MATH 430: INTRODUCTION TO TOPOLOGY
PROBLEM SET #1

SPRING 2025

Due Wednesday, January 22. You may consult books, papers, and websites as long as you cite all sources and write up your solutions in your own words.

Problem 1 (Munkres 83, #3)**.** Let X be any set. Show that the collection

$$\{\emptyset\} \cup \{U \subseteq X \mid X - U \text{ countable}\}$$

is a topology on X. Is the collection

$$\{\emptyset, X\} \cup \{U \subseteq X \mid X - U \text{ is infinite}\}$$

a topology on X?

Problem 2. We say that $U \subseteq \mathbf{Z}$ is *evenly-spaced* if and only if it is a (possibly empty) union of sets of the form

$$a\mathbf{Z} + b := \{aq + b \mid q \in \mathbf{Z}\}$$

for various $a, b \in \mathbf{Z}$ with $a \neq 0$. Prove that the collection of evenly-spaced sets is a topology on $\mathbf{Z}$.

Problem 3. Let $f : X \to Y$ be an arbitrary map between sets.

(1) Let $\{X_\alpha\}_\alpha$ be an arbitrary collection of subsets of X. Show that

$$f\left(\textstyle\bigcup_\alpha X_\alpha\right) = \textstyle\bigcup_\alpha f(X_\alpha) \quad \text{and} \quad f\left(\textstyle\bigcap_\alpha X_\alpha\right) \subseteq \textstyle\bigcap_\alpha f(X_\alpha).$$

(2) In the setup of (1), give an example where

$$f\left(\textstyle\bigcap_\alpha X_\alpha\right) \neq \textstyle\bigcap_\alpha f(X_\alpha).$$

(3) Let $\{Y_\beta\}_\beta$ be an arbitrary collection of subsets of Y. Show that

$$f^{-1}\left(\textstyle\bigcup_\beta Y_\beta\right) = \textstyle\bigcup_\beta f^{-1}(Y_\beta) \quad \text{and} \quad f^{-1}\left(\textstyle\bigcap_\beta Y_\beta\right) \subseteq \textstyle\bigcap_\beta f^{-1}(Y_\beta).$$

Problem 4. Endow $\mathbf{R}$ with the analytic topology. Give an example of a continuous map $f : \mathbf{R} \to \mathbf{R}$ and an open set $U \subseteq \mathbf{R}$ such that $f(U)$ is *not* open. *Hint:* Pick f to be polynomial.

Problem 5. Let X, Y be topological spaces, and let $f : X \to Y$ be a continuous bijection. Show that if $f(U)$ is open in Y for every open set U in X, then f is a homeomorphism.

Problem 6. Show that the following topological spaces are homeomorphic:

(1) $\mathbf{R}$.
(2) $(0, \infty)$.
(3) $(0, 1)$.

Above, (1) is endowed with the analytic topology; (2) and (3) are endowed with the subspace topology. You may assume that differentiable functions are continuous, and that a composition of homeomorphisms is a homeomorphism.

Problem 7. Let X be any topological space, and let $A \subseteq X$, endowed with its subspace topology. Prove that if A is open in X, then a subset of A is open in A if and only if it is open in X.

Problem 8. Endow $\mathbf{R}^2$ with the analytic topology. How is the subspace topology on $\mathbf{R}$, viewed as the x-axis of $\mathbf{R}^2$, related to the analytic topology on $\mathbf{R}$?