MATH 250: TOPOLOGY I

FALL 2025 LOOSE ENDS

Contents

1. Wednesday, 9/3

1.1. Here I try to dispel some potential confusion about bases.

Let X be any set. Let $\mathcal{B}$ be any collection of subsets of X. A useful general observation:

Lemma 1.1. *For any subset $Y \subseteq X$, the following conditions are equivalent:*

(1) *Y is the union of some elements of $\mathcal{B}$.*
(2) *For any $x \in Y$, there is some $B \in \mathcal{B}$ such that $x \in B \subseteq Y$.*

Now let $\mathcal{T}$ be the collection of all subsets of X that can be written as unions of elements of $\mathcal{B}$. Using the lemma, we see that

$$\mathcal{T} = \left\{ \text{subsets } U \subseteq X \;\middle|\; \begin{array}{l} \text{for any } x \in U, \text{ we have some } B \in \mathcal{B} \\ \text{such that } x \in B \subseteq U \end{array} \right\}.$$

Theorem 1.2. *Suppose that $\mathcal{B}$ satisfies the following conditions:*

(I) *Every point of X belongs to some element of $\mathcal{B}$.*
(II) *For any $B, B' \in \mathcal{B}$ and any point x of the intersection $B \cap B'$, we can find some $B'' \in \mathcal{B}$ such that $x \in B'' \subseteq B \cap B'$.*

Then $\mathcal{T}$ is a topology on X.

We proved this theorem at the start of the course, implicitly using Lemma 1.1. The only hard part is checking that finite intersections of elements of $\mathcal{T}$ are still elements of $\mathcal{T}$. To make this easier, I mentioned that it suffices by induction to check intersections between pairs of elements of $\mathcal{T}$.

Any collection $\mathcal{B}$ that satisfies hypotheses (I)–(II) in the theorem above is called a *basis*. In the situation of the theorem, we say that $\mathcal{B}$ *generates* or *induces* the topology $\mathcal{T}$, and that $\mathcal{B}$ is a *basis for $\mathcal{T}$* specifically.

1.2. Separately, if we are given $\mathcal{T}$ to start, then there is a way to check whether a subcollection $\mathcal{C} \subseteq \mathcal{T}$ is a basis that generates $\mathcal{T}$. In Munkres, this is Lemma 13.2.

Theorem 1.3. *Fix a topology $\mathcal{T}$ on X and a subset $\mathcal{C} \subseteq \mathcal{T}$. Suppose that for each $x \in X$ and $U \in \mathcal{T}$, there is some $C \in \mathcal{C}$ such that $x \in C \subseteq \mathcal{C}$. Then $\mathcal{C}$ is a basis, and moreover, the topology it generates is $\mathcal{T}$.*

2. Monday, 9/8

2.1. Let X be a set, and let $d : X \times X \to [0, \infty)$ be a metric on X. For all $x \in X$ and $\delta > 0$, we define the *d-ball* with center x and radius δ to be

$$B_d(x, \delta) = \{y \in X \mid d(x, y) < \delta\}.$$

Below is a cleaner version of a long proof from lecture.

Theorem 2.1. *The set* $\{B_d(x, \delta) \mid x \in X \text{ and } \delta > 0\}$ *forms a basis.*

Proof. Let $\mathcal{B}$ denote the set in question. We must check two axioms:

(I) Any point of X is contained in some element of $\mathcal{B}$.

(II) Given any two elements of $\mathcal{B}$ and a point in their intersection, we can find some other element of $\mathcal{B}$ containing that point and contained within the intersection as a subset.

(I) holds because for any $x \in X$, we have $x \in B(x, \delta)$ for any choice of δ.

To show (II): Pick balls $B_d(x, \epsilon)$ and $B_d(x', \epsilon')$ and a point z in their intersection $B_d(x, \epsilon) \cap B_d(x', \epsilon')$. We must exhibit some d-ball that contains z and is contained within the intersection as a subset.

It suffices to find some $\delta > 0$ such that

$$B_d(z, \delta) \subseteq B_d(x, \epsilon) \cap B_d(x', \epsilon').$$

Explicitly, this condition on δ means that

$$\text{if } y \in X \text{ satisfies } d(z, y) < \delta, \quad \text{then } d(x, y) < \epsilon \text{ and } d(x', y) < \epsilon'.$$

(Informally, this means that if y is close enough to z, then it is close enough to x and x' as well.) By drawing a picture of the situation, we get the idea that we need to use the triangle inequality to bound the distance $d(x, y)$ in terms of the distances $d(x, z)$ and $d(z, y)$.

Since $z \in B_d(x, \epsilon)$, we know that $d(x, z) < \epsilon$. Rearranging, $\epsilon - d(x, z) > 0$. So we can pick α such that $\epsilon - d(x, z) > \alpha > 0$. Then $d(x, z) + \alpha < \epsilon$. So if $y \in X$ satisfies $d(z, y) < \alpha$, then it also satisfies

$$\begin{aligned} d(x, y) &\leq d(x, z) + d(z, y) && \text{by the triangle inequality} \\ &< d(x, z) + \alpha && \text{by the hypothesis on } y \\ &< \epsilon. \end{aligned}$$

By analogous arguments, we can pick α' such that $\epsilon' - d(x', z) > \alpha' > 0$, and in this case, if y satisfies $d(z, y) < \alpha'$, then $d(x', y) < \epsilon'$.

Finally, set $\delta = \min(\alpha, \alpha')$. We see that if $y \in X$ satisfies $d(z, y) < \delta$, then we have both $d(x, y) < \epsilon$ and $d(x', y) < \epsilon'$. So we have found the desired δ. ☐

3. Wednesday, 9/10

3.1. Let X be a set, and let d be a metric on X. In lecture, I defined the metric topology for d as follows:

First, d gives rise to a collection of balls $B_d(x, \delta)$. By Theorem 2.1, it turns out to be a basis. Then, I defined the *metric topology* for d as the topology generated by this basis.

This is subtly different from what Munkres does. Munkres defines the metric topology for d to be

$$\{U \subseteq X \mid \text{for all } x \in U \text{, there is some } \delta > 0 \text{ such that } B_d(x, \delta) \subseteq U\}.$$

By contrast, if you unpack my definition from lecture, you get

$$\begin{aligned} &\{U \subseteq X \mid \text{for all } x \in U \text{, there is some } d\text{-ball } B \text{ such that } x \in B \subseteq U\} \\ &= \left\{ U \subseteq X \;\middle|\; \begin{array}{l} \text{for all } x \in U \text{, there are some } x' \in X \text{ and } \delta' > 0 \\ \text{such that } x \in B_d(x', \delta') \subseteq U \end{array} \right\}. \end{aligned}$$

If you think about it, any set that is open in Munkres's definition will be open in mine, but the converse is less obvious.

Happily, the converse turns out to be true, so our definitions do give the same topology. The key is to show:

Lemma 3.1. *If $x \in B_d(x', \delta')$ for some $x' \in X$ and $\delta' > 0$, then there is some $\delta > 0$ such that $B_d(x, \delta) \subseteq B_d(x', \delta')$.*

Proof. After unpacking the definitions, we want $\delta > 0$ such that if $y \in X$ satisfies $d(x, y) < \delta$, then $d(x', y) < \delta'$. As in the proof of Theorem 2.1, the key is the triangle inequality.

By hypothesis, $d(x, x') < \delta'$. Rearranging, $\delta' - d(x, x') > 0$. So we can pick δ such that $\delta' - d(x, x') > \delta > 0$. Then $d(x, x') + \delta < \delta'$. Altogether, if $y \in X$ satisfies $d(x, y) < \delta$, then (by using the triangle inequality)

$$d(x', y) \leq d(x, x') + d(x', y) < d(x, x') + \delta < \delta',$$

as needed. □