10.

More notes on homogeneous affine Springer fibers and (q,t)-Catalan numbers.

10.1.

Let G be a connected, complex semisimple algebraic group with Lie algebra $\mathfrak{g}$. Let $T \subseteq G$ be a maximal torus, and let $\Phi \subseteq \mathrm{X}^*(T)$, *resp.* $\Phi^\vee \subseteq \mathrm{X}_*(T)$, be the associated system of roots, *resp.* coroots. For each root $\alpha \in \Phi$, we fix a generator e_α of the corresponding root subspace of $\mathfrak{g}$.

Let $F = \mathbf{C}(\!(\varpi)\!)$ and $\mathcal{O} = \mathbf{C}[\![\varpi]\!]$. The roots of the loop Lie algebra $\mathfrak{g}(F)$ take the form $\alpha + n$ for $(\alpha, n) \in \Phi \times \mathbf{Z}$. The root subspace corresponding to $\alpha + n$ is generated by $\varpi^n e_\alpha$.

We recall the formalism used by Moy–Prasad to study $\mathfrak{g}(F)$. Let $A(T) = \mathrm{X}_*(T) \otimes \mathbf{R}$. For any $x \in A(T)$, let

$$\begin{aligned} \mathfrak{p}_x &= \mathfrak{t}(\mathcal{O}) \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha(\varpi^{\lceil -\langle \alpha, x \rangle \rceil} \mathcal{O}) \\ &= \mathfrak{t}(\mathcal{O}) \oplus \bigoplus_{\substack{\alpha \in \Phi \\ n \geq -\langle \alpha, x \rangle}} \mathbf{C}\varpi^n e_\alpha, \end{aligned}$$

a parahoric subalgebra of $\mathfrak{g}(F)$, and let

$$\mathfrak{l}_x = \mathfrak{t}(\mathbf{C}) \oplus \bigoplus_{\substack{\alpha \in \Phi \\ n = -\langle \alpha, x \rangle}} \mathbf{C}\varpi^n e_\alpha,$$

its Levi factor. We write $P_x \subseteq G(F)$ for the parahoric subgroup with Lie algebra $\mathfrak{p}_x$, and L_x for the Levi factor of P_x.

Recall that the affine hyperplane corresponding to $\alpha + n$ is

$$H_{\alpha+n} = \{x \in A(T) \mid \langle \alpha, x \rangle + n = 0\} \subseteq A(T).$$

The affine hyperplanes $H_{\alpha+n}$ define a polyhedral subdivision of $A(T)$, such that the structure of $\mathfrak{p}_x$ changes precisely when x crosses from the interior of one facet into another facet. We write $t_{\alpha+n}$ for the rigid transformation of $A(T)$ given by reflection across $H_{\alpha+n}$.

Let W and $\Lambda^\vee \subseteq A(T)$ be the Weyl group and coweight lattice defined by Φ. The group of rigid transformations of $A(T)$ generated by the reflections $t_{\alpha+0}$, *resp.* $t_{\alpha+n}$, can be identified with W, *resp.* the semidirect product $W^{\mathit{affext}} = W \ltimes \Lambda^\vee$, acting on $A(T)$ from the *right*. Explicitly, if $\omega^\vee$ is the fundamental coweight dual to α, then $t_{\alpha+n} = n\omega^\vee \circ t_\alpha \circ -n\omega^\vee$, where composition is *left-to-right* and the coweights act by translation.

Example 10.1. Take $G = \mathrm{SL}_2$. Let $(\alpha, \alpha^\vee)$ be a root-coroot pair for $\mathfrak{g}$, and let $\omega^\vee$ be the fundamental coweight dual to α. Then $H_{\alpha-2} = \{\alpha^\vee\}$, so we know that $t_{\alpha-2}$ fixes $\alpha^\vee$. Indeed, $\alpha^\vee = 2\omega^\vee$, from which

$$\alpha^\vee \cdot t_{\alpha-2} = \alpha^\vee \cdot (-2\omega^\vee \circ t_\alpha \circ 2\omega^\vee) = (\alpha^\vee - \alpha^\vee) \cdot t_\alpha + \alpha^\vee = \alpha^\vee,$$

as expected.

Let W_x be the Weyl group of L_x. Then W_x can be identified with the stabilizer of x under W^{affext}: that is, the subgroup of W^{affext} generated by the reflections $t_{\alpha+n}$ such that $x \in H_{\alpha+n}$.

Let $W^{\mathit{aff}} = W \ltimes \mathrm{X}_*(T) \subseteq W^{\mathit{affext}}$. The isomorphisms $W \simeq N_G(T)/T$ and $\mathrm{X}_*(T) \simeq \varpi^{\mathrm{X}_*(T)}$ extend to an isomorphism

$$W^{\mathit{aff}} \simeq N_{G(F)}(T(F))/T(\mathcal{O}).$$

(See Remark 10 of Haines's appendix to [PR08].) Just as the $N_G(T)$-action on the set of parabolic subgroups of G containing T factors through W, the $N_{G(F)}(T(F))$-action on the set of parahoric subgroups of $G(F)$ containing $T(\mathcal{O})$ factors through W^{aff}. Explicitly, if $\dot{w} \in N_{G(F)}(T(F))$ maps to $w \in W^{\mathit{aff}}$, then

$$\dot{w} \cdot_{\mathrm{Ad}} P_x = P_{x \cdot w^{-1}}. \tag{10.1}$$

The example below shows why the switch between left and right actions is necessary:

Example 10.2. Again take $G = \mathrm{SL}_2$. Let T be the diagonal torus, and let the coroot $\alpha^\vee : \mathbf{G}_m \to T$ be given by $c^{\alpha^\vee} = \left(\begin{smallmatrix} c & \\ & c^{-1} \end{smallmatrix}\right)$. We have $\mathfrak{p}_0 = \mathfrak{g}(\mathcal{O})$, from which

$$\mathfrak{p}_{-\alpha^\vee} = \varpi^{\alpha^\vee} \cdot_{\mathrm{Ad}} \mathfrak{p}_0 = \begin{pmatrix} \varpi & \\ & \varpi^{-1} \end{pmatrix} \cdot_{\mathrm{Ad}} \mathfrak{g}(\mathcal{O}) = \mathfrak{g}(F) \cap \begin{pmatrix} \mathcal{O} & \varpi^2\mathcal{O} \\ \varpi^{-2}\mathcal{O} & \mathcal{O} \end{pmatrix}.$$

This matches the defining formula

$$\mathfrak{p}_{-\alpha^\vee} = \mathfrak{t}(\mathcal{O}) \oplus \mathfrak{g}_\alpha(\varpi^2\mathcal{O}) \oplus \mathfrak{g}_{-\alpha}(\varpi^{-2}\mathcal{O}),$$

as expected.

10.2.

Fix a rational coweight $\lambda^\vee \in \mathbf{Q}\Phi^\vee$. We can write $\lambda^\vee = \frac{1}{m}\lambda_0^\vee$, where m is a positive integer and $\lambda_0^\vee$ is an *integral* coweight. Using m and $\lambda_0^\vee$, we can define an action of $\mathbf{G}_m$ on $G(F)$ by algebraic-group automorphisms:

$$c \cdot_{m,\lambda_0^\vee} g(\varpi) = c^{2\lambda_0^\vee} \cdot_{\mathrm{Ad}} g(c^{2m}\varpi).$$

This induces an action on the Lie algebra $\mathfrak{g}(F)$. We write $\mathfrak{g}(F)_{k/m}$ to denote the weight-$2k$ eigenspace of $\mathfrak{g}(F)$ under $\mathbf{G}_m$. With this convention, the grading only depends on $\lambda^\vee$, even though the $\mathbf{G}_m$-action itself depends on m and $\lambda_0^\vee$. In particular,

$$\varpi^n e_\alpha \in \mathfrak{g}(F)_{\langle \alpha, \lambda^\vee \rangle + n} \tag{10.2}$$

via the calculation $c \cdot_{m,\lambda_0^\vee} \varpi^n e_\alpha = c^{2(\langle \alpha, \lambda_0^\vee \rangle + mn)} \varpi^n e_\alpha$.

We observe that the parahoric subalgebra $\mathfrak{p}_{\lambda^\vee}$, *resp.* Levi factor $\mathfrak{l}_{\lambda^\vee}$, is precisely the nonnegative, *resp.* degree-zero, part of this eigengrading:

$$\begin{aligned} \mathfrak{p}_{\lambda^\vee} &= \mathfrak{t}(\mathcal{O}) \oplus \bigoplus_{\substack{\alpha \in \Phi \\ \langle \alpha, \lambda^\vee \rangle + n \geq 0}} \mathbf{C} \varpi^n e_\alpha = \mathfrak{g}(F)_{\geq 0}, \\ \mathfrak{l}_{\lambda^\vee} &= \mathfrak{t}(\varpi \mathcal{O}) \oplus \bigoplus_{\substack{\alpha \in \Phi \\ \langle \alpha, \lambda^\vee \rangle + n = 0}} \mathbf{C} \varpi^n e_\alpha = \mathfrak{g}(F)_{0}. \end{aligned}$$

In particular, the Levi factor $L_{\lambda^\vee}$ is precisely the connected component at the identity of $G^{\mathbf{G}_m}$, the subgroup of G of $\mathbf{G}_m$-invariants.

Let $w \mapsto \dot{w}$ be a set-theoretic lift of W^{aff} to $N_{G(F)}(T(F))$. For any $x \in A(T)$, the partial affine flag variety $\mathcal{F}l_x = G(F)/P_x$ admits the Bruhat decomposition

$$\mathcal{F}l_x = \bigsqcup_{[w] \in W_{\lambda^\vee} \backslash W^{\mathit{aff}} / W_x} P_{\lambda^\vee} \dot{w} P_x / P_x.$$

The action of $\mathbf{G}_m$ on $G(F)$ stabilizes P_x setwise, so it descends to an action on $G(F)/P_x$. Since $\dot{w} P_x$ is fixed by $\mathbf{G}_m$ and $P_{\lambda^\vee}^{\mathbf{G}_m} = L_{\lambda^\vee}$, we deduce that

$$\mathcal{F}l_x^{\mathbf{G}_m} = \bigsqcup_{[w] \in W_{\lambda^\vee} \backslash W^{\mathit{aff}} / W_x} L_{\lambda^\vee} \dot{w} P_x / P_x. \tag{10.3}$$

Now let $\gamma \in \mathfrak{g}(F)$ be regular semisimple. We can form the affine Springer fiber

$$\mathcal{F}l_x^\gamma = \{ g P_x \in \mathcal{F}l_x \mid g^{-1} \cdot_{\mathrm{Ad}} \gamma \in \mathfrak{p}_x \}.$$

The $\mathbf{G}_m$-action on $\mathcal{F}l_x$ interacts with affine Springer fibers in the following way:

Lemma 10.3. *For any $\lambda^\vee = \frac{1}{m} \lambda_0^\vee$ and $\gamma \in \mathfrak{g}(F)$ and $g \in G(F)$, we have*

$$c \cdot_{m,\lambda_0^\vee} (g^{-1} \cdot_{\mathrm{Ad}} \gamma) = (c \cdot_{m,\lambda_0^\vee} g^{-1}) \cdot_{\mathrm{Ad}} (c \cdot_{m,\lambda_0^\vee} \gamma). \tag{10.4}$$

Proof. Immediate from the definition of $\cdot_{m,\lambda_0^\vee}$. □

Lemma 10.4. *If γ is an eigenvector of the $\mathbf{G}_m$-action $\cdot_{m,\lambda_0^\vee}$, then $\mathcal{F}l_x^\gamma$ is stable under the same $\mathbf{G}_m$-action.*

Proof. Suppose that γ belongs to the weight-$2k$-eigenspace $\mathfrak{g}(F)_{k/m}$. By (10.4),

$$\begin{aligned}(c \cdot_{m,\lambda_0^\vee} g^{-1}) \cdot_{\mathrm{Ad}} \gamma &= (c \cdot_{m,\lambda_0^\vee} g^{-1}) \cdot_{\mathrm{Ad}} c^{-2k}(c \cdot_{m,\lambda_0^\vee} \gamma) \\ &= c^{-2k}((c \cdot_{m,\lambda_0^\vee} g^{-1}) \cdot_{\mathrm{Ad}} (c \cdot_{m,\lambda_0^\vee} \gamma)) \\ &= c^{-2k}(c \cdot_{m,\lambda_0^\vee} (g^{-1} \cdot_{\mathrm{Ad}} \gamma)).\end{aligned}$$

Now observe that

$$\begin{aligned} gP_y \in \mathcal{F}l_x^\gamma &\iff g^{-1} \cdot_{\mathrm{Ad}} \gamma \in \mathfrak{p}_x \\ &\iff c^{-2k}(c \cdot_{m,\lambda_0^\vee} (g^{-1} \cdot_{\mathrm{Ad}} \gamma)) \in \mathfrak{p}_x \quad \text{because } \mathfrak{p}_x \text{ is } (\cdot_{m,\lambda_0^\vee})\text{-stable} \\ &\iff (c \cdot_{m,\lambda_0^\vee} g^{-1}) \cdot_{\mathrm{Ad}} \gamma \in \mathcal{F}l_x^\gamma, \end{aligned}$$

as desired. ☐

10.3.

Let h be the sum of the Coxeter numbers of the irreducible root subsystems of Φ. Let $B_+ \supseteq T$ be a Borel subgroup of G, corresponding to a choice of positive subset $\Phi_{\geq 0} \subseteq \Phi$. We can then write

$$\Phi = \bigsqcup_{0<|\ell|\leq h-1} \Phi_\ell,$$

where Φ_ℓ is the set of roots of height ℓ with respect to B_+.

Let $\rho^\vee \in \frac{1}{2}\mathbf{Z}\Phi^\vee$ be the sum of the fundamental coweights, so that $\langle \rho^\vee, \alpha \rangle = \ell$ for all $\alpha \in \Phi_\ell$. We are most interested in the case where

$$(m, \lambda_0^\vee) = (h, -d\rho^\vee) \qquad \text{with } d \text{ coprime to } h. \tag{10.5}$$

In this case, (10.2) becomes

$$\varpi^n e_\alpha \in \mathrm{gr}_{-\frac{d\ell}{h}+n}\, \mathfrak{g}(F).$$

Since $d\ell$ is not divisible by h for any integer ℓ such that $0 < |\ell| \leq h-1$, we see that

$$\begin{aligned} \mathfrak{l}_{-d\rho^\vee} &= \mathfrak{t}, \\ W_{-d\rho^\vee} &= \{\mathrm{id}\}. \end{aligned}$$

Therefore, (10.3) becomes

$$\begin{aligned} \mathcal{F}l_x^{\mathbf{G}_m} &= \bigsqcup_{[w]\in W^{\mathit{aff}}/W_x} T\dot{w}P_x/P_x \\ &= \bigsqcup_{[w]\in W^{\mathit{aff}}/W_x} \dot{w}P_x/P_x. \end{aligned} \tag{10.6}$$

That is, the only $\mathbf{G}_m$-fixed points of $\mathcal{F}l_x$ in this case are the isolated cosets $\dot{w}P_x$.

Assuming (10.5), we can describe the eigenspaces of the $\cdot_{h,-d\rho^\vee}$ action on $\mathfrak{g}(F)$ fairly explicitly. Suppose that $\gamma \in \mathfrak{g}(F)_{k/h}$. The key is to observe that if $\varpi^n e_\alpha \in \varpi^n \Phi_\ell$ appears with nonzero coefficient in γ, then ℓ and n must satisfy $-\frac{d\ell}{h}+n = \langle -\frac{d}{h}\rho^\vee, \alpha\rangle + n = \frac{k}{h}$, or equivalently, $-d\ell + hn = k$.

When $k \not\equiv 0 \pmod h$, the general form of such γ looks like this. Fix a coefficient vector $\vec{c} = (c_\alpha)_\alpha \in \mathbf{C}^\Phi$. Let

$$\begin{aligned} e_{\vec{c},\ell} &= \sum_{\alpha \in \Phi_\ell} c_\alpha e_\alpha, \\ \gamma_{\vec{c},d,k} &= \sum_{\substack{(\ell,n)\in \mathbf{Z}^2 \\ |\ell| \le h-1 \\ -d\ell + hn = k}} \varpi^n e_{\vec{c},\ell}. \end{aligned}$$

Then $\gamma_{\vec{c},d,k} \in \mathfrak{g}(F)_{k/h}$, and every element of $\mathfrak{g}(F)_{k/h}$ takes this form.

We claim that above, there are exactly two values of ℓ that can appear in the outer sum. For, by the Chinese Remainder Theorem, we can write $k = ih + jd$ for some integers i, j that are determined uniquely once we require $0 < j < h$. So the possible values of ℓ are $-j$ and $h - j$. The respective values of n are i and $i + d$. So

$$\gamma_{\vec{c},d,k} = \varpi^i e_{\vec{c},-j} + \varpi^{i+d} e_{\vec{c},h-j}. \tag{10.7}$$

Note that if $c_\alpha \neq 0$ for all $\alpha \in \Phi_{-j} \cup \Phi_{h-j}$, then $\gamma_{\vec{c},d,k}$ is regular semisimple.

Example 10.5. If $c_\alpha = 1$ for all $\alpha \in \Phi_{-j} \cup \Phi_{h-j}$, then we abbreviate by writing

$$\begin{aligned} e_\ell &= e_{\vec{c},\ell}, \\ \gamma_{d,k} &= \gamma_{\vec{c},k}. \end{aligned}$$

Taking $d = 1$, we see that $\gamma_{1,k}$ recovers the element of $\mathfrak{g}(F)$ studied in the papers of Lusztig–Smelt, C.-K. Fan, and Sommers.

Proposition 10.6. *Let $\gamma_{\vec{c},d,k}$ be defined by* (10.7)*. Assume* (10.5)*, and assume that $c_\alpha \neq 0$ for all $\alpha \in \Phi_{-j} \cup \Phi_{h-j}$. Then we have bijections*

$$\begin{aligned} (\mathcal{F}l_x^\gamma)^{\mathbf{G}_m} &\simeq \{[w] \in W^{\mathit{aff}}/W_x \mid \mathfrak{p}_{x\cdot w^{-1}} \ni \gamma\} \\ &\simeq \left\{ [w] \in W^{\mathit{aff}}/W_x \;\middle|\; \begin{array}{l} \langle \alpha, x\cdot w^{-1}\rangle \geq -i \text{ for all } \alpha \in \Phi_{-j}, \\ \langle \alpha, x \cdot w^{-1}\rangle \geq -(i+d) \text{ for all } \alpha \in \Phi_{h-j} \end{array} \right\}. \end{aligned}$$

Proof. By (10.6) and (10.1),

$$\begin{aligned} (\mathcal{F}l_x^\gamma)^{\mathbf{G}_m} &= \{\dot{w}P_x \in \mathcal{F}l_x \mid \dot{w}^{-1} \cdot_{\mathrm{Ad}} \gamma \in \mathfrak{p}_x\} \\ &= \{\dot{w}P_x \in \mathcal{F}l_x \mid \gamma \in \dot{w} \cdot_{\mathrm{Ad}} \mathfrak{p}_x\} \\ &= \{\dot{w}P_x \in \mathcal{F}l_x \mid \gamma \in \mathfrak{p}_{x\cdot w^{-1}}\}, \end{aligned}$$

proving the first bijection. The second bijection follows from the definition of $\mathfrak{p}_{x\cdot w^{-1}}$. $\square$

Example 10.7. Let $d = k$, so that $(i, j) = (0, 1)$. Then

$$\gamma_{d,d} = e_{-1} + \varpi^d e_{h-1}.$$

Let $x = 0$, so that $P_x = G(\mathcal{O})$ and the W^{aff}-orbit of x is the cocharacter lattice $\mathrm{X}_*(T)$. Then Proposition 10.6 says that

$$(\mathcal{F}l_0^{\gamma_{d,d}})^{\mathbf{G}_m} \simeq \left\{ y \in \mathrm{X}_*(T) \,\middle|\, \begin{array}{l} \langle \alpha, y \rangle \leq 0 \text{ for all simple } \alpha, \\ \langle \alpha_{high}, y \rangle \geq -d \end{array} \right\},$$

where α_{high} denotes the highest root of Φ. (Above, we have used the fact that $\alpha \in \Phi_1$ iff $-\alpha \in \Phi_{-1}$.) That is, the $\mathbf{G}_m$-fixed points of $\mathcal{F}l_0^{\gamma_{d,d}}$ are in bijection with the anti-dominant (integral) cocharacters y for which $\langle \alpha_{high}, y \rangle \geq -d$.

If G is simply connected, so that $\mathrm{X}_*(T) = \mathbf{Z}\Phi^\vee$, and almost-simple, then it follows from the work of Oblomkov–Yun that the former set is enumerated by the rational Catalan number for (W, d). *Nota bene* that our $\mathbf{G}_m$-action actually differs from that of Oblomkov–Yun in that they set $\lambda_0^\vee = d\rho^\vee$, where we instead set $\lambda_0^\vee = -d\rho^\vee$, but this is immaterial for the combinatorics.

10.4.

Here is another way to understand the eigenspace $\mathfrak{g}(F)_{d/h}$, which is closer to the viewpoint of Oblomkov–Yun.

Let $\mathfrak{c} = \mathfrak{g} /\!\!/ G$, the adjoint quotient. Recall that the quotient map $\chi : \mathfrak{g} \to \mathfrak{c}$ transports the scaling action of $\mathbf{G}_m$ on $\mathfrak{g}$ to a weighted action on $\mathfrak{c}$. Suppose that χ admits a section $\sigma : \mathfrak{c} \to \mathfrak{g}$ satisfying the following equivariance condition:

$$\sigma(c^2 \cdot a) = c^2(c^{2\rho^\vee} \cdot_{\mathrm{Ad}} \sigma(a)). \tag{10.8}$$

Then the induced map $\sigma : \mathfrak{c}(F) \to \mathfrak{g}(F)$ satisfies

$$\sigma(c^{-2d} \cdot a(c^{2m}\varpi)) = c^{-2d}(c \cdot_{m,-d\rho^\vee} \sigma(a(\varpi))).$$

In particular, setting

$$\mathfrak{c}(F)_{d/m} = \{a \in \mathfrak{c}(F) \mid a(c^{2m}\varpi) = c^{2d} \cdot a(\varpi)\}$$

gives us the following (for a general denominator m, not just h):

Lemma 10.8. *Suppose that $\lambda = \frac{d}{m}\rho^\vee$. Then, for any map $\sigma : \mathfrak{c} \to \mathfrak{g}$ that satisfies (10.8) and any element $a \in \mathfrak{c}(F)$, we have*

$$a \in \mathfrak{c}(F)_{d/m} \iff \sigma(a) \in \mathfrak{g}(F)_{d/m}.$$

Example 10.9. Let $e_\pm$ be the nilpotent entries of a *regular* $\mathfrak{sl}_2$ triple in $\mathfrak{g}$. Our notation means e_+, *resp.* e_-, generates the weight-2, *resp.* weight-(-2) root space. Kostant showed that χ restricts to an isomorphism of affine spaces $e_- + \ker(\mathrm{Ad}(e_+)) \xrightarrow{\sim} \mathfrak{c}$. The inverse isomorphism induces a section

$$\kappa : \mathfrak{c} \to \mathfrak{g}$$

that we will call the Kostant section.

Note that e_+ is contained in a unique Borel subalgebra, so it determines a choice of B_+ and hence $\rho^\vee$. If we take $\sigma = \kappa$, then (10.8) holds.

Remark 10.10. In Oblomkov–Yun, the Kostant section is instead defined in terms of $e_+ + \ker(\mathrm{Ad}(e_-))$. This is related to how they choose $\lambda_0^\vee = d\rho^\vee$ while we choose $\lambda_0^\vee = -d\rho^\vee$.

Example 10.11. Take $\mathfrak{g} = \mathfrak{sl}_n$. Then we can choose coordinates $(a_2, \ldots, a_{n-1}, a_n) : \mathfrak{c} \xrightarrow{\sim} \mathbf{A}^{n-1}$ for which the weighted $\mathbf{G}_m$-action is

$$c \cdot a_i = c^i a_i.$$

Take e_+, *resp.* e_-, to be the $n \times n$ matrix with 1's along the superdiagonal, *resp.* subdiagonal, and 0's elsewhere. With respect to the resulting choice of $\rho^\vee$, the *syntrophic* or *companion-matrix section*

$$\sigma(a_2, \ldots, a_n) = \begin{pmatrix} & & & & -a_n \\ 1 & & & & -a_{n-1} \\ & \ddots & & & \vdots \\ & & 1 & & -a_2 \\ & & & 1 & 0 \end{pmatrix}$$

satisfies (10.8).

10.5.

We now set $G = \mathrm{SL}_n$, so that $h = n$, and $x = 0$, so that $P_x = G(\mathcal{O})$. Let $B_+ \subseteq G$ be the upper-triangular subgroup, and let

$$\gamma = \gamma_{d,d} = \begin{pmatrix} & & & \varpi^d \\ 1 & & & \\ & \ddots & & \\ & & 1 & \end{pmatrix} \in \mathfrak{g}(F)_{d/n}.$$

Recall that we have an identification:

$$G(F)/G(\mathcal{O}) \simeq \{\mathcal{O}\text{-submodules } L \subseteq F^n \mid L = g\mathcal{O}^n \text{ for some } g \in G(F)\}.$$

Observe that since γ is a companion matrix, we have a γ-equivariant isomorphism of F-vector spaces $F^n \xrightarrow{\sim} F\langle 1, \gamma, \dots, \gamma^{n-1}\rangle = F[\gamma]$ that transports the standard basis vectors to the powers of γ. In particular, the preceding identification induces:

$$
\begin{aligned}
(10.9)\quad \mathcal{F}l_0^{\gamma} &\simeq \{\mathcal{O}[\gamma]\text{-submodules } L \subseteq F^n \mid L = g\mathcal{O}^n \text{ for some } g \in G(F)\}\\
&\simeq \left\{ \mathcal{O}[\gamma]\text{-submodules } M \subseteq F[\gamma] \;\middle|\; \begin{array}{l} M = \alpha(\mathcal{O}[\gamma]) \\ \text{for some } \alpha \in \mathrm{Aut}_F(F[\gamma]) \\ \text{such that } \det(\alpha) = 1 \end{array} \right\}.
\end{aligned}
$$

Above, note that we have isomorphisms of algebras:

$$
\begin{aligned}
F[\gamma] &\simeq F[\gamma]/(\gamma^n - \varpi^d),\\
\mathcal{O}[\gamma] &\simeq \mathcal{O}[\gamma]/(\gamma^n - \varpi^d).
\end{aligned}
$$

There is a $\mathbf{G}_m$-action on $F[\gamma]$ given by

$$
(10.10)\qquad c \cdot \varpi = c^{2n}\varpi \quad \text{and} \quad c \cdot \gamma = c^{2d}\gamma.
$$

It preserves $\mathcal{O}[\gamma]$, so it induces a $\mathbf{G}_m$-action on the set of $\mathcal{O}[\gamma]$-submodules of $F[\gamma]$.

Proposition 10.12. *Suppose that under the γ-equivariant isomorphism $F^n \simeq F[\gamma]$, the action of $g \in G(F)$ on F^n corresponds to the action of $\alpha \in \mathrm{Aut}_F(F[\gamma])$ on $F[\gamma]$. Then, for any $c \in \mathbf{C}^\times$, the following $\mathcal{O}[\gamma]$-submodules correspond to one another:*

(1) $(c \cdot_{h,-d\rho^\vee} g)\mathcal{O}^n$.
(2) $c \cdot \alpha(\mathcal{O}[\gamma])$*, where c acts via* (10.10).

Proof. For all i, j, let $g_{i,j}$ be the (i, j)th entry of g as an $n \times n$ matrix, where we index starting at 0 rather than 1. Then $g\mathcal{O}^n$ is spanned over $\mathcal{O}$ by the columns of g. We deduce that $\alpha(\mathcal{O}[\gamma])$ is spanned over $\mathcal{O}$ by the elements

$$
\delta_j(\varpi) = g_{0,j}(\varpi) + g_{1,j}(\varpi)\gamma + \cdots + g_{n-1,j}(\varpi)\gamma^{n-1}
$$

for $0 \le j \le n-1$. Therefore, $c \cdot \alpha(\mathcal{O}[\gamma])$ is spanned by the elements

$$
c \cdot \delta_j(\varpi) = g_{0,j}(c^{2n}\varpi) + g_{1,j}(c^{2n}\varpi)(c^{2d}\gamma) + \cdots + g_{n-1,j}(c^{2n}\varpi)(c^{2d}\gamma)^{n-1}.
$$

At the same time, let $g' = c \cdot_{h,-d\rho^\vee} g$, and let $g'_{i,j}$ be the (i, j)th entry of g'. Then $g'\mathcal{O}^n$ is spanned over $\mathcal{O}$ by the columns of $c \cdot_{h,-d\rho^\vee} g'$, and

$$
g'_{i,j}(\varpi) = c^{2(i+j)d} g_{i,j}(c^{2n}\varpi).
$$

Let $\alpha' \in \mathrm{Aut}_F(F[\gamma])$ correspond to g'. Then $\alpha'(\mathcal{O}[\gamma])$ is spanned over $\mathcal{O}$ by the elements

$$
\begin{aligned}
\delta'_j(\varpi) &= g'_{0,j}(\varpi) + g'_{1,j}(\varpi)\gamma + \cdots + g'_{n-1,j}(\varpi)\gamma^{n-1}\\
&= c^{2jd}(c \cdot \delta_j(\varpi)).
\end{aligned}
$$

Therefore, $\alpha'(\mathcal{O}[\gamma]) = c \cdot \alpha(\mathcal{O}[\gamma])$, as needed. $\square$

Corollary 10.13. *Under* (10.9)*, the following correspond to one another:*

- *The* $\mathbf{G}_m$*-action given by* $\cdot_{h,-d\rho^\vee}$ *on* $\mathcal{F}l_0^\gamma$.
- *The* $\mathbf{G}_m$*-action given by* (10.10) *on the set of* $\mathcal{O}[\gamma]$*-submodules of* $F[\gamma]$ *of the form* $\alpha(\mathcal{O}[\gamma])$.

10.6.

We keep the setup of the preceding subsection. Since $\gamma = \gamma_{d,d}$, Proposition 10.6 says that the $\mathbf{G}_m$-fixed points of $\mathcal{F}l_0^\gamma$ are the points $\dot{w}G(\mathcal{O}) \in \mathcal{F}l_0$, as we run over $[w] \in W^{\mathit{aff}}/W$, such that

$$\langle \alpha, 0 \cdot w^{-1} \rangle \leq 0 \text{ for all } \alpha \in \Phi_1, \qquad \langle \alpha_{high}, 0 \cdot w^{-1} \rangle \geq -d.$$

As noted in Example 10.7, the points $0 \cdot w^{-1}$ all belong to $\mathrm{X}_*(T)$, and since $G = \mathrm{SL}_n$ is simply-connected, $\mathrm{X}_*(T)$ is the same as $\mathbf{Z}\Phi^\vee$. In particular, *we can and will choose the representative* $\dot{w}$ *to take the form* $\dot{w} = \varpi^w \in T(F)$.

Recall that B_+ is the upper-triangular Borel, so T is the diagonal torus. Recall also that we index rows and columns starting at 0. For $1 \leq i \leq n-1$, let $\alpha_i^\vee : \mathbf{G}_m \to T$ be the coroot such that $c^{\alpha_i^\vee}$ equals c in the $(i-1)$th entry and c^{-1} in the ith entry and 1 elsewhere. Below, we use this explicit labeling of the coroot lattice to describe, for each $\mathbf{G}_m$-fixed point $\dot{w}G(\mathcal{O}) \in \mathcal{F}l_0^\gamma$, the corresponding $\mathcal{O}[\gamma]$-submodule

$$M_{[w]} \subseteq F[\gamma].$$

Then the work of Piontkowski and Gorsky–Mazin describes how to assign certain Q- and T-statistics (*resp.*, "area" and "dinv") to each $M_{[w]}$. In place of the latter, we record the t^2*-statistic* given by $t^{2e} = T^{\frac{1}{2}(d-1)(n-1)-e}$.

It will be convenient to use the isomorphism

$$F[\gamma]/(\gamma^n - \varpi^d) \xrightarrow{\sim} F[\varrho^d, \varrho^n]$$

that sends $\gamma \mapsto \varrho^d$ and $\varpi \mapsto \varrho^n$. Since $\dot{w}$ is diagonal, we have

$$M_{[w]} = \mathcal{O}\langle \varrho^{d_1}, \ldots, \varrho^{d_{n-1}} \rangle$$

where $d_j = \mathrm{val}_\varrho(\dot{w}_{j,j}\gamma^j) = \mathrm{val}_\varpi(\dot{w}_{j,j})n + jd$.

Example 10.14. Taking $(n, d) = (3, 4)$ gives:

w	d_0	d_1	d_2	Q	t^2
0	0	4	8	3	3
$2\alpha_1^\vee + \alpha_2^\vee$	6	1	5	2	2
$\alpha_1^\vee + 2\alpha_2^\vee$	3	7	2	1	2
$2\alpha_1^\vee + 2\alpha_2^\vee$	6	4	2	1	1
$\alpha_1^\vee + \alpha_2^\vee$	3	4	5	0	0

Example 10.15. Taking $(n, d) = (3, 5)$ gives:

w	d_0	d_1	d_2	Q	t^2
0	0	5	10	4	4
$2\alpha_1^\vee + 3\alpha_2^\vee$	6	8	1	3	3
$3\alpha_1^\vee + 2\alpha_2^\vee$	9	2	4	2	3
$2\alpha_1^\vee + \alpha_2^\vee$	6	2	7	2	2
$\alpha_1^\vee + 2\alpha_2^\vee$	3	8	4	1	2
$\alpha_1^\vee + \alpha_2^\vee$	3	5	7	1	1
$2\alpha_1^\vee + 2\alpha_2^\vee$	6	5	4	0	0

Example 10.16. Taking $(n, d) = (4, 3)$ gives:

w	d_0	d_1	d_2	d_3	Q	t^2
0	0	3	6	9	3	3
$\alpha_1^\vee + 2\alpha_2^\vee + 2\alpha_3^\vee$	4	7	6	1	2	2
$2\alpha_1^\vee + 2\alpha_2^\vee + \alpha_3^\vee$	8	3	2	5	1	2
$\alpha_1^\vee + 2\alpha_2^\vee + \alpha_3^\vee$	4	7	2	5	1	1
$\alpha_1^\vee + \alpha_2^\vee + \alpha_3^\vee$	4	3	6	5	0	0

Example 10.17. Taking $(n, d) = (4, 5)$ gives:

w	d_0	d_1	d_2	d_3	Q	t^2
0	0	5	10	15	6	6
$3\alpha_1^\vee + 2\alpha_2^\vee + \alpha_3^\vee$	12	1	6	11	5	5
$2\alpha_1^\vee + 4\alpha_2^\vee + 2\alpha_3^\vee$	8	13	2	7	4	5
$\alpha_1^\vee + 2\alpha_2^\vee + 3\alpha_3^\vee$	4	9	14	3	3	5
$3\alpha_1^\vee + 4\alpha_2^\vee + 2\alpha_3^\vee$	12	9	2	7	4	4
$2\alpha_1^\vee + 4\alpha_2^\vee + 3\alpha_3^\vee$	8	13	6	3	3	4
$\alpha_1^\vee + \alpha_2^\vee + \alpha_3^\vee$	4	5	10	11	2	4
$2\alpha_1^\vee + 3\alpha_2^\vee + 3\alpha_3^\vee$	8	9	10	3	3	3
$\alpha_1^\vee + 2\alpha_2^\vee + \alpha_3^\vee$	4	9	6	11	2	3
$3\alpha_1^\vee + 3\alpha_2^\vee + 2\alpha_3^\vee$	12	5	6	7	1	3
$\alpha_1^\vee + 2\alpha_2^\vee + 2\alpha_3^\vee$	4	9	10	7	2	2
$2\alpha_1^\vee + 2\alpha_2^\vee + \alpha_3^\vee$	8	5	6	11	1	2
$2\alpha_1^\vee + 2\alpha_2^\vee + 2\alpha_3^\vee$	8	5	10	7	1	1
$2\alpha_1^\vee + 3\alpha_2^\vee + 2\alpha_3^\vee$	8	9	6	7	0	0