective, <u>without</u> picking explicit bases for the vector spaces involved.

Combining this with Problem Set 7, #6, we obtain a injective linear map

$$\mathrm{Hom}(W \otimes V, U) \to \mathrm{Hom}(W, \mathrm{Hom}(V, U))$$

independent of bases. It turns out to be an isomorphism for general U, V, W, though that is difficult to show starting from Axler's definition of $W \otimes V$. For details, see Atiyah–Macdonald, Introduction to Commutative Algebra, Ch. 2.

Problem 2. Let V be a real vector space. Using #5 from Problem Set 7, and the definition of $V_{\mathbf{C}}$ in Problem Set 2, give a linear isomorphism

$$V_{\mathbf{C}} \xrightarrow{\sim} \mathbf{C} \otimes_{\mathbf{R}} V$$

(and verify that it is one), <u>without</u> picking an explicit basis for V. Above, $\otimes_{\mathbf{R}}$ just means $\otimes$, but emphasizes that we view $\mathbf{C}$ as a real vector space.

Problem 3. Let $F \in \{\mathbf{R}, \mathbf{C}\}$. Recall that a bilinear form $\beta : V \times V \to F$ is called *symmetric* if and only if $\beta(w, v) = \beta(v, w)$ for all w, v, and *alternating* if and only if $\beta(v, v) = 0$ for all v. Let $\mathrm{Sym}^2(V)$ and $\mathrm{Alt}^2(V)$ denote the sets of symmetric and alternating bilinear forms on V, respectively. Give a linear isomorphism

$$\mathrm{Bil}(V, V) \xrightarrow{\sim} \mathrm{Sym}^2(V) \oplus \mathrm{Alt}^2(V)$$

(and verify that it is one), <u>without</u> picking an explicit basis for V. *Hint:* Relate the alternating property to *antisymmetry*. Reflect on how this problem resembles several earlier problems.

Problem 4. Show that

$$\langle f, g \rangle = \int_{-1}^{1} f(t)\overline{g(t)}\, dt$$

defines an inner product on the complex vector space $\mathbf{C}[t]$. You may take for granted that polynomials are continuous functions, and that nonzero polynomials have finitely many zeros.

Problem 5. Let V be an inner product space, and let $u, v \in V$.

(1) Show that if V is real, then u and v are orthogonal if and only if $\|u+v\|^2 = \|u\|^2 + \|v\|^2$.
(2) Show that if V is complex, then the conclusion to (1) can fail.

Problem 6. Let V be an inner product space, and let $u, v \in V$.

(1) Show that if V is real, then $u - v$ and $u + v$ are orthogonal if and only if $\|u\| = \|v\|$.
(2) Make a sketch of (1) where $V = \mathbf{R}^2$ and u, v are nonzero. What does it mean, in terms of the geometry of plane triangles?
(3) Suppose that V is complex. Does the conclusion to (1) still hold?

Problem 7. Let V be an inner product space, and let $r \in V$ be nonzero. A *reflection* over r is a linear operator $S_r : V \to V$ satisfying the following conditions:

(I) $S_r(r) = -r$.
(II) $S_r(v) = v$ if and only if $\langle r, v \rangle = 0$.

These conditions determine a unique linear operator. For arbitrary $v \in V$, give an explicit formula for $S_r(v)$ in terms of r, v, and various inner products. *Hint:* Write v as the sum of a scalar multiple of r and a vector orthogonal to r.

Problem 8. We say that a linear operator T on an inner product space V is *orthogonal* if and only if $\langle Tu, Tv \rangle = \langle u, v \rangle$ for all $u, v \in V$. Show that:

(1) In the sense of Problem 7, any reflection is orthogonal.
(2) If V is the real inner product space formed by $\mathbf{R}^2$ under the dot product, then any rotation in V is orthogonal.