Warmup analytic topology T_{an} on R:

generated by the basis

$\{B(x, \delta) \mid x \text{ in } R \text{ and } \delta > 0\}$
$= \{(x - \delta, x + \delta) \mid x \text{ in } R \text{ and } \delta > 0\}$
$= \{(a, b) \mid a < b\}$

[means: every open set is a union of (a, b)'s]

Df lower-limit topology T_ℓ on R:

generated by the basis

$\{[a, b) \mid a < b\}$

Q1 is $\{[a, b) \mid a, b \text{ in } R\}$ really a basis? [yes]
Q2 how does T_ℓ compare to T_{an}?

Prop T_ℓ is strictly finer than T_{an}

Pf T_ℓ is finer than T_{an}:
suppose U anlytc open in R
suppose x in U
pick a < b s.t. x in (a, b) sub U
then [x, b) sub (a, b)

strict because [0, 1) in T_ℓ but notin T_{an}

$T_{indisc} < T_f < T_{an} < T_\ell < T_{disc}$

will write R_ℓ to mean "R in the topology T_ℓ", etc.

(Munkres §18, 16) recall from real analysis:

$f : \mathbb{R}^n \to \mathbb{R}^m$ is <u>Bolzano continuous</u> iff

for all x in $\mathbb{R}^n$ and $\varepsilon > 0$, there exists $\delta > 0$ s.t.
$|x - x'| < \delta$ implies $|f(x) - f(x')| < \varepsilon$

equivalently
for all x in $\mathbb{R}^n$ and $\varepsilon > 0$, there exists $\delta > 0$ s.t.
x' in $B(x, \delta)$ implies $f(x')$ in $B(f(x), \varepsilon)$

equivalently
for all x in $\mathbb{R}^n$ and $\varepsilon > 0$, there exists $\delta > 0$ s.t.
$B(x, \delta) \subseteq f^{-1}(B(f(x), \varepsilon))$

<u>Goal</u> generalize to topological spaces

<u>Df</u> a function $f : X \to Y$ is <u>continuous</u> iff
V open in Y implies $f^{-1}(V)$ open in X

<u>Thm</u> $f : \mathbb{R}^n \to \mathbb{R}^m$ is Bolzano cts
iff f is cts wrt the analytic topologies

[<u>Pf</u> suppose f cts wrt analytic topologies:

for all x in $\mathbb{R}^n$ and $\varepsilon > 0$, want $\delta > 0$
s.t. $|x - x'| < \delta$ implies $|f(x) - f(x')| < \varepsilon$
know: $B(f(x), \varepsilon)$ is open in $\mathbb{R}^m$
so $f^{-1}(B(f(x), \varepsilon))$ is open in $\mathbb{R}^n$
so have $\delta > 0$ s.t. $B(x, \delta) \subseteq f^{-1}(B(f(x), \varepsilon))$

suppose f Bolzano cts:

for all V open in Y, want $f^{-1}(V)$ open in X

pick x in $f^{-1}(V)$

want $\delta > 0$ s.t. $B(x, \delta)$ sub $f^{-1}(V)$

pick $\varepsilon > 0$ s.t. $B(f(x), \varepsilon)$ sub V

pick $\delta > 0$ s.t. for all x' s.t. $|x - x'| < \delta$,

have $|f(x) - f(x')| < \varepsilon$

then $f(B(x, \delta))$ sub $B(f(x), \varepsilon)$ sub V

so $B(x, \delta)$ sub $f^{-1}(V)$]

Thm suppose that $\{C_i\}_{i \in I}$ is a basis for the topology on Y

then

$f : X$ to Y is cts

iff $f^{-1}(C_i)$ is open in X for all i in I

Pf suppose f is cts

each C_i is open in Y

so each $f^{-1}(C_i)$ is open in X

now suppose $f^{-1}(C_i)$ open in X for all i in I

pick V open in Y

know: $V = \bigcup_{i \in J} C_i$ for some J sub I

$$\begin{aligned} f^{-1}(V) &= f^{-1}\left(\bigcup_{i \in J} (C_i)\right) \\ &= \bigcup_{i \in J} f^{-1}(C_i) \end{aligned}$$

so $f^{-1}(V)$ is open in X

Ex which maps are continuous?

$f : \mathbb{R}_{an}$ to $\mathbb{R}_\ell$, $f(x) = x$ [no: $[0, 1)$]

$f : \mathbb{R}_\ell$ to $\mathbb{R}_{an}$, $f(x) = x$ [yes]

$f : \mathbb{R}_{an}$ to $\mathbb{R}_\ell$, $f(x) = 32$ [yes]

Df cts $f : X$ to Y is a homeomorphism
iff it has a two-sided inverse $g : Y$ to X
s.t. g is also cts

in this case, we say X and Y are homeomorphic

[“what is shape?” “X and Y have the same shape when there is a homeo between them”]

Rem any homeo is a cts bijection
but a cts bijection need not be a homeo:
[already have an example: which?]
$f : R_\ell$ to R_{an}, $f(x) = x$

Ex id : X to X is a homeo when we use the same topology for domain and range

Ex other homeo’s wrt analytic topologies:

$f : R$ to R, $f(x) = x^3$
$f : R^2$ to R^2 $f(x, y) = (x + y, (x - y)^3)$
[why?]

Q is there a homeo R to R^2? vice versa?

The Subspace Topology let A be a subset of X

Df the subspace topology on A induced by X is
$\{A \cap V \mid V \text{ is an open set in } X\}$

[check: $A \cap \bigcup_i U_i = \bigcup_i (A \cap U_i)$]

Ex take $X = R$ and $A = [0, \infty)$

open sets in subspace top on $[0, \infty)$
look like V cap $[0, \infty)$
what if $V = (a, b)$?
if $a \geq 0$, then (a, b) cap $[0, \infty) = (a, b)$
is open in both $[0, \infty)$ and R
if $a < 0$, then (a, b) cap $[0, \infty) = [0, b)$
is open in $[0, \infty)$, but not in R

Moral if U sub A is open in A,
then U may or may not be open in X

Thm if $\{B_i\}_{i \text{ in } I}$ is a basis for the top on X,
then $\{A \text{ cap } B_i\}_{i \text{ in } I}$ is a basis for
the subspace top on A

Pf by thm from last time, just need to show:
for all U open in A and x in U,
have i s.t. x in A cap B_i sub U

indeed, $U = A$ cap V for some V open in X
then x in V, so have i s.t. x in B_i sub V
so x in A cap B_i, and also, A cap B_i sub U

[how do subspaces interact with cts maps?]

Observations

- if A sub X, then the inclusion map $i : A$ to X
def by $i(a) = a$ is cts
- compositions of cts maps are cts
- so if $f : X$ to Y is cts, then $f|_A : A$ to Y is cts

Bonus Material

recall that Z is the set of integers

Df the evenly spaced topology on Z is generated by the basis

$\{aZ + b \mid a, b \text{ in } Z \text{ and } a \neq 0\}$

[will show on PS1 that this really is a basis]

Thm there are infinitely many prime numbers

Proof (Furstenberg, 1955)

assume finitely many primes p

then Z – bigcup_p pZ = bigcap_p (Z – pZ) is open
because Z – pZ is open for all p

but Z – bigcup_p pZ = {±1}
because if |a| > 1, then some prime divides a

so {±1} is open
but {±1} is not an evenly spaced set in Z □