Products, Coproducts, and Universal Properties

Df a category C consists of

- a class of objects
- for any objects X, Y, a set Hom(X, Y) of arrows from X to Y called morphisms
- for any objects X, Y, Z, a composition law

$\circ$: Hom(X, Y) × Hom(Y, Z) to Hom(X, Z)

s.t.
1) $\circ$ is associative
2) for all X, an elt Id_X in Hom(X, X) serving as (left and right) id elt for $\circ$

[note: no axiom of inversion]

Ex C = Top

objects are topological spaces
morphisms are continuous maps

Ex C = Grp

objects are groups
morphisms are group homomorphisms

Ex C = Ab

objects are abelian groups
morphisms are group homomorphisms

we say that Ab is a subcategory of Grp

it is <u>full</u> in the sense that $\mathrm{Hom}_{Ab}(A, B)$ is just $\mathrm{Hom}_{Grp}(A, B)$ for any abelian groups A, B

<u>Slogan</u> superlatives in natural language
become
<u>universal properties</u> of objects in cats

viz., defns asserting that
certain test objects and/or morphisms
give rise to certain unique maps

e.g., least upper bound s.t. …
coarsest topology s.t. …
largest quotient s.t. …

correspond to defns involving universal properties

<u>Ex</u> the product top is the coarsest top s.t.
$pr_\alpha : \prod_\alpha X_\alpha \to X_\alpha$ is cts for all α

another characterization:

[draw] given cts maps $f_\alpha : Y \to X_\alpha$ for all α
get a unique cts map $f : Y \to \prod_\alpha X_\alpha$
s.t. $f_\alpha = pr_\alpha \circ f$

<u>Ex</u> the product of groups G_α is a group
$\prod_\alpha G_\alpha$ s.t.

given hom's $\varphi_\alpha : H \to G_\alpha$ for all α
get a unique hom $\varphi : H \to \prod_\alpha G_\alpha$
s.t. $\varphi_\alpha = pr_\alpha \circ \varphi$

reversing the diagram gives the defn of free product:

Ex the free product of the G_α is a group $\bigast_\alpha G_\alpha$ s.t.

[draw] given hom's $\psi_\alpha : G_\alpha \to K$ for all α
get a unique hom $\psi : \bigast_\alpha G_\alpha \to K$
s.t. $\psi_\alpha = \psi \circ i_\alpha$ [for incl.'s i_α]

Df in a general category C

objects described by the pr_α property are called
products $\prod_\alpha X_\alpha$
objects described by the i_α property are called
coproducts $\coprod_\alpha X_\alpha$

Ex if A_α are abelian groups
then their product in Ab is isomorphic
as a group to their product in Grp

[still left: describe the coproducts in Top and Ab]

Ex the coproduct of A_α in Ab is not iso to
their free product, i.e., coproduct in Grp

it's isomorphic to the subgroup

$$\bigoplus_\alpha A_\alpha \subseteq \prod_\alpha A_\alpha$$

of elts $(x_\alpha)_\alpha$ s.t. $x_\alpha = e_{A_\alpha}$ for all but finitely many α

Ex given top spaces X_α
what is $\coprod_\alpha X_\alpha$?

given cts maps $g_\alpha : X_\alpha \to Z$ for all α
need a unique cts map $g : \coprod_\alpha X_\alpha \to Z$
s.t. $g_\alpha = g \circ i_\alpha$

turns out to be the disjoint union: $\coprod = \cup$

Rem related notion of a pushout
$X_1 \cup_Y X_2$

in Grp, this is the amalgamated prod
$G_1 \ast_H G_2$, a quotient of $G_1 \ast G_2$
in Top, this is gluing X_1 and X_2 along Y,
a quotient of $X_1 \cup X_2$

(Munkres §72–73)

Thm let X be Hausdorff
let $i : A \to X$ be inclusion of
a closed path-connected subspace

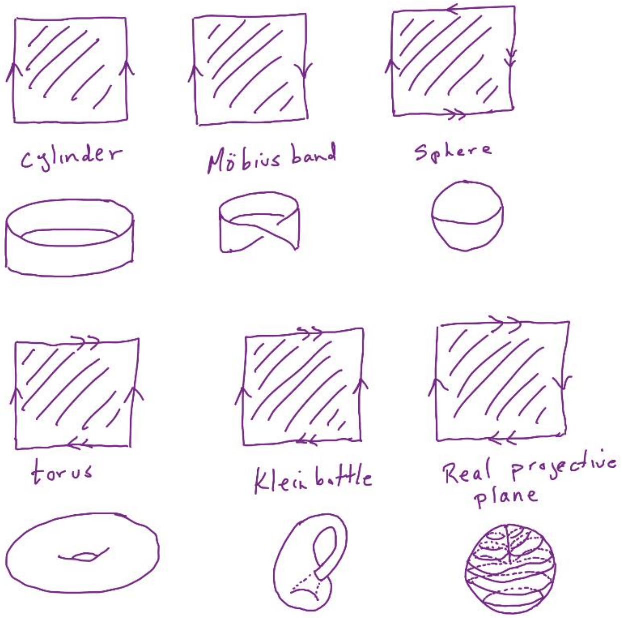
suppose there is cts $\zeta : D^2 \to X$ s.t.
ζ maps $\mathrm{Int}(D^2)$ bijectively onto $X - A$
ζ maps S^1 into A

let $a = \zeta(p)$ and $\eta = \zeta|_{S^1}$

then $\pi_1(X, a) = \pi_1(A, a)/\eta_*(\pi_1(S^1, p))$

[how to wield this thm efficiently?]

https://divisbyzero.com/2020/04/08/make-a-real-projective-plane-boys-surface-out-of-paper/



take X to be the quotient space of [0, 1]^2
resulting from the edge identifications

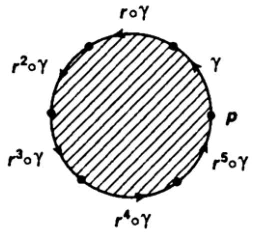
take A to be the image of the boundary square
take ζ to be a homeo from D^2 onto [0, 1]^2

for the torus and Klein bottle, A is a figure-eight

take a to be the path following “>”
take b to be the path following “>>”
then π_1(X) = <a, b | R>

where R is read off of a loop traversal of A

<u>Ex</u> π_1(n-fold dunce cap) = Z/nZ



Remarks on the Proof of Seifert–van Kampen

the full proof (Munkres §70) is tedious

recall our proof that $\pi_1(S^2)$ is trivial,
using open nbds of hemispheres intersecting in an annulus

that proof generalizes to a proof of the first part of Seifert–van Kampen:

if $i_j : U_j \to X$ for $j = 1, 2$ is an open inclusion
s.t. $X = U_1 \cup U_2$,
$U_1, U_2, U_1 \cap U_2$ are path-connected,
$x \in U_1 \cap U_2$,

then every elt of $\pi_1(X, x)$ is a (finite) iterated composition of elts of the images of

$$i_{1,*} : \pi_1(U_1, x) \to \pi_1(X, x),$$
$$i_{2,*} : \pi_1(U_2, x) \to \pi_1(X, x)$$