<u>Warmup/Interlude</u> recall: gen'lized μ-eigenspace
$\bigcup_n \ker((T - \mu)^n)$

three cases where V is a single gen'lized eigensp.

$$\begin{matrix} \mu & & \\ & \mu & \\ & & \mu \end{matrix} \qquad \begin{matrix} \mu & & \\ & \mu & 1 \\ & & \mu \end{matrix} \qquad \begin{matrix} \mu & & 1 \\ & \mu & 1 \\ & & \mu \end{matrix}$$

what's the dim of the actual μ-eigenspace?

3 2 2

smallest k s.t. $\ker((T - \mu)^k) = V$?

1 2 3

in general: $V = \sum_i W_i$
for gen'lized eigenspaces $W_1, \ldots, W_\ell$
with eigenvals $\lambda_1, \ldots, \lambda_\ell$

for each, k_i s.t. $\ker((T - \lambda_i)^k) = W_i$
=
k_i s.t. $(T - \lambda_i)|_{W_i}^k$ is zero

in this case $k_i \leq \dim W_i$

<u>Thm</u> above, $\prod_i (T - \lambda_i)^{k_i}$ is zero on V

<u>Pf</u> if $V = \sum_i W_i$, each W_i is T-stable,
and $T|_{W_i}$ is zero
then T is zero on V

(Axler, §5B)

Df the minimal polynomial of T is
the monic poly $p(z)$ of minimal deg s.t.
$p(T)$ is zero
denoted minpoly_T(z) in C[z]

(monic = the coeff of the highest power of z is 1)

Cor if $F = C$ and V is fin. dim., then
1) minpoly_T(z) exists
2) deg minpoly_T(z) $\leq$ dim V

Ex if T = id_V, then minpoly_T(z) = $z - 1$

Ex if T = zero op, then minpoly_T(z) = z

Rem these are examples where deg < dim V

Rem min poly still exists with deg $\leq$ dim V
even when $F = R$
(will defer proof to HW or later lecture)

(Axler, §8A) rest of today: structure of T|_W
for gen'lized eigenspace W
with eigenval λ
observe: T|_W – λ is nilpotent

Thm for any F and W fin. dim.:
if $S : W$ to W is nilpotent
then S has a matrix with
1's and 0's on the super-diagonal
0's everywhere else

<u>Cor</u> $T|_W$ has a matrix with

1's and 0's on the super-diagonal

λ's on the diagonal

0's everywhere else

e.g.

$$\begin{matrix} \lambda & 1 & & & & \\ & \lambda & & & & \\ & & \lambda & & & \\ & & & \lambda & 1 & \\ & & & & \lambda & 1 \\ & & & & & \lambda \end{matrix}$$

above: ↑ ↑ ↑ called <u>Jordan blocks</u>

each Jordan block J corresponds to
a T-stable linear subspace W_J sub W

W_J has a basis $e_1, \ldots, e_d$ s.t.

$T|_{W_J} e_1 = \lambda e_1$

$T|_{W_J} e_i = \lambda e_i + e_{i-1}$ for $1 < i \leq d$

<u>Df</u> a sequence $e_d \to \ldots \to e_1$ of max len
obeying these identities is called
a Jordan chain for T of length d
with eigenval λ

thus: T has a Jordan chain of len d, eigenval λ
iff
T has a matrix with a $d \times d$ Jordan block
with diagonal entries all λ

[now, can restate thm for nilpotent ops
in terms of Jordan chains]

<u>Thm</u> for any F and W fin. dim.:
if $S : W$ to W is nilpotent
then there is a basis for W that is
a disj union of Jordan chains for S,
all with eigenval 0

intuition:

$$\begin{pmatrix} 0 & 1 & & & & \\ & 0 & & & & \\ & & 0 & & & \\ & & & 0 & 1 & \\ & & & & 0 & 1 \\ & & & & & 0 \end{pmatrix}$$

Jordan chains: $e_2 \to e_1$,
e_3,
$e_6 \to e_5 \to e_4$

$\mathrm{im}(S) = \mathrm{span}(e_1, e_4, e_5)$

$\mathrm{im}(S)$ is T-stable:
Jordan chains for $T_{\mathrm{im}(S)}$: e_1
$e_5 \to e_4$

just need two steps to bootstrap from $\mathrm{im}(S)$ to W:
1) extend each Jordan chain in $\mathrm{im}(S)$ by one elt
2) add a Jordan chain of length 1
for each e_i in $\ker(S)$ s.t. $e_i \notin \mathrm{im}(S)$

then check that the result is indeed a basis for W

above: 1) adds e_2, e_6
2) adds e_3

<u>Pf</u> if $W = \{\mathbf{0}\}$, then done
induct on dim W

else: $\ker(S) \neq \{\mathbf{0}\}$ as some power of S is zero
so $\dim \operatorname{im}(S) < \dim W$
and $\operatorname{im}(S)$ is S-stable (by stability lem)
so by the inductive hypothesis,
im(S) has a basis that is a disj union
of Jordan chains all with eigenval 0

say,
$$w_1 \to S w_1 \to \dots \to S^{d_1} w_1,$$
$$w_2 \to S w_2 \to \dots \to S^{d_2} w_2,$$
$$\dots$$
$$w_s \to S w_s \to \dots \to S^{d_s} w_s$$

for some $w_1, w_2, \dots, w_s$ in $\operatorname{im}(S)$

note: since this is a basis for $\operatorname{im}(S)$,
$\{S^{d_1} w_1, \dots, S^{d_s} w_s\}$ is
a basis for $\ker(S|_{\operatorname{im}(S)})$
[why? linearly independent and span]

[steps described earlier:]

1) pick $f_1, \dots, f_s$ in W s.t. $Se_i = w_i$
2) extend above basis for $\ker(S|_{\operatorname{im}(S)})$ to a basis for $\ker(S)$

say, $\{S^{d_1} w_1, \dots, S^{d_s} w_s, u_1, \dots, u_t\}$

claim: $\bigcup_i \{f_i, S f_i, \dots, S^{d_1 + 1} f_i\}$
$\cup \{u_1, \dots, u_t\}$
is a basis for W

[the $S^k f_i$'s give the Jordan chains of len ≥ 2]
[the u_j's give the Jordan chains of len 1]

by PS3, #1, suffices to show:

- # of vectors = dim W
- linear independence

# of vectors: $(d_1 + 2) + \ldots + (d_s + 2) + t$
$= (d_1 + 1) + \ldots + (d_s + 1) + s + t$
$= \dim \mathrm{im}(S) + s + t$
$= \dim \mathrm{im}(S) + \dim \ker(S)$
$= \dim W$

linear independence: pick $a_{i, k}$, b_j s.t.

$$\sum_{i, k} a_{i, k} S^k f_i + \sum_j b_j u_j = \mathbf{0}$$

applying S to both sides:

$$\sum_{i, k} a_{i, k} S^k w_i + \sum_j b_j \mathbf{0} = \mathbf{0}$$

so by inductive hypothesis, $a_{i, k} = 0$ for all i, k
so now, $\sum_j b_j u_j = \mathbf{0}$
but the u_j's were constructed as part of a basis
so $b_j = 0$ for all j □