[promised last time:] (Axler, §5C)

<u>Thm</u> if $F = C$ and V is fin. dim.
then any linear op $T : V$ to V has
an upper-triangular matrix

<u>Q</u> can we remove the hypothesis $F = C$?

no: upper-triangular matrix implies
existence of eigenvector

we saw that rotations in R^2 have no eigenvector

<u>Q</u> can we remove the hypothesis V f.d.?

no: same issue of eigenvectors (recall $F[x]$)

<u>Pf</u> if $n = 0$ or 1, then done
induct on n, using two key ideas:

1) since $F = C$ and V fin. dim., T has an eigenline
let λ be the eigenvalue
then the λ-eigenspace $\ker(T - \lambda) = \{v \mid Tv = \lambda v\}$
is nonzero
so $\dim \operatorname{im}(T - \lambda) < \dim V$
want to apply inductive hypothesis to $\operatorname{im}(T - \lambda)$

set $W = \operatorname{im}(T - \lambda)$ [similar tactic as last time:]
W is T-stable!
if w in W
then $w = (T - \lambda)v$ for some v
so $Tw = T((T - \lambda)v) = (T - \lambda)(Tv)$ in W

so T restricts to an op $T|_W$
pick ordered basis for W making $T|_W$ triangular:
say, $(w_1, \ldots, w_m)$

2) extend ordered basis from W to V:
say $(w_1, \ldots, w_m, v_1, \ldots, v_\ell)$
claim that T is triangular wrt this extended basis[!]
suffices to check Tv_i's

$$Tv_i = (T - \lambda)v_i + \lambda v_i \text{ for all } i$$

[draw matrix] □

<u>Cor</u> any square matrix is conjugate to an upper-triangular matrix

<u>Cor</u> let $f : \mathrm{Mat}_2(F) \to F$ be a function def by a polynomial in matrix coords

i.e. $f = p(x_{11}, x_{12}, x_{21}, x_{22})$ $\begin{matrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{matrix}$

if f is conj-invariant
then f is a polynomial in tr and det

<u>Pf Sketch</u> if $\begin{matrix} X & Y \\ 0 & Z \end{matrix}$ conj to $\begin{matrix} a & b \\ c & d \end{matrix}$

then $p(a, b, c, d) = p(X, Y, 0, Z)$
so p must be independent of its third argument
so $p(x_{11}, x_{12}, x_{21}, x_{22}) = q(x_{11}, x_{12}, x_{22})$
for some q in three variables

claims: 1) q(x11, x12, x22) indep of x12

2) q(x11, x12, x22) = q(x22, x12, x11)

1) observe that $\begin{pmatrix} 1 & e \\ & 1 \end{pmatrix} \begin{pmatrix} X & Y \\ & Z \end{pmatrix} \begin{pmatrix} 1 & -e \\ & 1 \end{pmatrix}$

$= \begin{pmatrix} X & eY - eX \\ & Z \end{pmatrix}$

so q(X, Y, Z) = q(X, eY – eX, Z) for all e, X, Y, Z
so q invariant under these infin. many substit.'s

claim: this forces q to be indep. of its second arg.

[left as exercise]

2) observe that

$\begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \begin{pmatrix} X & Y \\ & Z \end{pmatrix} \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} = \begin{pmatrix} Z & \\ Y & X \end{pmatrix}$

so q(X, Y, Z) = q(Z, Y, X)

<u>Viète's Thm</u>
any poly in x, y invariant under swapping x and y
is a poly in x + y and xy

[left as exercise]

[look up "elementary symmetric functions"]

[another corollary of triangularity thm:]

<u>Cor</u> if $F = \mathbb{C}$ and V is fin. dim.
then $T = A + N$, where
A has a diagonal matrix
N is upper-triangular with 0's on the diagonal
in particular, N is nilpotent

$$\begin{matrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{matrix} = \begin{matrix} * & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{matrix} + \begin{matrix} 0 & * & * \\ 0 & 0 & * \\ 0 & 0 & 0 \end{matrix}$$

want to go further:
want the nilpotent part as simple as possible
[i.e., as many zero entries as possible]

recall: if $V = \sum_i W_i$ for T-stable W_i,
then T has a block-diagonal matrix
where blocks correspond to W_i's

e.g. $\begin{matrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & * \end{matrix}$ $F^3 = W_1 + W_2$,
$W_1 = \{(x, y, 0)\}$,
$W_2 = \{(0, 0, z)\}$

if each W_i is a line,
then they are eigenlines
then V is the sum of the eigenspaces $\ker(T - \lambda)$
as we run over eigenvals λ

[in general not so lucky:]
to proceed, weaken the notion of eigenspace

(Axler §8A) fix a linear op $T : V$ to V

observe that $\ker(T - \lambda)$ sub $\ker((T - \lambda)^2)$ sub …

Df the generalized λ-eigenspace for T is

$\{v \text{ in } V \mid (T - \lambda)^n v = \mathbf{0} \text{ for some } n\}$

equivalently, $\bigcup_{n > 0} \ker((T - \lambda)^n)$

its elts are called generalized λ-eigenvectors for T

Lem the generalized λ-eigenspace for T is the largest T-stable lin. sub. W sub V s.t. $(T - \lambda)|_W$ is nilpotent

Pf $\ker((T - \lambda)^n)$ is T-stable for all n
[same tactic as we used before]
so $\bigcup_{n > 0} \ker((T - \lambda)^n)$ is stable and nilpotence condition holds for it

conversely: easy to show that
if W is T-stable and $(T - \lambda)|_W$ is nilpotent,
then W sub $\ker((T - \lambda)^n)$

Lem if $\lambda \neq \lambda'$, then their gen'lized eigenspaces have intersection $\{\mathbf{0}\}$

Pf let W, W' be the gen'lized eigenspaces

since W, W' are T-stable, so is $W \cap W'$

since $(T - \lambda)|_W$, $(T - \lambda')|_{W'}$ are nilpotent,

so are $(T - \lambda)|_{W \cap W'}$,

$(T - \lambda')|_{W \cap W'}$

but sums/differences of nilpotent ops are nilpotent

so $(\lambda - \lambda')|_{W \cap W'}$ is nilpotent

so $W \cap W' = \{\mathbf{0}\}$

next time, we prove:

<u>Thm</u> suppose $F = \mathbb{C}$ and V is fin. dim.

then there exist a finite list of λ_i s.t.

$$V = \sum_i W_i$$

where W_i is the generalized λ_i-eigenspace for T