Hilb vs Quot vs HOMFLYPT

Minh-Tâm Quang Trinh

Howard University

O. Kivinen, M. Trinh. The Hilb-vs-Quot conjecture.
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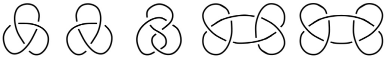
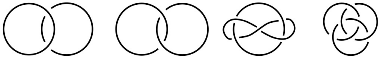
image credits: Chmutov–Duzhin–Mostovoy, Bar-Natan, Penpa+, Cherednik–Danilenko

1 Knots and Links

Some *knots* in $\mathbf{R}^3$ (or S^3).



Links allow multiple circles.



Knot theory studies *isotopy* invariants of links.

Trade-off between being *strong* and being *practical*.

$\pi_1(S^3 \setminus L)$ is a strong, but impractical, invariant.

More practical: the *HOMFLYPT polynomial* $\mathbb{P}_L(a, q)$, defined via skein relations.

$$\mathbb{P}_{\bigcirc} = 1 \tag{1}$$

$$a\mathbb{P}_{\times} - a^{-1}\mathbb{P}_{\times} = (q^{1/2} - q^{-1/2})\mathbb{P}_{)(} \tag{2}$$

The strange factor $q^{1/2} - q^{-1/2}$ will be useful later.

While $\pi_1(S^3 \setminus L)$ is intrinsic, $\mathbb{P}_L$ is *diagrammatic*:

A priori, it depends on the planar diagram.

$\pi_1(S^3 \setminus L) \simeq \mathbf{Z}$ implies $L = \bigcirc$.

Unknown whether $\mathbb{P}_L = 1$ implies $L = \bigcirc$.

Khovanov–Rozansky '07 A further refinement

$$\mathbf{P}_L(a, q, t)$$

such that $\mathbf{P}_L(a, q, -1) = \mathbb{P}_L(a, q)$.

The dimension of a triply-graded vector space called the *HOMFLYPT homology* of L.

Defined via *categorified* skein relations.

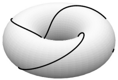
Khovanov '08 Can be done in certain monoidal triangulated categories $\mathsf{K}^b(\mathsf{SBim}_n)$.

Kronheimer–Mrowka '10 $\mathbf{P}_L = 1$ implies $L = \bigcirc$.

Proof used interpretation of $\mathbf{P}_L$ via gauge theory on $S^3 \setminus L$.

Mellit '16, Elias–Hogancamp–Mellit '15–19

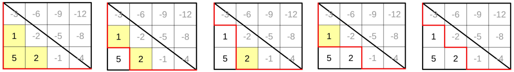
Recursions in $\mathsf{K}^b(\mathsf{SBim}_n)$ computing $\mathbf{P}$ for *torus links*.



$\implies$ Mellit '16 A closed formula for any torus *knot*.

$\implies$ Gorsky–Mazin–Vazirani '20 Another formula, valid for any torus *link*.

For torus knots, both formulas sum over *Dyck paths*.



Both formulas look like

$$\mathbf{P} \propto \sum_{D} q^{a(D)} t^{c(D)} \prod_{\square \in S_\bullet} f_{\bullet, D, \square}.$$

$a(D)$ counts shaded $\square$'s; $c(D)$ is messy.

$$S_{\text{Mellit}} = \{\square \mid \square \nearrow D\},$$
$$S_{\text{GMV}} = \{\square \mid D \swarrow \square \text{ with } \square \text{ shaded}\}.$$

Example For the $(3,4)$ torus knot:

q^a	t^c	$\prod f_{\text{Mellit}}$	$\prod f_{\text{GMV}}$
q^3	t^3	1	$1+aq^{-1}$
q^2	t^2	$1+at$	$(1+aq^{-1})(1+aq^{-1}t)$
q	t^2	$1+at$	$1+aq^{-1}$
q	t	$1+at$	$1+aq^{-1}$
1	1	$(1+at)(1+at^2)$	1

2 Plane Curve Singularities

Let

$$S^3_\epsilon = \{(x, y) \in \mathbf{C}^2 \mid |x|^2 + |y|^2 = \epsilon\}.$$

Let $C \subseteq \mathbf{C}^2$ be an algebraic curve through $(0, 0)$.

The *link* of C at the origin is

$$L_C = \{(x, y) \in S^3_\epsilon \mid f(x, y) = 0\},$$

independent of ϵ up to isotopy.

Example For $y^n = x^m$, it's the (m, n) torus link.

In general, *components* of L_C correspond to *branches* of C at the origin.

Example Take C parameterized by

$$(x(u), y(u)) = (u^4, u^6 + u^7).$$

Then L_C is the *link closure* of

In general, the completed local ring of C looks like

$$R_C = R_{C_1} \times \cdots \times R_{C_b},$$

and the branches look like $R_{C_i} \simeq \mathbf{C}[\![u^{n_i}, u^{m_i} + \cdots]\!]$ by Newton–Puiseux.

Puiseux exponents are *cabling* parameters of knots.

Oblomkov–Shende conjectured a formula for $\mathbb{P}_{L_C}$ in terms of the *intrinsic* ring R_C.

Later, with Rasmussen, upgraded to $\mathbf{P}_{L_C}$.

Form the *Hilbert schemes*

$$\mathcal{H}_C^\ell = \{\text{ideals } I \subseteq R_C \mid \dim(R_C/I) = \ell\}.$$

Conj (ORS '12) The lowest a-degree $\mathbf{P}_{L_C}^{\mathrm{lo}}$ satisfies

$$\boxed{\frac{\mathbf{P}_{L_C}^{\mathrm{lo}}(q, qt)}{1-q} \propto \sum_{\ell \geq 0} q^\ell \chi(t^{1/2}, \mathcal{H}_C^\ell),}$$

where χ denotes *virtual weight polynomials*.

Recall: $\chi(t^{1/2}, Z) = |Z(\mathbf{F}_t)|$ when t is a prime power and Z is especially nice.

Example $\mathbf{P}_{(2,3)\text{ torus}} \propto 1 + qt + at$, while

$$C = \{y^2 = x^3\} \implies \begin{cases} \mathcal{H}_C^0 = pt, \\ \mathcal{H}_C^1 = pt, \\ \mathcal{H}_C^\ell = \mathbf{CP}^1 \text{ for } \ell \geq 2, \end{cases}$$

giving $1 + q + \frac{q^2}{1-q}(1+t) = \frac{1}{1-q}(1 + q^2 t)$.

Next, form *nested Hilbert schemes*

$$\mathcal{H}_C^{\ell,k} = \{(I, J) \in \mathcal{H}_C^\ell \times \mathcal{H}_C^{\ell+k} \mid I \supseteq J \supseteq \langle x, y\rangle J\}.$$

The full conjecture:

$$\boxed{\frac{\mathbf{P}_{L_C}(a, q, qt)}{1-q} \propto \sum_{k,\ell \geq 0} a^k q^\ell t^{\binom{k}{2}} \chi(t^{1/2}, \mathcal{H}_C^{\ell,k}).}$$

Maulik '12 True at the level of $\mathbb{P}_L = \mathbf{P}_L|_{t \to -1}$.

Key idea is an analogue for $\mathbb{P}_L$ of a *wall-crossing identity* from DT theory.

Unknown how to upgrade to $\mathbf{P}_L$.

Maulik–Yun, Migliorini–Shende '11

Morally, why are the Hilbert schemes hard?

They know about a *perverse filtration* $\mathsf{P}_{\leq *}$:

$$\sum_\ell q^\ell \chi(s, \mathcal{H}_C^\ell) = \frac{\sum_i q^i \chi(s, \mathrm{gr}_i^{\mathsf{P}} \mathrm{H}^*(\bar{\mathcal{P}}_C/\mathbf{Z}^b))}{(1-q)^b}$$

where $\bar{\mathcal{P}}_C$ is the *compactified Picard* parametrizing full, finitely-gen. R_C-submodules of $\mathrm{Frac}(R_C)$.

3 Hilb vs Quot

R_C has a *normalization*

$$R_C \hookrightarrow \widetilde{R}_C = \mathbf{C}[\![u_1]\!] \times \cdots \times \mathbf{C}[\![u_b]\!].$$

Form the *Quot schemes*

$$\mathcal{Q}_C^\ell = \{R_C\text{-modules } M \subseteq \widetilde{R}_C \mid \dim(\widetilde{R}_C/M) = \ell\}$$

An initial motivation for these varieties:

Thm (Kivinen–T '23) We have

$$\sum_\ell q^\ell \chi(s, \mathcal{Q}_C^\ell) = \frac{\sum_i q^i \chi(s, \bar{\mathcal{P}}_C^{(i)}/\mathbf{Z}^b)}{(1-q)^b}$$

for an explicit $\mathbf{Z}^b$-stable stratification $\bar{\mathcal{P}}_C = \coprod_i \bar{\mathcal{P}}_C^{(i)}$.

Recall ORS:

$$\frac{\mathbf{P}_{L_C}(a,q,qt)}{1-q} \propto \sum_{k,\ell\geq 0} a^k q^\ell t^{\binom{k}{2}} \chi(t^{1/2}, \mathcal{H}_C^{\ell,k}).$$

Form *nested Quot schemes*

$$\mathcal{Q}_C^{\ell,k} = \{(M,N) \in \mathcal{Q}_C^{\ell} \times \mathcal{Q}_C^{\ell+k} \mid M \supseteq N \supseteq \langle x,y\rangle M\}.$$

"Quot ORS" Conj (Kivinen–T '23) For any C,

$$\boxed{\frac{\mathbf{P}_{L_C}(a,q,t)}{1-q} \propto \sum_{k,\ell\geq 0} a^k q^\ell t^{\binom{k}{2}} \chi(t^{1/2}, \mathcal{Q}_C^{\ell,k}).}$$

Thm (Kivinen–T '23) Quot ORS holds in full for:

- $y^n = x^m$ with m,n coprime.
- $y^n = x^{nk}$.

"Hilb-vs-Quot" Conj (Kivinen–T '23) For any C,

$$\boxed{\sum_\ell q^\ell \chi(t^{1/2}, \mathcal{H}_C^\ell) = \sum_\ell q^\ell \chi((qt)^{1/2}, \mathcal{Q}_C^\ell).}$$

Remarks on Hilb-vs-Quot

- Should really be an identity in $\mathrm{K}_0(\mathsf{Var})$.
- $t \mapsto qt$ because $\mathcal{Q}^\ell$ is *larger* than $\mathcal{H}^\ell$ for fixed ℓ.
- Unibranch case was proposed by Cherednik in a different form, without the $\mathcal{Q}^\ell$.

Example Take $C = \{y^3 = x^4\}$.

The $\mathbf{C}^\times$-action on C induces actions on the $\mathcal{H}^\ell$, $\mathcal{Q}^\ell$.

The $\mathbf{C}^\times$-orbits form affine pavings.

$\mathcal{H}^0$	$\mathcal{H}^1$	$\mathcal{H}^2$	$\mathcal{H}^3$	$\mathcal{H}^4$	$\mathcal{H}^5$	$\mathcal{H}^6$	$\cdots$
pt			$\mathbf{C}^2$	$\mathbf{C}^2$		$\mathbf{C}^3$	$\cdots$
		$\mathbf{C}$	$\mathbf{C}$		$\mathbf{C}^2$	$\mathbf{C}^2$	$\cdots$
	pt			$\mathbf{C}^2$	$\mathbf{C}^2$	$\mathbf{C}^2$	$\cdots$
		pt		$\mathbf{C}$	$\mathbf{C}$	$\mathbf{C}$	$\cdots$
			pt	pt	pt	pt	$\cdots$

The rows classify monomial ideals as R_C-modules.

The colors are $\mathcal{Q}^0$, $\mathcal{Q}^1$, $\mathcal{Q}^2$, $\mathcal{Q}^3$, $\ldots$

Similar picture for any $y^n = x^m$ with m, n coprime.

"Hilb ORS" is hard because Hilb-vs-Quot is hard.

Thm (Kivinen–T '23) Hilb-vs-Quot holds for

$$y^n = x^m \quad \text{with } m, n \text{ coprime and } n \leq 3.$$

Key idea is that for fixed n, we can compute

$$\lim_{\substack{m \to \infty \\ \gcd(m,n)=1}} \sum_\ell q^\ell \chi(t^{1/2}, \mathcal{H}^\ell_{y^n = x^m}),$$

$$\lim_{\substack{m \to \infty \\ \gcd(m,n)=1}} \sum_\ell q^\ell \chi(t^{1/2}, \mathcal{Q}^\ell_{y^n = x^m})$$

by combinatorics.

If $n \leq 3$, then the limits determine the series at finite m, by a Serre duality trick.

4 Generic Singularities

Beyond torus links: work of Gorsky–Mazin–Oblomkov and Caprau–González–Hogancamp–Mazin.

GMO '22 + CGHM '23

Quot ORS holds for the lowest a-degree $\mathbf{P}^{\mathrm{lo}}_{L_C}$, when C has a *generic* unibranch singularity.

For such a singularity,

$$R_C \simeq \mathbf{C}[\![u^{nd}, u^{md} + u^{md+1} + \cdots]\!].$$

for some m, n, d with m, n coprime.

Moreover, L_C is the $(mnd + 1, n)$ *cable* of the (m, n) torus knot.

For such knots, CGHM generalize the GMV formula for $\mathbf{P}$ to a sum over $md \times nd$ Dyck paths.

For such singularities, GMO exhibit an affine paving of $\mathcal{Q}^\ell$ indexed by *admissible* subsets of $\mathbf{Z}_{\geq 0}$:

If $\Delta \subseteq \mathbf{Z}_{\geq 0}$ is admissible and $|\mathbf{Z}_{\geq 0} \setminus \Delta| = \ell$, then

$$\mathcal{Q}^\ell_\Delta = \{M \in \mathcal{Q}^\ell \mid \Delta = \{\mathrm{ord}(f)\}_{f \in M}\}.$$

Gorsky–Mazin–Vazirani '17 + GMO '23 Bijection

$$\{\text{admissible sets } \Delta\} \xrightarrow{\sim} \{\text{Dyck paths } D\}$$

such that $q^{a(D)} t^{c(D)} = q^\ell t^{\dim \mathcal{Q}^\ell_\Delta}$.

Recall GMV:

$$\sum_D q^{a(D)} t^{c(D)} \prod_{\Box \in S_{\mathrm{GMV}}(D)} f_{\mathrm{GMV},D,\Box}.$$

But $\prod_\Box f_{\mathrm{GMV},D,\Box}$ does not match

$$\sum_{k,\Delta'} a^k t^{\binom{k}{2}} \chi(t^{1/2}, \mathcal{Q}^{\ell,k}_{\Delta \supseteq \Delta'}),$$

so CGHM cannot match higher a-degree terms.

Thm (T '25+) By contrast:

1 Mellit's formula for $\mathbf{P}$ generalizes to the knots of generic unibranch singularities.

2 f_{Mellit} does match nested Quot.

3 Quot ORS holds in full for such singularities.

5 Some Lie Theory

What got me into this?

Given a group G and subgroup K and $\gamma \in \mathrm{Lie}(G)$, the *Springer fiber* over γ is its fixed-point set

$$X_\gamma = \{gK \in G/K \mid \gamma \in \mathrm{Ad}(g)(\mathrm{Lie}(K))\}.$$

Laumon '02 Take C a branched n-cover of a line. Then its compactified Picard

$$\bar{\mathcal{P}}_C = \{\text{full, fin.-gen. modules } M \subseteq \mathrm{Frac}(R_C)\}$$

is a X_γ for $G = \mathrm{GL}_n(\mathbf{C}(\!(z)\!))$ and $K = \mathrm{GL}_n(\mathbf{C}[\![z]\!])$.

In this case, also called an *affine Springer fiber*.

Example Suppose that $R_C = \mathbf{C}[\![u^4, u^6 + u^7]\!]$.

Setting $u = z^{1/n}$ and fixing $\mathbf{C}(\!(u)\!) \xrightarrow{\sim} \mathbf{C}(\!(z)\!)^n$,

$$u^6 + u^7 \curvearrowright \mathbf{C}(\!(u)\!) \quad \rightsquigarrow \quad \gamma \curvearrowright \mathbf{C}(\!(z)\!)^n.$$

Two possibilities for γ:

$$\begin{pmatrix} & & & z^6 + z^7 \\ 1 & & & 4z^6 \\ & 1 & & 6z^6 \\ & & 1 & 4z^6 \end{pmatrix}, \begin{pmatrix} & u^2 & z^2 & \\ & & z^2 & z^2 \\ z & & & z^2 \\ z & z & & \end{pmatrix}.$$

Both give $\bar{\mathcal{P}}_C \simeq X_\gamma$, but different *positive truncations*

$$X_\gamma \cap \mathrm{Mat}_n(\mathbf{C}[\![z]\!])/\mathrm{GL}_n(\mathbf{C}[\![z]\!]).$$

Respectively, they are $\bigsqcup_\ell \mathcal{H}_C^\ell$ and $\bigsqcup_\ell \mathcal{Q}_C^\ell$.

This viewpoint also suggests:

1 Generalizing $K = \mathrm{GL}_n(\mathbf{C}[\![z]\!])$ to other *parahorics*.

2 Generalizing GL_n to other reductive groups.

(1) leads to flagged versions of $\mathcal{H}^\ell$, $\mathcal{Q}^\ell$ that indirectly encode the nested versions and more.

(2) leads to conjectures relating affine Springer fibers to q, t-traces on *generalized braid groups* Br_W.

For $\mathbf{G}$ with Lie algebra $\mathbf{g} \supseteq \mathbf{t}$ and Weyl group W,

$$\mathbf{g}(\!(z)\!) \to (\mathbf{g} /\!\!/ \mathbf{G})(\!(z)\!) = (\mathbf{t} /\!\!/ W)(\!(z)\!)$$

takes loops in $\mathbf{g}$ to conjugacy classes in $\widehat{\mathrm{Br}}_W$.

Thm (T '21) Formulas for such q, t-traces via

$$W \curvearrowright \sum_{j,k} q^j t^k \operatorname{gr}^{\mathsf{W}}_j \mathrm{H}^k_{\mathbf{G},c}(Z_\beta),$$

where Z_β is the *braid Steinberg variety* of $\beta \in \mathrm{Br}^+_W$:

$$Z_\beta = \tilde{\mathcal{U}} \times_{\mathcal{U}} \mathcal{U}_\beta,$$

where $\tilde{\mathcal{U}} \to \mathcal{U}$ is the Springer resolution and

$$\mathcal{U}_\beta \simeq \{(u\mathbf{B}_\ell u^{-1} \xrightarrow{s_1} \mathbf{B}_1 \xrightarrow{s_2} \cdots \xrightarrow{s_\ell} \mathbf{B}_\ell) \mid u \in \mathcal{U}\}$$

for Borel $\mathbf{B} \subseteq \mathbf{G}$ and $\beta = \sigma_{s_1} \cdots \sigma_{s_\ell}$.

Hope Nonabelian Hodge relates X_γ and $\mathcal{U}_\beta/\mathbf{G}$.

Thank you for listening.