PROBLEMS ON SYMPLECTIC REFLECTION ALGEBRAS

9. Commutativity and centers

Exercise 9.1. *Show that* $\{\cdot,\cdot\}_{t,c} = t\{\cdot,\cdot\}$, *where* $\{\cdot,\cdot\}$ *is the standard bracket on* $S(V)^\Gamma$.

Exercise 9.2. *Prove the commutativity theorem in the case when* V *is not necessarily symplectically irreducible.*

Exercise 9.3. *Let* $\mathcal{A}$ *be a* $\mathbb{Z}_{\geqslant 0}$*-filtered algebra. If* $\operatorname{gr}\mathcal{A}$ *is finitely generated, then so is* $\mathcal{A}$.

Problem 9.1. *Let* $p \in P_0$. *Equip* Z_p *with a filtration restricted from* H_p. *Show that* $\operatorname{gr} Z_p = S(V)^\Gamma$. *Deduce that* H_p *is a finitely generated module over* Z_p.

Problem 9.2. *Now let* $p \notin P_0$. *Show that the center of* H_p *coincides with* $\mathbb{C}$ *as follows:*

1. *Let* z *lie in the center of* H_p. *Show that* $\operatorname{gr} z \in \operatorname{gr} H_p = S(V)\#\Gamma$ *actually lies in* $S(V)^\Gamma$.
2. *Show that* $\operatorname{gr} z$ *lies in the Poisson center of* $S(V)^\Gamma$, *meaning that* $\{\operatorname{gr} z, S(V)^\Gamma\} = 0$.
3. *Show that the Poisson center of* $S(V)^\Gamma$ *coincides with* $\mathbb{C}$.

Problem 9.3. *In this problem we are going to equip* Z_c *with a structure of a Poisson algebra. Fix* c *and consider* $H_{t,c}$ *as an algebra over* $\mathbb{C}[t]$ *by making* t *an independent variable.*

1. *Let* $a, b \in Z_c$. *Lift* $a, b \in H_c = H_{t,c}/(t)$ *to elements* $\tilde{a}, \tilde{b} \in H_{t,c}$. *Show that* $[\tilde{a}, \tilde{b}] \in tH_{t,c}$ *and that the element* $\frac{1}{t}[\tilde{a}, \tilde{b}]$ *modulo* t *depends only on* a, b *and lies in* Z_c. *Let* $\{a, b\}$ *be that element. Show that* $\{\cdot,\cdot\}$ *is the Poisson bracket.*
2. *Show that* $\{Z_c^{\leqslant i}, Z_c^{\leqslant j}\} \subset Z_c^{i+j-2}$. *Show that the induced bracket on* $\operatorname{gr} Z_c = S(V)^\Gamma$ *is a nonzero multiple of the standard bracket. Can you identify the scalar factor?*

Problem 9.4. *Show that the scheme* C_p *is irreducible and normal (and, well, Cohen-Macaulay and Gorenstein, if you know what these words mean).*

Problem 9.5. *Show that if* C_p *is smooth, then* $H_p e$ *is a locally free* H_p*-module.*

Problem 9.6. *Let* $\mathcal{A}$ *be a filtered algebra. Show that if* $\operatorname{gr}\mathcal{A}$ *has finite global dimension, then* $\mathcal{A}$ *does.*

Problem 9.7. *Prove that if* C_p *is smooth, then* p *is spherical (we deal here with* $p \in P_0$*).*