13.

13.1.

The Birman–Murakami–Wenzl (BMW) algebra on n strands is the $\mathbf{Z}[a^{\pm 1}, z^{\pm 1}]$-algebra C_n generated by elements $\sigma_1^{\pm 1}, \dots, \sigma_{n-1}^{\pm 1}, e_1, \dots, e_{n-1}$ subject to these relations:

$$\begin{aligned}
\sigma_i \sigma_{i+1} \sigma_i &= \sigma_{i+1} \sigma_i \sigma_{i+1}, \\
\sigma_i \sigma_j &= \sigma_j \sigma_i \qquad \text{for } |i-j| > 1, \\
e_i^2 &= (1 + (a - a^{-1}) z^{-1}) e_i, \\
\sigma_i \sigma_{i\pm 1} e_i &= e_{i\pm 1} e_i, \\
e_i e_{i\pm 1} e_i &= e_i, \\
\sigma_i e_i = e_i \sigma_i &= -a e_i, \\
e_i \sigma_{i\pm 1} e_i &= -a^{-1} e_i, \\
\sigma_i - \sigma_i^{-1} &= z(1 - e_i).
\end{aligned}$$

The first two are “braiding” relations, the fourth and fifth are “tangling” relations, the sixth and seventh are “delooping” relations, and the last is Kauffman’s skein relation. It’s convenient to write the skein relation as

$$\sigma_i^2 = 1 + z\sigma_i + aze_i$$

when computing with positive braids.

Remark 13.1. Our variables are related to those of Birman–Wenzl by the substitutions $l = -a^{-1}$ and $m = z$. However, we’re also using the *Dubrovnik* sign conventions.

In general, the rank of C_n as a free $\mathbf{Z}[a^{\pm 1}, z^{\pm 1}]$-module equals $(2n)!/(2^n n!)$. For example, $\operatorname{rk} C_2 = 3$ and $\operatorname{rk} C_3 = 15$. Explicitly, C_3 is generated by $\sigma_1^{\pm 1}, \sigma_2^{\pm 1}, e_1, e_2$ subject to the following relations:

$$\begin{aligned}
\sigma_1 \sigma_2 \sigma_1 &= \sigma_2 \sigma_1 \sigma_2, \\
e_i^2 &= (1 + (a - a^{-1}) z^{-1}) e_i \qquad (\text{for } i = 1, 2), \\
\sigma_1 \sigma_2 e_1 &= e_2 e_1, \\
\sigma_2 \sigma_1 e_2 &= e_1 e_2, \\
e_1 e_2 e_1 &= e_1, \\
e_2 e_1 e_2 &= e_2, \\
\sigma_i e_i = e_i \sigma_i &= -a e_i \qquad (\text{for } i = 1, 2), \\
e_1 \sigma_2 e_1 &= -a^{-1} e_1, \\
e_2 \sigma_1 e_2 &= -a^{-1} e_2, \\
\sigma_i - \sigma_i^{-1} &= z(1 - e_i) \qquad (\text{for } i = 1, 2).
\end{aligned}$$

One checks that a $\mathbf{Z}[a^{\pm 1}, z^{\pm 1}]$-basis for C_3 is given by

$$1, \sigma_1, \sigma_2, \sigma_1\sigma_2, \sigma_2\sigma_1, \sigma_1\sigma_2\sigma_1, \quad e_1, e_1\sigma_2, e_1\sigma_2\sigma_1, \quad e_2, e_2\sigma_1, e_2\sigma_1\sigma_2,$$
$$\sigma_1 e_2\sigma_1, \sigma_2 e_1\sigma_2, \quad e_1 e_2.$$

Example 13.2. We calculate:

$$\begin{aligned}\sigma_1\sigma_2\sigma_1\sigma_2 &= \sigma_1^2\sigma_2\sigma_1\\ &= (1 + z\sigma_1 + aze_1)\sigma_2\sigma_1\\ &= \sigma_2\sigma_1 + z\sigma_1\sigma_2\sigma_1 + aze_2\sigma_2\sigma_1.\end{aligned}$$

In the cocenter $C_3/[C_3, C_3]$, this becomes

$$\begin{aligned}\sigma_1\sigma_2\sigma_1\sigma_2 &\sim \sigma_1\sigma_2 + z\sigma_1^2\sigma_2 + az\sigma_2\sigma_1 e_2\\ &= \sigma_1\sigma_2 + z(1 + z\sigma_1 + aze_1)\sigma_2 + aze_1e_2\\ &\sim z\sigma_1 + (1 + z^2)\sigma_1\sigma_2 + az^2 e_1\sigma_2 + aze_1e_2.\end{aligned}$$

13.2.

The Iwahori–Hecke algebra on n strands is the $\mathbf{Z}[z]$-algebra H_n generated by elements $\sigma_1^{\pm 1}, \ldots, \sigma_{n-1}^{\pm 1}$ subject to

$$\begin{aligned}\sigma_i\sigma_{i+1}\sigma_i &= \sigma_{i+1}\sigma_i\sigma_{i+1},\\ \sigma_i\sigma_j &= \sigma_j\sigma_i \qquad\qquad \text{for } |i-j| > 1,\\ \sigma_i - \sigma_i^{-1} &= z.\end{aligned}$$

For $n \geq 2$, there is an evident inclusion map $H_{n-1} \to H_n$. Ocneanu's theorem can be stated as follows: There is a family of $\mathbf{Z}[z]$-linear functions

$$\mathsf{tr}_n : H_n/[H_n, H_n] \to \mathbf{Z}[a^{\pm 1}, z^{\pm 1}],$$

known as Markov traces, uniquely determined by the following axioms:

(1) $\mathsf{tr}_1(1) = 1$.
(2) $\mathsf{tr}_n(\beta) = (a - a^{-1})z^{-1}\,\mathsf{tr}_{n-1}(\beta)$ for all $\beta \in H_{n-1}$.
(3) $\mathsf{tr}_n(\beta\sigma_{n-1}^{\pm 1}) = -a^{\mp 1}\,\mathsf{tr}_{n-1}(\beta)$ for all $\beta \in H_{n-1}$.

Note that our normalizations differ from those in Jones's *Annals* paper: For instance, the axioms imply that

$$\mathsf{tr}_n(1) = ((a - a^{-1})z^{-1})^{n-1}.$$

Our normalizations are convenient because (reduced) HOMFLY-PT satisfies

$$\bar{\mathbf{P}}(\hat{\beta}) = (-a)^{|\beta|}\mathsf{tr}_n(\beta)$$

for any braid β on n strands.

Birman–Wenzl proved the analogue of Ocneanu's theorem for the algebras C_n. Again, there are evident inclusion maps $C_{n-1} \to C_n$. A mild generalization of their argument shows that there is a family of $\mathbf{Z}[a^{\pm 1}, z^{\pm 1}]$-linear functions

$$\mathsf{tr}_n : C_n/[C_n, C_n] \to \mathbf{Z}[a^{\pm 1}, z^{\pm 1}, \delta]/(\delta - \delta^2)$$

uniquely determined by the following axioms:

(1) $\mathsf{tr}_1(1) = 1$.
(2) $\mathsf{tr}_n(\beta) = ((a - a^{-1})z^{-1} + \delta)\,\mathsf{tr}_{n-1}(\beta)$ for all $\beta \in C_{n-1}$.
(3) $\mathsf{tr}_n(\beta\sigma_{n-1}^{\pm 1}) = -a^{\mp 1}\,\mathsf{tr}_{n-1}(\beta)$ for all $\beta \in C_{n-1}$.
(4) $\mathsf{tr}_n(\beta e_{n-1}) = \delta\,\mathsf{tr}_{n-1}(\beta)$ for all $\beta \in C_{n-1}$.

These functions *specialize* to the Markov traces on the algebras H_n when we set $\delta = 0$ and quotient by the e_i. By comparison, the (reduced) Kauffman polynomial satisfies

$$\bar{\mathbf{F}}(\hat{\beta}) = (-a)^{|\beta|}\mathsf{tr}_n(\beta)|_{\delta=1}$$

for any braid β on n strands. In this way, the distinction between HOMFLY-PT and Kauffman becomes the distinction between $\delta = 0$ and $\delta = 1$.

Example 13.3. Set $\beta = \sigma_1\sigma_2\sigma_1\sigma_2$. From our earlier work,

$$\begin{aligned}
\mathrm{tr}_3(\beta) &= \mathrm{tr}_3(z\sigma_1 + (1 + z^2)\sigma_1\sigma_2 + az^2 e_1\sigma_2 + aze_1e_2) \\
&= -a^{-1}z((a - a^{-1})z^{-1} + \delta) + a^{-2}(1 + z^2) - z^2\delta + az\delta^2 \\
&= a^{-2}(2 + z^2) - 1 + (-a^{-1}z - z^2 + az)\delta.
\end{aligned}$$

Thus, $\mathrm{tr}_3(\beta)|_{\delta=1} = a^{-2}(2 + z^2) - a^{-1}z - 1 - z^2 + az$, from which

$$\bar{\mathbf{F}}(\widehat{\sigma_1\sigma_2\sigma_1\sigma_2}) = a^2(2 + z^2) - a^3z - a^4(1 + z^2) + a^5z.$$

At the same time, $\mathrm{tr}_3(\beta)|_{\delta=0} = a^{-2}(2 + z^2) - 1$, from which

$$\bar{\mathbf{P}}(\widehat{\sigma_1\sigma_2\sigma_1\sigma_2}) = a^2(2 + z^2) - a^4.$$

13.3.

In what follows, we write $\mathsf{tr}_n^a(\beta) = \mathsf{tr}_n(\beta)|_{\delta=a}$. We are interested in comparing

$$\mathsf{tr}_n^0(\beta) \quad \text{and} \quad \mathbf{F}(\bigcirc)(\mathsf{tr}_n^1(\beta) - \mathsf{tr}_n^0(\beta)).$$

Example 13.4. For $\beta = \sigma_1^2 \in Br_2$, we calculate

$$\begin{aligned}
\mathsf{tr}_2(\beta) &= \mathsf{tr}_2(1 + z\sigma_1 + aze_1) \\
&= (a - a^{-1})z^{-1} + \delta - a^{-1}z + az\delta \\
&= -a^{-1}(z^{-1} + z) + az^{-1} + (1 + az)\delta,
\end{aligned}$$

from which

$$\begin{aligned} \mathrm{tr}_2^0(\beta) &= -a^{-1}(z^{-1}+z)+az^{-1}, \\ \mathrm{tr}_2^1(\beta)-\mathrm{tr}_2^0(\beta) &= 1+az. \end{aligned}$$

From the latter,

$$\begin{aligned} \mathbf{P}(\bigcirc)(\mathrm{tr}_2^1(\beta)-\mathrm{tr}_2^0(\beta)) &= (a-a^{-1})(z^{-1}+a) \\ &= -a^{-1}z^{-1}-1+az^{-1}+a^2, \\ \mathbf{F}(\bigcirc)(\mathrm{tr}_2^1(\beta)-\mathrm{tr}_2^0(\beta)) &= -a^{-1}z^{-1}+a(z^{-1}+z)+a^2. \end{aligned}$$

Example 13.5. For $\beta=\sigma_1^3\in Br_2$, we calculate

$$\begin{aligned} \sigma_1^3 &= \sigma_1+z\sigma_1^2+aze_1\sigma_1 \\ &= z+(1+z^2)\sigma_1+(az^2-a^2z)e_1. \end{aligned}$$

Next we calculate

$$\begin{aligned} \mathrm{tr}_2(\beta) &= z((a-a^{-1})z^{-1}+\delta)-a^{-1}(1+z^2)+(az^2-a^2z)\delta \\ &= -a^{-1}(2+z^2)+a+(z+az^2-a^2z)\delta, \end{aligned}$$

from which

$$\begin{aligned} \mathrm{tr}_2^0(\beta) &= -a^{-1}(2+z^2)+a, \\ \mathrm{tr}_2^1(\beta)-\mathrm{tr}_2^0(\beta) &= z+az^2-a^2z. \end{aligned}$$

From the latter,

$$\begin{aligned} \mathbf{P}(\bigcirc)(\mathrm{tr}_2^1(\beta)-\mathrm{tr}_2^0(\beta)) &= (a-a^{-1})(1+az-a^2) \\ &= -a^{-1}-z+2a+a^2z-a^3, \\ \mathbf{F}(\bigcirc)(\mathrm{tr}_2^1(\beta)-\mathrm{tr}_2^0(\beta)) &= -a^{-1}+a(2+z^2)-a^3. \end{aligned}$$

Example 13.6. For $\beta=\sigma_1^4\in Br_2$, we calculate

$$\begin{aligned} \sigma_1^4 &= (1+z\sigma_1+aze_1)^2 \\ &= 1+2z\sigma_1+2aze_1+z^2\sigma_1^2+az^2(\sigma_1e_1+e_1\sigma_1)+a^2z^2e_1^2 \\ &\sim 1+2z\sigma_1+2aze_1+z^2(1+z\sigma_1+aze_1) \\ &\qquad -2a^2z^2e_1+a^2z^2(1+(a-a^{-1})z^{-1})e_1 \\ &= (1+z^2)+(2z+z^3)\sigma_1+(az+az^3-a^2z^2+a^3z)e_1. \end{aligned}$$

Next we calculate

$$\begin{aligned} \mathrm{tr}_2(\beta) &= (1+z^2)((a-a^{-1})z^{-1}+\delta)-a^{-1}(2z+z^3)+(az+az^3-a^2z^2+a^3z)\delta \\ &= -a^{-1}(z^{-1}+3z+z^3)+a(z^{-1}+z) \\ &\qquad +(1+z^2+az+az^3-a^2z^2+a^3z)\delta, \end{aligned}$$

from which

$$\begin{aligned} \mathrm{tr}_2^0(\beta) &= -a^{-1}(z^{-1} + 3z + z^3) + a(z^{-1} + z), \\ \mathrm{tr}_2^1(\beta) - \mathrm{tr}_2^0(\beta) &= 1 + z^2 + a(z + z^3) - a^2z^2 + a^3z. \end{aligned}$$

From the latter,

$$\begin{aligned} \mathbf{P}(\bigcirc)(\mathrm{tr}_2^1(\beta) - \mathrm{tr}_2^0(\beta)) &= (a - a^{-1})(z^{-1} + z + a + az^2 - a^2z + a^3) \\ &= -a^{-1}(z^{-1} + z) - (1 + z^2) + a(z^{-1} + 2z) + a^2z^2 - a^3z + a^4, \\ \mathbf{F}(\bigcirc)(\mathrm{tr}_2^1(\beta) - \mathrm{tr}_2^0(\beta)) &= -a^{-1}(z^{-1} + z) + a(z^{-1} + 3z + z^3) + a^4. \end{aligned}$$

Example 13.7. For $\beta = \sigma_1\sigma_2\sigma_1 \in Br_3$, we calculate

$$\begin{aligned} \sigma_1\sigma_2\sigma_1 &\sim \sigma_1^2\sigma_2 \\ &= (1 + z\sigma_1 + aze_1)\sigma_2 \\ &= \sigma_2 + z\sigma_1\sigma_2 + aze_1\sigma_2. \end{aligned}$$

Next we calculate

$$\begin{aligned} \mathrm{tr}_3(\beta) &= -a^{-1}((a - a^{-1})z^{-1} + \delta) + a^{-2}z - z\delta \\ &= a^{-2}(z^{-1} + z) - z^{-1} - (a^{-1} + z)\delta, \end{aligned}$$

from which

$$\begin{aligned} \mathrm{tr}_3^0(\beta) &= a^{-2}(z^{-1} + z) - z^{-1}, \\ \mathrm{tr}_3^1(\beta) - \mathrm{tr}_3^0(\beta) &= -a^{-1} - z. \end{aligned}$$

From the latter,

$$\begin{aligned} \mathbf{P}(\bigcirc)(\mathrm{tr}_2^1(\beta) - \mathrm{tr}_2^0(\beta)) &= -(a - a^{-1})z^{-1}(a^{-1} + z) \\ &= a^{-2}z^{-1} + a^{-1} - z^{-1} - a, \\ \mathbf{F}(\bigcirc)(\mathrm{tr}_2^1(\beta) - \mathrm{tr}_2^0(\beta)) &= a^{-2}z^{-1} - (z^{-1} + z) - a. \end{aligned}$$

Example 13.8. For $\beta = \sigma_1\sigma_2\sigma_1\sigma_2 \in Br_3$, our earlier work gives

$$\begin{aligned} \mathrm{tr}_n^0(\beta) &= a^{-2}(2 + z^2) + 1, \\ \mathrm{tr}_n^1(\beta) - \mathrm{tr}_n^0(\beta) &= -a^{-1}z - z^2 + az. \end{aligned}$$

From the latter,

$$\begin{aligned} \mathbf{P}(\bigcirc)(\mathrm{tr}_n^1(\beta) - \mathrm{tr}_n^0(\beta)) &= (a - a^{-1})(-a^{-1} - z + a) \\ &= a^{-2} + a^{-1}z - 2 - az + a^2, \\ \mathbf{F}(\bigcirc)(\mathrm{tr}_n^1(\beta) - \mathrm{tr}_n^0(\beta)) &= a^{-2} - (2 + z^2) + a^2. \end{aligned}$$

Example 13.9. For $\beta = \sigma_1^2\sigma_2^2\sigma_1\sigma_2 = (\sigma_1\sigma_2^{-1})(\sigma_2\sigma_1\sigma_2)^2 \in Br_3$, we calculate

$$
\begin{aligned}
&\sigma_1^2\sigma_2^2\sigma_1\sigma_2\\
&= (1 + z\sigma_1 + aze_1)(1 + z\sigma_2 + aze_2)\sigma_1\sigma_2\\
&= (1 + z(\sigma_1 + \sigma_2) + az(e_1 + e_2) + z^2\sigma_1\sigma_2 + az^2(\sigma_1 e_2 + e_1\sigma_2) + a^2z^2e_1e_2)\sigma_1\sigma_2\\
&\sim \sigma_1\sigma_2 + 2z\sigma_1\sigma_2\sigma_1 + az(e_1\sigma_1\sigma_2 + e_2\sigma_1\sigma_2) + z^2\sigma_1\sigma_2\sigma_1\sigma_2\\
&\qquad + az^2(\sigma_1e_2\sigma_1\sigma_2 + e_1\sigma_2\sigma_1\sigma_2) + a^2z^2e_1e_2\sigma_1\sigma_2\\
&\sim \sigma_1\sigma_2 + 2z\sigma_1\sigma_2\sigma_1 + az(\sigma_1\sigma_2e_1 + \sigma_2e_2\sigma_1) + z^2\sigma_1\sigma_2\sigma_1\sigma_2\\
&\qquad + az^2(\sigma_1\sigma_2\sigma_1e_2 + \sigma_2\sigma_1\sigma_2e_1) + a^2z^2\sigma_1\sigma_2e_1e_2\\
&= \sigma_1\sigma_2 + 2z\sigma_1\sigma_2\sigma_1 + az(e_2e_1 - ae_2\sigma_1) + z^2\sigma_1\sigma_2\sigma_1\sigma_2\\
&\qquad - a^2z^2(e_1e_2 + e_2e_1) + a^2z^2e_2e_1e_2\\
&\sim \sigma_1\sigma_2 + 2z\sigma_1^2\sigma_2 + az(e_1e_2 - ae_2\sigma_1) + z^2\sigma_1\sigma_2\sigma_1\sigma_2 - 2a^2z^2e_1e_2 + a^2z^2e_1e_2^2\\
&= \sigma_1\sigma_2 + 2z(1 + z\sigma_1 + aze_1)\sigma_2 - a^2ze_2\sigma_1 + z^2\sigma_1\sigma_2\sigma_1\sigma_2\\
&\qquad + (az - 2a^2z^2 + a^2z^2(1 + (a - a^{-1})z^{-1}))e_1e_2\\
&\sim 2z\sigma_1 + (1 + 2z^2)\sigma_1\sigma_2 + 2az^2e_1\sigma_2 - a^2ze_2\sigma_1 + (-a^2z^2 + a^3z)e_1e_2\\
&\qquad + z^2(z\sigma_1 + (1 + z^2)\sigma_1\sigma_2 + az^2e_1\sigma_2 + aze_1e_2)\\
&= (2z + z^3)\sigma_1 + (1 + 3z^2 + z^4)\sigma_1\sigma_2 + (2az^2 + az^4)e_1\sigma_2 - a^2ze_2\sigma_1\\
&\qquad + (az^3 - a^2z^2 + a^3z)e_1e_2.
\end{aligned}
$$

Next we calculate

$$
\begin{aligned}
\mathrm{tr}_3(\beta) &= (2z + z^3)(-a^{-1})((a - a^{-1})z^{-1} + \delta) + (1 + 3z^2 + z^4)a^{-2}\\
&\qquad - (2z^2 + z^4)\delta + az\delta + (az^3 - a^2z^2 + a^3z)\delta^2\\
&= a^{-2}(3 + 4z^2 + z^4) - (2 + z^2)\\
&\qquad + (-a^{-1}(2z + z^3) - 2z^2 - z^4 + az)\delta + (az^3 - a^2z^2 + a^3z)\delta^2\\
&= a^{-2}(3 + 4z^2 + z^4) - (2 + z^2)\\
&\qquad + (-a^{-1}(2z + z^3) - 2z^2 - z^4 + a(z + z^3) - a^2z^2 + a^3z)\delta.
\end{aligned}
$$

We arrive at

$$
\begin{aligned}
\mathrm{tr}_3^0(\beta) &= a^{-2}(3 + 4z^2 + z^4) - (2 + z^2),\\
\mathrm{tr}_3^1(\beta) - \mathrm{tr}_3^0(\beta) &= -a^{-1}(2z + z^3) - 2z^2 - z^4 + a(z + z^3) - a^2z^2 + a^3z.
\end{aligned}
$$

From the latter,

$$
\begin{aligned}
&\mathbf{P}(\bigcirc)(\mathrm{tr}_3^1(\beta) - \mathrm{tr}_3^0(\beta))\\
&= (a - a^{-1})(-a^{-1}(2 + z^2) - 2z - z^3 + a(1 + z^2) - a^2z + a^3)\\
&= -a^{-2}(2 + z^2) + a^{-1}(2z + z^3) - 3 - 2z^2 - a(z + z^3) + a^2z^2 - a^3z + a^4
\end{aligned}
$$

and therefore

$$
\mathbf{F}(\bigcirc)(\mathrm{tr}_3^1(\beta) - \mathrm{tr}_3^0(\beta)) = -a^{-2}(2 + z^2) - (3 + 4z^2 + z^4) + a^4.
$$

Example 13.10. For $\beta = (\sigma_1\sigma_2)^4 \in Br_3$, we calculate

$$\begin{aligned}
(\sigma_1\sigma_2)^4 &= (z\sigma_1 + (1+z^2)\sigma_1\sigma_2 + az^2e_1\sigma_2 + aze_1e_2)^2 \\
&\sim z^2\sigma_1^2 + 2(z+z^3)\sigma_1^2\sigma_2 + 2az^3\sigma_1e_1\sigma_2 + 2az^2\sigma_1e_1e_2 \\
&\quad + (1+z^2)^2\sigma_1\sigma_2\sigma_1\sigma_2 + 2az^2(1+z^2)\sigma_2\sigma_1\sigma_2e_1 + 2az(1+z^2)\sigma_1\sigma_2e_1e_2 \\
&\quad + a^2z^4e_1\sigma_2e_1\sigma_2 + 2a^2z^3e_1\sigma_2e_1e_2 + a^2z^2e_1e_2e_1e_2 \\
&= z^2\sigma_1^2 + 2(z+z^3)\sigma_1^2\sigma_2 + 2az^3(-ae_1)\sigma_2 + 2az^2(-ae_1)e_2 \\
&\quad + (1+z^2)^2\sigma_1\sigma_2\sigma_1\sigma_2 + 2az^2(1+z^2)\sigma_1\sigma_2(-ae_1) + 2az(1+z^2)e_2 \\
&\quad + a^2z^4(-a^{-1}e_1)\sigma_2 + 2a^2z^3(-a^{-1}e_1)e_2 + a^2z^2e_1e_2 \\
&= z^2(1+z\sigma_1+aze_1) + 2(z+z^3)(1+z\sigma_1+aze_1)\sigma_2 - 2a^2z^3e_1\sigma_2 \\
&\quad - 2a^2z^2e_1e_2 + (1+z^2)^2\sigma_1\sigma_2\sigma_1\sigma_2 - 2a^2z^2(1+z^2)e_2e_1 + 2az(1+z^2)e_2 \\
&\quad - az^4e_1\sigma_2 - 2az^3e_1e_2 + a^2z^2e_1e_2 \\
&\sim z^2 + (2z+3z^3)\sigma_1 + 2(z^2+z^4)\sigma_1\sigma_2 + az^3e_1 + 2az(1+z^2)e_2 \\
&\quad + (2az^2 + az^4 - 2a^2z^3)e_1\sigma_2 - (2az^3 + 3a^2z^2 + 2a^2z^4)e_1e_2 \\
&\quad + (1+z^2)^2\sigma_1\sigma_2\sigma_1\sigma_2.
\end{aligned}$$

Expanding the last term in the last expression,

$$\begin{aligned}
(\sigma_1\sigma_2)^4 &\sim z^2 + (2z+3z^3)\sigma_1 + 2(z^2+z^4)\sigma_1\sigma_2 + az^3e_1 + 2az(1+z^2)e_2 \\
&\quad + (2az^2 + az^4 - 2a^2z^3)e_1\sigma_2 - (2az^3 + 3a^2z^2 + 2a^2z^4)e_1e_2 \\
&\quad + (1+2z^2+z^4)(z\sigma_1 + (1+z^2)\sigma_1\sigma_2 + az^2e_1\sigma_2 + aze_1e_2) \\
&= z^2 + (2z+3z^3)\sigma_1 + 2(z^2+z^4)\sigma_1\sigma_2 + az^3e_1 + 2az(1+z^2)e_2 \\
&\quad + (2az^2 + az^4 - 2a^2z^3)e_1\sigma_2 - (2az^3 + 3a^2z^2 + 2a^2z^4)e_1e_2 \\
&\quad + (z+2z^3+z^5)\sigma_1 + (1+3z^2+3z^4+z^6)\sigma_1\sigma_2 \\
&\quad + a(z^2+2z^4+z^6)e_1\sigma_2 + a(z+2z^3+z^5)e_1e_2 \\
&= z^2 + (3z+5z^3+z^5)\sigma_1 + (1+5z^2+5z^4+z^6)\sigma_1\sigma_2 + az^3e_1 \\
&\quad + 2az(1+z^2)e_2 + (3az^2+3az^4+az^6-2a^2z^3)e_1\sigma_2 \\
&\quad + (az+az^5-3a^2z^2-2a^2z^4)e_1e_2.
\end{aligned}$$

Next we calculate

$$\begin{aligned}
\mathrm{tr}_3(\beta) &= z^2((a-a^{-1})z^{-1}+\delta)^2 + (3z+5z^3+z^5)(-a^{-1})((a-a^{-1})z^{-1}+\delta) \\
&\quad + a^{-2}(1+5z^2+5z^4+z^6) + (2az+3az^3)((a-a^{-1})z^{-1}+\delta)\delta \\
&\quad + (-3z^2-3z^4-z^6+2az^3)\delta + (az+az^5-3a^2z^2-2a^2z^4)\delta \\
&= (a^{-2}-2+a^2+2(a-a^{-1})z\delta+z^2\delta) + (a^{-2}-1)(3+5z^2+z^4) \\
&\quad - a^{-1}(3z+5z^3+z^5)\delta + a^{-2}(1+5z^2+5z^4+z^6) \\
&\quad + (-2-3z^2+2az+3az^3+2a^2+3a^2z^2)\delta \\
&\quad + (-3z^2-3z^4-z^6+az+2az^3+az^5-3a^2z^2-2a^2z^4)\delta.
\end{aligned}$$

Combining terms,

$$\begin{aligned}
\mathrm{tr}_3(\beta) &= a^{-2} - 2 + a^2 + (a^{-2} - 1)(3 + 5z^2 + z^4) + a^{-2}(1 + 5z^2 + 5z^4 + z^6)\\
&\quad + (-2a^{-1}z + z^2 + 2az)\delta - a^{-1}(3z + 5z^3 + z^5)\delta\\
&\quad + (-2 - 6z^2 - 3z^4 - z^6 + 3az + 5az^3 + az^5 + 2a^2 - 2a^2z^4)\delta\\
&= a^{-2}(5 + 10z^2 + 6z^4 + z^6) - (5 + 5z^2 + z^4) + a^2\\
&\quad - (a^{-1}(5z + 5z^3 + z^5) + 2 + 5z^2 + 3z^4 + z^6)\delta\\
&\quad + (a(5z + 5z^3 + z^5) + a^2(2 - 2z^4))\delta.
\end{aligned}$$

We arrive at

$$\begin{aligned}
\mathrm{tr}_3^0(\beta) &= a^{-2}(5 + 10z^2 + 6z^4 + z^6) - (5 + 5z^2 + z^4) + a^2,\\
\mathrm{tr}_3^1(\beta) - \mathrm{tr}_3^0(\beta) &= -a^{-1}(5z + 5z^3 + z^5) - (2 + 5z^2 + 3z^4 + z^6)\\
&\quad + a(5z + 5z^3 + z^5) + a^2(2 - 2z^4).
\end{aligned}$$

From the latter,

$$\begin{aligned}
&\mathbf{P}(\bigcirc)(\mathrm{tr}_3^1(\beta) - \mathrm{tr}_3^0(\beta))\\
&= a^{-2}(5 + 5z^2 + z^4) + a^{-1}(2z^{-1} + 5z + 3z^3 + z^5) - 10 - 10z^2 - 2z^4\\
&\quad + a(-4z^{-1} - 5z - z^3 - z^5) + a^2(5 + 5z^2 + z^4) + a^3(2z^{-1} - 2z^3).
\end{aligned}$$

and therefore

$$\begin{aligned}
&\mathbf{F}(\bigcirc)(\mathrm{tr}_3^1(\beta) - \mathrm{tr}_3^0(\beta))\\
&= a^{-2}(5 + 5z^2 + z^4) + a^{-1}(2z^{-1} - 2z^3) - (12 + 15z^2 + 5z^4 + z^6)\\
&\quad + a(-4z^{-1} + 4z^3) + a^2(7 + 5z^2 - z^4) + a^3(2z^{-1} - 2z^3)
\end{aligned}$$

A new phenomenon!