6.

Notes on Oscar's 4/28 talk at Yale.

6.1.

6.1. Recall that one of the most familiar descriptions of the Hilbert scheme of n points on $\mathbf{C}^2$ is the ADHM description due to Fogarty(?).

$$\mathrm{Hilb}^n(\mathbf{C}^2) \simeq \{(X, Y, v) \in \mathfrak{gl}_n^2 \times \mathbf{C}^n \mid [X, Y] = 0,\ \mathbf{C}[X, Y]v = \mathbf{C}^n\}.$$

Haiman gave a very different description. Let $\mathbf{C}[\vec{x}, \vec{y}] = \mathbf{C}[x_i, y_i \mid 1 \leq i \leq n]$, and let

$$\begin{aligned} A &= \mathbf{C}[\vec{x}, \vec{y}]^{\mathrm{sgn}}, \\ \Delta &= \prod_{i<j} (x_i - x_j). \end{aligned}$$

Then $\Delta A \subseteq \mathbf{C}[\vec{x}, \vec{y}]^{S_n}$. Haiman proves that there is an isomorphism

$$\mathrm{Hilb}^n(\mathbf{C}^2) \simeq \mathrm{Proj} \bigoplus_{d \geq 0} (\Delta A)^d,$$

and this isomorphism commutes with the natural maps on both sides to $\mathrm{Sym}^n(\mathbf{C}^2)$. (This also works with other antisymmetric polynomials in place of Δ.)

6.2. The isospectral Hilbert scheme is defined by a fiber product:

$$\mathrm{IHilb}^n(\mathbf{C}^2) = \left(\mathrm{Hilb}^n(\mathbf{C}^2) \times_{\mathrm{Sym}^n(\mathbf{C}^2)} \mathbf{C}^{2n}\right)^{\mathrm{red}}.$$

Haiman's theorem implies an analogous isomorphism

$$\mathrm{IHilb}^n(\mathbf{C}^2) \simeq \mathrm{Proj} \bigoplus_{d \geq 0} (\Delta I)^d, \quad \text{where } I = A\mathbf{C}[\vec{x}, \vec{y}].$$

There is a particularly interesting conical subset of $\mathbf{C}^{2n}$: The vanishing locus of the ideal

$$J = \bigcup_{i<j} \langle x_i - x_j, y_i - y_j \rangle.$$

Its image in $\mathrm{Sym}^n(\mathbf{C}^2)$ matches the image of the vanishing locus of I. So it's natural to ask: Is it true that $I = J$? The answer is yes, but this is deep, relying on tools from Haiman's proof of the $n!$/polygraph theorem. Haiman further shows that $J^d = J^{(d)}$, where the right-hand side is the "dth symbolic power"

$$J^{(d)} := \bigcup_{i<j} \langle x_i - x_j, y_i - y_j \rangle.$$

Altogether, the results above give

$$\mathrm{IHilb}^n(\mathbf{C}^2) \simeq \mathrm{Proj} \bigoplus_{d \geq 0} \Delta^d J^d = \mathrm{Proj} \bigoplus_{d \geq 0} \Delta^d J^{(d)}.$$

6.3. Why care about $\mathrm{Hilb}^n(\mathbf{C}^2)$ and $\mathrm{Hilb}^n(\mathbf{C}^2)$?

- (Fogarty) Hilb is smooth.
- (Haiman) IHilb is Gorenstein.
- Hilb is hyperkähler and diffeomorphic to Calogero–Moser space.
- Recent conjectures and results relating derived categories of coherent sheaves on the Hilbert schemes of $\mathbf{C}^2$ to knot invariants, affine Springer fibers, and more.

6.2.

6.4. We want to generalize these spaces in a Lie-theoretic direction, to a reductive Lie algebra $\mathfrak{g}$ with Cartan $\mathfrak{t}$ and Weyl group W. So now take

$$\begin{aligned} A &= \mathbf{C}[\mathfrak{t} \oplus \mathfrak{t}^*]^{\mathrm{sgn}}, \\ I &= A\mathbf{C}[\mathfrak{t} \oplus \mathfrak{t}^*], \\ J &= \bigcup_{\alpha \in \Phi^+} \langle \alpha, \alpha^\vee \rangle, \\ \Delta &= \prod_{\alpha \in \Phi^+} \alpha. \end{aligned}$$

We then set

$$Y_{\mathfrak{g},\mathrm{sgn}} = \mathrm{Proj} \bigoplus_{d \geq 0} (\Delta I)^d, \quad Y_{\mathfrak{g},\mathrm{diag}} = \mathrm{Proj} \bigoplus_{d \geq 0} (\Delta J)^d, \quad Y_{\mathfrak{g},\mathrm{symb}} = \mathrm{Proj} \bigoplus_{d \geq 0} \Delta^d J^{(d)}.$$

There are inclusions of ideals $I^d \subseteq J^d \subseteq J^{(d)}$. Consequently, there are maps $Y_{\mathfrak{g},\mathrm{symb}} \to Y_{\mathfrak{g},\mathrm{diag}} \to Y_{\mathfrak{g},\mathrm{sgn}}$.

6.5. More precisely, these schemes are the analogoues of the isospectral Hilbert schemes. Let $e = \frac{1}{|W|} \sum_w w$, and set

$$X_{\mathfrak{g},\mathrm{sgn}} = \mathrm{Proj} \bigoplus_{d \geq 0} e(\Delta I)^d, \quad X_{\mathfrak{g},\mathrm{diag}} = \mathrm{Proj} \bigoplus_{d \geq 0} e(\Delta J)^d, \quad X_{\mathfrak{g},\mathrm{symb}} = \mathrm{Proj} \bigoplus_{d \geq 0} e\Delta^d J^{(d)}.$$

These are the analogues of the usual Hilbert schemes.

6.6. In type A, taking $\mathfrak{g} = \mathfrak{sl}$ rather than $\mathfrak{g} = \mathfrak{gl}$ produces the "balanced" Hilbert schemes already appearing in the literature. In type BC_3, calculation with a computer shows that $Y_{\mathfrak{g},\mathrm{diag}} \neq Y_{\mathfrak{g},\mathrm{sgn}}$.

6.3.

6.7. Symbolic powers behave quite nicely. It's not obvious that the graded algebras you get are finitely-generated, but it turns out to be true. Moreover, the schemes $Y_{\mathfrak{g},\mathrm{symb}}$ and $X_{\mathfrak{g},\mathrm{symb}}$ turn out to be normal. The analogous facts for diag, sgn are unclear.

6.8. In the trigonometric setting, where we replacing $T^*\mathfrak{t}$ with $T^*T^\vee$, we can define $Y_{G,-}, X_{G,-}$ analogously to $Y_{\mathfrak{g},-}, X_{\mathfrak{g},-}$ by blowing up $T^*T^\vee /\!\!/ W$ (along analogous ideals). For instance, we replace $\langle \alpha, \alpha^\vee \rangle$ with $\langle \alpha, 1 - e^{\alpha^\vee} \rangle$ everywhere.

6.9. Gorsky–K–Oblomkov show: $X_{G,\mathrm{diag}}$ is the BFN Coulomb branch for adjoint matter, after a partial resolution using the flavor symmetry of the dilation(?) cocharacter. Here, finite generation follows from its description as a Hamiltonian reduction. We don't know an analogous interpretation of, say, $X_{G,\mathrm{symb}}$.

6.10. The natural map

$$X_{\mathfrak{g},\mathrm{symb}} \to (\mathfrak{t} \oplus \mathfrak{t}^*) /\!\!/ W$$

is a $(\mathbf{C}^\times)^2$-equivariant conical symplectic partial resolution. In particular, $X_{\mathfrak{g},\mathrm{symb}}$ has symplectic singularities. It is smooth if and only if $\mathfrak{g}$ is of type A.

To sketch the proof of the first claim: Formally locally, $X_{\mathfrak{g},\mathrm{symb}}$ and $X_{G,\mathrm{symb}}$ are the same. By Bellamy, the latter has symplectic singularities.

6.11. Conjecture: The $(\mathbf{C}^\times)^2$-fixed points of $X_{\mathfrak{g},\mathrm{symb}}$ are in bijection with two-sided cells for W. This is true in types ABC.

The idea of the proof in type BC: Rewrite $X_{\mathfrak{g},\mathrm{symb}}$ as a quiver variety $\mathscr{M}_{(0,1)}(Q)$, where Q has a bigon with vertices n, n and a tail on one vertex with vertex $\boxed{1}$. Then there is a $\mathrm{U}(1)$-equivariant homeomorphism from the latter to the Calogero–Moser space of type BC_n, given by the spectrum of the spherical $t = 0$ rational Cherednik algebra. There the fixed points for the Hamiltonian torus are in bijection with two-sided cells by the verification(?) of Bonnafé–Rouquier.

This line of thinking also suggests another conjecture: For general $\mathfrak{g}$, we still expect that $X_{\mathfrak{g},\mathrm{symb}}$ is $\mathrm{U}(1)$-equivariantly homeomorphic to a Calogero–Moser space.

6.12. Another conjecture: If $\mathfrak{g}$ is simply laced, then $X_{\mathfrak{g},\mathrm{symb}}$ is a $\mathbf{Q}$-factorial terminalization of $(\mathfrak{t} \oplus \mathfrak{t}^*) /\!\!/ W$. Ivan thinks this is actually easy.

6.13. Gorsky–K–Oblomkov show that each $\gamma \in \mathfrak{g}(\!(t)\!)^\circ$ (where $(-)^\circ$ means "regular semisimple") defines a $\mathbf{C}^\times$-equivariant quasi-coherent sheaf $\mathscr{F}_{G,\gamma}$ on $X_{G,\mathrm{symb}}$. Conjecturally it is coherent. In the case where γ is elliptic, they expect a degeneration to a $(\mathbf{C}^\times)^2$-equivariant coherent sheaf $\mathscr{F}_{\mathfrak{g},\gamma}$ on $X_{\mathfrak{g},\mathrm{symb}}$, such that

(1) We have a family of (compatible) isomorphisms

$$\mathrm{H}^*(\mathscr{F}_{\mathfrak{g},\gamma} \otimes \mathscr{O}(k)) \simeq \mathrm{gr}^{\mathrm{P}}_* \mathrm{H}^!_*(\mathrm{Sp}_{\gamma t^k}, \mathbf{C}),$$

where $\mathrm{Sp}_{(-)}$ denotes "affine Springer fiber".

(2) In the $(\mathbf{C}^\times)^2$-equivariant K-theory of $X_{\mathfrak{g},\mathrm{symb}}$, there is an expansion

$$[\mathscr{F}_{\mathfrak{g},\gamma}] = \sum_{x \in \{(\mathbf{C}^\times)^2\text{-fixed points of } X_{\mathfrak{g},\mathrm{symb}}\}} a_x(\gamma)[\delta_x],$$

where δ_x denotes the skyscraper at x and its coefficient $a_x(\gamma)$ belongs to $K^{(\mathbf{C}^\times)^2}(\mathrm{pt}) \simeq \mathbf{C}[q^{\pm 1}, t^{\pm t}]$. This expansion should correspond to a Shalika expansion for affine Springer fibers, via a bijection between two-sided cells and special nilpotent orbits.