

(Munkres \approx §71) last time, we saw:

$$\pi_1(\text{figure-eight}) = \mathbb{Z} * \mathbb{Z}$$

it was a corollary of Seifert–van Kampen:

given open $U_1, U_2 \subset X$ s.t.

$$X = U_1 \cup U_2,$$

U_1 and U_2 are path-connected,

$U_1 \cap U_2$ is path-connected,

$x \in U_1 \cap U_2$,

there is a surjective homomorphism

$$\pi_1(U_1, x) * \pi_1(U_2, x) \rightarrow \pi_1(X, x)$$

whose kernel depends on $\pi_1(U_1 \cap U_2, x)$

[how did it work?] let A_1, A_2 be copies of S^1
with resp. basepoints a_1, a_2

$$\text{figure-eight} = (A_1 \sqcup A_2)/(a_1 \sim a_2)$$

[how are the U_j 's related to the A_j 's?]

U_j is a small open nbd of the image of A_j

Df for general $(A_1, a_1), (A_2, a_2)$

$$A_1 \vee A_2 = (A_1 \sqcup A_2)/(a_1 \sim a_2)$$

is called the wedge sum of A_1 and A_2

$A_1 \vee A_2$ has a “natural” basept:

the common image of a_1, a_2

then wedge sum becomes an associative
binary operation on spaces with basepoints

Q let V_n denote the wedge sum
of n copies of S^1 [with any basepts]
[thus V_2 is the figure-eight]

what is $\pi_1(V_n)$? $Z * Z * \dots * Z$ with n copies
[note that $*$ is also an associative operation]
how to prove? induction

Rem as this line of thinking suggests:

Seifert–van Kampen can be restated,
more generally, for $X = U_1 \cup \dots \cup U_n$

here, it turns out that: [and we omit proofs]

- to get surjectivity of

$$\pi_1(U_1, x) * \dots * \pi_1(U_n, x) \rightarrow \pi_1(X, x)$$

we just need path-connectedness of $U_i \cap U_j$
for all i, j (including $i = j$)

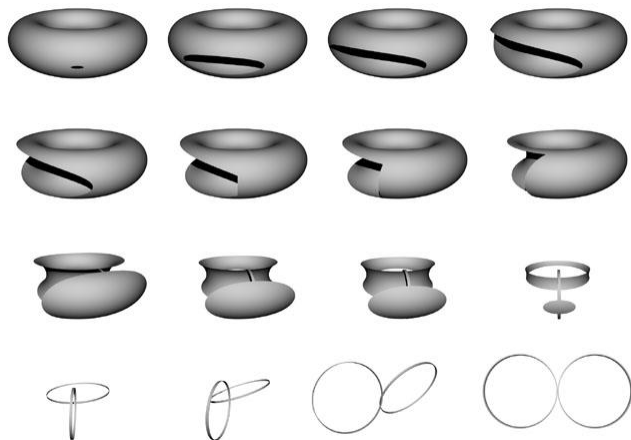
- to describe the kernel easily, we need
path-connectedness of $U_i \cap U_j \cap U_k$ for all
 i, j, k

it is the smallest normal subgp of the free product
containing, for all ordered pairs (i, j) ,
 $\text{im}(\pi_1(U_i \cap U_j) \rightarrow \pi_1(U_i) \rightarrow \pi_1(\text{product}))$

(Munkres §72) a cool trick about the torus T :

Prop $T - \{\text{point}\}$ deformation retracts onto the figure-eight

Pf <https://www.technomagi.com/josh/>



Cor $\pi_1(T - \{\text{point}\}) = \mathbb{Z} * \mathbb{Z}$

compare:

$$\begin{aligned}\pi_1(T) &= \pi_1(S^1) \times \pi_1(S^1) \\ &= \mathbb{Z} \times \mathbb{Z} \\ &= \langle a, b \mid [a, b] \rangle \\ &\quad \text{where } [a, b] = aba^{-1}b^{-1}\end{aligned}$$

$$\pi_1(T - \{\text{point}\}) = \langle a, b \rangle$$

somehow, puncturing T corresponds to adjoining $[a, b]$ to the set of relations

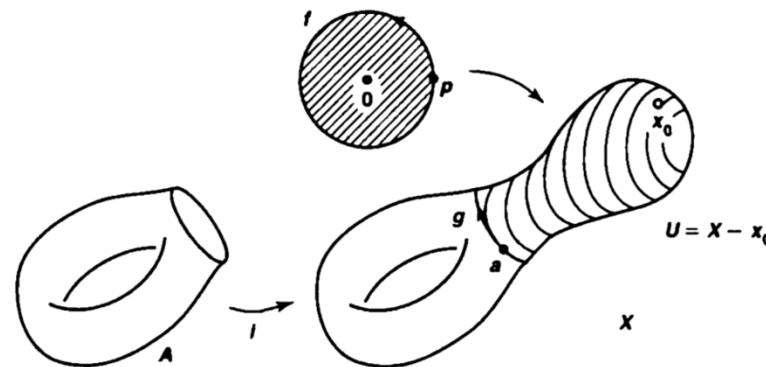
[ask: is there a way to systematize this?]

let D^2 be the closed unit disk with boundary S^1
 fix a basept p in S^1

Thm let X be Hausdorff
 let $i : A$ to X be inclusion of
 a closed path-connected subspace

suppose there is cts $\zeta : D^2$ to X s.t.
 ζ maps $\text{Int}(D^2)$ bijectively onto $X - A$
 ζ maps S^1 into A
 let $a = \zeta(p)$ and $\eta = \zeta|_{S^1}$
 then:

- 1) $i_* : \pi_1(A, a)$ to $\pi_1(X, a)$ is surjective
- 2) $\ker(i_*) = \text{im}(\eta_* : \pi_1(S^1, p) \text{ to } \pi_1(A, a))$



Pf Sketch let $x = \zeta(0, 0)$ in X and $U = X - \{x\}$

since $D^2 - \{(0, 0)\}$ deformation retracts onto S^1
 we can show U deformation retracts onto A
 let $[\gamma]$ be the image of 1 under

$$Z = \pi_1(S^1, p) \text{ to } \pi_1(A, a) = \pi_1(U, a)$$

remains to show:

- 1) $\pi_1(U, a) \rightarrow \pi_1(X, a)$ is surjective
- 2) its kernel is generated by $[\gamma]$

[want to use Seifert–van Kampen somewhere]

let $V = X - A =$ bijective image of D^2 under ζ

now see:

$X = U \cup V$,

U is path-connected bc A is,

V is path-connected bc D^2 is,

$U \cap V = (X - \{x\}) \cap V = V - \{x\}$

is also path-connected

so for any b in $U \cap V$:

surjective $\pi_1(U, b) * \pi_1(V, b) \rightarrow \pi_1(X, b)$

with kernel described in terms of $\pi_1(U \cap V, b)$

but V is also simply-connected bc D^2 is
so we get surjective

$\pi_1(U, b) \rightarrow \pi_1(X, b)$

with kernel the minimal normal subgroup containing
 $\text{im}(\pi_1(U \cap V, b) \rightarrow \pi_1(U, b))$

to finish: pick a path from a in A to b in $U \cap V$
translate results above into results about a , $[\gamma]$ \square

Gluing Diagrams for Surfaces

[how to wield this thm efficiently?]

[draw pictures]

<https://divisbyzero.com/2020/04/08/make-a-real-projective-plane-boys-surface-out-of-paper/>



take X to be the quotient space of $[0, 1]^2$ resulting from the edge identifications

take A to be the image of the boundary square

take ζ to be a homeo from D^2 onto $[0, 1]^2$

in the cases with both “>” and “>>”,

take a to be the path following “>”

take b to be the path following “>>”

then

$$\pi_1(X) = \langle a, b \mid R \rangle$$

where R is read off of a loop traversal of A