<u>Recall</u> for a top space X with <u>basepoint</u> x:

$\pi_1(X, x) = \{[\gamma] \mid \text{loops } \gamma \text{ in } X \text{ based at } x\}$

e_x is the constant path at x

<u>Df</u> loop γ at x is nulhomotopic iff $[\gamma] = [e_x]$

<u>Ex</u> let $\gamma(t) = (\cos(2\pi t), \sin(2\pi t))$

want γ nulhomotopic in R^2

but not in $R^2 – \{(0, 0)\}$ [picture]

<u>Thm</u> for any α, β, γ in $\pi_1(X, x)$:

1) $[\alpha * \beta] * [\gamma] = [\alpha] * [\beta * \gamma]$
2) $[e_x * \gamma] = [\gamma] = [\gamma * e_x]$
3) if β is the “reverse” of γ

then $[\beta * \gamma] = [e_x] = [\gamma * \beta]$

hence $(\pi_1(X, x), *)$ is a group with id elt $[e_x]$

will follow from the more general:

<u>Thm</u> fix v, w, x, y in X and paths

$$v \overset{\alpha}{\text{to}} w \overset{\beta}{\text{to}} x \overset{\gamma}{\text{to}} y$$

then

1) $[\alpha * \beta] * [\gamma] = [\alpha] * [\beta * \gamma]$
2) $[e_x * \gamma] = [\gamma] = [\gamma * e_y]$
3) if $w = y$ and β reverses γ

then $[\beta * \gamma] = [e_w]$

$[\gamma * \beta] = [e_x]$

[what does each statement mean, visually?]

[which proof will be hardest? easiest?]

Pf of 3) WLOG $\gamma : [0, 1]$ to X,
$\beta(s) = \gamma(1 - s)$

want $[\gamma * \beta] = [e\_x]$
proof that $[\beta * \gamma] = [e\_w]$ is analogous

want path homotopy $h : [0, 1] \times [0, 1]$ to X s.t.
$h(-, 0) = e\_x$ and $h(-, 1) = \gamma * \beta$

[draw picture]

[idea: $h(-, t)$ should “freeze” when it hits $\gamma(t)$]

$h(s, t)$: x to $\gamma(t)$ for s in $[0, t/2]$
stay at $\gamma(t)$ for s in $[t/2, 1 - t/2]$
$\gamma(t)$ back to x for s in $[1 - t/2, 1]$

$$h(s, t) = \begin{cases} \gamma(2s) & s \leq t/2 \\ \gamma(t) & t/2 \leq s \leq 1 - t/2 \\ \gamma(2 - 2s) = \beta(2s) & 1 - t/2 \leq s \end{cases}$$

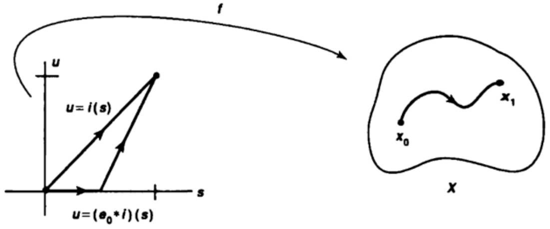
Pf of 2) again WLOG $\gamma : [0, 1]$ to X

want to prove $[e\_x * \gamma] = [\gamma]$
proof that $[\gamma * e\_y] = [\gamma]$ is analogous

want path homotopy $h : [0, 1] \times [0, 1]$ to X s.t.
$h(-, 0) = e\_x * \gamma$ and $h(-, 1) = \gamma$

[do a more abstract argument this time:]

notice: if X = [0, 1] and x = 0 and γ(t) = t
then much easier



reduce to this case using two identities:

Lem given paths φ, φ′, ψ : [0, 1] to A
cts map f : A to X

1) if j is a path homotopy φ to φ′
then f ◦ j is a path homotopy f ◦ φ to f ◦ φ′
2) if φ * ψ def., then f ◦ (φ * ψ) = (f ◦ φ) * (f ◦ ψ)

take A = [0, 1] and φ = e_0 * id
φ′ = id

pick a path homotopy j from φ to φ′
[uses convexity of [0, 1]]

now take f = γ

then γ ◦ j is a path homotopy γ ◦ φ to γ ◦ φ′

but γ ◦ (e_0 * id) = (γ ◦ e_0) * (γ ◦ id) = e_x * γ
γ ◦ id = γ

Pf of 1) recall

$$v \overset{\alpha}{\text{to}} w \overset{\beta}{\text{to}} x \overset{\gamma}{\text{to}} y$$

want [α * β] * [γ] = [α] * [β * γ]

similar idea as in 2):

if X = [0, 1] and α, β, γ all linear
then much easier

φ:	0 to 1/4	for s in [0, 1/2]	
	1/4 to 1/2	for s in [1/2, 3/4]	
	1/2 to 1	for s in [3/4, 1]	
id:	0 to 1/2	for s in [0, 1/2]	
	1/2 to 3/4	for s in [1/2, 3/4]	
	3/4 to 1	for s in [3/4, 1]	
f:	v to w	for s in [0, 1/4]	[!]
	w to x	for s in [1/4, 1/2]	
	x to y	for s in [1/2, 1]	

f = (α * β) * γ

check that f ∘ φ = α * (β * γ):

v to w	for s in [0, 1/2]
w to x	for s in [1/2, 3/4]
x to y	for s in [3/4, 1]

path homotopy φ to id yields
path homotopy f ∘ φ to f □

[so π_1(X, x) is a group under *]

Rem the subscript 1 in π_1 alludes to
“higher homotopy groups” π_n

roughly, π_n describes maps S^n to X
up to basepoint-fixing homotopy

Rem if P is the path comp. of X containing x
then π_1(X, x) = π_1(P, x)

so π_1 cannot “see” the other path components
so π_1(X, x) only interesting for X path-connected

Q how much does π_1(X, x) depend on x?

[recall what it means for groups to be “the same”]
from last time:
a homomorphism (G, •) and (K, ◦) is a map

φ : G to K s.t. φ(a • b) = φ(a) ◦ φ(b)

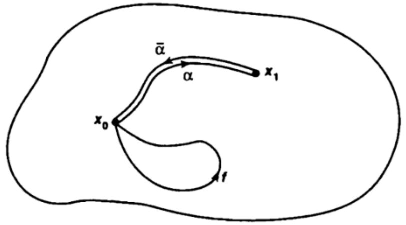
it’s an isomorphism iff it has an inverse

Thm for any x_0, x_1 in X
a choice of path α from x to y defines
an isomorphism

ᾰ : π_1(X, x_0) to π_1(X, x_1)

that actually only depends on [α]

Pf let ᾰ([γ]) = [α′] * [γ] * [α]
where α′ is the reverse of α



to get inverse map: switch α with α' □

Df write $f : (X, x)$ to (Y, y) to mean
$f : X$ to Y is cts,
$f(x) = y$

for such f, let $f_* : \pi_1(X, x)$ to $\pi_1(Y, y)$ be

$$f_*([\gamma]) = [f \circ \gamma]$$

earlier lemmas show this is well-defined

Thm given $(X, x)\ \overset{f}{\text{to}}\ (Y, y)\ \overset{g}{\text{to}}\ (Z, z)$

have $(g \circ f)_* = g_* \circ f_* : \pi_1(X, x)$ to $\pi_1(Z, z)$

Cor if f is a homeomorphism
then f_* is an isomorphism

so $\pi_1(X, x)$ is a *topological invariant* of X

Pf take g to be the inverse of f

next time: $\pi_1(R^2 - \{(0, 0)\}, (1, 0))$ is nontrivial