



# Zeta Functions as Knot Invariants

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O. Kivinen, M. Q. Trinh. The Hilb-vs-Quot Conjecture.  
*J. reine angew. Math. (Crelle)*, (2025). 44 pp.

## 1 The Riemann Hypothesis

(Euler ~1730s) Proof of the infinitude of primes:

If there were finitely many, then we'd have

$$\prod_{\text{prime } p > 0} \left(1 + \frac{1}{p} + \frac{1}{p^2} + \cdots\right) = \sum_{n=1}^{\infty} \frac{1}{n}.$$

But the series diverges—a contradiction.

He also implicitly studied the *zeta function*

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

For real  $s > 1$ , we have  $\zeta(s) = \prod_p \frac{1}{1 - p^{-s}}$ .

What if we allow  $s$  to be complex?

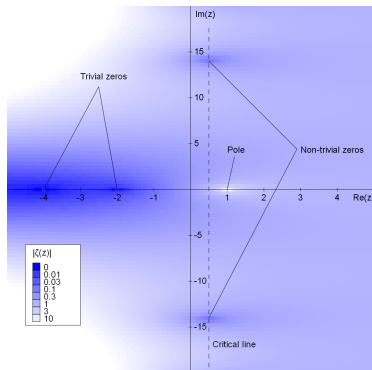
(Riemann 1859) A unique  $\mathbf{C}$ -valued function  $\zeta$  that is

- *holomorphic* (complex-differentiable) when  $s \neq 1$ .
- given by  $\sum_{n=1}^{\infty} \frac{1}{n^s}$  when  $\operatorname{Re}(s) > 1$ .

He checked that  $\zeta(n) = 0$  for  $n = -2, -4, -6, \dots$  by relating these zeros to poles of the  $\Gamma$  function.

He speculated from examples that all other zeros of  $\zeta$  live on the *critical line*  $\operatorname{Re}(s) = \frac{1}{2}$ .

Location of zeros  $\leftrightarrow$  distribution of prime numbers.



Wikipedia

(Hardy 1914) Infinitely many zeros on the line.

(Pratt–Robles–Zaharescu–Zeindler 2020) Among zeros  $s$  with  $0 < \operatorname{Re}(s) < 1$ , over five-twelfths on the line.

(Dedekind ~1860s) Generalize the formula

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

by replacing  $\mathbf{Z}$  with other *rings*  $R$ .

Thus  $R$  is a set with operations  $+$  and  $\cdot$  resembling addition and multiplication.

$$\zeta_R(s) = \sum_{\substack{\text{ideals } I \subseteq R \\ |R/I| < \infty}} \frac{1}{|R/I|^s}$$

An *ideal*  $I$  is the collection of all finite linear combos  $c_1 \cdot x_1 + \cdots + c_k \cdot x_k$  for some fixed  $x_1, x_2, \dots \in R$ .

The *quotient*  $R/I$  is the set of translates  $y + I \subseteq R$ .

**Note** For  $\zeta_R$  to make sense, the number of  $I$  such that  $|R/I| = n$  must be finite for each  $n > 0$ .

**Ex** Every ideal of  $\mathbf{Z}$  takes the form

$$n\mathbf{Z} = \{\text{multiples of } n\} \quad \text{for some integer } n \geq 0.$$

For instance, the ideal generated by 30 and 2025 is

$$\{c_1 \cdot 30 + c_2 \cdot 2025 \mid c_1, c_2 \in \mathbf{Z}\} = 15\mathbf{Z}.$$

We see that

$$\zeta_{\mathbf{Z}}(s) = \sum_{\substack{n\mathbf{Z} \subseteq \mathbf{Z} \\ n > 0}} \frac{1}{|\mathbf{Z}/n\mathbf{Z}|^s} = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

In other rings, ideals may not simplify so nicely.

Why care?

(Hilbert–Polyá ~1910s) The values  $e^{it}$  such that

$$\zeta(\tfrac{1}{2} + it) = 0 \quad \text{and} \quad 0 < \operatorname{Re}(\tfrac{1}{2} + it) < 1$$

behave like the eigenvalues of a random unitary matrix. Maybe this is what forces  $t$  to be real?

(Weil ~1940s) There is a class of rings  $R$ , coming from *algebraic geometry* over  $\mathbf{F}_p := \mathbf{Z}/p\mathbf{Z}$ , where an analogous fact for  $\zeta_R$  might be provable.

(Grothendieck–Deligne ~1960s–70s) Yes.

2 Weil's Rosetta Stone Algebraic geometry studies shapes cut out by polynomial equations.

For simplicity, we'll stick to (*affine*) *hypersurfaces*

$$V_f = \{\vec{a} = (a_1, \dots, a_n) \in \bar{\mathbf{F}}_p^n \mid f(\vec{a}) = 0\},$$

cut out by a single polynomial  $f \in \mathbf{F}_p[x_1, \dots, x_n]$ .

$V_f$  is *smooth* at a point  $\vec{a}$  when  $\frac{\partial f}{\partial x_i}(\vec{a}) \neq 0$  for some  $i$ . Else, *singular* at  $\vec{a}$ .

Ex Hypersurfaces in  $\bar{\mathbf{F}}_p^2$  are plane curves. Consider

$$f(x, y) = y^2 - x^3 - c \quad \text{for constant } c \in \mathbf{F}_p.$$

For which  $c$  is  $V_f$  smooth everywhere?

The *ring of polynomial functions* on  $V_f$  is

$$R_f := \mathbf{F}_p[x_1, \dots, x_n]/(f).$$

André Weil, in a letter to his sister Simone, described a dictionary:

$\mathbf{Z}$	$R_f$	$V_f$
$n\mathbf{Z}$	ideals	(closed) subvarieties
$p\mathbf{Z}$	maximal ideals	(closed) points

The first and last columns: a Rosetta stone between number theory and algebraic geometry.

Conj (Weil 1948) Assume  $V_f$  is smooth everywhere.

Then zeros of  $\zeta_{R_f}(s)$  have  $\operatorname{Re}(s) \in \{\frac{1}{2}, \frac{3}{2}, \dots, \frac{2n-1}{2}\}$ .

Set  $\zeta_f(s) := \zeta_{R_f}(s)$  for convenience.

(Grothendieck ~1964)  $\zeta_f(s)$  is a rational function in

$$\mathbf{q} := p^{-s}.$$

In fact: polynomials  $\phi_0, \phi_1, \dots, \phi_{2n-2}$  such that

$$\zeta_f(s) = \frac{\phi_1(\mathbf{q}) \cdots \phi_{2n-3}(\mathbf{q})}{\phi_0(\mathbf{q}) \cdot \phi_2(\mathbf{q}) \cdots \phi_{2n-2}(\mathbf{q})}.$$

$\phi_k$  is the charpoly of a certain operator  $F$  on a certain vector space  $H^k(V_f)$ .

Reduces Weil's conjecture to a "Hilbert–Polyá" claim:

Conj The eigenvalues of  $F$  on  $H^k(V_f)$  all have absolute value\*  $p^{k/2}$ .

\* With respect to any embedding into  $\mathbf{C}$ .

(Deligne 1974) True for all  $f$  (assuming  $V_f$  smooth).

*In fact, Weil conjectured—and Deligne proved—results for all smooth varieties, not just hypersurfaces.*

Ex Taking  $n = 2$  and  $f(x, y) = y^2 - x^3 - c$ :

$$\phi_0(t) = 1 - p\mathbf{q}$$

$$\phi_1(t) = 1 - \textcolor{red}{a}_p \mathbf{q} + p\mathbf{q}^2 \quad \text{for some integer } \textcolor{red}{a}_p,$$

giving  $\zeta_f(s) = \frac{1 - \textcolor{red}{a}_p p^{-s} + p^{1-2s}}{1 - p^{1-s}}$ . It turns out:

- $|a_p| \leq 2p^{1/2}$ .
- So the two roots of  $\phi_1(\mathbf{q})$  satisfy  $|\mathbf{q}| = p^{-1/2}$ .
- So the zeros of  $\zeta_f(s)$  satisfy  $\text{Re}(s) = \frac{1}{2}$ .

What if  $V_f$  has singularities?

Simplest case:  $V_f$  has a unique singularity at the origin  $(0, \dots, 0)$ . It turns out that here,

$$\zeta_f(s) = \zeta_f^\circ(s) \cdot \hat{\zeta}_f(s),$$

where:

- $\zeta_f^\circ$  satisfies Weil's Riemann Hypothesis.
- $\hat{\zeta}_f$  is the analogue of  $\zeta_f$  with the power-series ring

$$\hat{R}_f := \mathbf{F}_p[[x_1, \dots, x_n]]/(f)$$

in place of  $R_f$ .

Does  $\hat{\zeta}_f(s) = \sum_{\substack{I \subseteq \hat{R}_f \\ |\hat{R}_f/I| < \infty}} \frac{1}{|\hat{R}_f/I|^s}$  satisfy a RH?

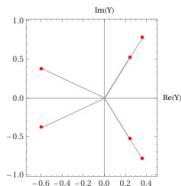
Ex If  $f = y^2 - x^3$ , then  $\hat{\zeta}_f(s) = \frac{1 + pq^2}{1 - q}$ .

Roots are  $q = \pm p^{-1/2}$ .

Ex If  $f = y^3 - x^4$ , then

$$\hat{\zeta}_f(s) = \frac{1 + pq^2 + p^2q^3 + p^2q^4 + p^3q^6}{1 - q}.$$

Two roots on the circle  $|q| = p^{-1/2}$ . The rest not on any circle  $|q| = p^{-k/2}$ .



WolframAlpha

3 From Curves to Knots In general, if  $V_f$  is a plane curve through the origin, then

$$\hat{\zeta}_f(s) = \frac{P_f(p^{1/2}, p^{-s})}{1 - p^{-s}}$$

for some  $P_f(t, q) \in \mathbf{Z}\left[t, q, \frac{1}{1-q}\right]$ .

The polynomials  $P_f$  are remarkably ubiquitous.

(Gorsky–Mazin 2013)

If  $f = y^n - x^{n+1}$ , then  $P_f(1, 1) = \frac{(2n)!}{(n+1)!n!}$ , the  $n$ th Catalan number.

For instance, if  $f = y^3 - x^4$ , then

$$P_f(t, q) = 1 + t^2q^2 + t^4q^3 + t^4q^4 + t^6q^6,$$

$$P_f(1, 1) = 5.$$



The  $P_f$  also arise from *knot/link invariants*.

A *knot* is a (tame) embedding of  $S^1$  into  $\mathbf{R}^3$  or  $S^3$ .



A *link* is similar, but can have multiple circles.



Two links are *isotopic* when they fit into a continuous family of embeddings.



Chmutov–Duzhin–Mostovoy

A *complex* plane curve  $V_f \subseteq \mathbf{C}^2$  through  $(0,0)$  defines a link

$L_f := V_f \cap S^3$ , where  $S^3$  is a 3-sphere around  $(0,0)$ .

Ex If  $f = y^n - x^m$ , then  $L_f$  is the  $(m,n)$  *torus link*. It's a knot when  $m$  and  $n$  are coprime.



Wolfram Language

Ex If  $f = y^4 - 2x^3y^2 - 4x^5y + x^6 - x^7$ , then  $L_f$  is the closure of the braid



Cherednik–Danilenko

Conj (Oblomkov–Shende ~2010)

$$P_f(-1, q) = \lim_{a \rightarrow 0} \left[ (a^{-1} q^{1/2})^\mu \mathbb{P}_{L_f}(a, q) \right],$$

where  $\mu \in \mathbf{Z}$  and  $\mathbb{P}$  is the *HOMFLYPT invariant*, discovered in 1986 and defined by the rules:

$$(1) \quad a \mathbb{P}_{\nearrow \searrow} - a^{-1} \mathbb{P}_{\nwarrow \nearrow} = (q^{1/2} - q^{-1/2}) \mathbb{P}_{\searrow \nearrow}$$

$$(2) \quad \mathbb{P}_{\bigcirc} = 1$$

Surprising, since  $P_f$  is *intrinsic* to  $f$ , while  $\mathbb{P}$  is defined *diagrammatically*.

Full statement incorporates  $a$  by upgrading  $P_f$ .

(Maulik 2012) True for all plane curves.

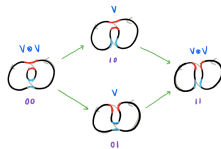
*Proof sketch* Blow up the singularity; control  $P_f$  via wall-crossing and  $L_f$  via skein algebra.

Conj (Oblomkov–Rasmussen–Shende ~2013)

$$P_f(t, q) = \lim_{a \rightarrow 0} \left[ (a^{-1} q^{1/2})^\mu \mathbf{P}_{L_f}(a, t, q) \right],$$

where  $\mathbf{P}$  is a refinement of  $\mathbb{P}$ , discovered in the 2000s by Khovanov–Rozansky.

$\mathbf{P}$  is defined by *categorifying* (1)–(2). Unknown how to categorify Maulik’s proof.



Melissa Zhang

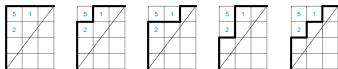
(Kivinen–T 2025) True for  $f = y^3 - x^m$  with  $3 \nmid m$ .

Cor (Kivinen–T) New closed formula for  $\mathbf{P}_{\text{torus}(m,3)}$ .

## Proof Sketch

1 Combinatorial recursions for  $\mathbf{P}_{\text{torus}(m,n)}(\mathbf{a}, \mathbf{t}, \mathbf{q})$  due to Mellit and Hogancamp–Mellit.

2 When  $m$  and  $n$  are coprime, get a formula summing over  $m \times n$  Dyck paths:



At the same time,  $\hat{R}_f \simeq \mathbf{F}_p[[u^m, u^n]]$ .

We relate the Dyck paths to the combinatorics of  $\hat{R}_f$ -submodules  $M \subseteq \mathbf{F}_p[[u]]$ .

3 We relate  $\sum_M \frac{1}{|\mathbf{F}[[u]]/M|^s}$  to  $\sum_I \frac{1}{|\hat{R}_f/I|^s}$ .

Uses *Serre duality*. For now, requires  $\min(m, n) \leq 3$ .

## Big Picture

I'm interested in special functions that appear in

- *algebraic geometry* (e.g., zeta functions)
- *knot theory* (e.g., HOMFLYPT polynomials)
- *combinatorics* (e.g., Dyck-path statistics)

One modern way to study such special functions is called *representation theory*.

T (2021) If  $L$  comes from a positive  $n$ -strand braid, then  $\mathbf{P}_L(\mathbf{a}, \mathbf{t}, \mathbf{q})$  can be recovered from a *representation* of  $S_n$  on the cohomology of an explicit variety  $\mathcal{Z}_L$ .

So if  $L = L_f$ , then  $\mathbf{P}_{L_f}$  relates to both  $V_f$  and  $\mathcal{Z}_{L_f}$ .

*Any direct relationship between these varieties?*

#### 4 Cherednik's New Hypothesis

Recall: For  $f = y^3 - x^4$  and prime  $p$ , the roots of

$$P_f(p^{1/2}, \mathbf{q}) = 1 + p\mathbf{q}^2 + p^2\mathbf{q}^3 + p^2\mathbf{q}^4 + p^3\mathbf{q}^6$$

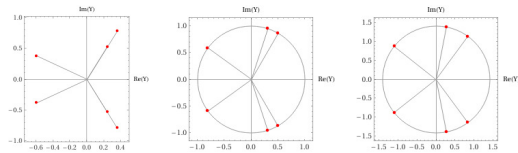
do not all satisfy  $|\mathbf{q}| = p^{-1/2}$ .

Conj (Cherednik 2018) For any plane curve  $f$ :

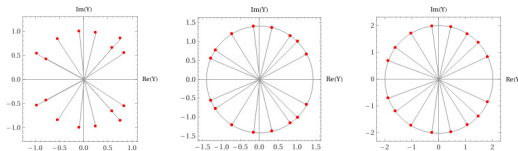
$$0 < t \leq \frac{1}{2} \implies \text{all zeros of } P_f(t, \mathbf{q}) \text{ satisfy } |\mathbf{q}| = t^{-1/2}.$$

Cherednik + ORS would imply mysterious arithmetic constraints on the link invariant  $\mathbf{P}$ .

$$y^3 - x^4 \quad t \in \{2, 1, 0.5\}:$$



$$y^4 - 2x^3y^2 - 4x^5y + x^6 - x^7 \quad t \in \{1, 0.5, 0.25\}:$$



*Thank you for listening.*