15.

We give an overview of Chapter 5 of Beilinson–Bernstein–Deligne–Gabber's book, which explains how perverse sheaves interact with Deligne's theory of weights, purity, and mixedness. Then we describe a categorification of the Iwahori–Hecke algebra by means of mixed, equivariant perverse sheaves. A possible reference is Bezrukavnikov–Yun's paper "On Koszul Duality for Kac–Moody Groups".

15.1.

Like last time, k is an algebraically closed field, ℓ is a prime distinct from its characteristic, X is a scheme of finite type over k, and G is a smooth algebraic group acting on X over k. For such data, we have sketched the formalism of the equivariant constructible derived category $\mathsf{D}_G(X)$, including the t-structures giving rise to the subcategories of equivariant constructible $\bar{\mathbf{Q}}_\ell$-sheaves $\mathsf{Shv}_G(X) = \mathsf{Shv}_G(X, \bar{\mathbf{Q}}_\ell)$ and equivariant perverse sheaves $\mathsf{Perv}_G(X)$.

The whole formalism generalizes to the setting where we replace k with a subfield k_1, the scheme X with a k_1-structure X_1, and the action of G on X over k with the action of a k_1-form G_1 on X_1 over k_1. Pullback from X_1 to $X = X_1 \otimes k$ induces a functor

$$\mathsf{D}_{G_1}(X_1) \to \mathsf{D}_G(X)$$

that commutes with the six operations on constructible derived categories that we discussed. Achar Proposition 5.3.3 says that the pullback functor is t-exact with respect to both the standard and the perverse t-structures.

There is one major difference from the setting over k. Let $pt_1 = \operatorname{Spec} k_1$ with the trivial G_1-action. Recall that we can identify $\mathsf{D}(pt_1)$ with the derived category of complexes of finite-dimensional $\bar{\mathbf{Q}}_\ell\mathrm{Gal}(k/k_1)$-modules. Under this identification, the forgetful functor $\mathsf{D}_{G_1}(pt_1) \to \mathsf{D}(pt_1)$ sends the constant object $(\bar{\mathbf{Q}}_\ell)_{pt_1}$ to the *trivial* 1-dimensional Galois module. So for any $a_1 : X_1 \to pt_1$ and $K_1 \in \mathsf{D}(X_1)$, the hypercohomology groups

$$\mathrm{H}^i_{G_1}(X_1, K_1) = \mathrm{Hom}_{\mathsf{D}_{G_1}(pt_1)}((\bar{\mathbf{Q}}_\ell)_{pt_1}, a_{1,*}K_1[i])$$

only see the Galois-invariant part of the pushforwards $a_{1,*}K_1[i]$. By contrast, the groups $\mathrm{H}^i_G(X, K)$ see the full pushforwards along with their Galois actions.

Remark 15.1. Recall that taking cohomology induces a functor

$$\mathsf{D}_G(pt) \to \mathsf{D}(\mathsf{Mod}^{\mathsf{fg}}_{\mathrm{H}^*_G(pt)}).$$

In stacky terms, we can view this functor as pushforward along $[pt/G] \to pt$. This is quite different from the equivalence

$$\mathsf{D}(pt_1) \xrightarrow{\sim} \mathsf{D}(\mathsf{Mod}^{\mathsf{fg}}_{\mathrm{Gal}(k/k_1)}),$$

which arises from the Galois action on a stalk, *i.e.*, pullback along $pt \to pt_1$.

15.2.

Henceforth, we assume that $k = \bar{\mathbf{F}}_q$ and $k_1 = \mathbf{F}_q$. We will first review the theory of weights in the non-equivariant case, then comment on the equivariant generalization.

15.2.1.

Let $\mathcal{F}_1 \in \mathsf{Shv}(X_1)$ be a constructible $\bar{\mathbf{Q}}_\ell$-sheaf on X_1, and let $\mathcal{F} \in \mathsf{Shv}(X)$ be its pullback. For any field isomorphism $\iota : \bar{\mathbf{Q}}_\ell \to \mathbf{C}$, we say that $\mathcal{F}_1$ is *pointwise ι-pure* if and only if there is a fixed number $\alpha \in \mathbf{R}$ such that, for every closed point $x \in X_1(\mathbf{F}_{q^d})$, geometric point $\bar{x} \in X_1(k) = X(k)$ over x, and eigenvalue $\lambda \in \bar{\mathbf{Q}}_\ell$ of the action of F^d on the stalk $\mathcal{F}_{1,\bar{x}} = \mathcal{F}_{\bar{x}}$, we have

$$|\iota(\lambda)| = q^{d\alpha/2}.$$

Then α is called the *pointwise weight* of $\mathcal{F}_1$. We say that $\mathcal{F}_1$ is *pointwise pure of weight α* iff it is ι-pure of weight α for all ι. It is *mixed* if and only if it admits a finite-length filtration where successive quotients are pure. When each has weight $\leq \alpha$, we say that $\mathcal{F}_1$ is *mixed of weight $\leq \alpha$*.

Let $K_1 \in \mathsf{D}(X_1)$. We say that K_1 is a *mixed complex* if and only if $\mathcal{H}^i(K)$ is mixed for all i. It is *mixed of weight $\leq \alpha$* if and only if $\mathcal{H}^i(K_1)$ is mixed of weight $\leq \alpha + i$ for all i. It is *mixed of weight $\geq \alpha$* if and only if $\mathbf{D}K$ is mixed of weight $\leq -\alpha$. It is *pure of weight α* if and only if it is both mixed of weight $\leq \alpha$ and mixed of weight $\geq \alpha$.

We write $\mathsf{D}_m(X_1) \subseteq \mathsf{D}(X_1)$ to denote the full subcategory of mixed complexes. For all $\alpha \in \mathbf{R}$, let $\mathsf{D}_{\leq\alpha}(X_1)$, *resp.* $\mathsf{D}_{\geq\alpha}(X_1)$, be the full subcategory of $\mathsf{D}_m(X_1)$ of complexes of weight $\leq \alpha$, *resp.* $\geq \alpha$. The notations $\mathsf{D}_{<\alpha}(X_1)$, $\mathsf{D}_{>\alpha}(X_1)$ are defined similarly.

Example 15.2. Recall the Tate twist $\bar{\mathbf{Q}}_\ell(1) = (\bar{\mathbf{Q}}_\ell)_{pt_1}(1) \in \mathsf{Shv}(pt_1)$, the 1-dimensional representation where F acts by q^{-1}. Its mth power $\bar{\mathbf{Q}}_\ell(m)$ is pointwise pure of weight $-2m$, hence mixed of weight $\leq -2m$. But also, $\mathbf{D}_{pt_1}(\bar{\mathbf{Q}}_\ell(m)) \simeq \bar{\mathbf{Q}}_\ell(-m)$, which is mixed of weight $\leq 2m$. Hence, as a complex supported in degree zero, $\bar{\mathbf{Q}}_\ell(1)$ is pure of weight $-2m$.

15.2.2.

Deligne's main theorem from his "Weil II" paper states how the six operations on constructible derived categories interact with weights. In Beilinson–Bernstein–Deligne–Gabber's book, it is Stabilités 5.1.14. Below, recall that $\mathsf{D}^b(X) \subseteq \mathsf{D}(X)$ denotes the full subcategory of objects with cohomology sheaves in bounded degrees. We define $\mathsf{D}^b_{\leq\alpha}$, $\mathsf{D}^b_{\geq\alpha}$, $\mathsf{D}^b_{<\alpha}$, $\mathsf{D}^b_{>\alpha}$ similarly.

Theorem 15.3 (Deligne). *Let p_1 be a separable morphism of schemes of finite type over $\mathbf{F}_q$. Then:*

(1) $p_{1,!}$, p_1^* *preserve* $\mathsf{D}^b_{\leq\alpha}$.
(2) $p_{1,*}$, $p_1^!$ *preserve* $\mathsf{D}^b_{\geq\alpha}$.
(3) $\mathcal{H}om(-,-)$ *sends* $\mathsf{D}^b_{\leq\alpha}\times\mathsf{D}^b_{\geq\beta}\to\mathsf{D}^b_{\geq-\alpha+\beta}$.
(4) $(-)\otimes(-)$ *sends* $\mathsf{D}^b_{\leq\alpha}\times\mathsf{D}^b_{\leq\beta}\to\mathsf{D}^b_{\leq\alpha+\beta}$.

Example 15.4. Deligne's theorem implies that if we keep track of weights, then the identity $p^! = p^*[2d]$ for a smooth morphism $p = p_1\otimes\mathrm{id}_{\mathrm{Spec}\,k}$ of relative dimension d can be refined. Writing $(\bar{\mathbf{Q}}_\ell)_{X_1}(m)$ for the pullback to X_1 of the Tate twist from Example 15.2, and (m) for the functor $(-)\otimes(\bar{\mathbf{Q}}_\ell)_{X_1}(m)$ on $\mathsf{D}(X_1)$, we find that:

Just as $[m]$ shifts weights up by m, so (m) shifts weights down by $2m$.

In particular, it turns out that if p_1 is smooth of relative dimension d, then $p_1^! = p_1^*[2d](d)$. For instance, if X_1 is smooth of dimension d, then for any m, the twist $(\bar{\mathbf{Q}}_\ell)_{X_1}(m)$ is pure of weight $-2m$, and the dualizing complex $\omega_{X_1}\simeq(\bar{\mathbf{Q}}_\ell)_{X_1}[2d](d)$ is pure of weight zero.

BBDG prove that the perverse truncation functors preserve mixed complexes. The following results are §5.17, 5.3.1–5.3.2 in their book. See also Achar Theorem 5.4.12.

Theorem 15.5 (BBDG). *The perverse t-structure on $\mathsf{D}^b(X_1)$ restricts to a t-structure on $\mathsf{D}^b_m(X_1)$, whose heart is necessarily the category of mixed perverse sheaves. For any $\alpha\in\mathbf{R}$, subquotients and intermediate extensions preserve mixed perverse sheaves of weight $\leq\alpha$ and mixed perverse sheaves of weight $\geq\alpha$.*

Using the classification of simple perverse sheaves in terms of IC complexes, one proves BBDG Corollaire 5.3.4:

Theorem 15.6 (Purity). *Every mixed simple perverse sheaf is pure.*

We also mention BBDG Théorème 5.4.1, which says that the weights of any complex can be characterized via the weights of its perverse cohomologies.

Theorem 15.7. *An object $K_1\in\mathsf{D}(X_1)$ is mixed of weight $\leq\alpha$ if and only if ${}^p\mathcal{H}^i(K)$ is mixed of weight $\leq\alpha+i$ for all i. The analogoue statement with $\geq$ in place of $\leq$ also holds.*

15.2.3.

There is a partial converse to the purity theorem. It is only partial because it requires pullback from X_1 to X.

First we need a digression about Frobenius actions on Hom spaces. Recall that $(\bar{\mathbf{Q}}_\ell)_{pt_1}$ corresponds to the trivial $\bar{\mathbf{Q}}_\ell\mathrm{Gal}(k/\mathbf{F}_q)$-module (in degree zero), and $\mathrm{Hom}((\bar{\mathbf{Q}}_\ell)_{pt_1}, -)$ to the functor that takes Galois invariants. Applying this functor to exact triangles of the form $\tau^{\leq -1}M \to M \to \tau^{\geq 0}M \to$, we obtain a short exact sequence

$$0 \to \mathrm{H}^{-1}(M)_F \to \mathrm{Hom}((\bar{\mathbf{Q}}_\ell)_{pt_1}, M_1) \to \mathrm{H}^0(M)^F \to 0$$

for any $M_1 \in \mathsf{D}(pt_1)$ with pullback $M \in \mathsf{D}(pt)$, where we have written F in place of $\mathrm{Gal}(k/\mathbf{F}_q)$, to follow the conventions of BBDG and Achar; the subscript denotes coinvariants; and the superscript denotes invariants. (Recall that here, the action of F is that of $\sigma_{\mathrm{Spec}\,k}^{-1} \in \mathrm{Gal}(k/\mathbf{F}_q)$.) In particular, we get BBDG (5.1.2.5) = Achar Proposition 5.3.4:

Lemma 15.8. *For any X_1 and $K_1, L_1 \in \mathsf{D}(X_1)$, we have a short exact sequence*

$$0 \to \mathrm{Hom}(K, L[-1])_F \to \mathrm{Hom}(K_1, L_1) \to \mathrm{Hom}(K, L)^F \to 0.$$

Proof. Take $M_1 = a_{1,*}\mathcal{H}om(K_1, L_1)$ for the unique map $a_1 : X_1 \to pt_1$. □

Applying Deligne's theorem to this exact sequence, we get BBDG Proposition 5.1.15:

Lemma 15.9. *In the setup above:*

1. *If $K_1 \in \mathsf{D}^b_{\leq\alpha}(\mathsf{X}_1)$ and $L_1 \in \mathsf{D}^b_{\geq\alpha}(X_1)$, then* $\mathrm{Hom}(K, L[i])^F = 0$ *for all $i > 0$.*
2. *If, more strongly, $L_1 \in \mathsf{D}^b_{>\alpha}(X_1)$, then* $\mathrm{Hom}(K, L[i]) = 0$ *for all $i > 0$.*

The following statements are BBDG Théorèmes 5.4.5 and 5.3.8.

Theorem 15.10 (Semisimplicity)**.** *(1) If K_1 is any* pure *complex over X_1, then there is a noncanonical isomorphism*

$$K \simeq \bigoplus_i {}^p\mathcal{H}^i(K)[-i].$$

(2) If E_1 is a pure *perverse sheaf over X_1, then E is a semisimple perverse sheaf over X.*

Proof. (1): By Theorem 15.7, if K is pure of weight α, then ${}^p\mathcal{H}^i(K)$ is pure of weight $\alpha + i$ for all i. Now consider the exact triangle

$${}^p\tau_{\leq i-1}K \to {}^p\tau_{\leq i}K \to {}^p\mathcal{H}^i(K)[-i] \xrightarrow{d} \dots$$

Using the purity of ${}^p\mathcal{H}^i(K)$ and Lemma 15.9, we can show that this extension splits, whence ${}^p\mathcal{H}^i(K)[-i]$ is a summand of ${}^p\tau_{\leq i} K$.

(2): The direct sum of all simple perverse subsheaves of E is F-stable, so it takes the form K for some $K_1 \subseteq E_1$. We see that $E \in \mathrm{Ext}^1(E/K, K)^F$. But these F-invariants vanish by Lemma 15.9, so K mut be a direct summand of E, which forces $K = E$. □

Corollary 15.11 (Decomposition)**.** *Let $p_1 : Y_1 \to X_1$ be a* proper *separable morphism of $\mathbf{F}_q$-schemes of finite type, and let E be a mixed simple perverse sheaf over Y_1. Then $p_* E$ is a (finite) direct sum of shifts mixed simple perverse sheaves over X.*

Proof. Purity, Deligne, semisimplicity. □

15.2.4.

In the equivariant setting, define weights, mixedness, and purity via the forgetful functor $\mathsf{D}_{G_1}(X_1) \to \mathsf{D}(X_1)$, just like we defined the perverse t-structure on $\mathsf{D}_G(X)$ via the forgetful functor $\mathsf{D}_G(X) \to \mathsf{D}(X)$.

Achar does not make this explicit: In Remark 6.4.12 of his book, all he says about the equivariant mixed derived category is that it exists. To my best knowledge, the earliest reference that discusses it (and its generalizations) is Bezrukavnikov–Yun's paper.

15.3.

Henceforth, we fix a square root $q^{1/2} \in \bar{\mathbf{Q}}_\ell^\times$. This allows us to define a *half Tate twist* $\bar{\mathbf{Q}}_\ell(\frac{1}{2})$. On any category of the form $\mathsf{D}_{G_1}(X_1)$, let

$$\langle m \rangle = (-) \otimes (\bar{\mathbf{Q}}_\ell)_{X_1}[m](\tfrac{m}{2}),$$

the so-called *shift-twist*. By the discussion in Example 15.4, it preserves the weights of mixed complexes.

Let W be the Weyl group of G. For simplicity in what follows, suppose that G_1 is the split form of G, so that F acts trivially on W. Let $\mathcal{B}_1$ be the flag variety of G_1. For each $w \in W$, the orbit $j_w : O_w \to \mathcal{B} \times \mathcal{B}$ then arises from some locally closed $j_{w,1} : O_{w,1} \to \mathcal{B}_1 \times \mathcal{B}_1$, and the IC complex IC_w arises from a corresponding object

$$IC_{w,1} := j_{w,1,!*}(\bar{\mathbf{Q}}_\ell)_{O_{w,1}}[\dim O_w] \in \mathsf{Perv}_{G_1}(\mathcal{B}_1 \times \mathcal{B}_1).$$

(Recall that $\dim O_w = \ell(w) + \dim \mathcal{B}$.) By Theorem 15.5, the twists

$$E_{w,1} = IC_{w,1}(\tfrac{1}{2} \dim O_w) = j_{w,1,!*}(\bar{\mathbf{Q}}_\ell)_{O_{w,1}} \langle \dim O_w \rangle$$

are all pure of weight 0. For all $m \in \mathbf{Z}$, let

$$\mathbf{B}_w(m) = \mathrm{H}^*_G(\mathcal{B} \times \mathcal{B}, E_w\langle m\rangle),$$

the intersection cohomology of the pullback of $E_{w,1}\langle m\rangle$ from $\mathbf{F}_q$ to $k = \bar{\mathbf{F}}_q$, viewed as a $\bar{\mathbf{Q}}_\ell[F]$-module. We abbreviate by writing $\mathbf{B}_w$ for $\mathbf{B}_w(0)$.

Here is where the use of equivariant objects, not merely orbit-constructible objects, becomes relevant. Fix an F-stable Borel $B \subseteq G$ corresponding to $B_1 \subseteq G_1$. Via the F-equivariant isomorphism of stacks

$$[G\backslash(\mathcal{B} \times \mathcal{B})] \simeq [B\backslash G/B],$$

we can identify $\mathsf{D}_{G_1}(\mathcal{B}_1 \times \mathcal{B}_1) \simeq \mathsf{D}_{B_1 \times B_1}(G_1)$. For any $K_1 \in \mathsf{D}_{B_1 \times B_1}(G_1)$, the hypercohomology $\mathrm{H}^*_{B \times B}(G, K)$ is endowed with an action of $\mathrm{H}^*_{B \times B}(pt, \bar{\mathbf{Q}}_\ell)$. Finally, by Künneth (for ordinary cohomology!),

$$\mathrm{H}^*_{B \times B}(pt, \bar{\mathbf{Q}}_\ell) \simeq R \otimes_{\bar{\mathbf{Q}}_\ell} R, \quad \text{where } R := \mathrm{H}^*_B(pt, \bar{\mathbf{Q}}_\ell).$$

Recall that if $T = B/[B, B]$, then $R \simeq \bar{\mathbf{Q}}_\ell \otimes \mathrm{X}^*(T)$, by reduction to the case where T is 1-dimensional. In this way, $\mathbf{B}_w$ forms an R-bimodule for all w. Note that R and $\mathbf{B}_w$ are moreover cohomologically graded. From calculating the weights of F on $\mathrm{H}^*_T(pt, \bar{\mathbf{Q}}_\ell)$, one finds that (1) shifts this grading by 1.

The following theorem is sometimes called the Erweiterungssatz. See also Bezrukavnikov–Yun Proposition 3.1.6.

Theorem 15.12 (Soergel)**.** *For any $v, w \in W$ and $m \in \mathbf{Z}$, we have*

$$\mathrm{Hom}_{\mathsf{D}_{G_1}(\mathcal{B}_1 \times \mathcal{B}_1)}(E_{v,1}, E_{w,1}\langle m\rangle) \simeq \mathrm{Hom}_{\mathsf{Mod}^{\mathsf{gr}}_{R \otimes R^{\mathsf{op}}}}(\mathbf{B}_v, \mathbf{B}_w(m)).$$

Moreover, $\mathbf{B}_w$ is free over R for all w, and the F-action on $\mathbf{B}_w(m)$ is semisimple for all w, m.

The following corollary, while implicit in the work of Soergel (and Ginzburg?), was first formalized in the setting of mixed complexes by Bezrukavnikov–Yun, in Proposition 3.2.5 and Remark 3.2.6 of their paper.

Corollary 15.13 (Bezrukavnikov–Yun)**.** *For any $v, w \in W$, the convolution $E_{v,1} * E_{w,1}$ is* already *isomorphic to a (finite) direct sum of shifts of semisimple (equivariant) mixed perverse sheaves, before pullback from $\mathbf{F}_q$ to $\bar{\mathbf{F}}_q$. (This is stronger than the decomposition theorem!)*

In fact, this convolution is isomorphic to a sum of objects of the form $E_{x,1}\langle m\rangle$ for varying $x \in W$ and $m \in \mathbf{Z}$.

Let C_W be the full additive subcategory of $\mathsf{D}_{G_1}(\mathcal{B}_1 \times \mathcal{B}_1)$ generated by the objects $E_{w,1}\langle m\rangle$ for all w, m. By Corollary 15.13, it is closed under convolution.

By Theorem 15.12, we can even embed it fully faithfully into the category of graded R-bimodules, $\mathsf{Mod}^{\mathsf{gr}}_{R \otimes R^{\mathrm{op}}}$.

We say that C_W is the *additive Hecke category*. Its essential image in $\mathsf{Mod}^{\mathsf{gr}}_{R \otimes R^{\mathrm{op}}}$ is called the category of *Soergel bimodules*. We say that

$$\mathsf{H}_W := \mathsf{K}^b(\mathsf{C}_W)$$

is the *triangulated Hecke category*, or just the *Hecke category*. Recall that $[\mathsf{H}_W]_\triangle \simeq [\mathsf{C}_W]_\oplus$.

Theorem 15.14 (Soergel)**.** *We have an isomorphism of* $\mathbf{Z}[\mathsf{x}^{\pm 1}]$*-modules*

$$[\mathsf{C}_W]_\oplus \simeq H_W(\mathsf{x}),$$

where x *acts on the left-hand side by* (-1)*. Under this isomorphism,* $[E_w]$ *corresponds to the Kazhdan–Lusztig element* c_w*, for all* $w \in W$*.*