MATH 340: ADVANCED LINEAR ALGEBRA
PROBLEM SET #6

SPRING 2025

Due Wednesday, March 26. You may consult books, papers, and websites as long as you cite all sources and write up your solutions in your own words.

Problem 1. Recall that for any finite-dimensional complex vector space V and linear operator $T : V \to V$, we defined the *characteristic polynomial* of T to be

$$p_T(z) = \prod_i (z - \lambda_i)^{d_i}$$

whenever T has a Jordan canonical form matrix in which the ith block has eigenvalue λ_i and size d_i. For any $\lambda \in \mathbf{C}$, we define the *multiplicity* of λ as an eigenvalue of T to be the sum of the d_i over all indices i such that $\lambda = \lambda_i$.

(1) Show that $p_T(z) = \det(z - T)$.

(2) Recall that the determinant of a triangular matrix is the product of its diagonal entries. Using this fact and (1), deduce that if M is <u>any</u> triangular matrix for T, then the multiplicity of λ as an eigenvalue of T is the number of times that λ occurs along the diagonal of M.

Problem 2. Keeping the setup of Problem 1:

(1) Show that if λ has multiplicity m as an eigenvalue of T, then λ^n has multiplicity <u>at least</u> m as an eigenvalue of T^n.

(2) Using (1), show that if $T^n = \mathrm{Id}_V$ for some $n > 0$, then all eigenvalues of T live on the unit circle $\{z \in \mathbf{C} \mid |z| = 1\}$.

Problem 3. Show that:

(1) If $T : \mathbf{C}^2 \to \mathbf{C}^2$ has <u>real</u> trace $\mathrm{tr}(T) \in [-2, 2]$ and $\det(T) = 1$, then its eigenvalues live on the unit circle.

(2) If $S : \mathbf{R}^2 \to \mathbf{R}^2$ satisfies $|\mathrm{tr}(S)| \leq 2$ and $\det(S) = 1$, then S is a rotation. You may use the fact that $S_{\mathbf{C}}$ is given by the "same" matrix as S, but operating on $\mathbf{C}^2$.

Problem 4 (Axler §5D, #21)**.** Define the *Fibonacci numbers* $F_0, F_1, F_2, \ldots$ by $F_0 = 0$ and $F_1 = 1$ and $F_n = F_{n-2} + F_{n-1}$ for all $n \geq 2$. Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be given by $T(x, y) = (y, x + y)$ in the standard basis.

(1) Show that $T^n(0, 1) = (F_n, F_{n+1})$ for all $n \geq 0$.

(2) Find the eigenvalues of T. *Hint:* Problem 1(1).

(3) Find a basis of $\mathbf{R}^2$ consisting of eigenvectors of T.

(4) Using (2)–(3), give a new expression for $T^n(0, 1)$: one that shows that

$$F_n = \frac{1}{\sqrt{5}} \left(\varphi_+^n - \varphi_-^n\right) \text{ for all } n \geq 0, \quad \text{where } \varphi_\pm = \frac{1 \pm \sqrt{5}}{2}.$$

(5) Deduce from (4) that F_n is the integer closest to $\frac{1}{\sqrt{5}}\varphi^n$, for all $n \geq 0$.

Problem 5. View $\mathbf{R}^4$ as column vectors and $(\mathbf{R}^4)^\vee$ as row vectors. Let

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \quad v_4 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}.$$

Given that (v_1, v_2, v_3, v_4) is an ordered basis for $\mathbf{R}^4$, what is the dual ordered basis for $(\mathbf{R}^4)^\vee$ in terms of row vectors?

Problem 6 (Axler, §3F, #32)**.** Let $\Lambda : V \to (V^\vee)^\vee$ be defined as follows:

$$\text{for all } v \in V, \quad \text{let } \Lambda v : V^\vee \to F \text{ be given by } (\Lambda v)(\theta) = \theta(v).$$

Show that:

(1) Λ is a linear map.
(2) For any linear operator $T : V \to V$, we have $(T^\vee)^\vee \circ \Lambda = \Lambda \circ T$.
(3) If V is finite-dimensional, then Λ is a linear isomorphism. *Hint:* Show that Λ is injective and that $\ker(\Lambda) = \{\vec{0}\}$.

In principle, the last two problems are solved in Axler's text. But I bet that it will be easier to think about them from scratch, than start with Axler.

Problem 7. Let V, W be finite-dimensional vector spaces and $T : V \to W$ a linear map. Show that the kernel $\ker(T^\vee)$ and the annihilator $\mathrm{Ann}_{W^\vee}(\mathrm{im}(T))$ are the same subspace of $W^\vee$.

Problem 8. Keep the setup of Problem 7.

(1) Show that

$$\dim W - \dim \ker(T^\vee) = \dim V - \dim \ker(T).$$

Hint: You'll need Problem 7, a dimension formula relating U and $\mathrm{Ann}_{W^\vee}(U)$ for some $U \subseteq W$, and a dimension formula relating $\ker(T)$ and $\mathrm{im}(T)$.

(2) Deduce from (1) that

$$\dim \mathrm{im}(T^\vee) = \dim \mathrm{im}(T).$$

(3) Using (2), show that the column rank and row rank of any square matrix M agree.

You may use the fact (Axler §3.132) that if M represents a linear operator $T : V \to V$ in some basis for V, then the *transpose* matrix M^t defined by $(M^t)_{j,i} = M_{i,j}$ represents $T^\vee : V^\vee \to V^\vee$ in the dual basis.