MATH 250: TOPOLOGY I PROBLEM SET #5

FALL 2025

Due Friday, November 14. Please attempt all of the problems. Six of them will be graded. You may consult books, papers, and websites as long as you cite all sources and write up your solutions in your own words.

Problem 1. Show that if $f, f' : X \to Y$ are homotopic continuous maps, and similarly, $g, g' : Y \to Z$ are homotopic, then $g \circ f$ and $g' \circ f'$ are homotopic. (In class, we discussed a similar result for path homotopy.)

Problem 2. A subset $A \subseteq \mathbf{R}^n$ is *star convex* if and only if there is some point $a_0 \in A$ such that the line segment between a_0 and any other point of A is contained in A.

(1) Show that if A is star convex, then any loop in A based at a_0 is path homotopic to the constant loop. Thus, A is *simply-connected*: $\pi_1(A, a_0)$ is trivial.
(2) Give a star convex subset of $\mathbf{R}^2$ that is not convex.

Problem 3. Let $s : A \to X$ and $r : X \to A$ be continuous maps such that $r \circ s$ is the identity map of A. Let $a \in A$ and $x = s(a) \in X$. Show that

$$\begin{aligned} s_* : \pi_1(A, a) \to \pi_1(X, x) &\text{ is injective} \\ \text{and} \quad r_* : \pi_1(X, x) \to \pi_1(A, a) &\text{ is surjective.} \end{aligned}$$

Problem 4. Let $X \subseteq \mathbf{R}^n$ be a subspace, and let $f : X \to Y$ be a continuous map. Suppose that $f = g|_X$ for some continuous map $g : \mathbf{R}^n \to Y$. Show that for any $x \in X$, the homomorphism

$$f_* : \pi_1(X, x) \to \pi_1(Y, f(x))$$

is *trivial*: It sends every element to the identity element in the target. *Hint:* $f = g \circ i$, where $i : X \to \mathbf{R}^n$ is the inclusion.

Problem 5. Let $x_0, x_1 \in X$. Recall that if $\alpha : [0, 1] \to X$ is a path from x_0 to x_1, and $\bar{\alpha}(s) = \alpha(1 - s)$ is the reverse path, then there is a homomorphism

$$\hat{\alpha} : \pi_1(X, x_0) \to \pi_1(X, x_1) \quad \text{defined by } \hat{\alpha}([\gamma]) = [\bar{\alpha} * \gamma * \alpha].$$

(1) Show that $\hat{\bar{\alpha}}$ is a two-sided inverse of $\hat{\alpha}$, and thus, both maps are isomorphisms. (This is written out in Munkres, but we want you to work through the details yourself.)
(2) Show that $\hat{\alpha}$ only depends on the path-homotopy class $[\alpha]$. That is, if β is path-homotopic to α, then $\hat{\alpha}$ and $\hat{\beta}$ are the same homomorphism.

Problem 6. Recall that X is *contractible* if and only if it has some point o such that the identity map on X is homotopic to the constant map at o: that is, the map $r_o : X \to X$ given by $r_o(x) = o$. Show that X is contractible if and only if X is homotopy equivalent to a one-point space.

Problem 7. Recall that the circle S^1 is (homeomorphic to) a quotient space: $S^1 = [0,1]/{\sim}$, where $0 \sim 1$ and there are no other identifications between distinct points of $[0,1]$. Similarly, we define the *Möbius band* to be the quotient space

$$\mathcal{M} = ([0,1] \times [0,1])/\overset{\bullet}{\sim} \quad (\texttt{\textbackslash overset\{\textbackslash bullet\}\{\textbackslash sim\}}),$$

where $(0,y) \overset{\bullet}{\sim} (1, 1-y)$ for all y and there are no other identifications between distinct points of $[0,1] \times [0,1]$.

Write down an explicit homotopy equivalence between S^1 and $\mathcal{M}$: *i.e.*, a pair of maps $f : S^1 \to \mathcal{M}$ and $g : \mathcal{M} \to S^1$ such that $g \circ f$ is homotopic to id_{S^1} and $f \circ g$ is homotopic to $\mathrm{id}_{\mathcal{M}}$. You do not need to check the homotopy conditions.

Problem 8. Classify the following letter shapes up to: (1) homeomorphism; (2) homotopy equivalence.

$$\mathsf{A},\ \mathsf{B},\ \mathsf{C},\ \mathsf{D},\ \mathsf{E},\ \mathsf{F},\ \mathsf{G},\ \mathsf{H},\ \mathsf{I},\ \mathsf{J}.$$

You do not need to write down explicit homeomorphisms or homotopy equivalences. Nonetheless, provide some informal reasoning for your classification.

Problem 9. Let G be a group equipped with a topology in which the

	group law $\mu_G : G \times G \to G$	defined by $\mu_G(g,h) = gh$
and	inversion $\iota_G : G \to G$	defined by $\iota_G(g) = g^{-1}$

are continuous. Such a structure is called a *topological group*.

(1) Show that $\mathbf{R}$ forms a topological group with respect to the addition law and the analytic topology.

(2) Show that $\mathbf{S} := \mathbf{R}/\mathbf{Z}$, as a quotient group and quotient space of $\mathbf{R}$, also forms a topological group. You may assume that the product topology on $\mathbf{S} \times \mathbf{S}$ matches its topology as a quotient space of $\mathbf{R} \times \mathbf{R}$.

Hint: For continuity of $\mu_{\mathbf{S}}$, we want to show that if $U \subseteq \mathbf{S}$ is open and $V = \mu_{\mathbf{S}}^{-1}(U) \subseteq \mathbf{S} \times \mathbf{S}$, then V is also open. The latter happens if and only if $(p \times p)^{-1}(V) \subseteq \mathbf{R} \times \mathbf{R}$ is open. Now observe that the diagram

$$\begin{tikzcd} \mathbf{R} \times \mathbf{R} \arrow[r, "\mu_{\mathbf{R}}"] \arrow[d, "p \times p"'] & \mathbf{R} \arrow[d, "p"] \\ \mathbf{S} \times \mathbf{S} \arrow[r, "\mu_{\mathbf{S}}"] & \mathbf{S} \end{tikzcd}$$

is *commutative*: that is, $\mu_{\mathbf{S}} \circ (p \times p) = p \circ \mu_{\mathbf{R}}$.

(In fact, $\mathbf{S}$ is homeomorphic as a space to the circle S^1.)

Problem 10. Show that:

(1) Any open subgroup of G is closed.
(2) Any closed subgroup of G of finite index is open (hence clopen).
(3) If G is compact, then a subgroup is open if and only if it is closed of finite index.

Observe that $\mathbf{S} \simeq S^1$ is compact and connected, since there is a continuous surjective map from $[0, 1]$ to S^1. Using (3), deduce that:

(4) $\mathbf{S}$ contains no open subgroups.