ANNULAR WEBS AND A CONJECTURE OF HAIMAN

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ABSTRACT. Haiman conjectured that when traces corresponding to monomial symmetric functions are evaluated on the Hecke-algebra elements denoted C'_w by Kazhdan–Lusztig, the resulting polynomials have nonnegative coefficients. We show that recent work on annular webs implies this for permutations w that are 321-hexagon-avoiding.

1. INTRODUCTION

1.1. Let $H_n(x)$ be the Iwahori–Hecke algebra of the symmetric group S_n over $\mathbf{Z}[x^{\pm 1}]$. As a quotient of the group algebra of a braid group, it has a standard basis $\{\sigma_w\}_{w\in S_n}$, consisting of the images of the positive permutation braids.

Kazhdan–Lusztig introduced two new bases for $H_n(x)$ with remarkable properties [KL79]. Taking our x to be their $q^{1/2}$, we will focus on the basis that they denote by $\{C'_w\}_w$, but write b_w in place of C'_w for simplicity. When the elements b_w are expanded in the standard basis, the coefficients are Laurent polynomials in x with nonnegative integer coefficients. Up to rescaling, these are the celebrated Kazhdan–Lusztig polynomials for S_n. Their positivity can be proved through a geometric interpretation of $H_n(x)$ in terms of sheaves on flag varieties.

The representation theory of S_n deforms to that of $H_n(x)$. In particular, each character $\chi : S_n \to \bar{\mathbf{Q}}$ defines a $\mathbf{Z}[x^{\pm 1}]$-linear function $\chi_x : H_n(x) \to \overline{\mathbf{Q}(x)}$ that still enjoys the trace property $\chi(\alpha\beta) = \chi(\beta\alpha)$. At the same time, the *irreducible* characters of S_n are indexed by integer partitions of n. Let χ^λ be the irreducible character indexed by $\lambda \vdash n$. A geometric argument, similar to that used in the positivity of the Kazhdan–Lusztig polynomials, proves that $\chi_x^\lambda(b_w) \in \mathbf{Z}_{\geq 0}[x^{\pm 1}]$ for all w and λ.

Haiman found evidence for a stronger positivity statement. Recall that for all λ, μ, the Kostka number $K_{\lambda,\mu}$ counts semistandard Young tableaux of shape λ and weight μ. The Kostka numbers can be assembled into a unitriangular matrix of nonnegative integers. In particular, this matrix has an inverse with integer entries, so there are functions $\phi_x^\mu : H_n(x) \to \mathbf{Z}[x^{\pm 1}]$ uniquely defined by requiring

$$\chi_x^\lambda = \sum_\mu K_{\lambda,\mu}\phi_x^\mu \quad \text{for all } \lambda \vdash n. \tag{1.1}$$

What follows is the main part of Conjecture 2.1 in [Hai93].

Conjecture 1.1 (Haiman)**.** *$\phi_x^\mu(b_w) \in \mathbf{Z}_{\geq 0}[x^{\pm 1}]$ for all $w \in S_n$ and $\mu \vdash n$.*

Abreu–Nigro observe that Conjecture 1.1 would imply several conjectures about the indifference graphs of Hessenberg functions in algebraic combinatorics: notably, the Stanley–Stembridge conjecture on the e-positivity of their chromatic symmetric functions, and Shareshian–Wachs's generalization of this conjecture to chromatic quasi-symmetric functions [AN24].

1.2. This note will show how recent work of Queffelec–Rose and Gorsky–Wedrich on the diagrammatics of $H_n(x)$ solves some cases of Conjecture 1.1.

For $1 \leq i \leq n-1$, let $b_i = b_{s_i}$, where $s_i \in S_n$ is the transposition that swaps i and $i+1$. The main theorem is:

Theorem 1.2. *$\phi_x^\mu(b_{i_1} \cdots b_{i_\ell}) \in \mathbf{Z}_{\geq 0}[x^{\pm 1}]$ for any sequence of indices $i_1, \dots, i_\ell$ that range between 1 and $n-1$ inclusive, and any $\mu \vdash n$.*

In what follows, suppose that $w \in S_n$ is given by $w = [w_1 w_2 \cdots w_n]$, meaning it sends i to w_i for $1 \leq i \leq n$. Fix $m \leq n$ and $v = [v_1 v_2 \cdots v_m] \in S_m$. We say that w is *$v_1 \cdots v_m$-avoiding* if and only if the sequence $(w_1, \dots, w_n)$ does *not* contain a subsequence of size m whose elements have the same relative order as $(v_1, \dots, v_m)$. More formally, this means we cannot find indices $1 \leq p_1 < \cdots < p_m \leq n$ such that $w_{p_i} < w_{p_j}$ whenever $i < j$ and $v_i < v_j$.

We write $S_n^{v_1 \cdots v_m} \subseteq S_n$ for the set of $v_1 \cdots v_m$-avoiding elements. Following Billey–Warrington, we say that w is *321-hexagon-avoiding* if and only if

$$w \in S_n^{321} \cap S_n^{46718235} \cap S_n^{46781235} \cap S_n^{56718234} \cap S_n^{56781234}.$$

In [BW01], Billey–Warrington prove that w is 321-hexagon-avoiding if and only if we have $b_w = b_{i_1} \cdots b_{i_\ell}$ whenever $w = s_{i_1} \cdots s_{i_\ell}$ and ℓ is the minimal length among such expressions. Via this result, Theorem 1.2 implies:

Corollary 1.3. *Conjecture 1.1 holds when w is 321-hexagon-avoiding.*

1.3. The key observation is that Remark 4.21 of [GW23], a refinement of the annular web evaluation algorithm of [QR18], provides a counterpart to Theorem 1.2 (in fact, a slightly stronger statement) in the setting of Murakami–Ohtsuki–Yamada (MOY) webs [MOY98, CKM14]. The passage from Hecke-algebra traces to web diagrammatics is best explained by assembling the cocenters of all the Hecke algebras into a direct sum that we identify with Macdonald's ring of symmetric functions $\Lambda(x)$ over $\mathbf{Z}[x^{\pm 1}]$ [Mac15], after extending scalars. There is a universal trace

$$\mathsf{tr} : \bigoplus_n H_n(x) \to \Lambda(x).$$

There is also a natural candidate for the diagrammatic counterpart to tr: the map ann that sends a rectangular web to its annular closure.

For any $\beta \in H_n(x)$, the value of $\phi_x^\mu(\beta)$ is just the μ-th coefficient when we expand $\mathsf{tr}(\beta)$ in the basis of complete homogeneous symmetric functions $\{h_\mu\}_\mu$: a fact already noted in [AN24]. Ultimately, we relate Theorem 1.2 to [GW23, Rem. 4.21] through a commutative diagram that relates tr to ann, and assigns simple webs to the b_i and h_μ.

In fact, there is another, inequivalent commutative diagram, where $\{b_w\}_w$ is replaced by Kazhdan–Lusztig's *other* basis for the Hecke algebra, and $\{h_\mu\}_\mu$ is replaced by the basis of elementary symmetric functions $\{e_\mu\}_\mu$. We present both diagrams together in Theorem 4.3. We prove their commutativity by comparing the compatibility of quantum Schur–Weyl duality with parabolic induction with the analogous property for two, mutually dual versions of the MOY web calculus: h_μ corresponds to the *symmetric* version where strand labels are symmetric powers, while e_μ corresponds to the *anti-symmetric* version where they are exterior powers. The original formalism of [MOY98] is the anti-symmetric one.

We have not found any explicit prior statement of Theorem 4.3 in the literature, though it seems to be folklore. Lemma 4.25 and Remark 4.26 of [GW23] establish closely related statements. We show that either of our commutative diagrams can be deduced from the other via a more general statement, Proposition 5.1, that holds for any finite Coxeter group. In particular, our treatment is *not* compatible with [RW20], where the skein relations for the symmetric and anti-symmetric web calculi are inconsistent.

1.4. **Acknowledgments.** I thank Elijah Bodish for many discussions that helped me to understand webs and the research literature on them. I also thank Mikhail Khovanov and Paul Wedrich for useful comments.

2. Hecke Algebras

2.1. This section reviews background that applies to any finite Coxeter group W with system of simple reflections S. Let $<$ denote the Bruhat order on W, and for any $w \in W$, let ℓ_w denote the Bruhat length of w [GP00, Ch. 1].

2.2. Formally, we define the *Iwahori–Hecke algebra* of W to be the $\mathbf{Z}[x^{\pm 1}]$-algebra $H_W(x)$ spanned as a free module by elements σ_w for $w \in W$, modulo the following relations:

$$\sigma_w \sigma_s = \begin{cases} \sigma_{ws} & ws > w, \\ \sigma_{ws} + (x - x^{-1})\sigma_w & ws < w. \end{cases} \tag{2.1}$$

Let D be the additive involution of $H_W(x)$ that sends $x \mapsto x^{-1}$ and $\sigma_w \mapsto \sigma_{w^{-1}}^{-1}$ for all $w \in W$.

2.3. Let $\mathbf{K} = \mathbf{F}(x)$, where $\mathbf{F} \supseteq \mathbf{Q}$ is a splitting field for W. If $W = S_n$, then we can take $\mathbf{F} = \mathbf{Q}$.

It turns out that $\mathbf{K}H_W(x) := \mathbf{K} \otimes_{\mathbf{Z}[x^{\pm 1}]} H_W(x)$ is split as a $\mathbf{K}$-algebra [GP00, Thm. 9.3.5]. At the same time, there is an isomorphism of rings $H_W(x)|_{x \to 1} \simeq \mathbf{Z}W$. So by Tits deformation [GP00, §7.4], the semisimplicity of $\mathbf{F}W$ implies the semisimplicity of $\mathbf{K}H_W(x)$, and moreover, there is a bijection between isomorphism classes of simple $\mathbf{K}H_W(x)$-modules and those of simple $\mathbf{F}W$-modules, compatible with the $x \to 1$ specialization from $H_W(x)$ to $\mathbf{Z}W$.

This induces the assignment from characters χ of W to $\mathbf{Z}[x^{\pm 1}]$-linear trace functions $\chi_x : H_W(x) \to \bar{\mathbf{K}}$ mentioned in the introduction. Explicitly, $\chi_x(\beta)$ is the trace of β on the $\mathbf{K}H_W(x)$-module that corresponds to the $\mathbf{F}W$-module with character χ.

2.4. Kazhdan–Lusztig proved that for all $w \in W$, there is a unique D-invariant element $b_w \in H_W(x)$ such that

$$b_w = \sum_{y \leq w} x^{\ell_y - \ell_w} P_{y,w}(x^2)\sigma_y$$

for some $P_{y,w}(q) \in \mathbf{Z}[q]$ satisfying

$$\begin{aligned} P_{w,w}(q) &= 1, \\ \deg P_{y,w}(q) &\leq \tfrac{1}{2}(\ell_w - \ell_y - 1) \quad \text{for all } w, y \in W \text{ with } y \leq w. \end{aligned} \tag{2.2}$$

Let j be the additive involution of $H_W(x)$ that sends $x \mapsto x^{-1}$ and $\sigma_w \mapsto (-1)^{\ell_w}\sigma_w$. Let $c_w = j(b_w)$. Then c_w is the unique D-invariant element of $H_W(x)$ such that

$$c_w = \sum_{y \leq w} (-1)^{\ell_y} x^{\ell_w - \ell_y} P_{y,w}(x^{-2})\sigma_y$$

for some $P_{y,w}(q) \in \mathbf{Z}[q]$ satisfying (2.2). They turn out to be the same polynomials as before.

The sets $\{b_w\}_{w \in W}$ and $\{c_w\}_{w \in W}$ form bases for $H_W(x)$ as a free $\mathbf{Z}[x^{\pm 1}]$-module, known as the two *Kazhdan–Lusztig bases* or *canonical bases*. The polynomials $P_{y,w}(q)$ are the *Kazhdan–Lusztig polynomials* for W. Note that in [KL79], b_w and c_w are respectively denoted C'_w and $-C_w$. (Also note the minus sign.)

2.5. For any $s \in S$, we have

$$\begin{aligned} b_s &= x^{-1} + \sigma_s &&= x + \sigma_s^{-1}, \\ c_s &= x - \sigma_s &&= x^{-1} - \sigma_s^{-1}. \end{aligned} \tag{2.3}$$

Just as $\{\sigma_s\}_{s \in S}$ generates $H_W(x)$ as a $\mathbf{Z}[x^{\pm 1}]$-algebra, so do $\{b_s\}_s$ and $\{c_s\}_s$. In fact, one can check using (2.1) and (2.3) that whether we take $\gamma_s = b_s$ for all s or take $\gamma_s = c_s$ for all s, the defining relations of $H_n(x)$ with respect to the generating set $\{\gamma_s\}_s$ remain the same.

Let η be the involution of $H_n(x)$ as a $\mathbf{Z}[x^{\pm 1}]$-algebra defined by swapping b_s and c_s for all $s \in S$. This is different from j, since j is not $\mathbf{Z}[x^{\pm 1}]$-linear.

2.6. In the next two sections, we will focus on $W = S_n$. Here, we will always take $S = \{s_1, s_2, \ldots, s_{n-1}\}$, where $s_i = (i, i+1)$ as in the introduction.

We will also write $H_n(x)$ in place of $H_{S_n}(x)$, and write σ_i, b_i, c_i in place of $\sigma_{s_i}, b_{s_i}, c_{s_i}$. Whether we take $\gamma_i = b_i$ or $\gamma_i = c_i$, the defining relations of $H_n(x)$ with respect to the generating set $\{\gamma_i\}_i$ are:

$$\begin{cases} \gamma_i\gamma_{i+1}\gamma_i - \gamma_i = \gamma_{i+1}\gamma_i\gamma_{i+1} - \gamma_{i+1}, \\ \gamma_i\gamma_j = \gamma_j\gamma_i & \text{for } |i-j| > 1, \\ \gamma_i^2 = (x + x^{-1})\gamma_i. \end{cases} \tag{2.4}$$

3. Symmetric Functions

3.1. Let Λ be the graded ring of symmetric functions over $\mathbf{Z}$ in (countably) infinitely many variables. For background on Λ, we refer to [Mac15, Ch. I]. In this note, we will need the following elements of Λ indexed by integer partitions λ:

the Schur functions	s_λ,
the monomial symmetric functions	$m_\lambda = m_{\lambda_1} m_{\lambda_2} \ldots$,
the complete homogeneous symmetric functions	$h_\lambda = h_{\lambda_1} h_{\lambda_2} \ldots$,
the elementary symmetric functions	$e_\lambda = e_{\lambda_1} e_{\lambda_2} \ldots$

Let Λ_n be the degree-n component of Λ. The Schur functions s_λ with $\lambda \vdash n$ form a basis for Λ as a free $\mathbf{Z}$-module. The same holds with m_λ or h_λ or e_λ in place of s_λ.

3.2. Recall the Kostka numbers $K_{\lambda,\mu} \in \mathbf{Z}$ from the introduction. As explained in [Mac15, §I.6], they relate the elements s_λ, m_μ, h_μ via the identities

$$s_\lambda = \sum_\mu K_{\lambda,\mu} m_\mu, \tag{3.1}$$

$$h_\mu = \sum_\lambda K_{\lambda,\mu} s_\lambda. \tag{3.2}$$

Comparing (1.1) to (3.1) shows the analogy: Haiman's character ϕ_x^μ is to m_μ as the irreducible character χ_x^λ is to s_λ.

Note that $K_{\lambda,\lambda} = 1$ for all λ, and that $K_{\lambda,\mu} = 0$ whenever $\mu > \lambda$ in the dominance order on partitions. This makes precise the unitriangularity mentioned earlier.

3.3. Let $\Lambda(x) = \mathbf{Z}[x^{\pm 1}] \otimes \Lambda$ and $\Lambda_n(x) = \mathbf{Z}[x^{\pm 1}] \otimes \Lambda_n$ for all n. The map tr mentioned in §1.3 is the sum of the $\mathbf{Z}[x^{\pm 1}]$-linear maps

$$\mathsf{tr}_n : H_n(x) \to \Lambda_n(x) \quad \text{defined by } \mathsf{tr}_n = \sum_{\lambda \vdash n} \chi_x^\lambda s_\lambda.$$

By construction, $\mathsf{tr}_n(\alpha\beta) = \mathsf{tr}_n(\beta\alpha)$ for all α, β. So the universal property of the cocenter of $H_n(x)$ defines a $\mathbf{Z}[x^{\pm 1}]$-linear map from the cocenter into $\Lambda(x)$, which turns out to be an isomorphism of $\mathbf{Z}[x^{\pm 1}]$-modules.

Let $\langle -, - \rangle : \Lambda(x) \times \Lambda(x) \to \mathbf{Z}[x^{\pm 1}]$ be the *Hall pairing*: the $\mathbf{Z}[x^{\pm 1}]$-linear pairing under which the Schur functions s_λ are orthonormal. By (1.1) and (3.2), we deduce:

Lemma 3.1. $\mathsf{tr}_n = \displaystyle\sum_{\mu \vdash n} \phi_x^\mu h_\mu$.

Altogether, Theorem 1.2 is claiming that for any sequence of indices $i_1, \ldots, i_\ell$ that range between 1 and $n-1$ inclusive, the expansion of $\mathsf{tr}_n(b_{i_1} \cdots b_{i_\ell})$ in the complete homogeneous basis of $\Lambda_n(x)$ will have coefficients in $\mathbf{Z}_{\geq 0}[x^{\pm 1}]$.

4. Webs

4.1. Let $R_n^{\mathsf{web}}(x)$ be the free $\mathbf{Z}[x^{\pm 1}]$-module generated by (isotopy classes of) strictly upward-oriented web diagrams in a rectangle, connecting n inputs with label 1 at the bottom to n outputs with label 1 at the top, modulo the relations of the $\mathbf{Z}[x^{\pm 1}]$-valued MOY bracket: that is, the bracket in [MOY98], except that our x is their $q^{1/2}$. Then $R_n^{\mathsf{web}}(x)$ forms a $\mathbf{Z}[x^{\pm 1}]$-algebra under concatenation of diagrams.

For $1 \leq i \leq n-1$, let $\mathsf{can}_i \in R_n^{\mathsf{web}}(x)$ denote the ith merge-split web. The notation can is intended to suggest *canonical*. Comparing (2.4) to Lemmas 2.2 and

2.5 of [MOY98], we obtain $\mathbf{Z}[x^{\pm 1}]$-algebra homomorphisms

$$\Theta_b, \Theta_c : H_n(x) \to R_n^{\mathsf{web}}(x) \quad \text{defined by } \Theta_b(b_i) = \Theta_c(c_i) = \mathsf{can}_i.$$

Note that $\Theta_c = \Theta_b \circ \eta$, where η is the $\mathbf{Z}[x^{\pm 1}]$-algebra involution from §2.5.

Remark 4.1. Recall that we can identify $H_n(x)$ with the skein algebra of a rectangle with n inputs and n outputs, by sending σ_i to the simple twist of the ith and $(i+1)$th strands. The skein relation in [MOY98, §3] defines a homomorphism from this skein algebra into $R_n^{\mathsf{web}}(x)$ that corresponds to our map Θ_c.

Meanwhile, the (rectangular analogue of) formula (17) in [GW23] decategorifies to a skein relation that corresponds to our map Θ_b. Their formula (16) does not quite correspond to Θ_c, even up to rescaling.

4.2. Let $R_{\bigcirc}^{\mathsf{web}}(x)$ be the free $\mathbf{Z}[x^{\pm 1}]$-module generated by positively-oriented web diagrams in an annulus. It forms a commutative $\mathbf{Z}[x^{\pm 1}]$-algebra under nesting of diagrams.

For any n and $\mu \vdash n$, let $o^\mu \in \Lambda^{\mathsf{web}}(x)$ be the diagram consisting of concentric essential circles with labels $\mu_1, \mu_2, \ldots$ Note that by the commutativity of $\Lambda^{\mathsf{web}}(x)$, the order of these circles does not matter. The *annular web evaluation algorithm* of Queffelec–Rose [QR18, Lem. 5.2] shows that the set $\bigcup_n \{o_\mu\}_{\mu \vdash n}$ forms a basis for $\Lambda^{\mathsf{web}}(x)$ as a free $\mathbf{Z}[x^{\pm 1}]$-module. The definition of the MOY bracket then implies, directly, that $\Lambda^{\mathsf{web}}(x)$ is freely generated *as an algebra* by the elements $o_n := o_{(n)}$ corresponding to single, labeled essential circles.

At the same time, display (2.4), *resp.* (2.8), in [Mac15] implies that $\Lambda(x)$ is freely generated as an algebra by the set $\{e_n\}_n$, *resp.* the set $\{h_n\}_n$. Thus, there are $\mathbf{Z}[x^{\pm 1}]$-algebra isomorphisms

$$\Xi_h, \Xi_e : \Lambda(x) \xrightarrow{\sim} \Lambda^{\mathsf{web}}(x) \quad \text{defined by } \Xi_e(e_\mu) = \Xi_h(h_\mu) = o_\mu.$$

They differ precisely by the $\mathbf{Z}[x^{\pm 1}]$-algebra involution of $\Lambda(x)$ that swaps h_μ and e_μ.

Prior to the introduction of webs, the analogous isomorphism onto the skein algebra of the annulus was first established by Turaev [Tur88].

4.3. Queffelec–Rose's annular web evaluation algorithm originally treated $\Lambda^{\mathsf{web}}(x)$ as the triangulated Grothendieck group of the bounded homotopy category of a graded, linear category of foams between positively-oriented annular webs. Gorsky–Wedrich observed that it could be refined, by instead treating $\Lambda^{\mathsf{web}}(x)$ as the additive Grothendieck group of the *Karoubi* or *idempotent completion* of this foam category [GW23, Rem. 4.21]. The refinement shows:

Theorem 4.2 (Queffelec–Rose + Gorsky–Wedrich). *The expansion of any annular web in the basis $\{o_\mu\}_\mu$ for $\Lambda^{\mathsf{web}}(x)$ will have coefficients in $\mathbf{Z}_{\geq 0}[x^{\pm 1}]$.*

4.4. There is a $\mathbf{Z}[x^{\pm 1}]$-linear map

$$\mathsf{ann} : \bigoplus_n R_n^{\mathsf{web}}(x) \to \Lambda^{\mathsf{web}}(x)$$

called *annular closure*. It is defined graphically, by embedding a rectangle into an annulus as a sector, so that the upward orientation in the rectangle becomes the positive orientation in the annulus, then wrapping the n outputs of the rectangle around the annulus, without crossing, back to the n inputs. For more on annular closure in the skein literature, see [MM08] and the references there.

Via Lemma 3.1 and Theorem 4.2, we conclude that Theorem 1.2 follows from the commutativity of diagram (I) below.

Theorem 4.3. *The following diagrams commute:*

$$\text{(I)}\qquad \begin{array}{ccc} H_n(x) & \xrightarrow{\mathsf{tr}} & \Lambda_n(x) \\ {\scriptstyle \Theta_b}\big\downarrow & & \big\downarrow{\scriptstyle \Xi_h} \\ R_n^{\mathsf{web}}(x) & \xrightarrow{\mathsf{ann}} & \Lambda^{\mathsf{web}}(x) \end{array}$$

$$\text{(II)}\qquad \begin{array}{ccc} H_n(x) & \xrightarrow{\mathsf{tr}} & \Lambda_n(x) \\ {\scriptstyle \Theta_c}\big\downarrow & & \big\downarrow{\scriptstyle \Xi_e} \\ R_n^{\mathsf{web}}(x) & \xrightarrow{\mathsf{ann}} & \Lambda^{\mathsf{web}}(x) \end{array}$$

Below, we handle (I) and (II) in parallel. As an alternative, we show in §5.2 how the commutativity of either diagram implies that of the other.

Proof. For convenience, set $(\star, \diamondsuit, \heartsuit) \in \{(\mathrm{I}, b, h), (\mathrm{II}, c, e)\}$.

Step 1. First, we reduce to checking specific central elements of $H_n(x)$. In what follows, $\mathsf{Z}(-)$ and $[-]$ denote center and cocenter, respectively. Let $\mathbf{K} = \mathbf{Q}(x)$. The map $\Theta_\diamondsuit$ induces $\mathbf{K}$-linear homomorphisms

$$\Theta_\diamondsuit : \mathsf{Z}(\mathbf{K}H_n(x)) \to \mathsf{Z}(\mathbf{K}R_n^{\mathsf{web}}(x)) \quad \text{and} \quad [\Theta_\diamondsuit] : [\mathbf{K}H_n(x)] \to [\mathbf{K}R_n^{\mathsf{web}}(x)].$$

It is enough to show the commutativity of diagram $(\star)$ where we extend scalars to $\mathbf{K}$ and replace $H_n(x)$, $R_n^{\mathsf{web}}(x)$, $\Theta_\diamondsuit$ with $[\mathbf{K}H_n(x)]$, $[\mathbf{K}R_n^{\mathsf{web}}(x)]$, $[\Theta_\diamondsuit]$. But the cocenters of $\mathbf{K}H_n(x)$ and $\mathbf{K}R_n^{\mathsf{web}}(x)$ are isomorphic to their centers. So it remains to show the commutativity of

$$\begin{array}{ccc} \mathsf{Z}(\mathbf{K}H_n(x)) & \xrightarrow{\mathsf{tr}} & \mathbf{K}\Lambda_n(x) \\ {\scriptstyle \Theta_\diamondsuit}\big\downarrow & & \big\downarrow{\scriptstyle \Xi_\heartsuit} \\ \mathsf{Z}(\mathbf{K}R_n^{\mathsf{web}}(x)) & \xrightarrow{\mathsf{ann}} & \mathbf{K}\Lambda^{\mathsf{web}}(x) \end{array}$$

where $\mathbf{K}\Lambda_n(x)$, $\mathbf{K}\Lambda^{\mathsf{web}}(x)$ are the $\mathbf{K}$-linear extensions of $\Lambda_n(x)$, $\Lambda^{\mathsf{web}}(x)$. As the top arrow is bijective, the basis $\{\heartsuit_\mu\}_\mu$ for $\mathbf{K}\Lambda_n(x)$ lifts to a basis $\{\heartsuit_\mu^\vee\}_\mu$ for $\mathsf{Z}(\mathbf{K}H_n(x))$. It will be convenient to write $\heartsuit_n^\vee := \heartsuit_{(n)}^\vee$ below. To show the commutativity of $(\star)$, we must show that $\mathsf{ann}(\Theta_\diamondsuit(\heartsuit_\mu^\vee)) = o_\mu$ for all μ.

Step 2. Next, we reduce to the case where μ is a trivial partition. Observe that for general $\mu = (\mu_1, \mu_2, \ldots, \mu_\ell)$, we have $\mathbf{K}$-linear inclusion maps

$$\begin{aligned} i_\mu &: \mathbf{K}H_{\mu_1}(x) \times \cdots \times \mathbf{K}H_{\mu_\ell}(x) \to \mathbf{K}H_n(x), \\ i_\mu^{\mathsf{web}} &: \mathbf{K}R_{\mu_1}^{\mathsf{web}}(x) \times \cdots \times \mathbf{K}R_{\mu_\ell}^{\mathsf{web}}(x) \to \mathbf{K}R_n^{\mathsf{web}}(x) \end{aligned}$$

compatible with $\Theta_\diamondsuit$. As we run over n and μ, the maps i_μ, *resp.* i_μ^{web}, endow the direct sum of the $\mathsf{Z}(\mathbf{K}H_n(x))$, *resp.* the $\mathsf{Z}(\mathbf{K}R_n^{\mathsf{web}}(x))$, with a new commutative algebra structure, for which

$$\bigoplus_n \mathsf{Z}(\mathbf{K}H_n(x)) \xrightarrow{\mathsf{tr}} \mathbf{K}\Lambda(x), \quad \textit{resp.} \quad \bigoplus_n \mathsf{Z}(\mathbf{K}R_n^{\mathsf{web}}(x)) \xrightarrow{\mathsf{ann}} \mathbf{K}\Lambda^{\mathsf{web}}(x),$$

is an algebra homomorphism.

Recall from §4.2 that $\Lambda(x) \xrightarrow{\Xi_\heartsuit} \Lambda^{\mathsf{web}}(x)$ is also an algebra homomorphism. Since $i_\mu(\heartsuit_{\mu_1}^\vee, \ldots, \heartsuit_{\mu_\ell}^\vee) = \heartsuit_\mu^\vee$, we deduce that if $\mathsf{ann}(\Theta_\diamondsuit(\heartsuit_{\mu_i}^\vee)) = o_{\mu_i}$ for all i, then $\mathsf{ann}(\Theta_\diamondsuit(\heartsuit_\mu^\vee)) = o_\mu$.

Step 3. By [MOY98, Lem. 2.6], there is a well-defined web $\hat{\mathsf{z}}_n \in R_n^{\mathsf{web}}(x)$ that merges all n strands into a single strand labeled n, then splits them up again. By [MOY98, Lem. A.1], $\hat{\mathsf{z}}_n$ is central in $\mathbf{K}R_n^{\mathsf{web}}(x)$, quasi-idempotent, and gets sent by ann to a scalar multiple of o_n. In particular, its idempotent rescaling z_n satisfies $\mathsf{ann}(\mathsf{z}_n) = o_\mu$. It remains to show that $\Theta_\diamondsuit(\heartsuit_n^\vee) = \mathsf{z}_n$.

For any free $\mathbf{K}$-module V of finite rank, let $\mathbf{K}H_n^V(x)$ be the commutant of the $\mathrm{U}_x(\mathfrak{gl}(V))$-action on $V^{\otimes n}$. These algebras admit analogues of $i_\mu, i_\mu^{\mathsf{web}}$:

$$i_\mu^V : \mathbf{K}H_{\mu_1}^V(x) \times \cdots \times \mathbf{K}H_{\mu_\ell}^V(x) \to \mathbf{K}H_n^V(x).$$

Let π_n^- and π_n^+ be the elements of $\mathbf{K}H_n^V(x)$ given by the projections onto the nth exterior and symmetric powers of V, respectively.

On the one hand, quantum Schur–Weyl duality defines isomorphisms $\Psi_n : \mathbf{K}H_n(x) \to \mathbf{K}H_n^V(x)$ that are injective for n less than or equal to the rank of V, send $e_n^\vee \mapsto \pi_n^-$ and $h_n^\vee \mapsto \pi_n^+$ for all n, and intertwine[1] i_μ with i_μ^V for all μ. On the other hand, the anti-symmetric web calculus of [MOY98] or [CKM14] defines homomorphisms $\Psi_n^- : \mathbf{K}R_n^{\mathsf{web}}(x) \to \mathbf{K}H_n^V(x)$ that send $\mathsf{z}_n \mapsto \pi_n^-$ for all n, and intertwine i_μ^{web} with i_μ^V for all μ. Its symmetric analogue shows the corresponding statements with $+$ in place of $-$ everywhere. So it suffices to show $\Psi_n = \Psi_n^- \circ \Theta_c$ and $\Psi_n = \Psi_n^+ \circ \Theta_b$, then take V of rank at least n.

Step 4. It suffices to check these identities on either the set $\{c_i\}_i$ or the set $\{b_i\}_i$. By the intertwiner properties above, we reduce[2] to the case where $n = 2$. The calculations

$$e_2^\vee = \tfrac{1}{x+x^{-1}}\, c_1, \qquad h_2^\vee = \tfrac{1}{x+x^{-1}}\, b_1, \qquad \mathsf{z}_2 = \tfrac{1}{x+x^{-1}}\, \hat{\mathsf{z}}_2 = \tfrac{1}{x+x^{-1}}\, \mathsf{can}_1$$

verify this case. □

[1] We view this intertwiner property as a compatibility with parabolic induction.

[2] Compare to the proof of Lemma 4.25 in [GW23].

4.5. Theorem 4.2 also suggests a categorification of Conjecture 1.1.

Let C be the category denoted $\mathsf{Kar}(A\mathbf{Foam}^+)$ in [GW23]: the Karoubi completion of a graded, linear category of foams between positively-oriented annular webs. Let H_n be the analogous category where we replace the annulus by a rectangle with n inputs and n outputs. By work of Mackaay–Vaz [MV10], H_n is a diagrammatic presentation of the category of Soergel bimodules for S_n.

Let $\mathbf{B}_w$ be the indecomposable object of H_n indexed by $w \in S_n$, so that the isomorphism from the Grothendieck group of H_n to $H_n(x)$ sends $[\mathbf{B}_w]$ to b_w. Let $\mathbf{O}_\mu$ be the object of C_n underlying the annular web o_μ.

Conjecture 4.4. *For all $w \in S_n$, the annular closure of $\mathbf{B}_w$ is isomorphic in C to a direct sum of objects of the form $\mathbf{O}_\mu$.*

5. Intertwining Dualities

5.1. We return to the generality of a finite Coxeter group W. Let ε be its sign character, defined by $\varepsilon(w) = (-1)^{\ell_w}$. The following result should be very well-known, but we have not found an explicit reference.

Proposition 5.1. *For any irreducible character χ of W, we have*

$$(\varepsilon\chi)_x = \chi_x \circ \eta$$

as functions on $H_W(x)$, where η is the $\mathbf{Z}[x^{\pm 1}]$-algebra involution from §2.5 that swaps the Kazhdan–Lusztig bases.

Proof. By Proposition 9.4.1 of [GP00],

$$(\varepsilon\chi)_x(\sigma_w) = (-1)^{\ell_w}\chi_x(\sigma_w)|_{x\to x^{-1}} \quad \text{for all } w \in W.$$

Using (2.3), we deduce that

$$(\varepsilon\chi)_x(b_{s^{(1)}}\cdots b_{s^{(\ell)}}) = \chi_x(c_{s^{(1)}}\cdots c_{s^{(\ell)}}) = \chi_x(\eta(b_{s^{(1)}}\cdots b_{s^{(\ell)}}))$$

for any sequence of elements $s^{(1)}, \ldots, s^{(\ell)} \in S$. But $H_W(x)$ is generated as an algebra by $\{b_s\}_{s\in S}$, so every element of $H_W(x)$ is a linear combination of elements of the form $b_{s^{(1)}}\cdots b_{s^{(\ell)}}$. $\square$

5.2. Using Proposition 5.1, we show that the commutativity of either diagram in Theorem 4.3 implies that of the other.

Proof. First, recall that the involution of $\Lambda(x)$ that swaps h_μ and e_μ also swaps s_λ and s_{λ^t}, where λ^t is the transpose of λ [Mac15, (3.8)]. So the map

$$\mathsf{tr}_n^t : H_n(x) \to \Lambda_n(x) \quad \text{defined by } \mathsf{tr}_n^t = \sum_{\lambda\vdash n} \chi_x^\lambda s_{\lambda^t}$$

satisfies $\Xi_e \circ \mathsf{tr}_n = \Xi_h \circ \mathsf{tr}_n^t$. Next, observe that

$$\mathsf{tr}_n^t = \sum_{\lambda\vdash n} \chi_x^{\lambda^t} s_\lambda = \sum_{\lambda\vdash n} (\varepsilon\chi^\lambda)_x s_\lambda.$$

So by Proposition 5.1, $\mathsf{tr}_n^t = \mathsf{tr}_n \circ \eta$.

Altogether, $\Xi_e \circ \mathsf{tr}_n = \Xi_h \circ \mathsf{tr}_n \circ \eta$, whereas $\Theta_c = \Theta_b \circ \eta$. So by the involutivity of η, we have $\mathsf{ann} \circ \Theta_c = \Xi_e \circ \mathsf{tr}_n$ if and only if $\mathsf{ann} \circ \Theta_b = \Xi_h \circ \mathsf{tr}_n$. $\square$

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