19.

We identify the Springer resolution within the Grothendieck alteration, and explain how to construct the Springer action of W using the functoriality of intermediate extension. Along the way, we explain how perverse sheaves interact with (semi)small maps. The main references are Z. Yun's PCMI notes and Achar's book.[1]

For most of these notes, k can be any algebraically closed field. As usual, G is a connected, smooth reductive algebraic group over k with flag variety $\mathcal{B}$ and Weyl group W, and we fix a Borel pair $T \subseteq B \subseteq G$. Note that G being smooth restricts the characteristic of k. We will further assume that the characteristic is *very good* in the sense of Kiehl–Weissauer Definition VI.1.6, going back to Slodowy, *Simple Singularities and Simple Algebraic Groups*. (Just assume that the characteristic is sufficiently large.)

19.1.

Let $\mathfrak{g}$ be the Lie algebra of G. The preimage of $[0]$ along the map $\mathfrak{g} \to \mathfrak{g} /\!\!/ G$ is called the *nilpotent cone* of $\mathfrak{g}$ and denoted $\mathcal{N}$. It forms a conical subvariety: *i.e.*, a closed subvariety stable under the $\mathbf{G}_m$-action contracting $\mathfrak{g}$ to its origin. One can check that its k-points are precisely the nilpotent elements of $\mathfrak{g}$: *i.e.*, those sent to nilpotent operators in any representation of $\mathfrak{g}$ as a Lie algebra.

Let $\tilde{\mathcal{N}}$ be the cotangent bundle of the flag variety $\mathcal{B}$. Once we fix a perfect, G-equivariant bilinear pairing $\langle -, - \rangle : \mathfrak{g} \times \mathfrak{g} \to k$, like the Killing form, the notation is justified by the following calculation:

$$\begin{aligned}
\tilde{\mathcal{N}}(k) &= \{(B, \xi) \in \mathcal{B}(k) \times \mathfrak{g}(k) \mid \xi \in (\mathfrak{g}/\mathfrak{b})^\vee(k)\} && \text{where } \mathfrak{b} = \mathrm{Lie}(B) \\
&= \{(B, \xi) \mid \xi \in \mathfrak{b}^\perp(k)\} \\
&\simeq \{(B, x) \mid x \in [\mathfrak{b}(k), \mathfrak{b}(k)]\} && \text{via } \langle -, - \rangle \\
&= \{(B, x) \in \mathcal{B}(k) \times \mathcal{N}(k) \mid x \in \mathfrak{b}(k)\} && \text{via } [\mathfrak{b}, \mathfrak{b}] = \mathfrak{b} \cap \mathcal{N}.
\end{aligned}$$

Henceforth, we identify $\tilde{\mathcal{N}}$ with the closed subscheme of $\mathcal{B} \times \mathcal{N}$ defined by the condition $x \in \mathfrak{b}$.

The calculation above suggests why this subscheme is smooth. With more work, one can verifiy that it is reduced, hence a variety. The map

$$\nu : \tilde{\mathcal{N}} \to \mathcal{N}$$

is a resolution of singularities called the *Springer resolution* of $\mathcal{N}$. Its fibers are known as *Springer fibers*.

[1]See also these notes by Yehao Zhou, which include several of the proofs: `https://www.math.toronto.edu/jkamnitz/seminar/perverse/YehaoNotes2.pdf`.

Example 19.1. Take $G = \mathrm{GL}_3$. There are three nilpotent orbits in $\mathfrak{g}$ (under the adjoint action of G), corresponding to the Jordan normal form representatives

$$\begin{pmatrix} 0 & 1 & \\ & 0 & 1 \\ & & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & \\ & 0 & \\ & & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & & \\ & 0 & \\ & & 0 \end{pmatrix}.$$

The first two are called the *regular*/*principal* and *subregular* orbits, respectively. If $x \in \mathcal{N}(k)$ is regular, then the Springer fiber above x is a point: For instance, the only Borel whose Lie algebra contains the leftmost element above is the upper-triangular subgroup. If x is subregular, then the Springer fiber above x turns out to form two projective lines intersecting transversely at a point. Finally, the Springer fiber above $x = 0$ is a copy of $\mathcal{B}$, because 0 lives in the Lie algebra of every Borel.

Springer made the remarkable discovery that W acts on the pushforward complex $\nu_*(\bar{\mathbf{Q}}_\ell)_{\tilde{\mathcal{N}}}$, even though it does not act on the fibers of ν themselves, and that every irreducible representation of W appears in the resulting representations of W formed by the cohomology of Springer fibers. His original construction was indirect, relying on a kind of harmonic analysis (*viz.*, Artin–Schreier sheaves).

We will present a later construction due to Lusztig in " Green Polynomials and Singularities of Unipotent Classes", which uses the functoriality of intermediate extension. Actually, Springer and Lusztig stay in the setting of the Lie algebra $\mathfrak{g}$, whereas we will present an analogue in the setting of the group G, where the ingredients are related to unipotent character sheaves.

Remark 19.2. Lusztig's action actually differs from Springer's by a sign twist. With care, this sign twist can be traced back to Poincaré/Verdier duality, as explained in a paper of Achar–Henderson–Juteau–Riche that also extends the possibilities for the coefficient rings involved.

19.2.

The preimage of $[1]$ along the map $G \to G /\!\!/ G$ is called the *unipotent variety* of G and denoted $\mathcal{U}$. Its k-points are the unipotent elements of $G(k)$.

By a theorem of Springer (Kiehl–Weissauer VI.3.3), the assumption that the characteristic of k is very good implies the existence of a G-equivariant isomorphism of varieties $\mathcal{N} \xrightarrow{\sim} \mathcal{U}$. Moreover, if we fix a Borel $B \subseteq G$ with unipotent radical $U = [B, B]$, and set $\mathfrak{n} = \mathrm{Lie}(U)$, then we can assume that the isomorphism restricts to a B-equivariant isomorphism $\mathfrak{u} \xrightarrow{\sim} U$.

Remark 19.3. The isomorphism $\mathcal{N} \xrightarrow{\sim} \mathcal{U}$ need not be unique. This is discussed thoroughly in Kiehl–Weissauer §VI.3.

The preceding discussion justifies replacing the Springer resolution of $\mathcal{N}$ with an isomorphic resolution $\nu : \tilde{\mathcal{U}} \to \mathcal{U}$. (We reuse the letter ν, as we will no longer use the original map.) Explicitly,

$$\tilde{\mathcal{U}}(k) = \{(u, B) \in \mathcal{U}(k) \times \mathcal{B}(k) \mid u \in B(k)\}.$$

This is a subset of (the k-points of) the Grothendieck alteration:

$$\tilde{G}(k) = \{(g, B) \in G(k) \times \mathcal{B}(k) \mid g \in B(k)\}.$$

There is a reason that we are being more careful than usual to distinguish sets from schemes. On schemes, we have a G-equivariant commutative diagram:

(19.1)
$$\begin{tikzcd} \tilde{\mathcal{U}} \arrow[r] \arrow[d, "\nu"'] & \tilde{G} \arrow[d, "\pi"] & \tilde{G}^{\mathrm{rs}} \arrow[l, hook'] \arrow[d, "\pi^{\mathrm{rs}}"] \\ \mathcal{U} \arrow[r, "i"] & G & G^{\mathrm{rs}} \arrow[l, hook', "j"'] \end{tikzcd}$$

Recall that the right-hand square is cartesian. The left-hand square is cartesian at the level of k-points, but actually fails to be cartesian at the level of schemes, because $\tilde{G} \times_G \mathcal{U}$ is not in fact reduced: *e.g.*, over any point of the regular orbit of $\mathcal{U}$, its fiber is a point of multiplicity $|W|$, just like its fiber over any point of a regular *semisimple* orbit is a W-torsor. (This seems to be an incarnation of Poincaré–Hopf.) See Exercise 1.7.2 in Yun's PCMI lectures.

However, it turns out that in all the sheaf-theoretic applications that interest us, we can ignore this discrepancy: As explained in, say, the Stacks Project, the étale site of a scheme is *invariant* under changes of non-reduced structure. As the closed embedding $\tilde{\mathcal{U}} \to \tilde{G} \times_G \mathcal{U}$ is G-equivariant, the analogous statement holds for the equivariant sites in our setup.

19.3.

We previously discussed how π^{rs} forms an étale cover with Galois group W. To transfer information from π^{rs} to ν, we need a further geometric fact refining the decomposition theorem.

A map $p : Y \to X$ with Y irreducible is called *semismall*, *resp.* *small*, if and only if it is proper, surjective, and satisfies $\dim Y \times_X Y \leq \dim Y$, *resp.* $< \dim Y$. In more concrete terms, this means: The locus in X over which p has relative dimension d forms a subscheme of codimension $\geq 2d$, *resp.* $> 2d$. Tautologically, smallness implies semismallness. Plugging in $d = 1$ shows that if p is semismall, then p is finite away from a codimension-1 locus of X. In other words, p being semismall is a slight weakening of p being proper.

Recall that if p is finite, then $p_! = p_*$ is perverse t-exact. If p is merely (semi)small, then a similar but weaker conclusion holds. Below, parts (1) and (2) are respectively Theorem 3.8.4 and Proposition 3.8.7 in Achar's book.

Theorem 19.4. *Consider a map $p : Y \to X$ of separated schemes of finite type over a field, where Y is smooth and irreducible, and a lisse or locally constant sheaf $\mathcal{L}$ on Y.*

(1) If p is semismall, then $p_*\mathcal{L}[\dim Y]$ is perverse.
(2) If p is small, and finite over some dense open $j : X' \to X$, then

$$p_*\mathcal{L}[\dim Y] \simeq j_{!*}p'_*(\mathcal{L}|_{Y'})[\dim Y],$$

where $p' : Y' \to X'$ is the restriction of p.

Remark 19.5. It is not true for an arbitrary perverse sheaf E on Y that if $p : Y \to X$ is small, then p_*E is a perverse sheaf on X.[2]

As usual, the statements in the theorem can be generalized to equivariant versions, whose details we omit. The relevance to Springer theory is:

Theorem 19.6. *The map $\nu : \tilde{\mathcal{U}} \to \mathcal{U}$ is semismall; the map $\pi : \tilde{G} \to G$ is small. (Hence, analogous statements hold for the stack quotients by G.)*

The semismallness of ν comes down to a dimension formula for Springer fibers due to Steinberg and Springer. Following §1.4.3 in Yun's PCMI notes: If $\mathcal{B}_u \subseteq \tilde{\mathcal{U}}$ is the Springer fiber above $u \in \mathcal{U}(k)$, then

$$2\dim \mathcal{B}_u = \dim \mathcal{N} - \dim \mathrm{Ad}(G)(u).$$

That is, the semismallness condition only barely holds, with the dimension inequality satisfied by an equality along each adjoint orbit.

As for the smallness of π, the argument is a little longer, and more easily explained in the setting of $\mathfrak{g}$ than the setting of G. Ultimately, it relies on Jordan decomposition to reduce from general elements of $\mathfrak{g}$ to nilpotent elements.

Remark 19.7. As a cotangent bundle, (the complex analogue of) the Springer resolution $\tilde{\mathcal{U}}$ naturally forms a symplectic variety. A theorem of Kaledin states that any proper birational map out of a smooth, symplectic, complex algebraic variety is semismall.

19.4.

We now take $k = \bar{\mathbf{F}}_q$, and fix a Frobenius map $F : G \to G$, corresponding to some $\mathbf{F}_q$-form G_1, such that B, T are F-stable and F acts trivially on W. Everything we discussed above descends from k to $k_1 = \mathbf{F}_q$.

Recall the W-isotypic summands $\mathcal{L}_{\chi,1} \subseteq \pi^{\mathrm{rs}}_{1,*}(\bar{\mathbf{Q}}_\ell)_{\tilde{G}^{\mathrm{rs}}_1}$ and perverse sheaves $A_{\chi,1} := j_{1,!*}\mathcal{L}_{\chi,1}\langle \dim G\rangle$ discussed in previous notes. Applying Theorem 19.4(2) and Theorem 19.6 to (19.1), we see that

$$\pi_{1,*}(\bar{\mathbf{Q}}_\ell)_{\tilde{G}_1} \simeq j_{1,!*}\pi^{\mathrm{rs}}_{1,*}(\bar{\mathbf{Q}}_\ell)_{\tilde{G}^{\mathrm{rs}}_1}.$$

[2]See https://mathoverflow.net/q/72872.

The functoriality of $j_{1,!*}$ transports the W-action on $(\bar{\mathbf{Q}}_\ell)_{\tilde{G}_1^{\mathrm{rs}}}$ to an action on $\pi_*(\bar{\mathbf{Q}}_\ell)_{\tilde{G}_1}$. Just as the $\mathcal{L}_{\chi,1}$ are the isotypic summands of the former, so the $A_{\chi,1}$ are the isotypics of the latter, up to shift-twist:

$$\pi_{1,*}(\bar{\mathbf{Q}}_\ell)_{\tilde{G}_1}\langle \dim G\rangle \simeq \bigoplus_{\chi \in \mathrm{Irr}(W)} \chi \otimes A_{\chi,1}$$

But by base change, together with the discussion below (19.1), we also have

$$\nu_{1,*}(\bar{\mathbf{Q}}_\ell)_{\tilde{\mathcal{U}}_1} \simeq i_1^*\pi_{1,*}(\bar{\mathbf{Q}}_\ell)_{\tilde{G}_1}.$$

By the functoriality of i_1^*, we arrive at a W-action on $\nu_{1,*}(\bar{\mathbf{Q}}_\ell)_{\tilde{\mathcal{U}}_1}$, for which the $i_1^* A_{\chi,1}$ are the isotypics:

$$\nu_{1,*}(\bar{\mathbf{Q}}_\ell)_{\tilde{\mathcal{U}}_1} \simeq \bigoplus_{\chi \in \mathrm{Irr}(W)} \chi \otimes i_1^* A_{\chi,1}.$$

We say that $\nu_{1,*}(\bar{\mathbf{Q}}_\ell)_{\tilde{\mathcal{U}}_1}$ is the (mixed, equivariant) *Springer sheaf* on $\mathcal{U}$.

19.5.

We do not need equivariance, mixedness, or even $k = \bar{\mathbf{F}}_q$ in the following result, though the conclusion remains the same with these refinements.

Theorem 19.8 (Springer)**.** *Every irreducible representation of W occurs in the cohomology of the stalks of $\nu_*\bar{\mathbf{Q}}_\ell$: more precisely, in the top étale cohomology of some Springer fiber.*

For $k = \mathbf{C}$, there is even a more elementary construction of the W-action on the Springer sheaf, due to Slodowy and explained in Section 4 of his book *Four Lectures on Simple Groups and Singularities*, that gives more: It gives a W-action on the *homotopy type* of each Springer fiber.