last time: inner products

today: how they interact with
- bases
- lin maps/ops

fix an $n \times n$ real matrix M and let

$$\beta(u, v) = u^t M v \quad \text{for all } u, v \text{ in } R^n$$

when is this bilinear form:

- positive?
- definite?
- symmetric?

[useful trick:] $\beta(e_j, e_i) = M_{j, i}$

$\beta(e_i, e_j) = M_{i, j}$

so if β is symmetric, then M is symmetric

[is the converse true?]

in general,

$u = \sum_j b_j e_j$

$v = \sum_i c_i e_i$

$\beta(u, v) = \sum_{j, i} b_j c_i \beta(e_j, e_i)$

if M is symmetric

then $\beta(e_j, e_i) = \beta(e_i, e_j)$ for all i, j

so $\beta(u, v) = \sum_{i, j} c_i b_j \beta(e_i, e_j) = \beta(v, u)$

altogether, β is symmetric <u>iff</u> M is symmetic

for positive-definiteness: need $\beta(v, v) > 0$ for $v \neq \mathbf{0}$

only have $\beta(v, v) = \sum_{j, i} c_j c_i \beta(e_j, e_i)$

so previous trick doesn't work here

(Axler §6B) let V be a vec. sp. over $F = R$,
resp. $F = C$
let $\langle\ ,\rangle$ be bilin., resp. skew-lin.

<u>Thm</u> if V has fin. dim. and
$\langle\ ,\ \rangle$ is an inner product,
then we can find a basis $(u_i)_i$ for V s.t.

$$\langle u_j, u_i\rangle = \begin{cases} 1 & \text{if } j = i \\ 0 & \text{if } j \neq i \end{cases}$$

<u>Df</u> a list of vec's satisfying these conditions is said to be orthonormal (wrt $\langle\ ,\ \rangle$)

<u>Ex</u> the std basis of F^n is orthonormal wrt $\langle u, v\rangle = u \bullet v$, resp. $\langle u, v\rangle = u \bullet v^*$

restate thm for R^n:
suppose $\langle u, v\rangle = u^t M v$ and $(u_i)_i$ is orthonormal wrt $\langle\ ,\ \rangle$
let $(e_i)_i$ be the standard basis
let $P : R^n$ to R^n be def by $Pe_i = u_i$

$\langle u_j, u_i\rangle = \langle Pe_j, Pe_i\rangle = e_j^t P^tMP\, e_i$
also $\langle u_j, u_i\rangle = e_j \bullet e_i = e_j^t e_i$

so $(P^tMP)_{j, i} = I_{j, i}$ for all j, i

<u>Thm for R^n</u> let $\langle u, v\rangle = u^t Mv$

if $\langle\ ,\ \rangle$ defines an inner product, then $P^tMP = I$
for some invertible P

[modify for $\mathbb{C}^n$?] need skew-linear $\langle\ ,\ \rangle$

<u>Thm for $\mathbb{C}^n$</u> let $\langle u, v\rangle = u^t M v^*$

if $\langle\ ,\ \rangle$ defines an inner product, then $P^tMP^* = I$
for some invertible P

<u>Pf of General Thm</u> $\langle\ ,\ \rangle$ inner product on V

pick an arbitrary ordered basis $(v_1, \ldots, v_n)$
want to build a new, orthonormal basis $(u_i)_i$
enough to build an orthogonal basis $(f_i)_i$:

$\langle f_j, f_i\rangle \neq 0$ if $j = i$
$\langle f_j, f_i\rangle = 0$ if $j \neq i$

[why?] can then set $u_i = f_i/\|f_i\|$ for all i

$\langle u_i, u_i\rangle = \langle f_i, f_i\rangle/\|f_i\|^2 = 1$
$\langle u_j, u_i\rangle = \langle f_j, f_i\rangle/(\|f_j\|\,\|f_i\|) = 0$ if $j \neq i$

<u>Gram–Schmidt Process</u>

define f_i's recursively:
$f_1 = v_1$
$f_k = v_k - \sum_{j < k} \mathrm{proj}_{f_j}(v_k)$
$= v_k - \sum_{j < k} (\langle v_k, f_j\rangle/\langle f_j, f_j\rangle)\, f_j$

check: if $i < k$, then $\langle f_k, f_i\rangle$
$= \langle v_k, f_i\rangle$
$- \sum_{j < k} (\langle v_k, f_j\rangle/\langle f_j, f_j\rangle)\,\langle f_j, f_i\rangle$
$= \langle v_k, f_i\rangle - \langle v_k, f_i\rangle$
$- \sum_{j \neq i} (\langle v_k, f_j\rangle/\langle f_j, f_j\rangle)\,\langle f_j, f_i\rangle$
$= 0 - 0 - 0$ by inductive hypothesis □

Rem process shows more:

- in any V, any finite orthonormal list can be extended to a longer (finite) list

- if V is fin.-dim'l, then have a direct sum $V = U + U^\perp$ for any linear U sub V, where $U^\perp = \{v \text{ in } V \mid \langle v, U\rangle = 0\}$ [why?]

$U^\perp$ is called the orthogonal complement to U in V

Rem if $u_1, \ldots, u_m$ in V is an orthonormal list then it is linearly indep, so $m \leq \dim(V)$

[why?] if $a_1u_1 + \ldots + a_mu_m = \mathbf{0}$ then applying $\langle u_i, –\rangle$ gives $a_i = \mathbf{0}$

Ex $\langle p, q\rangle = \int_{-1}^1 p(t)q(t)\,dt$ is an inner product on $R[x]$ [compare to PS8, #4]

$\langle 1, 1\rangle = 2$, $\langle 1, x\rangle = 0$, $\langle 1, x^2\rangle = 2/3$
$\langle x, x\rangle = 2/3$ $\langle x, x^2\rangle = 0$
$\langle x^2, x^2\rangle = 2/5$

let $V = \{p \text{ in } R[x] \mid p = 0 \text{ or } \deg(p) \leq 2\}$

$(v_1, v_2, v_3) = (1, x, x^2)$

$f_1 = v_1 = 1$

$f_2 = v_2 – \langle v_2, f_1\rangle/\langle f_1, f_1\rangle f_1 = x$

$f_3 = v_3 – \langle v_3, f_1\rangle/\langle f_1, f_1\rangle f_1 – \langle v_3, f_2\rangle/\langle f_2, f_2\rangle f_2$
$= x^2 – [(2/3)/2]\, 1 = x^2 – 1/3$

$(u_1, u_2, u_3) = (1/\sqrt{2}, x\sqrt{(3/2)}, (x^2 – 1/3)\sqrt{(5/2)})$

<u>Rem</u> let $K : \mathbb{R}$ to $\mathbb{R}$ be nice-enough that
$\langle p, q\rangle = \int_{-\infty}^{\infty} p(t)q(t)K(t)\,dt$
defines an inner product on $\mathbb{R}[t]$

we say p, q are orthogonal wrt K iff they are orthogonal wrt $\langle\ ,\ \rangle$

certain K give useful families of <u>orthogonal polys</u>
(often not orthonormal in literature)

<u>Legendre polys</u> P_n are orthogonal wrt $\delta_{[-1, 1]}$
but scaled so that $P_n(1) = 1$ for all n

$(P_1, P_2, P_3) = (1, x, (3x^2 - 1)/2)$

[see Wikipedia article “orthogonal polys”]

(Axler §7A) [Riesz representation, in Axler:]

<u>Lem</u> if V is finite-dim’l, $\langle\ ,\ \rangle$ is bi/skew-linear
and right-nondegenerate
[i.e., $\langle -, v\rangle$ is zero only if $v = \mathbf{0}$]
then v mapsto $\theta_v = \langle -, v\rangle$ is an iso V to $V^\vee$

<u>Pf</u> right-nondegen. means the map is inj.
but $\dim(V) = \dim(V^\vee)$

<u>Cor</u> given inner product spaces
$(V, \langle\ ,\ \rangle)$ and $(W, \{\ ,\ \})$,
a linear map $T : V$ to W,

there is a map $T^* : W$ to V s.t. $\{Tv, w\} = \langle v, T^*w\rangle$
for all v in V and w in W

explicitly: W to W^v $\overset{T^v}{\text{to}}$ V^v to V

[draw square]

Pf $\quad$ T^*w in V is the unique vector s.t.
$\theta_{T^*w} = T^v(\theta_w)$

now:, $\quad$ $\{Tv, w\} = \theta_w(Tv) = T^v(\theta_w)(v)$
$\langle v, T^*w\rangle = \theta_{T^*w}(v)$

Df $\quad$ a linear map $T : V$ to W is called
an isometry from $\langle\ ,\ \rangle$ to $\{\ ,\ \}$ iff either:

1) $\{Tv', Tv\} = \langle v', v\rangle$ for all v, v' in V
2) $T^*T = \mathrm{Id}_V$

Pf that 1) iff 2)

1) iff $\langle v', T^*Tv\rangle = \langle v', v\rangle$ for all v, v'
iff $\langle -, T^*Tv\rangle = \langle -, v\rangle$ for all v
iff $T^*Tv = v$ for all v [by Riesz]
iff 2)

Df $\quad$ suppose $W = V$ and $\{\ ,\ \} = \langle\ ,\ \rangle$
if $F = R$: T is called an orthogonal op
if $F = C$: T is called a unitary op