Recall      $\pi\_1$(one-holed solid donut) $\simeq \mathbb{Z}$

         one-holed solid donut    $= S^1 \times (a, b)$
                                    $\sim S^1$

Q          what is $\pi\_1$(two-holed solid donut)?

         two-hold solid donut = figure-eight

Df        given      nonempty spaces X, Y
                         x in X and y in Y,

the <u>wedge sum</u> of (X, x) and (Y, y) is the quotient

         $X \vee Y = (X \sqcup Y)/\sim$    where $x \sim y$
                                 but no other $\sim$'s

thus the figure-eight is $S^1 \vee S^1$
[here the basepoints don't matter]

Q          in general, how is $\pi\_1(X \vee Y)$ related
                to $\pi\_1(X, x)$ and $\pi\_1(Y, y)$?

Ex        consider "generating" loops $\gamma\_1, \gamma\_2$ for
                the two circles in the figure-eight

                [draw]

then $\gamma\_1 * \gamma\_2 \neq \gamma\_2 * \gamma\_1$ [though hard to prove]

Ex        by contrast, consider $\beta, \gamma$ generating
                $\pi\_1(S^1 \times S^1) \simeq \mathbb{Z}^2$

then β, γ correspond to (1, 0), (0, 1) in $\mathbb{Z}^2$
so β * γ = γ * β

another (independent) proof:

$$S^1 = [0, 1]/\sim$$
similarly, $S^1 \times S^1 = ([0, 1] \times [0, 1])/\sim$

[draw]

can draw β and γ in terms of $[0, 1] \times [0, 1]$

then draw β * γ and γ * β

[draw]

Thm    $\pi_1(X \times Y)$ is iso to
       the direct product of $\pi_1(X)$ and $\pi_1(Y)$

Df     the direct product of $\{G_i\}_i$ is
       a group $\prod_i G_i$

(Munkres §71)

Thm    if the gluing pt of $X \vee Y$ is a def retract
              of open nbd's in X and Y

then $\pi_1(X \vee Y)$ is iso to the free product
of $\pi_1(X)$ and $\pi_1(Y)$

Df     the free product of $\{G_i\}_i$ is
       a group $\bigast_i G_i$ [ \bigast ]

where elts are words over $\mathrm{coprod}_i G_i$,
the law is concatenation mod rels of each $G_i$

Ex  $\pi_1(\text{two-holed solid donut})$
   $\simeq$ $\pi_1(S^1 \vee S^1)$
   $\simeq$ $\mathbb{Z} \ast \mathbb{Z}$

$\mathbb{Z} \ast \mathbb{Z} = \{ \text{words in } a, b, a^{-1}, b^{-1} \}$

Df  the free group on $n$ letters is
   $F_n = \mathbb{Z} \ast \dots \ast \mathbb{Z}$
   with $n$ copies of $\mathbb{Z}$

Ex  $\pi_1(S^1 \vee \dots \vee S^1) \simeq F_n$
   when there are $n$ copies of $S^1$

Analogy

| | | |
|---|---|---|
| products | Top | product space $\mathrm{prod}_i X_i$ |
| | Grp | direct product $\mathrm{prod}_i G_i$ |
| | Mod | direct product $\mathrm{prod}_i M_i$ |
| coproducts | Top | wedge sum $\mathrm{bigvee}_i X_i$ |
| | Grp | free product $\mathrm{bigast}_i G_i$ |
| | Mod | direct sum $\mathrm{bigoplus}_i M_i$ |

Df  given spaces $A, X, Y$,
     maps $i : A \to X$ and $j : A \to Y$,

the gluing of $X$ and $Y$ along $A$ is the quotient

  $(X \sqcup Y)/\sim$ where $i(a) \sim j(a)$ for all $a$ in $A$

gluing generalizes wedge sum

Ex        let A = {a, b} and X = [0, 1] and Y = [1, 2]
            let i(a) = 0, i(b) = 1, j(a) = 1, j(b) = 2

            [draw]

the gluing is homeo to S^1

Ex        let A = S^1 × (−ε, ε)
            let X, Y be two copies of D^2
            let i, j be the boundary inclusions

            [draw]

the gluing is homeo to S^2
Q        how is $\pi_1$ of the gluing related to
           the $\pi_1$'s of A, X, Y?