

(Axler §1C)

last time:

$\text{Mag}(F) = \{3 \text{ by } 3 \text{ magic squares over } F\}$

viewed as a subset of F^9 via

$$\begin{array}{ccc} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{array} \quad \text{mapsto}$$

$(a_{11}, a_{12}, a_{13}, a_{21}, \dots, a_{33})$

suggests how to turn $\text{Mag}(F)$ into a vector space

Df given a vector space V over F

a subset W is an F -linear subspace of V iff

- 1) W is stable under the addition $+$ and scaling \bullet
- 2) $\mathbf{0}$ in W
- 3) W forms a vector space wrt $+$, \bullet , $\mathbf{0}$

Lem 1) and 2) are sufficient

Pf sketch associativity, commutativity,
distributive laws immediate from 1)

[only existence of inverses might be tricky]

suppose w in W :

$-w = (-1)w$ by previous class
 $(-1)w$ in W by 2)

Ex fix a, b in F

$\{(x, ax) \mid x \in F\}$ is a linear subspace of F^2

1) $(x, ax) + (x', ax') = (x + x', a(x + x'))$

2) $\mathbf{0} = (0, 0) = (0, a0)$

$\{(x, ax + 1) \mid x \in F\}$ not a linear subspace of F^2

[which conditions fail? both]

Ex $\{\mathbf{0}\}$ is a linear subspace of any V

Ex \emptyset is never a linear subspace of V
[why?] 1) holds but 2) fails

Ex R is an R -linear subspace of C

Ex R is not a C -linear subspace of C
[why?] 1) fails

Operations on Linear Subspaces

if W, W' are linear subspaces of V ,
then $W \cap W'$ is too [why?]

if W, W' are linear subspaces of V ,
then $W \cup W'$ need not be! [example?]

Ex $V = F^2$
 $W = \{(x, 0) \mid x \in F\}$, $W' = \{(0, y) \mid y \in F\}$

Df if W, W' are linear subspaces of V ,
then their sum is
 $W + W' = \{w + w' \mid w \text{ in } W \text{ and } w' \text{ in } W'\}$

Lem $W + W'$ is indeed a linear subspace of V

Pf exercise
 $[(v + v') + (w + w')] = (v + w) + (v' + w')]$
 $[a \cdot (w + w')] = a \cdot w + a \cdot w']$

more generally, for finitely many W_i sub V ,

$$W_1 + \dots + W_k = \{\text{sum}_i w_i \mid w_i \text{ in } W_i\}$$

(HW: sums of infinitely many linear subspaces)

Prop $W_1 + \dots + W_k$ is precisely
the minimal linear subspace of V
that contains $\bigcup_i W_i$

Pf by lemma, $W_1 + \dots + W_k$ is
a lin. sub. of V containing $\bigcup_i W_i$

take any lin. sub. U of V containing $\bigcup_i W_i$

take any element of $W_1 + \dots + W_k$
must look like $w_1 + \dots + w_k$
with w_i in W_i for all i
then w_i in U for all i
then $w_1 + \dots + w_k$ in U

Rem ["law of excluded middle"]

for any set V and subsets U, W, W' ,

$$U \cap (W \cup W') = (U \cap W) \cup (U \cap W')$$

by contrast, we can have

a vector space V and subspaces U, W, W' s.t.

$$U \cap (W + W') \neq (U \cap W) + (U \cap W')$$

Ex $V = F^2$

$$U = \{(x, x) \mid x \in F\}$$

$$W = \{(x, 0) \mid x \in F\}, \quad W' = \{(0, y) \mid y \in F\}$$

$$U \cap (W + W') = U, \quad U \cap W = U \cap W' = \{\mathbf{0}\}$$

classical logic:	propositions	sets
	"and"	cap
	"or"	cup
	[think: Venn diagrams]	

quantum logic:	propositions	vec. spaces
	"and"	cap
	"or"	+

Ex double-slit experiment

[draw picture]

U = "photon hits screen"

W = "photon passes through slit 1"

W' = "photon passes through slit 2"

Rem [sums of lin. sub.'s \neq unions]
[nor behave like sums of #'s]

in F^3 , consider

$$L = \{(x, x, 0) \mid x \in F\}$$

$$K = \{(x, -x, 0) \mid x \in F\}$$

$$M = \{(x, y, 0) \mid x, y \in F\}$$

$$N = \{(x, x, y) \mid x, y \in F\}$$

$$L + K = M \quad K + M = M \quad M + N = F^3$$

$$L + M = M \quad K + N = F^3$$

$$L + N = N$$

Q when do sums have “redundancy”?

Direct Sums

Df $\sum_i W_i$ is called a direct sum iff

any elt of $\sum_i W_i$ has a unique decomposition

$$w_1 + \dots + w_k$$

where $w_i \in W_i$ for all i

Rem some decomposition always exists
the point is a unique decomposition
[uniqueness = “no redundancy”]

Prop $W_1 + W_2$ is a direct sum
iff
 $W_1 \cap W_2 = \{\mathbf{0}\}$

Pf suppose $v_1 + v_2 = w_1 + w_2$
 where v_i, w_i in W_i

then $v_1 - w_1 = v_2 - w_2$ in $W_1 \cap W_2$

so $v_1 - w_1 = v_2 - w_2 = \mathbf{0}$

so $v_1 = w_1$ and $v_2 = w_2$

Rem no analogue for $W_1 + W_2 + W_3$:

in F^2 , consider $W_1 = \{(x, 0) \mid x \in F\}$

$W_2 = \{(0, x) \mid x \in F\}$

$W_3 = \{(x, x) \mid x \in F\}$

each pair is a direct sum, but

$(1, 0) + (0, 1) + (0, 0) = (0, 0) + (0, 0) + (1, 1)$

Ex in F^3 , consider [again]

$M = \{(x, y, 0) \mid x, y \in F\}$

$N = \{(x, x, y) \mid x, y \in F\}$

then $M \cap N = \{(x, x, 0) \mid x \in F\}$

$M + N$ is larger than either M or N , but

$M + N$ is not a direct sum