10-1-2024 ①

$\boxed{\text{Minh-Tam's Visit}}$

$$\mathcal{B}_x = \{ gB \in G/B \mid x \in gBg^{-1} \Leftrightarrow g^{-1}xg \in B \}$$

gB is a f.p. under the vector field on G/B induced by $x \in \mathfrak{g} := T_e(G)$.

$$\mathcal{B}_\gamma^{aff} = \{ gI \in G((z))/I \mid g^{-1}\gamma g \in I \}$$

$$\mathcal{B}^{aff} = G((z))/I \qquad G[[z]](\mathbb{C}) = G(\mathbb{C}[[z]])$$

$$\cap$$

$$G((z))(\mathbb{C})$$

$$\begin{array}{ccc} I & \subseteq & G[[z]] \\ \downarrow & & \downarrow {\scriptstyle z \to 0} \\ B & \subseteq & G \end{array}$$

$$I = \begin{pmatrix} \mathbb{C}[[z]]^* & \mathbb{C}[[z]] \\ z\mathbb{C}[[z]] & \mathbb{C}[[z]]^* \end{pmatrix} \subseteq GL_2$$

Corvallis volumes

Tits "Red gps over local fields"

$\updownarrow$

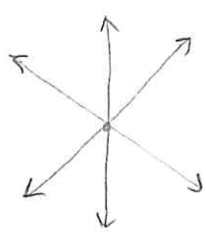
Moy-Prasad theory

Clay Math

Kottwitz - DeBacker

<u>Classical Picture:</u>

Φ root system:



$$w \cdot t = w t w^{-1}$$

$$\mathrm{Hom}(GL_1, GL_1) \simeq \mathbb{Z}$$

max'l torus Borel

$T \subset B \subset G \rightsquigarrow W := N_G(T)/T$

A such pairs (B,T) are conjugate in G

finite group acting on $X^*(T) = \mathrm{Hom}(T, GL_1) \overset{\text{char}}{\simeq} \mathbb{Z}^{rk(T)}$

$X_*(T) = \mathrm{Hom}(GL_1, T)$ cochar

} <u>lattices</u>

$\lambda \in X^*(T)$

<u>Ex</u>. $G = SL_2$; $T = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \middle| a \in \mathbb{C}^* \right\}$ ($\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} = t$)

$$(X^*(T), \cdot) \simeq (\mathbb{Z}, +)$$

$$\lambda : t \mapsto a^n \longleftarrow n$$

$X^*(T)$ and $X_*(T)$ are dual:

$X_* \times X^* \longrightarrow \mathbb{Z}$ composition ($\mathbb{Z} = \mathrm{Hom}(GL_1, GL_1)$)

$$\Rightarrow \underset{\text{Lie}(G)}{\mathfrak{g}} = \underset{\mathfrak{g}_{\mathbb{1}}}{\mathfrak{t}} \oplus \bigoplus_{\alpha \in \Phi} \underset{\text{Lie}(U_\alpha)}{\mathfrak{g}_\alpha} \; ; \quad \Phi \text{ root system; a finite set}$$

$$\mathfrak{g}_\alpha = \{ x \in \mathfrak{g} \mid t x t^{-1} = \alpha(t) x \}$$

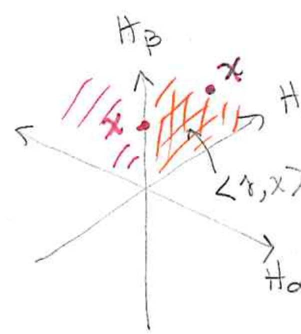
$T \curvearrowright \mathfrak{g}$ by the Adjoint action

For each char $\alpha : T \to GL_1$ in $X^*(T)$ $\Rightarrow \mathfrak{g}_\alpha$ is the α-generalized e-space of $\mathfrak{g}$.

<u>Thus</u>: $\Phi = \{ \alpha \in X^*(T) \mid \mathfrak{g}_\alpha \neq 0, \alpha \neq \mathbb{1} \}$.

Fix $T \subset G$.

$\underbrace{X_*(T)}_{\substack{\text{lattice} \\ = \mathbb{Z}\text{-mod}}} \otimes_{\mathbb{Z}} \mathbb{R}$ } $\mathbb{R}$-vector space



Cochar lattice for SL3

$\alpha \rightsquigarrow$ linear map $X_* \otimes \mathbb{R} \to \mathbb{R}$

$H_{\alpha+\beta} = \ker(\text{map})$

$x \in X_* \otimes \mathbb{R}$ defines a subgp $P_x \subseteq G$ that contains T.
Parabolic subgroup

$P_0 = G$; $P_x = B$ if x is in the interior part of the dom chamber

$$W = \langle s_\gamma \mid \gamma \in \Phi \rangle$$

$\underset{X_*(T) \otimes \mathbb{R}}{x} \rightsquigarrow W_x = \{ w \in W \mid w \cdot x = x \}$.
$= \mathrm{Stab}_W(x)$

$$\Rightarrow P_x = \coprod_{w \in W_x} BwB \subseteq G$$

$$\mathfrak{p}_x = \mathfrak{t} \oplus \bigoplus_{\substack{\alpha \text{ s.t.} \\ \langle \alpha, x \rangle \geq 0}} \mathfrak{g}_\alpha$$

<u>Why a subgp</u>? $BwB \cdot Bw'B \subset BW_xB$ $\forall w, w' \in W_x$

$= \bigoplus \mathfrak{g}_{x,r}$ $\quad \mathbb{C}\langle \alpha \mid \langle \alpha, x \rangle = r \rangle$

$W_x = N_P(T)/T$

Add a hyperplane that does not pass through the origin

Affine Story

$\Phi \rightsquigarrow \Phi^{aff} = \Phi \times \mathbb{Z}$

H_α $\qquad H_{\alpha,m} = \{x \in X_*(T) \otimes_{\mathbb{Z}} \mathbb{R} \mid \langle \alpha, x \rangle + k = 0\}$.

affine hyperplanes

G $\qquad G((z))$

$T_e(G) = \mathfrak{g}$ $\qquad T_e G((z)) =: \mathfrak{g}((z)) =$

$r \in \mathbb{R}$, x in the dom. chamber

For each $r \in \mathbb{R}$, define $\mathfrak{g}((z))_{x,r} = \langle z^k \mathfrak{g}_\alpha \mid \langle \alpha, x \rangle + k = r \rangle$

$\oplus \begin{cases} z^k \mathfrak{t} & r \in \mathbb{Z} \\ 0 & r \notin \mathbb{Z} \end{cases}$

allow infinite sums

Def: $\mathfrak{p}_x := \widehat{\bigoplus_{r \geq 0}} \mathfrak{g}((z))_{x,r}$

Lie subalg of $\mathfrak{g}((z))$ containing $\mathfrak{t}[[z]]$.

$W^{aff} \simeq W \ltimes X_*$

$\wr\!\!\shortparallel$

$N_{G((z))}(T[[z]])/T[[z]] \supseteq N_{G((z))}(T((z)))/T((z)) \simeq W$

$T((z)) \simeq (\mathbb{C}((z))^*)^d$ $\qquad$ and $\qquad T((z))/T[[z]] \simeq X_*(T)$

$T[[z]] \simeq (\mathbb{C}[[z]]^*)^d$

$G((z)) = \coprod_{w \in W^{aff}} \underbrace{IwI/I}_{\text{affine Schubert cells.}}$ $\qquad$ I is the Iwahori subgroup

are all conjugate if G simply connected.

$\mathbb{P}_x$; $\mathrm{Lie}(\mathbb{P}_x) = \mathfrak{p}_x$

for x generic

$\langle \alpha, x \rangle + k \neq 0$

$\forall k$

Ex. $G = SL_2$ $\qquad \langle \alpha, x \rangle \in \mathbb{R}$

$\xrightarrow{\quad \circ \quad \overset{x}{\circ} \quad} X_* \otimes \mathbb{R}$

If $0 < \langle \alpha, x \rangle < 1 \Rightarrow \mathcal{P}_x = \mathrm{Lie}(I) = \begin{bmatrix} \mathbb{C}[[z]] & \mathbb{C}[[z]] \\ z\,\mathbb{C}[[z]] & \mathbb{C}[[z]] \end{bmatrix}$ ← sum to zero

$GL_1 = \mathbb{C}^\times$
$\cong$
$\mathbb{G}_m \circlearrowright G((z)) \ni g(z)$ We define an action which depends on x. $\in X_* \otimes_{\mathbb{Z}} \mathbb{Q}$

($\mathbb{G}_m \circlearrowright I$ stab the Iwahori)

Lusztig, Sommers, O-Y: $\quad c \circ_x g(z) \overset{\cdot\cdot}{=} c^{dx_0} g(c^e z) c^{-dx_0}$

Can always write
$x = \frac{d}{e} x_0$; $\frac{d}{e} \in \mathbb{Q}^*$,
$x_0 \in X_*$

lattice $X_* \subseteq X_* \otimes_{\mathbb{Z}} \mathbb{Q}$
$\cap$
$X_* \otimes_{\mathbb{Z}} \mathbb{R}$

$x = \sum a_i x_i$
$a_i \in \mathbb{Q}$; $x_i \in X_*$
s.t. $a(x_i + x_j) = ax_i + ax_j$
$(a+b)x = ax + bx$

In general, $\lambda \in X_* \in \mathrm{Hom}(\mathbb{G}_m, T)$

c^λ = image of c in $T(\mathbb{C})$

The action $c \circ_x -$ is a group automorphism of $G((z))$
$\Rightarrow$ induces an action on $\mathfrak{g}((z))$ by Lie alg automor.

Lemma. $\mathfrak{g}((z))_{x,r}$ is the (e,r) wt eigenspace of the action.

$$(c \circ_x \xi) = c^{er} \xi \ ; \ r \in \tfrac{1}{e}\mathbb{Z}.$$

Lemma: If γ is an e-vector of $\mathbb{G}_m$ w.r.t. $c \circ_x -$ then
$\mathbb{G}_m \circlearrowright G((z))/I$ and stab the affine Springer fiber $\mathcal{B}^{aff}_\gamma = \{ gI \in \mathcal{B}^{aff} \mid g^{-1}\gamma g \in \mathrm{Lie}(I) \}$

In general,

$$(\mathcal{B}^{aff})^{\mathbb{G}_m^*} = \coprod_{w \in W^{aff}} L_x w I / I$$

where L_x is the connected red. alg gp

Lie $L_x = \mathfrak{g}((z))_{x,0}$ (zero e-space of x)

(α, k) s.t. $\langle \alpha, x \rangle + k \geq 0$

(α, k) s.t. $\langle \alpha, x \rangle + k = 0$ $\longrightarrow$ hyperplanes through x give the roots for L_x

Choice of γ, an eigenvector of $\mathbb{G}_m$

$\rightsquigarrow$ Not all choices are nice

$\rightsquigarrow$ Want γ to be reg ss. $\hookrightarrow$ diag. w/ distinct e-values in type A.

Sommers

$$\gamma = \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ z^d & & & 0 \end{pmatrix} \sim \begin{pmatrix} z^{d/4} & & & 0 \\ & iz^{d/4} & & \\ & & -z^{d/4} & \\ 0 & & & -iz^{d/4} \end{pmatrix}$$

char poly in $\mathbb{C}((z))[T]$.

elliptic case ; $G((z))_\gamma$ totally non-split torus

semisimple

Λ_γ is basically trivial

$\rightsquigarrow$ germ of a plane curve in z, T

$\mathbb{C}((z))$ is not alg closed!!

reg ss $\gamma \rightsquigarrow$ plane curve

Conversely, $T^n + a_1 T^{n-1} + \dots + a_n \tau$

$$\begin{bmatrix} 0 & 1 & & & \\ 0 & 0 & 1 & & \\ 0 & 0 & 0 & 1 & \\ \vdots & & & & \\ a_n & a_{n-1} & \cdots & a_1 \end{bmatrix} \leftarrow \text{matrix for } T\text{-action in the Quotien } \mathbb{C}[T]/(p(T))$$

reg ss. elt for your $G((z))$

Nilpotent analogies:

$Z_{G((z))}(\gamma) \circlearrowright \mathcal{B}_\gamma^{aff} \xrightarrow[H_c^*]{\text{Sommers}} \mathcal{B}_\gamma^{aff}$

structure of this group changes alot

γ reg nilp, $\longrightarrow$ centralizer "pro-unit" group

$$\left(G((z))_\gamma \backslash \mathcal{B}_\gamma^{aff} \right) \qquad \gamma g I = g I \quad \rightsquigarrow G((z))_{\gamma, gI}$$

$$= \sum_{gI \in \mathcal{B}_\gamma^{aff}} \frac{t^{\deg(gI)}}{\text{meas}\left(\text{size of } \underbrace{G((z))_{\gamma,gI}}_{q\text{-polyn}}\right)} < \infty$$

γ nilp $G((z))_\gamma$ is huge, but necessary

γ reg. ss $G((z))_\gamma$ is a nonsplit torus

field ext of the base pt.

<u>type A:</u> $T_{d_1} \times T_{d_2} \times \cdots$; $T_{d_i} = \mathbb{C}((z^{1/d_i}))^\times \subseteq GL_{d_i}(\mathbb{C}((z$
deg d_i

"Weil restriction of tori"

field ext of $k((t)) \longrightarrow$ Very hard problem

Interesting case $\left(\begin{array}{c|c} \begin{matrix} 0 & 1 \\ z^d & 0 \end{matrix} & \\ \hline & \begin{matrix} 0 & 1 \\ z^e & 0 \end{matrix} \end{array}\right)$

Springer $\Rightarrow$ Lusztig: Affine in title.

Oblomkov-Yun $\rightsquigarrow$ <u>Dot action</u>

$$\left(\begin{array}{c} \text{reg ss in} \\ \mathfrak{g}((x))^{reg,ss}_{\frac{d}{m}\rho^\vee}, \ d \underset{r}{\uparrow} \\ \uparrow \\ \text{lowest terms} \end{array} \Big/ L_{\frac{d}{m}\rho^\vee} \right) \overset{\text{O-Y}}{\cong} V_{d,m} \big/ W_{d,m}$$

vector space

reflection gp

m fixed $\qquad 3 = d^{m-1}$

$W_{d,m} \subset W$ is a complex reflection gp

$G = SL_2$

$\begin{pmatrix} 0 & 1 \\ z^3 & 0 \end{pmatrix} \to \mathcal{B}_\gamma^{aff} = \mathbb{P}^1 \amalg_{pt} \mathbb{P}^1$

Ex. $W = S_n$; $S_m \wr C_\ell$

$\begin{pmatrix} z & 0 \\ 0 & -z \end{pmatrix} \to \mathcal{B}_\gamma^{aff} =$ ... infinite "Hess paving"

lattice action

$$\left(\mathcal{B}_\gamma^{aff}\right)^{\mathbb{G}_m} = \coprod_{w \in W_{d/m\rho^\vee} \backslash W} \mathrm{Hess}_{d/m\rho^\vee, w, \gamma}$$

$\uparrow$ in $L_{\frac{d}{m}\rho^\vee} \Big/ \underbrace{P_{d/m\rho^\vee, w}}_{\parallel}$

$L_{\frac{d}{m}\rho^\vee} \cap w I w^{-1}$

$\left(\Lambda_\gamma \backslash \mathcal{B}_\gamma^{aff}\right)$ is more managable $\to$ Lusztig; same fundamental domain

Vilonen - Xue ; T.-H. Chen, Grinberg

totally split $\rightsquigarrow$ ~~total~~ ellipti

"split rank of centralizer increases"

m = Coxeter $\Rightarrow$ Hess are all pts $\rightsquigarrow$ Catalan comb.

$\Lambda_\gamma \circlearrowright \mathcal{B}_\gamma^{aff}$

$\parallel$

$\langle z^\lambda \mid \lambda : GL_1 \to G((z$

$\begin{smallmatrix} 0 & 1 & \\ & 0 & 1 \\ z^4 & & 0 \end{smallmatrix} \to$ block structures

$\begin{pmatrix} z^{d_1} & & & \\ & z^{d_2} & & \\ & & \ddots & \\ & & & z^{d_n} \end{pmatrix}$ "split case" $\to$ well-studied.

totally split case

$\Lambda_\gamma \cong X_*(T)$ $\leftarrow$ $G((z))_\gamma$ = diag matrices in $T((z))$