<u>Thm</u> if V is fin dim and $T : V$ to W is linear, then $\dim V = \dim \ker(T) + \dim \operatorname{im}(T)$

<u>Pf Outline</u>

pick basis $\{w_1, \ldots, w_r\}$ for $\operatorname{im}(T)$
pick $u_1, \ldots, u_r$ s.t. $T(v_i) = w_i$ for all i
let $U = \operatorname{span}(u_1, \ldots, u_r)$
pick basis $v_1, \ldots, v_k$ for $\ker(T)$
I) $\{u_1, \ldots, u_r\}$ is a basis for U
II) $\ker(T) + U = V$
II) $\ker(T) + U$ is a direct sum
then $\dim V = \dim \ker(T) + \dim \operatorname{im}(T)$ □

<u>Pf of I)</u> if $\sum_i a_i u_i = 0$, then
$\sum_i a_i T(u_i) = 0$ so $a_i = 0$ for all i

<u>Pf of II)</u> pick v in V

know $T(v) = \sum_i b_i w_i$ for some b_i
so $T(v) = \sum_i b_i T(u_i) = T(\sum_i b_i u_i)$
so $T(v - \sum_i b_i u_i) = \mathbf{0_W}$
so $v - \sum_i b_i u_i$ in $\ker(T)$
so v in $\ker(T) + U$

<u>Pf of III)</u> recall: $W + U$ is a direct sum iff
$W \cap U = \{\mathbf{0}\}$
so want $\ker(T) \cap U = \{\mathbf{0_V}\}$

pick v in $\ker(T) \cap U$
since v in U, have $v = \sum_i a_i u_i$ for some a_i
since v in $\ker(T)$, have $\sum_i a_i w_i = T(v) = \mathbf{0_W}$
but $\{w_i\}_i$ lin. indep. so $a_i = 0$ for all i

assuming V, W are both finite-dimensional:

(Axler §3D)

<u>Cor</u> if T : V to W is linear and dim V > dim W, then T is not injective

<u>Pf</u> dim im(T) ≤ dim W < dim V,
so
dim ker(T) = dim V – dim im(T) > 0

<u>Cor</u> if T : V to W is linear and dim V < dim W, then T is not surjective

<u>Pf</u> exercise

<u>Cor</u> if T : V to W is linear and bijective, then dim V = dim W [converse is false!]

<u>Thm</u> TFAE for a linear map T : V to W:

1) T is bijective
2) T takes any basis for V onto a basis for W
3) T takes some basis for V onto one for W
4) there is a linear map S : V to W s.t.
S(T(v)) = v and T(S(w)) = w

<u>Df</u> in the situation above, we say that T is a linear isomorphism from V onto W

<u>Pf of Thm</u> not hard to show that
2) implies 3) implies 4) implies 1)
remains to show 1) implies 2)

so suppose {e_i}_i is a basis for V

claim that {T(e_i)}_i spans W:

for all w in W, have w = T(v) for some v in V

by surjectivity of T

v = sum_i a_ie_i for some a_i

so w = sum_i a_iT(e_i)

claim that {T(e_i)}_i is lin. indep.

suppose sum_i b_i T(e_i) = **0_W**

then T(sum_i b_ie_i) = **0_W**

so sum_i b_ie_i in ker(T)

so sum_i b_ie_i = **0_V**

by injectivity of T

so b_i = 0 for all i

by lin. independence of {e_i}_i □

(Axler §3C) recap:

Slogan #1 if we know a basis for V

then a linear map V to W is det by

where it sends the basis

and any choices will do

in particular, a linear map F^n to W is det by

where it sends (1, 0, 0, …), (0, 1, 0, …), …

Slogan #2 a linear isomorphism is

a linear map taking bases to bases

in particular, a linear iso F^n to W is det by

an ordered basis for W

[images of (1, 0, 0, …), (0, 1, 0, …), …]

[in this sense, linear maps out of F^n are easy]
[what else can we do with linear maps?]

a composition of linear maps is linear: given linear

$A : V$ to W,
$B : W$ to U,

$B(A(v + v')) = B(A(v) + A(v')) = B(A(v)) + B(A(v'))$,
$B(A(c \bullet v)) = B(c \bullet A(v)) = c \bullet B(A(v))$

so $B \circ A : V$ to U is also linear

suppose $V = F^n$ with std basis $(v_1, \ldots, v_n)$,
$W = F^m$ with std basis $(w_1, \ldots, w_m)$,
$U = F^\ell$ with std basis $(u_1, \ldots, u_\ell)$
$A(v_i) = \sum_j a_{j,i} w_j$,
$B(w_j) = \sum_k b_{k,j} u_k$

$$\begin{aligned} B(A(v_i)) &= B\left(\sum_j a_{j,i} w_j\right) \\ &= \sum_j a_{j,i} B(w_j) \\ &= \sum_j a_{j,i} \sum_k b_{k,j} u_k \\ &= \sum_{k,j} a_{j,i} b_{k,j} u_k \\ &= \sum_k c_{k,i} u_k \\ &\qquad \text{where } c_{k,i} = \sum_j a_{j,i} b_{k,j} \end{aligned}$$

matrix multiplication = shorthand for these calc's

Df matrix of A wrt the ordered bases $(v_i)_{i = 1}^n, (w_j)_{j = 1}^m$:

a11	a12	…	a1n	# rows is n (dim domain),
a21	a22	…	a2n	# cols is m (dim target)
…	…		…	
am1	am2	…	amn	

henceforth write F^n, F^m, etc in <u>column notation</u>:

$$v = \sum_i c_i v_i: \quad \begin{matrix} c_1 \\ c_2 \\ \dots \\ c_n \end{matrix}$$

matrix × vector rule for Av:

$$\begin{matrix} a_{11} & \dots & a_{1n} \\ a_{21} & \dots & a_{2n} \\ \dots & & \dots \\ a_{m1} & \dots & a_{mn} \end{matrix} \quad \begin{matrix} c_1 \\ c_2 \\ \dots \\ c_n \end{matrix} \quad = \quad \begin{matrix} \dots \\ a_{j1} c_1 + \dots + a_{jn} c_n \\ \dots \\ \dots \end{matrix}$$

e.g.,

$$Av_i: \quad \begin{matrix} a_{1i} \\ a_{2i} \\ \dots \\ a_{mi} \end{matrix} \quad \text{(ith matrix col)}$$

matrix × matrix rule for B ○ A:

$$\begin{matrix} b_{11} & \dots & b_{1m} \\ b_{21} & \dots & b_{2m} \\ \dots & & \dots \\ b_{\ell 1} & \dots & b_{\ell m} \end{matrix} \quad \begin{matrix} a_{11} & \dots & a_{1n} \\ a_{21} & \dots & a_{2n} \\ \dots & & \dots \\ a_{m1} & \dots & a_{mn} \end{matrix}$$

$$= \quad \begin{matrix} \dots & \dots & \dots \\ \dots & \sum_j a_{\{j,i\}} b_{\{kj\}} & \dots \\ \dots & \dots & \dots \\ \dots & \dots & \dots \end{matrix}$$

so	vectors	become	columns,
	linear maps		matrices,
	○		matrix multiplication

if A iso, then dim V = dim W: so matrix is <u>square</u>