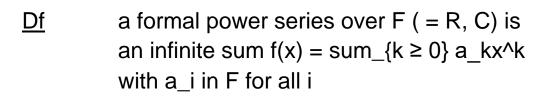
<u>Warmup</u>	let D : $F[x]$ to $F[x]$ be $D(p) = dp/dx$
what are some D-stable linear subspaces?	

e.g., for all n,
P n =
$$\{p \mid p = 0 \text{ or deg}(p) \le n\}$$
 is D-stable

non-example:

$$\{p \mid p(3) = 0\}$$
 is not D-stable [why?]



the <u>formal derivative</u> on F[[x]] is the F-linear operator D def by

$$D(sum_k a_kx^k) = sum_k ka_kx^{k-1}$$

$$\underline{Q}$$
 does D have eigenvectors in $F[[x]]$?

yes: $exp(\alpha x)$ where $exp(x) = sum_k (1/k!)x^k$

(Axler §5C) recap: fix T: V to V

- if $V = R^n$, then T might have no eigenvals [example? any rotation of $\theta \neq 0$, π radians]
- if V = F[x], then T might have no eigenvals [example? multiplication by x]
- if $V = \{0\}$, then T has no eigenvals by defin

[stated last time:]

<u>Thm</u> if F = C and V is fin. dim. and not {**0**} then T must have <u>some</u> eigenval

what does this fact say in terms of matrices?

Lem if V is fin. dim. and T has an eigenline then T has a matrix of the form

* *

0 * ... *

... ...

0 * ... *

Pf suppose v is the eigenvector for T
set v_1 = v
extend to a basis (v_i)_i for V

cor if F = C and n > 0
then any n x n matrix is conjugate to
one of this form

[something stronger holds:]
Thm if F = C and V is fin. dim., th

if F = C and V is fin. dim., then any T has an upper-triangular matrix:

* * ... *
0 * ... *
0
0 ... 0 *

idea: induct on dim V if dim V = 0 then done else T has some eigenvector v with eigenvalue λ [what next?] Tv = λ v means v in ker(T – λ) so dim ker(T – λ) > 0 so dim im(T – λ) < dim V so want to apply inductive hypothesis to im(T – λ)

Stability Lem for any T: V to V and p(z) in C[z], im(p(T)) is T-stable

 $\frac{Pf}{then w = p(T) v for some v in V}$ so Tw = T(p(T) v) = p(T)(T v) in im(p(T))

to see the last equality:

zp(z) = p(z)z as polynomials
so Tp(T) = p(T)T as operators on V
[even though general operators don't commute!]

<u>Pf of Thm</u> let $n = \dim V$; can assume n > 0

suppose $Tv = \lambda v$ where $v \neq 0$ let $W = im(T - \lambda)$

by lem, W is T-stable and dim W < n by inductive hypothesis, have ordered basis for W making T|_W triangular: say, (w 1, ..., w m)

extend ordered basis from W to V: say (w 1, ..., w m, v 1, ..., v \(\ext{\ell} \) claim that T is triangular wrt this extended basis[!] suffices to check Tv i's: for all i,

$$Tv_i = (T - \lambda)v_i + \lambda v_i$$
 in W + Fv_i

[draw matrix]

Cor any square matrix is conjugate to an upper-triangular matrix

let f: Mat 2 to F be a function def by Cor a polynomial in matrix coords

i.e. f = p(x11, x12, x21, x22)x11 x12 x21 x22 if f is conj-invariant

then f is a polynomial in tr and det

Pf Sketch Tri_2 = {upper-triangular matrices} Diag 2 = {diagonal matrices}

by thm, f is uniq. determ. by f[_{Tri_2}] observe: $f|_{Tri_2} = q(x11, x12, x22)$ for some q $tr|_{Tri_2} = x11 + x22$ $det|_{Tri_2} = x11 x22$

want $f|_{Tri_2}$ to be a poly in x11 + x22, x11 x22

```
Claim 1) q is indep of x12
          so q is uniq. determ. by q[_{Diag_2}]
Claim 2) q|_{Diag_2} invariant for x11 \leftrightarrow x22
to finish, use Viète's Thm:
     any poly in X, Y invariant under X \leftrightarrow Y
     is a poly in X + Y and XY
     [look up "elementary symmetric functions"]
shows q[_{Diag_2}] is a poly in x11 + x22, x11 x22
              x11 x12 1/a
                                 = x11 aa x12
   a
          1/a
                    x22
                                             x22
```

so $q(x11, x12, x22) = q(x11, a^2 x12, x22)$ for all a

[exercise:] forces q to be indep of x12

<u>Q</u> how block-diagonal can we make T?

next week:

Thm if W fin. dim. and S: W to W nilpotent then S has a matrix where the only nonzero entries are 1's on the "super-diagonal"

problem: in general, T = T' + T''

Q basis where T' is super-diagonal, T" is diagonal simultaneously?