**MATH 340: ADVANCED LINEAR ALGEBRA**
**PROBLEM SET #8**

SPRING 2025

**Due Wednesday, April 16.** You may consult books, papers, and websites as long as you cite all sources and write up your solutions in your own words.

**Problem 1.** In the notation of Problem Set 7, #6, let

$$\Phi : \mathrm{Bil}(W, V \mid U) \to \mathrm{Hom}(W, \mathrm{Hom}(V, U))$$

be the map such that, for all $\beta \in \mathrm{Bil}(W, V \mid U)$, we have $[\Phi(\beta)(w)](v) = \beta(w, v)$. Generalizing what we proved about $\mathrm{Bil}(W, V)$ in class, show that $\Phi$ is linear and bijective, <u>without</u> picking explicit bases for the vector spaces involved.

*Combining this with Problem Set 7, #6, we obtain a injective linear map*

$$\mathrm{Hom}(W \otimes V, U) \to \mathrm{Hom}(W, \mathrm{Hom}(V, U))$$

*independent of bases. It turns out to be an isomorphism for general $U, V, W$, though that is difficult to show starting from Axler's definition of $W \otimes V$. For details, see Atiyah–Macdonald, Introduction to Commutative Algebra, Ch. 2.*

**Problem 2.** Let $V$ be a real vector space. Using #5 from Problem Set 7, and the definition of $V_{\mathbf{C}}$ in Problem Set 2, give a linear isomorphism

$$V_{\mathbf{C}} \xrightarrow{\sim} \mathbf{C} \otimes_{\mathbf{R}} V$$

(and verify that it is one), <u>without</u> picking an explicit basis for $V$. Above, $\otimes_{\mathbf{R}}$ just means $\otimes$, but emphasizes that we view $\mathbf{C}$ as a real vector space. We view $\mathbf{C} \otimes_{\mathbf{R}} V$ as a complex vector space by setting $b \cdot (a \otimes v) := (ba) \otimes v$ for all $a, b \in \mathbf{C}$.

**Problem 3.** Let $F \in \{\mathbf{R}, \mathbf{C}\}$. Recall that a bilinear form $\beta : V \times V \to F$ is called *symmetric* if and only if $\beta(w, v) = \beta(v, w)$ for all $w, v$, and *alternating* if and only if $\beta(v, v) = 0$ for all $v$. Let $\mathrm{Sym}^2(V)$ and $\mathrm{Alt}^2(V)$ denote the sets of symmetric and alternating bilinear forms on $V$, respectively. Give a linear isomorphism

$$\mathrm{Bil}(V, V) \xrightarrow{\sim} \mathrm{Sym}^2(V) \oplus \mathrm{Alt}^2(V)$$

(and verify that it is one), <u>without</u> picking an explicit basis for $V$. *Hint:* Relate the alternating property to *antisymmetry*. Reflect on how this problem resembles several earlier problems.

**Problem 4.** Show that

$$\langle f, g \rangle = \int_{-1}^{1} f(t)\overline{g(t)}\, dt$$

defines an inner product on the complex vector space $\mathbf{C}[t]$. You may take for granted that polynomials are continuous functions, and that nonzero polynomials have finitely many zeros.

**Problem 5.** Let $V$ be an inner product space, and let $u, v \in V$.

(1) Show that if $V$ is real, then $u$ and $v$ are orthogonal if and only if $\|u+v\|^2 = \|u\|^2 + \|v\|^2$.
(2) Show that if $V$ is complex, then the conclusion to (1) can fail.

**Problem 6.** Let $V$ be an inner product space, and let $u, v \in V$.

(1) Show that if $V$ is real, then $u - v$ and $u + v$ are orthogonal if and only if $\|u\| = \|v\|$.
(2) Make a sketch of (1) where $V = \mathbf{R}^2$ and $u, v$ are nonzero. What does it mean, in terms of the geometry of plane triangles?
(3) Suppose that $V$ is complex. Does the conclusion to (1) still hold?

**Problem 7.** Let $V$ be an inner product space, and let $r \in V$ be nonzero. A *reflection* over $r$ is a linear operator $S_r : V \to V$ satisfying the following conditions:

(I) $S_r(r) = -r$.
(II) $S_r(v) = v$ if and only if $\langle r, v\rangle = 0$.

These conditions determine a unique linear operator. For arbitrary $v \in V$, give an explicit formula for $S_r(v)$ in terms of $r$, $v$, and various inner products. *Hint:* Write $v$ as the sum of a scalar multiple of $r$ and a vector orthogonal to $r$.

**Problem 8.** We say that a linear operator $T$ on an inner product space $V$ is *orthogonal* if and only if $\langle Tu, Tv\rangle = \langle u, v\rangle$ for all $u, v \in V$. Show that:

(1) In the sense of Problem 7, any reflection is orthogonal.
(2) If $V$ is the real inner product space formed by $\mathbf{R}^2$ under the dot product, then any rotation in $V$ is orthogonal.