Exam 1 in class on 1/26 [no fundamental groups]

Today homotopy equivalences
paths, path homotopies

(Munkres §54)

homotopy : equiv rel on cts maps
homotopy equivalence : equiv rel on spaces

Df a homotopy equivalence from X to Y is a pair of cts maps ($f : X$ to Y, $g : Y$ to X)

s.t. $g \circ f \sim id_X$ and $f \circ g \sim id_Y$

in this case, write $X \sim Y$

Ex if $f : X$ to Y is a homeomorphism then (f, f^{-1}) is a homotopy equiv

Ex R and R^2 not homeomorphic [why?]
but ($f : R$ to R^2, $g : R^2$ to R) def by

$f(x) = (x, 0)$ and $g(x, y) = x$

is a homotopy equivalence

$(g \circ f)(x) = x = id_R(x)$, so $g \circ f = id_R$

$(f \circ g)(x, y) = (x, 0)$, so $\varphi((x, y), t) = (x, ty)$ is a homotopy from $f \circ g$ to id_{R^2}

[draw]

in general, if $g \circ f = id$, then

f is called a <u>section</u>
g is called a <u>retraction</u>

<u>Lem</u> if $g \circ f = id$
then f is injective and g is surjective

a homotopy equivalence (f, g) is called a <u>deformation retraction</u> when f is an inclusion and $g \circ f = id$ [so that g is a retraction]

like in the example above

<u>Ex</u> there is a deformation retraction from $R^2 - \{(0, 0)\}$ onto S^1 [draw]

<u>Ex</u> no homotopy equivalence between R^2 and $R^2 - \{(0, 0)\}$ [hard]

<u>Prob</u> if $X \sim Y$ via (f, g) and $Y \sim Z$ via (j, k)
then $X \sim Z$ via $(j \circ f, g \circ k)$ [diagram]

need to prove: $j \circ f \sim id_X$ and $g \circ k \sim id_Z$

<u>Cor</u> $\sim$ is an equiv relation on spaces

[<u>Pf</u> $X \sim X$ via (id_X, id_X)
$X \sim Y$ via (f, g) iff $Y \sim X$ via (g, f)
$X \sim Y$ and $Y \sim Z$ imply $X \sim Z$ by prob]

Rem if $X \sim Y$, then X is connected iff Y is connected

Q true that X compact iff Y compact? [no]

(Munkres §51) [recall:] a path in X is a cts map $\gamma : [0, 1]$ to X

we say γ starts from $\gamma(0)$ and ends at $\gamma(1)$

Df X is path-connected iff, for any x, y in X, there is a path from x to y

the path components of X are the maximal path-connected subspaces of X

[compare to connectedness and connected components]

Thm path-connected implies connected

[Pf suppose U, V is a separation of X
pick x in U and y in V
pick a path γ in X from x to y

then $\gamma^{-1}(U), \gamma^{-1}(V)$ is a separation of $[0, 1]$

but $[0, 1]$ is connected (Munkres §24)]

Cor each path component of X is contained in some connected component of X

Rem if $X \sim Y$, then X is path-connected iff Y is path-connected

Df given paths $\gamma, \gamma' : [0, 1]$ to X s.t. $\gamma(0) = \gamma'(0)$ and $\gamma(1) = \gamma'(1)$

a path homotopy from γ to γ' is a homotopy

$$\varphi : [0, 1] \times [0, 1] \text{ to } X \text{ from } \gamma \text{ to } \gamma'$$

s.t. for all times t,

$$\varphi(0, t) = \gamma(0) = \gamma'(0) \text{ and } \varphi(1, t) = \gamma(1) = \gamma'(1)$$

in this case, write $\gamma \sim_p \gamma'$

Prob for any space X, points x, y in X, paths γ_1, γ_2, γ_3 from x to y,

given path homotopies φ from γ_1 to γ_2,
ψ from γ_2 to γ_3,

find path homotopy from γ_1 to γ_3 in terms of φ, ψ [related to previous prob]

so $\sim_p$ is an equiv rel on paths from x to y

[note:] $\gamma \sim_p \gamma'$ implies $\gamma \sim \gamma'$