(Axler §2B–2C)

<u>span</u> of $\{v_i\}_i$
[subspace of all linear combo's $\sum_i a_i v_i$]
<u>linear independence</u> of $\{v_i\}_i$
[each elt of span has a unique expr. $\sum_i a_i v_i$]
<u>basis</u> for V
[set that spans V and is linearly independent]

<u>Steinitz Thm</u>

if $\{v_i\}_i$ spans and $\{e_j\}_j$ is lin. indep.
then # of v_i's $\leq$ # of e_j's

<u>Cor</u> if V has a finite basis,
then all bases of V have the same size

<u>dimension</u> of V is the common size of its bases

<u>Ex</u> $V = \{(a, b, c, d) \mid a + b + c + d = 0\}$

what is dim V? [3, as $d = -a - b - c$ always works]

what is a basis for V?

$e_1 = (1, -1, 0, 0)$
$e_2 = (0, 1, -1, 0)$
$e_3 = (0, 0, 1, -1)$

why? $ae_1 + (a + b)e_2 + (a + b + c)e_3$

<u>Ex</u> in V, consider

$s = (1, -1, 0, 0)$
$t = (1, 0, -1, 0)$
$u = (1, 0, 0, -1)$
$v = (0, 1, 0, -1)$
$\mathbf{0} = (0, 0, 0, 0)$

which subsets are lin. indep.? [need ≤ 3 vec's]

Ø, {s}, {t}, {u}, {v}, {s, t},
{s, u}, {s, v}, {t, u}, {t, v},
$\boxed{\{s, t, u\}, \{s, t, v\}, \{t, u, v\}}$

which subsets span V? [need ≥ 3 vec's]

$\boxed{\{s, t, u\}, \{s, t, v\}, \{t, u, v\},}$
{s, t, u, v}, {s, t, u, **0**}, {s, t, v, **0**}, {t, u, v, **0**},
{s, t, u, v, **0**}

<u>Ex</u> recall F[x] = {polynomials in x over F}

is Q = {p in F[x] | p(–1) = p(1)} a linear subspace?
[yes]

what are some elements of Q?

0
polynomials of even degree
$x - x^3$, $x - x^7$, …

is Q finite-dimensional? [no]
how about Q′ = {p in Q | p = 0 or deg p ≤ 4} [yes]
[what is the degree of the zero polynomial?]
what is dim Q′? [tricky]

Thm if V is finite-dim, U sub V is a linear sub,
then U is finite-dim and $\dim U \leq \dim V$

would like to pick basis for V, then restrict to U
but this doesn't work!

Ex $V = F^2$ and $U = \{(x, x) \mid x \text{ in } F\}$
V has basis $\{(1, 0), (0, 1)\}$,
but no subset is a basis for U

Lem suppose $v_1, \ldots, v_n, w$ in V
s.t. $\{v_1, \ldots, v_n\}$ lin. indep.
if $w \notin \text{span}(v_1, \ldots, v_n)$,
then $\{v_1, \ldots, v_n, w\}$ lin. indep.

Pf suppose $\sum_i a_i v_i + bw = \mathbf{0}$
if $b \neq 0$ then w in $\text{span}(v_1, \ldots, v_n)$
so $b = 0$
by lin. indep. of v_i's, get $a_i = 0$ for all i

Pf of Thm starting from $\varnothing$:
build up basis for U by induction

at each step, have lin. indep. set of vectors in U
it is also a lin. indep. set in V
so # of vectors $\leq \dim V$ by Steinitz

if we continued to the $(\dim V)$th step,
our set would span V
so some set must span U after $\leq \dim V$ steps □

[return to example] recall

$Q = \{p \text{ in } F[x] \mid p(-1) = p(1)\}$
$Q' = \{p \text{ in } W \mid p = 0 \text{ or } \deg p \leq 4\}$

let $W = \{p \text{ in } F[x] \mid p = 0 \text{ or } \deg p \leq 4\}$
then $Q' = W$ cap Q
so $\dim Q' \leq \dim W = 5$

in fact $\dim Q' = 4$

similar application of lemma shows:

Thm — if V is finite-dim, U sub V is a linear sub,
then any basis for U
can be extended to a basis for V

Cor — if U, W are linear subspaces of V,
then any basis for U cap W
can be extended to a basis for U (or W),
then to a basis for U + W

[proof by swapping variables]
[can strengthen this result:]

Thm — suppose U + W is finite-dim,
$\{e_1, \ldots, e_\ell\}$ is a basis for U cap W,
$\{e_1, \ldots, e_\ell, f_1, \ldots, f_m\}$ for U,
$\{e_1, \ldots, e_\ell, g_1, \ldots, g_n\}$ for W

then $\{e_1, \ldots, e_\ell, f_1, \ldots, f_m, g_1, \ldots, g_n\}$
is a basis for U + W

<u>Pf</u> spanning is easy

suppose $\sum_i a_i e_i + \sum_j b_j f_j + \sum_k c_k g_k = \mathbf{0}$

let $G = \sum_k c_k g_k$ in W

have $G = \sum_i (-a_i)e_i + \sum_j (-b_j)f_j = U$

so G in $U \cap W$

so $G = \sum_i a'_i e_i$ for some a'_i

but $\{e_i\}_i \cup \{f_j\}_j$ is a basis for U

so $a'_i = -a_i$ for all i and $f_j = 0$ for all j

now $\sum_k c_k g_k = \sum_i (-a_i)e_i$

but $\{e_i\}_i \cup \{g_k\}_k$ is a basis for W

so $a_i = 0$ for all i, and $c_k = 0$ for all k □

<u>Cor (Dimension Formula)</u>

$\dim(U + W) = \dim(U) + \dim(W) - \dim(U \cap W)$

<u>Cor</u> suppose $U + W$ is finite-dim

then TFAE:

1) $U + W$ is a direct sum

2) $U \cap W = \{\mathbf{0}\}$

3) $\dim(U + W) = \dim(U) + \dim(W)$

[equivalence of 1) and 2) was shown last week]