Geometric Representation Theory — Ginzburg

Lecture 1 — 1/7/13

Main object: Hecke algebras

1) Hecke alg controls rep theory of reductive gps / $k = \mathbb{C}, \mathbb{F}_p, \mathbb{Q}_p$

2) Assoc. alg comes with a natural basis
There is another basis (Kazhdan-Lusztig basis) st. structure constants are in $\mathbb{Z}$ (often ≥ 0)

Categorification: Hecke alg H
Try to find a category $\mathcal{C}$ st. $H \simeq K(\mathcal{C})$, $\mathcal{C}$ monoidal category
mult in H induced by $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$

Canonical basis $\longleftrightarrow$ classes of distinguished objects of $\mathcal{C}$.

If struct constants ≥ 0, try to get $\mathcal{C}$ abelian cat.
otherwise " " triangulated

In many cases there are 2 natural candidates for $\mathcal{C}, \mathcal{C}^\vee$ st. $K(\mathcal{C}) \simeq K(\mathcal{C}^\vee) \simeq H$
turns out to be induced by an equivalence $\mathcal{C} \simeq \mathcal{C}^\vee$ in important cases (very mysterious)

G reductive $\rightsquigarrow$ $H = K(\mathcal{C}) = K(\mathcal{C}^\vee)$ where $\mathcal{C}^\vee$ defined in terms of $G^\vee$ Langlands dual of G.

Related to intertwining operators
~~Have~~ Have some category $\mathcal{A}$ of reps of reductive gp G
$\mathcal{A} \ni$ standard objects (principal series)

$F_{\lambda,w,y} : M_{w,\lambda} \longrightarrow M_{y,\lambda}$

For special values of λ, $F_{\lambda,w,y}$ may have poles.
basis from defn ; in KL basis, poles disappear

Hecke alg $=$ alg gen by intertwiners

Abstract framework all repn are in $\mathbb{C}$-vector spaces.

$G \supset K$ finite gps.

Want: decompose $\mathbb{C}(G/K)$ into G-irreps.

More generally let $G \curvearrowright X$ a finite set, $G \curvearrowright \mathbb{C}(X)$

$K \in \mathbb{C}(X \times X)$

$F_K : \mathbb{C}(X) \circlearrowleft \qquad (F_K f)(x) = \sum_{x' \in X} K(x,x') f(x')$

$$\mathrm{End}_G\, \mathbb{C}(X) \hookrightarrow \mathrm{End}_{\mathbb{C}}\, \mathbb{C}(X) = \mathbb{C}(X \times X)$$

$\mathrm{Id} \in \mathrm{End}_G\, \mathbb{C}(X)$, $\quad 1_{diag} \in \mathbb{C}^{G\text{-diag}}(X \times X) = \mathbb{C}(G \backslash (X \times X)) = \mathcal{H}$

Cor (Wedderburn thm)

1) $\mathcal{H}$ is s.s. alg $\quad \dim \mathcal{H} = \#(G \backslash (X \times X))$

2) There is a bijection $E \longleftrightarrow E'$

$$\left\{ \begin{matrix} E \in \mathrm{Irr}\, G \\ [\mathbb{C}(X) : E] \neq 0 \end{matrix} \right\} \overset{\sim}{\longleftrightarrow} \{ E' \in \mathrm{Irr}\, \mathcal{H} \}$$

Schur-Weyl duality

$$\mathbb{C}(X) \simeq \bigoplus_{E' \in \mathrm{Irr}\, \mathcal{H}} E \otimes E'$$

Cor $[\mathbb{C}(X) : E] \le 1 \ \forall E$ iff $\mathcal{H}$ is commutative

$\overset{\shortparallel}{\dim E'}$

Have bijections $\left\{ \begin{array}{l} G\text{-diag orbits} \\ \text{on } G/K \times G/K \end{array} \right\} = K\text{-orbits on } G/K = K\times K$ orbits on G wrt left & right trans.

$X = G/K$

$K \subset G$

$\mathcal{O} \longleftrightarrow \{gK \in G/K \mid (gK/K, K/K) \in \mathcal{O}\} \longleftrightarrow \ldots$

$$\mathbb{C}^G(G/K \times G/K) = \mathbb{C}(G\backslash(G/K \times G/K)) = \mathbb{C}(K\backslash G/K) = \mathbb{C}^{K\times K}(G) \subset \mathbb{C}(G)$$

Convolution on $\mathbb{C}^{K\times K}(G)$ corresponds to the alg structure on $\mathbb{C}^G(G/K \times G/K)$ if measure is normalized so $\mathrm{vol}(K) = 1$.

The unit of $\mathbb{C}^{K\times K}(G) \ni 1_K \cdot \frac{1}{\mathrm{vol}(K)}$

Which $E \in \mathrm{Irr}\, G$ occur in $\mathbb{C}(G/K)$?

$$\mathbb{C}(G) = \bigoplus_{E \in \mathrm{Irr}\, G} E \otimes E^* \qquad \mathbb{C}(K\backslash G) = {}^{K}\mathbb{C}(G) = \bigoplus_E E^K \otimes E^*$$

$$\mathbb{C}(G/K) = \mathbb{C}(G)^K = \bigoplus_E E \otimes (E^*)^K$$

$$[\mathbb{C}(G/K) : E] = \dim (E^*)^K = \dim(E^K)$$

__Language of categories:__ $e_K = \frac{1}{\mathrm{vol} K} 1_K \quad e_K^2 = e_K$

$\mathcal{H} = \mathbb{C}(K\backslash G/K) = e_K \mathbb{C}(G) e_K = \mathrm{End}_G\, \mathbb{C}(K\backslash G)$ Note: $\mathrm{End}_G(\mathbb{C}(G/K)) = (e_K \mathbb{C}(G) e_K)^{op}$

$\mathrm{Rep}\, G$ = f.dim reps of G

$V \longrightarrow V^K \circlearrowleft \mathcal{H} \qquad V^K = e_K V$

Get a functor $\mathrm{inv}^K : \mathrm{Rep}\, G \longrightarrow \mathcal{H}\text{-mod}$

induces an equivalence

$$\frac{\mathrm{Rep}\, G}{\mathrm{Ker}(\mathrm{inv}^K)} \xrightarrow{\sim} \mathcal{H}\text{-mod}$$

$E \longrightarrow (E')^*$ on simple objects

__Reps of p-adic groups__

k nonarchimedean local field (e.g. $\mathbb{Q}_p$, $\mathbb{F}_q((t))$)

k	$\mathbb{Q}_p$	$\mathbb{F}_q((t))$
$\mathcal{O}$	$\mathbb{Z}_p$	$\mathbb{F}_q[[t]]$
$\mathfrak{m}$	$p\mathbb{Z}_p$	$t\mathbb{F}_q[[t]]$

Fix $G \subset GL_n$ an alg subgroup defined over $\mathbb{Z}$.

$G(k) \subset GL_n(k)$ locally compact top. gp.

$G(\mathcal{O})$ is compact open in $G(k)$

__Notation__ $op(G) = \{$open compact subgps of $G(k)\}$

Lem 1) $1 \in G(k)$ has a countable basis of nbhds of forms $K, K \in op(G)$

2) $\forall K \in op(G)$ $\quad G(k)/K$ is countable set.

Pf $\forall N \geq 0$

(Iwasawa decomp for GL_n reduce to solvable reduce to $\mathbb{G}_a, \mathbb{G}_m$)

$$ev_N : G(\mathcal{O}) \longrightarrow G(\mathcal{O}/m^N)$$

$$op(G) \ni G_N = ev_N^{-1}(1), \quad G_N \subset G(\mathcal{O})$$

$G(k)/G_N$ countable set.

E.g. $G = GL_n$

$G_N = \{ g \in M_n(\mathcal{O}) \quad g \equiv 1 \mod m^N \}$

By a rep of $G(k) = G \longrightarrow GL(V)$

V a typically ∞-dim $\mathbb{C}$-vect sp.

Def $V^\infty = \{ v \in V \mid Stab_G(v) \text{ is open subgp} \}$

Lem TFAE:

1) $V = V^\infty$

2) $V = \bigcup_{K \in op(G)} V^K$

3) The action map $G \times V \longrightarrow V$ is continuous.

If this holds one says that V is <u>smooth</u>.

$Rep^\infty G$ = cat of all smooth G-reps.

Lem Rep^∞ is a Serre subcategory of $Rep\ G$.

G loc compact $\Rightarrow \exists$ Haar measure on G

If $K \in op G \Rightarrow vol(K) < \infty$

E.g. $G = (k, +) \supset (\mathcal{O}, +)$

$vol(\mathcal{O}) = 1.$

$vol(m^n) = \dfrac{1}{\#(\mathcal{O}/m^n)}$

Lem For a function f on G with compact support TFAE:

1) f locally constant

2) $\exists K \in op\ G$ st. f is K-biinvariant

Defn $\mathcal{H} = \{ \text{locally const functions } f \text{ on } G \text{ with compact support} \}$

0) $\mathcal{H}$ assoc. alg wrt. convolution wrt. Haar measure

1) $\mathcal{H}$ is $G \times G$-stable.

2) $K \in op\ G \Rightarrow e_K = \frac{1}{vol(K)} 1_K \in \mathcal{H} \qquad e_K^2 = e_K$

$\mathcal{H}_K = \{K\text{-biinvariant elts of } \mathcal{H}\} = e_K \mathcal{H} e_K \qquad e_K$ unit of $\mathcal{H}_K$

3) $\mathcal{H}_* = \varinjlim_{K \in op\ G} \mathcal{H}_K$ ← countable union (can find countable basis)

$\mathcal{H}$ has no unit ; e_K form a δ-sequence

4) $\dim \mathcal{H} \leq$ countable. <u>Pf</u> G/K is countable, $\mathcal{H}_K$ at most countable.

5) $\mathcal{H}$ acts naturaly on any $V \in Rep^\infty$

<u>Lem</u> Have an equiv of categories $Rep^\infty G \xrightarrow{\sim} \left\{ \begin{array}{l} \mathcal{H}\text{-modules } V \\ \text{st. } \mathcal{H}\cdot V = V \end{array} \right\}$

<u>Lecture 2</u> 1/11

<u>fixed</u> K open compact subgp of $G = G(k)$

$\mathcal{H}_K \curvearrowright V^K$ K-fixed vectors.

<u>Thm</u> The functor $inv^K : Rep^\infty G \longrightarrow \mathcal{H}_K\text{-mod} : V \mapsto V^K$ is exact, and moreover

$Rep^\infty V/_{Ker(inv^K)} \xrightarrow{\sim} \mathcal{H}_K\text{-mod}$

<u>Cor. 1</u> If $V \in Rep^\infty G$ is irred then V^K is either simple $\mathcal{H}_K$-mod or zero.

<u>Pf.</u> $V^K = e_K V = e_K \mathcal{H} \underset{\mathcal{H}}{\otimes} V$ $\qquad e = e_K$

functor $(-)^K$ is left exact, tensoring is right exact. Hence functor is exact. // Crucial that K is cpt

---

G/K inherits topology from G.

K open $\Rightarrow$ G/K discrete.

$X \subset G/K$ is compact iff it is finite

$C_c(G/K) =$ f-s with <u>finite supp</u>

$\underset{\text{Smooth rep}}{G \curvearrowright} C_c(G/K) \curvearrowleft \mathcal{H}_K$

$C_c(G) \supset \mathcal{H} \underset{\text{pull back}}{\overset{f \mapsto fe}{\rightleftarrows}} C_c(G/K) = \mathcal{H}\cdot e$

$\forall f \in \mathcal{H}$ $\quad f = fe + (f - fe)$

$\mathcal{H} = \mathcal{H}e \oplus \mathcal{H}'$ where $\mathcal{H}' = \{h \in \mathcal{H} \mid he = 0\}$

Define functor $\Phi : \mathcal{H}_K\text{-mod} \longrightarrow Rep^\infty G$

$$M \longmapsto \mathcal{H}e \underset{\mathcal{H}_K}{\otimes} M = C_c(G/K) \underset{\mathcal{H}_K}{\otimes} M$$

<u>Claim 1)</u> Φ is a left adjoint to inv^K.

$$Hom_G(\Phi(M), V) = Hom_{\mathcal{H}}(\mathcal{H}e \underset{e\mathcal{H}e}{\otimes} M, V) = Hom_{e\mathcal{H}e}(M, V) = Hom_{\mathcal{H}_K}(M, eV) = Hom_{\mathcal{H}_K}(M, V^K)$$

<u>Claim 2.</u> $\mathcal{H}e \underset{\mathcal{H}_K}{\otimes} M \longrightarrow \mathcal{H} \underset{e\mathcal{H}e}{\otimes} M$ is an isomorphism

$$\mathcal{H} \underset{e\mathcal{H}e}{\otimes} M = (\mathcal{H}e \oplus \mathcal{H}') \underset{e\mathcal{H}e}{\otimes} M = \mathcal{H}e \underset{e\mathcal{H}e}{\otimes} M \qquad \text{b/c } \mathcal{H}'e = 0$$

Have adjunction morphism

$$M \longrightarrow inv^K \Phi(M) = \left(\mathcal{H}e \underset{\mathcal{H}_K}{\otimes} M\right)^K$$

<u>Claim 3.</u> this is isomorphism

$$= e\mathcal{H}e \underset{e\mathcal{H}e}{\otimes} M = M$$

General result:

$\mathcal{C} \underset{\Phi}{\overset{F}{\rightleftarrows}} \mathcal{C}'$ abelian categories st.
1) F is exact
2) Φ left adjoint of F
3) $\mathrm{Id}_{\mathcal{C}'} \longrightarrow F\Phi$ is an isom.

Then $\mathcal{C}/\mathrm{Ker}\,F \xrightarrow{\sim} \mathcal{C}'$

This proves the theorem. □

Cor 2. inv^K gives a bijection $\left\{ \begin{matrix} V \text{ irrep}^\infty G \\ V^K \neq 0 \end{matrix} \right\} \xleftrightarrow{\sim}$ Simple $\mathcal{H}_K$-modules.

Direct proof (also follows from Thm):

$V_1^K, V_2^K \neq 0$ $\quad V_1, V_2$ irreps.

Let $\bar{f} : V_1^K \xrightarrow{\sim} V_2^K$ be an $\mathcal{H}_K$-isom

Claim $\bar{f}$ extends uniquely to $f : V_1 \longrightarrow V_2$ as G-reps.

$$\begin{array}{ccc} \Phi(V_1^K) & \xrightarrow[\sim]{\Phi(\bar{f})} & \Phi(V_2^K) \\ \| & & \| \\ \mathcal{H} \otimes_{\mathcal{H}_K} V_1^K & & \mathcal{H} \otimes_{\mathcal{H}_K} V_2^K \\ f_1 \downarrow & & f_2 \downarrow \\ V_1 & \longleftarrow \text{simple} \longrightarrow & V_2 \end{array}$$

Thm $\Rightarrow$ $\mathrm{Ker}\, f_i$, $\mathrm{Coker}\, f_i \in \mathrm{Ker}(\mathrm{inv}^K)$

$\mathrm{Coker}\, f_i = 0 \Rightarrow f_i$ surjective (otherwise $V_i^K = 0$)
apply inv^K again

$\Phi(\bar{f})$ takes $\mathrm{Ker}(\mathrm{inv}^K)$ to itself

$\Rightarrow$ $\Phi(\bar{f}) : V_1 \xrightarrow{\sim} V_2$. □

Schur lemma If $V \in \mathrm{Irrep}^\infty G$, then $\mathrm{End}_G V = \mathbb{C}$

Pf Let $f : V \longrightarrow V$ an intertwiner. $\exists K \in \mathrm{op}\, G$ st. $V^K \neq 0$

f induces an $\mathcal{H}_K$-morphism $\bar{f} : V^K \to V^K$.

Claim: $\bar{f} = f|_{V^K}$ is $\lambda \cdot \mathrm{Id}$.

Pf $\dim_{\mathbb{C}} \mathcal{H}_K$ countable. This forces $\mathrm{End}_{\mathcal{H}_K} M = \mathbb{C}$ for any simple $\mathcal{H}_K$-module M.

Since we know $\bar{f}$ lifts uniquely, $f = \lambda \cdot \mathrm{Id}_V$. □

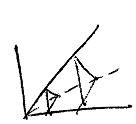
Important example $G = GL_n$

Motivation $k = \mathbb{R}$

$G := GL_n^+(\mathbb{R}) \supset SO(n, \mathbb{R}) =: K$ maximal compact subgroup.

$\cup$ positive det

$D^+ = \{ \mathrm{diag}(\lambda_1, \dots, \lambda_n) \mid \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n > 0 \} \subset \mathbb{R}^n$ abelian semigp



Simplicial cone

Cartan decomposition

$D^+ \xrightarrow{\sim} K \backslash G / K$ homeo of top spaces. $\qquad G = \coprod_{d \in D^+} KdK$ (ignoring top)

Pf: Want to classify K-orbits on G/K.

$X := \{ x \in M_n(\mathbb{R}) \mid x = x^t,\ x \text{ positive definite} \}$

- $G \curvearrowright X$ $\qquad g : x \longmapsto g x g^t$
- $1 \in X$ $\qquad \mathrm{Stab}_G(1) = SO_n(\mathbb{R})$

• $\forall x \in X \quad \exists g \in G \quad x = gg^t$ (classification of quadratic forms)

$\Rightarrow$ G-action on X is transitive

$\Rightarrow$ $X \cong G/K = \dfrac{GL_n^+(\mathbb{R})}{SO_n(\mathbb{R})}$

• If $x \in X$ $\exists v_1, \dots, v_n \in \mathbb{R}^n$ orthonormal basis st. $v_i^t x v_j = \lambda_i \delta_{ij}$ $\lambda_i > 0$ (Spectral thm)

$\exists!$ $k \in SO_n(\mathbb{R})$

$k : v_i \longmapsto e_i$ the standard basis

$k^{-1} x k = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$

$x = gK = k \cdot \mathrm{diag}(\lambda_1, \dots, \lambda_n) k^{-1} K \Rightarrow g \in K \,\mathrm{diag}(\lambda_1, \dots, \lambda_n) K$. $\square$

---

k	$\mathbb{Q}_p$ or	$\mathbb{F}_q((t))$
$\mathcal{O}$	$\mathbb{Z}_p$	$\mathbb{F}_q[[t]]$
t	p	t

generator of the max ideal $\mathfrak{m} \subset \mathcal{O}$. $F = \mathcal{O}/\mathfrak{m}$

F	$\mathbb{F}_p$	$\mathbb{F}_q$

$G = GL_n(k) \supset K := GL_n(\mathcal{O})$

$\mathbb{X}^+ = \{ t^\lambda = \mathrm{diag}(t^{\lambda_1}, t^{\lambda_2}, \dots, t^{\lambda_n}), \ \lambda = (\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n) \ \lambda_i \in \mathbb{Z} \}$

discrete abelian semigroup.

Cartan decomposition.

$\mathbb{X}^+ \xrightarrow{\sim} K\backslash G/K$ set theoretically $GL_n(k) = \coprod_{\lambda \in \mathbb{X}^+} GL_n(\mathcal{O}) t^\lambda GL_n(\mathcal{O})$

<u>Thm</u> $\mathcal{H}_K$ is a commutative algebra.

<u>Pf</u> (Gelfand) $\tau : g \mapsto g^t$, $GL_n \to GL_n$

$\tau(GL_n(\mathcal{O})) \subset GL_n(\mathcal{O})$

$\tau : \mathcal{H}_K \longrightarrow \mathcal{H}_K$ anti-involution $f \mapsto f^\tau(g) = f(g^t)$

$\mathcal{H}_K$ has a natural $\mathbb{C}$basis $\{ 1_{Kt^\lambda K}, \lambda \in \mathbb{X}^+ \}$ by Cartan.

$\tau(1_{Kt^\lambda K}) = 1_{Kt^\lambda K}$ (transpose doesn't change diagonal)

$\Rightarrow \tau = \mathrm{Id}_{\mathcal{H}_K}$ is an anti-involution $\Rightarrow$ $\mathcal{H}_K$ commutative. $\square$

This argument also works for $G = GL_n^+(\mathbb{R}) \supset K = SO_n(\mathbb{R})$
$GL_n(\mathbb{C})$ U_n

<u>Cor</u> Any simple $\mathcal{H}_K$-module is 1-dimensional $K = GL_n(\mathcal{O})$ [use Schur's lemma]

<u>Cor</u> $\left\{ \begin{matrix} V \in \mathrm{Irrep}^\infty GL_n(k) \\ V^{GL_n(\mathcal{O})} \neq 0 \end{matrix} \right\} \xleftrightarrow{\sim} \left\{ \begin{matrix} \text{max ideals} \\ \text{of } \mathcal{H}_K \end{matrix} \right\}$

Pf of Cartan decomp

Want to classify $K = GL_n(\mathcal{O})$-orbits on G/K.

This argument also works for $k = \mathbb{C}((t))$ // NT ⇢ AG

Defn A lattice $L \subset k^n$ is a finitely generated $\mathcal{O}$-submodule s.t. $k \cdot L = k^n$

1) $t^2 L \subset tL \subset L \subset t^{-1}L \subset t^{-2}L \cdots$

Lattice $\Rightarrow \bigcup_{m>0} \frac{1}{t^m} L = k^n$

2) L has no $\mathcal{O}$-torsion

$\mathcal{O}$ is a PID.

$\Rightarrow L$ is free over $\mathcal{O}$. So $\exists v_1, \dots, v_m \in k^n$ st. $L = \mathcal{O}v_1 \oplus \dots \oplus \mathcal{O}v_m$

$k \cdot L = kv_1 \oplus \dots \oplus kv_m \Rightarrow m = n$, v_i form k-basis of k^n.

Gr = set of all lattices in k^n

$G = GL_n(k) \curvearrowright Gr$ $\quad g: L \longrightarrow g(L)$

$\mathcal{O}v_1 \oplus \dots \oplus \mathcal{O}v_n \longrightarrow \mathcal{O}g(v_1) \oplus \dots \oplus \mathcal{O}g(v_n)$

G acts transitively on Gr.

$L_0 = \mathcal{O}e_1 \oplus \dots \oplus \mathcal{O}e_n$ $\quad \mathrm{Stab}_G(L_0) = K$

Conclusion is $Gr = GL_n(k)/GL_n(\mathcal{O})$

Observation $\forall L \in Gr$, $\exists m', m \geq 0$ s.t. $t^{m'} L_0 \subset L \subset t^{-m} L_0$

Modules over PID $\Rightarrow \exists$ basis $v_1, \dots, v_n$ of L_0 and $\mu_1 \geq \mu_2 \geq \dots \geq \mu_n \geq 0$

st. $L = \mathcal{O}t^{\mu_1}t^{-m}v_1 \oplus \mathcal{O}t^{\mu_2}t^{-m}v_2 \oplus \dots \oplus \mathcal{O}t^{\mu_n}t^{-m}v_n$

Put $\lambda_i := \mu_i - m$.

Lecture 3 1/14

$k^n \supset L_0 = \mathcal{O}^n$, Let $m', m \geq 0$ $\quad N := m' + m$. $\quad \mathcal{O}_N = \mathcal{O}/\mathfrak{m}^N$

$Gr_{m,m'} = \{ L \text{ lattice in } k^n \mid t^{m'}L_0 \subset L \subset t^{-m}L_0 \}$

$\wr\!\!\! \mid$

$Gr(\mathcal{O}_N^n) \overset{\text{not requiring injective on fibers}}{=} \{ \mathcal{O}_N\text{-submodules } M \subset \frac{t^{-m}L_0}{t^{m'}L_0} \simeq \frac{L_0}{t^N L_0} \} = \{ M \mid M \text{ is } t\text{-stable subspace } (F^N)^n \}$

$\mathcal{O} = F[\![t]\!]$ where $F = \mathbb{F}_q$ or $\mathbb{C}$

Now $\mathcal{O}_N \cong F^N \ni t$ $\quad L_0/t^N L_0 = \underbrace{F^N \oplus \dots \oplus F^N}_{n}{}^{\circlearrowleft t} = \mathcal{O}_N^n$

$$\boxed{\begin{matrix} Gr_{m',m} \hookrightarrow Gr_{\ell',\ell} \\ \ell' \leq m',\ \ell \leq m \end{matrix} \Rightarrow Gr = \varinjlim_{m',m} Gr_{m',m}.}$$

$Gr^+ = \varinjlim_{m',0} Gr_{m',0} = \{ L \mid L \subset L_0 \}$

$Gr = GL_n(k)/GL_n(\mathcal{O}) = \coprod_{\lambda\ (\lambda_1 \geq \lambda_2 \geq \dots \lambda_n)} GL_n(\mathcal{O})\, t^\lambda\, GL_n(\mathcal{O})$

$\cup$

$Gr^+ = G^+/GL_n(\mathcal{O}) = \coprod_{\lambda_1 \geq \dots \geq \lambda_n \geq 0}$ —— " ——

Hall algebras :

$\mathcal{C}$ finatary abelian category

- Any object has finite length.
- $\# \operatorname{Hom}(M, L) < \infty$

$\overline{Ob} := Ob\,\mathcal{C}/_{\text{Isom.}}$

$H_{\mathcal{C}}$ = $\mathbb{C}$-valued functions on $\overline{Ob}$ with finite supp.

If $M, L \in Ob\,\mathcal{C}$, let μ, λ denote iso-class in $\overline{Ob}$

For $\lambda, \mu, \nu \in \overline{Ob}$

Fix $M \in \mu$, $N \in \nu$, $L \in \lambda$

$$h^{\lambda}_{\mu,\nu} = \# \cancel{\left\{ \begin{array}{l} \text{Choose } L \in \lambda \\ 0 \to M \to L \to N \to 0 \\ \text{st. } M \in \mu,\ N \in \nu \end{array} \right\}} = \frac{\# \left\{ \begin{array}{l} 0 \to M \to L \to N \to 0 \\ M \in \mu,\ L \in \lambda,\ N \in \nu \end{array} \right\}}{\# \operatorname{Aut}(N) \cdot \# \operatorname{Aut}(M)}$$

Define an alg structure on $H_{\mathcal{C}}$

$$1_{\mu} * 1_{\nu} = \sum_{\lambda} h^{\lambda}_{\mu\nu} 1_{\lambda}$$

$f_1, f_2 \in H_{\mathcal{C}}$

$$(f_1 * f_2)(L) = \sum_{M \subset L} f_1(M) f_2(L/M)$$

$$(f_1 * f_2 * f_3)(L) = \sum_{N \subset M \subset L} f_1(N) f_2(M/N) f_3(L/M) \qquad \text{associative.}$$

$\mathcal{C} :=$ $\mathcal{O}$-modules of finite length with $\leq n$ generators

$$\overset{\cup}{M} = \frac{\mathcal{O}}{(t^{\lambda_1})} \oplus \cdots \oplus \frac{\mathcal{O}}{(t^{\lambda_n})} \qquad \lambda_1 \geq \cdots \geq \lambda_n$$

$$\overline{Ob} = \{ \lambda = (\lambda_1 \geq \cdots \geq \lambda_n) \}$$

$$\begin{tikzcd} & & 0 \arrow[d] \\ 0 \arrow[r] & L_0 \arrow[r, "y_j"] \arrow[d, "1"'] & L_0 \arrow[r] \arrow[d, "x_i"] & M_{\mu} \arrow[d] \\ 0 \arrow[r] & L_0 \arrow[r, "x_i y_j"] \arrow[d, "y_j"'] & L_0 \arrow[r] \arrow[d, "id"] & M_{\lambda} \arrow[d] \\ 0 \arrow[r] & L_0 \arrow[r, "x_i"] & L_0 \arrow[r] & M_{\nu} \arrow[d] \\ & & & 0 \end{tikzcd}$$

Prop $\quad \mathbb{C}[GL_n(\mathcal{O}) \backslash G^+ / GL_n(\mathcal{O})] \cong H_{\mathcal{C}}$

$\qquad \cap$

$\quad \mathbb{C}[K \backslash G / K]$

$$(1_{Kt^{\mu}K} * 1_{Kt^{\nu}K})(t^{\lambda}) = \int_{y \in G} 1_{Kt^{\mu}K}(t^{\lambda} y^{-1}) 1_{Kt^{\nu}K}(y)\, dy$$

$Kt^{\mu}K = \amalg K x_i$

$Kt^{\nu}K = \amalg K y_j$ consider $\amalg y_j^{-1} K$ for nicer formula

$$= \sum_j 1_{Kt^{\mu}K}(t^{\lambda} y_j^{-1}) = \# \{ (i,j) \mid t^{\lambda} \in K x_i y_j \} \leftarrow \text{this is Hall number}$$

$K^+(\mathcal{C})$ = Grothendieck semigroup

$$\overline{Ob} \longrightarrow K^+(\mathcal{C})$$
$$M \overset{JH}{\longmapsto} \overset{\cup}{ss(M)}$$

$$H_{\mathcal{C}} = \bigoplus_{s \in K^+(\mathcal{C})} H^{(s)}_{\mathcal{C}}$$ graded by Jordan-Holder algebra.

$$G^+ = GL_n(k) \cap M_n(\mathcal{O}) \supset GL_n(\mathcal{O})$$

$K^+(\mathcal{C}) = \mathbb{Z}^+ \qquad$ Grading = length of M

$$|\det| : GL_n(k) \longrightarrow t^{\mathbb{Z}}$$
$$g \longmapsto \det g \in k^{\times}/\mathcal{O}^{\times} \simeq \mathbb{Z}$$
$$GL_n(\mathcal{O}) \longrightarrow 1$$

$$|\det| : Gr \longrightarrow t^{\mathbb{Z}}$$

$\mathfrak{g} \overset{\text{def}}{=} \mathrm{Lie}\, G$

Let Killing form

$(\, , \,) : \mathfrak{g}(k) \times \mathfrak{g}(k) \to k$

invariant bilinear form

Def A lattice is a f.g. $\mathcal{O}$-submodule $L \subset \mathfrak{g}(k)$ st.

1) $kL = \mathfrak{g}(k)$
2) $[L, L] \subset L$ $\quad (\Rightarrow L \subset L^\vee)$
3) $L^\vee = L$

For any $\mathcal{O}$-submod $L \subset \mathfrak{g}(k)$

$L^\vee := \{ x \in \mathfrak{g}(k) \mid (x, L) \subset \mathcal{O} \}$

$L_0 := \mathfrak{g}(\mathcal{O})$

$Gr_{\mathfrak{g}} :=$ set of lattices $\qquad G(k) \curvearrowright Gr_{\mathfrak{g}}$

$g : L \mapsto \mathrm{Ad}_g(L)$

If G has no center then

$\mathrm{Stab}_{G(k)} L_0 = G(\mathcal{O})$

$G(k)/G(\mathcal{O}) \simeq Gr_{\mathfrak{g}}$

$\lambda \in X_*(T) \rightsquigarrow L_\lambda = \mathfrak{t}(\mathcal{O}) \oplus \bigoplus_{\alpha \in R} t^{\langle \lambda, \alpha^\vee \rangle} \cdot \mathfrak{g}_\alpha(\mathcal{O})$

$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha$

$\uparrow$

$t^\lambda \cdot G(\mathcal{O})/G(\mathcal{O}) \in G(k)/G(\mathcal{O})$

If $k = F((t))$, then $F \hookrightarrow F((t))$

$T(F) \hookrightarrow T(k)$

In particular $T(F) \curvearrowright G(k)/G(\mathcal{O})$

Lem $Gr_{\mathfrak{g}}^{T(F)} \simeq \{ L_\lambda, \ \lambda \in X_*(T) \}$

For each $m \geq 0$

$Gr_m := \{ L \in Gr_{\mathfrak{g}} \mid t^m L_0 \subset L \subset t^{-m} L_0 \}$

$\updownarrow$

$\left\{ V \subset \dfrac{t^{-m}\mathfrak{g}(\mathcal{O})}{t^m \mathfrak{g}(\mathcal{O})} := E \right\}$

~~Skew~~ sym bilinear form $\quad E \times E \xrightarrow{\beta} F \; : \; x, y \longmapsto \mathrm{Res}_0 (x, y)$

3-form on $\dfrac{t^{-m}\mathfrak{g}(\mathcal{O})}{t^{2m}\mathfrak{g}(\mathcal{O})} \xrightarrow{\pi} E \; : \; x \wedge y \wedge z \overset{\Omega}{\longmapsto} \mathrm{Res}_0 (x, [y, z])$

Lem $Gr_m = \{ V \subset E \mid$ st.

1) V is t-stable
2) V is maximal isotropic wrt β
3) $\Omega|_{\pi^{-1}(V)} \equiv 0$.

Lecture 4 $\quad$ 1/18

Formal Smoothness (Grothendieck)

A a comm ring

$\cup$

$\mathfrak{a}$ nilpotent ideal $\qquad f : X \to Y$ smooth mor of schemes

Thm If $f : X \to Y$ smooth, then ~~given $\mathrm{Hom}_Y(\mathrm{Spec}\, A, X) \twoheadrightarrow \mathrm{Hom}_Y(\mathrm{Spec}\, A/\mathfrak{a}, X)$~~

$$\begin{array}{ccc} \mathrm{Spec}\, A/\mathfrak{a} & \longrightarrow & X \\ \downarrow & \overset{\exists}{\nearrow} & \downarrow \\ \mathrm{Spec}\, A & \longrightarrow & Y \end{array}$$

$$\begin{array}{l} Gr \\ \cup \\ Gr^+ \ni L \longrightarrow \dfrac{L_0}{L} \longrightarrow \left|\det\left(\dfrac{L_0}{L}\right)\right| = t^{\lambda_1+\dots+\lambda_n} \end{array}$$

$$\frac{L_0}{L} = \frac{\mathcal{O}}{t^{\lambda_1}} \oplus \dots \oplus \frac{\mathcal{O}}{t^{\lambda_n}}$$

In general $\forall L \in Gr$, find $m \gg 0$ $L \subset t^{-m}L_0$, take $\left|\det\left(\frac{t^{-m}L_0}{L}\right)\right| \Big/ \left|\det \frac{t^{-m}L_0}{L_0}\right|$

$F = \mathbb{C}$ $\quad Gr(F) \xrightarrow{|\det|} \mathbb{Z}$

$\pi_0(Gr(\mathbb{C})) \xrightarrow{\sim} \mathbb{Z}$

$G = SL_n$ $\quad SL_n(k)/SL_n(\mathcal{O}) \simeq Gr_0(F)$

$Gr = \coprod_{i\in\mathbb{Z}} Gr_i$ where $Gr_i = \{L \in Gr \mid |\det L| = t^i\}$

$\cup$

Gr^+ $\quad Gr_0 \cap Gr^+ = \{L_0\}$

$G = PGL_n$ $\quad G(k)/G(\mathcal{O}) = Gr/t^{\mathbb{Z}} = \coprod Gr_{i \bmod n}$ $\quad \left|\det\left(\frac{t\cdot L_0}{L_0}\right)\right| = t^n$

$\pi_0\left(PGL_n(k)/PGL_n(\mathcal{O})\right) = \mathbb{Z}/n\mathbb{Z}$

$GL_n(\mathcal{O})L_0 = \{L_0\}$ $\quad e_1, \dots, e_n$ basis of k^n

$L_i = \mathcal{O}e_1 \oplus \dots \oplus \mathcal{O}e_{n-i} \oplus \mathfrak{m}e_{n-i+1} \oplus \dots \oplus \mathfrak{m}e_n$

Tori $\quad T \simeq (\mathbb{G}_m)^r$ alg gp

$X^*(T) = \mathrm{Hom}_{\text{alg gp}}(T, \mathbb{G}_m) \simeq \mathbb{Z}^r$ $\quad$ character lattice

$X_*(T) = \mathrm{Hom}_{\text{alg gr}}(\mathbb{G}_m, T) = \mathrm{Hom}(X^*(T), \mathbb{Z})$

$\overset{\shortparallel}{\mathbb{Z}^r}$ not canonical

Canonical pairing $\quad X^* \times X_* \longrightarrow \mathbb{Z}$

$\lambda^\vee, \mu \longmapsto \langle \lambda^\vee, \mu\rangle$ st. $\quad \mathbb{G}_m \xrightarrow{\mu} T \xrightarrow{\lambda^\vee} \mathbb{G}_m$

$z \longmapsto z^{\langle\lambda^\vee,\mu\rangle}$

$G \subset GL_n$ reductive $/k$

$\cup$

T max torus, split $\equiv$ eigenvalues of $\forall t \in T$ belong to k.

$\lambda \in X_*(T)$ $\quad \lambda : \mathbb{G}_m \to T$ $\quad R \subset X_*(T)$ root system

$t^\lambda := \lambda(t) \in T(k)$ $\quad R^\vee_+ \subset R^\vee \subset X^*(T)$

$X_* \hookrightarrow T(k)$

$\cup$

$X_*^+ = \left\{\lambda \in X_* \;\middle|\; \begin{array}{l}\langle\lambda,\alpha^\vee\rangle \geq 0 \\ \forall \alpha^\vee \in R^\vee_+\end{array}\right\}$

Cartan decomposition

$$G(k) = \coprod_{\lambda\in X_*^+(T)} G(\mathcal{O})t^\lambda G(\mathcal{O})$$

Concrete realization of $G(k)/G(\mathcal{O})$ (due to Lusztig)

Let $(\mathcal{O}, \mathfrak{m})$ local ring complete in $\mathfrak{m}$-adic topology. $F = \mathcal{O}/\mathfrak{m}$

$\varphi : X \to Y$ smooth morphism

$$\begin{array}{ccc} X(\mathcal{O}) & \xrightarrow{\tilde{\varphi}} & Y(\mathcal{O}) \\ \downarrow & & \downarrow \\ X(F) & \xrightarrow{\varphi} & Y(F) \end{array}$$

φ surjective $\Rightarrow$ $\tilde{\varphi}$ is surjective

G reductive connected gp

$\mathfrak{g} = \bar{\mathfrak{n}} \oplus \mathfrak{t} \oplus \mathfrak{n}$ triangular decomp.

$G \supset \bar{N}, T, N$

T max torus, $B = T \cdot N$, $\bar{B} = T \cdot \bar{N}$, $W = N(T)/T$ Weyl group

$X_*(T) = \mathrm{Hom}_{\text{alg gp}}(\mathbb{G}_m, T)$

Important decompositions

1) Bruhat decomp/K arb. field $\qquad G(K) = \coprod_{w \in W} N(K)\, w\, B(K)$

2) Cartan decomp/k loc field $\qquad G(k) = \coprod_{\lambda \in X_*^+} G(\mathcal{O})\, t^{\lambda}\, G(\mathcal{O})$

3) Iwasawa decomp $\qquad G(k) = G(\mathcal{O}) \cdot B(k)$

Congruence subgps

$G_\ell := \mathrm{Ker}(G(\mathcal{O}) \to G(\mathcal{O}/\mathfrak{m}^\ell)) \qquad \ell = 0, 1, 2, \dots$

$N_\ell, T_\ell, \bar{N}_\ell = N \cap G_\ell,\ T \cap G_\ell,\ \bar{N} \cap G_\ell$

Gauss (Iwahori) decomp For $\ell \geq 1$, $\quad G_\ell = \bar{N}_\ell T_\ell N_\ell$

Pf $\quad G_0 = G(\mathcal{O}) \supset G_1 \supset G_2 \cdots$

$G_\ell / G_{\ell+1} \simeq \mathfrak{g}(F)$ as groups

e.g. $G = GL_n$

$G_\ell = \begin{pmatrix} 1 + (t^\ell) & (t^\ell) \\ (t^\ell) & 1 + (t^\ell) \end{pmatrix}$ mult mod $G_{\ell+1}$ ~ add in Lie alg

Induct on ℓ. $\quad G_\ell / G_{\ell+1} = \mathfrak{g}$

Claim reduces to the triangular decomp of $\mathfrak{g}$.

$$\begin{array}{ccc} N_\ell \cdot T_\ell \cdot \bar{N}_\ell & \twoheadrightarrow & G_\ell \\ \downarrow & & \downarrow \\ N_\ell/N_{\ell+1} \cdot T_\ell/T_{\ell+1} \cdot \bar{N}_\ell/\bar{N}_{\ell+1} & \twoheadrightarrow & G_\ell/G_{\ell+1} \end{array}$$

surj

all varieties are smooth, and vertical map is smooth

Grothendieck thm $\Rightarrow$

(essentially Hensel's lem) $\square$

For $\ell = 0$ statement is false.

$$\begin{array}{ccc} N(\mathcal{O}) \cdot T(\mathcal{O}) \cdot \bar{N}(\mathcal{O}) & \xrightarrow{\text{surj?}} & G(\mathcal{O}) \\ \downarrow & & \downarrow \\ N(F)\, T(F)\, \bar{N}(F) & \longrightarrow & G(F) \end{array}$$

not surjective, principal minors must be nonzero [Gauss factorization]

$$\begin{array}{ccc} G_0 = G(\mathcal{O}) & \longrightarrow & G(F) \\ \cup & & \cup \\ I := \text{preimage of } B(F) & & B(F) \\ \cup & & \\ G_1 & & \end{array}$$

Iwahori

Gauss factorization for I:

$I = \bar{N}_1 \cdot T(\mathcal{O}) \cdot N(\mathcal{O})$

Pf $\quad 1 \to G_1 \to I \to B(F) \to 1$

$B(\mathcal{O})$

$I \ni g$

$\exists\, b \in B(\mathcal{O})$ st. $b \bmod G_1 = g \bmod G_1$ then $bg^{-1} \in G_1$ where we know factorization $\square$

Flag variety $\mathcal{B} = \{ \mathfrak{b} \in Gr(\mathfrak{g}) \mid \mathfrak{b}$ a Borel $\}$

Lie alg, solvable, max dim

$\mathcal{B}$ a projective variety

$\mathrm{Norm}_G(\mathfrak{b}) = B$

Choice of $B \subset G$ gives $G/B \longrightarrow \mathcal{B}$ $\quad g \mapsto Ad_g(\mathfrak{b})$

set-theoretic bijection

<u>Claim</u> the map $G/B \longrightarrow \mathcal{B}$, $\ g \longmapsto Ad_g(\mathfrak{b})$ is smooth (differential is surjective)

$\Downarrow$ isom of varieties by ZMT.

Motivation:

$k = \mathbb{R}$, $G = GL_n^+(\mathbb{R})$, $K = SO(n)$, $B = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$ Iwasawa says $GL_n^+(\mathbb{R}) = SO_n(\mathbb{R}) \cdot B$

i.e., Gram-Schmidt.

<u>Proof of ~~Iwasawa~~ Gram-Schmidt</u> $\varphi : K \xrightarrow[\text{surj}]{\text{want}} G/B$. $\quad \varphi$ is open

<u>Check</u> $\mathrm{Lie}\,K + \mathrm{Lie}\,B = \mathrm{Lie}\,G$ $\quad \mathrm{Im}(\varphi)$ open.

K compact $\Rightarrow \varphi$ surjective. □

Proof of Iwasawa:

$G(\mathcal{O}) \longrightarrow G(k)/B(k) = \mathcal{B}(k)$

<u>Claim</u> The map $G(\mathcal{O})/B(\mathcal{O}) \longrightarrow G(k)/B(k)$ is a bijection.

<u>Pf of Claim</u>

$$\begin{array}{ccc} G(\mathcal{O})/B(\mathcal{O}) & \twoheadrightarrow & \mathcal{B}(\mathcal{O}) \\ \downarrow & & \downarrow \\ G(F)/B(F) & \twoheadrightarrow & \mathcal{B}(F) \end{array}$$

Grothendieck $\Rightarrow$ surj

hence bijection

Now goal: $\mathcal{B}(\mathcal{O}) \longrightarrow \mathcal{B}(k)$ is surjective.

Consequence of valuative criterion of properness □

<u>Pf of Bruhat.</u> $\mathcal{B}(K) = G(K)/B(K) = \coprod_{w \in W} N(K)\, w\, B(K)/B(K)$ $\leftarrow$ WTS

<u>Claim</u> $\mathcal{B}^T = \{ wB/B,\ w \in W \}$ (all K-points) $\quad T$ fixed points

$X_*^{++} = \{ \lambda \mid \langle \lambda, \check{\alpha} \rangle > 0 \ \ \forall \check{\alpha} \in R^+ \}$

<u>Idea:</u> $\lambda \in X_*^{++}$ $\quad \mathcal{B}^T = \mathcal{B}^\lambda$

given $x \in \mathcal{B}$

$\gamma : \mathbb{G}_m \longrightarrow \mathcal{B} : t \to t^\lambda(x)$

γ extends to $0 \in \mathbb{k}$ by $\mathcal{B}$ projective ; $\quad \gamma(0) \in \mathcal{B}^\lambda$

Remains to show $x \in N(K) w$

$T_w \mathcal{B} = \mathfrak{g}/\mathrm{Lie}\, G_w$ $\quad$ ... shaky argument $\quad$ repulse/contract ... pos/neg

Hint of rigorous arg: $g \in \mathcal{B}(F)$ $\quad$ Choose $\lambda \in X_*^{++}$ $\quad t^\lambda \in T(k)$

$t^\lambda \cdot g \in \mathcal{B}(k) = G(k)/B(k) \simeq G(\mathcal{O})/B(\mathcal{O})$

$\exists\ y \in G(\mathcal{O})$ st. $t^\lambda g \in y B(\mathcal{O})$ now take limit $\square$

<u>Affine Bruhat decomp:</u> $W \curvearrowright T \Rightarrow W \curvearrowright X_*(T)$

$\tilde{W} := W \ltimes X_*(T)$

$$\boxed{\tilde{W} \xrightarrow{\sim} \cancel{{}_{N(k)T(\mathcal{O})}\backslash G(k)/G(\mathcal{O})}}$$

See Iwahori-Matsumoto

<u>Remark</u>

$X_*(T) \simeq T(k)/T(\mathcal{O})$

<u>Idea of proof:</u> $\cancel{Gr_{\mathfrak{g}} = G(k)/G(\mathcal{O})}$ Repeat argument above but using tori over smaller field?

$\cancel{\tilde{W} \xrightarrow{\sim} Gr_{\mathfrak{g}}^T}$

<u>Remind:</u> If K open compact in $G(k)$

$$\mathrm{inv}^K : \frac{\mathrm{Rep}^\infty G(k)}{\ker(\mathrm{inv}^K)} \xrightarrow{\sim} \mathcal{H}_K\text{-mod}$$

Take Iwahori subgp $= K = I$

$$\mathrm{Rep}^I G(k) := \{ V \in \mathrm{Rep}^\infty G(k) \mid V = \mathcal{H}\cdot V^K \}$$

<u>Thm</u> 1) $\mathrm{Rep}^I G(k)$ is an abelian subcategory

2) $\mathrm{inv}^I : \mathrm{Rep}^I G(k) \xrightarrow{\sim} \mathcal{H}_I\text{-mod}$

3) $\mathrm{Rep}^\infty G(k) = \mathrm{Rep}^I G(k) \oplus \ker(\mathrm{inv}^I)$

<u>Seminar Review 1/24</u>

preimage

$$\begin{tikzcd}
L_0 \arrow[r, "y_j", "②"'] \arrow[d, equal] & L_0 \arrow[r, "K"] \arrow[d, "③", "x_i"'] & L_0 \arrow[r] \arrow[d, phantom, "\cap"] & M \arrow[r] \arrow[d, hook] & 0 \\
L_0 \arrow[r] & L_0 \arrow[r, "K"] \arrow[d, equal] & L_0 \arrow[r, "①"] & L \arrow[r] \arrow[d] & 0 \\
 & L_0 \arrow[rr] & & N &
\end{tikzcd}$$

G split reductive alg gp $/\mathbb{Z}$

$\mathfrak{g} = \mathrm{Lie}\, G$ is a free $\mathbb{Z}$-module

$= \mathrm{Tan}_1(G) = \ker(G(\mathbb{Z}[\epsilon]) \to G(\mathbb{Z})) = \mathrm{Hom}_{\mathbb{Z}}(\Omega_{G/\mathbb{Z}}, \mathbb{Z})$

For any ring R, $\mathrm{Hom}_R(\Omega_{G/\mathbb{Z}} \otimes R, R) = \mathfrak{g} \otimes R$ since G smooth, so $\mathfrak{g}(R) := \mathfrak{g} \otimes R = \mathrm{Tan}_1(G_R)$

$\mathfrak{g}$ is Lie algebra (think of as subalg of $\mathfrak{g} \subset \mathfrak{gl}_n$)

Since G reductive, $\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha$ where $\mathfrak{t} = \mathrm{Lie}(T)$ and $\mathfrak{g}_\alpha$ line bundle on $\mathbb{Z}$.

$G \curvearrowright \mathfrak{g}$ by Ad. adjoint rep

Thm
Let G/S smooth affine gp scheme w/ connected fibers, B sm closed subgp with connected fibers such that $B = N_G(B)$.

Then quotient sheaf G/B is represented by smooth S-scheme.

If $(G/B)_{\bar{s}} = G_{\bar{s}}/B_{\bar{s}}$ are projective, then $G/B \to S$ admits canonical S-ample line bundle (projective Zar locally on base).

Flag variety

$\mathcal{B}(S') = \{\text{Borel subgroups of } G_{S'}\}$ is representable by smooth proper S-scheme

Ref: SGA3
B. Conrad Winter School Notes

Since G split reductive, B borel and we have

$$G/B \xrightarrow{\sim} \mathcal{B} \quad \text{by conjugation.}$$

In general $G(S)/B(S) \hookrightarrow (G/B)(S)$ and $G \to G/B$ is principal B-bundle (faithfully flat)

Since G reductive, minimal parabolic subgroups of G_K are $G(K)$ conjugate for any field K.

Lecture 5 1/23

$G/k \supset B/k$ alg subgp, M a rep of B

$$\mathrm{Ind}_B^G M = \left\{ \begin{array}{l} f : G \longrightarrow M \\ f(gb) = b^{-1} f(g) \quad \forall g \in G,\ b \in B \end{array} \right.$$

$\exists K_f \in \mathrm{op}\, G$ st. $f(kg) = f(g) \quad \forall k \in K_f$

$\mathrm{supp}\, f$ = finite union of $K_f\, gB/B$

From now on, assume G/B is compact $\Rightarrow$ compact in p-adic topology $\Rightarrow$ $\forall K \in \mathrm{op}\, G$ $K\backslash G/B$ is finite. (so second condition becomes redundant)

Lem 1) If M is a smooth B-rep, then $\mathrm{Ind}_B^G M$ is a smooth G-rep.

2) $\mathrm{Ind}_B^G : \mathrm{Rep}^\infty B \longrightarrow \mathrm{Rep}^\infty G$ is exact.

Pf

$$(\mathrm{Ind}_B^G M)^K \cong \bigoplus_{x \in K\backslash G/B} M^{B \cap xKx^{-1}} \qquad B \cap xKx^{-1} \in \mathrm{op}\, B$$

invariance exact, so $M \mapsto (\mathrm{Ind}_B^G M)^K$ exact.

If M smooth, $M = \varinjlim_K M^K \Rightarrow \mathrm{Ind}_B^G M = \varinjlim_K (\mathrm{Ind}_B^G M)^K$ prove 1) and 2). ∎

Frobenius reciprocity

$\mathrm{Hom}_G(V, \mathrm{Ind}_B^G M) \simeq \mathrm{Hom}_B(\mathrm{Res}_B^G V, M)$. Thus Res is left adjoint to Ind_B^G.

Parabolic Ind / Res

$$\underset{\text{reductive}}{G} \supset \underset{\text{Borel}}{B = TN} \xrightarrow{p} B/N = T$$

$\rho \in X^*(T)$ half-sum of positive roots

$h \in T$, $q = \#\mathcal{O}/m$ $\quad \mathrm{val}\, p(h)$ $\quad$ For us $\boxed{\underset{\rho}{\delta}: h \mapsto q^{\frac{1}{2}\mathrm{val}\, p(h)}}$ $\quad \delta_p : T \longrightarrow \mathbb{C}^*$, $\ T \to q^{\frac{1}{2}\mathbb{Z}} \subset \mathbb{C}^*$

$i_T^G : \mathrm{Rep}^\infty T \longrightarrow \mathrm{Rep}^\infty G$ $\qquad i_T^G(M) := \mathrm{Ind}_B^G\, p^*(\delta \otimes M)$

$\mathrm{Ind}_B^G\, p^*(\delta^2) =$ measures on G/B

$\delta^2 : T(\mathbb{C}) \longrightarrow \mathbb{C}^*$ $\qquad \mathrm{Ind}_B^G(\rho^2) =$ canonical bundle on G/B

Cor: If M is unitary T-rep, then $i_T^G M$ is unitary G-rep

$$(f_1, f_2) = \int_{G/B} \langle f_1(g), f_2(g) \rangle$$ Twist by ρ ensures that $(gf_1, gf_2) = (f_1, f_2)$ $\forall g \in G$

Left adjoint to i_T^G:

N any group, V any N-rep, $\qquad V_N := V/V(N)$ space of coinvariants wrt. N

$V \supset V(N) = \mathbb{C}\text{-span}\{ nv - v \mid v \in V, n \in N\}$

Let M a trivial N-module

$\mathrm{Hom}_N(V, M) = \{ f : V \to M \mid f(nv) = f(v)\} = \mathrm{Hom}_{\mathbb{C}}(V_N, M)$

V G-rep, M a T-rep: $\quad \mathrm{Hom}_G(V, i_T^G M) = \mathrm{Hom}_G(V, \mathrm{Ind}_B^G(\rho \otimes M)) = \mathrm{Hom}_B(V, \rho \otimes M)$

$= (\mathrm{Hom}_N(V, \rho \otimes M))^T = \mathrm{Hom}_{\mathbb{C}}(V_N, \rho \otimes M)^T = \mathrm{Hom}_T(\rho^{-1} \otimes V_N, M)$

Define Jacquet functor $r_T^G : V \mapsto \rho^{-1} \otimes V_N$

Lem $r_T^G : \mathrm{Rep}^\infty G \longrightarrow \mathrm{Rep}^\infty T$ is a left adjoint to i_T^G.
(check)

Rem $T = B/N \curvearrowright V_N$ naturally

Coinvariants.

N any group, V any N-rep $\qquad V^N \hookrightarrow V \longrightarrow V_N = V/V(N)$

N finite: $V = V^N \oplus V(N)$ $\quad$ so $\quad V^N = V_N$

N compact, V smooth : reduce to finite group so same is true

In general $V \mapsto V^N$ left exact, $V \mapsto V_N$ right exact. Whenever $V^N \simeq V_N$ both are exact.

Prop Let N be a unipotent grp, then the functor $\mathrm{Rep}^\infty N \longrightarrow \mathrm{Vect} : V \mapsto V_N$ is exact.

Pf Left exactness amounts to $V' \hookrightarrow V$ subrep:

$V'(N) = V' \cap V(N)$

Claim Any unipotent N is a union of compact subgroups.

Ex $N = k = \bigcup \frac{1}{t^n}\mathcal{O}$ $\qquad N = k^*$ cannot do this

By Lie-Kolchin, can embedd $N \hookrightarrow$ strict upper triangular matrices in GL_n

may assume $N =$ strict upper tri mat.

fix $\lambda = (\lambda_1 > \lambda_2 > \dots > \lambda_n) \in \mathbb{Z}^n$ strictly dominant $\qquad t^\lambda = \mathrm{diag}(t^{\lambda_1}, \dots, t^{\lambda_n})$

$t^\lambda E_{ij} t^{-\lambda} = t^{\lambda_i - \lambda_j} E_{ij}$

Define $N_k := t^{k\lambda} N(\mathcal{O}) t^{-k\lambda}$ $\qquad N_0 = N(\mathcal{O})$

$\cdots N_2 \subset N_1 \subset N_0 \subset N_{-1} \subset N_{-2}$

- N_k open compact in N
- $\bigcap N_k = \{1\}$ basis of nbhds of 1
- $\bigcup N_k = N$

For G general reductive gp, $\quad G \supset B = TN$

Take $\lambda \in X_*(T)$ strictly dominant, i.e. $\forall$ positive root $\alpha^\vee \in X^*(T)$, $\langle \lambda, \alpha^\vee \rangle > 0$

$t^\lambda \in T(k) \qquad N_k = Ad(t^\lambda)^k (N(\mathcal{O}))$

Completing proof of exactness: $N = \bigcup N_k$ compact subgps.

$$\begin{array}{ccccc} V' & = & (V')^{N_k} & \oplus & V'(N_k) \\ \downarrow & & \downarrow & & \downarrow \\ V & = & V^{N_k} & \oplus & V(N_k) \end{array} \quad \Rightarrow \quad V'(N_k) = V' \cap V(N_k)$$

$$\begin{aligned} V'(N) &= \varinjlim^{\cup} V'(N_k) \\ &= \bigcup (V' \cap V(N_k)) \\ &= V' \cap \Big(\bigcup_k V(N_k)\Big) \\ &= V' \cap V(N) \end{aligned}$$

□

<u>Cor</u> The Jacquet functor $r^G_T : Rep^\infty G \longrightarrow Rep^\infty T$ is exact.

<u>A different POV:</u>

For N finite $\displaystyle\int_N : C(N) \longrightarrow \mathbb{C}$ induces $C(N)_N \xrightarrow{\sim} \mathbb{C}$

Given an alg variety X/k $\qquad C(X) = C^\infty_c(X) =$ loc. const fns on X w/ compact support

Eg. $C(G) = \mathcal{H}_G$

$\displaystyle\int_N : f \longmapsto \int_N f(ng)\,dn \qquad \int_N : C(G) \longrightarrow C(N\backslash G)$

<u>Cor of proof before</u>: $\displaystyle\int_N : C(G)_N \xrightarrow{\sim} C(N\backslash G)$

$N\backslash G \xrightarrow{T} \underset{=B/N}{B}\backslash G$ principal (left) T-bundle $\Rightarrow T \curvearrowright N\backslash G \Rightarrow T \curvearrowright C(N\backslash G)$

But the above $\int_N$ is not compatible with this T-action (due to Haar measure)

So $T \curvearrowright C(N\backslash G) \otimes \rho^2$ and taking this into account:

<u>Cor</u>: 1) $i^G_T M = C(G/N) \underset{\mathcal{H}_T}{\otimes} M \qquad \forall M \in Rep^\infty T$

2) $r^G_T V = C(N\backslash G) \underset{\mathcal{H}_G}{\otimes} V \qquad \forall V \in Rep^\infty G$

<u>Sketch of proof</u> ① Check $i^G_T C(T) = C(G/N)$. $\quad$ Category $Rep^\infty T \simeq \mathcal{H}_T$-modules $\mathcal{H}_T M = M$

Since i^G_T exact, take free resolution of M by $\mathcal{H}_T$.

② is similar, check on $C(G)$. □

<u>Rem</u> i^G_T, r^G_T depend on the choice of Borel B.

Bruhat decomp: $G/B = \coprod_{w \in W} BwB/B$. $\quad$ Fix $w \in W$, define $B_w := wBw^{-1}$
$\qquad N_w = wNw^{-1}$

Horocycle space

$$\mathcal{X}_w := \left(G/N_w \times G/N\right)/T_{diag} \cong \left(G/N \times G/N\right)/T$$

w-conjugate action, then right mult on first factor.

$$\mathrm{Rep}^\infty T \underset{r^G_{T,w}}{\overset{i^G_{T,w}}{\rightleftarrows}} \mathrm{Rep}^\infty G$$

$$i^G_{T,w} r^G_T V = i^G_{T,w}\left(C(N\backslash G) \underset{\mathcal{H}_G}{\otimes} V\right) = C(G/N_w) \underset{\mathcal{H}_T}{\otimes} C(N\backslash G) \underset{\mathcal{H}_G}{\otimes} V$$

$$= C(\mathcal{X}_w) \underset{\mathcal{H}_G}{\otimes} V$$

Exercise: X alg variety / k padic

$$\underset{\text{Zar open}}{U} \xhookrightarrow{j} X \xhookleftarrow{i} X \setminus U = Y, \qquad 0 \to C(U) \hookrightarrow C(X) \to C(Y) \to 0$$

Claim: surj

Lecture 6 1/28

Alg. variety $X/_k = X^0 \supset X^1 \supset \dots X^n$
filtration by Zariski open subsets.

$\rightsquigarrow C(X^n) \hookrightarrow C(X^{n-1}) \hookrightarrow \dots C(X^1) \hookrightarrow C(X)$ which has subquotients $\cong C(X^i \setminus X^{i+1})$ by Exer.

Fix Borels $B \cap B' = T$.

$B = TN, \quad B' = TN'$

$G/B = \mathcal{B} \overset{\text{Bruhat}}{=} \coprod_{w\in W} N'\text{-orbits of } T\text{-fixed pts} = \coprod_{w\in W} N'wB/B$

$G/N = \coprod_{w\in W} X_w \qquad X_w = N'wB/N$ locally closed subvar. any orbit of alg gp is locally closed

Choose an ordering of W s.t. $\dim X_{w_j} \geq \dim X_{w_i} \Rightarrow w_j \geq w_i$

$X^i := \bigcup_{j \geq i} X_{w_j}$ open in G/N $\qquad G/N = X^0 \supset X^1 \supset \dots$ closure only contains smaller dim orbits

Get a filtration of $C(G/N)$ with quotients $C(N'wB/N)$

$\Rightarrow C(G/N)_{N'}$ has filtration with quotients $C(N'wB/B)_{N'}$ since $(\ )_{N'}$ is an exact functor

$N'\backslash N'wB/N = N'\backslash N'wTN/N = wT$ $\qquad$ pullback, res, $C(wT)$

Given a smooth T-rep M, $\left(C(G/N) \underset{\mathcal{H}_T}{\otimes} M\right)_{N'}$ has filtration with subquotients $C(G/N)_{N'} \underset{\mathcal{H}_T}{\otimes} M$

$T = B'/N'$

$$\| $$

$$C(wT) \underset{\mathcal{H}_T}{\otimes} M = M^w$$

the T-action is ${}^wT$ conj.

Cor $r^G_{T,B'} i^G_{T,B} M$ has canonical [functorial] filtration with quotients M^w, $w \in W$.

Special Cases ① $B' = B$ $\qquad B/N \hookrightarrow G/N$ closed embedding

can: $r^G_T i^G_T M \longrightarrow M$. This is the canonical adjunction

$$\mathrm{Hom}_G(V, i^G_T M) \xrightarrow{\sim} \mathrm{Hom}_T(r^G_T V, M) : f \longmapsto \left[r^G_T V \xrightarrow{r^G_T(f)} r^G_T i^G_T M \xrightarrow{\text{can}} M\right]$$

<u>Case</u> ② $B' = \overline{B}$ the opposite Borel. $\overline{N}TB/N \xhookrightarrow{\text{open}} G/N$

$\overline{\text{can}} : M \longrightarrow \overline{r}^G_T \, i^G_T M$

$\mathrm{Hom}_G(i^G_T M, V) \longrightarrow \mathrm{Hom}_T(M, \overline{r}^G_T V) \; : \; f \mapsto [\, M \xrightarrow{\overline{\text{can}}} \overline{r}^G_T i^G_T M \xrightarrow{\overline{r}(f)} \overline{r}^G_T(V) \,]$

<u>Second adjointness thm (J. Bernstein)</u>

$\overline{r}$ is a right adjoint of i. (3 proofs : Bernstein, Shalika, below)

<u>Approach</u> by Bezrukavnikov–Kazhdan.

<u>Idea:</u> Construct 2 maps ① $\alpha : C(G) \longrightarrow C(X_1)$ of $G\times G$ – reps

$X_1 = (G/N \times G/N)/T$ $\quad f \longmapsto \alpha(f)(g_1N, g_2N) = \int_N f(g_1 n g_2^{-1})\, dn$

$X' = X_{w_0}$ $\quad w_0 \in W$ the longest elt ② $\beta : C(X') \longrightarrow C(G)$ this map is main part of proof

Gives a morphism of functors

$$\begin{array}{ccc} i\overline{r}V & \xrightarrow{\text{def}} & V \\ \| & & \| \\ C(X') \otimes_{\mathcal{H}_G} V & \xrightarrow{\beta} & C(G) \otimes_{\mathcal{H}_G} V \end{array}$$

<u>Check</u> $i \xrightarrow{i(\overline{\text{can}})} i\overline{r}i \xrightarrow{\beta\circ i} i$
$\overline{r} \xrightarrow{\overline{\text{can}}\circ\overline{r}} \overline{r}i\overline{r} \xrightarrow{\overline{r}\circ\beta} \overline{r}$ are identity functors.

Construction of β is via p-adic specialization ($\sim$ nearby cycles) functor.

This will be a long digression:

Wonderful compactification of G (reductive alg group over field)

V_λ irrep of G with highest wt λ.
(over $\mathbb{C}$)

$G \longrightarrow GL(V_\lambda) \hookrightarrow \mathrm{End}_{\mathbb{C}} V_\lambda \setminus \{0\} \twoheadrightarrow \mathbb{P}(\mathrm{End}\, V_\lambda)$

<u>Assume:</u> 1) $\langle \lambda, \alpha^\vee \rangle > 0$ $\forall$ positive coroot $\alpha^\vee$ $\qquad$ Condition 1) $\Rightarrow G \hookrightarrow \mathbb{P}(\mathrm{End}\, V_\lambda)$ embedding

2) G has no center

$\overline{G} :=$ closure of the image of G in $\mathbb{P}(\mathrm{End}\, V_\lambda)$

$\underset{\text{set of roots}}{R} \supset \underset{\text{simple roots}}{S}$

Given $\Sigma \subset S$, $\underset{\substack{\uparrow \\ \text{parabolic}}}{P_\Sigma} = \underset{\substack{\uparrow \\ \text{Levi}}}{L_\Sigma} \cdot N_\Sigma$ unip rad. st. root vectors for Σ are in Lie N_Σ

$\overline{P}_\Sigma = L_\Sigma \cdot \overline{N}_\Sigma$ $\qquad$ $\Sigma = 0 \Rightarrow P_\Sigma = G$, $\quad \Sigma = S \Rightarrow P_\Sigma = B$ $\qquad$ $\underset{\substack{\uparrow \\ \text{torus}}}{Z(L_\Sigma)}$ = connected comp of center of L_Σ

<u>Thm</u> (DeConcini – Procesi)

1) $\overline{G}$ is <u>smooth</u> and independent of λ.

2) $\overline{G} = \coprod_{\Sigma \subset S} G_\Sigma$ where G_Σ is a $G\times G$-orbit

$G_\Sigma = \left(G/\overline{N}_\Sigma \times G/N_\Sigma \right)/H_\Sigma$ $\qquad$ $H_\Sigma = \{ (\overline{l}, l) \in L_\Sigma \times L_\Sigma \mid \overline{l}\cdot l^{-1} \in Z(L_\Sigma) \}$

$G_\emptyset = G \qquad G_S \simeq G/\overline{B} \times G/B$

$\overline{G}_\alpha$ smooth irreducible divisor $\qquad \overline{G} \setminus G = \bigcup_{\alpha \in S} \overline{G}_\alpha$ a normal crossing divisor

$\overline{G}_\Sigma = \bigcap_{\alpha \in \Sigma} \overline{G}_\alpha$

A different approach/construction : the homogeneous coordinate ring of $\overline{G}$

$\tilde{G}$ simply connected cover of G
$\cup$
$\tilde{T}$ max torus

$X^*(\tilde{T})$ = weight lattice
$\cup$
X^+ dominant weights $\qquad X^+ = \mathbb{Z}_{\geq 0}\omega_1 \oplus \dots \oplus \mathbb{Z}_{\geq}\omega_r \qquad \omega_1, \dots, \omega_r$ & fundamental wts

$r = \dim T = \mathrm{rk}\, G$

Define a partial order on $X^*(T)$ by $\lambda \leq \mu$ if $\mu - \lambda = \sum$ positive roots

<u>Rees alg</u> Let A an assoc alg. filtered by A_λ, $\lambda \in X^+$

- $\lambda \leq \mu \Rightarrow A_\lambda \subseteq A_\mu$
- $A_\lambda \cdot A_\mu \subseteq A_{\lambda+\mu}$.

Rees alg, $RA = \bigoplus_{\lambda \in X^+} A_\lambda$

$\mathbb{C}X^+ \ni e^\lambda$ basis elt, $\qquad \mathbb{C}[e^{\omega_1}, \dots, e^{\omega_r}] \simeq \mathbb{C}X^+ \hookrightarrow RA$ algebra embed.

$e^\lambda \longmapsto 1 \in A_\lambda$

$\mathbb{C}X^+ \hookrightarrow \mathbb{C}X^*(\tilde{T}) = \mathbb{C}[e^{\pm\omega_1}, \dots, e^{\pm\omega_r}]$

$\tilde{T} \hookrightarrow \mathbb{C}^r$

$t \longmapsto (\omega_1(t), \dots, \omega_r(t))$

Assume A comm: $\quad \pi : \mathrm{Spec}\, RA \longrightarrow \mathrm{Spec}\, \mathbb{C}X^+ = \mathbb{C}^r \hookleftarrow \tilde{T}$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \ni 0$

$\mathbb{C}[\pi^{-1}(0)] = \mathrm{gr} A = \bigoplus_{\lambda \in X^+} \left(A_\lambda / \sum_{\mu < \lambda} A_\mu \right)$

$$\begin{tikzcd}
\mathrm{Spec}\, RA \arrow[d, "\pi"] & \tilde{T} \times \mathrm{Spec} A \arrow[l, hook'] \arrow[d] \\
\mathrm{Spec}\, \mathbb{C}X^+ = \mathbb{C}^r & \tilde{T} \arrow[l, hook']
\end{tikzcd}$$

Let $A = \mathbb{C}[\tilde{G}] = \bigoplus_{\mu \in X^+} V_\mu^\vee \otimes V_\mu \qquad A_\lambda = \bigoplus_{\mu \leq \lambda} V_\mu^\vee \otimes V_\mu \qquad \mathrm{gr} A = \bigoplus_{\mu \in X^+} V_\mu^\vee \otimes V_\mu$

different alg structures

$\tilde{T} \curvearrowright \tilde{G}/N \xrightarrow{\ p\ } \tilde{G}/\tilde{B}$

$\mathbb{C}[\tilde{G}/N] = \bigoplus_{\lambda \in X^*(\tilde{T})} \mathbb{C}[\tilde{G}/N]_\lambda \qquad$ grading by $X^*(\tilde{T})$

$\mathbb{C}[\tilde{G}/N]_\lambda = \Gamma(\tilde{G}/B, \mathcal{O}_\lambda)$

~~Borel~~ <u>Borel-Weil</u> $\quad \Gamma(\tilde{G}/B, \mathcal{O}_\lambda) = \begin{cases} V_\lambda & \lambda \in X^+ \\ 0 & \text{otherwise} \end{cases}$

<u>Cor</u> $\quad \mathbb{C}[\tilde{G}/N] = \bigoplus_{\lambda \in X^+} V_\lambda \qquad \mathbb{C}[\tilde{G}/N] = \bigoplus_{\lambda \in X^+} V_\lambda^\vee$

<u>Cor</u> $\quad \mathbb{C}[(\tilde{G}/\tilde{N} \times \tilde{G}/N)/T] = \bigoplus_{\lambda \in X^+} V_\lambda^\vee \otimes V_\lambda = \mathrm{gr}\, A.$

$\quad\quad \| $

$\quad \mathbb{C}[\tilde{\mathfrak{X}}']$

Review 1/31

$G(k)$ reductive is unimodular $\qquad \Delta_G$ modulus character

$P(k)$ has Δ_P described by Lie algebra $\qquad \Delta_G^{-1}\mu_G = \nu_G$

These are all locally compact, tot disconnected spaces; open compact nbds.

Good ref is Bernstein-Zelevinsky

Say something about G/N filtration

<u>Lecture 7</u> 2/1

<u>Def</u> $\quad \tilde{G}_+ := \mathrm{Spec}\ \underbrace{R(\mathbb{C}[\tilde{G}])}_{\bigoplus_{\mu \in X^+} A_\mu}$ $\quad$ Rees alg

<u>Claim</u>

1) $\tilde{G}_+$ is a semigroup

2) For any $\tilde{G}$-rep V we have $\tilde{G}_+ \to \mathrm{End}_{\mathbb{C}} V$

<u>Pf</u> $\quad \mathbb{C}[\tilde{G}] = \bigoplus_{\lambda \in X^+} V_\lambda^\vee \otimes V_\lambda$

$$\mathbb{C}[\tilde{G}] \xrightarrow{\text{comult}} \mathbb{C}[\tilde{G}] \otimes \mathbb{C}[\tilde{G}]$$

$$\mathbb{C}[V_\lambda] \longrightarrow \mathbb{C}[\tilde{G}] \otimes \mathbb{C}[V_\lambda]$$

Choose a $\mathbb{C}$-basis $v_i^{(\lambda)} \in V_\lambda$, $\quad v_i^{(\lambda),*} \in V_\lambda^\vee$

$\varphi_{ij}^{(\lambda)} = v_j^{(\lambda)*} \otimes v_i^{(\lambda)}$

<u>Comult</u> $\quad \varphi_{ij}^{(\lambda)} \longmapsto \sum_\lambda \varphi_{ik}^{(\lambda)} \otimes \varphi_{kj}^{(\lambda)}, \quad v_i^{(\lambda),*} \longmapsto \sum_k \varphi_{ik}^{(\lambda)} \otimes v_k^{(\lambda,*)}$

Comult makes sense for RA since $A_\mu = \bigoplus_{\lambda \le \mu} V_\lambda^\vee \otimes V_\lambda$. ∎

$$\begin{tikzcd}
\tilde{G} \times \tilde{T} \arrow[r, hook] \arrow[d] & \mathrm{Spec}\, R\mathbb{C}[\tilde{G}] = \tilde{G}_+ \arrow[d, "\pi"] & \mathrm{Spec}\ \mathrm{gr}\,\mathbb{C}[\tilde{G}] = \tilde{\mathfrak{X}}' \arrow[l, hook'] \arrow[d] \\
\tilde{T} \arrow[r, hook] & \mathrm{Spec}\, \mathbb{C}X^+ = \mathbb{C}^r & \{0\} \arrow[l, hook']
\end{tikzcd}$$

$\tilde{G} \times \tilde{T} \curvearrowright V_\lambda \qquad (g,t) : v \longmapsto \lambda(t) \cdot g(v)$

$Z(\tilde{G}) \underset{\text{center}}{\hookrightarrow} \tilde{G}$

$Z(\tilde{G}) \hookrightarrow \tilde{G} \times \tilde{T}$ $\quad$ acts trivially on V_λ

$z \longmapsto (z, z^{-1})$

$G_+ := \tilde{G}_+ / Z(\tilde{G})$ $\quad$ Vinberg's semigroup

$G_+ \longrightarrow \mathrm{End}\, V_\lambda$

$T \hookrightarrow G_+$

$t \longmapsto (1, t)$ $\quad$ central imbedding

$G_+^\circ = \{ x \in G_+ \mid \text{image of } x \text{ in } \mathrm{End}\, V_{\omega_i} \text{ is } \neq 0 \text{ for each fundamental weight } \omega_i \}$

Thm (Vinberg) 1) G_+^0 is smooth

2) $\tilde{G}\times\tilde{G}\times\tilde{T}$ acts on G_+, G_+^0 with finitely many orbits

- $G^0_{+,\Sigma}$, $\Sigma\in S$ orbits of G_+^0 (not affine, $\neq$ same dim)

[morally $\mathbb{C}[G^0_{+,\Sigma}]\simeq\mathbb{C}[\tilde{G}]\otimes\mathbb{C}[\tilde{T}]$ reg functions "the same"] // forget more lower components as Σ bigger

3) $\tilde{T}$ action on G_+^0 is free

G_+^0 ← stable locus

$G_+^0/\tilde{T}\cong\overline{G}$ strata descend, smooth.

$Y := \overline{G}_S = G/\overline{B}\times G/B$ closed orbit $\subset\overline{G} \hookleftarrow G$

$y_0 = \overline{B}/\overline{B}\times B/B$

$\overline{T} \longleftarrow T$ ($\overline{T}\subset\overline{G}$, $T\subset G$)

$\overline{T}$ smooth toric variety

fan = Weyl chambers

$G\supset T\hookrightarrow\overline{T}$) Y, y_0

$\overline{T}$ meets Y transversely at y_0

Stratification by intersecting w/ " " of $\overline{G}$

$N := T_Y\overline{G}$ normal bundle to Y in $\overline{G}$.

(might have bigger intersection w/ bigger orbits)

$\mathrm{rk}\, N = \mathrm{rk}\, G$

$N = \bigoplus_{\alpha\in S} N_\alpha$ $T_Y\overline{G}_\Sigma$

$N_\alpha := T_Y\overline{G}_{S\setminus\{\alpha\}}$ line bundle inside N.

Define a fiberwise T-action on N by letting $t\in T$ act on N_α by $\alpha(t)$

$G\times G \curvearrowright Y, N$

$G\times G$ action on N commutes with T.

$N^0 := N\setminus\bigcup_{\Sigma\neq S} T_Y\overline{G}_\Sigma$ T action on N^0 is free

$N \xrightarrow{p} Y\ni y_0$, $N^0\subset N$, $x\in N^0$

$N^0\simeq (G\times G)/H$ $H=\mathrm{Stab}_{G\times G}(x)=\{(\bar{b},b)\in\overline{B}\times B \mid \bar{b} \bmod \overline{N}\simeq w_0(b \bmod N)\}$

$X' = (G/\overline{U}\times G/U)/T = N^0$

---

k p-adic field

Goal to construct $B: C(X')\to C(G)$

Idea: N^0 is piece of N corresponding to $\overline{G}\setminus$ strata $= G$

Would like to choose $\psi: G\to N^0 = X'$.

Argue that N locally looks like $\overline{G}$

Let $B:=\psi^*$. In general ψ^* need not be morphism of $G\times G$-modules.

$\psi: G\longrightarrow N\longrightarrow Y$ (suppose equivariant)

$T\subset G$, $y_0\in Y$

not $\overline{B}\times B$-stable, but satisfies transversality condition.

but not reductive. In fact, $\nexists$ $G\times G$-equiv ψ. slice.

Incomprehensible part: Iwasawa decomp $\Rightarrow$ $Y = G(k)/\overline{B}(k)\times G(k)/B(k) = G(\mathcal{O})\times G(\mathcal{O})/?$ (compact)

In diff geom, slice exists for compact manifolds. (Riemannian geom) $\dashrightarrow$ doesn't work b/c p-adic

But we don't need ψ on spaces, just need it on functions, which are $\mathbb{C}$-valued

Choose $\lambda \in X_*(T)$ strictly dominant

$\tau := t^\lambda \in \underset{T(k)}{\cancel{\quad}}$. Fix an open compact in $G(k)\times G(k) \supset K$

$$\mathcal{H}_K = C(K\backslash G(k)\times G(k)/K)$$

<u>Lemma</u> Let U small enough open nhd of Y in $\bar{G}$ (alg variety over p-adic ~~~~ analytic spaces)

Then $\exists$ analytic map $\psi : U \longrightarrow N$ s.t

1) $\psi|_Y = Id_Y$

2) $d\psi|_Y = Id_{T_Y\bar{G}}$

3) ~~xxxx~~ $\psi(G\cap U) \subset N^\circ$

$$\begin{tikzcd} & Y \arrow[dl] \arrow[dr] & \\ \bar{G} \arrow[rr, "\psi"] & & N \end{tikzcd}$$

Can assume U is K-stable. (K compact, shrink)

Choose $V \subset \psi(U)$ also K-stable.

<u>Thm</u> (B-K) $\exists!$ $\mathcal{H}_K$-module map $B_K : C(V) \longrightarrow C(U)$ s.t.

$\forall f \in C(V)$, $\exists n_f \gg 0$, s.t. $\forall n \geq n_f$ $B_K(f\circ\tau^{-n}) = \psi^*(f\circ\tau^{-n})$. (*) [magically this becomes $\mathcal{H}_K$-equiv]

~~1) supp$(f\circ\tau^{-n})$~~ (*) determines this B_K completely, so can glue together compatibly to get B.

<u>Analytic part</u>

$V = k^n$ $\tau = \mathrm{diag}(t^{\lambda_1}, \dots, t^{\lambda_n})$
$\lambda_i > 0$

Define a norm on V. $v = (c_1 t^{k_1}, \dots, c_n t^{k_n})$, $c_i \in \mathcal{O}^*$

$$|v| := \max\left\{\tfrac{1}{2^{k_i/\lambda_i}}, i = 1, \dots, n\right\}$$

$|\tau(v)| = \frac{1}{2}|v|$
$|\tau^{-1}(v)| = 2|v|$

<u>Rem.</u> In this norm, unit ball is compact.

$U \subset V$ open nhd of $0 \in V$. Let $\phi : U \longrightarrow V$
$\phi(v) = v + o(|v|)$ $|\phi(v)| > \frac{1}{2}|v|$

<u>Lem</u> Given $f \in C(V)$ $\exists\, n_0 = n_0(\varphi, f) \gg 0$ st. $\forall n \geq n_0$:

1) supports of $f\circ\tau^{-n}$, $f\circ\tau^{-n}\circ\varphi \subset U$

2) $f\circ\tau^{-n} = f\circ\tau^{-n}\circ\varphi$

<u>Pf</u> 1) $\mathrm{supp}(f\circ\tau^{-n}) = \tau^n(\mathrm{supp} f) \subset U$

2) ~~x~~ Let $v \in V$ st. $f(\tau^{-n}v) \neq 0$ or $f(\tau^{-n}\phi(v)) \neq 0$

Choose r st. $\mathrm{supp} f$, $\mathrm{supp}(f\circ\varphi) \subset B_r$

Choose n_1 $|v| < \frac{r}{2^n} \Rightarrow$ // next time

<u>Lecture 8</u> 2/4

V/k φ: open nhd $0 \in V \longrightarrow V$
$\varphi(x) = x + o(x)$

$|\tau(v)| = \frac{1}{2}|v|$ $|\tau^{-1}(v)| = 2|v|$

<u>Lemma</u>' $\forall f \in C(\text{nbhd of } 0)$. $\exists\, n_f$ st. $n \geq n_f$ then $f\circ\tau^{-n}\varphi = f\circ\tau^{-n}$

Pf $\quad f$ is loc. const with comp supp $\Rightarrow f$ "uniformly continuous" i.e. $\exists \varepsilon > 0 \quad |v - v'| < \varepsilon \Rightarrow f(v) = f(v')$

Choose a ball radius $r > 0$ so $\operatorname{supp}(f, f\circ\varphi) \subset B_r$

Choose n st. $|v| < \frac{1}{2^n} r \Rightarrow |o(v)| < \frac{\varepsilon}{2^n}$

If $f(\tau^{-n}(v)) \neq$ or $f(\varphi(\tau^{-n}(v))) \neq 0 \Rightarrow |\tau^{-n}(v)| < r. \Rightarrow |v| < \frac{1}{2^n} r$

$\Rightarrow |o(v)| < \frac{\varepsilon}{2^n} \Rightarrow |\tau^{-n}(o(v))| < \varepsilon$

$f(\tau^{-n}\varphi(v)) = f(\tau^{-n} v + \underbrace{\tau^{-n} o(v)}_{\hat{\varepsilon}}) = f(\tau^{-n} v).$ $\blacksquare$

Lemma 2 $\quad$ Can assume $f_1, \dots, f_k$

$\tau_1, \dots, \tau_m$ commuting automorphisms of V.

Statement: $\exists\ (n_0^1, \dots, n_0^m)$ st. $\forall n^1 > n_0^1, \dots, n^m > n_0^m$ then same conclusion holds. $\left(f_i \circ \tau_j^{-n_j} \varphi = f_i \circ \tau_j^{-n_j} \right)$

Upgraded version

$N \xrightarrow{p} Y = G/B \times G/B \qquad G\times G$ equiv vect. bundle. $\qquad U =$ open nhd of Y in N

$\varphi: U \longrightarrow N$

1) C^1-map
2) $\varphi|_Y = \mathrm{Id}$
3) $d\varphi|_Y = \mathrm{Id}$

$\lambda \in X_*(T)$

$\tau^\lambda : x \longrightarrow t^\lambda x$

$X_{++} \subset X_*(T)$

strictly dominant wts

Defn $\quad \mu > \lambda$ if $\mu - \lambda \in X_{++}$

Fix open compact subgp $K \subset G \times G$

Lemma 3 $\quad \exists K$-stable open nhd $V \subset N$ and $\mu \in X_*(T)$ s.t. $\forall \lambda > \mu$:

1) $\tau^\lambda(V) \subset V$

2) $\forall f \in C(V)^K, \quad \operatorname{supp}(f\circ\tau^{-\lambda}, f\circ\tau^{-\lambda}\circ\varphi) \subset U$

3) $f\circ\tau^{-\lambda} = f\circ\tau^{-\lambda}\circ\varphi$

Difference from before is that μ does not depend on f.

Pf $\quad$ Step 1. $\quad \dfrac{G(\mathcal{O}) \times G(\mathcal{O})}{K \cap (G(\mathcal{O})\times G(\mathcal{O}))} =$ finite set

Iwasawa decomp $\Rightarrow$ $G(\mathcal{O}) \times G(\mathcal{O})$ acts on Y transitively $\Rightarrow Y$ is finite union of K orbits.

Any K-invariant fn on N is determined by its restrictions to $p^{-1}(y)$ for finitely many $y \in Y$.

Step 2 $\quad p^{-1}(y) = k^S \circlearrowleft T$

$T_K = K \cap T$ open compact in T.

$X_{++} \hookrightarrow X_*(T) \subset T$

Pick for each of these y's, pick V_i a T_K stable open comp. nhd of $0 \in p^{-1}(y)$

Claim $\quad \bigcup V_i$ is a finite union of $T_K \cdot X_{++}$-orbits $\subset T$

Pf $\quad$ Can assume only one y. $\quad V \subset p^{-1}(y)$

$T \curvearrowright k^S \qquad \mathbb{G}_m^S \curvearrowright k^S$

$T \subset \mathbb{G}_m^S$ act on k^S with finitely many orbits $\qquad v = (1,\dots,1,0,\dots,0) \in k^S$

$X_*(T)/\mathrm{Stab}_{X_*(T)}(v)$ $\qquad$ k, $\; k^\times \supset t^{\mathbb{Z}}$ $\qquad$ V_i compact so $t^{\mathbb{Z}_+}$ and finitely many neg.

<u>Step 3</u> X_{++} has finite set of generators $\tau_1 = t^{\lambda_1}, \dots, \tau_m = t^{\lambda_m}$.

By Claim can choose finitely many $v_1, \dots, v_r \in Y$ st. $V \subset \bigcup_{i \in I}^{r} T_k X_{++} v_i$

$C(V)^{T_k} \subset \mathbb{C}$-linear span of $1_{T_k t^\lambda v_i} = 1_{T_k v_i} \circ t^{-\lambda}$

Have finite collection of fns $\left.\begin{matrix} 1_{T_k v_i}, \; i = 1,\dots,r \\ \tau_1, \dots, \tau_m \end{matrix}\right\}$ done by previous Lemma 2. ■

$$\begin{tikzcd}
 & Y = G/\overline{B} \times G/B \arrow[dl] \arrow[dr, hook] & \\
\overline{G} \supset U \arrow[rr, dashed, "\psi"] & & N \\
G \arrow[u, phantom, "\cup" rotate=90] \arrow[rr, "\psi"] & & N^\circ \arrow[u, phantom, "\cup" rotate=90]
\end{tikzcd}$$

<u>Assume</u> 1) $\psi(G) \subset N^\circ$
2) $\psi|_Y = \mathrm{Id}$
3) $d\psi|_Y = \mathrm{Id}$

Let $\mathcal{H} = \mathcal{H}_K := C(K \backslash G \times G / K)$, $\quad \mathcal{H} \curvearrowright C(N)^K$

For each $\lambda \in X_*(T)$, $\tau^\lambda \in \mathrm{Aut}_{\mathcal{H}} C(N)^K$

<u>Corollary</u> 1) Let $\tilde{\psi}$ another map like ψ. Then $\forall f \in C(N)^K \; \exists \lambda$ s.t. $\forall \mu > \lambda$

$\mathrm{supp}(f \circ \tau^{-\mu}) \subset \mathrm{Im}(\psi) \cap \mathrm{Im}(\tilde{\psi})$

moreover we have $\psi^*(f \circ \tau^{-\mu}) = \tilde{\psi}^*(f \circ \tau^{-\mu})$

2) Let h_j be a finite set in $\mathcal{H}$. $\exists V \subset N$, $\lambda \in X_*$ s.t. $\forall \mu > \lambda$, $f \in C(N)^K$

$\psi^*(h_j (f \circ \tau^{-\mu})) = h_j \psi^*(f \circ \tau^{-\mu})$

<u>Proof of 1.</u> Define $\varphi : N \xrightarrow{\psi^{-1}} \overline{G} \xrightarrow{\tilde{\psi}} N$ (defined on some open subsets)

Apply Lemma 3. to φ.

2) supp h_j is compact, the image of supp h_j in $(G \times G)/K$ is finite.

It suffices to prove for fns of the form $1_{K \cdot gK}$, finitely many $g \in G \times G$.

Consider $G \underset{\tilde{\psi} = g^{-1} \circ \psi \circ g}{\overset{\psi}{\rightrightarrows}} N$. Part 1) $\Rightarrow$ $\psi^*(g(f \circ \tau^{-\mu})) = g \psi^*(f \circ \tau^{-\mu})$. □

<u>Prop</u> 1) $\mathcal{H}_K$ is noetherian

2) $C(N)^K$ is a fin gen $\mathcal{H}_K$-module.

$N^\circ \subset N$

$G \times G$-stable open subset) $\quad C(N^\circ)^K$ is a finitely generated $\mathcal{H}_K$-module.

<u>Lem</u> Let $M = C(N^\circ)^K$. Let $L \subset M$ be a τ-stable ~~submodule~~ $\mathbb{C}$-subspace st. $M = \bigcup_{n \geq 0} \tau^{-n} L$

Then, $M = \mathcal{H} L$.

Pf $M' = \mathcal{H}L$ is τ-stable. $\mathcal{H}$-submodule.

$M' \subset \tau^{-1}M' \subset \tau^{-2}M' \subset \cdots$ an increasing chain of $\mathcal{H}$ submodules (stabilizes)

$\exists\, n$ s.t. $\tau^{-n}M' = \tau^{-(n+1)}M' \Rightarrow \tau^{-1}M' = M'$

$\Rightarrow \bigcup_{n\geq 0} \tau^{-n}M' = M'$. $\bigcup_{n\geq 0}\tau^{-n}M' \supset \bigcup_n \mathcal{H}\tau^{-n}L = M$ $\square$

Thm $\exists!$ $\mathcal{H}_K$-module map $\beta : C(N^0)^K \longrightarrow C(G)^K$ s.t. $\forall f \in C(N^0)^K$ $\exists\, n_f \gg 0$

$\beta(f\circ\tau^{-n}) = \psi^*(f\circ\tau^{-n})$ where $\tau = t^\lambda$

Pf Uniqueness $C(N^0)^K$ is finitely generated over $\mathcal{H}$

Let m_i' $i=1,\ldots,p_0$ set of generators.

By assumption $\exists\, l \gg 0$ st. $\beta(m_i'\circ\tau^{-l}) = \psi^*(m_i'\circ\tau^{-l})$

By alg lemma $m_1'\circ\tau^{-l}, \ldots, m_{p_0}'\circ\tau^{-l}$ still a set of generators.

$\Rightarrow C(N^0)^K = \sum \mathcal{H} m_i'\circ\tau^{-l} \Rightarrow$ must have $\beta(\sum a_i'(m_i'\circ\tau^{-l})) = \sum a_i' \psi^*(m_i'\circ\tau^{-l}))$

$\Rightarrow \beta$ uniquely determined.

Existence. Choose a presentation

$$\mathcal{H}^{p_1} \longrightarrow \mathcal{H}^{p_0} \longrightarrow M = C(N^0)^K \longrightarrow 0$$

$|h_i^j| \in M_{p_0 p_1}(\mathcal{H})$, $1_i \longmapsto m_i'$

finite collection. Let $V \subset N$ be an open set as in last Corollary

Let $L = \{f \in M \mid \operatorname{supp} f \subset V\}$. Cor says $\exists\, n \gg 0$ st.

1) $m_i = m_i'\circ\tau^{-n} \in L$

2) $h_i^j(m_i'\circ\tau^{-n}) \in L$

3) $\psi^*(h_i^j m_i) = h_i^j \psi^*(m_i)$

Alg lemma $\Rightarrow$ m_i's generate M as an $\mathcal{H}$-module

Define $\beta : M \longrightarrow \mathcal{C}(G)^K$ by $\beta : \sum a_i m_i \longmapsto \sum a_i \psi^*(m_i)$

Need to check well-defined.

$\sum a_i m_i = 0 \Rightarrow 0 = \sum a_i (\underbrace{m_i'\circ\tau^n}_{m_i'}) \xrightarrow{\text{Presentation}} a_i = \sum \alpha_j h_i^j$ for some $\alpha_j \in \mathcal{H}$.

$\sum_i a_i\psi^*(m_i) = \sum_{i,j} \alpha_j h_i^j \psi^*(m_i) = \sum_{i,j} \alpha_j \psi^*(h_i^j m_i) = 0$ (sum over $\sum_j (\underbrace{\sum_i}_{0})$). $\square$

2/7 Seminar review

$G = PGL_2$

$X' = (G/\bar{U} \times G/U)/T$ $\quad G_m \curvearrowright G/\bar{U} = \mathbb{A}^2 - \{0\}$ $\quad G/U = \mathbb{A}^2 - \{0\}$ $\quad \mathbb{A}^2 - 0 \to X'$, $X' \downarrow \mathbb{P}^1$

by $z(x,y) = (zx, z^{-1}y)$

$N = T_{\mathbb{P}^1\times\mathbb{P}^1}(\mathbb{P}^3)$ $\quad N^0 = N \setminus \underbrace{T_{\mathbb{P}^1\times\mathbb{P}^1}(\mathbb{P}^1\times\mathbb{P}^1)}_{(\text{zero section})}$ $\quad$ So $\mathbb{A}^2 - 0$ = "punctured disk"

Enough to define $C(U) \to C(N)$ on $K \subset G\times G$ invariance by smoothness.

2/8 $\quad K_+ = K\cap N_+,\ K_- = K\cap N_-,\ K_0 = K\cap T \qquad h(x) = \frac{1}{\mathrm{vol}} 1_{KxK}$ $\quad$ (Iwahori holds if $K = K_+K_0K_-$) true for $G_n, n\geq 1$, I

<u>Lemma</u> Assume that Iwahori holds for K. Then $\forall \mu,\lambda \in X_*^+(T)$, $\quad h(t^\lambda)h(t^\mu) = h(t^{\lambda+\mu})$

So have alg map $\quad h: \mathbb{C}X^+ \longrightarrow \mathcal{H}_K = C(K\backslash G/K)$

<u>Pf.</u> For $\nu \in X^+$, then
$$t^\nu K_+ t^{-\nu} \subset K_+$$
$$t^{-\nu}K_- t^\nu \subset K_-$$
$$t^\nu K_0 t^{-\nu} = K_0$$

Want to show $\quad Kt^\lambda K t^\mu K = Kt^{\lambda+\mu}K$

$\supseteq$ clear

$$\subseteq \quad Kt^\lambda K t^\mu K = Kt^\lambda K_+K_0K_- t^\mu K$$
$$= K(t^\lambda K_+ t^{-\lambda})(t^\lambda K_0 t^{-\lambda}) t^{\lambda+\mu}(t^{-\mu}K_- t^\mu)K$$
$$\subseteq KK_+K_0 t^{\lambda+\mu}K_-K \subseteq Kt^{\lambda+\mu}K$$

<u>Lem</u> If $\quad K = K_1\cdot K_2 \quad$ product of 2 subgroups,

$\varepsilon_K = \frac{1}{\mathrm{vol}(K)}\chi_K$, $\quad \varepsilon_{K_i} = \frac{1}{\mathrm{vol}(K_i)}\chi_{K_i}$ $\quad$ then $\quad \varepsilon_K = \varepsilon_{K_1}\cdot\varepsilon_{K_2}$

<u>Prop</u> The alg $\mathcal{H}_K$ is noetherian for any congruence subgroup $K \subset G(\mathcal{O})$

<u>Pf</u> $\quad K \triangleleft G(\mathcal{O})$ finite index subgroup

$\{x_i\}$ representatives of all elts of $G(\mathcal{O})/K$.

Cartan decomp: $\quad G = \coprod_{\lambda\in X_*^+} G(\mathcal{O})t^\lambda G(\mathcal{O}) = \coprod_{\lambda,i,j} Kx_i t^\lambda x_j K$

$\Rightarrow \left\{1_{Kx_it^\lambda x_jK}\right\}$ is a basis of $\mathcal{H}_K$ $\qquad h(x_it^\lambda x_j) = h(x_i)h(t^\lambda)h(x_j)$

For any $g\in G$, $\quad Kx_igK = Kx_iKKgK$

$\mathcal{H}_K = \mathcal{H}(G(\mathcal{O})/K)\cdot \mathbb{C}X^+\,\mathcal{H}(G(\mathcal{O})/K)$

<u>Lemma</u> If $A = A_1A_2A_3$ where $\dim A_1, \dim A_3 < \infty$ and A_2 noetherian $\Rightarrow$ A noetherian. □

<u>Cor</u> (Bezrukavnikov-Kazhdan) $\exists!$ alg hom. $\phi: \mathbb{C}X_*(T) \longrightarrow \mathcal{H}_K$ st. $\phi(t^\lambda) = h(t^\lambda)$ for all $\lambda \gg 0$.

<u>Idea</u>: $\mathcal{X}' = (G/N_- \times G/N_+)/T$

$T \hookrightarrow$

$h \longmapsto (h \bmod N_-,\ h^{-1} \bmod N_+)$

$h = t^\lambda \longmapsto i(t^\lambda)$

$x_\lambda := 1_{(K\times K)i(t^\lambda)} \in C(\mathcal{X}')^{K\times K}$

$\phi(t^\lambda) := \beta(x_\lambda)$ $\quad$ from Bezru-Kaz. $\qquad$ new result

---

<u>Thm</u> Let K be open comp in G. Then $\exists c(G,K) > 0$ s.t. $\dim M \leq c(G,K)$ for any simple $\mathcal{H}_K$-module.

Lemma Let $A \subset \operatorname{End}\mathbb{C}^m$ a comm. subalg with r generators.
Then $\dim A \le m^{2-\frac{1}{2^{r-1}}}$

Conj (Bernstein) Actually $\dim A \le m+r$.

Proof of thm. The lemma $\mathcal{H}_K = \text{Finite dim} \cdot \mathbb{C}X^+ \cdot \text{finite dim}$

$\Rightarrow$ any simple $\mathcal{H}_K$-module is finite dimensional.

Burnside's thm $\Rightarrow$ $\rho : \mathcal{H}_K \twoheadrightarrow \operatorname{End} M$ say $\dim M = m$.

$m^2 = \dim \operatorname{End}(M) \le \dim \rho(\mathcal{H}_K)$. Let $d = \# G(\mathcal{O})/K$

$\dim \rho(\mathcal{H}_K) \le d^2 \cdot \dim \rho(\mathbb{C}X^+) \le d^2 \cdot m^{2-\frac{1}{2^{r-1}}} \Rightarrow m^2 \le d^2 \cdot m^{2-\frac{1}{2^{r-1}}} \Rightarrow m < d^2 = c(G,K)$. $\square$

Let $V \in \operatorname{Rep}^\infty G$. $(V^*)^\infty =: V^\vee$ contragredient repn.

$B, \bar{B}$ opposite Borels.

$r^G_T, \bar{r}^G_T : \operatorname{Rep}^\infty G \longrightarrow \operatorname{Rep}^\infty T$

Cor $(r^G_T V)^\vee = \bar{r}^G_T(V^\vee)$ by adjoint functors

Recall $T(k) = X_*(T) \cdot T(\mathcal{O})$
$X_*(T) = \{t^\lambda, X_*(T)\}$

Prop $C(G(k)/T(\mathcal{O}) \cdot N(k))$ is a projective objective of $\operatorname{Rep}^\infty G$.

Pf $C(G/T(\mathcal{O})N(k)) = i^G_T\, C(T(k)/T(\mathcal{O}))$

Claim 1 $\operatorname{Rep}^\infty T(\mathcal{O})$ is a semisimple category with all simples 1-dim. (since $\operatorname{Rep} T(\mathcal{O}/\mathfrak{m}^k)$ is semisimple)

$\Rightarrow C(T(k)/T(\mathcal{O}))$ is projective in $\operatorname{Rep}^\infty T(k)$. (b/c the $X_*(T)$ contribution is free, $T(\mathcal{O})$ semisimple)

Claim 2 i^G_T : projectives $\longrightarrow$ projectives

Pf of Claim 2, Jacquet f-r $\bar{r}^G_T$ is exact. Second adjointness tells us. $\square$

Iwahori-Hecke algebras

Fix a root system $R \subset \mathfrak{h}^*$. $S \subset R$ simple roots, Cartan matrix $c_{ij} = \langle \check{\alpha}_j, \alpha_i \rangle$ $i,j \in S$.

If R comes from ss Lie alg, then $c_{ij} = \dfrac{2(\alpha_i, \alpha_j)}{(\alpha_j, \alpha_j)}$ using Killing form.

$c_{ii} = 2$. $c_{ij} = -2\cos\left(\frac{\pi}{m_{ij}}\right)$ where $m_{ij} = 2,3,4,6,\infty$ $i \ne j$

$W = W(\mathfrak{h}, R)$ Weyl group.

Lemma $W \simeq$ gr-p with generator s_i, $i \in S$ with relations
1) $s_i^2 = 1$
2) $(s_i s_j)^{m_{ij}} = 1$ $\forall\, i,j \in S$

Defn The braid group B_W = group with generators T_i, $i \in S$

Relations $\forall i,j \in S$ $\quad \underbrace{T_i T_j T_i \cdots}_{m_{ij} \text{ times}} = \underbrace{T_j T_i \cdots}_{m_{ij}}$ $\quad i \neq j$

Remark $\exists\ B_W \twoheadrightarrow W$

$\mathfrak{h}^{reg} = \{ h \in \mathfrak{h} \mid \alpha(h) \neq 0 \quad \forall \alpha \in R\}$ Zariski open subset

E.g. Type A_{n-1} $\quad W = S_n$

$B_n \longrightarrow S_n$

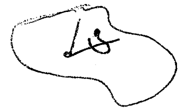
W acts on $\mathfrak{h}^{reg}$ freely.

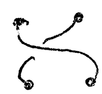
$\mathfrak{h}^{reg}/W = \{(z_1, \dots, z_n)\} = \mathrm{Conf}_n(\mathbb{C})$

Claim $\pi_1(\mathfrak{h}^{reg}/W) \simeq B_W$

image of half loop around $\alpha_i^{-1}(0) \subset \mathfrak{h}_{\mathbb{R}}$ $\longleftarrow$ T_i

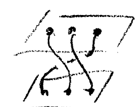
$\alpha_i^{-1}(0) \subset \mathfrak{h}^{reg} \longrightarrow \mathfrak{h}^{reg}/W$





$\mathbb{C}\times\mathbb{C} - \Delta$

$\mathbb{C}\times\mathbb{C} - \Delta$



$\pi_1 = B_n$

E.g. Type A_{n-1} $\quad W = S_n \curvearrowright \mathbb{C}^n$ $\quad \dfrac{\mathbb{C}^n \setminus \text{Big diagonal}}{S_n}$ = unordered n-tuples pairwise distinct complex numbers.

---

length fn $\ell : W \longrightarrow \mathbb{Z}_{\geq 0}$

$\ell(w)$ = length of shortest factorization $w = s_{i_1} \cdots s_{i_p}$ $\quad s_{ij} \in W$

$= \# \{ \alpha \in R_+ \mid w^{-1}(\alpha) \notin R_+ \}$

$= \# \{ \alpha_{i_1}, s_{i_1}(\alpha_{i_2}), \dots, s_{i_1} \cdots s_{i_{p-1}}(\alpha_{i_p}) \}$

Lemma If $s_{i_1} \cdots s_{i_p} = s_{j_1} \cdots s_{j_p}$ are reduced presentations, then $T_{i_1} \cdots T_{i_p} = T_{j_1} \cdots T_{j_p}$

Cor For any $w \in W$ there is a well-defined element $T_w = T_{i_1} \cdots T_{i_p}$

st. $\quad T_{wy} = T_w \cdot T_y$ whenever $\ell(wy) = \ell(w) + \ell(y)$

Fix comm ring R.

Defn Iwahori–Hecke algebra associated to W an $R[q,q^{-1}]$-algebra

$$H_W = \frac{R[q,q^{-1}][B_W]}{((T_i+1)(T_i - q) = 0, \forall i \in S)}$$

Mostly $R = \mathbb{C}$

Lemma The images of T_w, $w \in W$ form a free basis of H_W as an $R[q,q^{-1}]$-module.

Cor H_W is a flat deformation of $R[W]$ when $q \longrightarrow 1$.

Cor $H_W(q_0)$ is s.s. if $q_0 \in R$ not a root of unity

Rem Flat deformation of ss. will be ss. in Zar open.
There are some roots where it is ss.
not: divisors of $1, \dots, \mathrm{rank}(R)$

Case A_1 $\quad W = \mathbb{Z}/2$ $\quad H_W = \dfrac{\mathbb{C}[T,T^{-1}]}{((T+1)(T-q))}$ $\quad q_0 = -1 \rightsquigarrow \dfrac{\mathbb{C}[T]}{(T+1)^2}$ not s.s.

$$\begin{array}{c}\mathbb{C}B_W \twoheadrightarrow H_W \\ \| \\ \text{group alg} \\ \mathbb{C}\pi_1(\mathfrak{h}^{reg}/W)\end{array}$$

reps of π_1 = local system

H_W-mod $\rightsquigarrow$ local system on $\mathfrak{h}^{reg}/W$

In Kahler geometry, Gauss-Manin connection $\to$ local system

Can we find alg variety st. corresponding local system gives ~~rep of H_W~~ Gauss-Manin? Nobody knows.
family param by $\mathfrak{h}^{reg}/W$

Lecture 10 $\quad$ 2/11

Convolution: B alg gp, $\quad X \xrightarrow{B} X_1$ principal B-bundle, $\quad Y$ variety with B-action

$$\begin{tikzcd} X \times Y \arrow[r, "p"] \arrow[dr, "pr_1"'] & X \times_B Y \arrow[d, "Y"] \\ & X_1 \end{tikzcd}$$

$\varphi \in C(X_1)$ $\qquad$ Define $\varphi \boxtimes_B \varphi' \in C(X \times_B Y)$

$\varphi' \in C(B\backslash Y)$ $\qquad$! function st. $p^*(\varphi \boxtimes_B \varphi') = pr_1^*\varphi \boxtimes \varphi'$

$$\begin{array}{c} X = G \\ \downarrow B \\ X_1 = G/B \end{array}$$

~~Action map~~ Assume in addition that Y is a G-space.

Action: $G \times Y \longrightarrow Y$ descends to a well-defined map

$a: G \times_B Y \longrightarrow Y$.

$$\varphi * \varphi' := a_!(\varphi \boxtimes_B \varphi') \in C(Y)$$

This gives a convolution map $* : C(G/B) \times C(B\backslash Y) \longrightarrow C(Y)$

In general, a map $f: Z \to Z'$ with finite fibers, and $\varphi \in C(Z)$

$$(f_!\varphi)(z') = \sum_{z \in f^{-1}(z')} \varphi(z) \qquad f_! : C(Z) \to C(Z')$$

$$(\varphi * \varphi')(y) = \sum_{g \in G/B} \varphi(gB)\, \varphi'(Bg^{-1}y) \qquad (\text{for } \overset{Y=G}{\text{usual}} \text{ convolution})$$

An alternative view: $G \times_B Y \xrightarrow[\sim]{pr_1 \times a} G/B \times Y$ $\qquad$ (not true if Y only B-space)

$$\begin{tikzcd} G/B \times Y \arrow[d, "pr_2"] \\ Y \end{tikzcd}$$

$\varphi * \varphi' = (pr_2)_!(pr_1 \boxtimes a)_!(\varphi \boxtimes_B \varphi')$

Let in addition $\varphi \in C(B\backslash G/B)$.

$\varphi \boxtimes_B \varphi' \in C(B\backslash(G \times_B Y)) = C(G\backslash(G \times_B (G \times_B Y)))$

$$\begin{tikzcd} G \times_B Y \arrow[r, "a"] \arrow[d, hook, "i"] & Y \arrow[d] \\ G \times_B (G \times_B Y) \arrow[r, "Id \times a"] \arrow[d, equal] & G \times_B Y \arrow[d, equal] \\ G/B \times G/B \times Y \arrow[r, "pr_{13}"'] & G/B \times Y \end{tikzcd}$$

So if $\varphi, \varphi' \in C(B\backslash G/B) = C(G\backslash(G/B \times G/B))$

$$* : C^G(G/B \times G/B) \times C^G(G/B \times G/B) \longrightarrow C^G(G/B \times G/B)$$

$$\varphi * \varphi' = (pr_{13})_!(pr_{12}^*\varphi \cdot pr_{23}^*\varphi')$$

all horizontal maps have isomorphic fibers

Let $\mathbb{F}$ be a finite field. $\underset{B}{\overset{G}{\cup}}$ be a reductive gp. Borel. W = Weyl gp, H_W = Iwahori-Hecke alg

<u>Thm</u> The assignment $T_w \longrightarrow 1_{BwB/B}$ gives an algebra isomorphism.

$$H_W|_{q=\#\mathbb{F}} \xrightarrow{\sim} C(B(\mathbb{F})\backslash G(\mathbb{F})/B(\mathbb{F}))$$

<u>Pf</u> $R = R_+ \cup R_-$ set of roots. $\alpha \in R$ $N_\alpha \simeq \mathbb{G}_a$ one-parameter subgp in G corresponding $e_\alpha \in \mathrm{Lie}\, N$

$B = T \cdot N^+$ $\quad T \cdot N^-$ = opposite Borel. <u>Known</u>: $N \simeq \prod_{\alpha \in R^+} N_\alpha$

Given $w \in W$ $\quad N_w^\pm := N^+ \cap wN^\pm w^{-1}$

$$= \prod_{\{\alpha \in R^+ \cap w(R^\pm)\}} N_\alpha$$

<u>Lemma</u>
1) $N_w^+ \times N_w^- \xrightarrow{\sim} N$
2) $\dim N_w^- = \ell(w)$
3) $N_w^- \xrightarrow{\sim} BwB/B$

Pf of 3) $\quad BwB/B = N^+wB/B \simeq N^+/\mathrm{Stab}_{N^+}(wB/B)$

$\mathrm{Stab}_G(wB/B) = wBw^{-1}$, $\quad \mathrm{Stab}_{N^+}(wB/B) = N^+ \cap wBw^{-1} = N^+ \cap wN^+w^{-1} = N_w^+$

$N^+/\mathrm{Stab}_{N^+} = N^+/N_w^+ \cong N_w^-$. □

Have an augmentation hom $\varepsilon: C(B\backslash G/B) \longrightarrow \mathbb{C}$

$$\varepsilon(\varphi) = \sum_{x \in G/B} \varphi(x)$$

Now ground field is $\mathbb{F}$.

<u>Cor</u> $\#BwB/B = \#N_w^- = q^{\ell(w)}$ $\quad (q = \#\mathbb{F})$

$$\varepsilon(1_{BwB/B}) = q^{\ell(w)}$$

Case $\quad G = SL_2 \supset B = \left\{\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}\right\}$ $\quad K^2 \supset \ell$ B-stable line

$G/B = \mathbb{P}^1 = \underset{B/B}{\{\ell\}} \sqcup \underset{BsB/B}{\mathbb{A}^1}$

$W = \{e, s\}$

$$1_{BsB} * 1_{BsB} = a \cdot 1_{B/B} + b \cdot 1_{BsB/B}$$

Evaluating at B/B gives $a = q$

$$q^2 = \varepsilon(1_{BsB})^2 = \varepsilon(1_{BsB} * 1_{BsB}) = q\,\underset{1}{\underset{\|}{\varepsilon(1_{B/B})}} + b\,\varepsilon(1_{BsB/B}) = q + bq$$

$$\Rightarrow b = q-1$$

$$\boxed{1_{BsB/B}^2 = q \cdot 1_{B/B} + (q-1)1_{BsB/B}}$$

For G general, let $s = s_\alpha \in W$ be a simple reflection $\rightsquigarrow$ minimal parabolic $P_s = \left\{\begin{pmatrix} * & & * \\ & \boxed{\begin{smallmatrix} * & * \\ * & * \end{smallmatrix}} & \\ 0 & & * \end{pmatrix}\right\}$

$P_s/B \hookrightarrow G/B$ $\qquad$ Same identity still holds.

$\overset{\|}{B/B \sqcup BsB/B}$

Lemma 2 If $w, y \in W$ are s.t. $\ell(wy) = \ell(w) + \ell(y)$

1) $BwB \times_B ByB/B \xrightarrow{\sim} BwyB/B$

2) $1_{BwB/B} * 1_{ByB/B} = 1_{BwyB/B}$

Pf of 1 Write $y = s_1 \cdots s_p$. Reduce to the case $y = s$ a simple reflection

$$\begin{array}{ccc} G/B & \supset & BwsB/B \\ \downarrow {\scriptstyle P_s/B = \mathbb{P}^1} & & \downarrow {\scriptstyle BsB/B} \\ G/P_s & \supset & BwB/P \cong BwB/B \\ & & \| \\ & & BwsB/P \end{array}$$

only affine line shows up in restricted fiber.

$\Rightarrow BwsB/B \cong BwB/B \times \mathbb{A}^1$

Lemma 3 Let A be an alg with a basis $\{a_w, w \in W\}$ st.

i) $a_s^2 = q \cdot 1 + (q-1) a_s$

ii) $a_{wy} = a_w a_y$ whenever $\ell(wy) = \ell(w) + \ell(y)$

Then the map $T_w \longmapsto a_w$ is an alg isom $H_W \xrightarrow{\sim} A$.

Lemma 3 $\Rightarrow$ thm.

Recall defn of H_W

$$\frac{\text{Group alg } B_W}{(\text{Relations } T_s^2 = q \cdot 1 + (q-1) T_s)} = H_W$$

Claim i) & ii) hold in H_W.

i) by quotient, ii) Property of B_W.

On the other hand, $a_s \cdot a_w = \begin{cases} a_{sw}, & \ell(sw) = \ell(w) + 1 \\ q a_{sw} + (q-1) a_w, & \ell(sw) < \ell(w) \end{cases}$

If $\ell(sw) = \ell(w) - 1$, $\ell(w) = \ell(y) + 1$, $sy = w$
$\Rightarrow a_w = a_s a_y$
$a_s a_w = a_s^2 a_y$

Categorification:

Grothendieck: Let $X/\mathbb{F}$ $\qquad X(\overline{\mathbb{F}})^{Fr} = X(\mathbb{F})$ (Frobenius)

$Sh^{Fr}(X_{\overline{\mathbb{F}}})$ = category of Frob-equivariant sheaves of vector spaces on $X_{\overline{\mathbb{F}}}$ (in étale topology) vector spaces over $\mathbb{C} \supset \overline{\mathbb{Q}}_\ell$

Given $A \in Sh^{Fr}(X_{\overline{\mathbb{F}}})$ define a fn $tr_A : X(\mathbb{F}) \longrightarrow \mathbb{C}$

$$tr_A(x) = Tr(Fr \curvearrowright A_x)$$

$A \longrightarrow tr_A$ gives a homom $\mathbb{C} \otimes_{\mathbb{Z}} K(Sh^{Fr}(X_{\overline{\mathbb{F}}})) \xrightarrow{tr} \mathbb{C}(X(\mathbb{F}))$

Bad: tr does not commute with direct images

Let X projective var. $f : X \to pt.$ $\qquad tr(Fr, f_* A) \neq f_! \, tr_A$

Grothendieck-Lefschetz: $\sum (-1)^i tr(Fr, H^i(X, A)) = f_! \, tr_A$

Conclusion: we must take $K(D(Sh^{Fr}(X_{\overline{\mathbb{F}}}))) = K(Sh^{Fr})$ (a little sloppy)

Define $tr : K(D(\ldots)) \longrightarrow \mathbb{C}(X(\mathbb{F}))$
$$A \longrightarrow [tr_A(x) = \sum (-1)^i Tr(Fr, \mathcal{H}^i_x(A))]$$

$f: X \to Y$ proper

$$\begin{array}{ccc} K(D(X_{\mathbb{F}})) & \xrightarrow{f_*} & K(D(Y_{\mathbb{F}})) \\ \downarrow tr & & \downarrow tr \\ C(X(\mathbb{F})) & \xrightarrow{f_!} & C(Y(\mathbb{F})) \end{array}$$ ← but always too large

Lecture 11 2/15

X alg variety $/\mathbb{C}$

Defn A stratification of X is a decomp $X = \coprod_{\sigma \in \Sigma} X_\sigma$ disjoint union of finitely subsets:

- X_σ locally closed smooth connected alg variety
- $\overline{X_\sigma} = \bigcup_{\sigma' \in \Sigma'} X_{\sigma'}$

Eg. If alg group G acts on X with finitely many orbits, then G-orbits form a stratification (Verdier, Invenciones, Whitney condition automatically satisfied) ← extra axiom in C^∞ setting

might not be true for alg picture $x^2 - y^2 z = 0$, but probably true for G-orbits

$C(X, \Sigma) :=$ $\mathbb{C}$-vector space of constructible $\mathbb{C}$-valued fs constant on strata.

Basis: 1_{X_σ}, $\sigma \in \Sigma$

$C(X) = \bigcup_\Sigma C(X, \Sigma)$ constructible fns.

Define $\int_X : C(X) \to \mathbb{C}$; for $Z \subset X$ locally closed $\int 1_Z := \chi_c(Z)$

compactly supported Euler char.

Claim For $Z = Z_0 \amalg Z_1$, Z_0 open in Z, $Z_1 = Z \setminus Z_0$

then $\chi_c(Z) = \chi_c(Z_0) + \chi_c(Z_1)$

$\to H^i_c(Z_0) \to H^i_c(Z) \to H^i_c(Z_1) \to H^{i+1}_c(Z_0)$ (this is why we use comp. supp.)

$f: X \to Y$ morphism $\quad f^*: C(Y) \to C(X)$

$f_!: C(X) \to C(Y)$, $\varphi \in C(X)$ $\quad f_!(\varphi)(y) := \int_{f^{-1}(y)} \varphi|_{f^{-1}(y)}$

Clear: $\operatorname{supp}(f_! \varphi) \subset f(\operatorname{supp} \varphi)$

Fix G reductive $/\mathbb{C}$
$\quad \cup$
$\quad B$ Borel

$\mathcal{B} = G/B$ flag variety

$X = \mathcal{B} \times \mathcal{B}$

Let $G \curvearrowright X$ diagonally $\quad X_w = G \cdot (B/B, wB/B) \quad w \in W$

Bruhat decomposition $\Rightarrow X = \coprod_{w \in W} X_w$

$C(X, W) =$ Constructible fns wrt X_w

Define an assoc. alg. structure $C(X,W) \times C(X,W) \to C(X,W)$

Given $\varphi, \varphi' \in C(X,W)$, $\quad \varphi * \varphi' = (pr_{13})_! (pr_{12}^* \varphi \cdot pr_{23}^* \varphi')$

$$\begin{array}{ccccc} & & \mathcal{B} \times \mathcal{B} \times \mathcal{B} & & \\ & \swarrow pr_{12} & \downarrow pr_{13} & \searrow pr_{23} & \\ X & & X & & X \end{array}$$

If $Z, Z' \subset X$

<u>Def</u> $Z \circ Z' = pr_{13}(pr_{12}^{-1}(Z) \cap pr_{23}^{-1}(Z'))$

$\mathrm{supp}(\varphi * \varphi') \subset (\mathrm{supp}\,\varphi) \circ \mathrm{supp}(\varphi')$

<u>Prop</u> The map $1_{X_w} \longmapsto w$ yields an alg isom $C(X,W) \xrightarrow{\sim} \mathbb{C}W = H_W|_{q=1}$

$G = SL_2$ $\quad G/B = \mathbb{P}^1$ $\quad X = \mathcal{B} \times \mathcal{B} = \mathbb{P}^1 \times \mathbb{P}^1$

$X_e = $ diagonal $\subset \mathbb{P}^1 \times \mathbb{P}^1$ $\quad X_s = \mathbb{P}^1 \times \mathbb{P}^1 \setminus \mathrm{diag}$ $\quad 1_{\mathrm{diag}}$ is unit

$1_{X_s} * 1_{X_s} = ?$

$pr_{12}^{-1}(X_s) = (\mathbb{P}^1 \times \mathbb{P}^1 \setminus \mathrm{diag}) \times \mathbb{P}^1$

$pr_{12}^{-1}(X_s) \cap pr_{23}^{-1}(X_s) = \left\{ (\ell_1, \ell_2, \ell_3) \in (\mathbb{P}^1)^3 \;\middle|\; \begin{matrix} \ell_1 \neq \ell_2 \\ \ell_2 \neq \ell_3 \end{matrix} \right\}$

$\downarrow \qquad\qquad \mathbb{I}$

$pr_{13}(\cdots) \qquad (\ell_1, \ell_3)$

$pr_{13}^{-1}(\ell_1, \ell_3) = \begin{cases} \mathbb{A}^1 & \ell_1 = \ell_3 \\ \mathbb{A}^1 - pt & \ell_1 \neq \ell_3 \end{cases}$

over $\mathbb{F}_q$, $\# pr_{13}^{-1}(\ell_1, \ell_3) = \begin{cases} q & \ell_1 = \ell_3 \\ q-1 & \ell_1 \neq \ell_3 \end{cases}$

$T_s^2 = (q-1)T_s + q$

Over $\mathbb{C}$ again:

$H^i_c(\mathbb{A}^1) = \begin{cases} \mathbb{C} & i = 2 \\ 0 & i \neq 2 \end{cases}$

$H^i_c(\mathbb{A}^1 \setminus pt) = \begin{cases} \mathbb{C} & i = 1, 2 \\ 0 & i \neq 1, 2 \end{cases}$ $\qquad \chi_c(pr_{13}^{-1}(\ell_1, \ell_3)) = \begin{cases} 1 & \ell_1 = \ell_3 \\ 0 & \ell_1 \neq \ell_3 \end{cases}$

Therefore $1_{X_s} * 1_{X_s} = 1_{\mathrm{diag}}$

<u>Cor</u> For any G, and a simple reflection $s \in W$, $1_{X_s} * 1_{X_s} = 1_{\mathrm{diag}}$

For $w, w' \in W$ $\quad 1_{X_w} * 1_{X_{w'}} = ?$

$pr_{12}^{-1}(X_w) \cap pr_{23}^{-1}(X_{w'}) = X_w \times_{\mathcal{B}} X_{w'}$ $\quad \mathcal{B} \leftarrow$ middle copy $\qquad pr_{13} : X_w \times_{\mathcal{B}} X_{w'} \longrightarrow X$

Last time we showed that if $\ell(ww') = \ell(w) + \ell(w')$ then pr_{13} is an isom. $X_w \times_{\mathcal{B}} X_{w'} \xrightarrow{\sim} X_{ww'}$

Thus we have $1_{X_w} * 1_{X_{w'}} = 1_{X_{ww'}}$ for any w, w' st. $\ell(ww') = \ell(w) + \ell(w')$

Alg lemma from last time $\Rightarrow C(X,W) \xrightarrow{\sim} \mathbb{C}W$.

Reminder on constructible sheaves.

$X/\mathbb{C}$ alg. variety (analytic topology) $\quad Sh(X)$ sheaves of $\mathbb{C}$-v.s. $\quad X = \coprod_{\sigma \in \Sigma} X_\sigma$ a stratification

$Sh(X, \Sigma) \subset Sh(X)$ sheaves locally (fin. dim) constant on strata.

$D(Sh(X))$ derived cat

- Given any ab. cat $\mathcal{A}$, define $D(\mathcal{A})$: objects complexes in $\mathcal{A}$, morphisms of complexes with inverted quasi-isoms.

$D(\mathcal{A}) \longrightarrow D(\mathcal{A})$: $A \longmapsto A[1]$ shift functor

Fix a stratification $X = \coprod X_\sigma$

$D^b(X, \Sigma) \subset D(Sh(X))$ full subcategory with objects A st.

- $\mathcal{H}^i(A)$ are nonzero for only finitely many $i \in \mathbb{Z}$ (cohom of complex)
- $\mathcal{H}^i(A) \in Sh(X, \Sigma)$

6 operations (Grothendieck)

$D^b(X) = \bigcup_{\Sigma} D^b(X,\Sigma)$

Given $A, B \in D^b(X)$: $\quad R\mathcal{H}om(A,B) \in D^b(X)$ internal Hom.

$i : X \hookrightarrow Y$ closed embedding $\quad i^! A = \{$ sections of A supported on $X\}$

$i^! : D(Y) \longrightarrow D(X)$ derived.

Suppose Y is smooth. $\quad \mathbb{D}_X := i^! \mathbb{C}_Y[2\dim Y]$ the dualizing complex $\quad$ // all dim are $\mathbb{C}$

Lemma 1) $\mathbb{D}_X$ does not depend on the imbedding into a smooth var.

2) If X is smooth then $\mathbb{D}_X = \mathbb{C}_X[2\dim X]$

Duality on $D^b(X)$.

$A^\vee := \mathcal{H}om(A, \mathbb{D}_X)$ $\quad$ f^* just means f^{-1}

Let $f: X \longrightarrow Y$ a morphism $\quad$ • $D(X) \underset{f^*}{\overset{f_*}{\rightleftarrows}} D(Y)$ $\quad$ (f^*, f_*) adjoint pairs

• $f_! A := (f_*(A^\vee))^\vee$, $\quad f^!(A) := (f^*(A^\vee))^\vee$ $\quad$ $(f_!, f^!)$ adjoint pair

• $\Delta : X \hookrightarrow X \times X$ diag imbed

$A, B \in D(X)$ define $\quad A \otimes B := \Delta^*(A \boxtimes B)$

$A \overset{!}{\otimes} B := \Delta^!(A \boxtimes B)$ then $\mathcal{H}om(A,B) = A^\vee \overset{!}{\otimes} B$.

Assume X, Y locally compact, finite cohom dim

$f_!$ sections with proper support

$\overset{\cap}{f_*}$ $\quad R f_!$ has a right adjoint denote it by $f^!$

Remarks (i) $H^\bullet(X, \mathbb{C}_X) = H^\bullet(X)$ $\quad \leftarrow$ usual (co) homology

$H^{-i}(X, \mathbb{D}_X) = H_i(X \cup \{\infty\}, \{\infty\}) = H_i^{BM}(X)$ $\quad$ (Borel-Moore)

(ii) f is proper $\Rightarrow f_! = f_*$

f is smooth $\quad f^! = f^*$ (w/o shift)

(iii) $j : X \hookrightarrow Y$ open embedding $\quad j_! \mathbb{C}_X =$ extension by zero.

Fix stratification $X = \coprod_\sigma X_\sigma$

$K(D^b(X,\Sigma))$ generated by $\quad i_\sigma : X_\sigma \hookrightarrow X$

$\mathcal{L}_\sigma$ a local system on X_σ

$(i_\sigma)_! \mathcal{L}_\sigma$ generate $K(\dots)$

Special case: if $\pi_1(X_\sigma) = 0$, then $[(i_\sigma)_! \mathbb{C}_{X_\sigma}]$ is a free basis of $K(\dots)$

Define: $K(D^b(X,\Sigma)) \xrightarrow{\chi} C(X,\Sigma)$

$[(i_\sigma)_! \mathcal{L}_\sigma] \longmapsto \operatorname{rk} \mathcal{L}_\sigma \cdot 1_{X_\sigma}$

$\chi_A(x) = \sum_k (-1)^k \dim \mathcal{H}_x^k(A)$ $\quad$ (compare to last time: we are replacing Frob by Id)

Given morphism $f: X \longrightarrow Y$

$$\begin{tikzcd} K(D^b(X)) \arrow[r, shift left, "f_*"] \arrow[d, "\chi"] & K(D^b(Y)) \arrow[l, shift left, "f^*"] \arrow[d, "\chi"] \\ C(X) \arrow[r, shift left, "f_!"'] & C(Y) \arrow[l, shift left, "f^*"'] \end{tikzcd}$$

On $K(\dots)$, $f^* = f^!$

$f_* = f_!$

$$\chi_{A\otimes B} = \chi_A \cdot \chi_B = \chi_{A\overset{!}{\otimes} B}$$ (doesn't matter in Groth. gp)

$\chi^t_A = \sum_k t^k \mathcal{H}^k_x A$ not well-defined on Grothendieck gp.

$A \longmapsto \chi^t_A$ does not induce a map $K(D^b(X)) \longrightarrow C(X)[t,t^{-1}]$

Ultimate unsatisfactory approach: $X = \coprod_w X_w$ $\pi_1(X_w) = 0$ $\forall w$

H_W $X = \mathcal{B}\times\mathcal{B}$

$i_w : X_w \hookrightarrow X$

Take shift of ◯ to q-power) $(i_w)_! \mathbb{C}_{X_w} \longmapsto T_w$

Not Grothendieck group of anything. Not compatible w/ convolution.

- Idea: instead of $(i_w)_! \mathbb{C}_{X_w}$, take constructible cohomology, take Kazhdan-Lusztig basis instead of T_w.
  this is compatible w/ multiplication; but not category (yet...)

Lecture 12 2/18

Perverse sheaves

Fix a statification $X = \coprod_{\sigma\in\Sigma} X_\sigma$ ~~$D(X,\Sigma)$~~ $D^b(X,\Sigma)$ = derived cat of sheaves constr w.r.t. Σ

triangulated cat

Perverse sheaves $P(X,\Sigma) \subset D^b(X,\Sigma)$ an abelian subcategory

$\cup$

$Sh(X,\Sigma)$

Properties 1) $P(X,\Sigma)$ is abelian

2) Objects have finite length

3) Simple objects are (1-1) correspondence with $\{\sigma\in\Sigma, \mathcal{L}_\sigma$ an irred local system on $X_\sigma\}$

$IC(\mathcal{L}_\sigma)$ the IC-complex associated to $\mathcal{L}_\sigma$

$\mathrm{Supp}(IC(\mathcal{L}_\sigma)) = \overline{X_\sigma}$

$IC(\mathcal{L}_\sigma)\big|_{X_\sigma} = \mathcal{L}_\sigma[\dim X_\sigma]$

In particular, if X is smooth, $IC(\mathbb{C}_X) = \mathbb{C}_X[\dim X]$ $\longrightarrow$ $\left.\begin{array}{l}\mathbb{C}_X\\ \mathbb{D}_X = \mathbb{C}_X[2\dim X]\end{array}\right\}$ for X sm.

4) $IC(\mathcal{L}^*_\sigma) = IC(\mathcal{L}_\sigma)^\vee := R\mathcal{H}om(IC(\mathcal{L}_\sigma), \mathbb{D}_X)$

5) $P(X,\Sigma)$ is stable under duality (not true for $Sh(X,\Sigma)$)

Remark If $\pi_1(X_\sigma) = 0$ $\forall\sigma\in\Sigma$, then irreducible local system on X_σ is $\mathbb{C}_{X_\sigma}$.

$\Rightarrow$ simple perv. sheaves are $IC(\mathbb{C}_{X_\sigma})$

Decomposition theorem [BBD](G=Gabber)

Let $f: X\longrightarrow Y$ be a projective morphism of alg varieties, $Z\hookrightarrow X$ loc closed, smooth

Then $Rf_* IC(\mathbb{C}_Z) = \bigoplus IC(\mathcal{L})[n]$

$$\begin{array}{c} X \\ f\downarrow \\ Y \end{array}$$ fibration with smooth projective fibers and smooth X, Y

$$\left(R^i f_* \mathbb{C}_X\right)_y = H^i(f^{-1}(y), \mathbb{C})$$

Decomp thm says

$$Rf_*\mathbb{C}_X \cong \bigoplus_i R^i f_* \mathbb{C}_X$$

Deligne's degeneration of spectral sequence. (uses Hard Lefshetz) follows from wt decomp (in G-H)

$$H^i(X,\mathbb{C}) \Leftarrow E_2^{p,q} = H^p(Y, R^q f_* \mathbb{C}_X)$$

<u>Defn</u> $A \in D^b(X,\Sigma)$ is a perverse sheaf if

1) $\mathcal{H}^i A|_{X_\sigma} = 0$ for $i > \cancel{d}(\dim X_\sigma)$

2) (1) holds for $A^\vee$

(1) $\mathcal{H}^i(j_\sigma^* A) = 0$ for $i > -\dim X_\sigma \Leftrightarrow A \in D^b(X,\Sigma)^{\geq i}$

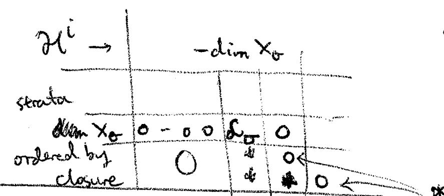
(2) $\mathcal{H}^i(j_\sigma^! A) = 0$ for $i > -\dim X_\sigma \Leftrightarrow A \in D^b(X,\Sigma)^{\leq i}$

<u>Lemma</u> An object $A \in D^b(X,\Sigma)$ is an IC-sheaf $= IC(\mathcal{L}_\sigma)$ iff 1+1') $\mathcal{H}^i A|_{X_{\sigma'}} = 0$ $\forall i \geq -\dim X_{\sigma'}$ and $\sigma \neq \sigma'$

2') = 2) ~~with~~ $\geq$

$f: X \longrightarrow Y$ proper

$R^i f_* \mathbb{C}_X$



(diagram for condition 1 of IC)

<u>Hecke algebra</u> H_W

($W \supset S$ simple reflections)

Basis T_w, $w \in W$ of H_W

$$T_s^2 = (q-1)T_s + q$$

$$T_s^{-1} = q^{-1}(T_s + (1-q))$$

Define an <u>involution</u> $*: h \longmapsto h^*$ on H_W

$$q \longmapsto q^{-1}$$

$$T_w \longmapsto (T_{w^{-1}})^{-1}$$

$$(T_s+1)^* = q^{-1}(T_s + (1-q) + q) = q^{-1}(T_s+1)$$

Put $t = \sqrt{q}$ i.e. $H := \mathbb{C}[t^{\pm 1}] \underset{\mathbb{C}[q^{\pm 1}]}{\otimes} H_W$ $\quad \mathbb{C}[q^{\pm 1}] \longrightarrow \mathbb{C}[t^{\pm 1}]$, $q \longmapsto t^2$

<u>Thm</u> (Kazhdan-Lusztig) $\forall w \in W$, $\exists!\ c_w \in H$ st.

1) $c_w^* = c_w$

2) $c_w = t^{-\ell(w)}\left(T_w + \sum_{x<w} \overset{\text{poly}}{P_{x,w}}(t^2) T_x\right)$ where $\deg P_{x,w} \leq \frac{1}{2}(\ell(w) - \ell(x) - 1)$

$P_{x,w} \in \mathbb{Z}[t]$ $\quad P_{w,w} = 1$

The c_w's form a $\mathbb{C}[t^{\pm 1}]$-basis of H called Kazhdan-Lusztig basis.

$P_{x,w}$ are KL polys.

<u>Pf</u> For $s \in S$, $c_s = t^{-1}(T_s + 1)$

<u>Lemma</u> $(T_{w^{-1}})^{-1} = t^{-2\ell(w)} \sum_{x \leq w} R_{x,w}(t^2) T_x$ where $R \in \mathbb{Z}[t]$, $\deg R_{x,w} \leq \ell(w) - \ell(x)$.

<u>Proof</u> Induction on $\ell(w)$. using $T_s T_w = \begin{cases} T_{sw} & \text{if } \ell(sw) > \ell(w) \\ (t^2-1)T_w + t^2 T_{sw} & \text{if } \ell(sw) < \ell(w) \end{cases}$

Proof of Existence

Induct on $\ell(w)$.

Suppose that we've defined c_x $\forall \ell(x) < \ell(w)$

- For any $x \le y < w$ $x,y,w \in W$, $\mu(x,y) :=$ coeff at $t^{\ell(y)-\ell(x)-1}$ in $P_{x,y}(t^2)$

Choose $s \in S$ st. $\ell(sw) < \ell(w)$

Put $$c_w := \underbrace{t^{-1}(T_s+1)}_{c_s} c_{sw} - \sum_{\substack{x<sw \\ sx<x}} \mu(x,sw)\, c_x$$

Uniqueness Let $c'_w = t^{-\ell(w)}\left(T_w + \sum_{x<w} Q_{x,w}(t) T_x\right)$ $\qquad$ $\deg Q_{x,w} \le \ell(w)-\ell(x)-1$

Must show $c'_w = c_w$.

Fix $w \in W$, do descending induction on $x \le w$, starting with $Q_{w,w} = P_{w,w} = 1$

By induct we know $Q_{y,w} = P_{y,w}$ $\forall$ $x < y \le w$

Want to show $Q_{x,w} = P_{x,w}$.

Condition 1) $c'_w = (c'_w)^*$. Using lemma, gives $$\overbrace{t^{\ell(x)-\ell(w)} Q_{x,w}(t)}^{\text{poly in } t^{-1}} - \overbrace{t^{-\ell(x)+\ell(w)} Q_{x,w}(t^{-1})}^{\text{a poly in } t \text{ without constant term.}}$$
$$= \sum_{x<y\le w} t^{\ell(w)+\ell(x)-2\ell(y)} R_{x,y}(t)\, Q_{y,w}(t)$$

We know the RHS. This completely determines each coeff of Q. $\square$

Geometric interpretation

$\mathcal{B} = G/B$, $X = \mathcal{B} \times \mathcal{B}$ $\qquad$ $X_w = G\cdot(B/B, wB/B)$ $\qquad$ $X = \coprod_{w\in W} X_w$

$X_w \hookrightarrow X \xrightarrow{pr_1} \mathcal{B}$ (composite p) is G-equivariant fibration $\qquad$ $p^{-1}(B/B) = NwB/B \simeq \mathbb{C}^{\ell(w)}$

$\Rightarrow X_w$ is homotopic to $\mathcal{B}$

$\pi_1(\mathcal{B}) = 0 \Rightarrow \pi_1(X_w) = 0$

$\uparrow$ Bruhat decomp w/ even dim'l cells (over $\mathbb{C}$) $\qquad$ $d := \dim \mathcal{B}$

Let $A \in D^b(X, W)$ $\qquad$ $\mathcal{H}^i A \in Sh(X,W)$ $\qquad$ $\chi(A) := \sum_{w\in W}\left(T_w \cdot \sum_{i\in\mathbb{Z}} \dim \mathcal{H}^{i-d}_w(A)\cdot t^i\right) \in H$

$$\begin{array}{ccc} s \in S \quad X_s \amalg B_{diag} = \overline{X}_s & \xhookrightarrow{j_s} & X \\ \mathbb{P}^1 \big\downarrow pr_1 & & \\ \mathcal{B} & & \end{array}$$

Notation: $IC_w := IC(\mathbb{C}_{X_w})$

$IC_s = (j_s)_* \mathbb{C}_{\overline{X}_s}[d+1]$ $\qquad$ $\chi(IC_s) = t^{-1}(T_s+1) = c_s$

Thm $\chi(IC_w) = c_w$ $\forall w \in W$

equivalently $P_{x,w}(t) = \sum_i \dim \mathcal{H}^{i-d}_x(IC_w)\, t^i$

Cor 1) $P_{x,w}$ has ≥ 0 coeff

2) $\mathcal{H}^i(IC_w) = 0$ unless $i \equiv \ell(w) \mod 2$

Proof The formula for χ insures that $\chi(A^\vee) = \chi(A)^*$

$$IC_w = IC(\mathbb{C}_{X_w}) \cong IC(\mathbb{C}^*_{X_w}) = (IC_w)^\vee. \quad \text{Therefore } \chi(IC_w)^* = \chi(IC_w)$$

Defn of IC $\Leftrightarrow$ bound on degree of $P_{x,w}$. (uniqueness requires this) ∎
doesn't need t^2

$$K^\#(D(X,W)) = \frac{\text{Free abelian gp on objects of } D^b(X,W)}{[A \oplus B] - [A] - [B]}$$

$\chi : K^\#(D(X,W)) \longrightarrow H$ well-defined $\mathbb{C}[t^{\pm 1}]$-module map

$H^\# = \mathbb{C}$-span of classes of $IC_w[n]$

Thm $H^\#$ is a subalg in $K^\#(D^b(X,W))$, and χ induces an alg isom $H^\# \xrightarrow{\sim} H$.

Lecture 2/22

Need to prove: $\forall w, w' \in W$

$IC_w = IC(X_w)$ $\quad X_w \subset X = \mathcal{B} \times \mathcal{B}$, $\quad X_w = G \cdot (B/B, wB/B)$

$IC_{w'}$

$$\begin{tikzcd}
 & & \mathcal{B}\times\mathcal{B}\times\mathcal{B} \arrow[dl, "pr_{12}"'] \arrow[dr, "pr_{23}"] & & \\
 & X & & X & \\
 & X_w \arrow[u, phantom, "\cup" description] \arrow[dl, "p_1"'] \arrow[dr, "q_1"] & & X_{w'} \arrow[u, phantom, "\cup" description] \arrow[dl, "p_2"'] \arrow[dr, "q_2"] & \\
\mathcal{B} & & \mathcal{B} & & \mathcal{B} \\
 & & B/B \arrow[u, phantom, "\in" description] & &
\end{tikzcd}$$

- $IC_w * IC_{w'} = \bigoplus_{y \in W} IC_y[n_y]$ (can repeat)

- $C_w \cdot C_{w'} = \sum t^{n_y} \cdot C_y$ (in H)

$q_1^{-1}(B/B) = Bw^{-1}B/B =: \mathcal{B}_{w^{-1}}$

$X_w \simeq G \times_B \mathcal{B}_{w^{-1}} \xrightarrow{q_1} G/B$

$X_{w'} \simeq G \times_B Bw'B/B \xrightarrow{p_2} G/B$

$(\mathcal{B}\times\mathcal{B}) \times_{\mathcal{B}} (\mathcal{B}\times\mathcal{B}) = \mathcal{B}\times\mathcal{B}\times\mathcal{B} \overset{i}{\hookrightarrow} X^2$

$X_w \times_{\mathcal{B}} X_{w'} = G \times_B (\mathcal{B}_{w^{-1}} \times \mathcal{B}_{w'})$

A, A' sheaves on $X = \mathcal{B}\times\mathcal{B}$

convolve means:

$X_w \times_{\mathcal{B}} X_{w'} = i^{-1}(\underbrace{X_w \times X_{w'}}_{\subset X^2})$

$$pr_{12}^* A \otimes pr_{23}^* A' := i^*(A \boxtimes A')$$

locally over $\mathcal{B}$, looks like

$G/B \times \mathcal{B}_{w^{-1}} \times \mathcal{B}_{w'} \overset{i}{\hookrightarrow} (G/B \times \mathcal{B}_{w^{-1}}) \times (G/B \times \mathcal{B}_{w'})$

Conclusion $pr_{12}^* IC_w \otimes pr_{23}^* IC_{w'} = IC(X_w \times_{\mathcal{B}} X_{w'})$

locally

$IC_w = \mathbb{C}_{G/B} \boxtimes IC(\overline{\mathcal{B}}_w)$, $IC_{w'} = \mathbb{C}_{G/B} \boxtimes IC(\mathcal{B}_{w'})$

Decomposition theorem:

$pr_{13} : \mathcal{B}\times\mathcal{B}\times\mathcal{B} \longrightarrow \mathcal{B}\times\mathcal{B}$

$IC_w * IC_{w'} = (pr_{13})_* IC(X_w \times_{\mathcal{B}} X_{w'}) = \bigoplus_y IC_y[n_y]$.

(to prove decomp thm, you need to go to positive char)

$\chi(A) \in H \qquad \chi(A) = \sum_{w,i} \left(\dim \mathcal{H}^i_w(A)\, t^i\right) T_w$

Need to show (I): $\chi(IC_w)\cdot \chi(IC_{w'}) = \chi(IC_w * IC_{w'})$

Reduction to positive char $k = \overline{\mathbb{F}_q}$ $\qquad \mathcal{B}_k,\ X_k,\ IC_w \circlearrowleft Fr$ = Frobenius.

$\mathcal{B}(\mathbb{F}_q) \hookrightarrow \mathcal{B}_k(k)$ $\qquad$ (defined same way using ℓ-adic)

$\chi_q(A) = \sum_w \left(\sum_i \dim \mathcal{H}^i_w A\ t^i\right) 1_{X_w(\mathbb{F}_q)} \in \mathbb{C}[X(\mathbb{F}_q)]$

(I) $\Leftrightarrow$ $\forall q,\ t := \sqrt{q}$

(II) $\chi_q(IC_w) * \chi_q(IC_{w'}) = \chi_q(IC_w * IC_{w'})$

Define $tr(A) \in \mathbb{C}[X(\mathbb{F}_q)]$

$tr(A) = \sum_w \sum_i (-1)^i\, tr(Fr, \mathcal{H}^i_w(A))\, 1_{X_w(\mathbb{F}_q)}$

III $\quad tr(A * A') = tr(A) * tr(A')$ by Grothendieck Lefschetz. [General fact]

<u>Key Lemma</u> $\quad \chi_q = tr$ in our particular situation

<u>Lemma</u> $\Rightarrow$ Thm

Amounts to saying that Fr acts on $\mathcal{H}^i_w(IC_y)$ by $q^{i/2} = t^i$ scalar mult.

Bott-Samelson resolution

$$\begin{tikzcd}[column sep=small] & \overline{X_{s_1}} \arrow[dl, "p_1"'] \arrow[dr, "q_1"] & & \overline{X_{s_2}} \arrow[dl, "p_2"'] \arrow[dr, "q_2"] & \cdots & \overline{X_{s_m}} \arrow[dl, "p_m"'] \arrow[dr, "q_m"] & \\ \mathcal{B} & & \mathcal{B} & & \mathcal{B} & & \mathcal{B} \end{tikzcd}$$

$s \in W$ simple reflection

$\overline{X_s} = X_s \cup \mathcal{B}_{diag}$ smooth

$s_1, \dots, s_m$ tuple of simple reflections.

$$\mathcal{X} = \overline{X_{s_1}} \times_{\mathcal{B}} \overline{X_{s_2}} \times \cdots \times_{\mathcal{B}} \overline{X_{s_m}}$$
$$\downarrow f = p_1 \times q_m$$
$$X = \mathcal{B} \times \mathcal{B}$$

① p_i, q_i are fibrations with fiber $\mathbb{P}^1$.

② $\mathcal{X}$ is smooth, projective

③ f projective morphism

④ Assume $w = s_1 \cdots s_m$ <u>reduced</u> decomp

Then $\cancel{Im(\mathcal{X})} = \overline{X_w}$, $\quad \cancel{Im(\mathcal{X})} = f(\mathcal{X})$

moreover $f : \mathcal{X} \to \overline{X_w}$ is Bott-Samelson resolution of singularities

⑤ Assuming reduced, $\forall x \in \overline{X}_w$

$f^{-1}(x) = \coprod_\alpha \mathbb{A}^{d_\alpha}$ a paving

$f : \mathcal{X} \longrightarrow \overline{X}_w$

$\mathcal{H}^i := R^i f_* \mathbb{C}_{\mathcal{X}}$

<u>Claim</u>: $\mathcal{H}^i = 0$ if i is odd.

$Fr|_{\mathcal{H}^{2i}} = q^i \cdot Id.$

<u>Follows</u> $\quad H^i_c(\mathbb{A}^1) = \begin{cases} \mathbb{C} & i = 2,\ Fr = q \\ 0 & i \neq 2 \end{cases}$

Since $\mathcal{X}$ is smooth,

$Rf_* \mathbb{C}_{\mathcal{X}}[\dim X_w] =$

$\qquad \shortparallel$

$Rf_* IC(\mathcal{X}) \overset{\text{decomp thm}}{=} IC_w \oplus \left(\bigoplus_{w' < w} IC_{w'}[n_{w'}]\right)$ allow repeat

Stalk $\mathcal{H}^i_y(IC_w)$ is a direct summand of $\mathcal{H}^i$

Implies $\mathcal{H}^i(IC_w) = 0$ $\quad i \not\equiv \dim X_w \mod 2$

Fr acts as q^{required}.

——— THM proved ———

$\overline{X}_s$ smooth $\quad IC_s = \mathbb{C}_{\overline{X}_s}[1+\dim \mathcal{B}]$

$Rf_* \mathbb{C}[\dim X] = IC_{s_1} * IC_{s_2} * \cdots * IC_{s_m}$

<u>Gap</u> from last time

Claimed the $\chi(IC_w^\vee) = \chi(IC_w)^*$

Proof by reduction to $\mathbb{F}_q$. It is true for any A, $tr(A^\vee) = tr(A)^*$. We have proved $tr = \chi$. $\square$

<u>Remark</u> $(\mathbb{A}^1 - \{0\}) \cup \{0\} = \mathbb{A}^1$ $\quad \# \mathbb{A}^1 = \#(\mathbb{A}^1 \setminus \{0\}) + \#\{0\}$. $\quad$ so reduce to >0 char ok

$\underset{\mathbb{C}}{\overset{\|}{H_c^2(\mathbb{A}^1)}} = \underbrace{H^2 \qquad H_c^1(\mathbb{A}^1 \setminus \{0\}) = \mathbb{C}}_{\text{cannot direct sum}}$ $\quad$ so doing directly not

Tempting to try: $j_w : X_w \hookrightarrow X$

$(j_w)_! \mathbb{C}_{X_w} \longrightarrow T_w$ $\quad$ breaks down when computing stalks.

Point w/ Bott-Samelson is that we work only with ~~closed~~ closure of (affine) cells, no open cells.

Let X be an alg variety $\quad$ [Sketch of how to categorify]

Saito defined an abelian cat

$\underset{\cup}{D_{\text{mixed}}(X)}$ of mixed $\mathcal{D}$-modules on X

$\mathcal{D}_{\text{pure}}(X)$ — " — ~~pure~~ pure — " ———

$Ob(\mathcal{D}_{\text{pure}}) = \bigoplus IC(\mathcal{L}^{\text{pure}})$ $\quad$ ~~pure~~ $\mathcal{L}^{\text{pure}}$ variation of pure Hodge struct.

1) $M \in D_{\text{mixed}}$ is a $\mathcal{D}$ module over $\mathcal{D}_X$ sheaf of alg diff operators, together with

2) $\mathbb{Z}$-filtration by $\mathcal{O}_X$-coherent subsheaves $\quad 0 \subset M_{-N} \subset M_{-N+1} \subset \cdots$ $\quad$ (bdd below)

$gr\, M$ a module over $gr\, \mathcal{D}_X = p_* \mathcal{O}_{T^*X}$ $\quad$ $p: T^*X \to X$ affine morphism

so $gr\, M$ is a coherent sheaf of T^*X st $\text{supp}(gr\, M) = $ Lagrangian in T^*X.

3) There is a finite filtration $0 \subset W_{-p} M \subset W_{-p+1} M \subset \cdots W_q M = M$

by $\mathcal{D}_X$-submodules (weight filtration) st. $gr_i^W M \in D_{\text{pure}}$

<u>Thm</u> (Saito) 1) Any morphism $M \to N$ in D_{mix} is <u>strictly</u> compatible with both filtrations, i.e., $\quad M \twoheadrightarrow Im\, M \hookrightarrow N$ $\quad$ (<u>defn</u>) morphism compatible w/ filtr, does not require equal)

two diff ways to give filtr, they always agree

2) All functors have natural mixed versions.

<u>Ex</u> $X = pt$. $D_{\text{mix}}(pt) = $ mixed Hodge structures on fd. v.s.

$\cup$

$D_{\text{pure}}(pt) = $ pure Hodge str

$\cup \qquad\qquad \cup$

~~Ex~~ $\qquad$ Tate struct

$X = \coprod_w X_w$

In general, we still can't categorify. It is a theorem about flag variety that allows us to do this.

Lecture 2/25

$X = \mathcal{B} \times \mathcal{B} = \coprod_{w \in W} X_w$

Thm $\exists$ an abelian subcat $\underset{\mathcal{D}\text{-mod}_{\text{Tate}}(X,W)}{\overset{\mathcal{C}}{\shortparallel}} \subset \mathcal{D}\text{-mod}_{\text{mixed}}(X,W)$

1) $IC(\mathbb{C}_{X_w}[n]) \in \mathcal{C}$ These are the simple objects of $\mathcal{C}$

2) $\forall M \in \mathcal{C}$ $gr_n^W M \in \overset{\text{some}}{\bigoplus} IC(\mathbb{C}_{X_w}[n])$

3) $j_w : X_w \hookrightarrow X$ $M_w := (j_w)_*(\mathbb{C}_{X_w}) \in \mathcal{C}$

$M_w^\vee := (j_w)_!(\mathbb{C}_{X_w}) \in \mathcal{C}$

4) The classes $[M_w[n]]$, $w \in W$, form a basis of $K(\mathcal{C})$, $[M_w^\vee[n]]$ — " —

5) The category $\mathcal{C}$ is stable under convolution and the assignment.

$[M_w[n]] \longmapsto q^n T_w$ gives an alg isom $K(\mathcal{C}) \xrightarrow{\sim} H_W$

$IC(\mathbb{C}_{X_w}) \longmapsto c_w$

Key Lemma For any $w, w' \in W$, then $\mathrm{Ext}^i_{D^b(X)}(IC(\mathbb{C}_{X_w}(0)), IC(\mathbb{C}_{X_{w'}}(0)))$ has pure Tate structure.

Pf Embed $\mathrm{Ext} \subset H^*($space related to Bott-Samuelson$)$ ~~[illegible]~~ then ...

Comments

$M_y = (j_y)_*(\mathbb{C}_{X_y}(0)) \in \mathcal{C}$ has weight filtration $W_p(M_y)/W_{p-1}(M_y) = \bigoplus IC(\mathbb{C}_{X_z})$

$(M_y^\vee)_x = 0$ $\forall x \in \overline{X}_y \setminus X_y$ but not true for IC. So stalk not compatible with grading/filtration (need to go to pos. char)

ext by zero

In general $\mathcal{D}\text{-mod}_{mix}(X) \xrightarrow{DR}$ Perverse sheaves on X

$\mathcal{D}\text{-mod}_{mix}(X) \xrightarrow{gr^F} \mathrm{Coh}(\coprod_\sigma T^*_{X_\sigma} X)$

DR ignores Hodge

$\begin{cases} DR(M) \overset{H^\bullet}{=} (\Omega^\bullet_X \otimes_{\mathcal{O}} M, \\ d : \Omega^{\bullet\, an}_X \otimes_{\mathcal{O}_X} M \longrightarrow \Omega^{\bullet+1\, an}_X \otimes_{\mathcal{O}_X} M) \end{cases}$

W weight filtration

$\dot{F}$ filtration by $\mathcal{O}_X$-coh subsh

$X = \coprod_\sigma X_\sigma$

Digression $X = \coprod_{\sigma \in \Sigma} X_\sigma$ stratification

$Sh(X,\Sigma) \hookrightarrow D^b(X,\Sigma) \subset D^b(Sh(X))$

full subcat

$\Downarrow$

A, B

$$\mathrm{Ext}^i_{Sh(X,\Sigma)}(A,B) \xrightarrow{?} \mathrm{Ext}^i_{D^b(Sh(X))}(A,B) = H^i(X, A^\vee \overset{!}{\otimes} B)$$

isom for $i=0,1$.

If $X_\sigma = X$: $\quad Sh(X,\{1\}) = $ local systems on X $= \mathrm{Rep}\,\pi_1(X)$ fin dim $\qquad \pi_1(X) = \pi$

$$\mathrm{Ext}^i_{\text{loc sys}}(\text{triv}, \mathcal{L}) \longrightarrow H^i(X,\mathcal{L}) = H^i(B\pi,\mathcal{L}) = H^i(\pi,\mathcal{L})$$

$$\| $$

$\mathrm{Ext}^i_{\mathrm{Rep}\,\pi}(\text{triv},\mathcal{L})$ Take $X = B\pi$ classifying space

$\mathrm{Ext}^i_\pi(\text{triv},\mathcal{L})$ <u>not f.d.</u>

Call π "good" if $\forall\, \mathcal{M},\mathcal{L} \in \mathrm{Rep}^{f.d.}\pi$

$$\mathrm{Ext}^i_{\mathrm{Rep}\,\pi}(\mathcal{M},\mathcal{L}) \xrightarrow{\sim} \mathrm{Ext}^i_{\mathbb{C}\pi}(\mathcal{M},\mathcal{L})$$

1) $\{1\} = \pi$ good

2) $\mathbb{Z}^n = \pi$ good

$$B(\mathbb{Z}^n) = (S^1)^n \sim (\mathbb{C}^*)^n$$

<u>Thm</u> Let $X = \coprod_{\sigma\in\Sigma} X_\sigma$ stratified alg variety s.t. $\forall \sigma \in \Sigma$

1) $\pi_i(X_\sigma) = 0 \quad \forall i \geq 2$

2) $\pi_1(X_\sigma)$ is good

Then $D^b(P(X,\Sigma)) \longrightarrow D^b(X,\Sigma)$ is an equivalence

Cohomology of strata will show up again in a totally diff way later.

G linear alg group (over $\mathbb{C}$)

X alg G-variety

$\mathrm{Coh}^G X$ $\quad$ G-equivariant coherent sheaves

$K^G(X) := K(\mathrm{Coh}^G X)$.

1) $K^G(pt) = K(\mathrm{Rep}\,G)$

$G = \mathbb{C}^\times$, $\quad K(\mathrm{Rep}\,G) = \mathbb{Z}[q,q^{-1}]$

2) If G unipotent, $K^G(pt) = \mathbb{Z}$

For any G, $\quad G_{red} := G/\text{Unip rad}$. Then $G \simeq G_{red} \ltimes \text{Unip. rad.}$ (general thm)

and $K^G(X) \xrightarrow{\sim} K^{G_{red}}(X)$

3) If G acts trivially on X, $\quad K^G(X) = K(\mathrm{Rep}\,G) \otimes K(X)$

4) Induction: $\quad G \supset H \curvearrowright X$

$$K^G(G \times^H X) \cong K^H(X) \qquad \text{In fact Coh are equiv.}$$

In particular $K^G(G/H) \simeq K(\mathrm{Rep}\,H)$

5) Functoriality $f: X \to Y$ G-equivariant mor

a) If f is proper, $f_*: K^G(X) \longrightarrow K^G(Y)$

b) If f has finite Tor dimension, then have $f^*: K^G(Y) \to K^G(X)$

(in particular f^* exists if X, Y are smooth)

6) <u>Thom</u> <u>isomorphism</u>

Let $\pi: E \to X$ be a G-equivariant vector bundle.

Then $\pi^*: K^G(X) \xrightarrow{\sim} K^G(E)$ is an isom.

More generally, also holds for G-equivariant affine bundles (fibers affine space)

Long exact sequence:

$$Y \underset{cl}{\overset{i}{\hookrightarrow}} X \underset{op}{\overset{j}{\hookleftarrow}} U = X \setminus Y$$

Then $0 \longrightarrow K^G(Y) \xrightarrow{i_*} K^G(X) \xrightarrow{j^*} K^G(U) \longrightarrow K_1^G(Y)$

~~claim~~ If U is affine space $\mathbb{A}^n$, then we get a SES, is 0 boundary map.

<u>Special case</u> $G = SL_2$, $W = \{1 = e, s\}$, $\mathcal{B} = G/B = \mathbb{P}^1$, $X = \mathbb{P}^1 \times \mathbb{P}^1$

$X_e = \Delta \cong \mathbb{P}^1$

$X_s = (\mathbb{P}^1 \times \mathbb{P}^1) \setminus \Delta \xrightarrow[\text{affine}]{pr_1} \mathbb{P}^1$ G-equiv bundle

$$T^*X \supset Z = T^*_\Delta X \amalg T^*_{X_s} X$$

conormal bundles

$T^*_\Delta X$ = line bundle on $\mathbb{P}^1$ — closed

$T^*_{X_s} X$ = zero section = affine bundle over $\mathbb{P}^1$ — open

$K(Z) = ?$ $\quad 0 \longrightarrow K(T^*_\Delta X) \longrightarrow K(Z) \longrightarrow K(T^*_{X_s} X) \longrightarrow 0$ (can reduce to $U = \mathbb{A}^1$)

By Thom isom $K(T^*_\Delta X) \simeq K(\Delta) \cong K(\mathbb{P}^1)$

$$K(T^*_{X_s} X) = K(X_s) = K(\mathbb{P}^1)$$

$\mathbb{C} \otimes_{\mathbb{Z}} K(\mathbb{P}^1) = H^*(\mathbb{P}, \mathbb{C}) = \mathbb{C}[t]/t^2$

(Chern character)

$$0 \to \mathbb{C}[t]/t^2 \longrightarrow \mathbb{C} \otimes_{\mathbb{Z}} K(Z) \longrightarrow \mathbb{C}[t]/t^2 \to 0$$

We have a functor $\mathcal{C} = \text{D-mod}_{Tate}(X, W) \xrightarrow{gr^F} Coh(Z)$.

$K(\mathcal{C}) \longrightarrow K(Z)$ (4-dim)

$K(\mathcal{C}) = H_W$, $\dim H_W = 2$

Same discrepancy of cohom of strata!

We get 2+2; want 1+1. No way to distinguish which of the 1-dim to take.

What naturally comes from geometry is $K(Z)$ with some modification ⇝ affine Hecke algebra.

Let G be a semisimple group, W Weyl gp, $X = \coprod_{w\in W} X_w$

$$Z = \coprod_{w\in W} T^*_{X_w}X$$

Define for any $w\in W$, $Z_{\le w} = \coprod_{X_y\subset \overline{X_w}} T^*_{X_y}X$ is a closed subvariety of Z.

$$0 \to K(Z_{<w}) \to K(Z_{\le w}) \to K(T^*_{X_w}X) \to 0$$

$$K(T^*_{X_w}X) \xrightarrow[\text{Thom}]{\sim} K(X_w) \underset{\text{affine bundle}}{\simeq} K(\mathcal{B})$$

Ariki 2/26

Kashiwara, easy to define, hard to prove crystal bases $V(\lambda) \rightsquigarrow (L(\lambda), B(\lambda))$

Lusztig, Perverse sheaves Construct inverse limit of these $\rightsquigarrow (L(\infty), B(\infty))$

Grojnowski, showed same bases.

Book (Ariki) gives 21 properties, combines 2 approaches.

Lectures on Quantum... gives overview, then follows Kashiwara.

Every $u\in U_v^-$ can be written as $u = \sum_{n\ge 0} f_i^{(n)} u_n$ where $u_n\in \ker(e_i)$

$$\tilde{e}_i u = \sum_{n\ge 0} f_i^{(n-1)} u_n \qquad \tilde{f}_i u = \sum_{n\ge 0} f_i^{(n+1)} u_n$$

$R := \mathbb{Q}[v]_{(v)}$

Define $L(\infty)$ as follows

$$L(\infty) = \sum R\, \tilde{f}_{i_1}\tilde{f}_{i_2}\cdots\tilde{f}_{i_N}\cdot 1$$

Lecture 3/1 Affine Weyl Group

$R\subset \mathfrak{h}^*$ (real vector sp) root system

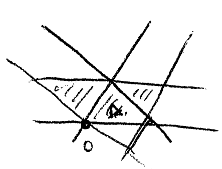
$R^{aff} \ni \{\alpha+k,\ \alpha\in R,\ k\in\mathbb{Z}\}$ affine linear f-s on $\mathfrak{h}$

$H_{\alpha+k} = \{h\in\mathfrak{h} \mid \alpha(h) = k\}$ called a wall

$\mathfrak{h}\setminus \bigcup_{\theta\in R^{aff}} H_\theta$ = union of connected components called alcoves

$R = A_2$

$\mathfrak{h} = \mathbb{R}^2$



$R^\vee \subset \mathfrak{h}$ · $\alpha \in R$

s_α reflection

$s_\alpha(h) = h - \alpha(h)\alpha^\vee$

$s_{\alpha-k}(h) = s_\alpha(h) + k\cdot\alpha^\vee$

<u>Define</u> $W^{aff} := \left\{ \text{group generated by } s_\theta \text{ , } \theta \in R^{aff} \right\}$

$Q^\vee \subset \mathfrak{h}$ coroot lattice

$\overset{\shortparallel}{\sum_{\alpha\in R}} \mathbb{Z}\alpha^\vee$

$s_{\alpha-1} \circ s_\alpha = s_\alpha \cdot s_{\alpha+1} = t_{\alpha^\vee} : h \mapsto h + \alpha^\vee$

<u>Cor</u> $W^{aff} \cong W \ltimes Q^\vee$

$R \supset S = \{\alpha_1, \dots, \alpha_r\}$ simple roots

For $\lambda, \mu \in \mathfrak{h}^*$, $\lambda \le \mu$ iff $\mu - \lambda \in \sum \mathbb{Z}_{\ge 0}\alpha_i$

α_0 = maximal root = highest weight in $\mathfrak{g}$

- α_0 is dominant : $\langle \alpha_0, \alpha_i^\vee \rangle \ge 0$
- $\alpha \le \alpha_0 \quad \forall \alpha \in R$

$$\Delta := \left\{ h \in \mathfrak{h} \;\middle|\; \begin{array}{l} \alpha_i(h) > 0 \quad \forall i = 1, \dots, r \\ \alpha_0(h) < 1 \end{array} \right\}$$

<u>Lemma</u> Δ is an alcove (called fundamental alcove)

If α is a positive root then $\alpha = \sum k_i \alpha_i$, $k_i \ge 0 \Rightarrow \alpha(\Delta) > 0$

Claim: $\alpha(\Delta) < 1 \Leftarrow \begin{cases} \alpha_0 - \alpha = \sum l_i \alpha_i \\ l_i \ge 0 \Rightarrow \alpha(\Delta) \le \alpha_0(\Delta) \end{cases}$

<u>Cor</u> 1) W^{aff} is generated by $s_0 = s_{-\alpha_0+1}, s_1, \dots, s_r$

2) $\overline{\Delta}$ is a fundamental domain of W^{aff}

3) W^{aff} acts freely and transitively on the set of alcoves

4) (fact) W^{aff} is a Coxeter group

<u>Terminology</u> 1) Δ fundamental alcove

2) $-\alpha_0 + 1, \alpha_1, \dots, \alpha_r$ simple affine roots

3) B_W^{aff} affine Braid group

$$B_W^{aff} \twoheadrightarrow W^{aff}$$

Length fn

$\ell : W^{aff} \longrightarrow \mathbb{Z}_{\ge 0}$ $\qquad \ell(w) = \#$ walls separating Δ and $w(\Delta)$

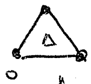
$P^\vee$ = coweight lattice $:= \{ h \in \mathfrak{h} \mid \alpha(h) \in \mathbb{Z} \;\; \forall \alpha \in R \}$ $\qquad P^\vee \supset Q^\vee$

$\widetilde{W} := W \ltimes P^\vee \supset W \ltimes Q^\vee = W^{aff}$

$\widetilde{W} \curvearrowright \mathfrak{h}$, $w \in \widetilde{W}$ takes alcoves to alcoves.

$\Gamma = \{ \gamma \in \widetilde{W} \mid \gamma(\Delta) = \Delta \}$, ℓ extends to $\widetilde{W}$, $\Gamma = \{ \gamma \in \widetilde{W} \mid \ell(\gamma) = 0 \}$

$\Rightarrow \Gamma$ acts on the set of $\{$faces of $\Delta\} \leftrightarrow S^{aff} :=$ simple affine roots

A_2:  $\Gamma = \mathbb{Z}/(3)$ $\Rightarrow \Gamma$ acts by automorphisms of the Dynkin diagram for R^{aff}.

affine Dynkin diag

Cor 1) $\widetilde{W} = \Gamma \ltimes W^{aff}$

2) $\Gamma = \widetilde{W}/W^{aff} = \dfrac{W \ltimes P^\vee}{W \ltimes Q^\vee} = P^\vee/Q^\vee$

Let $\mathfrak{g} = \mathfrak{g}(R)$ be a ss Lie alg associated to R

$G_{ad} = (\mathrm{Aut}\,\mathfrak{g})^\circ \longleftarrow$ gp w/ no center   Lie group picture

G simply connected covering of G_{ad}, $Z(G)$ finite abelian gp

$$\begin{array}{ccccccccc} 1 & \longrightarrow & Z(G) & \longrightarrow & G & \longrightarrow & G_{ad} & \longrightarrow & 1 \\ & & \| & & \cup & & \cup & & \\ 1 & \longrightarrow & Z(G) & \longrightarrow & T & \longrightarrow & T_{ad} & \longrightarrow & 1 \end{array}$$

$X_*(T) = \mathrm{Hom}(\mathbb{C}^\times, T) = P^\vee$

$X_*(T_{ad}) = \mathrm{Hom}(\mathbb{C}^\times, T_{ad}) = Q^\vee$

$\Gamma \cong P^\vee/Q^\vee = X_*(T)/X_*(T_{ad}) \cong Z(G) = \pi_1(G_{ad})$

$T \simeq \mathfrak{h}_{\mathbb{C}}/P^\vee$ $T^{reg} = T \setminus (\text{image} \bigcup_{\alpha \in R} H_\alpha)$ $W \curvearrowright T^{reg}$

all affine walls go to same thing

Fact: $\pi_1(T^{reg}/W) = \widetilde{B}_W := \Gamma \ltimes B_W^{aff}$

Affine Hecke alg $\mathcal{R} := \mathbb{C}[q^{\pm 1}]$

$$H(W^{aff}) = \mathcal{R}[B_W^{aff}] \Big/ \begin{pmatrix} (T_i+1)(T_i-q) \\ i=0,\dots,r \end{pmatrix}$$

$\widetilde{H} = H(\widetilde{W}) = \mathcal{R}[\widetilde{B}_W]/$ same

1) $\widetilde{H}$ has an $\mathcal{R}$-basis T_w, $w \in \widetilde{W}$
2) For $\ell(ww') = \ell(w) + \ell(w') \Rightarrow T_{ww'} = T_w \cdot T_{w'}$
3) Each T_w is invertible

also true for $H(W^{aff})$

Lemma Let $\lambda \in P^\vee$ be dominant
$\cap$
$\widetilde{W}$

$\ell(\lambda) = \sum_{i=1}^{r} \langle \alpha_i, \lambda \rangle$ every simple root you add $\rightsquigarrow$ cross a wall

Cor $\ell(\lambda+\mu) = \ell(\lambda) + \ell(\mu)$ if λ, μ are dominant

Therefore $T_{\lambda+\mu} = T_\lambda \cdot T_\mu$ (in particular they commute)

For λ dominant, $e_\lambda := q^{-\ell(\lambda)/2} \cdot T_\lambda$ // add $t = \sqrt{q}$ to $\mathcal{R}$

Defn (following Bernstein) For any $\lambda \in P^\vee$, define $e_\lambda := e_{\lambda_1}(e_{\lambda_2})^{-1}$ for $\lambda = \lambda_1 - \lambda_2$
(well-defined by Cor) λ_1, λ_2 dominant

Cor The map $\lambda \mapsto e_\lambda$ gives an alg. homomorphism
$$\mathcal{R}[P^\vee] \longrightarrow \tilde{H}$$

Prop (Bernstein) Let $\alpha \in S$, $\lambda \in P^\vee$
$$T_{s_\alpha} \cdot e_{s_\alpha(\lambda)} - e_\lambda T_{s_\alpha} = (1-q)\frac{e_\lambda - e_{s_\alpha(\lambda)}}{1-e_{-\alpha}}$$
$$\Leftrightarrow \quad T_{s_\alpha}(e_\lambda + e_{s_\alpha(\lambda)}) = (e_\lambda + e_{s_\alpha(\lambda)})T_{s_\alpha} \qquad (*)$$

Pf 1) If $(*)$ holds for λ and $\mu \Rightarrow$ holds for $\lambda+\mu$
2) If $\langle \alpha, \lambda \rangle = 0$ then $(*)$ says T_{s_α} and e_λ commute.
3) Prove for $\langle \lambda, \alpha \rangle = 1$

Cor The multiplication map in $\tilde{H}$ induces an $\mathcal{R}$-module isom $\mathcal{R}[P^\vee] \otimes_{\mathcal{R}} H_W \xrightarrow{\sim} \tilde{H}$
i.e. $\{e_\lambda \cdot T_w \mid \lambda \in P^\vee, w \in W\}$ form an $\mathcal{R}$-basis of $\tilde{H}$.

$$\mathcal{R}[P^\vee] \hookrightarrow \tilde{H}$$
$$\cup$$
$$A = \mathcal{R}[P^\vee]^W$$

Thm (Bernstein) $\mathcal{R}[P^\vee]^W = Z(\tilde{H})$ center of $\tilde{H}$

Pf ① A is central.
Basis consists of $a = \sum_{w \in W} e_{w(\lambda)}$, for some $\lambda \in P^\vee$

a commutes with $\mathcal{R}[P^\vee]$.

$\alpha \in S$, want to show a commutes with T_{s_α}

$W' = \{w \in W \mid \ell(s_\alpha w) > \ell(w)\}$ $\qquad W' \longleftrightarrow W \setminus W'$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad w \longmapsto s_\alpha w$

$$a = \sum_{w \in W'} \underbrace{\left(e_{w(\lambda)} + e_{(s_\alpha w)(\lambda)}\right)}_{} \quad \text{commutes with } T_{s_\alpha}.$$

Claim $\mathbb{C}[P^\vee] \hookrightarrow \mathbb{C}[\tilde{W}]$
$$\mathbb{C}[P^\vee]^W \xrightarrow{\sim} Z(\mathbb{C}[\tilde{W}])$$

$\tilde{W} = W \ltimes P^\vee$

$\mathbb{C}[\tilde{W}] = W \ltimes \mathbb{C}[P^\vee]$ smash product

$Z(W \ltimes \mathbb{C}[P^\vee]) = \mathbb{C}[P^\vee]^W$

Reduce (due to Lusztig)
Specialization map:
$$sp: \tilde{H} \longrightarrow \mathbb{C}[\tilde{W}]$$
$$q \longmapsto 1$$
$$T_w \longmapsto w \quad \forall w \in \tilde{W}$$

(in what space?)
$\mathrm{Ker}(sp) = (q-1) = \mathfrak{m}$

$sp(Z(\tilde{H})) \subset Z(\mathbb{C}[\tilde{W}]) = \mathbb{C}[P^\vee]^W$

If $z \in \mathrm{Ker}(sp) \cap Z(\tilde{H})$ then $z = (q-1)z' \Rightarrow z' \in Z(\tilde{H})$

$Z := Z(\tilde{H})$ $\qquad 0 \longrightarrow \mathfrak{m} Z \longrightarrow Z \xrightarrow{sp} \mathbb{C}[P^\vee]^W \longrightarrow 0$

$$\mathcal{R}[P^\vee]^W \hookrightarrow Z \longrightarrow \mathbb{C}[P^\vee]^W$$

$$0 \longrightarrow \mathfrak{m}\,\mathcal{R}[P^\vee]^W \longrightarrow \mathcal{R}[P^\vee]^W \longrightarrow \mathbb{C}[P^\vee]^W \to 0$$

$\Rightarrow$

$$\frac{\mathcal{R}[P^\vee]^W}{\mathfrak{m}\mathcal{R}[P^\vee]^W} \xrightarrow{\sim} \frac{Z}{\mathfrak{m}Z} \overset{\text{Nak}}{\Longrightarrow} (\mathcal{R}[P^\vee]^W)_{\mathfrak{m}} \xrightarrow{\sim} Z_{\mathfrak{m}}.$$

$\therefore$ $\underset{\tilde{H}}{\overset{\cap}{z \in Z}} \Rightarrow z = \sum_{\lambda \in \text{dominant}} c_\lambda \left(\sum_W e_{w(\lambda)}\right)$ where $c_\lambda \in \mathcal{R}_{\mathfrak{m}}$

$\uparrow$ these are basis elts $\Rightarrow c_\lambda \in \mathcal{R}$ $\square$

Lecture 3/4

Fix X $(\simeq \mathbb{Z}^n)$ a lattice. $X^* = \mathrm{Hom}(X, \mathbb{Z})$

$T := T(X) = \mathbb{G}_m \underset{\mathbb{Z}}{\otimes} X$ $\qquad T^\vee = T(X^*)$

$\pi_1(T) = X_*(T) = \mathrm{Hom}(\mathbb{G}_m, T) = X$

$X^*(T) = \mathrm{Hom}(T, \mathbb{G}_m) = X^*$

$T^\vee = \mathrm{Hom}(\pi_1(T), \mathbb{G}_m) = \mathcal{L}oc(T) :=$ rk 1 local systems on T.

// no direct way to go from T to $T^\vee$

// geom interpretation as rank 1 local systems.

Product on $T^\vee \leftrightarrow \otimes$ local systems

$$\mathrm{Lie}\, T^\vee = (\mathrm{Lie}\, T)^*$$
$$\mathbb{C} \underset{\mathbb{Z}}{\otimes} X^* \qquad \mathbb{C} \underset{\mathbb{Z}}{\otimes} X$$

root system $\quad R \subset Q(R) \subset P(R) \subset \mathfrak{t}^*$

$\quad R^\vee \subset Q(R^\vee) = P(R)^* \subset P(R^\vee) = Q(R)^*$

$W(R) = W(R^\vee)$

$\mathfrak{g}(R) \supset \mathfrak{t}$ Cartan $\qquad \mathfrak{g}^\vee = \mathfrak{g}(R^\vee) \supset \mathfrak{t}^\vee$ Cartan

<u>Ex</u> $\mathfrak{g}(R) = \mathfrak{sp}_{2n}$ $\qquad \mathfrak{g}^\vee = \mathfrak{so}_{2n+1}$

Given a lattice X s.t. $Q(R) \subset X^* \subset P(R)$, $\exists!$ ss. grp $G(R,X)$ s.t. root system of $G(R,X)$ is R.

$R^\vee$

$G^\vee(R,X) := G(R^\vee, X^*) \supset T^\vee = T(X^*)$ max'l torus $\qquad$ $\circ$ max torus of $G(R,X)$ is $T(X) = \mathbb{G}_m \underset{\mathbb{Z}}{\otimes} X$.

$$\underset{\text{simply connected}}{G(R, Q(R^\vee))} \twoheadrightarrow G(R, X) \twoheadrightarrow \underset{\text{adjoint}}{G(R, P(R^\vee))}$$

G is simply connected/adjoint $\longleftrightarrow$ $G^\vee$ adjoint/simply connected.

$\pi_1(G^\vee) = \cancel{\mathrm{Hom}(\pi_1(G), \mathbb{C}^\times)}$ $\mathrm{Hom}(Z(G), \mathbb{C}^\times)$ = Pontryagin dual of $Z(G)$

$Q^\vee \subset X \subset P^\vee$ a lattice

$W(X) = W \ltimes X \qquad W^{\mathrm{aff}} \subset W(X) \subset \tilde{W}$

$\widetilde{W} = \Gamma \ltimes W^{af}$ $\qquad \Gamma \simeq P^\vee / Q^\vee \supset X/Q^\vee$

$W(X) \simeq (X/Q^\vee) \ltimes W^{af}$

$H(W,X) = (X/Q^\vee) \ltimes H^{af}$

$e = \frac{1}{|W|} \sum_{w\in W} w \in \mathbb{C}W$ projector, central idempotent

$\mathbb{C}W \cdot e$ 1-dim rep $w \mapsto 1$ $\forall w \in W$.

$W_q := \sum_{w\in W} q^{\ell(w)} = \sum_i q^i \cdot \dim H^{2i}(G/B)$ (by Bruhat decomp)

If $q := \#\mathbb{F}_q$ a number (not variable), then $\# G(\mathbb{F}_q)/B(\mathbb{F}_q) = \#(G/B)(\mathbb{F}_q)$

$e_0 = \frac{1}{W_q} \sum_{w\in W} T_w \in \mathbb{C}(q) \otimes H_W$

__Lemma__ e_0 is a central idempotent in H_W.

$\dim H_W \cdot e_0 = 1$

$T_w \cdot e_0 = q^{\ell(w)} e_0$ $\qquad \varepsilon : T_w \longmapsto q^{\ell(w)}$ is an alg hom.

If $q = |\mathbb{F}_q|$ ~~then~~

$$\begin{array}{ccc} H_W & \xrightarrow[\mathcal{B}(\mathbb{F}_q)]{\varepsilon} & \mathbb{C} \\ \| & & \| \\ \int : \mathbb{C}^{B}(G(\mathbb{F}_q)/B(\mathbb{F}_q)) & \longrightarrow & \mathbb{C} \\ f & \longmapsto & \sum_{x\in \mathcal{B}(\mathbb{F}_q)} f(x) \end{array}$$

__Lemma__ The map $h \mapsto h \cdot e$ gives isomorphisms

1) $\mathbb{C}X \hookrightarrow \mathbb{C}W(X) \xrightarrow{\cdot e} \mathbb{C}W(X) \cdot e$ (composite $\cong$)

2) $(\mathbb{C}X)^W \xrightarrow{\sim} Z(\mathbb{C}W(X)) \xrightarrow[\sim]{\cdot e} e\,\mathbb{C}W(X)\,e$

__Pf__ $\{ \cancel{w\lambda}\ \lambda \cdot w,\ w \in W,\ \lambda \in X \}$ is a basis of $\mathbb{C} \cdot W(X) \Rightarrow \lambda \cdot e$ is a basis of $\mathbb{C}W(X)e$

2) $e(W \ltimes \mathbb{C}X)e \cong \mathbb{C}X^W$

__Prop__ The map $h \mapsto h \cdot e_0 \in H_W$ gives isom

1) $\mathbb{C}X \hookrightarrow H(W,X) \xrightarrow{\cdot e_0} H(W,X)e_0$ (composite $\sim$) $\qquad \lambda \mapsto e_\lambda \longrightarrow e_\lambda e_0 = e_\lambda$

2) $(\mathbb{C}X)^W \xrightarrow{\sim} Z(H(W,X)) \xrightarrow[\sim]{\cdot e_0} e_0 H(W,X) e_0$

Bernstein $\qquad$ Lusztig $\qquad$ (Asterique 101)

__Cor__ $\exists$ natural $H(W,X)$-action on $\mathbb{C}X$

$k \supset \mathcal{O} \longrightarrow \mathcal{O}/(t) = \mathbb{F}_q$

local field; $\mathcal{O}$ ring of int; t uniformizer

$G(k) \supset G(\mathcal{O}) \xrightarrow{pr} G(\mathbb{F}_q)$

$\cup \qquad\qquad \cup$

Iwahori $\to I = pr^{-1}(B) \longrightarrow B(\mathbb{F}_q)$

$C(-)$ = fs with compact support

~~$C(G(\mathcal{O})\backslash G(k)/G(\mathcal{O})) \hookrightarrow C(I\backslash G(k)/G(\mathcal{O})) \hookrightarrow C(I$~~

G split gp associated w/ $G(R,X)$

Cartan / Bruhat / Iwasawa decomp

$$I\backslash G(\mathcal{O})/I \simeq B(\mathbb{F}_q)\backslash G(\mathbb{F}_q)/B(\mathbb{F}_q) \cong W$$

$$\begin{array}{ccccc} I\backslash G(k)/I & \simeq & W(X) & & w\times\lambda \\ \downarrow & & \downarrow & & \downarrow \\ I\backslash G(k)/G(\mathcal{O}) & \simeq & X & & \lambda \\ \downarrow & & \downarrow & & \downarrow \\ G(\mathcal{O})\backslash G(k)/G(\mathcal{O}) & \simeq & X/W & & W\cdot\lambda \end{array}$$

$$C(I\backslash G(k)/I) \longleftrightarrow C(I\backslash G(k)/G(\mathcal{O})) \longleftrightarrow C(G(\mathcal{O})\backslash G(k)/G(\mathcal{O})) = C^{G(k)}(G(k)/G(\mathcal{O}) \times G(k)/G(\mathcal{O}))$$

acts by conv; acts; an alg; convolution easy

affine Grass

$Gr(G) := G(k)/G(\mathcal{O}) \longleftarrow$ I-orbits on $Gr(G) \longleftrightarrow X$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \cup \qquad\qquad \cup$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad I_\lambda \longleftrightarrow \lambda$

$\mathcal{F}(G) := G(k)/I \longleftarrow$ I-orbits on $\mathcal{F}(G) \longleftrightarrow W(X)$

affine flag variety

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \cup \qquad\qquad \cup$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \mathcal{F}_w \longleftrightarrow w$

<u>Thm</u> The map $1_{\mathcal{F}_w} \longmapsto T_w$ gives an alg isomorphism

1) $$C(I\backslash G(k)/I) \xrightarrow{\sim} H(W,X)$$

2) The map $1_{I_\lambda} \longmapsto e_\lambda$ for λ dominant

extends uniquely to a bijection $C(I\backslash G(k)/G(\mathcal{O})) \xrightarrow{\sim} \mathbb{C}X$

s.t. $C(I\backslash G(k)/I)$ convolution action on $C(I\backslash G(k)/G(\mathcal{O}))$ goes under 1) & 2) to the $H(W,X)$-action on $\mathbb{C}X \simeq H(W,X)\cdot e_0$.

$$\mathrm{End}_{H(W,X)} H(W,X)\cdot e_0 \cong e_0 H(W,X) e_0$$

$$\mathrm{End}_{C(I\backslash G(k)/I)} C(I\backslash G(k)/G(\mathcal{O})) \simeq C(G(\mathcal{O})\backslash G(k)/G(\mathcal{O}))$$

<u>Cor</u> $C(G(\mathcal{O})\backslash G(k)/G(\mathcal{O})) \simeq e_0 H(W,X) e_0 \longleftarrow$ spherical Hecke algebra

$\qquad\qquad\qquad\qquad\qquad\qquad \simeq \mathbb{C}X^W$

Satake-isomorphism

$G = G(R,X) \supset T(X)$ s.t. $X = X_*(T) = X^*(T^\vee)$, $R \subset X^*$

$G^\vee = G(R^\vee, X^*) \supset T^\vee$

$K_{\mathbb{C}}(\mathrm{Rep}\, T^\vee) = \mathbb{C}X^*(T^\vee) = \mathbb{C}X$

$K_{\mathbb{C}}(\mathrm{Rep}\, G^\vee) \simeq \mathbb{C}[G^\vee]^{\mathrm{Ad}\, G^\vee} \simeq \mathbb{C}[T^\vee]^W = (\mathbb{C}X)^W$

characters of rep; Chevalley restr. thm

$\mathcal{B} = G^\vee/B^\vee$ flag variety of $G^\vee$

$K^{G^\vee}(\mathcal{B}) = K^{G^\vee}(G^\vee/B^\vee) = K^{B^\vee}(pt) \cong K^{T^\vee}(pt) \cong \mathbb{C}X$

$\text{diag} \to \mathcal{B}\times\mathcal{B} = \coprod_W G^\vee \text{ orbits-} X_w$

$G^\vee$

$T^*_{\mathcal{B}_{diag}}(\mathcal{B}\times\mathcal{B}) \hookrightarrow Z = \coprod_{w\in W} T^*_{X_w}(\mathcal{B}\times\mathcal{B}) \subset T^*(\mathcal{B}\times\mathcal{B})$

$\mathcal{B}_{diag} = X_e$

Steinberg variety

$\downarrow$

$\mathcal{B}_{diag} \simeq \mathcal{B}$

$K^{G^\vee\times\mathbb{C}^*}(T^*_{\mathcal{B}_{diag}}(\mathcal{B}\times\mathcal{B})) \hookrightarrow K^{G^\vee\times\mathbb{C}^*}(Z)$

$\mathbb{C}^\times$ acts on $T^*(\mathcal{B}\times\mathcal{B})$

$z : \xi \longrightarrow z^{-2}\xi$ on fibers

(has to do with $\sqrt{q} = t$)

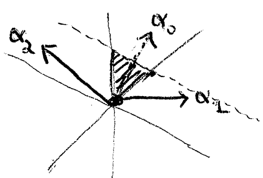
$K_{\mathbb{C}}^{G^\vee\times\mathbb{C}^*}(pt) = K_{\mathbb{C}}(Rep(G^\vee\times\mathbb{C}^\times))$

$= K_{\mathbb{C}}(Rep(G^\vee)) \otimes K_{\mathbb{C}}(\mathbb{C}^\times)$

$\mathbb{C}X^W \otimes \mathbb{C}[q^{\pm 1}]$

Review (seminar) 3/7/13

$R = A_2$



$\alpha_0 := \alpha_1 + \alpha_2$

Δ is shaded

$\alpha_1 = e_1 - e_2$

$\alpha_2 = e_2 - e_3$

$\alpha_3 = e_1 - e_3$

~~Start~~ with lattice X (f.g. free $\mathbb{Z}$ mod), $X^* \supset R$ root system $\qquad X^* = Hom_{\mathbb{Z}}(X, \mathbb{Z})$

$G = G(R, X)$ with maximal torus T $(= \mathbb{G}_m \otimes_{\mathbb{Z}} X)$

$X^* = X^*(T) := Hom_{alg\ gp}(T, \mathbb{G}_m) \qquad X = X_*(T) := Hom(\mathbb{G}_m, T)$

(by tensor adjunction)

$Q^\vee \subset X \subset P^\vee \qquad Q \subset X^* \subset P$

$$\begin{array}{llllll}
\text{Reductive Group:} & \tilde{G} & \twoheadrightarrow & G & \twoheadrightarrow & G_{ad} \\
& \cup & & \cup & & \cup \\
\text{Torus:} & \tilde{T} & \twoheadrightarrow & T & \twoheadrightarrow & T_{ad} \\
X^* : & P & \hookleftarrow & X^* & \hookleftarrow & Q \\
X_* : & Q^\vee & \hookrightarrow & X & \hookrightarrow & P^\vee \\
W(\ ): & W^{aff} & \hookrightarrow & W \ltimes X & \hookrightarrow & \tilde{W}
\end{array}$$

equiv of cat

X^* : diagonalizable group $\xrightarrow{\sim}$ f.g. ab gp

W^{aff} coxeter gp allows us to define $B_{W^{aff}}$ and then H_W^{aff} (gen and rel.)

$W(X) = X/Q^\vee \ltimes W^{aff} \rightsquigarrow$ define $H(W, X) = (X/Q^\vee) \ltimes H_W^{aff}$

$\tilde{W}$ and $\tilde{H}$ just adjoint case.

Example $\quad G = SO(3) \qquad R = B_1 \; (= \overset{C_1}{\overset{\shortparallel}{A_1}})$

$$Q = X^* = \mathbb{Z} \subset \tfrac{1}{2}\mathbb{Z} = P \qquad \tilde{G} = SU(2) = Spin(3) = Sp(1)$$

$$\alpha_1 = \begin{pmatrix} t_1 & & \\ & 1 & \\ & & t_1^{-1}\end{pmatrix} \longmapsto t_1 \qquad \omega_1 = \tfrac{1}{2}\alpha_1 \qquad \tilde{\alpha}_1 : \begin{pmatrix} t_1 & \\ & t_1^{-1}\end{pmatrix} \longmapsto t_1^2$$

so $\tilde{Q} = 2\mathbb{Z} \subset X^*(\tilde{T}) = \mathbb{Z}$

Lecture 3/8/13

local field $k \supset \mathcal{O} \longrightarrow \mathcal{O}/(t) = \mathbb{F}_{q_0} \qquad q_0 = \# \mathcal{O}/(t).$

$\mathrm{Irr}\, \mathbb{C}^* = \mathbb{Z} \qquad K^{\mathbb{C}^*}(pt) = \mathbb{C}[q^{\pm 1}] \qquad K := \mathbb{C} \otimes_{\mathbb{Z}} K$ complexified Grothendieck gp

$R \subset X^* \qquad R^\vee \subset X \qquad Q^\vee \subset X \subset P^\vee \qquad G(k) \supset G(\mathcal{O}) \supset I$

$T = \mathbb{C}^* \otimes_{\mathbb{Z}} X \qquad T^\vee = \mathbb{C}^* \otimes_{\mathbb{Z}} X^*$

$G = G(R, X) \cup T$ max torus $\qquad G^\vee = G(R^\vee, X^*) \cup T^\vee$

Remarks
// $W = W^\vee$

left side: p-adic $\qquad$ right side: $\mathbb{C}$ (or just alg. gp?)

$$I \backslash G(k) / G(\mathcal{O}) = X_*(T) = X = X^*(T^\vee) = \mathrm{Irr}\, T^\vee$$

$$I t^\lambda G(\mathcal{O}) \longleftrightarrow \lambda \longrightarrow \lambda \qquad \mathcal{B} = G^\vee / B^\vee$$

$$C(I \backslash G(k)/G(\mathcal{O})) = \mathbb{C}X_*(T) = \mathbb{C}X^*(T^\vee) = \mathbb{C}[T^\vee] = K^{T^\vee}(pt) \underset{\text{Induction}}{=} K^{G^\vee}(G^\vee/B^\vee) \overset{\text{Thom}}{=} K^{G^\vee}(T^*\mathcal{B})$$

$$1_{I t^\lambda G(\mathcal{O})} \longrightarrow [\mathcal{O}_{\mathcal{B}}(\lambda)] \longrightarrow [\mathcal{O}_{T^*\mathcal{B}}(\lambda)]$$

$$G(\mathcal{O}) \backslash G(k) / G(\mathcal{O}) = X_*(T)/W = X^*(T^\vee)/W = \text{dominant weights} = \mathrm{Irr}\, G^\vee$$

$$C(G(\mathcal{O}) \backslash G(k)/G(\mathcal{O})) = \mathbb{C}[G^\vee]^{G^\vee} = K^{G^\vee}(pt)$$

$$I \backslash G(k) / I = W(R, X) = W \ltimes X = W \ltimes X^*(T^\vee)$$

$$\uparrow \qquad\qquad \uparrow$$

$$I \backslash G(\mathcal{O}) / I = W = B^\vee \backslash G^\vee / B^\vee = G^\vee \backslash (\mathcal{B} \times \mathcal{B})$$

$$C(I \backslash G(\mathcal{O})/I) \simeq H_W|_{q_0} \overset{\neq}{\hookrightarrow} K^{\mathbb{C}^\times}(Z)|_{q_0} \quad K^{\mathbb{C}^*}(pt)\text{-alg}$$

$$1_{IwI} \longleftrightarrow T_w$$

$$1_{IsI \cup I} \longleftrightarrow T_s + 1 \longleftrightarrow \mathcal{O}_{\overline{Z}_s}$$

$$\xrightarrow{\wr} K^{\mathbb{C}^\times \times G^\vee}(Z)|_{q_0} \xrightarrow{\text{forget}} K^{\mathbb{C}^\times}(Z)|_{q_0}$$

w_0 = longest elt in W

X_{w_0} open dense in $\mathcal{B} \times \mathcal{B}$

$\overline{Z}_{w_0}$ = zero sect of $T^*(\mathcal{B} \times \mathcal{B}) \simeq \mathcal{B} \times \mathcal{B}$

$s \in W$ a simple refl.

$\overline{X}_s \xrightarrow{\mathbb{P}^1} \mathcal{B}$

$\overline{Z}_s = T^*_{\overline{X}_s}(\mathcal{B} \times \mathcal{B})$

$T^*(\mathcal{B} \times \mathcal{B}) \supset Z \quad$ Steinberg

$Z_w := T^*_{X_w}(\mathcal{B} \times \mathcal{B})$

$Z = \coprod_w Z_w$

$Z_e = T^*_{diag}(\mathcal{B} \times \mathcal{B}) \simeq T^*\mathcal{B}.$

$C(I\backslash G(k)/I) \cong H(W,X)|_{q_0}$

$1_{IwI} \longleftrightarrow T_w$

Thm $\exists$ alg isom $H(W,X)|_{q_0} \xrightarrow{\sim} K^{\mathbb{C}^\times \times G^\vee}(Z)|_{q_0}$ s.t.

1) extends $\imath$

2) $\mathbb{C}X_*(T) \hookrightarrow H(W,X) \longrightarrow K^{\mathbb{C}^\times \times G^\vee}(Z_e)$

$\lambda \longmapsto e_\lambda \longrightarrow \mathcal{O}_{Z_e}(\lambda)$

3) $\mathbb{C}[X_*(T)]^W \cong$ Center $H(W,X) \longrightarrow K^{\mathbb{C}^* \times G^\vee}(pt)$

4) It's compatible with module structures.

$H = H(W,X)|_{q_0} = C(I\backslash G(k)/I) = K^{\mathbb{C}^\times \times \check{G}}(Z)$

$\mathbb{C}X_*(T) = He = C(I\backslash G(k)/G(\mathcal{O})) = K^{\mathbb{C}^\times \times \check{G}}(T^*\mathcal{B})$

$e = \frac{1}{W_q}\sum T_w$ action by convolution

(function) (K-theoretic)

Rem 1

$$\begin{array}{ccccc} C(I\backslash G(k)/I) & = & H(W,X) & \cong & K^{\mathbb{C}^\times \times \check{G}}(Z) \\ \circlearrowleft & & \cup & & \circlearrowleft \\ C(I\backslash G(k)/G(\mathcal{O})) & & \mathbb{C}X & \cong & K^{\mathbb{C}^\times \times G^\vee}(Z_e) \end{array}$$

$e \longleftrightarrow \mathcal{O}_{\overline{Z}_{w_0}}$

$1_{It^\lambda G(\mathcal{O})} \xrightarrow[\lambda \text{ dominant}]{\text{only for}} e_\lambda \xleftrightarrow[\text{any } \lambda]{} \mathcal{O}(\lambda) \quad \forall \lambda \in X$

Rem 2

$C(G(\mathcal{O})\backslash G(k)/G(\mathcal{O})) \overset{?}{=} eH(W,X)e = eK^{\mathbb{C}^* \times G^\vee}(Z)e \simeq eK^{\mathbb{C}^* \times G^\vee}(\mathcal{B})$

$= eK^{\mathbb{C}^* \times T^\vee}(pt) = \mathbb{C}X^*(T^\vee)^W$

Satake

Center $C(I\backslash G(k)/I) \underset{\text{Gaitsgory}}{\overset{??}{=}}$ Center $H(W,X) = \mathbb{C}X^*(T^\vee)^W = K^{\mathbb{C}^\times \times G^\vee}(pt)$

$k = \mathbb{F}((t))$ where $F = \overline{\mathbb{F}}_q$ or $\mathbb{C}$, $\mathcal{O} = F[[t]]$ $\quad \pi_0\left\{ f: S^1 \to G \;\middle|\; f(1) = 1 \right\}$ based loop space

ind-scheme

$\pi_0 \left(G(F((t)))/G(F[[t]]) \right) \simeq \pi_0 \left(\frac{G(F((t)))}{G} \right) \cong \pi_0(\Omega G) = \pi_1(G) =$

loops on formal disk contract to center $\quad \cong \mathrm{Hom}(Z(G^\vee), \mathbb{C}^*)$

$Gr = G(F((t)))/G(F[[t]])$

$\cup$

$I_\lambda = It^\lambda G(\mathcal{O})/G(\mathcal{O})$

$Gr = \coprod_{\lambda \in X} I_\lambda \qquad I_\lambda \subset \overline{I_\mu} \Leftrightarrow \lambda \le \mu$ (i.e. $\mu - \lambda = \sum$ positive coroots)

$\Gamma = X/Q^\vee$ | I_λ and I_μ are in the same connected component of $Gr \Leftrightarrow \lambda - \mu \in Q^\vee$

In particular, λ, μ comparable $\Rightarrow \lambda - \mu \in Q^\vee$

$K^{\mathbb{C}^\times \times G^\vee}(\mathcal{B}) \ni \mathcal{O}(\lambda)$

$X^*(T^\vee) \ni \lambda$

$Z(G^\vee) \subset T^\vee$

$\lambda - \mu \in Q^\vee \Leftrightarrow \lambda|_{Z(G^\vee)} = \mu|_{Z(G^\vee)}$ by above π_0

$Gr = \coprod_{\gamma \in \Gamma} Gr^\gamma \qquad Gr^\gamma = \coprod G(\mathcal{O})\text{-orbits}$

$\exists!$ $G(\mathcal{O})$-orbit of minimal dimension $G(\mathcal{O})t^\mu G(\mathcal{O})/G(\mathcal{O})$ (aut. closed)

say $\{\mu$ minuscule$\} \simeq \Gamma$

G simply connected; then $\pi_1(G) = 1 \Rightarrow Gr$ is connected.

$G = PGL_n \twoheadleftarrow SL_n$ $\quad$ Center $(SL_n) = \mathbb{Z}/n = \pi_1(PGL_n)$

So $Gr = \coprod Gr^\gamma$, $\gamma = \{0, \ldots, n-1\}$

minuscule $\equiv$ fundamental weights (not in general true, just for A_{n-1})
$= \Lambda^i \mathbb{C}^n \quad i = 0, \ldots, n-1$

Categorification

$\mathcal{B} = \coprod_{w \in W} \mathcal{B}_w \qquad \mathcal{C}_{Tate}(\mathcal{B}, W)$

<u>Thm</u> $K(\mathcal{C}_{Tate}(\mathcal{B}, W)) = H_W$

} non-affine picture from before

$\mathcal{F} = G(F((t)))/I$

$\mathcal{F} = \coprod_{w \in W \ltimes X} \mathcal{F}_w \qquad \mathcal{F}_w = IwI/I$

We can define a category $\mathcal{C}_{Tate}(\mathcal{F}, W \ltimes X)$

<u>Thm</u> $K(\mathcal{C}_{Tate}(\mathcal{F}, W\ltimes X)) \xrightarrow{\sim} H(W, X)$

<u>Thm</u> (Bezrukavnikov) $\exists$ a triangulated ~~category~~ equivalence

$$D^b(\mathcal{C}_{Tate}(\mathcal{F}, W\ltimes X)) \overset{\beta}{\simeq} D^b(Coh^{\mathbb{C}^\times \times G^\vee}(Z))$$

s.t. it agrees with Grothendieck gp isomorphism $H(W, X) \cong K^{\mathbb{C}^\times \times G^\vee}(Z)$

// Cannot possibly be equiv of abelian categories

$\beta(\mathcal{C}_{Tate}(\mathcal{F}, W\ltimes X)) =$ exotic "coherent sheaves"

<u>Geometric Satake equivalence.</u> (work over $\mathbb{C}$)

$Gr = G(\mathbb{C}((t)))/G(\mathcal{O})$

$Gr_\lambda := G(\mathcal{O})t^\lambda G(\mathcal{O})/G(\mathcal{O}) \qquad Gr = \coprod_{\substack{\lambda \in X \\ \text{dominant}}} Gr_\lambda$

$P(Gr, \{Gr_\lambda\}) =$ perverse sheaves on Gr constant on each Gr_λ
(rem $\pi_1(Gr_\lambda) = 0$)

<u>Thm (G. Satake)</u> Convolution makes $P(Gr, \{Gr_\lambda\})$ a tensor category and there is a $\otimes$-equivalence $P(Gr, \{Gr_\lambda\}) \simeq Rep(G^\vee)$

$$IC_{Gr_\lambda} \longrightarrow V_\lambda$$

At Grothendieck gp level: $\mathbb{C}(G(\mathcal{O})\backslash G(k)/G(\mathcal{O})) \cong \mathbb{C}X^*(T^\vee)^W = \mathbb{C}[T^\vee]^W = K(Rep\, G^\vee)$

Lecture 3/11 $\quad Gr = \coprod_{\lambda \text{ dominant}} Gr^\lambda \qquad Gr^\lambda = G(\mathcal{O})\, t^\lambda\, G(\mathcal{O})/G(\mathcal{O})$

- Geometric Satake: $\mathrm{Perv}(Gr, \{Gr^\lambda\}) \simeq \mathrm{Rep}\, G^\vee = \mathrm{Coh}^{G^\vee}(pt) \;:\; M \longmapsto V(M)$
- $Gr = \coprod_{\lambda \in \mathbb{X}} I^\lambda \qquad I^\lambda = I \cdot t^\lambda G(\mathcal{O})/G(\mathcal{O}) \qquad \mathcal{C}_{Tate}(Gr, \{I^\lambda\}) \qquad D^b(\mathcal{C}_{Tate}(Gr, \{I^\lambda\})) \cong D^b(\mathrm{Coh}^{\mathbb{C}^* \times \check{G}}(T^*\mathcal{B}))$
- $\mathcal{F} = G(\mathbb{C}((t)))/I \qquad \mathcal{F} = \coprod_{w \in W(\mathbb{X})} \mathcal{F}_w \qquad \mathcal{F}_w = I t^\lambda I/I$

Bezrukavnikov: $D^b(\mathcal{C}_{Tate}(\mathcal{F}, \{\mathcal{F}_w\})) \simeq D^b(\mathrm{Coh}^{\mathbb{C}^\times \times G^\vee} Z)$

We sketch proof of 2nd pt. 3rd is similar with extra arg that we don't have time for.

Proj-construction: $\quad i: X \hookrightarrow \mathbb{P}^N \qquad \mathcal{O}(1) \qquad \mathcal{L} = i^* \mathcal{O}(1)$ ample line bundle on X.

$A := \bigoplus_{k \geq 0} \Gamma(X, \mathcal{L}^{\otimes k})$ homogeneous coord ring of X.

$\mathrm{Spec}\, A$ = affine cone over $X \qquad \mathrm{Spec}\, A \hookrightarrow \mathbb{C}^{N+1}$

$X \subset \mathbb{P}^N \qquad X = \mathrm{Proj}\, A$

A-grmod = $\mathbb{Z}$-graded fin. generated A-modules
$\cup$
A-tails = fin-dim'l $/\mathbb{C}$

$\mathbb{F}: A\text{-grmod} \longrightarrow \mathrm{Coh}(\mathrm{Proj}\, A)$

Thm (Serre) $\quad A\text{-tails} = \ker \mathbb{F}, \qquad \mathbb{F}: A\text{-grmod}/A\text{-tails} \xrightarrow{\sim} \mathrm{Coh}(\mathrm{Proj}\, A)$

If G alg gp acts on $\underbrace{X, \mathcal{L}}_{\text{gives a } G\text{-equiv embedding } X \hookrightarrow \mathbb{P}^N}$ is G-equiv ample $\rightsquigarrow$ G-action on A $\quad$ (A, G)-grmod $\quad$ then $(A,G)\text{-grmod} \to \mathrm{Coh}^G(X)$

$\mathcal{B} = G^\vee/B^\vee \qquad X = T^*\mathcal{B} \xrightarrow{p} \mathcal{B} \qquad \lambda \in X^*(T^\vee) \rightsquigarrow \mathcal{O}(\lambda)$ $\quad G^\vee$-equivariant line bundle on $\mathcal{B}$.

$\mathcal{O}_X(\lambda) = p^* \mathcal{O}(\lambda)$

multi-homogeneous coord ring of X:

$A = \bigoplus_{\substack{\lambda \in X^*(T^\vee) \\ \lambda \text{ dominant}}} \Gamma(X, \mathcal{O}_X(\lambda))$ $\quad$ $\shortparallel\ \mathbb{X}^d$

A-grmod = $\mathbb{X}$ graded f.gen. A-module
Ψ
M

$\mathrm{Spec}\, M = \{\lambda \in \mathbb{X} \mid M = \bigoplus_{\lambda \in \mathbb{X}^d} M_\lambda,\ M_\lambda \neq 0\}$

$A\text{-tails} = \{M \mid \exists \lambda \in \mathbb{X} \text{ s.t. } \mathrm{Spec}\, M \cap \lambda + \mathbb{X}^d = \emptyset\}$

Thm (version of Serre) $\quad A\text{-grmod}/A\text{-tails} \xrightarrow{\sim} \mathrm{Coh}\, X$

Strategy of pf of Thm

We'll find an object $P \in D^b(\mathcal{C}_{Tate})$ s.t.

1) $\mathrm{Ext}^\bullet_{\mathcal{C}}(P,P) = A$ grading comes from P; ignore cohomology grading

2) P is a generator, i.e. $\forall M \in D^b(\mathcal{C}_{Tate})$ $M \neq 0 \Rightarrow \mathrm{RHom}(P,M) \neq 0$.

3) $\mathrm{RHom}(P,P) =: \underset{dga}{D}$, is formal, i.e, D is quasi-isom to $H^\bullet(D) = \mathrm{Ext}(P,P)$

If 1)–3) hold $\Rightarrow D^b(\mathcal{C}_{Tate}) \overset{\Phi}{\simeq} D^b(A\text{-grmod})$

The equivalence Φ is constructed $D^b(\mathcal{C}_{Tate}) \xrightarrow{\sim} D^b(D\text{-mod})$

$$M \longmapsto \mathrm{RHom}(P,M)$$

Formality $\Rightarrow D^b(D\text{-mod}) \simeq D^b(A\text{-mod})$ // sketch, so details about A dga omit

Trick first used by Deligne: Pf of formality follows from:

Lemma Let $D = \bigoplus D^i$ be a dga equipped with a "weight filtration" $W_k D^\bullet$ s.t. (algebra)

$d: W_k D^i \longrightarrow W_k D^{i+1}$

If $\mathrm{gr}^W_k H^i(D) = 0$ unless $i=k$, then D is formal

Pf

$$D \longleftarrow \bigoplus_k W_k(D^k) \longrightarrow\!\!\!\!\to H^\bullet(D)$$

quasi-isom thanks to the purity assump

$\underset{\mathrm{Lie}\, G^\vee}{\mathfrak{g}} \supset \underset{\mathrm{Lie}\, B^\vee}{\mathfrak{b}} \supset \underset{\text{nilrad}}{\mathfrak{n}}$ $\qquad T_e(G^\vee/B^\vee) = \mathfrak{g}/\mathfrak{b}$ $\qquad T_e^*(G^\vee/B^\vee) = (\mathfrak{g}/\mathfrak{b})^* \underset{\text{Killing form}}{=} \mathfrak{n}$

$X = T^*\mathcal{B} = G^\vee \times_{B^\vee} \mathfrak{n} \longrightarrow \mathcal{B} = G^\vee/B^\vee$

$\mathfrak{b} = \underset{\text{Cartan sub}}{\mathfrak{h}} + \mathfrak{n}$ $\qquad \mathfrak{h} \ni h = 2\cdot\rho^\vee = \sum_{\alpha \in R^+} \alpha^\vee$ $\qquad Z_{G^\vee}(h) = H^\vee$ max torus in G

$\mathrm{Ad}\, B^\vee(h) = h + \mathfrak{n} \cong B^\vee/H^\vee$

$$\begin{array}{ccccc} X_h = G^\vee \times_{B^\vee} (h+\mathfrak{n}) & = & G^\vee \times_{B^\vee} (B^\vee/H^\vee) & = & G^\vee/H^\vee \\ \downarrow p \leftarrow G^\vee\text{-equiv affine bundle} & & & & \downarrow \text{natural projection} \\ G^\vee/B^\vee & & \overline{} & & G^\vee/B^\vee \end{array}$$

$\Gamma(X_h, p^*\mathcal{O}(\lambda)) = \Gamma(G^\vee/H^\vee, p^*\mathcal{O}(\lambda)) = \{ f \in \mathbb{C}[G^\vee] \mid f(gt^{-1}) = \lambda(t) f(g) \quad \forall g \in G^\vee, t \in H^\vee \} = \mathbb{C}[G^\vee](-\lambda)$

$= \left(\bigoplus_{V \in \mathrm{Irr}\, G^\vee} V \otimes V^* \right)(-\lambda) = \bigoplus_V V \otimes V^*_{-\lambda}$

$$\underbrace{\mathbb{C}[\mathbb{C}\cdot h + \mathfrak{n}]}_{\substack{\text{graded by degree}\\ \text{of the poly}}} \longrightarrow \underset{\substack{\uparrow \\ \text{induced}\\ \text{filtration}}}{\mathbb{C}[h+\mathfrak{n}]}$$

$$\mathrm{gr}\, \mathbb{C}[h+\mathfrak{n}] = \mathbb{C}[\mathfrak{n}]$$

$G^\vee \times_{B^\vee} (\mathbb{C}h+\mathfrak{n}) \longleftrightarrow X_h$ grading on $\Gamma(G^\vee \times_{B^\vee} (\mathbb{C}h+\mathfrak{n}), p^*\mathcal{O}(\lambda))$ induces filtration on $\Gamma(X_h, p^*\mathcal{O}(\lambda))$

<u>Claim</u> If λ is <u>dominant</u> then $\mathrm{gr}\,\Gamma(X_h, p^*\mathcal{O}(\lambda)) \simeq \Gamma(X, p^*\mathcal{O}(\lambda))$

<u>Cor</u> $\Gamma(X, \mathcal{O}_X(\lambda)) \simeq \bigoplus_{V \in \mathrm{Irr}\, G^\vee} V \otimes \mathrm{gr}(V^*_{-\lambda})$

Description of the filtration on $V^*_{-\lambda}$:

Let $\underset{\substack{\Psi \\ \alpha}}{S} \subset R^+$ set of simple roots. $e_\alpha \in \mathfrak{n}$ root vector $e = \sum_{\alpha \in S} e_\alpha$

(e.g. $\mathfrak{g} = \mathfrak{sl}_n$ $e = \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix}$ max Jordan block)

For any $V \in \mathrm{Rep}\, G^\vee$, $0 \subset \ker e \subset \ker e^2 \subset \cdots \ker(e^N) = V$

In particular for $V \in \mathrm{Irr}\, G^\vee$ for weight λ have a filtration $\ker(e^i) \cap V^*_{-\lambda}$ on $V^*_{-\lambda}$

<u>Thm</u> (Kostant–Brylinski) $\mathrm{gr}_i(V^*_{-\lambda}) = \dfrac{\ker(e^{i+1}, V^*_{-\lambda})}{\ker(e^i, V^*_{-\lambda})}$ for any dominant λ.

$\mathbb{C}((t)) \supset \mathbb{C}[[t]] =: \mathcal{O}$

$\mathrm{Gr} = G(\mathbb{C}((t)))/G(\mathcal{O})$ $j_\lambda : I^\lambda \hookrightarrow \overline{\mathrm{Gr}^\lambda} \hookrightarrow \mathrm{Gr}$ Iwahori orbit embedding

We'll assume geometric Satake (it is written down) $M \longmapsto V(M) \in \mathrm{Rep}\, G^\vee$

$H^*(j_\lambda^* M) = \mathrm{gr}\; V(M)_\lambda$ when λ dominant (extra check from geom Satake)

constant sheaf (grading is one above)

On Gr, have a distinguished line bundle <u>det</u> (ample, G-equiv)

Chern class $c_1(\det) \in H^2(\mathrm{Gr}, \mathbb{C})$

$H^\bullet(\mathrm{Gr}, \mathbb{C}) \curvearrowright H^\bullet(\mathrm{Gr}, M)$ graded module.

So c_1 acts on $\underset{\substack{\parallel \\ V(M)}}{H^\bullet(\mathrm{Gr}, M)}$ with degree 2. <u>Hard Lefschetz</u> (BBD): action of $c_1^{(\det)}$ comes from $\mathfrak{sl}_2$

$\rightsquigarrow$ this gives action of e and we deduce grading ...