Warmup

recall 2×2 inversion formula:

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad M^{(-1)} = (1/\det(M))\ \mathrm{adj}(M)$$

where $\det(M) = ad - bc$

$$\mathrm{adj}(M) = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

what are the inverses of:

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} ? \qquad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \text{[reflection]}$$

$$\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} ? \qquad \begin{pmatrix} 1 & -b \\ 0 & 1 \end{pmatrix} \qquad \text{[shear]}$$

in general $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}$ are shears

but there are others:

$$M = \begin{pmatrix} 5/2 & -3/2 \\ 3/2 & -1/2 \end{pmatrix}$$ is a shear too: why?

M sends $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ to $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ to $\begin{pmatrix} 4 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

[draw]

Idea have: qualitative defns of geometric ops
and examples given by matrices
want: matrix-independent defns

Q how to formalize “matrix-independent”?

fix linear map $T : V$ to W

$(e_1, \ldots, e_n)$ ordered basis of V,
$(f_1, \ldots, f_m)$ ordered basis of W
M matrix of T wrt $(v_i)_i$, $(w_j)_j$

$(e_1, \ldots, e_n)$ another ordered basis of V,
$(f_1, \ldots, f_m)$ another ordered basis of W
matrix of T wrt $(e_i)_i$, $(f_j)_j$? in terms of M?

$$\begin{array}{ccccccc} & id_V & & T & & id_W & \\ V & \text{to} & V & \text{to} & W & \text{to} & W \\ & & & & & & \\ e_i & & v_i & & w_j & & f_j \end{array}$$

let A be the matrix of id_V “from e_i to v_i”
B id_W “from w_j to f_j”

A, B are invertible [since they represent iso’s]
the matrix of T wrt $(e_i)_i$, $(f_j)_j$ is $B \bullet M \bullet A$

possibly confusing:
ith col of A expresses e_i as $\text{sum}_j A_{k, i} v_k$
jth col of B w_j as $\text{sum}_\ell B_{\ell, j} f_\ell$

[id_V sends e_i to e_i; id_W sends w_j to w_j]

suppose $W = V$,
$(w_j)_j = (v_i)_i$,
$(f_j)_j = (e_i)_i$

here T is a linear op on V,
M is its matrix "from $(v_i)_i$ to $(v_i)_i$"

A is the matrix of the id map "from e_i to v_i"
B is the matrix of the id map "from v_i to e_i"
so $A = B^{(-1)}$

<u>Thm</u> if M is the matrix of a lin op $T : V$ to V
wrt a basis $(v_i)_i$ for V,
then $BMB^{(-1)}$ is its matrix
wrt another basis $(e_i)_i$
where $v_j = \text{sum}_i B_{\{j, i\}} e_i$

<u>Rem</u> general case $T : V$ to W also useful,
but the statement is cumbersome –
easier to rederive from picture of maps

<u>Ex</u> $V = \{p \text{ in } F[x] \mid p = 0 \text{ or } \deg p \leq 3\}$
$T(p) = dp/dx$
$(v1, v2, v3, v4) = (1, x, x^2, x^3)$
$(e1, e2, e3, e4) = (1, x, x^2/2, x^3/6)$

$$M = \begin{matrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{matrix} \qquad B = \begin{matrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 6 \end{matrix}$$

[compute B^(-1)]
[compute BMB^(-1)]

Df matrices M, N are conjugate* iff,
for some matrix P,
$P \cdot M \cdot P^{(-1)}$ is well-defined,
equals N

* also say "M and N are conjugates of each other"

Rem almost always, we only use this notion in the context where M, N are square

Df a property of M is conjugation-invariant iff it is the same for all conjugates of M

Cor if M is the matrix of a lin op T,
then any property of T is
a conjugation-invariant property of M

and conversely!
any conjugation-invariant property of M
only depends on T

Ex entries of M are not conj-invariant

but tr(M) = sum of diagonal entries
and det(M) = ??? [to be discussed later]
are

for a 2 × 2 matrix $M = \begin{matrix} a & b \\ c & d \end{matrix}$

tr(M) = a + d, det(M) = ad – bc

[can check by brute force, if you want]

Ex fix an integer $k > 0$

the property ($M^k = 0_n$) is conj-invariant because

$$(BMB^{-1})^k = BM^kB^{-1} = 0_n$$

corresponds to having $T \circ \ldots \circ T =$ zero
where T is iterated k times

Df a lin op, resp. matrix, is nilpotent iff some iterate, resp. power, is zero

it is unipotent iff it takes the form $\mathrm{id} + T$, resp. $I_n + M$, where T, resp. M, is nilpotent

Ex since nilpotence is conj-invariant, unipotence is conj-invariant:

$$\begin{aligned} B(I_n + M)B^{-1} &= BI_nB^{-1} + BMB^{-1} \\ &= I_n + BMB^{-1} \end{aligned}$$

above, used the left & right distributive properties for matrix multiplication

Ex for fixed $k > 0$, the property $M^k = I_n$ is conj-invariant

M is called an involution iff $M^2 = I_n$

Rem 2×2 shears are unipotent
[hyperplane] reflections are involutions