MATH 430: INTRODUCTION TO TOPOLOGY
PROBLEM SET #8

SPRING 2025

Due Wednesday, April 16. You may consult books, papers, and websites as long as you cite all sources and write up your solutions in your own words.

Problem 1 (Munkres 366, #2(a)–(e))**.** For each of the following spaces, the fundamental group is either trivial, infinite cyclic, or isomorphic to the fundamental group of the figure eight. Determine, for each space, which option is the case. You do not need to give explicit homeomorphisms or homotopy equivalences, but you should at least give informal descriptions where needed.

(1) The "solid torus" $B^2 \times S^1$, where B^2 is the closed disk.
(2) The (surface) torus $S^1 \times S^1$ with one point removed.
(3) The cylinder $S^1 \times I$.
(4) The infinite cylinder $S^1 \times \mathbf{R}$.
(5) $\mathbf{R}^3$ with the nonnegative portions of the x, y, and z-axes deleted.

Problem 2 (Munkres 370, #3)**.** Show that:

(1) $\mathbf{R}^1$ and $\mathbf{R}^n$ are not homeomorphic if $n > 1$.
(2) $\mathbf{R}^2$ and $\mathbf{R}^n$ are not homeomorphic if $n > 2$.

Problem 3 (Munkres 412, #4(a))**.** Recall that a group $(G, \star)$ is *abelian* if and only if $a \star b = b \star a$ for all $a, b \in G$. In this case, we often write $+$, 0, and ma in place of $\star$, e_G, and a^m, respectively. The *order* of an element $a \in G$ is the smallest positive integer m such that $ma = 0$, if it exists, and infinity otherwise.

Show that if G is abelian, then its elements of finite order form a subgroup. It is called the *torsion subgroup* of G. If it is trivial, then we say that G is *torsion-free*.

Problem 4 (Munkres 412, #4(c))**.** A *free abelian group* is an (abelian) group isomorphic to a (possibly infinite) product of infinite cyclic groups. Show that the additive group of rational numbers $\mathbf{Q}$ is not free abelian, even though it is torsion-free.

Hint: Suppose that there is an isomorphism $\psi : \mathbf{Z}^I \xrightarrow{\sim} \mathbf{Q}$, for some index set I. Show that ψ cannot be injective and surjective at the same time.

Problem 5. Let $\{X_\alpha\}_\alpha$ be an arbitrary collection of path-connected topological spaces, and let $X = \prod_\alpha X_\alpha$. For each α, fix a basepoint $x_\alpha \in X_\alpha$, and set $x = (x_\alpha)_\alpha \in X$.

(1) Give mutually inverse isomorphisms between $\prod_\alpha \pi_1(X_\alpha, x_\alpha)$ and $\pi_1(X, x)$, and verify that they are isomorphisms.
(2) Use (1) to show that every free abelian group is a fundamental group.

With much more work, one can show that every group is the fundamental group of some path-connected topological space.

Problem 6 (Munkres 425, #2(a))**.** Let G_1, G_2 be groups. It is possible to define the free product $G_1 * G_2$ in terms of "reduced words in the elements of G_1 and G_2" under the operation of first concatenating, then reducing—much like how we defined free groups. See Munkres 415–416 for the definitions of *words* and *reduced words in the elements of a collection of groups*.

Using this approach, show that if G_1 and G_2 are nontrivial, then $G := G_1 * G_2$ is not abelian. *Hint:* If it were, then some nonempty reduced word in the elements of G_1 and G_2 would equal the empty word.

Problem 7 (Munkres 425, #2(b)–(c))**.** We keep the setup of Problem 6. For all $x \in G$, we define the *length* of x to be its length as a reduced word in the elements of G_1 and G_2.

(1) Show that if x has even length at least 2, then it cannot have finite order. Show that if x has odd length at least 3, then it is conjugate to some element of shorter length. Here, recall that the *conjugates* of x are the elements gxg^{-1}, where $g \in G$.

(2) Deduce that any element of G of finite order is conjugate to an element of G_1 or G_2 of finite order.

Problem 8 (Munkres 438, #2)**.** Let X be the joint union of closed subspaces $X_1, \ldots, X_n$. We say that X is their *wedge* if and only if it has a point p such that $X_i \cap X_j = \{p\}$ for any pair of distinct indices i and j. Suppose that this is the case, and for all i, the singleton $\{p\}$ is a deformation retract of an open subset W_i of X_i. Show that $\pi_1(X, p)$ is isomorphic to the (iterated) free product of the groups $\pi_1(X_i, p)$ for $1 \le i \le n$.