MATH 340: ADVANCED LINEAR ALGEBRA
PROBLEM SET #9

SPRING 2025

Due Friday, April 25. You may consult books, papers, and websites as long as you cite all sources and write up your solutions in your own words. **Last updated in red on 4/24 at 8:30 pm.**

Problem 1. Recall that in the two-element field $\mathbf{F}$ introduced on Problem Set 2, #8, we have $-1 = 1$: that is, 1 is its own additive inverse.

(1) Show that if $F \in \{\mathbf{R}, \mathbf{C}\}$, then a bilinear form on a vector space over F is alternating if and only if it is antisymmetric.
(2) What goes wrong in (1) when $F = \mathbf{F}$?

In the remainder of this problem set, we exclude this field, returning to our usual assumption that all vector spaces are defined over $\mathbf{R}$ *or* $\mathbf{C}$.

Problem 2. Let $\mathbf{E} \subseteq \mathfrak{sl}(2, \mathbf{C})$ be the real vector space of trace-zero, Hermitian 2×2 complex matrices. (See Problem Set 3, #6, and Problem Set 7, #2.)

(1) Using part (2) of Problem Set 3, #6, check that E_1, E_2, E_3 form a basis for $\mathbf{E}$, where

$$E_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad E_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

(2) Let $S : \mathbf{E} \xrightarrow{\sim} \mathbf{R}^3$ be the linear isomorphism that sends (E_1, E_2, E_3) onto the standard basis. Show that

$$S(X) \cdot S(Y) = \frac{1}{2}\,\mathrm{tr}(XY)$$

for all $X, Y \in \mathbf{E}$, where $- \cdot -$ is the dot product. Thus, the right-hand side defines an inner product on $\mathbf{E}$.

Problem 3. On a complex inner product space, operators T such that $\langle Tu, Tv\rangle = \langle u, v\rangle$ for all vectors u, v are usually called *unitary*, rather than orthogonal. Let

$$\mathrm{SO}(n) = \{n \times n \text{ real matrices } M \mid \det(M) = 1 \text{ and } M^t M = I\},$$
$$\mathrm{SU}(n) = \{n \times n \text{ complex matrices } M \mid \det(M) = 1 \text{ and } M^* M = I\},$$

where $M^*_{j,i} = \bar{M}_{i,j}$. The notations SO and SU stand for *special orthogonal* and *special unitary*. Using Problem Set 3, #8 and (a variant of) Problem Set 5, #8, give bijections

$$M : \{z \in \mathbf{C} \mid |z| = 1\} \xrightarrow{\sim} \mathrm{SO}(2), \qquad M : \{z \in \mathbf{H} \mid |z| = 1\} \xrightarrow{\sim} \mathrm{SU}(2)$$

such that $M(z_1 z_2) = M(z_1) \cdot M(z_2)$ in both cases. Above, the absolute value of a quaternion is given by $|a1 + bi + cj + dk| = \sqrt{a^2 + b^2 + c^2 + d^2}$.

Problem 4. Keep the notation of Problems 2–3. Show that if $M \in \mathrm{SU}(2)$, then

$$T_M : \mathbf{E} \to \mathbf{E} \quad \text{defined by } T_M(X) = MXM^{-1}$$

is a well-defined, linear, and orthogonal with respect to $\langle X, Y\rangle := \frac{1}{2}\operatorname{tr}(XY)$.

Together, Problem 3 and this problem encapsulate W. R. Hamilton's formalism for describing 3-dimensional rotations via quaternions.

Problem 5. Recall, from Problem Set 8, #4, the inner product on $\mathbf{C}[t]$ defined by

$$\langle f, g\rangle = \int_{-1}^{1} f(t)\overline{g(t)}\,dt.$$

(1) Show explicitly that the linear operator $D(p(t)) = \frac{d}{dt}p(t)$ is not self-adjoint with respect to $\langle -, -\rangle$.
(2) Is the linear operator $T(p(t)) = tp(t)$ self-adjoint?

Problem 6. Let T be an orthogonal linear operator on a real inner product space V of dimension n. In the language of Problem Set 8, #7, the problem below proves the *Cartan–Dieudonné theorem* that T is a composition of $\leq n$ reflections.

(1) Show that for all $v, w \in V$ such that $\|v\| = \|w\|$, the reflection $S_{v-w} : V \to V$ swaps v and w.
(2) Let $(e_1, \ldots, e_n)$ be a basis for V, and let $f_i = Te_i$ for all i. Suppose that for some k, there is an orthogonal linear operator $T_k : V \to V$ such that $T_k e_i = f_i$ for all $i \leq k$. Show that

$$\begin{aligned} \|T_k e_{k+1}\| &= \|f_{k+1}\|, \\ \|T_k e_{k+1} - f_i\| &= \|f_{k+1} - f_i\| \qquad \text{for all } i < k+1. \end{aligned}$$

Hint: Orthogonal operators preserve the norms of vectors.
(3) Let $T_0 = \mathrm{Id}_V$ and $T_{k+1} = S_{T_k e_{k+1} - f_{k+1}} \circ T_k$ for all $k \geq 0$. Use (1)–(2) to show that $T_{k+1} e_i = f_i$ for all $i \leq k+1$. Thus, $T_n = T$.

Hint: In the case where $i < k+1$, expand the second display from (2), then apply the first display to arrive at $\langle T_k e_{k+1}, f_i\rangle = \langle f_{k+1}, f_i\rangle$. This shows that f_i is orthogonal to $T_k e_{k+1} - f_{k+1}$.

Problem 7. Show that if $M \in \mathrm{Mat}_n(\mathbf{C})$ is invertible, then $M = QR$ for some unitary Q and invertible upper-triangular R. This is called the *QR decomposition* of M. *Hint:* Interpret the Gram–Schmidt process for the columns of M in terms of right multiplication by another matrix.

Problem 8. Let $A \in \mathrm{Mat}_n(\mathbf{C})$.

(1) Show that if the pairing $\langle u, v\rangle := u^t A \bar{v}$ is an inner product, then $A = B^* B$ for some invertible B. *Hint:* The spectral theorem.
(2) Use Problem 7 to deduce that in this case, $A = R^* R$ for some invertible and upper-triangular R. This is called the *Cholesky decomposition* of A. It is a special case of *singular value decomposition*. (The square roots of the eigenvalues of A are the *singular values* of R.)