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NOTES ON ABSOLUTE HODGE COHOMOLOGY

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INTRODUCTION. First a few words about the situation in étale cohomology to motivate what follows. Let $\pi: X \to \operatorname{Spec} K$ be a scheme over a field K, and let $\mathcal{F} \in D^b(X_{\acute{e}t})$ be a complex of sheaves on $X_{\acute{e}t}$; put $\underline{R\Gamma}(X,\mathcal{F}) := R\pi_*(\mathcal{F}) \in D^b((\operatorname{Spec} K)_{\acute{e}t})$. We have

$$\begin{aligned} R\Gamma(X_{\acute{e}t},\mathcal{F}) &= R\Gamma((\operatorname{Spec} K)_{\acute{e}t},\underline{R\Gamma}(X,\mathcal{F})) \\ &= R\operatorname{Hom}_{D^b((\operatorname{Spec} K)_{\acute{e}t})}(\mathbb{Z},\underline{R\Gamma}(X,\mathcal{F})). \end{aligned}$$

If $\overline{K}/K$ is a separable closure of K and $G = \operatorname{Gal} \overline{K}/K$, then the sheaves on $(\operatorname{Spec} K)_{\acute{e}t}$ are G-modules, and $\underline{R\Gamma}(X,\mathcal{F}) = R\Gamma((X \otimes_K \overline{K})_{\acute{e}t},\mathcal{F})$ is the geometric étale cochain complex of X with canonical G-action.

0.1. Now suppose that $K = \mathbb{C}$. Then, following Deligne [4] the role of sheaves on "arithmetic" Spec $\mathbb{C}$ should be played by (mixed) Hodge structures. This analogy suggests that for any scheme X there should be a canonical object $\underline{R\Gamma}(X,\mathbb{Z}) \in D^b(\mathcal{H})$ (where $\mathcal{H}$ = category of Hodge structures), whose underlying complex of abelian groups is the usual chain complex of topological space $X(\mathbb{C})$. We will see that this is indeed the case: The basic construction of Deligne [4] plus a bit of homological algebra do the job. For $i \in \mathbb{Z}$ define the absolute Hodge cochain complex of X with coefficients in $\mathbb{Z}(i)$

$$R\Gamma_{\mathcal{H}}(X,\mathbb{Z}(i)) := R\operatorname{Hom}_{D^b(\mathcal{H})}(\mathbb{Z},\underline{R\Gamma}(X,\mathbb{Z})(i)) .$$

Here (i) on the right-hand side means Tate twist in $D^b(\mathcal{H})$. The absolute Hodge (or simply $\mathcal{H}$-) cohomology groups $H^{\cdot}_{\mathcal{H}}(X,\mathbb{Z}(*)) = H^{\cdot}(R\Gamma_{\mathcal{H}}(X,\mathbb{Z}(*)))$ for m a twisted Poincaré duality theory in the sense of [3]. They may be easily computed in terms of Deligne-Hodge structure in $H^{\cdot}(X)$, e.g. we have canonical exact sequence

$$0 \to W_{2i}H^{j-1}(X,\mathbb{C})/[(2\pi\sqrt{1})^i W_{2i}H^{j-1}(X,\mathbb{Q})+(F^i \cap W_{2i})H^{j-1}(X,\mathbb{C})]$$

$$\xrightarrow{\alpha} H^j_{\mathcal{H}}(X,\mathbb{Z}(i))\otimes\mathbb{Q} \xrightarrow{\epsilon} (2\pi\sqrt{-1})^i W_{2i}H^j(X,\mathbb{Q}) \cap F^iH^j(X,\mathbb{C}) \to 0;$$

the arrow ϵ comes from the obvious map $H^{\cdot}_{\mathcal{H}}(X,\mathbb{Z}(*)) \xrightarrow{\epsilon} H^{\cdot}(X(\mathbb{C}),\mathbb{Z}(*))$ of cohomology theories. In particular, in contrast to the étale situation, the kernel of $H^{\cdot}_{\mathcal{H}}(X)\otimes\mathbb{Q} \to H^{\cdot}(X(\mathbb{C}))\otimes\mathbb{Q}$ is far from trivial: for $i > j$ this arrow is zero, and $H^j_{\mathcal{H}}(X,\mathbb{Z}(i))\otimes\mathbb{Q} = H^{j-1}(X(\mathbb{C}),\mathbb{C}/(2\pi\sqrt{-1})^i\mathbb{Q})$. This plays the crucial role in what follows.

Finally, we may define $\mathcal{H}$-groups with coefficients other than $\mathbb{Z}$ (e.g. with $\mathbb{Q}$- or $\mathbb{R}$-ones) replacing the category $\mathcal{H} = \mathcal{H}_{\mathbb{Z}}$ above of $\mathbb{Z}$-Hodge structures by $\mathcal{H}_{\mathbb{Q}}$, $\mathcal{H}_{\mathbb{R}}$ of $\mathbb{Q}$- or $\mathbb{R}$-ones (note that $H^{\cdot}_{\mathcal{H}}(X,\mathbb{Q}) = H^{\cdot}_{\mathcal{H}}(X,\mathbb{Z}(*))\otimes\mathbb{Q}$, but $H^{\cdot}_{\mathcal{H}}(X,\mathbb{R}(*)) \neq H^{\cdot}_{\mathcal{H}}(X,\mathbb{Q}(*))\otimes\mathbb{R}$). Also, considering Hodge structures over $\mathbb{R}$ (i.e., with the action of real Frobenius), we get $\mathcal{H}$-groups of schemes over $\mathbb{R}$.

0.2. For any scheme X let $H^j_{\mathcal{M}}(X,\mathbb{Q}(i))$ denote the i-th graded factor of the γ-filtration on Quillen's K-group $K_{2i-j}(X)\otimes\mathbb{Q}$; these groups - call them absolute motivic ones - also form a twisted duality theory (see [1], [8]; in [8] the notation $H^j(X,i)$ is used). Due to the theory of Chern characters and Riemann-Roch this theory is universal (among the theories with values in $\mathbb{Q}$-modules). For example, the Chern character in $\mathcal{H}$-theory defines for every X over $\mathbb{R}$ or $\mathbb{C}$ the natural morphism - call it regulator map - $r_{\mathcal{H}}:H^{\cdot}_{\mathcal{M}}(X,\mathbb{Q}(*)) \to H^{\cdot}_{\mathcal{H}}(X,\mathbb{Q}(*))$. This arrow is highly non-trivial: e.g. if $X = \operatorname{Spec}\mathbb{C}$ is a point and $i > 0$, then $r_{\mathcal{H}}:H^1_{\mathcal{M}}(X,\mathbb{Q}(i)) \to H^1_{\mathcal{H}}(X,\mathbb{R}(i)) = (2\pi\sqrt{-1})^{i-1}\mathbb{R}$ is Borel's regulator map. So, if V is the spectrum of a number field, then, according to Borel, one has $r_{\mathcal{H}}:H^1_{\mathcal{M}}(V,\mathbb{Q}(i))\otimes\mathbb{R} \xrightarrow{\sim} H^1_{\mathcal{H}}(V\otimes\mathbb{R},\mathbb{R}(i))$ and $\det r_{\mathcal{H}}$ is related to the value of $\zeta(V,s)$ at $s = i$. One expects that similar statements should be valid for any V smooth and proper over $\mathbb{Q}$; for precise conjectures see n° 8. These conjectures determine, up to multiplication by rational constant, the values (or principal terms of Taylor series expansion) of motivic L-functions at any integral point but the middle of the critical strip. In the middle, another construction - the height pairing - is needed; I will not touch this subject here.

0.3. I hope that the above picture should fit into the following general one. To any scheme X should correspond certain triangulated category $D_{\mathcal{M}}(X)$ of "motivic sheaves" on X s.t. all the formalism of mixed sheaf theory (the functors $f^{*,!}$, $f_{*,\cdot}$, weight filtrations and so on, see [2]) should be valid for $D_{\mathcal{M}}(X)$. One should have

$$H^{\cdot}_{\mathcal{M}}(X, \mathbb{Q}(*)) = \mathrm{Hom}_{D_{\mathcal{M}}(X)}(\mathbb{Q}, \mathbb{Q}(*)[\cdot]).$$

There should also be a parallel theory of Hodge sheaves $D_{\mathcal{H}}(X)$ for schemes over $\mathbb{R}$ or $\mathbb{C}$ s.t. the pure smooth Hodge sheaves are polarizable variations of Hodge structures, and $H^{\cdot}_{\mathcal{H}}(X, \mathbb{Q}(*)) = \mathrm{Hom}_{D_{\mathcal{H}}(X)}(\mathbb{Q}, \mathbb{Q}(*)[\cdot])$.

One should have natural realization functor $r_{\mathcal{H}}: D_{\mathcal{M}}(X) \to D_{\mathcal{H}}(X)$ and $r_{\mathcal{H}}$ from 0.2 should be the induced map between Hom's.

As for the values of L-functions, here is some amusing intuition. Let $\overline{\mathrm{Spec}\ \mathbb{Z}}$ be the spectrum of $\mathbb{Z}$ completed by ∞. For any irreducible motivic $\mathbb{Q}$-sheaf $\mathfrak{F}$ over $\mathbb{Z}$, not supported at a closed point, there corresponds to some esoteric prolongation $\overline{\mathfrak{F}}$ of $\mathfrak{F}$ to $\overline{\mathrm{Spec}\ \mathbb{Z}}$. The exact sequence of pairs $(\overline{\mathrm{Spec}\ \mathbb{Z}}, \mathrm{Spec}\ \mathbb{Z})$ reduces to something like $\to H^{\cdot}(\overline{\mathfrak{F}}) \to$ $\to H^{\cdot}_{\mathcal{M}}(\mathrm{Spec}\ \mathbb{Z}, \mathfrak{F}) \xrightarrow{r_{\mathcal{H}}} H^{\cdot}_{\mathcal{H}_{\mathbb{R}}}(\mathrm{Spec}\ \mathbb{R}, r_{\mathcal{H}_{\mathbb{R}}}(\mathfrak{F})) \to$. The groups $H^j(\overline{\mathfrak{F}})$ are topological groups with volume form, zero for $j \neq 0,1,2$; one has $H^0(\overline{\mathfrak{F}}) \neq 0 \Longrightarrow \overline{\mathfrak{F}} = \mathbb{Q}$, $H^1(\overline{\mathfrak{F}}) \neq 0 \Longrightarrow W(\mathfrak{F})$ = the weight of $\mathfrak{F} = -1$, $H^2(\overline{\mathfrak{F}}) \neq 0 \Longrightarrow W(\overline{\mathfrak{F}}) < -1$ and $W(\mathfrak{F})$ $-2 \Longrightarrow H^2(\overline{\mathfrak{F}})$ is compact. If $W(\mathfrak{F}) \geq -1$, then the principal term of $L(\mathfrak{F}, S)$ at $S = 0$ should correspond to either volume of $H^2(\overline{\mathfrak{F}}^0)$ if $W(\mathfrak{F}) \geq 0$, or to the determinant of a height pairing $H^1(\overline{\mathfrak{F}}) \otimes H^1(\overline{\mathfrak{F}}^0(1)) \to H^2(\overline{\mathbb{Q}(1)}) = \mathbb{R}$.

0.4. Certain variants of $\mathcal{H}$-groups were known for some time. If $X/\mathbb{C}$ is smooth proper and $i > 0$, then $H^{\cdot}_{\mathcal{H}}(X, \mathbb{Z}(i))$ $= H^{\cdot-1}(X_{an}, \mathcal{O}^* \to \Omega^1 \to \ldots \to \Omega^{i-1})$, where $\mathcal{O}^* \to \Omega^1$ is $(2\pi\sqrt{-1})^{i-1} \cdot d\log$; these groups were introduced long ago by Deligne. For arbitrary X certain less convenient and more ad hoc constructions may be found in [1] and [7] (in this paper I call the later groups weak $\mathcal{H}$-ones, see (3.13) and (5.7)). Finally, a construction of characteristic classes with values in Deligne's groups $H^{\cdot}(X_{an}, \mathcal{O}^* \to \ldots \to \Omega^{i-1})$ was found independently by Karoubi.

0.5. Here is a brief content of the paper. In n° 1 the basic properties of derived category $D(\mathcal{H})$ are presented; we introduce $\mathcal{H}$-groups and compute them explicitly together with any Ext's; these computations ought to be known to specialists. In n° 2 the polarizable version - the more precise one - is presented. Hodge complexes - the objects far more flexible then the complexes of Hodge structures - are introduced in n° 3. The basic result is that the corresponding derived category coincides with $D^e(\mathcal{H})$. We will see in n° 4 that the construction of Deligne [4] associates with any X a canonical object $\underline{R\Gamma}(X,\mathbb{Z})$ in the derived category of polarizable Hodge complexes - and so, by n° 3, the one in $D(\mathcal{H})$; one has also the variants of this construction: cohomology with compact support, etc. This permits us to define as in 0.1. $\mathcal{H}$-groups of schemes; the basic properties are listed in n° 5; n° 6 contains certain extensions of Hodge conjectures, and in n° 7 the variant of $\mathcal{H}$-groups for schemes over $\mathbb{R}$ is mentioned. Finally the conjectures about the values of L-functions are presented in n° 8; this n° repeats more or less §3 of [1].

0.6. The reader may see how much this note owes to Pierre Deligne; above all he taught me how to use polarizable structures. I would like to thank him cordially.

1. DERIVED CATEGORY OF HODGE STRUCTURES. From now on $A \subset \mathbb{R}$ is a noetherian subring such that $A \otimes \mathbb{Q}$ is a field ([4], III, 0.3). Recall that an A-(mixed) Hodge structure M consists of a finitely generated A-module M_A, a weight filtration $W_{\mathbb{Q}}.$ on $M_{\mathbb{Q}} := M_A \otimes \mathbb{Q}$, and a Hodge fibration $F^{\cdot}$ on $M_{\mathbb{C}} := M_A \underset{A}{\otimes} \mathbb{C}$ subject to the axioms of [4] ( ). If M is a Hodge structure, put $W_{\mathbb{C}}. := W_{\mathbb{Q}}. \underset{A}{\otimes} \mathbb{C}$ - this is a filtration on $M_{\mathbb{C}}$; let $W.$ be the filtration on the $A \otimes \mathbb{Q}$-mixed Hodge structure $M \otimes \mathbb{Q}$ such that $W_i(M\otimes\mathbb{Q})_{A\otimes\mathbb{Q}} := W_{\mathbb{Q}i}(M)$.

Let $\mathcal{H}_A$, or simply $\mathcal{H}$, be the category of A-Hodge structures. This is an abelian A-category with inner Hom (denoted $\mathcal{H}$om) and tensor product $\otimes$. For $i \in \mathbb{Z}$ one has Tate's structure $A(i) \in \mathrm{Ob}\,\mathcal{H}$ (cf. [4] ( )). For $M \in \mathrm{Ob}\,\mathcal{H}$ put $M(i) = M \otimes A(i) = \mathcal{H}om(A(-i),M)$. Denote by $\Gamma_{\mathcal{H}}:\mathcal{H}_A \to$ A-mod the functor $\Gamma_{\mathcal{H}}(M) = \mathrm{Hom}_{\mathcal{H}}(A(0),M)$. This is left-exact A-functor; one has canonically $\mathrm{Hom}_{\mathcal{H}}(M,N) = \Gamma_{\mathcal{H}}\mathcal{H}om(M,N)$. Say that a Hodge structure M is torsion if $M \otimes \mathbb{Q} = 0$; the category of such structures is equivalent (via $M \mapsto M_A$) to the category of finitely-generated torsion A-modules.

Let $C^{\cdot}(\mathcal{H})$ be the category of complexes of Hodge structures, and let $K^{\cdot}(\mathcal{H})$, $D^{\cdot}(\mathcal{H})$ be the corresponding homotopy and derived categories (here $\cdot = +, -$, b or $\emptyset$ is boundedness condition). Let $\underline{H}: D^{\cdot}(\mathcal{H}) \to \mathcal{H}$ be standard cohomological functor. One has derived functors $R\,\mathcal{H}om: D^-(\mathcal{H})^0 \times D^+(\mathcal{H}) \to D^+(\mathcal{H})$ (or $D^b(\mathcal{H})^0 \times D(\mathcal{H}) \to D(\mathcal{H})$), $\overset{L}{\otimes}: D^-(\mathcal{H}) \times D^-(\mathcal{H}) \to D^-(\mathcal{H})$. The condition on A implies

Lemma 1.1. If $\mathcal{F}$, G are complexes of Hodge structures, then canonical arrows $R\,\mathcal{H}om(\mathcal{F},G)_A \to R\,\mathrm{Hom}_{A\text{-mod}}(\mathcal{F}_A, G_A)$, $\mathcal{F}_A \overset{L}{\underset{A}{\otimes}} G_A \to (\mathcal{F} \overset{L}{\otimes} G)_A$ are isomorphisms. So, in particular, if $M, N \in \mathrm{Ob}\,\mathcal{H}$ and $i > 0$ then $R^i\mathcal{H}om(M,N)$ is torsion and equals $\mathrm{Ext}^i_{A\text{-mod}}(M_A, N_A)$.

Define the absolute Hodge cohomology functor to be $R\Gamma_{\mathcal{H}}: D^+(\mathcal{H}) \to D^+(A\text{-mod})$; put $H^i_{\mathcal{H}}(\mathcal{F}) := H^i(R\Gamma_{\mathcal{H}}(\mathcal{F}))$. Our first aim is to compute $R\Gamma_{\mathcal{H}}$ explicitly.

1.2. We will often use the following construction. Consider a diagram of complexes of A-modules

$$\mathfrak{D} = \left(\begin{tikzcd} & B_1 & & B_2 & & B_n & \\ A_1 \arrow[ur, "f_1"] & & A_2 \arrow[ul, "g_1"'] \arrow[ur, "f_2"] & & \cdots & & A_{n+1} \arrow[ul, "g_n"'] \end{tikzcd} \right)$$

Put $\tilde{\Gamma}^0(\mathfrak{D}) := \oplus A_i$, $\tilde{\Gamma}^1(\mathfrak{D}) := \oplus B_i$. One has two morphisms $\varphi_1, \varphi_2 : \tilde{\Gamma}^0(\mathfrak{D}) \to \tilde{\Gamma}^1(\mathfrak{D})$, $\varphi_i := \Sigma f_i$, $\varphi_i = \Sigma g_i$. Put $\tilde{\Gamma}(\mathfrak{D}) := \mathrm{Cone}(\varphi_1 - \varphi_2 : \tilde{\Gamma}^0(\mathfrak{D}) \to \tilde{\Gamma}^1(\mathfrak{D}))[-1]$, $\Gamma(\mathfrak{D}) := \mathrm{Ker}(\varphi_1 - \varphi_2)$, $\Gamma^1(\mathfrak{D}) := \mathrm{Coker}(\varphi_1 - \varphi_2)$ We have a distinguished triangle $\Gamma(\mathfrak{D}) \to \tilde{\Gamma}(\mathfrak{D}) \to \Gamma^1(\mathfrak{D})[-1] \to \dots$ in $D(A\text{-mod})$.

Now for $\mathcal{F} \in \mathrm{Ob}\,C^{\cdot}(\mathcal{H})$ consider the following obvious diagram of complexes of A-modules:

$$\mathfrak{D}_{\mathcal{H}}(\mathfrak{F}) := \left(\begin{tikzcd}[column sep=small] & \mathfrak{F}_{\mathbb{Q}} & & W_{\mathbb{C}O}(\mathfrak{F}) & \\ \mathfrak{F}_A \arrow[ur] & & W_{\mathbb{Q}O}(\mathfrak{F}) \arrow[ul] \arrow[ur] & & (F^O \cap W_{\mathbb{C}O})(\mathfrak{F}) \arrow[ul] \end{tikzcd}\right)$$

Put $\tilde{\Gamma}_{\mathcal{H}}(\mathfrak{F}) := \tilde{\Gamma}(\mathfrak{D}_{\mathcal{H}}(\mathfrak{F}))$ and so on; clearly, if $M \in \operatorname{Ob} \mathcal{H} \subset \operatorname{Ob} C(\mathcal{H})$ then $\tilde{\Gamma}^0_{\mathcal{H}}(M)$, $\tilde{\Gamma}^1_{\mathcal{H}}(M)$ are simply A-modules. The map $f \mapsto (f(1),f(1),f(1))$ identifies the old $\Gamma_{\mathcal{H}}(\mathfrak{F})$ with our $\Gamma_{\mathcal{H}}(\mathfrak{F}) := \Gamma(\mathfrak{D}_{\mathcal{H}}(\mathfrak{F}))$. One has an obvious

Lemma 1.3. The functors $\tilde{\Gamma}^0_{\mathcal{H}}, \tilde{\Gamma}^1_{\mathcal{H}}: \mathcal{H} \to$ A-mod are exact. The functors $\tilde{\Gamma}^0_{\mathcal{H}}, \tilde{\Gamma}^1_{\mathcal{H}}, \tilde{\Gamma}_{\mathcal{H}}: C^{\cdot}(\mathcal{H}) \to C^{\cdot}(\text{A-mod})$ transform Qis to Qis; so in particular $\tilde{\Gamma}_{\mathcal{H}}$ induces the functor $\tilde{\Gamma}_{\mathcal{H}}: D^{\cdot}(\mathcal{H}) \to D^{\cdot}(\text{A-mod.})$ $\square$

Let us compute the cohomology groups of $\tilde{\Gamma}_{\mathcal{H}}(\mathfrak{F})$. Consider the $\tilde{\Gamma}_{\mathcal{H}}$-transform of the canonical filtration $\tau_{\leq \cdot}(\mathfrak{F})$ on $\mathfrak{F}$ - by 1.3 $\tilde{\Gamma}_{\mathcal{H}}(\tau_{\leq \cdot}(\mathfrak{F}))$ is a filtration on $\tilde{\Gamma}_{\mathcal{H}}(\mathfrak{F})$, with i-th graded factor quasi-isomorphic to $\tilde{\Gamma}_{\mathcal{H}}(\underline{H}^i(\mathfrak{F}))[-i]$. The spectral sequence of this filtration degenerates to give

Lemma 1.4. One has canonical short exact sequences

$$0 \to \Gamma^1_{\mathcal{H}}(\underline{H}^{i-1}(\mathfrak{F})) \to H^i\tilde{\Gamma}_{\mathcal{H}}(\mathfrak{F}) \to \Gamma_{\mathcal{H}}H^i(\mathfrak{F}) \to 0. \qquad \square$$

Remark 1.5. Sometimes it is convenient to factor $\tilde{\Gamma}_{\mathcal{H}}$ by the cone of identity map $W_{\mathbb{Q}O} \to W_{\mathbb{Q}O}$ to get a smaller complex $\tilde{\Gamma}'_{\mathcal{H}}$ quasi-isomorphic to $\tilde{\Gamma}_{\mathcal{H}}$; the components of $\tilde{\Gamma}'_{\mathcal{H}}(\mathfrak{F})$ are $\mathfrak{F}_A \oplus (W_{\mathbb{C}O} \cap F^O)(\mathfrak{F})$ and $\mathfrak{F}_{\mathbb{Q}} + W_{\mathbb{C}O}(\mathfrak{F}) \subset \mathfrak{F}_{\mathbb{Q}}$. $\square$

Now consider canonical injection $\Gamma_{\mathcal{H}}(\mathfrak{F}) \hookrightarrow \tilde{\Gamma}_{\mathcal{H}}(\mathfrak{F})$. Passing to the derived functors this defines morphism $R\Gamma_{\mathcal{H}} \to \tilde{\Gamma}_{\mathcal{H}}$ of D^+ (A-mod)-valued exact functors on $D^+(\mathcal{H})$.

Lemma 1.6. This arrow $R\Gamma_{\mathcal{H}} \to \tilde{\Gamma}_{\mathcal{H}}$ is an isomorphism.

We will prove a slightly more general fact. If $\mathfrak{F} \in \operatorname{Ob} C^+(\mathcal{H})$, $G \in \operatorname{Ob} C^b(\mathcal{H})$ are two complexes, then $\Gamma_{\mathcal{H}}\mathcal{H}om^{\cdot}(G,\mathfrak{F}) = \operatorname{Hom}^{\cdot}(G,\mathfrak{F})$, so one has an injection $\operatorname{Hom}^{\cdot}(G,\mathfrak{F}) \hookrightarrow \tilde{\Gamma}_{\mathcal{H}}\mathcal{H}om(G,\mathfrak{F})$. Passing to derived functors one gets an arrow $R\operatorname{Hom}^{\cdot}(G,\mathfrak{F}) \to \tilde{\Gamma}_{\mathcal{H}}R\,\mathcal{H}om^{\cdot}(G,\mathfrak{F})$ in D^+ (A-mod) (since $R\tilde{\Gamma}_{\mathcal{H}}\mathcal{H}om^{\cdot} = \tilde{\Gamma}_{\mathcal{H}}\ R\,\mathcal{H}om^{\cdot}$ by 1.3). The following lemma coincides with 1.6 if $G = A(0)$.

Lemma 1.7. The arrow $R\,\mathrm{Hom}^\cdot(G,\mathfrak{F}) \to \tilde{\Gamma}_{\mathfrak{H}} R\,\mathcal{H}om^\cdot(G,\mathfrak{F})$ is an isomorphism.

Before proving 1.7 here are a few preliminaries. Let $\mathfrak{F},\mathcal{M} \in \mathrm{Ob}\,C(\mathfrak{H})$ be two complexes. Define a rigidified extension of $\mathcal{M}$ by $\mathfrak{F}$ to be an extension $0 \to \mathfrak{F} \to \mathfrak{L} \to \mathcal{M} \to 0$ together with sections $\gamma_\cdot := (\gamma_A,\gamma_{\mathbb{Q}},\gamma_{\mathbb{C}}): \gamma_A:\mathcal{M}_A \to \mathfrak{L}_A$, $\gamma_{\mathbb{Q}}:(\mathcal{M}_{\mathbb{Q}},W_{\mathbb{Q}}.) \to (\mathfrak{L}_{\mathbb{Q}},W_{\mathbb{Q}}.)$ and $\gamma_{\mathbb{C}}:(\mathcal{M}_{\mathbb{C}},W_{\mathbb{C}}.,F^\cdot) \to (\mathfrak{L}_{\mathbb{C}},W_{\mathbb{C}}.,F^\cdot)$. The isomorphism classes of rigidified extensions as usual form an A-module $\widetilde{\mathrm{Ext}}^1(\mathcal{M},\mathfrak{F})$. Define an arrow $cl:\widetilde{\mathrm{Ext}}^1(\mathcal{M}^\cdot,\mathfrak{F}^\cdot) \to Z^0\tilde{\Gamma}^1_{\mathfrak{H}}(\mathcal{H}om(\mathcal{M},\mathfrak{F})) = \mathrm{Hom}_{C^\cdot(A\otimes\mathbb{Q}\text{-mod})}(\mathcal{M}_{\mathbb{Q}},\mathfrak{F}_{\mathbb{Q}}) \oplus \mathrm{Hom}_{C^\cdot F(\mathbb{C}\text{-mod})}((\mathcal{M}_{\mathbb{C}},W_{\mathbb{C}}.),(\mathfrak{F}_{\mathbb{C}},W_{\mathbb{C}}.))$ by the formula $cl(\gamma_\cdot) := (\gamma_A\otimes\mathbb{Q}-\gamma_{\mathbb{Q}},\gamma_{\mathbb{Q}}\otimes\mathbb{C}-\gamma_{\mathbb{C}})$. We have a simple

Lemma 1.8. The arrow cl is isomorphism of A-modules. $\square$

Now suppose that $\mathcal{M} = \mathrm{Cone}(\mathrm{id}_{G^\cdot}:G \to G)[-1]$. Then $\mathrm{Hom}_{C^\cdot(A\otimes\mathbb{Q}\text{-mod})}(\mathcal{M}_{\mathbb{Q}},\mathfrak{F}_{\mathbb{Q}}) = \mathrm{Hom}^0(G_{\mathbb{Q}},\mathfrak{F}_{\mathbb{Q}}),\ldots$, and $\widetilde{\mathrm{Ext}}^1(\mathcal{M},\mathfrak{F}) = \tilde{\Gamma}^1_{\mathfrak{H}}(\mathcal{H}om(G,\mathfrak{F}))^0$. If $x \in \tilde{\Gamma}^1_{\mathfrak{H}}(\mathcal{H}om(G,\mathfrak{F}))^0$ is class of rigidified extension $(0 \to \mathfrak{F} \to \mathfrak{L} \to \mathcal{M} \to 0,\gamma_\cdot)$ then it is easy to see that $x \in \mathrm{Im}[\varphi_1-\varphi_2:\tilde{\Gamma}^0_{\mathfrak{H}}(\mathcal{H}om(G,\mathfrak{L}))^0 \to \tilde{\Gamma}^1_{\mathfrak{H}}(\mathcal{H}om(G,\mathfrak{L}))^0]$; namely, $x = (\varphi_1-\varphi_2)(\gamma_A(y),\gamma_{\mathbb{Q}}(y),\gamma_{\mathbb{C}}(y))$ where $y \in \mathrm{Hom}^0(G,\mathcal{M})$ is standard inclusion (y is not morphism of complexes!). Note that $\mathfrak{F} \hookrightarrow \mathfrak{L}$ is a quasi-isomorphism, so we have proven (use translation by j):

Lemma 1.9. For any element $x \in \tilde{\Gamma}^1_{\mathfrak{H}}(\mathcal{H}om^\cdot(G,\mathfrak{F}))^j$ there exists a quasi-isomorphism $\alpha:\mathfrak{F} \to \mathfrak{F}'$ s.t. $\alpha(x) \in (\varphi_1-\varphi_2)(\tilde{\Gamma}^0_{\mathfrak{H}}(\mathcal{H}om^\cdot(G,\mathfrak{F}'))^j$.

Now we may prove 1.7 (and 1.6). Consider the system $S = \{\mathfrak{F} \to \mathfrak{F}'\}$ of quasi-isomorphisms $\mathfrak{F} \to \mathfrak{F}'$. For any $\mathfrak{F}'$ we have a distinguished triangle $\mathrm{Hom}^\cdot(G,\mathfrak{F}') \to \tilde{\Gamma}_{\mathfrak{H}}\mathcal{H}om(G,\mathfrak{F}') \to \tilde{\Gamma}^1_{\mathfrak{H}}\mathcal{H}om(G^\cdot,\mathfrak{F}'^\cdot) \to \ldots$. By 1.9 one has $\varinjlim_S \tilde{\Gamma}^1_{\mathfrak{H}}\mathcal{H}om(G,\mathfrak{F}') = 0$. This implies that $R\,\mathrm{Hom}^\cdot(G,\mathfrak{F}) := \varinjlim_S \mathrm{Hom}^\cdot(G,\mathfrak{F}') = \tilde{\Gamma}_{\mathfrak{H}} R\,\mathcal{H}om(G,\mathfrak{F})$, q.e.d.

Corollary 1.10. Let M, N be Hodge structures. Then for $i > 1$ one has $\mathrm{Ext}^i_{\mathfrak{H}}(M,N) = \mathrm{Ext}^i_A(M_A,N_A)$. In particular, if A is $\mathbb{Z}$ or A is a field, then $\mathrm{Ext}^i_{\mathfrak{H}} = 0$ for $i > 1$. So in this case for any $\mathfrak{F} \in \mathrm{Ob}\,D^+(\mathfrak{H})$ there exists a quasi-isomorphism $\mathfrak{F} \simeq \oplus\, \underline{H}^i(\mathfrak{F})[-i]$ inducing the identity on $H^\cdot(\mathfrak{F})$. $\square$

1.11. I will finish this section with an explicit formula for multiplication in $\mathfrak{H}$-cohomology. Let $\mathfrak{D}^\cdot$, $\mathfrak{D}'$ be some diagrams of type 1.2; put

$$\mathfrak{D} \otimes \mathfrak{D}' := \left(\begin{tikzcd}[column sep=small] & B_1 \otimes B_1' & & \\ A_1 \otimes A_1' \arrow[ur, "f_1 \otimes f_2"] & & A_2 \otimes A_2' \arrow[ul, "g_1 \otimes g_2"'] \arrow[ur] & \end{tikzcd} \right)$$

Fix any $\alpha \in A$. Define the arrow $\underset{\alpha}{*}: \tilde{\Gamma}(\mathfrak{D}) \otimes \tilde{\Gamma}(\mathfrak{D}') \to \tilde{\Gamma}(\mathfrak{D} \otimes \mathfrak{D}')$ by formulas $a \underset{\alpha}{*} a' := a \otimes a'$, $b \underset{\alpha}{*} b' = 0$, $a \underset{\alpha}{*} b' := (-1)^{\deg a}(\alpha\varphi_1(a) + (1-\alpha)\varphi_2(a)) \otimes b'$, $b \underset{\alpha}{*} a' := b \otimes ((1-\alpha)\varphi_1(a') + \alpha\varphi_2(a'))$; here $a = (a_i) \in \tilde{\Gamma}^0(\mathfrak{D}) = \oplus A_i$, $b = (B_i) \in \tilde{\Gamma}^1(\mathfrak{D}) = \oplus B_i, \ldots$, and $a \otimes a' = (a_i \otimes a_i') \in \tilde{\Gamma}^0(\mathfrak{D} \otimes \mathfrak{D}'), \ldots$.

Lemma 1.11. a) $\underset{\alpha}{*}$ is morphism of complexes.

b) All $\underset{\alpha}{*}$ are homotopic; namely the homotopy between $\underset{\alpha_1}{*}$ and $\underset{\alpha_2}{*}$ is given by the formula $b_1 \otimes b_2 \in \tilde{\Gamma}(\mathfrak{D}) \otimes \tilde{\Gamma}(\mathfrak{D}')$ $\mapsto (\alpha_1 - \alpha_2) b_1 \otimes b_2 \in \tilde{\Gamma}(\mathfrak{D} \otimes \mathfrak{D}')$; other components are zero.

c) Under the canonical isomorphism $X_1 \otimes X_2 \simeq X_2 \otimes X_1$ multiplication $\underset{\alpha}{*}$ transforms to $\underset{1-\alpha}{*}$. The multiplications $\underset{0}{*}$ and $\underset{1}{*}$ are associative. $\square$

In particular, if $\mathfrak{D} \otimes \mathfrak{D}' \to \mathfrak{D}''$ is any morphism of diagrams, we have canonical homotopy class $*: \Gamma(\mathfrak{D}) \otimes \Gamma(\mathfrak{D}') \to \Gamma(\mathfrak{D}'')$.

Now, if $\mathfrak{F}$, $\mathfrak{F}'$ and $\mathfrak{F}''$ are complexes of Hodge structures and $\mathfrak{F} \otimes \mathfrak{F}' \to \mathfrak{F}''$ is a morphism of complexes, then the above construction defines the homotopy class $*: \tilde{\Gamma}_{\mathfrak{H}}(\mathfrak{F}) \underset{A}{\otimes} \tilde{\Gamma}_{\mathfrak{H}}(\mathfrak{F}') \to \tilde{\Gamma}_{\mathfrak{H}}(\mathfrak{F}'')$ (since one has an obvious inclusion $\mathfrak{D}_{\mathfrak{H}}(\mathfrak{F}) \otimes \mathfrak{D}_{\mathfrak{H}}(\mathfrak{F}') \hookrightarrow \mathfrak{D}_{\mathfrak{H}}(\mathfrak{F} \otimes \mathfrak{F}')$). Since $*$ being restricted to $\Gamma_{\mathfrak{H}} \subset \tilde{\Gamma}_{\mathfrak{H}}$ coincides with an obvious map, we see that $*$ defines a canonical multiplication on $\mathfrak{H}$-cohomology under the identification 1.6.

Remark 1.12. This formula helps to describe the category structure of $\mathfrak{D}^b(\mathfrak{H})$ in a very explicit way. E.g. if A is a field, then $\mathfrak{H}om$ is exact. By 1.7 one has $\mathrm{Hom}_{\mathfrak{D}(\mathfrak{H})}(G, \mathfrak{F}) = H^0\tilde{\Gamma}(\mathfrak{H}om^{\cdot}(G, \mathfrak{F}))$ and the multiplication formula defines composition of morphisms.

2. VARIANT: POLARIZABLE STRUCTURES. Say that a Hodge structure M is polarizable if the $A \otimes \mathbb{Q}$-pure structure $\mathrm{Gr}_i^W(M \otimes \mathbb{Q})$ are. Denote by $\mathfrak{H}^p \subset \mathfrak{H}$ the subcategory of polarizable structures (and arbitrary morphisms), and by $\mathfrak{H}_i^p \subset \mathfrak{H}^p_{A\otimes\mathbb{Q}}$ the subcategory of $A \otimes \mathbb{Q}$-pure ones of weight i. Clearly $\mathfrak{H}^p$ is an abelian A-subcategory,

closed under $\mathcal{H}om$, $\otimes$ and taking sub- and factor-objects. The categories $\mathcal{H}^p_i$ are semisimple - this is the basic property of $\mathcal{H}^p$.

Let $C^\cdot(\mathcal{H}^p)$, $K^\cdot(\mathcal{H}^p$, $D^\cdot(\mathcal{H}^p)$ be the category of complexes over $\mathcal{H}^p$ and its homotopy and derived category respectively. We have derived functors $R\,\mathcal{H}om: D^-(\mathcal{H}^p)^0 \times D^+(\mathcal{H}^p) \to D^+(\mathcal{H}^p)$, $\overset{L}{\otimes}: D^-(\mathcal{H}^p) \times D^-(\mathcal{H}^p) \to D^-(\mathcal{H}^p)$, $R\,\mathrm{Hom}: D^-(\mathcal{H}^p)^0 \times D^+(\mathcal{H}^p) \to D^+(A\text{-mod})$ and cohomology functor $\underline{H}: D(\mathcal{H}^p) \to \mathcal{H}^p$.

In this n$^\mathrm{o}$ we will study the obvious functor $D(\mathcal{H}^p) \to D(\mathcal{H})$. First, since for $R\,\mathcal{H}om$ and $\overset{L}{\otimes}$ one has an analog of 1.1, we have

Lemma 2.1. The functor $D(\mathcal{H}^p) \to D(\mathcal{H})$ commutes with $R\,\mathcal{H}om$, $\overset{L}{\otimes}$ (and $\underline{H}$). □

Define the absolute polarizable Hodge cohomology functor $R\Gamma_{\mathcal{H}^p}: D^+(\mathcal{H}^p) \to D^+(A\text{-mod})$ to be right derived functor of the functor $\Gamma_{\mathcal{H}^p} := \mathrm{Hom}_{\mathcal{H}^p}(A(0), \cdot)$ ($= \Gamma_{\mathcal{H}}$ restricted to $\mathcal{H}^p$). Let us compute it the same way we did for $R\Gamma_{\mathcal{H}}$.

Let M be a polarizable structure. Consider the subgroup $X(M) := \Gamma_{\mathcal{H}_{A\otimes\mathbb{Q}}}(\mathrm{Gr}^W_0(M \otimes \mathbb{Q})) \subset W_{\mathbb{Q}0}(M)/W_{\mathbb{Q}-1}(M) \subset W_{\mathbb{C}0}(M)/W_{\mathbb{C}-1}(M)$. Define the subgroups $W^{(p)}_{\mathbb{Q}0}(M) \subset W_{\mathbb{Q}0}(M), W^{(p)}_{\mathbb{C}0}(M) \subset W_{\mathbb{C}0}(M)$ to be images of $X(M)$ under the projections $W_{\mathbb{Q}0} \to W_{\mathbb{Q}0}/W_{\mathbb{Q}-1}$, $W_{\mathbb{C}0} \to W_{\mathbb{C}0}/W_{\mathbb{C}-1}$. Note that $X(M)$ depends in an exact way on M (since $\mathcal{H}^p_0$ is semisimple), and so therefore do $W^{(p)}_{\mathbb{Q}0}$, $W^{(p)}_{\mathbb{C}0}$, and $F^0 \cap W^{(p)}_{\mathbb{C}0}$.

Now for a complex $\mathcal{F} \in \mathrm{Ob}\, C^\cdot(\mathcal{H}^p)$ consider the subdiagram $\mathcal{D}_{\mathcal{H}^p}(\mathcal{F}^\cdot) \subset \mathcal{D}_{\mathcal{H}}(\mathcal{F}^\cdot)$ (cf 1.2):

$$\mathcal{D}_{\mathcal{H}^p}(\mathcal{F}) := \left(\begin{tikzcd}[column sep=small] & \mathcal{F}_{\mathbb{Q}} & & W^{(p)}_0(\mathcal{F}) & \\ \mathcal{F}_A \arrow[ur] & & W^{(p)}_{\mathbb{Q}0}(\mathcal{F}) \arrow[ul] \arrow[ur] & & (F^0 \cap W^{(p)}_{\mathbb{C}0})(\mathcal{F}) \arrow[ul] \end{tikzcd} \right)$$

For $\tilde{\Gamma}_{\mathcal{H}^p}(\mathcal{F}) := \tilde{\Gamma}(\mathcal{D}_{\mathcal{H}^p}(\mathcal{F}))$ and so on; these are subcomplexes of $\tilde{\Gamma}_{\mathcal{H}}(\mathcal{F}), \ldots$. If $M \in \mathcal{H}^p \subset C^\cdot(\mathcal{H}^p)$ then $\tilde{\Gamma}^0_{\mathcal{H}^p}(M)$, $\tilde{\Gamma}^1_{\mathcal{H}^p}(M)$, $\Gamma_{\mathcal{H}^p}(M)$ and $\Gamma^1_{\mathcal{H}^p}(M)$ are A-modules. We have distinguished triangles $\Gamma_{\mathcal{H}^p}(\mathcal{F}) \to \tilde{\Gamma}_{\mathcal{H}^p}(\mathcal{F}) \to \Gamma^1_{\mathcal{H}^p}(\mathcal{F})[-1] \to \dots$ and one has an analog of 1.3, 1.4:

<u>Lemma 2.2</u>. a) The functors $\tilde{\Gamma}^0_{\mathcal{H}^p}, \tilde{\Gamma}^1_{\mathcal{H}^p} : \mathcal{H} \to A\text{-mod}$ are exact. The functors $\tilde{\Gamma}^0_{\mathcal{H}^p}, \tilde{\Gamma}^1_{\mathcal{H}^p}, \tilde{\Gamma}_{\mathcal{H}^p} : C^{\cdot}(\mathcal{H}^p) \to C^{\cdot}(A\text{-mod})$ transform Qis to Qis, so $\tilde{\Gamma}_{\mathcal{H}^p}$ factors trivially through the (same notation) exact functor $\tilde{\Gamma}_{\mathcal{H}^p} : D^{\cdot}(\mathcal{H}^p) \to D^{\cdot}(A\text{-mod})$.

b) One has $\Gamma_{\mathcal{H}^p}(\mathcal{F}) = \Gamma_{\mathcal{H}}(\mathcal{F})$ and $\Gamma^1_{\mathcal{H}^p}(\mathcal{F})$ injects into $\Gamma^1_{\mathcal{H}}(\mathcal{F}^{\cdot})$; the factor complex $\Gamma^1_{\mathcal{H}}(\mathcal{F})/\Gamma^1_{\mathcal{H}^p}(\mathcal{F})$ is canonically isomorphic to $\Gamma^1_{\mathcal{H}_{A\otimes\mathbb{Q}}}(Gr^W_0(\mathcal{F}\otimes\mathbb{Q}))$.

c) One has a canonical short exact sequence $0 \to \Gamma^1_{\mathcal{H}^p}(\underline{H}^{i-1}(\mathcal{F})) \to H^i\tilde{\Gamma}_{\mathcal{H}^p}(\mathcal{F}) \to \Gamma_{\mathcal{H}}(\underline{H}^i(\mathcal{F})) \to 0$ which injects into the sequence of 1.4. One has $H^i\tilde{\Gamma}_{\mathcal{H}}(\mathcal{F})/H^i\tilde{\Gamma}_{\mathcal{H}^p}(\mathcal{F})$ $= \Gamma^1_{\mathcal{H}}(\underline{H}^{i-1}(\mathcal{F}))/\Gamma^1_{\mathcal{H}^p}(\underline{H}^{i-1}(\mathcal{F})) = \Gamma^1_{\mathcal{H}_{A\otimes\mathbb{Q}}}(Gr^W_0\underline{H}^{i-1}(\mathcal{F}\otimes\mathbb{Q}))$. In particular, $H^{\cdot}\tilde{\Gamma}_{\mathcal{H}^p} = H^{\cdot}\tilde{\Gamma}_{\mathcal{H}}$ in case $A = \mathbb{R}$. □

Passing to derived functors we get an arrow $R\Gamma_{\mathcal{H}^p} \to \tilde{\Gamma}_{\mathcal{H}^p}$ between $D^+(A\text{-mod})$-valued exact functors on $D^+(\mathcal{H}^p)$; the same way we get an arrow $R\,Hom^{\cdot}_{\mathcal{H}^p} \to \tilde{\Gamma}_{\mathcal{H}^p} R\,\mathcal{H}om^{\cdot}$.

<u>Lemma 2.3</u>. This arrow $R\Gamma_{\mathcal{H}^p} \to \tilde{\Gamma}_{\mathcal{H}^p}$ and $R\,Hom^{\cdot}_{\mathcal{H}^p} \to \tilde{\Gamma}_{\mathcal{H}^p} R\,\mathcal{H}om^{\cdot}$ are isomorphisms in $D^+(A\text{-mod})$. □

The proof of 2.3 goes the same pattern as the one of 1.6, 1.7. One has only to change the definition of rigidified extension, demanding in the polarizable case that $Gr_{\cdot}^{W_{\mathbb{Q}}}(\gamma_{\mathbb{Q}})$ and $Gr_{\cdot}^{W_{\mathbb{C}}}(\gamma_{\mathbb{C}})$ should be morphisms of $A\otimes\mathbb{Q}$-Hodge structures.

<u>Remark 2.4</u>. a) One has also word-by-word analogy of 1.10. The construction of multiplication on Hodge cohomology of 1.11 applies also to polarizable cases.

b) Lemma 2.3 implies that in case $A = \mathbb{R}$ one has $R\Gamma_{\mathcal{H}^p} = R\Gamma_{\mathcal{H}}$, $R\,Hom_{\mathcal{H}^p} = R\,Hom_{\mathcal{H}}$. The lemmas 2.1-2.3 imply that $D(\mathcal{H}^p)$ is a subcategory of $D(\mathcal{H})$; let us construct $D(\mathcal{H}^p)$ directly in terms of $D(\mathcal{H})$. Note that for any $\mathcal{F} \in D(\mathcal{H}^p)$ one has <u>canonical</u> quasi-isomorphisms

$$\varphi_i(\mathfrak{F}):\mathrm{Gr}_i^W(\mathfrak{F}\otimes\mathbb{Q}) \to \bigoplus_j \underline{H}^j(\mathrm{Gr}_i^W(\mathfrak{F}\otimes\mathbb{Q}))[-j] \quad (= \bigoplus_j \mathrm{Gr}_i^W\underline{H}^j(\mathfrak{F}\otimes\mathbb{Q})[-j])$$

in $D(\mathcal{H}^p)$, since $\mathcal{H}_i^p$ is semisimple. Now 2.1-2.3 imply

Corollary 2.5. For $\mathfrak{F}, G \in \mathrm{Ob}\, D^b(\mathcal{H}^p)$ one has

$\mathrm{Hom}_{D^b(\mathcal{H}^p)}(G,\mathfrak{F}) = \{f \in \mathrm{Hom}_{D^b(\mathcal{H})}(G,\mathfrak{F})$ s.t. for any i the morphism $\varphi_i(\mathfrak{F})\mathrm{Gr}_i^W(f\otimes\mathbb{Q})\varphi_i(G)^{-1}: \bigoplus_j \underline{H}^j(\mathrm{Gr}_i^W(G\otimes\mathbb{Q}))[-j] \to \bigoplus_j \underline{H}^j(\mathrm{Gr}_i^W(\mathfrak{F}\otimes\mathbb{Q}))[-j]$ is diagonal, i.e., coincides with the direct sum of $\underline{H}^j\mathrm{Gr}_i^W(f\otimes\mathbb{Q}):\underline{H}^j(\mathrm{Gr}_i^W(G\otimes\mathbb{Q})) \to \underline{H}^j(\mathrm{Gr}_i^W(\mathfrak{F}\otimes\mathbb{Q}))\}$. □

Lemma-Definition 2.6. Let $\mathfrak{F}$ e Ob $D^b(\mathcal{H})$ be a complex s.t. any $\underline{H}^j(\mathfrak{F})$ are polarizable. The following sets of structure on $\mathfrak{F}$ are in natural 1-1-correspondence.

a) Isomorphisms $\varphi_{\mathfrak{F}}^{\cdot}:\mathrm{Gr}_{\cdot}^W(\mathfrak{F}\otimes\mathbb{Q}) \to \bigoplus_j \underline{H}^j\mathrm{Gr}_{\cdot}^W(\mathfrak{F}\otimes\mathbb{Q})[-j]$ in $D^b(\mathcal{H}_{A\otimes\mathbb{Q}})$ that induce identity on cohomology.

b) Isomorphisms $\varphi_j^{\cdot}:\mathrm{Gr}_{\cdot}^W(\mathfrak{F}\otimes\mathbb{Q}) \to \bigoplus_j M_{\cdot}^j[-j]$ in $D^b(\mathcal{H}_{A\otimes\mathbb{Q}})$, where $M_{\cdot}^j$ are certain polarizable weight $\cdot$ $A\otimes\mathbb{Q}$-structures, up to the following equivalence; say that $\varphi_{\mathfrak{F}}^{'\cdot} \sim \varphi_{\mathfrak{F}}^{''\cdot}$ if there exists (unique) set of (iso)morphisms: $\psi_{\cdot}^j:M_{\cdot}^{'j} \to M_{\cdot}^{''j}$ s.t. $(\bigoplus_j \psi_{\cdot}^j)\circ\varphi_{\mathfrak{F}}' = \varphi_{\mathfrak{F}}''$.

c) Isomorphisms $\varphi_j^{\cdot}:\mathrm{Gr}_{\cdot}^W(\mathfrak{F}\otimes\mathbb{Q}) \to N_{\cdot}$ in $D^b(\mathcal{H}_{A\otimes\mathbb{Q}})$, where $N_{\cdot} \in \mathrm{Ob}\, D^b(\mathcal{H}_{A\otimes\mathbb{Q}}^p)$, up to the following equivalence; say that $\varphi_{\mathfrak{F}}^{'\cdot} \sim \varphi_{\mathfrak{F}}^{''\cdot}$ if there exists (unique) set of (iso)morphisms $\psi_{\cdot}:N_{\cdot}' \to N_{\cdot}''$ in $D^b(\mathcal{H}_{A\otimes\mathbb{Q}}^p)$ s.t. $\psi_{\cdot}\circ\varphi_{\mathfrak{F}}' = \varphi_{\mathfrak{F}}''$.

Call such a structure on $\mathfrak{F}^{\cdot}$ a p-structure; a complex with p-structure is a p-complex. □

Since for any $M \in \mathrm{Ob}\,\mathcal{H}$ the map $\Gamma_{\mathcal{H}_A}^1(M) \to \Gamma_{\mathcal{H}_{A\otimes\mathbb{Q}}}^1(M\otimes\mathbb{Q})$ is surjective, the lemmas 1.4 and 1.8 imply

Corollary 2.7. Let $\mathfrak{F} \in \mathrm{Ob}\, D^b(\mathcal{H})$ be a complex s.t. all $\underline{H}^j(\mathfrak{F}^{\cdot})$ are polarizable. Then $\mathfrak{F}$ admits a p-structure, and the group $\mathrm{Aut}^0(\mathfrak{F}) := \{g \in \mathrm{Aut}(\mathfrak{F}):g$ acts as identity on cohomology groups$\}$ acts transitively on the set of all possible p-structures on $\mathfrak{F}$. □

A morphism $f:(G,\{\psi_G^{\cdot}\}) \to (\mathfrak{F},\{\varphi_{\mathfrak{F}}^{\cdot}\})$ between p-complexes is a morphism $f:G \to \mathfrak{F}$ in $D^b(\mathcal{H})$ s.t. $\varphi_{\mathfrak{F}}^j\mathrm{Gr}_i^W(f\otimes\mathbb{Q})(\varphi_G^i)^{-1}$ is diagonal (see 2.5). Denote by $D^b(\mathfrak{H}_p)'$ the category of p-complexes. One has canonical functor $D^b(\mathfrak{H}^p) \to D^b(\mathfrak{H}_p)', \mathfrak{F} \to (\mathfrak{F}, \{\varphi(\mathfrak{F})\})$ (cf. above).

Lemma 2.8. This functor is equivalence of categories.

Proof. By 2.5 this functor is faithful imbedding, so, by (2.7), it suffices to show that any $\mathfrak{F} \in \mathrm{Ob}\, D^b(\mathfrak{H})$, s.t. all the $\underline{H}^\cdot(\mathfrak{F})$ are polarizable, is isomorphic to certain $G = \mathrm{Ob}\, D^b(\mathfrak{H}^p)$. Use induction on the length of $\mathfrak{F}$. Suppose that $\mathfrak{F}$ lies in degrees $[a,b]$. By induction $\tau_{<b}(\mathfrak{F}) \simeq G' \in \mathrm{Ob}\, D^b(\mathfrak{H}^p)$. One has $\mathrm{Hom}_{D^b(\mathfrak{H})}(M, G'[b+1]) = \mathrm{Hom}_{D^b(\mathfrak{H}^p)}(M, G'[b+1])$ $(= \mathrm{Hom}_{D^b(A\text{-mod})}(M_A, G'_A[b+1]))$ for any $M \in \mathrm{Ob}\, \mathfrak{H}^p$. So $\mathfrak{H} \simeq \mathrm{Cone}(\underline{H}^b(\mathfrak{F})[b-1] \to G') \in \mathrm{Ob}(D^b(\mathfrak{H}^p))$. $\square$

3. HODGE COMPLEXES. In this n$^\circ$ we will study Hodge complexes - objects more flexible than complexes of Hodge structures. We will see that derived category of Hodge complexes is equivalent to $D(\mathfrak{H})$ - this will be of use in the next section.

In the following all the filtrations on objects of abelian categories will be assumed to have finite length (and be exhaustive and separated). Filtered complex means complex in the category of filtered objects. The following technical lemma will be of use.

Lemma 3.1. Let $f: (\mathfrak{F}_1, W_{1\cdot}) \to (\mathfrak{F}_2, W_{\cdot 2})$ be a morphism of filtered complexes in some abelian category. Suppose that differentials of $\mathfrak{F}_1$ and $\mathfrak{F}_2$ are strictly compatible with filtrations, and the morphism $H^\cdot(f): H^\cdot(\mathfrak{F}_1) \to H^\cdot(\mathfrak{F}_2)$ is strictly compatible with the filtrations induced on cohomology groups. Then the differential of the filtered complex $\mathrm{Cone}(f)$ is also strictly compatible with filtration. $\square$

Definition 3.2. An A-Hodge complex is a diagram

$$\begin{tikzcd}
 & \mathfrak{F}'_{\mathbb{Q}} & & (\mathfrak{F}'_{\mathbb{C}}, W'_{\mathbb{C}\cdot}) & \\
\mathfrak{F}_A \arrow[ur, "\alpha_1"] & & (\mathfrak{F}_{\mathbb{Q}}, W_{\mathbb{Q}\cdot}) \arrow[ul, "\alpha_2"'] \arrow[ur, "\alpha_3"] & & (\mathfrak{F}_{\mathbb{C}}, W_{\mathbb{C}\cdot}, F^\cdot) \arrow[ul, "\alpha_4"']
\end{tikzcd}$$

Here $\mathcal{F}_A$ is a complex of A-modules, $\mathcal{F}'_{\mathbb{Q}}$ is one of $A\otimes\mathbb{Q}$-modules, $(\mathcal{F}_{\mathbb{Q}}, W_{\mathbb{Q}}.)$ is filtered complex of $A\otimes\mathbb{Q}$-modules, $(\mathcal{F}'_{\mathbb{C}}, W'_{\mathbb{C}}.)$ is one of $\mathbb{C}$-modules, and $(\mathcal{F}_{\mathbb{C}}, W_{\mathbb{C}}., F^{\cdot})$ is bifiltered complex of $\mathbb{C}$-modules. The arrows in diagram $\mathcal{F}_A\otimes\mathbb{Q}\to\mathcal{F}'_{\mathbb{Q}}$, $\mathcal{F}_{\mathbb{Q}}\to\mathcal{F}'_{\mathbb{Q}}$, $(\mathcal{F}_{\mathbb{Q}}, W_{\mathbb{Q}}.)\underset{A}{\otimes}\mathbb{C}\to(\mathcal{F}'_{\mathbb{C}}, W'_{\mathbb{C}}.)$, $(\mathcal{F}_{\mathbb{C}}, W_{\mathbb{C}}.)\to(\mathcal{F}'_{\mathbb{C}}, W'_{\mathbb{C}}.)$ are (filtered) quasi-isomorphisms.

The following conditions should be satisfied:

(i) $H^{\cdot}(\mathcal{F}_A)$ are finitely generated A-modules.

(ii) For any $a\in\mathbb{Z}$ consider the filtered complex $(Gr_a^{W_{\mathbb{C}}}\mathcal{F}_{\mathbb{C}}, Gr_a^{W_{\mathbb{C}}}F^{\cdot})$. The differential of this complex is strictly compatible with filtration.

(iii) This filtration, together with isomorphism $H^{\cdot}(Gr_a^{W_{\mathbb{Q}}}\mathcal{F}_{\mathbb{Q}})\otimes\mathbb{C}\xrightarrow{\sim}H^{\cdot}Gr_a^{W_{\mathbb{C}}}\mathcal{F}_{\mathbb{C}}$ that comes from the diagram, define on $H^{\cdot}(Gr_a^{W_{\mathbb{Q}}}\mathcal{F}_{\mathbb{Q}})$ a pure $A\otimes\mathbb{Q}$-Hodge structure of weight a. $\square$

<u>Lemma 3.3</u>. Let $\mathcal{F}$ be any Hodge complex. Then the spectral sequences of complexes $\mathcal{F}_{\mathbb{Q}}$ and $\mathcal{F}_{\mathbb{C}}$ relative to filtrations $W_{\mathbb{Q}}.$, $W_{\mathbb{C}}.$ and $F^{\cdot}$ degenerate at E_1. These filtrations together with isomorphisms $H^{\cdot}(\mathcal{F}_A)\otimes\mathbb{Q}\xrightarrow{\sim}H^{\cdot}(\mathcal{F}_{\mathbb{Q}})$, $H^{\cdot}(\mathcal{F}_{\mathbb{Q}})\underset{A}{\otimes}\mathbb{C}\xrightarrow{\sim}H^{\cdot}(\mathcal{F}_{\mathbb{C}})$ that come from the diagram, define on $H^{\cdot}(\mathcal{F}_A)$ an A-Hodge structure. $\square$

Denote this Hodge structure by $\underline{H}^{\cdot}(\mathcal{F})\in\mathrm{Ob}\,\mathcal{H}$.

A morphism $f:\mathcal{F}_1\to\mathcal{F}_2$ of Hodge complexes is morphism of corresponding diagrams; and in the same way one defines a homotopy between a pair of morphisms. Let $C^{\cdot}_{\mathcal{H}}$ be the category of Hodge complexes, and $K^{\cdot}_{\mathcal{H}}$ be the corresponding homotopy category (here $\cdot$ is the boundedness condition on $H^{\cdot}(\mathcal{F}_A)$). Lemma 3.1 shows that the cone of a morphism of Hodge complexes is a Hodge complex itself. This supplies $K^{\cdot}_{\mathcal{H}}$ with a triangulated category structure, and $\underline{H}:K^{\cdot}_{\mathcal{H}}\to\mathcal{H}$ becomes a cohomological functor. Say that a morphism is a quasi-isomorphism if it induces isomorphisms on any $\underline{H}^i$. Following Verdier we may localize $K^{\cdot}_{\mathcal{H}}$ by Qis to get the derived category $D^{\cdot}_{\mathcal{H}}$.

We have an obvious faithful embedding $C^{\cdot}(\mathcal{H})\hookrightarrow C^{\cdot}_{\mathcal{H}}$ that commutes with $\underline{H}^{\cdot}$. It defines exact functors $K^{\cdot}(\mathcal{H})\to K^{\cdot}_{\mathcal{H}}$ and $D^{\cdot}(\mathcal{H})\to D^{\cdot}_{\mathcal{H}}$. The basic result of this n° is

<u>Theorem 3.4</u>. The functor $D^b(\mathcal{H})\to D^b_{\mathcal{H}}$ is an equivalence of categories.

To prove the theorem first note that for any Hodge complex $\mathfrak{F}^\cdot$ the truncated diagram $\tau_{\leq i}(\mathfrak{F}^\cdot)$ is also a Hodge complex. This immediately implies

Lemma 3.5. The functor $\underline{H}: D^\cdot_{\mathcal{H}} \to \mathcal{H}$ is the cohomological functor of a certain (unique) non-degenerate t-structure on $D^\cdot_{\mathcal{H}}$ and $\mathcal{H} \hookrightarrow D^\cdot_{\mathcal{H}}$ (by $\mathcal{H} \hookrightarrow D^\cdot(\mathcal{H}) \to D^\cdot_{\mathcal{H}}$) is equivalence with the heart of $D^\cdot_{\mathcal{H}}$ (see [2]). $\square$

To prove 3.4 it remains to show that $\mathrm{Hom}^i_{D^b(\mathcal{H})}(M,N) = \mathrm{Hom}^i_{D^b_{\mathcal{H}}}(M,N)$ for any $M,N \in \mathrm{Ob}\,\mathcal{H}$, $i \in \mathbb{Z}$. To do this we will compute $R\,\mathrm{Hom}^\cdot_{D^b_{\mathcal{H}}}$ the same way we computed $R\,\mathrm{Hom}_{D^b(\mathcal{H})}$ in 1.7.

One defines the functor $\tilde{\Gamma}_{\mathcal{H}}: C^\cdot_{\mathcal{H}} \to C^\cdot(A\text{-mod})$ and the related functors the same way we did in 1.2, $\tilde{\Gamma}_{\mathcal{H}}(\mathfrak{F}) = \tilde{\Gamma}(\mathfrak{D}_{\mathcal{H}}(\mathfrak{F}))$ where

$$\mathfrak{D}_{\mathcal{H}}(\mathfrak{F}) := \left(\begin{array}{ccccccc} & \mathfrak{F}^!_{\mathbb{Q}} & & & W'_{\mathbb{C}0}(\mathfrak{F}^!_{\mathbb{C}}) & & \\ \nearrow & & \nwarrow & \nearrow & & \nwarrow & \\ \mathfrak{F}_A & & W_{\mathbb{Q}0}(\mathfrak{F}_{\mathbb{Q}}) & & & & (W_{\mathbb{C}0} \cap F^0)(\mathfrak{F}_{\mathbb{C}}) \end{array} \right)$$

If $\mathfrak{F} \in C^\cdot(\mathcal{H})$ then this $\tilde{\Gamma}_{\mathcal{H}}(\mathfrak{F})$ coincides with $\tilde{\Gamma}_{\mathcal{H}}(\mathfrak{F})$ of 1.1. Clearly $\tilde{\Gamma}_{\mathcal{H}}$ transforms Qis to Qis, so it factors trivially through the functor $\tilde{\Gamma}_{\mathcal{H}}: D^\cdot_{\mathcal{H}} \to D^\cdot(A\text{-mod})$. Clearly $\Gamma_{\mathcal{H}}(\mathfrak{F}) = \mathrm{Hom}^\cdot(A(0),\mathfrak{F})$. Passing to derived functors we get a distinguished triangle $R\Gamma_{\mathcal{H}}(\mathfrak{F}) \to \tilde{\Gamma}_{\mathcal{H}}(\mathfrak{F}) \to R\Gamma^1_{\mathcal{H}}(\mathfrak{F})[-1] \to \ldots$ of exact $D^\cdot(A\text{-mod})$-valued functors on $D^\cdot_{\mathcal{H}}$.

Lemma 3.6. The arrow $R\Gamma_{\mathcal{H}} \to \tilde{\Gamma}_{\mathcal{H}}$ is an isomorphism.

Before proving 3.6 let us formulate an analog of 1.7. To do this one has to define the Hodge complex $\underline{\mathcal{H}om}(\mathfrak{F},G)$ of inner Hom's for $\mathfrak{F},G \in C^b_{\mathcal{H}}$.

Put $\mathfrak{F}''_{\mathbb{Q}} := \mathrm{Cone}(\alpha_1-\alpha_2: \mathfrak{F}_A \otimes \mathbb{Q} \oplus \mathfrak{F}_{\mathbb{Q}} \to \mathfrak{F}^!_{\mathbb{Q}})[-1]$, $(\mathfrak{F}''_{\mathbb{C}}, W''_{\mathbb{C}}\cdot) := \mathrm{Cone}(\alpha_3-\alpha_1)$: $(\mathfrak{F}_{\mathbb{Q}}, W_{\mathbb{Q}}) \underset{A}{\otimes} \mathbb{C} \oplus (\mathfrak{F}_{\mathbb{C}}, W_{\mathbb{C}}\cdot) \to (\mathfrak{F}^!_{\mathbb{C}}, W^!_{\mathbb{C}}\cdot))[-1]$. We have obvious arrows

$$\mathfrak{F}_A \otimes \mathbb{Q} \leftarrow \mathfrak{F}''_{\mathbb{Q}} \to \mathfrak{F}_{\mathbb{Q}}, (\mathfrak{F}_{\mathbb{Q}}, W_{\mathbb{Q}}) \underset{A}{\otimes} \mathbb{C} \leftarrow (\mathfrak{F}''_{\mathbb{C}}, W''_{\mathbb{C}}\cdot) \to (\mathfrak{F}_{\mathbb{C}}, W_{\mathbb{C}}\cdot)$$

which are (filtered) quasi-isomorphisms. Now, if G is another Hodge complex, define the Hodge complex $\mathfrak{L} := \underline{\mathcal{H}om}^\cdot(\mathfrak{F},G)$ as follows. Put $\mathcal{L}_A := \mathrm{Hom}^{\cdot}_A(\mathfrak{F}_A, G_A)$, $\mathcal{L}'_{\mathbb{Q}} := \mathrm{Hom}_{A\otimes\mathbb{Q}}(\mathfrak{F}''_{\mathbb{Q}}, G'_{\mathbb{Q}})$, $(\mathcal{L}_{\mathbb{Q}}, W_{\mathbb{Q}}) := \mathrm{Hom}^{\cdot}_{A\otimes\mathbb{Q}}((\mathfrak{F}_{\mathbb{Q}}, W_{\mathbb{Q}}), (G_{\mathbb{Q}}, W_{\mathbb{Q}}))$, $(\mathcal{L}'_{\mathbb{C}}, W_{\mathbb{C}\cdot})$ $:= \mathrm{Hom}^{\cdot}_{\mathbb{C}}((\mathfrak{F}''_{\mathbb{C}}, W''_{\mathbb{C}}), (G'_{\mathbb{C}}, W'_{\mathbb{C}}))$, $(\mathcal{L}_{\mathbb{C}}, W_{\mathbb{C}\cdot}, F^{\cdot})$ $:= \mathrm{Hom}^{\cdot}_{\mathbb{C}}((\mathfrak{F}_{\mathbb{C}}, W_{\mathbb{C}\cdot}, F^{\cdot}), (G_{\mathbb{C}}, W_{\mathbb{C}\cdot}, F^{\cdot}))$. All the needed arrows are constructed in an obvious way from the arrows above and the arrows of the diagram of G. One verifies trivially that this way we get a Hodge complex $\underline{\mathcal{H}om}^{\cdot}(\mathfrak{F}, G)$. If $\mathfrak{F}, G \in \mathrm{Ob}\, C^b(\mathcal{H})$, then one has an obvious epimorphism $\underline{\mathcal{H}om}^{\cdot}(\mathfrak{F}, G) \to \mathcal{H}om^{\cdot}(\mathfrak{F}, G)$, which is quasi-isomorphism. Passing to derived functors we get a complex $R\,\underline{\mathcal{H}om}^{\cdot}(\mathfrak{F}, G) \in D^r_{\mathcal{H}}$; clearly $R\,\underline{\mathcal{H}om}^{\cdot}(\mathfrak{F}, G)_A = R\,\mathrm{Hom}^{\cdot}(\mathfrak{F}_A, G_A)$. This implies that for $\mathfrak{F}, G \in D^b(\mathcal{H})$ the natural arrow $R\,\mathcal{H}om^{\cdot}(\mathfrak{F}, G) \to R\,\underline{\mathcal{H}om}(\mathfrak{F}, G)$ is isomorphism. It is easy to see that $\widetilde{\Gamma}_{\mathcal{H}}\,\underline{\mathcal{H}om}(\mathfrak{F}, G) = \widetilde{\Gamma}(\mathfrak{D}\,\mathcal{H}om(\mathfrak{F}, G))$ (cf. 1.2), where $\mathfrak{D}\,\mathcal{H}om(\mathfrak{F}, G)$ is the following diagram:

$$\begin{tikzcd}[column sep=tiny]
& \mathrm{Hom}^{\cdot}(\mathfrak{F}_A, G'_{\mathbb{Q}}) & & \mathrm{Hom}^{\cdot}(\mathfrak{F}_{\mathbb{Q}}, G'_{\mathbb{Q}}) & & \mathrm{Hom}^{\cdot}((\mathfrak{F}_{\mathbb{Q}}, W_{\mathbb{Q}}), (G'_{\mathbb{C}}, W'_{\mathbb{C}})) & & \mathrm{Hom}^{\cdot}((\mathfrak{F}_{\mathbb{C}}, W_{\mathbb{C}}), (G'_{\mathbb{C}}, W'_{\mathbb{C}})) & \\
\mathrm{Hom}^{\cdot}(\mathfrak{F}_A, G_A) \arrow[ur] & & \mathrm{Hom}(\mathfrak{F}'_{\mathbb{Q}}, G'_{\mathbb{Q}}) \arrow[ul] \arrow[ur] & & \mathrm{Hom}^{\cdot}((\mathfrak{F}_{\mathbb{Q}}, W_{\mathbb{Q}}), (G_{\mathbb{Q}}, W_{\mathbb{Q}})) \arrow[ul] \arrow[ur] & & \mathrm{Hom}^{\cdot}((\mathfrak{F}'_{\mathbb{C}}, W'_{\mathbb{C}}), (G'_{\mathbb{C}}, W'_{\mathbb{C}})) \arrow[ul] \arrow[ur] & & \mathrm{Hom}^{\cdot}((\mathfrak{F}_{\mathbb{C}}, W_{\mathbb{C}}, F^{\cdot}), (G_{\mathbb{C}}, W_{\mathbb{C}}, F^{\cdot})) \arrow[ul]
\end{tikzcd}$$

Clearly one has $\mathrm{Hom}^{\cdot}(\mathfrak{F}^{\cdot}, G') = \Gamma(\mathfrak{D}\,\mathcal{H}om(\mathfrak{F}, G)) \hookrightarrow \widetilde{\Gamma}_{\mathcal{H}}\,\underline{\mathcal{H}om}(\mathfrak{F}, G)$. Passing to derived functors we get an arrow $R\,\mathrm{Hom}^{\cdot}(\mathfrak{F}^{\cdot}, G^{\cdot}) \to \widetilde{\Gamma}_{\mathcal{H}}\, R\,\underline{\mathcal{H}om}(\mathfrak{F}^{\cdot}, G^{\cdot})$ in $D^+(A\text{-mod})$.

Lemma 3.7. This arrow is isomorphism.

The proof of 3.6, 3.7 goes the same way as the one of 1.6, 1.7. One defines a rigidified extension of Hodge complex $\mathfrak{F}$ by Hodge complex G as extension trivialized independently at each point of the top of the diagram 3.2; the group of classes of such extensions coincides with $\widetilde{\Gamma}^1\mathfrak{D}\underline{\mathcal{H}om}(\mathfrak{F}, G)$. Then one does the same thing as in the proof of 1.7 to show that $R(\Gamma^1\mathfrak{D}\,\underline{\mathcal{H}om}(\mathfrak{F}, G)) = 0$. This proves 3.7.

Now we may prove 3.4. The thing we have to verify is $R\,\mathrm{Hom}_{D^b(\mathcal{H})}(M_1, M_2) = R\,\mathrm{Hom}_{D^b_{\mathcal{H}}}(M_1, M_2)$ for $M_1, M_2 \in \mathrm{Ob}\,\mathcal{H}$. By 1.7 and 3.7 this is equivalent to $\widetilde{\Gamma}_{\mathcal{H}}\, R\,\mathcal{H}om(M_1, M_2) = \widetilde{\Gamma}_{\mathcal{H}}\, R\,\underline{\mathcal{H}om}(M_1, M_2)$ which is true, since $R\,\mathcal{H}om = R\,\underline{\mathcal{H}om}$. $\square$

Remark 3.8. Any construction of $n^{\circ}1$ may be done for Hodge complexes. In particular we have tensor product $\otimes$ (obviously defined), inner Hom (defined above) and a formula for multiplication in complexes $\widetilde{\Gamma}_{\mathfrak{H}}$.

Now let me describe the polarizable situation. Here is the third way to define $D^b(\mathfrak{H}^p)$.

<u>Definition 3.9</u>. A $\tilde{p}$-Hodge complex $\widetilde{\mathfrak{F}}$ is a diagram

$$\begin{tikzcd}[column sep=small] & \widetilde{\mathfrak{F}}'_{\mathbb{Q}} & & (\widetilde{\mathfrak{F}}'_{\mathbb{C}}, \widetilde{W}'_{\mathbb{C}}) & \\ \widetilde{\mathfrak{F}}_A \arrow[ur] & & (\widetilde{\mathfrak{F}}_{\mathbb{Q}}, \widetilde{W}_{\mathbb{Q}}) \arrow[ul] \arrow[ur] & & (\widetilde{\mathfrak{F}}_{\mathbb{C}}, \widetilde{W}_{\mathbb{C}}, \widetilde{F}^{\cdot}) \arrow[ul] \end{tikzcd}$$

of the objects of the same sort as in 3.2. The arrows in the diagram $\mathfrak{F}_A \otimes \mathbb{Q} \to \mathfrak{F}'_{\mathbb{Q}}$, $\mathfrak{F}_{\mathbb{Q}} \to \mathfrak{F}'_{\mathbb{Q}}$, $(\mathfrak{F}_{\mathbb{Q}}, \widetilde{W}_{\mathbb{Q}\cdot}) \underset{A}{\otimes} \mathbb{C} \to (\mathfrak{F}'_{\mathbb{C}}, \widetilde{W}'_{\mathbb{C}})$, $(\mathfrak{F}_{\mathbb{C}}, \widetilde{W}_{\mathbb{C}}) \to (\mathfrak{F}'_{\mathbb{C}}, \widetilde{W}'_{\mathbb{C}})$ are (filtered) quasi-isomorphisms. The following conditions should be satisfied.

(i) $H^{\cdot}(\mathfrak{F}_A)$ are finitely generated A-modules and only a finite number of them are non-zero.

(ii) For any $a \in \mathbb{Z}$ the differential in complex $\mathrm{Gr}_a^{\widetilde{W}_{\mathbb{C}}}(\mathfrak{F}_{\mathbb{C}})$ is strictly compatible with filtration induced by $F^{\cdot}$.

(iii) This filtration together with the isomorphism $H^i(\mathrm{Gr}_a^{\widetilde{W}_{\mathbb{Q}}}(\mathfrak{F}_{\mathbb{Q}})) \underset{A}{\otimes} \mathbb{C} \xrightarrow{\sim} H^i(\mathrm{Gr}_a^{\widetilde{W}_{\mathbb{C}}}(\mathfrak{F}_{\mathbb{C}}))$ define on $H^i(\mathrm{Gr}_a^{\widetilde{W}_{\mathbb{Q}}}(\mathfrak{F}_{\mathbb{Q}}))$ a pure polarizable $A \otimes \mathbb{Q}$-Hodge structure of weight $a+i$. □

<u>Remark 3.10</u>. The notion of $\tilde{p}$-Hodge complex is similar to the notion mixed Hodge complex of [4] (8.1.5).

A morphism of $\tilde{p}$-complexes is a morphism of corresponding diagrams; we have category $C_{\widetilde{\mathfrak{H}}^p}$ of $\tilde{p}$-complexes. If $\widetilde{\mathfrak{F}}$ is a $\tilde{p}$-complex and $a \in \mathbb{Z}$, then one defines translation $\widetilde{\mathfrak{F}}[a]$ to be the translation of the diagram of $\widetilde{\mathfrak{F}}$ with renumerated $\widetilde{W}.:\widetilde{W}.(\widetilde{\mathfrak{F}}[a]) := \widetilde{W}_{\cdot+a}(\widetilde{\mathfrak{F}})[a]$. This is a $\tilde{p}$-complex. If $f:\widetilde{\mathfrak{F}} \to \widetilde{G}$ is a morphism, then its cone is the cone of the morphism of diagrams, but the filtration should be defined by the formula $\widetilde{W}.(\mathrm{Cone}(f)) = \widetilde{W}.(G) \oplus \widetilde{W}.(\widetilde{\mathfrak{F}}[1])$. Clearly $\mathrm{Cone}(f)$ is also a $\tilde{p}$-complex (by 3.1). Having defined Cone, we may define homotopy between two morphisms; we have a triangulated category structure on homotopy category $K_{\widetilde{\mathfrak{H}}^p}$ of $\tilde{p}$-complexes.

Remark 3.11. If $\tilde{\mathfrak{F}}^{\cdot *} = (\ldots \to \tilde{\mathfrak{F}}^{\cdot 0} \to \tilde{\mathfrak{F}}^{\cdot 1} \to \ldots)$ is a complex in $C_{\tilde{\mathcal{H}}^p}$, then the corresponding simple complex $s\,\tilde{\mathfrak{F}}^{\cdot *}$ equipped with a filtration $W. := \underset{i}{\oplus} \tilde{W}_{\cdot +i}(\mathfrak{F}^{\cdot i})$ is also an object of $C_{\tilde{\mathcal{H}}^p}$. Similarly, we get an ordinary $\tilde{p}$-complex from a simplicial or cosimplicial one. This construction, generalizing the construction of cone (and taken from [4] §8), will be needed in n°4.

Say that a morphism $\tilde{\mathfrak{F}}^{\cdot} \to \tilde{G}^{\cdot}$ in $K_{\tilde{\mathcal{H}}^p}$ is a quasi-isomorphism if it induces isomorphisms $H^{\cdot}(\mathfrak{F}_A) \to H^{\cdot}(G_A)$. (Note that Quis may not be filtered Quis relative to $\tilde{W}.$.)

One may see that Quis form multiplicative system, so we may localize by them to get derived category $D^b_{\tilde{\mathcal{H}}^p}$. Note that we have an obvious exact functor $D^b(\mathcal{H}^p) \to D^b_{\tilde{\mathcal{H}}^p}$, that transforms a complex $\mathfrak{F}^{\cdot}$ to the $\tilde{p}$-complex $\tilde{\mathfrak{F}}$ with $\tilde{\mathfrak{F}}_A := \mathfrak{F}_A$, $\tilde{W}_{\mathbb{Q}i}(\tilde{\mathfrak{F}}^j_{\mathbb{Q}}) = W_{i+j}(\mathfrak{F}^j_{\mathbb{Q}})$, $\tilde{F}^{\cdot} = F^{\cdot}$.

Lemma 3.11. This functor is equivalence of categories.

I will confine myself with construction of inverse functor; this will suffice for our needs.

Recall that if $(G^{\cdot}, V.)$ is any filtered complex, then the decalage of $V.$ is the filtration defined by the formula $(\mathrm{Dec}\ V)_a(G^i) := \mathrm{Ker}(d{:}V_{a-i}(G^i) \to V_{a-i}(G^{i+1})/V_{a-i-1}(G^{i+1}))$ (see [4]). Define the complexes $H^a(G^{\cdot},V)^{\cdot}$ by formula $H^a(\ )^i := H^i\mathrm{Gr}^V_{a-i}(G^{\cdot})$; the differential $H^a(\ )^i \to H^a(\ )^{i+1}$ induced by d (this complex is first term of spectral sequence of $V.$). We have an obvious canonical arrow $\mathrm{Gr}_a^{\mathrm{Dec}\ V}(G^{\cdot}) \to H^a(G,V.)^{\cdot}$ which is a quasi-isomorphism.

Remark 3.11.1. Let A be an abelian category. Then on the filtered derived category $D^bF(A)$ one has the non-degenerate t-structure st. $D^{\leq i} = \{(G,V.){:}H^b\mathrm{Gr}^V_a(b) = 0 \text{ for } a+b > 0\}$. The heart of it is the category of stupid filtered complexes, canonically equivalent to $C^b(A)$ by [2] (3.1.8); under this equivalence the corresponding cohomological functor is $H^{\cdot}$ above. One has $\tau_{\leq a}(G,V) = ((\mathrm{Dec}\ V)_a, V. \cap (\mathrm{Dec}\ V)_a)$. In fact, one has the exact functor $D^b(C^b(A))$, which maps a bicomplex to the corresponding simple complex filtered stupidly by the second degree; this functor is t-equivalence between t-categories ($D^b(C^b(A))$ is assumed to have the canonical t-structure). $\square$

Let us apply this remark to our situation

Lemma 3.12. Let $\tilde{\mathfrak{F}}$ be a $\tilde{p}$-Hodge complex. Then

$$\mathfrak{F} = (\mathfrak{F}, W, F^{\cdot}) := \left(\begin{tikzcd}[column sep=small] & \tilde{\mathfrak{F}}^{!}_{\mathbb{Q}} & & (\tilde{\mathfrak{F}}^{!}_{\mathbb{C}}, \mathrm{Dec}\ \tilde{W}^{!}_{\mathbb{C}}) & \\ \tilde{\mathfrak{F}}_A \arrow[ur] & & (\tilde{\mathfrak{F}}_{\mathbb{Q}}, \mathrm{Dec}\ \tilde{W}_{\mathbb{Q}}) \arrow[ul] \arrow[ur] & & (\tilde{\mathfrak{F}}_{\mathbb{C}}, \mathrm{Dec}\ \tilde{W}_{\mathbb{C}}, \tilde{F}) \arrow[ul] \end{tikzcd}\right)$$

is an ordinary Hodge complex. $\square$

Consider the complex $C_a = H^a(\mathfrak{F}_{\mathbb{Q}}, \tilde{W}_{\mathbb{Q}})$. According to 3.8 (iii) this is a complex of pure polarizable $A \otimes \mathbb{Q}$-Hodge structures of weight a. So the canonical morphism $\mathrm{Gr}_a^{\mathrm{Dec}\ W}(\mathfrak{F} \otimes \mathbb{Q}) \to C_a$ defines a p-structure $\{\varphi_i(\tilde{\mathfrak{F}})\}$ on $\mathfrak{F}$ (cf. 2.8). Clearly $\tilde{\mathfrak{F}} \mapsto (\mathfrak{F}, \{\varphi_i(\tilde{\mathfrak{F}})\})$ is an exact functor $D^b_{\tilde{\mathfrak{H}}^p} \to D^p(\mathfrak{H}^p)$. This combined with 2.8 defines the desired inverse functor $D^b_{\tilde{\mathfrak{H}}^p} \to D^b(\mathfrak{H}^p)$.

3.13. Weak $\mathfrak{H}$-cohomology. Sometimes it is convenient to use less precise invariants of Hodge complexes then $\tilde{\Gamma}_{\mathfrak{H}}$. For example, you may forget about the weight filtration and consider instead of $\mathfrak{D}_{\mathfrak{H}}(\mathfrak{F})$ the diagram

$$\mathfrak{D}_{\mathfrak{H}\mathrm{weak}}(\mathfrak{F}) := \left(\begin{tikzcd}[column sep=small] & \mathfrak{F}^{!}_{\mathbb{Q}} & & \mathfrak{F}^{!}_{\mathbb{C}} & \\ \mathfrak{F}_A \arrow[ur] & & \mathfrak{F}_{\mathbb{Q}} \arrow[ul] \arrow[ur] & & F^0(\mathfrak{F}_{\mathbb{C}}) \arrow[ul] \end{tikzcd}\right)$$

Put $R\Gamma_{\mathfrak{H}W}(\mathfrak{F}) := \tilde{\Gamma}(\mathfrak{D}_{\mathfrak{H}\mathrm{weak}}(\mathfrak{F}))$. One has the exact sequence $0 \to H^{i-1}(\mathfrak{F}_{\mathbb{C}})/(F^0H^{i-1}(\mathfrak{F}_{\mathbb{C}}) + H^{i-1}(\mathfrak{F}_A)) \to H^i_{\mathfrak{H}W}(\mathfrak{F}) \to H^i(\mathfrak{F}_A) \cap F^0H^i(\mathfrak{F}_{\mathbb{C}}) \to 0$. The diagram $\mathfrak{D}_{\mathfrak{H}\mathrm{weak}}(\mathfrak{F})$ is filtered by a filtration $W.$:

$$W_i\mathfrak{D}_{\mathfrak{H}\mathrm{weak}}(\mathfrak{F}) := \left(\begin{tikzcd}[column sep=small] & \mathfrak{F}^{!}_{\mathbb{Q}} & & W_i(\mathfrak{F}^{!}_{\mathbb{C}}) & \\ \mathfrak{F}_A \arrow[ur] & & W_i(\mathfrak{F}_{\mathbb{Q}}) \arrow[ul] \arrow[ur] & & (W_i \cap F^0)(\mathfrak{F}_{\mathbb{C}}) \arrow[ul] \end{tikzcd}\right)$$

The complex $\tilde{\Gamma}(\mathfrak{D}_{\mathfrak{H}\mathrm{weak}})$ is filtered by $W_{\mathfrak{H}} := \tilde{\Gamma}(W_i\mathfrak{D}_{\mathfrak{H}\mathrm{weak}})$. One has $W_{\mathfrak{H}0}(\tilde{\Gamma}(\mathfrak{D}_{\mathfrak{H}\mathrm{weak}})) = \tilde{\Gamma}\mathfrak{D}_{\mathfrak{H}}$ so we have canonical arrow

$R\Gamma_{\mathfrak{H}}(\mathfrak{F}) \to R\Gamma_{\mathfrak{H}^W}(\mathfrak{F}.)$. Define $\cup$-products on $R\Gamma_{\mathfrak{H}^W}$ following 1.11; clearly the above arrow commutes with $\cup$-products.

Suppose that $A \supset \mathbb{Q}$. Then the spectral sequence of $W_{\mathfrak{H}}.$ degenerates at E_1, so the arrow $H^{\cdot}_{\mathfrak{H}}(\mathfrak{F}^{\cdot}) \to H^{\cdot}_{\mathfrak{H}^W}(\mathfrak{F})$ is injective. Since $\mathfrak{F}_A \to \mathfrak{F}'_{\mathbb{Q}}$ is quis, we also have

$$R\Gamma_{\mathfrak{H}^W}(\mathfrak{F}^{\cdot}) = \tilde{\Gamma}\left(\mathfrak{F}_{\mathbb{Q}} \nearrow \mathfrak{F}'_{\mathbb{C}} \nwarrow F^0(\mathfrak{F}_{\mathbb{C}})\right)$$

(if A is arbitrary, then

$$R\Gamma_{\mathfrak{H}^W}(\mathfrak{F}) = \tilde{\Gamma}\left(\mathfrak{F}_{A} \nearrow \mathfrak{F}''_{\mathbb{C}} \nwarrow F^0(\mathfrak{F}_{\mathbb{C}})\right) \quad ,$$

where $\mathfrak{F}''_{\mathbb{C}} := \mathrm{Cone}(\mathfrak{F}_{\mathbb{Q}} \to \mathfrak{F}'_{\mathbb{Q}} \oplus \mathfrak{F}'_{\mathbb{C}})$ is quasi-isomorphic to $\mathfrak{F}'_{\mathbb{C}}$).

3.13.1. Now suppose that $A = \mathbb{R}$. Choose a quis of the diagram

$$\mathfrak{F}_{\mathbb{Q}} \nearrow \mathfrak{F}'_{\mathbb{C}} \nwarrow (\mathfrak{F}_{\mathbb{C}}, F^{\cdot})$$

with a diagram

$$\mathfrak{F}^{\#}_{\mathbb{Q}} \nearrow \mathfrak{F}^{\#'}_{\mathbb{C}} \nwarrow (\mathfrak{F}^{\#}_{\mathbb{C}}, F^{\cdot})$$

of the same type (i.e., $\mathfrak{F}^{\#}_{\mathbb{Q}}$ is a complex of $\mathbb{R}$-modules, and $\mathfrak{F}^{\#}_{\mathbb{C}}$, $\mathfrak{F}^{\#'}_{\mathbb{C}}$ are ones of $\mathbb{C}$-modules) together with isomorphism $\mathfrak{F}^{\#}_{\mathbb{C}} = \mathfrak{F}^{\#'}_{\mathbb{R}} \otimes \mathbb{C}$ for certain complex $\mathfrak{F}^{\#'}_{\mathbb{R}}$ of $\mathbb{R}$-modules s.t. $\mathfrak{F}^{\#}_{\mathbb{Q}} \to \mathfrak{F}^{\#'}_{\mathbb{C}}$ maps $\mathfrak{F}^{\#}_{\mathbb{Q}}$ to $\mathfrak{F}^{\#'}_{\mathbb{R}}$. Put $\mathrm{Re}\,\mathfrak{F}^{\#'}_{\mathbb{C}} = \mathfrak{F}^{\#'}_{\mathbb{R}}$, $\mathrm{Im}\,\mathfrak{F}^{\#'}_{\mathbb{C}} = \sqrt{-1}\,\mathfrak{F}^{\#'}_{\mathbb{R}}$ and let $\mathfrak{F}^{\#}_{\mathbb{C}} \xrightarrow{\mathrm{Re}} \mathrm{Re}\,\mathfrak{F}^{\#'}_{\mathbb{C}}$, $\mathfrak{F}^{\#}_{\mathbb{C}} \xrightarrow{\mathrm{Im}} \mathrm{Im}\,\mathfrak{F}^{\#'}_{\mathbb{C}}$ be $\mathfrak{F}^{\#}_{\mathbb{C}} \to \mathfrak{F}^{\#'}_{\mathbb{C}}$ composed with projection on real or imaginary part. Since $\mathfrak{F}^{\#}_{\mathbb{Q}} \to \mathfrak{F}^{\#'}_{\mathbb{R}}$ is quis, we have canonical isomorphism

$$R\Gamma_{\mathfrak{H}^W}(\mathfrak{F}) = \tilde{\Gamma}\left(\mathfrak{F}^{\#}_{\mathbb{Q}} \nearrow \mathfrak{F}^{\#'}_{\mathbb{C}} \nwarrow F^0\mathfrak{F}^{\#}_{\mathbb{C}}\right) = \mathrm{Cone}(F^0\mathfrak{F}^{\#}_{\mathbb{C}} \xrightarrow{\mathrm{Im}} \mathrm{Im}\,\mathfrak{F}^{\#}_{\mathbb{C}})[-1]$$

in $D^b(\mathbb{R}\text{-mod})$. It is easy to see that $\cup$-product $R\Gamma_{\mathcal{H}^W}(\mathfrak{F}) \otimes R\Gamma_{\mathcal{H}^W}(G) \to R\Gamma_{\mathcal{H}^W}(\mathfrak{F} \otimes G)$ in this presentation may be given by the formula $f \cup g = f \otimes g$, $s \cup t = 0$, $f \cup t = (-1)^{\deg f}\mathrm{Re}(f) \otimes t$, $s \cup g = s \otimes \mathrm{Re}(g)$; here $f \in F^0(\mathfrak{F}^{\#}_{\mathbb{C}})$, $g \in F^0(G^{\#}_{\mathbb{C}})$, $s \in \mathrm{Im}\, \mathfrak{F}^{\#'}_{\mathbb{C}}$, $t \in \mathrm{Im}\, G^{\#}_{\mathbb{C}}$. This formula is quite useful in computations.

4. <u>HODGE COMPLEXES OF ALGEBRAIC VARIETIES</u>. In [4] Deligne defined a Hodge structure on cohomology groups of algebraic varieties over $\mathbb{C}$. In fact his construction gives Hodge structure not only on cohomology, but on the cochain complex itself. Let me describe briefly Deligne's construction.

4.1. We are going to define the functor $\underline{R\Gamma}(\cdot, A): Sch/\mathbb{C} \to D^b(\mathcal{H}_A)$ s.t. $\underline{R\Gamma}(\cdot, A)_A$ coincides with usual cochain complex. If X is smooth, proper then one defines $\underline{R\Gamma}(X, A)$ by means of the diagram $(A_X \to (\Omega_X, F^\cdot))$ of complexes of sheaves on X with classical topology (here A_X is the constant sheaf, and $(\Omega^\cdot, F^\cdot)$ is the holomorphic de Rham complex with stupid filtration). Namely, this diagram gives rise to the diagram.

$$\mathfrak{D}[X] = \left(\begin{tikzcd}[column sep=small] & A_X \otimes \mathbb{Q} & & (\Omega^\cdot, \widetilde{W}.) & \\ A_X \arrow[ur] & & (A_X \otimes \mathbb{Q}, \widetilde{W}.) \arrow[ul] \arrow[ur] & & (\Omega, \widetilde{W}., F^\cdot) \arrow[ul] \end{tikzcd}\right)$$

where $\widetilde{W}.$ is trivial filtration: $\widetilde{W}_{-1} = 0$, $\widetilde{W}_0 =$ whole complex. Hodge theory shows that $R\Gamma(X, \mathfrak{D}[X])$ is a $\widetilde{p}$-Hodge complex; put $\underline{R\Gamma}(X, A) := R\Gamma(X, \mathfrak{D}[X]) \in \mathrm{Ob}\, D^b(\mathcal{H}^p)$ (cf. 3.11).

If X is smooth, then let $j: X \hookrightarrow \overline{X}$ be an open embedding s.t. $\overline{X}$ is proper and $\mathfrak{D} := \overline{X} \setminus X$ is the divisor with normal crossings. On $\overline{X}$ one has the bifiltered complex $(\Omega^\cdot_{\overline{X}}(\log \mathfrak{D}), W., F^\cdot)$; here $\Omega^\cdot_{\overline{X}}(\log \mathfrak{D})$ is the complex of holomorphic forms with logarithmic singularities along $\mathfrak{D}$, $W.$ is filtration by the order of pole, and $F^\cdot$ is the stupid filtration. The canonical filtration $\tau_{\leq .}$ on $\Omega^\cdot_{\overline{X}}(\log \mathfrak{D})$ is finer than $W.$ and the identity map $(\Omega^\cdot_{\overline{X}}(\log \mathfrak{D}), \tau_{\leq .}) \to (\Omega^\cdot_{\overline{X}}(\log \mathfrak{D}), W.)$ is a filtered quasi-isomorphism. Choose some flask resolvants $A_X \to A^{\#}_X, \ldots,$ that fit into the commutative diagram

$$(*)\qquad \begin{array}{ccccccc} A_X^{\#} & \longrightarrow & A_X\otimes\mathbb{Q}^{\#} & \longrightarrow & \mathbb{C}_X^{\#} & \longrightarrow & \Omega_X^{\cdot\,\#} \\ \uparrow & & \uparrow & & \uparrow & & \uparrow \\ A_X & \longrightarrow & A_X\otimes\mathbb{Q} & \longrightarrow & \mathbb{C}_X & \longrightarrow & \Omega_X^{\cdot} \end{array}$$

of complexes of sheaves on X. Consider the diagram

$$\begin{array}{ccccc} (j_*\mathbb{C}_X^{\#},\tau_{\leq}.) & \to & (j_*\Omega_X^{\#},\tau_{\leq}.) & \xleftarrow{\ i_1\ } & (\Omega^{\cdot}_{\overline{X}}(\log \mathfrak{D}),\tau_{\leq}.) \\ & & & & \big\downarrow{\scriptstyle i_2} \\ & & & & (\Omega^{\cdot}_{\overline{X}}(\log \mathfrak{D}),W.) \end{array}$$

of filtered complexes on $\overline{X}$ whose arrows are filtered quasi-isomorphisms. Put $(\widetilde{\Omega}^{\cdot},W.) := \mathrm{Cone}(i_1-i_2\colon(\Omega^{\cdot}_{\overline{X}}(\log \mathfrak{D}),\tau_{\leq}.) \to (j_*\Omega_X^{\#},\tau_{\leq}.)\oplus(\Omega^{\cdot}_{\overline{X}}(\log \mathfrak{D}),W.))$; we have obvious filtered quasi-isomorphisms $(j_*\mathbb{C}_X^{\#},\tau_{\leq}.)\to(\widetilde{\Omega}^{\cdot},W.)\leftarrow(\Omega^{\cdot}_{\overline{X}}(\log \mathfrak{D}),W.)$. This together with (*) defines a diagram

$$\mathfrak{D}[X,\overline{X}] := \begin{array}{ccccccc} & j_*(A_X\otimes\mathbb{Q})^{\#} & & & & (\widetilde{\Omega}^{\cdot},W.) & \\ \nearrow & & \nwarrow & & \nearrow & & \nwarrow \\ j_*A_X^{\#} & & & (j_*(A_X\otimes\mathbb{Q})^{\#},\tau_{\leq}.) & & & (\Omega^{\cdot}_{\overline{X}}(\log \mathfrak{D}),W.,F^{\cdot}) \end{array}$$

of complexes of sheaves on $\overline{X}$ (the arrows have an obvious sense: e.g. $(\Omega^{\cdot}_{\overline{X}}(\log \mathfrak{D}),W.,F^{\cdot})\to(\widetilde{\Omega},W.)$ means the filtered morphism $(\Omega^{\cdot}_{\overline{X}}(\log \mathfrak{D}),W.)\to(\widetilde{\Omega}^{\cdot},W.)$. The diagram $R\Gamma(\overline{X},\mathfrak{D}[X,\overline{X}])$ is of type (3.8). Deligne shows in [5] that this is in fact a $\widetilde{p}$-Hodge complex. Put $\underline{R\Gamma}(X,A) := R\Gamma(\overline{X},\mathfrak{D}[X,\overline{X}])$; this complex, viewed as an object of $D^b_{\widetilde{\mathfrak{H}}^p}$ depends only on X (and not on $X\hookrightarrow\overline{X}$) and depends in an obvious functorial way.

To define $\underline{R\Gamma}(X,A)$ for arbitrary scheme X, Deligne replaces X by appropriate smooth simplicial scheme $\widetilde{X}.$, compactified by $j\colon\widetilde{X}_{\cdot}\hookrightarrow\widetilde{\overline{X}}.$ the same way as above. This way we get a cosimplicial $\widetilde{p}$-Hodge complex; the corresponding simple complex (cf. 3.10) is an ordinary $\widetilde{p}$-Hodge complex. Denote this complex, viewed as an object of $D^b_{\widetilde{\mathfrak{H}}^p}\simeq D^b(\mathfrak{H})^p$ by $\underline{R\Gamma}(X,A)$. It depends (in a functorial way) only on X.

This way we get canonical functor $\underline{R\Gamma}(\ ,A):\mathrm{Sch}/\mathbb{C} \to D^b(\mathcal{H}_A^p)$; in fact $\underline{R\Gamma}(\cdot,A)$ defined also on the category of (poly) simplicial schemes and so on. Put $\underline{H}^\cdot(X,A) := \underline{H}^\cdot R\Gamma(X,A) \in \mathrm{Ob}\, \mathcal{H}_A^p$ these are usual cohomology groups of X with Deligne-Hodge structure. Clearly we have a natural cup-product $\cup:\underline{R\Gamma}(X,A) \overset{L}{\underset{A}{\otimes}} \underline{R\Gamma}(X,A) \to \underline{R\Gamma}(X,A)$ that comes from an obvious map $\mathfrak{D}[X,\overline{X}]^{\otimes 2} \to \mathfrak{D}[X,\overline{X}]$ of diagrams of (filtered) sheaves.

4.2. Let us define cohomology with compact support; this will be the functor $\underline{R\Gamma_c}(\cdot,A):\mathrm{Sch}_*^0/\mathbb{C} \to D^b(\mathcal{H}_A^p)$ (here $\mathrm{Sch}_*/\mathbb{C}$ is the category of schemes over $\mathbb{C}$ and proper morphisms) s.t. $\underline{R\Gamma_c}(X,A)_A$ is usual complex of compactly supported cochains with coefficients in A. First suppose that X is smooth. Consider the simplicial scheme $\pi:\mathfrak{D}_\cdot \to \overline{X}$ (we preserve the notations of 4.1) where $\mathfrak{D}_0$ is normalized of $\mathfrak{D}$ and $\mathfrak{D}_i := \underbrace{\mathfrak{D}_0 \times_{\overline{X}} \cdots \times \mathfrak{D}_0}_{i+1 \text{ times}}$.

The schemes $\mathfrak{D}_i$ and $\overline{X}$ are smooth proper. Consider the augmented cosimplicial diagram $\mathfrak{D}[\overline{X}] \to \pi_*(\mathfrak{D}[\mathfrak{D}_\cdot])$ of (filtered) complexes of sheaves on $\overline{X}$. Let $\mathfrak{D}[X,\overline{X}]_c$ be the corresponding diagram of simple complexes; the filtration $\widetilde{W}.$ is defined according to the rule of 3.10. The diagram $R\Gamma(\overline{X},\mathfrak{D}[X,\overline{X}]_c)$ is a $\tilde{p}$-Hodge complex; put $\underline{R\Gamma_c}(X,A) := R\Gamma(\overline{X},\mathfrak{D}[X,\overline{X}]_c)$ viewed as an object of $D^b(\mathcal{H}_A^p)$. To define $\underline{R\Gamma_c}$ for arbitrary singular X, one proceeds exactly the same way as with $\underline{R\Gamma}(X,A)$.

Let me sketch the construction of cup-product map $\smile:\underline{R\Gamma}(X,A) \overset{L}{\underset{A}{\otimes}} R\Gamma_c(X,A) \to R\Gamma_c(X,A)$. Consider the diagram $\mathfrak{D}[X,\overline{X}] \mathbin{\tilde{\otimes}} \mathfrak{D}[X,\overline{X}]_c$ of the same type as $\mathfrak{D}[X,\overline{X}]$ and $\mathfrak{D}[X,\overline{X}]_c$ defined as follows. Its first three terms are tensor products (over A) of the corresponding terms of $\mathfrak{D}[X,\overline{X}]$ and $\mathfrak{D}[X,\overline{X}]_c$. The fourth term is $\mathrm{Cone}((\Omega^{\cdot}_{\overline{X}}(\log \mathfrak{D}),\tau_{\leq\cdot}) \underset{\mathbb{C}}{\otimes} (\mathcal{L},\widetilde{W}^\cdot)$ $\to (j_*\Omega_X^{\#},\tau_{\leq\cdot}) \underset{\mathbb{C}}{\otimes} (\mathcal{L},\widetilde{W}) \oplus (\Omega^{\cdot}_{\overline{X}}(\log \mathfrak{D}),\widetilde{W}.) \underset{\Omega^{\cdot}_{\overline{X}}}{\otimes} (\mathcal{L}^\cdot,\widetilde{W}.)$ and the fifth one is $(\Omega^{\cdot}_{\overline{X}}(\log \mathfrak{D}),\widetilde{W}.,F^\cdot) \underset{\Omega^{\cdot}_{\overline{X}}}{\otimes} (\mathcal{L}^\cdot,\widetilde{W}.,F^\cdot)$; here $(\mathcal{L}^\cdot,\widetilde{W}^\cdot,F^\cdot)$ is the fifth term of $\mathfrak{D}[X,\overline{X}]_c$. It is easy to see that graded factors of $\mathfrak{D}[X,\overline{X}] \mathbin{\tilde{\otimes}} \mathfrak{D}[X,\overline{X}]_c$ with respect to $\widetilde{W}.$ are, appropriately shifted, complexes of the varieties $\mathfrak{D}_i$ and $\overline{X}$: so $\mathfrak{D}[X,\overline{X}] \mathbin{\tilde{\otimes}} \mathfrak{D}[X,\overline{X}]_c$ is a $\tilde{p}$-Hodge complex. The obvious morphism of diagrams $\mathfrak{D}[X,\overline{X}]_c = \mathfrak{D}[\overline{X}] \otimes \mathfrak{D}[X,\overline{X}]_c \to \mathfrak{D}[X,\overline{X}] \mathbin{\tilde{\otimes}} \mathfrak{D}[X,\overline{X}]_c$

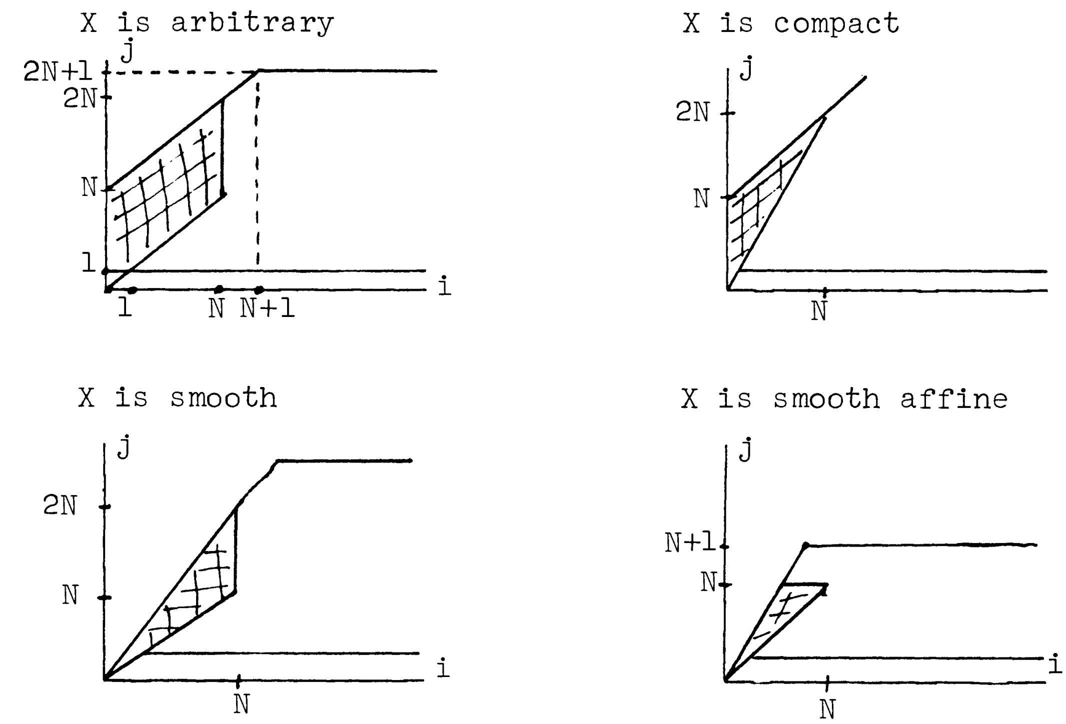
is compatible with any filtration, so we get a morphism of $\tilde{p}$-Hodge complexes $R\Gamma(\overline{X},\mathfrak{D}[X,\overline{X}]_c) \to R\Gamma(\overline{X},\mathfrak{D}[X,\overline{X}] \tilde{\otimes} \mathfrak{D}[X,\overline{X}]_c)$. Clearly this is a quasi-morphism. We have an obvious pairing $R\Gamma(\overline{X},\mathfrak{D}[X,\overline{X}]) \otimes R\Gamma(\overline{X},\mathfrak{D}[X,\overline{X}]_c) \to R\Gamma(\overline{X},\mathfrak{D}[X,\overline{X}] \tilde{\otimes} \mathfrak{D}[X,\overline{X}]_c)$ of $\tilde{p}$-complexes; this defines the desired cup-product $\underline{R\Gamma}(X,A) \overset{L}{\underset{A}{\otimes}} \underline{R\Gamma_c}(X,A) \to \underline{R\Gamma_c}(X,A)$.

In fact, all the standard functoriality for $R\Gamma$, $R\Gamma_c$ may be done on the level of Hodge complexes. E.g. we have canonical morphism $\underline{R\Gamma_c}(X,A) \to \underline{R\Gamma}(X,A)$ which is isomorphism if X is proper; if $i:Y \hookrightarrow X$ is a closed subscheme, then we have a canonical distinguished triangle $\underline{R\Gamma_c}(X\setminus Y,A) \to \underline{R\Gamma_c}(X,A) \xrightarrow{i^*} \underline{R\Gamma_c}(Y,A) \to$; also one has various Gysin maps. Passing to duals, one defines homology Hodge complexes (with compact supports and Borel-Moore ones) with all their standard functorialities.

5. ABSOLUTE HODGE COHOMOLOGY OF ALGEBRAIC VARIETIES; CHARACTERISTIC CLASSES. Let X be a scheme over $\mathbb{C}$. Put $R\Gamma_{\mathcal{H}}(X,A(i)) := R\Gamma_{\mathcal{H}}(\underline{R\Gamma}(X,A)(i))$, $H^{\cdot}_{\mathcal{H}}(X,A(i)) := H^{\cdot}_{\mathcal{H}}(\underline{R\Gamma}(X,A)(i))$. In the same way one defines $R\Gamma_{\mathcal{H}^p}(X,A(i)) := R\Gamma_{\mathcal{H}^p}(\underline{R\Gamma}(X,A)(i))$, $R\Gamma_{c\mathcal{H}^p}(X,A(i)) := R\Gamma_{\mathcal{H}^p}(\underline{R\Gamma_c}(X,A)(i))$, $H^{BM}_{\mathcal{H}\cdot}(X,A(i)) = H^{\cdot}R\Gamma_{\mathcal{H}}(R\,\mathcal{H}om(\underline{R\Gamma_c}(X,A),A(i)) = \mathrm{Hom}^{\cdot}(\underline{R\Gamma_c}(X,A),A(i))$ and so on. These absolute Hodge (or simply $\mathcal{H}$-) homology and cohomology groups may be computed following 1.4 and 2.2. The functorialities, cup-products, etc. that may be done on the level of Hodge complexes induce the ones for absolute Hodge groups.

Here are some properties of $\mathcal{H}$-cohomology.

5.1. In the following pictures, the large regions show where non-zero $H^j_{\mathcal{H}^p}(X,A(i))$ may happen. The hatched regions show where $\Gamma^c_{\mathcal{H}^p}\underline{H}^j(X,A)(i)$ may be non-zero. These pictures follow from [4] (8.24) and (2.2). Put $N = \dim X$.



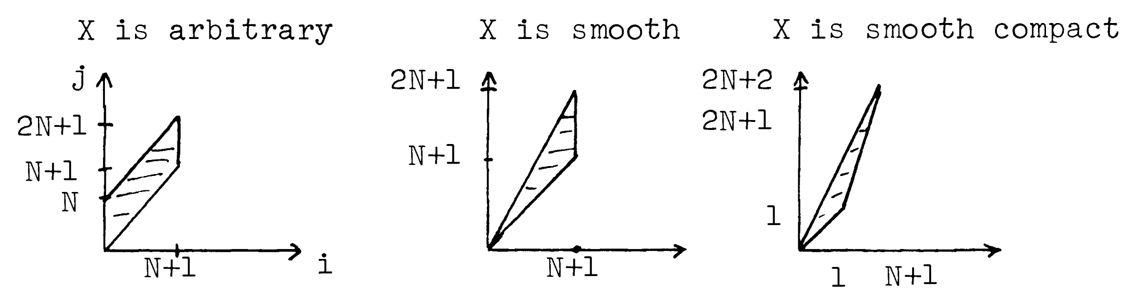
5.2. Cycles. One has $H^0_{c\mathcal{H}}(X,A(0)) = H^0_c(X,A(0))$, $H^{BM}_{\mathcal{H}\,2\cdot\dim X}(X,A(-\dim X)) = H^{BM}_{2\cdot\dim X}(X,A(-\dim X))$. This defines a fundamental cycle $cl_X \in H^{BM}_{2\cdot\dim X}(X,A(-\dim X))$ and a cycle map $CH_i(Y) \to H^{BM}_{2i}(Y,A(-i))$. If Y is smooth compact, then $H^{BM}_{\mathcal{H}2(\dim Y-j)}(Y,\mathbb{Z}(j-\dim Y)) = H^{2j}_{\mathcal{H}}(Y,\mathbb{Z}(j))$ coincides with the standard extension of integral (j,j)-cocycles by means of the j-th Griffiths Jacobian, and the cycle map coincides with Griffiths-Abel-Jacobi period map.

5.3. Absolute Hodge homology and cohomology form a Poincare duality theory with supports in the sense of [3].

5.4. Chern classes. For any scheme, or simplicial scheme X one has a canonical morphism $R\Gamma(X_{Zar},\mathcal{O}^*[-1]) \to R\Gamma_{\mathcal{H}}(X,A(1))$; if X is smooth and $A = \mathbb{Z}$ this is an isomorphism. This arrow defines the 1-st Chern class $c_1:\mathrm{Pic}(X) \to H^2_{\mathcal{H}p}(X,A(1)) = H^2_{\mathcal{H}}(X,A(1))$. If $\mathcal{E}$ is any N-dimensional vector bundle over X, then one has canonical isomorphism $\sum\limits_{0\leq i\leq N-1} c_1(\mathcal{O}(1))^i : \sum\limits_{0\leq i\leq N-1} \underline{R\Gamma}(X,A)(-i)[-2i] \to \underline{R\Gamma}(\mathbb{P}(\mathcal{E}),A)$. Applying $R\Gamma_{\mathcal{H}}$ we get a corresponding isomorphism of absolute Hodge groups. This permits us to define higher Chern classes in absolute Hodge cohomology in a usual way. The other way to define them is to look at cohomology of classifying spaces: one has $H^{2i}_{\mathcal{H}}(BGL_{\cdot}, A(i)) = H^{2i}(BGL_{\cdot}, A(i))$ (here $BGL.$ is standard simplicial classifying space of GL), and so Chern classes in $\mathcal{H}$-groups may be recovered in a unique way from Chern classes in usual cohomology.

5.5. Chern classes in higher K-theory. One defines them according to e.g. [5]. One has a corresponding Riemann-Roch formalism and thus the morphism $r_{\mathcal{H}} H^{\cdot}_{\mathcal{M}}(X, \mathbb{Q}(*)) \to H^{\cdot}_{\mathcal{H}}(X, A(X))$, $H_{\mathcal{M}}.(X, \mathbb{Q}(*)) \to H^{BM}_{\mathcal{H}^p}(X, \mathbb{Q}(*))$ (for $A \supset \mathbb{Q}$) from "absolute motivic" cohomology and homology groups (see Introduction) to Hodge ones. If X is a point, then $r_{\mathcal{H}}$ coincides with Borel's regulator map (cf [1]).

5.6. Rigidity. For a scheme X of dimension N consider the following regions on (i,j)-plane:



If a pair (i,j) lies outside this region, then the elements of $H^j_{\mathcal{H}^p}(X, A(i))$ are rigid. This means that for any connected scheme S, $s_1, s_2 \in S(\mathbb{C})$ and $y \in H^j_{\mathcal{H}^p}(X \times S, A(i))$ the elements $s_1^*(y), s_2^*(y) \in H^j_{\mathcal{H}^p}(X, A(i))$ coincide. (To show this notice that we may suppose that S is a smooth curve. In this case either $H^j_{\mathcal{H}^p}(X, A(i))$ is zero, or $H^j_{\mathcal{H}^p}(X \times S, A(i)) = \Gamma^1_{\mathcal{H}^p}(\underline{H}^{j-1}(X \times S, A(i)))$. Since $\underline{H}^{j-1}$ is rigid, this implies the fact.) In the usual way this fact implies that for such (i,j) the image of $H^i_{\mathcal{M}}(X, \mathbb{Q}(i)) \to H^j_{\mathcal{H}^p}(X, A(i))$ is countable.

5.7. Weak $\mathcal{H}$-cohomology of schemes. Define weak $\mathcal{H}$-cohomology of a scheme X by formula $R\Gamma_{\mathcal{H}^w}(X, A(i)) := R\Gamma_{\mathcal{H}^w}(\underline{R\Gamma}(X, A(i)))$ (cf. 3.12). These complexes are the ones studied in [1] and [7]. According to 3.13 we have a natural arrow

$R\Gamma_{\mathcal{H}}(X,A(i)) \to R\Gamma_{\mathcal{H}^W}(X,A(i))$ that with $\cup$-product and induces injection on cohomology modulo torsion. Clearly one has $H^j_{\mathcal{H}}(X,A(i)) = H^j_{\mathcal{H}^W}(X,A(i))$ for $j \leq i$, or, if X is compact, for $j \leq 2i$.

Now suppose that $A = \mathbb{R}$. Denote by $\pi_i : \mathbb{C} = \mathbb{R}(i) \oplus \mathbb{R}(i-1) \to \mathbb{R}(i)$ the projection. Let X be a smooth variety; choose $j: X \hookrightarrow \overline{X}$ as in 4.1. Let $C^\infty\Omega^{\cdot}_X$ be the complex of $\mathbb{R}$-valued C^∞-class forms on X, $C^\infty\Omega^{\cdot}_X(i)$ be the complex of $\mathbb{R}(i)$-valued ones, $F^{\cdot}$ be stupid filtration on $\Omega^{\cdot}_{\overline{X}}(\log \mathcal{D})$, and $\pi_i : \Omega^{\cdot}_{\overline{X}}(\log \mathcal{D}) \to j_*C^\infty\Omega^{\cdot}_X(i)$ be the injection $\Omega^{\cdot}_{\overline{X}}(\log \mathcal{D}) \to j_*C^\infty\Omega^{\cdot}_X(\mathbb{C}) = j_*C^\infty\Omega^{\cdot}_X(i) \oplus j_*C^\infty\Omega^{\cdot}_X(i-1)$ followd by projection π_i. Put $C(i) := \mathrm{Cone}(F^i \xrightarrow{\pi_{i-1}} j_*C^\infty\Omega^{\cdot}_X(i-1))[-1]$. Clearly 3.13.1 implies that one has canonical isomorphism $R\Gamma_{\mathcal{H}^W}(X,\mathbb{R}(i)) = R\Gamma(\overline{X},C(i))$. The $\cup$-product on $R\Gamma_{\mathcal{H}^W}(X,\mathbb{R}(\cdot))$ is induced by the arrow $\cup: C(i) \underset{\mathbb{R}}{\otimes} C(j) \to C(i+j)$ defined by formula $f_i \cup f_j := f_i \wedge f_j$, $s_i \cup s_j = 0$, $f_i \cup s_j = (-1)^{\deg f_i} \pi_i(f_i) \wedge s_j$, $s_i \cup f_j = s_i \wedge s_j(f_j)$.

<u>Example 5.7.1</u>. If X is smooth, then $H^i_{\mathcal{H}}(X,\mathbb{R}(i)) = H^i_{\mathcal{H}^W}(X,\mathbb{R}(i)) = \{\varphi \in \Gamma(X, C^\infty\Omega^{i-1}_X(i-1)) : d\varphi = \omega \pm \overline{\omega}, \omega$ is holomorphic i-form with logarithmic singularities as $\infty\}$/exact ones. The $\cup$-product is given by the formula $\varphi_1 \cup \varphi_2 = \varphi_1 \wedge \widetilde{d\varphi_2} + (-1)^{i_1}\widetilde{d\varphi_1} \wedge \varphi_2$. Here $\varphi_{1,2} \in H^{i_{1,2}}_{\mathcal{H}^W}(X,\mathbb{R}(i_{1,2}))$ and $\widetilde{d\varphi} := \omega + (-1)^i\overline{\omega}$ if $d\varphi = \omega + (-1)^{i-1}\overline{\omega}$. The arrow $\mathcal{O}^*(X) \to H^1(X,\mathbb{R}(1))$ from 5.4 is $f \to \log|f|$.

6. <u>HODGE CONJECTURE IN $\mathcal{H}$-COHOMOLOGY</u>. Let X be a smooth scheme over $\mathbb{C}$. The groups $H^j_{\mathcal{H}}(X,\mathbb{Q}(i))$ carry a natural topology. One defines it via the exact sequence 1.4. $0 \to \Gamma^1_{\mathcal{H}}(\underline{H}^{j-1}(X,\mathbb{Q}(i)) \to H^j_{\mathcal{H}}(X,\mathbb{Q}(i)) \to \Gamma^0_{\mathcal{H}}(H^j(X,\mathbb{Q}(i)) \to 0$ so that $\Gamma^1_{\mathcal{H}}(\quad)$ with an obvious (non-separated) topology is a connected component of $H^j_{\mathcal{H}}$.

<u>Conjecture 6</u>. The image of $H^j_{\mathcal{M}}(X,\mathbb{Q}(i)) \to H^j_{\mathcal{H}}(X,\mathbb{Q}(i))$ is dense. This conjecture divides into two parts. First it claims that $H^j_{\mathcal{M}}(X,\mathbb{Q}(i)) \to \Gamma^0_{\mathcal{H}}(\underline{H}^j(X,\mathbb{Q}(i)) = \mathrm{Hom}_{\mathcal{H}}(\mathbb{Q}(-i),\underline{H}^j(X,\mathbb{Q}))$ is surjective. If X is compact, this is exactly the usual Hodge conjecture. Consider instead the case when X coincides with

its "generic point". Then, by Suslin's theorem ([9], cf. also [3]), one has $H^i_{\mathcal{M}}(X,\mathbb{Q}(i)) = K^{\mathrm{Milnor}}_i(X) \otimes \mathbb{Q}$. In particular $H^i_{\mathcal{M}}(X,\mathbb{Q}(i))$ is generated by symbols of elements of $\mathcal{O}^*(X) \otimes \mathbb{Q} = H^1_{\mathcal{M}}(X,\mathbb{Q}(1))$. So in this case Conjecture 6 implies the following.

Conjecture 6.1. Let ω be a meromorphic form of degree i with logarithmic singularities at infinity. If any period of ω is rational multiple of $(2\pi i)^i$, then ω is rational linear combination of forms of type $d\log f_1 \wedge \dots \wedge d\log f_i$.

Secondly, Conjecture 6 claims that $\operatorname{Ker}(H^j_{\mathcal{M}}(X,\mathbb{Q}(i)) \to H^j_B(X,\mathbb{Q}(i)))$ has dense image in $\Gamma^1_{\mathcal{H}^p_{\mathbb{Q}}}(\underline{H}^{j-1}(X,\mathbb{Q}(i)))$. This is equivalent to the group having dense image in the $\mathbb{R}$-vector space $\Gamma^1_{\mathcal{H}^p_{\mathbb{Q}}}(\underline{H}^{j-1}(X,\mathbb{R}(i)))$ $= [W_{2i}(\underline{H}^{j-1}(X,\mathbb{R}))/\operatorname{Re}(F^i \cap W_{2i})(\underline{H}^{j-1}(X,\mathbb{R}))](i-1)$. Here is a corollary of this. Suppose that $\overline{X}$ is smooth. By another theorem of Suslin ([9] and also [8]) the groups $H^j_{\mathcal{M}}(X,\mathbb{Q}(i))$ vanish outside the codimension $j-i$. So Conjecture 6 implies that $\Gamma^1_{\mathcal{H}^p_{\mathbb{R}}}(\underline{H}^{j-1}(X,\mathbb{R}(i)))$ also vanishes outside codimension $j-i$. This means that for certain $Y \subset X$ of codimension $j-i$ the image of $W_{2i}(H^{j-1}(X,\mathbb{C}))$ in $W_{2i}(H^{j-1}(X\setminus Y,\mathbb{C}))$ belongs to the sum of $(F^i \wedge W_{2i})(H^{j-1}(X\setminus Y,\mathbb{C}))$ plus complex conjugate. So if X is compact and $\bigoplus_{\substack{p+q=j-1\\ p,q>i}} H^{p,q}(X,\mathbb{C})$ is not defined over $\mathbb{Q}$, then there exists $Y \subset X$ of codimension $j-i$ s.t. mixed Hodge structure $\underline{H}^{j-1}(X\setminus Y,\mathbb{R})$ does not split over $\mathbb{R}$.

Let me describe explicitly the conjecture in case X is compact and $j = 2i-1$. According to Bloch and Quillen elements of $H^j_{\mathcal{M}}(X,\mathbb{Q}(i))$ may be represented by finite formal sums $\varphi = \Sigma(c_\ell, f_\ell)$, where c_ℓ is irreducible subscheme of codimension $i-1$ and f_ℓ is invertible function at generic point of c_ℓ s.t. $\sum_\ell \operatorname{div} f_\ell$ is zero as codimension i cycle on X. The image of φ in $H^{2i-2}(X,\mathbb{R}(i-1))/\pi_{i-1}F^i$ may be constructed as follows. Consider the current $\delta_\varphi := \Sigma \log|f_\ell|\delta_{c_\ell}$ of type $(i-1,i-1)$ (note that $\log|f_\ell|$ is L^1 on C_ℓ, so this definition makes sense). The condition on φ above means that both $(i,i-1)$ and $(i-1,i)$-components of $d\delta_\varphi$ are closed currents. Since $d\delta_\varphi$ is exact, both $(d\delta_\varphi)^{(i,i-1)}$ and $(d\delta_\varphi)^{(i-1,i)}$ are exact. So there exist some currents $a_\varphi = \sum_{p\geq i} a_\varphi^{(p,q)}$ and $\bar a_\varphi = \sum_{q\geq i} \bar a_\varphi^{(p,q)}$ of degree $2i-2$ s.t. $da_\varphi = -(d\delta_\varphi)^{(i,i-1)}$, $d\bar a_\varphi = -(d\delta_\varphi)^{(i-1,i)}$ so $\tilde\delta_\varphi := a_\varphi + a_\varphi + \bar a_\varphi$ is a closed current. Clearly the cohomology class of $\tilde\delta_\varphi$ up to an element $F^i + \bar F^i$ does not depend on the choice of a_φ, $\bar a_\varphi$, so its class in $H^{2i-2}(X,\mathbb{C})/F^i + \bar F^i = H^{i-1,i-1}(X)$ depends only on φ. It is easy to see that it is just the image of φ under $H^{2i-1}_{\mathcal{M}}(X, \mathbb{Q}(i)) \to H^{2i-1}_{\mathcal{H}}(X, \mathbb{R}(i))$. So the Conjecture 6 claims that any $(i-1,i-1)$-cycle is $\mathbb{C}$-linear combination of such.

7. HODGE STRUCTURES OVER $\mathbb{R}$. The Hodge structures we spoke about were ones over $\mathbb{C}$. Let me say a few words about how to translate all above to the case of Hodge structures over $\mathbb{R}$. Recall that an A-Hodge structure over $\mathbb{R}$ is a pair (M,S), where M is a Hodge structure over $\mathbb{C}$ and $S \in \operatorname{End} M_A$ is an involution s.t. $S_{\mathbb{Q}} := S \otimes \mathbb{Q}$ respects $W_{\mathbb{Q}}$ and $S_{\mathbb{C}}$ - the anti-linear prolongation of S to $M_{\mathbb{C}}$ - respects $F^{\cdot}$ (and $W_{\cdot\mathbb{C}}$). Denote by $\mathcal{H}_{A/\mathbb{C}}$ the category (previously denoted by $\mathcal{H}_A$) of Hodge structures over $\mathbb{C}$ and by $\mathcal{H}_{A/\mathbb{R}}$ of ones over $\mathbb{R}$. If $M,N \in \operatorname{Ob} \mathcal{H}_{A/\mathbb{R}}$, then $\operatorname{Hom}_{\mathcal{H}_{A/\mathbb{C}}}(M,N)$ is $\mathbb{Z}/2\mathbb{Z}$-module and $\operatorname{Hom}_{\mathcal{H}/\mathbb{R}}(M,N) = \operatorname{Hom}_{\mathcal{H}/\mathbb{C}}(M,N)^{\mathbb{Z}/2\mathbb{Z}}$. This implies that $R\operatorname{Hom}_{\mathcal{H}_{A/\mathbb{R}}}(\mathcal{F},G) = R\Gamma^{\mathbb{Z}/\mathbb{Z}}\, R\operatorname{Hom}^{\cdot}_{\mathcal{H}_{A/\mathbb{C}}}(\mathcal{F},G)$ for any complexes $\mathcal{F}^{\cdot}$, $G^{\cdot}$ of Hodge structures over $\mathbb{R}$; here $R\Gamma^{\mathbb{Z}/2\mathbb{Z}}$ is the right derived functor of taking invariants of real Frobenius. In particular $R\Gamma_{\mathcal{H}/\mathbb{R}}(\mathcal{F}) := R\operatorname{Hom}_{\mathcal{H}/\mathbb{R}}(A,\mathcal{F}) = R\Gamma_{\mathbb{Z}/2\mathbb{Z}} R\Gamma_{\mathcal{H}/\mathbb{C}}(\mathcal{F})$. If $1/2 \in A$, then $R\Gamma^{\mathbb{Z}/2\mathbb{Z}}(\mathcal{M}) = \mathcal{M}^{\mathbb{Z}/2\mathbb{Z}}$ for any complex $\mathcal{M}$ of $A[\mathbb{Z}/2\mathbb{Z}]$-modules, so in this case $R\Gamma_{\mathcal{H}/\mathbb{R}}(\mathcal{F}) = \tilde\Gamma_{\mathcal{H}/\mathbb{C}}(\mathcal{F})^{\mathbb{Z}/2\mathbb{Z}} =: \tilde\Gamma_{\mathcal{H}/\mathbb{R}}(\mathcal{F})$ and $H^{\cdot}_{\mathcal{H}/\mathbb{R}}(\mathcal{F}) = H^{\cdot}_{\mathcal{H}/\mathbb{C}}(\mathcal{F})^{\mathbb{Z}/2\mathbb{Z}}$.

Similarly, one defines polarizable complexes, Hodge complexes and so on over $\mathbb{R}$. It is easy to see that for any scheme X over $\mathbb{R}$ the complex $\underline{R\Gamma}(X \otimes \mathbb{C}, A)$ is a Hodge complex over $\mathbb{R}$; this permits us to define absolute Hodge cohomology of schemes over $\mathbb{R}$ and so on. All the results of the paper remain valid in this context.

For further use let me mention the following. Let M be a pure $\mathbb{R}$-Hodge structure over $\mathbb{R}$ of weight j. One has corresponding Γ-factor $L_\infty(M,S)$ (cf. e.g. [5] n°5). This is a meromorphic function without zeros; the poles of $L_\infty(M,S)$ are integers.

Lemma 7.1. For $n \in \mathbb{Z}$ the order of pole of $L_\infty(M,S)$ at $s = n$ equals to $\dim \Gamma^1_{\mathcal{H}/\mathbb{Q}}(M(j+1-n))$. $\square$

The same is true for Hodge structures over $\mathbb{C}$.

8. VALUES OF L-FUNCTIONS. (cf. [1] §3) First some preliminaries. For a scheme $X_{\mathbb{R}}$ over $\mathbb{R}$ let $R\Gamma_B(X_{\mathbb{R}}, A(i)) := R\Gamma^{\mathbb{Z}/2\mathbb{Z}}(\underline{R\Gamma}(X_{\mathbb{R}}, A)(i)_A)$, $R\Gamma_B(X_{\mathbb{R}}, \mathbb{C}) := R\Gamma^{\mathbb{Z}/2\mathbb{Z}}(\underline{R\Gamma}(X_{\mathbb{R}}, A)_{\mathbb{C}})$ denote the usual topological chain complexes, and $H^{\cdot}_B(X_{\mathbb{R}}, A(i))$, $H^{\cdot}_B(X_{\mathbb{R}}, \mathbb{C})$ be corresponding cohomology groups. Clearly $R\Gamma_B(X_{\mathbb{R}}, \mathbb{C})$ coincides with de Rham complex of X over $\mathbb{R}$, and for any $i \in \mathbb{Z}$ one has $R\Gamma_B(X, \mathbb{C}) = R\Gamma_B(X, \mathbb{R}(i)) \oplus R\Gamma_B(X, \mathbb{R}(i-1))$; denote by $\pi_i : H^{\cdot}_{DR}(X_{\mathbb{R}}) \to H^{\cdot}_B(X_{\mathbb{R}}, \mathbb{R}(i))$ the corresponding projection. If X is smooth and proper, then one has canonical exact sequence

$$(8.1) \qquad 0 \to F^i H^{j-1}_{DR}(X_{\mathbb{R}}) \xrightarrow{\pi_{i-1}} H^{j-1}_B(X_{\mathbb{R}}, \mathbb{R}(i-1)) \to \Gamma^1_{\mathcal{H}/\mathbb{R}}(\underline{H}^{j-1}(X_{\mathbb{R}}, \mathbb{R}(i))) \to 0$$

and the isomorphism $\Gamma^1_{\mathcal{H}/\mathbb{R}} \underline{H}^{j-1}(X_{\mathbb{R}}, \mathbb{R}(i)) \xrightarrow[\sim]{\varepsilon} H^j_{\mathcal{H}}(X_{\mathbb{R}}, \mathbb{R}(i))$ (cf. 1.4).

8.2. For a scheme X over $\mathbb{Q}$ put $H^{\cdot}_{\mathcal{H}}(X, A(i)) := H^{\cdot}_{\mathcal{H}}(X \otimes \mathbb{R}, A(i))$ and so on. Suppose from now on that is smooth and proper. Then, for $j < 2i$, the left two terms of (8.1) have natural rational structures: $F^i H^{j-1}_{DR}(X \otimes \mathbb{R}/\mathbb{R}) = F^i H^{j-1}_{DR}(X) \otimes \mathbb{R}$ and $H^{j-1}_B(X, \mathbb{R}(i-1)) = H^{j-1}_B(X, \mathbb{Q}(i-1)) \otimes \mathbb{R}$. Consider the 1-dimensional $\mathbb{Q}$-vector space $V(x,j,i) := \det H^{j-1}_B(X, \mathbb{Q}(i-1)) \otimes \det^{\otimes -1} F^j H^{j-1}_{DR}(X)$; then, by (8.1), one has a canonical isomorphism $\det H^j_{\mathcal{H}}(X, \mathbb{R}(i)) = V(X,j,i) \otimes \mathbb{R}$ this defines natural $\mathbb{Q}$-structure on $\det H^j_{\mathcal{H}}(X, \mathbb{R}(i))$.

8.3. Now consider absolute motivic cohomology $H^{\cdot}_{\mathcal{M}}(X, \mathbb{Q}(*))$. Assume that X has a regular model $X_{\mathbb{Z}}$ over $\mathbb{Z}$, i.e., $X_{\mathbb{Z}}$ is regular scheme proper over $\mathbb{Z}$ s.t. $X_{\mathbb{Z}} \otimes \mathbb{Q} = X$.

Lemma-Definition 8.3.1. (cf. [1] (2.4))* Put

$$H^{\cdot}_{\mathcal{M}}(X,\mathbb{Q}(*))_{\mathbb{Z}} := \mathrm{Im}(H^{\cdot}_{\mathcal{M}}(X_{\mathbb{Z}},\mathbb{Q}(*)) \to H^{\cdot}_{\mathcal{M}}(X,\mathbb{Q}(*)).$$

This group does not depend on the choice of particular model $X_{\mathbb{Z}}$.

Proof. Let $X'_{\mathbb{Z}}$ be another model and $p,p' : X_{\mathbb{Z}} \underset{\mathbb{Z}}{\otimes} X'_{\mathbb{Z}} \to X_{\mathbb{Z}}, X'_{\mathbb{Z}}$ be projections. The lemma follows from commutative diagram

$$\begin{tikzcd}
 & & K_{\cdot}(X) & \\
K_{\cdot}(X_{\mathbb{Z}}) \arrow[r, "p'^*"] \arrow[urr] & K_{\cdot}(X_{\mathbb{Z}} \otimes X'_{\mathbb{Z}}) \arrow[r] & K'_{\cdot}(X_{\mathbb{Z}} \otimes X'_{\mathbb{Z}}) \arrow[r, "p_*"] & K'_{\cdot}(X'_{\mathbb{Z}}) = K_{\cdot}(X'_{\mathbb{Z}}) \arrow[ul]
\end{tikzcd}$$

□

Clearly $H^{\cdot}_{\mathcal{M}}(X,\mathbb{Q}(*))_{\mathbb{Z}} \subset H^{\cdot}_{\mathcal{M}}(X,\mathbb{Q}(*))$ is a subring.

Conjecture 8.3.2. a) One has $H^{j}_{\mathcal{M}}(X,\mathbb{Q}(i))_{\mathbb{Z}} = H^{j}_{\mathcal{M}}(X,\mathbb{Q}(i))$ for any (i,j) that does not satisfy $i \leq j \leq 2i-1$ and $i-1 \leq \dim X$.

b) If X has potentially good reduction at any prime, then the above inequalities may be replaced by $2i-2 = j-1 \leq 2 \dim X$.

Note that 8.3.2 is implied (by use of localization sequence) by the following.

Conjecture 8.3.3. a) If F is any field of finite characteristic p, then $H^{j}_{\mathcal{M}}(F,\mathbb{Q}(i)) = 0$ for $j \neq i$ or for $j > \mathrm{tr.deg}\, F$. (Note that by [9] and [8] the group $H^{i}_{\mathcal{M}}(F,\mathbb{Q}(i))$ coincides with i-th Milnor K-functor of F tensored by $\mathbb{Q}$.)

b) One has $K_j(Y) \otimes \mathbb{Q} = 0$ for any smooth proper Y over $\mathbb{F}_p$ and $j \neq 0$. □

Remark 8.3.4. a) The conjectures 8.3.3 are valid for 1-dimensional schemes over $\mathbb{F}_p$ according to results of Harder. So 8.3.2 is true when X is a curve.

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* It was Bloch who has found that one needs to use $H^{\cdot}_{\mathcal{M}}(X)_{\mathbb{Z}}$ rather than $H^{\cdot}_{\mathcal{M}}(X)$ in the conjectures from 8.4. Namely, he had shown (jointly with Grayson) that for certain elliptic curves X over $\mathbb{Q}$ one has $H^{2}_{\mathcal{M}}(X,\mathbb{Q}(2)) \neq H^{2}(X,\mathbb{Q}(2))_{\mathbb{Z}}$ and $\mathrm{rk}\, H^{2}_{\mathcal{M}}(X,\mathbb{Q}(2)) > 1$ (letter to Soule, March 1981).

b) Clearly 8.3.3 b) follows from generalized Tate conjecture, due to Friedlander, it claims that for any smooth Y over $\mathbb{F}_p$ one has $H^{\cdot}_{\mathcal{M}}(Y,\mathbb{Q}(*)) \otimes \mathbb{Q}_\ell \overset{\sim}{\to} H^{\cdot}_{ét}(Y \otimes \overline{\mathbb{F}}_p, \mathbb{Q}_\ell(*))^F$. Compare with the situation over $\mathbb{Q}$.

8.4. Now we may begin to formulate the conjectures about the values of L-functions.

Conjecture 8.4.1. Consider the arrow $r_{\mathcal{H}}$:

$$H^j_{\mathcal{M}}(X,\mathbb{Q}(i)) \to H^j_{\mathcal{H}}(X,\mathbb{R}(i)) = H^{j-1}_B(X,\mathbb{R}(i-1))/\pi_{i-1}F^i.$$

a) For $j \leq 2i-2$ it induces isomorphism
$$H^j_{\mathcal{M}}(X,\mathbb{Q}(i))_{\mathbb{Z}} \otimes \mathbb{R} \overset{\sim}{\to} H^{j-1}_B(X,\mathbb{R}(i-1))/\pi_{i-1}F^iH^{j-1}_{DR}(X \otimes \mathbb{R}) = H^j_{\mathcal{H}}(X,\mathbb{R}(i)).$$

b) Put $Z^a(X) = \{$a-codimensional cycles on X modulo homological equivalence$\} \otimes \mathbb{Q} \hookrightarrow H^{2a}_B(X,\mathbb{Q}(a))$. Then $ch_{\mathcal{H}}$ together with this injection induces isomorphism
$$(H^{2i-1}_{\mathcal{M}}(X,\mathbb{Q}(i))_{\mathbb{Z}} \oplus Z^{i-1}(X)) \otimes \mathbb{R} \overset{\sim}{\to} H^{2i-2}_B(X,\mathbb{R}(i-1))/\pi_{i-1}F^iH^{j-1}_{DR}(X \otimes \mathbb{R})$$
$$= H^{2i-1}_{\mathcal{H}}(X,\mathbb{R}(i)).$$ □

This conjecture determines for $j < 2i$ a certain $\mathbb{Q}$-structure on $H^j_{\mathcal{H}}(X,\mathbb{R}(i))$, and so the one denoted by $W(X,j,i)$ on the determinant: $W(X,j,i) \otimes \mathbb{R} = \det H^j_{\mathcal{H}}(X,\mathbb{R}(i))$. By 8.2 we have another $\mathbb{Q}$-structure $V(X,j,i)$ on the same 1-dimensional $\mathbb{R}$-spaces. One has $W(X,j,i) = C(X,j,j-i)\cdot V(X,j,i) \subset \det H^j_{\mathcal{H}}(X,\mathbb{R}(i))$ for certain $C(X,i,j-i) \in \mathbb{R}^*$; this equality determines $C(X,j,j-i)$ up to an element of $\mathbb{Q}^*$.

Now consider L-function $L(H^{j-1}(X),s)$; we assume that all the standard conjectures about it are valid. In particular $\Lambda(X,s) := L_\infty(\underline{H}^{j-1}(X \otimes \mathbb{R}),s)\cdot L(H^{j-1}(X),s)$ is meromorphic on the whole complex plane with the only possible pole at $s = \frac{j+1}{2}$ of degree $\operatorname{rk} Z^{\frac{j-1}{2}}(X)$ in case of odd j (Tate), and satisfies functional equation for $s \leftrightarrow j-s$. Note that by the Weil conjectures, proved by Deligne, and standard conjectures on degenerate local L-multiples, the Euler product for $L(\underline{H}^{j-1}(X),s)$ converges absolutely for $\operatorname{Re} S > \frac{j+1}{2}$. This, together with 8.4.1 and 7.1, implies that for any $n \in \mathbb{Z}$, $n < j/2$, the order of zero of $L(H^{j-1}(X),s)$ at $s = n$ equals to $d(j,n) = \operatorname{rk} H^j_{\mathcal{M}}(X,\mathbb{Q}(j-n))_{\mathbb{Z}}$. One has $L(H^{j-1}(X),s) = \tilde{C}(X,j,n)(s-n)^{d(X,j,n)} + O(s-n)d(X,j,n)+1$ for certain $\tilde{C}(X,j,n) \in \mathbb{R}^*$.

Conjecture 8.4.2. One has $C(X,j,n) = \tilde{C}(X,j,n)$ (up to an element of $\mathbb{Q}^*$). $\square$

This conjecture + functional equation determines, up to multiplication by non-zero rational, the values, or principal terms, of L-functions at any integer but the middle of the critical strip (see the end of 8.5).

8.5. Here are motivic versions of the above conjectures. First let me show how to define all the needed functors for motives. Put $\tilde{H}^{\cdot}_{\mathcal{M}}(X,\mathbb{Q}(*))_{\mathbb{Z}} := \mathrm{Ker}(H^{\cdot}_{\mathcal{M}}(X,\mathbb{Q}(*))_{\mathbb{Z}} \to H^{\cdot}_{B}(X,\mathbb{Q}(*)))$. Clearly this is an ideal in $H^{\cdot}_{\mathcal{M}}(X,\mathbb{Q}(*))$. Since we assume X to be smooth and proper, one has $\tilde{H}^{j}_{\mathcal{M}}(X,\mathbb{Q}(i))_{\mathbb{Z}} = H^{j}_{\mathcal{M}}(X,\mathbb{Q}(i))_{\mathbb{Z}}$ for $j \neq 2i$ and $H^{2i}_{\mathcal{M}}(X,\mathbb{Q}(i))_{\mathbb{Z}} / \tilde{H}^{2i}_{\mathcal{M}}(X,\mathbb{Q}(i))_{\mathbb{Z}} = Z^{i}(X)$.

Conjecture 8.5.1. $\tilde{H}^{\cdot}_{\mathcal{M}}(X,G(*))_{\mathbb{Z}}$ is an ideal of square zero. $\square$

This implies that $\cup$-product action of $H^{2}_{\mathcal{M}}(X,\mathbb{Q}(*))_{\mathbb{Z}}$ on $\oplus \tilde{H}^{j}(X,\mathbb{Q}(i))_{\mathbb{Z}}$ factors through $Z^{\cdot}(X)$ (for $\bigoplus_{j\neq 2i} H^{j}(X,\mathbb{Q}(i))$ - the case we need - this fact also follows from 8.4.1; if you do not want to assume 8.5.1 or 8.4.1, in the following discussion you have to replace the groups $H^{j}(X,\mathbb{Q}(i))_{\mathbb{Z}}$, $j \neq 2i$, by their images in $H^{j}_{\mathcal{H}}(X,\mathbb{R}(i)) = H^{j-1}_{B}(X,\mathbb{R}(i-1)))$. The groups $H^{j}_{\mathcal{M}}(X,\mathbb{Q}(i))_{\mathbb{Z}}$, are naturally functors on category of correspondences. The above implies that $\tilde{H}$-groups are naturally functors on the category C of correspondences modulo homological equivalence (the objects of C are smooth proper schemes over $\mathbb{Q}$ and $\mathrm{Hom}_{C}(X,Y) := Z^{\dim Y}(X\times Y))$. It will be more convenient to consider the category $C_{\overline{\mathbb{Q}}} := C \otimes \overline{\mathbb{Q}}$ of correspondences with $\overline{\mathbb{Q}}$-coefficients; the $\overline{\mathbb{Q}}$-modules $\tilde{H}^{j}_{\mathcal{M}}(X,\overline{\mathbb{Q}}(i))_{\mathbb{Z}} := \tilde{H}^{j}_{\mathcal{M}}(X,\mathbb{Q}(i)) \underset{\mathbb{Z}}{\otimes} \overline{\mathbb{Q}}$ are naturally $\overline{G}$-functors on $C_{\overline{\mathbb{Q}}}$. Let $\mathcal{M}^{\mathrm{eff}}$ be the category of effective Grothendieck $\overline{\mathbb{Q}}$-motives over $\mathbb{Q}$ = pseudoabelian envelope of $C_{\overline{\mathbb{Q}}}$ and let $\tilde{H}^{j}_{\mathcal{M}}(\cdot,\mathbb{Q}(i))_{\mathbb{Z}}$ be canonical prolongations of the above functors to $\mathcal{M}^{\mathrm{eff}}$ (so, if M is motive of a scheme X, then $H^{j}_{\mathcal{M}}(X,\overline{\mathbb{Q}}(i))_{\mathbb{Z}} = H^{j}_{\mathcal{M}}(M,\mathbb{Q}(i))_{\mathbb{Z}}$). Finally the theorem about cohomology of projective space implies that $\tilde{H}^{j}_{\mathcal{M}}(M(\ell),\mathbb{Q}(i))_{\mathbb{Z}} = \tilde{H}^{j+2\ell}_{\mathcal{M}}(M,\mathbb{Q}(i+\ell))_{\mathbb{Z}}$ for any $\ell \in \mathbb{Z}$, $\ell < 0$ (here (ℓ) means motivic Tate twist). So we may prolong the above functors on the category $\mathcal{M}$ of all motives demanding that this equality should be valid for any $\ell \in \mathbb{Z}$. In the same way one defines $\overline{\mathbb{Q}}$-modules $Z^{\cdot}(M)$ and $\mathbb{R} \otimes \overline{\mathbb{Q}}$ modules $\tilde{H}^j_{\mathcal{H}}(M, \mathbb{R}(i))$, for any motive M. We have standard functorial morphisms between these functors.

8.5.2. Now you may reformulate Conjecture 8.4.1 word-by-word replacing X by arbitrary motive M. According to Deligne-Gross conjecture one has $L(H^{j-1}(M),s) = \tilde{C}(M,j,n)(s-n)^{d(j,n)} + O(s-n)^{d(j,n)+1}$, where $\tilde{C}(M,j,n) \in (\overline{\mathbb{Q}} \otimes \mathbb{R})^*$ for certain $d(j,n) \in \mathbb{Z}$ (recall that L-functions for $\overline{\mathbb{Q}}$-motives take values in $\overline{\mathbb{Q}} \otimes \mathbb{C}$ of [5]). The order of zero-counterpart of Conjecture 8.4.1 claims that $d(j,n) = \dim H^j_{\mathcal{M}}(M, \mathbb{Q}(j-n))$ for $n < j/2$. To formulate the conjecture about the values, say that $v_1, v_2 \in (\mathbb{Q} \otimes \mathbb{R})^*$ are equivalent, $v_1 \sim v_2$ if $v_1 \cdot v_2^{-1} \in \mathbb{Q}^*$. Define the equivalence class $C(M,j,n) \in (\overline{\mathbb{Q}} \otimes \mathbb{R})^* / \overline{\mathbb{Q}}^*$ the same way we defined $C(X,j,n)$ in 8.4: you have only to consider $\overline{\mathbb{Q}}$ and $\overline{\mathbb{Q}} \otimes \mathbb{R}$-modules instead of $\mathbb{Q}$- and $\mathbb{R}$-ones respectively (note that all the $\overline{\mathbb{Q}} \otimes \mathbb{R}$-modules you encounter are free ones by [5] (2.5). The conjecture 8.4.2 for motives claims that for $n < j/2$ one has $\tilde{C}(M^0,j,n) \sim C(M,j,n)$. (Here M^0 is any motive s.t. there exists a pairing $M \otimes M^0 \to \mathbb{Q}(1-j)$ that induces non-degenerate pairing between H^{j-1}_{DR}. One may reformulate it as a conjecture about the values of L-function in the half-plane of absolute convergence. Namely, use the functional equation to see that above (for $n < \frac{j-1}{2}$ for simplicity) is equivalent to the equality $C(M,j,j-m) \sim \mathcal{E}^{-1} \cdot (2\pi\sqrt{-1})^{d_n + (j-1-2m)\cdot d/2} L(H^{j-1}(M), m)$ for $m > \frac{j+1}{2}$. Here $d_m = \dim H^{j-1}_B(M, \mathbb{R}(1+m))$, $d = \dim H^{j-1}_{DR}(M)$ and $\mathcal{E}$ is the equivalence class of values of (finite) $\mathcal{E}$-factors in the functional equation at any integer (this does not depend on the integer) (cf. [5] §5).

8.6. The current status of this conjecture is the following. If $d(j,n) = 0$, they reduce to Deligne's conjectures [5]. The conjectures are true for the φ-functions of number fields (Borel's theorem), and for Dirichlet's L-functions ([1] §7). There are partial results for L-functions over $\mathbb{Q}$ of elliptic curves with complex multiplication (Bloch; values of L-functions at $S = 2$), and for L-functions over $\mathbb{Q}$ of modular curves (values at arbitrary integers, cf. my subsequent paper); also there are similar facts about the values at $s = 2$ of L-functions of product of two modular curves (Bloch; case of $X_0(37) \times X_0(37)$; for general case see [1] §6).

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