<u>Wrap-up</u> (from Mon) topologies on $\mathbb{R}$, so far:

- analytic
- indiscrete
- discrete
- finite-complement

<u>Df</u> given topologies T, T' on the same X

if $T \subseteq T'$, then we say that

T is <u>coarser</u> than T'

T' is <u>finer</u> than T

[T' is more refined: it sees more open sets]

<u>Ex</u> $T_{indisc} \subseteq T_f \subseteq T_{an} \subseteq T_{disc}$

<u>Rem</u> topologies can be <u>incomparable</u>:

think about $X = \{a, b, c\}$

[notice: most q's about T_{an} ≈ q's about balls]

(Munkres §13) $\{B_i\}_{i \in I}$ any collection of subsets of X

<u>Df</u> $\{B_i\}_{i \in I}$ is a <u>basis</u> (for a top) on X iff

1) $X = \bigcup_{i \in I} B_i$
2) for all $i, j \in I$, and $x \in B_i \cap B_j$,

have $k \in I$ s.t. $x \in B_k \subseteq B_i \cap B_j$

["$B_i \cap B_j$ is covered by B_k's"]

<u>Rem</u> Munkres also discusses "subbases"

any basis <u>generates</u> a topology:

<u>Thm</u> if $\{B_i\}_{i \in I}$ is a basis on X, then

$$T = \{U \subseteq X \mid \text{for all } x \in U, \text{ have } i \in I \text{ s.t. } x \in B_i \subseteq U\}$$

is a topology on X [note! formula includes $J = \varnothing$]

<u>Pf</u> 1) $\varnothing$, X in T [why?]

2) T is closed under unions [why?]

to show 3) T is closed under finite intersections:

by induction, just need to show: if U, V in T, then $U \cap V$ in T

$U = \bigcup_{j \in J} B_i$ for some $J \subseteq I$
$V = \bigcup_{k \in K} B_i$ for some $K \subseteq I$
$U \cap V = \bigcup_{j \in J, k \in K} (B_j \cap B_k)$

for all j in J and k in K and x in $B_j \cap B_k$,
have ℓ in I s.t. $x \in B_\ell \subseteq B_j \cap B_k$
so $x \in B_\ell \subseteq U \cap V$

<u>Ex</u> $\{B(x, \delta) \mid x \in \mathbb{R} \text{ and } \delta > 0\}$ is a basis for the analytic top on $\mathbb{R}$ [why?]

same as $\{(a, b) \mid a < b\}$ [why?]

<u>Rem</u> different bases can generate the same topology

[criterion to check when a subcoll is a basis]

Thm suppose T is a topology on X,

{B_i}_{i in I} a subcollection of T

s.t. for all U in T, and x in U,

there is some k in I s.t. x in B_k sub U

then {B_i}_{i in I} is a basis, and it generates T

Pf 1) X = bigcup_{i in I} B_i [why?]

2) pick i, j, and x in B_i cap B_j

B_i cap B_j in T since B_i, B_j in T

get k s.t. x in B_k sub B_i cap B_k

so {B_i}_i is a basis

and T is generated by {B_i}_{i in I} tautologically

Ex recall: {(a, b) | a < b} is a basis for the analytic top on R

is {(a, b) | a < b and a, b in Q} a basis? [yes]

is {(a, b) | a < b and a, b in Z} a basis? [no]

Ex the lower-limit topology T_ℓ on R is generated by the basis

{[a, b) | a < b}

Q1 is {[a, b) | a, b in R} really a basis? [yes]

Q2 how does T_ ℓ compare to T_{an}?

Thm T_ℓ is strictly finer than T_{an}

Pf T_ ℓ is finer than T_{an}:

suppose U anlytc open in R

suppose x in U

pick a < b s.t. x in (a, b) sub U

then [x, b) sub (a, b)

strict because [0, 1) in T_ℓ but notin T_{an}

T_indisc < T_f < T_{an} < T_ℓ < T_disc

will write R_ℓ to mean "R in the topology T_ℓ", etc.

(Munkres §18, 16) recall from real analysis:

f : R^n to R^m is <u>Bolzano continuous</u> iff

for all x in R^n and ε > 0, there exists δ > 0 s.t.

|x – x′| < δ implies |f(x) – f(x′)| < ε

equivalently

for all x in R^n and ε > 0, there exists δ > 0 s.t.

x′ in B(x, δ) implies f(x′) in B(f(x), ε)

equivalently

for all x in R^n and ε > 0, there exists δ > 0 s.t.

B(x, δ) sub f^{-1}(B(f(x), ε))

Goal generalize to topological spaces

Df a function $f : X$ to Y is continuous iff
V open in Y implies $f^{-1}(V)$ open in X

Thm $f : R^n$ to R^m is Bolzano cts
iff f is cts wrt the analytic topologies

[Pf suppose f cts wrt analytic topologies:

for all x in R^n and $\varepsilon > 0$, want $\delta > 0$
s.t. $|x - x'| < \delta$ implies $|f(x) - f(x')| < \varepsilon$
know: $B(f(x), \varepsilon)$ is open in R^m
so $f^{-1}(B(f(x), \varepsilon))$ is open in R^n
so have $\delta > 0$ s.t. $B(x, \delta)$ sub $f^{-1}(B(f(x), \varepsilon))$

suppose f Bolzano cts:

for all V open in Y, want $f^{-1}(V)$ open in X
pick x in $f^{-1}(V)$
want $\delta > 0$ s.t. $B(x, \delta)$ sub $f^{-1}(V)$
pick $\varepsilon > 0$ s.t. $B(f(x), \varepsilon)$ sub V
pick $\delta > 0$ s.t. for all x' s.t. $|x - x'| < \delta$,
have $|f(x) - f(x')| < \varepsilon$
then $f(B(x, \delta))$ sub $B(f(x), \varepsilon)$ sub V
so $B(x, \delta)$ sub $f^{-1}(V)$]

Thm suppose that $\{C_i\}_{i \in I}$ is a basis for
the topology on Y
then
$f : X$ to Y is cts
iff $f^{-1}(C_i)$ is open in X for all i in I

Pf suppose f is cts
each C_i is open in Y
so each $f^{-1}(C_i)$ is open in X

now suppose $f^{-1}(C_i)$ open in X for all i in I
pick V open in Y
know: $V = \bigcup_{i \in J} C_i$ for some $J \subset I$
$f^{-1}(V) = f^{-1}(\bigcup_{i \in J} (C_i))$
$= \bigcup_{i \in J} f^{-1}(C_i)$
so $f^{-1}(V)$ is open in X

Ex which maps are continuous?

$f : \mathbb{R}_{an} \to \mathbb{R}_\ell$, $f(x) = x$ [no: [0, 1)]
$f : \mathbb{R}_\ell \to \mathbb{R}_{an}$, $f(x) = x$ [yes]
$f : \mathbb{R}_{an} \to \mathbb{R}_\ell$, $f(x) = 32$ [yes]

Df cts $f : X \to Y$ is a homeomorphism
iff it has a two-sided inverse $g : Y \to X$
s.t. g is also cts

in this case, we say X and Y are homeomorphic

[“what is shape?” “X and Y have the same shape when there is a homeo between them”]

Rem any homeo is a cts bijection
but a cts bijection need not be a homeo:
[already have an example: which?]
$f : \mathbb{R}_\ell \to \mathbb{R}_{an}$, $f(x) = x$

Ex id : $X \to X$ is a homeo when we use the same topology for domain and range

Ex other homeo's wrt analytic topologies:

$f : R$ to R, $f(x) = x^3$

$f : R^2$ to R^2 $f(x, y) = (x + y, (x - y)^3)$

[why?]

Q is there a homeo R to R^2? vice versa?

Bonus Material

recall that Z is the set of integers

Df the evenly spaced topology on Z is generated by the basis

$$\{aZ + b \mid a, b \text{ in } Z \text{ and } a \neq 0\}$$

[will show on PS1 that this really is a basis]

Thm there are infinitely many prime numbers

Proof (Furstenberg, 1955)

assume finitely many primes p

then Z – bigcup_p pZ = bigcap_p (Z – pZ) is open
because Z – pZ is open for all p

but Z – bigcup_p pZ = {±1}
because if |a| > 1, then some prime divides a

so {±1} is open
but {±1} is not an evenly spaced set in Z □