(Axler §1C)

last time:

Mag(F) = {3 by 3 magic squares over F}

viewed as a subset of F^9 via

a_{11} a_{12} a_{13}
a_{21} a_{22} a_{23}   mapsto
a_{31} a_{32} a_{33}

(a_{11}, a_{12}, a_{13}, a_{21}, …, a_{33})

suggests how to turn Mag(F) into a vector space

Df   given a vector space V over F

a subset W is an F-linear subspace of V iff
1) W is stable under the addition + and scaling •
2) **0** in W
3) W forms a vector space wrt +, •, **0**

Lem   1) and 2) are sufficient

Pf sketch   associativity, commutativity,
distributive laws immediate from 1)
[only existence of inverses might be tricky]
suppose w in W:
–w = (–1)w by previous class
(–1)w in W by 2)

Ex fix a, b in F

{(x, ax) | x in F} is a linear subspace of F^2

1) (x, ax) + (x′, ax′) = (x + x′, a(x + x′))

2) **0** = (0, 0) = (0, a0)

{(x, ax + 1) | x in F} not a linear subspace of F^2

[which conditions fail? both]

Ex {**0**} is a linear subspace of any V

Ex Ø is never a linear subspace of V

[why?] 1) holds but 2) fails

Ex R is an R-linear subspace of C

Ex R is not a C-linear subspace of C

[why?] 1) fails

Operations on Linear Subspaces

if W, W′ are linear subspaces of V,
then W cap W′ is too [why?]

if W, W′ are linear subspaces of V,
then W cup W′ need not be! [example?]

Ex V = F^2

W = {(x, 0) | x in F} , W′ = {(0, y) | y in F}

Df — if W, W' are linear subspaces of V, then their sum is
$W + W' = \{w + w' \mid w \text{ in } W \text{ and } w' \text{ in } W'\}$

Lem — $W + W'$ is indeed a linear subspace of V

Pf — exercise
$[(v + v') + (w + w') = (v + w) + (v' + w')]$
$[a \bullet (w + w') = a \bullet w + a \bullet w']$

more generally, for finitely many W_i sub V,

$$W_1 + \ldots + W_k = \{\text{sum}_i\, w_i \mid w_i \text{ in } W_i\}$$

(HW: sums of infinitely many linear subspaces)

Prop — $W_1 + \ldots + W_k$ is precisely the minimal linear subspace of V that contains $\bigcup_i W_i$

Pf — by lemma, $W_1 + \ldots + W_k$ is a lin. sub. of V containing $\bigcup_i W_i$

take any lin. sub. U of V containing $\bigcup_i W_i$

take any element of $W_1 + \ldots + W_k$
must look like $w_1 + \ldots + w_k$
with w_i in W_i for all i
then w_i in U for all i
then $w_1 + \ldots + w_k$ in U

Rem [“law of excluded middle”]

for any set V and subsets U, W, W′,

$\boxed{U \cap (W \cup W') = (U \cap W) \cup (U \cap W')}$

by contrast, we can have
a vector space V and subspaces U, W, W′ s.t.

$\boxed{U \cap (W + W') \neq (U \cap W) + (U \cap W')}$

Ex $V = F^2$
$U = \{(x, x) \mid x \in F\}$
$W = \{(x, 0) \mid x \in F\}$, $W' = \{(0, y) \mid y \in F\}$

$U \cap (W + W') = U$, $U \cap W = U \cap W' = \{\mathbf{0}\}$

classical logic:	propositions	sets
	“and”	cap
	“or”	cup
	[think: Venn diagrams]	
quantum logic:	propositions	vec. spaces
	“and”	cap
	“or”	+

Ex double-slit experiment

[draw picture]

U = “photon hits screen”
W = “photon passes through slit 1”
W′ = “photon passes through slit 2”

Rem [sums of lin. sub.'s ≠ unions]
[nor behave like sums of #'s]

in F^3, consider

$L = \{(x, x, 0) \mid x \in F\}$

$K = \{(x, -x, 0) \mid x \in F\}$

$M = \{(x, y, 0) \mid x, y \in F\}$

$N = \{(x, x, y) \mid x, y \in F\}$

$L + K = M$ $K + M = M$ $M + N = F^3$

$L + M = M$ $K + N = F^3$

$L + N = N$

Q when do sums have "redundancy"?

Direct Sums

Df $\sum_i W_i$ is called a direct sum iff

any elt of $\sum_i W_i$ has a unique decomposition
$w_1 + \ldots + w_k$
where $w_i \in W_i$ for all i

Rem some decomposition always exists
the point is a unique decomposition
[uniqueness = "no redundancy"]

Prop $W_1 + W_2$ is a direct sum
iff
$W_1 \cap W_2 = \{\mathbf{0}\}$

Pf suppose $v_1 + v_2 = w_1 + w_2$
where $v_i, w_i \in W_i$

then $v_1 - w_1 = v_2 - w_2$ in $W_1 \cap W_2$
so $v_1 - w_1 = w_2 - v_2 = \mathbf{0}$
so $v_1 = w_1$ and $v_2 = w_2$

Rem no analogue for $W_1 + W_2 + W_3$:

in F^2, consider
$W_1 = \{(x, 0) \mid x \in F\}$
$W_2 = \{(0, x) \mid x \in F\}$
$W_3 = \{(x, x) \mid x \in F\}$

each pair is a direct sum, but
$(1, 0) + (0, 1) + (0, 0) = (0, 0) + (0, 0) + (1, 1)$

Ex in F^3, consider [again]
$M = \{(x, y, 0) \mid x, y \in F\}$
$N = \{(x, x, y) \mid x, y \in F\}$

then $M \cap N = \{(x, x, 0) \mid x \in F\}$

$M + N$ is larger than either M or N, but
$M + N$ is not a direct sum