

(Axler §2B–2C)

span of  $\{v_i\}_i$

[subspace of all linear combo's  $\sum_i a_{iv_i}$ ]

linear independence of  $\{v_i\}_i$

[each elt of span has a unique expr.  $\sum_i a_{iv_i}$ ]

basis for  $V$

[set that spans  $V$  and is linearly independent]

Steinitz Thm

if  $\{v_i\}_i$  spans and  $\{e_j\}_j$  is lin. indep.  
then  $\#$  of  $v_i$ 's  $\leq \#$  of  $e_j$ 's

Cor if  $V$  has a finite basis,  
then all bases of  $V$  have the same size

dimension of  $V$  is the common size of its bases

Ex  $V = \{(a, b, c, d) \mid a + b + c + d = 0\}$

what is  $\dim V$ ? [3, as  $d = -a - b - c$  always works]

what is a basis for  $V$ ?

$$e_1 = (1, -1, 0, 0)$$

$$e_2 = (0, 1, -1, 0)$$

$$e_3 = (0, 0, 1, -1)$$

why?  $ae_1 + (a + b)e_2 + (a + b + c)e_3$

Ex in  $V$ , consider

$$\begin{aligned}s &= (1, -1, 0, 0) \\ t &= (1, 0, -1, 0) \\ u &= (1, 0, 0, -1) \\ v &= (0, 1, 0, -1) \\ \mathbf{0} &= (0, 0, 0, 0)\end{aligned}$$

which subsets are lin. indep.? [need  $\leq 3$  vec's]

$\emptyset, \{s\}, \{t\}, \{u\}, \{v\}, \{s, t\},$   
 $\{s, u\}, \{s, v\}, \{t, u\}, \{t, v\},$   
 $\{s, t, u\}, \{s, t, v\}, \{t, u, v\}$

which subsets span  $V$ ? [need  $\geq 3$  vec's]

$\{s, t, u\}, \{s, t, v\}, \{t, u, v\},$   
 $\{s, t, u, v\}, \{s, t, u, \mathbf{0}\}, \{s, t, v, \mathbf{0}\}, \{t, u, v, \mathbf{0}\},$   
 $\{s, t, u, v, \mathbf{0}\}$

Ex recall  $F[x] = \{\text{polynomials in } x \text{ over } F\}$

is  $Q = \{p \text{ in } F[x] \mid p(-1) = p(1)\}$  a linear subspace?  
 [yes]

what are some elements of  $Q$ ?

$0$   
 polynomials of even degree  
 $x - x^3, x - x^7, \dots$

is  $Q$  finite-dimensional? [no]

how about  $Q' = \{p \text{ in } Q \mid p = 0 \text{ or } \deg p \leq 4\}$  [yes]  
 [what is the degree of the zero polynomial?]

what is  $\dim Q'$ ? [tricky]

Thm if  $V$  is finite-dim,  $U \subseteq V$  is a linear sub,  
then  $U$  is finite-dim and  $\dim U \leq \dim V$

would like to pick basis for  $V$ , then restrict to  $U$   
but this doesn't work!

Ex  $V = F^2$  and  $U = \{(x, x) \mid x \in F\}$   
 $V$  has basis  $\{(1, 0), (0, 1)\}$ ,  
but no subset is a basis for  $U$

Lem suppose  $v_1, \dots, v_n, w \in V$   
s.t.  $\{v_1, \dots, v_n\}$  lin. indep.  
if  $w \notin \text{span}(v_1, \dots, v_n)$ ,  
then  $\{v_1, \dots, v_n, w\}$  lin. indep.

Pf suppose  $\sum_i a_i v_i + bw = \mathbf{0}$   
if  $b \neq 0$  then  $w \in \text{span}(v_1, \dots, v_n)$   
so  $b = 0$   
by lin. indep. of  $v_i$ 's, get  $a_i = 0$  for all  $i$

Pf of Thm starting from  $\emptyset$ :  
build up basis for  $U$  by induction  
  
at each step, have lin. indep. set of vectors in  $U$   
it is also a lin. indep. set in  $V$   
so # of vectors  $\leq \dim V$  by Steinitz

so we must get a basis after  $\leq \dim V$  steps  $\square$

[return to example] recall

$$Q = \{p \text{ in } F[x] \mid p(-1) = p(1)\}$$

$$Q' = \{p \text{ in } W \mid p = 0 \text{ or } \deg p \leq 4\}$$

$$\text{let } W = \{p \text{ in } F[x] \mid p = 0 \text{ or } \deg p \leq 4\}$$

$$\text{then } Q' = W \cap Q$$

$$\text{so } \dim Q' \leq \dim W = 5$$

$$\text{in fact } \dim Q' = 4$$

similar application of lemma shows:

Thm if  $V$  is finite-dim,  $U \text{ sub } V$  is a linear sub,  
then any basis for  $U$   
can be extended to a basis for  $V$

Cor if  $U, W$  are linear subspaces of  $V$ ,  
then any basis for  $U \cap W$   
can be extended to a basis for  $U$  (or  $W$ ),  
then to a basis for  $U + W$

[proof by swapping variables]  
[can strengthen this result:]

Thm suppose  $U + W$  is finite-dim,  
 $\{e_1, \dots, e_\ell\}$  is a basis for  $U \cap W$ ,  
 $\{e_1, \dots, e_\ell, f_1, \dots, f_m\}$  for  $U$ ,  
 $\{e_1, \dots, e_\ell, g_1, \dots, g_n\}$  for  $W$   
then  $\{e_1, \dots, e_\ell, f_1, \dots, f_m, g_1, \dots, g_n\}$   
is a basis for  $U + W$

Pf spanning is easy  
suppose  $\sum_i a_{ie_i} + \sum_j b_{jf_j} + \sum_k c_{kg_k} = \mathbf{0}$

if  $c_k \neq 0$  for some  $k$   
then have  $g_k$  in  $\text{span}(e_i\text{'s}, f_j\text{'s}) = U$   
so  $g_k$  in  $U \cap W = \text{span}(e_i\text{'s})$   
contradicts basis for  $W$ , so  $c_k = 0$  for all  $k$

now if  $b_j \neq 0$  for some  $j$   
then have  $f_j$  in  $\text{span}(e_i\text{'s})$   
contradicts basis for  $U$ , so  $b_j = 0$  for all  $j$

now if  $a_i \neq 0$  for some  $i$   
contradicts basis for  $U \cap W$ , so  $a_i = 0$  for all  $i$   $\square$

### Cor (Dimension Formula)

$$\dim(U + W) = \dim(U) + \dim(W) - \dim(U \cap W)$$

Cor suppose  $U + W$  is finite-dim  
then TFAE:  
1)  $U + W$  is a direct sum  
2)  $U \cap W = \{\mathbf{0}\}$   
3)  $\dim(U + W) = \dim(U) + \dim(W)$

[equivalence of 1) and 2) was shown last week]