KIW 52 (Dec. 2025)

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$G = \mathrm{GL}_n$

$\mathcal{U} = \{\text{unipotent } n \times n \text{ matrices}\} \subseteq G$

<u>Ex</u> for $n = 2$, have $\mathcal{U} \simeq \left\{ \begin{pmatrix} 1+x & y \\ z & 1-x \end{pmatrix} \,\middle|\, x^2 + yz = 0 \right\}$

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$B \subseteq G$ upper-triangular

<u>Bruhat, Chevalley</u> understand G via B:

$$G = \bigsqcup_{w \in S_n} B\dot{w}B$$

can we understand $\mathcal{U}$ via $U := B \cap \mathcal{U}$?

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$U_- \subseteq B_- \subseteq G$ lower-triangular

<u>Fine–Herstein '58, Steinberg '65</u>

$$|\mathcal{U}(\mathbb{F}_q)| = q^{n(n-1)} = |U(\mathbb{F}_q)|^2 = |UU_-(\mathbb{F}_q)|$$

<u>Kawanaka '75</u> for any $w \in S_n$,

$$|(\overbrace{\mathcal{U} \cap B\dot{w}B}^{\mathcal{U}_w})(\mathbb{F}_q)| = |(\overbrace{UU_- \cap B\dot{w}B}^{\mathcal{V}_w})(\mathbb{F}_q)|$$

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<u>Ex</u> for any n, have $\mathcal{U}_{\mathrm{id}} = U = \mathcal{V}_{\mathrm{id}}$

<u>Ex</u> for $n = 3$ and $\dot{w} = \begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix}$, not even homeo over $\mathbb{C}$

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diagonal $T \curvearrowright \mathcal{U}_w, \mathcal{V}_w$ by conjugation

<u>Thm (T)</u> $\mathrm{gr}^{\mathsf{W}}_* \mathrm{H}^*_{c,T}(\mathcal{U}_w(\mathbb{C})) \simeq \mathrm{gr}^{\mathsf{W}}_* \mathrm{H}^*_{c,T}(\mathcal{V}_w(\mathbb{C}))$
where W is the weight filtration on $\mathrm{H}^*_{c,T}$

implies (Kawanaka) via results of Katz

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<u>Conj (T)</u> the T-equivariant map

$$UU_- \to \mathcal{U} \quad \text{given by } xy \mapsto xyx^{-1}$$

restricts to a homotopy equivalence $\mathcal{V}_w(\mathbb{C}) \to \mathcal{U}_w(\mathbb{C})$

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for $\beta = \sigma_{w_1} \cdots \sigma_{w_k} \in Br_n^+$,

$$\begin{array}{ccc} X_\beta^C & \to & X_\beta := B\dot{w}_1 B \times^B B\dot{w}_2 B \times^B \cdots \times^B B\dot{w}_k B \\ \downarrow & & \downarrow \\ C & \to & G \end{array}$$

$\mathcal{U}_w = X_{\sigma_w}^{\mathcal{U}}$ and $\mathcal{V}_w = X_{\sigma_w \pi}^1$, where π is the full twist

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<u>Thm (T)</u> for any $\beta \in Br_n^+$,

$$|X_\beta^{\mathcal{U}}(\mathbb{F}_q)| = |X_{\beta\pi}^1(\mathbb{F}_q)|,$$

$$\mathrm{gr}^{\mathsf{W}}_* \mathrm{H}^*_{c,T}(X_\beta^{\mathcal{U}}(\mathbb{C})) \simeq \mathrm{gr}^{\mathsf{W}}_* \mathrm{H}^*_{c,T}(X_{\beta\pi}^1(\mathbb{C}))$$

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<u>Kálmán '09</u> writing P for HOMFLYPT,

$$P(\widehat{\beta})[a^{|\beta|-n+1}] = P(\widehat{\beta\pi})[a^{|\beta|+n-1}]$$

<u>Gorsky–Hogancamp–Mellit–Nakagane '19</u>

true with KhR superpolynomial $\mathbb{P}$ in place of P

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Springer resolution $\tilde{\mathcal{U}} = \{(u, gB) \in \mathcal{U} \times G/B \mid ugB = gB\}$

$$\begin{array}{ccccccc} X_\beta^1 \times G/B & \to & \tilde{X}_\beta^{\mathcal{U}} & \to & X_\beta^{\mathcal{U}} & \to & X_\beta \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \{1\} \times G/B & \to & \tilde{\mathcal{U}} & \to & \mathcal{U} & \to & G \end{array}$$

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<u>Thm (T)</u> if $\beta \in Br_n^+$, then $S_n \curvearrowright \mathrm{gr}_*^{\mathsf{W}} \mathrm{H}_{c,T}^*(\tilde{X}_\beta^{\mathcal{U}})$ and

$$\mathbb{P}(\hat{\beta}) \propto (\Lambda^*(\mathbb{V}), \mathrm{gr}_*^{\mathsf{W}} \mathrm{H}_{c,T}^*(\tilde{X}_\beta^{\mathcal{U}}(\mathbb{C})))_{S_n}, \quad \text{where } \mathbb{V} = \mathbb{C}^{n-1}$$

$$\begin{aligned} a \text{ tracks } \Lambda^*\text{-grading } &\implies \mathbb{P}(\widehat{\beta})[a^{\mathrm{lo}}] \propto \mathrm{gr}_*^{\mathsf{W}} \mathrm{H}_{c,T}^*(X_\beta^{\mathcal{U}}(\mathbb{C})) \\ &\implies \mathbb{P}(\widehat{\beta})[a^{\mathrm{hi}}] \propto \mathrm{gr}_*^{\mathsf{W}} \mathrm{H}_{c,T}^*(X_\beta^1(\mathbb{C})) \end{aligned}$$

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Jordan type: $\mathcal{U} = \bigcup_{\lambda \vdash n} \mathcal{U}_\lambda$

$$\tilde{X}_\beta^{\mathcal{U}} = \bigcup_\lambda (X_\beta^{\mathcal{U}_\lambda} \times \tilde{\mathcal{U}}_{u_\lambda}) \quad \text{for fixed } u_\lambda \in \mathcal{U}_\lambda$$

$$P(\hat{\beta})[a^{\mathrm{lo}+2k}] = \sum_\lambda \underbrace{(\Lambda^k(\mathbb{V}), [\mathrm{H}^*(\tilde{\mathcal{U}}_{u_\lambda}(\mathbb{C}))]_q)_{S_n}}_{\text{known}} |X_\beta^{\mathcal{U}_\lambda}(\mathbb{F}_q)|$$

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just as P arises from traces on Hecke algebras,

$$\begin{aligned} \mathsf{D}^b_{mix,G}\mathsf{Perv}(\mathcal{U}) &\simeq \mathsf{D}^b\mathsf{Mod}(\mathbb{C}S_n \ltimes \mathrm{Sym}) && \text{(Rider)} \\ &\simeq \mathsf{hTr}(\mathsf{Hecke}(S_n)) && \text{(Gorsky–Wedrich)} \end{aligned}$$