Thm if V is fin dim and T : V to W is linear,
then dim V = dim ker(T) + dim im(T)

Pf Outline

pick basis {w_1, …, w_r} for im(T)
pick u_1, …, u_r s.t. T(v_i) = w_i for all i
let U = span(u_1, …, u_r)
pick basis v_1, …, v_k for ker(T)

I) show that ker(T) + U = V
II) show that ker(T) + U is a direct sum

then dim V = dim ker(T) + dim U
= dim ker(T) + dim im(T) □

Pf of I) pick v in V

know T(v) = sum_i b_iw_i for some b_i
so T(v) = sum_i b_iT(u_i) = T(sum_i b_iu_i)
so T(v – sum_i b_iu_i) = **0_W**
so v – sum_i b_iu_i in ker(T)
so v in ker(T) + U

Pf of II) recall: W + U is a direct sum iff
W cap U = {**0**}
so want ker(T) cap U = {**0_V**}

pick v in ker(T) cap U
since v in U, have v = sum_i a_iu_i for some a_i
since v in ker(T), have sum_i a_iw_i = T(v) = **0_W**
but {w_i}_i lin. indep. so a_i = 0 for all i

assuming V, W are both finite-dimensional:

Cor — if T : V to W is linear and dim V > dim W, then T is not injective

Pf — dim im(T) ≤ dim W < dim V,
so
dim ker(T) = dim V – dim im(T) > 0

Cor — if T : V to W is linear and dim V < dim W, then T is not surjective

Pf — exercise

Cor — if T : V to W is linear and bijective, then dim V = dim W [converse is false!]

(Axler §3D)

Thm — TFAE for a linear map T : V to W:

1) T is bijective
2) T takes any basis for V onto a basis for W
3) T takes some basis for V onto one for W
4) there is a linear map S : V to W s.t.
   S(T(v)) = v and T(S(w)) = w

Df — in the situation above, we say that T is a linear isomorphism from V onto W

Pf of Thm — not hard to show that
2) implies 3) implies 4) implies 1)
remains to show 1) implies 2)

so suppose {e_i}_i is a basis for V

claim that {T(e_i)}_i spans W:

for all w in W, have w = T(v) for some v in V

by surjectivity of T

v = sum_i a_ie_i for some a_i

so w = sum_i a_iT(e_i)

claim that {T(e_i)}_i is lin. indep.

suppose sum_i b_i T(e_i) = **0**_**W**

then T(sum_i b_ie_i) = **0**_**W**

so sum_i b_ie_i in ker(T)

so sum_i b_ie_i = **0**_**V**

by injectivity of T

so b_i = 0 for all i

by lin. independence of {e_i}_i □

(Axler §3C) recap:

Slogan #1 if we know a basis for V

then a linear map V to W is det by

where it sends the basis

and any choices will do

in particular, a linear map F^n to W is det by

where it sends (1, 0, 0, …), (0, 1, 0, …), …

Slogan #2 a linear isomorphism is

a linear map taking bases to bases

in particular, a linear iso F^n to W is det by

an ordered basis for W

[images of (1, 0, 0, …), (0, 1, 0, …), …]

[in this sense, linear maps out of F^n are easy]

[what else can we do with linear maps?]

a composition of linear maps is linear: given linear

$A : V$ to W,

$B : W$ to U,

$B(A(v + v')) = B(A(v) + A(v')) = B(A(v)) + B(A(v'))$,

$B(A(c \bullet v)) = B(c \bullet A(v)) = c \bullet B(A(v))$

so $B \circ A : V$ to U is also linear

suppose $V = F^n$ with std basis $(v_1, \ldots, v_n)$,

$W = F^m$ with std basis $(w_1, \ldots, w_m)$,

$U = F^\ell$ with std basis $(u_1, \ldots, u_\ell)$

$A(v_i) = \mathrm{sum}_j\, a_{j,i} w_j$,

$B(w_j) = \mathrm{sum}_k\, b_{k,j} u_k$

$$
\begin{aligned}
B(A(v_i)) &= B(\mathrm{sum}_j\, a_{j,i} w_j) \\
&= \mathrm{sum}_j\, a_{j,i} B(w_j) \\
&= \mathrm{sum}_j\, a_{j,i}\, \mathrm{sum}_k\, b_{k,j}\, u_k \\
&= \mathrm{sum}_{k,j}\, a_{j,i} b_{k,j}\, u_k \\
&= \mathrm{sum}_k\, c_{k,i}\, u_k
\end{aligned}
$$

where $c_{k,i} = \mathrm{sum}_j\, a_{j,i} b_{k,j}$

matrix multiplication = shorthand for these calc’s

<u>Df</u> matrix of A wrt the ordered bases $(v_i)_{i = 1}^n, (w_j)_{j = 1}^m$:

a11	a12	…	a1n	# rows is n (dim domain),
a21	a22	…	a2n	# cols is m (dim target)
…	…		…	
am1	am2	…	amn	

henceforth write F^n, F^m, etc in <u>column notation</u>:

$$v = \sum_i c_i v_i: \quad \begin{matrix} c1 \\ c2 \\ \dots \\ cn \end{matrix}$$

matrix × vector rule for Av:

$$\begin{matrix} a11 & \dots & a1n \\ a21 & \dots & a2n \\ \dots & & \dots \\ am1 & \dots & amn \end{matrix} \quad \begin{matrix} c1 \\ c2 \\ \dots \\ cn \end{matrix} \quad = \quad \begin{matrix} \dots \\ aj1\, c1 + \dots + ajn\, cn \\ \dots \\ \dots \end{matrix}$$

e.g., Av_i: $\begin{matrix} a1i \\ a2i \\ \dots \\ ami \end{matrix}$ (ith matrix col)

matrix × matrix rule for B ○ A:

$$\begin{matrix} b11 & \dots & b1m \\ b21 & \dots & b2m \\ \dots & & \dots \\ b\ell 1 & \dots & b\ell m \end{matrix} \quad \begin{matrix} a11 & \dots & a1n \\ a21 & \dots & a2n \\ \dots & & \dots \\ am1 & \dots & amn \end{matrix}$$

$$= \quad \begin{matrix} \dots & \dots & \dots \\ \dots & \sum_j a_{\{j,i\}}b_{\{kj\}} & \dots \\ \dots & \dots & \dots \\ \dots & \dots & \dots \end{matrix}$$

so	vectors	become	columns,
	linear maps		matrices,
	○		matrix multiplication

if A iso, then dim V = dim W: so matrix is <u>square</u>