CENTRAL ELEMENTS, CELL DECOMPOSITIONS, AND PARTIAL SPRINGER RESOLUTIONS

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Abstract. For any finite Weyl group W and parabolic subgroup W_J, arising from a finite reductive group G and parabolic subgroup P_J, we prove identities relating the partial Springer resolutions of type J to central elements in the Hecke algebra, given by sums of terms $q^{-\ell(v)}T_{v^{-1}}T_v$ as v runs over minimal- or maximal-length representatives of the right cosets of W_J in W. We thereby obtain formulas for Hecke traces arising from these central elements, generalizing work of Lascoux and Wan–Wang beyond type A, and cell decompositions of new braid varieties involving J, generalizing work of Shende–Treumann–Zaslow. From the latter, we construct noncrossing sets that interpolate between rational Catalan and parking objects, generalizing our work with Galashin–Lam, and new formulas for arbitrary a-degrees of the HOMFLYPT polynomials of positive braid closures.

1. Introduction

1.1. Fix a finite Coxeter system (W, S) and a subset $J \subseteq S$ generating a subgroup $W_J \subseteq W$. Let H_W and H_{W_J} be the Hecke algebras over $\mathbf{Z}[\boldsymbol{q}^{\pm 1}]$ corresponding to W and W_J. We take the convention where the Hecke operators $T_s \in H_W$, for $s \in S$, obey the relations $T_s^2 = (\boldsymbol{q} - 1)T_s + \boldsymbol{q}$. We identify H_{W_J} with the subalgebra of H_W generated by the elements T_s with $s \in J$.

Under this embedding, the center $Z(H_{W_J})$ need not embed into the center $Z(H_W)$. Nonetheless, Hoefsmit–Scott constructed an injective, linear *relative norm* map

$$N_J^S : Z(H_{W_J}) \to Z(H_W),$$

that they, and L. K. Jones, used to study induction from H_{W_J} to H_W [Jon90]. To define N_J^S, recall that each right coset of W_J in W contains a unique representative of minimal Bruhat length. Let W^J be the set of such representatives. Then

$$N_J^S(\alpha) = \sum_{v \in W^J} \boldsymbol{q}^{-\ell(v)} T_{v^{-1}} \alpha T_v \quad \text{for all } \alpha \in Z(H_{W_J}),$$

where T_v and $\ell(v)$ denote the Hecke operator for and Bruhat length of v.

When W is crystallographic, we can interpret it as the Weyl group of a split finite reductive group G. We can then interpret the above algebras geometrically, as convolution algebras of functions on the flag variety of G or its square. The main observation of this paper is that in the geometric framework, the relative norm N_J^S is related to the two partial Springer resolutions for J, defined in (1.1).

From this relationship, we obtain applications to traces on H_W, generalizing work of Lascoux [Las06] and Wan–Wang [WW15]; cell decompositions of *partial* *braid Steinberg varieties*, generalizing work of Shende–Treumann–Zaslow [STZ17]; and the rational parking combinatorics of (W, S), generalizing our prior work with Galashin–Lam [GLTW24].

1.2. **Partial Resolutions.** Let $\mathbf{F}$ be a finite field of order q. Let $\mathbf{G}$ be a connected reductive algebraic group over $\bar{\mathbf{F}}$, equipped with a Frobenius map $F : \mathbf{G} \to \mathbf{G}$. We assume that the characteristic of $\mathbf{F}$ is a good prime for $\mathbf{G}$ [Car93, 28].

Fix an F-stable maximal torus in an F-stable Borel: $\mathbf{T} \subseteq \mathbf{B} \subseteq \mathbf{G}$. Let $\mathbf{W} = N_{\mathbf{G}}(\mathbf{T})/\mathbf{T}$. We now take W to be the finite Coxeter group $\mathbf{W}^F$. Similarly, we write G, B, *etc.* for the groups formed by the F-fixed points of $\mathbf{G}$, $\mathbf{B}$, *etc.*

The G-invariant, $\mathbf{Z}[q^{\pm 1}]$-valued functions on $(G/B)^2$ form a convolution algebra H_B^G. If G is *split*, meaning $W = \mathbf{W}$, then H_B^G is the specialization at $\boldsymbol{q} \to q$ of the algebra H_W presented earlier. Explicitly, T_w specializes to the indicator function on the set of pairs (hB, gB) such that $Bh^{-1}gB = BwB$. In Section 2, we review the presentation of H_B^G for general G. *In the rest of this introduction, we assume that G is split, for simplicity.*

We take S to be the system of simple reflections arising from $\mathbf{B}$. Let $\mathbf{P}_J = \mathbf{B}W_J\mathbf{B}$, a parabolic subgroup of $\mathbf{G}$. Let $\mathbf{U}_J$ be its unipotent radical and $\mathbf{V}_J$ the variety of all unipotent elements in $\mathbf{P}_J$. If $J = \emptyset$, then $\mathbf{P}_J = \mathbf{B}$ and $\mathbf{U}_J = \mathbf{V}_J$; otherwise, $\mathbf{V}_J$ is larger than $\mathbf{U}_J$. At the level of points, the two *partial Springer resolutions* of type J are defined by

$$\begin{aligned} \mathbf{Spr}_J^+ &= \{(u, y\mathbf{P}_J) \in \mathbf{G} \times \mathbf{G}/\mathbf{P}_J \mid u \in y\mathbf{V}_J y^{-1}\}, \\ \mathbf{Spr}_J^- &= \{(u, y\mathbf{P}_J) \in \mathbf{G} \times \mathbf{G}/\mathbf{P}_J \mid u \in y\mathbf{U}_J y^{-1}\}. \end{aligned} \tag{1.1}$$

The $+$ case is a partial resolution of singularities of the unipotent variety $\mathbf{V} \subseteq \mathbf{G}$, while the $-$ case is a resolution of the closure of the Richardson orbit for J.

It will be convenient to set $\mathbf{E}_J^\pm := \mathbf{G}/\mathbf{B} \times \mathbf{Spr}_J^\pm$ and $E_J^\pm = (\mathbf{E}_J^\pm)^F$. There is a natural left G-action on $E_J^\pm$, under which the map $f : E_J^\pm \to (G/B)^2$ defined by

$$f(hB, u, yP_J) = (hB, uhB)$$

is equivariant. For any set E equipped with a G-action and an equivariant map $f : E \to (G/B)^2$, we write $f_! \delta_E \in H_B^G$ to denote the function whose value at a point in $(G/B)^2$ is the size of its preimage in E.

Let $w_\circ$ and $w_{J\circ}$ respectively denote the longest elements of W and W_J. For convenience, we set $\ell_S = \ell(w_\circ)$ and $\ell_J = \ell(w_{J\circ})$. Recall that $w_\circ, w_{J\circ}$ are involutions, and that $T_{w_{J\circ}}^2$ is central in H_{W_J} [BMR98]. We can now state the split case of our main result, proven for general G in Section 3.

Theorem 1.1. *For any $J \subseteq S$, we have*

$$\begin{aligned} f_! \delta_{E_J^-} &= q^{\ell_S - \ell_J} N_J^S(1)|_{\boldsymbol{q} \to q}, \\ f_! \delta_{E_J^+} &= q^{\ell_S - \ell_J} N_J^S(T_{w_{J\circ}}^2)|_{\boldsymbol{q} \to q}. \end{aligned}$$

Let $W^{J,-} = W^J$, and by analogy, let $W^{J,+}$ of *maximal-length* representatives for the right cosets of W_J in W, so that multiplication by $w_{J\circ}$ interchanges $W^{J,-}$ with $W^{J,+}$. Then the identities above can be rewritten as:

$$\begin{aligned} f_! \delta_{E_J^-} &= q^{\ell_S - \ell_J} \sum_{w \in W^{J,-}} q^{-\ell(v)} T_{v^{-1}} T_v, \\ f_! \delta_{E_J^+} &= q^{\ell_S} \sum_{w \in W^{J,+}} q^{-\ell(v)} T_{v^{-1}} T_v. \end{aligned}$$

We emphasize that the $+$ case is deeper than the $-$ case. The $-$ case only uses standard results about Bruhat decomposition. Under the assumption that G is split, we can refine it to an algebro-geometric statement: essentially, that $\mathbf{E}_J^-$ can be partitioned into fiber bundles over appropriate varieties. See Proposition 3.3 for details. By contrast, the $+$ case relies on a difficult theorem of Kawanaka [Kaw75]. We expect that it can only be refined to a statement at the level of cohomology. This issue seems related to the sheafification of Kawanaka's work discussed in [Tri22].

1.3. **Traces.** Recall that a *trace* on an algebra is a linear map that vanishes on commutators. We write $R(H_W)$ to denote the vector space of $\mathbf{Q}(\boldsymbol{q})$-valued traces on H_W. Our first application of Theorem 1.1 is to identify certain elements of $R(H_W)$ arising from $N_J^S(1)$ and $N_J^S(T_{w_{J\circ}}^2)$.

Let $e \in W$ be the identity. Let $\tau : H_W \to \mathbf{Z}[\boldsymbol{q}^{\pm 1}]$ be the trace given by $\tau(T_e) = 1$ and $\tau(T_w) = 0$ for all $w \neq e$. Then any central element $\zeta \in Z(H_W)$ gives rise to a trace $\tau[\zeta] : H_W \to \mathbf{Z}[\boldsymbol{q}^{\pm 1}] \subseteq \mathbf{Q}(\boldsymbol{q})$: namely,

$$\tau[\zeta](\beta) = \tau(\beta\zeta).$$

These traces are best understood when W is a symmetric group.

Let S_n, the symmetric group on n letters, and let Λ_n be the vector space of symmetric functions over $\mathbf{Q}(\boldsymbol{q})$ of degree n in variables $X = (X_1, X_2, \ldots, X_n)$. Then $R(H_{S_n})$ is isomorphic to Λ_n, as both of these vector spaces have bases indexed by the integer partitions of n. Let $ch_{\boldsymbol{q}} : R(H_{S_n}) \xrightarrow{\sim} \Lambda_n$ be the ***q**-deformed Frobenius characteristic* isomorphism that sends the irreducible character $\chi_{\boldsymbol{q}}^\lambda$ to the Schur function $s_\lambda(X)$, for any partition $\lambda \vdash n$.

For $W = S_n$, we take $S = \{s_1, \ldots, s_{n-1}\}$, where $s_i = (i, i+1)$. This choice sets up a bijection between subsets $J \subseteq S$ and integer *compositions* ν of n. Let $e_\nu(X)$ and $h_\nu(X)$ respectively denote the elementary and complete homogeneous symmetric functions in Λ_n indexed by ν. Wan–Wang [WW15], recasting work of Lascoux [Las06], show that if J corresponds to ν, then

$$\begin{aligned} ch_{\boldsymbol{q}}(\tau[N_J^S(1)]) &= (\boldsymbol{q}-1)^n e_\nu\left(\frac{X}{\boldsymbol{q}-1}\right), \\ ch_{\boldsymbol{q}}(\tau[N_J^S(T_{w_{J\circ}}^2)]) &= \boldsymbol{q}^{\ell_J}(\boldsymbol{q}-1)^n h_\nu\left(\frac{X}{\boldsymbol{q}-1}\right). \end{aligned} \tag{1.2}$$

Using these formulas, they show that the maps N_J^S give rise to a ring structure on the direct sum of the centers $Z(\mathbf{Q}(\boldsymbol{q}) \otimes H_{S_n})$, isomorphic to the ring of symmetric functions over $\mathbf{Q}(\boldsymbol{q})$. We will generalize the formulas to any crystallographic W.

Recall that Springer constructed a W-action on the étale cohomologies of the fibers of his resolution of the unipotent variety, now called *Springer fibers*. In [Tri21], the first author used this action to construct a trace on H_W valued in $\mathbf{Q}(\boldsymbol{q})$-linear traces on W, or equivalently, a *bitrace*

$$\tau_G : \mathbf{Q}W \otimes H_W \to \mathbf{Q}(\boldsymbol{q}),$$

which refines the Markov traces on H_W studied by Gomi [Gom06] and Webster–Williamson [WW11]. These, in turn, were motivated by the traces used by Ocneanu to construct the HOMFLYPT link polynomial [Jon87].

In this paper, we use a normalization for τ_G characterized by the formula

$$\tau_G(z \otimes T_w)|_{\boldsymbol{q}\to q} = \frac{1}{|G|} \sum_{\substack{u \in G \\ \text{unipotent}}} |O(w)_u| \chi_u(z) \quad \text{for all } z, w \in W,$$

where χ_u is the total Springer character for u, reviewed in §4.3, and $O(w)_u$ is the set of pairs (hB, gB) such that $h^{-1}gB = BwB$ and $gB = uhB$. Let $e_{J,-}$, *resp.* $e_{J,+}$, denote the antisymmetrizer, *resp.* symmetrizer, in $\mathbf{Q}W_J$, reviewed in §4.4. By combining Theorem 1.1 with work of Borho–MacPherson [BM83], we show the following theorem, and how it recovers (1.2) when $G = \mathrm{GL}_n(\mathbf{F})$.

Theorem 1.2. *For any $J \subseteq S$, we have*

$$\begin{aligned} \tau[N_J^S(1)] &= (\boldsymbol{q} - 1)^{\mathrm{rk}(G)}\, \tau_G(e_{J,-} \otimes -), \\ \tau[N_J^S(T_{w_{J\circ}}^2)] &= \boldsymbol{q}^{\ell_J} (\boldsymbol{q} - 1)^{\mathrm{rk}(G)}\, \tau_G(e_{J,+} \otimes -) \end{aligned}$$

as traces on H_W, where $\mathrm{rk}(G)$ is the rank of the (split) maximal torus T.

1.4. **Cell Decompositions.** Our second application of Theorem 1.1 is to provide point-counting formulas for new kinds of braid varieties, by means of Deodhar-type decompositions. In what follows, we write $h\mathbf{B} \xrightarrow{w} g\mathbf{B}$ to mean $\mathbf{B}h^{-1}g\mathbf{B} = \mathbf{B}w\mathbf{B}$.

Let $\vec{s} = (s^{(1)}, s^{(2)}, \ldots, s^{(\ell)})$ be a word in S. Recall that in [Deo85], Deodhar showed how to partition a certain *Richardson variety* for $\vec{s}$ into strata of the form $\mathbf{A}^{\mathbf{d}} \times \mathbf{G}_m^{\mathbf{e}}$, now called *Deodhar cells*. As in [GLTW24], we will work with a variant definition depending on an element $v \in W$:

$$\mathbf{R}^{(v)}(\vec{s}) = \{\vec{g}\mathbf{B} = (g_i\mathbf{B})_i \in (\mathbf{G}/\mathbf{B})^\ell \mid vw_\circ\mathbf{B} \xrightarrow{s^{(1)}} g_1\mathbf{B} \xrightarrow{s^{(2)}} \cdots \xrightarrow{s^{(\ell)}} g_\ell\mathbf{B} \xleftarrow{vw_\circ} \mathbf{B}\}.$$

The geometry of Theorem 1.1 shows how to relate the disjoint union of the varieties $\mathbf{R}^{(v)}(\vec{s})$ for $v \in W^{J,\mp}$ to the variety

$$\mathbf{Z}_J^\pm(\vec{s}) = \{(\vec{g}\mathbf{B}, u, y\mathbf{P}_J) \in (\mathbf{G}/\mathbf{B})^\ell \times \mathbf{Spr}_J^\pm \mid u^{-1}g_\ell\mathbf{B} \xrightarrow{s^{(1)}} g_1\mathbf{B} \xrightarrow{s^{(2)}} \cdots \xrightarrow{s^{(\ell)}} g_\ell\mathbf{B}\}.$$

Note the sign flip above in the preceding statement. It arises because the element $w_\circ$ in the formula for $\mathbf{R}^{(v)}(\vec{s})$ interchanges $W^{J,-}$ with $W^{J,+}$.

Note that $\mathbf{Z}_\emptyset^+(\vec{s})$ and $\mathbf{Z}_\emptyset^-(\vec{s})$ coincide, and match the *braid Steinberg variety* of $\vec{s}$ introduced in [Tri21]. At the other extreme, $\mathbf{Z}_S^+(\vec{s})$ and $\mathbf{Z}_S^-(\vec{s})$ are the varieties respectively denoted $\mathcal{U}(\vec{s})$ and $\mathcal{X}(\vec{s})$ in *ibid.* The latter was studied even earlier by Shende–Treumann–Zaslow [STZ17], who described a partition of it into subvarieties resembling Deodhar's cell decompositions.

To sketch Deodhar's results, recall that a *subword* of $\vec{s}$ is a sequence $\vec{\omega}$ of the same length with $\omega^{(i)} \in \{e, s^{(i)}\}$ for all i. We set $\omega_{(i)} := \omega^{(1)} \cdots \omega^{(i)}$. For any $v \in W$, a *v-distinguished subword* of $\vec{s}$ is a subword $\vec{\omega}$ such that

$$v\omega_{(i)} \leq v\omega_{(i-1)}s^{(i)} \quad \text{for all } i.$$

Let $\mathcal{D}^{(v)}(\vec{s})$ be the set of v-distinguished subwords $\vec{\omega}$ of $\vec{s}$ for which $\omega_{(\ell)} = e$. Then the Deodhar cells of $\mathbf{R}^{(v)}(\vec{s})$ are indexed by $\mathcal{D}^{(v)}(\vec{s})$. The Deodhar cell for a given element $\vec{\omega}$ is isomorphic to $\mathbf{A}^{\mathsf{d}_{\vec{\omega}}} \times \mathbf{G}_m^{\mathsf{e}_{\vec{\omega}}}$ for certain disjoint subsets $\mathsf{d}_{\vec{\omega}}, \mathsf{e}_{\vec{\omega}} \subseteq \{1, 2, \ldots, \ell\}$, reviewed in Section 5. In this way, we can count $R^{(v)}(\vec{s}) := \mathbf{R}^{(v)}(\vec{s})^F$:

$$|R^{(v)}(\vec{s})| = \sum_{\vec{\omega} \in \mathcal{D}^{(v)}(\vec{s})} q^{|\mathsf{d}_{\vec{\omega}}|}(q-1)^{|\mathsf{e}_{\vec{\omega}}|}. \tag{1.3}$$

Using essentially the same arguments that allow us to prove Theorem 1.2 from Theorem 1.1, we show:

Theorem 1.3. *For any word $\vec{s}$, we have*

$$\frac{|Z_J^-(\vec{s})|}{|G|} = \frac{1}{q^{\ell_J}(q-1)^{\mathrm{rk}(G)}} \sum_{v \in W^{J,+}} \sum_{\vec{\omega} \in \mathcal{D}^{(v)}(\vec{s})} q^{|\mathsf{d}_{\vec{\omega}}|}(q-1)^{|\mathsf{e}_{\vec{\omega}}|},$$

$$\frac{|Z_J^+(\vec{s})|}{|G|} = \frac{1}{(q-1)^{\mathrm{rk}(G)}} \sum_{v \in W^{J,-}} \sum_{\vec{\omega} \in \mathcal{D}^{(v)}(\vec{s})} q^{|\mathsf{d}_{\vec{\omega}}|}(q-1)^{|\mathsf{e}_{\vec{\omega}}|}.$$

(Note the sign flip between the left and right sides of each identity.)

We will explain in Section 5 that when $G = \mathrm{PGL}_n(\mathbf{F})$ and $J = S$, the $-$ case of Theorem 1.3 appears to recover [STZ17, Prop. 6.31].

1.5. **Combinatorics.** Our third application of Theorem 1.1, by way of Theorem 1.2, is to construct noncrossing sets of interest in the Catalan combinatorics of (W, S). *In the rest of this introduction, W is irreducible with Coxeter number h.*

Let $d_1, \ldots, d_{|S|}$ be the fundamental degrees of the action of W on its (irreducible) reflection representation. For each i, let $e_i = d_i - 1$. For any positive integer p coprime to h, the *rational Catalan number* of (W, p) is

$$\mathrm{Cat}_{W,p} := \prod_i \frac{p + e_i}{d_i},$$

while the *rational parking number* of (W, p) is $p^{|S|}$. These numbers enumerate disparate families of combinatorial objects. Most are constructed from root-theoretic data generalizing nonnesting partitions and parking functions, respectively. The collective study of these families and the bijections between them is the "nonnesting" side of rational Catalan/parking combinatorics. In [GLTW24], we instead sought, and constructed, "noncrossing" families: those depending on a chosen ordering of S, or *Coxeter word*.

For any word $\vec{s}$ in S, let $\mathcal{M}^{(v)}(\vec{s}) \subseteq \mathcal{D}^{(v)}(\vec{s})$ be the subset of elements $\vec{\omega}$ such that $|\mathbf{e}_{\vec{\omega}}| = |S|$, the minimum possible value [GLTW24, Cor. 4.9]. Let $\vec{c}$ be a Coxeter word for (W, S). The main results of [GLTW24] are the identities

$$\mathrm{Cat}_{W,p} = |\mathcal{M}^{(e)}(\vec{c}^{p})| \quad \text{and} \quad p^{|S|} = \sum_{v \in W} |\mathcal{M}^{(v)}(\vec{c}^{p})|,$$

proved by way of $\boldsymbol{q}$-deformed identities involving $\mathcal{D}^{(v)}(\vec{c}^{p})$ and taking $\boldsymbol{q} \to 1$.

In Section 6, we prove an identity that interpolates between the two above. Let $d_1^J, \ldots, d_{|J|}^J$ be the fundamental degrees of W_J. Let $e_1^J, \ldots, e_{|J|}^J$ be the exponents of the W_J-action on the reflection representation of W. We define the *rational parabolic parking numbers* of (W, p, J) to be

$$\mathrm{Park}_{W,p}^{J,\pm} = \prod_i \frac{p \pm e_i^J}{d_i^J}.$$

Then $\mathrm{Park}_{W,p}^{S,+} = \mathrm{Cat}_{W,p}$ and $\mathrm{Park}_{W,p}^{\emptyset,+} = \mathrm{Park}_{W,p}^{\emptyset,-} = p^{|S|}$. We relate these numbers to τ_G via a result from [Tri21], which describes $\tau_G(- \otimes T_{\vec{c}}^{p})$ for a certain $T_{\vec{c}} \in H_W$ as the graded character of a *rational parking space* for (W, p), in the sense of [ARR15] and [ALW16]. Ultimately, we obtain:

Corollary 1.4. *For any Coxeter word $\vec{c}$ and integer $p > 0$ coprime to h, we have*

$$\mathrm{Park}_{W,p}^{J,\pm} = \sum_{v \in W^{J,\mp}} |\mathcal{M}^{(v)}(\vec{c}^{p})|.$$

(Note the sign flip.) That is, $\coprod_{v \in W^{J,\pm}} \mathcal{M}^{(v)}(\vec{c}^{p})$ is a $\vec{c}$-noncrossing set enumerated by the $\mp$-rational parabolic parking number of (W, p, J).

1.6. **a-Degrees in Markov Traces.** In Section 7, we explain how Theorem 1.2 and Corollary 1.4 resemble results about Markov traces and rational Kirkman numbers that follow from work of Bezrukavnikov–Tolmachov [BT22].

First, for any $v \in W$, recall the *left ascent set* $\mathsf{Asc}(v) = \{s \in S \mid \ell(sv) > \ell(v)\}$ and *descent set* $\mathsf{Des}(v) = \{s \in S \mid \ell(sv) < \ell(v)\}$. Observe that $W^{J,-}$, *resp.* $W^{J,+}$, consists of those v such that $\mathsf{Asc}(v) \supseteq J$, *resp.* $\mathsf{Des}(v) \supseteq J$. Hence, $N_J^S(1)$ and $\boldsymbol{q}^{-\ell_J} N_J^S(T_{w_{J\circ}}^2)$ respectively decompose as sums, over supersets $I \supseteq J$, of elements

$$\zeta_I^+ := \sum_{\mathsf{Asc}(v) = I} \boldsymbol{q}^{-\ell(v)} T_{v^{-1}} T_v \quad \text{and} \quad \zeta_I^- := \sum_{\mathsf{Des}(v) = I} \boldsymbol{q}^{-\ell(v)} T_{v^{-1}} T_v.$$

Note that $\zeta_S^+ = \zeta_\emptyset^- = 1$ and $\zeta_\emptyset^+ = \zeta_S^- = \boldsymbol{q}^{-\ell_S} T_{w_\circ}^2$. By inclusion-exclusion, the elements $\zeta_I^\pm$ are again central in H_W.

Question 1.5. For general W and I, is there a more familiar description of the traces on H_W of the form $\tau[\zeta_I^{\pm}]$?

We now take $W = S_n$ and $S = \{s_1, \ldots, s_{n-1}\}$. The HOMFLYPT Markov trace on H_{S_n} can be written as a $\mathbf{Q}(\boldsymbol{q}^{1/2})[a^{\pm 1}]$-valued trace μ_n. For $0 \leq k \leq n-1$, let $\mu_n^{(k)} : H_W \to \mathbf{Q}(\boldsymbol{q}^{1/2})$ be the coefficient of the kth highest power of a in μ_n. We will show how to recast [BT22, Cor. 6.1.2] as the identity

$$\tau[\zeta_{I_k}^-] = (\boldsymbol{q}-1)^{n-1}\mu_n^{(k)}, \quad \text{where } I_k = \{s_1, s_2, \ldots, s_{n-1-k}\}. \tag{1.4}$$

Let $e_{\Lambda^k} \in \mathbf{Q}W$ be the Young symmetrizer of the hook partition $(n-k, 1, \ldots, 1) \vdash n$, which indexes the kth exterior power of the reflection representation of S_n. By combining (1.4) with the result in [Tri21] relating the Markov trace to τ_G, we get the following analogue of Theorem 1.2:

Theorem 1.6. *For G split semisimple of type A_{n-1}, and any integer k, we have*

$$\tau[\zeta_{I_k}^-] = (\boldsymbol{q}-1)^{n-1}\,\tau_G(e_{\Lambda^k} \otimes -)$$

as traces on H_{S_n}.

For general W and $0 \leq k \leq |S| - 1$, we can use the rational parking space for (W, p) mentioned earlier to define rational generalizations $\mathrm{Kirk}_{W,p}^{(k)}$ of the Kirkman numbers studied in [ARR15]. For $W = S_n$, the preceding result implies the following analogue of Corollary 1.4:

Corollary 1.7. *Take $W = S_n$ and $S = \{s_1, \ldots, s_{n-1}\}$. Then for any Coxeter word $\vec{c}$, any integer $p > 0$ coprime to n, and any k, we have*

$$\mathrm{Kirk}_{W,p}^{(k)} = \sum_{\mathsf{Asc}(v) = I_k} |\mathcal{M}^{(v)}(\vec{c}^p)|.$$

That is, $\coprod_{\mathsf{Asc}(v)=I_k} \mathcal{M}^{(v)}(\vec{c}^p)$ is a $\vec{c}$-noncrossing set enumerated by the kth rational Kirkman number of (W, p).

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2. Geometry of the Hecke Algebra

2.1. In this section, we review the general definition of the convolution algebra H_B^G without assuming G to be split, following [Car95, §3.3]. At the end, we explain how to adapt N_J^S to this generality. Along the way, we review Bruhat decomposition and related facts about Borel subgroups. Our geometric setup is essentially Kawanaka's in [Kaw75]. See also [Car93, Car95].

We keep $\mathbf{F}$, q, $\mathbf{G}$, $\mathbf{B}$, $\mathbf{T}$, $\mathbf{W}$ as in §1.2. Let $S_{\mathbf{B}}$ be the system of simple reflections of $\mathbf{W}$ arising from $\mathbf{B}$, and let $\ell_{\mathbf{B}}$ be the Bruhat length function on $\mathbf{W}$ defined by $S_{\mathbf{B}}$.

2.2. **Bruhat Decomposition.** Note that $w\mathbf{B}$ and $\mathbf{B}w$ are well-defined for any $w \in \mathbf{W}$. Bruhat decomposition says that as we run over all w, the double cosets $\mathbf{B}w\mathbf{B}$ are pairwise disjoint and partition $\mathbf{G}$.

Let $\mathbf{U}$ be the unipotent radical of $\mathbf{B}$, so that $\mathbf{B} = \mathbf{T} \ltimes \mathbf{U}$. Let $\mathbf{U}_-$ be the unipotent radical of the opposed Borel $\mathbf{B}_-$. Note that $w\mathbf{U}w^{-1}$ and $w\mathbf{U}_-w^{-1}$ are well-defined for all $w \in \mathbf{W}$. Let

$$\begin{aligned}\mathbf{U}_w &= \mathbf{U} \cap w\mathbf{U}w^{-1},\\ \mathbf{U}_w^- &= \mathbf{U} \cap w\mathbf{U}_-w^{-1}.\end{aligned}$$

Then $\mathbf{U}_w, \mathbf{U}_w^-$ are stable under the conjugation action of $\mathbf{T}$ on $\mathbf{U}$. The following results are proved in [Car93, §2.5]:

Lemma 2.1. *For all $w \in \mathbf{W}$:*

1. *If $\ell_{\mathbf{B}}(wv) = \ell_{\mathbf{B}}(w) + \ell_{\mathbf{B}}(v)$, then $\mathbf{U}_{wv}^- = \mathbf{U}_w^-\mathbf{U}_v^-$, and $\mathbf{U}_w^- \cap \mathbf{U}_v^- = \{1\}$.*
2. *$\mathbf{U} = \mathbf{U}_w\mathbf{U}_w^- = \mathbf{U}_w^-\mathbf{U}_w$, and $\mathbf{U}_w \cap \mathbf{U}_w^- = \{1\}$.*
3. *$\mathbf{B}w\mathbf{B} = \mathbf{U}_w^-w\mathbf{B}$, and the map $\mathbf{U}_w^- \to \mathbf{U}_w^-w\mathbf{B}/\mathbf{B}$ is an isomorphism.*
4. *As an algebraic variety (but not group), $\mathbf{U}_w^-$ is the product of the root subgroups inverted by w, hence an affine space of dimension $\ell_{\mathbf{B}}(w)$.*

2.3. **Bott–Samelson Varieties.** The double cosets of $\mathbf{B}$ in $\mathbf{G}$ are in bijection with the set of diagonal $\mathbf{G}$-orbits on $(\mathbf{G}/\mathbf{B})^2$. As in the introduction, we write $h\mathbf{B} \xrightarrow{w} g\mathbf{B}$ to mean $\mathbf{B}h^{-1}g\mathbf{B} = \mathbf{B}w\mathbf{B}$. Such pairs $(h\mathbf{B}, g\mathbf{B})$ form the points of the $\mathbf{G}$-orbit of $(\mathbf{G}/\mathbf{B})^2$ corresponding to w, which we will denote by $\mathbf{O}(w)$.

More generally, for any sequence of elements $\vec{w} = (w^{(1)}, w^{(2)}, \dots, w^{(k)})$ in $\mathbf{W}$, let $\mathbf{O}(\vec{w})$ be the subvariety of $(\mathbf{G}/\mathbf{B})^{1+k}$ defined on points by

$$\mathbf{O}(\vec{w}) = \{\vec{g}\mathbf{B} = (g_0\mathbf{B}, g_1\mathbf{B}, \dots, g_k\mathbf{B}) \mid g_0\mathbf{B} \xrightarrow{w^{(1)}} g_1\mathbf{B} \xrightarrow{w^{(2)}} \cdots \xrightarrow{w^{(k)}} g_m\mathbf{B}\}.$$

The Zariski closure of $\mathbf{O}(\vec{w})$ is sometimes called the *Bott–Samelson variety* of $\vec{w}$. For this reason, $\mathbf{O}(\vec{w})$ is sometimes called the *open Bott–Samelson variety*.

For any subset $I \subseteq \{1, \dots, k\}$, we write $pr_I : \mathbf{O}(\vec{w}) \to (\mathbf{G}/\mathbf{B})^I$ to denote the map that sends $pr_I(\vec{g}\mathbf{B}) = (g_i\mathbf{B})_{i\in I}$. When writing out $\vec{w}$, *resp.* I, explicitly, we will omit the parentheses, *resp.* brackets, where convenient.

Lemma 2.1(1) implies that if $\ell_{\mathbf{B}}(wv) = \ell_{\mathbf{B}}(w) + \ell_{\mathbf{B}}(v)$, then $pr_{0,2}$ induces an explicit isomorphism $\mathbf{O}(w, v) \xrightarrow{\sim} \mathbf{O}(wv)$. By induction, any variety of the form $\mathbf{O}(\vec{w})$ is explicitly isomorphic to one of the form $\mathbf{O}(\vec{s})$, where $\vec{s}$ is a word in $S_{\mathbf{B}}$.

2.4. **Frobenius Maps.** For algebraic varieties over $\bar{\mathbf{F}}$ equipped with Frobenius maps, we use italics to denote the corresponding sets of Frobenius-fixed points.

As in §1.2, we fix a Frobenius map $F : \mathbf{G} \to \mathbf{G}$ arising from an $\mathbf{F}$-form, such that $\mathbf{B}$ and $\mathbf{T}$ are F-stable. Then $\mathbf{W}$ and $S_{\mathbf{B}}$ are also F-stable. The group $W := \mathbf{W}^F$ is again a Coxeter group, which can be identified with $N_G(T)/T$.

Remark 2.2. When $\mathbf{G}$ is almost-simple, the options for G and W are listed in [Car95, §1.5–1.6]. Notably, W is crystallographic except when it has factors of type ${}^2F_4$.

There is a system of simple reflections for W, which we will denote S, indexed by the F-orbits on $S_{\mathbf{B}}$: Each element $s \in S$ is the product of all the elements in the given F-orbit, which pairwise commute and form a reduced word in $S_{\mathbf{B}}$ in any order. Let ℓ be the Bruhat length function on W defined by S.

By Lang's theorem, $g\mathbf{B}$ is F-stable if and only if $g \in G$, and in this case, $gB = (x\mathbf{B})^F$. Similarly, $\mathbf{B}w\mathbf{B}$ is F-stable if and only if $w \in W$, and in this case, $BwB = (\mathbf{B}w\mathbf{B})^F$. Thus, the double cosets BwB for $w \in W$ partition G, while the G-orbits on $(G/B)^2$ are the sets $O(w)$ for $w \in W$. As explained in [Car93], parts (1)–(3) of Lemma 2.1 have exact analogues with $\mathbf{W}$ replaced by W. See also [Kaw75, §1].

Lemma 2.3. *For all $w \in W$:*

1. *If $\ell(wv) = \ell(w) + \ell(v)$, then $U^-_{wv} = U^-_w U^-_v$, and $U^-_w \cap U^-_v = \{1\}$.*
2. *$U = U_w U^-_w = U^-_w U_w$, and $U_w \cap U^-_w = \{1\}$.*
3. *$BwB = U^-_w wB$, and the map $U^-_w \to U^-_w wB/B$ is a bijection.*

The one point where caution is needed concerns the sizes of U_w and U^-_w, as they use $\ell_{\mathbf{B}}(w)$, not $\ell(w)$, in general [Car93, 74].

Lemma 2.4. *For all $w \in W$, we have $|U^-_w| = q^{\dim(\mathbf{U}^-_w)} = q^{\ell_{\mathbf{B}}(w)}$.*

2.5. **Operations on Functions.** For any finite set X equipped with the action of a finite group G, we write $\mathcal{C}_G(X)$ to denote the free module of $\mathbf{Z}$-valued, G-invariant functions on X. For any G-stable subset $Y \subseteq X$, we write $\delta_Y \in \mathcal{C}_G(X)$ to denote the indicator function on Y.

For a G-equivariant map $f : Y \to X$, the *pullback* of functions along f is the linear map $f^* : \mathcal{C}_G(X) \to \mathcal{C}_G(Y)$ given by $f^*(\varphi)(y) = \varphi(f(y))$. The *pushforward*, or *integral*, of functions along f is the linear map $f_! : \mathcal{C}_G(Y) \to \mathcal{C}_G(X)$ given by

$$f_!(\psi)(x) = \sum_{y \in f^{-1}(x)} \psi(y).$$

When f can be understood from context, we omit $f_!$ from our notation.

Let $*$ denote the *convolution product* on $\mathcal{C}(X \times X)$ defined in terms of the three projection maps $pr_{i,j} : X^3 \to X^2$ by

$$\varphi_1 * \varphi_2 = pr_{1,3,!}(pr^*_{1,2}(\varphi_1) \cdot pr^*_{2,3}(\varphi_2)),$$

where $\cdot$ denotes pointwise multiplication. Explicitly,

$$(\varphi_1 * \varphi_2)(y, x) = \sum_{z \in X} \varphi_1(y, z)\varphi_2(z, x).$$

The indicator function of the diagonal $X \subseteq X^2$ is the identity element for this operation. If X is equipped with a G-action, and G acts on X^2 diagonally, then $*$ restricts to an operation on $\mathcal{C}_G(X \times X)$ with the same identity element.

Iwahori proved that the ring formed by $\mathcal{C}_G(G/B \times G/B)$ under convolution is freely generated by the elements $\delta_w := \delta_{O(w)}$ for $w \in W$ modulo the following relations for all $w \in W$ and $s \in S$:

$$\delta_s * \delta_w = \begin{cases} \delta_{sw} & \ell(sw) > \ell(w), \\ |U_s^-|\,\delta_{sw} + (|U_s^-| - 1)\,\delta_w & \ell(sw) < \ell(w) \end{cases}$$

See [Car95, §3.3] or [Kaw75, Thm. 2.6]. We define H_B^G to be the $\mathbf{Z}[\frac{1}{q}]$-algebra

$$H_B^G = \mathcal{C}_G(G/B \times G/B)[\tfrac{1}{q}].$$

If G is *split*, meaning $W = \mathbf{W}$, then $\ell_{\mathbf{B}}(s) = \ell(s) = 1$ and $|U_s^-| = q$ for all $s \in S$. This is the case on which the introduction focused. Here, W is crystallographic, and H_B^G is a specialization of the $\mathbf{Z}[\boldsymbol{q}^{\pm 1}]$-algebra H_W freely generated by elements T_w for $w \in W$ modulo the following relations for all $w \in W$ and $s \in S$:

$$T_s T_w = \begin{cases} T_{sw} & \ell(sw) = \ell(w) + 1, \\ \boldsymbol{q} T_{sw} + (\boldsymbol{q} - 1) T_w & \ell(sw) = \ell(w) - 1 \end{cases}$$

2.6. **Parabolic Subgroups.** Fix an F-stable subset $J_{\mathbf{B}} \subseteq S_{\mathbf{B}}$, corresponding to a subset $J \subseteq S$. Let $\mathbf{W}_J \subseteq \mathbf{W}$, *resp.* $W_J \subseteq W$, be the subgroup generated by $J_{\mathbf{B}}$, *resp.* J. Then $\mathbf{W}_J$ is F-stable and $W_J = \mathbf{W}_J^F$.

Let $\mathbf{P}_J = \mathbf{B}\mathbf{W}_J\mathbf{B} \subseteq \mathbf{G}$. We can write $\mathbf{P}_J = \mathbf{L}_J \ltimes \mathbf{U}_J$, where $\mathbf{L}_J$ is reductive with Weyl group $\mathbf{W}_J$ and $\mathbf{U}_J$ the unipotent radical of $\mathbf{P}_J$. These subgroups are F-stable, and on F-fixed points, we have $P_J = L_J \ltimes U_J$.

By construction, $\mathbf{B}_J = \mathbf{L}_J \cap \mathbf{B}$ is a Borel subgroup of $\mathbf{L}_J$. The inclusion $L_J \subseteq P_J$ descends to an L_J-equivariant bijection $L_J/B_J \simeq P_J/B$, which in turn yields an isomorphism of algebras

$$\mathcal{C}_{L_J}(L_J/B_J \times L_J/B_J) \simeq \mathcal{C}_{L_J}(P_J/B \times P_J/B).$$

Once we adjoin $\frac{1}{q}$, the left-hand side becomes $H_{B_J}^{L_J}$, and the right-hand side becomes the subalgebra of H_B^G generated by the elements δ_w with $w \in W_J$. Henceforth, we identify these $\mathbf{Z}[\frac{1}{q}]$-algebras with each other.

As in the introduction, let $W^{J,-} \subseteq W$ be the set of minimal-length right coset representatives for W_J. By Lemma 2.1, the split case of the following definition recovers the $\boldsymbol{q} \to q$ specialization of the relative norm map in §1.1.

Definition 2.5. The *relative norm* map $N_J^S : H_{B_J}^{L_J} \to H_B^G$ is defined by

$$N_J^S(\alpha) = \sum_{v \in W^{J,-}} \frac{1}{|U_v^-|}\, \delta_{v^{-1}} * \alpha * \delta_v.$$

We have implicitly used Lemma 2.4 to ensure that $|U_v^-|$ is a power of q.

2.7. Let $w_\circ$ and $w_{J\circ}$ respectively denote the longest elements of W and W_J with respect to S. Then $U = U_{w_\circ}$ and $U_J = U_{w_{J\circ}}$. The following fact will be useful:

Lemma 2.6. *For any $J \subseteq S$ and $v \in W^{J,-}$, we have*

$$U_J \cap U_v = U_{w_{J\circ} v} \quad \text{and} \quad U_J \cap U_v^- = U_v^-.$$

In particular, $U_J = U_{w_{J\circ}v}U_v^- = U_v^- U_{w_{J\circ}v}$ *and* $U_{w_{J\circ}v} \cap U_v^- = \{1\}$. *In the split case, the analogous identities hold with* $\mathbf{U}_J$, $\mathbf{U}_v$, etc. *in place of* U_J, U_v, etc..

Proof. To show $U_J \cap U_v = U_{w_{J\circ}v}$: In general, if $w, v \in W$ satisfy $\ell(wv) = \ell(w) + \ell(v)$, then $U_{wv}^- = U_w^- U_v^-$ and $U_w^- \cap U_v^- = \{1\}$ by Lemma 2.3(1), which implies that $U_{wv} = U_w \cap U_v$ by Lemma 2.3(2).

To show $U_J \cap U_v^- = U_v^-$, meaning $U_v^- \subseteq U_J$: In general, if $w \in W_J$ and $v \in W^{J,-}$, then the F-orbits of root subgroups of $\mathbf{U}_J$ inverted by wv are precisely those inverted by w. Taking $w = e$ gives the result.

In the split case, $\ell_{\mathbf{B}} = \ell$, and thus, v minimizes $\ell_{\mathbf{B}}$ in $\mathbf{W}_J v$. So we can repeat all the arguments above with the varieties in place of the sets. □

3. Partial Springer Resolutions

3.1. Recall the partial Springer resolutions $\mathbf{Spr}_J^{\pm} \subseteq \mathbf{G} \times \mathbf{G}/\mathbf{P}_J$ and the varieties $\mathbf{E}_J^{\pm} = \mathbf{G}/\mathbf{B} \times \mathbf{Spr}_J^{\pm}$ from §1.2. The latter are stable under the left $\mathbf{G}$-action on $\mathbf{G}/\mathbf{B} \times \mathbf{G} \times \mathbf{G}/\mathbf{P}_J$ defined by

$$g \cdot (h\mathbf{B}, u, y\mathbf{P}_J) = (gh\mathbf{B}, gug^{-1}, gy\mathbf{P}_J). \tag{3.1}$$

Let $f : \mathbf{G}/\mathbf{B} \times \mathbf{G} \times \mathbf{G}/\mathbf{P}_J \to (\mathbf{G}/\mathbf{B})^2$ be the $\mathbf{G}$-equivariant map defined by

$$f(h\mathbf{B}, u, y\mathbf{P}_J) = (h\mathbf{B}, uh\mathbf{B}).$$

On F-fixed points, it restricts to G-equivariant maps $f : E_J^{\pm} \to (G/B)^2$. These recover the maps f in §1.2. The goal of this section is to prove the identities

$$\begin{aligned} f_! \delta_{E_J^-} &= |U_J|\, N_J^S(1), \\ f_! \delta_{E_J^+} &= |U_J|\, N_J^S(\delta_{w_{J\circ}}^2), \end{aligned} \tag{3.2}$$

where N_J^S is now given by Definition 2.5. They recover Theorem 1.1 in the split case.

3.2. **Reduction to Strata.** Observe that $\mathbf{E}_J^{\pm}$ is a union of $\mathbf{G}$-stable subvarieties $\mathbf{E}_{J,v}^{\pm}$ for $\mathbf{W}_J v \in \mathbf{W}_J \backslash \mathbf{W}$, where on points,

$$\mathbf{E}_{J,v}^{\pm} = \{(h\mathbf{B}, u, y\mathbf{P}_J) \in \mathbf{G}/\mathbf{B} \times \mathbf{Spr}_J^{\pm} \mid \mathbf{P}_J y^{-1} h\mathbf{B} = \mathbf{P}_J v\mathbf{B}\}.$$

From §2.4, we see that $\mathbf{P}_J v\mathbf{B}$ is F-stable if and only if $v \in W$, and in this case, $P_J vB = (\mathbf{P}_J v\mathbf{B})^F$. Therefore, $E_J^{\pm}$ is the union of its G-stable subsets $E_{J,v}^{\pm}$ as v runs over a full set of right coset representatives for W_J: for instance, $W^{J,-}$. As Lemma 2.6 shows that $U_J \simeq U_{w_{J\circ}v} \times U_v^-$, we reduce (3.2) to:

Theorem 3.1. *If* $v \in W^{J,-}$, *then:*

1. $f_! \delta_{E_{J,v}^-} = |U_{w_{J\circ}v}|\, \delta_{v^{-1}} * \delta_v$.
2. $f_! \delta_{E_{J,v}^+} = |U_{w_{J\circ}v}|\, \delta_{v^{-1}} * \delta_{w_{J\circ}}^2 * \delta_v$.

3.3. **Reduction to the Borel.** Let $\check{\mathbf{E}}^{\pm}_{J,v} \subseteq \mathbf{G}/\mathbf{B} \times \mathbf{G} \times \mathbf{G}/\mathbf{B}$ be the subvariety defined on points by

$$\check{\mathbf{E}}^{\pm}_{J,v} = \{(h\mathbf{B}, u, y\mathbf{B}) \mid (u, y\mathbf{B}) \in \mathbf{Spr}^{\pm}_J \text{ and } y\mathbf{B} \xrightarrow{v} h\mathbf{B}\}$$

The forgetful map $\mathbf{G}/\mathbf{B} \to \mathbf{G}/\mathbf{P}_J$ induces a map $\check{\mathbf{E}}^{\pm}_{J,v} \to \mathbf{E}^{\pm}_{J,v}$.

Lemma 3.2. *If $v \in W^{J,-}$, then $\check{E}^{\pm}_{J,v} \to E^{\pm}_{J,v}$ is a bijection. In the split case, this bijection arises from an isomorphism $\check{\mathbf{E}}^{\pm}_{J,v} \to \mathbf{E}^{\pm}_{J,v}$.*

Proof. The first claim is just the fact that if v minimizes ℓ in $W_J v$, then there are compatible bijections from U_v^- to the Schubert cells BvB/B and BvP_J/P_J.

For the second claim: As in the proof of Lemma 2.6, v minimizes $\ell_{\mathbf{B}}$ in $\mathbf{W}_J v$. So we can repeat the argument above, but with the varieties $\mathbf{U}_v^-$, $\mathbf{B}$, $\mathbf{P}_J$ in place of the sets U_v^-, B, P_J, and isomorphisms in place of bijections. □

The varieties $\check{\mathbf{E}}^{\pm}_J$ are stable under the $\mathbf{G}$-action on $\mathbf{G}/\mathbf{B} \times \mathbf{G} \times \mathbf{G}/\mathbf{B}$ analogous to (3.1). Let $\check{f} : \mathbf{G}/\mathbf{B} \times \mathbf{G} \times \mathbf{G}/\mathbf{B} \to (\mathbf{G}/\mathbf{B})^3$ be the equivariant map defined by

$$\check{f}(h\mathbf{B}, u, y\mathbf{B}) = (h\mathbf{B}, y\mathbf{B}, uh\mathbf{B}).$$

The proofs of the two parts of Theorem 3.1 will use $\check{f}$ in different ways.

3.4. **Proof of (1).** In the notation of Section 2,

$$pr_{0,2,!}\delta_{O(v^{-1},v)} = \delta_{v^{-1}} * \delta_v.$$

This suggests comparing $\mathbf{E}^-_{J,v}$ to a bundle over $\mathbf{O}(v^{-1}, v)$. It turns out that $\check{\mathbf{E}}^-_{J,v}$ is the bundle we seek.

Observe that if $(h\mathbf{B}, u, y\mathbf{B})$ is a point of $\check{\mathbf{E}}^-_{J,v}$, then $\mathbf{B}y^{-1}uh\mathbf{B} = \mathbf{B}y^{-1}h\mathbf{B} = \mathbf{B}v\mathbf{B}$. Therefore, $\check{f}$ restricts to a map from $\check{\mathbf{E}}^-_{J,v}$ into $\mathbf{O}(v^{-1}, v)$, giving an equivariant commutative diagram:

$$\begin{tikzcd}
\check{\mathbf{E}}^-_{J,v} \arrow[r] \arrow[d, "\check{f}"'] & \mathbf{E}^-_{J,v} \arrow[ddl, bend left, "f"] \\
\mathbf{O}(v^{-1}, v) \arrow[d, "pr_{0,2}"'] & \\
(\mathbf{G}/\mathbf{B})^2 &
\end{tikzcd}$$

By Lemma 3.2 and this diagram, we reduce case (1) of Theorem 3.1 to:

Proposition 3.3. *If $v \in W^{J,-}$, then*

$$\check{f}_! \delta_{\check{E}^-_{J,v}} = |U_{w_{J\circ}v}|\, \delta_{O(v^{-1},v)}$$

in $\mathcal{C}_G(O(v^{-1}, v))$. In the split case, this identity arises from $\check{\mathbf{E}}^-_{J,v} \xrightarrow{\check{f}} \mathbf{O}(v^{-1}, v)$ being a smooth fiber bundle with fiber $\mathbf{U}_{w_{J\circ}v}$ above $(\mathbf{B}, v\mathbf{B})$.

Proof. For the first claim: Recall that the G-action on pairs $(yB, hB) \in O(v)$ is transitive. So by equivariance of $\check{f}$ and homogeneity, it suffices to compute $\check{f}$ over any fixed choice of such yB and hB.

We take $(yB, hB) = (B, vB)$. Over this pair, the fiber of $\check{E}_J^-$ consists of (vB, u, B) with $u \in U_J$, the fiber of $O(v^{-1}, v)$ consists of (vB, B, gB) with $gB \in BvB/B$, and $\check{f}$ is given by $u \mapsto uvB$. Therefore, under the bijections $U_J \simeq U_{w_{J\circ}v} \times U_v^-$ of Lemma 2.6 and $BvB/B \simeq U_v^-$ of Lemma 2.3(3), $\check{f}$ corresponds to the projection $U_{w_{J\circ}v} \times U_v^- \to U_v^-$. This proves the claim.

For the second claim: As in the proof of Lemma 2.6, we observe that v minimizes $\ell_{\mathbf{B}}$ in $\mathbf{W}_J v$. So we can repeat the arguments above with the varieties $\mathbf{G}$, $\mathbf{O}(v)$, *etc.* in place of the sets G, $O(v)$, *etc.*, and Lemma 2.1 in place of Lemma 2.3. ☐

3.5. **Proof of (2).** In the notation of Section 2 (*nota bene* §2.7),

$$pr_{0,4,!}\delta_{O(v^{-1}, w_{J\circ}, w_{J\circ}, v)} = \delta_{v^{-1}} * \delta_{w_{J\circ}}^2 * \delta_v.$$

This suggests comparing $\mathbf{E}_{J,v}^+$ to a bundle over $\mathbf{O}(v^{-1}, w_{J\circ}, w_{J\circ}, v)$. But unlike the situation in case (1), there is no obvious map from $\check{\mathbf{E}}_{J,v}^+$ into the latter variety.

We do know that $\check{f}$ restricts to a map from $\check{\mathbf{E}}_{J,v}^+$ into $\mathbf{O}(v^{-1}) \times \mathbf{G}/\mathbf{B}$, giving an equivariant commutative diagram:

$$\begin{tikzcd}
\check{\mathbf{E}}^+_{J,v} \arrow[r] \arrow[d, "\check{f}"'] & \mathbf{E}^+_{J,v} \arrow[ddl, bend left, "f"] \\
\mathbf{O}(v^{-1}) \times \mathbf{G}/\mathbf{B} \arrow[d, "pr_0 \times \mathrm{id}"'] & \\
(\mathbf{G}/\mathbf{B})^2 &
\end{tikzcd}$$

At the same time, we have a map

$$\mathbf{O}(v^{-1}, w_{J\circ}, w_{J\circ}, v) \xrightarrow{pr_{0,1,4}} \mathbf{O}(v^{-1}) \times \mathbf{G}/\mathbf{B}.$$

So by Lemma 3.2 and the discussion above, we reduce case (2) of Theorem 3.1 to:

Proposition 3.4. *If $v \in W^{J,-}$, then*

$$\check{f}_! \delta_{E_{J,v}^+} = |U_{w_{J\circ}v}|\, pr_{0,1,4,!}\delta_{O(v^{-1}, w_{J\circ}, w_{J\circ}, v)}$$

in $\mathcal{C}_G(O(v^{-1}) \times G/B)$.

Proof. For any $w \in W$, let

$$O(v^{-1}) \times_w G/B = \{(hB, yB, gB) \in O(v^{-1}) \times G/B \mid Bh^{-1}gB = BwB\},$$

the preimage of $O(w)$ along $pr_0 \times \mathrm{id}$. Recall that the G-action on $O(w)$ is transitive. So by equivariance and homogeneity, the fibers of $\check{E}_{J,v}^+$ and $O(v^{-1}, w_{J\circ}, w_{J\circ}, v)$ have constant size over $O(v^{-1}) \times_w G/B$. So it suffices to compare them over any fixed choice of $(hB, gB) \in O(w)$, for each $w \in W$. Moreover, to do this, it suffices to fix hB and average over $gB \in hBwB/B$.

We take $hB = B$. Then we must compare the preimages of

$$\{(B, yB, gB) \in O(v^{-1}) \times_w G/B\} \tag{3.3}$$

in $\check{E}_J^+$ and $O(v^{-1}, w_{J\circ}, w_{J\circ}, v)$. Since $v \in W^{J,-}$, we can trade the latter set and the map $pr_{0,1,4}$ for the set $O(v^{-1}, w_{J\circ}, w_{J\circ}v)$ and the map $pr_{0,1,3}$.

The preimage of (3.3) in $\check{E}_J^+$ consists of (B, u, yB) such that $u \in yV_Jy^{-1}$ and $u \in BwB$. Hence it has size

$$|yV_Jy^{-1} \cap BwB|. \tag{3.4}$$

The preimage of (3.3) in $O(v^{-1}, w_{J\circ}, w_{J\circ}v)$ consists of (B, yB, zB, gB) such that

$$yB \xleftarrow{w_{J\circ}} zB \xrightarrow{w_{J\circ}v} gB$$

and $gB \in BwB/B$. Observe that $yB \in Bv^{-1}B/B$, so homogeneity under left multiplication by B lets us count the preimage for a given yB by averaging over the preimages for all $yB \in Bv^{-1}B/B$. Since $v \in W^{J,-}$, Lemma 2.3(1) shows that the union of these preimages is parametrized by (zB, gB) such that

$$B \xleftarrow{w_{J\circ}v} zB \xrightarrow{w_{J\circ}v} gB \tag{3.5}$$

and $gB \in BwB/B$. It also shows that there is a bijection from $U^-_{(w_{J\circ}v)^{-1}} \times U^-_{w_{J\circ}v}$ to the set of pairs (zB, gB) satisfying (3.5), given by

$$(u, u') \mapsto (u(w_{J\circ}v)^{-1}B, u(w_{J\circ}v)^{-1}u'w_{J\circ}vB).$$

So the set of (zB, gB) satisfying (3.5) and $gB \in BwB/B$ is parametrized by

$$(U^-_{(w_{J\circ}v)^{-1}}(w_{J\circ}v)^{-1}U^-_{w_{J\circ}v}w_{J_\circ}v) \cap BwB.$$

Since $U^-_{(w_{J\circ}v)^{-1}} \subseteq B$, this last set can be identified with

$$U^-_{(w_{J\circ}v)^{-1}} \times ((w_{J\circ}v)^{-1}U^-_{w_{J\circ}v}w_{J_\circ}v \cap BwB).$$

By Lemma 2.3(3), we have $|U^-_{v^{-1}}|$ many choices for $yB \in Bv^{-1}B/B$, and since $v \in W^{J,-}$, we also have $|U^-_{(w_{J\circ}v)^{-1}}| = |U^-_{w_{J\circ}}||U^-_{v^{-1}}|$. Altogether, we conclude that the size of the preimage of (3.3) in $O(v^{-1}, w_{J\circ}, w_{J\circ}v)$ is

$$|U^-_{w_{J\circ}}||(w_{J\circ}v)^{-1}U^-_{w_{J\circ}v}w_{J_\circ}v \cap BwB|. \tag{3.6}$$

Finally, we compare (3.4) and (3.6). Theorem 4.1 of [Kaw75] says

$$|yV_Jy^{-1} \cap BwB| = |U_{v^{-1}}||(w_{J\circ}v)^{-1}U^-_{w_{J\circ}v}w_{J_\circ}v \cap BwB|.$$

Again using $|U^-_{(w_{J\circ}v)^{-1}}| = |U^-_{w_{J\circ}}||U^-_{v^{-1}}|$, we see that $|U_{v^{-1}}| = |U_{(w_{J\circ}v)^{-1}}||U^-_{w_{J\circ}}| = |U_{w_{J\circ}v}||U^-_{w_{J_\circ}}|$, giving the desired identity. □

Remark 3.5. The asymmetry of the variety $\mathbf{O}(v^{-1}) \times \mathbf{G}/\mathbf{B}$ may seem defective. To make the geometry more symmetrical, one might try to replace the diagram

$$\mathbf{E}_J^+ \xrightarrow{\check{f}} \mathbf{O}(v^{-1}) \times \mathbf{G}/\mathbf{B} \xleftarrow{pr_{0,1,4}} \mathbf{O}(v^{-1}, w_{J\circ}, w_{J\circ}, v)$$

with the diagram

$$\mathbf{E}_J^+ \xrightarrow{\check{f}'} \mathbf{O}(v^{-1}) \times \mathbf{O}(v) \xleftarrow{pr_{0,1,3,4}} \mathbf{O}(v^{-1}, w_{J\circ}, w_{J\circ}, v)$$

in which $\check{f}'(h\mathbf{B}, u, x\mathbf{B}) = (h\mathbf{B}, x\mathbf{B}, ux\mathbf{B}, uh\mathbf{B})$. Then one would hope that

$$\check{f}'_! \delta_{E^+_{J,v}} = |U_J|\, pr_{0,1,3,4,!} \delta_{O(v^{-1}, w_{J\circ}, w_{J\circ}, v)}$$

in $\mathcal{C}_G(O(v^{-1}) \times O(v))$. However, Kawanaka's work does not seem to establish this stronger identity.

4. Traces on the Hecke Algebra

4.1. The goal of this section is to prove a form of Theorem 1.2 for general G, and also, to prove that it recovers (1.2) when $G = \mathrm{GL}_n$. We keep the general setup of Section 2.

4.2. **Traces from Relative Norms.** As in §1.3, let $\tau : H_B^G \to \mathbf{Z}[\frac{1}{q}]$ be the trace given by $\tau(\delta_e) = 1$ and $\tau(\delta_w) = 0$ for all $w \neq e$, and for any central element $\zeta \in Z(H_B^G)$, let $\tau[\zeta] : H_B^G \to \mathbf{Z}[\frac{1}{q}]$ be the trace given by $\tau[\zeta](\beta) = \tau(\beta * \zeta)$.

Lemma 4.1. *For all $J \subseteq S$ and $w \in W$ and $\alpha \in Z(H_{B_J}^{L_J})$, we have*

$$\frac{1}{|B|}\, \tau[N_J^S(\alpha)](\delta_w) = \frac{1}{|G|} \sum_{(hB, gB) \in O(w)} N_J^S(\iota(\alpha))(hB, gB),$$

where ι is the additive anti-involution of $H_{B_J}^{L_J}$ given by $\iota(\delta_w) = \delta_{w^{-1}}$.

Proof. For any $\beta \in H_B^G$ and $xB \in G/B$, we have $\tau(\beta) = \beta(xB, xB)$. Moreover, $|G/B| = |G|/|B|$. So for any $\zeta \in Z(H_B^G)$, we have

$$\frac{|G|}{|B|}\, \tau[\zeta](\beta) = \sum_{xB \in G/B} (\beta * \zeta)(xB, xB).$$

Next, for any $w, v, z \in W$, observe that there is a bijection

$$\begin{aligned} &\{(x_0B, x_1B, x_2B, x_3B, x_4B) \in O(w, v^{-1}, z, v) \mid x_0B = x_4B\} \\ &\quad \xrightarrow{\sim} \{(g_0B, g_1B, g_2B, g_3B) \in O(v^{-1}, z^{-1}, v) \mid g_0B \xrightarrow{w} g_3B\} \end{aligned}$$

given by $(g_0B, g_1B, g_2B, g_3B) = (x_4B, x_3B, x_2B, x_1B)$. This shows the identity

$$\sum_{gB \in G/B} (\delta_w * \delta_{v^{-1}} * \delta_z * \delta_v)(gB, gB) = \sum_{(hB, gB) \in O(w)} (\delta_{v^{-1}} * \delta_{z^{-1}} * \delta_v)(hB, gB).$$

By expanding α in the basis $(\delta_z)_{z \in W_J}$ for $H_{B_J}^{L_J}$, and summing over all $v \in W^{J,-}$, we deduce that

$$\sum_{xB \in G/B} (\beta * N_J^S(\alpha))(xB, xB) = \sum_{(hB, gB) \in O(w)} N_J^S(\iota(\alpha))(hB, gB),$$

concluding the proof. □

4.3. **Springer Fibers.** A reference for this subsection is [Sho88]. In order to work with étale cohomology, we fix a prime ℓ invertible in $\mathbf{F}$. The notation $\mathrm{H}^*(-, \bar{\mathbf{Q}}_\ell)$ will always mean étale cohomology with coefficients in the constant $\bar{\mathbf{Q}}_\ell$-sheaf. Henceforth, let $\mathbf{V} = \mathbf{V}_\emptyset$ and

$$\mathbf{Spr} = \mathbf{Spr}_\emptyset^+ = \mathbf{Spr}_\emptyset^- \subseteq \mathbf{V} \times \mathbf{G}/\mathbf{B}.$$

By the *Springer resolution*, we mean either $\mathbf{Spr}$ or the projection map from $\mathbf{Spr}$ onto $\mathbf{V}$. For any $u \in \mathbf{V}$, the *Springer fiber* over u is the (reduced) fiber of this map over u, viewed as a subvariety $\mathbf{Spr}_u$ of $\mathbf{G}/\mathbf{B}$. On points,

$$\mathbf{Spr}_u = \{y\mathbf{B} \in \mathbf{G}/\mathbf{B} \mid u \in y\mathbf{U}y^{-1}\}.$$

Springer showed that this is a projective variety with no odd cohomology. For $u \in V := \mathbf{V}^F$, he constructed an action of W on $\mathrm{H}^*(\mathbf{Spr}_u)$ through a type of Fourier transform. Later, other authors gave independent constructions, generalizing to other base fields like the complex numbers.

In this paper, we use the W-action on $\mathrm{H}^*(\mathbf{Spr}_u)$ constructed through perverse sheaf theory, which differs from Springer's original action by a sign twist. Let $\chi_u : \mathbf{Q}W \to \bar{\mathbf{Q}}_\ell$ be the trace defined by

$$\chi_u(w) = \mathrm{tr}(Fw \mid \mathrm{H}^*(\mathbf{Spr}_u)).$$

For our choice of action, the sign character of W only occurs in χ_1.

As reviewed in [Sho88, §15], it is now known χ_u arises from the specialization at $\boldsymbol{q} \to q$ of a $\mathbf{Z}[\boldsymbol{q}]$-valued trace on $\mathbf{Z}W$. In particular, $\chi_u(w) \in \mathbf{Z}$ for all $w \in W$.

4.4. **Partial Springer Fibers.** For all $J \subseteq S$, the *symmetrizer* and *antisymmetrizer* in $\mathbf{Q}W_J$ are respectively defined by

$$e_{J,+} = \frac{1}{|W_J|} \sum_{w \in W_J} w \quad \text{and} \quad e_{J,-} = \frac{1}{|W_J|} \sum_{w \in W_J} (-1)^{\ell(w)} w.$$

These are central elements of $\mathbf{Q}W_J$, such that $\mathbf{Q}W_J e_{J,+}$ and $\mathbf{Q}W_J e_{J,-}$ respectively afford the trivial and sign representations of W_J.

Borho–MacPherson related $e_{J,-}$ and $e_{J,+}$ to the *partial Springer fibers*

$$\mathbf{Spr}_{J,u}^- = \{y\mathbf{P}_J \in \mathbf{G}/\mathbf{P}_J \mid u \in y\mathbf{U}_J y^{-1}\},$$
$$\mathbf{Spr}_{J,u}^+ = \{y\mathbf{P}_J \in \mathbf{G}/\mathbf{P}_J \mid u \in y\mathbf{V}_J y^{-1}\}.$$

By §2.4, the set of F-fixed points $Spr_{J,u}^-$, *resp.* $Spr_{J,u}^+$, is the set of $yP_J \in G/P_J$ such that $u \in yU_J y^{-1}$, *resp.* $u \in yV_J y^{-1}$. For our choice of Springer action, the main result of [BM83] implies that for all $J \subseteq S$ and $u \in V$, we have

$$\tag{4.1} \begin{aligned} \frac{1}{|U_{w_{J\circ}}^-|}\,\chi_u(e_{J,-}) &= |Spr_{J,u}^-|,\\ \chi_u(e_{J,+}) &= |Spr_{J,u}^+|. \end{aligned}$$

More precisely, these results come from transferring Borho–MacPherson's arguments from sheaves in the analytic topology over $\mathbf{C}$ to sheaves in the étale topology over $\bar{\mathbf{F}}$, and keeping track of Tate twists arisng from the $\mathbf{F}$-structure. The factor of $|U_{w_{J\circ}}^-| = q^{\dim(\mathbf{L}_J/\mathbf{B}_J)}$ in the $-$ case arises from a Tate twist of order $2\dim(\mathbf{L}_J/\mathbf{B}_J)$ that accompanies the cohomological shift in case (b) of [BM83, §3.4].

4.5. **The Bitrace.** As in §1.3, let $O(w)_u$ be the subset of $O(w)$ of pairs taking the form (hB, uhB). Let $\tau_G : \mathbf{Q}W \otimes H_B^G \to \mathbf{Q}$ be defined by

$$\tau_G(z \otimes \delta_w) = \frac{1}{|G|} \sum_{u \in V} |O(w)_u| \chi_u(z).$$

The framework of [Tri21] shows that this is, indeed, a bitrace, meaning $\tau_G(z \otimes -)$ and $\tau_G(- \otimes \delta_w)$ are traces for all $z, w \in W$. In the split case, it recovers the $\boldsymbol{q} \to q$ specialization of the trace denoted τ_G in the introduction.

Lemma 4.2. *For all $J \subseteq S$ and $w \in W$, we have*

$$\begin{aligned} \frac{1}{|U_{w_{J\circ}}^-|}\,\tau_G(e_{J,-} \otimes \delta_w) &= \frac{1}{|G|} \sum_{(hB,gB) \in O(w)} f_! \delta_{E_J^-}(hB, gB),\\ \tau_G(e_{J,+} \otimes \delta_w) &= \frac{1}{|G|} \sum_{(hB,gB) \in O(w)} f_! \delta_{E_J^+}(hB, gB), \end{aligned}$$

where $E_J^{\pm}$ and f are defined as in Section 3.

Proof. Apply (4.1) to the formula for τ_G. Then observe that

$$\begin{aligned} \coprod_{u \in V} O(w)_u \times Spr_{J,u}^{\pm} &= \{(hB, u, yP_J) \in E_J^{\pm} \mid (hB, uhB) \in O(w)\}\\ &= \coprod_{(hB,gB) \in O(w)} f^{-1}(hB, gB). \end{aligned}$$ □

The split case of the following result is the $\boldsymbol{q} \to q$ specialization of Theorem 1.2. Since it amounts to a family of identities of Laurent polynomials in q, and it holds for infinitely many q, we can lift it to a result for $\boldsymbol{q}$.

Theorem 4.3. *For any $J \subseteq S$, we have*

$$\begin{aligned} \tau[N_J^S(1)] &= |T|\,\tau_G(e_{J,-} \otimes -),\\ \tau[N_J^S(\delta_{w_{J\circ}}^2)] &= |B_J|\,\tau_G(e_{J,+} \otimes -) \end{aligned}$$

as traces on H_W.

Proof. Combine Lemmas 4.1–4.2 with (3.2), noting that 1 and $\delta^2_{w_{J\circ}}$ are invariant under ι. Doing so gives

$$\frac{1}{|B|}\,\tau[N_J^S(1)] = \frac{1}{|U_J||U^-_{w_{J\circ}}|}\,\tau_G(e_{J,-}\otimes -) = \frac{1}{|U|}\,\tau_G(e_{J,-}\otimes -),$$
$$\frac{1}{|B|}\,\tau[N_J^S(\delta^2_{w_{J\circ}})] = \frac{1}{|U_J|}\,\tau_G(e_{J,+}\otimes -).$$

Then recall that $B = T \ltimes U = B_J \ltimes U_J$. □

4.6. **Recovering Lascoux–Wan–Wang.** In this subsection, we assume that $\mathbf{G} = \mathbf{GL}_n$ and F is the standard Frobenius that raises each matrix coordinate to its qth power. Then $G = \mathrm{GL}_n(\mathbf{F})$ and $W = \mathbf{W} = S_n$.

As in §1.3, we take $S = \{s_1, \ldots, s_{n-1}\}$, where $s_i \in S_n$ is the transposition swapping i and $i+1$. We will use the bijection between integer compositions of n and subsets of S that matches $\nu = (\nu_1, \nu_2, \ldots) \vdash n$ with

$$J = S \setminus \{s_{\nu_1}, s_{\nu_1+\nu_2}, \ldots\}$$

For this J, we find that $W_J \subseteq W$ is the *Young subgroup* $S_\nu \simeq S_{\nu_1} \times S_{\nu_2} \times \ldots$

Proposition 4.4. *If the subset J corresponds to the integer composition ν, then*

$$ch_{\boldsymbol{q}}(\tau_G(e_{J,-}\otimes -)) = e_\nu\left(\frac{X}{\boldsymbol{q}-1}\right),$$
$$ch_{\boldsymbol{q}}(\tau_G(e_{J,+}\otimes -)) = h_\nu\left(\frac{X}{\boldsymbol{q}-1}\right).$$

Thus, Theorem 1.2 (hence also Theorem 4.3) recovers (1.2).

As preparation, let $R(S_n)$ be the vector space of $\mathbf{Q}(\boldsymbol{q})$-valued traces on $\mathbf{Q}S_n$. For any partition $\lambda \vdash n$, let $\chi^\lambda \in R(S_n)$ be the irreducible character indexed by λ. Let $ch : R(S_n) \xrightarrow{\sim} \Lambda_n$ be the *(undeformed) Frobenius characteristic* isomorphism that sends χ^λ to the Schur function $s_\lambda(X)$. It sends the multiplicity pairing on $R(S_n)$ to the Hall pairing $\langle -, - \rangle$ on Λ_n.

Proof. In [Tri21], the first author gave a character-theoretic formula for the trace on H_W corresponding to the bitrace τ_G. For $G = \mathrm{GL}_n(\mathbf{F})$, it specializes to the formula

$$\tau_G(z \otimes -) = \sum_{\lambda \vdash n} \frac{\chi^\lambda(z)}{\det(\boldsymbol{q} - z \mid \mathsf{perm})} \cdot \chi^\lambda_{\boldsymbol{q}}, \tag{4.2}$$

where perm denotes the permutation representation of S_n on $\mathbf{Q}^n$. Above, recall that $\chi^\lambda_{\boldsymbol{q}} \in R(H_{S_n})$ corresponds to χ^λ under Tits deformation [GP00, Ch. 7].

The map ch sends $\chi^\lambda/\det(\boldsymbol{q} - (-) \mid \mathsf{perm})$ to a plethystically transformed Schur:

$$ch\left(\frac{\chi^\lambda}{\det(\boldsymbol{q} - (-) \mid \mathsf{perm})}\right) = s_\lambda\left(\frac{X}{\boldsymbol{q}-1}\right).$$

At the same time, since $W_J = S_\lambda$, it sends the induced character of $W = S_n$ arising from the trivial, *resp.* sign, character of W_J to the symmetric function $h_\nu(X)$, *resp.* $e_\nu(X)$. Using Frobenius reciprocity, we get

$$\begin{aligned}\frac{\chi^\lambda(e_{J,+})}{\det(\boldsymbol{q} - e_{J,+} \mid \mathsf{perm})} &= \left\langle h_\nu(X), s_\lambda\left(\frac{X}{\boldsymbol{q}-1}\right)\right\rangle \\ &= \left\langle h_\nu\left(\frac{X}{\boldsymbol{q}-1}\right), s_\lambda(X)\right\rangle,\end{aligned}$$

and similarly with $e_{J,-}$, e_ν in place of $e_{J,+}$, h_ν. So we arrive at

$$\begin{aligned}ch_{\boldsymbol{q}}(\tau_G(e_{J,+} \otimes -)) &= \sum_{\lambda \vdash n} \left\langle h_\nu\left(\frac{X}{\boldsymbol{q}-1}\right), s_\lambda(X)\right\rangle s_\lambda(X) \\ &= h_\nu\left(\frac{X}{\boldsymbol{q}-1}\right),\end{aligned}$$

and similarly with $e_{J,-}$, e_ν in place of $e_{J,+}$, h_ν. □

Remark 4.5. The generalization of (4.2) beyond $W = S_n$ involves the *exotic Fourier transform*: a certain pairing introduced by Lusztig to relate irreducible characters of W to unipotent irreducible characters of G. This pairing remains fairly mysterious; notably, its definition in [Lus84] is case by case. For this reason, it seems that Theorem 4.3 is the closest we can get to a uniform generalization of Proposition 4.4 to Weyl groups.

5. Braid Varieties and Cell Decompositions

5.1. *For the rest of the paper, we assume that G is split.* In this section, we prove Theorem 1.3, relating partial braid Steinberg varieties to the cell decompositions of open braid Richardson varieties. We actually prove a stratum-wise refinement, parallel to Theorem 3.1.

We will freely use the terminology from Coxeter combinatorics that we reviewed in §1.4. Throughout, we fix a word $\vec{s} = (s^{(1)}, \ldots, s^{(\ell)})$ in S.

5.2. **Richardson Varieties.** Recall that for any $v \in W$, we defined the *v-twisted open Richardson variety* of $\vec{s}$ on points by

$$\mathbf{R}^{(v)}(\vec{s}) = \{\vec{g}\mathbf{B} \in \mathbf{O}(\vec{s}) \mid g_0\mathbf{B} = vw_\circ\mathbf{B} \text{ and } \mathbf{B} \xrightarrow{vw_\circ} g_\ell\mathbf{B}\}.$$

Below, we give further detail about the cell decomposition mentioned in §1.4. For any v-distinguished subword $\vec{\omega}$ of $\vec{s}$, let $\mathbf{R}^{(v)}(\vec{s})_{\vec{\omega}} \subseteq \mathbf{R}^{(v)}(\vec{s})$ be the subvariety

$$\mathbf{R}^{(v)}(\vec{s})_{\vec{\omega}} = \{\vec{g}\mathbf{B} \in \mathbf{R}^{(v)}(\vec{s}) \mid \mathbf{B} \xrightarrow{v\omega_{(i)}w_\circ} g_i\mathbf{B}\}.$$

As before, let $\mathcal{D}^{(v)}(\vec{s})$ be the set of v-distinguished subwords $\vec{\omega}$ of $\vec{s}$ such that $\omega_{(\ell)} = e$. For any $\vec{\omega} \in \mathcal{D}^{(v)}(\vec{s})$, let

$$\begin{aligned}\mathbf{d}_{\vec{\omega}} &= \{i \mid v\omega_{(i)} < v\omega_{(i-1)}\}, \\ \mathbf{e}_{\vec{\omega}} &= \{i \mid \omega^{(i)} = e\},\end{aligned}$$

The main results of [Deo85] show that for any word $\vec{s}$ in S:

(1) $\mathbf{R}^{(v)}(\vec{s})_{\vec{\omega}}$ is nonempty if and only if $\omega \in \mathcal{D}^{(v)}(\vec{s})$. In this case,

$$\mathbf{R}^{(v)}(\vec{s})_{\vec{\omega}} \simeq \left\{ \vec{t} \in \mathbf{A}^\ell \;\middle|\; \begin{array}{ll} t_i \neq 0 & \text{for } i \in \mathbf{e}_{\vec{\omega}}, \\ t_i = 0 & \text{for } i \notin \mathbf{d}_{\vec{\omega}} \cup \mathbf{e}_{\vec{\omega}} \end{array} \right\} \tag{5.1}$$

from which $R^{(v)}(\vec{s})_{\vec{\omega}} := \mathbf{R}^{(v)}(\vec{s})^F_{\vec{\omega}}$ satisfies

$$|R^{(v)}(\vec{s})_{\vec{\omega}}| = q^{|\mathbf{d}_{\vec{\omega}}|}(q-1)^{|\mathbf{e}_{\vec{\omega}}|}.$$

(2) The subvarieties $\mathbf{R}^{(v)}(\vec{s})_{\vec{\omega}}$ are pairwise disjoint and partition $\mathbf{R}^{(v)}(\vec{s})$ as we run over $\vec{\omega} \in \mathcal{D}^{(v)}(\vec{s})$.

In light of (5.1), the varieties $\mathbf{R}^{(v)}(\vec{s})_{\vec{\omega}}$ are called *Deodhar cells*.

5.3. **Change of Structure Group.** To compare them to the geometry in previous sections, we need a more symmetrical version of the open Richardson varieties. Let $\mathbf{X}^{(v)}$, $\mathbf{X}^{(v)}_\square$, $\mathbf{R}^{(v)}$ be defined on points by

$$\begin{aligned} \mathbf{X}^{(v)} &= \{(h\mathbf{B}, x\mathbf{B}, g\mathbf{B}) \in (\mathbf{G}/\mathbf{B})^3 \mid h\mathbf{B} \xleftarrow{vw_\circ} x\mathbf{B} \xrightarrow{vw_\circ} g\mathbf{B}\} \\ &\simeq \mathbf{O}((vw_\circ)^{-1}, vw_\circ), \\ \mathbf{X}^{(v)}_\square &= \{(h\mathbf{B}, g\mathbf{B}) \in (\mathbf{G}/\mathbf{B})^2 \mid h\mathbf{B} \xleftarrow{vw_\circ} \mathbf{B} \xrightarrow{vw_\circ} g\mathbf{B}\}, \\ \mathbf{R}^{(v)} &= \{vw_\circ\mathbf{B}\} \times \mathbf{B}vw_\circ\mathbf{B}/\mathbf{B}. \end{aligned}$$

By construction, $\mathbf{R}^{(v)}(\vec{s})$ is the preimage of $\mathbf{R}^{(v)}$ along $\mathbf{O}(\vec{s}) \xrightarrow{pr_{0,\ell}} (\mathbf{G}/\mathbf{B})^2$. We will relate the varieties above to one another, thereby relating $\mathbf{R}^{(v)}(\vec{s})$ and its Deodhar cells to analogous varieties built from $\mathbf{X}^{(v)}$, $\mathbf{X}^{(v)}_\square$.

Observe that $\mathbf{X}^{(v)}$ is stable under the $\mathbf{G}$-action on $(\mathbf{G}/\mathbf{B})^3$. The action of $\mathbf{G}$ on $\mathbf{X}^{(v)}$ restricts to an action of $\mathbf{B}$ on $\mathbf{X}^{(v)}_\square$, which in turn restricts to an action of

$$\mathbf{B}^-_v := \mathbf{B} \cap (vw_\circ)\mathbf{B}(vw_\circ)^{-1}$$

on $\mathbf{R}^{(v)}$. Note that $\mathbf{B} = \mathbf{B}^-_v\mathbf{U}_v = \mathbf{U}_v\mathbf{B}^-_v$ and $\mathbf{B}^-_v \cap \mathbf{U}_v = \{1\}$ by Lemma 2.1(2).

Proposition 5.1. *For any $v \in W$, let $\mathbf{B}$ act on $\mathbf{G} \times \mathbf{X}^{(v)}_\square$ from the left by*

$$b \cdot (x, h\mathbf{B}, g\mathbf{B}) = (xb^{-1}, bh\mathbf{B}, bg\mathbf{B}).$$

Then:

(1) *The map $(\mathbf{G} \times \mathbf{X}^{(v)}_\square)/\mathbf{B} \to \mathbf{X}^{(v)}$ that sends $[x, h\mathbf{B}, g\mathbf{B}] \mapsto (xh\mathbf{B}, x\mathbf{B}, xg\mathbf{B})$ is an isomorphism.*

(2) *The quotient $\mathbf{X}^{(v)}_\square/\mathbf{U}_v$ forms an algebraic variety. The composition of maps*

$$\mathbf{R}^{(v)} \to \mathbf{X}^{(v)}_\square \to \mathbf{X}^{(v)}_\square/\mathbf{U}_v$$

is an isomorphism.

Proof. (1): $\mathbf{X}_\square^{(v)}$ is the closed subvariety of $\mathbf{X}^{(v)}$ cut out by the condition $x\mathbf{B} = \mathbf{B}$. The $\mathbf{G}$-action on $\mathbf{X}^{(v)}$ is transitive on the coordinate $x\mathbf{B}$, and the stabilizer of the point $\mathbf{B}$ is itself.

(2): $\mathbf{R}^{(v)}$ is the closed subvariety of $\mathbf{X}_\square^{(v)}$ cut out by the condition $h\mathbf{B} = vw_\circ\mathbf{B}$. By Lemma 2.1(3), the $\mathbf{B}$-action on $\mathbf{X}_\square^{(v)}$ restricts to an action of $\mathbf{U}_{vw_\circ}^- = \mathbf{U}_v$ that is simply transitive on the coordinate $h\mathbf{B}$. □

As usual, let $X^{(v)}$, $X_\square^{(v)}$, $R^{(v)}$ denote the F-fixed point sets of $\mathbf{X}^{(v)}$, $\mathbf{X}_\square^{(v)}$, $\mathbf{R}^{(v)}$, respectively.

Corollary 5.2. *The maps $(G \times X_\square^{(v)})/B \to X^{(v)}$ and $R^{(v)} \to X_\square^{(v)}/U_v$ induced by the isomorphisms above are bijections.*

Proof. Immediate from Lang's theorem, since $\mathbf{B}$, *resp.* $\mathbf{U}_v$, is connected and acts freely on $\mathbf{G} \times \mathbf{X}_\square^{(v)}$, *resp.* $\mathbf{X}_\square^{(v)}$. □

Let $\mathbf{X}^{(v)}(\vec{s}) = \mathbf{O}(\vec{s}) \times_{(\mathbf{G}/\mathbf{B})^2} \mathbf{X}^{(v)}$ and $\mathbf{X}_\square^{(v)}(\vec{s}) = \mathbf{O}(\vec{s}) \times_{(\mathbf{G}/\mathbf{B})^2} \mathbf{X}_\square^{(v)}$, where the fiber products are formed with respect to the maps $pr_{0,\ell}$ on the left factors and the coordinate pairs $(h\mathbf{B}, g\mathbf{B})$ on the right factors. On points,

$$\begin{aligned}\mathbf{X}^{(v)}(\vec{s}) &= \{(\vec{g}\mathbf{B}, x\mathbf{B}) \in \mathbf{O}(\vec{s}) \times \mathbf{G}/\mathbf{B} \mid g_0\mathbf{B} \xleftarrow{vw_\circ} x\mathbf{B} \xrightarrow{vw_\circ} g_\ell\mathbf{B}\},\\ \mathbf{X}_\square^{(v)}(\vec{s}) &= \{\vec{g}\mathbf{B} \in \mathbf{O}(\vec{s}) \mid g_0\mathbf{B} \xleftarrow{vw_\circ} \mathbf{B} \xrightarrow{vw_\circ} g_\ell\mathbf{B}\}\end{aligned}$$

For any subword $\vec{\omega} \in \mathcal{D}^{(v)}(\vec{s})$, let

$$\begin{aligned}\mathbf{X}^{(v)}(\vec{s})_{\vec{\omega}} &= \{(\vec{g}\mathbf{B}, x\mathbf{B}) \in \mathbf{O}(\vec{s}) \times \mathbf{G}/\mathbf{B} \mid x\mathbf{B} \xrightarrow{v\omega_{(i)}w_\circ} g_i\mathbf{B}\},\\ \mathbf{X}_\square^{(v)}(\vec{s})_{\vec{\omega}} &= \{\vec{g}\mathbf{B} \in \mathbf{X}^{(v)}(\vec{s}) \mid \mathbf{B} \xrightarrow{v\omega_{(i)}w_\circ} g_i\mathbf{B}\}.\end{aligned}$$

Note that $\mathbf{X}^{(v)}(\vec{s})_{\vec{\omega}}$ is stable under the $\mathbf{G}$-action on $\mathbf{X}^{(v)}(\vec{s})$, as are $\mathbf{X}_\square^{(v)}(\vec{s})_{\vec{\omega}}$, *resp.* $\mathbf{R}^{(v)}(\vec{s})_{\vec{\omega}}$, under $\mathbf{B}$, *resp.* $\mathbf{B}_v^-$. Pulling back to $\mathbf{O}(\vec{s})$, we see:

Corollary 5.3. *For any word $\vec{s}$, the analogues of Proposition 5.1 and Corollary 5.2 hold with $\diamondsuit(\vec{s})$ in place of $\diamondsuit$ for each*

$$\diamondsuit \in \{\mathbf{X}^{(v)}, \mathbf{X}_\square^{(v)}, \mathbf{R}^{(v)}\}.$$

For any subword $\vec{\omega} \in \mathcal{D}^{(v)}(\vec{s})$, they hold with $\diamondsuit(\vec{s})_{\vec{\omega}}$ in place of each $\diamondsuit$.

5.4. **Steinberg Varieties.** Fix $J \subseteq S$. As in §1.4, we define the *partial Steinberg varieties* of $\vec{s}$ of type J by

$$\mathbf{Z}_J^\pm(\vec{s}) = \{(\vec{g}\mathbf{B}, u, y\mathbf{P}_J) \in \mathbf{O}(\vec{s}) \times \mathbf{Spr}_J^\pm \mid g_\ell\mathbf{B} = ug_0\mathbf{B}\}.$$

Note that $\mathbf{G}$ acts on $\mathbf{Z}_J^\pm(\vec{s})$ via its actions on $\mathbf{Spr}_J^\pm$ and $\mathbf{O}(\vec{s})$. As this action need not be free, it is not obvious from Lang's theorem whether $Z_J^\pm(\vec{s}) := \mathbf{Z}_J^\pm(\vec{s})^F$ yields $Z_J^\pm(\vec{s})/G = (\mathbf{Z}_J^\pm(\vec{s})/\mathbf{G})^F$.

The coordinate triple $(g\mathbf{B}, u, y\mathbf{P}_J)$ defines a map $\mathbf{Z}_J^{\pm}(\vec{s}) \to \mathbf{E}_J^{\pm}$. Pulling back the partition of $\mathbf{E}_J^{\pm}$ by subvarieties $\mathbf{E}_{J,v}^{\pm}$ in Section 3, we get a partition of $\mathbf{Z}_J^{\pm}(\vec{s})$ by $\mathbf{G}$-stable subvarieties

$$\mathbf{Z}_{J,v}^{\pm}(\vec{s}) = \{(\vec{g}\mathbf{B}, u, y\mathbf{P}_J) \in \mathbf{Z}_J^{\pm}(\vec{s}) \mid \mathbf{P}_J y^{-1} h\mathbf{B} = \mathbf{P}_J v\mathbf{B}\}$$

as $W_J v$ runs over $W_J\backslash W$. On F-fixed points, we get an analogous partition of $Z_J^{\pm}(\vec{s})$ by its G-stable subsets $Z_{J,v}^{\pm}(\vec{s}) := \mathbf{Z}_{J,v}^{\pm}(\vec{s})^F$.

Theorem 5.4. *For all $J \subseteq S$ and $v \in W^{J,-}$ we have*

$$|Z_{J,v}^{-}(\vec{s})| = |U_{w_{J\circ}v}||X^{(vw_\circ)}(\vec{s})|,$$
$$|Z_{J,v}^{+}(\vec{s})| = |U_{w_{J\circ}v}||X^{(w_{J\circ}vw_\circ)}(\vec{s})|.$$

Proof. Just from the definitions (and the involutivity of $w_\circ$), we see that

$$|X^{(vw_\circ)}(\vec{s})| = \sum_{\vec{g}B \in O(\vec{s})} (\delta_{v^{-1}} * \delta_v)(g_0B, g_\ell B), \tag{5.2}$$

$$|Z_{J,v}^{\pm}(\vec{s})| = \sum_{\vec{g}B \in O(\vec{s})} f_! \delta_{E_{J,v}^{\pm}}(g_0B, g_\ell B) \tag{5.3}$$

for any $v \in W$. Now apply Theorem 3.1. □

Since multiplication by $w_\circ$ or $w_{J\circ}$ swaps $W^{J,-}$ with $W^{J,+}$, the following result implies Theorem 1.3.

Corollary 5.5. *For all $J \subseteq S$ and $v \in W^{J,-}$ we have*

$$\frac{|Z_{J,v}^{-}(\vec{s})|}{|G|} = \frac{1}{q^{\ell_J}(q-1)^{\mathrm{rk}(G)}} \sum_{\vec{\omega} \in \mathcal{D}^{(vw_\circ)}(\vec{s})} q^{|\mathsf{d}_{\vec{\omega}}|}(q-1)^{|\mathsf{e}_{\vec{\omega}}|},$$
$$\frac{|Z_{J,v}^{+}(\vec{s})|}{|G|} = \frac{1}{(q-1)^{\mathrm{rk}(G)}} \sum_{\vec{\omega} \in \mathcal{D}^{(w_{J\circ}vw_\circ)}(\vec{s})} q^{|\mathsf{d}_{\vec{\omega}}|}(q-1)^{|\mathsf{e}_{\vec{\omega}}|}.$$

Proof. We only do the $-$ case, as the $+$ case is similar. Observe that

$$\frac{|U_{w_{J\circ}v}||X^{(vw_\circ)}(\vec{s})|}{|G|} = \frac{|U_{w_{J\circ}v}||U_{vw_\circ}||R^{(vw_\circ)}(\vec{s})|}{|B|} = \frac{|R^{(vw_\circ)}(\vec{s})|}{q^{\ell_J}(q-1)^{\mathrm{rk}(G)}}$$

by Corollary 5.3 and Lemma 2.4. Then apply Theorem 5.4 on the left and (1.3) on the right. □

Remark 5.6. Let $\delta_{\vec{s}} = \delta_{s^{(1)}} * \cdots * \delta_{s^{(\ell)}}$. Then (5.2) yields

$$\frac{|X^{(vw_\circ)}(\vec{s})|}{|G|} = \frac{1}{|B|}\tau(\delta_{\vec{s}} * \delta_{v^{-1}} * \delta_v), \tag{5.4}$$

while summing (5.3) over $W_J v \in W_J\backslash W$ yields

$$\frac{|Z_J^{\pm}(\vec{s})|}{|G|} = \tau_G(e_{J,\pm} \otimes \delta_{\vec{s}}). \tag{5.5}$$

For the purpose of proving Theorem 1.3, we do not actually need the comparison between τ_G and τ in Theorem 4.3. It is nonetheless useful to record the identity

$$|R^{vw_\circ}(\vec{s})| = q^{-\ell(v)}\tau(\delta_{\vec{s}} * \delta_{v^{-1}} * \delta_v) \tag{5.6}$$

that follows from (5.4), Corollary 5.3, and Lemma 2.4. This identity is a simpler version of Corollary 5.3 in [GLTW24].

Remark 5.7. Recall that in Section 3, we discussed a refinement of case (1) of Theorem 3.1 to the level of algebraic varieties. The corresponding refinement of the $-$ case of Theorem 5.4 says that, for $v \in W^{J,-}$:

(1) The isomorphism $\check{\mathbf{E}}_J^- \xrightarrow{\sim} \mathbf{E}_J^-$ lifts to an isomorphism $\check{\mathbf{Z}}_{J,v}^-(\vec{s}) \xrightarrow{\sim} \mathbf{Z}_{J,v}^-(\vec{s})$.

(2) The map $\check{f} : \check{\mathbf{E}}_J^- \to \mathbf{O}(v^{-1}, v) = \mathbf{X}^{(vw_\circ)}$ in §3.3 fits into a cartesian diagram

$$\begin{array}{ccc} \check{\mathbf{Z}}_{J,v}^-(\vec{s}) & \longrightarrow & \check{\mathbf{E}}_{J,v}^- \\ \downarrow & & \downarrow \\ \mathbf{X}^{(vw_\circ)}(\vec{s}) & \longrightarrow & \mathbf{X}^{(vw_\circ)} \end{array} \tag{5.7}$$

where the vertical arrows are smooth fiber bundles, with geometric fibers that form torsors over conjugates of $\mathbf{U}_{w_{J\circ}v}$.

In particular, the decomposition of $\mathbf{X}^{(vw_\circ)}(\vec{s})$ into subvarieties $\mathbf{X}^{(vw_\circ)}(\vec{s})_{\vec{\omega}}$ lifts to an analogous decomposition of $\check{\mathbf{Z}}_{J,v}^-(\vec{s})$.

Remark 5.8. For $J = S$, we see that $W^{J,-} = \{e\}$ and the vertical arrows in (5.7) are trivial, giving isomorphisms $\mathbf{E}_S^- \simeq \check{\mathbf{E}}_S^- \simeq \mathbf{G}/\mathbf{B}$ and $\mathbf{Z}_S^-(\vec{s}) \simeq \mathbf{X}^{(w_\circ)}(\vec{s})$.

When $G = \mathrm{PGL}_n(\mathbf{F})$, so that $W = S_n$, and β is the positive braid on n strands defined by $\vec{s}$, the stack denoted $\mathcal{M}(\beta^\circ)$ in [STZ17] is precisely $[\mathbf{X}^{(w_\circ)}(\vec{s})/\mathbf{G}]$. Their Proposition 6.31 gives a decomposition of another stack $\mathcal{M}(\beta^\succ)$ into substacks indexed by rulings of a Legendrian link $\beta^\succ$. At the same time,

$$\mathcal{M}(\beta^\succ) \simeq \mathcal{M}((\Delta\beta\Delta)^\circ) \simeq \mathcal{M}((\beta\Delta^2)^\circ),$$

where Δ is the *half-twist*: the minimal positive braid that lifts $w_\circ \in S_n$. (Note that $\mathcal{M}((\Delta\beta\Delta)^\circ)$ is also isomorphic to $[\mathbf{X}^{(e)}(\vec{s})/\mathbf{G}]$.)

In this way, the varieties in our work generalize the stacks $\mathcal{M}(\beta^\circ)$ and $\mathcal{M}(\beta^\succ)$ in [STZ17]. The Deodhar-type decompositions of $\mathbf{Z}_{J,v}^-(\vec{s})$ and $\mathbf{X}^{(v)}(\vec{s})$ seem to recover the ruling decomposition of $\mathcal{M}(\beta^\succ)$ in [STZ17]. The precise relation of our decomposition to theirs will be left to future work.

Remark 5.9. In §5.3, the passage from $\mathbf{X}^{(v)}$ to $\mathbf{X}_\square^{(v)}$ to $\mathbf{R}^{(v)}$ encoded a passage from $\mathbf{G}$-symmetry to $\mathbf{B}$-symmetry to $\mathbf{B}_v^-$-symmetry. Instead of relating the Steinberg varieties and their strata to the $\mathbf{B}$-varieties $\mathbf{X}^{(v)}$, we could have used $\mathbf{B}$-varieties

$$\mathbf{Z}_{J,\square}^-(\vec{s}) = \{(\vec{g}\mathbf{B}, u) \in \mathbf{O}(\vec{s}) \times \mathbf{U}_J \mid g_\ell\mathbf{B} = ug_0\mathbf{B}\},$$
$$\mathbf{Z}_{J,\square}^+(\vec{s}) = \{(\vec{g}\mathbf{B}, u) \in \mathbf{O}(\vec{s}) \times \mathbf{V}_J \mid g_\ell\mathbf{B} = ug_0\mathbf{B}\}$$

and corresponding strata $\check{\mathbf{Z}}^{\pm}_{J,\square,v}(\vec{s})$ cut out by conditions of the form $\mathbf{P}_J h\mathbf{B} = \mathbf{P}_J v\mathbf{B}$. This is the approach in our FPSAC 2025 abstract. In analogy to the last observation in Remark 5.7, the decomposition of $\mathbf{X}^{(vw_\circ)}_{\square}(\vec{s})$ into subvarieties $\mathbf{X}^{(vw_\circ)}_{\square}(\vec{s})_{\vec{\omega}}$ lifts to an analogous decomposition of $\check{\mathbf{Z}}^{-}_{J,\square,v}(\vec{s})$.

6. Parking Numbers

6.1. *For the rest of the paper, we assume not only that G is split, but also that W is irreducible with Coxeter number h.*

For any integer $k \geq 0$, let $[k]_{\boldsymbol{q}} = 1 + \boldsymbol{q} + \cdots + \boldsymbol{q}^{k-1}$. Keeping the notation of §1.5, we define the *rational parabolic $\boldsymbol{q}$-parking numbers* of (W, p, J) to be

$$\mathrm{Park}^{J,\pm}_{W,p}(\boldsymbol{q}) = \prod_i \frac{[p \pm e_i^J]_{\boldsymbol{q}}}{[d_i^J]_{\boldsymbol{q}}}.$$

Recall that a Coxeter word in S is a word $\vec{c}$ formed by placing the elements of S in any order. We write $\vec{c}^p$ for the concatenation of p copies of $\vec{c}$. In this section, we prove the following identity, which implies Corollary 1.4 in the $\boldsymbol{q} \to 1$ limit.

Theorem 6.1. *For any Coxeter word $\vec{c}$ and integer $p > 0$ coprime to h, we have*

$$\mathrm{Park}^{J,\pm}_{W,p}(\boldsymbol{q}) = \sum_{v \in W^{J,\mp}} \sum_{\vec{\omega} \in \mathcal{D}^{(v)}(\vec{c}^p)} \boldsymbol{q}^{|\mathsf{d}_{\vec{\omega}}|}(\boldsymbol{q}-1)^{|\mathsf{e}_{\vec{\omega}}|}.$$

(Note the sign flip.)

6.2. **From Products to Traces.** Let refl be the reflection representation of W, which can be defined over $\mathbf{Q}$ because W is a split Weyl group. There is a graded representation $\mathsf{L}_{p/h}$ of W whose graded character is given by

$$\sum_i \boldsymbol{q}^i \operatorname{tr}(w \mid \mathsf{L}^i_{p/h}) = \frac{\det(1 - \boldsymbol{q}^p w \mid \mathsf{refl})}{\det(1 - \boldsymbol{q} w \mid \mathsf{refl})} \quad \text{for all } w \in W.$$

In the spirit of [ARR15, ALW16], it could be called the *rational parking space* for (W, p). Explicitly, $\mathsf{L}_{p/h}$ is the representation of W underlying the simple spherical module of the rational Cherednik algebra of W at parameter p/h, equipped with a shift of the W-stable grading arising from the Euler element. We view the graded character of $\mathsf{L}_{p/h}$ as a $\mathbf{Q}[\boldsymbol{q}]$-valued trace on $\mathbf{Q}W$.

Proposition 6.2. *For any Coxeter word $\vec{c}$ and integer $p > 0$ coprime to h, we have*

$$\mathrm{Park}^{J,\pm}_{W,p}(\boldsymbol{q}) = \frac{\det(1 - \boldsymbol{q}^p e_{J,\pm} \mid \mathsf{refl})}{\det(1 - \boldsymbol{q} e_{J,\pm} \mid \mathsf{refl})}.$$

Proof. **blah**

blah

blah

blah □

6.3. **From Traces to Cells.** Let $T_{\vec{c}} = T_{s^{(1)}} \cdots T_{s^{(r)}} \in H_W$, where $r = |S|$ and $\vec{c} = (s^{(1)}, \ldots, s^{(r)})$. In [Tri21], the first author showed that the value at $T_{\vec{c}}$ of the trace on H_W corresponding to τ_G is the graded character of $\mathsf{L}_{p/h}$, up to shift. In our notation, this corresponds to the identity

$$\tau_G(w \otimes T_{\vec{c}}^p) = \frac{\det(1 - \boldsymbol{q}^p w \mid \mathsf{refl})}{\det(1 - \boldsymbol{q} w \mid \mathsf{refl})}. \tag{6.1}$$

Altogether, we find that

$$\begin{aligned}
\mathrm{Park}_{W,p}^{J,\pm}(\boldsymbol{q}) &= \frac{\det(1 - \boldsymbol{q}^p e_{J,\pm} \mid \mathsf{refl})}{\det(1 - \boldsymbol{q} e_{J,\pm} \mid \mathsf{refl})} && \text{by Proposition 6.2} \\
&= \tau_G(e_{J,\pm} \otimes T_{\vec{c}}^p) && \text{by (6.1)} \\
&= \frac{|Z_J^{\pm}(\vec{c}^p)|}{|G|} && \text{by (5.5).}
\end{aligned}$$

Applying Theorem 1.3 to the last expression, we get the $\boldsymbol{q} \to q$ specialization of Theorem 6.1. But like in the argument above Theorem 4.3, it holds for infinitely many q, so we can lift it to a result for $\boldsymbol{q}$.

7. Markov Traces and Kirkman Numbers

7.1. In this section, we prove (1.4), Theorem 1.6, and Corollary 1.7. Along the way, we review Markov traces, the HOMFLYPT polynomial, and rational Kirkman polynomials. Unless otherwise specified, we take $W = S_n$ and $S = \{s_1, \ldots, s_{n-1}\}$, as in §4.6.

7.2. **Markov Traces and HOMFLYPT.** As explained in [Jon87] (in a different normalization), there is a unique family of traces $\mu_n : H_{S_n} \to \mathbf{Q}(\boldsymbol{q}^{1/2})[a^{\pm 1}]$ such that:

(1) $\mu_1(1) = 1$.
(2) For all $\beta \in H_{S_{n-1}}$, we have

$$\mu_{n+1}(\beta T_{s_n}^{\pm 1}) = (-a^{-1}\boldsymbol{q}^{1/2})^{\pm 1}\, \mu_n(\beta).$$

In particular, $\mu_{n+1}(\beta) = \dfrac{a - a^{-1}}{\boldsymbol{q}^{1/2} - \boldsymbol{q}^{-1/2}}\, \mu_n(\beta)$, due to the quadratic relation on T_{s_n}.

These traces give rise to an isotopy invariant of (tame) topological links. Namely: Any topological braid on n strands β defines an element of H_{S_n}, which we again denote by β, via the map from the braid group to H_{S_n} that sends the positive simple twist of strands i and $i+1$ to the element $\boldsymbol{q}^{-1/2}T_{s_i}$. At the same time, closing up β by wrapping it around a solid torus, then embedding it into 3-space, defines a link $\hat{\beta}$ up to isotopy, called the *closure* of β. Ocneanu showed that if $e(\beta) \in \mathbf{Z}$ is the *writhe* of β, meaning its length with respect to positive simple twists, then

$$\mathbf{P}(\hat{\beta}) := (-a)^{e(\beta)}\mu_n(\beta) \in \mathbf{Q}(\boldsymbol{q}^{1/2})[a^{\pm 1}]$$

only depends on $\hat{\beta}$.

The Laurent polynomial $\mathbf{P}(\hat{\beta})$ is now called its *reduced HOMFLYPT polynomial*, after its discoverers. (The "O" stands for Ocneanu; the adjective "reduced" means that the normalization satisfies $\mathbf{P}(\text{unknot}) = 1$.) The traces μ_n are called *Markov traces*, as condition (2) in their definition corresponds to the so-called second Markov move on braids. For further details, see [Jon87].

In [Gom06], Y. Gomi introduced a uniform generalization of the traces μ_n to finite Coxeter groups W. In [WW15], Webster–Williamson gave a construction of Gomi's traces from weight filtrations on the cohomology of mixed sheaves. Building on their work, the main result of [Tri21] relates a categorification of Gomi's traces to a Springer-type action of W on the weight-filtered, $\mathbf{G}$-equivariant cohomology of the Steinberg varieties $\mathbf{Z}_\emptyset^-(\vec{s}) = \mathbf{Z}_\emptyset^+(\vec{s})$.

7.3. **Individual a-Degrees.** Induction on $|e(\beta)|$ shows that if $\beta \in H_{S_n}$ arises from a topological braid, then the only exponents of a that can occur in $\mu(\beta)$ are

$$-n+1, \quad -n+3, \quad \ldots, \quad n-1.$$

For $0 \le k \le n-1$, we define $\mu_n^{(k)} : H_{S_n} \to \mathbf{Q}(\boldsymbol{q}^{1/2})$ by

$$\mu_n^{(k)}(\beta) = \mathbf{Q}(\boldsymbol{q}^{1/2})\text{-coefficient of } a^{-n+1+2k} \text{ in } \mu_n(\beta).$$

By linearity, this is still a trace.

When G is (split) semisimple of type A_{n+1}, the formula for categorified traces in [Tri21] decategorifies to a formula relating $\mu_n^{(k)}$ to τ_G. To state it, let $e_{\Lambda^k} \in \mathbf{Q}S_n$ be the symmetrizer corresponding to $\Lambda^k(\mathsf{refl})$, the kth exterior power of the reflection representation. For any finite, irreducible Coxeter group W of rank $r = |S|$, such elements $e_{\Lambda^k} \in \mathbf{Q}W$ may be defined for $0 \le k \le r$ through the formal identity

$$\frac{1}{|W|}\sum_{w \in W} \det(1 + tw \mid \mathsf{refl}) = \sum_{k=0}^{r} t^k e_{\Lambda^k}. \tag{7.1}$$

Note that $e_{\Lambda^0} = e_{S,+}$ and $e_{\Lambda^{n-1}} = e_{S,-}$, in the notation of §4.4. For G (split) semisimple of type A_{n+1}, we have:

$$\mu_n^{(k)} = \textbf{blah}\, \tau_G(e_{\Lambda^k} \otimes -). \tag{7.2}$$

To summarize the proof: One starts from the analogue of (4.2) with G, refl in place of GL_n, perm, then rearranges terms using (7.1) to arrive at the character-theoretic formula for $\mu_n^{(k)}$ in [Gom06, §4.3].

Meanwhile, in [BT22], Bezrukavnikov–Tolmachov gave a formula that (in our normalization) relates $\mu_n^{(k)}$ to $\mu_n^{(n-1)}$. To state it, we need the *multiplicative Jucys–Murphy elements* $JM_k \in H_{S_n}$ defined by

$$JM_k = \boldsymbol{q}^{1-k} T_{s_{k-1}} \cdots T_{s_2} T_{s_1}^2 T_{s_2} \cdots T_{s_{k-1}} \quad \text{for } 1 \leq k \leq n.$$

Let $e_i(X_1, \ldots, X_{n-1})$ be the elementary symmetic polynomial of degree i in variables $X_1, \ldots, X_{n-1}$. Then [BT22, Cor. 6.1.1] is the identity

$$\mu_n^{(k)}(\beta) = \mu_n^{(n-1)}(\beta e_{n-1-k}(JM_1, \ldots, JM_{n-1})). \tag{7.3}$$

It turns out that $\mu_n^{(n-1)}$ is precisely the trace denoted τ in §1.3, as one can also deduce from (7.2) and Theorem 1.2. Therefore, identity (1.4) and Theorem 1.6 in §1.6 reduce to:

Proposition 7.1. *For all k, we have*

$$e_{n-1-k}(JM_1, \ldots, JM_{n-1}) = \sum_{\mathsf{Des}(v) = I_k} \boldsymbol{q}^{-\ell(v)} T_{v^{-1}} T_v,$$

where $I_k = \{s_1, \ldots, s_{n-1-k}\} \subseteq S$.

Proof. **blah** $\square$

Remark 7.2. Taking $k = 0$ above, we get the identity

$$JM_1 \cdots JM_{n-1} = \boldsymbol{q}^{-\ell_S} T_{w_\circ}^2.$$

Here, (7.3) says that the "lowest" and "highest" a-degrees of μ_n are related by the *full twist* $\Delta^2 := \boldsymbol{q}^{-\ell_S} T_{w_\circ}^2$: explicitly,

$$\mu_n^{(0)}(\beta) = \mu_n^{(n-1)}(\beta \Delta^2),$$

an identity originally discovered by Kálmán [Kál09]. Compare to Remark 5.8.

Remark 7.3. Jucys–Murphy elements were originally defined in the context of the group rings $\mathbf{Z}S_n$. One can show [IO05, (3)] that

$$\frac{JM_k - 1}{\boldsymbol{q} - 1} = \sum_{i=1}^{k-1} \boldsymbol{q}^{i-k} T_{s_{k-1}} \cdots T_{s_{i+1}} T_{s_i} T_{s_{i+1}} \cdots T_{s_{k-1}}.$$

At $\boldsymbol{q} \to 1$, the right-hand side specializes to the kth classical Jucys–Murphy element in $\mathbf{Z}S_n$. These elements generate a maximal commutative subalgebra of $\mathbf{Z}S_n$. Similarly, the JM_k generate a maximal commutative subalgebra of H_{S_n} [IO05, Prop. 1].

7.4. **Kirkman Numbers.** For any finite, irreducible Coxeter group W of rank r and Coxeter number h, and integer $p > 0$ coprime to h, we define the *rational Kirkman polynomials* of (W, p) to be

$$\mathrm{Kirk}_{W,p}^{(k)}(\boldsymbol{q}) = \frac{\det(1 - \boldsymbol{q}^p e_{\Lambda^k} \mid \mathsf{refl})}{\det(1 - \boldsymbol{q} e_{\Lambda^k} \mid \mathsf{refl})} \quad \text{for } 0 \leq k \leq r.$$

Equivalently, by (7.1),

$$\sum_{k=0}^{r} t^k \,\mathrm{Kirk}_{W,p}^{(k)}(\boldsymbol{q}) = \frac{1}{|W|} \sum_{w \in W} \frac{\det(1 + tw \mid \mathsf{refl}) \det(1 - \boldsymbol{q}^p w \mid \mathsf{refl})}{\det(1 - \boldsymbol{q} w \mid \mathsf{refl})}.$$

When $p = h + 1$, this definition recovers the *Kirkman polynomials* of W introduced in [ARR15, §9.2]. We define the *rational Kirkman numbers* of (W, p) by

$$\mathrm{Kirk}_{W,p}^{(k)} := \mathrm{Kirk}_{W,p}^{(k)}(1)$$

Now the following identity implies Corollary 1.7 in the $\boldsymbol{q} \to 1$ limit:

Theorem 7.4. *Take $W = S_n$ and $S = \{s_1, \ldots, s_{n-1}\}$. Then for any Coxeter word $\vec{c}$, any integer $p > 0$ coprime to n, and any k, we have*

$$\mathrm{Kirk}_{W,p}^{(k)}(\boldsymbol{q}) = \sum_{\mathsf{Asc}(v) = I_k} \sum_{\vec{\omega} \in \mathcal{D}^{(v)}(\vec{c}^p)} \boldsymbol{q}^{|\mathsf{d}_{\vec{\omega}}|} (\boldsymbol{q} - 1)^{|\mathsf{e}_{\vec{\omega}}|}.$$

Proof. Observe that

$$\begin{aligned}
\mathrm{Kirk}_{W,p}^{(k)}(\boldsymbol{q}) &= \tau_G(e_{\Lambda^k} \otimes T_{\vec{c}}^p) && \text{by (6.1)} \\
&= \frac{1}{(\boldsymbol{q} - 1)^{n-1}} \,\tau[\zeta_{I_k}^-](T_{\vec{c}}^p) && \text{by Theorem 1.6} \\
&= \frac{1}{(\boldsymbol{q} - 1)^{n-1}} \sum_{\mathsf{Des}(v) = I_k} \boldsymbol{q}^{-\ell(v)} \,\tau(T_{\vec{c}}^p T_{v^{-1}} T_v).
\end{aligned}$$

At $\boldsymbol{q} \to q$, and in the notation of Remark 5.6, we have

$$\begin{aligned}
\sum_{\mathsf{Des}(v) = I_k} q^{-\ell(v)} \,\tau(\delta_{\vec{c}}^p * \delta_{v^{-1}} * \delta_v) &= \sum_{\mathsf{Des}(v) = I_k} |R^{(vw_\circ)}(\vec{c}^p)| && \text{by (5.6)} \\
&= \sum_{\mathsf{Des}(v) = I_k} \sum_{\vec{\omega} \in \mathcal{D}^{(vw_\circ)}(\vec{c}^p)} q^{|\mathsf{d}_{\vec{\omega}}|} (q - 1)^{|\mathsf{e}_{\vec{\omega}}|} && \text{by Deodhar} \\
&= \sum_{\mathsf{Asc}(v) = I_k} \sum_{\vec{\omega} \in \mathcal{D}^{(v)}(\vec{c}^p)} q^{|\mathsf{d}_{\vec{\omega}}|} (q - 1)^{|\mathsf{e}_{\vec{\omega}}|},
\end{aligned}$$

where the last step is the fact that $\ell(sv) < \ell(v)$ if and only if $\ell(svw_\circ) > \ell(svw_\circ)$. This gives the $\boldsymbol{q} \to q$ specialization of the desired identity. But as in the arguments above Theorem 4.3 and at the end of §6.3, it holds for infinitely many q, so we can lift it from q to $\boldsymbol{q}$. □

References

[ALW16] Drew Armstrong, Nicholas A. Loehr, and Gregory S. Warrington. Rational parking functions and Catalan numbers. *Ann. Comb.*, 20(1):21–58, 2016. doi:10.1007/s00026-015-0293-6.

[ARR15] Drew Armstrong, Victor Reiner, and Brendon Rhoades. Parking spaces. *Adv. Math.*, 269:647–706, 2015. doi:10.1016/j.aim.2014.10.012.

[BM83] Walter Borho and Robert MacPherson. Partial resolutions of nilpotent varieties. In *Analysis and topology on singular spaces, II, III (Luminy, 1981)*, volume 101-102 of *Astérisque*, pages 23–74. Soc. Math. France, Paris, 1983.

[BMR98] M. Broué, G. Malle, and R. Rouquier. Complex reflection groups, braid groups, Hecke algebras. *J. Reine Angew. Math.*, 500:127–190, 1998.

[BT22] Roman Bezrukavnikov and Kostiantyn Tolmachov. Monodromic model for Khovanov-Rozansky homology. *J. Reine Angew. Math.*, 787:79–124, 2022. doi:10.1515/crelle-2022-0008.

[Car93] Roger W. Carter. *Finite groups of Lie type*. Wiley Classics Library. John Wiley & Sons, Ltd., Chichester, 1993. Conjugacy classes and complex characters, Reprint of the 1985 original, A Wiley-Interscience Publication.

[Car95] R. W. Carter. On the representation theory of the finite groups of Lie type over an algebraically closed field of characteristic 0 [MR1170353 (93j:20034)]. In *Algebra, IX*, volume 77 of *Encyclopaedia Math. Sci.*, pages 1–120, 235–239. Springer, Berlin, 1995. doi:10.1007/978-3-662-03235-0_1.

[Deo85] Vinay V. Deodhar. On some geometric aspects of Bruhat orderings. I. A finer decomposition of Bruhat cells. *Invent. Math.*, 79(3):499–511, 1985. doi:10.1007/BF01388520.

[GLTW24] Pavel Galashin, Thomas Lam, Minh-Tâm Trinh, and Nathan Williams. Rational noncrossing Coxeter-Catalan combinatorics. *Proc. Lond. Math. Soc. (3)*, 129(4):Paper No. e12643, 50, 2024. doi:10.1112/plms.12643.

[Gom06] Yasushi Gomi. The Markov traces and the Fourier transforms. *J. Algebra*, 303(2):566–591, 2006. doi:10.1016/j.jalgebra.2005.09.034.

[GP00] Meinolf Geck and Götz Pfeiffer. *Characters of finite Coxeter groups and Iwahori-Hecke algebras*, volume 21 of *London Mathematical Society Monographs. New Series*. The Clarendon Press, Oxford University Press, New York, 2000.

[IO05] A. P. Isaev and O. V. Ogievetsky. On representations of Hecke algebras. *Czechoslovak J. Phys.*, 55(11):1433–1441, 2005. doi:10.1007/s10582-006-0022-9.

[Jon87] V. F. R. Jones. Hecke algebra representations of braid groups and link polynomials. *Ann. of Math. (2)*, 126(2):335–388, 1987. doi:10.2307/1971403.

[Jon90] Lenny K. Jones. Centers of generic Hecke algebras. *Trans. Amer. Math. Soc.*, 317(1):361–392, 1990. doi:10.2307/2001467.

[Kál09] Tamás Kálmán. Meridian twisting of closed braids and the Homfly polynomial. *Math. Proc. Cambridge Philos. Soc.*, 146(3):649–660, 2009. doi:10.1017/S0305004108002016.

[Kaw75] Noriaki Kawanaka. Unipotent elements and characters of finite Chevalley groups. *Osaka Math. J.*, 12(2):523–554, 1975. URL: http://projecteuclid.org/euclid.ojm/1200757873.

[Las06] Alain Lascoux. The Hecke algebra and structure constants of the ring of symmetric polynomials, 2006. arXiv:0602379.

[Lus84] George Lusztig. *Characters of reductive groups over a finite field*, volume 107 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1984. doi:10.1515/9781400881772.

[Sho88] Toshiaki Shoji. Geometry of orbits and Springer correspondence. Number 168, pages 9, 61–140. 1988. Orbites unipotentes et représentations, I.

[STZ17] Vivek Shende, David Treumann, and Eric Zaslow. Legendrian knots and constructible sheaves. *Invent. Math.*, 207(3):1031–1133, 2017. doi:10.1007/s00222-016-0681-5.

[Tri21] Minh-Tâm Quang Trinh. From the Hecke category to the unipotent locus, 2021. v2. arXiv:2106.07444.

[Tri22] Minh-Tâm Quang Trinh. Unipotent elements and Kálmán–Serre duality, 2022. v2. arXiv:2210.09051.

[WW11] Ben Webster and Geordie Williamson. The geometry of Markov traces. *Duke Math. J.*, 160(2):401–419, 2011. doi:10.1215/00127094-1444268.

[WW15] Jinkui Wan and Weiqiang Wang. Frobenius map for the centers of Hecke algebras. *Trans. Amer. Math. Soc.*, 367(8):5507–5520, 2015. doi:10.1090/S0002-9947-2014-06211-9.

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