PROBLEMS ON SYMPLECTIC REFLECTION ALGEBRAS

1. Kleinian singularities

Problem 1.1. *Let G be a finite subgroup of $\mathrm{SO}_3(\mathbb{R})$. Consider its action on the unit sphere. Show that any non-unit element of G fixes a unique pair of opposite points and that the stabilizer of each point P is cyclic of some order, say, n_P. Choose representatives $P_1, \ldots, P_k$ of orbits with non-trivial stabilizers, one in each orbit. Show that $2\left(1-\frac{1}{n}\right) = \sum_{i=1}^k \left(1-\frac{1}{n_{P_i}}\right)$. Use this to show that the finite subgroups of $\mathrm{SO}_3(\mathbb{R})$ are precisely the following:*

1. *The cyclic group of order n – generated by a rotation by the angle of $2\pi/n$.*
2. *The dihedral group of order $2n$ with $n \geqslant 2$: the group of rotational symmetries of a regular n-gon on the plane inside of the 3D space (a regular 2-gon= a segment).*
3. *The group of rotational symmetries of the regular tetrahedron isomorphic to the alternating group A_4.*
4. *The group of rotational symmetries of the regular cube/octahedron isomorphic to the symmetric group S_4.*
5. *The group of rotational symmetries of the regular dodecahedron/icosahedron isomorphic to A_5.*

Problem 1.2. *Use the previous problem to deduce that the complete list of finite subgroups $\mathrm{SL}_2(\mathbb{C})$ up to conjugacy is as follows.*

- (A_r) *The cyclic group of order $r+1$, i.e., $\{\begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon^{-1} \end{pmatrix} | \epsilon^{r+1} = 1\}$.*
- (D_r) *The dihedral group of order $4(r-2), r \geqslant 4$: $\{\begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon^{-1} \end{pmatrix}, \begin{pmatrix} 0 & \epsilon \\ -\epsilon^{-1} & 0 \end{pmatrix} | \epsilon^{2(r-2)} = 1\}$.*
- (E_6) *The double cover of $A_4 \subset \mathrm{SO}_3(\mathbb{R})$.*
- (E_7) *The double cover of $S_4 \subset \mathrm{SO}_3(\mathbb{R})$.*
- (E_8) *The double cover of $A_5 \subset \mathrm{SO}_3(\mathbb{R})$.*

Problem 1.3. *Compute the McKay graph for $\Gamma \subset \mathrm{SL}_2(\mathbb{C})$ of type D_r.*

Problem 1.4. *This problem discusses the Kleinian group of type E_6.*

1) We start with a construction. Take the group Q_8 of unit quaternions. It has elements $\{\pm 1, \pm i, \pm j, \pm k\}$. Show that the cyclic group $\mathbb{Z}_3$ acts on Q_8 by automorphisms in such a way that the generator ω acts as follows: $\omega(-1) = -1, \omega(i) = j, \omega(j) = k, \omega(k) = i$. Embed the semi-direct product $\Gamma := \mathbb{Z}_3 \ltimes Q_8$ into $\mathrm{SL}_2(\mathbb{C})$. Further, show that $\Gamma/\{\pm 1\} \cong A_4$.

2) Show that Γ has 3 one-dimensional, 3 two-dimensional and 1 three-dimensional irreducible representations.

3) Compute the McKay graph for Γ.

Problem 1.5. *Show that for $\Gamma \subset \mathrm{SL}_2(\mathbb{C})$ of type D_r we have $\mathbb{C}[x,y]^\Gamma / \cong \mathbb{C}[x_1, x_2, x_3]/(x_1^{r-1} + x_1 x_2^2 + x_3^2)$.*