



Character Formulas from Lusztig Varieties and Affine Springer Fibers

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This talk is about...

- 1 Braids
- 2 Lusztig Varieties
- 3 Springer Fibers
- 4 Affine Springer Fibers

1 Braids The braid group $Br_n =$

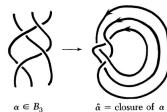
$$\left\langle \sigma_1, \dots, \sigma_{n-1}, \left| \begin{array}{l} \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \\ \sigma_i \sigma_j = \sigma_j \sigma_i \text{ when } |i - j| > 1 \end{array} \right. \right\rangle$$

appears in knot theory and representation theory.

$$\begin{array}{c} \sigma_i \\ \left[\dots \right] \underset{i}{\times} \underset{i+1}{\left[\dots \right]} \end{array}$$

A *link* is a collection of circles (tamely) embedded in \mathbf{R}^3 . Knot theory is about isotopy invariants of links.

(Alexander) Every link is the *closure* of some braid.



Let $G = \mathrm{SL}_n$ and B its upper-triangular subgroup.

Let $R(q) = \{\mathbf{Z}\text{-valued functions on } G(\mathbf{F}_q)/B(\mathbf{F}_q)\}.$

(Iwahori) There is a surjective homomorphism

$$\mathbf{Z}Br_n \twoheadrightarrow H_n(q) := \mathrm{End}_{G(\mathbf{F}_q)}(R(q)).$$

To describe it, recall the Bruhat decomposition

$$G = \bigsqcup_{w \in S_n} B\dot{w}B.$$

Let $h_w \curvearrowright R(q)$ be the *Hecke operator*

$$h_w(\mathbf{1}_{xB(\mathbf{F}_q)}) = \sum_{y^{-1}x \in B\dot{w}B} \mathbf{1}_{yB(\mathbf{F}_q)}.$$

Iwahori sends $\sigma_i \mapsto h_{s_i}$, where $s_i = (i, i+1).$

The *Hecke algebra*

$$H_n := \frac{\mathbf{Z}[q]Br_n}{\langle \sigma_i^2 - (q-1)\sigma_i - q \rangle}$$

is a generic version: $H_n|_{q \rightarrow q} = H_n(q).$

Jones–Ocneanu used traces $H_n \xrightarrow{\mu_n} \mathbf{Q}(q)[a^{\pm 1}]$ to construct the HOMFLYPT *link invariant*.

If $L = \hat{\beta}$ for some n and $\beta \in Br_n$, then

$$\mathrm{HOMFLYPT}(\hat{\beta}) = (-a)^{e(\beta)} \mu_n(\beta),$$

where $e : Br_n \rightarrow \mathbf{Z}$ is the *writhe* map $\sigma_i \mapsto 1.$

Surprisingly, special values of HOMFLYPT are famous polynomials in combinatorics: q -Catalan numbers, q -Kirkman numbers, *etc*.

2 Lusztig Varieties We can geometrize Iwahori.

Fix a *positive* braid $\beta = \sigma_{i_1} \cdots \sigma_{i_\ell}$.

(Deligne) The variety

$$O(\beta) = \left\{ (g_0 B, g_1 B, \dots, g_\ell B) \left| \begin{array}{l} g_{j-1}^{-1} g_j \in B \dot{s}_{i_j} B \\ \text{for } j = 1, \dots, \ell \end{array} \right. \right\}$$

only depends on β , not $(i_1, i_2, \dots, i_\ell)$, up to isomorphisms that fix $g_0 B$ and $g_\ell B$.

In fact, if we fix \bar{g}_0, \bar{g}_ℓ such that $\bar{g}_0^{-1} \bar{g}_\ell \in B \dot{s} B$, then

$$\left| \left\{ \bar{g} B \in O(\beta)(\mathbf{F}_q) \left| \begin{array}{l} g_0 B = \bar{g}_0 B, \\ g_\ell B = \bar{g}_\ell B \end{array} \right. \right\} \right|$$

is the coefficient of h_w when we expand β in the Hecke operator basis of $H_n(q)$.

For any $x \in G(\mathbf{F}_q)$, form the *braid Lusztig variety*

$$\mathcal{B}(\beta)_x = \{ \bar{g} B \in O(\beta) \mid g_\ell B = x g_0 B \}.$$

(Shende–Treumann–Zaslow) Up to a monomial in q ,

$$\frac{|\mathcal{B}(\beta)_1(\mathbf{F}_q)|}{|\mathrm{PGL}_n(\mathbf{F}_q)|}$$

is the “highest” a -degree of $\mathrm{HOMFLYPT}(\hat{\beta})$ at $\mathbf{q} \rightarrow q$.

Example Let $n = 2$ and $\beta = \sigma_1^3 \in Br_2$.

$$\begin{aligned} O(\beta) &\simeq \{ \bar{g} \in (\mathbf{P}^1)^4 \mid g_0 \neq g_1 \neq g_2 \neq g_3 \}, \\ \mathcal{B}(\beta)_1 &\simeq \{ \bar{g} \in (\mathbf{P}^1)^3 \mid g_1 \neq g_2 \neq g_3 \}. \end{aligned}$$

PGL_2 acts simply transitively on the latter.

Indeed, $\mathrm{HOMFLYPT}(\hat{\sigma}_1^3) = a^2(\mathbf{q} + \mathbf{q}^{-1}) - a^4$.

3 Springer Fibers How to access other a -degrees?

One way uses Springer theory. Observe that

$$\mathcal{B}_x := \mathcal{B}(1)_x = \{gB \mid gB = xgB\}$$

is the usual Springer fiber over z , whose cohomology defines a character of S_n : namely,

$$Q_x(w) := \sum_i q^{i \operatorname{tr}(w \mid H^{2i}(\mathcal{B}_x))}.$$

Most interesting over the unipotent variety $\mathcal{U} \subseteq G$.

Thm 1 (T) Let

$$Q_\beta(w) = \frac{1}{|\operatorname{PGL}_n(\mathbf{F}_q)|} \sum_{u \in \mathcal{U}(\mathbf{F}_q)} |\mathcal{B}(\beta)_u(\mathbf{F}_q)| Q_u(w).$$

Then $(\chi_{(n-k, 1, \dots, 1)}, Q_\beta)_{S_n}$ sees the k th a -degree.

Example Again, let $n = 2$ and $\beta = \sigma_1^3 \in Br_2$.

$$|\mathcal{B}(\beta)_u| = \begin{cases} |\operatorname{PGL}_2| & u = 1, \\ q^3 & u \neq 1, \end{cases}$$

$$Q_u = \begin{cases} 1 + q \operatorname{sgn} & u = 1, \\ 1 & u \neq 1. \end{cases}$$

Moreover, $\operatorname{PGL}_2(\mathbf{F}_q) \curvearrowright \mathcal{U}(\mathbf{F}_q) - \{1\}$ transitively, with stabilizer of size q .

$$\begin{aligned} Q_\beta &= \frac{|\operatorname{PGL}_2|}{|\operatorname{PGL}_2|} \cdot (1 + q \operatorname{sgn}) + \frac{q^3}{q} \cdot 1 \\ &= 1 + q^2 + q \operatorname{sgn}. \end{aligned}$$

Thm 2 (T) The cohomology of $\mathcal{U}(\beta) \times_{\mathcal{U}} \mathcal{U}(1)$ sees finer invariants of $\hat{\beta}$, where

$$\mathcal{U}(\beta) = \{(u, \vec{g}B) \mid u \in \mathcal{U}, \vec{g}B \in \mathcal{B}(\beta)_u\}.$$

The *full twist* $\pi = (\sigma_1 \cdots \sigma_{n-1})^n$ generates $Z(Br_n)$.



Thm 3 (T) Suppose $\beta^m = \pi^d$ for some $d, m > 0$.

Then up to a monomial, $Q_\beta(w)$ equals

$$\frac{\text{sgn}(w)}{\det(1 - qw \mid \mathfrak{h})} \sum_{\lambda \vdash n} q^{c(\lambda)d/m} D_\lambda(e^{2\pi i d/m}) \chi_\lambda(w)$$

where:

- \mathfrak{h} is the *reflection representation*.
- $c(\lambda)$ is the sum of *contents* of λ .
- $D_\lambda(t) = K_{\lambda, (1^n)}(t)$ is the *fake degree* of λ .

Proof uses the character theory of S_n and H_n .

Thm 3 generalizes to any reductive G .

Replace S_n with the Weyl group W .

Replace c with $c(\chi) = \sum_{t \text{ refl.}} \frac{\chi(t)}{\chi(1)}$ and fake degrees with *generic degrees*:

$$D_\chi(t) \in \mathbb{Q}[t] \quad \text{such that} \quad R(q) = \bigoplus_{\chi \in \text{Irr}(W)} \chi_q^{\oplus D_\chi(q)}.$$

When $m = n$ and $\gcd(d, n) = 1$, the formula simplifies:

$$(\text{monomial}) \cdot \left. \frac{\det(1 - q^d w \mid \mathfrak{h})}{\det(1 - qw \mid \mathfrak{h})} \right\} =: \Pi_q^{(d)}.$$

$\Pi_q^{(d)}$ is the character of a *rational parking space*.

$(\text{triv}, \Pi_q^{(d)})_W$ is a *rational q -Catalan number*.

4 Affine Springer Fibers Now work over \mathbf{C} .

Rational parking spaces form modules over *rational Cherednik algebras* = *rational DAHAs*:

$$\mathfrak{H}_W = \frac{\mathbf{C}W \ltimes (\mathrm{Sym}(\mathfrak{h}) \otimes \mathrm{Sym}(\mathfrak{h}^\vee))}{\langle \text{relations} \rangle}.$$

finite Springer	affine Springer
G	$G((z))$
G/B	$G((z))/I$
W	$\widetilde{W} = W \ltimes X^\vee$
$\mathbf{C}W \ltimes \mathrm{Sym}(\mathfrak{h})$	$\mathbf{C}\widetilde{W} \ltimes \mathrm{Sym}(\mathfrak{h})$ or \mathfrak{H}_W

Above:

- $G((z))$ is the loop group $G((z))(R) := G(R((z)))$.
- I is the preimage of B in $G[[z]]$.
- X^\vee is the cocharacter lattice of B .

Henceforth, we consider Springer fibers over the Lie algebras \mathfrak{g} and $\mathfrak{g}((z))$, not the groups.

$$x : \quad \mathcal{B}_x = \{gB \in G/B \mid g^{-1}xg \in \mathfrak{b}\},$$

$$\gamma = \gamma(z) : \quad \mathcal{B}_\gamma^{\mathrm{aff}} = \{gI \in G((z))/I \mid g^{-1}\gamma g \in \mathfrak{I}\}.$$

The table hides some key differences:

In the **finite** case, \mathcal{B}_x is **interesting** for x nilpotent, and eas(ier) for x regular semisimple.

In the **affine** case, $\mathcal{B}_\gamma^{\mathrm{aff}}$ is *terribly infinite* for $\gamma = \gamma(z)$ nilpotent, but **interesting** for $\gamma(z)$ regular semisimple.

Example If $G = \mathrm{SL}_2$ and $\gamma(z) = \begin{pmatrix} 0 & 1 \\ z^3 & 0 \end{pmatrix}$, then

$$\mathcal{B}_\gamma^{\mathrm{aff}} \simeq \mathbf{P}^1 \sqcup_{\mathrm{pt}} \mathbf{P}^1.$$

Fixing $\nu = d/m > 0$ in lowest terms, $\mathbf{C}^\times \curvearrowright \mathfrak{g}((z))$:

$$c \cdot_\nu \gamma(z) = c^{2d\rho^\vee} \gamma(c^{2m} z) c^{-2d\rho^\vee},$$

where $2\rho^\vee = \sum_{\alpha \in \Phi^+} \alpha^\vee$.

Let $\mathfrak{g}((z))_{\nu,k}$ be the **weight- $2k$ eigenspace**.

Lemma If γ is an **eigenvector** for \cdot_ν , then the induced action on $G((z))/I$ fixes $\mathcal{B}_\gamma^{\text{aff}}$.

Lemma $\mathfrak{g}((z))_{\nu,0}$ is the Lie algebra of a connected reductive group L_ν . Moreover,

$$(G((z))/I)^{\mathbf{C}^\times} = \bigsqcup_{w \in W_\nu \setminus \widetilde{W}} L_\nu w I / I,$$

where W_ν is the Weyl group of L_ν .

Henceforth, $\gamma \in \mathfrak{g}((z))_{\nu,d}$.

By Springer, $\widetilde{W} \curvearrowright H_c^*(\mathcal{B}_\gamma^{\text{aff}}), H_{c,\mathbf{C}^\times}^*(\mathcal{B}_\gamma^{\text{aff}})$.

(Sommers) If m is the Coxeter number of W , then:

- L_ν is a torus, so $(\mathcal{B}_\gamma^{\text{aff}})^{\mathbf{C}^\times} \hookrightarrow \widetilde{W}$.
- Writing $H_{\mathbf{C}^\times}^*(\text{pt}) = \mathbf{C}[\epsilon]$, we have

$$H_c^*(\mathcal{B}_\gamma^{\text{aff}}) \simeq H_{c,\mathbf{C}^\times}^*(\mathcal{B}_\gamma^{\text{aff}})|_{\epsilon \rightarrow 1} \simeq H^0((\mathcal{B}_\gamma^{\text{aff}})^{\mathbf{C}^\times}).$$

- For $w \in W$, we have

$$\text{tr}(w \mid H_c^*(\mathcal{B}_\gamma^{\text{aff}})) = \lim_{q \rightarrow 1} \frac{\det(1 - q^d w \mid \mathfrak{h})}{\det(1 - qw \mid \mathfrak{h})}.$$

Example In the previous SL_2 example, $\gamma \in \mathfrak{g}((z))_{\nu,3}$.

Recall $\mathcal{B}_\gamma^{\text{aff}} = \mathbf{P}^1 \sqcup_{\text{pt}} \mathbf{P}^1$. It turns out $|(\mathcal{B}_\gamma^{\text{aff}})^{\mathbf{C}^\times}| = 3$.

(Goresky–Kottwitz–MacPherson) For general ν ,

$$(\mathcal{B}_\gamma^{\text{aff}})^{\mathbf{C}^\times} = \bigsqcup_{w \in W_\nu \setminus \widetilde{W}} \text{Hess}_{\gamma,w},$$

a disjoint union of *partial Hessenberg varieties*

$$\text{Hess}_{\gamma,w} = \{gP_{\nu,w} \in L_\nu/P_{\nu,w} \mid g^{-1}\gamma g \in \mathfrak{P}_{\nu,w}\},$$

where $P_{\nu,w} := L_\nu \cap \dot{w}I\dot{w}^{-1}$ and $\mathfrak{P}_{\nu,w} = \text{Lie}(P_{\nu,w})$.

They are smooth. They can be empty.

If $\text{Hess}_{\gamma,w} \neq \emptyset$, then its codim in $L_\nu/P_{\nu,w}$ is

$$\left| \left\{ \begin{array}{l} \text{hyperplanes } H \\ \text{in } X^\vee \otimes \mathbf{R} \text{ between} \\ \nu\rho^\vee \text{ and } w \cdot \frac{1}{n}\rho^\vee \end{array} \left| \begin{array}{l} H(\xi) = \langle \alpha, \xi \rangle + k, \\ \langle \alpha, \nu\rho^\vee \rangle = \nu, \\ \alpha \in \Phi, k \in \mathbf{Z} \end{array} \right. \right\} \right|.$$

Proof uses *Moy–Prasad theory*.

Conj (T) For general ν , the representation

$$W \curvearrowright H_{c,\mathbf{C}^\times}^*(\mathcal{B}_\gamma^{\text{aff}})|_{\epsilon \rightarrow 1}$$

contains a summand whose character is the $q \rightarrow 1$ limit of our earlier formula:

$$\frac{\text{sgn}(w)}{\det(1 - qw \mid \mathfrak{h})} \sum_{\chi \in \text{Irr}(W)} q^{c(\chi)\nu} D_\chi(e^{2\pi i\nu}) \chi(w).$$

Dream For certain choices $\gamma \leftrightarrow \beta$,

$$\mathcal{B}_\gamma^{\text{aff}} \quad \text{and} \quad [(\mathcal{U}(\beta) \times_{\mathcal{U}} \tilde{\mathcal{U}})/G]$$

have the “same” Springer theory.

Thank you for listening.