# MATH 250: TOPOLOGY I

FALL 2025 LOOSE ENDS

## Contents

## 1. Wednesday, 9/3

1.1. Here I try to dispel some potential confusion about bases.

Let $X$ be any set. Let $\mathcal{B}$ be any collection of subsets of $X$. A useful general observation:

**Lemma 1.1.** *For any subset $Y \subseteq X$, the following conditions are equivalent:*

(1) *$Y$ is the union of some elements of $\mathcal{B}$.*
(2) *For any $x \in Y$, there is some $B \in \mathcal{B}$ such that $x \in B \subseteq Y$.*

Now let $\mathcal{T}$ be the collection of all subsets of $X$ that can be written as unions of elements of $\mathcal{B}$. Using the lemma, we see that

$$\mathcal{T} = \left\{ \text{subsets } U \subseteq X \;\middle|\; \begin{array}{l} \text{for any } x \in U \text{, we have some } B \in \mathcal{B} \\ \text{such that } x \in B \subseteq U \end{array} \right\}.$$

**Theorem 1.2.** *Suppose that $\mathcal{B}$ satisfies the following conditions:*

(I) *Every point of $X$ belongs to some element of $\mathcal{B}$.*
(II) *For any $B, B' \in \mathcal{B}$ and any point $x$ of the intersection $B \cap B'$, we can find some $B'' \in \mathcal{B}$ such that $x \in B'' \subseteq B \cap B'$.*

*Then $\mathcal{T}$ is a topology on $X$.*

We proved this theorem at the start of the course, implicitly using Lemma 1.1. The only hard part is checking that finite intersections of elements of $\mathcal{T}$ are still elements of $\mathcal{T}$. To make this easier, I mentioned that it suffices by induction to check intersections between pairs of elements of $\mathcal{T}$.

Any collection $\mathcal{B}$ that satisfies hypotheses (I)–(II) in the theorem above is called a *basis*. In the situation of the theorem, we say that $\mathcal{B}$ *generates* or *induces* the topology $\mathcal{T}$, and that $\mathcal{B}$ is a *basis for $\mathcal{T}$* specifically.

1.2. Separately, if we are given $\mathcal{T}$ to start, then there is a way to check whether a subcollection $\mathcal{C} \subseteq \mathcal{T}$ is a basis that generates $\mathcal{T}$. In Munkres, this is Lemma 13.2.

**Theorem 1.3.** *Fix a topology $\mathcal{T}$ on $X$ and a subset $\mathcal{C} \subseteq \mathcal{T}$. Suppose that for each $x \in X$ and $U \in \mathcal{T}$, there is some $C \in \mathcal{C}$ such that $x \in C \subseteq \mathcal{C}$. Then $\mathcal{C}$ is a basis, and moreover, the topology it generates is $\mathcal{T}$.*

## 2. Monday, 9/8

2.1. Let $X$ be a set, and let $d : X \times X \to [0, \infty)$ be a metric on $X$. For all $x \in X$ and $\delta > 0$, we define the *d-ball* with center $x$ and radius $\delta$ to be

$$B_d(x, \delta) = \{y \in X \mid d(x, y) < \delta\}.$$

Below is a cleaner version of a long proof from lecture.

**Theorem 2.1.** *The set $\{B_d(x, \delta) \mid x \in X$ and $\delta > 0\}$ forms a basis.*

*Proof.* Let $\mathcal{B}$ denote the set in question. We must check two axioms:

(I) Any point of $X$ is contained in some element of $\mathcal{B}$.
(II) Given any two elements of $\mathcal{B}$ and a point in their intersection, we can find some other element of $\mathcal{B}$ containing that point and contained within the intersection as a subset.

(I) holds because for any $x \in X$, we have $x \in B(x, \delta)$ for any choice of $\delta$.

To show (II): Pick balls $B_d(x, \delta)$ and $B_d(x', \delta')$ and a point $z$ in their intersection $B_d(x, \delta) \cap B_d(x', \delta')$. We must exhibit some $d$-ball that contains $z$ and is contained within the intersection as a subset.

It suffices to find some $\epsilon > 0$ such that

$$B_d(z, \epsilon) \subseteq B_d(x, \delta) \cap B_d(x', \delta').$$

Explicitly, this condition on $\epsilon$ means that

$$\text{if } y \in X \text{ satisfies } d(z, y) < \epsilon, \quad \text{then } d(x, y) < \delta \text{ and } d(x', y) < \delta'.$$

(Informally, this means that if $y$ is close enough to $z$, then it is close enough to $x$ and $x'$ as well.) By drawing a picture of the situation, we get the idea that we need to use the triangle inequality to bound the distance $d(x, y)$ in terms of the distances $d(x, z)$ and $d(z, y)$.

Since $z \in B_d(x, \delta) \cap B_d(x', \delta')$, we know that $d(x, z) < \delta$ and $d(x', z) < \delta'$. Let $\alpha = \delta - d(x, z)$ and $\alpha' = \delta' - d(x', z)$, the respective distances from $z$ to the boundaries of the balls $B_d(x, \delta)$ and $B_d(x', \delta')$. Now observe that if $y \in X$ satisfies $d(z, y) < \alpha$, then $y$ also satisfies

$$\begin{aligned} d(x, y) &\leq d(x, z) + d(z, y) && \text{by the triangle inequality} \\ &< d(x, z) + \alpha && \text{by the hypothesis on } y \\ &= \delta. \end{aligned}$$

An analogous argument shows that if $y$ satisfies $d(z, y) < \alpha'$, then $d(x', y) < \delta'$.

So let $\epsilon = \min(\alpha, \alpha')$. We see that if $y \in X$ satisfies $d(z, y) < \epsilon$, then we have both $d(x, y) < \delta$ and $d(x', y) < \delta'$. So we have found the desired $\epsilon$. $\square$

## 3. Problem Set 2, #9

**Problem.** Let $X$ be arbitrary, and let $d : X \times X \to [0, \infty)$ be an arbitrary metric. Assume that the function $e : X \times X \to [0, \infty)$ defined by

$$e(x, y) = \frac{d(x, y)}{1 + d(x, y)}.$$

is a bounded metric. Show that $d$ and $e$ induce the same topology on $X$.

*Solution.* Let $\mathcal{T}_d$ and $\mathcal{T}_e$ denote the topologies respectively induced by $d$ and $e$.

We first show that $\mathcal{T}_d$ is finer than $\mathcal{T}_e$, meaning $\mathcal{T}_e \subseteq \mathcal{T}_d$. Since elements of $\mathcal{T}_e$ are unions of $e$-balls, it is suffices to check that any $e$-ball is an element of $\mathcal{T}_d$. So fix an $e$-ball $B_e(x, \delta)$. It suffices to show that for $y \in B_e(x, \delta)$, we can find some $\epsilon > 0$ such that $B_d(y, \epsilon) \subseteq B_e(x, \delta)$.

As a warmup, ignore $d$: Can we find some $\epsilon > 0$ such that $B_e(y, \epsilon) \subseteq B_e(x, \delta)$? Explicitly, for any $y$ satisfying $e(x, y) < \delta$, we have to exhibit some $\epsilon > 0$ such that, if $z$ satisfies $e(y, z) < \epsilon$, then $z$ also satisfies $e(x, z) < \delta$. The argument will be similar to the latter half of the proof of Theorem 2.1. Namely, let $\epsilon = \delta - e(x, y) > 0$, the distance from $y$ to the boundary of the ball $B_e(x, \delta)$. If $z$ satisfies $e(y, z) < \epsilon$, then by the triangle inequality, it satisfies

$$e(x, z) \leq e(x, y) + e(y, z) < e(x, y) + \epsilon = \delta,$$

as needed. So this choice of $\epsilon$ does give $B_e(y, \epsilon) \subseteq B_e(x, \delta)$.

Now go back to the original problem involving $d$. By combining $B_e(y, \epsilon) \subseteq B_e(x, \delta)$ with the following observation, we get $B_d(y, \epsilon) \subseteq B_e(x, \delta)$, as needed.

**Lemma 3.1.** *For any $y \in X$ and $\epsilon > 0$, we have $B_d(y, \epsilon) \subseteq B_e(y, \epsilon)$.*

*Proof.* Left as an exercise. □

Now we show the reverse inclusion: $\mathcal{T}_d \subseteq \mathcal{T}_e$. So fix a $d$-ball $B_d(x, \delta)$. We must show that for any $y \in B_d(x, \delta)$, we can find some $\epsilon > 0$ such that $B_e(x, \epsilon) \subseteq B_d(x, \delta)$. Explicitly, for any $y$ satisfying $d(x, y) < \delta$, we have to exhibit some $\epsilon > 0$ such that, if $z$ satisfies $e(y, z) < \epsilon$, then $z$ also satisfies $d(x, z) < \delta$.

Since the roles of $d$ and $e$ in Lemma 3.1 cannot be switched, we cannot just replicate our earlier argument with $d$ and $e$ switched. But we still expect to use the triangle inequality that $d(x, z) \leq d(x, y) + d(y, z)$. Letting $\alpha = \delta - d(x, y) > 0$ gives us $d(x, y) + \alpha = \delta$. So we just need to exhibit $\epsilon > 0$ such that $e(y, z) < \epsilon$ implies $d(y, z) < \alpha$, because for such $\epsilon$,

$$e(y, z) < \epsilon \quad \text{will imply} \quad d(x, z) \leq d(x, y) + d(y, z) < d(x, y) + \alpha = \delta,$$

as needed.

Rearranging the identity $e(y, z) = \frac{d(y,z)}{1+d(y,z)}$ gives $d(y, z) = \frac{e(y,z)}{1-e(y,z)}$. Moreover, rearranging $e(y, z) < \epsilon$ gives $\frac{e(y,z)}{1-e(y,z)} < \frac{\epsilon}{1-\epsilon}$. So the following lemma finishes the argument:

**Lemma 3.2.** *For any $\alpha > 0$, there is some $\epsilon > 0$ such that $\frac{\epsilon}{1-\epsilon} < \alpha$. (Moreover, we can pick $\epsilon < 1$, so that $\frac{\epsilon}{1-\epsilon}$ is well-defined.)*

*Proof.* Left as an exercise. *Hint:* If $0 < \epsilon < \frac{1}{2}$, then $\frac{1}{1-\epsilon} < 2$. $\square$

Since we have shown that $\mathcal{T}_d$ and $\mathcal{T}_e$ contain each other, they coincide. That is, $d$ and $e$ induce the same topology. $\square$