(Munkres §26–27) last time:

X is compact iff every open cover of X admits
a finite subcover

R is not compact in the analytic topology
what about…
the indiscrete topology? compact

the finite-complement topology? compact
if $\{U_i\}_i$ covers R
then some U_i is nonempty
pick fin many U_j to cover R – U_i

the countable-complement topology? no
$U_n = R - \{m \text{ in } Z \mid m \geq n\}$ [draw]

in fact:

Prop in the indiscrete and fin-comp. top's,
every subset of R becomes compact

Pf suffices to consider nonempty A sub R

indiscrete top:
the only way to cover A is with {R}

finite-complement top:
proof that R is compact also works for any A

Q by comparison, what are the closed sets
in the fin-comp. topology on R?

A R itself and its finite subsets

so here: all subsets of R are compact,
but most are not closed!

Thm if X is Hausdorff and A is compact
then A is closed in X

in fact: for any x in X – A
there exist disjoint open U, V s.t.
x in U and A sub V

[note: indiscrete, fin-comp., and countable-comp.
are not Hausdorff]

Pf use the Hausdorff condition:

for all a in A, get disj open U_a, V_a sub X s.t.
x in U_a and a in V_a
then A sub bigcup_a V_a
so there is a finite subcollection {V_a}_{a in B} s.t.
A sub bigcup_{a in B} V_a

since B is finite, bigcap_{a in B} U_a is still open
set U = bigcap_{a in B} U_a
V = bigcup_{a in B} V_a
then U, V disjoint, x in U, and A sub V

[why does Munkres introduce
connectedness and compactness together?]

many theorems about connectedness have
parallel theorems for compactness

Thm 1 (Heine–Borel) [0, 1] is compact

Thm 2 if $f : X \to S$ is cts and X is compact
then $f(X)$ is compact in S

Thm 3 if X, Y are compact, then $X \times Y$ is too
(hence finite products of cpts are cpt)

Thms 1 + 2 imply: images of paths are compact;
S^1 is compact
Thms 1 + 3 imply: $[0, 1]^n$ compact for any n

[all facts still true with "conn." replacing "compact"]

[since Thm 1 is so standard in intro analysis,
we only prove Thms 2 and 3 in detail]

Pf Sketch of Thm 1

is $[0, 1] \cap \mathbb{Q}$ cpt as a subspace? [no, but tricky]
so again, need a proof that uses the LUBP

let $\{U_i\}_i$ be an open cover of $[0, 1]$

Lem for any x in $[0, 1]$, can find $x < y \leq 1$ s.t.
$[x, y]$ is covered by a finite union of U_i

let $C = \{x \in (0, 1] \mid [0, x]$ cov. by fin union of $U_i\}$

I) show $C \neq \varnothing$ via lemma
II) set $c = \sup C$
show $c \notin C$ is a contradiction, via LUBP
III) show $c < 1$ is a contradiction, via lemma

cf. the proof of Munkres Thm 27.1
[slightly more general setup using order topology]

Pf of Thm 2

pick $\{V_i\}_i$ covering $f(X)$ in S
then $\{f^{-1}(V_i)\}_i$ is a cover of X
so has a finite subcover $\{f^{-1}(V_i)\}_{i \in J}$
then $\{V_i\}_{i \in J}$ still covers $f(X)$ in S

Cor if X is compact
Y is Hausdorff
$f : X \to Y$ is cts and bijective
then f is a homeomorphism

Pf have set-theoretic inverse $f^{-1} : Y \to X$
need f^{-1} cts:
i.e., U open in X implies $f(U)$ open in Y

to use Thm 2, show instead:
Z closed in X implies $f(Z)$ closed in Y

indeed: X is compact, so Z is compact
[by a thm from last time]
so $f(Z)$ is compact by Thm 2
but Y is Hausdorff, so $f(Z)$ is closed
[by first thm today]

Cor any cts self-bijection of $[0, 1]$ (or S^1)
has a cts inverse

<u>Pf</u> check that $[0, 1]$ and S^1 are Hausdorff
$[0, 1]$ is cpt by Thm 1
so S^1 is cpt by Thm 2
so their cts self-bijections are homeos
by the corollary to Thm 2

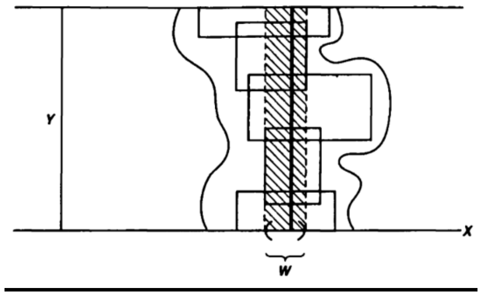
<u>Pf of Thm 3</u> fix open cover $\{U_i\}_i$ of $X \times Y$

idea: slices $\{a\} \times Y$ are compact
so some fin. subcoll. $\{U_i\}_{i \in I_a}$
covers $\{a\} \times Y$

problem: $\{U_i\}_{i \in I_a}$ depends on a
what if the variation among the U_i's
"in the Y-dir" varies a lot with a?

<u>Tube Lem</u> for all a in X, open N sub $X \times Y$ s.t.
$\{a\} \times Y$ sub N,

there is open W sub X s.t. a in W,
$W \times Y$ sub N



<u>Tube Lem implies Thm 3</u>

given $\{U_i\}_{i \in I_a}$ finite and covering $\{a\} \times Y$
set $N_a = \bigcup_{i \in I_a} U_i$

pick open $W_a \subseteq X$ s.t. $a \in W_a$,

$W_a \times Y \subseteq N_a$

then $\{U_i\}_{i \in I_a}$ covers $W_a \times Y$

but $X \times Y = \bigcup_{a \in X} (W_a \times Y)$

and each $W_a \times Y$ is open

so there is a finite set $J \subseteq X$ s.t.

$$X \times Y = \bigcup_{a \in J} (W_a \times Y) = \bigcup_{a \in J,\, i \in I_a} U_i$$

remove repetitions of U_i's if needed

this is the desired finite subcover of $\{U_i\}_i$ □