

1.

Notes on truncated convolution. The goal is to prove a theorem that would:

- (1) Imply a theorem of Bonnafé–Dudas–Rouquier [BDR]: roughly, by extending to the half-twist $\Delta(w_\circ)$ what they establish for the full twist $\Delta(w_\circ)^{*2}$.
- (2) Prove, and in fact refine, a conjecture of Deligne–Lusztig about the structure of $H_c^*(X_{w_\circ})$ as a representation of G^F [DL25].

1.1. Fix an algebraically closed field \mathbf{k} . Fix a connected reductive algebraic group G over \mathbf{k} . Let \mathcal{B} be the flag variety of G . Let W be the Weyl group of G , and for each $w \in W$, let $j_w : \mathcal{O}_w \rightarrow \mathcal{B} \times \mathcal{B}$ be the inclusion of the corresponding G -orbit. We write e for the identity element of W , and w_\circ for the longest element.

1.2. Suppose that either (I) $\mathbf{k} = \mathbf{C}$, or (II) $\mathbf{k} = \bar{\mathbf{F}}$ for a finite field \mathbf{F} whose characteristic is a good prime for G . In the latter case, we fix a split \mathbf{F} -form of G in order to work with Frobenius weights.¹ Let $\mathbf{D} = \mathbf{D}_{G,m}(\mathcal{B} \times \mathcal{B})$ be the G -equivariant, bounded, mixed, constructible derived category of $\mathcal{B} \times \mathcal{B}$, defined in terms of either (I) mixed Hodge modules, or (II) mixed complexes of $\bar{\mathbf{Q}}_\ell$ -sheaves, for a fixed prime ℓ invertible in \mathbf{F} . Recall that \mathbf{D} is endowed with a convolution operation $*$. We write $\langle 1 \rangle$ for the shift-twist $[1](\frac{1}{2})$, where $(\frac{1}{2})$ is a formal half-Tate twist.

Let \mathbf{C}_w be the constant sheaf over \mathcal{O}_w . For each $w \in W$, form the objects

$$L(w) = j_{w,!} \mathbf{C}_w \langle \dim \mathcal{O}_w \rangle, \quad \Delta(w) = j_{w,!} \mathbf{C}_w \langle \dim \mathcal{O}_w \rangle$$

in \mathbf{D} . Observe that we have normalized $L(w), \Delta(w)$ to be perverse and pure of weight 0. The simple perverse sheaves in \mathbf{D} are precisely the objects $L(w)$.

1.3. Let \leq be Kazhdan–Lusztig’s partial order on two-sided cells of W in which $\{e\}$ is maximal and $\{w_\circ\}$ is minimal. Fix a two-sided cell \mathbf{c} . Let $\mathbf{D}^{\leq \mathbf{c}}$, *resp.* $\mathbf{D}^{< \mathbf{c}}$, be the thick (additive) subcategory of \mathbf{D} generated by objects K such that for all integers i , the composition factors of the perverse sheaf ${}^p\mathcal{H}^i(K)$ are objects $L(w)$ with $w \in \mathbf{c}'$ for some $\mathbf{c}' \leq \mathbf{c}$, *resp.* $\mathbf{c}' < \mathbf{c}$. Form the Serre quotient category

$$\mathbf{D}^{\mathbf{c}} := \mathbf{D}^{\leq \mathbf{c}} / \mathbf{D}^{< \mathbf{c}},$$

and let $E \mapsto \underline{E} : \mathbf{D}^{\leq \mathbf{c}} \rightarrow \mathbf{D}^{\mathbf{c}}$ denote the quotient functor.

Lusztig showed (*e.g.*, in [L15, Lemma 1.4(b)]) that $\mathbf{D}^{\leq \mathbf{c}}$ and $\mathbf{D}^{< \mathbf{c}}$ are stable under left and right convolution with any object of \mathbf{D} . Thus, $\mathbf{D}^{\mathbf{c}}$ forms a bimodule category over \mathbf{D} with respect to actions induced by convolution. For instance, the left action by an object $K \in \mathbf{D}$ sends $\underline{E} \mapsto \underline{K * E}$ for all $E \in \mathbf{D}^{\leq \mathbf{c}}$.

Let $a_{\mathbf{c}}$ be Lusztig’s a -invariant for \mathbf{c} , a nonnegative integer. We are interested in the (invertible) endofunctor $\Xi_{\mathbf{c}} : \mathbf{D}^{\mathbf{c}} \rightarrow \mathbf{D}^{\mathbf{c}}$ defined by

$$\Xi_{\mathbf{c}}(\underline{E}) = (\underline{\Delta(w_\circ)} * \underline{E})[a_{\mathbf{c}}].$$

¹We hope to generalize to nonsplit forms later.

Mathas proved [M96] at the level of the triangulated, graded Grothendieck group—*i.e.*, the Hecke algebra—that there is a left-cell-preserving involution $w \mapsto w^!$ such that

$$[\Xi_{\mathbf{c}}(L(w))] = [L(w^!)\langle a_{w_{\mathbf{c}}}\rangle] \quad \text{for all } w \in \mathbf{c}.$$

Moreover, as explained in [BDR], it follows from [BFO12, Remark 4.3] that $\Xi_{\mathbf{c}}^2$ is involutive: *i.e.*, isomorphic to the identity functor on $\mathcal{D}^{\mathbf{c}}$.

1.4. Let \mathbf{SBim}_W be the category of Soergel bimodules for (W, V) , where V is the representation of W on the cocharacter lattice in the root datum of G .

Via the weight realization functor from $K^b\mathbf{SBim}_W$ into \mathcal{D} , recent work of Elias–Hogancamp in [EH24] implies that $\Delta(w_{\circ})$ lifts to a twisted Drinfeld center of \mathcal{D} , in the following sense. First, [EH24] defines a monoidal involution of $K^b\mathbf{SBim}_W$. An analogous construction yields an involution of \mathcal{D} , which we again denote by Φ . We assume $\Phi \circ \Phi = \text{id}$ from now on. The arguments of [EH24] show that there is an isomorphism of functors

$$\tau : \Delta(w_{\circ}) * (-) \xrightarrow{\sim} \Phi(-) * \Delta(w_{\circ})$$

such that if τ_K is its value at $K \in \mathcal{D}$, then we have

$$(\text{id}_{\Phi(K)} * \tau_L) \circ (\tau_K * \text{id}_L) = \tau_{K*L} \quad \text{for all } K, L \in \mathcal{D}.$$

Loosely, we will refer to τ or similar data on related categories as *Φ -central structures*. Since Φ preserves the thick subcategories $\mathcal{D}^{\leq \mathbf{c}}$, $\mathcal{D}^{< \mathbf{c}}$ and commutes with the shift $[a_{\mathbf{c}}]$, we obtain Φ -central structures on the involutions $\Xi_{\mathbf{c}}$.

Note that if w_{\circ} is central in W , then Φ is the identity map on objects. This occurs, for instance, in types B, C, D, E_7, E_8 .

1.5. Let $\mathcal{C}^{\mathbf{c}}$ be the full subcategory of $\mathcal{D}^{\leq \mathbf{c}}$ whose objects are direct sums of the objects $L(w)$ for $w \in \mathbf{c}$. By construction, $\mathcal{C}^{\mathbf{c}}$ is a semisimple abelian category. Moreover, any morphism in $\mathcal{C}^{\mathbf{c}}$ that factors through $\mathcal{D}^{< \mathbf{c}}$ is already zero, so the composition of functors $\mathcal{C}^{\mathbf{c}} \rightarrow \mathcal{D}^{\leq \mathbf{c}} \rightarrow \mathcal{D}^{\mathbf{c}}$ is fully faithful. When convenient, we will identify $\mathcal{C}^{\mathbf{c}}$ with its essential image in $\mathcal{D}^{\mathbf{c}}$. We expect to prove:

Conjecture 1.1. *$\Xi_{\mathbf{c}}$ is exact in the perverse t -structure that $\mathcal{D}^{\mathbf{c}}$ inherits from $\mathcal{D}^{\leq \mathbf{c}}$. Equivalently, $\mathcal{C}^{\mathbf{c}}$ is stable under $\Xi_{\mathbf{c}}$.*

Let \otimes be the *truncated convolution* operation on objects of $\mathcal{C}^{\mathbf{c}}$ defined by

$$\underline{E'} \otimes \underline{E} := \underline{{}^p\mathcal{H}^{a_{\mathbf{c}}}(E' * E)}.$$

In [L97], Lusztig showed that the associativity constraint on $*$ descends to one on \otimes . In this way, $\mathcal{C}^{\mathbf{c}}$ forms a tensor category. We see that if Conjecture 1.1 holds, then

$$\text{for all } E \in \mathcal{C}^{\mathbf{c}}, \quad \text{we have } \Xi_{\mathbf{c}}(\underline{E}) \simeq \underline{{}^p\mathcal{H}^0(\Delta(w_{\circ}) * E[a_{\mathbf{c}}])} \simeq \underline{{}^p\mathcal{H}^{a_{\mathbf{c}}}(\Delta(w_{\circ}) * E)}.$$

This leads us to speculate:

Conjecture 1.2. *There exist a \otimes -invertible object $J_{\mathbf{c}} \in \mathbf{C}^{\mathbf{c}}$ and an isomorphism*

$$\Xi_{\mathbf{c}}|_{\mathbf{C}^{\mathbf{c}}} \simeq (\underline{J}_{\mathbf{c}} \otimes -) \langle a_{w_{\circ} \mathbf{c}} \rangle$$

in the category of endofunctors of $\mathbf{C}^{\mathbf{c}}$. Moreover, this endofunctor categorifies Mathas's involution in the sense that $\underline{J}_{\mathbf{c}} \otimes \underline{L}(w) \simeq \underline{L}(w^!)$ for all $w \in \mathbf{c}$.

If Conjecture 1.2 holds, then the Φ -central structure on $\Xi_{\mathbf{c}}$ can be transported to a Φ -central structure on $J_{\mathbf{c}}$, where we again write Φ for the induced involution on $\mathbf{C}^{\mathbf{c}}$.

1.6. For any finite group \mathcal{G} , acting (from the left) on a finite set \mathcal{X} , we write $\text{Coh}_{\mathcal{G}}(\mathcal{X})$ to denote the category of \mathcal{G} -equivariant coherent \mathbf{K} -sheaves on \mathcal{X} , where either (I) $\mathbf{K} = \mathbf{C}$, or (II) $\mathbf{K} = \bar{\mathbf{Q}}_{\ell}$. Recall that an object of this category is a \mathbf{K} -vector space V equipped with:

- (1) A grading $V = \bigoplus_{x \in \mathcal{X}} V_x$.
- (2) A (left) action $G \rightarrow \text{GL}(V)$ such that $g \cdot V_x = V_{gx}$ for all $x \in \mathcal{X}$ and $g \in G$.

Assume for now that \mathbf{c} is not an exceptional cell. Let $\mathcal{G}_{\mathbf{c}}$ be the finite group that Lusztig attaches to \mathbf{c} . Then by [BFO09, Theorem 4], there exist a finite $\mathcal{G}_{\mathbf{c}}$ -set $\mathbf{X}_{\mathbf{c}}$ and an equivalence of tensor categories

$$(\text{Coh}_{\mathcal{G}_{\mathbf{c}}}(\mathbf{X}_{\mathbf{c}} \times \mathbf{X}_{\mathbf{c}}), *) \xrightarrow{\sim} (\mathbf{C}^{\mathbf{c}}, \otimes).$$

In what follows, we write Φ for any endofunctor induced by Φ via an equivalence of tensor categories. We then get an isomorphism of twisted centers:

$$(1.1) \quad Z_{\Phi}(\text{Coh}_{\mathcal{G}_{\mathbf{c}}}(\mathbf{X}_{\mathbf{c}} \times \mathbf{X}_{\mathbf{c}})) \xrightarrow{\sim} Z_{\Phi}(\mathbf{C}^{\mathbf{c}}).$$

If Conjecture 1.2 holds, then $J_{\mathbf{c}}$ lifts to an object of $Z_{\Phi}(\mathbf{C}^{\mathbf{c}})$, hence defines an object of $Z_{\Phi}(\text{Coh}_{\mathcal{G}_{\mathbf{c}}}(\mathbf{X}_{\mathbf{c}} \times \mathbf{X}_{\mathbf{c}}))$. We expect that in many situations, we can simplify the above Z_{Φ} 's to Z 's:

Conjecture 1.3. *If w_{\circ} commutes with all of \mathbf{c} , then Φ is the identity functor on $\mathbf{C}^{\mathbf{c}}$.*

1.7. *Henceforth, we assume that w_{\circ} commutes with all of \mathbf{c} .* By Morita equivalence for module categories over tensor categories, as explained in [EGNO, Example 7.12.19 and Corollary 7.16.2], we have a tensor equivalence

$$(1.2) \quad \text{Coh}_{\mathcal{G}_{\mathbf{c}}}(\mathcal{G}_{\mathbf{c}}) \xrightarrow{\sim} Z(\text{Coh}_{\mathcal{G}_{\mathbf{c}}}(\mathbf{X}_{\mathbf{c}} \times \mathbf{X}_{\mathbf{c}})).$$

We can now make contact with the recent preprint [DL25] of Deligne–Lusztig.

Let $\{-, -\}$ be the exotic Fourier transform on the Grothendieck group $K_{0, \mathcal{G}_{\mathbf{c}}}(\mathcal{G}_{\mathbf{c}})$. To describe it explicitly, recall that the isomorphism classes of simple objects in $\text{Coh}_{\mathcal{G}_{\mathbf{c}}}(\mathcal{G}_{\mathbf{c}})$ are indexed by conjugacy classes of pairs (g, η) , where $g \in \mathcal{G}_{\mathbf{c}}$ and η is a \mathbf{K} -valued irreducible character of the centralizer $Z(g) = Z_{\mathcal{G}_{\mathbf{c}}}(g)$. In this indexing,

$$\{[g, \eta], [g', \eta']\} = \frac{1}{|Z(g)||Z(g')|} \sum_{\substack{h \in \mathcal{G}_{\mathbf{c}} \\ h^{-1}ghg' = g'h^{-1}gh}} \eta(hg'h^{-1})\eta'(h^{-1}g^{-1}h).$$

Lusztig previously observed that

$$\{[g, \eta], [g', \eta'] * [g'', \eta'']\} = \frac{|Z(g)|}{\eta(1)} \{[g, \eta], [g', \eta']\} \{[g, \eta], [g'', \eta'']\}.$$

That is, for fixed $[g, \eta]$, the linear map $K_{0, \mathcal{G}_c}(\mathcal{G}_c) \rightarrow \mathbf{K}$ defined by

$$[g', \eta'] \mapsto \frac{|Z(g')|}{\eta'(1)} \{[g, \eta], [g', \eta']\}$$

is a ring homomorphism.

Theorem 1.4 (Deligne–Lusztig). *For any W and two-sided cell $\mathbf{c} \subseteq W$, there is an invertible simple object $m_{\mathbf{c}}$ of $\text{Coh}_{\mathcal{G}_c}(\mathcal{G}_c)$ such that, for any $\chi \in \text{Irr}^c(W)$ corresponding to $[g, \eta] \in K_{0, \mathcal{G}_c}(\mathcal{G}_c)$, we have*

$$\{[g, \eta], [m_{\mathbf{c}}]\} = (-1)^{b_{\chi} - a_{\mathbf{c}}} \frac{\eta(1)}{|Z(g)|} \quad \text{for all } [g, \eta],$$

where b_{χ} is the valuation of the fake degree of χ . If W is irreducible and \mathbf{c} is not exceptional, then $j_{\mathbf{c}}$ is unique up to isomorphism.

One checks that tensoring with $m_{\mathbf{c}}$ is involutive on $\text{Coh}_{\mathcal{G}_c}(\mathcal{G}_c)$. Since $m_{\mathbf{c}}$ is simple and invertible, we get an involution of the set of isomorphism classes of simple objects of $\text{Coh}_{\mathcal{G}_c}(\mathcal{G}_c)$.

Let $j_{\mathbf{c}} \in Z(\text{Coh}_{\mathcal{G}_c}(\mathbf{X}_{\mathbf{c}} \times \mathbf{X}_{\mathbf{c}}))$ be the image of $m_{\mathbf{c}}$. Then tensoring with $j_{\mathbf{c}}$ defines an involution of $\text{Coh}_{\mathcal{G}_c}(\mathbf{X}_{\mathbf{c}} \times \mathbf{X}_{\mathbf{c}})$. As before, we get an involution of the set of isomorphism classes of simple objects of $\text{Coh}_{\mathcal{G}_c}(\mathbf{X}_{\mathbf{c}} \times \mathbf{X}_{\mathbf{c}})$.

At the same time, recall that under Conjecture 1.2, $J_{\mathbf{c}}$ defines an object of $Z(\mathbf{C}^c)$, and tensoring with $J_{\mathbf{c}}$ is an involution of \mathbf{C}^c .

Conjecture 1.5. *Assume that w_{\circ} commutes with all of \mathbf{c} . Then (1.1) takes $j_{\mathbf{c}}$ (with its central structure) to $J_{\mathbf{c}}$ (with its central structure), up to isomorphism.*

1.8. Finally, we explain the application to Deligne–Lusztig’s conjecture at the end of [DL25]. Henceforth, we assume that we are in setting (II), so that $\mathbf{k} = \bar{\mathbf{F}}$ for a finite field \mathbf{F} , and $\mathbf{K} = \bar{\mathbf{Q}}_{\ell}$, and \mathbf{D} is defined in terms of mixed complexes of $\bar{\mathbf{Q}}_{\ell}$ -sheaves relative to a split Frobenius $F : G \rightarrow G$.

Let $\text{Rep}_u^c(G^F)$ be the full additive subcategory of $\text{Rep}(G^F)$ generated by the unipotent representations in the family indexed by \mathbf{c} . The Harish-Chandra transform

$$\text{HC}_F : \text{Rep}(G^F) = \mathbf{D}_G^b(G^F, \bar{\mathbf{Q}}_{\ell}) \rightarrow \mathbf{D}_{G, m}^b(\mathcal{B} \times \mathcal{B}, \bar{\mathbf{Q}}_{\ell}) =: \mathbf{D}$$

restricts to a functor $\text{HC}_F : \text{Rep}_u^c(G^F) \rightarrow \mathbf{D}^{\leq c}$. As explained in [BDR], the essential image of the latter is right-orthogonal to $\mathbf{D}^{< c}$, in the sense that $\text{Hom}_{\mathbf{D}}(K, \text{HC}_F(\rho)) = 0$ for all $\rho \in \text{Rep}_u^c(G^F)$ and $K \in \mathbf{D}^{< c}$. Moreover, Lusztig showed that the composition

$$\underline{\text{HC}}_F : \text{Rep}_u^c(G^F) \xrightarrow{\text{HC}_F} \mathbf{D}^{\leq c} \rightarrow \mathbf{D}^c$$

factors through a tensor equivalence

$$\mathrm{Rep}_u^c(G^F) \xrightarrow{\sim} Z(\mathbb{C}^c)$$

for a certain monoidal product on $\mathrm{Rep}_u^c(G^F)$, introduced in [L15] by means of weight filtrations. Altogether, we get tensor equivalences

$$\mathrm{Rep}_u^c(G^F) \xleftarrow{[L15]} Z(\mathbb{C}^c) \xleftarrow{(1.1)} Z(\mathrm{Coh}_{\mathcal{G}_c}(\mathbf{X}_c \times \mathbf{X}_c)) \xleftarrow{(1.2)} \mathrm{Coh}_{\mathcal{G}_c}(\mathcal{G}_c).$$

Let $\mathrm{Uch}^c(G^F)$ be the set of unipotent irreducible characters of G^F in the family indexed by \mathbf{c} . We then get a bijection between $\mathrm{Uch}^c(G^F)$ and the set of \mathcal{G}_c -conjugacy classes of pairs (g, η) with $g \in \mathcal{G}_c$ and $\eta \in \mathrm{Irr} Z(g)$. This is precisely the bijection described by Lusztig in [L84].

Recall that tensoring with m_c induces an involution of the set of classes $[g, \eta]$. The corresponding involution on $\mathrm{Uch}^c(G^F)$ is denoted $(-)^!$ in [DL25].

If all of our conjectures above hold, then for any $\rho \in \mathrm{Uch}^c(G^F)$, we have

$$(1.3) \quad \Xi_c(\underline{\mathrm{HC}}_F(\rho)) = (\underline{J}_c \otimes \underline{\mathrm{HC}}_F(\rho)) \langle a_{w_\circ \mathbf{c}} \rangle \simeq \underline{\mathrm{HC}}_F(\rho^!) \langle a_{w_\circ \mathbf{c}} \rangle = \underline{\mathrm{HC}}_F(\rho^!) \langle a_{w_\circ \mathbf{c}} \rangle.$$

To give the applications of this identity, let

$$\mathrm{CH}_F : \mathcal{D} \rightarrow \mathrm{Rep}(G^F)$$

denote the left adjoint to $\underline{\mathrm{HC}}_F$. For any sequence $\vec{w} = (w^{(1)}, w^{(2)}, \dots, w^{(k)})$ of elements of W , let $X(\vec{w})$ be the associated Deligne–Lusztig variety over \mathbf{k} , and let

$$\Delta(\vec{w}) = \Delta(w^{(1)}) * \Delta(w^{(2)}) * \dots * \Delta(w^{(k)})$$

in \mathcal{D} . We get

$$\begin{aligned} (\mathrm{H}_c^i(X(\vec{w})), \rho)_{G^F} &= \mathrm{Hom}_{G^F}(\mathrm{CH}_F(\Delta(\vec{w}))[i], \rho) \\ &\simeq \mathrm{Hom}_{\mathcal{D}}(\Delta(\vec{w})[i], \underline{\mathrm{HC}}_F(\rho)) \quad \text{by adjunction} \\ &\simeq \mathrm{Hom}_{\mathcal{D}^c}(\underline{\Delta}(\vec{w})[i], \underline{\mathrm{HC}}_F(\rho)) \quad \text{by orthogonality of } \underline{\mathrm{HC}}_F \text{ to } \mathcal{D}^{<c}. \end{aligned}$$

If (1.3) holds, then

$$\begin{aligned} \mathrm{Hom}_{\mathcal{D}^c}(\underline{\Delta}(\vec{w})[i], \underline{\mathrm{HC}}_F(\rho)) &\simeq \mathrm{Hom}_{\mathcal{D}^c}(\underline{\Delta}(w_\circ, \vec{w})[i], \underline{\Delta}(w_\circ) * \underline{\mathrm{HC}}_F(\rho)) \\ &\simeq \mathrm{Hom}_{\mathcal{D}^c}(\underline{\Delta}(w_\circ, \vec{w})[i], \Xi_c(\underline{\mathrm{HC}}_F(\rho))[-a_c]) \\ &\simeq \mathrm{Hom}_{\mathcal{D}^c}(\underline{\Delta}(w_\circ, \vec{w})[i], \underline{\mathrm{HC}}_F(\rho^!) \langle a_{w_\circ \mathbf{c}} \rangle [-a_c]) \\ &\simeq \mathrm{Hom}_{\mathcal{D}^c}(\underline{\Delta}(w_\circ, \vec{w}) \langle -a_{w_\circ \mathbf{c}} \rangle [i + a_c], \underline{\mathrm{HC}}_F(\rho^!)). \end{aligned}$$

Altogether, writing W for weight filtrations on cohomology, we would get

$$(\mathrm{gr}_j^W \mathrm{H}_c^i(X(\vec{w})), \rho)_{G^F} \simeq (\mathrm{gr}_{j-a_{w_\circ \mathbf{c}}}^W \mathrm{H}_c^{i+a_c}(X(w_\circ, \vec{w})), \rho^!)_{G^F}.$$

This would imply both Theorem B of [BDR] (by taking \vec{w} to be first empty, then a reduced word for w_\circ), and the conjecture in [DL25] (by taking \vec{w} to be empty).