MATH 340: ADVANCED LINEAR ALGEBRA
PROBLEM SET #3

SPRING 2025

Due Wednesday, February 5. You may consult books, papers, and websites as long as you cite all sources and write up your solutions in your own words.

Problem 1. Let V be a vector space of dimension n. Let E be a set of exactly n vectors. Show that E is linearly independent if and only if E spans V.

Problem 2 (Axler §2C, #10)**.** Fix an integer $m \geq 1$. Let $\mathcal{P}_m$ be the vector space of polynomials over F that are either zero or have degree $\leq m$. Show that $\{p_0, p_1, \ldots, p_m\}$ is a basis of $\mathcal{P}_m$, where

$$p_k(x) = x^k(1-x)^{m-k}.$$

Hint: Since $\dim \mathcal{P}_m = m + 1$, it suffices by Problem 1 to check that they span.

Problem 3. Linear maps of the form $T : V \to V$ are also called *linear operators* on V. Show that if V is finite-dimensional, then

$$V = \ker(T) + \operatorname{im}(T),$$

and that the right-hand side is a direct sum. *Hint:* Use the latter claim to prove the former.

Problem 4. Let $p \in F[x]$.

(1) Check that multiplication by p is a linear operator $T_p : F[x] \to F[x]$.
(2) Show that if $p \neq 0$ and $p(0) = 0$, then

$$F[x] \neq \ker(T_p) + \operatorname{im}(T_p).$$

Why doesn't this contradict Problem 3?

Problem 5. Find a linear operator $T : \mathbf{R}^3 \to \mathbf{R}^3$ such that $\operatorname{im}(T) \not\subseteq \ker(T)$, but $\operatorname{im}(T \circ T) \subseteq \ker(T)$. What do these conditions imply about $T \circ T$ and $T \circ T \circ T$?

Problem 6. We say that a square matrix M with complex entries is *hermitian* if and only if $M_{i,j} = \bar{M}_{j,i}$ for all i, j, where the bar denotes complex conjugation. Let

$$\mathrm{Herm}_n = \{n \times n \text{ hermitian matrices}\} \subseteq \mathrm{Mat}_n(\mathbf{C}).$$

(1) Show that Herm_n forms an $\mathbf{R}$-linear subspace of $\mathrm{Mat}_n(\mathbf{C})$ under the

addition $(M + M')_{i,j} = M_{i,j} + M'_{i,j}$ and scaling $(\lambda \cdot M)_{i,j} = \lambda M_{i,j}$,

but not a $\mathbf{C}$-linear subspace.
(2) Find a basis for Herm_2 as a vector space over $\mathbf{R}$. What is its dimension?

Problem 7. We say that a square matrix M with complex entries is *skew-hermitian* if and only if $M_{i,j} = -\bar{M}_{j,i}$ for all i, j. Let

$$\mathrm{Skew}_n = \{n \times n \text{ skew-hermitian matrices}\} \subseteq \mathrm{Mat}_n(\mathbf{C}).$$

(1) Show that $\mathrm{Mat}_n(\mathbf{C}) = \mathrm{Herm}_n + \mathrm{Skew}_n$, as real vector spaces, and that the right-hand side is a direct sum.
(2) Using Problem 6, deduce the dimension of Skew_2.

Problem 8. For any complex number $z = a + bi$, let

$$M_z = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \in \mathrm{Mat}_2(\mathbf{R}).$$

(1) Check that $z = \cos\theta + i\sin\theta$ if and only if (the underlying linear operator of) M_z rotates the standard basis of $\mathbf{R}^2$ in column notation by θ radians counterclockwise.

    Hint: For the "if" direction, observe that $z \mapsto M_z$ is injective.

    (Below, you may take for granted that the linear operator rotates any other vector in $\mathbf{R}^2$ by the same angle.)
(2) Check that $M_{z_1 z_2} = M_{z_1} \cdot M_{z_2}$ for all $z_1, z_2 \in \mathbf{C}$.
(3) Now take $z_j = \cos\theta_j + i\sin\theta_j$ for $j = 1, 2$. Using (1)–(2), deduce the classical formulas for $\cos(\theta_1 + \theta_2)$ and $\sin(\theta_1 + \theta_2)$.