

(Munkres §53) main idea of covering spaces:
generalize the structure of

$$p : \mathbb{R} \rightarrow S^1 \quad p(t) = (\cos(2\pi t), \sin(2\pi t))$$

specifically, the homotopy lifting property for paths
that we used to prove $\pi_1(S^1) = \mathbb{Z}$

first attempt:

Df a continuous map $p : E \rightarrow X$ is
locally a homeomorphism iff,

for all e in E ,
there is an open neighborhood V of e s.t.
 $p|_V$ is a homeomorphism from V onto $p(V)$

Ex $p : \{0 < t < 1\} \rightarrow S^1$
again given by $p(t) = (\cos(2\pi t), \sin(2\pi t))$
is locally a homeomorphism

objection: p is not surjective

e.g., cannot lift loops in S^1 based at $(x, y) = (1, 0)$

Ex $p : \{-1 < t < 1\} \rightarrow S^1$
again given by $p(t) = (\cos(2\pi t), \sin(2\pi t))$
is both surjective and locally a homeo

objection?

still cannot lift all loops...

Df suppose U is an open subset of X ,
 $V = p^{-1}(U)$ for some $p : E \rightarrow X$

we say that U is evenly covered by p iff both:

- 1) V is homeomorphic to a nonempty(!)
disjoint union of copies of U
- 2) p restricts to a homeomorphism
from each copy onto U

Df we say that $p : E \rightarrow X$ is a covering map
iff,
for all x in X ,
 there's an open neighborhood of x
 evenly covered by p

Ex claim that $p : \{-1 < t < 1\} \rightarrow S^1$ is not
a covering

if $(x, y) \neq (1, 0)$

then (x, y) does have an open neighborhood
evenly covered by p

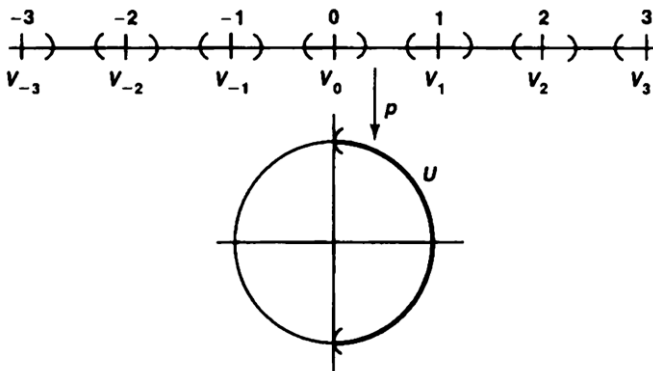
what goes wrong when $(x, y) = (1, 0)$?

Lem if p is a covering
then p is surjective and locally a homeo

(example shows that the converse can fail)

Ex

$p : \mathbb{R} \rightarrow S^1$ is indeed a covering



Ex

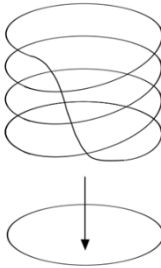
paths in S^1 cannot be covering maps:
why?

Ex

identity maps $\text{Id} : X \rightarrow X$ are always
covering maps

Ex

more generally:



<https://ahilado.wordpress.com/2017/04/14/covering-spaces/>

for any $n > 0$, there is a covering $p : S^1 \rightarrow S^1$ s.t.
the fiber $p^{-1}(x)$ has cardinality n for all x in S^1

we say that the covering is of degree n , or n -fold

we say that R and S^1 form covering spaces, or covers,
of S^1

will show later:

- R and S^1 are the only **connected** covers of S^1
- S^2 is the only **connected** cover of S^2

yet S^2 is a cover for another topological space X :
what is X ?

Ex consider the relation \sim on S^2 identifying
all pairs of antipodal points:

$$(x, y, z) \sim (-x, -y, -z)$$

the quotient map S^2 to S^2/\sim is a 2-fold covering map

have we seen S^2/\sim before?

General Properties of Covering Maps

if $p : E \rightarrow X$ is a covering, then:

- the fibers $p^{-1}(x)$ are discrete sets for all x in X

(because for some open nbd U of x ,
 $p^{-1}(U)$ is a bunch of copies of U mapped onto U
homeomorphically by p)

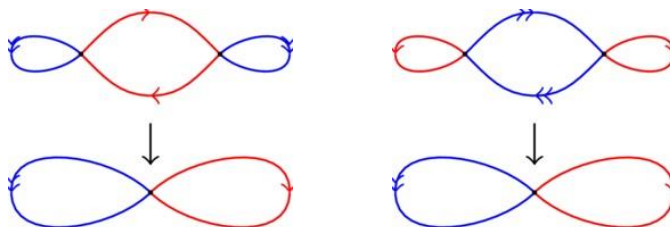
- p is a quotient map

(true for any surjective map that's locally a homeo)

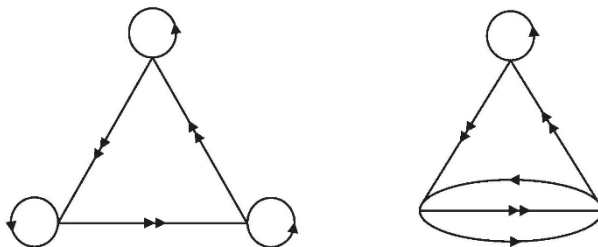
- if $p' : E' \rightarrow X'$ is another covering
then $(p, p') : E \times E' \rightarrow X \times X'$ is a covering

Ex any pair of integers $m, n > 0$ defines
a $\deg\text{-(mn)}$ covering $T \rightarrow T$, where $T = S^1 \times S^1$

Ex coverings of the figure-eight can be weird:



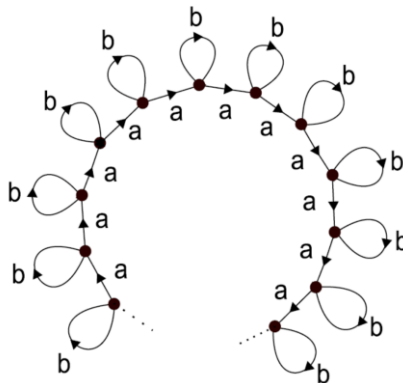
or weirder:



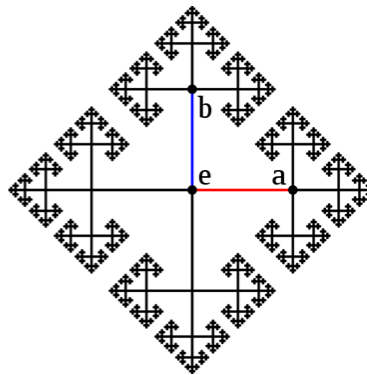
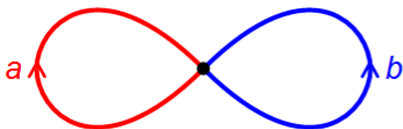
(1) <https://www.homepages.ucl.ac.uk/~ucahjde/tg/html/cov-01.html>

(2) <https://groupoids.org.uk/images/fig10-3.jpg>

even weirder:



<https://www.math.cmu.edu/~nkomarov/NK-NormalSubFreeGrp.pdf>



<https://math.stackexchange.com/a/3762676>

(Munkres §54) the most important property:

Thm if $p : E \rightarrow X$ is a covering, then:

1) for any path $\gamma : [0, 1] \rightarrow X$ and e in E s.t.

$$p(e) = \gamma(0),$$

there's a unique $\Gamma : [0, 1] \rightarrow E$ s.t.

$$\Gamma(0) = e \text{ and } \gamma = p \circ \Gamma,$$

which we call the lift of γ to E

2) for any path homotopy $h : [0, 1]^2 \rightarrow X$ and e in E s.t.

$$p(e) = h(0, 0),$$

there's a unique(!) path homotopy $H : [0, 1]^2 \rightarrow E$ s.t.

$$H(0, 0) = e \text{ and } h = p \circ H$$

(slightly stronger than our version from 3/26)

we'll do 2) on Mon, 4/21

Pf of 1) can pick an open cover $\{U_\alpha\}_\alpha$ of X s.t.
each U_α is evenly covered by p

recall from our proof that S^2 is simply-connected:
an argument showing that we can find

$$0 = s_0 < s_1 < \dots < s_n = 1$$

s.t. $\gamma : [0, 1] \rightarrow X$ maps each segment $[s_i, s_{i+1}]$
into a single U_α at a time

(Munkres calls this the Lebesgue number lemma)

we build the lift $\Gamma : [0, 1] \rightarrow E$ inductively:

set $\Gamma(0) = e$

assume that for some i , we've defined Γ for $s \leq s_i$

to define Γ on $[s_i, s_{i+1}]$:

we know that γ maps $[s_i, s_{i+1}]$ into a single U_α ,

that U_α is evenly covered by p

and $\Gamma(s_i)$ is in $p^{-1}(U_\alpha)$, which is a bunch of copies of U_α

let V_α be the copy containing $\Gamma(s_i)$

via the homeomorphism $V_\alpha \approx U_\alpha$,

$\gamma|_{[s_i, s_{i+1}]}$ has a unique lift into V_α

define $\Gamma|_{[s_i, s_{i+1}]}$ to be this lift

by the pasting lemma, we've now defined Γ for $s \leq s_{i+1}$ \square