



# HOMFLYPT Twisting as a Homotopy Equivalence

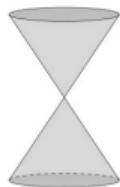
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$G = \mathrm{GL}_n$

$\mathcal{U} = \{\text{unipotent } n \times n \text{ matrices}\} \subseteq G$



Ex For  $n = 2$ ,

$$\mathcal{U} = \left\{ \begin{pmatrix} 1+a & b \\ c & 1-a \end{pmatrix} \middle| a^2 + bc = 0 \right\}$$

because  $u$  is unipotent  $\iff u - I$  is nilpotent  
 $\iff a + d = ad - bc = 0$ .

$B$  upper-triangular subgroup of  $G$

Bruhat, Cartan, Chevalley

$$G = \bigsqcup_{w \in S_n} B\dot{w}B$$

where  $\dot{w}$  is the permutation matrix lifting  $w$ .

Lets us study  $G$  in terms of the *Borel subgroup*  $B$  and the *Weyl group*  $S_n$ .

What can we say about  $\mathcal{U}$  in terms of

$$\mathcal{U} = B \cap \mathcal{U}?$$

B lower-triangular subgroup of  $G$

Fine–Herstein '58, Steinberg '65

$$\begin{aligned} |\mathcal{U}(\mathbb{F}_q)| &= q^{n(n-1)} \\ &= |U(\mathbb{F}_q)|^2 \\ &= |UU_-(\mathbb{F}_q)|. \end{aligned}$$

where  $U_- = B_- \cap \mathcal{U}$ .

$\mathcal{U}_w$  =  $\mathcal{U} \cap BwB$ ,  $\mathcal{V}_w$  =  $UU_- \cap BwB$

Kawanaka '75 For any  $w \in S_n$ ,

$$|\mathcal{U}_w(\mathbb{F}_q)| = |\mathcal{V}_w(\mathbb{F}_q)|.$$

Ex For any  $n$ , we have  $\mathcal{U}_{\text{id}} = U = \mathcal{V}_{\text{id}}$ .

Ex For  $n = 3$  and  $w = \begin{pmatrix} & 1 & 1 \\ & 1 & \end{pmatrix}$ ,

$$\mathcal{U}_w \simeq U \times \{(a, b, c, d) \mid a, b \neq 0, (1+ab)^3 = abcd\}$$

$$\mathcal{V}_w \simeq U \times \{(a, b, c, d) \mid a, b \neq 0, 1+ab = abcd\}$$

These are not even homeomorphic over  $\mathbb{C}$ .

$T$  diagonal subgroup

$T \curvearrowright \mathcal{U}_w, \mathcal{V}_w$  by conjugation.

Thm (T) For any  $w \in S_n$ ,

$$\mathrm{gr}_*^W H_{c,T}^*(\mathcal{U}_w(\mathbb{C})) \simeq \mathrm{gr}_*^W H_{c,T}^*(\mathcal{V}_w(\mathbb{C})).$$

$H_{c,T}^*$  is *equivariant cohomology with compact support*.

$W_{\leq *}$  is its *weight filtration*.

This (almost!) implies Kawanaka's identity

$$|\mathcal{U}_w(\mathbb{F}_q)| = |\mathcal{V}_w(\mathbb{F}_q)|$$

via results of Katz.

The map

$$UU_- \rightarrow \mathcal{U} \text{ given by } xy \mapsto xyx^{-1}$$

is  $T$ -equivariant.

Conj (T) It restricts to a homotopy equivalence

$$\mathcal{V}_w(\mathbb{C}) \rightarrow \mathcal{U}_w(\mathbb{C}).$$

This would imply the theorem about cohomology.

(The original map is itself a homotopy equivalence:  
 $UU_-$  and  $\mathcal{U}$  are both contractible.)

Each  $w \in S_n$  lifts to some *braid*  $\sigma_w \in Br_n^+$ .

For a positive braid  $\beta = \sigma_{w_1} \cdots \sigma_{w_k}$ , let

$$X_\beta = B\dot{w}_1 B \times^B B\dot{w}_2 B \times^B \cdots \times^B B\dot{w}_k B$$

where  $\times^B$  means “ $\times$  mod antidiagonal  $B$ .”

For any conjugation-stable set  $C \subseteq G$ , form

$$\begin{array}{ccc} X_\beta^C & \rightarrow & X_\beta \\ \downarrow & & \downarrow \\ C & \rightarrow & G \end{array}$$

$\mathcal{U}_w, \mathcal{V}_w$  are special cases of the  $X_\beta^C$ .

The *full twist*  $\pi$ :



Turns out:  $\mathcal{U}_w = X_{\sigma_w}^{\mathcal{U}}$  and  $\mathcal{V}_w = X_{\sigma_w \pi}^1$ .

Thm (T) For any  $\beta \in Br_n^+$ ,

$$\begin{aligned} |X_\beta^{\mathcal{U}}(\mathbb{F}_q)| &= |X_{\beta \pi}^1(\mathbb{F}_q)|, \\ \text{gr}_*^W H_{c,T}^*(X_\beta^{\mathcal{U}}(\mathbb{C})) &\simeq \text{gr}_*^W H_{c,T}^*(X_{\beta \pi}^1(\mathbb{C})). \end{aligned}$$

Surprisingly, the proof uses knot theory.

Previous theorem is the case  $\beta = \sigma_w$ .

HOMFLYPT poly     $P$  :  $\{\text{links}\} \rightarrow \mathbf{Z}[[q][a, q^{-1/2}],$   
 KhR superpoly     $\mathbf{P}$  :  $\{\text{links}\} \rightarrow \mathbf{Z}[[q][a, q^{-1/2}, t]$

Kálmán '09   Writing  $\widehat{\beta}$  for the closure of  $\beta$ ,

$$P(\widehat{\beta})[a^{|\beta|-n+1}] = P(\widehat{\beta\pi})[a^{|\beta|+n-1}]$$

where  $P[a^i]$  means “ $q$ -coefficient of  $a^i$ .”

Gorsky–Hogancamp–Mellit–Nakagane '19

True with  $\mathbf{P}$  in place of  $P$ .

We will relate  $X_\beta^{\mathcal{U}}, X_\beta^1$  to these pieces of  $\mathbf{P}$ .

$X_\beta^{\mathcal{U}}, X_\beta^1$  are pieces of a larger variety  $\tilde{X}_\beta^{\mathcal{U}}$ .

*Springer resolution* of  $\mathcal{U}$ :

$$\tilde{\mathcal{U}} = \{(u, gB) \in \mathcal{U} \times G/B \mid ugB = gB\}.$$

Pullback squares:

$$\begin{array}{ccccccc} X_\beta^1 \times G/B & \rightarrow & \tilde{X}_\beta^{\mathcal{U}} & \rightarrow & X_\beta^{\mathcal{U}} & \rightarrow & X_\beta \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \{1\} \times G/B & \rightarrow & \tilde{\mathcal{U}} & \rightarrow & \mathcal{U} & \rightarrow & G \end{array}$$

To relate  $X_\beta^{\mathcal{U}}, X_\beta^1$  to pieces of  $\mathbf{P}$ , relate  $\tilde{X}_\beta^{\mathcal{U}}$  to  $\mathbf{P}$ .

Thm (T) For any  $\beta \in Br_n^+$ ,

$$S_n \curvearrowright \text{gr}_*^W H_{c,T}^*(\tilde{X}_\beta^{\mathcal{U}}(\mathbb{C})).$$

Moreover,

$$\mathbf{P}(\widehat{\beta}) \propto (\Lambda^*(\mathbb{V}), \text{gr}_*^W H_{c,T}^*(\tilde{X}_\beta^{\mathcal{U}}(\mathbb{C})))_{S_n}$$

where  $\mathbb{V} = \mathbb{C}^{n-1}$  is the standard irrep of  $S_n$ .

The variable  $a$  corresponds to the  $\Lambda^*$ -grading.

$$\implies \mathbf{P}(\widehat{\beta})[a^{\text{lo}}] = \dim \text{gr}_*^W H_{c,T}^*(X_\beta^{\mathcal{U}}(\mathbb{C})),$$

$$\implies \mathbf{P}(\widehat{\beta})[a^{\text{hi}}] = \dim \text{gr}_*^W H_{c,T}^*(X_\beta^1(\mathbb{C})).$$

For  $P$ , we can use *symmetric functions* to be more concrete.

Hotta–Springer '77 For  $u \in \mathcal{U}$  of Jordan type  $\mu \vdash n$ ,

$$S_n \curvearrowright H^*(\tilde{\mathcal{U}}_u(\mathbb{C}))$$

is given by the *Hall–Littlewood polynomial*  $\tilde{H}_\mu(q)$ .

Consequently:

$$P(\widehat{\beta})[a^{\text{lo}+2k}] \propto \sum_{\mu \vdash n} \langle s_{\Lambda^k}, \tilde{H}_\mu(q) \rangle \cdot |X_\beta^\mu(\mathbb{F}_q)|.$$

$s_{\Lambda^k}$  is the *Schur function* of type  $(n-k, 1^k)$ .

$X_\beta^\mu \subseteq X_\beta^{\mathcal{U}}$  is the preimage of  $\mathcal{U}_\mu \subseteq \mathcal{U}$ .

Why should  $\mathcal{U}$  be related to knot invariants?

$P$  arises from traces on *Hecke algebras*.

$\mathbf{P}$  arises from traces on *Hecke categories*.

$$\mathrm{D}_{mix,G}^b \mathrm{Perv}(\mathcal{U})$$

$$\simeq \mathrm{D}^b \mathrm{Mod}(\mathbb{C} S_n \ltimes \mathrm{Sym}(\mathbb{V})) \quad (\text{Rider})$$

$$\simeq \textcolor{red}{\mathrm{hTr}}(\mathrm{Hecke}(S_n)) \quad (\text{Gorsky--Wedrich})$$

*Thank you for listening.*

Passing from  $X_\beta$  to  $\tilde{X}_\beta^{\mathcal{U}}$  is like passing from  $\beta$  to its  
*annular closure*.