



## Hilb vs Quot vs HOMFLYPT

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O. Kivinen, M. Trinh. [The Hilb-vs-Quot conjecture.](#)  
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*image credits:* Chmutov–Duzhin–Mostovoy, Bar-Natan,  
Penpa+, Cherednik–Danilenko

## 1 Knots and Links

Some *knots* in  $\mathbf{R}^3$  (or  $S^3$ ).



*Links* allow multiple circles.



Knot theory studies *isotopy* invariants of links.

Trade-off between being *strong* and being *practical*.  
 $\pi_1(S^3 \setminus L)$  is a strong, but impractical, invariant.

More practical: the *HOMFLYPT polynomial*  $\mathbb{P}_L(a, q)$ , defined via skein relations.

$$(1) \quad \mathbb{P}_{\bigcirc} = 1$$

$$(2) \quad a\mathbb{P}_{\nearrow} - a^{-1}\mathbb{P}_{\nwarrow} = (q^{1/2} - q^{-1/2})\mathbb{P}_{\swarrow}$$

The strange factor  $q^{1/2} - q^{-1/2}$  will be useful later.

While  $\pi_1(S^3 \setminus L)$  is intrinsic,  $\mathbb{P}_L$  is *diagrammatic*:

A priori, it depends on the planar diagram.

$\pi_1(S^3 \setminus L) \simeq \mathbf{Z}$  implies  $L = \bigcirc$ .

Unknown whether  $\mathbb{P}_L = 1$  implies  $L = \bigcirc$ .

**Khovanov–Rozansky '07** A further refinement

$$\mathbf{P}_L(a, q, t)$$

such that  $\mathbf{P}_L(a, q, -1) = \mathbb{P}_L(a, q)$ .

The dimension of a triply-graded vector space called the *HOMFLYPT homology* of  $L$ .

Defined via *categorified* skein relations.

**Khovanov '08** Can be done in certain monoidal triangulated categories  $\mathbf{K}^b(\mathbf{SBim}_n)$ .

**Kronheimer–Mrowka '10**  $\mathbf{P}_L = 1$  implies  $L = \bigcirc$ .

Proof used interpretation of  $\mathbf{P}_L$  via gauge theory on  $S^3 \setminus L$ .

Mellit '16, Elias–Hogancamp–Mellit '15–19

Recursions in  $K^b(\text{SBim}_n)$  computing  $\mathbf{P}$  for *torus links*.



⇒ Mellit '16 A closed formula for any torus *knot*.

⇒ Gorsky–Mazin–Vazirani '20 Another formula, valid for any torus *link*.

For torus knots, both formulas sum over *Dyck paths*.



Both formulas look like

$$\mathbf{P} \propto \sum_D q^{a(D)} t^{c(D)} \prod_{\square \in S_\bullet} f_{\bullet, D, \square}.$$

$a(D)$  counts shaded  $\square$ 's;  $c(D)$  is messy.

$$S_{\text{Mellit}} = \{\square \mid \square \nearrow D\},$$

$$S_{\text{GMV}} = \{\square \mid D \swarrow \square \text{ with } \square \text{ shaded}\}.$$

Example For the  $(3, 4)$  torus knot:

$q^a$	$t^c$	$\prod f_{\text{Mellit}}$	$\prod f_{\text{GMV}}$
$q^3$	$t^3$	1	$1 + aq^{-1}$
$q^2$	$t^2$	$1 + at$	$(1 + aq^{-1})(1 + aq^{-1}t)$
$q$	$t^2$	$1 + at$	$1 + aq^{-1}$
$q$	$t$	$1 + at$	$1 + aq^{-1}$
1	1	$(1 + at)(1 + at^2)$	1

## 2 Plane Curve Singularities

Let

$$S_\epsilon^3 = \{(x, y) \in \mathbf{C}^2 \mid |x|^2 + |y|^2 = \epsilon\}.$$

Let  $C \subseteq \mathbf{C}^2$  be an algebraic curve through  $(0, 0)$ .

The *link* of  $C$  at the origin is

$$L_C = \{(x, y) \in S_\epsilon^3 \mid f(x, y) = 0\},$$

independent of  $\epsilon$  up to isotopy.

**Example** For  $y^n = x^m$ , it's the  $(m, n)$  torus link.

In general, *components* of  $L_C$  correspond to *branches* of  $C$  at the origin.

**Example** Take  $C$  parameterized by

$$(x(u), y(u)) = (u^4, u^6 + u^7).$$

Then  $L_C$  is the *closure* of



In general, the completed local ring of  $C$  looks like

$$R_C = R_{C_1} \times \cdots \times R_{C_b},$$

and the branches look like  $R_{C_i} \simeq \mathbf{C}[[u^{n_i}, u^{m_i} + \cdots]]$  by Newton–Puiseux.

Puiseux exponents are *cabling* parameters of knots.

Oblomkov–Shende conjectured a formula for  $\mathbb{P}_{L_C}$  in terms of the *intrinsic* ring  $R_C$ .

Later, with Rasmussen, upgraded to  $\mathbf{P}_{L_C}$ .

Form the *Hilbert schemes*

$$\mathcal{H}_C^\ell = \{\text{ideals } I \subseteq R_C \mid \dim(R_C/I) = \ell\}.$$

**Conj (ORS '12)** The lowest  $a$ -degree  $\mathbf{P}_{L_C}^{\text{lo}}$  satisfies

$$\frac{\mathbf{P}_{L_C}^{\text{lo}}(q, qt)}{1-q} \propto \sum_{\ell \geq 0} q^\ell \chi(t^{1/2}, \mathcal{H}_C^\ell),$$

where  $\chi$  denotes *virtual weight polynomials*.

Recall:  $\chi(t^{1/2}, Z) = |Z(\mathbf{F}_t)|$  when  $t$  is a prime power and  $Z$  is especially nice.

**Example**  $\mathbf{P}_{(2, 3) \text{ torus}} \propto 1 + qt + at$ , while

$$C = \{y^2 = x^3\} \implies \begin{cases} \mathcal{H}_C^0 = pt, \\ \mathcal{H}_C^1 = pt, \\ \mathcal{H}_C^\ell = \mathbf{CP}^1 \text{ for } \ell \geq 2, \end{cases}$$

$$\text{giving } 1 + q + \frac{q^2}{1-q}(1+t) = \frac{1}{1-q}(1+q^2t).$$

Next, form *nested Hilbert schemes*

$$\mathcal{H}_C^{\ell, k} = \{(I, J) \in \mathcal{H}_C^\ell \times \mathcal{H}_C^{\ell+k} \mid I \supseteq J \supseteq \langle x, y \rangle J\}.$$

The full conjecture:

$$\frac{\mathbf{P}_{L_C}(a, q, qt)}{1-q} \propto \sum_{k, \ell \geq 0} a^k q^\ell t^{\binom{k}{2}} \chi(t^{1/2}, \mathcal{H}_C^{\ell, k}).$$

**Maulik '12** True at the level of  $\mathbb{P}_L = \mathbf{P}_L|_{t \rightarrow -1}$ .

Key idea is an analogue for  $\mathbb{P}_L$  of a *wall-crossing identity* from DT theory.

Unknown how to upgrade to  $\mathbf{P}_L$ .

**Maulik–Yun, Migliorini–Shende '11**

Morally, why are the Hilbert schemes hard?

They know about a *perverse filtration*  $\mathbf{P}_{\leq *}$ :

$$\sum_{\ell} q^{\ell} \chi(s, \mathcal{H}_C^{\ell}) = \frac{\sum_i q^i \chi(s, \text{gr}_i^{\mathbf{P}} H^*(\bar{\mathcal{P}}_C / \mathbf{Z}^b))}{(1-q)^b}$$

where  $\bar{\mathcal{P}}_C$  is the *compactified Picard* parametrizing full, finitely-gen.  $R_C$ -submodules of  $\text{Frac}(R_C)$ .

**3 Hilb vs Quot**  $R_C$  has a *normalization*

$$R_C \hookrightarrow \widetilde{R}_C = \mathbf{C}[[u_1]] \times \cdots \times \mathbf{C}[[u_b]].$$

Form the *Quot schemes*

$$\mathcal{Q}_C^{\ell} = \{R_C\text{-modules } M \subseteq \widetilde{R}_C \mid \dim(\widetilde{R}_C/M) = \ell\}$$

An initial motivation for these varieties:

**Thm (Kivinen–T '23)** We have

$$\sum_{\ell} q^{\ell} \chi(s, \mathcal{Q}_C^{\ell}) = \frac{\sum_i q^i \chi(s, \bar{\mathcal{P}}_C^{(i)} / \mathbf{Z}^b)}{(1-q)^b}$$

for an explicit  $\mathbf{Z}^b$ -stable stratification  $\bar{\mathcal{P}}_C = \coprod_i \bar{\mathcal{P}}_C^{(i)}$ .

Recall ORS:

$$\frac{\mathbf{P}_{L_C}(a, q, qt)}{1 - q} \propto \sum_{k, \ell \geq 0} a^k q^\ell t^{\binom{k}{2}} \chi(t^{1/2}, \mathcal{H}_C^{\ell, k}).$$

Form *nested Quot schemes*

$$\mathcal{Q}_C^{\ell, k} = \{(M, N) \in \mathcal{Q}_C^\ell \times \mathcal{Q}_C^{\ell+k} \mid M \supseteq N \supseteq \langle x, y \rangle M\}.$$

“Quot ORS” Conj (Kivinen–T ’23) For any  $C$ ,

$$\frac{\mathbf{P}_{L_C}(a, q, t)}{1 - q} \propto \sum_{k, \ell \geq 0} a^k q^\ell t^{\binom{k}{2}} \chi(t^{1/2}, \mathcal{Q}_C^{\ell, k}).$$

**Thm (Kivinen–T ’23)** Quot ORS holds in full for:

- $y^n = x^m$  with  $m, n$  coprime.
- $y^n = x^{nk}$ .

“Hilb-vs-Quot” Conj (Kivinen–T ’23) For any  $C$ ,

$$\sum_{\ell} q^\ell \chi(t^{1/2}, \mathcal{H}_C^\ell) = \sum_{\ell} q^\ell \chi((qt)^{1/2}, \mathcal{Q}_C^\ell).$$

Remarks on Hilb-vs-Quot

- Should really be an identity in  $\mathbf{K}_0(\mathbf{Var})$ .
- $t \mapsto qt$  because  $\mathcal{Q}^\ell$  is *larger* than  $\mathcal{H}^\ell$  for fixed  $\ell$ .
- Unibranch case was proposed by Cherednik in a different form, without the  $\mathcal{Q}^\ell$ .

**Example** Take  $C = \{y^3 = x^4\}$ .

The  $\mathbf{C}^\times$ -action on  $C$  induces actions on the  $\mathcal{H}^\ell$ ,  $\mathcal{Q}^\ell$ .

The  $\mathbf{C}^\times$ -orbits form affine pavings.

$\mathcal{H}^0$	$\mathcal{H}^1$	$\mathcal{H}^2$	$\mathcal{H}^3$	$\mathcal{H}^4$	$\mathcal{H}^5$	$\mathcal{H}^6$	$\dots$
$pt$		$\mathbf{C}^2$	$\mathbf{C}^2$		$\mathbf{C}^3$		$\dots$
	$\mathbf{C}$	$\mathbf{C}$		$\mathbf{C}^2$	$\mathbf{C}^2$		$\dots$
$pt$		$\mathbf{C}^2$	$\mathbf{C}^2$	$\mathbf{C}^2$		$\mathbf{C}^2$	$\dots$
$pt$			$\mathbf{C}$	$\mathbf{C}$	$\mathbf{C}$		$\dots$
	$pt$	$pt$	$pt$	$pt$	$pt$		$\dots$

The rows classify ideals as  $R_C$ -modules.

The colors are  $\mathcal{Q}^0$ ,  $\mathcal{Q}^1$ ,  $\mathcal{Q}^2$ ,  $\mathcal{Q}^3$ ,  $\dots$

Similar picture for any  $y^n = x^m$  with  $m, n$  coprime.

“Hilb ORS” is hard because Hilb-vs-Quot is hard.

**Thm (Kivinen–T '23)** Hilb-vs-Quot holds for

$$y^n = x^m \quad \text{with } m, n \text{ coprime and } n \leq 3.$$

Key idea is that for fixed  $n$ , we can compute

$$\lim_{\substack{m \rightarrow \infty \\ \gcd(m,n)=1}} \sum_{\ell} q^\ell \chi(t^{1/2}, \mathcal{H}_{y^n=x^m}^\ell),$$

$$\lim_{\substack{m \rightarrow \infty \\ \gcd(m,n)=1}} \sum_{\ell} q^\ell \chi(t^{1/2}, \mathcal{Q}_{y^n=x^m}^\ell)$$

by combinatorics.

If  $n \leq 3$ , then the limits determine the series at finite  $m$ , by a Serre duality trick.

## 4 Generic Singularities

Beyond torus links: work of Gorsky–Mazin–Oblomkov and Caprau–González–Hogancamp–Mazin.

GMO '22 + CGHM '23

Quot ORS holds for the lowest  $a$ -degree  $\mathbf{P}_{L_C}^{\text{lo}}$ , when  $C$  has a *generic* unibranch singularity.

For such a singularity,

$$R_C \simeq \mathbf{C}[[u^{nd}, u^{md} + u^{md+1} + \dots]].$$

for some  $m, n, d$  with  $m, n$  coprime.

Moreover,  $L_C$  is the  $(mnd + 1, n)$  *cable* of the  $(m, n)$  torus knot.

For such knots, CGHM generalize the GMV formula for  $\mathbf{P}$  to a sum over  $md \times nd$  Dyck paths.

For such singularities, GMO exhibit an affine paving of  $\mathcal{Q}^\ell$  indexed by *admissible subsets* of  $\mathbf{Z}_{\geq 0}$ :

If  $\Delta \subseteq \mathbf{Z}_{\geq 0}$  is admissible and  $|\mathbf{Z}_{\geq 0} \setminus \Delta| = \ell$ , then

$$\mathcal{Q}_\Delta^\ell = \{M \in \mathcal{Q}^\ell \mid \Delta = \{\text{ord}(f)\}_{f \in M}\}.$$

Gorsky–Mazin–Vazirani '17 + GMO '23 Bijection

$$\{\text{admissible sets } \Delta\} \xrightarrow{\sim} \{\text{Dyck paths } D\}$$

such that  $q^{a(D)} t^{c(D)} = q^\ell t^{\dim \mathcal{Q}_\Delta^\ell}$ .

Recall GMV:

$$\sum_D q^{a(D)} t^{c(D)} \prod_{\square \in S_{GMV}(D)} f_{GMV, D, \square}.$$

But  $\prod_{\square} f_{GMV, D, \square}$  does not match

$$\sum_{k, \Delta'} a^k t^{\binom{k}{2}} \chi(t^{1/2}, Q_{\Delta \supseteq \Delta'}^{\ell, k}),$$

so CGHM cannot match higher  $a$ -degree terms.

Thm (T '25+) By contrast:

- 1 Mellit's formula for  $\mathbf{P}$  generalizes to the knots of generic unibranch singularities.
- 2  $f_{Mellit}$  does match nested Quot.
- 3 Quot ORS holds in full for such singularities.

## 5 Some Lie Theory What got me into this?

Given a group  $G$  and subgroup  $K$  and  $\gamma \in \text{Lie}(G)$ , the *Springer fiber* over  $\gamma$  is its fixed-point set

$$X_\gamma = \{gK \in G/K \mid \gamma \in \text{Ad}(g)(\text{Lie}(K))\}.$$

Laumon '02 Take  $C$  a branched  $n$ -cover of a line.

Then its compactified Picard

$$\bar{\mathcal{P}}_C = \{\text{full, fin.-gen. modules } M \subseteq \text{Frac}(R_C)\}$$

is a  $X_\gamma$  for  $G = \mathbf{GL}_n(\mathbf{C}((x)))$  and  $K = \mathbf{GL}_n(\mathbf{C}[[x]])$ .

In this case, also called an *affine Springer fiber*.

**Example** Suppose that  $R_C = \mathbf{C}[[u^4, u^6 + u^7]]$ .

Setting  $x = u^4$  and fixing  $\mathbf{C}((u)) \simeq \mathbf{C}((x))^4$ ,

$$u^6 + u^7 \curvearrowright \mathbf{C}((u)) \quad \rightsquigarrow \quad \gamma \curvearrowright \mathbf{C}((x))^4.$$

Two possibilities for  $\gamma$ :

$$\begin{pmatrix} & & x^6 + x^7 \\ 1 & & 4x^6 \\ & 1 & 6x^6 \\ & & 1 & 4x^6 \end{pmatrix}, \begin{pmatrix} x^2 & x^2 & & \\ & x^2 & x^2 & \\ x & & & x^2 \\ x & x & & x^2 \end{pmatrix}.$$

Both give  $\bar{\mathcal{P}}_C \simeq X_\gamma$ , but different *positive truncations*

$$X_\gamma \cap \mathrm{Mat}_n(\mathbf{C}[[x]])/\mathrm{GL}_n(\mathbf{C}[[x]]).$$

Respectively, they are  $\bigsqcup_\ell \mathcal{H}_C^\ell$  and  $\bigsqcup_\ell \mathcal{Q}_C^\ell$ .

This viewpoint also suggests:

- 1 Generalizing  $K = \mathrm{GL}_n(\mathbf{C}[[x]])$  to other *parahorics*.
- 2 Generalizing  $\mathrm{GL}_n$  to other reductive groups.

(1) leads to flagged versions of  $\mathcal{H}^\ell$ ,  $\mathcal{Q}^\ell$  that indirectly encode the nested versions and more.

(2) leads to conjectures relating affine Springer fibers to  $q, t$ -traces on *generalized braid groups*  $\widehat{\mathrm{Br}}_W$ .

For  $\mathbf{G}$  with Lie algebra  $\mathbf{g} \supseteq \mathbf{t}$  and Weyl group  $W$ ,

$$\mathbf{g}((z)) \rightarrow (\mathbf{g} // \mathbf{G})((z)) = (\mathbf{t} // W)((z))$$

takes loops in  $\mathbf{g}$  to conjugacy classes in  $\widehat{\mathrm{Br}}_W$ .

**Def (T '21)** Suppose  $\beta = \sigma_{s_1} \cdots \sigma_{s_\ell} \in \text{Br}_W^+$ .

The *braid Steinberg variety* of  $\beta$  is

$$Z_\beta = \tilde{\mathcal{U}} \times_{\mathcal{U}} \mathcal{U}_\beta,$$

where  $\tilde{\mathcal{U}} \rightarrow \mathcal{U}$  is the unipotent Springer resolution and

$$\mathcal{U}_\beta \simeq \{(ug_\ell \mathbf{B} \xrightarrow{s_1} g_1 \mathbf{B} \xrightarrow{s_2} \cdots \xrightarrow{s_\ell} g_\ell \mathbf{B}) \mid u \in \mathcal{U}\}$$

for a fixed Borel  $\mathbf{B}$ .

**Thm (T '21)** Suppose  $\mathbf{G} = \text{PGL}_n$ .

If  $L$  is the closure of  $\beta$ , then we can recover  $\mathbf{P}_L$  from the Springer action

$$W = S_n \quad \curvearrowright \quad H_{c,\mathbf{G}}^*(Z_\beta).$$

$$\begin{array}{ccccc} & & & & \\ & \text{Iwahori} & X_\gamma^\mathbf{B} & \xleftarrow[\approx]{} & Z_\beta/\mathbf{G} \\ & & \downarrow & & \downarrow \\ & \text{spherical} & X_\gamma & \xleftarrow[\approx]{} & \mathcal{U}_\beta/\mathbf{G} \\ & & & & ? \end{array}$$

Thank you for listening.