



Zeta Functions as Knot Invariants

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[Rational noncrossing Coxeter–Catalan combinatorics.](#)
Proc. London Math. Soc. (2024), 50 pp.

+ Pavel Galashin, Thomas Lam, Nathan Williams

[The Hilb-vs-Quot conjecture.](#)
J. reine angew. Math. (Crelle) (2025), 44 pp.

+ Oscar Kivinen

[From the Hecke category to the unipotent locus.](#) 88 pp.

1 The Riemann Hypothesis

(Euler ~1730s) Proof of the infinitude of primes:

If there were finitely many, then we'd have

$$\prod_{\text{prime } p > 0} \left(1 + \frac{1}{p} + \frac{1}{p^2} + \cdots\right) = \sum_{n=1}^{\infty} \frac{1}{n}.$$

But the series diverges—a contradiction.

He also implicitly studied the [zeta function](#)

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

For $s > 1$, we have $\zeta(s) = \prod_p \left(1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \cdots\right).$

What if we allow s to be complex?

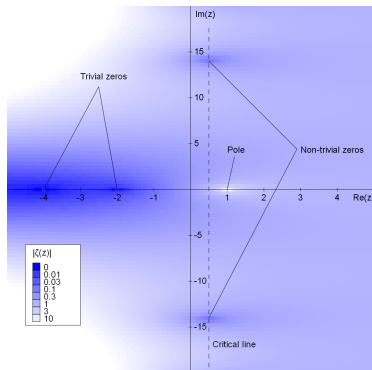
(Riemann 1859) A unique \mathbf{C} -valued function ζ that is

- *holomorphic* (complex-differentiable) when $s \neq 1$.
- given by $\sum_{n=1}^{\infty} \frac{1}{n^s}$ when $\operatorname{Re}(s) > 1$.

He checked that $\zeta(n) = 0$ for $n = -2, -4, -6, \dots$ by relating these zeros to poles of the *gamma function*.

He speculated from examples that all other zeros of ζ live on the *critical line* $\operatorname{Re}(s) = \frac{1}{2}$.

Location of zeros \leftrightarrow distribution of prime numbers.



[Wikipedia](#)

(Hardy 1914) Infinitely many zeros on the line.

(Pratt–Robles–Zaharescu–Zeindler 2020) Among zeros s with $0 < \operatorname{Re}(s) < 1$, over five-twelfths on the line.

(Dedekind ~1860s) Generalize the formula

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

by replacing \mathbf{Z} with other *rings* R .

Thus R is a set with operations $+$ and \cdot resembling addition and multiplication.

$$\zeta_R(s) = \sum_{\substack{\text{ideals } I \subseteq R \\ |R/I| < \infty}} \frac{1}{|R/I|^s}$$

An *ideal* I is the collection of all finite linear combos $c_1 \cdot x_1 + \cdots + c_k \cdot x_k$ for some fixed $x_1, x_2, \dots \in R$.

The *quotient* R/I is the set of translates $y + I \subseteq R$.

Note For ζ_R to make sense, the number of I such that $|R/I| = n$ must be finite for each $n > 0$.

Ex Every ideal of \mathbf{Z} takes the form

$$n\mathbf{Z} = \{\text{multiples of } n\} \quad \text{for some integer } n \geq 0.$$

For instance, the ideal generated by 30 and 2025 is

$$\{c_1 \cdot 30 + c_2 \cdot 2025 \mid c_1, c_2 \in \mathbf{Z}\} = 15\mathbf{Z}.$$

We see that

$$\zeta_{\mathbf{Z}}(s) = \sum_{\substack{n\mathbf{Z} \subseteq \mathbf{Z} \\ n > 0}} \frac{1}{|\mathbf{Z}/n\mathbf{Z}|^s} = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

In other rings, ideals may not simplify so nicely.

Why care?

(Hilbert–Polyá ~1910s) The values e^{it} such that

$$\zeta(\tfrac{1}{2} + it) = 0 \quad \text{and} \quad 0 < \operatorname{Re}(\tfrac{1}{2} + it) < 1$$

behave like the eigenvalues of a generic unitary matrix.

$$\text{RH} \iff t \text{ always real}$$

$$\iff e^{it} \text{ always on the unit circle.}$$

(Weil ~1940s) There is a class of rings R ,
coming from *algebraic geometry* over $\mathbf{Z}/p\mathbf{Z}$,
where analogous facts for ζ_R might be provable.

(Grothendieck–Deligne ~1960s–70s) Yes.

2 Weil's Rosetta Stone Algebraic geometry studies
varieties: shapes cut out by polynomial equations.

For simplicity, we'll stick to (*affine*) *hypersurfaces*

$$V_f = \{\vec{a} = (a_0, a_1, \dots, a_d) \mid f(\vec{a}) = 0\},$$

cut out by a single polynomial $f \in \mathbf{Z}[x_0, x_1, \dots, x_d]$.

V_f is *smooth* at $\vec{a} \bmod p$ when $\frac{\partial f}{\partial x_i}(\vec{a}) \not\equiv 0 \pmod{p}$
for some i . Else, *singular*.

Ex For $d = 1$, hypersurfaces are plane curves, like

$$f(x, y) = y^2 - x^3 - c \quad \text{for constant } c.$$

For which c is V_f smooth everywhere mod p ?

The *ring of polynomial functions* on $V_f \bmod p$ is

$$R_{f,p} := \frac{\mathbf{F}_p[x_0, \dots, x_d]}{\mathbf{F}_p[x_0, \dots, x_d] \cdot f}, \quad \text{where } \mathbf{F}_p = \mathbf{Z}/p\mathbf{Z}.$$

In a letter to his sister, Weil described a dictionary:

\mathbf{Z}	$R_{f,p}$	$V_f \bmod p$
$n\mathbf{Z}$	ideals	subvarieties
$p\mathbf{Z}$	maximal ideals	points

The first and last columns: a Rosetta stone between number theory and algebraic geometry.

Conj (Weil 1948) Assume V_f is smooth everywhere.

Then zeros of $\zeta_{R_{f,p}}(s)$ have $\operatorname{Re}(s) \in \{\frac{1}{2}, \frac{3}{2}, \dots, \frac{2d-1}{2}\}$.

Thm (Weil) True for $f = c_0 x_0^{n_0} + \dots + c_d x_d^{n_d} - c$.

Set $\zeta_{f,p}(s) := \zeta_{R_{f,p}}(s)$ for convenience.

(Grothendieck ~1964) $\zeta_{f,p}(s)$ is a rational function in

$$q := p^{-s}.$$

In fact: polynomials $\phi_0, \phi_1, \dots, \phi_{2d-1}$ such that

$$\zeta_{f,p}(s) = \frac{\phi_1(q) \cdot \phi_3(q) \cdots \phi_{2d-1}(q)}{\phi_0(q) \cdot \phi_2(q) \cdots \phi_{2d-2}(q)}.$$

ϕ_k is the charpoly of a certain operator on a certain vector space, describing the *étale topology* of V_f .

Conj For all k , the roots of $\phi_k(q)$ live on the circle

$$|q| = p^{k/2}.$$

\implies Weil's Riemann Hypothesis.

(Deligne 1974) True for all f (smooth mod p).

In fact, Weil conjectured—and Deligne proved—results for all varieties, not just hypersurfaces.

Ex Taking $d = 1$ and $f(x, y) = y^2 - x^3 - c$:

$$\phi_0(t) = 1 - p\mathbf{q}$$

$$\phi_1(t) = 1 - \textcolor{red}{a}_p \mathbf{q} + p\mathbf{q}^2 \quad \text{for some integer } \textcolor{red}{a}_p,$$

giving $\zeta_{f,p}(s) = \frac{1 - \textcolor{red}{a}_p p^{-s} + p^{1-2s}}{1 - p^{1-s}}$. It turns out:

- $|a_p| \leq 2p^{1/2}$.
- So the two roots of $\phi_1(\mathbf{q})$ satisfy $|\mathbf{q}| = p^{-1/2}$.
- So the zeros of $\zeta_{f,p}(s)$ satisfy $\text{Re}(s) = \frac{1}{2}$.

What if V_f has singularities?

Simplest case: $f(x, y)$ with unique singularity at $(0, 0)$.

It turns out that here,

$$\zeta_{f,p}(s) = \zeta_{f,p}^\circ(s) \cdot \hat{\zeta}_{f,p}(s),$$

where:

- $\zeta_{f,p}^\circ$ satisfies Weil's Riemann Hypothesis.
- $\hat{\zeta}_{f,p}$ is analogous to $\zeta_{f,p}$, with the power-series ring

$$\hat{R}_{f,p} := \frac{\mathbf{F}_p[[x, y]]}{\mathbf{F}_p[[x, y]] \cdot f}$$

in place of $R_{f,p}$.

Does $\hat{\zeta}_{f,p}(s) = \sum_{\substack{I \subseteq \hat{R}_{f,p} \\ |\hat{R}_{f,p}/I| < \infty}} \frac{1}{|\hat{R}_{f,p}/I|^s}$ satisfy a RH?

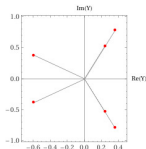
Ex For $f = y^2 - x^3$,

$$\hat{\zeta}_{f,p}(s) = \frac{1 - p^{1-2s}}{1 - p^{-s}} = \frac{1 + pq^2}{1 - q}.$$

Ex For $f = y^3 - x^4$,

$$\hat{\zeta}_{f,p}(s) = \frac{1 + pq^2 + p^2q^3 + p^2q^4 + p^3q^6}{1 - q}.$$

Here, not all roots satisfy $|q| = p^{-1/2}$.



WolframAlpha

3 From Curves to Knots For general $f(x, y)$,

it turns out there's $P_f(\mathbf{t}, \mathbf{q}) \in \mathbf{Z}\left[\mathbf{t}, \mathbf{q}, \frac{1}{1-\mathbf{q}}\right]$ such that

$$\hat{\zeta}_{f,p}(s) = \frac{P_f(p, p^{-s})}{1 - p^{-s}} \quad \text{for almost all } p.$$

These polynomials have many surprising features.

(Gorsky–Mazin 2013) If $f = y^n - x^{n+1}$,

then $P_f(1, 1) = \frac{(2n)!}{(n+1)!n!}$, the n th Catalan number.

Ex If $f = y^3 - x^4$, then

$$P_f(\mathbf{t}, \mathbf{q}) = 1 + \mathbf{t}\mathbf{q}^2 + \mathbf{t}^2\mathbf{q}^3 + \mathbf{t}^2\mathbf{q}^4 + \mathbf{t}^3\mathbf{q}^6,$$

$$P_f(1, 1) = 5.$$

The P_f also arise from *knot/link invariants*.

A *knot* is an embedding of a circle into \mathbf{R}^3 or S^3 .



A *link* is a generalization allowing multiple circles.



Two links are *isotopic* when we can deform one into the other without self-intersections.



Chmutov–Duzhin–Mostovoy

Let $S_\epsilon^3 = \{(x, y) \in \mathbf{C}^2 \mid |x|^2 + |y|^2 = \epsilon\}$. The subset

$$L_f = \{(x, y) \in S_\epsilon^3 \mid f(x, y) = 0\}$$

is a link in S_ϵ^3 when $\epsilon > 0$ is small enough.

Ex If $f = y^n - x^m$, then L_f is the (m, n) torus link. It's a knot when m and n are coprime.



Wolfram Language

Ex If $f = y^4 - 2x^3y^2 - 4x^5y + x^6 - x^7$, then L_f is the closure of the braid



Cherednik–Danilenko

Conj (Oblomkov–Shende ~2010)

$$P_f(1, q^2) = \lim_{a \rightarrow 0} \left[(q/a)^\mu \mathbb{P}_{L_f}(a, q) \right],$$

where $\mu \in \mathbf{Z}$ and \mathbb{P} is the *HOMFLYPT invariant*, discovered in 1986 and defined by the *skein rules*:

$$(1) \quad a \mathbb{P}_{\nearrow \searrow} - a^{-1} \mathbb{P}_{\nwarrow \swarrow} = (q - q^{-1}) \mathbb{P}_{\smile \frown}$$

$$(2) \quad \mathbb{P}_{\bigcirc} = 1$$

Full statement incorporates a , by upgrading P_f .

(Maulik 2012) True for all plane curves.

Proof sketch Blow up the singularity repeatedly.

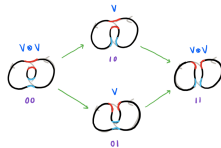
Control P_f via *wall crossing* and L_f via skein algebra.

Conj (Oblomkov–Rasmussen–Shende ~2013)

$$P_f(t^2, q^2) = \lim_{a \rightarrow 0} \left[(q/a)^\mu \mathbf{P}_{L_f}(a, t, q) \right],$$

where \mathbf{P} is a refinement of \mathbb{P} , discovered in the 2000s by Khovanov–Rozansky.

\mathbf{P} is defined by *categorifying* (1)–(2). Unknown how to categorify Maulik’s proof.



Melissa Zhang

(Kivinen–T 2025) True for $f = y^3 - x^m$ with $3 \nmid m$.

Cor (Kivinen–T) New closed formula for $\mathbf{P}_{\text{torus}(m,3)}$.

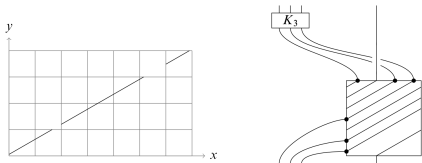
Proof Sketch

1 Recursions that compute $\mathbf{P}_{\text{torus}(m,n)}(\mathbf{a}, \mathbf{t}, \mathbf{q})$, due to Elias–Hogancamp–Mellit.

$$\begin{array}{c} \text{Diagram with } K_{n-1} \end{array} \simeq \left((TQ^{-1})^{1-n} \begin{array}{c} \text{Diagram with } K_n \end{array} \rightarrow Q^2 \begin{array}{c} \text{Diagram with } K_{n-1} \end{array} \right)$$

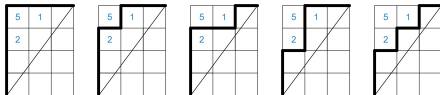
Elias–Hogancamp

Encoding of braids into grid diagrams, due to Mellit.



Mellit

2 For m, n coprime, yields a sum over $\frac{(m+n-1)!}{m!n!}$ many *rational Dyck paths*.



At the same time, $\hat{R}_{f,p} \simeq \mathbf{F}_p[[u^m, u^n]]$.

We relate Dyck paths to $\hat{R}_{f,p}$ -submodules $M \subseteq \mathbf{F}_p[[u]]$.

3 We then relate

$$\sum_M \frac{1}{|\mathbf{F}_p[[u]]/M|^s} \quad \text{and} \quad \sum_I \frac{1}{|\hat{R}_{f,p}/I|^s}$$

Uses *Serre duality*. For now, requires $\min(m, n) \leq 3$.

Big Picture I study special functions that appear in

- *algebraic geometry*
- *knot theory*
- *combinatorics*

I use *representation theory* to decompose them into simpler functions, like in Fourier analysis.

Hikita (2016) Dyck-path formula \longleftrightarrow

decomposition of *diagonal harmonics* into *LLT polys.*

T (2021) Generalizations of \mathbb{P} , \mathbf{P} to Coxeter groups, explicitly decomposing into *irreducible characters*.

Galashin–Lam–T–Williams (2024) Ideas from T (2021) solve conjectures about Coxeter groups from 2012.

4 Cherednik's New Hypothesis

Recall: For $f = y^3 - x^4$ and prime p , the roots of

$$P_f(p, \mathbf{q}) = 1 + p\mathbf{q}^2 + p^2\mathbf{q}^3 + p^2\mathbf{q}^4 + p^3\mathbf{q}^6$$

do not all satisfy $|\mathbf{q}| = p^{-1/2}$.

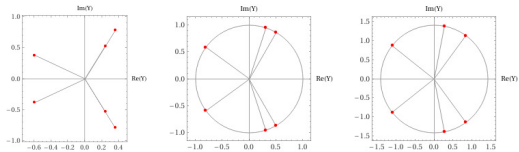
Conj (Cherednik 2018) For any plane curve f :

$$0 < t \leq \frac{1}{2} \implies \begin{array}{l} \text{all roots of } P_f(t, \mathbf{q}) \text{ satisfy} \\ |\mathbf{q}| = t^{-1/2}. \end{array}$$

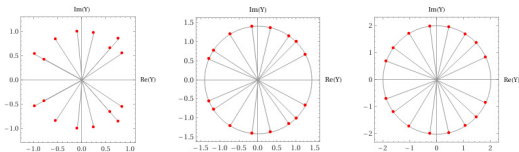
Would imply *arithmetic* constraints on $\mathbf{P}_{L_f}(\mathbf{a}, \mathbf{t}, \mathbf{q})$.

Dream (Cherednik) Related to the Lee–Yang theorem on zeros of *partition functions* in statistical mechanics.

$$f = y^3 - x^4 \quad t \in \{2, 1, \frac{1}{2}\}:$$



$$f = y^4 - 2x^3y^2 - 4x^5y + x^6 - x^7 \quad t \in \{1, \frac{1}{2}, \frac{1}{4}\}:$$



Thank you for listening.