1.

Notes on K-theoretic realizations of tensor representations of the affine Hecke algebra.

1.1. Fix integers $d, n > 0$ such that $d \geq n$. When we introduce sets and varieties that depend on both d and n, we suppress d from the notation, but not n. Let

$$\begin{aligned}\Pi_n &= \{\nu \vdash d \mid \nu_i \leq n \text{ for each part } \nu_i\},\\ \Pi_n^t &= \{\nu \vdash d \mid \nu \text{ has at most } n \text{ parts}\}.\end{aligned}$$

Where convenient, we identify $\nu \in \Pi_n$ with the n-tuple of integers that we obtain by appending zeros to the end of ν. That is, we regard Π_n not only as a set of integer partitions of d, but as a subset of $\mathbf{Z}^n_{\geq 0}$.

1.2. Let $G = \mathrm{GL}_d$ and $\mathfrak{g} = \mathfrak{gl}_d$ and $\mathscr{K} = \mathrm{GL}_n[[z]]$. We need the following varieties:

- $\mathscr{N}$ is the nilpotent cone in $\mathfrak{g}$.
- $\mathscr{O}_\nu \subseteq \mathscr{N}$ is the orbit of elements with Jordan type ν, for any partition $\nu \vdash d$.
- $\mathscr{N}_n = \bigcup_{\nu \in \Pi_n} \mathscr{O}_\nu$.
- $\mathscr{N}_n^t = \bigcup_{\nu \in \Pi_n^*} \mathscr{O}_\nu$.
- $\mathscr{B}$ is the complete flag variety of G. Thus $T^*\mathscr{B}$ is the Springer resolution of $\mathscr{N}$.
- $\mathscr{F}_n$ is the n-step partial flag variety of G. Thus $T^*\mathscr{F}_n$ is a disjoint union of partial Springer resolutions of $\mathscr{N}$.
- $\mathscr{X}_n$ is the reduced scheme underlying the $\mathscr{K}$-Schubert variety attached to the coweight $d\omega_1 \in \mathbf{Z}^n$ in the affine Grassmannian $\mathrm{GL}_n((z))/\mathscr{K}$. On points,

$$\mathscr{X}_n(\mathbf{C}) = \left\{ \text{full lattices } M \subseteq \mathbf{C}((z))^n \;\middle|\; \begin{array}{l} M \subseteq \mathbf{C}[[z]]^n \text{ of codimension } d, \\ M \text{ is } z\text{-stable} \end{array} \right\}.$$

The adjoint G-action and scaling $\mathbf{G}_m$-action on $\mathscr{N}$ together form a $(G \times \mathbf{G}_m)$-action that stabilizes $\mathscr{O}_\nu$ for all ν. It lifts equivariantly to $(G \times \mathbf{G}_m)$-actions on $T^*\mathscr{B}$ and $T^*\mathscr{F}_n$. At the same time, the left $\mathscr{K}$-action and loop-rotation $\mathbf{G}_m$-action on $\mathrm{GL}_n((z))/\mathscr{K}$ together form a $(\mathscr{K} \rtimes \mathbf{G}_m)$-action that stabilizes $\mathscr{X}_n$.

There is a map $\mathscr{X}_n \to [\mathscr{N}/G]$: on points, it is given by $[M] \mapsto (z \curvearrowright \mathbf{C}[[z]]^n/M)$. This is $\mathbf{G}_m$-equivariant and factors through $[\mathscr{X}_n/\mathscr{K}]$.

1.3. Recall the Steinberg scheme

$$\mathscr{Z} := T^*\mathscr{B} \times^{\mathrm{R}}_{\mathscr{N}} T^*\mathscr{B}.$$

Its stack quotient under the diagonal action of $G \times \mathbf{G}_m$ can be rewritten:

$$[\mathscr{Z}/(G \times \mathbf{G}_m)] \simeq [T^*\mathscr{B}/(G \times \mathbf{G}_m)] \times^{\mathrm{R}}_{[\mathscr{N}/(G \times \mathbf{G}_m)]} [T^*\mathscr{B}/(G \times \mathbf{G}_m)].$$

Via pullback and pushforward along a diagram involving copies of this stack, the $(G \times \mathbf{G}_m)$-equivariant K-theory of $\mathscr{Z}$ forms a convolution algebra. Kazhdan–Lusztig observed that it is isomorphic to the affine Hecke algebra of G.

1.4. Similarly, form the scheme

$$\mathscr{W}_n := T^*\mathscr{F}_n \times^{\mathrm{R}}_{\mathscr{N}} T^*\mathscr{B}.$$

The analogous stack quotient is:

$$[\mathscr{W}_n/(G \times \mathbf{G}_m)] \simeq [T^*\mathscr{F}_n/(G \times \mathbf{G}_m)] \times^{\mathrm{R}}_{[\mathscr{N}/(G\times\mathbf{G}_m)]} [T^*\mathscr{B}/(G \times \mathbf{G}_m)].$$

Again via convolution, the $(G \times \mathbf{G}_m)$-equivariant K-theory of $\mathscr{Z}$ acts on that of $\mathscr{W}_n$. Ginzburg–Reshetikhin–Vasserot [GRV] observe that the resulting module corresponds to the polynomial tensor representation of the affine Hecke algebra.

1.5. We claim that the action of $K^{G\times\mathbf{G}_m}(\mathscr{Z})$ on $K^{G\times\mathbf{G}_m}(\mathscr{W}_n)$ is related to $\mathfrak{sl}_n$ link homology through Langlands duality.

First, as already mentioned at the end of [GRV], this action on (coherent) K-theory has a constructible counterpart, arising from the realization of the affine Hecke algebra and its polynomial tensor representation in terms of mixed ℓ-adic complexes on the affine flag variety and its partial analogues, respectively. The equivalence of the coherent and constructible realizations should be categorified by a parahoric specialization of Bezrukavnikov's equivalence of affine Hecke categories.

On both sides, there is an embedding of finite Schur–Weyl duality into affine Schur–Weyl duality, essentially by restricting the structure group from $G(\!(z)\!)$ to $\mathscr{K}$. On the constructible side, the resulting action of the finite Hecke category on mixed perverse sheaves on n-step partial flag varieties is equivalent, via Soergel's functor, to the action of parabolic functors on parabolic categories O for G, or rather $G^{\mathrm{der}} = \mathrm{SL}_d$. The latter is precisely what gives rise to $\mathfrak{sl}_n$ link homology in the sense of Khovanov. For instance, when $n = 2$, this is discussed by Bernstein–Frenkel–Khovanov [BFK].

1.6. We plan to compare $\mathscr{W}_n$ to the scheme

$$\mathscr{Y}_n := \mathscr{X}_n \times^{\mathrm{R}}_{[\mathscr{N}/G]} [T^*\mathscr{B}/G].$$

Let $\mathbf{G}_m$ act on $\mathscr{Y}_n$ diagonally, and let $\mathscr{K}$ act on $\mathscr{Y}_n$ through $\mathscr{X}_n$. We have the following identifications of stacks:

$$[\mathscr{Y}_n/\mathbf{G}_m] \simeq [\mathscr{X}_n/\mathbf{G}_m] \times^{\mathrm{R}}_{[\mathscr{N}/(G\times\mathbf{G}_m)]} [T^*\mathscr{B}/(G \times \mathbf{G}_m)],$$
$$[\mathscr{Y}_n/(\mathscr{K} \rtimes \mathbf{G}_m)] \simeq [\mathscr{X}_n/(\mathscr{K} \rtimes \mathbf{G}_m)] \times^{\mathrm{R}}_{[\mathscr{N}/(G\times\mathbf{G}_m)]} [T^*\mathscr{B}/(G \times \mathbf{G}_m)].$$

Again via convolution, $K^{G\times\mathbf{G}_m}(\mathscr{Z})$ acts on $K^{\mathbf{G}_m}(\mathscr{Y}_n)$ and $K^{\mathscr{K}\rtimes\mathbf{G}_m}(\mathscr{Y}_n)$.

1.7. The stacks $[\mathscr{Y}_n/\mathbf{G}_m]$ are precisely those used by Cautis–Kamnitzer to construct a bigraded isotopy invariant of tangles in [CK1] (for $n = 2$) and [CK2] (for general n, also allowing colored refinements). More precisely, they construct functors between the $\mathbf{G}_m$-equivariant coherent derived categories of the varieties $\mathscr{Y}_n$, and a graphical calculus that assigns, to any oriented, planar tangle diagram, a functor that provides an isotopy invariant of the tangle.

Cautis–Kamnitzer show that their functors decategorify to the action of the finite Hecke algebra on its tensor representation. More precisely, they show this for $n = 2$ in

[CK1], conjecture a colored refinement for general n in [CK2], and jointly with Licata, prove the latter for the "trivial" and "sign" colors in [CKL].

Let $K^{G\times\mathbf{G}_m}_{\mathrm{fin}}(\mathscr{Z})$ be the subalgebra of $K^{G\times\mathbf{G}_m}(\mathscr{Z})$ corresponding to the finite Hecke algebra of G: that is, the Hecke algebra of S_d. The action of $K^{G\times\mathbf{G}_m}_{\mathrm{fin}}(\mathscr{Z})$ on $K^{\mathbf{G}_m}(\mathscr{Y}_n)$ is weakly categorified by a collection of endofunctors of $\mathsf{D}^b\mathsf{Coh}^{\mathbf{G}_m}(\mathscr{Y}_n)$: namely, pull-push along correspondence diagrams involving the top-dimensional components of the Steinberg stack $[\mathscr{Z}/(G\times\mathbf{G}_m)]$. It is natural to expect:

Conjecture 1.1. *There is a* 2*-equivalence between:*

(1) The category of endofunctors of $\mathsf{D}^b\mathsf{Coh}^{\mathbf{G}_m}(\mathscr{Y}_n)$ *arising from correspondences along top-dimensional Steinberg components.*
(2) The category of endofunctors of $\mathsf{D}^b\mathsf{Coh}^{\mathbf{G}_m}(\mathscr{Y}_n)$ *generated by the braiding functors in [CK1, CK2] (in the uncolored case).*

Thus, (1) also decategorifies to the action of the finite Hecke algebra on its tensor representation.

1.8. It is not yet clear to me how to upgrade either $K^{\mathbf{G}_m}(\mathscr{Y}_n)$ or $K^{\mathscr{K}\rtimes\mathbf{G}_m}(\mathscr{Y}_n)$ from finite to affine level. Yet we might hope:

Conjecture 1.2. *There is a fully faithful functor*

$$\mathsf{D}^b\mathsf{Coh}^{\mathbf{G}_m}(\mathscr{Y}_n)\to\mathsf{D}^b\mathsf{Coh}^{G\times\mathbf{G}_m}(\mathscr{W}_n).$$

It is compatible with the action of correspondences along top-dimensional Steinberg components on both the source and the target. Hence, it decategorifies to the inclusion of the tensor representation of the finite Hecke algebra into the polynomial tensor represntation of the affine Hecke algebra.

This conjecture would essentially match Cautis–Kamnitzer's link homology with Khovanov's by way of Langlands duality. A colored refinement of this conjecture would likely prove the full, colored version of Conjecture 7.1 in [CK2].

1.9. We mention an interesting duality between the constructions of $\mathscr{W}_n$ and $\mathscr{Y}_n$. The map $T^*\mathscr{F}_n\to\mathscr{N}$ factors through $\mathscr{N}_n$, while the map $\mathscr{X}_n\to\mathscr{N}$ factors through $\mathscr{N}_n^t$. As the notation suggests, $\mathscr{N}_n$ and $\mathscr{N}_n^t$ are interchanged under the involution of $\mathscr{N}$ that sends any nilpotent $d\times d$ matrix to its transpose.