Warmup recall:

R^ω = all seq's $(a_1, a_2, \ldots)$

R^∞ = seq's eventually zero

[why do we use R^∞ to refer to the latter?]

for all n, an injective map

R^n to R^ω (extend by zero's past n)

what is the union of their images? R^∞

moreover:

there is a metric $\rho : R^\infty \times R^\infty$ to $[0, \infty)$ def by

$\rho(x, y) = \max_{i = 1, 2, \ldots} |x_i - y_i|$

that restricts to the square metric on R^n for all n

ρ is not well-defined on R^ω

Df the uniform metric on R^ω is defined by

$u(x, y) = \sup_i \min\{1, |x_i - y_i|\}$

[is that the same as $\min\{1, \sup_i |x_i - y_i|\}$? no]

is this a metric?

1) $u(x, y) = 0$ implies $x = y$? [yes]

2) $u(x, y) = u(y, x)$? [yes]

3) $u(x, y) + u(y, z) \geq u(x, z)$?

$\sup_i \min\{1, |x_i - y_i|\} + \sup_i \min\{1, y_i - z_i|\}$

$\geq \sup_i (\min\{1, |x_i - y_i|\} + \min\{1, |y_i - z_i|\})$

$\geq \sup_i \min\{1, |x_i - z_i|\}$

[thus far, topologies gen’d by metrics]
[now, topologies gen’d by collections of subsets]

(Munkres §13, 19)

X any set,
{B_i}_i any collection of subsets of X

<u>Df</u> {B_i}_i is a subbasis for a topology on X
iff
X = bigcup_i B_i

{B_i}_i is a basis for a topology on X
iff
X = bigcup_i B_i and
for all i, j, can find B_k sub B_i cap B_j

a subbasis {C_j}_j gives rise to a basis {B_i}_i:

{B_i}_i = {finite intersections of the C_j’s,
including Ø}

a basis {B_i}_i gives rise to a topology T:

T = {arbitrary unions of the B_i’s, incl. Ø}
= {U sub X | for all x in U,
have x in B_i sub U for some i}
[why are the two defn’s of T equivalent?]

<u>Rem</u> different bases can induce
the same topology

[just like different metrics inducing the same top]

__Ex__ bases in R^n:

{balls $B(x, \delta)$ | x in R^n and $\delta > 0$}

but also, $\{(a_1, b_1) \times \ldots \times (a_n, b_n) \mid a_i, b_i\}$

__Lem__ suppose T is a topology on X,
$\{C_i\}_i$ __any__ subcollection of T

s.t. for all U in T, and x in U,
there is some i s.t. x in C_i sub U

then $\{C_i\}_i$ is a basis, and it gives rise to T

__Pf__ X in T, so $X = \text{bigcup}_i C_i$
C_i's in T, so C_i cap C_j in T
whence C_k sub C_i cap C_j
immediate by hypothesis that $\{C_i\}_i$ induces T

__Warning__ a subbasis is not a special kind of basis
[prefix “sub-” is misleading]
a subbasis generates a basis,
and forms a subset of that basis

__Warning__ bases for topologies T
have nothing to do with
bases for vector spaces V

if we fix a basis for T, then an open set in T can be a union of basis open sets in many ways

if we fix a basis for V, then a vector in V can be a linear combo of basis vectors in only one way

[return to R^ω:]

Df the box topology on R^ω is gen'd by the basis of "boxes"

$(a_1, b_1) \times (a_2, b_2) \times \dots$ [and so on forever]

[is this really a basis?]

Df the product topology on R^ω is gen'd by the subbasis of sets

$C_{i, a, b} = \{x = (x_1, x_2, \dots) \mid a < x_i < b\}$
as we run over $i > 0$ and $a < b$

Q how do these topologies compare?

the basis generated by the product subbasis:

{finite intersections of the $C_{i, a, b}$}

these intersections look like

$B_{J, \mathbf{a}, \mathbf{b}} = \{x = (x_1, x_2, \dots) \mid a_i < x_i < b_i$ for i in $J\}$

as we run over
finite sets J sub $\{1, 2, \dots\}$
$\mathbf{a} = (a_i)_{i \text{ in } J}$, $\mathbf{b} = (b_i)_{i \text{ in } J}$

conclude:
each product basis open set is a union of some box basis open sets

Lem suppose S is gen'd by a basis $\{B_i\}_i$,
T is gen'd by a basis $\{C_j\}_j$,
and each B_i is a union of C_j's

then T is finer than S

Pf exercise

Cor the box topology is finer than the product topology [but not the same?]

why care? [surprisingly, the product topology is "better behaved"] we will discuss later:

the product topology is the coarsest topology that makes pr_i R^ω to R continuous for all i

what about the uniform topology gen'd by

$$u(x, y) = \sup_i \min\{1, |x_i - y_i|\} ?$$

Q1 what's a basis for the uniform topology?
Q2 how does it compare to box, product?

A1 the collection of balls $B_u(x, \delta)$

suffices to use balls where $\delta \leq 1$ [why?] for such,

$B_u(x, \delta) = \{y \mid \sup_i |x_i - y_i| < \delta\}$ [why?]
$\neq \{y \mid |x_i - y_i| < \delta\}$ [why??]

A2 compare bases:
box supset uniform supset product
[do any of them coincide?]