Last time A sub X

the subspace topology on A induced by X is
{A cap V | V is an open set in X}

if U sub A sub X
then (U open in X) implies (U open in A) [why?]
but converse can fail

Rem [how do subsp's interact with cts maps?]

1. if A sub X, then the inclusion map i : A to X def by i(a) = a is cts
2. compositions of cts maps are cts
3. so if f : X to Y is cts, then f|_A : A to Y is cts (compare to PS2, #6)

Rem if A is open in X itself
then (U open in A) iff (U open in X)
(see PS2, #3)

Thm if {B_i}_{i in I} is a basis for the top on X,
then {A cap B_i}_{i in I} is a basis for
the subspace top on A

Pf by earlier criterion, just need to show:
for all U open in A and x in U,
have i s.t. x in A cap B_i sub U

indeed, U = A cap V for some V open in X
then x in V, so have i s.t. x in B_i sub V
so x in A cap B_i, and also, A cap B_i sub U

(Munkres §20) recall from real analysis:

Df a metric on a set X is a function
$d : X \times X$ to $[0, \infty)$

s.t., for all x, y, z in X,

1) $d(x, y) = 0$ implies $x = y$
2) $d(x, y) = d(y, x)$
3) $d(x, y) + d(y, z) \geq d(x, z)$

given $\delta > 0$, let $B_d(x, \delta) = \{y \text{ in } X \mid d(x, y) < \delta\}$

Df the metric topology on X induced by d:

U is open in the metric topology iff
for all x in U, there is a $\delta > 0$ s.t. $B_d(x, \delta)$ sub U

Idea metric topology on X generalizes
analytic topology on R^n

Thm the metric topology really is a topology

Pf exactly like the proof that
the anlytic topology is a topology

[so how much weirder can it be?]

Ex in any X: the discrete metric defined by
$d(x, x) = 0$
$d(x, y) = 1$ when $x \neq y$

1) and 2) easy

3) [how many cases to check? 5 but can combine]

if $x = z$:

$d(x, y) + d(y, z) \geq 0 = d(x, z)$

[because $d(-, -) \geq 0$]

if $x \neq z$:

either $y \neq x$ or $y \neq z$

so $d(x, y) + d(y, z) \geq 1 = d(x, z)$

observe $B_d(x, 1) = \{x\}$ for all x. thus:

the discrete metric induces the discrete topology

Rem metric ↦ basis of balls ↦ topology
but recall:
different bases can give the same top

Thm suppose d induces T on X,
d' induces T' on X

then T is finer than T' iff
for all x in X and $\varepsilon > 0$, there is $\delta > 0$ s.t.
$B_d(x, \delta)$ sub $B_{d'}(x, \varepsilon)$

Pf exercise (Munkres Lem 20.2)

Ex [picture of $B_d(x, \delta)$ versus $B_\rho(x, \delta)$]

euclidean metric:
$d(x, y) = \mathrm{sqrt}((x_1 - y_1)^2 + \ldots + (x_n - y_n)^2)$

square metric:
$\rho(x, y) = \max(|x_1 - y_1|, \ldots, |x_n - y_n|)$

observe:

$d(x, y) \le \sqrt{n \max_i (x_i - y_i)^2}$
$= \sqrt{n}\, \rho(x, y)$

$\rho(x, y) = \sqrt{\max_i |x_i - y_i|^2}$
$\le d(x, y)$

shows [note reverse directions!]:

$B_\rho(x, \varepsilon/\sqrt{n}) = \{y \mid \rho(x, y) < \varepsilon/\sqrt{n}\}$
$= \{y \mid \sqrt{n}\, \rho(x, y) < \varepsilon\}$
$\subseteq \{y \mid d(x, y) < \varepsilon\}$
$= B_d(x, \varepsilon)$

similarly, $B_d(x, \varepsilon) \subseteq B_\rho(x, \varepsilon)$

<u>Thm</u> Euclidean and square metrics both induce the analytic topology on R^n

in general:

<u>Df</u> metrics d, d' are called <u>equivalent</u> iff there exist $A, B > 0$ s.t. $d(x, y) \le A\, d'(x, y)$ and $d'(x, y) \le B\, d(x, y)$ <u>uniformly</u> in x and y

<u>Thm</u> if two metrics are equivalent, then their metric topologies coincide

<u>Rem</u> converse is false: see PS2, #9–10

<u>Q</u> what about infinite-dim'l space?

$R^\omega = \{(x_1, x_2, \ldots) \mid x_i \in R \text{ for all } i\}$
$R^\infty = \{(x_1, x_2, \ldots) \mid x_i \neq 0 \text{ for only fin. many } i\}$

Euclidean and square metrics don't work on $\mathbb{R}^\omega$

<u>Q</u> do they still work on $\mathbb{R}^\infty$?

<u>Q</u> are there other metrics on $\mathbb{R}^\omega$?

(Munkres §15, 19) given $\{X_i\}_{i \in I}$

we write $\prod_{i \in I} X_i$ for the set of sequences $(x_i)_{i \in I}$ such that $x_i \in X_i$ for all i

<u>Q</u> if each X_i has a topology, do we get a natural topology on $\prod_{i \in I} X_i$?