last time: a group consists of

a set G

a function $- \bullet -: G \times G$ to G

obeying

1) $(a \bullet b) \bullet c = a \bullet (b \bullet c)$
2) some e in G s.t. $a \bullet e = a = e \bullet a$
3) for all a in G, some b in G s.t. $a \bullet b = e = b \bullet a$

<u>Thm</u> if H is a subgroup of $(Z, +)$
then $H = mZ$ for some m

[we had shown:]

<u>Lem</u> if H is a subgroup of $(Z, +)$
and H contains some elt n
then H supset $nZ := \{nk \mid k \text{ in } Z\}$

<u>Pf of Thm</u> if $H = \{0\}$, done
else let $m = \min\{n \text{ in } H \mid n > 0\}$

by lem, know mZ sub H
want to show: if n in H, then n in mZ
enough to consider $n > 0$
long division gives $n = mq + r$ with $0 \leq r < m$
[what next?]

observe $r = n + (-mq)$ in H
so contradiction unless $r = 0$
so m divides n
so nZ sub mZ □

observe: if $m \neq 0$, then mZ “looks like” Z
if $m = 0$, then $mZ = \{0\}$, which doesn’t

Df suppose $(G, \bullet)$ and $(K, \circ)$ are groups
a homomorphism from $(G, \bullet)$ and $(K, \circ)$
is a map $\varphi : G$ to K
s.t. $\varphi(a \bullet b) = \varphi(a) \circ \varphi(b)$

i.e., φ transforms/transports the group law of G into the group law of K

Lem if a homomorphism is bijective
then the inverse is also a hom

in this case, we say the maps are isomorphisms

Ex $\varphi : Z$ to mZ given by $\varphi(k) = mk$
is a homomorphism for any m:
indeed, $\varphi(k + \ell) = m(k + \ell) = mk + m\ell = \varphi(k) + \varphi(\ell)$

when is it an isomorphism? only for $m \neq 0$

Ex recall: Sym(X) = {self-bijections of X}

if Y sub X, then get φ : Sym(Y) to Sym(X)
namely,

$$\varphi(f)(x) = \begin{cases} f(x) & \text{if } x \text{ in } Y \\ x & \text{if } x \text{ notin } Y \end{cases}$$

Ex for any G and K,
the trivial hom from G to K
sends every elt of G to the id of K

Exercise find a nontrivial hom…
from Sym({1, 2, …, n}) to Sym({1, 2})
from Sym({1, 2, 3, 4}) to Sym({1, 2, 3})

(Munkres §51) let X be a top space

today's goal: a group to study "holes" in X

Df a loop in X is
a path $\gamma : [0, 1]$ to X s.t. $\gamma(0) = \gamma(1)$

in this case, we say $\gamma(0)$ is the basepoint of γ

idea: fix x in X
compose loops based at x using pasting lem

$\beta, \gamma : [0, 1]$ to X yield $\beta * \gamma : [0, 1]$ to X def by

$$(\beta * \gamma)(s) = \begin{cases} \beta(2s) & \text{if } s \leq 1/2 \\ \gamma(2s - 1) & \text{if } s \geq 1/2 \end{cases}$$

[left-to-right composition!] [draw picture]

problem: all of the group axioms fail
no associativity
no id elt
no inverses

idea: id elt should be the const. loop γ s.t. $\gamma(s) = x$
[so, consider paths only up to some equiv. rel.?]

Df let ψ, ψ': S to X be cts
a homotopy from ψ to ψ' is
a cts map $h : S \times [0, 1]$ to X s.t.
$h(s, 0) = \psi(s)$
$h(s, 1) = \psi'(s)$
for all s in S

[draw picture for paths]

for paths, want a more restrictive notion:

Df let $\gamma, \gamma' : [a, b]$ to X be paths s.t.

γ and γ' have the same endpts

a path homotopy from γ to γ' is

a cts map $h : [a, b] \times [0, 1]$ s.t.

$h(s, 0) = \gamma(s)$

$h(s, 1) = \gamma'(s)$

for all s in $[a, b]$

$h(a, t) = \gamma(a) = \gamma'(a)$

$h(b, t) = \gamma(b) = \gamma'(b)$

for all t in $[0, 1]$

[draw new picture?]

Lem fix paths $\gamma, \gamma', \gamma''$ with the same endpts

1) γ has a path homotopy to itself [?]
2) if there's a path homotopy from γ to γ'
   then there's a path homotopy from γ' to γ [?]
3) if there are path homotopies from γ to γ',
   from γ' to γ'',
   then there's one from γ to γ'' [?]

Df for all x, y in X

let ~_p be the equivalence relation on paths

from x to y

in which γ ~_p γ'

iff

there's a path homotopy from γ to γ'

here, we say γ and γ' are <u>path homotopic</u>

<u>Lem</u> fix x, y, z in X

fix paths β, β' from x to y,
γ, γ' from y to z,
s.t. $\beta \sim_p \beta'$ and $\gamma \sim_p \gamma'$

then $\beta * \gamma \sim_p \beta' * \gamma'$

<u>Pf</u> pasting lem
[draw picture]

therefore * descends to an operation
on path-homotopy classes
which we will denote by $[\gamma]$, etc.

<u>Df</u> whenever $\beta * \gamma$ is well-defined
we set $[\beta] * [\gamma] := [\beta * \gamma]$

let $\pi_1(X, x) = \{[\gamma] \mid \text{loops } \gamma \text{ in } X \text{ based at } x\}$
let e_x be the constant loop at x

<u>Thm</u> for any α, β, γ in $\pi_1(X, x)$:

1) $[\alpha * \beta] * [\gamma] = [\alpha] * [\beta * \gamma]$
2) $[e_x * \gamma] = [\gamma] = [\gamma * e_x]$
3) if β "reverses" γ
then $[\beta * \gamma] = [e_x] = [\gamma * \beta]$

<u>Df-Cor</u> $\pi_1(X, x)$ forms a group under *
with id elt $[e_x]$
called the <u>fundamental group</u> of X at x