

take $X = \mathbb{R}$ and $A = [0, \infty)$ analytic

U open in A iff $U = A \cap V$ for some V open in X

Claim $[0, b)$ open in A [but not in X]

Pf $[0, b) = A \cap (-1, b)$

Claim if $a > 0$, then $[a, b)$ not open in A

Pf must show: no V open in X s.t.
 $[a, b) = A \cap V$

if V exists, then there is $\delta > 0$ s.t. $B(a, \delta) \subset V$
after shrinking δ , can assume $B(a, \delta) \subset A$
but then $B(a, \delta) \subset A \cap V = [a, b)$

Warmup X topological space, x in X

Df $Y \subset X$ is a neighborhood of x iff
 $x \in U \subset Y$ for some open $U \subset X$

Df the interior of $Y \subset X$ is the set

$\text{Int}(Y) = \bigcup \{U \subset Y, \text{ open in } X\}$
 $= \{x \in X \mid Y \text{ is a nbd of } x\}$ [why?]

in the analytic topology on \mathbb{R} , compute:

$\text{Int}([0, 1)) = (0, 1)$

$\text{Int}(\{0, 1, 2, \dots\}) = \emptyset$

$\text{Int}(\mathbb{R} - \{0, 1, 2, \dots\}) = \mathbb{R} - \{0, 1, 2, \dots\}$

$\text{Int}(\{1/n \mid n \text{ is a positive integer}\}) = \emptyset$

[draw picture]

Df $Z \subset X$ is closed iff $X - Z$ is open in X

Df the closure of $Y \subset X$ is the set

$$\begin{aligned}\bar{Y} = \text{Cl}(Y) &= \bigcap \{Z \supset Y, \text{ closed in } X\} \\ &= X - \bigcup \{V \subset X - Y, \text{ open in } X\} \\ &= X - \text{Int}(X - Y)\end{aligned}$$

in analytic \mathbb{R} , compute:

$$\text{Cl}([0, 1)) = [0, 1]$$

$$\text{Cl}(\{0, 1, 2, \dots\}) = \{0, 1, 2, \dots\}$$

$$\text{Cl}(\mathbb{R} - \{0, 1, 2, \dots\}) = \mathbb{R}$$

$$\text{Cl}(\{1/n \mid n \text{ is a positive integer}\}) = \{1/n\}_n \cup \{0\}$$

(Munkres §20–21) recall from real analysis:

Df a metric on a set X is a function
 $d : X \times X$ to $[0, \infty)$

s.t., for all x, y, z in X ,

$$1) d(x, y) = 0 \text{ implies } x = y$$

$$2) d(x, y) = d(y, x)$$

$$3) d(x, y) + d(y, z) \geq d(x, z)$$

given $\delta > 0$, let $B_d(x, \delta) = \{y \in X \mid d(x, y) < \delta\}$

[note that $x \in B_d(x, \delta)$, because $d(x, x) = 0 < \delta$]

Df the metric topology on X induced by d :

U is open in the metric topology iff
for all x in U , there is a $\delta > 0$ s.t. $B_d(x, \delta) \subset U$

Idea metric topology on X generalizes
analytic topology on \mathbb{R}^n

Thm the metric topology really is a topology

Pf exactly like the proof that
the analytic topology is a topology

[so how much weirder can it be?]

Ex in any X : the discrete metric defined by

$$d(x, x) = 0, \quad d(x, y) = 1 \text{ when } x \neq y$$

1) and 2) easy

3) [how many cases to check? 5 but can combine]

if $x = z$:

$$d(x, y) + d(y, z) \geq 0 = d(x, z)$$

[because $d(-, -) \geq 0$]

if $x \neq z$:

either $y \neq x$ or $y \neq z$

$$\text{so } d(x, y) + d(y, z) \geq 1 = d(x, z)$$

observe $B_d(x, 1) = \{x\}$ for all x . thus:

Prop metric topology from the discrete metric is the discrete topology

Df we say a topology or topological space is metrizable iff
the topology is induced by some metric

sometimes, different metrics induce the same topology

Lem suppose d induces T on X ,
 d' induces T' on X

then T' is finer than T iff
for all x in X and $\varepsilon > 0$, there is $\delta > 0$ s.t.
 $B_{d'}(x, \delta) \subset B_d(x, \varepsilon)$.

Pf exercise (Munkres Lem 20.2)

Ex [picture of $B_d(x, \delta)$ versus $B_\rho(x, \delta)$]

euclidean metric:

$$d(x, y) = \sqrt{(x_1 - y_1)^2 + \dots + (x_n - y_n)^2}$$

square metric:

$$\rho(x, y) = \max(|x_1 - y_1|, \dots, |x_n - y_n|)$$

observe:

$$\begin{aligned} d(x, y) &\leq \sqrt{n \max_i (x_i - y_i)^2} \\ &= \sqrt{n} \rho(x, y) \end{aligned}$$

$$\begin{aligned} \rho(x, y) &= \sqrt{\max_i |x_i - y_i|^2} \\ &\leq d(x, y) \end{aligned}$$

shows:

$$B_\rho(x, \varepsilon/\sqrt{n}) \subset B_d(x, \varepsilon)$$

$$B_d(x, \varepsilon) \subset B_\rho(x, \varepsilon)$$

[in general:]

Df metrics d, d' are called equivalent iff
there exist $A, B > 0$ s.t.
 $d(x, y) \leq A d'(x, y)$ and $d'(x, y) \leq B d(x, y)$
uniformly in x and y

Lem if two metrics are equivalent,
then their metric topologies coincide

Cor Euclidean and square metrics
both induce the analytic topology on \mathbb{R}^n

Rem converse is false: given a metric d , let

$$d'(x, y) = d(x, y)/(1 + d(x, y))$$

then 1) d' is still a metric

2) metric topologies for d, d' coincide

3) d and d' need not be equivalent

[reason: equivalence involves uniformity in x, y]

Ex $\mathbb{R}^\omega = \{(x_1, x_2, \dots) \mid x_i \text{'s in } \mathbb{R}\}$
 $\mathbb{R}^\infty = \{(x_1, x_2, \dots) \mid x_i \text{ eventually } 0\}$

Euclidean and square metrics still work on \mathbb{R}^∞ ,
but not on \mathbb{R}^ω

but $u(x, y) = \min \{1, \sup_i |x_i - y_i|\}$ works...