MATH 340: ADVANCED LINEAR ALGEBRA
PROBLEM SET #6

SPRING 2025

Due Friday, March 28 (NEW). You may consult books, papers, and websites as long as you cite all sources and write your solutions in your own words. **Updated on 3/13, in red.**

Problem 1. Recall that for any finite-dimensional complex vector space V and linear operator $T : V \to V$, we defined the *characteristic polynomial* of T to be

$$p_T(z) = \prod_i (z - \lambda_i)^{d_i}$$

whenever T has a Jordan canonical form matrix where the ith block has eigenvalue λ_i and size d_i. For any scalar λ, we define the *multiplicity* of λ as an eigenvalue of T to be the sum of the d_i's over indices i such that $\lambda = \lambda_i$.

Assume that the determinant of <u>any</u> triangular matrix is the product of its diagonal entries. Deduce that if M is <u>any</u> triangular matrix for T, then the multiplicity of λ as an eigenvalue of T is the number of times that λ occurs along the diagonal of M. *Hint:* Show that $p_T(z)$, as a function of z, can also be expressed as a determinant.

Problem 2. Keeping the setup of Problem 1:

(1) Show that if λ has multiplicity m as an eigenvalue of T, then λ^n has multiplicity <u>at least</u> m as an eigenvalue of T^n.
(2) Using (1), show that if $T^n = \mathrm{Id}_V$ for some $n > 0$, then all eigenvalues of T live on the unit circle $\{z \in \mathbf{C} \mid |z| = 1\}$.

Problem 3. Show that:

(1) If $T : \mathbf{C}^2 \to \mathbf{C}^2$ has <u>real</u> trace $\mathrm{tr}(T) \in [-2, 2]$ and $\det(T) = 1$, then its eigenvalues live on the unit circle.
(2) If $S : \mathbf{R}^2 \to \mathbf{R}^2$ satisfies $|\mathrm{tr}(S)| \leq 2$ and $\det(S) = 1$, then S is a rotation. You may use the fact that $S_{\mathbf{C}}$ is given by the "same" matrix as S, but operating on $\mathbf{C}^2$.

Problem 4 (Axler §5D, #21)**.** Define the *Fibonacci numbers* $F_0, F_1, F_2, \ldots$ by

$$\begin{aligned} &F_0 = 0, \\ &F_1 = 1, \\ &F_n = F_{n-2} + F_{n-1} \text{ for all } n \geq 2. \end{aligned}$$

Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be given by $T(x, y) = (y, x + y)$ in the standard basis.

(1) Show that $T^n(0, 1) = (F_n, F_{n+1})$ for all $n \geq 0$.
(2) Find the eigenvalues of T. *Hint:* Problem 1(1).

(3) Find a basis of $\mathbf{R}^2$ consisting of eigenvectors of T.

(4) Using (2)–(3), give a new expression for $T^n(0,1)$: one that shows that

$$F_n = \tfrac{1}{\sqrt{5}}\left(\varphi_+^n - \varphi_-^n\right) \text{ for all } n \geq 0, \quad \text{where } \varphi_\pm = \tfrac{1\pm\sqrt{5}}{2}.$$

(5) Deduce from (4) that F_n is the integer closest to $\frac{1}{\sqrt{5}}\varphi^n$, for all $n \geq 0$.

Problem 5. View $\mathbf{R}^4$ as column vectors and $(\mathbf{R}^4)^\vee$ as row vectors. Let

$$v_1 = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \quad v_2 = \begin{pmatrix}1\\1\\0\\0\end{pmatrix}, \quad v_3 = \begin{pmatrix}1\\1\\1\\0\end{pmatrix}, \quad v_4 = \begin{pmatrix}1\\1\\1\\1\end{pmatrix}.$$

Given that (v_1, v_2, v_3, v_4) is an ordered basis for $\mathbf{R}^4$, what is the dual ordered basis for $(\mathbf{R}^4)^\vee$ in terms of row vectors? Recall that it is the ordered basis $(\theta_1, \theta_2, \theta_3, \theta_4)$ such that $\theta_j(v_i)$ equals 1 when $i = j$ and 0 otherwise.

Problem 6 (Axler, §3F, #32)**.** Let $\Lambda : V \to (V^\vee)^\vee$ be defined as follows:

$$\text{for all } v \in V, \quad \text{let } \Lambda : V^\vee \to F \text{ be given by } (\Lambda v)(\theta) = \theta(v).$$

Show that:

(1) Λ is a linear map.

(2) For any linear operator $T : V \to V$, we have $(T^\vee)^\vee \circ \Lambda = \Lambda \circ T$.

(3) If V is finite-dimensional, then Λ is a linear isomorphism. *Hint:* Show that Λ is injective and that $\ker(\Lambda) = \{\vec{0}\}$.

Problem 7. For each linear subspace in $\mathbf{R}^4$ below, determine a basis for its annihilator in $(\mathbf{R}^4)^\vee$. Again, it may help to view $\mathbf{R}^4$ as column vectors and $(\mathbf{R}^4)^\vee$ as row vectors.

(1) $U = \{(a, b, c, d) \in \mathbf{R}^4 \mid c = d = 0\}$.

(2) $W = \{(a, b, c, d) \in \mathbf{R}^4 \mid a + b = c + d = 0\}$.

(3) $U \cap W$.

(4) $U + W$.

Hint: In (3)–(4), simplify the subspace before calculating its annihilator.

Problem 8. For each map $\beta : \mathbf{R}[x] \times \mathbf{R}[x] \to \mathbf{R}$ below, determine whether β is *bilinear*: That is, whether

$$\beta(-, q) : \mathbf{R}[x] \to \mathbf{R} \quad \text{and} \quad \beta(p, -) : \mathbf{R}[x] \to \mathbf{R}$$

are linear for all $p, q \in \mathbf{R}[x]$.

(1) $\beta(p, q) = \displaystyle\int_0^1 p(x)q(x)\,dx$.

(2) $\beta(p, q) = p(1) + q(1)$.

(3) $\beta(p, q) = p(1)q(1)$.

(4) $\beta(p, q) = p(1)q'(1)$, where $q'(x)$ is the derivative of $q(x)$.