MATH 250: TOPOLOGY I PROBLEM SET #5

FALL 2025

Due Friday, November 14. Please attempt all of the problems. Six of them will be graded. You may consult books, papers, and websites as long as you cite all sources and write up your solutions in your own words.

Problem 1. Show that if $f, f' : X \to Y$ are homotopic continuous maps, and similarly, $g, g' : Y \to Z$ are homotopic, then $g \circ f$ and $g' \circ f'$ are homotopic. (In class, we discussed a similar result for path homotopy.)

Problem 2. A subset $A \subseteq \mathbf{R}^n$ is *star convex* if and only if there is some point $a_0 \in A$ such that the line segment between a_0 and any other point of A is contained in A.

(1) Show that if A is star convex, then any loop in A based at a_0 is path homotopic to the constant loop. Thus, A is *simply-connected*: $\pi_1(A, a_0)$ is trivial.
(2) Give a star convex subset of $\mathbf{R}^2$ that is not convex.

Problem 3. Let $s : A \to X$ and $r : X \to A$ be continuous maps such that $r \circ s$ is the identity map of A. Let $a \in A$ and $x = s(a) \in X$. Show that

$$\begin{aligned} & s_* : \pi_1(A, a) \to \pi_1(X, x) \text{ is injective} \\ \text{and} \quad & r_* : \pi_1(X, x) \to \pi_1(A, a) \text{ is surjective.} \end{aligned}$$

Problem 4. Let $X \subseteq \mathbf{R}^n$ be a subspace, and let $f : X \to Y$ be a continuous map. Suppose that $f = g|_X$ for some continuous map $g : \mathbf{R}^n \to Y$. Show that for any $x \in X$, the homomorphism

$$f_* : \pi_1(X, x) \to \pi_1(Y, f(x))$$

is *trivial*: It sends every element to the identity element in the target. *Hint:* $f = g \circ i$, where $i : X \to \mathbf{R}^n$ is the inclusion.

Problem 5. Let $x_0, x_1 \in X$. Recall that if $\alpha : [0, 1] \to X$ is a path from x_0 to x_1, and $\bar{\alpha}(s) = \alpha(1 - s)$ is the reverse path, then there is a homomorphism

$$\hat{\alpha} : \pi_1(X, x_0) \to \pi_1(X, x_1) \quad \text{defined by } \hat{\alpha}([\gamma]) = [\bar{\alpha} * \gamma * \alpha].$$

(1) Show that $\hat{\bar{\alpha}}$ is a two-sided inverse of $\hat{\alpha}$, and thus, both maps are isomorphisms. (This is written out in Munkres, but we want you to work through the details yourself.)
(2) Check that $\hat{\alpha}$ only depends on the path-homotopy class $[\alpha]$. That is, if β is path-homotopic to α, then $\hat{\alpha}$ and $\hat{\beta}$ are the same homomorphism.

Problem 6. Let $\{X_i\}_{i \in I}$ be an arbitrary collection of topological spaces, and let $X = \prod_i X_i$. For each i, fix a basepoint $x_i \in X_i$. Let $x = (x_i)_i \in X$.

(1) Give mutually inverse isomorphisms between $\prod_i \pi_1(X_i, x_i)$ and $\pi_1(X, x)$, and verify that they are isomorphisms.
(2) Use (1) to show that $\mathbf{Z}^I$ is the fundamental group of an explicit topological space.

Problem 7. Recall that X is *contractible* if and only if it has some point o such that the identity map on X is homotopic to the constant map at o: that is, the map $r_o : X \to X$ given by $r_o(x) = o$. Show that X is contractible if and only if X is homotopy equivalent to a one-point space.

Problem 8. Recall that the circle S^1 is (homeomorphic to) a quotient space: $S^1 = [0,1]/{\sim}$, where $0 \sim 1$ and there are no other identifications between distinct points of $[0,1]$. Similarly, we define the *Möbius band* to be the quotient space

$$\mathcal{M} = ([0,1] \times [0,1])/\overset{\bullet}{\sim} \quad (\texttt{\textbackslash overset\{\textbackslash bullet\}\{\textbackslash sim\}}),$$

where $(0, y) \overset{\bullet}{\sim} (1, 1-y)$ for all y and there are no other identifications between distinct points of $[0,1] \times [0,1]$.

Write down an explicit homotopy equivalence between S^1 and $\mathcal{M}$: *i.e.*, a pair of maps $f : S^1 \to \mathcal{M}$ and $g : \mathcal{M} \to S^1$ such that $g \circ f$ is homotopic to id_{S^1} and $f \circ g$ is homotopic to $\mathrm{id}_{\mathcal{M}}$. You do not need to check the homotopy conditions.

Problem 9. Classify the following letter shapes up to: (1) homeomorphism; (2) homotopy equivalence.

A, B, C, D, E, F, G, H, I, J.

You do not need to write down explicit homeomorphisms or homotopy equivalences. Nonetheless, provide some informal reasoning for your classification.

Problem 10. A *topological group* is a group G equipped with a topology in which

$$\begin{aligned} &\text{the group law } \mu_G : G \times G \to G && \text{defined by } \mu_G(g,h) = gh \\ \text{and}\quad &\text{inversion } \iota_G : G \to G && \text{defined by } \iota_G(g) = g^{-1} \end{aligned}$$

are continuous. Show that $\mathbf{R}$ forms a topological group with respect to the addition law and the analytic topology.

Problem 11. Let G be a topological group. Show that:

(1) Any subgroup of G is a topological group in its subspace topology.
(2) Any quotient group of G is a topological group in its quotient topology.

You may assume that for a quotient $p : G \to Q$, the product topology on $Q \times Q$ matches the quotient topology it gets from $p \times p$.

Hint: For continuity of μ_Q, we want to show that if $U \subseteq Q$ is open and $V = \mu_Q^{-1}(U) \subseteq Q \times Q$, then V is also open. Next, V is open if and only if

$(p \times p)^{-1}(V) \subseteq G \times G$ is open. Finally, observe that the diagram

$$\begin{tikzcd} G \times G \arrow[r, "\mu_G"] \arrow[d, "p\times p"'] & G \arrow[d, "p"] \\ Q \times Q \arrow[r, "\mu_Q"] & Q \end{tikzcd}$$

is *commutative*: that is, $\mu_Q \circ (p \times p) = p \circ \mu_G$.

Problem 12. Recall the set of p-adic integers

$$\mathbf{Z}_p \subseteq \prod_{i \geq 0} \mathbf{Z}/p^i\mathbf{Z}$$

and its topology from #11–12 on Problem Set 4.

(1) Show that if $\{G_i\}_{i\in I}$ is a collection of topological groups, then $\prod_i G_i$ forms a topological group with respect to the coordinate-wise group law and the product topology. *Hint:* Munkres Theorem 19.6.

(2) Use (1) and Problem 11(1) to show that $\mathbf{Z}_p$ forms a topological group with respect to addition.

(3) Use #12 from Problem Set 4 to show that $p^j\mathbf{Z}_p$ is a clopen proper subgroup of $\mathbf{Z}_p$ for all integers $j \geq 1$.

(4) Show that by contrast, $\mathbf{R}$ contains no proper open subgroups.

Hint: Using cosets, show that any open subgroup of any topological group is clopen. But $\mathbf{R}$ is connected.

(3)–(4) suggest the remarkable topological differences between p-adic numbers and real numbers.