Warmup suppose $T : F^2 \to F^2$ sends

$v1 = \begin{matrix} 1 \\ 1 \end{matrix}$ to $\begin{matrix} 2 \\ 2 \end{matrix}$ $v2 = \begin{matrix} -1 \\ 1 \end{matrix}$ to $\begin{matrix} 1 \\ -1 \end{matrix}$

matrix of T wrt std basis (e1, e2) of F^2?

1) matrix wrt (v1, v2): $M = \begin{matrix} 2 & 0 \\ 0 & -1 \end{matrix}$

2) $v1 = e1 + e2$ $B = \begin{matrix} 1 & -1 \\ 1 & 1 \end{matrix}$
$v2 = -e1 + e2$

3) $\det(B) = 2$ $[B^{(-1)} = \begin{matrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{matrix}]$
$\text{adj}(B) = \begin{matrix} 1 & 1 \\ -1 & 1 \end{matrix}$

$BMB^{(-1)} = (1/2)\, B\, M\, \text{adj}(B)$
$= (1/2) \begin{matrix} 2 & 1 \\ 2 & -1 \end{matrix} \quad \begin{matrix} 1 & 1 \\ -1 & 1 \end{matrix}$
$= (1/2) \begin{matrix} 1 & 3 \\ 3 & 1 \end{matrix}$
$= \begin{matrix} 1/2 & 3/2 \\ 3/2 & 1/2 \end{matrix}$ [works]

Df given a linear op $T : V \to V$
a subspace W sub V is T-stable iff
w in W implies Tw in W

Ex above: Fv1, Fv2 are T-stable lines
Fe1, Fe2 are not T-stable

matrix wrt (v1, v2) easier to understand

Ex for any linear op T : V to V,
{**0**} and V are are [trivially] T-stable

Ex suppose that wrt some basis,
T : F^3 to F^3 has the matrix

$$\begin{matrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & * \end{matrix}$$

nontrivial T-stable subspaces? {(x, y, 0) | x, y}
{(0, 0, z) | z}

when a matrix for T has nontrivial diagonal blocks,
the blocks indicate nontrivial T-stable subspaces
[why care?]

when V = sum_i W_i for T-stable W_i,
studying T reduces to studying T|_{W_i} for all i

best situation: each W_i is a line
[what means?]
in this case: there is a basis (w_i)_i s.t.
W_i = Fw_i,
Tw_i = a_iw_i for some i
matrix of T wrt w_i is diagonal

Df for any linear op T : V to V

an eigenline of T is a T-stable line [dim-1 sub.]
an eigenvector of T is v in V s.t. Fv is an eigenline
(which forces v ≠ **0**)
if Tv = λv, then we call λ the eigenvalue of the line

Ex $\quad T : F^2$ to F^2 given by the matrix

$$\begin{matrix} 1/2 & 3/2 \\ 3/2 & 1/2 \end{matrix}$$

wrt the std basis

$\left\{\begin{matrix} (x \\ x) \end{matrix}\right\}$ is an eigenline with eigenvalue 2

$\left\{\begin{matrix} (x \\ -x) \end{matrix}\right\}$ is an eigenline with eigenvalue –1

Ex $\quad P : F^2$ to F^2 given by the matrix

$$\begin{matrix} 0 & 0 \\ 0 & 1 \end{matrix}$$

wrt the std basis

[does P have any eigenvectors?]

$$\begin{matrix} \lambda x \\ \lambda y \end{matrix} = \begin{matrix} 0 & 0 \\ 0 & 1 \end{matrix} \quad \begin{matrix} x \\ y \end{matrix} = \begin{matrix} 0 \\ y \end{matrix}$$

[take $\lambda = 0$:] Fe1 is an eigenline with eigenvalue 0
[take $\lambda \neq 0$:] Fe2 is an eigenline with eigenvalue 1

Ex $\quad N : F^2$ to F^2 given by the matrix

$$\begin{matrix} 0 & 0 \\ 1 & 0 \end{matrix}$$

wrt the std basis

$$\begin{matrix} \lambda x \\ \lambda y \end{matrix} = \begin{matrix} 0 & 0 \\ 1 & 0 \end{matrix} \quad \begin{matrix} x \\ y \end{matrix} = \begin{matrix} 0 \\ x \end{matrix}$$

[regardless of λ:] $x = 0$
Fe2 is the only eigenline with eigenvalue 0
[same regardless of whether F = R or F = C]

Ex $\quad H : \mathbb{R}^2 \to \mathbb{R}^2$ given by the matrix $\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ wrt the std basis

$$\begin{pmatrix} \lambda x \\ \lambda y \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x - y \\ x + y \end{pmatrix}$$

messy to solve…

notice: $(1/\sqrt{2})\, H = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$

$= \begin{pmatrix} \cos(\pi/4) & -\sin(\pi/4) \\ \sin(\pi/4) & \cos(\pi/4) \end{pmatrix}$

so H is the composition of: rotate by $\pi/4$
scale by $\sqrt{2}$

no H-stable lines through **0**

compare to $H' : \mathbb{C}^2 \to \mathbb{C}^2$ given by $\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$

claim: $1 + i$ and $1 - i$ are eigenvalues

$(1 + i)x = x - y$
$(1 + i)y = x + y$ imply $ix = -y$ and $iy = x$

so $\{\begin{pmatrix} x \\ ix \end{pmatrix}\}$ is an eigenline with eigenvalue $1 + i$

similarly

$\{\begin{pmatrix} x \\ -ix \end{pmatrix}\}$ is an eigenline with eigenvalue $1 - i$

Rem — C^2 is iso to the complex'n of R^2

$$\begin{tikzcd} (R^2)_C \arrow[r, "H_C"] \arrow[d, phantom, "="] & (R^2)_C \\ C^2 \arrow[r, "H'"'] & C^2 \end{tikzcd}$$

Moral — choice of R vs C affects eigenstuff

even if $T : R^n$ to R^n has no eigenvals,
$T_C : C^n$ to C^n may have eigenvals

[later: we will show it has at least one]

Summary

if W sub V is stable under $T : V$ to V
then we can study T in terms of smaller op's $T|_W$

nicer when	W is a line
nicest when	V is a sum of eigenlines

over R	T may have no eigenlines
over C	T will have some eigenline [later], but V need not be sum(eigenlines)

Q for next time

let $D : F[x]$ to $F[x]$ be $D(p) = dp/dx$
what are the D-stable linear subspaces?