12.

We introduce the notion of categorification. Then we discuss some naive attempts to categorify the Iwahori–Hecke algebra by means of constructible sheaves. A possible reference for the latter topic is Lecture 24 in Romanov–Williamson's lecture notes. Along the way, we introduce the function-sheaf dictionary, partly following the book by Kiehl–Weissauer.

12.1.

Categorification of an additive group A means constructing an additive category C such that A is the Grothendieck group of C in an appropriate sense.

There are several different kinds of additive category, each with its own notion of Grothendieck group. In each case, we assume that C admits a small skeleton; the Grothendieck group is generated by the isomorphism classes of objects in the skeleton modulo certain relations.

(1) For any C, the *split Grothendieck group* $[\mathsf{C}]_\oplus$ is given by the relations

$$[c] = [c'] + [c''] \quad \text{for any } c \simeq c' \oplus c''.$$

(2) For C abelian, the usual *Grothendieck group* $[\mathsf{C}]$ is given by the relations

$$[c] = [c'] + [c''] \quad \text{for any exact sequence } 0 \to c' \to c \to c'' \to 0.$$

(3) For C triangulated, the *triangulated Grothendieck group* $[\mathsf{C}]_\triangle$ is given by the relations

$$[c] = [c'] + [c''] \quad \text{for any exact triangle } c' \to c \to c'' \to c'[1].$$

Note that for any c, the triangle $c \to 0 \to c[1] \to c[1]$ is exact, giving $[c[1]] = -[c]$. That is, the shift $[1]$ must decategorify to scaling by -1.

It appears to be well-known that if C is abelian and $\mathsf{D}^b(\mathsf{C})$ is the bounded derived category of complexes of objects in C, then $[\mathsf{D}^b(\mathsf{C})]_\triangle = [\mathsf{C}]$. Seemingly less-known, but important for our goals, is a result recorded by David Rose in "A Note on the Grothendieck Group…" Below, for any additive C, let $\mathsf{K}^b(\mathsf{C})$ be the bounded homotopy category of complexes of objects in C.

Theorem 12.1 (Rose)**.** *We have* $[\mathsf{K}^b(\mathsf{C})]_\triangle = [\mathsf{C}]_\oplus$.

Remark 12.2. For C abelian, $[\mathsf{C}]_\oplus$ is usually larger than $[\mathsf{C}]$. This corresponds to the fact that a short exact sequence of complexes in C will give rise to an exact triangle in $\mathsf{D}^b(\mathsf{C})$ but not necessarily in $\mathsf{K}^b(\mathsf{C})$.[1]

[1]Thank-you to David B. for spotting an error here during the lecture.

Categorification of a ring R begins with categorification of the underlying additive group to some category C. We then need to construct some monoidal product $*$ on C that distributes over the direct sum $\oplus$, such that the relations

$$[c] = [c'][c''] \quad \text{for any } c \simeq c' * c''$$

define the multiplication on R.

12.2.

Fix an algebraically closed field k and a prime ℓ invertible in k. Fix a scheme X of finite type over k and a finite stratification $\mathcal{S}$ of X by (pairwise-disjoint) constructible subschemes.[2] A sheaf over $\mathcal{F}$ is *constructible* with respect to $\mathcal{S}$ if and only if it is étale-locally constant along each stratum: *i.e.*, constant after pullback along a finite étale cover of the stratum. Let $\mathsf{Shv}(X) = \mathsf{Shv}(X, \bar{\mathbf{Q}}_\ell; \mathcal{S})$ be the category of étale sheaves of finite-dimensional $\bar{\mathbf{Q}}_\ell$-vector spaces over X that are constructible with respect to $\mathcal{S}$.

Example 12.3. Set $pt := \operatorname{Spec} k$. Then all stratifications of pt are the same, and $\mathsf{Shv}(pt)$ is simply the category of finite-dimensional $\bar{\mathbf{Q}}_\ell$-vector spaces.

If H is an algebraic group over k, acting on X with finitely many orbits, then it is natural to take the stratification $\mathcal{S}$ to be the H-orbit stratification. In particular, if G is a reductive algebraic group over k and $\mathcal{B}$ is its flag variety, then it is interesting to take $X = \mathcal{B} \times \mathcal{B}$ and $H = G$. Alternatively, if $B \subseteq G$ is a fixed Borel subgroup, then we might take $X = G$ and $H = B \times B$.

How can we recover the Iwahori–Hecke algebra $H_W = H_W(\mathsf{x})$ from sheaves in this setup? Recall that if $k = \bar{\mathbf{F}}_q$ and $F : G \to G$ corresponds to the split $\mathbf{F}_q$-form, then we can identify $H_W(q^{1/2})$ with G^F-invariant functions on $\mathcal{B}^F \times \mathcal{B}^F$, and under this identification, the rescaled standard basis elements $q^{\ell(w)/2}\sigma_w$ correspond to the indicator functions on G^F-orbits.

The sheafy analogue of an indicator function is the extension-by-zero of a constant sheaf. This leads to a naive guess: Since $\mathsf{Shv}(X)$ is abelian, we can form either of the Grothendieck groups $[\mathsf{Shv}(X)]_\oplus$ or $[\mathsf{Shv}(X)]$. For $X = \mathcal{B} \times \mathcal{B}$, we might hope that the extensions-by-zero to X of the constant sheaves on its G-orbits decategorify to a rescaled standard basis of $H_W(\mathsf{x})$.

Unfortunately, this can't work. Take $G = \mathrm{GL}_1$ (or any other torus). Then $\mathcal{B} = pt$, so $X = pt$. By Example 12.3, $[\mathsf{Shv}(pt)]_\oplus = [\mathsf{Shv}(pt)] = \mathbf{Z}$, whereas we want $H_W(\mathsf{x}) = \mathbf{Z}[\mathsf{x}^{\pm 1}]$.

[2]The definition of a stratification is something technical, but I will not go into this.

12.3.

In trying to fix this issue, we might set $k = \bar{\mathbf{F}}_q$ and try to specialize x to a square root of q, like we did to compare the Hecke algebra to functions on $\mathcal{B}^F \times \mathcal{B}^F$. However, we immediately realize that k itself does not see q, but only the underlying prime of which q is a power. This suggests working with sheaves equipped with extra structure coming from an $\mathbf{F}_q$-structure on X, and using this structure to enrich our Grothendieck groups.

Henceforth, $k = \bar{\mathbf{F}}_q$, so that $\ell \nmid q$. Suppose that $X = X_1 \otimes k$ and $\mathcal{F} = \mathcal{F}_1|_X$ for some scheme X_1 over $\mathbf{F}_q$ and sheaf $\mathcal{F}_1$ over X_1. We will describe the *fonctions-faisceaux* (or *function-sheaf*) *dictionary*, which sends $\mathcal{F}$ to a collection of *trace of Frobenius* functions

$$\mathbf{t}_{\mathcal{F}} = \mathbf{t}_{\mathcal{F},d} : X_1(\mathbf{F}_{q^d}) \to \bar{\mathbf{Q}}_\ell \quad \text{for } d \geq 1.$$

Theorem 1.1.2 in Laumon's "Transformation de Fourier…" states a precise sense in which the functions $\mathbf{t}_{\mathcal{F},d}$ determine the class of $\mathcal{F}$ in an appropriate Grothendieck group. We will return to this point in a later lecture.

Recall that given Z over $\mathbf{F}_q$, the absolute Frobenius map $\sigma_Z : Z \to Z$ is the map over $\mathbf{F}_q$ that fixes the underlying topological space and sends $f \mapsto f^q$ on sections of the structure sheaf $\mathcal{O}_Z$. We also worked with the relative Frobenius maps $F = \sigma_{X_1} \otimes \mathrm{id} : X \to X$, which are maps over k. We claim that if $\mathcal{F} = \mathcal{F}_1|_X$, then there is an isomorphism

$$F^*\mathcal{F} \xrightarrow{\sim} \mathcal{F} \tag{12.1}$$

induced from $\mathcal{F}_1$ via étale descent. The idea is that $\mathcal{F}_1$ is built up from étale algebraic spaces $E_1 \to X_1$. The relative Frobenius of $E = E_1 \otimes k$, given by $\sigma_{E_1} \otimes \mathrm{id}$, factors through an isomorphism $E \xrightarrow{\sim} (\sigma_{X_1} \otimes \mathrm{id})^* E = F^*E$, and the inverse isomorphisms $F^*E \xrightarrow{\sim} E$ give rise to (12.1).

An arbitrary sheaf $\mathcal{F}$ equipped with an isomorphism of the form (12.1) is called a *Weil sheaf* with respect to $F : X \to X$. We may regard (12.1) as specifying an action of a *pro-generator* of the Galois group $\mathrm{Gal}(k/\mathbf{F}_q)$. It defines a descent datum from X to X_1 if and only if it extends to an action of $\mathrm{Gal}(k/\mathbf{F}_q)$ itself. We will focus on the Weil sheaves for which this occurs: *i.e.*, those that take the form $\mathcal{F}_1|_X$ for some $\mathcal{F}_1$ on X_1.

Example 12.4. A Weil sheaf on $pt := \operatorname{Spec} k$ is equivalent to a vector space over $\bar{\mathbf{Q}}_\ell$ equipped with an invertible operator F. The sheaf comes from $\operatorname{Spec} \mathbf{F}_q$ precisely when the F-action extends to a continuous $\mathrm{Gal}(k/\mathbf{F}_q)$-action.

In particular, if the vector space is 1-dimensional, then such a Galois action is specified by a continuous homomorphism $\mathrm{Gal}(k/\mathbf{F}_q) \to \bar{\mathbf{Q}}_\ell^\times$. Continuity forces the image of F to be an element of $\bar{\mathbf{Z}}_\ell^\times$, the maximal compact subgroup of $\bar{\mathbf{Q}}_\ell^\times$.

We want to use (12.1) to define a Frobenius action on stalks. Note that for any $\bar{x} \in X(k)$, we have an identification

$$\mathcal{F}_{F(\bar{x})} = \varinjlim_{U \ni F(\bar{x})} \mathcal{F}(U) = \varinjlim_{V \ni \bar{x}} \mathcal{F}(F(V)) = (F^*\mathcal{F})_{\bar{x}}.$$

One can check that if $\bar{x}$ factors through a point $x \in X_1(\mathbf{F}_{q^d})$, then F^d fixes $\bar{x}$. So for such $\bar{x}$ and x, we obtain a morphism of $\bar{\mathbf{Q}}_\ell$-vector spaces

$$F^d : \mathcal{F}_{\bar{x}} = F^*\mathcal{F}_{\bar{x}} \xrightarrow{(12.1)} \mathcal{F}_{\bar{x}}.$$

The trace of this morphism only depends on x, so we can set

$$\mathbf{t}_{\mathcal{F},d}(x) = \operatorname{tr}(F^d \mid \mathcal{F}_{\bar{x}}).$$

Remark 12.5. In Kiehl–Weissauer's book, they take a different approach that results in the same action. To explain: Observe that

$$\sigma_X = F_X \circ F = F \circ F_X, \quad \text{where } F_X = \mathrm{id}_{X_1} \otimes \sigma_{\operatorname{Spec} k}.$$

We get two actions on the set of k-points $X(k)$ that turn out to be equivalent. Namely, given $\bar{x} : \operatorname{Spec} k \to X$:

(1) One action sends $\bar{x} \mapsto F \circ \bar{x}$.
(2) Another action sends $\bar{x} \mapsto F_X \circ \bar{x} \circ \sigma_{\operatorname{Spec} k}^{-1}$.

The composition of (1) and (2) in either order sends $\bar{x} \mapsto \sigma_X \circ \bar{x} \circ \sigma_{\operatorname{Spec} k}^{-1} = \bar{x}$. Therefore, (1) and (2) are mutually inverse. Kiehl–Weissauer present most of the stalk construction in terms of (the inverse of) map (2).

In our earlier discussion of Galois actions on a sheaf $\mathcal{F}$, we may regard $\sigma_{\operatorname{Spec} k}^{-1}$ as the pro-generator of $\operatorname{Gal}(k/\mathbf{F}_q)$ whose action on $\mathcal{F}$ coincides with that of F.

12.4.

The above discussion gives the impression that we should replace $\mathsf{Shv}(X)$ with an analogue $\mathsf{Shv}(X_1)$, understood as a full subcategory of the category of Weil sheaves on X.

Unfortunately, we have now overshot the size of the Grothendieck groups. Again taking G to be a torus, so that $X = pt$, we find from Example 12.4 that $[\mathsf{Shv}(pt_1)]_\oplus = [\mathsf{Shv}(pt_1)] = \mathbf{Z}[\bar{\mathbf{Z}}_\ell^\times]$.

So we should only use a certain subcategory of $\mathsf{Shv}(X_1)$. Based on our example, we might try to restrict the possible eigenvalues that occur in the action of F^d on $\mathcal{F}_{\bar{x}}$ discussed above. For instance, we could restrict the eigenvalues to be powers of $q^{1/2}$.

But more issues arise when we try to define a monoidal product $*$ that corresponds to the multiplication in H_W. From previous lectures, we expect $*$ to arise from some kind of convolution on X that should preserve our subcategory. A priori, it is not clear how to ensure that restrictions on eigenvalues would be preserved by this convolution.

Finally, we also want our subcategory to contain objects that, under the function-sheaf dictionary, recover the two bases that we have studied in detail: the standard basis $(\sigma_w)_w$ and the Kazhdan–Lusztig basis $(c_w)_w$. Next week, we will find that the approach that works is:

(1) First, categorify H_W to a merely additive category C preserved by a convolution $*$, the element x to a *new* kind of shift functor $\langle 1 \rangle$, and the basis $(c_w)_w$ to a collection of objects that generate this category under $\oplus$ and $\langle 1 \rangle$.
(2) Next, form objects in the triangulated category $\mathsf{K}^b(\mathsf{C})$ that categorify the standard basis $(\sigma_w)_w$.

We can give a preview of how $\langle 1 \rangle$ arises.

Recall that the *Tate twist* $\bar{\mathbf{Q}}_\ell(1)$ is the 1-dimensional vector space on which F acts by $q^{-1} \in \bar{\mathbf{Z}}_\ell^\times$. (Here we use the hypothesis that $\ell \nmid q$.) Fixing a square root of q allows us to define a *half Tate twist* $\bar{\mathbf{Q}}_\ell(\frac{1}{2})$, on which F acts by $q^{-1/2}$. We will construct C inside a larger triangulated category called the constructible derived category, then set $\langle 1 \rangle = (- \otimes \bar{\mathbf{Q}}_\ell(\frac{1}{2}))[1]$.