Last time A sub X

the subspace topology on A induced by X is
{A cap V | V is an open set in X}

if U sub A sub X
then (U open in X) implies (U open in A) [why?]
but converse can fail

Ex give R the analytic topology

	{0} open	{1} open
A = {0, 1}	y	y
A = Q	n	n
A = {0} cup {1/n | n in Z_+}	n	y

Rem [how do subsp's interact with cts maps?]

1. if A sub X, then the inclusion map i : A to X def by i(a) = a is cts
2. compositions of cts maps are cts
3. so if f : X to Y is cts, then f|_A : A to Y is cts (compare to PS2, #6)

Rem if A is open in X itself
then (U open in A) iff (U open in X)
(see PS2, #3)

Thm if {B_i}_{i in I} is a basis for the top on X,
then {A cap B_i}_{i in I} is a basis for
the subspace top on A

Pf by earlier criterion, just need to show:
for all U open in A and x in U,
have i s.t. x in A cap B_i sub U

indeed, U = A cap V for some V open in X
then x in V, so have i s.t. x in B_i sub V
so x in A cap B_i, and also, A cap B_i sub U

(Munkres §20) recall from real analysis:

Df a metric on X is a fn d : X × X to [0, ∞)

s.t., for all x, y, z in X,

1) d(x, y) = 0 implies x = y
2) d(x, y) = d(y, x)
3) d(x, y) + d(y, z) ≥ d(x, z)

given δ > 0, let B_d(x, δ) = {y in X | d(x, y) < δ}

Df the metric topology on X induced by d:

U is open in the metric topology iff
for all x in U, there is a δ > 0 s.t. B_d(x, δ) sub U

Idea metric topology on X generalizes
analytic topology on R^n

Thm the metric topology really is a topology

Pf exactly like the proof that
the anlytic topology is a topology

[so how much weirder can it be?]

Ex in any X: the discrete metric defined by

$$d(x, x) = 0$$
$$d(x, y) = 1 \text{ when } x \neq y$$

1) and 2) easy

3) [how many cases to check? 5 but can combine]

if $x = z$:

$d(x, y) + d(y, z) \geq 0 = d(x, z)$

[because $d(-, -) \geq 0$]

if $x \neq z$:

either $y \neq x$ or $y \neq z$

so $d(x, y) + d(y, z) \geq 1 = d(x, z)$

observe $B_d(x, 1) = \{x\}$ for all x. thus:

the discrete metric induces the discrete topology

Rem metric $\mapsto$ basis of balls $\mapsto$ topology

but recall:

different bases can give the same top

Thm suppose d induces T on X,

d' induces T' on X

then T is finer than T' iff

for all x in X and $\varepsilon > 0$, there is $\delta > 0$ s.t.

$$B_d(x, \delta) \subseteq B_{d'}(x, \varepsilon)$$

Pf exercise (Munkres Lem 20.2)

Ex [picture of $B_d(x, \delta)$ versus $B_\rho(x, \delta)$]

euclidean metric:

$d(x, y) = \sqrt{(x_1 - y_1)^2 + \dots + (x_n - y_n)^2}$

square metric:

$\rho(x, y) = \max(|x_1 - y_1|, \dots, |x_n - y_n|)$

observe:

$$\begin{aligned} d(x, y) &\le \sqrt{n \max_i (x_i - y_i)^2} \\ &= \sqrt{n}\, \rho(x, y) \\ \rho(x, y) &= \sqrt{\max_i |x_i - y_i|^2} \\ &\le d(x, y) \end{aligned}$$

shows [note reverse directions!]:

$$\begin{aligned} B_\rho(x, \varepsilon/\sqrt{n}) &= \{y \mid \rho(x, y) < \varepsilon/\sqrt{n}\} \\ &= \{y \mid \sqrt{n}\, \rho(x, y) < \varepsilon\} \\ &\subseteq \{y \mid d(x, y) < \varepsilon\} \\ &= B_d(x, \varepsilon) \end{aligned}$$

similarly, $B_d(x, \varepsilon) \subseteq B_\rho(x, \varepsilon)$

<u>Thm</u> Euclidean and square metrics both induce the analytic topology on $\mathbb{R}^n$

in general:

<u>Df</u> metrics d, d' are called <u>equivalent</u> iff there exist $A, B > 0$ s.t.

$d(x, y) \le A\, d'(x, y)$ and $d'(x, y) \le B\, d(x, y)$

<u>uniformly</u> in x and y

<u>Thm</u> if two metrics are equivalent, then their metric topologies coincide

<u>Rem</u> converse is false: see PS2, #9–10

Q what about infinite-dim'l space?

$\mathbb{R}^\omega = \{(x_1, x_2, \ldots) \mid x_i \in \mathbb{R} \text{ for all } i\}$

$\mathbb{R}^\infty = \{(x_1, x_2, \ldots) \mid x_i \neq 0 \text{ for only fin. many } i\}$

Euclidean and square metrics don't work on $\mathbb{R}^\omega$

Q do they still work on $\mathbb{R}^\infty$?

Q are there other metrics on $\mathbb{R}^\omega$?

(Munkres §15, 19) given $\{X_i\}_{i \in I}$,

$\prod_{i \in I} X_i$

$= \{\text{sequences } (x_i)_{i \in I} \mid x_i \in X_i \text{ for all } i\}$

Q if each X_i has a topology, do we get a natural topology on $\prod_{i \in I} X_i$?