Warmup how many topologies on $\varnothing$? [one]

Q is $\varnothing$ connected? [yes!]

Q is $\mathbb{R}^\omega$ connected in the box topology? [no]

say that (x_i) in $\mathbb{R}^\omega$ is bounded iff there is a fixed $C > 0$ s.t. $|x_i| < C$ for all i

$U = \{(x_i)_i \text{ is bounded}\}$
$V = \{(x_i)_i \text{ is not bounded}\}$

disjoint, nonempty, and satisfy $\mathbb{R}^\omega = U \cup V$

to prove that U is open:

Lem if $(x_i)_i$ in U, then for any fixed $\delta > 0$, we have $\prod_i (x_i - \delta, x_i + \delta) \subseteq U$

Pf pick $(y_i)_i$ in $\prod_i (x_i - \delta, x_i + \delta)$
pick C s.t. $|x_i| < C$ for all i

then $|y_i| < |x_i + \delta| \leq |x_i| + \delta < C + \delta$ for all i
so $(y_i)_i$ in U

a similar proof shows that V is open, so:

Thm $\mathbb{R}^\omega$ is disconnected in the box top

Q for integer n, is $\mathbb{R}^n$ connected in the box = product = analytic topology? [yes!]

Facts

1) top on R^n = product top arising from analytic R^{n-1} and R
   [more general: Munkres 118, #4]
2) R is connected
3) if X, Y are connected, then $X \times Y$ is too

Thm R^n is connected

Pf if $n = 0$, then $R^n = \{0\}$, so true
suppose $n > 0$ and induct:

by 1), enough to show $R^{n-1} \times R$ connected in product top
by 2), enough to show analytic R^{n-1} and R are connected

by 3), R is connected
by inductive hypothesis, R^{n-1} is connected

Q examples of connected subspaces of R?
[$\varnothing$, singleton sets, intervals]
how about connected subspaces of Q?

Thm the only connected subsets of Q are $\varnothing$ and singleton sets

Pf suppose A sub Q contains distinct a, b
pick irrational α s.t. $a < \alpha < b$

then take $U = A \cap (-\infty, \alpha)$ and $V = A \cap (\alpha, \infty)$
then U, V are disjoint, nonempty, open sets of A s.t. $A = U \cup V$

Df a top space is totally disconnected iff its only connected subspaces are $\varnothing$ and singletons

so Q is totally disconnected, but not discrete, in the subspace top it inherits from analytic R

[we'll use the same proof strategy to show:]

Thm (Intermediate Value Thm) suppose
X connected,
$f : X$ to R cts,
x, y in X

if $f(x) \leq \alpha \leq f(y)$, then there is z in X s.t. $f(z) = \alpha$

Pf if no such z, then
$U = f^{-1}((-\infty, \alpha))$ and $V = f^{-1}((\alpha, \infty))$
would be a separation of X

Moral to study general top spaces, helpful to compare them to R via cts functions

Df a path in X from x to y is a cts map $\gamma : [0, 1]$ to X s.t. $\gamma(0) = x$ and $\gamma(1) = y$, where we give [0, 1] the analytic top

[sometimes replace [0, 1] with [a, b], where $a < b$]

Df X is path-connected iff, for all x, y in X, there is a path from x to y