(Munkres §24)

<u>Q</u> is R connected in the analytic topology?
why is this tricky?

last time, showed:
Q as a subspace of R is disconnected,
even though it is <u>dense</u> in R (i.e. $Cl_R(Q) = R$)

any proof that R is connected must use
some fact about R that fails for Q

<u>Df</u> given a set X with a total order $<$
it has the least upper bound property iff
every subset of X bounded above
has a least upper bound (= <u>sup</u>)

<u>Ex</u> in an intro analysis course,
we define R from scratch

the defn shows that R has the LUBP
but Q does not: e.g. $\{x \text{ in } Q \mid x^2 < 2\}$ has no LUB

the LUBP is not enough to prove R is connected:
why?

<u>Ex</u> if X sub R finite and $|X| \geq 2$
then X disconnected but has the LUBP

<u>Df</u> a set X with a total order $<$ is called
a linear continuum iff
1) it has the LUBP
2) for all $x < y$ in X, there is z s.t. $x < z < y$

thus $\mathbb{R}$ is a linear continuum

Thm R is connected in the analytic topology

Pf suppose U, V form a separation of $\mathbb{R}$
pick a in U and b in V
WLOG we can assume $a < b$

let $A = [a, b] \cap U$ and $B = [a, b] \cap V$
then A, B form a separation of $[a, b]$
in its subspace top
but A is bounded above by b
so by the LUBP, $\sup A$ exists and is in $[a, b]$

remains to show 1) $\sup A \notin B$
2) $\sup A \notin A$

1) assume $\sup A$ in B
then $\sup A > a$: else $A = \emptyset$
but B is open in $[a, b]$, so $(\sup A - \delta, \sup A] \subseteq B$
for some $\delta > 0$
but then, $\sup A - \varepsilon$ is a smaller upper bound for A
whenever $0 < \varepsilon \leq \delta$

2) assume $\sup A$ in A
then $\sup A < b$: else, $B = \emptyset$
but A is open in $[a, b]$, so $[\sup A, \sup + \delta) \subseteq A$
for some $\delta > 0$
but then $\sup A + \varepsilon$ is an elt of A above $\sup A$
whenever $0 < \varepsilon < \delta$ □

Rem thm generalizes to any linear continuum
in its order topology (we did not define)

for this course, just note:
the proof still works if we replace R with
    (a, b), (a, b], [a, b), [a, b]
    (–∞, b), (–∞, b], (a, ∞), [a, ∞)

Thm (Intermediate Value)

suppose X connected and f : X to R cts
    wrt the analytic top on R
if a, b in X and f(a) ≤ α ≤ f(b)
then there is some c in X s.t. f(c) = α

Lem   if X is connected, f : X to Y cts,
    then f(X) sub Y is connected

Pf   preimage of separation is a separation

Pf of Thm   suppose no c in X s.t. f(c) = α

then f(X) has a separation: {f(x) < α | x in X},
    {f(x) > α | x in X}

but X is connected, so contradiction with lemma

Moral   the connectedness of intervals in R
    is very well-understood

Idea   study connectedness in other spaces
    by comparing them to intervals in R

Df   for any X and x, y in X
    a path from x to y is a cts map
    γ : [a, b] to X s.t. γ(a) = x and γ(b) = y

we say X is <u>path-connected</u> iff
there is a path between every pair of pts in X

<u>Rem</u> we require $a \leq b$
but otherwise a, b can be any numbers

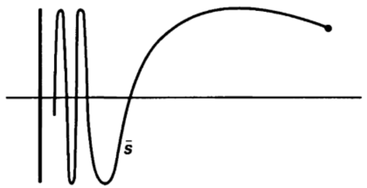
<u>Lem</u> if X is path-connected
then X is connected

<u>Pf</u> if $X = \emptyset$, then done
else, can fix x in X

for all y in X, pick a path γ_y from x to y
then $X = \bigcup_y \gamma_y([0, 1])$
each $\gamma_y([0, 1])$ is connected and contains x
so X is connected

[famous non-example:]

<u>Ex</u> in analytic R^2: consider
$S = \{(x, \sin(1/x)) \text{ in } R^2 \mid 0 < x \leq 1\}$
$A = \{(0, y) \mid -1 \leq y \leq 1\}$
$\check{S} = S \cup A$ = the closure of S in R^2



the <u>topologist's sine curve</u>: $\check{S}$ in its subspace top.

not hard: S, A are path-connected
claim: $\check{S}$ is not

sketch: fix a in A and s in S

suppose γ : [0, 1] to Š a path from a to s

claim there is a largest t_0 in [0, 1] s.t. γ(t_0) in A:

A is closed in R, hence in Š

so γ^{-1}(A) is closed in [0, 1]

so γ^{-1}(A) is its own closure

so it contains its sup

so for all t in (t_0, 1], have γ(t) in S

let (t_n)_n be a decreasing seq converging to t_0

for all n: pick 0 < x_n < x-coord(γ(t_n)) s.t.

sin(1/x_n) = (–1)^n

by IVT: x-coord(γ(t_0)) < x_n < x-coord(γ(t_n))

means there is t′_n in [t_0, t_n] s.t.

γ(t′_n) = x_n

(Munkres §25)

<u>Lem</u> for any X, the following are equiv. rel's:

1) x ~ y iff there is a connected subspace of X that contains both x and y

2) x ↔ y iff there is a path between x and y in X

<u>Pf</u> transitivity for 1): if x, y in A & y, z in B, then A cap B ni y

transitivity for 2): [Munkres 18.3:]

<u>Pasting Lemma</u> let Y = Y_1 cup Y_2

f_i : Y_i to X for i = 1, 2

if f_1, f_2 cts and f_1(x) = f_2(x) on Y_1 cap Y_2

then f : Y to X def by f|_{Y_i} = f_i is cts

Pf boring

Df the connected components of X are
the equiv. classes under ~
the path components of X are
the equiv. classes under ↔

Ex with some work, we can show:

S and A are the conn. components of Š,
and also, its path components