KIW 52 (Dec. 2025)

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$G = \mathrm{GL}_n$

$\mathcal{U} = \{\text{unipotent } n \times n \text{ matrices}\} \subseteq G$

Ex for $u \in \mathrm{GL}_2$, write $u - I = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$\begin{aligned} u \text{ is unipotent } &\iff u - I \text{ is nilpotent} \\ &\iff a + d = ad - bc = 0 \end{aligned}$$

so for $n = 2$, have $\mathcal{U} = \left\{ \begin{pmatrix} 1+a & b \\ c & 1-a \end{pmatrix} \,\middle|\, a^2 + bc = 0 \right\}$

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B upper-triangular subgroup of G

$U = B \cap \mathcal{U}$

Bruhat, Cartan, Chevalley understand G via

$$G = \bigsqcup_{w \in S_n} B\dot{w}B, \quad \text{where } \dot{w}\text{'s are permutation matrices}$$

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how to understand $\mathcal{U}$ via U?

Thm (Steinberg '65) $|\mathcal{U}(\mathbb{F}_q)| = |U(\mathbb{F}_q)|^2 \ (= q^{n(n-1)})$

Thm (Kawanaka '75)

$$|(\overbrace{\mathcal{U} \cap B\dot{w}B}^{\mathcal{U}_w})(\mathbb{F}_q)| = |(\overbrace{UU_- \cap B\dot{w}B}^{\mathcal{V}_w})(\mathbb{F}_q)| \quad \text{for any } w$$

where $U_- \subseteq B_-$ are lower-triangular

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<u>Ex</u> for any n, have $\mathcal{U}_{\mathrm{id}} = U = \mathcal{V}_{\mathrm{id}}$

<u>Ex</u> for $n = 3$ and $\dot{w} = \begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix}$,

$$\mathcal{U}_w \simeq U \times \{(a, b, c, d) \mid a, b \neq 0,\ (1 + \tfrac{1}{ab})^3 = \tfrac{cd}{ab}\}$$
$$\mathcal{V}_w \simeq U \times \{(a, b, c, d) \mid a, b \neq 0,\ 1 + ab = abcd\}$$

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$B = T \ltimes U$ and $B_- = T \ltimes U_-$, where T is diagonal

$$\begin{array}{lcll} T & \curvearrowright & \mathcal{U}, & t \cdot u = tut^{-1} \\ T & \curvearrowright & UU_-, & t \cdot xy = (txt^{-1})(tyt^{-1}) \end{array}$$

<u>Thm (T)</u>

$$\mathrm{gr}^{\mathsf{W}}_* \mathrm{H}^*_{c,T}(\mathcal{U}_w(\mathbb{C})) \simeq \mathrm{gr}^{\mathsf{W}}_* \mathrm{H}^*_{c,T}(\mathcal{V}_w(\mathbb{C})) \quad \text{for all } w$$

where $\mathsf{W}_{\leq *}$ is the weight filtration on $\mathrm{H}^*_{c,T}$
implies Thm (Kawanaka) via results of Katz

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a T-equivariant map:

$$\Phi \colon UU_- \to \mathcal{U} \quad \text{def by } \Phi(xy) = xyx^{-1}$$

<u>Conj (T)</u> Φ restricts to a homotopy equivalence

$$\mathcal{V}_w(\mathbb{C}) \to \mathcal{U}_w(\mathbb{C}) \quad \text{for any } w$$

would imply Thm (T)

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recall: $w \in S_n$ lifts to $\sigma_w \in Br_n^+$
for $\beta = \sigma_{w_1} \cdots \sigma_{w_k} \in Br_n^+$,

$$\begin{array}{rl} & M_\beta := B\dot{w}_1 B \times^B B\dot{w}_2 B \times^B \cdots \times^B B\dot{w}_k B \\ m_\beta & \downarrow \\ & G \end{array}$$

turns out: $\mathcal{U}_w = m_{\sigma_w}^{-1}(\mathcal{U})$ and $\mathcal{V}_w = m_{\sigma_w \sigma_{w_\circ}^2}^{-1}(1)$
where $\sigma_{w_\circ}^2$ is the full-twist braid

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HOMFLYPT poly $P\colon \{\text{links in } \mathbb{R}^3\} \to \mathbf{Z}[\![q]\!][a, q^{-1/2}]$
KhR superpoly $\mathbb{P} : \{\text{links in } \mathbb{R}^3\} \to \mathbf{Z}[\![q]\!][a, q^{-1/2}, t]$

recall that $\mathbb{P}|_{t \to -1} = P$

<u>Thm (Kálmán '09)</u> if $\beta \in Br_n$ has closure $\hat{\beta}$, then

$$P(\hat{\beta})[a^{e(\beta)-n+1}] = P(\widehat{\beta\sigma_{w_\circ}^2})[a^{e(\beta)+n-1}]$$

<u>Thm (GHMN '19)</u> true with $\mathbb{P}$ in place of P

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<u>Thm (T)</u> if $\beta \in Br_n^+$, then

$$|\mathcal{U}_\beta(\mathbb{F}_q)| = |X_{\beta\sigma_{w_\circ}^2}(\mathbb{F}_q)|,$$

$$\mathrm{gr}_*^{\mathsf{W}} \mathrm{H}_{c,T}^*(\mathcal{U}_\beta(\mathbb{C})) \simeq \mathrm{gr}_*^{\mathsf{W}} \mathrm{H}_{c,T}^*(X_{\beta\sigma_{w_\circ}^2}(\mathbb{C}))$$

where $\mathcal{U}_\beta = m_\beta^{-1}(\mathcal{U})$ and $X_\beta = m_\beta^{-1}(1)$

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$\mathcal{U}_\beta, X_\beta$ are pieces of a larger variety:
recall the Springer resolution

$$\tilde{\mathcal{U}} = \{(u, gB) \in \mathcal{U} \times G/B \mid u \in gBg^{-1}\} \to \mathcal{U}$$

pullback squares:

$$\begin{array}{ccccc} \tilde{\mathcal{U}}_\beta & \to & \mathcal{U}_\beta & \to & M_\beta \\ \downarrow & & \downarrow & & \downarrow \\ \tilde{\mathcal{U}} & \to & \mathcal{U} & \to & G \end{array}$$

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<u>Thm (≈ Springer)</u> if u has Jordan type λ, then

$$\mathrm{H}^*(\tilde{\mathcal{U}}_u) \simeq \mathrm{Ind}_{S_\lambda}^{S_n}(1)$$

$$\text{implies} \begin{cases} \mathrm{H}^*_{c,B}(\tilde{\mathcal{U}}_\beta)[\mathrm{triv}] = \mathrm{H}^*_{c,B}(\mathcal{U}_\beta) \\ \mathrm{H}^*_{c,B}(\tilde{\mathcal{U}}_\beta)[\mathrm{sgn}] = \mathrm{H}^*_{c,B}(\mathcal{X}_\beta) \end{cases}$$

<u>Thm (T)</u> if $\beta \in Br_n^+$, then

$$\mathbb{P}(\hat{\beta}) \propto \sum_{i,j,k} a^i q^{j/2} t^k \; \mathrm{gr}_j^{\mathsf{W}} \, \mathrm{H}^k_{c,B}(\tilde{\mathcal{U}}_\beta)[\textstyle\bigwedge^i(V)]$$

this + (Springer) + (GHMN) $\implies$ thm about $\mathcal{U}_w, \mathcal{V}_w$