11.

Notes on parabolic Hessenberg varieties inside the affine flag variety.

11.1.

11.1. Let G be a connected reductive algebraic group over $\mathbf{C}$. Fix a maximal torus $T \subseteq G$ and Borel $B \subseteq G$ containing T. Let $X \supseteq \Phi \supseteq \Phi^{\pm}$ be the corresponding character lattice, root system, and subsets of positive/negative roots. Let

$$\rho^\vee = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha^\vee \in X^\vee \otimes \mathbf{Q}.$$

Let $\mathfrak{g}$ and $\mathfrak{t}$ be the Lie algebras of G and T.

Let $W = N_G(T)/T$ and $\widetilde{W} = N_{G(\!(z)\!)}(T[\![z]\!])/T[\![z]\!]$. At the level of $\mathbf{C}$-points, we have isomorphisms $W \simeq N_{G(\!(z)\!)}(T(\!(z)\!))/T(\!(z)\!)$ and $X^\vee \simeq T(\!(z)\!)/T[\![z]\!]$. They induce an isomorphism $\widetilde{W} \simeq W \ltimes X^\vee$.

11.2. We picture each affine root $(\alpha, k) \in \Phi \times \mathbf{Z}$ as a hyperplane in $X^\vee \otimes \mathbf{R}$ with a fixed orientation: namely, pointing toward vectors x such that $\langle \alpha, x\rangle + k > 0$. For all $x \in X^\vee \otimes \mathbf{R}$ and $r \in \mathbf{R}$, we set

$$\mathfrak{g}(\!(z)\!)_{x,\kappa} = \prod_{\substack{(\alpha,k)\in\Phi\times\mathbf{Z}\\ \langle\alpha,x\rangle+k=r}} z^k \mathfrak{g}_\alpha \times \begin{cases} z^r\mathfrak{t} & r \in \mathbf{Z}, \\ 0 & r \notin \mathbf{Z}. \end{cases}$$

(The product symbol is an abuse of notation for the completion of a direct sum.) Let $\mathfrak{l}_x \subseteq \mathfrak{p}_x \subseteq \mathfrak{g}(\!(z)\!)$ be defined by

$$\begin{aligned} \mathfrak{l}_x &= \mathfrak{g}(\!(z)\!)_{x,0}, \\ \mathfrak{P}_x &= \prod_{r \geq 0} \mathfrak{g}(\!(z)\!)_{x,r}. \end{aligned}$$

Let $\mathbf{P}_x$ be the parahoric subgroup of $G(\!(z)\!)$ with Lie algebra $\mathfrak{P}_x$. Then its Levi factor L_x is a connected (finite-dimensional) reductive algebraic group with Lie algebra $\mathfrak{l}_x$. Note that $\mathbf{P}_x$ contains $T[\![z]\!]$, whereas L_x contains T. Let $W_x = N_{L_x}(T)/T$.

It helps to picture the hyperplane arrangement of the root system of L_x as the set of hyperplanes that pass through x. For example, taking $x = 0$, we find that $\mathfrak{l}_0 = G$ and $\mathfrak{P}_0 = G[\![z]\!]$, giving $W_0 = W$.

The conjugation action of W on T induces a right action of W on $X^\vee$. Under it,

$$\mathfrak{g}(\!(z)\!)_{x,r} \cdot w = \mathfrak{g}(\!(z)\!)_{x\cdot w,r}$$

Therefore, $\mathfrak{l}_x \cdot w = \mathfrak{l}_{x\cdot w}$ and $\mathfrak{P}_x \cdot w = \mathfrak{P}_{x\cdot w}$. For instance, if $G = \mathrm{SL}_2$, then

$$\mathfrak{p}_0 \cdot \alpha^\vee = z^{-\alpha^\vee}\mathfrak{g}[\![z]\!]z^{\alpha^\vee} = z^{-2}\mathfrak{g}_\alpha[\![z]\!] \oplus \mathfrak{t}[\![z]\!] \oplus z^2\mathfrak{g}_{-\alpha}[\![z]\!].$$

11.3. If x is rational, meaning $x = (d/e)x_0$ for some $d, e \in \mathbf{Q}$ and $x_0 \in X^\vee$, then it defines a $\mathbf{G}_m$-action on $G(\!(z)\!)$ as follows: For all $g = g(z) \in G(\!(z)\!)$ and $c \in \mathbf{G}_m$, let

$$c \cdot_x g(z) = \mathrm{Ad}(c^{dx_0})g(c^m z).$$

Taking the differential of the action above, we get a $\mathbf{G}_m$-action on $\mathfrak{g}(\!(z)\!)$.

Proposition 11.1. *Above, $\mathfrak{g}(\!(z)\!)_{x,r}$ is the eigenspace of $\cdot_x$ of weight er.*

Corollary 11.2. *The adjoint action of $G(\!(z)\!)$ on $\mathfrak{g}(\!(z)\!)$ restricts to an L_x-action on $\mathfrak{g}(\!(z)\!)_{x,r}$ for each r.*

11.2.

11.4. Recall that the fundamental alcove is the region of points $x \in X^\vee \otimes \mathbf{R}$ such that $0 < \langle \alpha, x \rangle < 1$ for all $\alpha \in \Phi^+$. Let $\mathfrak{I} = \mathfrak{P}_x$ and $\mathbf{I} = \mathbf{P}_x$ for any choice of x inside the fundamental alcove. For example, if $G = \mathrm{SL}_2$, so that $\Phi^\pm = \{\pm\alpha\}$, then

$$\mathfrak{I} = \mathfrak{g}_\alpha[\![z]\!] \oplus \mathfrak{t}[\![z]\!] \oplus z\mathfrak{g}_{-\alpha}[\![z]\!].$$

11.5. The affine flag variety with respect to $\mathbf{I}$ is (the underlying reduced ind-scheme of) the (fpqc) quotient $\mathcal{F}\ell = G(\!(z)\!)/\mathbf{I}$. We have an affine Bruhat decomposition

$$\mathcal{F}\ell = \coprod_{w \in W_x \backslash \widetilde{W}} \mathbf{P}_x \dot{w} \mathbf{I}/\mathbf{I}.$$

The action $\cdot_x$ descends from $G(\!(z)\!)$ to $\mathcal{F}\ell$. The fixed points are given by

$$\mathcal{F}\ell^{\mathbf{G}_m} = \coprod_{w \in W_x \backslash \widetilde{W}} L_x \dot{w} \mathbf{I}/\mathbf{I}.$$

For any $\gamma \in \mathfrak{g}(\!(z)\!)$, the affine Springer fiber above γ of type $\mathbf{I}$ is

$$\mathcal{F}\ell_\gamma = \{g\mathbf{I} \in \mathcal{F}\ell \mid \mathrm{Ad}(g^{-1})\gamma \in \mathfrak{I}\}.$$

Proposition 11.3. *If γ is an eigenvector of $\cdot_x$, then $\mathcal{F}\ell_\gamma$ is stable under the $\mathbf{G}_m$-action induced on $\mathcal{F}\ell$.*

We will focus on the case where γ is regular semisimple, since $\mathcal{F}\ell^\gamma$ is locally of finite type for such γ, as shown by Kazhdan–Lusztig. To this end, let $\mathfrak{t}^{\mathrm{reg}} \subseteq \mathfrak{t}$ be the locus where W acts freely. Let $\mathfrak{g}(\!(z)\!)^{\mathrm{rs}} \subseteq \mathfrak{g}(\!(z)\!)$ be the regular semisimple locus, meaning the preimage of $(\mathfrak{t}^{\mathrm{reg}} /\!/ W)(\!(z)\!)$ along the map $\mathfrak{g} \to \mathfrak{t} /\!/ W$ induced by Chevalley restriction. Let $\mathfrak{g}(\!(z)\!)^{\mathrm{rs}}_{x,r} \subseteq \mathfrak{g}(\!(z)\!)_{x,r}$ be defined similarly.

11.6. For all $w \in \widetilde{W}$ and $r \in \mathbf{R}$, let

$$\begin{aligned} V_{x,r,w} &= \mathfrak{g}(\!(z)\!)_{x,r} \cap w\mathfrak{I}w^{-1}, \\ \mathfrak{p}_{x,w} &= V_{x,0,w} = \mathfrak{l}_x \cap w\mathfrak{I}w^{-1}, \\ P_{x,w} &= L_x \cap w\mathbf{I}w^{-1}. \end{aligned}$$

Then $P_{x,w}$ is a parabolic subgroup of L_x with Lie algebra $\mathfrak{p}_{x,w}$. The adjoint L_x-action on $\mathfrak{g}(\!(z)\!)_{x,r}$ restricts to a $P_{x,w}$-action on $V_{x,r,w}$. Thus, for all $\gamma \in \mathfrak{g}(\!(z)\!)_{x,r}$, we can form

$$\mathrm{Hess}_{x,w,\gamma} = \{gP_{x,w} \in L_x/P_{x,w} \mid \mathrm{Ad}(g^{-1})\gamma \in V_{x,r,w}\},$$

a parabolic Hessenberg variety.

We may regard $\mathrm{Hess}_{x,w,\gamma}$ as the zero locus of a section of the vector bundle

$$(L_x \times \mathfrak{g}(\!(z)\!)_{x,r}/V_{x,r,w})/P_{x,w} \to L_x/P_{x,w}$$

defined by γ. If $\gamma \in \mathfrak{g}(\!(z)\!)^{\mathrm{rs}}_{x,r}$, then it is transverse to the zero section, giving:

Proposition 11.4 (Goresky–Kottwiz–MacPherson)**.** *If* $\gamma \in \mathfrak{g}(\!(z)\!)^{\mathrm{rs}}_{x,r}$*, then* $\mathrm{Hess}_{x,w,\gamma}$ *is smooth and*

$$\begin{aligned} \dim \mathrm{Hess}_{x,w,\gamma} &= \dim L_x/P_{x,w} - \dim \mathfrak{g}(\!(z)\!)_{x,r}/V_{x,\nu,w} \\ &= \{(\alpha, k) \mid \langle \alpha, y \cdot w\rangle + k < 0 \leq \langle \alpha, x\rangle + k = r\} \end{aligned}$$

for any choice of y *such that* $\mathbf{I} = \mathbf{P}_y$.

The inclusion $L_x \subseteq G(\!(z)\!)$ descends to an inclusion $L_x/P_{x,w} \subseteq \mathcal{F}\ell$. For all $w \in \widetilde{W}$, we identify $\mathrm{Hess}_{x,w,\gamma}$ with its image in $\mathcal{F}\ell$.

11.3.

11.7. Fix a fraction $\nu = d/m > 0$ in lowest terms, where m is a regular number for W. Then ν is an admissible slope for W in the sense of Oblomkov–Yun. Henceforth,

$$x = \nu\rho^\vee \in X^\vee \otimes \mathbf{Q}.$$

Theorem 11.5 (Goresky–Kottwitz–MacPherson)**.** *Suppose that* $\gamma \in \mathfrak{g}(\!(z)\!)^{\mathrm{rs}}_{x,\nu}$*. Then*

$$\mathcal{F}\ell_\gamma^{\mathbf{G}_m} = \coprod_{w \in W_x \backslash \widetilde{W}} \mathrm{Hess}_{x,w,\gamma}.$$

Moreover, the locus that contracts onto a given Hessenberg variety under the $\mathbf{G}_m$*-action forms an affine-space bundle over that variety. As we run over* $w \in W_x\backslash\widetilde{W}$*, these loci form a stratification of* $\mathcal{F}\ell_\gamma$*.*

Following Varagnolo–Vasserot, connected components of

$$X^\vee \otimes \mathbf{R} - \bigcup_{\substack{(\alpha,k)\in\Phi\times\mathbf{Z} \\ \langle\alpha,x\rangle+k=\nu}} \{y \mid \langle\alpha, y\rangle + k = 0\}$$

are called ν-clans.

Proposition 11.6 (Varagnolo–Vasserot)**.** *If the w- and w'-translates of the fundamental alcove are contained in the same clan, then $V_{x,\nu,w} = V_{x,\nu,w'}$ and $P_{x,w} = P_{x,w'}$, whence* $\mathrm{Hess}_{x,w,\gamma} \simeq \mathrm{Hess}_{x,w',\gamma}$ *for all $\gamma \in \mathfrak{g}(\!(z)\!)_{x,\nu}$.*