

## MATH 665 PROBLEM SET 2

FALL 2024

**Due Thursday, October 17.** You may consult books, papers, and websites as long as you cite all sources and write up your solutions in your own words.

**Problem 1.** Let  $G$  be a (connected, smooth) reductive algebraic group over  $k$  with Frobenius  $F$ , compatible with the group law. Let  $(B, T)$  be an  $F$ -stable Borel pair. For  $w \in N_G(T)/T$  lifting to  $\dot{w} \in N_G(T)$ , recall the variety

$$\tilde{X}_{\dot{w}} = \{g \in G \mid g^{-1}F(g) \in U\dot{w}U\}/U, \quad \text{where } U = [B, B].$$

For any  $F$ -stable maximal torus  $T' \subseteq G$  and Borel  $B' \subseteq G$  containing  $T'$ , let

$$\tilde{X}_{U'} = \{h \in G \mid h^{-1}F(h) \in F(U')\}/(U' \cap F(U')), \quad \text{where } U' = [B', B'].$$

Show that for fixed  $T'$ , we can exhibit  $B', w, \dot{w}$ , and

$$f : \tilde{X}_{U'} \rightarrow \tilde{X}_{\dot{w}}$$

such that:

- (1)  $f$  is bijective on  $k$ -points.
- (2)  $f$  is  $G^F$ -equivariant, and transports the right  $(T')^F$ -action on  $\tilde{X}_{U'}$  to the right  $T^{wF}$ -action on  $\tilde{X}_{\dot{w}}$ .

*Hint:* Consider the intermediate variety  $\{h \in G \mid h^{-1}F(h) \in F(U)\}/U$ .

*Extra:* If you are bored, show that  $f$  is in fact an isomorphism of sheaves in the smooth topology, hence of varieties.

This problem suggests how, given an  $F$ -stable maximal torus  $T' \subseteq G$  of type  $w$  and character  $\theta' : (T')^F \rightarrow \bar{\mathbf{Q}}_\ell^\times$ , we can construct a class function  $R_{T', \theta'}$  directly from  $T'$  and  $\theta'$  such that  $R_{T', \theta'} = R_{T, \theta}$  for some  $\theta : T^{wF} \rightarrow \bar{\mathbf{Q}}_\ell^\times$ .

**Problem 2.** Fix an integer  $n > 0$ , invertible in  $k = \bar{\mathbf{F}}_q$ . Let  $\mathcal{B}$  be the variety of Borel subgroups of  $\mathrm{GL}_n$ , which we identify with the variety of complete flags in  $\mathbf{A}^n$ . Below, we work with the standard Frobenius on  $\mathcal{B}$ , induced by the map  $F$  on  $\mathbf{A}^n$  that sends  $F(x_i) = x_i^q$  on coordinates.

Recall that if a pair of flags  $(\vec{V}, \vec{V}')$  corresponds to a pair of Borels in relative position  $w \in S_n$ , then  $k^n$  admits a basis  $e_1, \dots, e_n$  such that  $V_i = \langle e_1, \dots, e_i \rangle$  and  $V'_i = \langle e_{w(1)}, \dots, e_{w(i)} \rangle$ . Consider the  $n$ -cycle

$$c = (1, 2, \dots, n) \in S_n.$$

- (1) Identifying the Deligne–Lusztig variety  $X_c$  with a subvariety of  $\mathcal{B}$ , write the defining condition on  $X_c$  in terms of flags.
- (2) Show that the map  $\vec{V} \mapsto V_1 : \mathcal{B} \rightarrow \mathbf{P}^n$  restricts to a bijection

$$X_c(k) \xrightarrow{\sim} \{L \in \mathbf{P}^n(k) \mid L + F(L) + \dots + F^{n-1}(L) = k^n\}.$$

Deduce that  $X_c$  is the nonvanishing locus of an explicit determinant.

- (3) Can you guess a similar description for  $\tilde{X}_c$ ? *Hint:* Look back at #4 on Problem Set 1.

**Problem 3.** Let  $A$  be the localization of  $\mathbf{Z}[x]$  at the prime  $x - 1$ . Let

$$H_2 = \frac{A[\sigma]}{\langle \sigma^2 - (x - x^{-1})\sigma - 1 \rangle},$$

viewed as the Hecke algebra of  $S_2$  over  $A$ .

- (1) Show that the ring isomorphism  $H_2/(x - 1) \xrightarrow{\sim} \mathbf{Z}S_2$  lifts to an  $A$ -algebra isomorphism  $H_2 \xrightarrow{\sim} AS_2$ .
- (2) Bootstrap (1) to show the analogous result with

$$H_3 = \frac{A[\sigma_1, \sigma_2]}{\langle \sigma_1^2 - (x - x^{-1})\sigma_1 - 1, \sigma_2^2 - (x - x^{-1})\sigma_2 - 1, \sigma_1\sigma_2\sigma - \sigma_2\sigma_1\sigma_2 \rangle}$$

and  $S_3$  in place of  $H_2$  and  $S_2$ .

- (3) For  $n = 2, 3$ , determine the elements of  $H_n$  that correspond, under the  $A$ -algebra isomorphisms above, to the central idempotents  $\sum_{w \in S_n} \chi(w)w$  with  $\chi \in \text{Irr}(S_n)$ .

In the problems that follow,  $\mathbf{P}$  denotes the reduced HOMFLY-PT polynomial in variables  $\mathbf{a}$  and  $\mathbf{x}$ .

**Problem 4.** Let  $\beta_1 = \sigma_1^3 \in Br_2$  and  $\beta_2 = (\sigma_1\sigma_2)^2 \in Br_3$ .

- (1) Using only pictures, show that the link closures  $\hat{\beta}_1$  and  $\hat{\beta}_2$  are isotopic.
- (2) Using only the Jones–Ocneanu construction of  $\mathbf{P}$  from Markov traces, show that  $\mathbf{P}(\hat{\beta}_1) = \mathbf{P}(\hat{\beta}_2)$ .
- (3) Prove that  $\hat{\beta}_1, \hat{\beta}_1^{-1}$  cannot be isotopic.

**Problem 5.** Let  $\sigma_1$  denote the positive generator of  $Br_2$ .

- (1) Show that the polynomials  $\mathbf{P}(\widehat{\sigma_1^n})$  satisfy a linear recurrence for  $n \geq 0$ .
- (2) Deduce that the (positive)  $(2, n)$  torus links are pairwise non-isotopic.

**Problem 6.** Consider the varieties

$$\mathcal{X}_m = \{ \vec{L} = (L_1, \dots, L_m) \in (\mathbf{P}^1)^m \mid L_{i-1} \neq L_i \text{ for all } i \bmod m \},$$

which can be defined over any field. Show that:

- (1) For each  $m \geq 2$ , there is a polynomial  $P_m(\mathbf{q}) \in \mathbf{Z}[\mathbf{q}]$  satisfying  $P_m(q) = |\mathcal{X}_m(\mathbf{F}_q)|$  for all (odd) prime powers  $q$ .  
*Hint:* Partition  $\mathcal{X}_m$  into subvarieties  $\mathcal{X}_{\vec{e}}$  indexed by binary sequences  $\vec{e} \in \{0, 1\}^{m-1}$ , where  $\mathcal{X}_{\vec{e}}$  consists of the tuples  $\vec{L}$  in which  $L_i = L_m$  if and only if  $e_i = 0$ .
- (2) The polynomials  $P_m(\mathbf{q})$  satisfy a linear recurrence. Can you guess how they are related to the polynomials  $\mathbf{P}(\widehat{\sigma_1^n})$  in Problem 5?