4.

Throughout, G is a connected, reductive algebraic group over $k = \bar{\mathbf{F}}_q$ with a Frobenius map $F : G \to G$ corresponding to an $\mathbf{F}_q$-form, and (B, T) is an F-stable Borel pair. Today we will discuss the virtual characters of G^F arising from Deligne–Lusztig varieties, largely following Bonnafé's book (and notes I took from a WARTHOG course by Dudas).

4.1.

First, we review étale cohomology as a "black-box" formalism. This also serves as a warm-up for a later lecture about derived categories of complexes of sheaves with constructible cohomology. Throughout, $[d]$ means the degree-d shift functor on $\mathbf{Z}$-graded vector spaces V, so that $(V[d])^i = V^{i+d}$ for all i.

Fix a prime ℓ invertible in k. For our purposes, the *ℓ-adic étale cohomology* of a scheme X of finite type over k consists of $\mathbf{Z}$-graded $\bar{\mathbf{Q}}_\ell$-vector spaces

$$\mathrm{H}^*(X) = \bigoplus_i \mathrm{H}^i(X) \quad \text{and} \quad \mathrm{H}^*_c(X) = \bigoplus_i \mathrm{H}^i_c(X)$$

satisfying these properties, where all maps of graded vector spaces are assumed to be grading-preserving:

(1) Any map $f : Y \to X$ induces

$$\text{a } \textit{pullback } f^* : \mathrm{H}^*(X) \to \mathrm{H}^*(Y).$$

If f is smooth of relative dimension d, then it induces

$$\text{a } \textit{!-pushforward } f_! : \mathrm{H}^*_c(Y)[2d] \to \mathrm{H}^i_c(X).$$

Similarly, if f is proper, then it induces

$$\text{a } \textit{pushforward } f_! = f_* : \mathrm{H}^*_c(Y) \to \mathrm{H}^*_c(X).$$

All of these constructions are functorial in f. In particular, if a group Γ acts on X, then it acts on $\mathrm{H}^*(X)$ contravariantly. If Γ acts by proper maps, then it also acts on $\mathrm{H}^*_c(X)$ covariantly.

(2) There are functorial maps $\mathrm{H}^*_c(X) \to \mathrm{H}^*(X)$. They are isomorphisms for proper X.

(3) For X connected and smooth of dimension n, there is a perfect pairing

$$\mathrm{H}^*(X) \otimes \mathrm{H}^*_c(X) \to \bar{\mathbf{Q}}_\ell[-2n].$$

called *Poincaré duality*. Note that the grading-preserving condition means that it restricts to a perfect pairing between $\mathrm{H}^i(X)$ and $\mathrm{H}^{2n-i}_c(X)$.

(4) For any closed embedding $i : Z \to X$ with complement $j : U \to X$, we have a long exact sequence

$$\cdots \to \mathrm{H}_c^*(U) \xrightarrow{j_!} \mathrm{H}_c^*(X) \to \mathrm{H}_c^*(Z) \to \mathrm{H}_c^*(U)[1] \to \cdots$$

When X is proper, so that Z is also proper, the map $\mathrm{H}_c^*(X) \to \mathrm{H}_c^*(Z)$ is dual via item (2) and Poincaré to the map $i_! = i_*$.

(5) Pullback induces functorial isomorphisms

$$\mathrm{H}^*(X \sqcup Y) \simeq \mathrm{H}^*(X) \oplus \mathrm{H}^*(Y) \quad \text{and} \quad \mathrm{H}^*(X \times Y) \simeq \mathrm{H}^*(X) \otimes \mathrm{H}^*(Y),$$

and similarly with H_c^* in place of H^* (by Poincaré).

(6) For the affine n-space $\mathbf{A}^n$, we have

$$\begin{aligned} \mathrm{H}^*(\mathbf{A}^n) &\simeq \bar{\mathbf{Q}}_\ell \text{ (in degree zero)}, \\ \mathrm{H}_c^*(\mathbf{A}^n) &\simeq \bar{\mathbf{Q}}_\ell[-2n] \text{ (by Poincaré)}. \end{aligned}$$

(7) If $d = \dim X$, then $\mathrm{H}^i(X) = 0$ for $i > 2d$ and $i < 0$. If X is moreover affine, then $\mathrm{H}_c^i(X) = 0$ for $i < d$.

We say that $\mathrm{H}^*(X)$ is the *ordinary cohomology* and $\mathrm{H}_c^*(X)$ the *compactly-supported cohomology*.

Now instead of schemes of finite type over k, consider the category of pairs (X, F), where X is of finite type over k and $F : X \to X$ is a Frobenius map corresponding to an $\mathbf{F}_q$-rational structure on X, where morphisms of such pairs are the k-morphisms that commute with the Frobenius maps.

Let $\bar{\mathbf{Q}}_\ell(m)$ be the *m-fold Tate twist*: the one-dimensional representation of $\langle F \rangle$ given by $F \cdot 1 = q^{-m}$. Then:

(8) The maps in items (1)–(6) are F-equivariant after we replace $[2m]$ with $[2m](m)$.

(9) For smooth X, we have the *Lefschetz fixed-point formula*

$$|X^F| = \sum_i \mathrm{tr}(F \mid \mathrm{H}_c^i(X)).$$

Note that the right-hand side uses H_c^i, not H^i.

Example 4.1. The formula for the ℓ-adic cohomology of affine space implies the formula for that of projective space, via Lefschetz. First, use the partition $\mathbf{P}^n = \mathbf{A}^n \sqcup \mathbf{P}^{n-1}$ and induction to show that $\mathrm{H}^i(\mathbf{P}^n)$ vanishes for i odd and that F acts on $\mathrm{H}^{2j}(\mathbf{P}^n)$ by q^j. Next, since $|\mathbf{P}^n(\mathbf{F}_q)| = 1 + q + \cdots + q^n$, Lefschetz forces $\dim \mathrm{H}^{2j} = 1$ for $0 \leq j \leq n$.

4.2.

Last time we defined the varieties X_w and $\tilde{X}_w$. Let us now present a slightly different viewpoint on X_w.

Recall that $\mathcal{B}$ is the flag variety of G, isomorphic to G/B for any choice of Borel B, but itself independent of that choice. Let $O_w \subseteq \mathcal{B} \times \mathcal{B}$ be the G-orbit indexed by w. Explicitly, if we fix a Borel B, then the k-points of O_w are the pairs (gB, gwB) for $g \in G(k)$. We see that X_w can be defined through a cartesian square:

$$\begin{tikzcd} X_w \arrow[r] \arrow[d] & \mathcal{B} \arrow[d, "F \times \mathrm{id}"] \\ O_w \arrow[r] & \mathcal{B} \times \mathcal{B} \end{tikzcd}$$

It turns out that O_w is smooth of dimension $\ell(w) + \dim \mathcal{B}$ and intersects the image of $\mathrm{id} \times F$, *i.e.*, the graph of F, transversely: The latter claim can be verified by calculating differentials. Thus X_w is a smooth variety of dimension $\ell(w)$. Here, $\ell(w) = \dim(BwB)/B$.

Recall that we also defined a scheme $\tilde{X}_w$ with commuting actions of G^F and T^{wF}, such that T^{wF} acts freely and the quotient by T^{wF} defines a G^F-equivariant finite cover

$$\pi_w : \tilde{X}_w \to X_w.$$

We draw the following conclusions:

(1) The map π_w is finite étale. Thus $\tilde{X}_w$ is also a smooth variety of dimension $\ell(w)$.

(2) The compactly-supported cohomology $\mathrm{H}_c^*(\tilde{X}_w)$ forms a graded (G^F, T^{wF})-bimodule. In particular, if we write $V[\theta]$ for the θ-isotypic component of a representation V of T^{wF}, then the $\bar{\mathbf{Q}}_\ell$-vector space

$$\mathbf{R}_{w,\theta} = \mathbf{R}_{T^{wF}}^{G^F}(\theta) := \mathrm{H}_c^*(\tilde{X}_w)[\theta]$$

is a graded representation of G^F for any character $\theta : T^{wF} \to \bar{\mathbf{Q}}_\ell^\times$.

(3) Pushforward defines a map

$$\pi_{w,!} = \pi_{w,*} : \mathrm{H}_c^*(\tilde{X}_w) \to \mathrm{H}_c^*(X_w).$$

With more work, one can show that it factors through an isomorphism $\mathbf{R}_{T^{wF}}^{G^F}(1) = \mathrm{H}_c^*(\tilde{X}_w)^{T^{wF}} \xrightarrow{\sim} \mathrm{H}_c^*(X_w)$.

We refer to the operation $\mathbf{R}_{T^{wF}}^{G^F}$ as *Deligne–Lusztig induction* from T^{wF} to G^F.

In their original paper, Deligne–Lusztig focused on the virtual character of G^F defined by

$$R_{w,\theta} = R_{T^{wF}}^{G^F}(\theta) := \sum_i (-1)^i \mathrm{H}_c^i(\tilde{X}_w)[\theta].$$

Indeed this alternating sum resembles that appearing in the Lefschetz formula, which suggests that $R_{w,\theta}$ is related to point-counting, hence more tractable than $\mathbf{R}_{w,\theta}$ itself.

4.3.

Take $G = \mathrm{SL}_2$ and F the standard Frobenius, so that we can write $W = \{e, s\}$. We saw last time that

$$X_s = \mathbf{P}^1 \setminus Z,$$

where Z is a set of $q + 1$ points. In particular, X_s is affine of dimension 1, so we know that $\mathrm{H}^0_c(X_s) = 0$ and the remaining compactly-supported cohomology of X_s is supported in degrees 1 and 2. Similarly, the compactly-supported cohomology of Z is supported in degree 0, where it is $\bar{\mathbf{Q}}_\ell^{\oplus(q+1)}$. The long exact sequence from the inclusion $j : X_s \to \mathbf{P}^1$ gives

$$\cdots \to 0 = \mathrm{H}^1_c(Z) \to \mathrm{H}^2_c(X_s) \xrightarrow{j_!} \mathrm{H}^2_c(\mathbf{P}^1) \to \mathrm{H}^2_c(Z) = 0 \to \cdots$$

from which $\mathrm{H}^2_c(X_s) \simeq \mathrm{H}^2_c(\mathbf{P}^1) \simeq \bar{\mathbf{Q}}_\ell(-1)$, and

$$\cdots \to 0 = \mathrm{H}^0_c(X_s) \xrightarrow{j_!} \mathrm{H}^0_c(\mathbf{P}^1) \to \mathrm{H}^0_c(Z) \to \mathrm{H}^1_c(X_s) \xrightarrow{j_!} \mathrm{H}^1_c(\mathbf{P}^1) = 0 \to \cdots$$

from which $\mathrm{H}^1_c(X_s) \simeq \mathrm{H}^0_c(Z)/\mathrm{H}^0_c(\mathbf{P}^1) \simeq \bar{\mathbf{Q}}_\ell^{\oplus q}$. In particular,

$$\mathrm{tr}(F \mid \mathrm{H}^1_c(X_s)) = \mathrm{tr}(F \mid \mathrm{H}^2_c(X_s)) = q.$$

This agrees with the sanity check from Lefschetz: $|X_s^F| = 0$ by construction, matching $0 - q + q = 0$.

Note that $\mathrm{H}^1_c(X_s)$ and $\mathrm{H}^2_c(X_s)$ individually define representations of G^F. With more work, one can show that their respective characters are ρ, the Steinberg character, and 1, the trivial character, using the notation from the previous set of notes. Unfortunately, this means that $\mathbf{R}_{s,1} = \mathrm{H}^*(X_s)$ fails to see anything new: We have only reproduced the principal series from last time. Even so, we see something interesting on virtual characters:

$$R_{e,1} = 1 + \rho,$$
$$R_{s,1} = 1 - \rho,$$

so under the pairing $(-,-)_{G^F}$ on class functions induced by the Hom_{G^F}-pairing on isomorphism classes of representations, we have $(R_{e,1}, R_{s,1})_{G^F} = 1 - 1 = 0$: *i.e.*, $R_{e,1}$ and $R_{s,1}$ are orthogonal.