Séminaire Lotharingien de Combinatoire **XX** (2025)
Article #YY, 12 pp.

Proceedings of the 37th Conference on Formal Power Series and Algebraic Combinatorics (Sapporo)

Level-Rank Bijections for Finite Reductive Groups

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Abstract. We propose a generalization, for objects from the character theory of finite reductive groups, of the dualities discovered by Uglov between Ariki–Koike algebras at roots of unity, which provide an incarnation of the level-rank duality discovered by Frenkel. At the most concrete level, our conjectures imply remarkable bijections between certain sets of irreducible characters of complex reflection groups, defined via associated Hecke algebras. In type A, we show that the bijections come from Uglov's via the combinatorics of charged partitions. In general, they arise from finite reductive groups via the l-Harish-Chandra series of Broué–Malle–Michel, while the Hecke algebras conjecturally arise from Deligne–Lusztig varieties.

Keywords: charged partition, level-rank duality, finite reductive group, Harish-Chandra series, complex reflection group, Hecke algebra

1 The Original Story

Our goal is to explain some evidence for a family of surprising coincidences within the representation theory of a finite reductive group G. We wish to emphasize the parts of these coincidences that are concrete and combinatorial. In particular, the motivation for our conjectures, and our main source of evidence, comes from the linear cases $G = \mathrm{GL}_n(\mathbb{F}_q)$, where the whole story can be stated in terms of integer partitions.

1.1 Partitions and Abaci

It will be convenient for us to regard an integer partition as an infinite sequence $\pi = (\pi_1 \geq \pi_2 \geq \cdots)$ such that $\pi_i = 0$ for all i large enough. As usual, we define its size to be $|\pi| = \pi_1 + \pi_2 + \ldots$ We define an *m-partition* to be an m-tuple of partitions

$$\vec{\pi} = (\pi^{(0)}, \pi^{(1)}, \ldots, \pi^{(m-1)}),$$

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indexed from 0 through $m-1$, and the *size* of $\vec{\pi}$ to be $\sum_{i=0}^{m-1} |\pi^{(i)}|$. Sometimes we will regard these indices as elements of $\mathbb{Z}_m := \mathbb{Z}/m\mathbb{Z}$. We define the empty partition by $\varnothing = (0,0,\ldots)$, and the empty m-partition by $\vec{\varnothing} = (\varnothing,\varnothing,\ldots,\varnothing)$.

For any fixed $m > 0$, there is a relationship between partitions and m-partitions of purely combinatorial interest. To give more detail, recall that a partition is an *m-core* if and only if it contains no hook lengths divisible by m. For instance, the only 1-core is the empty partition, while the 2-cores are the "staircase" partitions $(k,\ldots,2,1)$ for $k > 0$. However, these cases are deceptively simple; m-cores for $m \geq 3$ are more complicated. Let Π be the set of all integer partitions, and $\Pi_{m\text{-}cor} \subseteq \Pi$ the subset of m-cores. Then there is a bijection

$$m\text{-core} \times m\text{-quotient} : \Pi \xrightarrow{\sim} \Pi_{m\text{-}cor} \times \Pi^m,$$

called the *core-quotient bijection* at level m. It provides, for partitions, an analogue of the division algorithm for integers. One way to describe it uses a more general bijection

$$v_m = (v_m^{(0)}, v_m^{(1)}, \ldots, v_m^{(m-1)}) : \mathbb{B} \xrightarrow{\sim} \mathbb{B}^m,$$

where $\mathbb{B}$ is the collection of subsets $\beta \subseteq \mathbb{Z}$ with the property that $n \in \beta$ when n is a sufficiently negative integer, but $n \notin \beta$ when n is sufficiently positive. We can identify $\Pi \times \mathbb{Z}$ with $\mathbb{B}$ through the correspondence

$$\begin{aligned} \Pi \times \mathbb{Z} &\xrightarrow{\sim} \mathbb{B}, \\ (\pi, s) &\mapsto \beta_{\pi,s}, \end{aligned} \qquad \text{where } \beta_{\pi,s} = \{\pi_i - i + s + 1 \mid i = 1,2,\ldots\}.$$

We define v_m by setting $v_m^{(r)}(\beta) = \{q \in \mathbb{Z} \mid mq + r \in \beta\}$ for $r = 0,1,\ldots,m-1$. Now, the composition of bijections

$$\Upsilon_m : \Pi \times \mathbb{Z} \xrightarrow{\sim} \mathbb{B} \xrightarrow{v_m} \mathbb{B}^m \xrightarrow{\sim} (\Pi \times \mathbb{Z})^m = \Pi^m \times \mathbb{Z}^m$$

determines the core-quotient bijection as follows: If $\Upsilon_m(\pi, 0) = (\vec{\pi}, \vec{s})$, then the *$m$-quotient* of π is the m-partition $\vec{\pi}$, while the *m-core* of π is the unique partition λ such that $\Upsilon_m(\lambda, 0) = (\vec{\varnothing}, \vec{s})$.

In the literature, elements of $\Pi^m \times \mathbb{Z}^m$ are called *charged m-partitions* and written with the "ket" notation $|\vec{\pi}, \vec{s}\rangle$, to allude to their meaning in the Fock spaces of quantum mechanics. The vector $\vec{s}$ is called the *m-charge*. Elements of $\mathbb{B}^m$ are sometimes called *m-runner abacus configurations*. They can be pictured as configurations of beads on the set $\mathbb{Z} \times \mathbb{Z}_m$, *i.e.*, on an abacus with m runners. Under the bijection $\Pi^m \times \mathbb{Z}^m \simeq \mathbb{B}^m$, the operation $|\vec{\pi}, \vec{s}\rangle \mapsto |\vec{\varnothing}, \vec{s}\rangle$ corresponds to sliding all beads as far left as possible.

In what follows, we will use generalizations of v_m and Υ_m that involve fixing another integer $l > 0$. Namely, let

$$v_m^l = (v_m^{l,(0)}, v_m^{l,(1)}, \ldots, v_m^{l,(m-1)}) : \mathbb{B}^l \xrightarrow{\sim} \mathbb{B}^m$$

be defined by setting

$$v_m^{l,(r)}(\vec{\beta}) = \left\{ lq' + r' \;\middle|\; \begin{array}{l} (q', r') \in \mathbb{Z} \times \mathbb{Z}_l, \\ mq + r \in \beta^{(r')} \end{array} \right\} \quad \text{for } r = 0, 1, \ldots, m-1,$$

and form the composition of bijections

$$\Upsilon_m^l : \Pi^l \times \mathbb{Z}^l = \mathbb{B}^l \xrightarrow{v_m^l} \mathbb{B}^m = \Pi^m \times \mathbb{Z}^m.$$

It is also possible to rewrite v_m^l as a composition $\mathbb{B}^l \xrightarrow{v_l^{-1}} \mathbb{B} \xrightarrow{v_m^*} \mathbb{B}^m$, where v_m^* is some modified version of v_m.

1.2 Groups and Algebras

One of the best-known results of representation theory is the bijection, due to Frobenius, Schur, and Young, between integer partitions of size N and irreducible characters of the symmetric group S_N. More generally, by Clifford, there is a bijection between m-partitions of size N and irreducible characters of the wreath product $S_{N,m} := S_N \ltimes \mathbb{Z}_m^N$. (For all m, we set $S_{0,m} = \{1\}$.) The maps Υ_m^l hold significance in the representation theory of the groups $S_{N,m}$, or rather, certain deformations of the rings $\mathbb{Z}S_{N,m}$.

Recall that the Coxeter presentation of S_N shows that the *Hecke algebra*

$$H_N(x) = \frac{\mathbb{C}[x^{\pm 1}]Br_N}{\langle (\sigma_i - 1)(\sigma_i - x) \mid i = 1, \ldots, N-1 \rangle}$$

specializes to $\mathbb{C}S_N$ at $x = -1$, where Br_N is the braid group on n strands:

$$Br_N = \left\langle \sigma_1, \ldots, \sigma_{N-1} \;\middle|\; \begin{array}{rl} \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} & \text{for } 1 \leq i \leq N-2, \\ \sigma_i\sigma_j = \sigma_j\sigma_i & \text{for } |j-i| > 1 \end{array} \right\rangle.$$

For any $\xi \in \mathbb{C}$, we set $H_N(\xi) = H_N(x)/(x - \xi)$. For sufficiently generic ξ, there is an isomorphism $H_N(\xi) \simeq \mathbb{C}S_N$, and hence, a bijection between the isomorphism classes of (finite-dimensional) simple modules over $H_N(\xi)$ and the same over $\mathbb{C}S_N$. But if ξ is special, then $H_N(\xi)$ can even fail to be semisimple.

We set $A_1 = \mathbb{C}[x^{\pm 1}]$ and $Br_{N,1} = Br_N$ and $H_{N,1} = H_N$. For all $m > 0$, let $\zeta_m \in \mathbb{C}^\times$ be a primitive mth root of unity. For $m > 1$, the group $S_{N,m}$ remains a complex reflection group with a Coxeter-like presentation, and the *Ariki–Koike algebra*

$$H_{N,m}(\vec{x}) = \frac{A_m Br_{N,m}}{\left\langle \begin{array}{l} (\sigma_i - 1)(\sigma_i - x_\sigma) \text{ for } i = 1, \ldots, N-1, \\ (\tau - 1)(\tau - x_{\tau,1}) \cdots (\tau - x_{\tau,m-1}) \end{array} \right\rangle},$$

where $A_m = \mathbb{C}[\overbrace{x_\sigma, x_{\tau,1}, \ldots, x_{\tau,m-1}}^{\vec{x}}]$ and

$$Br_{N,m} = \left\langle Br_N, \tau \;\middle|\; \begin{array}{cc} \tau\sigma_1\tau\sigma_1 = \sigma_1\tau\sigma_1\tau, & \\ \tau\sigma_j = \sigma_j\tau & \text{for } j \neq 1 \end{array} \right\rangle,$$

specializes to $\mathbb{C}S_{N,m}$ at $x_\sigma = -1$ and $x_{\tau,i} = \zeta_m^i$. Note that $Br_{N,m}$ is the same for all $m > 1$. For any $\vec{\xi} \in \mathbb{C}^m$, we define $H_{N,m}(\vec{\xi})$ similarly to how we defined $H_N(\xi)$. Like before, this algebra is isomorphic to $\mathbb{C}S_{N,m}$ at generic parameters, while failing to be semisimple at special ones. We are especially interested in parameters of the form $(\zeta, \zeta^{\vec{s}}) := (\zeta, \zeta^{s_1}, \ldots, \zeta^{s_m})$ for a fixed root of unity ζ and m-charge $\vec{s}$.

Given an associative algebra H, we write $\mathrm{K}_0(H)$ for the Grothendieck group of the category of finitely-generated H-modules, and $\mathrm{K}_0^+(H) \subseteq \mathrm{K}_0(H)$ for its non-negative part. Let $H_{N,m}^\circ(\vec{x}) = K \otimes_{A_m} H_{N,m}(\vec{x})$ for some field K such that A_m is an integrally closed subring of K and $H_{N,m}^\circ(\vec{x}) \simeq KS_{N,m}$ as K-algebras. As explained in [6, Ch. 7], the specialization map $H_{N,m}(\vec{x}) \to H_{N,m}(\vec{\xi})$ induces a *decomposition map*

$$\mathrm{d}_{\vec{\xi}} : \mathrm{K}_0^+(H_{N,m}^\circ(\vec{x})) \to \mathrm{K}_0^+(H_{N,m}(\vec{\xi})). \tag{1.1}$$

At the same time, every character of $S_{N,m}$ corresponds to a module over $H_{N,m}^\circ(\vec{x})$ up to isomorphism. Altogether, we get maps

$$\Pi^m \xrightarrow{\sim} \mathrm{Irr}(S_{N,m}) \to \mathrm{K}_0^+(H_{N,m}^\circ(\vec{x})) \xrightarrow{\mathrm{d}_{\vec{\xi}}} \mathrm{K}_0^+(H_{N,m}(\vec{\xi})).$$

We say that two m-partitions of N, or irreducible characters of $S_{N,m}$, belong to the same *$\vec{\xi}$-block* if and only if their images χ, ψ in $\mathrm{K}_0^+(H_{N,m}(\vec{\xi}))$ are linked by a sequence of classes $\chi = \chi_0, \chi_1, \ldots, \chi_k = \psi$ such that consecutive χ_i's always share a Jordan–Hölder factor. When $H_{N,m}(\vec{\xi})$ is semisimple, each block is a singleton; in general, they can be more complicated. The *decomposition matrix* of a $\vec{\xi}$-block is a matrix with rows indexed by isomorphism classes of simple $H_{N,m}(\vec{\xi})$-modules, columns indexed by m-partitions of N, and entries given by the Jordan–Hölder multiplicities of the rows in (the classes of) the $H_{N,m}(\vec{\xi})$-modules arising via $\mathrm{d}_{\vec{\xi}}$ from the columns.

1.3 Level-Rank Duality

Fix $s \in \mathbb{Z}$. It turns out that the map $\mathrm{Y}_m^l : \Pi^l \times \mathbb{Z}^l \to \Pi^m \times \mathbb{Z}^m$ relates $(\zeta_m, \zeta_m^{\vec{r}})$-blocks of l-partitions, as we run over l-charges $\vec{r}$ with $|\vec{r}| := r_1 + \cdots + r_l = s$, and $(\zeta_l, \zeta_l^{\vec{s}})$-blocks of m-partitions, as we run over m-charges $\vec{s}$ with $|\vec{s}| = s$. To state this precisely, we define an *Uglov datum of rank l and level m* to be a triple $(M, \vec{r}, \mathbf{b})$, where $N \geq 0$ is any integer, $\vec{r}$ is an l-charge, and $\mathbf{b}$ is a $(\zeta_m, \zeta_m^{\vec{r}})$-block of the l-partitions of size N. The following result is implicit in the paper [18]:

Theorem 1 (Uglov)**.** *Any rank-l, level-m Uglov datum $(M, \vec{r}, \mathbf{b})$ defines a rank-m, level-l Uglov datum $(N, \vec{s}, \mathbf{c})$ such that:*

1. *This assignment defines a bijection between rank-l, level-m Uglov data and rank-m, level-l Uglov data.*

2. *Υ^l_m restricts to a bijection $\{|\vec{\pi}, \vec{r}\rangle \mid |\vec{\pi}| = M,\ \vec{\pi} \in \mathbf{b}\} \xrightarrow{\sim} \{|\vec{\omega}, \vec{s}\rangle \mid |\vec{\omega}| = N,\ \vec{\omega} \in \mathbf{c}\}$. In fact, this property uniquely determines the bijection in (1).*

3. *There is a precise "inversion formula" relating the decomposition matrices of $\mathbf{b}$ and $\mathbf{c}$, arising from the formulas of [18, §5].*

Theorem 1 is remarkable because there is no direct relationship between the algebras $H_{M,l}(\zeta_m, \zeta_m^{\vec{r}})$ and $H_{N,m}(\zeta_l, \zeta_l^{\vec{s}})$ beyond the numerics of their parameters. As the nomenclature suggests, the theorem is a version of the *level-rank duality* discovered by Frenkel between affine Lie algebras, or rather, between their quantizations [5].

In fact, Uglov actually shows the existence of commuting actions of quantum affine algebras $\mathrm{U}_v(\widehat{\mathfrak{sl}}_l)_{\mathbb{Q}}$ and $\mathrm{U}_v(\widehat{\mathfrak{sl}}_m)_{\mathbb{Q}}$ on certain vector spaces with distinguished bases. For any integer $l > 0$ and m-charge $\vec{s}$, there is a module over the $\mathbb{Q}(v)$-algebra $\mathrm{U}_v(\widehat{\mathfrak{sl}}_m)_{\mathbb{Q}}$ called the *Fock space of level l and charge $\vec{s}$*, whose underlying vector space takes the form

$$\Lambda_v^{\vec{s}} = \bigoplus_{\vec{\pi} \in \Pi^m} \mathbb{Q}(v)|\vec{\pi}, \vec{s}\rangle.$$

It controls the representation theory of the algebras $H_{N,m}(\zeta_l, \zeta_l^{\vec{s}})$ through a theorem of Ariki, which relates their simple modules and decomposition matrices to the structure of irreducible, highest-weight submodules of $\Lambda_v^{\vec{s}}$. For fixed $s \in \mathbb{Z}$, Uglov's commuting actions arise from the isomorphisms of vector spaces

$$\bigoplus_{|\vec{r}|=s} \Lambda_v^{\vec{r}} \xleftarrow{v_l} \Lambda_v^s \xrightarrow{v_m^*} \bigoplus_{|\vec{s}|=s} \Lambda_v^{\vec{s}}$$

induced by the maps v_l and v_m^* such that $v_m^l = v_m^* \circ v_l^{-1}$ (see §1.1).

1.4 Hecke Algebras and Cherednik Algebras

Following Uglov, several teams of researchers worked to categorify Theorem 1 via the representation theory of *rational Cherednik algebras*. To motivate these algebras, recall that in general, a finite complex reflection group C with reflection representation V defines a *braid group* Br_C, generalizing $Br_{N,m}$, and a *Hecke algebra* $H_C(\vec{x})$, generalizing $H_{N,m}(\vec{x})$. The braid group is defined as the fundamental group of V^{reg}/C, where $V^{\mathrm{reg}} \subseteq V$ is the so-called regular locus where C acts freely. The Hecke algebra, as defined in [4], is a certain quotient of $\mathbb{C}[\vec{x}^{\pm 1}][Br_C]$, where $\vec{x}$ refers to a collection of indeterminates indexed in terms of the reflecting hyperplanes of the C-action on V and the orders of certain corresponding complex reflections. Specializing $\vec{x}$ to a vector of complex numbers $\vec{\xi}$, we obtain an algebra $H_C(\vec{\xi})$ closely related to the monodromy around these hyperplanes, a decomposition map $\mathrm{d}_{\vec{\xi}}$ generalizing the map in (1.1), and a notion of $\vec{\xi}$-blocks.

By comparison, the rational Cherednik algebra of C, at a parameter vector $\vec{\nu}$ with the same indices as $\vec{x}$, is an algebra closely related to polynomial *differential operators* on V/C. In brief, it takes the form

$$D_C^{rat}(\vec{\nu}) = (\mathbb{C}C \ltimes (\mathrm{Sym}(V) \otimes \mathrm{Sym}(V^*)))/I(\vec{\nu})$$

for some ideal of relations $I(\vec{\nu})$ deforming the Heisenberg–Weyl relations $xy - yx = \langle x, y\rangle$ for $x \in V$ and $y \in V^*$. As shown in [7], the rational Cherednik algebra shares many features with the universal enveloping algebras of semisimple Lie algebras: It has a triangular decomposition, where $\mathbb{C}C$ plays the role of the Cartan subalgebra, and an analogue of the Bernstein–Gelfand–Gelfand category O, which we will denote $\mathsf{O}_C(\vec{\nu})$. In particular, the simple objects of $\mathsf{O}_C(\vec{\nu})$ are indexed by $\mathrm{Irr}(C)$. For any $\chi \in \mathrm{Irr}(C)$, let $\Delta_{\vec{\nu}}(\chi)$ be the *Verma module* that has the corresponding simple quotient.

Essentially by localizing to V^{reg}/W and taking monodromy, one can define the so-called *Knizhnik–Zamolodchikov functor*

$$\mathsf{KZ} : \mathsf{O}_C(\vec{\nu}) \to \mathsf{Mod}(H_C(\vec{\xi})), \quad \text{where } \vec{\xi} \text{ depends on } \vec{\nu}.$$

When C is a *real* reflection group (*i.e.*, Coxeter), $\vec{\xi} = \exp(2\pi i\vec{\nu})$; in general, $\vec{\xi}$ remains related to $\vec{\nu}$ by some exponentiation formula. It turns out that the class of $\mathsf{KZ}(\Delta_{\vec{\nu}}(\chi))$ in $\mathrm{K}_0(H_C(\vec{\xi}))$ is precisely $\mathrm{d}_{\vec{\xi}}(\chi)$. In fact, the functor KZ induces a bijection between the (categorical) blocks of $\mathsf{O}_C(\vec{\nu})$ and those of $\mathsf{Mod}(H_C(\vec{\xi}))$.

Chuang–Miyachi conjectured that Uglov's bijections could be categorified by *Koszul duality* equivalences between category-O blocks of the rational Cherednik algebras of the groups $C = S_{N,m}$ at appropriate parameters. This *categorical* level-rank duality was proved by Shan–Varagnolo–Vasserot in [15], using equivalences between such categories O and truncated parabolic categories O of the affine Lie algebras $\widehat{\mathfrak{sl}}_l$ that were proved by Losev [8], Rouquier–Shan–Varagnolo–Vasserot [14], and Webster [19] independently.

2 Finite Reductive Groups

We will propose a generalization of Theorem 1, replacing the groups $S_{N,m}$ with complex reflection groups arising from the representation theory of finite groups of Lie type. The original story by Uglov *et al.* will be the case of our story for *general linear groups*.

Below, we consider a prime power q and a (connected, smooth) reductive algebraic group $\mathbf{G}$ over $\bar{\mathbb{F}}_q$, split over $\mathbb{F}_q$, equipped with a q-Frobenius map $F : \mathbf{G} \to \mathbf{G}$. This defines a finite reductive group $G = \mathbf{G}^F$. Throughout, we reserve **boldface** uppercase letters for spaces over $\bar{\mathbb{F}}_q$, and ordinary *italics* for their loci of F-fixed points.

2.1 Deligne–Lusztig Induction

By the work of Deligne–Lusztig and Lusztig, the irreducible representations of G can all be obtained from induction functors

$$\mathsf{R}_{\mathbf{L}}^{\mathbf{G}} : \mathsf{Rep}(L) \to \mathsf{Rep}(G),$$

constructed from the compactly-supported étale cohomology of algebraic varieties $\mathbf{Y}_{\mathbf{L}\subseteq\mathbf{P}}^{\mathbf{G}}$ over $\bar{\mathbb{F}}_q$, now called *Deligne–Lusztig varieties*. Here, $L = \mathbf{L}^F$, as $\mathbf{L}$ runs over F-stable Levi subgroups of $\mathbf{G}$; and to define the variety, we must choose a parabolic $\mathbf{P} \subseteq \mathbf{G}$ containing $\mathbf{L}$. For a fixed prime ℓ invertible in $\mathbb{F}_q$, the compactly-supported étale cohomology $\mathrm{H}_c^*(\mathbf{Y}_{\mathbf{L}\subseteq\mathbf{P}}^{\mathbf{G}}, \bar{\mathbb{Q}}_\ell)$ admits commuting actions of G and L, which thereby define the functor. (It is conjectured that $\mathsf{R}_{\mathbf{L}}^{\mathbf{G}}$ is always independent of $\mathbf{P}$.)

In fact, to construct the irreducibles of G, it suffices to run over maximal tori, not all Levis. An irreducible character is *unipotent* if and only if it occurs in $\mathsf{R}_{\mathbf{T}}^{\mathbf{G}}(1)$ for some maximal torus $\mathbf{T}$, where 1 denotes the trivial character. We write $\mathrm{Uch}(G)$ for the set of unipotent irreducible characters of G. Lusztig shows that $\mathrm{Uch}(G)$ can be indexed in a way independent of q, depending only on the Weyl group W of $\mathbf{G}$ itself.

2.2 (Generalized) Harish-Chandra Series

An irreducible character $\lambda \in \mathrm{Irr}(L)$ is *cuspidal* if and only if it does not occur in $\mathsf{R}_{\mathbf{M}}^{\mathbf{L}}(\mu)$ for some smaller F-stable Levi $\mathbf{M} \subseteq \mathbf{L}$ and $\mu \in \mathrm{Irr}(M)$. For any such $\mathbf{L}, \lambda$, we define the corresponding *Harish-Chandra series* of G to be

$$\mathrm{Uch}(G)_{\mathbf{L},\lambda} = \{\rho \in \mathrm{Uch}(G) \mid (\rho, \mathsf{R}_{\mathbf{L}}^{\mathbf{G}}(\lambda))_G \neq 0\}.$$

Then, as discovered by Harish-Chandra, the sets $\mathrm{Uch}(G)_{\mathbf{L},\lambda}$ are pairwise disjoint and partition $\mathrm{Uch}(G)$ as we run over $\mathbf{G}$-conjugacy classes of *cuspidal pairs* $(\mathbf{L}, \lambda)$ for which $\mathbf{L}$ is F-maximally split. This last condition means that $\mathbf{L} = Z_{\mathbf{G}}(\mathbf{T})^\circ$ for some F-stable torus $\mathbf{T}$ such that T is a power of $\mathbb{F}_q^\times$. Moreover, there are bijections

$$\chi_{\mathbf{L},\lambda} : \mathrm{Uch}(G)_{\mathbf{L},\lambda} \xrightarrow{\sim} \mathrm{Irr}(W_{\mathbf{L},\lambda}^{\mathbf{G}})$$

satisfying various compatibilities, where $W_{\mathbf{L},\lambda}^{\mathbf{G}}$ is the stabilizer of λ under the action of $N_G(L)/L$, arising from comparing $\bar{\mathbb{Q}}_\ell W_{\mathbf{L},\lambda}^{\mathbf{G}}$ to the algebra of G-equivariant endomorphisms of $\mathsf{R}_{\mathbf{L}}^{\mathbf{G}}(\lambda)$. We say that $W_{\mathbf{L},\lambda}^{\mathbf{G}}$ is the *relative Weyl group* of $(\mathbf{L}, \lambda)$ in G.

Motivated by observations from the ℓ-modular representation theory of G at large primes ℓ, Broué–Malle–Michel discovered a generalization of Harish–Chandra theory depending on an integer $l > 0$, not necessarily prime [3]. The precise statements of their results use the notion of the *generic finite reductive group* $\mathbb{G}$ that interpolates the groups G as we keep the root datum and its Frobenius automorphism fixed, but vary the prime power q. We will similarly need generic versions of the Levi subgroups of $\mathbf{G}$ and the unipotent irreducible characters of the groups L. We will use this formalism without comment, save to note that any unipotent irreducible character ρ has a *generic degree* $\mathrm{Deg}_\rho(x) \in \mathbb{Q}[x]$, which recovers the actual character degree at $x = q$. We will also write $\mathbb{G}(q)$ for the finite group G recovered at a specific q. We leave the notation for unipotent irreducible characters unchanged.

Broué–Malle–Michel define an arbitrary F-stable torus $\mathbb{T}$ to be *l-split* if and only if $|\mathbb{T}(q)|$ is a power of $\Phi_l(q)$, the value at $x = q$ of the lth cyclotomic polynomial $\Phi_l(x)$. They define an F-stable Levi subgroup $\mathbb{L} \subseteq \mathbb{G}$ to be *l-split* if and only if $\mathbb{L} = Z_{\mathbb{G}}(\mathbb{T})$ for some l-split torus $\mathbb{T}$. For such $\mathbb{L}$, we say that $\lambda \in \mathrm{Uch}(\mathbb{L})$ is *l-cuspidal* if and only if it does not occur in $\mathsf{R}_{\mathbb{M}}^{\mathbb{L}}(\mu)$ for any smaller l-split Levi $\mathbb{M}$ and $\mu \in \mathrm{Uch}(\mathbb{M})$. In this case, we say that $(\mathbb{L}, \lambda)$ is a *l-cuspidal pair*.

Taking $l = 1$ above, we recover the usual notions of maximally split tori, maximally split Levis, and cuspidal pairs. Moreover, the notions of the *l-Harish-Chandra series* $\mathrm{Uch}(\mathbb{G})_{\mathbb{L},\lambda}$ and the *relative Weyl group* $W_{\mathbb{L},\lambda}^{\mathbb{G}}$ still make sense for l-cuspidal pairs $(\mathbb{L}, \lambda)$. Significantly for us, these relative Weyl groups remain complex reflection groups. (In the classical case, they are always Coxeter groups.)

Theorem 2 (Broué–Malle–Michel [3])**.** *For any fixed integer $l > 0$, the l-Harish-Chandra series $\mathrm{Uch}(\mathbb{G})_{\mathbb{L},\lambda}$ are disjoint and partition $\mathrm{Uch}(\mathbb{G})$ as we run over appropriate equivalence classes of l-cuspidal pairs $(\mathbb{L}, \lambda)$. For any fixed l-cuspidal pair $(\mathbb{L}, \lambda)$, there is a map*

$$(\varepsilon_{\mathbb{L},\lambda}, \chi_{\mathbb{L},\lambda}) : \mathrm{Uch}(\mathbb{G})_{\mathbb{L},\lambda} \to \{\pm 1\} \times \mathrm{Irr}(W_{\mathbb{L},\lambda}^{\mathbb{G}})$$

such that:

1. *$\chi_{\mathbb{L},\lambda}$ is bijective; $\varepsilon_{\mathbb{L},\lambda}\chi_{\mathbb{L},\lambda}$ defines an isometry with respect to $(-,-)_{\mathbb{G}(q)}$ and $(-,-)_{W_{\mathbb{L},\lambda}^{\mathbb{G}}}$.*
2. *These maps are transitive along inclusions of l-split Levis, and in a precise sense, intertwine Deligne–Lusztig induction with ordinary induction between relative Weyl groups.*
3. *$\chi_{\mathbb{G},\lambda}^{\mathbb{G}}(\lambda)$ is the trivial character of the trivial group $W_{\mathbb{G},\lambda}^{\mathbb{G}}$.*
4. *For all $\rho \in \mathrm{Uch}(\mathbb{G})_{\mathbb{L},\lambda}$, we have $\mathrm{Deg}_\rho(\zeta_l) = \varepsilon_{\mathbb{L},\lambda}(\rho) \deg \chi_{\mathbb{L},\lambda}(\rho)$.*

Example 1. *Let $\mathbb{GL}_n$ be the generic finite reductive group such that $\mathbb{GL}_n(q) = \mathrm{GL}_n(\mathbb{F}_q)$. Then every unipotent irreducible character of $\mathbb{GL}_n(q)$ arises from a principal series representation attached to an irreducible character of S_n. Hence there is a bijection*

$$\mathrm{Uch}(\mathbb{GL}_n) \simeq \{\textit{partitions of } n\}. \tag{2.1}$$

It gives rise to the following dictionary between the notions from Section 1 and the notions we have just introduced. First, every l-split Levi of $\mathbb{GL}_n$ takes the form $\mathbb{L} \simeq \mathbb{GL}_{n-lN} \times \mathbb{S}_l^N$, where

$\mathbb{S}(q) = \mathbb{F}_{q^l}^{\times}$. *For such* $\mathbb{L}$, *the analogue of* (2.1) *for* $\mathbb{L}$ *identifies*

$$\{l\text{-cuspidals } \lambda \in \mathrm{Uch}(\mathbb{L})\} \simeq \{l\text{-cores } \lambda \vdash n - lN\}.$$

For l*-cuspidal* $\lambda \in \mathrm{Uch}(\mathbb{L})$ *as above,* (2.1) *identifies*

$$\mathrm{Uch}(\mathbb{GL}_n)_{\mathbb{L},\lambda} \simeq \{\pi \vdash n \mid l\text{-core}(\pi) = \lambda\}.$$

Moreover, $W_{\mathbb{L},\lambda} \simeq S_{N,l}$. *The bijection* $\chi_{\mathbb{L},\lambda} : \mathrm{Uch}(\mathbb{GL}_n)_{\mathbb{L},\lambda} \xrightarrow{\sim} \mathrm{Irr}(W^{\mathbb{G}}_{\mathbb{L},\lambda})$ *is essentially the map sending a partition to its* l*-quotient. For the precise statement, see [17, §6].*

2.3 Hecke Algebras of Cuspidal Pairs

In [2], Broué–Malle conjectured that the algebras of G-equivariant endomorphisms of the Deligne–Lusztig modules $\mathrm{H}^*_c(Y^{\mathbf{G}}_{\mathbf{L}\subseteq\mathbf{P}}, \bar{\mathbb{Q}}_\ell)$ can be also made generic. The statement requires the generic Hecke algebras $H_C(\vec{x})$ for complex reflection groups C that we had mentioned in §1.4.

Conjecture 1 (Broué–Malle [2])**.** *For any integer* $l > 0$, *and* l*-cuspidal pair* $(\mathbb{L}, \lambda)$ *of* $\mathbb{G}$, *there is a homomorphism of* $\mathbb{C}$*-algebras* $\varphi^{\mathbb{G}}_{\mathbb{L},\lambda} : \mathbb{C}[\vec{x}^{\pm 1}] \to \mathbb{C}[x^{\pm 1/\infty}]$ *such that the algebra*

$$H^{\mathbb{G}}_{\mathbb{L},\lambda}(x) := \mathbb{C}[x^{\pm 1/\infty}] \otimes_{\mathbb{C}[\vec{x}^{\pm 1}]} H_{W^{\mathbb{G}}_{\mathbb{L},\lambda}}(\vec{x})$$

satisfies the following properties:

1. $H^{\mathbb{G}}_{\mathbb{L},\lambda}(\zeta_l) \simeq \mathbb{C}W^{\mathbb{G}}_{\mathbb{L},\lambda}$.
2. *For any prime power* $q > 1$, *there is an isomorphism of algebras*

$$\bar{\mathbb{Q}}_\ell \otimes_{\mathbb{C}[x^{\pm 1/\infty}]} H^{\mathbb{G}}_{\mathbb{L},\lambda}(x) \simeq \mathrm{End}_{\bar{\mathbb{Q}}_\ell G}(\overbrace{\mathrm{H}^*_c(\mathbf{Y}^{\mathbf{G}}_{\mathbf{L}\subseteq\mathbf{P}}, \bar{\mathbb{Q}}_\ell)[\lambda]}^{R^{\mathbf{G}}_{\mathbf{L}}(\lambda)}),$$

where on the left, we fix $\mathbb{C} \simeq \bar{\mathbb{Q}}_\ell$, *and the base change sends* $x^{1/n}$ *to an* n*th root of* q *in* $\bar{\mathbb{Q}}_\ell$.

Moreover, $\varphi^{\mathbb{G}}_{\mathbb{L},\lambda}$ *sends each entry of* $\vec{x}$ *to a monomial parameter of the form* ζx^α, *where* $\zeta \in \mathbb{C}$ *is a root of unity and* $\alpha \in \mathbb{Q}$. *The text of [2] predicts these parameters explicitly for essentially every choice of* $\mathbb{G}, \mathbb{L}, \lambda$.

Several cases of Conjecture 1, including the explicit paameter predictions, have been verified. In particular, the cases of 1-split Levis and so-called *Coxeter tori* were resolved by work of Lusztig [9, 10], and various other cases by work of Digne–Michel and Digne–Michel–Rouquier, as reviewed in [17, §6]. Otherwise, the conjecture remains open. In *loc. cit.*, we verify:

Lemma 1. *If* $\mathbb{G} = \mathbb{GL}_n$, *then the parameter predictions in [2] for the algebras* $H^{\mathbb{G}}_{\mathbb{L},\lambda}(x)$ *hold for the cases where* $\mathbb{L}$ *is* F*-maximally split, where* $\mathbb{L}$ *is any maximal torus, and where* $\mathbb{L} \simeq \mathbb{GL}_{n-l} \times \mathbb{S}$ *for some* l*-split Coxeter torus* $\mathbb{S}$.

3 The Story for Finite Reductive Groups

Fix a generic finite reductive group $\mathbb{G}$, as in the previous section, and integers $l, m > 0$. Fix an l-cuspidal pair $(\mathbb{L}, \lambda)$ and an m-cuspidal pair $(\mathbb{M}, \mu)$ of $\mathbb{G}$. Let

$$\mathrm{Uch}(\mathbb{G})_{\mathbb{L},\lambda,\mathbb{M},\mu} = \mathrm{Uch}(\mathbb{G})_{\mathbb{L},\lambda} \cap \mathrm{Uch}(\mathbb{G})_{\mathbb{M},\mu}$$

in $\mathrm{Uch}(\mathbb{G})$. Let $\mathrm{Irr}(W_{\mathbb{G},\mathbb{L},\lambda})_{\mathbb{M},\mu} \subseteq \mathrm{Irr}(W_{\mathbb{G},\mathbb{L},\lambda})$ and $\mathrm{Irr}(W_{\mathbb{G},\mathbb{M},\mu})_{\mathbb{L},\lambda} \subseteq \mathrm{Irr}(W_{\mathbb{G},\mathbb{M},\mu})$ be the images of the maps

$$\mathrm{Irr}(W^{\mathbb{G}}_{\mathbb{L},\lambda}) \xleftarrow{\chi_{\mathbb{L},\lambda}} \mathrm{Uch}(\mathbb{G})_{\mathbb{L},\lambda,\mathbb{M},\mu} \xrightarrow{\chi_{\mathbb{M},\mu}} \mathrm{Irr}(W^{\mathbb{G}}_{\mathbb{M},\mu}).$$

Assume that Conjecture 1 holds, so that $H^{\mathbb{G}}_{\mathbb{L},\lambda}(x)$ and $H^{\mathbb{G}}_{\mathbb{M},\mu}(x)$ are meaningful. We define $H^{\mathbb{G},\circ}_{\mathbb{L},\lambda}(x)$ and d_ζ by analogy with the localized Ariki–Koike algebra $H^{\circ}_{N,m}(\vec{x})$ and its decomposition map in §1.2. Moreover, we define a *$(\mathbb{G}, \mathbb{L}, \lambda, \zeta_m)$-block* of $\mathrm{Irr}(W^{\mathbb{G}}_{\mathbb{L},\lambda})$ by analogy with the definitions of $\vec{\xi}$-blocks in §1.2 and §1.4, via the maps

$$\mathrm{Irr}(W^{\mathbb{G}}_{\mathbb{L},\lambda}) \to \mathrm{K}_0(H^{\mathbb{G},\circ}_{\mathbb{L},\lambda}(x)) \xrightarrow{\mathrm{d}_\zeta} \mathrm{K}_0(H^{\mathbb{G}}_{\mathbb{L},\lambda}(\zeta)).$$

Our main conjecture, like Uglov's Theorem 1, has three parts.

Conjecture 2. *With the setup above:*

1. $\mathrm{Irr}(W^{\mathbb{G}}_{\mathbb{L},\lambda})_{\mathbb{M},\mu}$ *is a (disjoint) union of* $(\mathbb{G}, \mathbb{L}, \lambda, \zeta_m)$*-blocks. Dually,* $\mathrm{Irr}(W^{\mathbb{G}}_{\mathbb{M},\mu})_{\mathbb{L},\lambda}$ *is a union of* $(\mathbb{G}, \mathbb{M}, \mu, \zeta_l)$*-blocks.*

2. *The bijection from* $\mathrm{Irr}(W^{\mathbb{G}}_{\mathbb{L},\lambda})_{\mathbb{M},\mu}$ *onto* $\mathrm{Irr}(W^{\mathbb{G}}_{\mathbb{M},\mu})_{\mathbb{L},\lambda}$ *induced by* $\chi_{\mathbb{L},\lambda}$ *and* $\chi_{\mathbb{M},\mu}$ *matches each* $(\mathbb{G}, \mathbb{L}, \lambda, \zeta_m)$*-block with a corresponding* $(\mathbb{G}, \mathbb{M}, \mu, \zeta_l)$*-block.*

3. *When blocks match in (2), their decomposition matrices are related by an inversion formula generalizing Uglov's in [18]. Moreover, the bijection between the blocks themselves is categorified by an equivalence between the derived categories of appropriate category-*O *blocks of the rational Cherednik algebras of* $W^{\mathbb{G}}_{\mathbb{L},\lambda}$ *and* $W^{\mathbb{G}}_{\mathbb{M},\mu}$ *(see §1.4).*

The proofs of our main results, gathered below, are purely combinatorial. They rely on the abacus methods discussed in Section 1, as well as the combinatorial classification of $\vec{\xi}$-blocks of Ariki–Koike algebras by Lyle–Mathas [11].

Theorem 3. *Suppose that* $\mathrm{G} = \mathbb{GL}_n$*, and that* ℓ *and* m *are coprime. Then, under the parameter predictions in [2] for the algebras* $H^{\mathbb{G}}_{\mathbb{L},\lambda}(x)$*:*

1. $\mathrm{Irr}(W^{\mathbb{G}}_{\mathbb{L},\lambda})_{\mathbb{M},\mu}$ *and* $\mathrm{Irr}(W^{\mathbb{G}}_{\mathbb{M},\mu})_{\mathbb{L},\lambda}$ *would each consist of a single block.*

2. *All three parts of Conjecture 2 would hold.*

Moreover, the bijections in Conjecture 2(2) are essentially the same as Uglov's bijections Υ^l_m, in the sense that they fit into a commutative diagram

$$\begin{tikzcd}
\mathrm{Irr}(W^{\mathbb{G}}_{\mathbb{L},\lambda}) \arrow[d] & \mathrm{Uch}(\mathbb{G})_{\mathbb{L},\lambda,\mathbb{M},\mu} \arrow[l, "\chi_{\mathbb{L},\lambda}"'] \arrow[r, "\chi_{\mathbb{M},\mu}"] \arrow[d] & \mathrm{Irr}(W^{\mathbb{G}}_{\mathbb{M},\mu}) \arrow[d] \\
\Pi^l \times \mathbb{Z}^l \arrow[d] & \Pi \arrow[l, "\Upsilon^1_l(|\rho,\ell_\lambda\rangle) \mapsfrom \rho"'] \arrow[r, "\rho \mapsto \Upsilon^1_m(|\rho,\ell_\mu\rangle)"] & \Pi^m \times \mathbb{Z}^m \arrow[d] \\
\Pi^l \times \mathbb{Z}^l \arrow[rr, "\Upsilon^l_m"] & & \Pi^m \times \mathbb{Z}^m
\end{tikzcd}$$

where the vertical arrows in the bottom half essentially arise from affine permutations. Above, ℓ_λ, ℓ_μ refer to the lengths of the l-core λ and m-core μ as partitions.

We also checked in almost all cases where $\mathbb{G}$ has exceptional type that Conjecture 2(1) is compatible with the sizes of the sets $\mathrm{Uch}(\mathbb{G})_{\mathbb{L},\lambda,\mathbb{M},\mu}$ and the sizes of the blocks of $\mathrm{Irr}(W^{\mathbb{G}}_{\mathbb{L},\lambda})_{\mathbb{M},\mu}$ and $\mathrm{Irr}(W^{\mathbb{G}}_{\mathbb{M},\mu})_{\mathbb{L},\lambda}$, where these are known: See [17, §8].

A particular exciting aspect of this work, not mentioned in this abstract, is that we expect several different *geometric* incarnations of the dualities in Conjecture 2, via bimodules constructed from the cohomology of various algebraic varieties. These include the *affine Springer fibers* studied by Oblomkov–Yun [12] and Boixeda-Alvarez–Losev [1], as well as the *braid Steinberg varieties* studied by Trinh [16].

Acknowledgements

We thank Pablo Boixeda-Alvarez, Olivier Dudas, Ivan Losev, George Lusztig, Gunther Malle, Andrew Mathas, Jean Michel, Kari Vilonen, Ben Webster, and Zhiwei Yun for useful discussions. We also thank Ian Grojnowski and Raphaël Rouquier for their interest.

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