



Character Formulas from Lusztig Varieties and Affine Springer Fibers

Minh-Tâm Quang Trinh

Yale University

This talk is about...

- 1 Braids
- 2 Lusztig Varieties
- 3 Springer Fibers
- 4 Affine Springer Fibers

1 Braids The braid group $Br_n =$

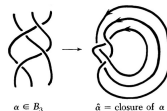
$$\left\langle \sigma_1, \dots, \sigma_{n-1}, \left| \begin{array}{l} \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \\ \sigma_i \sigma_j = \sigma_j \sigma_i \text{ when } |i - j| > 1 \end{array} \right. \right\rangle$$

appears in knot theory and representation theory.

$$\begin{array}{c} \sigma_i \\ \left[\dots \right] \underset{i}{\times} \underset{i+1}{\left[\dots \right]} \end{array}$$

A *link* is a collection of circles (tamely) embedded in \mathbf{R}^3 . Knot theory is about isotopy invariants of links.

(Alexander) Every link is the *closure* of some braid.



Let $G = \mathrm{GL}_n$ and B its upper-triangular subgroup.

$$V_n(q) = \{\text{functions } G(\mathbf{F}_q)/B(\mathbf{F}_q) \rightarrow \mathbf{C}\},$$

$$H_n(q) = \mathrm{End}_{G(\mathbf{F}_q)}(V_n(q)).$$

$$(\text{Iwahori}) \quad H_n(q) \simeq \frac{\mathbf{C}Br_n}{\langle \sigma_i^2 - (q-1)\sigma_i - q \rangle}.$$

To explain, recall Bruhat: $G = \coprod_{w \in S_n} B\dot{w}B$.

Then $\mathbf{C}Br_n \hookrightarrow V_n(q)$ via

$$\sigma_i \cdot \mathbf{1}_{xB(\mathbf{F}_q)} = \sum_{yB \xrightarrow{i} xB} \mathbf{1}_{yB(\mathbf{F}_q)},$$

where $yB \xrightarrow{i} xB$ means $By^{-1}xB = B\dot{w}_{(i,i+1)}B$.

Motivates a *Hecke algebra* $H_n(\mathbf{q})$ over $\mathbf{C}[\mathbf{q}^{\pm 1}]$.

Ocneanu used functions $Br_n \rightarrow H_n(\mathbf{q}) \rightarrow \mathbf{C}(\mathbf{q})[a^{\pm 1}]$ to construct a link invariant

$$\text{HOMFLYPT} : \{\text{links}\}/\text{isotopy} \rightarrow \mathbf{C}(\mathbf{q})[\mathbf{a}^{\pm 1}].$$

Jones computed it for torus knots. Remarkably, the values encode \mathbf{q} -Catalan (and \mathbf{q} -Kirkman) numbers.

On the other hand, Iwahori suggests that HOMFLYPT is related to the geometry of G/B .

We'll discuss a Springer-theoretic function of β that refines the HOMFLYPT invariant of $\hat{\beta}$.

2 Lusztig Varieties Suppose that β is *positive*:

$$\beta = \sigma_{i_1} \cdots \sigma_{i_\ell}.$$

(Deligne) The variety $O(\beta) =$

$$\left\{ (g_0 B, g_1 B, \dots, g_\ell B) \left| g_{j-1} B \xrightarrow{i_j} g_j B \text{ for all } j \right. \right\}$$

only depends on β , up to isomorphisms that keep $g_0 B$ and $g_\ell B$ fixed.

For any positive β, β' , we have

$$O(\beta\beta') \simeq O(\beta) \times_{G/B} O(\beta'),$$

where $\times_{G/B}$ means the variety of pairs $(\vec{g}B, \vec{g}'B)$ such that $g_\ell B = g'_0 B$.

A literal geometric representation of positive braids.

For any $x \in G(\mathbf{F}_q)$, form the *braid Lusztig variety*

$$\mathcal{B}(\beta)_x = \{\vec{g}B \in O(\beta) \mid g_\ell B = xg_0 B\}.$$

(Shende–Treumann–Zaslow) Up to a monomial in q ,

$$\frac{|\mathcal{B}(\beta)_1(\mathbf{F}_q)|}{|\mathrm{PGL}_n(\mathbf{F}_q)|}$$

is the “highest” a -degree of $\mathrm{HOMFLYPT}(\hat{\beta})$ at $\mathbf{q} \rightarrow q$.

Example Let $n = 2$ and $\beta = \sigma_1^3 \in Br_2$.

$$O(\beta) \simeq \{\vec{g} \in (\mathbf{P}^1)^4 \mid g_0 \neq g_1 \neq g_2 \neq g_3\},$$

$$\mathcal{B}(\beta)_1 \simeq \{\vec{g} \in (\mathbf{P}^1)^3 \mid g_1, g_2, g_3 \text{ pairwise distinct}\}.$$

PGL_2 acts simply transitively on the latter.

Indeed, $\mathrm{HOMFLYPT}(\hat{\sigma}_1^3) = a^2(\mathbf{q} + \mathbf{q}^{-1}) - a^4$.

3 Springer Fibers

How to access other a -degrees? Observe that

$$\mathcal{B}_x := \mathcal{B}(1)_x = \{gB \mid gB = xgB\}$$

is the usual Springer fiber over x , whose cohomology defines a character of S_n : namely,

$$\Psi_x(w) := \sum_i \mathbf{q}^{i \operatorname{tr}(w \mid H^{2i}(\mathcal{B}_x))}.$$

Thm 1 (T) Let $\mathcal{U} \subseteq G$ be the unipotent variety,

$$\Psi_\beta(w) = \sum_{u \in \mathcal{U}(\mathbf{F}_q)} \frac{|\mathcal{B}(\beta)_u(\mathbf{F}_q)|}{|\operatorname{PGL}_n(\mathbf{F}_q)|} \Psi_u(w)|_{\mathbf{q} \rightarrow q}.$$

Then $(\chi_{(n-k, 1, \dots, 1)}, \Psi_\beta)_{S_n}$ sees the k th a -degree.

Think of $\beta \mapsto \Psi_\beta$ as a function

$$Br_n \rightarrow H_n(q) \rightarrow \{\text{characters of } S_n\}.$$

Example Again, let $n = 2$ and $\beta = \sigma_1^3 \in Br_2$.

$$\Psi_u = \begin{cases} 1 + \mathbf{q} \operatorname{sgn} & u = 1, \\ 1 & u \neq 1. \end{cases}$$

$$\Psi_\beta = q^2 + 1 + q \operatorname{sgn}.$$

Recall $\operatorname{HOMFLYPT}(\widehat{\sigma_1^3}) = a^2(\mathbf{q} + \mathbf{q}^{-1}) - a^4$.

Thm 2 (T) The cohomology of $\mathcal{U}(\beta) \times_{\mathcal{U}} \mathcal{U}(1)$, where

$$\mathcal{U}(\beta) = \{(u, \vec{g}B) \mid u \in \mathcal{U} \text{ and } \vec{g}B \in \mathcal{B}(\beta)_u\},$$

encodes finer invariants of $\hat{\beta}$.

The *full twist* $\pi = (\sigma_1 \cdots \sigma_{n-1})^n$:



Thm 3 (T) Suppose $\beta^m = \pi^d$ for some $d, m > 0$.

Then up to a monomial, $\Psi_\beta(w)$ is the $\mathbf{q} \rightarrow q$ limit of

$$\frac{\text{sgn}(w)}{\det(1 - \mathbf{q}w \mid \mathfrak{h})} \sum_{\lambda \vdash n} \mathbf{q}^{c(\lambda)d/m} D_\lambda(e^{2\pi i d/m}) \chi_\lambda(w)$$

where:

- \mathfrak{h} is the *reflection representation*.
- $c(\lambda)$ is the sum of *contents* of λ .
- $D_\lambda(t) = K_{\lambda, (1^n)}(t)$ is the *fake degree* of λ .

Recovers Jones's HOMFLYPT formula for torus knots.

Thm 3 generalizes to any reductive G , once we replace:

- S_n with the Weyl group W .
- $c(\lambda)$ with $c(\chi) = \sum_{\text{refl. } t} \frac{\chi(t)}{\chi(1)}$.
- fake degrees D_λ with *generic degrees* D_χ .

If $\gcd(d, m) = 1$ and m is the Coxeter number of W , then the formula simplifies:

$$(\text{monomial}) \cdot \left. \frac{\det(1 - \mathbf{q}^d w \mid \mathfrak{h})}{\det(1 - \mathbf{q}w \mid \mathfrak{h})} \right\} =: \Pi_{\mathbf{q}}^{(d)}.$$

$\Pi_{\mathbf{q}}^{(d)}$ is the character of a *rational parking space*.

$(\text{triv}, \Pi_{\mathbf{q}}^{(d)})_W$ is a *rational q -Catalan number*.

Example If $W = S_n$, then $(\text{triv}, \Pi_{\mathbf{q}}^{(d)})_W = \frac{[n+d-1]!}{[n]![d]!}$.

4 Affine Springer Fibers

Rational parking spaces appear in a loop or *affine* analogue of Springer theory.

finite Springer	affine Springer
G	$G((z))$
G/B	$G((z))/I$
W	$\widetilde{W} = W \ltimes X^\vee$

Above:

- $G((z))$ is the loop group $G((z))(R) := G(R((z)))$.
- I is the preimage of B in $G[[z]]$.
- X^\vee is the cocharacter lattice of B .

We now study Springer fibers over the Lie algebras, not the groups, and over \mathbf{C} , not \mathbf{F}_q .

$$x : \quad \mathcal{B}_x = \{gB \in G/B \mid g^{-1}xg \in \mathfrak{b}\},$$

$$\gamma = \gamma(z) : \quad \mathcal{B}_\gamma^{\text{aff}} = \{gI \in G((z))/I \mid g^{-1}\gamma g \in \mathfrak{I}\}.$$

The table hides key differences:

In the **finite** case, \mathcal{B}_x is most **interesting** for x nilpotent.

In the **affine** case, $\mathcal{B}_\gamma^{\text{aff}}$ is *terribly infinite* for $\gamma = \gamma(z)$ nilpotent, but **interesting** for $\gamma(z)$ regular semisimple.

Example If $G = \text{SL}_2$ and $\gamma(z) = \begin{pmatrix} 0 & 1 \\ z^3 & 0 \end{pmatrix}$, then

$$\mathcal{B}_\gamma^{\text{aff}} \simeq \mathbf{P}^1 \sqcup_{\text{pt}} \mathbf{P}^1.$$

Dream Braid Lusztig varieties see the *finite* part of affine Springer representations.

Fix $\nu = d/m > 0$ in lowest terms. Let $\mathbf{C}^\times \curvearrowright \mathfrak{g}((z))$:

$$c \cdot_\nu \gamma(z) = c^{2d\rho^\vee} \gamma(c^{2m} z) c^{-2d\rho^\vee},$$

where $2\rho^\vee = \sum_{\alpha \in \Phi^+} \alpha^\vee$.

Let $\mathfrak{g}((z))_{\nu, k}$ be the weight- $2k$ eigenspace.

Lemma If γ is an eigenvector for \cdot_ν , then the induced action on $G((z))/I$ preserves $\mathcal{B}_\gamma^{\text{aff}}$.

Lemma $\mathfrak{g}((z))_{\nu, 0}$ is the Lie algebra of a connected reductive group L_ν . Moreover,

$$(G((z))/I)^{\mathbf{C}^\times} = \coprod_{w \in W(L_\nu) \backslash \widetilde{W}} L_\nu \dot{w} I / I.$$

Fix $\gamma \in \mathfrak{g}((z))_{\nu, d}$. In the SL_2 example, $\gamma \in \mathfrak{g}((z))_{3/2, 3}$.

$$\text{Springer : } \widetilde{W} \curvearrowright H^*(\mathcal{B}_\gamma^{\text{aff}}), H_{\mathbf{C}^\times}^*(\mathcal{B}_\gamma^{\text{aff}}).$$

(Sommers) If m is the Coxeter number, then:

- L_ν is a torus, and $L_\nu \dot{w} I = \dot{w} I$.
- $(\mathcal{B}_\gamma^{\text{aff}})^{\mathbf{C}^\times}$ is a finite subset of the $\dot{w} I$.
- Writing $H_{\mathbf{C}^\times}^*(\text{pt}) = \mathbf{C}[\epsilon]$, we have

$$\begin{aligned} H^*(\mathcal{B}_\gamma^{\text{aff}}) &= H_{\mathbf{C}^\times}^*(\mathcal{B}_\gamma^{\text{aff}})|_{\epsilon \rightarrow 1} \\ &= \{\text{functions on } (\mathcal{B}_\gamma^{\text{aff}})^{\mathbf{C}^\times}\}. \end{aligned}$$

- $\Pi_q^{(d)}(w)|_{q \rightarrow 1}$ is the W -character of $H_c^*(\mathcal{B}_\gamma^{\text{aff}})$.

(Oblomkov–Yun) Filtration on $H_{\mathbf{C}^\times}^*|_{\epsilon \rightarrow 1}$ that sees \mathfrak{q} .

(Goresky–Kottwitz–MacPherson) For general ν ,

$$(\mathcal{B}_\gamma^{\text{aff}})^{\mathbf{C}^\times} = \coprod_{w \in W(L_\nu) \setminus \widetilde{W}} \text{Hess}_{\gamma, w},$$

a disjoint union of *partial Hessenberg varieties*

$$\text{Hess}_{\gamma, w} = \{gP_{\nu, w} \in L_\nu/P_{\nu, w} \mid g^{-1}\gamma g \in \dot{w}\mathfrak{I}\dot{w}^{-1}\},$$

where $P_{\nu, w} := L_\nu \cap \dot{w}I\dot{w}^{-1}$. They are smooth.

(Oblomkov–Yun) If $\text{Hess}_{\gamma, w} \neq \emptyset$, then:

The codimension in $L_\nu/P_{\nu, w}$ is the number of affine roots $\alpha + k$ such that:

- $\langle \alpha, \nu\rho^\vee \rangle + k = \nu$.
- $\{\alpha + k = 0\}$ separates $\nu\rho^\vee$ and $w \cdot \frac{1}{n}\rho^\vee$ in $X^\vee \otimes \mathbf{R}$.

Conj (T) For general ν , the representation

$$W \curvearrowright H_{c, \mathbf{C}^\times}^*(\mathcal{B}_\gamma^{\text{aff}})|_{\epsilon \rightarrow 1}$$

contains a summand whose character is the $\mathbf{q} \rightarrow 1$ limit of our earlier formula:

$$\frac{\text{sgn}(w)}{\det(1 - \mathbf{q}w \mid \mathfrak{h})} \sum_{\chi \in \text{Irr}(W)} \mathbf{q}^{c(\chi)\nu} D_\chi(e^{2\pi i\nu}) \chi(w).$$

Moreover, the Oblomkov–Yun filtration restores \mathbf{q} .

Thank you for listening.