MATH 250: Topology I

[intros]

[recap: website
LaTeX
textbook
late hw policy
exam dates]

[initial reading]

I) algebraic topology:

how to tell shapes apart?

Euler: Königsberg (1736), “V – E + F = 2” (1750)
Poincaré: Analysis situs (1890s–1900s)

II) point-set topology:

what IS shape?

Weyl: rigorous defn of “surface” (1913)
Hausdorff: topological spaces (1914)

[we first do (II), then do (I)]

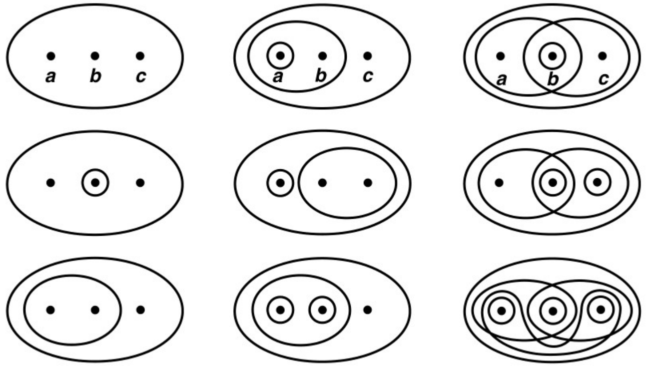
(Munkres §12) fix a set X

Def a topology on X is
a collection T of subsets of X s.t.

1) Ø, X in T
2) if {U_i}_i is a subcollection of T, then the union of the U_i in X is in T
3) if {U_i}_i is a finite subcollection of T, then the intersection of the U_i in X is in T

we say that
(X, T) is a topological space
the elements of T are its open sets

Ex X = {a, b, c}



[in each case, Ø is not depicted]

[example collection of subsets that isn't a topology?]

Ex $\quad X = R^n$ with $|x| = \sqrt{\sum_i x_i^2}$

write $B(x, \delta) = \{y \in R^n \mid |y - x| < \delta\}$

we say $U \subseteq R^n$ is analytically open iff
for all x in U,
there is some $\delta > 0$ s.t. $B(x, \delta) \subseteq U$

Thm $\quad$ {analytically open sets} is a topology on R^n

1) easy
2) suppose U_i anlytc opens, $U = \bigcup_i U_i$
pick x in U
x belongs to x in U_j for some j
have $\delta > 0$ s.t. $B(x, \delta) \subseteq U_j \subseteq U$

3) suppose finitely many i,
U_i analytic opens, $V = \bigcap_i U_i$
pick x in V
for all i, pick $\delta_i > 0$ s.t. $B(x, \delta_i) \subseteq U_i$

[what next?] let $\delta = \min_i \delta_i$
then $B(x, \delta) \subseteq B(x, \delta_i) \subseteq U_i$ for all i
therefore $B(x, \delta) \subseteq V$

[observe: 3) wouldn't work for infinite $\delta_i \to 0$]

we call this the analytic topology T_{an} on R^n

Rem $\quad$ not the only topology possible:
also discrete and indiscrete topologies

Ex X arbitrary

$T_f = \{\emptyset\} \cup \{U \subset X \text{ s.t. } X - U \text{ is finite}\}$

Thm T_f is a topology on X

1) easy
2) suppose $\{U_i\}_i$ is a subcollection of T_f,
   $U = \bigcup_i U_i$
   [what happens if $U_i = \emptyset$ for all i?]
   [what if $U_i \neq \emptyset$ for some i?]
3) suppose finitely many i,
   U_i subcollection of T_f
   $V = \bigcap_i U_i$
   [what happens if $U_i = \emptyset$ for some i?]
   [what if $U_i \neq \emptyset$ for all i?]

we call this the finite complement topology

Df given topologies T, T' on the same X s.t. T is a subcollection of T',

we say that
   T is coarser than T'
   T' is finer than T

[T' is more refined: it sees more open sets]

Ex [how do the topologies on R^n compare? analytic, discrete, indiscrete, finite-comp]

$T_{indisc} \subset T_f \subset T_{an} \subset T_{disc}$

Rem topologies can be incomparable:
think about $X = \{a, b, c\}$

(Munkres §13) $\{B_i\}_{i \in I}$ any collection of subsets of X

Df $\{B_i\}_{i \in I}$ is a basis for a top on X iff

1) $X = \bigcup_{i \in I} B_i$
2) for all $i, j \in I$, and $x \in B_i \cap B_j$,
have $k \in I$ s.t. $x \in B_k \subset B_i \cap B_j$
["$B_i \cap B_j$ is covered by B_k's"]

Rem Munkres also discusses "subbases"

[idea: reduce q's abt topologies to q's abt bases]

any basis generates a topology:

Thm if $\{B_i\}_{i \in I}$ is a basis on X, then

$$T = \{\bigcup_{i \in I'} B_i \mid \text{any } I' \subset I\}$$

is a topology on X [note! formula includes $I' = \varnothing$]

Pf 1) $\varnothing$, X in T [why?]
2) T is closed under unions [why?]

to show 3) T is closed under finite intersections:
suppose $\{U_j\}_{j = 1}^r$ finite subcoll where
$$U_j = \bigcup_{i_j \in I'_j} B_{i_j}$$

$\bigcap_{j=1}^r U_j$
$= \bigcup_{i_1, \ldots, i_r} (B_{i_1} \cap \ldots \cap B_{i_r})$

use induction to show: for all $i_1, \ldots, i_r$,
and $x \in B_{i_1} \cap \ldots \cap B_{i_r}$,
have $k \in I$ s.t.
$x \in B_k \subseteq B_{i_1} \cap \ldots \cap B_{i_r}$
so $B_{i_1} \cap \ldots \cap B_{i_r}$ is union of B_k's
so $\bigcap_{j=1}^r U_j$ is too

Ex $\{B(x, \delta) \mid x \in \mathbb{R} \text{ and } \delta > 0\}$ is a basis for the analytic top on $\mathbb{R}$ [why?]
[what if we drop the condition $\delta > 0$?]

Rem different bases can generate the same topology

[criterion to check when a subcoll is a basis]

Thm suppose T is a topology on X,
$\{B_i\}_{i \in I}$ a subcollection of T
s.t. for all $U \in T$, and $x \in U$,
there is some $k \in I$ s.t. $x \in B_k \subseteq U$

then $\{B_i\}_{i \in I}$ is a basis, and it generates T

Pf 1) $X = \bigcup_{i \in I} B_i$ [why?]
2) pick i, j, and $x \in B_i \cap B_j$
$B_i \cap B_j \in T$ since $B_i, B_j \in T$
get k s.t. $x \in B_k \subseteq B_i \cap B_k$
so $\{B_i\}_i$ is a basis

finally, $T = \{\bigcup_{i \in I'} B_i \mid I' \subseteq I\}$ [why?]

<u>Ex</u> recall: $\{B(x, \delta) \mid x, \delta \text{ in } R\}$ is a basis for the analytic top on R

same as $\{(a, b) \mid a, b \text{ in } R\}$ [why?]

is $\{(a, b) \mid a, b \text{ in } Q\}$ a basis? [yes]
is $\{(a, b) \mid a, b \text{ in } Z\}$ a basis? [no]