<u>last time</u> V, W fin. dim. vector spaces over F

given $\{v_1, \ldots, v_n\}$ a basis for V,
$\{w_1, \ldots, w_m\}$ a basis for W

any linear map $T : V$ to W rep by $m \times n$ matrix M:

in "column" notation,
entry $M_{j, i}$ in the jth row and ith col is def by

$$Tv_i = \sum_j M_{j, i} w_j$$

so ith col of M describes coeffs of Tv_i wrt $(w_j)_j$
[draw]

<u>Ex</u> suppose 1st and 2nd cols of M match
what sort of lin map does M represent?

$Tv_1 = Tv_2$ because same w_j expansion
so M represents T s.t. $v_1 - v_2$ in $\ker(T)$

<u>Ex</u> suppose M is a single row
[same question]

means $m = 1$, i.e., W is 1-dimensional
fixing $\{w_1\}$ equiv to fixing a linear iso $W \simeq F$
[it sends aw_1 to a]
so M represents a linear map V to F

also known as <u>linear functionals</u> [when inj? surj?]

U another vector space over F, $T' : W$ to U,

can always form $T' \circ T : V$ [to W] to U

given $\{u_1, \ldots, u_\ell\}$ a basis for U

if T rep by M wrt $(v_i)_i, (w_j)_j$,
T' rep by M' wrt $(w_j)_j, (u_k)_k$,
then $T' \circ T$ rep by $M' \bullet M$

Q suppose instead of M: only know
T rep by M_ω wrt $(v_i)_i, (\omega_j)_j$

how to represent $T' \circ T$ in terms of M_ω, M'?

define N so that $\omega_j = \sum_i N_{j, i} w_i$:

T rep by $N \bullet M_\omega$ wrt $(v_i), (w_j)_j$
so $M = N \bullet M_\omega$
so $T' \circ T$ rep by $M' \bullet N \bullet M_\omega$ wrt $(v_i)_i, (u_k)_k$

Properties of Compositions

how does ker(T) compare to ker($T' \circ T$)?
im(T') im($T' \circ T$)?

Ex if $W = F$, then $T' \circ T$ "bottlenecked" by F:

so dim im($T' \circ T$) ≤ 1, dim ker($T' \circ T$) $\geq n - 1$
[using the dim formula to get dim ker]

<u>Prop</u> 1) ker(T) sub ker(T′ ○ T)
2) im(T) supset im(T′ ○ T)

<u>Pf</u> if Tv = **0_W**
then (T′ ○ T)v = T′(**0_W**) = **0_U**

if u = (T′ ○ T)v then u = T′(Tv)

<u>Cor</u> 1) if T′ ○ T is inj, then T is inj [why?]
2) if T′ ○ T is surj, then T′ is surj [why?]

<u>Rem</u> if T′ ○ T is bij, then T, T′ need <u>not</u> be bij
[example?]
V = F, W = F^2, U = F,
T(1) = (1, 0), T′(x, y) = x

<u>Principle</u> "coordinate free"
=
define stuff without matrices or bases

<u>Ex 1</u> for any m × n matrix M,
span(cols of M) is a lin. sub. of F^m

<u>col rank</u> of M is def by dim span(cols of M)

iso F^m to W : jth std basis vec to w_j
ith col of M to Tv_i
[= sum_M_{j, i}w_j]

restricted iso: span(cols of M) to im(T)

so col rank of M = dim im(T) [RHS basis-indep]

Rem can also define row rank

turns out that col rank = row rank

[but tricky; we will defer for now]

Ex 2 $\mathrm{Mat}_{m \times n}(F) = \{m \times n \text{ matrices}\}$

is a vector space under obvious +, •

define the hom space from V to W to be

$$\mathrm{Hom}(V, W) = \{\text{linear maps from } V \text{ to } W\}$$

forms a [basis-indep] vector space under:

$(T + T')v = Tv + T'v$,

$(a \cdot T)v = a \cdot Tv$

$\mathbf{0_{Hom(V, W)}}$ = zero map

Prop if dim V = n and dim W = m

then a choice of bases $(v_i)_i$, $(w_j)_j$

defines a linear iso

$$\mathrm{Hom}(V, W) \text{ to } \mathrm{Mat}_{m \times n}(F)$$

Pf send T to its matrix wrt $(v_i)_i$, $(w_j)_j$

compare previous formulas with

$(M + M')v = Mv + M'v$,

$(aM)v = a \cdot Mv$

Rem Hom(V, W) exists even when

V, W are not finite-dimensional

[example of how coord-free ideas can be easier]

<u>Recall</u> if $V = F^n$ and $W = F^m$, then we take $(v_i)_i$, $(w_j)_j$ std [as a convention]

henceforth suppose that $V = W = F^n$

what is the matrix of

$T(v) = v$?	identity $I = I_n$
$T(v) = \mathbf{0}_V$?	zero 0_n
$T(a_1, a_2 \ldots) = (a_2, a_3, \ldots)$?	nilpotent [draw]

take $n = 2$

what is the matrix of

scaling y-axis by 2?	diagonal [draw]
reflect across $y = -x$?	[draw]
projection onto x-axis?	trick question

<u>Ex</u> $P = \begin{matrix} 1 & 0 \\ 0 & 0 \end{matrix}$ is a proj on the x-axis

$Q = \begin{matrix} 0 & 0 \\ 0 & 1 \end{matrix}$ is a proj on the y-axis

note that $I = P + Q$

<u>Q</u> what is $\mathrm{span}(P, Q)$ sub $\mathrm{Hom}(F^2, F^2)$?

[draw]

all lin op's given by rescaling x, y-axes

<u>Q</u> which elts of $\mathrm{span}(P, Q)$ are lin isos?

$aP + bQ$ s.t. $a \neq 0$ and $b \neq 0$

Df a matrix is invertible iff it reps a lin iso

Ex $\begin{matrix} a & 0 \\ 0 & b \end{matrix}$ is invertible: [what inverse?]

inverse $\begin{matrix} 1/a & 0 \\ 0 & 1/b \end{matrix}$

[other invertible maps?]

$\begin{matrix} 1 & 1 \\ 0 & 1 \end{matrix}$ with inverse $\begin{matrix} 1 & -1 \\ 0 & 1 \end{matrix}$

any $\begin{matrix} a & b \\ c & d \end{matrix}$ s.t. $ad - bc \neq 0$

[what inverse?]