Last time (Seifert–van Kampen)

U, V open in X,
X = U cup V,
x in U cap V,
U, V, U cap V are all path-connected

inclusions i : U cap V to U and j : U cap V to V

embed i_*(π_1(U cap V, x)), j_*(π_1(U cap V, x))
into π_1(U, x) * π_1(V, x)

let N sub π_1(U, x) * π_1(V, x) be
the smallest normal subgrp containing

{i_*([γ]) * j_*([γ])^{-1} | [γ] in π_1(U cap V, x)}

then (π_1(U, x) * π_1(V, x))/N ≃ π_1(X, x)

Analogy set A = U cap V: compare [“colimits”]

U
U cap V X
V

largest quotient of U ⊔ V s.t. i(x) ~ j(x) for x in A

π_1(U, x)
π_1(U cap V, x) π_1(X, x)
π_1(V, x)

largest quotient of π_1(U, x) * π_1(V, x) s.t.
i_*([γ]) ~ j_*([γ]) for [γ] in π_1(U cap V, x)

Today let X be the hollow two-holed donut
will compute $\pi_1(X)$

let U, V be (open) punctured one-holed donuts s.t.
X = U cup V,
U cap V is an open annulus,
x in U cap V

[draw]

pick generator $[\gamma]$ for $\pi_1(U \cap V, x) \simeq Z$

need to compute
1) π_1(punctured hollow one-holed donut),
2) $i_*([\gamma])$ in $\pi_1(U, x)$
$j_*([\gamma])$ in $\pi_1(V, x)$

Step 1 punctured hollow one-holed donut
~ figure-eight

[draw]

know π_1(figure-eight) is free on two generators
how to see generators in donut?

punctured square
to
punctured hollow one-holed donut

[draw]

generators: a from left-to-right $[\alpha]$ at bottom
b from bottom-to-top $[\beta]$ at right

Step 2 $\pi_1(U, x)$, $\pi_1(V, x)$ are two copies of the free group on two generators $[\alpha]$, $[\beta]$

write $\pi_1(U, x) = \langle a, b \rangle$
$\pi_1(V, x) = \langle a', b' \rangle$

$i_*([\gamma])$, $j_*([\gamma])$ will be loops around the puncture with opposite orientations

counterclockwise: $a * b * a^{-1} * b^{-1}$
$= [a, b]$

clockwise: $b' * a' * b'^{-1} * a'^{-1}$
$= [a', b']^{-1}$

Conclusion N sub $\pi_1(U, x) * \pi_1(V, x)$ is the smallest normal subgrp having

$$[a, b] * ([a', b']^{-1})^{-1} = [a, b][a', b']$$

therefore,

$$\begin{aligned} &\pi_1(\text{hollow two-holed donut}, x) \\ &\simeq (\pi_1(U, x) * \pi_1(V, x))/N \\ &\simeq \langle a, b, a', b' \mid [a, b][a', b'] \rangle \end{aligned}$$

Rem in general,

$$\begin{aligned} &\pi_1(\text{hollow g-holed donut}) \\ &\simeq \langle a1, b1, \ldots, ag, bg \mid [a1, b1]\ldots [ag, bg] \rangle \end{aligned}$$

Rem [a, b] does trivialize in the unpunctured square

thus, π_1(one-holed donut)

$\simeq \langle a, b \mid [a, b] \rangle$

$\simeq$ abelianization of $\langle a, b \rangle$,

i.e., free abelian group on a, b

illustrates:

π_1(punctured one-holed donut) $\simeq$ $Z * Z$

to   to

π_1(one-holed donut) $\simeq$ $Z \times Z$