Last time fix a cts map p : E to X

U sub X is evenly covered by p iff:

1) p^{-1}(U) is homeomorphic to a nonempty(!) disjoint union of copies of U
2) p restricts to a homeomorphism from each copy onto U

p : E to X is a covering map iff each pt of X has an open nbd evenly covered by p

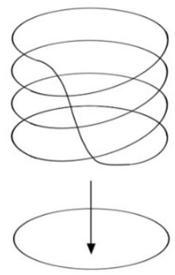
we say that E is a covering space or cover of X

Ex consider the formula
p(x) = (cos(2πx), sin(2πx))

p : (–1, 1) to S^1 is not a covering map [why?]
but
p : R to S^1 is a covering map

Ex identity maps Id : X → X are always covering maps

Ex more generally:



https://ahilado.wordpress.com/2017/04/14/covering-spaces/

for any $n > 0$, a covering $p_n : S^1 \to S^1$ s.t. the fiber $p_n^{-1}(x)$ has cardinality n for all x

we say that the covering is of degree n, or n-fold

Ex $\quad p_m \times p_n : S^1 \times S^1 \to S^1 \times S^1$ is a covering: of what degree? [mn]

more generally:

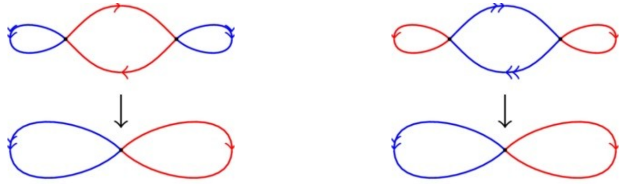
if $p : E \to X$ and $p' : E' \to X'$ are covering maps, then so is $p \times p'$

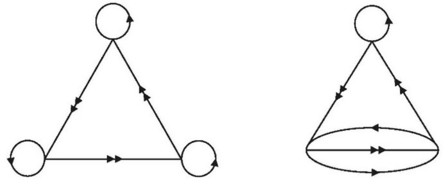
Ex $\quad$ consider the relation ~ on S^2 given by $(x, y, z) \sim (-x, -y, -z)$

$S^2/\sim$ is called the real projective plane, or RP^2

$S^2 \to RP^2$ is a 2-fold covering map

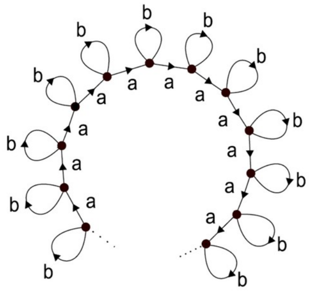
Ex $\quad$ coverings of the figure-eight:



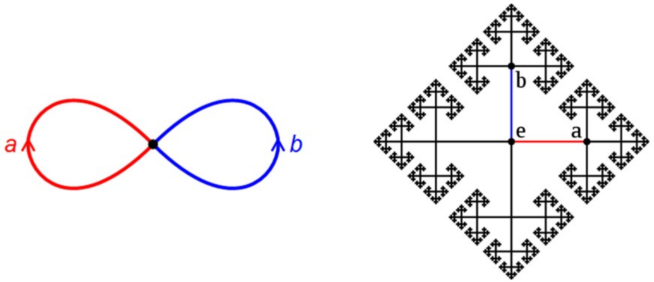


(1) https://www.homepages.ucl.ac.uk/~ucahjde/tg/html/cov-01.html
(2) https://groupoids.org.uk/images/fig10-3.jpg

weirder:



https://www.math.cmu.edu/~nkomarov/NK-NormalSubFreeGrp.pdf



https://math.stackexchange.com/a/3762676

Thm if $p : E$ to X is a covering, then:

I) for any path $\gamma : [0, 1]$ to X and e in E s.t.
$p(e) = \gamma(0)$,

a unique $\Gamma : [0, 1]$ to R s.t.
$\Gamma(0) = e$ and $\gamma = p \circ \Gamma$, called a lift of γ

II) for any path homotopy $h : [0, 1]^2$ to X s.t.
$h(–, 0) = \gamma$, and lift Γ of γ,

a unique path-homotopy $H : [0, 1]^2$ to R s.t.
$H(–, 0) = \Gamma$ and $h = p \circ H$, called a lift of h

Pf Lemmas 54.1 and 54.2 in Munkres

<u>Cor</u> if $\gamma_0 \sim_p \gamma_1$ in X and e in E s.t.
$p(e) = \gamma_0(0) = \gamma_1(0)$,

then the unique lifts Γ_0, Γ_1 starting at e satisfy
$\Gamma_0 \sim_p \Gamma_1$

<u>Cor</u> if $p : E \to X$ is a covering,
$p(e) = x$,

then $p_* : \pi_1(E, e) \to \pi_1(X, x)$ is injective

<u>Pf</u> let Γ_0, Γ_1 be loops in E based at e s.t.
$p_*([\Gamma_0]) = p_*([\Gamma_1])$

since $p_*(\Gamma_i) = [p \circ \Gamma_i]$ and Γ_i lifts $p \circ \Gamma_i$,
we require $[\Gamma_0] = [\Gamma_1]$

<u>Ex</u> recall the covering $p_n : S^1 \to S^1$

under $\pi_1(S^1) \simeq \mathbb{Z}$, we have $\mathrm{im}(p_{n, *}) \simeq n\mathbb{Z}$

[draw]

<u>Df</u> a <u>pointed covering</u> of X is a pair (p, e)
s.t. $p : E \to X$ is a path-conn. covering,
e in E

a <u>pointed equivalence</u> from (p, e) to (p', e') is
a homeo $f : (E, e) \to (E', e')$ s.t. $p = p' \circ f$

[draw]

Thm (Galois Correspondence)

if X is conn. & locally simply-conn., then

$$(p : E \to X, e) \mapsto p_*(\pi_1(E, e))$$

is a bijection

$$\{\text{pointed coverings of } X\}/\sim \quad \text{to } \{\text{subgroups of } \pi_1(X, x)\}$$

leads to topological analogue of Galois theory

Rem if $E = E'$ but $e \neq e'$
then (p, e), (p', e') may not be equivalent