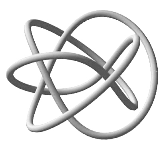
PROJECT DESCRIPTION

QUOT SCHEMES, LINK HOMOLOGY, AND q, t-SYMMETRIC FUNCTIONS

1. Background

1.1. One of the richest isotopy invariants of (oriented) links in $\mathbf{R}^3$ that we can still compute is Khovanov–Rozansky (KhR) homology [KR]. It sends a link to a triply-graded vector space. Its Euler characteristic with respect to one grading is the celebrated HOMFLYPT polynomial, discovered nearly four decades ago by multiple authors inspired by work of Jones [HOMFLY]. For a general link, all known approaches to computing its KhR homology involve choosing a planar diagram and applying rules that categorify the HOMFLYPT skein relations. In this generality, it is unclear whether KhR has an intrinsic—not diagram-dependent—interpretation.

Let p be a point on a complex algebraic plane curve C. The interection of C with any small-enough 3-sphere $S^3 \subseteq \mathbf{C}^2$ around p is a link $L_{C,p} \subseteq S^3$, well-defined up to isotopy. For instance, if C is the cusp $y^3 = x^4$, and p the origin, then $L_{C,p}$ will be the $(3, 4)$ torus knot, shown on the left. One decade ago, Oblomkov–Rasmussen–Shende conjectured a surprising identity expressing the KhR homology of $L_{C,p}$ in terms of the algebraic geometry of (C, p), or more precisely, its Hilbert schemes, which parametrize ideals of its completed local ring $\mathcal{O}_{C,p}$ [OS, ORS]. To state the conjecture, let



$$\mathsf{KhR}_{L_{C,p}}(A, q, t) \in \mathbf{Z}[A, t][\![q]\!]$$

be the triply-graded dimension of the KhR homology shifted so that its lowest degrees in A and q are zero. For the precise grading conventions, see Section 2. Let $\mathfrak{m}_{C,p}$ be the maximal ideal of $\mathcal{O}_{C,p}$, and for any integers $\ell, r \geq 0$, form the scheme

$$\mathcal{H}^{\ell,r}_{C,p} = \{\text{ideals } I \subseteq \mathcal{O}_{C,p} \mid (\dim_{\mathbf{C}}(\mathcal{O}_{C,p}/I), \dim_{\mathbf{C}}(I/\mathfrak{m}_{C,p}I)) = (\ell, r)\}.$$

Finally, let us write $\chi(X, t) \in \mathbf{Z}[t]$ for the virtual weight polynomial of a scheme of finite type X, a certain t-deformation of its Euler characteristic. Then [ORS, Conj. 2] states that

$$\mathsf{KhR}_{L_{C,p}}(A, q, q^{\frac{1}{2}}t) = \sum_{\ell,r} q^{\ell} \chi(\mathcal{H}^{\ell,r}_{C,p}, t) \prod_{0 \leq k \leq r-1} (1 + At^{2k}). \tag{1}$$

The conjecture is known in the HOMFLYPT limit where $t = -1$, by work of Diaconescu–Hua–Soibelman [DHS] and Maulik [Mau], and for the cusps $y^2 = x^{2k+1}$, but few other cases.

The full conjecture not only gives an intrinsic geometric interpretation to $\mathsf{KhR}_{L_{C,p}}(A, q, t)$, but also shows that the polynomials $\chi(\mathcal{H}^{\ell,r}_{C,p}, t)$ are explicitly computable, as long as the KhR side is. These polynomials also appear in other applications: most notably, as generalized orbital integrals for GL_n in the Langlands program, inviting generalizations to other G.

1.2. In a 2013 grant proposal [S13], Shende sketched a program to prove the ORS conjecture through the nonabelian Hodge theory of curves.

The usual nonabelian Hodge correspondence for a complex smooth curve X, due principally to Corlette and Simpson [Co, Si90, Si92], concerns homeomorphisms between a *Dolbeault* moduli space of Higgs bundles on X, a *de Rham* moduli space of flat conections on X, and a *Betti* moduli space of local systems on X. The first depends on the algebro-geometric structure of X, and by work of Hitchin [Hit], forms the total space of an algebraic completely integrable system. The second and third depend only on the smooth and topological structure of X, respectively. All these homeomorphisms are nonabelian analogues of comparison results in classical Hodge theory;

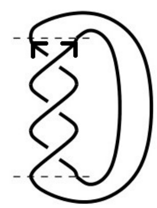
crucially, they are transcendental. The $\mathsf{P} = \mathsf{W}$ conjecture of [dCHM], now proved for compact X in [MS, HMMS], says the Dolbeault-to-Betti comparison matches the halved weight filtration on the cohomology of the Betti side with a *perverse* filtration on the cohomology of the Dolbeault side, defined via the perverse truncation of the Hitchin sheaf complex in the sense of [BBD].

By work of Biquard, Boalch, Sabbah, and others [BB, B11, B14, S], there is a wild variant in which the Higgs bundles and connections develop poles with diagonal Laurent tails ("irregular types"), and the local systems are augmented by unipotent monodromy data ("Stokes structures"). Moreover, there is a twisted generalization, discussed in [BoYa, Mo11, Mo13], where the irregular types need only diagonalize after pullback along local cyclic covers.

Shende–Treumann–Williams–Zaslow made progress on relating the HOMFLYPT polynomial of $L_{C,p}$ to the virtual weight polynomial of a twisted, wild Betti moduli space. In more detail: For any integer n and braid β on n strands, one can form a link $\hat{\beta}$ called the closure of β. For positive β, STZ construct a variety $\mathcal{X}(\beta)$ with an action of $G = \mathrm{GL}_n$, and show that

$$\mathsf{KhR}^{\mathrm{hi}}_{\hat{\beta}}(q, -q^{\frac{1}{2}}) = \pm\chi_G(\mathcal{X}(\beta), q^{\frac{1}{2}}), \quad \text{where } \mathsf{KhR}^{\mathrm{hi}}_L(q,t) = \lim_{A \to 0} A^{-n}\mathsf{KhR}_L(A,q,t), \tag{2}$$

and $\chi_G(-,t)$ is a G-equivariant version of $\chi(-,t)$ [STZ]. When $n = 2$ and $\hat{\beta}$ is a knot, as in the example to the right, STWZ check that the stacks $[\mathcal{X}(\beta)/G]$ recover cases of the twisted, wild Betti moduli spaces previously mentioned [STWZ]. Later, Galashin–Lam gave a partial refinement of (2): For so-called Richardson braids $\beta_{w,v}$ indexed by pairs of permutations $w, v \in S_n$, they show



$$\mathsf{KhR}^{\mathrm{hi}}_{\widehat{\beta_{v,w}}}(q, q^{\frac{1}{2}}t) = \pm\sum_{j,k} q^{\frac{j}{2}} t^k \dim \mathrm{gr}^{\mathsf{W}}_j \mathrm{H}^k_{c,G}(\mathcal{X}(\beta_{v,w})), \tag{3}$$

where $\mathrm{H}^*_{c,G}$ is G-equivariant, compactly-supported cohomology with rational coefficients and $\mathsf{W}_{\leq *}$ its weight filtration [GL].

Around the same time, Gorsky–Oblomkov–Rasmussen–Shende showed that when C is a planar branched cover of a smooth curve X, one can express the cohomology of the Hilbert schemes $\mathcal{H}^{\ell,r}_{C,p}$ in terms of the cohomology of fibers in various parabolic Hitchin systems built from X [GORS]. Significantly, the q variable in (1) corresponds to the perverse degree on the Hitchin side, by a formula of [MY, MiSh]. However, for general (C,p), these fibers are not obviously homotopic to the total spaces of twisted, wild Hitchin systems.

1.3. Independently, the past decade has seen breakthroughs in the calculation of KhR homology for torus links, phrased in the language of q,t-symmetric functions. We write $\Lambda_{q,t} = \bigoplus_n \Lambda^n_{q,t}$ for the graded ring formed by such functions. For each n, there is an isomorphism

$$\mathsf{Frob} : \mathbf{Q}(q,t) \otimes K_0(\mathsf{Rep}(S_n)) \xrightarrow{\sim} \Lambda^n_{q,t} \tag{4}$$

called the Frobenius character, which sends the irreducible representation indexed by a partition $\lambda \vdash n$ to the Schur function s_λ. We write $\langle -, - \rangle$ for the q,t-linear Hall inner product under which the Schur functions are orthonormal.

Let $T_{n,d}$ be the positive (n,d) torus link. In a series of works, Elias, Gorsky–Neguţ, Hogancamp, Mellit, and Wilson related $\mathsf{KhR}_{T_{n,d}}$ to expressions originally introduced in purely combinatorial conjectures about "shuffle" operators acting on $\Lambda_{q,t}$ [EH, GN, HM, M21, M22, Wi], motivated by the algebra of Macdonald polynomials [M]. These shuffle conjectures, initiated in [HHLRU] and generalized in [BGLX], have since been proved by Carlsson and Mellit [CM18, M21].

Though results exist for all (n, d), the story is most settled for coprime pairs, where the link is a knot. Here, there is an explicit element $\mathsf{F}_{d/n}(q,t) \in \Lambda^n_{q,t}$, defined via parking functions, such that

$$\text{(5)} \quad \mathsf{KhR}_L(A,q,t) = \frac{\mathsf{Hook}(A, \mathsf{F}_{d/n}(q,t))}{1-q}, \quad \text{where } \mathsf{Hook}(A,-) = \sum_{1\leq i\leq n} A^i \langle s_{n-i+1,i-1} + s_{n-i,i}, -\rangle$$

Hikita related $\mathsf{F}_{d/n}$ to algebraic geometry, though not of the Hilbert schemes discussed earlier.

In general, suppose that $C \subseteq \mathbf{A}^2_{x,y}$ forms an n-fold branched cover of the x-axis, fully ramified at p. Let $K_{C,p}$ be the ring of fractions of $\mathcal{O}_{C,p}$. For each partition $\mu \vdash n$, there is an ind-scheme $\mathcal{F}_{C,p}(\mu)$ defined on points by

$$\text{(6)} \qquad \mathcal{F}_{C,p}(\mu)(\mathbf{C}) = \left\{ (M, F) \;\middle|\; \begin{array}{l} M \subseteq K_{C,p} \text{ is a finite-type } \mathcal{O}_{C,p}\text{-submodule, } K_{C,p}M = K_{C,p}, \\ F \text{ is a } y\text{-stable partial flag on } M/xM \text{ of parabolic type } \mu \end{array} \right\}.$$

The ind-schemes $\mathcal{F}_{C,p}(\mu)$ embed into the partial affine flag varieties studied in Lie theory, where they are known as affine Springer fibers for $G = \mathrm{GL}_n$ [Y17]. Corresponding to $\mathrm{SL}_n \subseteq \mathrm{GL}_n$ is a connected component $\mathcal{F}^0_{C,p}(\mu) \subseteq \mathcal{F}_{C,p}(\mu)$. Laumon observed [L] that the point counts of (quotients of) these ind-schemes over finite fields compute stable orbital integrals for GL_n and SL_n; soon after, Ngô observed in his proof of the Fundamental Lemma [N] that they are local analogues of the Hitchin fibers mentioned above. We set

$$\mathcal{F}^0_{d/n}(\mu) = \mathcal{F}^0_{C,p}(\mu) \quad \text{when } C = \{y^n = x^d\} \text{ and } p = (0,0), \text{ with } n, d \text{ coprime.}$$

There is a paving of $\mathcal{F}^0_{d/n}(\mu)$ by affine spaces, also studied in [LS, So, GKM]. Hikita implicitly gave a filtration of $\mathcal{F}^0_{d/n}(\mu)$ by unions of paving strata. Writing $\mathcal{F}^0_{d/n}(\mu, c) \subseteq \mathcal{F}^0_{d/n}(\mu)$ for the union of the strata added in degree c, we can state Hikita's result as

$$\langle h_\mu, \mathsf{F}_{d/n}(q,t)\rangle = \sum_c q^c \chi(\mathcal{F}^0_{d/n}(\mu, c), t) \qquad \text{for all } \mu,$$

where h_μ is the complete homogeneous symmetric function for μ [H]. For trivial μ, the left-hand side is a rational q,t-Catalan number, and the identity is due to Gorsky–Mazin [GM].

From Gorsky–Oblomkov–Rasmussen–Shende, one might expect Hikita's filtration on $\mathcal{F}^0_{d/n}(\mu)$ to be related to the cohomological perverse filtration on a Hitchin fiber. This would be striking because the former is elementary in nature: No elementary definition of the latter is known.

2. Results from Prior NSF Support: MSPRF

At MIT, I was supported by an NSF Mathematical Sciences Postdoctoral Fellowship (MSPRF), 2020–2023 (grant DMS-2002238, "Algebraic Braids in Geometric Representation Theory", \$150 000).

2.1. Intellectual Merit.

Intellectual Merit. Below, I describe two broad directions in which I contributed to the topics of Section 1 while supported by the MSPRF. Throughout, $\mathsf{KhR}_L(A,q,t) \in \mathbf{Z}[A,t][\![q]\!]$ is a series normalized so that if $L = \hat{\beta}$ for a braid β of writhe e on n strands, then

$$\mathsf{KhR}_U(A,q,t) = \frac{1+A}{1-q}, \qquad \mathcal{P}^{\mathrm{DGR}}_L(a,q,t) = (aq^{-1})^{e-n+1} \frac{\mathsf{KhR}_L(a^2t, q^2, q^2t^2)}{\mathsf{KhR}_U(a^2t, q^2, q^2t^2)},$$

where U is the unknot, and $\mathcal{P}^{\mathrm{DGR}}_L(a,q,t)$ is the reduced version of the link invariant described by Dunfield–Gukov–Rasmussen in [DGR].

2.1.1. *KhR via Springer Theory.* Let $G = \mathrm{GL}_n$. In [T21], I upgraded (2) and (3) to a formula for the *full* KhR homology of *any* positive braid closure, in terms of the G-equivariant cohomology of a (derived) scheme whose G-quotient is essentially a Betti moduli space. The only previous results in this direction had been work of Khovanov [Kh], computing KhR in terms of the Hochschild (co)homology of Soergel bimodules, and work of Webster–Williamson [WW08, WW11, WW17], expressing the latter in terms of the cohomology of character sheaves on G. My work was the first to embed the full KhR homology into the cohomology of a single space.

To state the result, let $\mathcal{B}$ be the variety of complete flags in $\mathbf{C}^n$. By Bruhat, G-orbits on $\mathcal{B} \times \mathcal{B}$ are indexed by elements $w \in S_n$. We write $F \xrightarrow{w} F'$ to indicate that a pair of flags (F, F') belongs to the wth orbit, *a.k.a.*, relative position w. We write $\sigma_w \in Br_n$ for the positive braid lift of w in which no two strands cross more than once. Let $\mathcal{U} \subseteq G$ be the subvariety of unipotent elements. The appearance of $\mathcal{U}$ is a new feature of my work, missing in [STZ, STWZ, GL].

Definition 1 ([T21])**.** For any sequence of simple reflections $\vec{s} = (s_1, \ldots, s_\ell)$ in S_n, defining a positive braid $\beta = \sigma_{s_1} \cdots \sigma_{s_\ell}$, let

$$\mathcal{U}(\beta) = \mathcal{U}(\vec{s}) := \{(u, \vec{F}) \in \mathcal{U} \times \mathcal{B}^\ell \mid u^{-1}F_\ell \xrightarrow{s_1} F_1 \xrightarrow{s_2} \cdots \xrightarrow{s_\ell} F_\ell\}.$$

If sequences $\vec{s}$ and $\vec{t}$ differ by a braid relation, then by a result stated in [D, BM97], there is an explicit, G-equivariant isomorphism $\mathcal{U}(\vec{s}) \simeq \mathcal{U}(\vec{t})$ commuting with their projections to $\mathcal{U}$. Writing $\tilde{\mathcal{U}} = \{(u, F) \in \mathcal{U} \times \mathcal{B} \mid uF = F\}$ for the Springer resolution, let

$$\mathcal{Z}(\beta) = \tilde{\mathcal{U}} \times^{\mathrm{L}}_{\mathcal{U}} \mathcal{U}(\beta),$$

where $\times^{\mathrm{L}}_{\mathcal{U}}$ is the (derived) fiber product with respect to the forgetful maps to $\mathcal{U}$.

Theorem 2 ([T21])**.** *For any positive braid $\beta \in Br_n$, there is a Springer-type action of S_n on the GL_n-equivariant cohomology of $\mathcal{Z}(\beta)$ with compact support, such that*

$$\begin{aligned} \textit{if} \quad & \mathsf{Tr}_\beta(v, t) := \mathsf{Frob}\left(\sum_{j,k} v^j t^k [\mathrm{gr}^{\mathsf{W}}_j \, \mathrm{H}^k_{c,G}(\mathcal{Z}(\beta), \mathbf{C})]\right) \in \Lambda^n_{v,t}, \\ \textit{then} \quad & \mathsf{KhR}_{\hat{\beta}}(A, q, q^{\frac{1}{2}}t) = \mathsf{Hook}(A, \mathsf{Tr}_\beta(q^{\frac{1}{2}}, t)), \end{aligned}$$

where Frob *and* Hook *are defined by* (4) *and* (5)*. Above, we normalize the cohomological and weight gradings to avoid monomial prefactors.*

In [T21], I conjecture that $\mathsf{Tr}_\beta(q, t)$ matches the *underived horizontal trace* of the Rouquier complex of β in $\mathsf{K}^b(\mathsf{SBim}_n)$, in the notation of Gorsky–Wedrich [GW].

The bulk of the proof is bridging the Khovanov–Webster–Williamson approach with the sheaf theory that computes the desired cohomology. The former takes place in a homotopy category $\mathsf{K}^b(\mathsf{C}(G))$, where $\mathsf{C}(G)$ is a graded additive category built from semisimplified character sheaves over G and their Tate twists; the latter takes place in the usual mixed derived category $\mathsf{D}^b_{mix,G}(\mathcal{U})$. The bridge is a *weight realization functor*, inspired by works of Bezrukavnikov–Yun [BY] and Rider [R], in turn inspired by Beilinson's work [B]. The precondition for such a functor is an Ext-purity that fails for the relevant sheaves over G but holds for their restrictions to $\mathcal{U}$.

Building on my work, Bezrukavnikov–Boixeda Alvarez–McBreen–Yun have recently shown how to interpet (torus bundles over) the stacks $[\mathcal{Z}(\beta)/G]$ as twisted, wild Betti moduli spaces [BBMY]. For braids β such that $\hat{\beta} = T_{n,d}$, they also introduce moduli spaces of Higgs bundles that they conjecture to be homeomorphic to the Betti moduli spaces, on $\mathbf{C}$-points. When n and d are coprime, they show that the former retract onto the affine Springer fibers $\mathcal{F}_{d/n}(1^n)$ described earlier. Their work also extends to the setting of Section 3.1.1 below, where G can be any connected reductive group. However, their proposed homeomorphism is not induced by a nonabelian Hodge correspondence, but by a one-parameter twistor deformation, and their moduli spaces are not obviously the twisted, wild moduli spaces of the usual story.

Conjecture 3 (T)**.** *If $\hat{\beta} = L_{C,p}$ for some (C, p), then the weight-filtered cohomology of $[\mathcal{Z}(\beta)/G]$ is isomorphic to the perverse-filtered cohomology of a twisted, wild Dolbeault moduli space.*

The variety $\mathcal{X}(\beta)$ studied in [STZ, STWZ, GL], and more recently, by several research groups in cluster algebra [CGGS, CGGLSS, GLS, GLSS, SW], turns out to be the fiber of $\mathcal{U}(\beta) \to \mathcal{U}$ above $1 \in \mathcal{U}$. There is a remarkable duality between these two types of variety. As in (2), set

$$\mathsf{KhR}^{\mathrm{hi}}_L(q,t) = \lim_{A\to 0} A^{-n}\mathsf{KhR}_L(A,q,t), \quad \text{and similarly,} \quad \mathsf{KhR}^{\mathrm{lo}}_L(q,t) = \mathsf{KhR}_L(0,q,t).$$

Using Theorem 2 and Springer theory [Sp76, Y17], I showed:

Theorem 4 ([T21])**.** *In the setting of Theorem 2, we have*

$$\mathsf{KhR}^{\mathrm{hi}}_{\hat{\beta}}(q, q^{\frac{1}{2}}t) = \sum_{j,k} q^{\frac{j}{2}} t^k \dim \mathrm{gr}^{\mathsf{W}}_j \mathrm{H}^k_{c,G}(\mathcal{X}(\beta)), \quad \mathsf{KhR}^{\mathrm{lo}}_{\hat{\beta}}(q, q^{\frac{1}{2}}t) = \sum_{j,k} q^{\frac{j}{2}} t^k \dim \mathrm{gr}^{\mathsf{W}}_j \mathrm{H}^k_{c,G}(\mathcal{U}(\beta))$$

with the same normalizations as there. The second identity generalizes (3) *to all positive braids.*

These identities have a surprising corollary. Let $\Delta^2 = \Delta_n^2 \in Br_n$ be the full twist, a positive braid generating the center of Br_n. Gorsky–Hogancanp–Mellit–Nakagane [GHMN], categorifying a theorem of Kálmán for the HOMFLYPT polynomial [K], showed that $\mathsf{KhR}^{\mathrm{hi}}_{\beta\hat{\Delta}^2} = \mathsf{KhR}^{\mathrm{lo}}_{\hat{\beta}}$ for any integer n and braid $\beta \in Br_n$. We deduce:

Corollary 5 ([T21])**.** *There is a bigraded isomorphism* $\mathrm{gr}^{\mathsf{W}}_* \mathrm{H}^*_{c,G}(\mathcal{X}(\beta\Delta^2)) \simeq \mathrm{gr}^{\mathsf{W}}_* \mathrm{H}^*_{c,G}(\mathcal{U}(\beta))$.

In [T22], I explain how this isomorphism categorifies a point-counting identity due to Kawanaka [Kaw]. Without the "$\mathrm{gr}^{\mathsf{W}}_*$" on both sides, it is also a Betti analogue of an isomorphism that Oblomkov–Rasmussen–Shende observed in the setting of the Hilbert schemes $\mathcal{H}^{\ell,r}_{C,p}$ [ORS, §2.2], where insertion of a full twist corresponds to blowup of (C, p).

As the isomorphism in Corollary 5 preserves weights, it is natural to wonder if it is induced by an algebraic map. In [T22], I showed that the following conjecture implies that a specific G-equivariant, algebraic map $\mathcal{X}(\beta\Delta^2) \to \mathcal{U}(\beta)$ would induce a homotopy equivalence on $\mathbf{C}$-points. Fix opposed flags $F_\pm$ with stabilizers $B_\pm \subseteq G$, and let $U_\pm \subseteq B_\pm$ be the unipotent radicals.

Conjecture 6 ([T22])**.** *For any flag $F \in \mathcal{B}$, the map*

$$\Phi : \{(x,y) \in U_+ \times U_- \mid xyF_+ = F\} \to \{u \in \mathcal{U} \mid uF_+ = F\}$$

given by $\Phi(x,y) = xyx^{-1}$ induces a homotopy equivalence on $\mathbf{C}$-points.

The domain of Φ is smooth, whereas the target is generally singular. For $n = 3$ and $F = F_-$, the map Φ fails to be even a homeomorphism, yet remains a homotopy equivalence.

2.1.2. *KhR via Quot Schemes.* I now discuss contributions that involve a plane curve germ (C, p). In work with Kivinen, we initiate the study of a new kind of Quot scheme for (C, p), and use it to formulate a more tractable analogue of the ORS conjecture [KT].

For convenience, let $R = \mathcal{O}_{C,p}$. As in the discussion of Hikita's work in Section 1, suppose that C is presented as an n-fold branched cover of the x-axis, fully ramified at p. Let $\tilde{R}$ be the normalization of R. For instance, if $R = \mathbf{C}[\![x,y]\!]/(y^3 - x^4) \simeq \mathbf{C}[\![t^3,t^4]\!]$, as in the example on the first page, then $\tilde{R} = \mathbf{C}[\![t]\!]$. For any partition $\mu \vdash n$, let

$$\mathcal{H}^{\ell}_{C,p}(\mu) = \left\{ (I,F) \;\middle|\; \begin{array}{l} I \subseteq R \text{ is an ideal such that } \dim_{\mathbf{C}}(R/I) = \ell, \\ F \text{ is a } y\text{-stable partial flag on } I/xI \text{ of type } \mu \end{array} \right\},$$
$$\mathcal{Q}^{\ell}_{C,p}(\mu) = \left\{ (M,F) \;\middle|\; \begin{array}{l} M \subseteq \tilde{R} \text{ is an } R\text{-submodule such that } \dim_{\mathbf{C}}(\tilde{R}/M) = \ell, \\ F \text{ is a } y\text{-stable partial flag on } M/xM \text{ of type } \mu \end{array} \right\}.$$

There exist symmetric functions $\mathsf{Hilb}_{C,p}(q,t), \mathsf{Quot}_{C,p}(q,t) \in \Lambda_{q,t}$ characterized by

$$\left.\begin{array}{r} \langle h_\mu, \mathsf{Hilb}_{C,p}(q,t)\rangle = \sum_\ell q^\ell \chi(\mathcal{H}^\ell_{C,p}(\mu), t), \\ \langle h_\mu, \mathsf{Quot}_{C,p}(q,t)\rangle = \sum_\ell q^\ell \chi(\mathcal{Q}^\ell_{C,p}(\mu), t) \end{array}\right\} \quad \text{for all } \mu \vdash n.$$

It was essentially observed in [GORS] that the ORS conjecture can be reformulated in terms of the flag Hilbert schemes $\mathcal{H}^{\ell}_{C,p}(\mu)$, instead of the schemes $\mathcal{H}^{\ell,r}_{C,p}$, as the identity

$$\mathsf{KhR}_{L_{C,p}}(A, q, q^{\frac{1}{2}}t) = \mathsf{Hook}(A, \mathsf{Hilb}_{C,p}(q,t)). \tag{7}$$

Thus, the ORS conjecture would be implied by the following two conjectures combined:

Conjecture 7 ([KT], "Hilb-vs-Quot")**.** *In the setup above,*

$$\mathsf{Hilb}_{C,p}(q,t) = \mathsf{Quot}_{C,p}(q, q^{\frac{1}{2}}t).$$

Conjecture 8 ([KT], "Quot-vs-KhR")**.** *In the setup above,*

$$\mathsf{KhR}_{L_{C,p}}(A,q,t) = \mathsf{Hook}(A, \mathsf{Quot}_{C,p}(q,t)).$$

Even though Conjectures 7–8 are apparently unrelated to Shende's nonabelian Hodge program, there are several reasons why we advocate refining the ORS conjecture in this way.

First, note that Conjecture 7 is a statement purely about two kinds of Quot scheme, without reference to link homology. We expect it to be a wall-crossing identity in the sense of [DHS, Mau] in disguise, and we believe it invites generalization beyond plane curves.

Recall the ind-schemes $\mathcal{F}_{C,p}(\mu)$ from (6) and their sub-ind-schemes $\mathcal{F}^0_{C,p}(\mu)$. The level sets of

$$c : \mathcal{F}_{C,p}(\mu)(\mathbf{C}) \to \mathbf{Z}_{\geq 0} \qquad \text{defined by } c(M) = \dim_{\mathbf{C}}((\tilde{R}M)/M)$$

define strata $\mathcal{F}_{C,p}(\mu, c) \subseteq \mathcal{F}_{C,p}(\mu)$ that generalize Hikita's strata $\mathcal{F}^0_{d/n}(\mu, c) \subseteq \mathcal{F}^0_{d/n}(\mu)$. Hence, the following identity relates $\mathsf{Quot}_{C,p}(q,t)$ to his work. As we explain in [KT], it also shows that Conjecture 7 would imply a conjecture of Cherednik's about unibranch germs [C, Conj. 4.5].

Theorem 9 ([KT])**.** *If b is the number of branches of (C,p), then*

$$\sum_\ell q^\ell [\mathcal{Q}^\ell_{C,p}(\mu)] = \frac{1}{(1-q)^b} \sum_c q^c [\mathcal{F}_{C,p}(\mu,c)/\Lambda]$$

in the Grothendieck ring of varieties, where Λ is a lattice of rank b acting freely on $\mathcal{F}_{C,p}(\mu)$ and stabilizing $\mathcal{F}_{C,p}(\mu, c)$ for all c.

Moreover, this theorem shows that Conjecture 7 would imply a purely elementary construction of the perverse filtration $\mathsf{P}_{\leq *}$ mentioned at the end of Section 1. To be more precise, let $\mathcal{F}_{C,p} = \mathcal{F}_{C,p}((n))$ and $\mathcal{F}_{C,p}(c) = \mathcal{F}_{C,p}((n), c)$. Let $\mathsf{Q}^{\geq *}$ be the decreasing filtration

$$\mathsf{Q}^{\geq c}\mathrm{H}^*(\mathcal{F}_{C,p}/\Lambda) = \ker(\mathrm{H}^*(\mathcal{F}_{C,p}/\Lambda) \to \mathrm{H}^*(\mathcal{F}_{C,p}(\leq c)/\Lambda)), \quad \text{where } \mathcal{F}_{C,p}(\leq c) = \textstyle\bigcup_{c' \leq c} \mathcal{F}_{C,p}(c').$$

The perverse filtration for Hitchin systems has an analogue for $\mathcal{F}_{C,p}/\Lambda$, defined via [MY, Thm. 3.11]. Under Theorem 9, the Hilb-vs-Quot Conjecture 7 would imply:

Conjecture 10. $\mathrm{H}^k(\mathcal{F}_{C,p}/\Lambda) = \mathsf{P}_{\leq j}\,\mathrm{H}^k(\mathcal{F}_{C,p}/\Lambda) \oplus \mathsf{Q}^{\geq j-k}\,\mathrm{H}^k(\mathcal{F}_{C,p}/\Lambda)$ *for all* (C,p), j, k.

Yet another motivation to introduce Conjectures 7–8 is that we can prove Conjecture 8 in full for many cases of the form $y^n = x^d$. No analogous result keeping all three of the variables A, q, t is known for the original ORS conjecture, despite erroneous claims in the literature.

Theorem 11 ([KT])**.** *Suppose that* (C,p) *is (étale) locally isomorphic as a branched cover of the* x*-axis to* $y^n = x^d$ *for some* d*, over* $x = 0$*. If* d *is either coprime to or a multiple of* n*, then Conjecture 8 holds for* (C,p)*.*

We give two proofs in the coprime case: For instance, one relies on Theorem 9 to relate the Quot side to Hikita's series $\mathsf{F}_{d/n}$, and hence, to KhR homology via the long chain of works [EH, GN, HM, M21, Wi]. The proof in the case where n divides d is totally separate, relying instead on [GH, CM20, CM21]. In particular, this case requires a formula from [GH] whose proof, in turn, requires the K-theory of the Hilbert scheme of the affine plane.

A final motivation for Conjecture 8, explained in [KT], is that it can be augmented to include actions of $\mathbf{C}[\vec{x}] = \mathbf{C}[x_1, \ldots, x_b]$ on both the KhR side and the Quot side. Moreover, the KhR side can be extended to the y-ified homology introduced by Gorsky–Hogancamp in [GH], and the Quot side to a torus-equivariant refinement; then the $\mathbf{C}[\vec{x}]$-action can be extended to a $\mathbf{C}[\vec{x}, \vec{y}]$-action. None of these refinements seems to have an analogue for the original ORS conjecture.

2.2. **Broader Impacts.** Throughout most of 2021 and 2023, I volunteered with the MIT PRIMES program, mentoring two high-school students and co-advising three others. In each case, I designed an original research project. In 2021, one mentee and one co-advisee entered the Regeneron STS competition. The co-advisee, whose project concerned a braid invariant sketched in Gauss's *Handbuch*, was named a 2022 Regeneron Scholar, indicating a spot among the top 300 entries. The mentee, whose project [RT] concerned the topological entropy of simple braids, won Tenth Place in the competition. The mentee was later named a 2022 Davidson Fellow for the same project.

In 2022–23, I mentored three undergraduates through the MIT Undergraduate Research Opportunties Program (UROP), each for a semester, again on research projects I designed myself. In 2023, I mentored two undergraduates in the Summer Program in Undergraduate Research (SPUR), on a number-theory project based on ideas from B. Poonen and A. Sutherland.

I also designed and advised a research project for the 2023 PROMYS summer math camp. This project, about generalizing rational tangles and their $\mathrm{SL}_2(\mathbf{Z})$-action by increasing the number of ends of rope, was chosen by an all-female team of students who completed an 11-page report about their results within the six weeks of the camp.

In 2021, I gave an survey talk at the undergraduate level about Gauss's work anticipating knot theory, for the MIT Independent Activities Period (IAP) Mathematics Lecture Series. That year, I also gave multiple expository talks online in the AIM Link Homology research community, on topics at the early-career-researcher level ranging from the Hecke category to affine Springer theory. Since then, I have also given expository service talks to informal student seminars at the University of Edinburgh and UChicago about geometric representation theory.

I care strongly about clear, accessible expository writing. When I taught an elementary number theory course in Spring 2023, I voluntarily posted to my website 90 pages of typed notes for my students. Although my course was based on a textbook by Stillwell, the content of my notes was largely independent, offering many numerical examples not in Stillwell's text. Also at MIT, in Spring 2022, I volunteered beyond my required teaching to give a two-week mini-course about Jones's work on von Neumann algebras.

2.3. **Publications from Prior NSF Support.** For URLs, please see the References page.

[GLTW] P. Galashin, T. Lam, M. Trinh, N. Williams. Rational Noncrossing Coxeter–Catalan Combinatorics. Preprint (2022). Available at `arXiv:2208.00121`

[T21] M. Trinh. From the Hecke Category to the Unipotent Locus. Preprint (2021). Available at `arXiv:2106.07444`

[T22] M. Trinh. Unipotent Elements and Twisting in Link Homology. Preprint (2022). Available at `arXiv:2210.09051`

Evidence and Availability of Research Products. Links to these papers and the slides of accompanying talks are also available on my MIT website.

3. Results from Prior NSF Support: GRFP

At UChicago, I was supported by an NSF Graduate Research Fellowship (GRFP), 2014–15 (award DGE-1144082, no title, \$34 000) and 2017–19 (award DGE-1746045, no title, \$68 000).

3.1. **Intellectual Merit.** The contributions discussed in Section 2.1.1 were inspired by a Lie-theoretic viewpoint that I developed as graduate student at UChicago, prior to MIT. Below I discuss the results I obtained at UChicago.

3.1.1. *DAHAs and Coxeter–Catalan Combinatorics.* As motivation, I note that Theorem 2 and Theorem 4 generalize to any connected reductive G: Replace the symmetric group S_n with the Weyl group W, and the braid group Br_n with the Artin–Tits braid group Br_W. This replaces $\mathsf{Tr}_\beta(v,t) \in \Lambda^n_{v,t}$ with an element $\mathsf{Tr}_\beta(v,t) \in \mathbf{Q}(v,t) \otimes K_0(\mathsf{Rep}(W))$. The link $\hat{\beta}$ does not generalize, but Khovanov's construction of KhR homology via Soergel bimodules generalizes to yield a $\mathbf{Z}[A,t][\![q]\!]$-valued invariant of the *conjugacy class* of $\beta \in Br_W$, decategorifying to Y. Gomi's trace in [G].

The full twist has a generalization $\Delta^2 = \Delta^2_W \in Br_W$ [BMR]. The invariant $\mathsf{Tr}_\beta(v,t)$ is easiest to calculate when β is a power of a root of Δ^2.

Definition 12 ([T21])**.** A braid $\beta \in Br_W$ is *periodic of slope* $\frac{d}{m}$ iff $\beta^m = (\Delta^2)^d$. One can show that this condition only depends on the ratio $\frac{d}{m}$, not on the integers d, m.

These Artin–Tits braids generalize the topological braids whose closures form torus links. In [J], Jones computed the Markov traces of the latter using the character theory of S_n and its Hecke algebra, thereby computing HOMFLYPT for torus knots. Using work of Springer [Sp74] and Lusztig [Lu84], I extended his argument from S_n to W, and from the Markov trace to Tr_β.

The resulting theorem in [T21] involves certain data attached to the irreducible characters of W. For each χ, let $\kappa(\chi) = \frac{1}{\chi(1)} \sum_t \chi(t)$, where t runs over all reflections in W. Let $\mathrm{Deg}_\chi(x) \in \mathbf{Q}[x]$ be the generic degree of the unipotent principal series attached to χ (see Section 4.6).

Theorem 13 ([T21])**.** *If $\beta \in Br_W$ is periodic of slope $\alpha = \frac{d}{m}$, then*

$$\mathsf{Tr}_\beta(v,-1) = \left(\sum_\chi v^{2\alpha(\kappa(1)+\kappa(\chi))} \mathrm{Deg}_\chi(e^{2\pi i \alpha})[V(\chi)]\right) \cdot \left(\sum_{\ell \geq 0} v^\ell [\mathrm{H}^\ell_T(pt)]\right),$$

where $V(\chi)$ is the representation underlying χ, for all χ, and T is the Cartan torus of G.

This strange formula generalizes Jones's calculation and implies, as we explain below, relationships between Tr_β, rational Cherednik algebras, and Catalan combinatorics that generalize observations made in [GORS] for $W = S_n$.

Let $V = \mathrm{H}^2_T(pt)$. The rational Cherednik algebra or double affine Hecke algebra (DAHA) of (W, α) is a certain $\mathbf{C}$-algebra $\mathfrak{H}_{W,\alpha}$ formed from $\mathbf{C}[W] \otimes \mathrm{Sym}^*(V) \otimes \mathrm{Sym}^*(V^\vee)$. It behaves like a quantized ring of differential operators, and holds applications to Calogero–Moser integrable systems and Knizhnik–Zamolodchikov connections [EG]. The BGG category O of a semisimple Lie algebra admits an analogue for $\mathfrak{H}_{W,\alpha}$, where every object M defines a graded character

$$[M]_v = \sum_i v^i [M^i] \in K_0(\mathsf{Rep}(W))[\![v]\!][v^{-1}],$$

and the simple objects are indexed by irreducibles for W [GGOR]. Let $L_\alpha(\chi)$ and $\Delta_\alpha(\chi)$ denote the simple and Verma objects indexed by χ.

Notably, Varagnolo–Vasserot [VV] and Oblomkov–Yun [OY16] gave constructions of $L_\alpha(1)$ that use the K-theory and cohomology, respectively, of specific affine Springer fibers $\tilde{\mathcal{F}}_\gamma$ for G. When $G = \mathrm{SL}_n$ and $\alpha = \frac{d}{n}$, they recover $\mathcal{F}^0_{d/n}(1^n)$. Refining [GORS, Thm. 1.1], I proposed [T20]:

Conjecture 14 ([T20])**.** *If α is a regular elliptic slope for W à la [VV, OY16], then $\mathsf{Tr}_\beta(q, q^{\frac{1}{2}}t)$ is the* bigraded *character of the $\mathfrak{H}_{W,\alpha}$-module $\mathrm{gr}^{\mathsf{P}}_* \, \mathrm{H}^*_{\mathbf{G}_m}(\tilde{\mathcal{F}}_\gamma)^{G_{0,\gamma}}_{\epsilon \to 1}$ in [OY16], which contains $L_\alpha(1)$. (For the notation, see Section 4.6.)*

Theorem 15 ([T20])**.** *In the setting of Theorem 13,*

$$\mathsf{Tr}_\beta(v, -1) = \sum_\chi \mathrm{Deg}_\chi(e^{2\pi i \alpha})[\Delta_\alpha(\chi)]_v \quad \textit{up to an overall prefactor.}$$

On the right-hand side, $[L_\alpha(1)]_v$ occurs with nonzero multiplicity. If W is irreducible and the lowest denominator of α is its Coxeter number, then it is precisely $[L_\alpha(1)]_v$.

When W is irreducible with Coxeter number n, and $\alpha = \frac{d}{n}$ in lowest terms, $[L_\alpha(\chi)]_v$ is the rational parking space for (W, d) studied in [ARR]. The q-series formed by its W-invariants upon setting $v = q^{\frac{1}{2}}$ is the rational q-Catalan number introduced by Gordon–Griffeths in [GG].

Much later, Galashin, Lam, Williams and I used Theorem 15 to solve a problem in Catalan combinatorics going back at least to an AIM workshop held in 2012: the construction, uniformly in (W, d), of a set depending only a Coxeter element $c \in W$ and explicitly enumerated by the $(q = 1)$ rational Catalan number for (W, d), also known as a *c-noncrossing family*. The enumeration is proved by relating a q-deformation of the family to $\mathsf{Tr}_\beta(q^{\frac{1}{2}}, -1)$, for $\beta = \sigma_c^d$ [GLTW].

3.2. **Broader Impacts.** At UChicago, I mentored five undergraduates through the Directed Reading Program on topics including Lie theory, the representation theory of finite groups, analytic number theory, harmonic analysis over number fields, and the combinatorics of finite geometries. I also mentored three undergraduates through the UChicago REU program (actually closer to an RTG). Each of my REU students completed an expository paper more than twenty pages in length, now available on the REU website, which I helped to edit and revise.

In 2016 and 2019, I volunteered as a judge for an event called QED Day, organized by the Math Circles of Chicago, in which middle-school and high-school students gather in an auditorium to present posters about original research projects they pursued. This event offers a supportive venue for many students from underrepresented backgrounds, especially female students. I adjudicated roughly a half-dozen projects each time.

As service, I gave two expository talks—at a large conference at UChicago, in 2017, and a summer school at the Simons Center in Stony Brook, in 2019—and co-authored a 21-page chapter for an introductory textbook on Soergel bimodules based on an MSRI summer school [TT].

3.3. Publications from Prior NSF Support.

[T20] M. Trinh. Algebraic Braids and Geometric Representation Theory. Ph.D. thesis. University of Chicago (2020). 238 pages.

Evidence and Availability of Research Products. This thesis is available on ProQuest.

4. Proposed Research

4.1. It is natural to promote the ORS conjecture (7) and Conjecture 8 to identities in $\Lambda_{q,t}$.

Conjecture 16 (T)**.** *Let (C, p) be an n-fold branched cover of a disk, fully ramified at p, and $\beta \in Br_n$ the positive braid formed by tracing a circle around the disk, so that $\hat{\beta} = L_{C,p}$. Then:*

$$\mathsf{Hilb}_{C,p}(q,t) = \mathsf{Tr}_\beta(q, q^{\frac{1}{2}}t), \tag{8}$$

$$\mathsf{Quot}_{C,p}(q,t) = \mathsf{Tr}_\beta(q,t). \tag{9}$$

4.2. Below are some directions of new research that seem rich in applications, yet still actionable:

- Pushing Conjectures 3, 7, 16, as far as possible for the "toric" cases $y^n = x^d$. In the case of (9), unifying the frameworks of [H, M21, Wi] and [GH, CM20, CM21].
- Reducing Conjecture 8 to a "Macdonald-polynomial" limit, via Shalika expansions of orbital integrals and categorified Jones–Wenzl weights.
- A new family of dualities for the categories O of rational DAHAs, emerging from work on Conjecture 14 and bearing applications to the representations of finite groups of Lie type.

4.3. **The Toric Case.** In what follows, $C = \{y^n = x^d\}$ and $p = (0, 0)$.

4.3.1. $\mathsf{P} = \mathsf{W}$ *for Toric Cases.* With Maulik and Shen, we are studying the $\mathsf{P} = \mathsf{W}$ conjecture in a twisted, wild setting where the Dolbeault moduli spaces retract onto the schemes $\tilde{\mathcal{F}}_{d/n}$ with n, d coprime. This work will build on a separate note by myself that will establish the isomorphism for general $L_{C,p}$, but without the filtrations. The Dolbeault moduli problem for the toric case is close to that of [CDDNP, FN], but with a further tame pole at infinity.

As in [dCMS, MS], two of the key steps are to show that the cohomology on the Dolbeault side is generated by the tautological classes arising from the universal bundle, and that the kth tautological class sides in weight $2k$ on the Betti side, for all k. In the compact setting of *ibid.*, these steps had previously been shown by Markman [Ma] and Shende [S17], respectively. Here the generation argument is mostly supplied by Oblomkov–Yun in [OY17]. By contrast, the statement about weights requires new arguments via cohomological transgression, inspired by [M19].

This project is more ambitious than, though independent of, the others in this subsection: It would completely solve the ORS conjecture for $y^n = x^d$ with n, d coprime.

4.3.2. *Hilb-vs-Quot for $y^3 = x^d$.* The original ORS conjecture is not even known in the toric cases where $n = 3$ and $d \not\equiv 0 \pmod 3$. Here the authors of [ORS] only computed the Hilbert-scheme side, and matched it with conjectural closed formulas for the KhR homology of Dunfield-Gukov–Rasmussen in [DGR]. These conjectural formulas have remained open.

I expect that it is actually easier to show the Hilb-vs-Quot Conjecture 7 in these cases than to verify the link homology formulas directly. For $y^n = x^d$ with n, d coprime, both the Hilbert schemes $\mathcal{H}^\ell_{C,p}(\mu)$ and the Quot schemes $\mathcal{Q}^\ell_{C,p}(\mu)$ are paved by affine spaces whose dimensions are given by explicit combinatorics. After rewriting these schemes in terms of the positive parts of affine Springer fibers for GL_n, as in [GK] or Section 4.4.1 below, one can use results of Goresky–Kottwitz–MacPherson [GKM] to recast $\mathsf{Hilb}_{C,p}$ and $\mathsf{Quot}_{C,p}$ as generating series for the numbers of facets, or more precisely perturbed lattice points, in explicit polytopal regions of a Bruhat–Tits apartment. In this setting, Conjecture 7 becomes a statement about how the regions involved on the two sides can be rearranged into one another by rigid motions. For $n \leq 3$, the regions can be visualized, and their combinatorics are tractable.

Solving these cases would both verify the conjectural closed formulas for $\mathsf{KhR}_{T_{3,d}}$ in [DGR] and prove the corresponding cases of the ORS conjecture, via Theorem 11.

4.3.3. *Hilb-vs-Quot beyond Planar Curves.* We can weaken Conjecture 7 to a non-parabolic version that is more easily generalized. Let $\mathcal{H}^\ell_{C,p}, \mathcal{Q}^\ell_{C,p}$ be the analogues of $\mathcal{H}^\ell_{C,p}(\mu), \mathcal{Q}^\ell_{C,p}(\mu)$ without flags: These can be defined without presenting (C,p) as a branched cover of the x-axis.

Conjecture 17. *For any planar curve germ (C,p), we have $\sum_\ell q^\ell \chi(\mathcal{H}^\ell_{C,p}, t) = \sum_\ell q^\ell \chi(\mathcal{Q}^\ell_{C,p}, q^{\frac{1}{2}} t)$.*

Question 18 (Kivinen–T)**.** What other varieties exhibit an analogue of Conjecture 17?

Surprisingly, Kivinen and I found that for the *nonplanar* curve germ formed by the meeting of the three axes in xyz-space, the Hilb and Quot sides almost match [KT]: They are respectively

$$\frac{1-2q+q^2(t^4+t^2+1)+q^3(t^4-2t^2)}{(1-q)^3} \quad \text{and} \quad \frac{1-2q+q^2(t^2+1)+q^3(t^4-2t^2)+q^4t^4}{(1-q)^3}.$$

This suggests, as a starting point, other nonplanar curve germs admitting open dense torus orbits.

4.3.4. *Equation (9) for all (n,d).* Even though Theorem 11 is proved differently for n,d coprime versus for $n \mid d$, Kivinen and I expect a uniform proof of the stronger claim (9) for all (n,d):

In the coprime case, the key to one of the proofs is recent work of Wilson [Wi], who observed that a recursion developed by Mellit in [M21] for the Hikita polynomial $\mathsf{F}_{d/n}$ specializes under Hook to a recursion developed by Hogancamp–Mellit in [HM] for $\mathsf{KhR}_{T_{n,d}}$. The latter generalizes to the non-coprime setting. Wilson similarly generalized the former, but in a way not matching the non-coprime case of Mellit's setup.

However, we expect that Wilson's expression can be matched with the operator-theoretic expressions in Gorsky–Hogancamp's calculation of the y-ified homology of powers of Δ^2, and in Carlsson–Mellit's study of the corresponding affine Springer fibers in [CM20, CM21]. These are precisely the sources we use in the n-dividing-d case of Theorem 11. This matching would thus unify the proofs we currently have, and suggest that general (n,d) is within reach.

4.4. **Macdonald Polynomials.** The (modified) Macdonald polynomials $\tilde{H}_\lambda(v,t)$, introduced in [M] and indexed by integer partitions λ, form a combinatorially rich basis for $\Lambda_{v,t}$ over $\mathbf{Q}(v,t)$. Most of their constructions are either recursive, inducting along the dominance order on partitions [H03, §3.5], or algebro-geometric, as in Haiman's work expressing them via the K-theory of the Hilbert scheme of the affine plane [H01]. I am interested in two further geometric constructions of these polynomials that bridge the two sides of (9) in Conjecture 16. These should also be related to Haiman's picture, via [GKO] and [GNR, HL], though I omit those details below.

Let $\omega : \Lambda_{v,t} \to \Lambda_{v,t}$ be the involution exchanging $s_\lambda \leftrightarrow s_{\lambda^t}$, where λ^t is the conjugate of λ. Keeping the hypotheses of Conjecture 16, there exist $\Gamma_\lambda(C,p), \Gamma_\lambda(\beta) \in \mathbf{Q}(v,t)$ such that

$$\mathsf{Quot}_{C,p}(v^2,t) = \sum_{\lambda \vdash n} \Gamma_\lambda(C,p)\, \omega \tilde{H}_\lambda(v,t^{-1}), \qquad \mathsf{Tr}_\beta(v^2,t) = \sum_{\lambda \vdash n} \Gamma_\lambda(\beta)\, \omega \tilde{H}_\lambda(v,t^{-1}).$$

With this notation, (9) amounts to showing $\Gamma_\lambda(C,p) = \Gamma_\lambda(\beta)$ for all λ. This is likely out of reach. However, it raises more approachable questions: (1) interpreting $\tilde{H}_\lambda(v,t^{-1})$ geometrically in both identities, and (2) using these interpretations to guess formulas for $\Gamma_\lambda(C,p), \Gamma_\lambda(\beta)$.

4.4.1. *The Quot Side.* I propose that the Macdonald polynomials should be interpreted via certain strata in nilpotent affine Springer fibers, implicit in [M20], and that the coefficients $\Gamma_\lambda(C,p)$ are v-deformations of the so-called Shalika germs of an orbital integral built from (C,p).

In what follows, I work with $G = \mathrm{GL}_n$ and $\mathfrak{g} = \mathfrak{gl}_n$ over a finite field $\mathbf{F}$ rather than over $\mathbf{C}$. The virtual weight polynomials of varieties over $\mathbf{C}$ are given by the point counts of their reductions over $\mathbf{F}$ when the latter are strongly polynomial-count in the sense of [Kat], as is expected here.

Let LG and L^+G be the loop group and arc group respectively defined by $LG(A) = G(A(\!(x)\!))$ and $L^+G(A) = G(A[\![x]\!])$ for any $\mathbf{F}$-algebra A. (We define $L\mathfrak{g}, L^+\mathfrak{g}$ similarly.) For each $\mu \vdash n$, the standard parabolic subgroup of G of type μ lifts to a subgroup $P_\mu \subseteq L^+G$ called the corresponding parahoric. For any $\gamma \in \mathfrak{g}(\mathbf{F}[\![x]\!])$ such that $\det(y-\gamma) = 0$ is the equation defining (C,p), the $\mathbf{F}$-analogue of $\mathcal{F}_{C,p}(\mu)$ is isomorphic to the ind-scheme $\mathcal{F}_\gamma(\mu) \subseteq LG/P_\mu$ defined by

$$\mathcal{F}_\gamma(\mu) = \{[g] \in LG/P_\mu \mid \mathrm{Ad}(g^{-1})\gamma \in \mathrm{Lie}(P_\mu)\},$$

This is the usual definition of the affine Springer fiber over γ of type μ.

The set above admits a left action by LG_γ, the centralizer of γ in LG. It is explained in [Y17, §3] that the mass of the groupoid $[LG_\gamma \backslash \mathcal{F}_\gamma(\mu)](\mathbf{F})$ encodes the value of the orbital integral of γ, a Schwartz distribution on $\mathfrak{g}(\mathbf{F}(\!(x)\!))$ supported on the adjoint orbit of γ, on the indicator function of $\mathrm{Lie}(P_\mu)$. In [Sh], Shalika showed that any such orbital integral, as a functional on the space spanned by such indicator functions, has a linear expansion in terms of the orbital integrals of the *nilpotent* elements of $\mathfrak{g}(\mathbf{F}(\!(x)\!))$. The coefficients of Shalika's expansion are called Shalika germs. Via Theorem 9, the preceding discussion relates $\mathsf{Quot}_{C,p}$ to the Shalika germs of γ.

Indeed, using recent work of Kivinen–Tsai [KTs], in turn based on works of Waldspurger and Howe, it is now possible to match the Shalika germs of γ with the values of the functions $\Gamma_\lambda(C,p)$ at $(v,t) = (1,|\mathbf{F}|)$. It remains to upgrade these formulas to incorporate v.

For nilpotent $\eta \in \mathfrak{g}(\mathbf{F}) \subseteq \mathfrak{g}(\mathbf{F}(\!(x)\!))$, Mellit introduced a condition on points of the ind-schemes $\mathcal{F}_\eta(\mu)$ called *kernel-strictness*. For η of Jordan type λ, he gave a formula for $\tilde{H}_\lambda(v,t)$ in terms of the L^+G_η-orbits of the kernel-strict loci of the $\mathcal{F}_\eta(\mu)$, in which $t = |\mathbf{F}|$ as before, and v tracks the value of $\det(g)$ on any representative g of the given orbit [M20]. I expect that:

Conjecture 19 (T)**.** *Removing the kernel-strictness condition in* [M20, Cor. 5.12 and Thm. 5.13] *corresponds to passing from $\tilde{H}_\lambda(v,t^{-1})$ to $\omega\tilde{H}_\lambda(v,t^{-1})$.*

At the same time, for a particular choice of γ, one can rewrite $\mathsf{Quot}_{C,p}$ in terms of the loci of points $[g] \in \mathcal{F}_\gamma(\mu)$ where $g \in L\mathrm{GL}_n \cap L^+\mathfrak{gl}_n$, also known as the *positive* loci of the $\mathcal{F}_\gamma(\mu)$. In the resulting expression, v once again tracks $\det(g)$, making this proof strategy look promising.

4.4.2. *The* Tr *Side.* I intend to categorify the decomposition of $\mathsf{Tr}_\beta(q,-1)$ in the ring $\mathbf{Q}(v) \otimes K_0(\mathsf{Rep}(W))$ that was described in [T21] for any Weyl group W, and takes the form:

$$\mathsf{Tr}_\beta(v^2,-1) = \left(\sum_\chi (K_\beta : A_\chi)_{v,t}|_{t\to -1}[V(\chi)]\right) \cdot \left(\sum_{\ell \geq 0} v^\ell [\mathrm{H}^\ell_T(pt)]\right).$$

Above, A_χ is the G-equivariant unipotent character sheaf over G with monodromy $V(\chi)$ over the regular semisimple locus, and $(K_\beta : A_\chi)_{v,t}$ is its bigraded multiplicity, in the homotopy category

$\mathsf{K}^b(\mathsf{C}(G))$ mentioned in Section 2.1.1, in a certain complex K_β also used in [STZ, WW17]. Here, t tracks the degree coming from K^b, and v tracks the weight grading on each term of K_β.

This formula is the geometric origin of the formula in Theorem 13. For $W = S_n$, it specializes to Jones–Wenzl's decomposition of their Markov trace into Hecke characters with weights [J].

When $\lambda \vdash n$ indexes χ, the Frobenius character of the term coming from $V(\chi) \otimes \mathrm{H}^*_T(pt)$ is the v^2-plethystic transform of the Schur function s_{λ^t}, in turn known to be $\omega \tilde{H}_\lambda(v, v^{-1})$. Geometrically, this term arises as $\mathrm{Ext}^*_{\mathcal{U}/G}(\mathcal{S}_\chi, \mathcal{S})$, where $\mathcal{S}$ is the Springer sheaf over $\mathcal{U}$ and $\mathcal{S}_\chi = A_\chi|_{\mathcal{U}/G}$ its χ-isotypic summand. But this Ext-group is pure, so its weight filtration is not rich enough to give $\tilde{H}_\lambda(v,t)$. Instead, it seems the right geometric object takes the form $\mathrm{Ext}^*_{\mathcal{U}/G}(A_{\Delta^{2\infty},\chi}|_{\mathcal{U}/G}, \mathcal{S})$, where $A_{\Delta^{2\infty},\chi}$ is some kind of χ-isotypic of $K_{\Delta^{2\infty}} := \varinjlim_k K_{\Delta^{2k}}$ in an ind-completion of $\mathsf{K}^b(\mathsf{C}(G))$, implicit in [HL], and Δ^{2k} refers to the kth power of the full twist. The last formula is motivated by the fact that the Macdonald polynomials are eigenfunctions under an operator ∇ [H03, §3.5.5], and the belief that ∇ corresponds to insertion of a full twist in $\mathsf{K}^b(\mathsf{C}(G))$.

Conjecture 20 (T). *If $W = S_n$ and $\lambda \vdash n$ indexes χ, then $\Gamma_\lambda(\beta) = (K_\beta : A_{\Delta^{2\infty},\chi})_{v,t}$.*

4.5. **Big Diagram.** Keeping the hypotheses of Conjecture 16, (I)–(II) can be pictured as follows.

$$\begin{tikzcd}
\mathsf{P}_{\leq j}\,\mathrm{H}^k(\tilde{\mathcal{M}}^{\mathrm{Dol}}_{C,p}) \arrow[r, leftrightarrow, dotted] \arrow[ddd, leftrightarrow, dashed, "\text{Conjecture 3}"] & \mathsf{P}_{\leq j}\,\mathrm{H}^k(\tilde{\mathcal{F}}_{C,p}) \arrow[rr, leftrightarrow, dashed, "\text{Conjecture 10}"] \arrow[d, leftrightarrow, "\text{[MY, MiSh]}"'] & & \mathsf{Q}^{\geq j-k}\,\mathrm{H}^k(\tilde{\mathcal{F}}_{C,p}) \arrow[d, leftrightarrow, "\text{Theorem 9}"] & \\
& \mathsf{Hilb}_{C,p}(q,t) \arrow[rr, leftrightarrow, dashed, "\text{Conjecture 7}"] \arrow[ddr, leftrightarrow, dashed, "(8)"'] & & \mathsf{Quot}_{C,p}(q, q^{\frac{1}{2}}t) \arrow[r, dashed, "\text{Shalika}"] \arrow[ddl, leftrightarrow, dotted, "(9)"] & \Gamma_\lambda(C,p) \arrow[d, leftrightarrow, dashed, "\text{[M20]?}"'] \\
& & & & \text{Macdonald} \arrow[d, leftrightarrow, dashed] \\
\mathsf{W}_{\leq j}\,\mathrm{H}^k_{c,G}(\mathcal{Z}(\beta)) \arrow[rr, leftrightarrow, "\text{Theorem 2}"] & & \mathsf{Tr}_\beta(q, q^{\frac{1}{2}}t) \arrow[rr, dashed, "\text{categorified Jones–Wenzl}"] & & \Gamma_\lambda(\beta)
\end{tikzcd}$$

The solid arrows are assertions whose proofs are fully available; the dotted (not dashed) arrows are those whose proofs are partially available for $C = \{y^n = x^d\}$. At the top, $\tilde{\mathcal{F}}_{C,p}$ is $\mathcal{F}_{C,p}(1^n)$, equipped with an action of S_n on its cohomology via Springer, and $\tilde{\mathcal{M}}_{\mathrm{Dol}}(C,p)$ is the Dolbeault space of Section 4.3.1. The top left horizontal arrow is a Hitchin-to-affine-Springer comparison, whose analogue for the stack in [BBMY] was constructed for $y^n = x^d$ using a $\mathbf{G}_m$-action.

Conjecture 6 is also hidden in this diagram. More precisely, the top left horizontal arrow and a nonabelian Hodge proof of the leftmost vertical arrow would together explain the cohomological isomorphism of Corollary 5 for β, though not the matching of weights.

4.6. **Dualities for DAHA Blocks.** Finally, I describe work with T. Xue building on Section 3.1.1, with the goal of proving Conjecture 14 among other applications.

4.6.1. First, we review the main result of [OY16]. Let G be a complex, simply-connected, almost-simple group, and let $LG, L\mathfrak{g}$ be as in Section 4.4.1. Let $\alpha = \frac{d}{m}$ be a regular elliptic slope in the sense of [VV, OY16], given in lowest terms. Fixing a maximal torus and Borel, say $T \subseteq B \subseteq G$, we get a $\mathbf{G}_m$-action on LG defined by $t \cdot g(\varpi) = \mathrm{Ad}(t^{-2d\rho^\vee})g(t^{2m}\varpi)$, where $2\rho^\vee$ is the sum of the positive coroots. Let $L\mathfrak{g}_\alpha$ be the weight-$2d$ eigenspace of $L\mathfrak{g}$ under this action, and $L\mathfrak{g}_\alpha^{\mathrm{rs}} \subseteq L\mathfrak{g}_\alpha$ its regular semisimple locus. Let

$$\tilde{\mathcal{F}}_\gamma = \{[g] \in LG/I \mid \mathrm{Ad}(g^{-1})\gamma \in \mathrm{Lie}(I)\} \quad \text{for any } \gamma \in L\mathfrak{g}_\alpha^{\mathrm{rs}},$$

where $I \subseteq LG$ is the Iwahori subgroup lifting B. Then $\tilde{\mathcal{F}}_\gamma$ is a projective variety, isomorphic to Hikita's $\mathcal{F}^0_{d/n}(1^n)$ when $G = \mathrm{SL}_n$ and $m = n$.

Let $G_0 \subseteq LG$ be the connected reductive group whose Lie algebra is the weight-0 eigenspace of $L\mathfrak{g}$, and $G_{0,\gamma}$ the centralizer of γ in G_0. Then $G_{0,\gamma}$ is finite, and the commuting actions of $\mathbf{G}_m$ and G_0 on LG descend to commuting actions of $\mathbf{G}_m$ and $G_{0,\gamma}$ on $\tilde{\mathcal{F}}_\gamma$.

Via a Hitchin-to-affine-Springer comparison, the equivariant cohomology $\mathrm{H}^*_{\mathbf{G}_m}(\tilde{\mathcal{F}}_\gamma)$ is endowed with a perverse filtration $\mathsf{P}_{\leq *}$. Writing Br_α for the fundamental group of $L\mathfrak{g}^{\mathrm{rs}}_\alpha$ based at γ, we can extend the $G_{0,\gamma}$-action on $\tilde{\mathcal{F}}_\gamma$ to a $(G_{0,\gamma} \ltimes Br_\alpha)$-action on $\mathrm{H}^*_{\mathbf{G}_m}(\tilde{\mathcal{F}}_\gamma)$. What Oblomkov–Yun construct is a $\mathfrak{H}_{W,\alpha}$-action, commuting with the Br_α-action, on the bigraded vector space

$$\tilde{\Omega}_\alpha(v,t) := \bigoplus_{j,k} v^j t^k \operatorname{gr}^{\mathsf{P}}_j \mathrm{H}^k_{\mathbf{G}_m}(\tilde{\mathcal{F}}_\gamma)^{G_{0,\gamma}}_{\epsilon \to 1}, \quad \text{where } \epsilon \text{ generates } \mathrm{H}^2_{\mathbf{G}_m}(pt),$$

such that the Br_α-invariants form the simple $\mathfrak{H}_{W,\alpha}$-module $L_\alpha(1)$. When the denominator of α in lowest terms is the Coxeter number of W, the whole module is Br_α-invariant. Thus, Theorem 15 confirms the $t \to -1$ limit of Conjecture 14 in this case.

Xue and I plan to prove Conjecture 14 in full by: (1) determining the Br_α-action explicitly, and (2) proving, via a double-centralizer argument, that $\tilde{\Omega}_\alpha$ manifests a case of a much more general duality, relating the blocks of different rational DAHAs.

4.6.2. *The Braid Action on $\tilde{\Omega}_\alpha$.* For step (1), we use work of Lusztig–Yun and Vilonen–Xue:

Let $\pi_\alpha : \tilde{\mathcal{F}} \to L\mathfrak{g}^{\mathrm{rs}}_\alpha$ be the fibration formed by the affine Springer fibers $\tilde{\mathcal{F}}_\gamma$. Let $\tilde{W}$ be the affine Weyl group of G, and $W_{\alpha\rho^\vee}$ the stabilizer of $\alpha\rho^\vee$ in $\tilde{W}$. By localization [OY16, Cor. 5.4.4], we can decompose $\pi_{\alpha,*}\mathbf{C}$ into a direct sum of terms $h_{\alpha,w,*}\mathbf{C}$, where w runs over finitely many cosets in $W_{\alpha\rho^\vee}\backslash\tilde{W}$ and each map $h_{\alpha,w}$ is a fibration in Hessenberg varieties. There is a Fourier duality between $\mathsf{D}^b_{G_0}(L\mathfrak{g}_\alpha)$ and $\mathsf{D}^b_{G_0}(L\mathfrak{g}_{-\alpha})$ that exchanges each $h_{\alpha,w,*}\mathbf{C}$ with a *spiral induction* complex on $L\mathcal{N}_{-\alpha}$ in the sense of [LY17, LY18], where $L\mathfrak{g}_{-\alpha}$ is the weight-$(-2d)$ eigenspace of $L\mathfrak{g}$ and $L\mathcal{N}_{-\alpha}$ the preimage of zero along the map $L\mathfrak{g}_{-\alpha} \to L\mathfrak{g}_{-\alpha} /\!\!/ G_0$.

Meanwhile, [VX] studies (for $d = 1$) perverse sheaves P_ψ on $L\mathcal{N}_{-\alpha}$ that arise from local systems on $L\mathfrak{g}^{\mathrm{rs}}_{-\alpha}$ via a nearby cycle construction. From [GVX], the Fourier transform of P_ψ is known to be the IC extension of a local system M_ψ on $L\mathfrak{g}_{-\alpha}$ whose monodromy can be computed explicitly.

Conjecture 21 (T–Xue)**.** *The Lusztig–Yun spiral induction complexes can be expressed in terms of the perverse sheaves P_ψ, and hence, $\pi_{\alpha,*}\mathbf{C}$ can be expressed in terms of the local systems M_ψ, all equivariantly with respect to $\mathbf{G}_m \times G_0$. Hence the Br_α-action on $\tilde{\Omega}_\alpha$ is computable.*

4.6.3. *Double Centralizers for DAHA Dualities.* Vilonen–Xue found that the monodromy of M_ψ always factors through a Hecke algebra $H_\alpha(v)$ at $v = 1$. This led Xue and I to propose a curious connection between $\tilde{\Omega}_\alpha$ and works of Broué–Malle–Michel and Uglov.

The root datum of G defines a generic reductive group $\mathbb{G}$ in the sense of [BMM]: a list of features in the corresponding finite Chevalley groups that do not depend on the underlying finite field. In particular, the *unipotent* irreducible characters of these groups, in the sense of Lusztig [Lu78, Lu84], behave generically: They are always indexed by the same finite set $\mathrm{Uch}(\mathbb{G})$. Their degrees are also generic: Each $\rho \in \mathrm{Uch}(\mathbb{G})$ defines a *generic degree* polynomial $\mathrm{Deg}_\rho(x) \in \mathbf{Q}[x]$.

Via his induction and restriction functors, Lusztig partitioned the irreducible characters of finite Chevalley groups into Harish-Chandra series indexed by cuspidal pairs for possibly-nonsplit Levis. In [BMM], Broué–Malle–Michel observed that their effect on unipotent characters is also generic. For each integer $m > 0$, BMM construct a partition $\mathrm{Uch}(\mathbb{G}) = \coprod_{\{(\mathbb{L},\lambda)\}/\sim} \mathrm{Uch}_{\mathbb{G},\mathbb{L},\lambda}$, where the indices are pairs $(\mathbb{L}, \lambda)$ consisting of a generic Levi $\mathbb{L}$ of minimal splitting degree m and a cuspidal

$\lambda \in \mathrm{Uch}(\mathbb{L})$, up to conjugacy. Each $(\mathbb{L}, \lambda)$ defines a finite complex reflection group $W_{\mathbb{G},\mathbb{L},\lambda}$, called its relative Weyl group, and a *Howlett–Lehrer bijection* $\chi_{\mathbb{L},\lambda} : \mathrm{Uch}_{\mathbb{G},\mathbb{L},\lambda} \xrightarrow{\sim} \mathrm{Irr}(W_{\mathbb{G},\mathbb{L},\lambda})$ such that

$$\mathrm{Deg}_\rho(e^{2\pi i/m}) = \varepsilon_{\mathbb{L},\lambda}(\rho) \deg \chi(\rho), \quad \text{where } \varepsilon_{\mathbb{L},\lambda}(\rho) \in \{\pm 1\}.$$

For a split maximal torus $\mathbb{A}$, the set $\mathrm{Uch}_{\mathbb{G},\mathbb{A},1}$ indexes the unipotent principal series characters, and $W_{\mathbb{G},\mathbb{A},1} = W$. Here, $\mathrm{Deg}_\rho = \mathrm{Deg}_{\chi(\rho)}$ in the notation of Conjecture 14.

In [BM93], Broué–Malle defined a Hecke algebra $H_{\mathbb{G},\mathbb{L},\lambda}(v)$ for each cuspidal pair, deforming the group algebra of $W_{\mathbb{G},\mathbb{L},\lambda}$ and generalizing the usual Hecke algebra of W. In [BM97, Br], it is conjectured that $H_{\mathbb{G},\mathbb{L},\lambda}(v)$ forms the ring of $\mathbb{G}$-equivariant endomorphisms of $\mathrm{Ind}_{\mathbb{L}}^{\mathbb{G}}(\lambda)$.

The braid group Br_α in Oblomkov–Yun is precisely the Artin–Tits braid group for $W_\alpha := W_{\mathbb{G},\mathbb{T},1}$, where $\mathbb{T}$ is a generic m-split maximal torus of $\mathbb{G}$. In particular, it maps into $H_\alpha(v) := H_{\mathbb{G},\mathbb{T},1}(v)$. This inspired Xue and I to predict:

Conjecture 22 (T–Xue)**.** *The Hecke algebras in* [VX] *are precisely the algebras $H_\alpha(v)$. Also:*

(1) The actions of $\mathfrak{H}_{W,\alpha}$ and Br_α on $\tilde{\Omega}_\alpha$ centralize each other.
(2) The Br_α-action on $\tilde{\Omega}_\alpha$ factors through $H_\alpha(1)$.
(3) If χ_v is the character of $H_\alpha(v)$ corresponding to χ, then

$$\tilde{\Omega}_\alpha(v, -1) = \sum_{\rho \in \mathrm{Uch}_{\mathbb{G},\mathbb{A},1} \cap \mathrm{Uch}_{\mathbb{G},\mathbb{T},1}} [\Delta_\alpha(\chi_{\mathbb{A},1}(\rho))]_v \otimes \varepsilon_{\mathbb{T},1}(\rho) \chi_{\mathbb{T},1,x}(\rho)|_{x \to 1}$$

in $K_0(\mathbf{C}[W] \otimes H_\alpha(1))[\![v]\!][v^{-1}]$.

Part (1) of this conjecture may follow from recent work of Liu [Liu21, Liu22].

As $\chi_{\mathbb{A},1}$ defines a bijection $\mathrm{Uch}_{\mathbb{G},\mathbb{A},1} \cap \mathrm{Uch}_{\mathbb{G},\mathbb{T},1} \xrightarrow{\sim} \{\chi \in \mathrm{Irr}(W) \mid \mathrm{Deg}_\chi(e^{2\pi i \alpha}) \neq 0\}$, the sum above is consistent with that in Theorem 15. Perhaps surprisingly, Xue and I believe this bijection is a case of a much more general duality, existing for general Levis and not merely tori:

Conjecture 23 (T–Xue)**.** *Let* $(\mathbb{L}, \lambda)$, resp. $(\mathbb{M}, \mu)$ *be an Φ_ℓ-,* resp. *Φ_m-Harish-Chandra series.*

(1) $\chi_{\mathbb{L},\lambda}(\mathrm{Uch}_{\mathbb{G},\mathbb{L},\lambda} \cap \mathrm{Uch}_{\mathbb{G},\mathbb{M},\mu})$ and $\chi_{\mathbb{M},\mu}(\mathrm{Uch}_{\mathbb{G},\mathbb{L},\lambda} \cap \mathrm{Uch}_{\mathbb{G},\mathbb{M},\mu})$ are respectively unions of Φ_m-blocks in $\mathrm{Irr}(W_{\mathbb{G},\mathbb{L},\lambda})$ and Φ_ℓ-blocks in $\mathrm{Irr}(W_{\mathbb{G},\mathbb{M},\mu})$, in the sense of Brauer [GP, Ch. 7]*.*
(2) $\chi_{\mathbb{L},\lambda}$ and $\chi_{\mathbb{M},\mu}$ induce a matching between the two kinds of blocks in (1).
(3) The bijections of (2) categorify to equivalences between the corresponding blocks of the highest-weight covers of the Hecke algebras, which are categories O of rational DAHAs [R08]*.*

Xue and I are writing up proofs of parts (1)–(2) of Conjecture 23 for $\mathbb{G} = \mathbb{GL}_n$. Using abacus combinatorics, we reduce the bijections to the level-rank bijections between bases of higher-level Fock spaces studied by Uglov in [U]. Using similar methods, we are also working to extend the result to other $\mathbb{G}$. We expect to prove part (3) using the ideas of [RSVV, Lo, W].

5. Broader Impacts

Mentoring. I expect the projects in Sections 4.3.2–4.3.3 to lead to (sub)-projects very accessible to younger mathematicians. I hope to continue to expand the breadth of my mentoring, in both subject matter and student background. Going forward, I also plan to devote greater attention to serving underrepresented groups, as in programs like MIT MathRoots next summer.

Service Talks. I will continue to seek opportunities for service talks/courses in the future. I believe they can be deeply grounding for students still adrift in the community—as I once was.

Expository Writing. I am currently working on preparing the notes from my volunteer mini-course on Jones's work (see Section 2.2) to be made available online.