MATH 430: INTRODUCTION TO TOPOLOGY
FINAL EXAM

SPRING 2025

You have till 4:30 pm to do these problems in any order. You do not need to write in complete sentences, but please show your work. Calculators, other electronic devices, and study aids are prohibited. If you need to use the bathroom, please hand in any electronic devices, especially phones, to the proctor. **Throughout: $\mathbf{R}$ has the analytic topology unless otherwise specified.**

Problem 1 (4 points)**.** Let $A = \{0\} \cup \{\frac{1}{n} \mid n = 1, 2, 3, \ldots\}$ as a subspace of $\mathbf{R}$. Show that A is compact. Do not simply cite that it is closed and bounded in $\mathbf{R}$.

Solution. Pick an open cover $\mathcal{U}$ of A. Then some element $U \in \mathcal{U}$ contains 0. By definition, $U = A \cap V$ for some open $V \subseteq \mathbf{R}$. Pick $\epsilon > 0$ small enough that $(-\epsilon, \epsilon) \subseteq V$. Then U contains $\frac{1}{n}$ for all n such that $n > \frac{1}{\epsilon}$. There are finitely many n for which this inequality fails. So there are finitely many elements of $\mathcal{U}$ needed to cover them. These together with U form the desired finite subcover of $\mathcal{U}$. □

Problem 2 (4 points)**.** Let $\mathbf{Q}$ be the set of rational numbers. Let $X = \mathbf{R}^2/{\sim}$, where $p \sim q$ if and only if $p, q \in \mathbf{Q}^2$. Show that X is not Hausdorff. You may assume that every nonempty open set of $\mathbf{R}$ intersects $\mathbf{Q}$.

Solution. By definition, the quotient map $\mathbf{R}^2 \to X$ sends $\mathbf{Q}^2$ to a single point: say, o. It suffices to show that every nonempty open set of X contains o. Indeed, if $U \subseteq X$ is such a set, then its preimage $V \subseteq \mathbf{R}^2$ is also nonempty and open. Hence it contains a set of the form $V' \times V''$, where V', V'' are nonempty and open in $\mathbf{R}$. Pick rational $a \in V'$ and $b \in V''$. Then $(a, b) \in V' \times V'' \subseteq V$, whence $o \in U$. □

Problem 3 (5 points). (1) Show that for any space X, the map

$$\Delta : X \to X^\omega \quad \text{defined by } \Delta(x) = (x, x, x, \ldots)$$

is continuous with respect to the product topology on X^ω.

(2) Let I be any indexing set. Show that if $g_i : Y_i \to Z_i$ is a continuous map for all $i \in I$, then

$$g : \prod_i Y_i \to \prod_i Z_i \quad \text{defined by } g((y_i)_{i\in I}) = (g_i(y_i))_{i\in I}$$

is continuous with respect to the product topologies.

Solution. (1) Any open set of $\prod_i X_i$ takes the form $\prod_i U_i$, where U_i is open in X_i for all i and $U_i = X_i$ for all but finitely many i. The preimage of this set under Δ is $\bigcap_i U_i$. In this intersection, we can ignore i such that $U_i = X_i$. We are left with a finite intersection of open sets in X, which is open.

[Alternatively: Let $\mathrm{pr}_n : X^\omega \to X$ be the nth projection. Then $\mathrm{pr}_n \circ \Delta$ is the identity map of X, so it is continuous for all n. By the defining property of the product topology, Δ is also continuous.]

(2) Any open set of $\prod_i Z_i$ takes the form $\prod_i V_i$, where V_i is open in Z_i for all i, and $V_i = Z_i$ for all but finitely many i. Its preimage under g is $\prod_i g_i^{-1}(V_i)$. Here, $g_i^{-1}(V_i)$ is open in Y_i for all i, and if $V_i = Z_i$, then $g_i^{-1}(V_i) = Y_i$. Therefore the preimage is open in $\prod_i Y_i$. Therefore g is continuous. □

Problem 4 (4 points). Let $Z = \prod_{n=1}^\infty Z_n$. Suppose that Z_n is contractible for all n. That is, there is a homotopy $h_n : Z_n \times [0,1] \to Z_n$ such that $h_n(-,0) = \mathrm{id}_{Z_n}$ and $h_n(-,1)$ is constant. Show that Z is contractible in the product topology. You may (and should) assume both parts of Problem 3.

Solution. By Problem 3 part (2), the map

$$g : \prod_n (Z_n \times [0,1]) \to \prod_n Z_n = Z \quad \text{defined by } g((z_n, t_n)_n) = (h_n(z_n, t_n))_n$$

is also continuous. The source is homeomorphic to $Z \times [0,1]^\omega$ by permuting factors. So the map $h : Z \times [0,1]^\omega \to Z$ defined by $h((z_n)_n, (t_n)_n) = (h_n(z_n, t_n))_n$ is continuous.

By Problem 3 part (1), the map $\Delta : [0,1] \to [0,1]^\omega$ sending $\Delta(t) = (t, t, t, \ldots)$ is continuous, so by Problem 3 part (2), the map $\mathrm{id}_Z \times \Delta$ is continuous. So the map

$$H = h \circ (\mathrm{id}_Z \times \Delta) : Z \times [0,1] \to Z \quad \text{defined by } H((z_n)_n, t) = (h_n(z_n, t))_n$$

is continuous. This is a homotopy between id_Z and the constant map valued at $(p_n)_n$, where $p_n = h_n(z_n, 1)$ for any $z_n \in Z_n$. □

Problem 5 (5 points)**.** Let $M = \mathbf{R}^n - \{\text{origin}\}$, where $n \geq 2$. Show that in M, any point has an explicit path to some point in the hyperplane

$$\{(1, x_2, \ldots, x_n) \mid x_2, \ldots, x_n \in \mathbf{R}\} \subseteq M.$$

You may leave any path concatenations as they are.

Solution. Fix a point $p = (a_1, a_2, \ldots, a_n) \in M$. If $a_i \neq 0$ for some $i \neq 1$, then take the path $\gamma_p(s) = ((1-s)a_1 + s, a_2, \ldots, a_n)$.

Else, $a_2 = \cdots = a_n = 0$. Let $\delta(s) = (a_1, s, 0, \ldots, 0)$, then take the path $\delta * \gamma_{\delta(1)}$, where $\gamma_{\delta(1)}$ is analogous to γ_p, but with $\delta(1)$ in place of p. $\square$

Problem 6 (3 points)**.** Show that for any space X, there is an explicit homotopy equivalence between $\mathbf{R} \times X$ and X. You may assume that X is homeomorphic to $\{0\} \times X$.

Solution. It suffices to give a homotopy equivalence between $\mathbf{R} \times X$ and $\{0\} \times X$. Let $i : \{0\} \times X \to \mathbf{R} \times X$ be the inclusion, and let $r : \mathbf{R} \times X \to \{0\} \times X$ be defined by $r(s, x) = r(0, x)$. Then $r \circ i = \mathrm{id}_{\{0\} \times X}$. It remains for us to show that $i \circ r$ is homotopic to $\mathrm{id}_{\mathbf{R} \times X}$.

Let $h : (\mathbf{R} \times X) \times [0, 1] \to \mathbf{R} \times X$ be defined by $h((s, x), t) = (st, x)$. Then $h(s, x, 0) = (0, x)$ and $h(s, x, 1) = (s, x)$ for all $(s, x) \in \mathbf{R} \times X$. Thus h is the desired homotopy. $\square$

Problem 7 (5 points)**.** Let $S^1 \subseteq \mathbf{R}^2$ be the unit circle, and let $p = (1, 0) \in S^1$.

(1) Show that the paths $\alpha, \beta : [0, 1] \to S^1$ defined by

$$\alpha(s) = (\cos(\pi s), \sin(\pi s)) \quad \text{and} \quad \beta(s) = (\cos(\pi s), -\sin(\pi s))$$

are not path-homotopic in S^1. You may assume the explicit description of $\pi_1(S^1, p)$ from lecture.

(2) Are α and β at least homotopic in S^1? Just state yes or no.

Solution. (1) If α and β are path-homotopic, then $\alpha * \bar{\beta}$ and $\beta * \bar{\beta}$ are path-homotopic, where $\bar{\beta}(s) = \beta(1 - s)$. But $[\alpha * \bar{\beta}]$ generates the nontrivial group $\pi_1(S^1, p) \simeq \mathbf{Z}$, while $[\beta * \bar{\beta}]$ is its identity element.

(2) Yes. [Both α and β are homotopic, just not path-homotopic, to the constant loop at p.] □

Problem 8 (3 points)**.** Let $U \subseteq \mathbf{R}^3$ be the complement of $\{(x, y, 0) \mid y = x^2\}$.

(1) Show that U is homeomorphic to the complement of the x-axis.

(2) Assuming the path-connectedness of U and part (1), compute $\pi_1(U)$. You may also assume Problem 6, and any homotopy equivalences or π_1 formulas from lecture.

Solution. (1) Let $V = \mathbf{R}^3 - (x\text{-axis})$. By definition,

$$U = \{(x, y, z) \mid \text{either } y \neq x^2 \text{ or } z \neq 0\},$$
$$V = \{(x, y, z) \mid \text{either } y \neq 0 \text{ or } z \neq 0\}.$$

Let $f : V \to U$ be defined by $f(x, y, z) = (x, y + x^2, z)$, and let $g : U \to V$ be defined by $g(x, y, z) = (x, y - x^2, z)$. Then f and g are continuous and mutually inverse, so U and V are homeomorphic.

(2) By part (1), $\pi_1(U) \simeq \pi_1(V)$. But V is homeomorphic to $\mathbf{R} \times X$, where $X = \{(y, z) \neq (0, 0)\} \in \mathbf{R}^2$. By Problem 6, $\pi_1(V) \simeq \pi_1(\mathbf{R} \times X) \simeq \pi_1(X)$. Finally, X is homotopy equivalent to the circle $S^1 \subseteq \mathbf{R}^2$. Therefore, $\pi_1(V) \simeq \pi_1(S^1) \simeq \mathbf{Z}$. □

Problem 9 (3 points)**.** Give open sets U, V in the unit sphere $S^2 \subseteq \mathbf{R}^3$ such that:

- $S^2 = U \cup V$.
- U, V, and $U \cap V$ are path-connected.
- U, V, and $U \cap V$ are not simply connected.

You may use labeled pictures to do so.

Justify that your answer satisfies the third condition above. You may assert (true) homotopy equivalences without proof, and you may use any π_1 formula from lecture.

Solution. Pick four (pairwise-distinct) points $p_1, p_2, p_3, p_4 \in S^2$. Let

$$U = S^2 - \{p_1, p_2\} \quad \text{and} \quad V = S^2 - \{p_3, p_4\}.$$

Then $p_3, p_4 \in U$ and $p_1, p_2 \in V$, so $S^2 = U \cup V$. Moreover, U, V, and $U \cap V$ are all path-connected.

We see that U and V are each homotopy equivalent to a circle, so $\pi_1(U), \pi_1(V) \simeq \mathbf{Z}$. Meanwhile, $U \cap V = S^2 - \{p_1, p_2, p_3, p_4\}$. This is homotopy equivalent to a triply-punctured plane, which is in turn homotopy equivalent to $S^1 \vee S^1 \vee S^1$, a wedge of three circles. So $\pi_1(U \cap V) = \mathbf{Z} * \mathbf{Z} * \mathbf{Z}$. □

Problem 10 (2 points)**.** Fix spaces X, Y, and continuous maps $f : X \to Y$ and $g : Y \to X$ such that $g \circ f$ is a self-homeomorphism of X. Show that

$$f_* : \pi_1(X, x) \to \pi_1(Y, f(x))$$

is injective for any $x \in X$.

Solution. For any $[\gamma_1], [\gamma_2] \in \pi_1(X, x)$ such that $f_*([\gamma_1]) = f_*([\gamma_2])$, we see that $g_* f_*([\gamma_1]) = g_* f_*([\gamma_2])$. But $g \circ f$ is a homeomorphism, so $(g \circ f)_* = g_* \circ f_*$ is an isomorphism of fundamental groups. So $g_* \circ f_*$ is bijective, giving $[\gamma_1] = [\gamma_2]$. □

Problem 11 (2 points)**.** Do one of the following:

(I) Write down the "existence" part of the universal property of the free product of two groups: the part asserting that certain data imply the existence of a certain homomorphism.

(II) Write down the path-lifting property satisfied by a covering map.

Solution. (I) If G_1, G_2 are groups, then for any group K and homomorphisms $\phi_1 : G_1 \to K$ and $\phi_2 : G_2 \to K$, there is a unique homomorphism $\phi : G_1 * G_2 \to K$ such that $\phi = \phi_j \circ i_j$ for $j = 1, 2$, where $i_J : G_j \to G_1 * G_2$ is the inclusion.

(II) If $p : E \to X$ is a covering map and $\gamma : [0, 1] \to X$ is a path and $e \in E$, then there is a unique path $\Gamma : [0, 1] \to E$ such that $\Gamma(0) = e$ and $\gamma = p \circ \Gamma$. □

Extra Credit (2 points)**.** Let $P^2 = S^2/{\sim}$, where $p \sim q$ if and only if p and q are antipodal. Determine the integers n for which there is an n-fold covering map onto the four-dimensional manifold $P^2 \times P^2$ with path-connected covering space.

Solution. The quotient map $p : S^2 \to P^2$ is a 2-fold covering map. Since S^2 is simply-connected, we deduce from the lifting correspondence that $|\pi_1(P^2)| = 2$, which forces $\pi_1(P^2) \simeq \mathbf{Z}/2\mathbf{Z}$. Therefore,

$$|\pi_1(P^2 \times P^2)| = 4.$$

So for any subgroup $H \subseteq \pi_1(P^2 \times P^2)$, the number of cosets $|\pi_1(P^2 \times P^2)/H|$ must be a divisor of 4. So by the lifting correspondence, the only possible n for which there is an n-fold covering map of $P^2 \times P^2$ with path-connected covering space are the divisors of 4: namely, $1, 2, 4$. Each option can occur: Use the covering spaces

$$P^2 \times P^2, \quad S^2 \times P^2, \quad S^2 \times S^2.$$

(Note that we did not need the Galois correspondence to see that each option occurs.) □