Warmup

Df X is Hausdorff iff, for all $x \neq y$ in X, there are disjoint opens U, V s.t. x in U and y in V

in which topologies is R Hausdorff?

- analytic [yes]
- discrete [yes]
- indiscrete [no]
- finite complement [no]
- T_ℓ [yes]

[we build more non-Hausdorff spaces by gluing]

Ex let $X = R \times \{a, b\}$ in the analytic top

let ~ be the equivalence relation on X where:

1) ~ is reflexive and symmetric
2) $(x, a) \sim (x, b)$ for all $x \neq 0$
3) $(0, a) \nsim (0, b)$

let A = X/~, the corresponding quotient space

then points of A can be labeled:

- x for real $x \neq 0$
- (0, a), (0, b)

[draw picture]

[what are some open sets in A containing (0 a)?]

A is not Hausdorff because of (0, a) and (0, b)

(Munkres §17, 21) for A sub X, recall:

interior Int_X(A)

= {a in A | have open U s.t. y in U sub A}

[largest open set of X contained inside A]

closure Cl_X(A)

= X – Int_X(X – A) [unpack this]

= X – {x | have open U s.t. x in U sub X – A}

= {x in X | for all open U ni x, U intersects A}

[smallest closed set of X containing A]

Int(A) sub A sub Cl(A)

<u>Rem</u> suffices to check conditions on a basis

[Int does not play well with subspaces]

<u>Ex</u> let X = analytic R and A = [0, 1]

Int_X(A) = (0, 1)

Int_A(A) = [0, 1]

[nonetheless:]

<u>Prop</u> for any Y sub X, still true that

Int_Y(A cap Y) supset Int_X(A) cap Y

<u>Pf</u> Int_X(A) open in X

so Int_X(A) cap Y open in Y

but also A cap Y subset Int_X(A) cap Y

so Int_Y(A cap Y) supset Int_X(A) cap Y

Cor for any Y sub X, true that
Cl_Y(A cap Y) sub Cl_X(A) cap Y

Pf let B = X – A
Cl_X(A) = X – Int_X(B)
Cl_X(A) cap Y = Y – (Int_X(B) cap Y)
Cl_Y(A cap Y) = Y – Int_Y(B cap Y)

thus Cl_Y(A cap Y) sub Cl_X(A) cap Y

[further case where Cl does not play well with sub]

Ex let X = analytic R and Y = [0, 1]
let A = (1, 2)
Cl_Y(A cap Y) = Ø
Cl_X(A) cap Y = {1}

[another salvage:] Munkres Thm 17.4:

Thm if A sub Y, then get equality:
Cl_Y(A) = Cl_X(A) cap Y

Rem no analogous thm for Int!:
saw that in general, Int_A(A) ≠ Int_X(A)

Pf remains to show
Cl_Y(A) supset Cl_X(A) cap Y

pick y in Cl_X(A) cap Y
want to show y in Cl_Y(A): i.e.,
any open V sub Y containing y intersxts A
know V = U cap Y for some open U sub X
but U intersxts A, and A sub Y, so V intersxts A □

Summary

1) $\mathrm{Int}_Y(A \cap Y) \supset \mathrm{Int}_X(A) \cap Y$
2) $\mathrm{Cl}_Y(A \cap Y) \subset \mathrm{Cl}_X(A) \cap Y$
3) if $A \subset Y$, then $\mathrm{Cl}_Y(A) = \mathrm{Cl}_X(A) \cap Y$

Df we say x is a limit point of A in X iff $x \in \mathrm{Cl}_X(A - \{x\})$

equivalently, every open set of X containing x intersects A in a point distinct from x

Df we say $x_1, x_2, \ldots$ converges to x iff, for all open U containing x,
there is N s.t. $x_n \in U$ for all $n \geq N$

Ex $X = \mathbb{R}$ and $K = \{1/n \mid n = 1, 2, 3, \ldots\}$
[draw picture]

$\mathrm{Cl}_X(K) = \{0\} \cup K$; 0 is the only limit point

Ex $X = \mathbb{R}$ and $I = [0, 1]$
[draw picture]

$\mathrm{Cl}_X(I) = I$; every point of I is a limit point

Thm in general,
$\mathrm{Cl}_X(A) = A \cup \{\text{limit points of } A \text{ in } X\}$

Pf boring

Q how does our intuition for limit pts fail beyond the analytic topology?

Ex recall $X = R \times \{a, b\}$ and $A = X/\sim$ where $(x, a) \sim (x, b)$ for all $x \neq 0$
[draw picture]

here a sequence of pts can converge to two distinct limit pts simultaneously

[the Hausdorff condition rules this out]

Thm if X is Hausdorff
then a sequence of pts in X converges to at most one limit pt

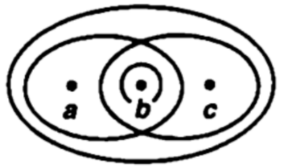
Pf suppose $(x_n)_n$ converges to x

want to show:
if $y \neq x$, then $(x_n)_n$ cannot converge to y
pick disjoint open U, V s.t. x in U and y in V
then there is N s.t. $n \geq N$ forces x_n in U
hence $n \geq N$ forces x_n notin V

Hierarchy of Separation Conditions

Hausdorff	for all $x \neq y$, disjoint open U, V s.t. x in U and y in V
T_1	for all $x \neq y$, open U s.t. x in U and y notin U
T_0	for all $x \neq y$, have open U s.t. either x in U, y notin U or vice versa

Ex T_0 but not T_1:



Munkres Thm 17.9:

Thm if X is T_1 and A sub X
then x is a limit point of A iff
every open U containing x contains
infinitely many points of A