

2.

I motivate representation theory from scratch, focusing on the Maschke–Peter–Weyl theorem. All vector spaces will be over the complex numbers \mathbf{C} .

Throughout mathematics, we find the motif of “atoms” or “building blocks.” For instance, every whole number can be decomposed as a product of prime numbers. A fancier example: If a linear operator T acts on a vector space V , then to *diagonalize* T means to decompose V into a direct sum of eigenlines for T . Doing so clarifies the geometry of T , since it scales each eigenline by the associated eigenvalue.

Representation theory can be viewed as a generalization of the problem of finding eigenspaces for linear operators. But, instead of starting with a single linear operator on a fixed vector space, we start with an abstract *group*, i.e., a set G equipped with a composition law $\star : G \times G \rightarrow G$ that satisfies certain axioms (associativity, identity, inverses). A *representation* of G is a choice of vector space V , together with a linear operator $\rho(g) : V \rightarrow V$ for each $g \in G$, such that

$$\rho(g \star h) = \rho(g) \circ \rho(h)$$

for all $g, h \in G$. That is, ρ transforms the composition law of G into the composition of operators. If G is equipped with a topology that isn’t discrete, then ρ is required to satisfy a continuity property that we won’t discuss here.

If G is abelian, meaning $g \star h = h \star g$ for all g, h , then we can simultaneously diagonalize all the operators $\rho(g)$. In the language of representation theory, we’ve decomposed (V, ρ) into *subrepresentations*: namely, the simultaneous eigenlines. In general, a subrepresentation is defined as a subspace $V' \subseteq V$ such that $\rho(g)V' \subseteq V'$ for all g , which needn’t be a simultaneous eigenspace of the $\rho(g)$. A (nonzero) representation (V, ρ) is *irreducible*, or *simple*, iff its only subrepresentations are the zero subspace and itself.

If G is not abelian, then it may not be possible to diagonalize the $\rho(g)$ simultaneously. Nonetheless, there’s a more general statement where irreducible subrepresentations take the place of eigenlines.

Theorem 2.1 (Semisimplicity). *Suppose that G is compact (e.g., finite).*

- (1) (Maschke). *If V is finite-dimensional, then (V, ρ) is a direct sum of irreducible subrepresentations.*
- (2) (Peter–Weyl). *More generally, if V is a Hilbert space and $\rho(g)$ is unitary for all $g \in G$, then there is a dense subspace of (V, ρ) that is a direct sum of finite-dimensional irreducible subrepresentations.*

Below, the first two examples illustrate Maschke, while the last two illustrate Peter–Weyl. The first and third are abelian, while the second and fourth are nonabelian.

Example 2.2. *If $T : V \rightarrow V$ is an operator on a finite-dimensional vector space, and T^m is the identity map for some integer $m > 0$, then T is diagonalizable.*

Here, we set $G = \mathbf{Z}/m\mathbf{Z}$, the additive group of integers modulo m . We can turn V into a representation of G by setting $\rho(n) = T^n$. Using a famous result called *Schur's lemma*, we can check that any irreducible representation of an abelian group must be 1-dimensional. Since G is, moreover, finite, Theorem 2.1 now shows that (V, ρ) decomposes into a direct sum of 1-dimensional subrepresentations, i.e., lines that are stable under the operation of G . Each such line is an eigenline of T .

Example 2.3. Let $X \subseteq \mathbf{C}^d$ be a set of vectors that is stable under any permutation σ of the coordinates, i.e., $(x_1, \dots, x_d) \in X \implies (x_{\sigma(1)}, \dots, x_{\sigma(d)}) \in X$. If the span of X is nonzero, then it must be one of the following:

- (1) The subspace where $x_1 = \dots = x_d$.
- (2) The subspace where $x_1 + \dots + x_d = 0$.
- (3) \mathbf{C}^d itself.

Here, we take G to be the group of permutations of the set $\{1, \dots, d\}$, also known as the d th symmetric group, and $V = \mathbf{C}^d$. For every permutation σ , we define the linear operator $\rho(\sigma)$ by setting

$$\rho(\sigma)(x_1, \dots, x_d) = (x_{\sigma(1)}, \dots, x_{\sigma(d)}).$$

Let V' and V'' be the subspaces in items (1) and (2), respectively. Then $V = V' \oplus V''$ as vector spaces, and in fact, V' and V'' are both stable under $\rho(\sigma)$ for all σ , so they form subrepresentations of (V, ρ) . Since V' is 1-dimensional, it must be irreducible. It is harder, but still possible, to show that V'' is also irreducible. These results amount to the statement about X . They also imply that $V = V' \oplus V''$ is the decomposition predicted by Theorem 2.1.

Example 2.4 (Fourier Series). If $f : [0, 1] \rightarrow \mathbf{C}$ is square-integrable in the sense that $\|f\|_2 = \int_0^1 |f|^2 dx < \infty$, then there are constants $a_n \in \mathbf{C}$ such that

$$\sum_{-N \leq n \leq N} a_n e_n \xrightarrow{\|\cdot\|_2} f \quad \text{as } N \rightarrow \infty,$$

where $e_n(x) = e^{2\pi i n x}$.

Here, we set $G = \mathbf{R}/\mathbf{Z}$, the additive group of real numbers modulo 1, and $V = L^2(\mathbf{R}/\mathbf{Z})$, the Hilbert space of square-integrable functions on G . (Note that $[0, 1]$ is a fundamental domain for \mathbf{R}/\mathbf{Z} .) We define ρ by

$$(\rho(y)f)(x) = f(x - y)$$

for all $x, y \in G$ and $f \in L^2(G)$. Then the 1-dimensional subspace $\mathbf{C}e_n \subseteq V$ forms a subrepresentation of (V, ρ) because $\rho(y)e_n = e^{-2\pi i ny}e_n$ for all y . It turns out that the direct sum $V^\circ = \bigoplus_n \mathbf{C}e_n$ is dense in V with respect to the norm $\|\cdot\|_2$, so the function $f : [0, 1] \rightarrow \mathbf{C}$, viewed as a vector in V , is the limit of a sequence of vectors $f_N \in V^\circ$. The scalar a_n is the projection of f_N onto $\mathbf{C}e_n$ when $N \gg n$.

To state the next example, we need more terminology. In coordinates x, y, z on \mathbf{R}^3 , the *Laplacian* is the differential operator $\Delta = \partial_x^2 + \partial_y^2 + \partial_z^2$. A twice-differentiable function $f : \mathbf{R}^3 \rightarrow \mathbf{C}$ is *harmonic* iff $\Delta f = 0$. We let A_ℓ be the vector space of harmonic polynomials in x, y, z of homogeneous degree ℓ :

$$A_0 = \mathbf{C}, \quad A_1 = \mathbf{C}\langle x, y, z \rangle, \quad A_2 = \mathbf{C}\langle xy, xz, yz, x^2 - y^2, x^2 - z^2 \rangle, \quad \text{etc.}$$

Next, we let H_ℓ be the vector space of functions on the 2-dimensional sphere $S^2 \subseteq \mathbf{R}^3$ that arise from elements of A_ℓ by restriction of domain. The elements of H_ℓ are known as the *spherical harmonics of degree ℓ* .

It turns out that $\dim A_\ell = \dim H_\ell = 2\ell + 1$. Laplace introduced explicit functions $Y_\ell^m : S^2 \rightarrow \mathbf{C}$, with $-\ell \leq m \leq \ell$, that together form a basis of H_ℓ .

Example 2.5 (Spherical Harmonics). (1) *For any choice of ℓ, m with $|m| \leq \ell$, the vector space H_ℓ is spanned by the functions we get by rotating Y_ℓ^m . Conversely, any such function belongs to H_ℓ .*

(2) *If $f : S^2 \rightarrow \mathbf{C}$ is square-integrable, then there are constants $a_\ell^m \in \mathbf{C}$ such that*

$$\sum_{0 \leq \ell \leq N} \sum_{-\ell \leq m \leq \ell} a_\ell^m Y_\ell^m \xrightarrow{\|\cdot\|_2} f \quad \text{as } N \rightarrow \infty,$$

where $\|f\|_2 = \int_{S^2} |f|^2 dA$.

Here, we set $G = \mathrm{SO}(3)$, the group of 3×3 rotation matrices. Explicitly,

$$\mathrm{SO}(3) = \{g \in \mathrm{Mat}^{3 \times 3}(\mathbf{R}) : g^t g = I\},$$

where I is the identity matrix and g^t is the transpose of g . We take $V = L^2(S^2)$ and

$$(\rho(g)f)(x, y, z) = f(g^{-1}(x, y, z)),$$

where $g^{-1}(x, y, z)$ means we apply the matrix g^{-1} to the vector (x, y, z) . As every continuous function on S^2 is square-integrable, H_ℓ is a subspace of V for all $\ell \geq 0$. Item (1) says that, more strongly, H_ℓ is an irreducible subrepresentation of (V, ρ) . Item (2) says that the direct sum $\bigoplus_\ell H_\ell$ is dense in V .

Here are two examples where G is not compact and the conclusion of Theorem 2.1 fails, but a weaker conclusion does hold.

- (1) Take $G = \mathbf{Z}$. If $\rho(1)$ is not diagonalizable, then V won't be a direct sum of irreducible subrepresentations. But the *Jordan normal form* of $\rho(1)$ essentially encodes how the generalized eigenspaces of $\rho(1)$ are *filtered* by irreducibles.
- (2) Take $G = \mathbf{R}$ and $V = L^2(\mathbf{R})$. Then V is not a direct sum, but a *direct integral* of 1-dimensional subrepresentations. This integral is essentially described by the *Fourier transform* on $L^2(\mathbf{R})$.