



# Character Formulas from Lusztig Varieties and Affine Springer Fibers

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This talk is about...

- 1 Braids
- 2 Lusztig Varieties
- 3 Springer Fibers
- 4 Affine Springer Fibers

1 Braids The braid group  $Br_n =$

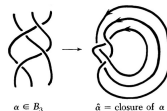
$$\left\langle \sigma_1, \dots, \sigma_{n-1}, \left| \begin{array}{l} \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \\ \sigma_i \sigma_j = \sigma_j \sigma_i \text{ when } |i - j| > 1 \end{array} \right. \right\rangle$$

appears in knot theory and representation theory.

$$\begin{array}{c} \sigma_i \\ \boxed{\dots} \sum_{i \atop i+1} \boxed{\dots} \end{array}$$

A *link* is a collection of circles (tamely) embedded in  $\mathbf{R}^3$ . Knot theory is about isotopy invariants of links.

(Alexander) Every link is the *closure* of some braid.



Let  $G = \mathrm{GL}_n$  and  $B$  its upper-triangular subgroup.

$$V_n(q) = \{\text{functions } G(\mathbf{F}_q)/B(\mathbf{F}_q) \rightarrow \mathbf{C}\},$$

$$H_n(q) = \mathrm{End}_{G(\mathbf{F}_q)}(V_n(q)).$$

$$(\text{Iwahori}) \quad H_n(q) \simeq \frac{\mathbf{C}Br_n}{\langle \sigma_i^2 - (q-1)\sigma_i - q \rangle}.$$

To explain, recall Bruhat:  $G = \coprod_{w \in S_n} B\dot{w}B$ .

Then  $\mathbf{C}Br_n \hookrightarrow V_n(q)$  via

$$\sigma_i \cdot \mathbf{1}_{xB(\mathbf{F}_q)} = \sum_{yB \xrightarrow{i} xB} \mathbf{1}_{yB(\mathbf{F}_q)},$$

where  $yB \xrightarrow{i} xB$  means  $By^{-1}xB = B\dot{w}_{(i,i+1)}B$ .

Motivates a *Hecke algebra*  $H_n(\mathbf{q})$  over  $\mathbf{C}[\mathbf{q}^{\pm 1}]$ .

Ocneanu used functions  $Br_n \rightarrow H_n(\mathbf{q}) \rightarrow \mathbf{C}(\mathbf{q})[a^{\pm 1}]$  to construct a link invariant

$$\text{HOMFLYPT} : \{\text{links}\}/\text{isotopy} \rightarrow \mathbf{C}(\mathbf{q})[\mathbf{a}^{\pm 1}].$$

Jones computed it for torus knots. Remarkably, the values encode  $\mathbf{q}$ -Catalan (and  $\mathbf{q}$ -Kirkman) numbers.

On the other hand, Iwahori suggests that HOMFLYPT is related to the geometry of  $G/B$ .

We'll discuss a Springer-theoretic function of  $\beta$  that refines the HOMFLYPT invariant of  $\hat{\beta}$ .

2 Lusztig Varieties Suppose that  $\beta$  is *positive*:

$$\beta = \sigma_{i_1} \cdots \sigma_{i_\ell}.$$

(Deligne) The variety  $O(\beta) =$

$$\left\{ (g_0 B, g_1 B, \dots, g_\ell B) \left| g_{j-1} B \xrightarrow{i_j} g_j B \text{ for all } j \right. \right\}$$

only depends on  $\beta$ , up to isomorphisms that keep  $g_0 B$  and  $g_\ell B$  fixed.

For any positive  $\beta, \beta'$ , we have

$$O(\beta\beta') \simeq O(\beta) \times_{G/B} O(\beta'),$$

where  $\times_{G/B}$  means the variety of pairs  $(\vec{g}B, \vec{g}'B)$  such that  $g_\ell B = g'_0 B$ .

A literal geometric representation of positive braids.

For any  $x \in G(\mathbf{F}_q)$ , form the *braid Lusztig variety*

$$\mathcal{B}(\beta)_x = \{\vec{g}B \in O(\beta) \mid g_\ell B = xg_0 B\}.$$

(Shende–Treumann–Zaslow) Up to a monomial in  $q$ ,

$$\frac{|\mathcal{B}(\beta)_1(\mathbf{F}_q)|}{|\mathrm{PGL}_n(\mathbf{F}_q)|}$$

is the “highest”  $a$ -degree of  $\mathrm{HOMFLYPT}(\hat{\beta})$  at  $\mathbf{q} \rightarrow q$ .

**Example** Let  $n = 2$  and  $\beta = \sigma_1^3 \in Br_2$ .

$$O(\beta) \simeq \{\vec{g} \in (\mathbf{P}^1)^4 \mid g_0 \neq g_1 \neq g_2 \neq g_3\},$$

$$\mathcal{B}(\beta)_1 \simeq \{\vec{g} \in (\mathbf{P}^1)^3 \mid g_1, g_2, g_3 \text{ pairwise distinct}\}.$$

$\mathrm{PGL}_2$  acts simply transitively on the latter.

Indeed,  $\mathrm{HOMFLYPT}(\hat{\sigma}_1^3) = a^2(\mathbf{q} + \mathbf{q}^{-1}) - a^4$ .

### 3 Springer Fibers

How to access other  $a$ -degrees? Observe that

$$\mathcal{B}_x := \mathcal{B}(1)_x = \{gB \mid gB = xgB\}$$

is the usual Springer fiber over  $x$ , whose cohomology defines a character of  $S_n$ : namely,

$$\Psi_x(w) := \sum_i \mathbf{q}^{i \operatorname{tr}(w \mid H^{2i}(\mathcal{B}_x))}.$$

Thm 1 (T) Let  $\mathcal{U} \subseteq G$  be the unipotent variety,

$$\Psi_\beta(w) = \sum_{u \in \mathcal{U}(\mathbf{F}_q)} \frac{|\mathcal{B}(\beta)_u(\mathbf{F}_q)|}{|\operatorname{PGL}_n(\mathbf{F}_q)|} \Psi_u(w)|_{\mathbf{q} \rightarrow q}.$$

Then  $(\chi_{(n-k, 1, \dots, 1)}, \Psi_\beta)_{S_n}$  sees the  $k$ th  $a$ -degree.

Think of  $\beta \mapsto \Psi_\beta$  as a function

$$Br_n \rightarrow H_n(q) \rightarrow \{\text{characters of } S_n\}.$$

Example Again, let  $n = 2$  and  $\beta = \sigma_1^3 \in Br_2$ .

$$\Psi_u = \begin{cases} 1 + \mathbf{q} \operatorname{sgn} & u = 1, \\ 1 & u \neq 1. \end{cases}$$

$$\Psi_\beta = q^2 + 1 + q \operatorname{sgn}.$$

Recall  $\operatorname{HOMFLYPT}(\widehat{\sigma_1^3}) = a^2(\mathbf{q} + \mathbf{q}^{-1}) - a^4$ .

Thm 2 (T) The cohomology of  $\mathcal{U}(\beta) \times_{\mathcal{U}} \mathcal{U}(1)$ , where

$$\mathcal{U}(\beta) = \{(u, \vec{g}B) \mid u \in \mathcal{U} \text{ and } \vec{g}B \in \mathcal{B}(\beta)_u\},$$

encodes finer invariants of  $\hat{\beta}$ .

The *full twist*  $\pi = (\sigma_1 \cdots \sigma_{n-1})^n$ :



Thm 3 (T) Suppose  $\beta^m = \pi^d$  for some  $d, m > 0$ .

Then up to a monomial,  $\Psi_\beta(w)$  is the  $\mathbf{q} \rightarrow q$  limit of

$$\frac{\text{sgn}(w)}{\det(1 - \mathbf{q}w \mid \mathfrak{h})} \sum_{\lambda \vdash n} \mathbf{q}^{c(\lambda)d/m} D_\lambda(e^{2\pi i d/m}) \chi_\lambda(w)$$

where:

- $\mathfrak{h}$  is the *reflection representation*.
- $c(\lambda)$  is the sum of *contents* of  $\lambda$ .
- $D_\lambda(t) = K_{\lambda, (1^n)}(t)$  is the *fake degree* of  $\lambda$ .

Recovers Jones's HOMFLYPT formula for torus knots.

Thm 3 generalizes to any reductive  $G$ , once we replace:

- $S_n$  with the Weyl group  $W$ .
- $c(\lambda)$  with  $c(\chi) = \sum_{\text{refl. } t} \frac{\chi(t)}{\chi(1)}$ .
- fake degrees  $D_\lambda$  with *generic degrees*  $D_\chi$ .

If  $\gcd(d, m) = 1$  and  $m$  is the Coxeter number of  $W$ , then the formula simplifies:

$$(\text{monomial}) \cdot \left. \frac{\det(1 - \mathbf{q}^d w \mid \mathfrak{h})}{\det(1 - \mathbf{q}w \mid \mathfrak{h})} \right\} =: \Pi_{\mathbf{q}}^{(d)}.$$

$\Pi_{\mathbf{q}}^{(d)}$  is the character of a *rational parking space*.

$(\text{triv}, \Pi_{\mathbf{q}}^{(d)})_W$  is a *rational  $q$ -Catalan number*.

Example If  $W = S_n$ , then  $(\text{triv}, \Pi_{\mathbf{q}}^{(d)})_W = \frac{[n+d-1]!}{[n]![d]!}$ .

## 4 Affine Springer Fibers

Rational parking spaces appear in a loop or *affine* analogue of Springer theory.

finite Springer	affine Springer
$G$	$G((z))$
$G/B$	$G((z))/I$
$W$	$\tilde{W} = W \ltimes X^\vee$

Above:

- $G((z))$  is the loop group  $G((z))(R) := G(R((z)))$ .
- $I$  is the preimage of  $B$  in  $G[[z]]$ .
- $X^\vee$  is the cocharacter lattice of  $B$ .

We now study Springer fibers over the Lie algebras, not the groups, and over  $\mathbf{C}$ , not  $\mathbf{F}_q$ .

$$x : \quad \mathcal{B}_x = \{gB \in G/B \mid g^{-1}xg \in \text{Lie}(B)\},$$

$$\gamma = \gamma(z) : \quad \mathcal{B}_\gamma^{\text{aff}} = \{gI \in G((z))/I \mid g^{-1}\gamma g \in \text{Lie}(I)\}.$$

The table hides key differences:

In the **finite** case,  $\mathcal{B}_x$  is most **interesting** for  $x$  nilpotent.

In the **affine** case,  $\mathcal{B}_\gamma^{\text{aff}}$  is *terribly infinite* for  $\gamma = \gamma(z)$  nilpotent, but **interesting** for  $\gamma(z)$  regular semisimple.

**Example** If  $G = \text{SL}_2$  and  $\gamma(z) = \begin{pmatrix} 0 & 1 \\ z^3 & 0 \end{pmatrix}$ , then

$$\mathcal{B}_\gamma^{\text{aff}} \simeq \mathbf{P}^1 \sqcup_{\text{pt}} \mathbf{P}^1.$$

**Dream** Braid Lusztig varieties see the *finite* part of affine Springer representations.

Fix  $\nu = d/m > 0$  in lowest terms. Let  $\mathbf{C}^\times \curvearrowright \mathfrak{g}((z))$ :

$$c \cdot_\nu \gamma(z) = c^{2d\rho^\vee} \gamma(c^{2m} z) c^{-2d\rho^\vee},$$

where  $2\rho^\vee = \sum_{\alpha \in \Phi^+} \alpha^\vee$ .

Let  $\mathfrak{g}((z))_{\nu, k}$  be the weight- $2k$  eigenspace.

**Lemma** If  $\gamma$  is an eigenvector for  $\cdot_\nu$ , then the induced action on  $G((z))/I$  preserves  $\mathcal{B}_\gamma^{\text{aff}}$ .

**Lemma**  $\mathfrak{g}((z))_{\nu, 0}$  is the Lie algebra of a connected reductive group  $L_\nu$ . Moreover,

$$(G((z))/I)^{\mathbf{C}^\times} = \coprod_{w \in W(L_\nu) \setminus \widetilde{W}} L_\nu \dot{w} I / I.$$

Fix  $\gamma \in \mathfrak{g}((z))_{\nu, d}$ . In the  $\text{SL}_2$  example,  $\gamma \in \mathfrak{g}((z))_{3/2, 3}$ .

$$\text{Springer : } \widetilde{W} \curvearrowright H^*(\mathcal{B}_\gamma^{\text{aff}}), H_{\mathbf{C}^\times}^*(\mathcal{B}_\gamma^{\text{aff}}).$$

(Sommers) If  $m$  is the Coxeter number, then:

- $L_\nu$  is a torus, and  $L_\nu \dot{w} I = \dot{w} I$ .
- $(\mathcal{B}_\gamma^{\text{aff}})^{\mathbf{C}^\times}$  is a finite subset of the  $\dot{w} I$ .
- Writing  $H_{\mathbf{C}^\times}^*(\text{pt}) = \mathbf{C}[\epsilon]$ , we have

$$\begin{aligned} H^*(\mathcal{B}_\gamma^{\text{aff}}) &= H_{\mathbf{C}^\times}^*(\mathcal{B}_\gamma^{\text{aff}})|_{\epsilon \rightarrow 1} \\ &= \{\text{functions on } (\mathcal{B}_\gamma^{\text{aff}})^{\mathbf{C}^\times}\}. \end{aligned}$$

- $\Pi_q^{(d)}(w)|_{q \rightarrow 1}$  is the  $W$ -character of  $H_c^*(\mathcal{B}_\gamma^{\text{aff}})$ .

(Oblomkov–Yun) Filtration on  $H_{\mathbf{C}^\times}^*|_{\epsilon \rightarrow 1}$  that sees  $\mathfrak{q}$ .



(Goresky–Kottwitz–MacPherson) For general  $\nu$ ,

$$(\mathcal{B}_\gamma^{\text{aff}})^{\mathbf{C}^\times} = \coprod_{w \in W(L_\nu) \setminus \widetilde{W}} \text{Hess}_{\gamma,w},$$

a disjoint union of *partial Hessenberg varieties*

$$\text{Hess}_{\gamma,w} = \{gP_{\nu,w} \in L_\nu/P_{\nu,w} \mid g^{-1}\gamma g \in \text{Lie}(P_{\nu,w})\},$$

where  $P_{\nu,w} := L_\nu \cap \dot{w}I\dot{w}^{-1}$ .

(Oblomkov–Yun) They are smooth. Finitely many are nonempty.

For such  $\text{Hess}_{\gamma,w}$ , the codimension in  $L_\nu/P_{\nu,w}$  is the number of affine roots  $\alpha + k$  such that:

- $\langle \alpha, \nu\rho^\vee \rangle + k = \nu$ .
- $\{\alpha + k = 0\}$  separates  $\nu\rho^\vee$  and  $w \cdot \frac{1}{n}\rho^\vee$  in  $X^\vee \otimes \mathbf{R}$ .

Conj (T) For general  $\nu$ , the representation

$$W \curvearrowright H_{c,\mathbf{C}^\times}^*(\mathcal{B}_\gamma^{\text{aff}})|_{\epsilon \rightarrow 1}$$

contains a summand whose character is the  $\mathbf{q} \rightarrow 1$  limit of our earlier formula:

$$\frac{\text{sgn}(w)}{\det(1 - \mathbf{q}w \mid \mathfrak{h})} \sum_{\chi \in \text{Irr}(W)} \mathbf{q}^{c(\chi)\nu} D_\chi(e^{2\pi i\nu}) \chi(w).$$

Moreover, the Oblomkov–Yun filtration restores  $\mathbf{q}$ .

Thank you for listening.