<u>Today</u> the n-sphere

$$S^n = \{(x_0, x_1, \ldots, x_n) \mid \sum_i x_i^2 = 1\}$$

as a subspace of analytic $R^{n + 1}$

<u>Thm 1</u> $\pi_1(S^1) \simeq Z$ via explicit iso
<u>Thm 2</u> $\pi_1(S^n)$ trivial for all $n \geq 2$

<u>Q</u> why did I omit the basept?

S^n is path-connected for any $n \geq 1$ [$n = 0$?]
so $\pi_1(S^n, x) \simeq \pi_1(S^n, y)$ for any x, y in S^n

<u>Df</u> X is simply connected iff
X is path connected and $\pi_1(X)$ is trivial

<u>Ex</u> any star-convex subspace of $R^{n + 1}$
is simply connected

but S^n is not star-convex

<u>Naive Pf of Thm 2</u> [Munkres 370, #2]

pick basept x and loop γ in S^n at x
pick p in S^n – image(γ)

there is a homeo S^n – $\{p\}$ to R^n:
namely, stereographic projection

R^n is simply-connected
so $[\gamma] = [e_x]$ [i.e., γ nulhomotopic] in S^n – $\{p\}$
hence $[\gamma] = [e_x]$ in S^n

[where did we use $n \geq 2$?]

<u>Problem</u> how do we know that image(γ) $\neq S^n$?

definitely false when $n = 1$
but also false when $n \geq 2$:

<u>Thm</u> (Peano) there are "space-filling curves"(!)

<u>Cor</u> there are loops γ in S^n that are surjective as maps $[0, 1]$ to S^n

<u>Solution</u> if $n \geq 2$, then any loop in S^n is path-homotopic to a non-surjective loop

[proof is hard so we merely sketch:]

<u>Pf sketch of Claim</u>

suppose the loop is γ, based at x
pick open hemispheres B, C s.t.

- $B \cup C = S^n$
- $B \cap C$ homeo to $S^{n-1} \times (-\varepsilon, \varepsilon)$
- x in $B \cap C$

then $C - B$ is nonempty
idea: path-homotope γ to a loop avoiding $C - B$

for all s in $[0, 1]$, know $\gamma(s)$ in B or $\gamma(s)$ in C
but B, C open, so must have $\delta_s > 0$ s.t.

- either $\gamma((s - \delta_s, s + \delta_s)) \subseteq B$
- or $\gamma((s - \delta_s, s + \delta_s)) \subseteq C$

let $U_s = (s - \delta_s, s + \delta_s)$
then $\{U_s\}_s$ is an open cover of $[0, 1]$
so it has a finite subcover $\{U_{s_j}\}_j$

[draw]

pick t_j in U_{s_j} cap $U_{s_{j + 1}}$ for all j

$$0 = t_0 < t_1 < \dots < t_N = 1$$

s.t. for all j, either $\gamma([t_j, t_{j + 1}])$ sub B
or $\gamma([t_j, t_{j + 1}])$ sub C

path-homotope the sub-paths in C
to sub-paths inside B cap C

Pf of Thm 1 will use covering map

$$\mathbb{R} \text{ to } S^1$$

for now, just mention: an explicit iso

$$\Phi : \mathbb{Z} \text{ to } \pi_1(S^1, (1, 0))$$

given by $\Phi(n) = [\omega_n]$,
where $\omega_n : [0, 1]$ to S^1 is defined by

$$\omega_n(s) = (\cos(2\pi ns), \sin(2\pi ns))$$

claiming: Φ is a homomorphism,
i.e., $[\omega_{m + n}] = [\omega_m * \omega_n]$
Φ is bijective

Q what is π_1(solid donut)?

A solid donut $\sim S^1$, so $\pi_1 \simeq \mathbb{Z}$

Q what is π_1(hollow donut)?

A hollow donut $\sim S^1 \times S^1$

PS5, #6(1): $\pi_1(X \times Y) \simeq \pi_1(X) \times \pi_1(Y)$

thus $\pi_1(S^1 \times S^1) \simeq \mathbb{Z} \times \mathbb{Z}$

Q what is π_1(two-holed solid donut)?