MATH 251: TOPOLOGY II

SPRING 2026 PRACTICE PROBLEMS

Contents

Note: All citations are to Munkres's textbook, *Topology*, 2nd Edition. When a problem statement has a proof in Munkres, try your best to find your own proof, before comparing with his.

0. Review of Topology I

Problem 1. Suppose that $A \subseteq X$. Recall that the *interior* of A in X is

$$\mathrm{Int}_X(A) := \{x \in X \mid x \in U \text{ for some } U \text{ open in } X \text{ such that } U \subseteq A\}$$

and the *closure* of A in X is

$$\mathrm{Cl}_X(A) := \{x \in X \mid x \in K \text{ for all } K \text{ closed in } X \text{ such that } K \supseteq A\}.$$

Show the identity $\mathrm{Cl}_X(A) = X \setminus \mathrm{Int}_X(X \setminus A)$.

Problem 2. Suppose that $A \subseteq Y \subseteq X$, where Y is a subspace of X.

(1) Show that $\mathrm{Int}_Y(A) \supseteq \mathrm{Int}_X(A)$.
(2) Give an example where $\mathrm{Int}_Y(A) \neq \mathrm{Int}_X(A)$. *Hint:* You can assume $Y = A$.

Problem 3. Let X be the real line $\mathbf{R}$ in its finite complement, or *cofinite*, topology. Show that every sequence of points in X converges to every point of X simultaneously. Deduce that X is not Hausdorff.

Problem 4. Show that any metric space is Hausdorff.

Problem 5. Show that for any integer $n \geq 1$, the analytic topology on $\mathbf{R}^n$ matches the product topology on $\mathbf{R} \times \cdots \times \mathbf{R}$, where there are n factors.

Problem 6. Let $p \colon \mathbf{R} \to S^1$ be the map

$$p(t) = (\cos(2\pi t), \sin(2\pi t)).$$

For any $a, b \in \mathbf{R}$, we define the *(open) arc* $J_{a,b} \subseteq S^1$ to be

$$J_{a,b} = \{p(t) \mid a < t < b\}.$$

Show that the collection of arcs $\{J_{a,b} \mid a, b \in \mathbf{R}\}$ satisfies the definition of a basis (Munkres page 78).

Problem 7. Show that the following topologies on S^1 are all the same:

- The topology generated by the basis $\{J_{a,b} \mid a, b \in \mathbf{R}\}$ in Problem 6.
- The subspace topology that S^1 inherits from its inclusion into $\mathbf{R}^2$.
- The quotient topology that S^1 inherits from the surjective map $p \colon \mathbf{R} \to S^1$.

Problem 8. Let $f_1, f_2, f_3 \colon A \to X$ be continuous maps. Suppose that φ is a homotopy from f_1 to f_2 and ψ is a homotopy from f_2 to f_3. Construct a homotopy from f_1 to f_3 explicitly in terms of φ and ψ.

Problem 9. Let $f, g \colon X \to Y$ and $F, G \colon Y \to Z$ be continuous. Show that

$$\text{if } f \sim g \text{ and } F \sim G, \quad \text{then } F \circ f \sim G \circ g.$$

Problem 10. Use the $f = g$ case of Problem 9 to show: If X, Y are nonempty and Y is contractible, then any continuous map from X into Y is nulhomotopic.

Problem 11. Suppose that:

- $f\colon X \to Y$ and $g\colon Y \to X$ define a homotopy equivalence between X and Y.
- $j\colon Y \to Z$ and $k\colon Z \to Y$ define a homotopy equivalence between Y and Z.

Show that $j \circ f\colon X \to Z$ and $g \circ k\colon Z \to X$ define a homotopy equivalence between X and Z.

Hint: You will need to show, for instance, that $g \circ k \circ j \circ f \sim \mathrm{id}_X$. But you know that $k \circ j \sim \mathrm{id}_Y$. Use Problem 9 to deduce the former statement from the latter.

Problem 12. In the setup of Problem 8, suppose that

$$A = [0,1] \text{ and } f_1(0) = f_2(0) = f_3(0) \text{ and } f_1(1) = f_2(1) = f_3(1).$$

Show that in this case, if φ and ψ are <u>path</u> homotopies, then we can choose the solution of (1) to be a <u>path</u> homotopy as well.

Problem 13. Show that if X is contractible—say, with id_X homotopic to the constant map at a point $x \in X$—then any loop in X based at x is path homotopic to the *constant loop* e_x defined by $e_x(s) = x$ for all $s \in [0,1]$.

1. Exam 1 Topics

Exam 1 will be a closed-book exam. It will be held in-class, 12:40–2:00 pm, on January 26.

You should know by heart the definitions of the following terms. You should be able to state multiple examples of each (with justification), and in some cases, non-examples.

- Munkres §13. Bases
- §18. Continuous maps
- §15. The product topology (only on finite direct products)
- §16. Subspaces
- §22. Quotient spaces
- §17. Interiors and closures
- §23–24. Separations, connectedness
- §23. Paths, path-connectedness
- §26–27 Compactness
- §51. Homotopies, path homotopies
- §58. Homotopy equivalences

2. Fundamental Groups and Covering Spaces

3. Separation Theorems in the Plane

4. Exam 2 Topics

5. Simplicial Complexes and Surfaces

6. Exam 3 Topics

7. Homology