REPRESENTATIONS OF THE GROUP OF UNIMODULAR MATRICES OF ORDER 2 WITH ELEMENTS FROM A LOCALLY COMPACT TOPOLOGICAL FIELD

2

In this chapter we study the representations of the group G of unimodular matrices of order 2 with elements from a locally compact topological field $\mathbf{K}$. The complete classification of all such fields is well known (see **§ 1**).

In **§§ 3** and **4** we construct the irreducible unitary representations of G.

The representation operators $T(g)$ are given by their kernels, which are generalized functions. The question is: what are the functions from which these kernels are formed?

Two types of functions on a locally compact field play the fundamental role—additive characters, which are generalizations of the exponential function, and multiplicative characters, which are generalizations of the power function.

An additive character on **K** is a continuous complex-valued function $\chi(x)$ satisfying the condition

$$\chi(x+y) = \chi(x)\chi(y)$$

for arbitrary elements x and y from **K**.

For the field of real numbers these functions have the form $\chi(x) = e^{\alpha x}$, where α is a complex number; for the field of complex numbers $z = x + iy$ they have the form $\chi(z) = e^{\alpha x + \beta y}$, where α and β are complex numbers.

A multiplicative character on **K** is a continuous complex-valued function $\pi(x)$ on $\mathbf{K} \setminus 0$ satisfying the condition

$$\pi(xy) = \pi(x)\pi(y)$$

for arbitrary nonzero elements x and y from **K**. For the field of real numbers these functions have the form $\pi(x) = |x|^{\alpha}$ or $\pi(x) = |x|^{\alpha} \operatorname{sign} x$, where α is an arbitrary complex number; for the field of complex numbers $z = re^{i\varphi}$ they have the form $\pi(z) = r^{\alpha}e^{in\varphi}$, where α is a complex and n a real number.

The entire stock of functions needed in the theory of representations (Gamma-functions, Beta-functions, Bessel functions, the hypergeometric function) are formed from additive and multiplicative characters by rational transformations of the independent variables and by integration with respect to parameters. In particular, we shall see in **§ 3** that the kernels of the operators of irreducible unitary representations of G can be expressed in terms of Bessel functions, or after transition to another basis in the representation space, by the hypergeometric function.

The group G has several series of irreducible unitary representations. One of these (the continuous series) is connected with the ground field **K**; each of the remaining (discrete) series is connected with a certain quadratic extension of **K**. Thus, if **K** is the field of complex numbers, there is only one series, because the field of complex numbers has no proper algebraic extensions; if **K** is the field of real numbers, there are two series of representations, because the field of real numbers has only one quadratic extension, and, if **K** is a disconnected field, then there are four series of representations because a disconnected field has three quadratic extensions.†

Within each series a representation is given by a certain multiplicative character. More accurately, a representation of the continuous series is given by a multiplicative character π on **K**, and

† Apart from certain special cases when the number of quadratic extensions of **K** is greater than three (see **§ 1**).

to the characters π and π^{-1} there correspond equivalent representations. A representation of the discrete series corresponding to the quadratic extension $\mathbf{K}(\sqrt{\tau})$ of $\mathbf{K}$ is given by a character on the unit circle in $\mathbf{K}(\sqrt{\tau})$, that is, on the multiplicative group of elements $t = x + \sqrt{\tau}\, y$ for which $t\bar{t} \equiv x^2 - \tau y^2 = 1$. Again, to the characters π and π^{-1} there correspond equivalent representations.

So there is complete duality between the irreducible representations of G and the Cartan subgroups of G: every irreducible representation of G is given by a character on one of the Cartan subgroups.

In the construction of the representations of the discrete series the following interesting fact emerges: these representations are realized not in the space of all functions on $\mathbf{K}$, but in a space of functions that resemble analytic functions. (For a disconnected field the concept of a complex-valued analytic function does not exist. Nevertheless there is a natural way of defining the concept of a function resembling an analytic function in the upper half-plane, see **§ 2.8.**)

In **§ 5** we compute the traces (characters) of the irreducible representations. We obtain a single formula for them, independent of the structure of $\mathbf{K}$. In fact, we shall see that the trace of the representation of the continuous series corresponding to the character $\pi(t)$ is expressed by the following formula:

$$\operatorname{Tr} T_\pi(g) = \int_K \delta(\lambda_g + \lambda_g^{-1} - t - t^{-1})\pi(t)\, |t|^{-1}\, dt,$$

where λ_g and λ_g^{-1} are the eigenvalues of the matrix g, and $\delta(t)$ is the Delta-function.

It is convenient to combine the representations of the discrete series corresponding to a quadratic extension $\mathbf{K}(\sqrt{\tau})$ of $\mathbf{K}$ into pairs. Then the trace of the sum of the related representations of the discrete series is expressed by the following formula:

$$\operatorname{Tr} T_\pi(g) = 2 \int_{t\bar{t}=1} \frac{\operatorname{sign}_\tau (\lambda_g + \lambda_g^{-1} - t - t^{-1})}{|\lambda_g + \lambda_g^{-1} - t - t^{-1}|}\, \pi(t)\, d^*t.$$

The meaning of the notation $|t|$ and $\operatorname{sign}_\tau t$ for a disconnected field will be explained in **§ 1.**

In **§ 6** we obtain the Plancherel formula, which gives the decomposition of the regular representation of G into representations of the continuous and the discrete series. Specifically, when we associate with each representation $T_\pi(g)$ of these series the operator

$$T_\pi(f) = \int f(g)\, T_\pi(g)\, dg,$$

where f is a function on G of integrable square, then we have the inversion formula

$$f(g) = \int \mu(\pi) \operatorname{Tr} (T_\pi(f) T_\pi^{-1}(g)) \, d\pi$$

and the Plancherel formula

$$\int |f(g)|^2 \, dg = \int \mu(\pi) \operatorname{Tr} (T_\pi(f) T_\pi^*(f)) \, d\pi.$$

It will be shown that the "Plancherel measure" $\mu(\pi)$ occurring in these formulae may be given by the following single formula:

$$\mu(\pi) = c \int \pi(t) \, |1 - t|^{-2} \, dt.$$

For representations of the continuous series the integration here is taken over $\mathbf{K}$, and, for representations of the discrete series corresponding to a quadratic extension $\mathbf{K}(\sqrt{\tau})$ of $\mathbf{K}$, over the unit circle $t\bar{t} \equiv x^2 - \tau y^2 = 1$. The integral must be understood in the sense of the regularizing value.

This integral can be computed without difficulty when $\mathbf{K}$ is the field of complex or real numbers.

For the field of complex numbers we find

$$\mu(\pi) = c(\rho^2 + n^2), \qquad \text{where } \pi(re^{i\varphi}) = r^{i\rho} e^{in\varphi}.$$

For the field of real numbers we have: for representations of the continuous series

$$\mu(\pi) = c\rho \tanh \frac{\pi\rho}{2}, \qquad \text{when } \pi(x) = |x|^{i\rho},$$

$$\mu(\pi) = c\rho \coth \frac{\pi p}{2}, \qquad \text{when } \pi(x) = |x|^{i\rho} \operatorname{sign} x;$$

for representations of the discrete series

$$\mu(\pi) = c\,|n|, \qquad \text{when } \pi(t) = t^n, \quad |t| = 1.$$

§ 1. STRUCTURE OF LOCALLY COMPACT FIELDS

In this section we give an account of essentially well-known results on the structure of locally compact fields. Some of the results will be only stated. Their detailed proof can be found, for example, in [8] and [61].

1. Classification of Locally Compact Fields. We discuss only topological fields (that is, fields with a nondiscrete topology).†

† Hence, the field of rational numbers is excluded from this discussion.

Here are the classical examples of locally compact topological fields:

1. The field R of real numbers.
2. The field C of complex numbers.
3. The field Q_p of p-adic numbers, where p is any prime number. Let us recall the definition of the field Q_p.

The elements of Q_p are formal power series

$$x = \sum_{i=k}^{+\infty} a_i p^i, \tag{1}$$

where k is an integer, and the a_i are integers satisfying the condition $0 \leq a_i < p$. Thus, the series (1) may contain an arbitrary finite number of terms with negative integral powers. The sum of the two p-adic numbers $x = \sum_{i=k}^{\infty} a_i p^k$ and $y = \sum_{i=l}^{\infty} b_i p^i$ is the p-adic number $z = \sum_{i=m}^{\infty} c_i p^i$, $m = \min(k, l)$ such that

$$\sum_{i=k}^{n} a_i p^i + \sum_{i=l}^{n} b_i p^i \equiv \sum_{i=m}^{n} c_i p^i (\mathrm{mod}\, p^{n+1}) \tag{2}$$

for every positive integer n. Obviously, the coefficients c_i can be found successively from the relations (2). The product of p-adic numbers is defined in a similar fashion. A neighborhood of a p-adic number $x = \sum_{i=k}^{\infty} a_i p^i$ is the set U_n of p-adic numbers $y = \sum_{i=k}^{\infty} b_i p^i$ for which $b_i = a_i$ for $i \leq n$. It is not hard to verify that under this topology Q_p becomes a locally compact space and that the operations of addition and multiplication are continuous in this topology.

We mention that the field Q_p can also be obtained by completing the field of rational numbers relative to a suitable topology.

For let $n(r)$ be the power to which the prime number p occurs as a factor in the rational number r. We call p^{-r} the p-norm of r. A sequence of rational numbers is called fundamental if it is fundamental in the sense of the p-norm.

Thus, Q_p contains the field of rational numbers as an everywhere dense subset.

4. The field $\mathbf{K}_p(t)$ of power series over the residue class field modulo p (p a prime number). By definition, the elements of $\mathbf{K}_p(t)$ are power series

$$x = \sum_{i=k}^{\infty} a_i t^i,$$

which may contain a finite number of terms with negative powers of t; the coefficients of these series lie in the residue class field modulo p. Addition and multiplication of two power series is defined in the natural way. A neighborhood of the power series $x = \sum_{i=k}^{\infty} a_i t^i$ is the set of power series in which all the coefficients up to a certain fixed index coincide with the a_i.

Now we can give the classification of all locally compact nondiscrete fields (theorem of Koval'skii-Pontryagin).

The field R of real numbers and the field C of complex numbers are the only connected locally compact fields.

Every disconnected locally compact field of characteristic 0 is a finite extension of the field Q_p of p-adic numbers.

Every locally compact field of characteristic $p \neq 0$ is a finite extension of the field $K_p(t)$ of power series over the residue class field modulo p.

For fields of characteristic $p \neq 0$ there is an even stronger result. *Every locally compact field of characteristic $p \neq 0$ is isomorphic to the field of power series*

$$x = \sum_{i=k}^{\infty} a_i t_i,$$

whose coefficients belong to a finite field of characteristic p. The algebraic operations and the topology in this field are defined just as in the case of $\mathbf{K}_p(t)$.

2. The Norm in K. For an arbitrary locally compact field **K** we can introduce the concept of a norm. For this purpose we consider a measure dx on **K** invariant under addition:

$$d(x + a) = dx$$

for every a from **K**. Such a measure on **K** is known to be uniquely determined to within a constant factor.

Let x_0 be an arbitrary element from **K**. It is easy to see that the measure $d_{x_0}x = d(xx_0)$ is also invariant under addition, so that it differs from dx only by a factor depending on x_0, which we denote by $|x_0|$:

$$d_{x_0}x = |x_0| \, dx.$$

Thus, we have introduced in **K** a continuous function $|x|$, which obviously has the following properties:

1. $|x| > 0$ for $x \neq 0$; $|0| = 0$,
2. $|xy| = |x| \cdot |y|$.

It can be shown that for a disconnected field **K** also the following property holds:

3. $|x + y| \leq \max(|x|, |y|)$.

We call $|x|$ the *norm* of x in **K**.

Clearly, for the field of real numbers $|x|$ is the absolute value of x; for the field of complex numbers $|x|$ is the square of the modulus of x.

Let us see what values $|x|$, $x \neq 0$, may assume. For this purpose we observe that the map

$$x \to |x|$$

is a continuous homomorphism of the multiplicative group of **K** into the multiplicative group of positive real numbers. *From this it follows easily that for a connected field* $|x|$ $(x \neq 0)$ *ranges over all positive real numbers; but for a disconnected field* $|x|$ $(x \neq 0)$ *assumes only the discrete set of values* q^n, *where* q *is a fixed number and* n *is an integer.*

From the result just stated it follows that in a disconnected field **K** the set of points x for which $|x| = c$, $c > 0$, and the set of points x for which $|x| < c$, $c > 0$, are both open in **K**.

It can be shown that the sets of points x of a disconnected field **K** for which $|x| < c$ (when c ranges over the positive numbers) form a complete system of neighborhoods of the zero element. Hence, *the topology in a disconnected field* **K** *is completely determined by the norm in* **K**. For connected fields the last result is obvious.

3. The Structure of Disconnected Fields. Using the concept of norm we can describe the detailed structure of disconnected fields. Let **K** be a disconnected field with norm $|x|$. Then the following facts hold:

1. The set O of elements of **K** for which $|x| \leq 1$ is compact and open in **K**. Obviously O is a subring whose elements are called the *integers* of **K**.

2. The set of elements x in O for which $|x| < 1$ forms a prime ideal P of O. The residue class field $\mathscr{K} = O/P$ consists of a finite number q of elements, where q is a power of a prime number.

3. P is a principal ideal, that is, P contains an element $\mathfrak{p}$ such that $P = \mathfrak{p}O$. The norm of $\mathfrak{p}$ is

$$|\mathfrak{p}| = q^{-1},$$

where q is the order of the residue class field O/P.

Here are some examples:

1. **K** is the field of p-adic numbers. Here O consists of the elements of the form $\sum_{i=0}^{\infty} a_i p^i$ and its prime ideal P of the elements of the form $\sum_{i=1}^{\infty} a_i p^i$. Obviously P is generated by the number p, and $|p| = p^{-1}$.

2. **K** is the field of power series over the residue class field modulo p. Here O consists of the elements of the form $\sum_{i=0}^{\infty} a_i t^i$ and its prime ideal P of the elements of the form $\sum_{i=1}^{\infty} a_i t^i$. Obviously P is generated by the element t, and $|t| = p^{-1}$.

3. The multiplicative group of **K** contains an element ε of finite order $q - 1$ (where q is the order of the residue class field O/P). Clearly, $|\varepsilon| = 1$, that is, ε belongs to O but not to P. The elements $0, \varepsilon, \varepsilon^2, \ldots, \varepsilon^{q-1} = 1$ form a complete set of representatives of the residue classes of O/P.

4. Every element of **K** has a unique representation as a convergent series

$$x = \mathfrak{p}^n(a_0 + a_1\mathfrak{p} + a_2\mathfrak{p}^2 + \cdots), \qquad a_0 \neq 0, \tag{1}$$

where $\mathfrak{p}$ is a generating element† of P, n an integer, and the coefficient a_i may assume the values $0, \varepsilon, \varepsilon^2, \ldots, \varepsilon^{q-1} = 1$.

4. Additive and Multiplicative Characters of K. As an algebraic object the field **K** functions on two planes: it is a group under addition, and at the same time the set of elements of **K** other than 0 forms a group under multiplication. Henceforth we denote by $\mathbf{K}^+$ the additive group of **K**, and by $\mathbf{K}^*$ its multiplicative group. The most important functions on **K** are additive and multiplicative characters of **K**. Later we shall see that on the basis of these functions we can construct the theory of group representations, and in particular, the theory of special functions.

An additive character of **K** is a character of $\mathbf{K}^+$, that is, a continuous complex-valued function $\chi(x)$ satisfying the conditions:

1. $\chi(x + y) = \chi(x)\chi(y)$ for arbitrary elements x and y from K.
2. $|\chi(x)| = 1$.

A multiplicative character of **K** is a character of its multiplicative group $\mathbf{K}^*$, that is, a continuous complex-valued function $\pi(x)$ on **K** satisfying the conditions:

1. $\pi(xy) = \pi(x)\pi(y)$ for arbitrary elements x and y from $\mathbf{K}^*$.
2. $|\pi(x)| = 1$.

The additive and multiplicative characters themselves constitute topological groups, which we shall now describe.

The group of additive characters of a locally compact topological field‡ **K** *is isomorphic to its additive group* $\mathbf{K}^+$. This isomorphism is realized as follows. Let $\chi(x) \not\equiv 1$ be a fixed nontrivial additive character. Then it can be shown that every character on $\mathbf{K}^+$ is of the form

$$\chi_u(x) = \chi(ux),$$

† Thus, $\mathfrak{p} = p$ for the field Q_p of p-adic numbers; for the field $\mathbf{K}_q(t)$ of power series over a finite field we have $\mathfrak{p} = t$ (see the examples above).

We emphasize that for Q_p the representation (1) is not equivalent to the usual representation of a p-adic number in the form of a series (see **§ 1.1**). There the coefficients a_i were integers, $0 \leq a_i < p$; here, they are p-adic integers such that either $a_i^{p-1} = 1$, or $a_i = 0$.

‡ If **K** is a discrete field, then the group of additive characters is compact, and therefore not isomorphic to $\mathbf{K}^+$.

where u is an element of $\mathbf{K}$. The correspondence $u \rightarrow \chi_u(x)$ gives the required isomorphism of $\mathbf{K}^+$ with its character group.

We mention that for the field Q_p of p-adic numbers every character $\chi(ux)$ can be written in explicit form

$$\chi(ux) = e^{2\pi iux}.$$

The expression $e^{2\pi iux}$ has the following meaning. Since $e^{2\pi in} = 1$ for every integer n, the integral part of the p-adic number ux in the exponent on the right can be ignored. However, for extensions of Q_p such an expression for the characters is not available.

Now we proceed to a description of the multiplicative group $\mathbf{K}^*$ of $\mathbf{K}$ and its character group.

In accordance with **§ 1.3** (Proposition 4) we write every element of the field in the form

$$x = \mathfrak{p}^n \varepsilon^k (1 + a_1 \mathfrak{p} + a_2 \mathfrak{p}^2 + \cdots), \tag{1}$$

where $\mathfrak{p}$ is a generating element of the prime ideal P (in the ring of integers O), and the a_i take the values 0 or ε^l ($\varepsilon^{q-1} = 1$). The elements $\mathfrak{p}^n$ form an infinite cyclic subgroup of $\mathbf{K}^*$, and the elements ε^k a finite subgroup of order $q - 1$. It is clear that the elements $1 + a_1\mathfrak{p} + a_2\mathfrak{p}^2 + \cdots$ also form a subgroup of $\mathbf{K}^*$, and this is compact. Note that this subgroup can be described succinctly in terms of the norm: its elements are precisely those elements of $\mathbf{K}$ for which $|x - 1| < 1$.

Thus, the multiplicative group $\mathbf{K}^*$ *of* $\mathbf{K}$ *is a direct product*

$$\mathbf{K}^* = Z \times Z_{q-1} \cdot A$$

of three groups: the infinite cyclic group Z *of the elements* $\mathfrak{p}^n$, *the finite cyclic group* Z_{q-1} *of order* $q - 1$ *of the elements* ε^k, *and the compact group* A *of the elements* x *for which* $|x - 1| < 1$.

From this we can deduce the structure of the group multiplicative characters of $\mathbf{K}$. *The group of multiplicative characters of* $\mathbf{K}$ *is a direct product of three groups: the group of rotations of a circle, a cyclic group of order* $q - 1$, *and a certain infinite discrete group* (*the group dual to* A). Thus, every multiplicative character $\pi(x)$ is given by three quantities: a real number ρ, which is determined modulo 1; an integer α, which is determined modulo $q - 1$; and a character $\theta(a)$ of A. It is expressed by the following formula: if

$$x = \mathfrak{p}^n \varepsilon^k a, \tag{2}$$

where a belongs to A, then

$$\pi(x) = e^{2\pi i n\rho} e^{2\pi i\, \alpha k/(q-1)} \theta(a). \tag{3}$$

In what follows we also consider nonunitary characters $\pi(x)$, that is, continuous functions satisfying only the condition

$$\pi(xy) = \pi(x)\pi(y).$$

It is easy to verify that every such character $\pi(x)$, as before, is given by (3), in which ρ may now be an arbitrary complex number.

5. The Structure of the Subgroup A. **The functions** $\exp x$ **and** $\ln x$. Here we consider a disconnected field $\mathbf{K}$ of characteristic 0. Our aim is to study in detail the structure of the multiplicative group A of the elements x for which $|x - 1| < 1$. We show that under certain additional restrictions on $\mathbf{K}$ this subgroup is isomorphic to the additive group P of elements x for which $|x| < 1$.

The isomorphism $A \cong P$ is established by means of the functions $\exp x$ and $\ln x$, which we define via the sums of the power series:

$$\exp x = 1 + x + \frac{x^2}{2!} + \dots,$$

$$\ln (1 + x) = x - \frac{x^2}{2} + \dots.$$

First we find out for what x these series are convergent.

Note that $\mathbf{K}$ is a finite extension of the field Q_p of p-adic numbers. The prime p is uniquely determined by $\mathbf{K}$: it is the only prime number for which $|p| < 1$; the norms of all other prime numbers are equal to 1.

The series for $\exp x$ *converges if and only if* $|x| < |p|^{1/(p-1)}$.

To prove this we begin by estimating $|n!|$. Let $p^k \leq n < p^{k+1}$. Then it is easy to verify that the power to which p occurs as a factor in $n!$ is†

$$\left[\frac{n}{p}\right] + \left[\frac{n}{p^2}\right] + \dots + \left[\frac{n}{p^k}\right].$$

Consequently,

$$|n!| = |p|^{[n/p]+\dots+[n/p^k]},$$

and therefore,

$$|n!| \geq |p|^{n(1-p^{-k})/(p-1)}. \tag{1}$$

Suppose now that $|x| < |p|^{1/(p-1)}$ that is, $|x| = |p|^{(1+\varepsilon)/(p-1)}$, where $\varepsilon > 0$. Then we find on the basis of the estimate for $|n!|$

$$\left|\frac{x^n}{n!}\right| \leq |p|^{n(\varepsilon+p^{-k})/(p-1)}. \tag{2}$$

The convergence of the series $\exp x$ for $|x| < |p|^{1/(p-1)}$ follows immediately from this.

On the other hand, let $|x| \geq |p|^{1/(p-1)}$. Then for $n = p^k$ we have

$$|n!| = |p|^{n(1-p^{-k})/(p-1)},$$

therefore,

$$\left|\frac{x^n}{n!}\right| \geq |p|^{p^{-k}/(p-1)}.$$

From this estimate it is clear that for $|x| \geq |p|^{1/(p-1)}$ the series $\exp x$ diverges.

Let x belong to the domain of convergence of $\exp x$, that is, $|x| < |p|^{1/(p-1)}$.

† The symbol $[a]$ denotes the integral part of a.

Then

$$|\exp x - 1 - x| < |x|. \tag{3}$$

For from (2) we have, since $n \geq p^k$,

$$\left|\frac{x^n}{n!}\right| \leq |x|\,|p|^{(n-1)\varepsilon/(p-1)}$$

Consequently, $\left|\frac{x^n}{n!}\right| < |x|$ for $n \geq 2$. Hence, (3) follows immediately.

The series for $\ln(1 + x)$ *converges if and only if* $|x| < 1$.

If $|1 - y| < |p|^{1/(p-1)}$, *then* $|\ln y| = |1 - y|$.

We leave the verification of these statements to the reader.

It is easy to show that the functions $\exp x$ and $\ln x$ have the usual properties: $\exp(x_1 + x_2) = \exp x_1 \exp x_2$ $\ln(x_1 x_2) = \ln x_1 + \ln x_2$ (provided that the elements x_1 and x_2 lie in the domain of definition of the corresponding function).

The function $y = \exp x$ *effects an isomorphic map of the additive group* B *of elements for which* $|x| < |p|^{1/(p-1)}$ *onto the multiplicative group* A_1 *of elements* y *for which* $|1 - y| < |p|^{1/(p-1)}$. *The inverse isomorphism is given by the function* $x = \ln y$.

For let $x \in B$. Then the series $y = \exp x$ converges for x. From (3) it follows that $|\exp x - 1| = |x|$ so that $|1 - y| < 1$. But then the series $\ln y = \ln(\exp x)$ converges. By formal operations on series we verify that

$$\ln(\exp x) = x \tag{4}$$

for every x of B.

Conversely, let $y \in A_1$. Then the series $x = \ln y$ converges, and $|\ln y| = |1 - y|$; consequently, the series $\exp x = \exp(\ln y)$ also converges. By formal operations on series we verify that

$$\exp(\ln y) = y$$

for every y in A_1.

From (4) and (5) it follows that the function $y = \exp x$ effects a one-to-one map of B onto A_1. The fact that this is an isomorphism follows from the relation $\exp(x_1 + x_2) = \exp x_1 \exp x_2$.

Now let us find out under what conditions the subgroup A_1 coincides with the multiplicative group A of all elements x of the field for which $|1 - x| < 1$.

Let O be the ring of integers of $\mathbf{K}$, P the maximal ideal in O, $\mathfrak{p}$ a generating element of P, and $q^{-1} = |\mathfrak{p}|$ its norm.

Clearly, A consists of precisely those elements x for which $|1 - x| \leq q^{-1}$, where the equality sign may hold.

Consequently, the condition that $A = A_1$ can be expressed in the form

$$q^{-1} < |p|^{\frac{1}{p-1}}. \tag{6}$$

Let $|p| = q^{-(s-1)}$. This means that p belongs to P^{s-1}, but not to P^s. Then (6) can be rewritten in the form $q^{-1} < q^{\frac{s-1}{p-1}}$. From this we obtain the condition on s: $s < p$.

We state the final result. *Let* $\mathbf{K}$ *be a disconnected field of characteristic zero,* O *the subring of integers of* $\mathbf{K}$, P *the maximal ideal in* O, p *the characteristic of the residue class field* O/P. *We assume that* p *does not belong to* P^{p-1}. *Then the multiplicative group* A *of elements* x *of the field for which* $|1 - x| < 1$ *is isomorphic to the additive group* P *of elements* y *for which* $|v| < 1$. *The isomorphism is effected by the function* $y = \ln x$.

In general this statement is not true: A may contain elements of finite order p^n; then it is not isomorphic to any of the subgroups of the additive group P (because all elements of P are of infinite order).

6. Quadratic Extensions of a Disconnected Field. Let τ be an element of $\mathbf{K}$ that is not a square of another element of the field. By adjoining to $\mathbf{K}$ the square root $\sqrt{\tau}$, we obtain a quadratic extension $\mathbf{K}(\sqrt{\tau})$ of $\mathbf{K}$. The elements of $\mathbf{K}(\sqrt{\tau})$ have the form $z = x + \sqrt{\tau}\, y$, where x and y belong to $\mathbf{K}$. Addition and multiplication of such elements proceed in the usual way. Let us find out how many distinct quadratic extensions a disconnected field $\mathbf{K}$ has.

Obviously, two quadratic extensions $\mathbf{K}(\sqrt{x})$ and $\mathbf{K}(\sqrt{y})$ of $\mathbf{K}$ coincide if and only if the quotient xy^{-1} is a square in $\mathbf{K}$. In other words, there are as many quadratic extensions of $\mathbf{K}$ as there are nontrivial cosets of the multiplicative group $\mathbf{K}^*$ with respect to the subgroup of all squares $(\mathbf{K}^*)^2$.

We determine the index $\mathbf{K}^*:(\mathbf{K}^*)^2$. From **§ 1.4** we know that $\mathbf{K}^*$ is a direct product

$$\mathbf{K}^* = Z \times Z_{q-1} \times A$$

of an infinite cyclic group Z, a finite cyclic group Z_{q-1} of order $q - 1$, and the subgroup A of the element x in K for which $|x - 1| < 1$. Therefore,

$$\mathbf{K}^*:(\mathbf{K}^*)^2 = (Z:Z^2) \times (Z_{q-1}:Z_{q-1}^2) \times (A:A^2),$$

where Z^2, Z_{q-1}^2, A^2 are the subgroups consisting of the squares of the elements of the corresponding groups.

We now assume that q is an odd number, so that the residue class field O/P is of characteristic $p \neq 2$. In this case Z_{q-1} is a cyclic group of even order, and $Z_{q-1}:Z_{q-1}^2 = 2$. Also $Z:Z^2 = 2$.

We show, finally, that $A = A^2$, that is, that the equation $x^2 = a$ has a solution in A for every $a \in A$. For let

$$a = 1 + a_1\mathfrak{p} + a_2\mathfrak{p}^2 + \dots .$$

We are looking for a solution x of the equation $x^2 = a$ in the form of a series $x = 1 + x_1\mathfrak{p} + x_2\mathfrak{p}^2 + \dots$, where $x_i = 0$ or $x_i = \varepsilon^n$ (ε an element of O^* of order $q - 1$). The equation $x^2 = a$ reduces to a system of congruences

$$2x_1 \equiv a_1 (\mathrm{mod}\ P), \qquad 2x_1 + (2x_2 + x_1^2)\mathfrak{p} \equiv a_1 + a_2\mathfrak{p}\ (\mathrm{mod}\ P^2)$$

and so forth. Clearly, under the assumption made on q, $x_1, x_2, \dots$ can be found successively from these congruences.

So we obtain: *if the characteristic of the residue class field O/P is different from 2, then the squares of the elements $x \neq 0$ of $\mathbf{K}$ form a subgroup of index 4 of the multiplicative group of $\mathbf{K}$. There are then three distinct quadratic extensions of $\mathbf{K}$. Clearly, these quadratic extensions are $\mathbf{K}(\sqrt{\mathfrak{p}})$, $\mathbf{K}(\sqrt{\varepsilon\mathfrak{p}})$, and $\mathbf{K}(\sqrt{\varepsilon})$.*†

† Observe that the cases $\tau = \varepsilon\mathfrak{p}$ and $\tau = \mathfrak{p}$ are not distinct, because the element $\varepsilon\mathfrak{p}$ may play the role of p.

This result is not true when the characteristic of O/P is 2. For example, if $\mathbf{K}$ is of characteristic 2, then $A:A^2 = \infty$.

7. The Multiplicative Characters $\operatorname{sign}_\tau x$. Let $\mathbf{K}$ be a locally compact disconnected field. We assume, as before, that the finite residue class field O/P associated with it is not of characteristic 2. In this subsection we associate with every quadratic extension $\mathbf{K}(\sqrt{\tau})$ of $\mathbf{K}$ a certain multiplicative character $\operatorname{sign}_\tau x$ assuming the values ± 1 on $\mathbf{K}$.

Suppose then that $\mathbf{K}(\sqrt{\tau})$ is a quadratic extension of $\mathbf{K}$. We consider the product

$$z\bar{z} = x^2 - \tau y^2$$

of the element $z = x + \sqrt{\tau}\, y$ from $\mathbf{K}(\sqrt{\tau})$ by its conjugate† $\bar{z} = x - \sqrt{\tau}\, y$.

The set of elements $z\bar{z}$, $z \neq 0$, forms a subgroup $\mathbf{K}_\tau^*$ of the multiplicative group $\mathbf{K}^*$; and $\mathbf{K}_\tau^*$ obviously contains $(\mathbf{K}^*)^2$.

Let us show that *the index* $\mathbf{K}^*:\mathbf{K}_\tau^*$ *is* 2.

It is enough to verify that $\mathbf{K}_\tau^* \neq \mathbf{K}^*$ and $\mathbf{K}_\tau^* \neq (\mathbf{K}^*)^2$. Our assertion then follows immediately from the fact that $\mathbf{K}^*:(\mathbf{K}^*)^2 = 4$ (see **§ 1.6**).

We begin by showing that $\mathbf{K}_\tau^* \neq (\mathbf{K}^*)^2$. For if $\tau = \mathfrak{p}$ or $\tau = \varepsilon\mathfrak{p}$, then $-\tau$ is not a square of an element from $\mathbf{K}^*$, but belongs to $\mathbf{K}_\tau^*$. Now let $\tau = \varepsilon$. It can be shown that there exists integers x and y such that $x^2 - \varepsilon y^2 \equiv \varepsilon (\operatorname{mod} P)$‡ It is obvious that then $x^2 - \varepsilon y^2$ is not a square, but belongs to $\mathbf{K}_\tau^*$. Hence $\mathbf{K}_\tau^* \neq (\mathbf{K}^*)^2$. Now we show that $\mathbf{K}_\tau^* \neq \mathbf{K}^*$. For in the case $\tau = \mathfrak{p}$ or $\tau = \varepsilon\mathfrak{p}$ the element ε does not belong to $\mathbf{K}_\tau^*$. But for $\tau = \varepsilon$ the subgroup $\mathbf{K}_\tau^*$ cannot contain $\mathfrak{p}$ (otherwise we have a congruence $x^2 - \varepsilon y^2 \equiv 0$ $(\operatorname{mod} P)$ for certain $x \not\equiv 0 \ (\operatorname{mod} P)$ and $y \not\equiv 0 \ (\operatorname{mod} P)$, which is impossible).

Hence $\mathbf{K}_\tau^* \neq (\mathbf{K}^*)^2$, $\mathbf{K}_\tau^* \neq \mathbf{K}^*$, and $\mathbf{K}^*:\mathbf{K}_\tau^* = 2$.

Now we introduce the function $\operatorname{sign}_\tau x$ on $\mathbf{K}^*$. We set

$$\operatorname{sign}_\tau x = 1,$$

when $x \in \mathbf{K}_\tau^*$, that is, when x is representable in the form $x = x_1^2 - \tau x_2^2$ and

$$\operatorname{sign}_\tau x = -1,$$

when x is not so representable.

† The expression $z\bar{z}$ is often called the norm of z relative to $\mathbf{K}$.

‡ This results from the following theorem. Let F be a finite field, and ε an element of the field that is not a square; then every element x of the field can be represented in the form $x = x_1^2 - \varepsilon x_2^2$, where $x_1, x_2 \in F$ (see [42]).

From the fact that $\mathbf{K}_\tau^*$ is a subgroup of index 2 in $\mathbf{K}^*$ it follows immediately that $\operatorname{sign}_\tau x$ *is a character on* $\mathbf{K}^*$, *that is*,

$$\operatorname{sign}_\tau (xy) = \operatorname{sign}_\tau x \cdot \operatorname{sign}_\tau y$$

for arbitrary x and y of $\mathbf{K}^*$.

We call elements x of $\mathbf{K}$ positive or negative (it would be more accurate to say τ-positive or τ-negative) according to the sign of $\operatorname{sign}_\tau x$.

It can be shown that the functions $\operatorname{sign}_\tau x$, where $\tau = \mathfrak{p}, \varepsilon\mathfrak{p}, \varepsilon$, are independent. Therefore, together with $\pi_0(x) = 1$ they form a complete system of characters on the factor group $\mathbf{K}^*/(\mathbf{K}^*)^2$.

8. Circles in $\mathbf{K}(\sqrt{\tau})$. Let $\mathbf{K}(\sqrt{\tau})$ be a quadratic extension of a disconnected field $\mathbf{K}$. The set of elements z of $\mathbf{K}(\sqrt{\tau})$ that satisfy the equation

$$z\bar{z} = c, \qquad c \neq 0,$$

is called a *circle* in $\mathbf{K}(\sqrt{\tau})$ (with center at 0).

Observe that in contrast to the field of real numbers there are two types of circles: circles of "real" radius for which c is a square of an element of $\mathbf{K}$, and circles of "imaginary" radius for which c is not a square.

A special role is played by the circle

$$z\bar{z} = x^2 - \tau y^2 = 1$$

whose elements form a group under multiplication and which we denote by C_τ.

We now give a parametric equation for the circle

$$x^2 - \tau y^2 = 1.$$

We use the parameter $\dfrac{y}{x+1} = t$. From the equation of the circle it follows that

$$\frac{x-1}{x+1} = \tau \frac{y^2}{(x+1)^2} = \tau t^2,$$

hence,

$$x = \frac{1+\tau t^2}{1-\tau t^2} \qquad y = (x+1)t = \frac{2t}{1-\tau t^2}.$$

Thus, *the circle $x^2 - \tau y^2 = 1$ is given by the following parametric representation;*

$$x = \frac{1+\tau t^2}{1-\tau t^2} \qquad y = \frac{2t}{1-\tau t^2}, \tag{1}$$

Next we show that *all circles are compact.*

It is sufficient to consider the unit circle, because every other circle consists of the points $w = az$, where z ranges over $z\bar{z} = 1$. Obviously the set of points of the circle $z\bar{z} = 1$ is closed. On the other hand, from the parametric equations (1) it follows that $|x| \leq 1$, $|y| \leq 1$; consequently, the set of points of the circle lies in a bounded domain and is therefore compact.

Let us study the detailed structure of the group C_τ of elements z, $z\bar{z} = x^2 - \tau y^2 = 1$. To begin with, let $\tau = \mathfrak{p}$ or $\tau = \varepsilon\mathfrak{p}$. In this case we have $|x^2| = 1$, $|\tau y^2| < 1$. Consequently, $|1 - x^2| < 1$, and therefore either $|1 - x| < 1$ or $|1 + x| < 1$. Hence we conclude: for $\tau = \mathfrak{p}$ or $\tau = \varepsilon\mathfrak{p}$ the group C_τ is the direct product

$$C_\tau = Z_2 \times C'_\tau$$

of a cyclic group of order 2, $Z_2 = \{1, -1\}$, and the subgroup C'_τ of elements of C_τ for which $|z - 1| < 1$.†

Now we take the case $\tau = \varepsilon$. The elements of the circle $z\bar{z} = 1$ may be written in the form of a series

$$z = (a_0 + \sqrt{\varepsilon}b_0)[1 + (a_1 + \sqrt{\varepsilon}b_1)\mathfrak{p} + (a_2 + \sqrt{\varepsilon}b_2)\mathfrak{p}^2 + \cdots],$$

where a_i and b_i take the values 0 and ε^k, $k = 0, 1, \ldots, q - 1$. From $z\bar{z} = 1$ it follows that

$$a_0^2 - \varepsilon b_0^2 \equiv 1 \pmod{P}.$$

It can be shown that this congruence has $q + 1$ solutions, where q is the order of the residue class field O/P.‡

We conclude: *let C'_ε be the subgroup of C_ε that consists of the elements z for which $|z - 1| < 1$; then the index of C'_ε in C_ε is $q + 1$.*

9. Cartesian and Polar Coordinates in $\mathbf{K}(\sqrt{\tau})$. Every element of $\mathbf{K}(\sqrt{\tau})$ has a unique representation in the form

$$z = x + \sqrt{\tau}y,$$

where $x, y \in \mathbf{K}$. Hence, it is given by a pair of elements x and y from $\mathbf{K}$, which we call the Cartesian coordinates of z.

Now we introduce polar coordinates of z. Let $z\bar{z} = c$. Then two cases are possible: either c is a square of an element of $\mathbf{K}$, or it is not a square.

First, let $c = r^2$, $r \in \mathbf{K}$. Then we define the polar coordinates of z as a pair: the element $\rho = r \in \mathbf{K}$ and the element $t = \rho^{-1}z$, which belongs to the circle $t\bar{t} = 1$. Clearly the point z is uniquely determined by its polar coordinates.

---

† We define the norm $|z|$ on $\mathbf{K}(\sqrt{\tau})$ relative to the norm $|x|$ on $\mathbf{K}$ by the following formula: $|z| = |z\bar{z}|^{1/2}$.

‡ This follows from a theorem on finite fields: the equation $x^2 - \omega y^2 = 1$, where ω is not a square has $q + 1$ solutions in the finite field of order q. (The theorem then follows immediately from the parametric equations of the circle.)

Observe that ρ is determined to within its sign. Consequently, $(-\rho, -t)$ can equally well be regarded as the polar coordinates of z. Thus, the polar coordinates of z are determined to within a sign.

Now we take the case when c is not a square. In $\mathbf{K}(\sqrt{\tau})$ we fix an arbitrary element ν such that $\nu\bar{\nu}$ is not a square in $\mathbf{K}$. Then c can be represented in the form $c = (\nu r)(\bar{\nu} r)$, where $r \in \mathbf{K}$. We now define the polar coordinates of z as the pair of elements $\rho = \nu r$ and $t = \nu^{-1}z$, of which the latter is again a point of the unit circle. As in the first case, we have

$$(\rho, t) = (-\rho, -t).$$

10. Invariant Measures on K and Its Quadratic Extension $\mathbf{K}(\sqrt{\tau})$. There are two invariant measures on $\mathbf{K}$: a measure dx, invariant under addition ($d(x + a) = dx$), and a measure d^*x, invariant under multiplication ($d^*(xa) = d^*x$). These measures are very simply connected:

$$d^*x = |x|^{-1}\, dx. \tag{1}$$

For by definition of $|x|$, we have $d(xa) = |a|\, dx$. Consequently, $|xa|^{-1}\, d(xa) = |x|^{-1}\, dx$, so that the measure $|x|^{-1}\, dx$ is invariant under multiplication.

We always normalize dx by the following condition:

$$\int\limits_{|x|\leq 1} dx = 1.$$

We now consider the measures dz and d^*z, $z = x + \sqrt{\tau}\, y$, on $\mathbf{K}(\sqrt{\tau})$ that are invariant under addition and multiplication, respectively. Expressing these measures in terms of the Cartesian coordinates x and y of z we find

$$dz = dx\, dy, \qquad d^*z = \frac{dx\, dy}{|x^2 - \tau y^2|}.$$

Next we express the measures dz and d^*z in terms of the polar coordinates (ρ, t) of z. We recall that the coordinate ρ is determined to within its sign from the equation $\rho\bar{\rho} = z\bar{z}$, and that the second coordinate $t = \rho^{-1}z$ is a point on the circle $t\bar{t} = 1$.

Since the circle $t\bar{t} = 1$ is a group under multiplication, there is an invariant measure d^*t on it, and we normalize the measure by the condition

$$\int\limits_{t\bar{t}=1} d^*t = 1.$$

It is easy to check that in polar coordinates the measures dz

and d^*z are expressed by the following formulae:

$$dz = a_\tau\, d(zz)\, \bar{d}^*t, \qquad d^*z = a_\tau \frac{d(z\bar{z})\, d^*t}{|z\bar{z}|},$$

where $d(z\bar{z})$ is the measure on $\mathbf{K}$ and $a_\tau = 2(1 + q^{-1})(1 + |\tau|)^{-1}$.

11. Additive and Multiplicative Characters on the "Plane" $\mathbf{K}(\sqrt{\tau})$. The additive group of $\mathbf{K}(\sqrt{\tau})$ is the direct sum of two groups isomorphic to the additive group of $\mathbf{K}$. From this it follows that every additive character $\chi(z)$, $z = x + \sqrt{\tau}y$, of $\mathbf{K}(\sqrt{\tau})$ is of the form

$$\chi(z) = \chi_1(x)\chi_2(y), \tag{1}$$

where χ_1 and χ_2 are additive characters on $\mathbf{K}$.

Now we proceed to a description of the multiplicative characters on $\mathbf{K}(\sqrt{\tau})$.

For this purpose we study in detail the multiplicative group of $\mathbf{K}(\sqrt{\tau})$. In accordance with § **1.8** we represent every element of the field in the form $z = rt$ or $z = \nu rt$, where $r \in \mathbf{K}$, $t\bar{t} = 1$, and ν is a fixed element from $\mathbf{K}(\sqrt{\tau})$ for which $\nu\bar{\nu}$ is not a square of an element from $\mathbf{K}$.

Let $\pi(z)$ be a multiplicative character on $\mathbf{K}(\sqrt{\tau})$. We denote by π_1 and π_2 its restrictions to $\mathbf{K}$ and to the circle $t\bar{t} = 1$, respectively. Then we have

$$\pi(rt) = \pi_1(r)\pi_2(t). \tag{2}$$

From the equation $rt = (-r)(-t)$ we obtain a condition connecting π_1 and π_2:

$$\pi_1(-1) = \pi_2(-1). \tag{3}$$

Furthermore, since $\nu\bar{\nu} = r_0 \in K$, we have $\nu^2 = r_0t_0$, where $t_0\bar{t}_0 = 1$. Consequently, $\pi(\nu^2) = \pi_1(r_0)\pi_2(t_0)$, that is,

$$\pi^2(\nu) = \pi_1(\nu\bar{\nu})\pi_2\left(\frac{\nu}{\bar{\nu}}\right). \tag{4}$$

Suppose, conversely, that π_1 and π_2 are arbitrary multiplicative characters on $\mathbf{K}$ and on $t\bar{t} = 1$, respectively, and connected by (3). We define $\pi(\nu)$ so that (4) is satisfied and then give the function $\pi(z)$ on $\mathbf{K}(\sqrt{\tau})$ by the following formulae:

$$\pi(rt) = \pi_1(r)\pi_2(t), \tag{5}$$

$$\pi(\nu rt) = \pi(\nu)\pi(rt). \tag{6}$$

Obviously, this function is a multiplicative character on $\mathbf{K}(\sqrt{\tau})$. Thus, *a multiplicative character of* $\mathbf{K}(\sqrt{\tau})$ *is given by its values on the* *ground field* **K** *and on the circle* $t\bar{t} = 1$ *and its value at a fixed point* v *such that* $v\bar{v}$ *is not a square in* **K**.† These values are linked by the relations (3) and (4).

§ 2. TEST AND GENERALIZED FUNCTIONS ON A LOCALLY COMPACT DISCONNECTED FIELD K

In this section we discuss certain problems of analysis on a locally compact disconnected topological field **K**.

1. The Space of Test Functions. Let **K** be a locally compact disconnected field. We recall that **K** contains a decreasing sequence of subrings

$$P \supset P^2 \supset \cdots \supset P^n \supset \cdots$$

(where P is the maximal ideal of the ring of integers in **K**) that are open compact sets and form a complete system of neighborhoods of zero.

We wish to lay down a set of sufficiently well-behaved functions on **K**. For this purpose we consider the set S of all complex-valued functions $f(x)$ on **K** that satisfy the following two requirements:

1. The function $f(x)$ is finite, that is, equal to 0 outside some compact open set.

2. There exists a positive integer n (depending on $f(x)$) such that $f(x)$ is constant on the cosets $\mathbf{K}/P^n$.

From 2. it follows automatically that $f(x)$ is a continuous function on **K**. Clearly, the set S of these functions $f(x)$ forms a linear space. We now introduce a topology in S.

We say that a sequence of functions $f_i(x)$ tends to zero if:

1. The functions $f_i(x)$ are zero outside some fixed compact set (independent of i).

2. There exists a positive integer n such that all the functions $f_i(x)$ are constant on the cosets $\mathbf{K}/P^n$.

3. The sequence $f_i(x)$ tends to zero uniformly in x, as $i \to \infty$.

It is easy to verify that *with this topology* S *becomes a complete linear space*, which we call the space of *test* functions. A *generalized function* $\varphi(x)$ is a continuous functional (φ, f) on S.

By analogy with S we can also define the space S_n of functions $f(x_1, \ldots, x_n)$ of n variables from **K**.

---

† Observe that the value of the character π at the point v is determined to within its sign, in accordance with (4), by its values on **K** and on the circle $t\bar{t} = 1$.

2. Generalized Functions Concentrated at a Point. As usual, we define the generalized function $\delta(x)$ by the following formula:

$$(\delta(x), f(x)) = f(0).$$

It is not hard to see that *every generalized function concentrated at* $x = 0$ *is, to within a factor, the function* $\delta(x)$.

This follows immediately from the fact that every test function $f(x) \in S$ is constant in a neighborhood of $x = 0$.

Of course, this statement is also true for generalized functions of n variables.

3. Homogeneous Generalized Functions. Let $\pi(x)$ be a multiplicative character on $\mathbf{K}$, that is,

$$\pi(xy) = \pi(x)\pi(y)$$

for arbitrary x and y from $\mathbf{K}$ (we do not require that $|\pi(x)| = 1$). We call a generalized function $\varphi(x)$ *homogeneous of degree* π if for every function $f \in S$ and $t \neq 0$ we have

$$(\varphi, f(t^{-1}x)) = \pi(t)\ |t|\ (\varphi, f(x)). \tag{1}$$

Our task is to describe all the homogeneous generalized functions. According to **§ 1.4** the multiplicative character $\pi(x)$ can be represented in the form

$$\pi(x) = |x|^{s-1}\,\theta(x), \tag{2}$$

where s is a complex number, and $\theta(x)$ is another character on $\mathbf{K}$ such that

$$|\theta(x)| = 1, \tag{3}$$

$$\theta(\mathfrak{p}) = 1. \tag{4}$$

By (4), θ is given by its values on the compact subgroup of elements x of norm $|x| = 1$. Consequently, the set of these characters θ is discrete.

With the character $\pi(x)$ we associate a generalized function $\pi(x)$ defined by the formula

$$(\pi(x), f(x)) = \int \pi(x) f(x)\, dx \equiv \int |x|^{s-1}\,\theta(x) f(x)\, dx. \tag{5}$$

For $\operatorname{Re} s > 0$ this integral converges in the usual sense and is an analytic function of s. For $\operatorname{Re} s < 0$ we define it by means of analytic continuation.

It is not difficult to see that $\pi(x)$ *is a homogeneous generalized function of degree* π, *provided* s *is a nonsingular point of the integral* (5).

We show now that *the only singularity of the generalized function* $\pi(x) = |x|^{s-1}\theta(x)$, *regarded as a function of the discrete argument* θ *and* *as an analytic function of s, is the point* $\theta \equiv 1$, $s = 0$. *At this point* $\pi(x)$, *as a function of s, has a simple pole with the residue* $\dfrac{q-1}{q \ln q}\,\delta(x)$.

Proof. Without loss of generality we may assume that $f(x)$ is concentrated in the domain $|x| \leq 1$. We rewrite the expression (5) in the form

$$(\pi, f) = \int\limits_{|x|\leq 1} |x|^{s-1}\theta(x)[f(x) - f(0)]\,dx + f(0) \int\limits_{|x|\leq 1} |x|^{s-1}\,\theta(x)\,dx.$$

The first integral converges for arbitrary s, because the function $f(x) - f(0)$ is equal to 0 in the neighborhood of $x = 0$. Therefore we examine the second integral. We split it into the sum of the integrals

$$\int\limits_{|x|\leq 1} |x|^{s-1}\,\theta(x)\,dx = \sum_{k=0}^{\infty} q^{-k(s-1)} \int\limits_{|x|=q^{-k}} \theta(x)\,dx.$$

If $\theta(x) \not\equiv 1$, then $\int\limits_{|x|=q^{-k}} \theta(x)\,dx = 0$ for every k. Thus we are left with the case $\theta(x) \equiv 1$, that is, we are led to compute the integral

$$\int\limits_{|x|\leq 1} |x|^{s-1}\,dx = \sum_{k=0}^{\infty} q^{-k(s-1)} \int\limits_{|x|=q^{-k}} dx.$$

We recall that the measure dx is normalized so that

$$\int\limits_{|x|\leq 1} dx = 1.$$

It follows that

$$\int\limits_{|x|\leq q^{-k}} dx = \int\limits_{|y|\leq 1} d(\mathfrak{p}^k y) = q^{-k}.$$

Consequently,

$$\int\limits_{|x|=q^{-k}} dx = \int\limits_{|x|\leq q^{-k}} dx - \int\limits_{|x|\leq q^{-k-1}} dx = q^{-k}(1 - q^{-1}).$$

Thus,

$$\int\limits_{|x|\leq 1} |x|^{s-1}\,dx = (1 - q^{-1}) \sum_{k=0}^{\infty} q^{-ks} = \frac{1 - q^{-1}}{1 - q^{-s}}.$$

So we see that the only singularity of this expression is a simple pole at $s = 0$, and here

$$\operatorname*{Res}_{s=0} \frac{1 - q^{-1}}{1 - q^{-s}} = \frac{q-1}{q \ln q}.$$

Thus

$$\operatorname*{Res}_{s=0}\,(|x|^{s-1}, f(x)) = \frac{q-1}{q\ln q} f(0),$$

that is,

$$\operatorname*{Res}_{s=0} |x|^{s-1} = \frac{q-1}{q\ln q}\,\delta(x).$$

This proves the assertion. We formulate the final result.

To every multiplicative character $\pi(x)$, *except* $\pi_0(x) = |x|^{-1}$ *there corresponds a homogeneous generalized function* $\pi(x)$ *of degree* π, *defined by* (5). *Obviously the function* $\delta(x)$ *is homogeneous of degree* π_0.

We show now that *other homogeneous generalized functions do not exist.*

Let $\varphi(x)$ be a homogeneous generalized function of degree π, $\pi(x) \neq |x|^{-1}$. It is not difficult to check that for functions $f(x)$ that are zero in a neighborhood of $x = 0$ we have

$$(\varphi, f) = c(\pi, f),$$

where c is some constant. Consequently, the function $\varphi(x) - c\pi(x)$ is concentrated at $x = 0$, and so $\varphi(x) - c\pi(x) = c_1\,\delta(x)$. But the functions $\varphi(x) - c\pi(x)$ and $\delta(x)$ have different degrees of homogeneity. Consequently, $c_1 = 0$, that is, $\varphi(x) = c\pi(x)$.

Now let $\varphi(x)$ be a homogeneous generalized function of degree π_0, $\pi_0(x) = |x|^{-1}$. We show that $\varphi(x)$ is concentrated at $x = 0$ and therefore that $\varphi(x) = c\,\delta(x)$. For suppose that $\varphi(x)$ is not concentrated at $x = 0$. Then we can easily show that

$$(\varphi, f) = c\int |x|^{-1} f(x)\,dx, \qquad c \neq 0,$$

for every function f that is zero in a neighborhood of 0. We introduce the generalized function φ_1:

$$(\varphi_1, f) = c\int_{|x|\leq 1} |x|^{-1}\,[f(x) - f(0)]\,dx + c\int_{|x|>1} |x|^{-1} f(x)\,dx. \quad (6)$$

For every function f that is equal to zero in a neighborhood of $x = 0$ we have $(\varphi, f) = (\varphi_1, f)$. Consequently, the function $\varphi - \varphi_1$ is supported at $x = 0$, and so $\varphi - \varphi_1 = c\,\delta(x)$. But the functions $\varphi(x)$ and $\delta(x)$ are homogeneous of one and the same degree π_0. Hence φ_1 must also be homogeneous. However, as is easily verified from (6), φ_1 is not homogeneous. This contradiction shows that $\delta(x)$ is the only homogeneous function of degree π_0, $\pi_0(x) = |x|^{-1}$.

4. The Fourier Transform of Test Functions. Let $\chi(x) \not\equiv 1$ be an additive character on **K**. We define the Fourier transform of $f(x)$ by the formula

$$\tilde{f}(u) = \int \chi(ux) f(x)\, dx. \tag{1}$$

The Fourier transform is defined for every function $f(x)$ of integrable square modulus; the integral (1) must then be understood in the sense of the mean square value. It is known that $f(x)$ is expressed in terms of its Fourier transform by the formula

$$f(x) = c \int \chi(-ux) \tilde{f}(u)\, du. \tag{2}$$

where c is a positive constant dependent on the choice of the character χ. Moreover, the Plancherel formula holds:

$$\int |f(x)|^2\, dx = c \int |\tilde{f}(u)|^2\, du. \tag{3}$$

Let us find out how the constant c depends on the choice of the character χ. From the continuity of χ it follows that $\chi(x) \equiv 1$ on the subgroup $\mathfrak{p}^k O$ for sufficiently large k, where O is the subgroup of elements x of norm $|x| \leq 1$. We define the *rank* of the character χ as the least integer n such that $\chi(x) \equiv 1$ on $\mathfrak{p}^n O$. Clearly, if χ is of rank n, then the character $\chi'(x) = \chi(\mathfrak{p}^k x)$ is of rank $n - k$.

We show that the constant c in the inversion formula (2) and the Plancherel formula (3) can be expressed in terms of the rank of χ as follows:

$$c = q^n, \tag{4}$$

with $q^{-1} = |\mathfrak{p}|$. For this purpose we denote by ψ the characteristic function of O and compute its Fourier transform. We find

$$\tilde{\psi}(u) = \int_K \psi(x) \chi(ux)\, dx = \int_O \chi(ux)\, dx.$$

We represent the element u in the form $\mathfrak{p}^k v$, $|v| = 1$. Then

$$\tilde{\psi}(u) = \int_O \chi(\mathfrak{p}^k v x)\, dx = |\mathfrak{p}|^{-k} \int_{\mathfrak{p}^k O} \chi(y)\, dy. \tag{5}$$

Suppose that the rank of χ is n. For $k \geq n$ the function under the integral sign in (5) is equal to 1, and we obtain

$$\tilde{\psi}(u) = |\mathfrak{p}|^{-k} \int_{\mathfrak{p}^k O} dy = \int_O dx = 1.$$

But if $k < n$, then χ is a nontrivial character on $p^k O$; and therefore, the integral is equal to 0.

This result can be written as follows:

$$\tilde{\varphi}(u) = \begin{cases} 1, & \text{when} \quad |u| \leq q^{-n}, \\ 0, & \text{when} \quad |u| > q^{-n}, \end{cases} \tag{6}$$

that is, $\tilde{\varphi}$ is the characteristic function of $\mathfrak{p}^n O$.

Substituting φ and $\tilde{\varphi}$ in the Plancherel formula we obtain the required equation (4). In particular, if the rank of χ is zero, then $c = 1$.

Henceforth we always assume that the character χ in the definition of the Fourier transform is of rank 1 so that

$$c = 1.$$

First, we discuss the Fourier transforms of the test functions.

The Fourier transform of a function $f \in S$ is also a function in S.

Proof. Let $f(x)$ be a function in S. This means that:

1. There is an m such that $f(x) = 0$ for $|x| \geq q^m$.

2. There is an n such that for every t of norm $|t| \leq q^{-n}$ we have $f(x + t) = f(x)$.

Consider the Fourier transform of $f(x)$:

$$\tilde{f}(u) = \int \chi(ux) f(x)\, dx. \tag{7}$$

First let us show that $\tilde{f}(u)$ is a finite function. For this purpose we replace x by $x + t$ under the integral, where $|t| \leq q^{-n}$. By 2. we obtain

$$\tilde{f}(u) = \chi(ut) \int \chi(ux) f(x)\, dx,$$

that is,

$$\tilde{f}(u) = \chi(ut) \tilde{f}(u). \tag{8}$$

If $|u| > q^n$, then $|ut| > 1$ and hence $\chi(ut) \neq 1$. But then it follows from (8) that $\tilde{f}(u) = 0$ when $|u| > q^n$. This shows that $\tilde{f}(u)$ is a finite function.

Next, we show that $\tilde{f}(u)$ satisfies condition 2.

Since $\tilde{f}(x) = 0$ for $|x| \geq q^m$, we have

$$\tilde{f}(u) = \int_{|x| \leq q^m} \chi(ux) f(x)\, dx.$$

Consequently, for $|t| \leq q^{-m}$ we find

$$\tilde{f}(u + t) = \int_{|x| \leq q^m} \chi(tx) \chi(ux) f(x)\, dx = \tilde{f}(u),$$

because $\chi(tx) = 1$. Hence $\tilde{f}(u)$ satisfies condition 2, and the proposition is proved.

Note that $\tilde{\tilde{f}}(x) = f(-x)$. Hence it follows immediately that:

The Fourier transform effects a one-to-one map of the space S of test functions onto itself.

Now we give a definition of the Fourier transform of a generalized function. As a basis for this definition we use the Plancherel formula

$$\int \varphi(x)\overline{f(x)}\, dx = \int \tilde{\varphi}(u)\overline{\tilde{f}(u)}\, du, \tag{9}$$

which holds for arbitrary test functions f and φ. It is not difficult to see that the function $\tilde{f}(u)$ is the Fourier transform of $\overline{f(-x)}$. Thus, if in (9) we replace $f(x)$ by $\overline{f(-x)}$, we obtain

$$\int \varphi(x) f(-x)\, dx = \int \tilde{\varphi}(u)\tilde{f}(u)\, du. \tag{10}$$

The equation (1) means that the function $\tilde{\varphi}(u)$, regarded as a functional, satisfies the following relation:

$$(\tilde{\varphi}, \tilde{f}(u)) = (\varphi, f(-x)). \tag{11}$$

We take this relation as the definition of the Fourier transform of the generalized functions $\varphi(x)$. Thus, *the Fourier transform of the generalized function $\varphi(x)$ is the generalized function $\tilde{\varphi}(u)$ defined by* (11).

5. The Fourier Transform of Homogeneous Generalized Functions. The Gamma-Function and Beta-Function. From the definition of the Fourier transform we deduce immediately that

$$\tilde{1} = \delta(x), \qquad \widetilde{\delta(x)} = 1. \tag{1}$$

Now we show that *the Fourier transform of a homogeneous generalized function of degree π is homogeneous of degree $\pi^{-1}\pi_0^{-1}$, where $\pi_0(x) = |x|^{-1}$.*

For let φ be a homogeneous function of degree π. This means that for every $t \neq 0$ from K we have

$$(\varphi, f(t^{-1}x)) = \pi\pi_0(t)(\varphi, f(x)),$$

where

$$\pi_0(t) = |t|(\pi\pi_0(t) \equiv \pi(t)\pi_0(t)).$$

Now we observe that when $f_1(x) = |t| f(tx)$, then $\tilde{f}_1(u) = \tilde{f}(t^{-1}u)$. Consequently,

$$(\tilde{\varphi}, \tilde{f}(t^{-1}u)) = (\varphi, |t| f(-tx)) = \pi^{-1}(t)(\varphi, f(-x)),$$

that is,

$$(\tilde{\varphi}, \tilde{f}(t^{-1}u)) = \pi^{-1}(t)(\tilde{\varphi}, \tilde{f}(u)).$$

This equation means that $\tilde{\varphi}$ is a homogeneous function of degree $\pi^{-1}\pi_0^{-1}$.

Thus, the Fourier transform of the homogeneous generalized function $\pi(x)\ |x|^{-1}$ is, to within a factor, the homogeneous generalized function $\pi^{-1}(u)$. We denote the factor arising here by $\Gamma(\pi)$ and call it the *Gamma-function*. So we have

$$\widetilde{\pi(x)\ |x|^{-1}} = \Gamma(\pi)\pi^{-1}(u). \tag{2}$$

Let us find an integral representation of $\Gamma(\pi)$. For this purpose we write $\widetilde{\pi(x)\ |x|^{-1}}$ in the form of an integral

$$\Gamma(\pi)\pi^{-1}(u) = \int \chi(ux)\pi(x)\ |x|^{-1}\, dx.$$

By taking $u = 1$ we find

$$\Gamma(\pi) = \int \chi(x)\pi(x)\ |x|^{-1}\, dx. \tag{3}$$

Clearly, this expression is reminiscent of the formula for the classical Gamma-function.†

We can give a meaning to the integral (3) by writing it as the sum of the two integrals

$$\Gamma(\pi) = \int\limits_{|x|\leqslant 1} \chi(x)\pi(x)\ |x|^{-1}\, dx + \int\limits_{|x|>1} \chi(x)\pi(x)\ |x|^{-1}\, dx.$$

Each of these integrals converges in a certain domain of values of π and is in this domain an analytic function of π. By analytic continuation we define these integrals for arbitrary π.

By splitting the integrals (3) into a sum of integrals over domains $|x| = \text{const}$ we find, after a suitable change of variables, the following expression for the function $\Gamma(\pi)$ (*expansion in a Fourier series*):

$$\Gamma(\pi) = \sum_{k=-\infty}^{+\infty} \pi(\mathfrak{p}^k) \int\limits_{|x|=1} \chi(\mathfrak{p}^k x)\pi(x)\, dx. \tag{4}$$

---

† We mention that in the case of the field of real numbers the Gamma-function we have introduced does not coincide with the classical one, but differs from it by a factor. For example, if $\pi(x) = |x|^s$, then

$$\Gamma(\pi) = \int_{-\infty}^{+\infty} |x|^{s-1} e^{ix}\, dx = 2\cos\frac{\pi s}{2}\,\Gamma(s),$$

where $\Gamma(s)$ is the classical Gamma-function. Similarly, for $\pi(x) = |x|^s \operatorname{sign} x$ we have $\Gamma(\pi) = 2i \sin\dfrac{\pi s}{2}\,\Gamma(s)$.

The following properties of the Gamma-function are immediate consequences of the definition:

1. The only singular point of $\Gamma(\pi)$ is $\pi \equiv 1$.
2. The only zero of $\Gamma(\pi)$ is $\pi_0(x) = |x|$.
3. $\Gamma(\pi)\,\Gamma(\pi_0\pi^{-1}) = \pi(-1)$. (5)

To obtain (5) we apply the Fourier transform to both sides of (2).

Note that the formula (5) is reminiscent of the relation (for the classical Gamma-function) connecting $\Gamma(t)$ and $\Gamma(1-t)$. Now we give a definition of the Beta-function.

The Beta-function of the multiplicative characters π_1 and π_2 of $\mathbf{K}$ is the following expression:

$$B(\pi_1, \pi_2) = \int \pi_1(x)\,|x|^{-1}\,\pi_2(1 - x|^{-1}\,dx. \tag{6}$$

The integral diverges and must be understood in the following sense. We split (6) into two integrals:

$$B(\pi_1, \pi_2) = \int\limits_{|x|\leqslant 1} \pi_1(x)\,|x|^{-1}\,\pi_2(1-x)\,|1-x|^{-1}\,dx + \int\limits_{|x|>1} \pi_1(x)\,|x|^{-1}\,\pi_2(1-x)\,|1-x|^{-1}\,dx.$$

Each of these integrals converges in a certain domain of the characters π_1 and π_2 and is, in this domain, an analytic function of π_1 and π_2. By analytic continuation we define these integrals for all π_1 and π_2. In this way we define $B(\pi_1, \pi_2)$ as an analytic function of π_1 and π_2.

It is easy to show that the function $B(\pi_1, \pi_2)$ has the following expression in terms of the Gamma-function

$$B(\pi_1, \pi_2) = \frac{\Gamma(\pi_1)\,\Gamma(\pi_2)}{\Gamma(\pi_1\pi_2)}\,. \tag{7}$$

The derivation of this formula proceeds just as for the classical Beta- and Gamma-functions.

6. Additional Information on the Gamma-Function. We recall that the multiplicative group $\mathbf{K}^*$ of a disconnected topological field $\mathbf{K}$ is the direct product of the infinite cyclic group generated by $\mathfrak{p}$ and the compact group O^* consisting of all elements of norm 1. Therefore the group Π of all (not necessarily unitary) characters of $\mathbf{K}^*$ is the direct product of the multiplicative group C^* of complex numbers $\lambda \neq 0$ and the group $\hat{O}^*$ of all characters θ of O^*. Thus, every character π on $\mathbf{K}^*$ can be given by a pair (λ, θ), where a $\lambda \in C^*$, $\theta \in \hat{O}^*$.

Every element $x \in \mathbf{K}^*$ has a unique expression in the form

$$x = \mathfrak{p}^k y,$$

where $y \in O^*$. The value of the character π on x is equal to

$$\pi(x) = \lambda^k \theta(y), \tag{1}$$

The following expression for π, which is equivalent to (1), is also convenient. We extend the character θ to the whole of $\mathbf{K}^*$ by setting $\theta(\mathfrak{p}) = 1$. Then we have

$$\pi(x) = |x|^s \, \theta(x), \tag{1'}$$

where s is a complex number connected with λ by the relation $\lambda = |\mathfrak{p}|^s = q^{-s}$.

We note that the set $\hat{O}^*$ of characters θ is countable and discrete so that Π is the union of a countable number of complex planes with zero deleted.

From the integral representation of the Gamma-function

$$\Gamma(\pi) = \int \chi(x) \pi(x) \, d^*x \tag{2}$$

we can derive the expansion of the Gamma-function in a Laurent series in λ. For this purpose we represent $\mathbf{K}^*$ as the union of the sets $\mathfrak{p}^k O^*$, $k = 0, \pm 1, \pm 2, \ldots$.

Then we find

$$\Gamma(\pi) = \sum_k \int_{|y|=1} \chi(\mathfrak{p}^k y) \pi(\mathfrak{p}^k y) \, dy$$

$$= \sum_k \lambda^k \int_{|y|=1} \chi(\mathfrak{p}^k y) \theta(y) \, dy.$$

Thus the coefficients of the expansion of $\Gamma(\pi)$ in a Laurent series in λ are of the form

$$\Gamma_k(\theta) = \int_{|y|=1} \chi(\mathfrak{p}^k y) \theta(y) \, dy. \tag{3}$$

Observe that the integrals (3) converge, in contrast to the integral (2), which must be understood in the sense of generalized functions.

We shall see presently that almost all coefficients $\Gamma_k(\theta)$ can be computed explicitly. From this computation it follows that $\Gamma(\pi) \equiv \Gamma(\lambda, \theta)$ is a rational function of λ for every fixed θ.

First we recall that we subjected the character χ that occurs in the definition of the Gamma-function to the following condition: $\chi(x) = 1$, when $|x| \leq 1$; $\chi(x) \not\equiv 1$ on the set $|x| \leq q$.

We now introduce the concept of the rank of a character θ. We consider the group O^* of elements of norm 1. This group has a decreasing sequence of open subgroups O_n^* consisting of the elements of the form $1 + \mathfrak{p}^n x$, $|x| \leq 1$. From the continuity of θ it follows that when n is sufficiently large, $\theta(x) \equiv 1$ on O_n^*. We define the *rank* of θ as the smallest number n for which $\theta(x) \equiv 1$ on O_n^*. Obviously, the set of characters of a rank not exceeding n is finite.† In particular, there is only one character of rank 0, namely $\theta_0 \equiv 1$.

We now prove the following proposition.

If the rank of the character θ is equal to m, $m > 0$, then $\Gamma_k(\theta) = 0$ for $k \neq -m$; moreover,

$$|\Gamma_{-m}(\theta)| = q^{-m/2}. \tag{4}$$

If the rank of character θ equal to zero, that is, $\theta(x) \equiv 1$, then

$$\Gamma_k(\theta) = \begin{cases} 0 & \text{for } k < -1 \\ -q^{-1} & \text{for } k = -1 \\ 1 - q^{-1} & \text{for } k > -1. \end{cases} \tag{5}$$

Thus, on a disconnected field $\mathbf{K}$ the Gamma-function is of a very simple type. Specifically, if $\theta(x) \not\equiv 1$, then

$$\Gamma(\lambda, \theta) = \Gamma_{-m}(\theta)\lambda^{-m}, \tag{6}$$

where m is the rank of θ $(m > 0)$, and $|\Gamma_{-m}(\theta)| = q^{-m/2}$. If $\theta = \theta_0 \equiv 1$, then

$$\Gamma(\lambda, \theta_0) = (1 - q^{-1}) \sum_{k=0}^{\infty} \lambda^k - q^{-1}\lambda^{-1} = \frac{1 - q^{-1}\lambda^{-1}}{1 - \lambda}. \tag{7}$$

Proof. We begin by discussing the case when θ is a character of rank $m > 0$. Let us show that then $\Gamma_k(\theta) = 0$ for $k \neq -m$.

If $k \geq 0$, then we have, since $\chi(\mathfrak{p}^k y) \equiv 1$,

$$\Gamma_k(\theta) = \int_{|y|=1} \theta(y)\, dy = 0.$$

Now take $k < 0$. We write y in the form

$$y = \alpha_0 + \alpha_1 \mathfrak{p} + \ldots + \alpha_n \mathfrak{p}^n + \ldots$$

(see § **1**). Since the rank of θ is m, $\theta(y)$ depends only on $\alpha_0, \alpha_1, \ldots, \alpha_{m-1}$, and the dependence on α_{m-1} is nontrivial. On the other hand, the function $\chi(\mathfrak{p}^k y)$ depends only on $\alpha_0, \alpha_1, \ldots, \alpha_{-k-1}$, and the dependence on α_{-k-1} is nontrivial.

If $0 > k > -m$, then $-k - 1 < m - 1$, and so $\chi(\mathfrak{p}^k y)$ does not depend on α_{m-1}. When we split the integrals (3) into integrals over

† This set is the group dual to the finite group O^*/O_n^*.

the cosets of O^*_{m-1}, we find that each of these integrals is equal to zero. Thus, if $0 > k > -m$, then $\Gamma_k(\theta) = 0$.

Finally, if $k < -m$, then $m - 1 < -k - 1$, and so $\theta(y)$ does not depend on α_{-k-1}. When we split the integrals (3) into integrals over sets of the form $y + \mathfrak{p}^{-k-1}O$, and bear in mind that the character $\chi(x)$ is nontrivial on $\mathfrak{p}^{-1}O$, we see again that each of these integrals is equal to zero.† So if $k < -m$, then $\Gamma_k(\theta) = 0$.

We have now shown that if θ is a character of rank $m > 0$, then $\Gamma_k(\theta) = 0$ for $k \neq -m$. It remains to compute $\Gamma_{-m}(\theta)$.

For this purpose we consider the function

$$\varphi(x) = \begin{cases} \theta(x), & \text{when } |x| = 1, \\ 0, & \text{when } |x| \neq 1, \end{cases} \tag{8}$$

and compute its Fourier transform. Let $u = \mathfrak{p}^k v$, $|v| = 1$; then

$$\tilde{\varphi}(u) = \int_K \varphi(x)\chi(ux)\,dx = \int_{|y|=1} \theta(y)\chi(\mathfrak{p}^k vy)\,dy = \Gamma_k(\theta)\theta^{-1}(v).$$

But $\Gamma_k(\theta)$ is different from zero only for $k = -m$. So we have

$$\tilde{\varphi}(u) = \begin{cases} \Gamma_{-m}(\theta)\theta^{-1}(v), & \text{when } u = \mathfrak{p}^{-m}v,\ |v| = 1, \\ 0, & \text{when } |u| \neq q^m. \end{cases} \tag{9}$$

By substituting this value of $\tilde{\varphi}$ in the Plancherel formula for the Fourier transform we obtain the required equation $|\Gamma_{-m}(\theta)|^2 = q^{-m}$.

Now we proceed to the case $\theta = \theta_0 \equiv 1$. The integral to be investigated has the form

$$\begin{aligned} \Gamma_k(\theta_0) &= \int_{|y|=1} \theta_0(y)\ \chi(\mathfrak{p}^k y)\,dy = \int_{|y|=1} \chi(\mathfrak{p}^k y)\,dy \\ &= \int_{|y|\leqslant 1} \chi(\mathfrak{p}^k y)\,dy - \int_{|y|<1} \chi(\mathfrak{p}^k y)\,dy \\ &= q^k \int_{|y|\leqslant q^{-k}} \chi(y)\,dy - q^k \int_{|y|\leqslant q^{-k-1}} \chi(y)\,dy. \end{aligned}$$

We note that

$$\int_{|y|\leqslant q^{-1}} \chi(y)\,dy = \begin{cases} q^{-k} & \text{for } k \geq 0 \\ 0 & \text{for } k < 0. \end{cases}$$

The formula (5) for $\Gamma_k(\theta_0)$ follows immediately from this, and the proposition is proved.

---

† Note that the splitting of O^* into domains of the form $y = p^{-k-1}O$, which we have used here, is possible only under the condition that $-k - 1 > 0$, that is, for $k < -1$. This condition is automatically satisfied for $m > 0$, because we have assumed that $k < -m$.

We mention that the relation (4) may be presented in the following form. If $\pi(x) = |x|^s \theta(x)$, where $\theta(\mathfrak{p}) = 1$, and the rank of θ is $m > 0$, then

$$|\Gamma(\pi)| = q^{m(\mathrm{Re}\, s - \frac{1}{2})}.$$

Thus, in this instance it follows that on the set of characters of the form $\pi(x) = |x|^{\frac{1}{2}+rP} \theta(x)$ we have

$$|\Gamma(\pi)| = 1.$$

(The character $\pi_{\frac{1}{2}+rp}(x) = |x|^{\frac{1}{2}+rp}$ is not excluded, because by (7) we have $|\Gamma(\pi_{\frac{1}{2}+rp})| = 1$.)

Now we show that *if the rank of* θ *is* m, $m > 0$, *then the following relation holds*:

$$\Gamma_{-m}(\theta)\, \Gamma_{-m}(\theta^{-1}) = q^{-m}\theta(-1). \tag{10}$$

To prove this we consider the function $\varphi(x)$ defined by (8). Its Fourier transform $\tilde{\varphi}(u)$ is expressed by (9). We now compute the Fourier transform $\tilde{\tilde{\varphi}}(x)$ of $\tilde{\varphi}(u)$. Let $x = \mathfrak{p}^k y$, $|y| = 1$. Then

$$\begin{aligned}\tilde{\tilde{\varphi}}(x) &= \int\limits_{K} \tilde{\varphi}(u)\chi(ux)\, du \\ &= \Gamma_{-m}(\theta)q^m \int\limits_{|v|=1} \theta^{-1}(v)\,\chi(\mathfrak{p}^{k-m}vy)\, dv \\ &= \Gamma_{-m}(\theta)q^m\,\Gamma_{k-m}(\theta^{-1})\,\theta(y)\end{aligned}$$

But $\Gamma_{k-m}(\theta^{-1})$ is different from zero only for $k = 0$. So we have

$$\tilde{\tilde{\varphi}}(x) = \begin{cases} \Gamma_{-m}(\theta)\,\Gamma_{-m}(\theta^{-1})q^m\theta(x), & \text{when } |x| = 1 \\ 0 & \text{when } |x| \neq 1 \end{cases}$$

that is,

$$\tilde{\tilde{\varphi}}(x) = \Gamma_{-m}(\theta)\,\Gamma_{-m}(\theta^{-1})q^m\,\varphi(x) \tag{11}$$

On the other hand, from general properties of the Fourier transform it follows that

$$\tilde{\tilde{\varphi}}(x) = \varphi(-x) = \theta(-1)\,\varphi(x) \tag{12}$$

Comparing (11) and (12) we obtain the required relation (10).

The relation (10) enables us to compute the value

$$\Gamma(\pi) = \Gamma(\lambda, \theta),$$

to within its sign, when $\theta^2 \equiv 1$, and $\theta \not\equiv 1$. For in this case the rank of θ is 1, and by (10), we have

$$\Gamma_{-1}^2(\theta) = q^{-1}\theta(-1),$$

whence $\Gamma_{-1}(\theta) = \pm\sqrt{\theta(-1)}\,q^{-\frac{1}{2}}$. Consequently, by (6),

$$\Gamma(\lambda, \theta) = \pm\sqrt{\theta(-1)}\,q^{-\frac{1}{2}}\lambda^{-1}. \tag{13}$$

The values of $\Gamma(\pi)$ for certain special values of the character π are as follows:

1. $\pi(x) = |x|^s$. In this case $\lambda = q^{-s}$, $\theta \equiv 1$. Consequently, by (7)

$$\Gamma(\pi) = \frac{1 + q^{s-1}}{1 - q^{-s}}. \tag{14}$$

2. $\pi(x) = |x|^s \operatorname{sign}_\varepsilon x$. In this case† $\lambda = -q^{-s}$, $\theta \equiv 1$. Consequently,

$$\Gamma(\pi) = \frac{1 + q^{s-1}}{1 + q^{-s}}. \tag{15}$$

3. $\pi(x) = |x|^s \sin_\tau x$, where $\tau = \mathfrak{p}$ or $\varepsilon\mathfrak{p}$. In this case $\lambda = q^{-s}$, $\theta(y) = \operatorname{sign}_\tau y$, that is, $\theta^2 \equiv 1$. Consequently, by (13) we have

$$\Gamma(\pi) = \pm\sqrt{\operatorname{sign}_\tau(-1)}\, q^{s-\frac{1}{2}}. \tag{16}$$

We give two other relations for the Gamma-function. These can be obtained as consequences of (10). Let π be a character of rank $m > 0$ on O^*. Then we have

$$\Gamma(\pi)\Gamma(\pi^{-1}) = q^{-m}\pi(-1). \tag{17}$$

Further, by comparing this relation with

$$\Gamma(\pi\pi_0)\Gamma(\pi^{-1}) = \pi(-1),$$

where $\pi_0(x) = |x|$ (see **§ 25** (4)), we obtain

$$\Gamma(\pi\pi_0) = q^m\Gamma(\pi). \tag{18}$$

The relation (18) may be regarded as an analogue of the relation $\Gamma(x - 1) = x\Gamma(x)$ for the classical Gamma-function.

In the excluded case, when $\pi(x) \equiv 1$ on O^*, that is,

$$\pi(x) = |x|^s,$$

we obtain from (14):

$$\Gamma(\pi)\Gamma(\pi^{-1}) = \frac{(1 - q^{s-1})(1 - q^{-s-1})}{(1 - q^{-s})(1 - q^s)}; \tag{17'}$$

$$\Gamma(\pi\pi_0) = \frac{(1 - q^{-s})(1 - q^s)}{(1 - q^{s-1})(1 - q^{-s-1})}\Gamma(\pi). \tag{18'}$$

We now introduce the concept of the *incomplete Gamma-function*, and define it by the formula:

$$\Gamma^{(k)}(\pi) \equiv \Gamma^{(k)}(\lambda, \theta) = \int_{|x| \leq q^k} \chi(x)\pi(x)\, d^*x.$$

On the basis of (4) and (5) we have the following result. *If the rank of the character θ is equal to $m > 0$, then*

$$\Gamma^{(k)}(\pi) = \begin{cases} 0 & \text{for } k < m, \\ \Gamma(\pi) & \text{for } k \leq m. \end{cases}$$

† The case 2 is obtained from 1 when s is replaced by $s + \dfrac{2\pi i}{\ln q}$.

In the excluded case, when $\theta = \theta_0 \equiv 1$,

$$\Gamma^{(k)}(\pi) = \begin{cases} \dfrac{(1 - q^{-1})\,\lambda^k}{1 - \lambda} & \text{for } k \le 0 \\[2ex] \dfrac{1 - q^{-1}\lambda^{-1}}{1 - \lambda} & \text{for } k > 0. \end{cases}$$

So we see that every fixed π the sequence

$$\Gamma^{(0)}(\pi),\ \Gamma^{(1)}(\pi), \ldots, \Gamma^{(k)}(\pi), \ldots$$

stabilizes at a sufficiently large index k.

7. The Integral $\int \chi(ut\bar{t})\,dt$. In what follows we need the integral

$$F(u) = \int \chi(ut\,\bar{t})\,dt\,, \tag{1}$$

where the integration is taken over the plane $\mathbf{K}(\sqrt{\tau})$, which we shall now compute. First of all, (1) can be rewritten as an integral over $\mathbf{K}$ (see the formula on p. 136):

$$\begin{aligned} F(u) &= a_\tau \int\limits_{\operatorname{sign}_\tau x = 1} \chi(ux)\,dx \\ &= \frac{a_\tau}{2} \int\limits_K \chi(ux)\,dx + \frac{a_\tau}{2} \int\limits_K \chi(ux) \operatorname{sign}_\tau x\,dx, \end{aligned}$$

where $a_\tau = 2(1 + q^{-1})(1 + |\tau|)^{-1}$. According to **§ 2.5** we have

$$\int \chi(ux)\,dx = \delta(u), \qquad \int \chi(ux) \operatorname{sign}_\tau x\,dx = \Gamma(\pi)\,\frac{\operatorname{sign}_\tau u}{|u|}\,,$$

where $\pi(x) = |x| \operatorname{sign}_\tau x$. Thus,

$$\int \chi(ut\bar{t})\,dt = c_\tau^{-1}\,\frac{\operatorname{sign}_\tau u}{|u|} + \frac{a_\tau}{2}\,\delta(u), \tag{2}$$

where we have set

$$c_\tau^{-1} = \frac{a_\tau}{2}\,\Gamma(\pi) = \frac{1 + q^{-1}}{1 + |\tau|} \int \chi(x) \operatorname{sign}_\tau x\,dx. \tag{3}$$

Note that the coefficient c_τ satisfies the relation

$$\bar{c}_\tau = c_\tau \operatorname{sign}_\tau(-1). \tag{4}$$

Hence c_τ is real when $\operatorname{sign}_\tau(-1) = 1$, c_τ is purely imaginary when $\operatorname{sign}_\tau(-1) = -1$.

The coefficient c_τ can be calculated to within its sign on the basis of the results in **§ 2.6**. For by **§ 2.6** (15) we have

$$c_\tau = 1 \quad \text{when} \quad \tau = \varepsilon \tag{5}$$

and by **§ 2.6** (16) we have

$$c_\tau = \pm[\operatorname{sign}_\tau(-1)]^{1/2} q^{-1/2} \quad \text{when} \quad \tau = \mathfrak{p} \text{ or } \varepsilon\mathfrak{p} \tag{6}$$

8. On Functions Resembling Analytic Functions in the Upper and the Lower Half-Plane. Let $\mathbf{K}(\sqrt{\tau})$ be a quadratic extension of a disconnected field $\mathbf{K}$. We define the *upper half-plane* of $\mathbf{K}(\sqrt{\tau})$ as the set of points $z = x + \sqrt{\tau}\, y$, $\operatorname{sign}_\tau y = 1$; and the *lower half-plane* as the set of points $z = x + \sqrt{\tau}\, y$, $\operatorname{sign}_\tau y = -1$.

It is easy to verify that *the upper and the lower half-planes are homogeneous spaces relative to the group of fractional-linear transformations* $z' = \dfrac{\alpha z + \gamma}{\beta z + \delta}$, $\alpha\delta - \beta\gamma = 1$.

For disconnected fields the concept of a complex-valued function analytic in the upper or the lower half-plane does not exist. However, we may introduce the concept of a function resembling an analytic function.

With this aim we introduce on $\mathbf{K}$ generalized functions analogous to $(x + i0)^{-1}$ and $(x - i0)^{-1}$ in the case of the real field.

We define the generalized function $(x + \sqrt{\tau}\, 0)^{-1}$ as the Fourier transform of the generalized function

$$f_\tau^+(u) = \tfrac{1}{2}(1 + \operatorname{sign}_\tau u),$$

which is equal to 1 when $\operatorname{sign}_\tau u = 1$, and to 0 when $\operatorname{sign}_\tau u = -1$. Similarly we define the generalized function $(x - \sqrt{\tau}\, 0)^{-1}$ as the Fourier transform of the generalized function

$$f_\tau^-(u) = \tfrac{1}{2}(1 - \operatorname{sign}_\tau u),$$

which is equal to 1 when $\operatorname{sign}_\tau u = -1$, and to 0 when $\operatorname{sign}_\tau u = 1$.

On the basis of results in **§ 2.6** and **§ 2.7** we may express these functions in terms of the generalized functions $\delta(x)$ and $\dfrac{\operatorname{sign}_\tau x}{|x|}$:

$$(x + \sqrt{\tau}\, 0)^{-1} = \tfrac{1}{2}\delta(x) + a_\tau^{-1} c_\tau^{-1} \frac{\operatorname{sign}_\tau x}{|x|}, \tag{1}$$

$$(x - \sqrt{\tau}\, 0) = \tfrac{1}{2}\delta(x) - a_\tau^{-1} c_\tau^{-1} \frac{\operatorname{sign}_\tau x}{|x|}, \tag{2}$$

The coefficients c_τ were calculated in 7.

We call a function $f(x)$ *resembling an analytic function in the upper half-plane* if its convolution with $(x - \sqrt{\tau}\, 0)^{-1}$ is identically zero:

$$\int (t - \sqrt{\tau}\, 0)^{-1} f(x - t)\, dt = 0$$

(or, what is equivalent, if its Fourier transform is concentrated on the half-line $\operatorname{sign}_\tau u = 1$).

The concept of a function resembling an analytic function in the lower half-plane is defined similarly.

9. The Mellin Transform. We define the Mellin transform of a function $f(x)$ by the formula

$$F(\pi) = \int \pi(x) f(x) \, d^*x, \tag{1}$$

where π ranges over the unitary multiplicative characters,† $d^*x = |x|^{-1} \, dx$.

Thus, the Mellin transform may be regarded as the Fourier transform on the multiplicative group $\mathbf{K}^*$ of $\mathbf{K}$.

The Mellin transform is defined for every function $f(x)$ for which

$$\int |f(x)|^2 \, d^*x < \infty.$$

The integral (1) must be understood in the sense of the mean square value.

The inversion formula

$$f(x) = c \int \pi^{-1}(x) F(\pi) \, d\pi \tag{2}$$

and the Plancherel formula

$$\int |f(x)|^2 \, d^*x = c \int |F(\pi)|^2 \, d\pi \tag{3}$$

are valid. The integration is taken here with respect to the invariant measure $d\pi$ on the character group; c is a positive constant depending on the normalization of $d\pi$, which we shall take in what follows so that $c = 1$.

In studying the Mellin transform we have to take as the space of test functions not S, but another space S^*, which we now define.

By S^* we denote the set of functions $f(x)$ satisfying the following requirements:

1. The function $f(x)$ is finite on $\mathbf{K}^*$; in other words, there are positive numbers a and b, $a > b$, such that $f(x) = 0$ for $|x| > a$ and for $|x| < b$.

2. There exists a sufficiently small open subgroup of $\mathbf{K}^*$ such that $f(x)$ is constant on its cosets. In other words,

$$f(xa) = f(x), \tag{4}$$

if the norm $|1 - a|$ is sufficiently small.

† That is, $|\pi(x)| = 1$.

A topology in S^* is introduced in the natural manner.

It is not hard to verify that S^* consists of precisely those functions $f(x)$ for which

$$f(x) \in S \quad \text{and} \quad f(x^{-1}) \in S.$$

Now we define the Mellin transform of a generalized function. As the basis of this definition we take the Plancherel formula (3). Using the notation

$$(\varphi(x), f(x)) = \int \varphi(x) f(x)\, d^*x.$$

$$(\Phi(\pi), F(\pi)) = \int \Phi(\pi) F(\pi)\, d\pi,$$

we can rewrite the Plancherel formula in the form

$$(\varphi(x), \overline{f(x)}) = (\Phi(\pi), \overline{F(\pi)}). \tag{5}$$

Observe that the Mellin transform of $\overline{f(x^{-1})}$ is $\overline{F(\pi)}$. Consequently, by replacing in (5) $f(x)$ by $\overline{f(x^{-1})}$ we find

$$(\varphi(x), f(x^{-1})) = (\Phi(\pi), F(\pi)). \tag{6}$$

Formula (6) defines the Mellin transform as a functional in the function space of the test functions. And so we take this formula as definition of the Mellin transform of a generalized function.

Definition of the Generalized Function $\Gamma(\pi)$. We use the name *generalized Gamma-function* for the Mellin transform of the generalized function $\chi(x)$. Thus, formally, $\Gamma(\pi)$ can be written as an integral

$$\Gamma(\pi) = \int \pi(x) \chi(x)\, |x|^{-1}\, dx. \tag{7}$$

Definition of the Generalized Bessel Function $J(\pi; u)$. We use the name *generalized Bessel function* $J(\pi; u)$ for the Mellin transform of the generalized function $\chi(u(x + x^{-1}))$, that is,

$$J(\pi; u) = \int \pi(x) \chi(u(x + x^{-1}))\, |x|^{-1}\, dx. \tag{8}$$

We write down another integral representation of $J(\pi; u)$. We know that $\chi(t)$ is inverse Mellin transform of $\Gamma(\pi)$, that is,

$$\chi(t) = \int \Gamma(\pi_1) \pi_1^{-1}(t)\, d\pi_1.$$

It follows that

$$\chi(ux) = \int \Gamma(\pi_1) \pi_1^{-1}(u) \pi_1^{-1}(x)\, d\pi_1,$$

$$\chi(ux^{-1}) = \int \Gamma(\pi_2) \pi_2^{-1}(u) \pi_2(x)\, d\pi_2,$$

and therefore,

$$\chi(u(x + x^{-1})) = \int \Gamma(\pi_1)\Gamma(\pi_2)\pi_1^{-1}\pi_2^{-1}(u)\pi_1^{-1}\pi_2(x)\, d\pi_1\, d\pi_2.$$

Substituting this expression in (8) for $J(\pi; u)$ and integrating with respect to x and to π_1 we find

$$J(\pi; u) = \int \Gamma(\pi'^{-1})\Gamma(\pi\pi'^{-1})\pi^{-1}\pi'^{2}(u)\, d\pi'. \tag{9}$$

10. The Relation Between the Gamma-Function Connected with the Ground Field K and the Gamma-Function Connected with the Quadratic Extension $\mathbf{K}(\sqrt{\tau})$ of K. Apart from the Gamma-function connected with **K** we consider the Gamma-function $\Gamma_\tau(\pi)$, connected with the field $\mathbf{K}(\sqrt{\tau})$:

$$\Gamma_\tau(\pi) = \int_{\mathbf{K}(\sqrt{\tau})} \chi_\tau(t)\pi(t)\, d^*t, \tag{1}$$

where π ranges over the set of multiplicative characters on $\mathbf{K}(\sqrt{\tau})$. We assume that the additive character $\chi_\tau(t)$ on $\mathbf{K}(\sqrt{\tau})$ is given by the following formula:

$$\chi_\tau(t) = \chi(t + \bar{t}), \tag{2}$$

where χ is the given additive character (of rank 0) on **K**. We note that the rank of $\chi_\tau(t)$ on $\mathbf{K}(\sqrt{\tau})$ is zero when $\tau = \varepsilon$, and 1 when $\tau = \mathfrak{p}$ or $\varepsilon\mathfrak{p}$.

We show that *the following relation holds:*

$$\Gamma_\tau(\pi\bar{\pi}) = |\tau|^{-1} c_\tau \Gamma(\pi)\Gamma(\pi\pi_\tau), \tag{3}$$

where $\bar{\pi}(t)$ denotes the character $\bar{\pi}(t) = \pi(\bar{t})$

$$\pi_\tau(x) \equiv \operatorname{sign}_\tau x \quad \text{and} \quad c_\tau^{-1} = \frac{a_\tau}{2}\int \chi(x)\pi_\tau(x)\, dx.$$

We set

$$f(\pi) = \frac{\Gamma(\pi)\Gamma(\pi\pi_\tau)}{\Gamma_\tau(\pi\bar{\pi})}. \tag{4}$$

Since $\Gamma_\tau^{-1}(\pi\bar{\pi}) = |\tau|\ \Gamma_\tau(\pi^{-1}\bar{\pi}^{-1}\pi_0^2)$, where $\pi_0(t) = |t\bar{t}|^{1/2}$, (see **§ 2.5** (4)), we have

$$\begin{aligned} f(\pi) &= |\tau|\ \Gamma(\pi)\Gamma(\pi\pi_\tau)\Gamma_\tau(\pi^{-1}\bar{\pi}^{-1}\pi_0^2) \\ &= |\tau| \int \chi(x + y + t + \bar{t})\pi\left(\frac{xy}{t\bar{t}}\right) \operatorname{sign}_\tau y\, |x|^{-1}\, |y|^{-1}\, dx\, dy\, dt; \end{aligned}$$

the integration is taken with respect to the variables $x, y \in \mathbf{K}$ and $t \in \mathbf{K}(\sqrt{\tau})$. By making the change of variables $x = \dfrac{t\bar{t}}{y} s$ and $t = yt'$

the integral reduces to the form

$$f(\pi) = |\tau| \int \chi(y(st\bar{t} + 1 + t + \bar{t}))\pi_0\pi_\tau(y)\pi\pi_0^{-1}(s)\, dy\, ds\, dt$$
$$= \int \chi(y(st\bar{t} + 1 - s^{-1}))\pi_0\pi_\tau(y)\pi\pi_0^{-1}(s)\, dy\, ds\, dt.$$

(The transition to the last integral is effected by the substitution: $t = t' - s^{-1}$.) Integrating with respect to y, we find

$$f(\pi) = |\tau|\ \Gamma(\pi_0^2\pi_\tau) \int \pi_0^{-2}\pi_\tau(st\bar{t} + 1 - s^{-1})\pi\pi_0^{-1}(s)\, dt\, ds$$
$$= |\tau|\ \Gamma(\pi_0^2\pi_\tau) \int \pi_0^{-2}\pi_\tau(t\bar{t} + s - 1)\pi\pi_0^{-1}\pi_\tau(s)\, dt\, ds. \tag{5}$$

(Change of variable $t = s^{-1}t'$.)

We compute separately the integral

$$\varphi(x) = \int \pi_0^{-2}\pi_\tau(t\bar{t} + x)\, dt.$$

Passing from $\varphi(x)$ to its Fourier transform we obtain

$$\tilde{\varphi}(u) = \int \chi(ux)\pi_0^{-2}\pi_\tau(t\bar{t} + x)\, dt\, dx$$
$$= \int \chi(ux)\pi_0^{-2}\pi_\tau(x)\, dx \int \chi(-ut\bar{t})\, dt$$
$$= \Gamma(\pi_0^{-1}\pi_\tau)\pi_0\pi_\tau(u)\left[c_\tau^{-1}\pi_0^{-1}\pi_\tau(-u) + \frac{a_\tau}{2}\,\delta(u)\right]$$

(see **§ 2.7** (2)).

So we have

$$\tilde{\varphi}(u) = c_\tau^{-1}\Gamma(\pi_0^{-1}\pi_\tau)\pi_\tau(-1),$$

hence,

$$\varphi(x) = c_\tau^{-1}\Gamma(\pi_0^{-1}\pi_\tau)\pi_\tau(-1)\delta(x).$$

Thus, we have established that

$$\int \pi_0^{-2}\pi_\tau(t\bar{t} + s - 1)\, dt = c_\tau^{-1}\Gamma(\pi_0^{-1}\pi_\tau)\pi_\tau(-1)\ \delta(s - 1).$$

Substituting this expression in (5) we find that

$$f(\pi) = |\tau|\ c_\tau^{-1}\Gamma(\pi_0^{-1}\pi_\tau)\Gamma(\pi_0^2\pi_\tau)\pi_\tau(-1).$$

Finally, observing that

$$\Gamma(\pi_0^{-1}\pi_\tau) = \Gamma(\pi_0\pi_\tau), \quad \Gamma(\pi_0^2\pi_\tau) = \Gamma^{-1}(\pi_0\pi_\tau)\pi_\tau(-1),$$

we obtain the end result: $f(\pi) = |\tau|\ c_\tau^{-1}$, that is

$$\Gamma_\tau(\pi\bar{\pi}) = |\tau|^{-1}\, c_\tau\, \Gamma(\pi)\, \Gamma(\pi\pi_\tau).$$

§ 3. IRREDUCIBLE REPRESENTATIONS OF THE GROUP OF MATRICES OF ORDER 2 WITH ELEMENTS FROM A LOCALLY COMPACT FIELD (THE CONTINUOUS SERIES)

In this and the subsequent sections we study representations of the group G of matrices $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$, $\alpha\delta - \beta\gamma = 1$, whose elements α, β, γ, δ belong to a locally compact topological field. In **§ 3** we give a description of the continuous series of irreducible unitary representations of G. Other (discrete) series of irreducible unitary representations of G will be discussed in **§ 4**. In **§ 5** we compute the traces (characters) of the irreducible representations of G, and in **§ 6** we obtain an expansion of a function $f(g)$ on G as a Fourier integral (Plancherel's theorem).

Instead of G we frequently consider groups related to it:

1. The factor group $G_1 = G/\mathfrak{Z}$ of G over its center $\mathfrak{Z}$ ($\mathfrak{Z}$ consists of the two elements e and $-e$, where e is the unit matrix).

2. The group G_2 of all fractional-linear transformations $x' = \dfrac{\alpha x + \gamma}{\beta x + \delta}$.

It is easy to show that G_2 is isomorphic to the group of all automorphisms of G, and that G_1 is isomorphic to the group of all inner automorphisms of G. Thus, G_1 is a (normal) subgroup of G_2. It is not hard to show that

$$G_2/G_1 \cong \mathbf{K}^*/(\mathbf{K}^*)^2.$$

Thus, $G_2 = G_1$ when $\mathbf{K}$ is the field of complex numbers, $G_2 : G_1 = 2$ when $\mathbf{K}$ is the field of real numbers, $G_2 : G_1 = 4$ when $\mathbf{K}$ is a disconnected field.†

All the results to be expounded below carry over to the groups G_1 and G_2 without any essential modifications.

1. The Continuous Series of Unitary Representations of G. We begin with a description of the continuous series of representations of G. For the field of complex numbers this series of representations was discovered by Gel'fand and Naimark. The construction given by them carries over directly to the case of a locally compact topological field $\mathbf{K}$.

A representation of the continuous series is given by a unitary multiplicative character $\pi(x)$ on $\mathbf{K}$.

The representation is constructed in the space of complex-valued functions $\varphi(x)$ on $\mathbf{K}$ for which

$$(\varphi, \varphi) = \int |\varphi(x)|^2 \, dx < \infty.$$

† The special case when the characteristic of the finite field O/P associated with $\mathbf{K}$ is 2 is excluded (see **§ 1.5**).

The representation operator T_π corresponding to the matrix $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ has the following form:

$$T_\pi(g)\,\varphi(x) = \varphi\left(\frac{\alpha x + \gamma}{\beta x + \delta}\right)\pi(\beta x + \delta)\ |\beta x + \delta|^{-1}. \tag{1}$$

The fact that the operators $T_\pi(g)$ form a representation, that is, that $T_\pi(g_1 g_2) = T_\pi(g_1)\,T_\pi(g_2)$, is established by direct verification.

A similar construction of representations is available for the group of matrices with elements from a *finite* field $\mathbf{K}_q$ of order q. To describe these representations it is convenient to go over from the functions $\varphi(x)$ to homogeneous functions $f(x_1, x_2)$ of two variables. Then we have the following description of the representations. Each representation is given by a multiplicative character $\pi(t)$ on $\mathbf{K}_q$. It is constructed in the space of functions $f(x_1, x_2)$ on $\mathbf{K}_q$, $(x_1, x_2) \neq (0, 0)$, that satisfy the condition of homogeneity

$$f(tx_1, tx_2) = \pi(t) f(x_1, x_2).$$

The representation operator $T_\pi(g)$ has the form

$$T_\pi(g) f(x_1, x_2) = f(\alpha x_1 + \gamma x_2, \beta x_1 + \delta x_2). \tag{2}$$

These representations are irreducible except when $\pi \equiv 1$ (in this case a one-dimensional representation splits off), and when $\pi(t)$ assumes only the values ± 1 (in this case the representation decomposes into two representations of equal dimension). As in the case of an arbitrary locally compact topological field $\mathbf{K}$, the representations $T_\pi(g)$ and $T_{\pi^{-1}}(g)$ turn out to be equivalent. The traces of the representations $T_\pi(g)$ were first computed by Frobenius.

Apart from the representations (2), the group of matrices over a finite field $\mathbf{K}_q$ also has an "analytic" series of representations, which we shall discuss in **§ 4**.

Let us show that the operators $T_\pi(g)$ are unitary, that is,

$$(T_\pi(g)\,\varphi, T_\pi(g)\,\varphi) = (\varphi, \varphi). \tag{3}$$

Indeed, we have

$$(T_\pi(g)\,\varphi, T_\pi(g)\,\varphi) = \int \left|\varphi\left(\frac{\alpha x + \gamma}{\beta x + \delta}\right)\right|^2 |\beta x + \delta|^{-2}\, dx.$$

By making the change of variable $x' = \dfrac{\alpha x + \gamma}{\beta x + \delta}$ and using the equation $dx' = |\beta x + \delta|^{-2}\, dx$ we obtain equation (3) at once.

Now we give a derivation of the formula $dx' = |\beta x + \delta|^{-2}\, dx$. We set $dx' = a(x, g)\, dx$. An immediate consequence of the definition of $a(x, g)$ is the functional equation

$$a(x, g_1 g_2) = a(x, g_1) a(x g_1, g_2). \tag{4}$$

(Here xg is the result of applying to x the fractional linear transformation corresponding to g.) The same relation (4) is easily seen to be satisfied by the function $|\beta x + \delta|^{-2}$.

Now we observe that every matrix g can be represented as a product of matrices of the following types:

$$\begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix}, \quad \begin{pmatrix} \delta^{-1} & 0 \\ 0 & \delta \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{5}$$

Therefore, by virtue of (4), the function $a(x, g)$ is uniquely determined by its values on the matrices g of the form (5). So it is sufficient to verify that

$$a(x, g) = |\beta x + \delta|^{-2} \tag{6}$$

for matrices g of the form (5). But for these matrices (6) is obvious. For if $g = \begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix}$, then $x' = x + \gamma$, hence, $dx' = dx$. If $g = \begin{pmatrix} \delta^{-1} & 0 \\ 0 & \delta \end{pmatrix}$, then $x' = \delta^{-2}x$, hence, $dx' = |\delta|^{-2}\, dx$. Finally, if $g = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, then $x' = -\dfrac{1}{x}$; since the multiplicatively invariant measure $d^*x = |x|^{-1}\, dx$ is preserved under this transformation, we have $|x'|^{-1}\, dx' = |x|^{-1}\, dx$, hence $dx' = |x|^{-2}\, dx$.

2. Another Realization of the Representations of the Continuous Series. We obtain another realization of the representations of the continuous series by going over from the functions $\varphi(x)$ to their Fourier transforms

$$\tilde{\varphi}(u) = \int \varphi(x)\, \chi(-ux)\, dx, \tag{1}$$

Let us find the expression for the operator $T_\pi(g)$ in this realization. The operator $T_\pi(g)$ acts on the function $\tilde{\varphi}(u)$ by the following formula:

$$\begin{aligned} T_\pi(g)\, \tilde{\varphi}(u) &= \int [T_\pi(g)\, \varphi(x)]\, \chi(-ux)\, dx \\ &= \int \varphi\left(\frac{\alpha x + \gamma}{\beta x + \delta}\right) \pi(\beta x + \delta)\, |\beta x + \delta|^{-1}\, \chi(-ux)\, dx. \end{aligned} \tag{2}$$

It remains for us to express the right-hand side of (2) again in terms of $\tilde{\varphi}(u)$.

By the formula for the inverse Fourier transform we have

$$\varphi(x) = \int \tilde{\varphi}(v)\, \chi(vx)\, dv.$$

Consequently,

$$T_\pi(g)\, \tilde{\varphi}(u) = \int \chi\left(-ux + v\, \frac{\alpha x + \gamma}{\beta x + \delta}\right) \pi(\beta x + \delta)\, |\beta x + \delta|^{-1}\, \varphi(v)\, dv\, dx.$$

Henceforth we call this realization of the continuous series the *χ-realization.*

Thus, *in the χ-realization a representation of the continuous series is constructed in the space of functions $\varphi(u)$ for which*

$$(\varphi, \varphi) = \int |\varphi(u)|^2\, du < \infty.$$

The representation operator $T_\pi(g)$, $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$, is given by the formula

$$T_\pi(g)\, \varphi(u) = \int K_\pi(g \mid u, v)\, \varphi(v)\, dv, \tag{3}$$

where

$$K_\pi(g \mid u, v) = \int \chi\left(-ux + v\frac{\alpha x + \gamma}{\beta x + \delta}\right) \pi(\beta x + \delta) \, |\beta x + \delta|^{-1} \, dx. \quad (4)$$

Let us examine the expression (4) in detail. To begin with we assume that $\beta \neq 0$. Then it is convenient to write (4) in a somewhat different form, by making the change of variables $\beta x + \delta = t$. After an elementary transformation we find

$$K_\pi(g \mid u, v) = |\beta|^{-1} \chi\left(\frac{\delta u + \alpha v}{\beta}\right) \int \chi\left(-\frac{1}{\beta}(ut + vt^{-1})\right) \pi(t) \, |t|^{-1} \, dt. \quad (5)$$

We mention a peculiarity of this formula. It tells us that $K_\pi(g \mid u, v)$ is a product of two functions—the function

$$|\beta|^{-1}\chi\left(\frac{\delta u + \alpha v}{\beta}\right),$$

one and the same for all the representations of the series, and the Bessel function

$$\int \chi\left(-\frac{1}{\beta}(ut + vt^{-1})\right) \pi(t) \, |t|^{-1} \, dt,$$

which does not depend essentially on g.

Now we discuss the special case when $\beta = 0$. Then we have

$$\begin{aligned} K_\pi(g \mid u, v) &= \chi\left(\frac{\gamma}{\delta} v\right) \pi(\delta) \, |\delta|^{-1} \int \chi\left(\left(-u + \frac{\alpha}{\delta} v\right) x\right) dx \\ &= \chi\left(\frac{\gamma}{\delta} v\right) \pi(\delta) \, |\delta|^{-1} \, \delta\left(-u + \frac{\alpha}{\delta} v\right). \end{aligned}$$

Bearing in mind that in this case $\alpha = \delta^{-1}$, we may rewrite this formula as follows:

$$K_\pi(g \mid u, v) = \pi(\delta) \, |\delta| \, \chi(\delta \gamma u) \, \delta(v - \delta^2 u). \quad (6)$$

So the operator $T_\pi(g)$ corresponding to the matrix $g = \begin{pmatrix} \alpha & 0 \\ \gamma & \delta \end{pmatrix}$ has the following form in the χ-realization:

$$T_\pi(g) \varphi(u) = \pi(\delta) \, |\delta| \, \chi(\delta \gamma u) \varphi(\delta^2 u). \quad (7)$$

There is one further convenient realization of the representations of the continuous series, which we call the *π-realization.* We obtain it by going over from the functions $\varphi(x)$ to their Mellin transforms

$$F(\pi_1) = \int \varphi(x) \pi_1(x) \, |x|^{-1/2} \, dx; \quad (8)$$

where π_1 ranges over the unitary multiplicative characters. From the inversion formula for the Mellin transform (see **§ 2.9**) it follows that

$$\int |\varphi(x)|^2 \, dx = \int |F(\pi_1)|^2 \, d\pi_1.$$

Let us find the expression for the operator $T_\pi(g)$ in the π-realization. By definition,

$$T_\pi(g)F(\pi_1) = \int [T_\pi(g)\,\varphi(x)]\pi_1(x)\ |x|^{-1/2}\,dx$$

$$= \int \varphi\left(\frac{\alpha x + \gamma}{\beta x + \delta}\right)\pi(\beta x + \delta)\ |\beta x + \delta|^{-1}\ \pi_1(x)\ |x|^{-1/2}\,dx. \quad (9)$$

It remains to express the right-hand side of (9) again in terms of the function $F(\pi_1)$.

By the inversion formula for the Mellin transform we have

$$\varphi(x) = \int F(\pi_2)\pi_2^{-1}(x)\ |x|^{-1/2}\,d\pi_2.$$

Consequently,

$$T_\pi(g)F(\pi_1) = \int \pi_2^{-1}\left(\frac{\alpha x + \gamma}{\beta x + \delta}\right)\left|\frac{\alpha x + \gamma}{\beta x + \delta}\right|^{-1/2}\pi(\beta x + \delta)$$

$$|\beta x + \delta|^{-1}\ \pi_1(x)\ |x|^{-1/2}\,F(\pi_2)\ dx\,d\pi_2.$$

Thus, *in the π-realization a representation of the continuous series is constructed in the space of functions $F(\pi_1)$ on the group of multiplicative characters π_1 for which*

$$(F, F) = \int |F(\pi_1)|^2\,d\pi_1 < \infty.$$

The representation operator $T_\pi(g)$, $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$, is given by the formula

$$T_\pi(g)F(\pi_1) = \int K_\pi(g \mid \pi_1, \pi_2)F(\pi_2)\,d\pi_2, \quad (10)$$

where

$$K_\pi(g \mid \pi_1, \pi_2) = \int \pi\pi_2(\beta x + \delta)\ |\beta x + \delta|^{-1/2}$$

$$\pi_2^{-1}(\alpha x + \gamma)\ |\alpha x + \gamma|^{-1/2}\ \pi_1(x)\ |x|^{-1/2}\,dx. \quad (11)$$

The expression (11), which gives the matrix element of the operator $T_\pi(g)$ in the π-representation, can appropriately be called the *hypergeometric function* of π, π_1, π_2. For the field of real numbers $K_\pi(g \mid \pi_1, \pi_2)$ can be expressed immediately in terms of the hypergeometric function of Gauss.

Note that if one of the elements of g is zero, then the hypergeometric function (11) degenerates into a Beta-function. For example, if $\alpha = 0$, we have

$$K_\pi(g \mid \pi_1, \pi_2)$$
$$= \pi_2^{-1}(\gamma)\,|\gamma|^{-\frac{1}{2}} \int \pi\pi_2(\beta x + \delta)\,|\beta x + \delta|^{-\frac{1}{2}}\,\pi_1(x)\,|x|^{-\frac{1}{2}}\,dx.$$

The formulae for the representation operators corresponding to the matrices

$$\delta = \begin{pmatrix} \delta^{-1} & 0 \\ 0 & \delta \end{pmatrix}, \qquad z = \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix}, \qquad z \neq 0,$$
$$\text{and } \zeta = \begin{pmatrix} 1 & \zeta \\ 0 & 1 \end{pmatrix}, \qquad \zeta \neq 0$$

have a very simple form.

For on the basis of (10) and (11) we find after elementary transformations that

$$T_\pi(\delta)F(\pi_1) = \pi\pi_1^2(\delta)F(\pi_1), \tag{12}$$

$$T_\pi(z)F(\pi_1) = \int \frac{\Gamma(\pi_1\pi_0)}{\Gamma(\pi_2\pi_0)}\,\Gamma(\pi_1^{-1}\pi_2)\,\pi_1\pi_2^{-1}(z)F(\pi_2)\,d\pi_2, \tag{13}$$

$$T_\pi(\zeta)F(\pi_1) = \int \frac{\Gamma(\pi\pi_2\pi_0)}{\Gamma(\pi\pi_1\pi_0)}\,\Gamma(\pi_1\pi_2^{-1})\,\pi_1^{-1}\pi_2(-\zeta)F(\pi_2)\,d\pi_2. \tag{14}$$

It is convenient to go over from $T_\pi(g)$ to the equivalent representation $T'_\pi(g) = A^{-1}T_\pi(g)A$, where A is the operator of multiplication by $\Gamma(\pi_1\pi_0)$. Clearly, the kernels of the operators $T'_\pi(g)$ are obtained from those of $T_\pi(g)$ by multiplying by $\dfrac{\Gamma(\pi_2\pi_0)}{\Gamma(\pi_1\pi_0)}$. So we obtain

$$T'_\pi(\delta)F(\pi_1) = \pi\pi_1^2(\delta)F(\pi_1), \tag{15}$$

$$T'_\pi(z)F(\pi_1) = \int \Gamma(\pi_1^{-1}\pi_2)\,\pi_1\pi_2^{-1}(z)F(\pi_2)\,d\pi_2, \tag{16}$$

$$T'_\pi(\zeta)F(\pi_1) = \int \frac{\Gamma(\pi_2\pi_0)\,\Gamma(\pi\pi_2\pi_0)}{\Gamma(\pi_1\pi_0)\,\Gamma(\pi\pi_1\pi_0)}\,\Gamma(\pi_1\pi_2^{-1})\,\pi_1^{-1}\pi_2(-\zeta)F(\pi_2)\,d\pi_2. \tag{17}$$

This realization of the representation has the advantage that the operator $T'_\pi(z)$ in it does not depend on the "index" of the representation. Since the matrices z and ζ generate the whole group G, the representation $T'_\pi(g)$ is completely determined by the formula for the operator $T'_\pi(\zeta)$. In this formula the only factor depending on the index of the representation is

$$a(\pi \mid \pi_1, \pi_2) = \frac{\Gamma(\pi_2\pi_0)\,\Gamma(\pi\pi_2\pi_0)}{\Gamma(\pi_1\pi_0)\,\Gamma(\pi\pi_1\pi_0)},$$

under the integral sign, and this then gives us our representation.

In **§ 4.5** we shall show that similar formulae hold for the representations of the discrete series.

3. Equivalence of Representations of the Continuous Series. We are going to show that *the representations of the continuous series $T_\pi(g)$ and $T_{\pi^{-1}}(g)$ are equivalent.*

To prove this we consider the operator $T_\pi(g)$ in the χ-realization. The kernel $K_\pi(g \mid u, v)$ of $T_\pi(g)$ is given by the following formula:

$$K_\pi(g \mid u, v) = |\beta|^{-1}\chi\left(\frac{\delta u + \alpha v}{\beta}\right)\int \chi\left(-\frac{1}{\beta}(ut + vt^{-1})\right)\pi(t)\ |t|^{-1}\,dt.$$

Making the change of variable $t = vu^{-1}t'^{-1}$ under the integral, we obtain

$$K_\pi(g \mid u, v)$$
$$= \frac{\pi(v)}{\pi(u)}\,|\beta|^{-1}\,\chi\left(\frac{\delta u + \alpha v}{\beta}\right) \times \int \chi\left(-\frac{1}{\beta}(ut + vt^{-1})\right)\pi^{-1}(t)\ |t|^{-1}\,dt,$$

that is,

$$K_\pi(g \mid u, v) = \frac{\pi(v)}{\pi(u)}\,K_{\pi^{-1}}(g \mid u, v).$$

So we have shown that

$$T_\pi(g) = A^{-1}T_{\pi^{-1}}(g)A,$$

where A is the operator of multiplying by $\pi(u)$:

$$A\varphi(u) = \pi(u)\,\varphi(u).$$

Hence, the representations $T_\pi(g)$ and $T_{\pi^{-1}}(g)$ are equivalent.

In **§ 5** we shall see that there are no other pairs of equivalent representations in the continuous series.

4. Irreducibility of the Representations of the Continuous Series. We show that the representations $T_\pi(g)$ of the continuous series are irreducible, apart from certain special values of π.

We recall that a unitary representation $T(g)$ is called irreducible if the representation space contains no invariant subspace other than zero. The following is an equivalent definition: *a unitary representation $T(g)$ is called irreducible if every bounded operator in the representation space that commutes with all the operators $T(g)$ is a multiple of the unit operator.*

The following propositions hold.

1. *For the field of complex numbers all the representations $T_\pi(g)$ are irreducible.*

2. *For the field of real numbers all the representations* $T_\pi(g)$ *are irreducible, except* $\pi(x) = \operatorname{sign} x$. *In this special case* $\pi(x) = \operatorname{sign} x$, *the representation* $T_\pi(g)$ *splits into two irreducible representations.*

3. *For a disconnected field all the representations* $T_\pi(g)$ *are irreducible, except when* $\pi(x) = \operatorname{sign}_\tau x$, $\tau = \mathfrak{p}, \varepsilon\mathfrak{p}$, *or* ε (*see* **§ 1.7**). *In each of these special cases the representation splits into two irreducible representations.*

We give a proof only for the case of a disconnected field; for connected fields the proof is similar.†

We consider the representation $T_\pi(g)$ in the χ-realization. Our aim is to describe all the bounded operators A that commute with the $T_\pi(g)$;

$$AT_\pi(g) = T_\pi(g)A.$$

First, we see what we can derive from the permutability of A with the operators $T_\pi(g)$, where $g = \begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix}$. These operators are of the form

$$T_\pi(g)\,\varphi(u) = \chi(\gamma u)\,\varphi(u). \tag{1}$$

Hence, A commutes with the operators of multiplication by $\chi(\gamma u)$, and therefore, with every operator of multiplication by a bounded function. From this it follows that A itself is an operator of multiplication by a bounded function $a(u)$:

$$A\varphi(u) = a(u)\,\varphi(u).$$

Next, we see what we can derive from the permutability of A with the operators $T_\pi(g)$ where $g = \begin{pmatrix} \delta^{-1} & 0 \\ 0 & \delta \end{pmatrix}$. These operators are of the form

$$T_\pi(g)\,\varphi(u) = \pi(\delta)\;|\delta|\;\varphi(\delta^2 u). \tag{2}$$

The condition that A commutes with the operators $T_\pi(g)$ can be written in the form

$$a(\delta^2 u) = a(u)$$

for every $\delta \neq 0$. So the function $a(u)$ is constant on every coset of $\mathbf{K}^*/(\mathbf{K}^*)^2$. We can show, in fact, that in the nonexceptional cases $a(u)$ is constant on the whole of $\mathbf{K}$, and that in the exceptional cases it can assume only two distinct values. This then will prove the theorem.

---

† Another proof of the irreducibility of the representations for the field of complex and the field of real numbers can be found in Gel'fand, et al. [27].

We now write the condition that A commutes with the operator $T_\pi(s)$, where $s = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. This condition takes the form

$$\mathbf{K}_\pi(s \mid u, v)a(v) = a(u)\mathbf{K}_\pi(s \mid u, v), \tag{3}$$

where

$$\mathbf{K}_\pi(s \mid u, v) = \int \chi(ut + vt^{-1})\pi(t)\, d^*t. \tag{4}$$

Let $\mathbf{K}^{(1)}$ and $\mathbf{K}^{(2)}$ be distinct cosets of $\mathbf{K}^*/(\mathbf{K}^*)^2$. If we can show that $\mathbf{K}_\pi(s \mid u, v) \not\equiv 0$ when $u \in \mathbf{K}^{(1)}$, $v \in \mathbf{K}^{(2)}$, then it follows from (3) that $a(u)$ assumes identical values on $\mathbf{K}^{(1)}$ and $\mathbf{K}^{(2)}$.

Let $u \in \mathbf{K}^{(1)}$, $v \in \mathbf{K}^{(2)}$. We assume that $|u|$, $|v|$ are sufficiently small so that $\chi(ut) \equiv 1$, $\chi(vt) \equiv 1$ for $|t| \leq 1$. Then the expression for $K_\pi(s \mid u, v)$ can be written in the following form:

$$K_\pi(s \mid u, v)$$

$$= \int_{|t|<1} \chi(vt^{-1})\pi(t)\, d^*t + \int_{|t|>1} \chi(ut)\pi(t)\, d^*t + \int_{|t|=1} \pi(t)\, d^*t. \tag{5}$$

First, we take the case $\pi(t) \equiv 1$. Here we find, after a change of variables,

$$K_\pi(s \mid u, v) = \int_{|t|>|v|} \chi(t)\, d^*t + \int_{|t|>|u|} \chi(t)\, d^*t + (1 - q^{-1}).$$

Since $\chi(t) \equiv 1$ when $|t|$ is sufficiently small, this expression cannot possibly be constant, and hence $K_\pi(s \mid u, v) \not\equiv 0$. So we have shown that for $\pi \equiv 1$ the function $a(u)$ is a constant and therefore the representation $T_\pi(g)$ is irreducible.

Now let $\pi(t) \not\equiv 1$. Then we have

$$\int_{|t|\geq 1} \chi(vt^{-1})\pi(t)\, d^*t + \int_{|t|\leq 1} \chi(ut)\pi(t)\, d^*t - \int_{|t|=1} \pi(t)\, d^*t$$

$$= \int \pi(t)\, d^*t = 0.$$

Adding this to (5) we find

$$K_\pi(s \mid u, v) = \int \chi(vt^{-1})\pi(t)\, d^*t + \int \chi(ut)\pi(t)\, d^*t,$$

where the integrals are taken over the whole of K. By a change of variable we obtain

$$K_\pi(s \mid u, v) = \Gamma(\pi^{-1})\pi(v) + \Gamma(\pi)\pi^{-1}(u).$$

Here the coefficients $\Gamma(\pi)$ and $\Gamma(\pi^{-1})$ are different from zero. We assume that $\pi(u)$ is not constant on $(\mathbf{K}^*)^2$. Then $\pi(v)$ is not constant when v ranges over a coset of $\mathbf{K}^*/(\mathbf{K}^*)^2$, and hence $K_\pi(s \mid u, v) \not\equiv 0$. This shows that $a(u)$ is constant and therefore the representations $T_\pi(g)$ are irreducible when $\pi(t) \not\equiv 1$ for $t \in (\mathbf{K}^*)^2$.

Finally, we discuss the special case $\pi(t) \equiv 1$ for $t \in (\mathbf{K}^*)^2$. Such characters π have the form $\pi_\tau(t) = \operatorname{sign}_\tau t$, where $\tau = \mathfrak{p}$, $\varepsilon\mathfrak{p}$, or ε (the case $\pi \equiv 1$ was considered above).

Here we have, because $\pi_\tau = \pi_\tau^{-1}$,

$$\mathbf{K}_\pi(s \mid u, v) = \Gamma(\pi_\tau)[\operatorname{sign}_\tau u + \operatorname{sign}_\tau v].$$

Hence, if $\operatorname{sign}_\tau u = \operatorname{sign}_\tau v$, then $\mathbf{K}_\pi(s \mid u, v) \neq 0$.

This shows that for $\pi(t) = \operatorname{sign}_\tau t$ the function $a(v)$ is constant on the set of elements v where $\operatorname{sign}_\tau v$ is constant, that is, $a(v)$ assumes not more than two distinct values. Hence, the representation $T_\pi(g)$, $\pi(t) = \operatorname{sign}_\tau t$, if it splits at all, splits into no more than two irreducible representations. This fact will be shown in **§ 3.5.**

5. The Decomposition of the Representations $T_{\pi_\tau}(g)$, $\pi_\tau(t) = \operatorname{sign}_\tau t$, into Irreducible Representations. We decompose the space of functions $\varphi(u)$ into the direct sum of two subspaces: the subspace H^+ of functions $\varphi(u)$ that are zero for $\operatorname{sign}_\tau u = -1$ and the subspace H^- of functions $\varphi(u)$ equal to zero for $\operatorname{sign}_\tau u = 1$. We show now that *these subspaces H^+ and H^- are invariant under the operators $T_{\pi_\tau}(g)$, $\pi_\tau(t) = \operatorname{sign}_\tau t$.*

First, we recall that every matrix g can be represented as a product of matrices of the form $\begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix}$, $\begin{pmatrix} \delta^{-1} & 0 \\ 0 & \delta \end{pmatrix}$, and the matrix $s = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Therefore, it is sufficient to verify that H^+ and H^- are invariant under the operators corresponding to these matrices.

Clearly, the operators $T_{\pi_\tau}(g)$ corresponding to the matrices $\begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix}$ and $\begin{pmatrix} \delta^{-1} & 0 \\ 0 & \delta \end{pmatrix}$ preserve the spaces H^+ and H^- (because the first reduces to multiplying $\varphi(u)$ by the function $\chi(\gamma u)$ and the second carries $\varphi(u)$ into $\pi(\delta)\,|\delta|\,\varphi(\varphi^2 u)$ (see **§ 3.4** (1) and (2)). Thus, it remains to show the invariance of H^+ and H^- under the operator $T_{\pi_\tau}(s)$:

$$T_{\pi_\tau}(s)\,\varphi(u) = \int \chi(ut + vt^{-1}) \operatorname{sign}_\tau t\,\varphi(v)\, d^*t\, dv.$$

To prove this we pass from the functions $\varphi(u)$ to their Mellin transforms

$$F(\pi) = \int \varphi(u)\, |u|^{1/2}\, \pi(u)\, d^*u.$$

We write down the action of $T_{\pi_\tau}(s)$ on $F(\pi)$:

$$T_{\pi_\tau}(s)F(\pi) = \int \chi(ut + vt^{-1}) \operatorname{sign}_\tau t\varphi(v) \, |u|^{1/2} \, \pi(u) \, d^*t \, dv \, d^*u.$$

Integrating first with respect to u and then with respect to t we obtain

$$T_{\pi_\tau}(s)F(\pi) = \Gamma(\pi\pi_1)\,\Gamma(\pi\pi_1\pi_\tau)\int \pi^{-1}\pi_1^{-1}\pi_\tau(v)\,\varphi(v)\,dv,$$

where $\pi_1(v) = |v|^{1/2}$. On the right-hand side we substitute for $\varphi(v)$ its expression in terms of the Mellin transform $F(\pi)$:

$$\varphi(v) = \pi_1^{-1}(v)\int \pi'^{-1}(v)F(\pi')\,d\pi'.$$

After integrating with respect to v and to π' we obtain the following formula for $T_{\pi_\tau}(s)$:

$$T_{\pi_\tau}(s)F(\pi) = \Gamma(\pi\pi_1)\,\Gamma(\pi\pi_1\pi_\tau)F(\pi^{-1}\pi_\tau). \tag{1}$$

Let us show that the operator $T_{\pi_\tau}(s)$ defined by (1) preserves the subspaces H^+ and H^-. For this purpose we write down the condition that $F(\pi)$ belongs to H^+. The condition that $\varphi(u)$ belongs to H^+ can be written as follows:

$$\varphi(u)\,\pi_\tau(u) = \varphi(u),$$

where $\pi_\tau(u) = \operatorname{sign}_\tau u$. Obviously in the Mellin transform this condition can be written:

$$F(\pi\pi_\tau) = F(\pi). \tag{2}$$

Suppose now that $F(\pi)$ belongs to H^+, that is, that it satisfies (2), and let $F_1(\pi) = T_{\pi_\tau}(s)F(\pi)$. Bearing in mind that $\pi_\tau^2 = 1$ we then obtain from (1):

$$\begin{aligned} F_1(\pi\pi_\tau) &= \Gamma(\pi\pi_1\pi_\tau)\,\Gamma(\pi\pi_1)F(\pi^{-1}) \\ &= \Gamma(\pi\pi_1\pi_\tau)\,\Gamma(\pi\pi_1)F(\pi^{-1}\pi_\tau) \\ &= F_1(\pi). \end{aligned}$$

Hence, together with $F(\pi)$ the function $F_\pi(\pi) = T_{\pi_\tau}(s)F(\pi)$ also belongs to H^+. And so the invariance of the subspace H^+ is proved.

6. The Quasiregular Representation of G and its Decomposition into Irreducible Representations. *Quasiregular representation* of G is the name we use for the representation in the space of functions $f(x_1, x_2)$, $x_1, x_2 \in K$, for which

$$(f,f) = \int |f(x_1, x_2)|^2 \, dx_1 \, dx_2 < \infty.$$

The representation operator $T(g)$ corresponding to the matrix $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ is given by the formula

$$T(g)f(x_1, x_2) = f(\alpha x_1 + \gamma x_2, \beta x_1 + \delta x_2). \tag{1}$$

It is obvious that $T(g_1 g_2) = T(g_1)T(g_2)$ for arbitrary g_1 and g_2 from G and that

$$(T(g)f, T(g)f) = (f, f).$$

So the operators $T(g)$ form a unitary representation of G.

We now obtain the decomposition of $T(g)$ into irreducible representations of the principal series.

We call a function $f(x_1, x_2)$ *homogeneous* of degree π, where π is a multiplicative character on **K**, if it satisfies the condition

$$f(tx_1, tx_2) = \pi(t)\,|t|^{-1} f(x_1, x_2) \tag{2}$$

for every $t \neq 0$.

Every function $f(x_1, x_2)$ can be decomposed into homogeneous functions: we assign to every multiplicative character π the function

$$f_\pi(x_1, x_2) = \int f(tx_1, tx_2)\pi^{-1}(t)\, dt. \tag{3}$$

Clearly, $f_\pi(x_1, x_2)$ is a homogeneous function of degree π.

By the formula for the inverse Mellin transform we have

$$f(tx_1, tx_2) = |t|^{-1} \int f_\pi(x_1, x_2)\pi(t)\, d\pi, \tag{4}$$

where $d\pi$ is a suitably normed invariant measure on the character group. From (4) we obtain for $t = 1$ the required expansion of $f(x_1, x_2)$ into homogeneous functions

$$f(x_1, x_2) = \int f_\pi(x_1, x_2)\, d\pi. \tag{5}$$

The functions $f_\pi(x_1, x_2)$ being homogeneous are uniquely determined by their values on the line $x_2 = 1$. We set

$$\varphi_\pi(x) = f_\pi(x, 1). \tag{6}$$

We show now that the following Plancherel formula holds:

$$\int |f(x_1, x_2)|^2\, dx_1\, dx_2 = \int |\varphi_\pi(x)|^2\, dx\, d\pi. \tag{7}$$

For according to the Plancherel formula for the Mellin transform we have by (4)

$$\int |f(tx_1, tx_2)|^2\, |t|\, dt = \int |f_\pi(x_1, x_2)|^2\, d\pi.$$

Substituting $x_2 = 1$ we find

$$\int |f(tx, t)|^2 \, |t| \, dt = \int |\varphi_\pi(x)|^2 \, d\pi. \tag{8}$$

Integrating both sides of (8) with respect to x we obtain the Plancherel formula (7).

Let us see how the operator $T(g)$ acts on the function $\varphi_\pi(x)$. We have

$$\begin{aligned} T(g)\,\varphi_\pi(x) &= T(g) f_\pi(x_1, x_2)\big|_{\substack{x_1=x\\ x_2=1}} \\ &= f_\pi(\alpha x + \gamma, \beta x + \delta) \\ &= f_\pi\left(\frac{\alpha x + \gamma}{\beta x + \delta}, 1\right)\pi(\beta x + \delta)\,|\beta x + \delta|^{-1}. \end{aligned}$$

Thus,

$$T(g)\,\varphi_\pi(x) = \varphi_\pi\left(\frac{\alpha x + \gamma}{\beta x + \delta}\right)\pi(\beta x + \delta)\,|\beta x + \delta|^{-1}.$$

So we see that the functions $\varphi_\pi(x)$ are transformed according to the representation of the continuous series corresponding to the character π.

Hence, the formulae (5) and (7) give us the decomposition of the quasiregular representation of G into irreducible unitary representations of the continuous series.

7. The Supplementary Series of Irreducible Unitary Representations of G. Here we give a description of yet another series of irreducible unitary representations of G. The series is defined by analogy to the case of the field of complex or the field of real numbers (see Gel'fand et al. [27]).

Each representation of this series is given by a real number $\rho \neq 0$ in the interval $-1 < \rho < 1$, and is constructed in the space of functions $\varphi(x)$ on $\mathbf{K}$ for which

$$(\varphi, \varphi) = \frac{1}{\Gamma(\pi_\rho)} \int |x_1 - x_2|^{\rho - 1} \, \varphi(x_1)\overline{\varphi(x_2)} \, dx_1 \, dx_2 < \infty.$$

Here π_ρ denotes the character $\pi_\rho(x) = |x|^\rho$,

$$\Gamma(\pi_\rho) = \int \chi(x) \, |x|^{\rho - 1} \, dx. \tag{1}$$

The representation operator $T_\rho(g)$ corresponding to the matrix $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ is given by the following formula:

$$T_\rho(g)\,\varphi(x) = \varphi\left(\frac{\alpha x + \gamma}{\beta x + \delta}\right) |\beta x + \delta|^{-\rho - 1}. \tag{2}$$

An immediate verification shows that

$$T_\rho(g_1 g_2) = T_\rho(g_1) T_\rho(g_2)$$

for arbitrary matrices g_1 and g_2 from G and that

$$(T_\rho(g)\varphi, T_\rho(g)\varphi) = (\varphi, \varphi).$$

So the operators $T_\rho(g)$ provide a unitary representation of G.

We call the series of representations so constructed the *supplementary series.*

It can be shown that the representations $T_\rho(g)$ and $T_{-\rho}(g)$ are equivalent (for a proof in the case of the field of real or of complex numbers see [28]; for disconnected fields, the proof is similar). Therefore we may henceforth assume that $0 < \rho < 1$.

Another realization of the representations of the supplementary series is obtained by passing from the functions $\varphi(x)$ to their Fourier transforms

$$\tilde{\varphi}(u) = \int \varphi(x)\chi(-ux)\,dx.$$

In this realization the formula for the operator $T_\rho(g)$ takes the following from:

$$T_\rho(g)\varphi(u) = \int K_\rho(g \mid u, v)\varphi(v)\,dv, \tag{3}$$

where

$$K_\rho(g \mid u, v) = \int \chi\left(-ux + v\,\frac{\alpha x + \gamma}{\beta x + \delta}\right) |\beta x + \delta|^{-\rho-1}\,dx. \tag{4}$$

The expression for the kernel $K_\rho(g \mid u, v)$ can be written in a somewhat different form. Namely,

$$K_\rho(g \mid u, v) = |\beta|^{-1}\chi\left(\frac{\delta u + \alpha v}{\beta}\right)\int \chi\left(-\frac{1}{\beta}(ut + vt^{-1})\right)|t|^{-\rho-1}\,dt,$$

when $\beta \neq 0$, and

$$K_\rho(g \mid u, v) = |\delta|^{-\rho+1}\chi(\delta\gamma u)\,\delta(\delta^2 u - v),$$

when $\beta = 0$.

Let us find the expression for the scalar product (φ, φ) in the new realization. We observe that the Fourier transform carries the convolution of functions into their product. Since the Fourier transform of $|x|^{\rho-1}$ is

$$|x|^{\rho-1} = \Gamma(\pi_\rho)\,|u|^{-\rho},$$

where $\pi_\rho(x) = |x|^\rho$, we have

$$(\varphi, \varphi) = \int |u|^{-\rho}\,|\varphi(u)|^2\,du. \tag{5}$$

So the representation $T_\rho(g)$ of the supplementary series $(0 < \rho < 1)$ may be realized in the space of functions $\varphi(u)$ on $\mathbf{K}$ with the scalar product (5). The representation operator in this realization is given by the formulae (3) and (4).

All the representations of the supplementary series are irreducible.

The proof of this proposition is word for word the same as in the case of the principal continuous series (see **§ 3.4**).

8. The Special Representation of G**.** In **§ 3.7** we constructed the supplementary series of irreducible unitary representations $T_\rho(g)$, where $0 < \rho < 1$. Let us see what representations arise in the limiting case when $\rho = 0$ or $\rho = 1$.

Obviously, for $\rho = 0$ we obtain the representation of the principal continuous series corresponding to the character $\pi \equiv 1$.

We shall see presently that for $\rho = 1$ a new representation of G arises.

So we examine the representation

$$T_1(g)\,\varphi(x) = \varphi\left(\frac{\alpha x + \gamma}{\beta x + \delta}\right) |\beta x + \delta|^{-2}. \tag{1}$$

Let us clarify how we must define the space of functions $\varphi(x)$ so that the operators $T_1(g)$ are unitary operators in this space.

We note that the formula for the scalar product

$$(\varphi, \varphi) = \frac{1}{\Gamma(\pi_\rho)} \int |x_1 - x_2|^{\rho-1}\, \varphi(x_1)\overline{\varphi(x_2)}\, dx_1\, dx_2 \tag{2}$$

is meaningless for $\rho = 1$, because $\Gamma(\pi_\rho)\big|_{\rho=1} = 0$. Therefore we impose on $\varphi(x)$ the additional condition

$$\int \varphi(x)\, dx = 0. \tag{3}$$

For such functions we have

$$\int |x_1 - x_2|^{\rho-1}\, \varphi(x_1)\overline{\varphi(x_2)}\, dx_1\, dx_2\big|_{\rho=1} = 0,$$

and the expression (2) tends, as $\rho \to 1$, to the finite limit†

$$(\varphi, \varphi) = c\int \ln|x_1 - x_2|\, \varphi(x_1)\overline{\varphi(x_2)}\, dx_1\, dx_2. \tag{4}$$

We show now that the functions $\varphi(x)$ satisfying condition (3) form an invariant space under the operators $T_1(g)$. For we have

$$\int T_1(g)\,\varphi(x)\, dx = \int \varphi\left(\frac{\alpha x + \gamma}{\beta x + \delta}\right) |\beta x + \delta|^{-2}\, dx = \int \varphi(x)\, dx.$$

† $\ln|x|$ is the adjoint homogeneous function of degree of homogeneity $\pi \equiv 1$.

Consequently, if $\int \varphi(x)\,dx = 0$, then also

$$\int T_1(g)\,\varphi(x)\,dx = 0.$$

So we have obtained a representation in the space of functions $\varphi(x)$ for which

$$\int \varphi(x)\,dx = 0,$$

$$(\varphi, \varphi) = \int \ln|x_1 - x_2|\,\varphi(x_1)\overline{\varphi(x_2)}\,dx_1\,dx_2 < \infty.$$

The representation operator $T_1(g)$ is given by the following formula:

$$T_1(g)\,\varphi(x) = \varphi\left(\frac{\alpha x + \gamma}{\beta x + \delta}\right)|\beta x + \delta|^{-2}.$$

We call this representation $T_1(g)$ the *singular representation.*†

If we go over from the functions $\varphi(x)$ to their Fourier transforms $\tilde{\varphi}(u) = \int \varphi(x)\chi(-ux)\,dx$, we obtain another realization of the singular representation, in which it is constructed in the space of functions $\varphi(u)$ with

$$(\varphi, \varphi) = \int |u|^{-1}\,|\varphi(u)|^2\,du < \infty$$

(and so $\varphi(0) = 0$). The representation operator $T_1(g)$ has the form

$$T_1(g)\,\varphi(u) = \int K_1(g \mid u, v)\,\varphi(v)\,dv,$$

where

$$K_1(g \mid u, v) = \int \chi\left(-ux + v\frac{\alpha x + \gamma}{\beta x + \delta}\right)|\beta x + \delta|^{-2}\,dx.$$

9. Representations in the Spaces $\mathscr{D}_\pi$. In this subsection we give a brief description of nonunitary representations‡ of G.

With every multiplicative character $\pi(x)$ of $\mathbf{K}$ (here we do not require that $|\pi(x)| \equiv 1$) we associate a function space $\mathscr{D}_\pi$. This space consists of the functions $f(x_1, x_2)$, $x_1, x_2 \in \mathbf{K}$, that satisfy the following two requirements:

1. For a connected field $\mathbf{K}$ the functions $f(x_1, x_2)$ are continuous and infinitely differentiable everywhere except at $(0, 0)$. If $\mathbf{K}$ is disconnected, then for a matrix $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ sufficiently close to the unit matrix

$$f(\alpha x_1 + \gamma x_2, \beta x_1 + \delta x_2) = f(x_1, x_2). \tag{1}$$

† In the case of a connected field the representation $T_1(g)$ is one of the representations of the continuous or the discrete series. Therefore the term "singular representation" refers only to disconnected fields.

‡ For details on these representations in the case of a connected field $\mathbf{K}$ see [28].

2. The functions $f(x_1, x_2)$ are homogeneous of weight π, that is,

$$f(tx_1, tx_2) = \pi(t)\ |t|^{-1} f(x_1, x_2) \tag{2}$$

for every $t \neq 0$.

There is a natural way of introducing a topology in $\mathscr{D}_\pi$ under which it becomes a complete space.

Now we give a representation of G in $\mathscr{D}_\pi$. If $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$, then we define the corresponding representation operator $T_\pi(g)$ by the formula

$$T_\pi(g) f(x_1, x_2) = f(\alpha x_1 + \gamma x_2, \beta x_1 + \delta x_2). \tag{3}$$

The question of the irreducibility and equivalence of the representations $T_\pi(g)$ arises.† The problem for the field of complex numbers and the field of real numbers is discussed in detail in [28]. Here we state, without proof, the analogous results for a disconnected field.

We defined the *singular points* in the group of multiplicative characters π as the characters $\pi(x) = |x|$ and $\pi(x) = |x|^{-1}$.

1. Two representations $T_{\pi_1}(g)$ and $T_{\pi_2}(g)$, where π_1 is a nonsingular point, are equivalent if and only if $\pi_1 = \pi_2$ or $\pi_1 = \pi_2^{-1}$.

2. For nonsingular points π the representations $T_\pi(g)$ are irreducible, except when $\pi(x) = \mathrm{sign}_\tau\, x$. In this case $\pi(x) = \mathrm{sign}_\tau\, x$, $T_\pi(g)$ splits into the direct sum of two irreducible representations.

3. Let $\pi(x) = |x|$. Then the space $\mathscr{D}_\pi$ contains a one-dimensional invariant subspace $\mathscr{G}_\pi$. It consists of the functions $f(x_1, x_2) = \text{const}$. The space $\mathscr{D}_{\pi^{-1}}$ also contains an invariant subspace $\mathscr{F}_{\pi^{-1}}$ consisting of the functions $f(x_1, x_2)$ for which

$$\int f(x, 1)\, dx = 0.$$

Clearly, the factor space $\mathscr{D}_{\pi^{-1}}/\mathscr{F}_{\pi^{-1}}$ is one-dimensional and consequently, $\mathscr{D}_{\pi^{-1}}/\mathscr{F}_{\pi^{-1}} \cong \mathscr{G}_\pi$. It can be shown that $\mathscr{D}_\pi/\mathscr{G}_\pi \cong \mathscr{F}_{\pi^{-1}}$ that is, the representation in $\mathscr{F}_{\pi^{-1}}$ is equivalent to that in the factor space $\mathscr{D}_\pi/\mathscr{G}_\pi$.

Now let us find out for what π we can introduce in $\mathscr{D}_\pi$ a scalar product invariant under the representation operators. When this is possible, we can complete $\mathscr{D}_\pi$ relative to this scalar product and obtain a unitary representation of G. For connected fields the problem was investigated in [28]. Here, without proof, are the analogous results for disconnected fields.

An invariant scalar product exists in $\mathscr{D}_\pi$ if and only if one of the following conditions is satisfied:

† For the definitions of irreducibility and equivalence in the spaces see $\mathscr{D}_\pi$ [28].

1. $|\pi(x)| \equiv 1$; the corresponding unitary representation of G is the representation of the principal continuous series discussed in **§ 3.1**.

2. $\pi(x) = |x|^\rho$, where ρ is a real number, $0 < |\rho| < 1$; the corresponding unitary representations of G are the representations of the supplementary series discussed in **§ 3.7.**

Furthermore, for $\pi(x) = |x|^{-1}$ an invariant scalar product exists in the subspace $\mathscr{F}_\pi$ of functions from $\mathscr{D}_\pi$ satisfying the condition

$$\int f(x, 1)\, dx = 0;$$

the corresponding unitary representation of G is the singular representation discussed in **§ 3.8.**

In the classification of all irreducible representations of G, we mention one essential difference between the case of a connected field and a disconnected field **K**. In the case of a connected field it is sufficient to consider the space $\mathscr{D}_\pi$ and all its invariant subspaces and factor spaces, if $\mathscr{D}_\pi$ is reducible. It can be shown that in this way we obtain, to within equivalence, all the irreducible representations of G [27]. For a disconnected field **K** this is not so: the representations of the discrete series, which will be constructed in **§ 4**, are inequivalent in the spaces $\mathscr{D}_\pi$.

10. Spherical Functions. We say that an irreducible representation of G is of class I if the representation space contains a vector that is invariant under the subgroup U of integral matrices, that is, matrices whose elements are all $\mathfrak{p}$-adic integers.

Let us find that representations of the continuous series that belong to class I. As we know, a representation of the continuous series $T_\pi(g)$ can be realized in the space of functions $f(x) = f(x_1, x_2)$ satisfying the condition of homogeneity

$$f(tx) = \pi(t)\, |t|^{-1} f(x)$$

for every $t \neq 0$.

In this space we look for a function invariant under the operators $T(u)$ with $u \in U$.

We define the norm $|x|$ of a vector $x = (x_1, x_2)$ as the maximum of the norms of its coordinates:

$$|x| = \max(|x_1|, |x_2|). \tag{1}$$

It is easy to verify that any vector x' can be carried into another x'' of equal norm by a transformation from U. Hence it follows immediately that every function invariant under the compact subgroup U is of the form

$$f = F(|x|).$$

From the condition of homogeneity we obtain that

$$f = C\pi(|x|)\, |x|^{-1} = C\, |x|^{s-1}.$$

From this we conclude: the irreducible representations of the continuous series having a vector invariant under the subgroup U of integral matrices are precisely those that correspond to the character

$$\pi(x) = |x|^s.$$

This vector is uniquely determined to within a constant factor and has the following form:

$$f_0 = \sqrt{\frac{q}{1+q}}\,|x|^{s-1},$$

where $|x|$ is the norm of the vector $x = (x_1, x_2)$ defined by (1). The factor $\sqrt{\frac{q}{1+q}}$ is adjusted so that $\|f_0\| = 1$.

We define an elementary spherical function on G corresponding to an irreducible representation of class I as a function $\varphi(g)$ on G determined by the following formula:

$$\varphi(g) = (T(g)f_0, f_0)$$

where f_0 is a vector in the representation space that is invariant under U and such that $\|f_0\| = 1$, and the parentheses denote the scalar product. From the definition it follows immediately that the function $\varphi(g)$ is constant on the double cosets of U, that is,

$$\varphi(u_1 g u_2) = \varphi(g) \qquad \text{for arbitrary } u_1, u_2 \in U.$$

It can be shown that every matrix $g \in G$ can be represented in the following form:

$$g = u_1 \delta u_2,$$

where $u_1, u_2 \in U$ and δ is a diagonal matrix of the form

$$\delta = \begin{pmatrix} \mathfrak{p}^{-n} & 0 \\ 0 & \mathfrak{p}^{n} \end{pmatrix}, \qquad n \geq 0.$$

Thus, a spherical function $\varphi(g)$ is completely determined by its values on the matrices δ.

Let us compute $\varphi(\delta)$. Suppose, for the sake of definiteness, that $T(g)$ is a representation of the principal series, that is, $s = i\rho$ is a purely imaginary number. Then the scalar product is given by the following formula:

$$(f_1, f_2) = \int f_1(t, 1)\overline{f_2(t, 1)}\,dt.$$

So we have

$$\varphi(\delta) = \frac{q}{q+1}\int [\max\,(q^n\,|t|, q^{-n})]^{s-1}[\max\,(|t|, 1)]^{-s-1}\,dt.$$

We split this integral into three parts—the integral over the domain $|t| \leq q^{-2n}$, the integral over the domain $q^{-2n} < |t| \leq 1$, and the integral over the domain $|t| > 1$. Then we have

$$\frac{q}{q+1}\,\varphi(\delta)$$
$$= q^{-n(s-1)} \int\limits_{|t| \leq q^{-2n}} dt + q^{n(s-1)} \int\limits_{q^{-2n}<|t|\leq 1} |t|^{s-1}\,dt + q^{n(s-1)} \int\limits_{|t|>1} |t|^{-2}\,dt.$$

All the integrals in this expression are easily computed. Namely,

$$\int\limits_{|t| \leqslant q^{-2n}} dt = q^{-2n}; \quad \int\limits_{|t|>1} |t|^{-2}\,dt = q^{-1},$$

$$\int\limits_{q^{-2n}<|t|\leq 1} |t|^{s-1}\,dt = (1 - q^{-1})(1 + q^{-s} + q^{-2s} + \cdots + q^{-(2n-1)s})$$
$$= (1 - q^{-1})\,\frac{1 - q^{-2ns}}{1 - q^{-s}}.$$

As a result we find

$$\frac{q+1}{q}\,\varphi(\delta) = q^{-ns-n} + (1 - q^{-1})q^{-n}\,\frac{q^{ns} - q^{-ns}}{1 - q^{-s}} + q^{ns-n-1}.$$

After elementary transformations we obtain the following final form for a spherical function:

$$\varphi(\delta) = q^{-n}\,\frac{q^{1/2}(q^{(n+1/2)s} - q^{-(n+1/2)s}) - q^{-1/2}(q^{(n-1/2)s} - q^{-(n-1/2)s})}{(q^{s/2} - q^{-s/2})(q^{1/2} + q^{-1/2})}.$$

11. The Operator of the Horospherical Automorphism. Following Chapter 1 we define the horospherical subgroups of G as the subgroup Z of matrices of the form $\begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix}$ and all subgroups conjugate to Z. Horospheres in a homogeneous space X relative to G are orbits of horospherical subgroups. Thus, every horosphere on X consists of the points of the form

$$x_z = x_0 g_1 z g_2, \tag{1}$$

where x_0 is a fixed point in X, g_1 and g_2 fixed elements of G, and z ranges over Z.

From the definition it follows that every transitive family of horospheres on X either coincides with the space of cosets $\Omega = Z \setminus G$, or is obtained from Ω by an additional identification of points. We call Ω the *space of horospheres*. This space Ω is isomorphic to a two-dimensional affine space over $\mathbf{K}$ from which the origin has been deleted.

Let us find all the horospheres in Ω. We give them by formula (1), where x_0 is the point of Ω corresponding to the unit class. We consider the matrix

$$g_1 = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

in (1). Let us show that *if* $\beta = 0$, *then the horosphere* (1) *degenerates to a point*. For in this case we have $g_1 z = z' g_1$, where $z' \in Z$. Consequently, $x_z = x_0 g_1 g_2$ for every z, because $x_0 z = x_0$.

Now let $\beta \neq 0$. Then g_1 may be represented in the form

$$g_1 = z_1 s\, \delta z_2.$$

where $z_1, z_2 \in Z$, δ is a diagonal matrix, and

$$s = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{2}$$

Thus, the equation of the horosphere (1) takes the following form:

$$x_z = x_0 s z\, \delta g_2. \tag{3}$$

So we see that *nondegenerate horospheres in* Ω *form a homogeneous family*. For they are all obtained by group translations from $x_z = x_0 s z$.

When we go over in (3) to coordinates and bear in mind that $x_0 = (1, 0)$, we obtain the following equation of horospheres:

$$x_1 = \alpha z + \gamma, \quad x_2 = \beta z + \delta, \quad z \in \mathbf{K}, \quad \alpha\delta - \beta\gamma = 1. \tag{4}$$

Thus, *the horospheres in the space* Ω *of points* $x = (x_1, x_2)$, $x \neq 0$, *are all the lines that do not pass through the origin of coordinates.*

Let $\varphi(x)$ be a test function in Ω. We associate with it integrals of $\varphi(x)$ over all the horospheres (that is, lines) in Ω:

$$\psi(g) = \int_Z \varphi(x_0 s z g)\, dz. \tag{5}$$

Observe that $\psi(zg) = \psi(g)$ for every $z \in Z$. Thus, ψ may be regarded as a function in the space $\Omega = Z \setminus G$, and we can write $\psi(x)$ instead of $\psi(g)$.

Hence, the map

$$B: \varphi(x) \to \psi(x) \tag{6}$$

carries functions on Ω again into functions on Ω. We call this map B the *horospherical automorphism.*

In coordinate notation the operator B is given, as is easy to see, by the following formula:

$$B\varphi(x_1, x_2) = \int_K \varphi(x_1 z + y_1, x_2 z + y_2)\, dz. \tag{7}$$

where y_1 and y_2 are arbitrary elements from $\mathbf{K}$ connected with x_1 and x_2 by the relation

$$x_1 y_2 - x_2 y_1 = 1.$$

For example, when $x_1 \neq 0$, (7) can be written in the form

$$B\varphi(x_1, x_2) = \int_K \varphi(x_1 z, x_2 z + x_1^{-1})\, dz. \tag{8}$$

The fundamental properties of B are the following:

1. *The operator B commutes with the operators of group translation $f(x) \to f(xg)$.*

This follows immediately from (5).

2. *The operator B carries homogeneous functions of weight π into homogeneous functions*† *of weight π^{-1}.*

This follows immediately from (8).

From property 2 of B it follows that *the operator B^2 carries every space $\mathscr{D}_\pi$ of homogeneous functions into itself.* Since $\mathscr{D}_\pi$ is irreducible and B^2 commutes with the representation operators, this operator is a multiple of the unit operator on each space $\mathscr{D}_\pi$.

$$B^2 \varphi_\pi = \lambda(\pi)\varphi_\pi$$

for every function $\varphi_\pi \in \mathscr{D}_\pi$.

Our main task is to compute the factor of proportionality $\lambda(\pi)$. In this subsection we find $\lambda(\pi)$ for the field of real numbers and for a disconnected field; see formulae (15) and (30).

We introduce two homogeneous function of weight π. To construct them we extend the character π to a multiplicative character on the quadratic extension $\mathbf{K}(\sqrt{\varepsilon})$ of $\mathbf{K}$. We denote the character so obtained, as before, by π. We set

$$\varphi_\pi^{(1)}(x, y) = \pi(x + \sqrt{\varepsilon}\, y)\, |x + \sqrt{\varepsilon}\, y|^{-1},$$

$$\varphi_\pi^{(2)}(x, y) = \pi(x - \sqrt{\varepsilon}\, y)\, |x - \sqrt{\varepsilon}\, y|^{-1},$$

where $x, \gamma \in \mathbf{K}$.

---

† That is, functions satisfying the relation

$$\varphi(tx) = \pi(t)\, |t|^{-1}\varphi(x), \qquad t \in \mathbf{K}$$

Let us compute $B\varphi_\pi^{(1)}$ and $B\varphi_\pi^{(2)}$.

To begin with we consider the field **K** of real numbers. Here we have ($\sqrt{\varepsilon} = i$):

$$= \int_{-\infty}^{+\infty} \pi(xz + i(yz + x^{-1}))\, |xz + i(yz + x^{-1})|^{-1}\, dz.$$

We transform this integral. We have

$$xz + i(yz + x^{-1}) = (x + iy)\left(z + \frac{i + x^{-1}y}{x^2 + y^2}\right)$$
$$= (x - iy)^{-1}\left[(x^2 + y^2)\left(z + \frac{x^{-1}y}{x^2 + y^2}\right) + i\right].$$

Consequently, after a suitable change of variable we find

$$B\varphi_\pi^{(1)}(x, y) = \pi^{-1}(x - iy)\, |x - iy|^{-1} \int_{-\infty}^{+\infty} \pi(z + i)\, |z + i|^{-1}\, dz.$$

Thus,

$$B\varphi_\pi^{(1)}(x, y) = \lambda^{(1)}(\pi)\, \varphi_{\pi^{-1}}^{(2)}(x, y), \tag{9}$$

where

$$\lambda^{(1)}(\pi) = \int_{-\infty}^{+\infty} \pi(z + i)\, |z + i|^{-1}\, dz. \tag{10}$$

The integral (10) can be expressed directly in terms of the classical Beta-function. For let

$$\pi(x) = |x|^s \operatorname{sign}^\nu x, \qquad \nu = 0, 1.$$

We extend the character π to the field of complex numbers by the following formula:

$$\pi(z) = |z|^s e^{\nu \arg z}. \tag{11}$$

It is not hard to check that then

$$\lambda^{(1)}(\pi) = \begin{cases} B\left(-\dfrac{s}{2}, \dfrac{1}{2}\right), & \text{when } \nu = 0, \\ iB\left(-\dfrac{s-1}{2}, \dfrac{1}{2}\right), & \text{when } \nu = 1. \end{cases} \tag{12}$$

Similarly we find that

$$B\varphi_\pi^{(2)}(x, y) = \lambda^{(2)}(\pi)\, \varphi_{\pi^{-1}}^{(1)}(x, y), \tag{13}$$

where

$$\lambda^{(2)}(\pi) = (-1)^\nu \lambda^{(1)}(\pi).$$

From this formulae it follows that

$$B^2\varphi_\pi^{(1)} = \lambda(\pi)\,\varphi_\pi^{(1)}, \qquad B^2\varphi_\pi^{(2)} = \lambda(\pi)\,\varphi_\pi^{(2)}, \tag{14}$$

where

$$\lambda(\pi) = \lambda^{(1)}(\pi)\,\lambda^{(2)}(\pi^{-1}) = \begin{cases} -\dfrac{2\pi}{s}\cot\dfrac{\pi s}{2}, & \text{when } \nu = 0. \\[2ex] \dfrac{2\pi}{s}\tan\dfrac{\pi s}{2}, & \text{when } \nu = 1. \end{cases} \tag{15}$$

Now we consider a disconnected field **K**. In this case we have

$$B\varphi_\pi^{(1)}(x,y)$$

$$= \int_{\mathbf{K}} \pi(xz + \sqrt{\varepsilon}\,(yz + x^{-1}))\,|xz + \sqrt{\varepsilon}\,(yz + x^{-1})|^{-1}\,dz.$$

As in the case of the field of real numbers, we find

$$B\varphi_\pi^{(1)}(x,y) = \lambda^{(1)}(\pi)\,\varphi_{\pi^{-1}}^{(2)}(x,y), \tag{16}$$

where

$$\lambda^{(1)}(\pi) = \int_{\mathbf{K}} \pi(z + \sqrt{\varepsilon})\,|z + \sqrt{\varepsilon}|^{-1}\,dz. \tag{17}$$

Now we compute the integral (17).

The character $\pi(x)$ is given by the following formula:

$$\pi(x) = |x|^s\,\theta(x), \tag{18}$$

where s is a complex number and $\theta(\mathfrak{p}) = 1$.

We begin with $\theta(x) \equiv 1$. Then we have

$$\lambda^{(1)}(\pi) = \int_{|x|>1} |x|^{s-1}\,dx + \int_{|x|\le 1} dx = \frac{1 - q^{s-1}}{1 - q^s}. \tag{19}$$

Now we consider the case $\theta \not\equiv 1$. Let n be the rank of θ in **K**.

We recall that the rank is the smallest natural number n for which

$$\theta(1 + \mathfrak{p}^n x) \equiv 1, \qquad |x| \le 1\,.$$

The rank of θ is the same, whether regarded as a character on **K** or on $\mathbf{K}(\sqrt{\varepsilon})$.

First we show that

$$I_k = \int_{|z|=q^k} \pi(z + \sqrt{\varepsilon})\,|z + \sqrt{\varepsilon}|^{-1}\,dz = 0 \qquad \text{for } k > 0.$$

For we have

$$I_k = q^{sk} \int_{|t|=1} \theta(\mathfrak{p}^{-k}t + \sqrt{\varepsilon})\, dt = p^{sk} \int_{|t|=1} \theta(t + \mathfrak{p}^k\sqrt{\varepsilon})\, dt.$$

Hence it follows that for every $x \in \mathbf{K}$, $|x| \leq 1$,

$$I_k\theta(1 + \mathfrak{p}^{n-1}x) = q^{sk} \int_{|t|=1} \theta(t(1 + \mathfrak{p}^{n-1}x) + \mathfrak{p}^k\sqrt{\varepsilon})\, dt$$

$$= q^{sk} \int_{|t|=1} \theta(t + \mathfrak{p}^k\sqrt{\varepsilon}) = I_k.$$

But $\theta(1 + \mathfrak{p}^{n-1}x) \not\equiv 1$. Consequently, $I_k = 0$. By what we have proved we obtain the following expression for $\lambda^{(1)}(\pi)$:

$$\lambda^{(1)}(\pi) = \int_{|z|\leq 1} \theta(z + \sqrt{\varepsilon})\, dz = q^{-n} \sum_{z\in O/\mathfrak{p}^nO} \theta(z + \sqrt{\varepsilon}). \tag{20}$$

On the basis of this formula we can show that

$$|\lambda^{(1)}(\pi)|^2 = q^{-n}. \tag{21}$$

Proof. We have

$$|\lambda^{(1)}(\pi)|^2 = q^{-2n} \sum_{z,u\in O/\mathfrak{p}^nO} \theta\left(\frac{z + \sqrt{\varepsilon}}{u + \sqrt{\varepsilon}}\right).$$

(z and u range over a set consisting of one representative from each coset of $O/\mathfrak{p}^nO$.) In this sum we separate the term with $z = u$. We find

$$|\lambda^{(1)}(\pi)|^2 = q^{-n} + q^{-2n} \sum_{z\neq u} \theta\left(\frac{z + \sqrt{\varepsilon}}{u + \sqrt{\varepsilon}}\right).$$

We show now that the second term is zero. For this purpose we consider the set of values mod $\mathfrak{p}^n$ of $\dfrac{z + \sqrt{\varepsilon}}{u + \sqrt{\varepsilon}}$. It is not hard to verify that this set is preserved under multiplication by elements of the form $x_s = 1 + \mathfrak{p}^{n-1}s$, $|s| \leq 1$. Hence,

$$\sum_{z\neq u} \theta\left(\frac{z + \sqrt{\varepsilon}}{u + \sqrt{\varepsilon}}\right) = \sum_{z\neq u} \theta\left(x_s \frac{z + \sqrt{\varepsilon}}{u + \sqrt{\varepsilon}}\right) = \pi(x_s) \sum_{z\neq u} \theta\left(\frac{z + \sqrt{\varepsilon}}{u + \sqrt{\varepsilon}}\right).$$

Since $\theta(x_s) \not\equiv 1$, it follows immediately that

$$\sum_{z\neq u} \theta\left(\frac{z + \sqrt{\varepsilon}}{u + \sqrt{\varepsilon}}\right) = 0.$$

We give another derivation of the formula (21) based on results of **§ 2.6.**

We introduce the Gamma-function $\Gamma_\varepsilon(\pi)$ in $\mathbf{K}(\sqrt{\varepsilon})$ by the following formula:

$$\Gamma_\varepsilon(\pi) = \int_{K(\sqrt{\varepsilon})} \chi\left(\frac{t - \bar{t}}{2\sqrt{\varepsilon}}\right) \pi(t)\, d^*t, \tag{22}$$

and show that

$$\lambda^{(1)}(\pi) = \frac{\Gamma_\varepsilon(\pi\pi_0)}{\Gamma(\pi\pi_0)}, \tag{23}$$

where $\pi_0(x) = |x|$.

By the substitution $t = xy + \sqrt{\varepsilon}\, y$, where $x, y \in \mathbf{K}$, the integral (22) reduces to the form:

$$\Gamma_\varepsilon(\pi) = \int \chi(y)\pi(y)\,|y|^{-1}\,dy \cdot \int \pi(x + \sqrt{\varepsilon})\,|x + \sqrt{\varepsilon}|^{-2}\,dx.$$

Equation (23) follows immediately from this.

To obtain (21) from (23) we use the following formula, which was derived in **§ 2.6**:

$$|\Gamma(\pi)| = q^{n(\operatorname{Re} s - \frac{1}{2})}. \tag{24}$$

We show that

$$|\Gamma_\varepsilon(\pi)| = q^{n(\operatorname{Re} s - 1)}. \tag{24'}$$

Indeed in passing from $\mathbf{K}$ to $\mathbf{K}(\sqrt{\varepsilon})$ the number q (the order of the residue class field O/P) is replaced by q^2, and the rank of the character π is preserved. Hence, by (24), we have $|\Gamma_\varepsilon(\pi)| = q^{2n(\operatorname{Re} s/2 - 1/2)} = q^{n(\operatorname{Re} s - 1)}$.

(21) is an immediate consequence of (23), (24), and (24').

We have now obtained the final formula for the operator B: If $\pi(x) = |x|^s\,\theta(x)$, $\theta(\mathfrak{p}) = 1$, then

$$B\varphi_\pi^{(1)}(x, y) = \lambda^{(1)}(\pi)\,\varphi_{\pi^{-1}}^{(2)}(x, y), \tag{25}$$

where

$$\lambda^{(1)}(\pi) = \begin{cases} \dfrac{1 - q^{s-1}}{1 - q^s}, & \text{when } \theta(x) \equiv 1, \\ q^{-n/2}\mu^{(1)}(\pi), & \text{when } \theta(x) \not\equiv 1. \end{cases} \tag{26}$$

Here n is the rank of θ, and

$$|\mu^{(1)}(\pi)| = 1.$$

The computation of $\mu^{(1)}(\pi)$ is a task of considerably greater complexity. However, for our purposes the value of $\mu^{(1)}(\pi)$ is not required.

Similarly, we have

$$B\varphi_\pi^{(2)}(x, y) = \lambda^{(2)}(\pi)\,\varphi_{\pi^{-1}}^{(1)}(x, y). \tag{27}$$

Next we show that the functions $\lambda^{(2)}(\pi)$ and $\lambda^{(1)}(\pi)$ are connected by the following relation:

$$\lambda^{(1)}(\pi)\,\lambda^{(2)}(\pi^{-1}) = \pi(-1)q^{-n}. \tag{28}$$

For by analogy with (20) we have

$$\begin{aligned} \lambda^{(2)}(\pi^{-1}) &= \int\limits_{|z| \le 1} \theta^{-1}(z - \sqrt{\varepsilon})\,dz \\ &= \theta(-1) \int\limits_{|z| \le 1} \theta^{-1}(z + \sqrt{\varepsilon})\,dz = \theta(-1)\,\overline{\lambda^{(1)}(\pi)}. \end{aligned}$$

Consequently,

$$\lambda^{(1)}(\pi)\lambda^{(2)}(\pi^{-1}) = \theta(-1)\,|\lambda^{(1)}(\pi)|^2 = \theta(-1)q^{-n}.$$

On the basis of these results we have

$$B^2\varphi_\pi^{(1)} = \lambda(\pi)\,\varphi_\pi^{(1)}, \qquad B^2\varphi_\pi^{(2)} = \lambda(\pi)\,\varphi_\pi^{(2)}, \tag{29}$$

where

$$\lambda(\pi) = \lambda^{(1)}(\pi)\lambda^{(2)}(\pi^{-1}) = \begin{cases} \dfrac{(1-q^{s-1})(1-q^{-s-1})}{(1-q^s)(1-q^{-s})}, & \text{when } \theta \equiv 1, \\ q^{-n}\pi(-1), & \text{when the rank of } \theta \text{ is } n \neq 0, \end{cases} \tag{30}$$

that is, in accordance with **§ 2.6**

$$\lambda(\pi) = \Gamma(\pi)\,\Gamma(\pi^{-1}). \tag{31}$$

§ 4. THE DISCRETE SERIES OF IRREDUCIBLE UNITARY REPRESENTATIONS OF G

1. Description of the Representations of the Discrete Series. Here we show that with every quadratic extension $\mathbf{K}(\sqrt{\tau})$ of $\mathbf{K}$, a certain discrete series of irreducible unitary representations of G is connected. Thus, for the field of real numbers there is one discrete series, and for a disconnected field there are three discrete series of irreducible unitary representations of G.

As a preliminary we recall the formula for the operators of the continuous series in the χ-realization. If $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$, then the corresponding representation operator $T_\pi(g)$ is given by the formula

$$T_\pi(g)\,\varphi(u) = \int K_\pi(g \mid u, v)\,\varphi(v)\,dv,$$

where

$$K_\pi(g \mid u, v) = |\beta|^{-1}\,\chi\left(\frac{\delta u + \alpha v}{\beta}\right)\int \chi\left(-\frac{1}{\beta}(ut + vt^{-1})\right)\pi(t)\,d^*t,$$

when $\beta \neq 0$;

$$K_\pi(g \mid u, v) = \pi(\delta)\,|\delta|\,\chi(\delta\gamma u)\,\delta(\delta^2 u - v),$$

when $\beta = 0$.

Here π is the multiplicative character on $\mathbf{K}$ that generates the representation.

We define the representations of the discrete series by similar formulae.

Let $\mathbf{K}(\sqrt{\tau})$ *be a quadratic extension of* $\mathbf{K}$, $\pi(t)$ *a multiplicative character on* $\mathbf{K}(\sqrt{\tau})$. *We consider the space* H *of functions* $\varphi(u)$ *on* $\mathbf{K}$ *for which*

$$(\varphi, \varphi) = \int |\varphi(u)|^2 \, du < \infty.$$

With every matrix $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ *we associate the operator* $T_\pi(g)$ *that is defined in* H *by the following formula:*

$$T_\pi(g)\,\varphi(u) = \int K_\pi(g \mid u, v)\,\varphi(v)\,dv, \tag{1}$$

where

$$K_\pi(g \mid u, v) = a_\tau c_\tau \frac{\operatorname{sign}_\tau \beta}{|\beta|} \operatorname{sign}_\tau u\,\chi\left(\frac{\delta u + \alpha v}{\beta}\right)$$

$$\int\limits_{t\bar{t}=vu^{-1}} \chi\left(-\frac{1}{\beta}(ut + vt^{-1})\right)\pi(t)\,d^*t, \tag{2}$$

when $\beta \neq 0$, $\operatorname{sign}_\tau u = \operatorname{sign}_\tau v$;

$$K_\pi(g \mid u, v) = 0, \tag{3}$$

when $\operatorname{sign}_\tau u \neq \operatorname{sign}_\tau v$;

$$K_\pi(g \mid u, v) = \operatorname{sign}_\tau \delta \cdot \pi(\delta)\,|\delta|\,\chi(\delta\gamma u)\,\delta(\delta^2 u - v), \tag{4}$$

when $\beta = 0$.

Here d^*t denotes the measure that is uniquely determined on the circle $t\bar{t} = vu^{-1}$ by the condition $d^*(tt_0) = d^*t$ for every t_0 with $t_0\bar{t}_0 = 1$; $\int d^*t = 1$; $a_\tau = 2(1 + q^{-1})(1 + |\tau|)^{-1}$. The coefficient c_τ is determined by the formula

$$c_\tau^{-1} = \int \chi(t\bar{t})\,dt, \tag{5}$$

where the integration is taken over the plane $\mathbf{K}(\sqrt{\tau})$. The precise meaning and the value of this integral were indicated in **§ 2.7.**

In **§ 4.3** and **§ 4.4** we shall show that *the operators* $T_\pi(g)$ *form a unitary representation of* G.

We make some preliminary remarks on the representations $T_\pi(g)$:

1. We see that the operators $T_\pi(g)$ are defined essentially by the same formulae as those for the representation operators of the continuous series. The only important difference is that the integration in (2) taken not over a "line," but over the circle

$t\bar{t} = vu^{-1}$ on the plane $\mathbf{K}(\sqrt{\tau})$. The points t of this circle are characterized by the condition that $ut + vt^{-1}$ must belong to $\mathbf{K}$.

2. In **§ 4.6** we shall show that if $\pi_1 = \pi_2$ on the circle $t\bar{t} = 1$, then the representations $T_{\pi_1}(g)$ and $T_{\pi_2}(g)$ are equivalent. Hence the representations $T_\pi(g)$ are, in fact, given by the characters on $t\bar{t} = 1$, and so the set of these representations is discrete. This is the explanation of the name "discrete series."

3. The representations $T_\pi(g)$ are reducible. For let H^+ be the subspace of functions $\varphi(u)$ for which $\varphi(u) = 0$ when $\operatorname{sign}_\tau u = -1$; H^- the subspace of functions for which $\varphi(u) = 0$ when $\operatorname{sign}_\tau u = 1$.

From the formulae for the representation operators it is immediately clear that H^+ and H^- are invariant subspaces.

From now on we denote the representations in H^+ and H^-, respectively, by $T_\pi^+(g)$ and $T_\pi^-(g)$. These representations are irreducible (see **§ 4.6**).

So we see that *every discrete series of irreducible unitary representations consists of two halves—the representations* $T_\pi^+(g)$ *and* $T_\pi^-(g)$. *The first are realized in the subspace of functions* $\varphi(u)$ *for which* $\varphi(u) = 0$ *when* $\operatorname{sign}_\tau u = -1$; *the second in the supplementary subspace.*

A similar series of representations arises in the case of a finite field $\mathbf{K}_q$. We assume that the characteristic of $\mathbf{K}_q$ is different from two. Then $\mathbf{K}_q$ has precisely one quadratic extension. The series of representations connected with this extension is realized on the functions $\varphi(u)$, where u ranges over the elements of $\mathbf{K}_q$ other than zero. The representation operator has the form

$$T_\pi(g)\varphi(u) = \sum_{v \neq 0} K_\pi(g \mid u, v)\varphi(v),$$

where

$$K_\pi(g \mid u, v) = -\chi\left(\frac{\delta u + \alpha v}{\beta}\right) \sum_{t\bar{t}=vu^{-1}} \chi\left(-\frac{1}{\beta}(ut + vt^{-1})\right)\pi(t),$$

when $\beta \neq 0$;

$$K_\pi(g \mid u, v) = \pi(\delta)\chi(\delta\gamma u)\delta(\delta^2 u - v),$$

when $\beta = 0$. Here $\delta(u)$ is the Delta-function: $\delta(u) = 0$ for $u \neq 0$, $\delta(0) = 1$.

In contrast to an infinite field, the representations $T_\pi(g)$ turn out to be irreducible (except when $\pi(x) = \pm 1$). It can be shown that $T_\pi(g)$ and $T_{\pi^{-1}}(g)$ are equivalent representations.

2. The Continuous Dependence of the Operators $T_\pi(g)$ on g.

The operators $T_\pi(g)$ corresponding to the matrices $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ were defined by different formulae in the cases $\beta \neq 0$ and $\beta = 0$.

We show now that *the formula for* $T_\pi(g)$ *in the special case* $\beta = 0$ *is obtained by a limit process from the formula for* $T_\pi(g)$ *corresponding to a matrix in general position.* In this way we establish that the operators $T_\pi(g)$ depend continuously on g.

As a preliminary we put the formulae for the operators into a somewhat different form.

By **§ 4.1** the operator $T_\pi(g)$, $\beta \neq 0$, is given by the following formula:

$$T_\pi(g)\,\varphi(u) = a_\tau c_\tau \frac{\operatorname{sign}_\tau \beta}{|\beta|} \operatorname{sign}_\tau u \int\limits_{t\bar{t}=vu^{-1}} \chi\left(\frac{\delta u + \alpha v}{\beta} - \frac{1}{\beta}(ut + vt^{-1})\right) \pi(t)\,\varphi(v)\, d^*t\, dv, \tag{1}$$

where the integration is with respect to t is taken over the circle $t\bar{t} = vu^{-1}$. When we substitute $v = ut\bar{t}$ in the integral, we may rewrite the formula as follows:

$$T_\pi(g)\,\varphi(u) = c_\tau \frac{\operatorname{sign}_\tau \beta}{|\beta|} |u| \operatorname{sign}_\tau u \int \chi\left(\frac{u}{\beta}(\delta + \alpha t\bar{t} - t - \bar{t})\right) \pi(t)\,\varphi(ut\bar{t})\, dt, \tag{2}$$

where the integration is taken over the whole plane $\mathbf{K}(\sqrt{\tau})$. We make the change of variable: $t = \beta t' + \delta$. After elementary transformations we find

$$T_\pi(g)\,\varphi(u) = c_\tau\, |\beta| \operatorname{sign}_\tau \beta |u| \operatorname{sign}_\tau u\, \chi(\delta\gamma u)$$
$$\int \chi[(\alpha\beta t\bar{t} + \gamma\beta(t + \bar{t}))u] \times \pi(\beta t + \delta)\,\varphi(u(\beta t + \delta)(\beta\bar{t} + \delta))\, dt. \tag{3}$$

Let us see what the limiting value of this expression is, as $\beta \to 0$. We assume that $\operatorname{sign}_\tau \beta$ remains constant. Let β_0 be a fixed element such that $\operatorname{sign}_\tau \beta_0 = \operatorname{sign}_\tau \beta$. Then we have

$$\beta = \beta_0 \sigma \bar{\sigma},$$

where σ is an element from $\mathbf{K}(\sqrt{\tau})$. In the integral (3) we make the change of variable $t = \bar{\sigma}^{-1} t'$, and find

$$T_\pi(g)\,\varphi(u) = c_\tau\, |\beta_0| \operatorname{sign}_\tau \beta_0\, |u| \operatorname{sign}_\tau u \cdot \chi(\delta\gamma u)$$
$$\int \chi[u(\alpha\beta_0 t\bar{t} + \gamma\beta_0(\sigma t + \bar{\sigma}\bar{t}))]\pi(\beta_0\sigma t + \delta)\,\varphi(u(\beta_0\sigma t + \delta)(\beta_0\bar{\sigma}\bar{t} + \delta))\, dt. \tag{4}$$

We are interested in the limit of this expression, as $\sigma \to 0$. Let us perform a formal limit passage under the integral sign. Then we obtain

$$T_\pi(g)\,\varphi(u) = c_\tau\, |\beta_0| \operatorname{sign}_\tau \beta_0\, |u| \operatorname{sign}_\tau u \cdot \chi(\delta\gamma u)\pi(\delta)\,\varphi(\delta^2 u) \int \chi(u\alpha\beta_0 t\bar{t})\, dt. \tag{5}$$

However,

$$\int \chi(u\alpha\beta_0 t\bar{t})\,dt = c_\tau^{-1}\frac{\operatorname{sign}_\tau(u\alpha\beta_0)}{|u\alpha\beta_0|} + \frac{a_\tau}{2}\,\delta(\alpha\beta_0 u)$$

(see **§ 2.7**). Substituting this expression in (5) and bearing in mind that in the limit matrix $\alpha = \delta^{-1}$, we find

$$T_\pi(g)\,\varphi(u) = \operatorname{sign}_\tau \delta \cdot \pi(\delta)\,|\delta|\,\chi(\delta\gamma u)\,\varphi(\delta^2 u),$$

that is,

$$T_\pi(g)\,\varphi(u) = \operatorname{sign}_\tau \delta \cdot \pi(\delta)\,|\delta|\,\chi(\delta\gamma u)\int \delta(\delta^2 u - v)\,\varphi(v)\,dv.$$

So we have obtained precisely the formula (4) of **§ 4.1** for the operator $T_\pi(g)$ corresponding to the matrix $g = \begin{pmatrix}\alpha & 0\\ \gamma & \delta\end{pmatrix}$.

The limiting process, as $\sigma \to 0$, as we have carried it out, is not completely rigorous. To make the argument rigorous we have to introduce instead of the $T_\pi(g)$ auxiliary operators by adding the factor $|u(\alpha\beta_0 t\bar{t} + \gamma\beta_0(\sigma t + \bar{\sigma}\bar{t}))|^\lambda$ under the integral (4) (λ is a complex number). We split the integral so obtained into two terms: over $|t| < 1$ and over $|t| \geq 1$. It is easy to see that for each of these integrals there is a domain of values λ for which it converges absolutely and uniformly in σ, as $\sigma \to 0$; and then the limit passage, as $\sigma \to 0$, under the integral sign is possible. We do not wish to go into details of this common technique in generalized functions here.

3. Proof of the Relation $T_\pi(g_1 g_2) = T_\pi(g_1)\cdot T_\pi(g_2)$**.** We show that the operators $T_\pi(g)$ actually give a representation of G, that is,

$$T_\pi(g_1 g_2) = T_\pi(g_1)\,T_\pi(g_2) \tag{1}$$

for arbitrary matrices g_1 and g_2 from G.

The operators $T_\pi(g_1)$, $T_\pi(g_2)$, and $T_\pi(g_1 g_2)$ are given, respectively, by the kernels $K_\pi(g_1 \mid u, v)$, $K_\pi(g_2 \mid u, v)$, and $K_\pi(g_1 g_2 \mid u, v)$. So we must show that

$$\int K_\pi(g_1 \mid u, w)K_\pi(g_2 \mid w, v)\,dw = K_\pi(g_1 g_2 \mid u, v). \tag{2}$$

Let

$$g_1 = \begin{pmatrix}\alpha_1 & \beta_1\\ \gamma_1 & \delta_1\end{pmatrix},\qquad g_2 = \begin{pmatrix}\alpha_2 & \beta_2\\ \gamma_2 & \delta_2\end{pmatrix},\qquad g_1 g_2 = \begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}.$$

It is sufficient to discuss the case $\beta_1 \neq 0$, $\beta_2 \neq 0$, $\beta \neq 0$. For the special cases when at least one of the elements β_1, β_2, β is equal to zero, the relation (1) can be obtained later by a limit process.

When we substitute in (2) the expressions for the kernels from **§ 4.1** we find

$$\mathscr{T} = \int K_\pi(g_1 \mid u, w) K_\pi(g_2 \mid w, v)\, dw$$

$$= a_\tau^2 c_\tau^2 \operatorname{sign}_\tau (\beta_1\beta_2)\, |\beta_1\beta_2|^{-1}\, \chi\left(\frac{\delta_1 u}{\beta_1} + \frac{\alpha_2 v}{\beta_2}\right)$$

$$\int \int_{t\bar{t}=w/u} \int_{s\bar{s}=v/w} \chi\left(-\frac{u}{\beta_1}(t+\bar{t}) - \frac{w}{\beta_2}(s+\bar{s}) + \frac{\alpha_1 w}{\beta_1} + \frac{\delta_2 w}{\beta_2}\right) \pi(ts)\, d^*s\, d^*t\, dw. \tag{3}$$

We make the change of variable $s = t^{-1}\sigma$ and obtain

$$\mathscr{T} = a_\tau^2 c_\tau^2 \operatorname{sign}_\tau (\beta_1\beta_2)\, |\beta_1\beta_2|^{-1}\, \chi\left(\frac{\delta_1 u}{\beta_1} + \frac{\alpha_2 v}{\beta_2}\right)$$

$$\int_{\sigma\bar{\sigma}=vu} \int \int_{t\bar{t}=wu} \chi\left(-\frac{u}{\beta_1}(t+\bar{t}) - \frac{u}{\beta_2}(\sigma\bar{t}+\bar{\sigma}t) + \frac{\beta}{\beta_1\beta_2} w\right) \pi(\sigma)\, d^*t\, dw\, d^*\sigma. \tag{4}$$

We compute the inner integral

$$I = \int \int_{\bar{t}t=w/u} \chi\left(-\frac{u}{\beta_1}(t+\bar{t}) - \frac{u}{\beta_2}(\sigma\bar{t}+\bar{\sigma}t) + \frac{\beta}{\beta_1\beta_2} w\right) d^*t\, dw$$

separately.

Substituting $w = ut\bar{t}$ in the integral we may rewrite it as an integral over the plane $\mathbf{K}(\sqrt{\tau})$:

$$I = a_\tau^{-1} |u| \int \chi\left(-u(\bar{a}t + a\bar{t}) + u \frac{\beta}{\beta_1\beta_2} t\bar{t}\right) dt,$$

where $a = \dfrac{1}{\beta_1} + \dfrac{\sigma}{\beta_2}$.

We make the substitution $t = t' + \dfrac{\beta_1\beta_2}{\beta} a$ and obtain

$$I = a_\tau^{-1} |u|\, \chi\left(-u \frac{\beta_1\beta_2}{\beta} a\bar{a}\right) \int \chi\left(u \frac{\beta}{\beta_1\beta_2} t\bar{t}\right) dt.$$

However,

$$\int \chi\left(u \frac{\beta}{\beta_1\beta_2} t\bar{t}\right) dt = c_\tau^{-1} \operatorname{sign}_\tau \left(\frac{u\beta}{\beta_1\beta_2}\right) \left|\frac{u\beta}{\beta_1\beta_2}\right|^{-1} + \frac{a_\tau}{2}\, \delta\left(\frac{\beta}{\beta_1\beta_2} u\right).$$

Consequently,

$$I = a_\tau^{-1} c_\tau^{-1} \operatorname{sign}_\tau \left(\frac{u\beta}{\beta_1\beta_2}\right) \left|\frac{\beta_1\beta_2}{\beta}\right| \chi\left(-u \frac{\beta_1\beta_2}{\beta} a\bar{a}\right)$$

$$= a_\tau^{-1} c_\tau^{-1} \operatorname{sign}_\tau \left(\frac{u\beta}{\beta_1\beta_2}\right) \left|\frac{\beta_1\beta_2}{\beta}\right| \chi\left(-\frac{\beta_2}{\beta_1\beta_2} u - \frac{u}{\beta}(\sigma - \bar{\sigma}) - \frac{\beta_1}{\beta_2\beta} v\right).$$

We substitute this expression in (4) and use the easily verified relations

$$\frac{\delta_1}{\beta_1} - \frac{\beta_2}{\beta_1\beta} = \frac{\delta}{\beta}, \qquad \frac{\alpha_2}{\beta_2} - \frac{\beta}{\beta_2\beta} = \frac{\alpha}{\beta}.$$

So we find

$$\mathscr{T} = a_\tau c_\tau \frac{\operatorname{sign}_\tau \beta}{|\beta|} \operatorname{sign}_\tau u \cdot \chi\left(\frac{\delta u + \alpha v}{\beta}\right) \int_{\sigma\bar{\sigma} = v/u} \chi\left(-\frac{u}{\beta}(\sigma + \bar{\sigma})\right) \pi(\sigma)\, d^*\sigma$$

$$= K_\pi(g_1 g_2 \mid u, v).$$

The relation (2) is now proved.

There is a certain lack of rigor in our arguments, because we have computed the integral (2), which diverges in the usual sense. This can be avoided by considering instead of $K_\pi(g \mid u, v)$ the auxiliary kernels

$$K_\pi(g \mid u, v \mid \lambda) = K_\pi(g \mid u, v)\, |v|^\lambda,$$

where λ is a complex number. We form the integral

$$\mathscr{T}_\lambda = \int K_\pi(g_1 \mid u, w \mid \lambda) K_\pi(g_2 \mid w, v \mid \lambda)\, dw.$$

We split it into two integrals—over $|w| < 1$ and over $|w| \geq 1$. It is easy to see that each of these integrals converges in a certain domain of values λ and is in this domain an analytic function of λ. So the integral $\mathscr{T}_\lambda$ is defined as an analytic function of λ. It can be shown that at $\lambda = 0$ the function $\mathscr{T}_\lambda$ is regular and that $\mathscr{T}_0 = K_\pi(g_1 g_2 \mid u, v)$.

4. Unitariness of the Operators $T_\pi(g)$. Let us show that *the operators $T_\pi(g)$ of the representations of the discrete series are unitary, that is,*

$$T_\pi^*(g) = T_\pi^{-1}(g)$$

where the asterisk denotes the adjoint operator.

For $T_\pi^*(g)$ is given by the kernel†

$$\overline{K_\pi(g \mid v, u)}$$

$$= a_\tau \bar{c}_\tau \frac{\operatorname{sign}_\tau \beta}{|\beta|} \operatorname{sign}_\tau v \cdot \chi\left(-\frac{\delta v + \alpha u}{\beta}\right) \int \chi\left(\frac{1}{\beta}(vt + ut^{-1})\right) \pi^{-1}(t)\, d^*t$$

$$= a_\tau c_\tau \frac{\operatorname{sign}_\tau(-\beta)}{|-\beta|} \operatorname{sign}_\tau u \cdot \chi\left(\frac{\alpha u + \delta v}{-\beta}\right) \int \chi\left(-\frac{1}{-\beta}(ut + vt^{-1})\right) \pi(t)\, d^*t^*).$$

† We use the relation $\bar{c}_\tau = c_\tau \operatorname{sign}_\tau(-1)$, see **§ 2.7**.

So we see that $\overline{K_\pi(g \mid v, u)} = K_\pi(g^{-1} \mid u, v)$, that is, $T_\pi^*(g) = T_\pi(g^{-1})$.

On the other hand, since the operators $T_\pi(g)$ form a representation, we have $T_\pi(g^{-1}) = T_\pi^{-1}(g)$. Therefore, $T_\pi^*(g) = T_\pi^{-1}(g)$, as required.

5. The π-Realization of the Representations of the Discrete Series. In this subsection and the next, we give two other realizations of the representations of the discrete series. We obtain the π-realization by going over from the functions $\varphi(u)$ to their Mellin transforms:

$$F(\pi_1) = \int \varphi(u)\pi_1^{-1}(u)\ |u|^{-\frac{1}{2}}\ du. \tag{1}$$

It is not hard to see that the kernels $\tilde{K}_\pi(g \mid \pi_1, \pi_2)$ of the operators $T_\pi(g)$ in the π-realization can be expressed in terms of their kernels $K_\pi(g \mid u, v)$ in the original realization of the representation by the following formula:

$$\tilde{K}_\pi(g \mid \pi_1, \pi_2) = \int K_\pi(g \mid u, v)\pi_1^{-1}(u)\ |u|^{-\frac{1}{2}}\pi_2(v)\ |v|^{-\frac{1}{2}}\ du\, dv. \tag{2}$$

Let us find the formulae in the π-realization for the representation operators corresponding to the matrices

$$\delta = \begin{pmatrix} \delta^{-1} & 0 \\ 0 & \delta \end{pmatrix}, \qquad z = \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix}, \qquad z \neq 0$$

and

$$\zeta = \begin{pmatrix} 1 & \zeta \\ 0 & 1 \end{pmatrix}, \qquad \zeta \neq 0.$$

For this purpose we need, apart from the Gamma-function connected with $\mathbf{K}$, the Gamma-function $\Gamma_\tau(\pi)$ connected with $\mathbf{K}(\sqrt{\tau})$,

$$\Gamma_\tau(\pi) = \int_{\mathbf{K}^*(\sqrt{\tau})} \chi_\tau(t)\pi(t)\ d^*t.$$

Here π ranges over the set of multiplicative characters on $\mathbf{K}(\sqrt{\tau})$, and $\chi_\tau(t)$ is an additive character on $\mathbf{K}(\sqrt{\tau})$, which can be expressed in terms of the character $\chi(x)$ on $\mathbf{K}$ by the formula

$$\chi_\tau(t) = \chi(t + \bar{t}).$$

We assume that all multiplicative characters on $\mathbf{K}$ are extended to multiplicative characters on $\mathbf{K}(\sqrt{\tau})$; we denote the latter by the same letters as those used before. Furthermore, let $\bar{\pi}(t)$ denote the character corresponding to the formula

$$\bar{\pi}(t) = \pi(\bar{t}).$$

We show that *the representation operators corresponding to the matrices* δ, z, *and* ζ *are given in the* π*-realization by the following formulae*:

$$T_\pi(\delta)F(\pi_1) = \pi\pi_1^2\pi_\tau(\delta)F(\pi_1); \tag{3}$$

$$T_\pi(z)F(\pi_1) = \pi_1\pi_2^{-1}(z)\int\Gamma(\pi_1^{-1}\pi_2)F(\pi_2)\,d\pi_2; \tag{4}$$

$$T_\pi(\zeta)F(\pi_1) = \int\frac{\Gamma_\tau(\pi\pi_2\bar{\pi}_2\pi_0^2)}{\Gamma_\tau(\pi\pi_1\bar{\pi}_0\pi_0^2)}\,\Gamma(\pi_1\pi_2^{-1})\pi_1^{-1}\pi_2(-\zeta)F(\pi_2)\,d\pi_2, \tag{5}$$

where $\pi_0^2(t) = |t\bar{t}|^{1/2}$.

These formulae are similar to the formulae for the representations of the principal series obtained in **§ 3.2**. Indeed, according to **§ 3.2** the formula for the operator $T_\pi(z)$ of the representation of the principal series is precisely the same as (4), but the operator $T_\pi(\zeta)$ of the representation of the principal series is given by the following formula:

$$T_\pi(\zeta)F(\pi_1) = \int\frac{\Gamma(\pi_2\pi_0)\,\Gamma(\pi\pi_2\pi_0)}{\Gamma(\pi_1\pi_0)\,\Gamma(\pi\pi_1\pi_0)}\,\Gamma(\pi_1\pi_2^{-1})\pi_1^{-1}\pi_2(-\zeta)F(\pi_2)\,d\pi_2.$$

(3) and (4) are easily obtained on the basis of the formula (4) in **§ 4.1** for the kernel $K_\pi(g \mid u, v)$. We give a derivation for (5). The kernel of $T_\pi(\zeta)$ is given in the π-realization by the following formula:

$$\tilde{K}_\pi(\zeta \mid \pi_1, \pi_2) = a_\tau c_\tau\frac{\operatorname{sign}_\tau\zeta}{|\zeta|}$$

$$\iint_{t\bar{t}=v/u}\chi\left(\frac{1}{\zeta}\,(u+v-u(t+\bar{t}))\right)\operatorname{sign}_\tau u\cdot\pi_1^{-1}(u)|u|^{-1/2}\pi_2(v)\;|v|^{-1/2}\pi(t)\;d^*t\,du\,dv.$$

After elementary transformations we find

$$\tilde{K}_\pi(\zeta \mid \pi_1, \pi_2) = c_\tau\pi_1^{-1}\pi_2(\zeta)$$

$$\iint\chi(u(1-t)(1-\bar{t}))\;\operatorname{sign}_\tau u\pi_1^{-1}\pi_2(u)\pi\pi_2\bar{\pi}_2(t)\;du\,dt.$$

By integrating over u we obtain

$$\tilde{K}_\pi(\zeta \mid \pi_1, \pi_2) = c_\tau\pi_1^{-1}\pi_2(\zeta)\,\Gamma(\pi_1^{-1}\pi_2\pi_\tau\pi_0^2)$$

$$\int\pi\pi_2\bar{\pi}_2(t)\pi_1\bar{\pi}_1\pi_1^{-1}(1-t)\;|(1-t)(1-\bar{t})|^{-1}\,dt.$$

where

$$\pi_\tau(x) = \operatorname{sign}_\tau x, \qquad x \in \mathbf{K}^*,$$
$$\pi_0(t) = |t\bar{t}|^{1/4}, \qquad t \in \mathbf{K}^*(\sqrt{\tau}).$$

The last integral is the Beta-function connected with $\mathbf{K}(\sqrt{\tau})$ and can therefore be expressed in terms of Γ_τ. As a result we find

$$\tilde{K}_\pi(\zeta \mid \pi_1, \pi_2) = c_\tau \pi_1^{-1}\pi_2(\zeta) \cdot \Gamma(\pi_1^{-1}\pi_2\pi_\tau\pi_0^2) \frac{\Gamma_\tau(\pi\pi_2\bar{\pi}_2\pi_0^2)\,\Gamma_\tau(\pi_1\bar{\pi}_1\pi_2^{-1}\bar{\pi}_2^{-1})}{\Gamma_\tau(\pi\pi_1\bar{\pi}_1\pi_0^2)}.$$

Since according to **§ 2.10**

$$\Gamma_\tau(\pi_1\bar{\pi}_1\pi_2^{-1}\bar{\pi}_2^{-1}) = |\tau|^{-1}c_\tau\Gamma(\pi_1\pi_2^{-1}\pi_\tau)\,\Gamma(\pi_1\pi_2^{-1}) = |\tau|^{-1}c_\tau\pi_1\pi_2^{-1}\pi_\tau(-1)\,\frac{\Gamma(\pi_1\pi_2^{-1})}{\Gamma(\pi_1^{-1}\pi_2\pi_\tau\pi_0^2)},$$

we finally obtain (because $|\tau|^{-1}c_\tau^2\pi_\tau(-1) = 1$):

$$\tilde{K}_\pi(\zeta \mid \pi_1, \pi_2) = \pi_1^{-1}\pi_2(-\zeta)\,\frac{\Gamma_\tau(\pi\pi_2\bar{\pi}_2\pi_0^2)}{\Gamma_\tau(\pi\pi_1\bar{\pi}_1\pi_0^2)}\,\Gamma(\pi_1\pi_2^{-1}).$$

So the formula (5) for $T_\pi(\zeta)$ is proved.

Now we derive the formula for the operator $T_\pi(s)$, where $s = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. We have

$$\tilde{K}_\pi(s \mid \pi_1, \pi_2) = a_\tau c_\tau \pi_\tau(-1)\int\int_{t\bar{t}=v/u} \chi(u(t+\bar{t}))\,\pi(t)\,\pi_1^{-1}\pi_\tau(u)\,\pi_2(v)\,|u|^{-1/2}\,|v|^{-1/2}\,d^*t\,du\,dv$$

$$= c_\tau\pi_\tau(-1)\int\int_{K(\sqrt{\tau})} \chi(u(t+\bar{t}))\,\pi_1^{-1}\pi_2\pi_\tau(u)\,\pi\pi_2\bar{\pi}_2\pi_0^{-2}(t)\,dt\,du.$$

By the change of variable $t = u^{-1}t'$ this integral reduces to the form

$$\tilde{K}_\pi(s \mid \pi_1, \pi_2) = c_\tau\pi_\tau(-1)\int\chi(t+\bar{t})\,\pi\pi_2\bar{\pi}_2\pi_0^{-2}(t)\,dt \cdot \int \pi_1^{-1}\pi_2^{-1}\pi_\tau\pi^{-1}\pi_0^{-2}(u)\,du,$$

that is,

$$\tilde{K}_\pi(s \mid \pi_1, \pi_2) = c_\tau\pi_\tau(-1)\,\Gamma_\tau(\pi\pi_2\bar{\pi}_2\pi_2^0)\,\delta(\pi_1^{-1}\pi_2^{-1}\pi_\tau\pi^{-1}), \tag{6}$$

where $\delta(\pi)$ is the Delta-function on the group of multiplicative characters of $\mathbf{K}$.

On the basis of (6) we obtain the following expression for $T_\pi(s)$:

$$T_\pi(s)F(\pi_1) = c_\tau\pi_\tau(-1)\,\Gamma_\tau(\bar{\pi}^{-1}\pi_1^{-1}\bar{\pi}_1^{-1}\pi_0^2)F(\hat{\pi}^{-1}\pi_\tau\pi_1^{-1}), \tag{7}$$

where π is the restriction of $\pi(t)$ to $\mathbf{K}$.

Incidentally, the operators $T_\pi(s)$ of the principal series are given in the π-realization by the similar formula

$$T_\pi(s)F(\pi_1) = \Gamma(\pi_1^{-1})\,\Gamma(\pi^{-1}\pi_1^{-1}\pi_0^2)F(\pi^{-1}\pi_1^{-1}).$$

6. Another Realization of the Representations of the Discrete Series. We now examine another realization of the representations of the discrete series, which we obtain by going over from the functions $\varphi(u)$ to their Fourier transforms

$$\tilde{\varphi}(x) = \int \varphi(u)\,\chi(ux)\,du.$$

In this realization the representation operator $T_\pi(g)$ is given by the kernel

$$K'_\pi(g \mid x, y) = \int K_\pi(g \mid u, v)\,\chi(ux - vy)\,du\,dv, \tag{1}$$

where $K_\pi(g \mid u, v)$ is the kernel of $T_\pi(g)$ in the original representation. We shall find an explicit expression for this kernel.

Let $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$, where $\beta \neq 0$. When we substitute in (1) the expression (2) in **§ 4.1** for the kernel $K_\pi(g \mid u, v)$, we obtain

$$K'_\pi(g \mid u, v) = a_\tau c_\tau \frac{\operatorname{sign}_\tau \beta}{|\beta|} \int \operatorname{sign}_\tau u\,\chi\left(\frac{\delta u + \alpha v}{\beta} - \frac{1}{\beta}(ut + vt^{-1})\right) \times \pi(t)\,\chi(ux - vy)\,d^*t\,du\,dv. \tag{2}$$

Here the integration with respect to t is taken over the circle $t\bar{t} = vu^{-1}$. Substituting under the integral $v = ut\bar{t}$ we may rewrite this formula as follows:

$$K'_\pi(g \mid u, v) = c_\tau \frac{\operatorname{sign}_\tau \beta}{|\beta|} \int |u| \operatorname{sign}_\tau u \times \chi\left[u\left(\frac{\delta + \alpha t\bar{t}}{\beta} - \frac{1}{\beta}(t + \bar{t}) + x - t\bar{t}y\right)\right] \pi(t)\,dt\,du, \tag{3}$$

where the integration with respect to t is taken over the whole plane $\mathbf{K}(\sqrt{\tau})$. Now we integrate with respect to u.

On the basis of the formula

$$\int \pi(u)\,|u|^{-1}\,\chi(ux)\,du = \Gamma(\pi)\pi^{-1}(x)$$

we obtain

$$K'_\pi(g \mid x, y) = c_1 \frac{\operatorname{sign}_\tau \beta}{|\beta|} \int \frac{\operatorname{sign}_\tau\left(\dfrac{\delta + \alpha t\bar{t}}{\beta} - \dfrac{1}{\beta}(t + \bar{t}) + x - t\bar{t}y\right)}{\left|\dfrac{\delta + \alpha t\bar{t}}{\beta} - \dfrac{1}{\beta}(t + \bar{t}) + x - t\bar{t}y\right|^2}\,\pi(t)\,dt, \tag{4}$$

where

$$c_1 = c_\tau \int |u| \operatorname{sign}_\tau u x(u)\,du.$$

Thus, *in the new realization the representations of the discrete series are constructed in the space of functions $\varphi(x)$ on K for which*

$$(\varphi, \varphi) = \int |\varphi(x)|^2\,dx < \infty.$$

The representation operator $T_\pi(g)$ corresponding to the matrix $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$, with $\beta \neq 0$, has the form

$$T_\pi(g)\,\varphi(x) = \int K'_\pi(g \mid x, y)\,\varphi(y)\,dy, \tag{5}$$

where the kernel $K\,(g \mid x, y)$ is given by (4).

The formula for the kernels of the operators $T_\pi(g)$ corresponding to the triangular matrices $g = \begin{pmatrix} \alpha & 0 \\ \gamma & \delta \end{pmatrix}$ may be obtained from (4) by a limit process. However, it is more convenient to obtain it directly from the formulae for $T_\pi(g)$ in the original realization:

$$T_\pi(g)\,\varphi(u) = \operatorname{sign}_\tau \delta\pi(\delta)\;|\delta|\;\chi(\delta\gamma u)\,\varphi(\delta^2 u).$$

By applying the Fourier transform we easily find: the operator $T_\pi(g)$ corresponding to the matrix $g = \begin{pmatrix} \alpha & 0 \\ \gamma & \delta \end{pmatrix}$ has the following form in the new realization:

$$T_\pi(g)\,\varphi(x) = \operatorname{sign}_\tau \delta\pi(\delta)\;|\delta|\;\varphi\left(\frac{\alpha x + \gamma}{\delta}\right).$$

7. Equivalence of Representations of the Discrete Series. Each representation of the discrete series is given by a multiplicative character $\pi(t)$ on the plane $\mathbf{K}(\sqrt{\tau})$, and also by $\operatorname{sign}_\tau u$ (because it is realized either in the space of functions $\varphi(u)$ that are zero for $\operatorname{sign}_\tau u = -1$ or in the complementary space). We shall now find out which representations of the discrete series are equivalent.

1. If $\pi_1(t) \equiv \pi_2(t)$ on the circle $t\bar{t} = 1$, then the representations $T^+_{\pi_1}(g)$ and $T^+_{\pi_2}(g)$ (or $T^-_{\pi_1}(g)$ and $T^-_{\pi_2}(g)$, respectively) are equivalent.†

2. If $\pi_1(t) = \pi_2^{-1}(t)$, then the representations $T^+_{\pi_1}(g)$ and $T^+_{\pi_2}(g)$, or $T^-_{\pi_1}(g)$ and $T^-_{\pi_2}(g)$ respectively, are equivalent.

The converse statement follows from results in **§ 5.4**: if $\pi_1(t) \neq \pi_2(t)$ and $\pi_1(t) \neq \pi_2^{-1}(t)$ on the circle $t\bar{t} = 1$, then the representations $T^+_{\pi_1}(g)$ and $T^+_{\pi_2}(g)$ (or $T^-_{\pi_1}(g)$ and $T^-_{\pi_2}(g)$, respectively) are inequivalent.

3. The representations $T^+_{\pi_1}(g)$ and $T^-_{\pi_2}(g)$ are not equivalent for any π_1 and π_2.

---

† We recall that we denote by $T^+_\pi(g)$ the representations realized in the subspace of functions $\varphi(u)$ that are zero for sign $u = -1$, and by $T^-_\pi(g)$ the representations realized in the complementary subspace.

Proof of Proposition 1. The kernel of the operator $T_\pi^+(g)$ has the form

$$K_\pi^+(g \mid u, v) \\ = a_\tau c_\tau \frac{\operatorname{sign}_\tau \beta}{|\beta|} \chi\left(\frac{\delta u + \alpha v}{\beta}\right) \int_{t\bar{t}=u/v} \chi\left(-\frac{1}{\beta}(ut + vt^{-1})\right) \pi(t)\, d^*t. \quad (1)$$

Here $\operatorname{sign}_\tau u = \operatorname{sign}_\tau v = 1$. Hence each of the elements u and v is either a square of an element from $\mathbf{K}$ or is of the form $\nu\bar{\nu}s^2$, where s is an element from $\mathbf{K}$ and ν a fixed element from $\mathbf{K}(\sqrt{\tau})$ such that $\nu\bar{\nu}$ is not a square of an element from $\mathbf{K}$.

We transform the formula for $K_\pi^+(g \mid u, v)$ and treat the cases $\pi(-1) = 1$ and $\pi(-1) = -1$ separately. Thus, let $\pi(-1) = 1$. If $u = s_1^2$, $v = s_2^2$, $s_1, s_2 \in \mathbf{K}$, then by the change of variable $t = \sqrt{\frac{v}{u}}\, t'$. we find

$$K_\pi^*(g \mid u, v) = \frac{\pi(\sqrt{v})}{\pi(\sqrt{u})} a_\tau c_\tau \frac{\operatorname{sign}_\tau \beta}{|\beta|} \chi\left(\frac{\delta u + \alpha v}{\beta}\right) \\ \times \int \chi\left(-\frac{\sqrt{uv}}{\beta}(t + \bar{t})\right) \pi(t)\, d^*t. \quad (2)$$

Since $\pi(x) = \pi(-x)$, all the factors in this expression are single-valued functions of u and v.

Similarly, if $u = \nu\bar{\nu}s_1^2$, $v = s_2^2$, $s_1, s_2 \in \mathbf{K}$, then $\left(\text{by the change of variable } t = \dfrac{\sqrt{v}}{\nu\sqrt{(\nu\bar{\nu})^{-1}u}}\, t'\right)$

$$K_\pi^+(g \mid u, v) = \frac{\pi(\sqrt{v})}{\pi(\nu\sqrt{(\nu\bar{\nu})^{-1}u})} a_\tau c_\tau \frac{\operatorname{sign}_\tau \beta}{|\beta|} \chi\left(\frac{\delta u + \alpha v}{\beta}\right) \\ \times \int_{t\bar{t}=1} \chi\left(-\frac{\sqrt{(\nu\bar{\nu})uv}}{\beta}\left(\frac{t}{\nu} + \frac{\bar{t}}{\bar{\nu}}\right)\right) \pi(t)\, d^*t; \quad (3)$$

if $u = s_1^2$, $v = \nu\bar{\nu}s_2^2$ $(s_1, s_2 \in \mathbf{K})$, then

$$K_\pi^+(g \mid u, v) = \frac{\pi(\nu\sqrt{(\nu\bar{\nu})^{-1}v})}{\pi(\sqrt{u})} a_\tau c_\tau \frac{\operatorname{sign}_\tau \beta}{|\beta|} \chi\left(\frac{\delta u + \alpha v}{\beta}\right) \\ \times \int_{t\bar{t}=1} \chi\left(-\frac{\sqrt{(\nu\bar{\nu})^{-1}uv}}{\beta}(\nu t + \bar{\nu}\bar{t})\right) \pi(t)\, d^*t. \quad (4)$$

Finally, if $u = \nu\bar{\nu}s_1^2$, $v = \nu\bar{\nu}s_2^2$, then

$$K_\pi^+(g \mid u, v) = \frac{\pi(\nu\sqrt{(\nu\bar{\nu})^{-1}v})}{\pi(\nu\sqrt{(\nu\bar{\nu})^{-1}u})} a_\tau c_\tau \frac{\operatorname{sign}_\tau \beta}{|\beta|} \chi\left(\frac{\delta u + \alpha v}{\beta}\right) \times \int_{t\bar{t}=1} \chi\left(-\frac{\sqrt{uv}}{\beta}(t + \bar{t})\right)\pi(t)\, d^*t. \quad (5)$$

In the representation space we consider the operator A_π of multiplication by the function $a(u)$

$$A_\pi \varphi(u) = a(u)\varphi(u) \quad (6)$$

where $a(u) = \pi(\sqrt{u})$ when $u = s^2$, $s \in \mathbf{K}$, and $a(u) = \pi(\nu\sqrt{(\nu\bar{\nu})^{-1}u})$ when $u = \nu\bar{\nu}s^2$, $s \in \mathbf{K}$.

We transform $T_\pi^+(g)$ to the equivalent representation

$$\hat{T}_\pi^+(g) = A_\pi^{-1}T_\pi^+(g)A_\pi.$$

Clearly, the formulae for the kernels of the operators $T_\pi^+(g)$ are obtained from (2)–(5) by omitting the first factors. Hence, these kernels depend only on the values assumed by the character $\pi(t)$ on the circle $t\bar{t} = 1$. So we have shown that if $\pi_1(t) = \pi_2(t)$ on $t\bar{t} = 1$ and $\pi_1(-1) = 1$, then the representations $T_{\pi_1}^+(g)$ and $T_{\pi_2}^+(g)$ are equivalent.

Now we take the case $\pi(-1) = -1$. Let $\pi_0(t)$ be a fixed character such that $\pi_0(-1) = -1$. Just as in the first case, we then transform the formula for the kernel of $T_\pi^+(g)$ to the following form.

If $u = s_1^2$, $v = s_2^2$, $s_1, s_2 \in \mathbf{K}$, then

$$K_\pi^+(g \mid u, v) = \frac{\pi\pi_0^{-1}(\sqrt{v})}{\pi\pi_0^{-1}(\sqrt{u})} a_\tau c_\tau \frac{\operatorname{sign}_\tau \beta}{|\beta|} \chi\left(\frac{\delta u + \alpha v}{\beta}\right) \times \pi_0\left(\frac{\sqrt{v}}{\sqrt{u}}\right) \int_{t\bar{t}=1} \chi\left(-\frac{u}{\beta}\frac{\sqrt{v}}{\sqrt{u}}(t + \bar{t})\right)\pi(t)\, d^*t.$$

(The expression $\pi_0\left(\frac{\sqrt{v}}{\sqrt{u}}\right)\int_{t\bar{t}=1} \chi\left(-\frac{u}{\beta}\frac{\sqrt{v}}{\sqrt{u}}(t + \bar{t})\right)\pi(t)\, d^*t$ is a single-valued function of u and v, because it does not depend on the choice of the sign of $\sqrt{u}$ and $\sqrt{v}$.)

If $u = \nu\bar{\nu}s_1^2$, $v = s_2^2$, $s_1, s_2 \in \mathbf{K}$, then

$$K_\pi^+(g \mid u, v) = \frac{\pi\pi_0^{-1}(\sqrt{v})}{\pi\pi_0^{-1}(\nu\sqrt{(\nu\bar{\nu})^{-1}u})} a_\tau c_\tau \frac{\operatorname{sign}_\tau \beta}{|\beta|} \chi\left(\frac{\delta u + \alpha v}{\beta}\right) \times \int_{t\bar{t}=1} \chi\left(-\frac{u}{\beta}\frac{\sqrt{v}}{\sqrt{(\nu\bar{\nu})^{-1}u}}\left(\frac{t}{\nu} + \frac{\bar{t}}{\bar{\nu}}\right)\right)\pi(t)\, d^*t$$

and so forth.

We go over from $T_\pi^+(g)$ to the equivalent representation $\hat{T}_\pi^+(g) = A_{\pi\pi_0^{-1}}^{-1} T_\pi^+(g) A_{\pi\pi_0^{-1}}$, where the operator A_π is given by (6).

Again, the kernel of the operators $\hat{T}_\pi^+(g)$ depends only on the values assumed by $\pi(t)$ on $t\bar{t} = 1$. Hence, if $\pi_1(t) = \pi_2(t)$ on $t\bar{t} = 1$, the representations $T_{\pi_1}^+(g)$ and $T_{\pi_2}^+(g)$ are equivalent. Proposition 1 is now proved.

Proof of Proposition 2. In the formula for the kernel of the operator $T_\pi^+(g)$

$$K_\pi^+(g \mid u, v) = a_\tau c_\tau \frac{\operatorname{sign}_\tau \beta}{|\beta|} \chi\left(\frac{\delta u + \alpha v}{\beta}\right) \int_{t\bar{t}=v/u} \chi\left(-\frac{1}{\beta}(ut + vt^{-1})\right) \pi(t)\, d^*t$$

we make the change of variable $t = vu^{-1}t'^{-1}$, and obtain

$$K_\pi^+(g \mid u, v) = \frac{\pi(v)}{\pi(u)} a_\tau c_\tau \frac{\operatorname{sign}_\tau \beta}{|\beta|} \chi\left(\frac{\delta u + \alpha v}{\beta}\right) \times \int_{t\bar{t}=v/u} \chi\left(-\frac{1}{\beta}(vt^{-1} + ut)\right) \pi^{-1}(t)\, d^*t,$$

that is,

$$K_\pi^+(g \mid u, v) = \frac{\pi(v)}{\pi(u)} K_{\pi^{-1}}^+(g \mid u, v). \tag{7}$$

The equivalence of $T_\pi^+(g)$ and $T_{\pi^{-1}}^+(g)$ follows immediately from (7).

Proof of Proposition 3. Let A be a bounded operator map the representation space of $T_{\pi_1}^+(g)$ into that of $T_{\pi_2}^-(g)$ and commuting with the representations:

$$T_{\pi_2}^-(g)A = AT_{\pi_1}^+(g). \tag{8}$$

Our task is to show that $A = 0$. We examine the operators $T_{\pi_1}^+(g)$ and $T_{\pi_2}^-(g)$ corresponding to the matrices $g = \begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix}$. These operators have the form

$$T_{\pi_1}^+(g)\varphi(u) = \chi(\gamma u)\varphi(u), \qquad T_{\pi_2}^-(g)\psi(u) = \chi(\gamma u)\varphi(u)$$

We set $\psi(u) = A\varphi(u)$. Then the condition (8) can be written in the form

$$\chi(\gamma u)\psi(u) = A[\chi(\gamma u)\varphi(u)]$$

for every γ in $\mathbf{K}$. Hence, it follows immediately that

$$f(u)\psi(u) = A[f(u)\varphi(u)] \tag{9}$$

for every bounded function $f(u)$ on $\mathbf{K}$. In particular, we consider the function

$$f(u) = \begin{cases} 1, & \text{when } \operatorname{sign}_\tau u = 1, \\ 0, & \text{when } \operatorname{sign}_\tau u = -1. \end{cases}$$

Since the functions $\varphi(u)$ are concentrated in the domain $\mathrm{sign}_\tau u = 1$ and $\psi(u)$ in the domain $\mathrm{sign}_\tau u = -1$, we have: $f(u)\varphi(u) = \varphi(u)$, $f(u)\psi(u) = 0$. Consequently, equation (9) gives us $A\varphi(u) = 0$, that is, $A = 0$.

All the representations of the discrete series $T_\pi^+(g)$ and $T_\pi^-(g)$ are irreducible.

The proof of this proposition follows the same line as that in the case of representations of the continuous series (see **§ 3.4**).

8. Discrete Series for the Field of 2-adic Numbers. In the preceding account we have assumed everywhere that the characteristic of the residue class field O/P is different from 2. However, in Chapter III we need the representations of the group of unimodular matrices of order 2 with elements from the field Q_2 of 2-adic numbers.

This case differs only insignificantly from the general case treated above. In fact, the constructions of the principal series, the supplementary series, and the special representation carry over to Q_2 without change. Some modifications are required only in the description of the discrete series, which we now indicate.

In the case $K = Q_2$ the factor group $\mathbf{K}^*/(\mathbf{K}^*)^2$ is of order 8 and can be represented as a direct sum of three cyclic groups of order 2. As generators of these groups we can take the cosets of $\mathbf{K}^*/(\mathbf{K}^*)^2$ containing the numbers 2, 3, and 5.

For from the arguments in **§ 1.5** it follows that the subgroup $A_2 \subset \mathbf{K}^*$ consisting of the elements of the form $1 + 8x$, $|x| \leq 1$, is contained in $(O^*)^2$. Also a direct computation shows that when $|x| = 1$, then $x^2 \in A_2$. Our statement on the structure of $\mathbf{K}^*/(\mathbf{K}^*)^2$ follows from this.

Thus, the field $K = Q_2$ has seven distinct quadratic extensions $\mathbf{K}(\sqrt{\tau})$, $\tau = 2, 3, 5, 6, 10, 15, 30$. It can be verified that in each of these extensions the subgroup $\mathbf{K}_\tau^*$ consisting of the elements of the form $z\bar{z}$, $z \in \mathbf{K}(\sqrt{\tau})$, is of index 2 in $\mathbf{K}^*$. Therefore, we may define the functions $\mathrm{sign}_\tau x$ that assume the values ± 1 and give a complete set of characters on $\mathbf{K}^*/(\mathbf{K}^*)^2$. The construction of the discrete series as described in this section may now be extended to the field Q_2, and here we obtain not three but seven discrete series of representations.

§ 5. THE TRACES OF IRREDUCIBLE REPRESENTATIONS OF G

1. Statement of the Problem. Let $T_\pi(g)$ be a representation of G belonging to the continuous (principal or supplementary) or discrete series. With every finite function† $f(g)$ on G we associate the operator

$$T_\pi(f) = \int f(g) T_\pi(g)\, dg. \tag{1}$$

Then the following proposition holds.

The operator $T_\pi(f)$ *has a trace, which we denote by* Tr $T_\pi(f)$, *and this trace is a continuous functional in the space of finite functions* $f(g)$. So the trace Tr $T_\pi(g)$ of $T_\pi(g)$ is defined as a generalized function on G:

$$(\mathrm{Tr}\ T_\pi(g), f(g)) = \mathrm{Tr}\ T_\pi(f).$$

For the classical groups over the field of complex numbers this result was first obtained by Gel'fand and Naimark. Later it was proved by Godement and Harish-Chandra for the irreducible unitary representations of every real semisimple Lie group.

In the Appendix to this Chapter we give a proof of this proposition for the group of matrices of order 2 with elements from a disconnected locally compact topological field.

Our object is to compute the traces Tr $T_\pi(g)$ of the operators of the irreducible representations.

In this section we compute the traces Tr $T_\pi(g)$ on the basis of a unified method for all fields **K**.

We use the formula

$$\mathrm{Tr}\ T_\pi(g) = \int K_\pi(g \mid u, u)\, du.$$

where $K_\pi(g \mid u, v)$ is the kernel of this operator.

2. The Traces of the Representations of the Continuous Series. The representation operator of the continuous series corresponding to the matrix $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ is given by the following formula (see **§ 3.1**):

$$T_\pi(g) f(x) = f\left(\frac{\alpha x + \gamma}{\beta x + \delta}\right) \pi(\beta x + \delta)\, |\beta x + \delta|^{-1}.$$

Thus, $T_\pi(g)$ may be regarded as an integral operator whose kernel is the generalized function

$$K_\pi(g \mid x, y) = \pi(\beta x + \delta)\, |\beta x + \delta|^{-1}\, \delta\left(\frac{\alpha x + \gamma}{\beta x + \delta} - y\right). \tag{1}$$

† In the case of a connected field **K** we always assume that the function $f(g)$ is infinitely differentiable; in the case of a disconnected field $f(g)$ is assumed to be piecewise constant.

We compute the trace of $T_\pi(g)$ by the formula

$$\begin{aligned}\operatorname{Tr} T_\pi(g) &= \int K_\pi(g \mid x, x)\, dx \\ &= \int \pi(\beta x + \delta)\, |\beta x + \delta|^{-1}\, \delta\left(\frac{\alpha x + \gamma}{\beta x + \delta} - x\right) dx. \qquad (2)\end{aligned}$$

We may assume that $\beta \neq 0$ (otherwise we pass from g to any matrix conjugate to it). We make the change of variables $\beta x + \delta = t$, and obtain

$$\begin{aligned}\operatorname{Tr} T_\pi(g) &= \int \delta(\alpha + \delta - t - t^{-1}) \pi(t)\, |t|^{-1}\, dt \\ &= \int \delta(\lambda_g + \lambda_g^{-1} - t - t^{-1}) \pi(t)\, |t|^{-1}\, dt, \qquad (3)\end{aligned}$$

where λ_g and λ_g^{-1} are the eigenvalues of g. From (3) it is clear that $\operatorname{Tr} T_\pi(g)$ *is concentrated on the matrices* g *whose eigenvalues* λ_g *and* λ_g^{-1} *lie in* $\mathbf{K}$, because the expression $\lambda_g + \lambda_g^{-1} - t - t^{-1}$, which is the argument of the Delta-function, vanishes at zero only for $t = \lambda_g$ and $t = \lambda_g^{-1}$.

The integral (3) is easy to evaluate. For this purpose it is sufficient to use the following property of the Delta-function:†

$$\delta((t - a)(t - b)) = \frac{1}{|a - b|} (\delta(t - a) + \delta(t - b)) \qquad (4)$$

(provided $a \neq b$). Suppose that λ_g and λ_g^{-1} lie in $\mathbf{K}$ and that $\lambda_g \neq \lambda_g^{-1}$. Then we have

$$\begin{aligned}|t|^{-1}\, \delta(\lambda_g + \lambda_g^{-1} - t - t^{-1}) &= \delta((t - \lambda_g)(t - \lambda_g^{-1})) \\ &= \frac{1}{|\lambda_g - \lambda_g^{-1}|} (\delta(t - \lambda_g) + \delta(t - \lambda_g^{-1})).\end{aligned}$$

Substituting this expression in (3) we find

$$\operatorname{Tr} T_\pi(g) = \frac{\pi(\lambda_g) + \pi(\lambda_g^{-1})}{|\lambda_g - \lambda_g^{-1}|}. \qquad (5)$$

Thus, *the trace of the operator* $T_\pi(g)$ *of a representation of the continuous series is expressed by* (5), *provided the eigenvalues* λ_g *and* λ_g^{-1} *of* g *lie in* $\mathbf{K}$;

$$\operatorname{Tr} T_\pi(g) = 0,$$

when λ_g *and* λ_g^{-1} *do not lie in* $\mathbf{K}$.

From (5) it follows that the traces $\operatorname{Tr} T_{\pi_1}(g)$ and $\operatorname{Tr} T_{\pi_2}(g)$ of two representations of the continuous series coincide if and only if

† A proof of (4) for the field of real numbers is given in volume 7. We recommend that the reader prove (4) as an easy exercise in analysis in disconnected fields.

either $\pi_1 = \pi_2$ or $\pi_1 = \pi_2^{-1}$. Hence, we conclude: if $\pi_1 \neq \pi_2$ and $\pi_1 \neq \pi_2^{-1}$, *then the representations* $T_{\pi_1}(g)$ *and* $T_{\pi_2}(g)$ *of the continuous series are inequivalent.*

3. The Trace of the Singular Representation. The arguments in **§ 5.2** and the formula for the trace remain valid for the representations of the supplementary series, and also for the nonunitary representations in the spaces $\mathscr{D}_\pi$ (see **§ 3.8**).

We make use of this fact to compute the trace of the singular representation $T_0(g)$ in the case of a disconnected field.

We recall how the singular representation is constructed. We consider the space $\mathscr{D}_\pi$, $\pi(x) = |x|^{-1}$, of functions $f(x_1, x_2)$ satisfying the following condition of homogeneity:

$$f(tx_1, tx_2) = |t|^{-2} f(x_1, x_2) \tag{1}$$

for every $t \neq 0$. The representation operator $T_\pi(g)$ in $\mathscr{D}_\pi$ is given by the formula

$$T_\pi(g) f(x_1, x_2) = f(\alpha x_1 + \gamma x_2, \beta x_1 + \delta x_2). \tag{2}$$

If transfer from the homogeneous functions of two variables $f(x_1, x_2)$ to the functions of a single variable $\varphi(x) = f(x, 1)$, we obtain another realization of the space $\mathscr{D}_\pi$. In this realization the representation operator has the form

$$T_\pi(g)\,\varphi(x) = \varphi\left(\frac{\alpha x + \gamma}{\beta x + \delta}\right) |\beta x + \delta|^{-2}. \tag{3}$$

The space $\mathscr{D}_\pi$ contains an invariant subspace $\mathscr{F}_\pi$ consisting of the functions $\varphi(x)$ for which

$$\int \varphi(x)\, dx = 0.$$

The singular representation of G is a representation in the subspace† $\mathscr{F}_\pi$.

Clearly, the factor space $\mathscr{D}_\pi / \mathscr{F}_\pi$ is one-dimensional and the unit representation of the group acts in it. So the matrix of $T_\pi(g)$ in $\mathscr{D}_\pi$ has the form

$$\begin{pmatrix} 1 & * \\ 0 & T_0(g) \end{pmatrix},$$

where $T_0(g)$ is the operator of the singular representation.

Hence, it follows that we obtain the trace of $T_0(g)$ by computing the trace of the unit representation $\operatorname{Tr} T(g) = 1$ from the trace of

† More accurately, not in the space $\mathscr{F}_\pi$ itself, but in its completion relative to the invariant scalar product.

the operator $T_\pi(g)$, $\pi(x) = |x|^{-1}$ defined by **§ 5.2** (5). As a result we find that: *the trace of the operator $T_0(g)$ of the singular representation is expressed by the following formula:*

$$\operatorname{Tr} T_0(g) = \frac{|\lambda_g| + |\lambda_g^{-1}|}{|\lambda_g - \lambda_g^{-1}|} - 1, \tag{4}$$

if the eigenvalues λ_g and λ_g^{-1} of g belong to $\mathbf{K}$;

$$\operatorname{Tr} T_0(g) = -1,$$

when λ_g and δ_g^{-1} do not belong to $\mathbf{K}$.

4. The Traces of the Representations of the Discrete Series. We recall that the operators $T_\pi^+(g)$ and $T_\pi^-(g)$ of the representations of the discrete series are given by the following formulae:

$$T_\pi^+(g)\,\varphi(u) = \int K_\pi(g \mid u, v)\,\varphi(v)\,dv, \quad \operatorname{sign}_\tau u = \operatorname{sign}_\tau v = 1,$$

$$T_\pi^-(g)\,\varphi(u) = \int K_\pi(g \mid u, v)\,\varphi(v)\,dv, \quad \operatorname{sign}_\tau u = \operatorname{sign}_\tau v = -1,$$

where,

$$K_\pi(g \mid u, v) = a_\tau c_\tau \frac{\operatorname{sign}_\tau \beta}{|\beta|} \operatorname{sign}_\tau u \cdot \chi\left(\frac{\delta u + \alpha v}{\beta}\right)$$
$$\int\limits_{tt = vu^{-1}} \chi\left(-\frac{1}{\beta}(ut + vt^{-1})\right)\pi(t)\,d^*t. \tag{1}$$

The representation $T_\pi^+(g)$ is realized in the space of functions on the half-line $\operatorname{sign}_\tau u = 1$, and $T_\pi^-(g)$ in the space of functions on the half-line $\operatorname{sign}_\tau u = -1$.

We compute the traces of the representations by the formulae

$$\operatorname{Tr} T_\pi^+(g) = \int\limits_{\operatorname{sign}_\tau u = 1} K_\pi(g \mid u, u)\,du$$
$$= a_\tau c_\tau \frac{\operatorname{sign}_\tau \beta}{|\beta|} \int\limits_{\operatorname{sign}_\tau u = 1} \int\limits_{tt = 1} \chi\left(\frac{u}{\beta}(\alpha + \delta - t - t^{-1})\right)\pi(t)\,d^*t\,du, \tag{2}$$

$$\operatorname{Tr} T_{\bar\pi}(g) = \int\limits_{\operatorname{sign}_\tau u = -1} K_\pi(g \mid u, u)\,du$$
$$= -a_\tau c_\tau \frac{\operatorname{sign}_\tau \beta}{|\beta|} \int\limits_{\operatorname{sign}_\tau u = -1} \int\limits_{tt = 1} \chi\left(\frac{u}{\beta}(\alpha + \delta - t - t^{-1})\right)\pi(t)\,d^*t\,du \tag{2'}$$

It is convenient to compute not the trace of $T_\pi^+(g)$ and $T_\pi^-(g)$, but of their sum and difference.

First, we evaluate the difference of the traces. We have

$$\operatorname{Tr} T_\pi^+(g) - \operatorname{Tr} T_\pi^-(g) = a_\tau c_\tau \frac{\operatorname{sign}_\tau \beta}{|\beta|} \int_K \int_{t\bar{t}=1} \chi\left(\frac{u}{\beta}(\alpha + \delta - t - t^{-1})\right) \pi(t)\, d^*t\, du.$$

Since

$$\int \chi(ux)\, du = \delta(x),$$

we obtain

$$\operatorname{Tr} T_\pi^+(g) - \operatorname{Tr} T_\pi^-(g) = a_\tau c_\tau \operatorname{sign}_\tau \beta \int_{t\bar{t}=1} \delta(\lambda_g + \lambda_g^{-1} - t - t^{-1}) \pi(t)\, d^*t, \qquad (3)$$

where λ_g and λ_g^{-1} are the eigenvalues of g. From this formula it is clear that *the difference of the traces* $\operatorname{Tr} T_\pi^+(g) - \operatorname{Tr} T_\pi^-(g)$ *is concentrated on those matrices g whose eigenvalues lie on the circle $t\bar{t} = 1$ on* $\mathbf{K}(\sqrt{\tau})$.

This holds because the argument of the Delta-function vanishes only for $t = \lambda_g$ and $t = \lambda_g^{-1}$.

Let us compute $\operatorname{Tr} T_\pi^+(g) - \operatorname{Tr} T_\pi^-(g)$ for these matrices.

For this purpose we rewrite (3) as an integral over the whole plane $\mathbf{K}(\sqrt{\tau})$:

$$\operatorname{Tr} T_\pi^+(g) - \operatorname{Tr} T_\pi^-(g) = c_\tau \operatorname{sign}_\tau \beta \int \delta((t - \lambda_g) + (\bar{t} - \bar{\lambda}_g))\, \delta(1 - t\bar{t}) \pi(t)\, dt), \qquad (4)$$

where the integral is taken over $\mathbf{K}(\sqrt{\tau})$.

We use the following relation:

$$\delta((t - \lambda_g) + (\bar{t} - \bar{\lambda}_g))\, \delta(1 - t\bar{t}) = \frac{1}{|\tau|^{1/2}\, |\lambda_g - \lambda^{-1}|} (\delta_\tau(t - \lambda_g) + \delta_\tau(\bar{t} - \bar{\lambda}_g)), \qquad (5)$$

where $\delta_\tau(t)$ is the Delta-function on $\mathbf{K}(\sqrt{\tau})$:

$$\delta_\tau(x + \sqrt{\tau} y) = \delta(x)\, \delta(y).$$

For if we set $t = x + \sqrt{\tau} y, \lambda_g = \alpha + \sqrt{\tau}\, \beta$, $\alpha^2 - \tau\beta^2 = 1$, we have

$$\begin{aligned}\delta((t - \lambda_g) + (\bar{t} - \bar{\lambda}_g))\delta(1 - t\bar{t}) &= \delta(x - \alpha)\delta(1 - x^2 + \tau y^2) = \delta(x - \alpha)\delta(\tau(y^2 - \beta^2)) \\ &= \frac{1}{|\tau|\, |\beta|} \delta(x - \alpha)(\delta(y - \beta) + \delta(y + \beta)).\end{aligned}$$

Hence (5) follows immediately.

Substituting (5) in (4) we obtain

$$\mathrm{Tr}\ T_\pi^+(g) - \mathrm{Tr}\ T_\pi^-(g) = c_\tau\,|\tau|^{-1/2}\,\mathrm{sign}_\tau\,\beta\,\frac{\pi(\lambda_g) + \pi(\lambda_g^{-1})}{|\lambda_g - \lambda_g^{-1}|}. \quad (6)$$

Thus, *the difference of the traces of the representations* $T_\pi^+(g)$ *and* $T_\pi^-(g)$ *of the discrete series corresponding to the quadratic extension* $\mathbf{K}(\sqrt{\tau})$ *of* $\mathbf{K}$ *is expressed by* (6), *if the eigenvalues* λ_g *and* λ_g^{-1} *of* g *lie on the circle* $t\bar{t} = 1$ *on* $K(\sqrt{\tau})$; *and*

$$\mathrm{Tr}\ T_\pi^+(g) - \mathrm{Tr}\ T_\pi^-(g) = 0,$$

when λ_g *and* λ_g^{-1} *do not lie on this circle.*

Now we compute the trace of the sum $T_\pi(g) = T_\pi^+(g) \oplus T_\pi^{-1}(g)$ of the representations $T_\pi^+(g)$ and $T_\pi^-(g)$.

We have

$$\mathrm{Tr}\ T_\pi(g)$$
$$= a_\tau c_\tau\,\frac{\mathrm{sign}_\tau\,\beta}{|\beta|}\int_K\int_{t\bar{t}=1}\mathrm{sign}_\tau\,u\ \chi\left(\frac{u}{\beta}\,(\lambda_g + \lambda_g^{-1} - t - t^{-1})\right)\pi(t)\,d^*t\,du.$$

Using the formula

$$\int \mathrm{sign}_\tau\,u\cdot\chi(ux)\,du = 2a_\tau^{-1}c_\tau^{-1}\,\frac{\mathrm{sign}_\tau\,x}{|x|}$$

(see **§ 2.7**), we find that *the trace of the sum* $T_\pi(g) = T_\pi^+(g) \oplus T_\pi^-(g)$ *of the representations of the discrete series is expressed by the following formula:*†

$$\begin{aligned}\mathrm{Tr}\ T_\pi(g) &= \mathrm{Tr}\ T_\pi^+(g) + \mathrm{Tr}\ T_\pi^-(g)\\ &= 2\int_{t\bar{t}=1}\frac{\mathrm{sign}_\tau\,(\lambda_g + \lambda_g^{-1} - t - t^{-1})}{|\lambda_g + \lambda_g^{-1} - t - t^{-1}|}\,\pi(t)\,d^*t. \quad (7)\end{aligned}$$

This formula is similar to the formula for the traces of the representations of the continuous series (see **§ 5.2**):

$$\mathrm{Tr}\ T_\pi(g) = \int_K \delta(\lambda_g + \lambda_g^{-1} - t - t^{-1})\,\pi(t)\,d^*t$$

It is often useful to consider not the traces $\mathrm{Tr}\ T_\pi(g)$ themselves, but their Mellin transforms with respect to π, which we denote by $S(g; t)$. These Mellin

† The integral (7) converges if λ_g and λ_g^{1-} do not lie on the circle $t\bar{t} = 1$. But if λ_g and λ_g^{-1} lie on this circle, then the integral must be understood in the sense of the regularizing value, namely as the value of the analytic function in ν:

$$f(\nu) = 2\int_{t\bar{t}=1}\frac{\mathrm{sign}_\tau\,(\lambda_g + \lambda_g^{-1} - t - t^{-1})}{|\lambda_g + \lambda_g^{-1} - t - t^{-1}|^\nu}\,\pi(t)\,d^*t$$

at the point $\nu = 1$.

transforms have the following form. For representations of the continuous series

$$S(g;t) = \delta(\lambda_g + \lambda_g^{-1} - t - t^{-1}), \qquad \text{where } t \in \mathbf{K}.$$

For representations of the discrete series corresponding to the quadratic extension $\mathbf{K}(\sqrt{\tau})$ of $\mathbf{K}$,

$$S(g;t) = 2\,\frac{\operatorname{sign}_\tau(\lambda_g + \lambda_g^{-1} - t - t^{-1})}{|\lambda_g + \lambda_g^{-1} - t - t^{-1}|},$$

where t is a point of the circle $t\bar{t} = 1$ on the plane $\mathbf{K}(\sqrt{\tau})$.

Let us rewrite formula (7) for the field of real numbers in more detail. Here we have $t = e^{i\varphi}$, $d^*t = \dfrac{1}{2\pi}\,d\varphi$, $\pi(t) = e^{in\varphi}$ and (7) can easily be transformed as follows:

$$\operatorname{Tr} T_\pi(g) = \frac{1}{\pi i}\int_C \frac{\zeta^n\,d\zeta}{(\zeta - \lambda_g)(\zeta - \lambda_g^{-1})}, \tag{8}$$

where the integration is taken over the unit circle C: $\zeta\bar{\zeta} = 1$. It is easy to evaluate this integral (for the final formula see **§ 5.5**). The integral (8) turns out to be different from zero both for complex and for real λ_g.

A different result holds for the case of a disconnected field $\mathbf{K}$. *Suppose that the eigenvalues λ_g and λ_g^{-1} of g do not lie on the circle $t\bar{t} = 1$ on $\mathbf{K}(\sqrt{\tau})$. Then*

$$\operatorname{Tr} T_\pi(g) = 0$$

for all π, except possibly a finite number of characters π (depending on g).

Proof. We expand $\lambda_g + \lambda_g^{-1}$ in a series (see **§ 1.3**)

$$x \equiv \lambda_g + \lambda_g^{-1} = \sum_{i=k}^{\infty} a_i \mathfrak{p}^i.$$

If $|x| > 1$, then

$$\operatorname{sign}_\tau(x - t - t^{-1}) = \operatorname{sign}_\tau x,\ |x - t - t^{-1}| = |x|$$

for every t on $t\bar{t} = 1$. Consequently,

$$\operatorname{Tr} T_\pi(g) = 2\,\frac{\operatorname{sign}_\tau x}{|x|}\int_{t\bar{t}=1} \pi(t)\,d^*t = 0.$$

There remains the case† $|x| \le 1$, that is, $k \ge 0$.

By hypothesis, for every t on $t\bar{t} = 1$ we have $t + t^{-1} \ne \lambda_g + \lambda_g^{-1}$. Therefore we can find a natural number m with the following property: if $t + t^{-1} = b_0 + b_1\mathfrak{p} + \cdots$, where t is an arbitrary point on $t\bar{t} = 1$, then $b_i \ne a_i$ for at least one index $i < m$.

We divide the circle $t\bar{t} = 1$ into a finite number of subsets

† We mention that if -1 is not a square in $\mathbf{K}$, then $|\lambda_g + \lambda_g^{-1}| \ge 1$.

$A_{b_0,\dots,b_{m-1}}$; the subset $A_{b_0,\dots,b_{m-1}}$ consists of all points t of the circle at which $t + t^{-1}$ has the first m given terms of the expansion: $b_0 + \cdots + b_{m-1}\mathfrak{p}^{m-1}$.

It is easy to see that $\operatorname{sign}_\tau(\lambda_g + \lambda_g^{-1} - t - t^{-1})$ and

$$|\lambda_g + \lambda_g^{-1} - t - t^{-1}|$$

are constant on each of these subsets. Therefore we consider the integrals

$$I_{b_0,\dots,b_{m-1}} = \int\limits_{A_{b_0,\dots,b_{m-1}}} \pi(t)\, d^*t.$$

and check that they are equal to zero for all π except a finite set.

On the circle $t\bar{t} = 1$ we consider the set A_m of points t of the form $t = 1 + \mathfrak{p}^m s$, where $|s| \leq 1$. It is not hard to see that A_m is a subgroup of finite index in the group of all points of the circle. Hence, there are only finitely characters on the circle that are identically equal to unity on A_m.

Suppose that the character π is not identically equal to unity on A_m. Then we show that for it $I_{b_0,\dots,b_{m-1}} = 0$. For let $\pi(t_0) \neq 1$ for some $t_0 \in A_m$. Since the transformation $t \to tt_0$ preserves the set $A_{b_0,\dots,b_{m-1}}$, we have

$$\pi(t_0)I_{b_0,\dots,b_{m-1}} = \int\limits_{A_{b_0,\dots,b_{m-1}}} \pi(tt_0)\, d^*t = \int\limits_{A_{b_0,\dots,b_{m-1}}} \pi(t)\, d^*t = I_{b_0,\dots,b_{m-1}}$$

Consequently, $I_{b_0,\dots,b_{m-1}} = 0$, and the proposition is proved.

In this section we have computed the traces of the irreducible representations without detailed proofs. But there is no difficulty in giving a rigorous foundation to all the preceding calculations.

For example, let us look at the derivation of the formula for the trace of the sum $T_\pi(g) = T_\pi^+(g) \oplus T_\pi^-(g)$ of two representations of the discrete series. We assume $\mathbf{K}$ to be disconnected.

Let S be the space of finite piecewise constant functions on G. For every $f \in S$ the operator

$$T_\pi(f) = \int f(g)\, T_\pi(g)\, dg$$

is completely continuous (and positive if f is a function of the form $\varphi * \varphi^*$) and has a trace. We have to show that the trace of $T_\pi(f)$ is expressed by the formula

$$\operatorname{Tr} T_\pi(f) = 2 \int\limits_G \int\limits_{t\bar{t}=1} f(g) \frac{\operatorname{sign}_\tau(\lambda_g + \lambda_g^{-1} - t - t^{-1})}{|\lambda_g + \lambda_g^{-1} - t - t^{-1}|}\, \pi(t)\, d^*t\, dg. \tag{9}$$

Since the kernel of $T_\pi(f)$ is

$$\int f(g) K_\pi(g \mid u, v)\, dg,$$

we have

$$\operatorname{Tr} T_\pi(f) = \lim_{k\to\infty} \int\limits_{|u|\le q^k} \int\limits_G f(g) K_\pi(g \mid u, u)\, dg\, du$$

$$= \lim_{k\to\infty} \int\limits_G \int\limits_{|u|\le q^k} f(g) K_\pi(g \mid u, u)\, du\, dg.$$

The interchange of the order of integration is permissible, because the integration with respect to G and to u is taken over a compact domain.

Substituting the explicit expression for $K_\pi(g \mid u, u)$ and integrating with respect to u we find

$\operatorname{Tr} T_\pi(f)$

$$= a_\tau c_\tau \lim_{k\to\infty} \int\limits_G f(g)\, \Gamma^{(k+s)}(\pi_\tau) \frac{\operatorname{sign}_\tau (\lambda_g + \lambda_g^{-1} - t - t^{-1})}{|\lambda_g + \lambda_g^{-1} - t - t^{-1}|} \pi(t)\, d^*t\, dg. \quad (10)$$

where $\pi_\tau(x) = |x| \operatorname{sign}_\tau x$, and $\Gamma^{(n)}(\pi_\tau)$ is the incomplete Gamma-function:

$$\Gamma^{(n)}(\pi_\tau) = \int\limits_{|x|\leqslant q^n} \chi(x) \operatorname{sign}_\tau x\, dx.$$

The number s is defined by $|\lambda_g + \lambda_g^{-1} - t - t^{-1}| = q^s$.

It is easy to verify that the limit, as $k \to \infty$, of the sequence of generalized functions†

$$\varphi_k(g) = \int \Gamma^{(k+s)}(\pi_\tau) \frac{\operatorname{sign}_\tau (\lambda_g + \lambda_g^{-1} - t - t^{-1})}{|\lambda_g + \lambda_g^{-1} - t - t^{-1}|} \pi(t)\, d^*t$$

is the generalized function

$$\Gamma(\pi_\tau) \int \frac{\operatorname{sign}_\tau(\lambda_g + \lambda_g^{-1} - t - t^{-1})}{|\lambda_g + \lambda_g^{-1} - t - t^{-1}|} \pi(t)\, d^*t.$$

Thus, by passing to the limit in (10) and bearing in mind that $a_\tau c_\tau = 2\Gamma^{-1}(\pi_\tau)$ we obtain the required formula (9).

5. The Traces of the Representations of the Discrete Series for the Field of Real Numbers.

For the field of real numbers a character on the unit circle is given by the formula

$$\pi(t) = e^{in\varphi}, \qquad t = e^{i\varphi}, \qquad 0 \le \varphi \le 2\pi. \quad (1)$$

Hence, a representation of the discrete series is given by an integer n, which we may assume to be positive (for negative n we obtain equivalent representations). We denote the representation operators by $T_n^+(g)$ and $T_n^-(g)$.

---

† The existence of the limit of the sequence $\varphi_n(g)$ follows from the existence of the trace $\operatorname{Tr} T_\pi(g)$, as a generalized function in S. Besides, it is not hard to prove the existence of this limit directly.

Formula (6) of **§ 5.4** gives us:

$$\operatorname{Tr} T_n^+(g) - \operatorname{Tr} T_n^-(g) = -i \operatorname{sign} \beta \frac{\lambda_g^n + \lambda_g^{-n}}{|\lambda_g - \lambda_g^{-1}|}. \tag{2}$$

when λ_g and λ_g^{-1} are complex numbers;

$$\operatorname{Tr} T_n^+(g) - \operatorname{Tr} T_n^-(g) = 0. \tag{3}$$

when λ_g and λ_g^{-1} are real numbers.

On the other hand, by formula (8) of **§ 5.4** we have:

$$\begin{aligned}\operatorname{Tr} T_n^+(g) + \operatorname{Tr} T_n^-(g) &= \frac{1}{\pi i} \int_C \frac{\zeta^n \, d\zeta}{(\zeta - \lambda_g)(\zeta - \lambda_g^{-1})} \\ &= \frac{2}{\lambda_g - \lambda_g^{-1}} \left(\frac{1}{2\pi i} \int_C \frac{\zeta^n \, d\zeta}{\zeta - \lambda_g^{-1}} - \frac{1}{2\pi i} \int_C \frac{\zeta^n \, d\zeta}{\zeta - \lambda_g} \right),\end{aligned} \tag{4}$$

where the integration is taken over the unit circle C: $\zeta\bar{\zeta} = 1$.

When λ_g and λ_g^{-1} are real numbers, one of them lies inside C, and the other outside. In this case we find by the Cauchy formula that

$$\operatorname{Tr} T_n^+(g) + \operatorname{Tr} T_n^-(g) = \frac{2\lambda_g^{-n}}{\lambda_g - \lambda_g^{-1}}, \tag{5}$$

where λ_g is the eigenvalue of g of greater absolute value.

When λ_g and λ_g^{-1} are complex numbers and, hence, lie on the unit circle, the integrals in (4) diverge and must be interpreted in the sense of regularlizing values.

We sketch this regularization without proof. We note that

$$\frac{1}{2\pi i} \int_C \frac{\zeta^n \, d\zeta}{\zeta - \lambda} = \begin{cases} \lambda^n, & \text{when } |\lambda| < 1, \\ 0, & \text{when } |\lambda| > 1. \end{cases}$$

Naturally, on the limit set $|\lambda| = 1$ the value of this integral must be defined by

$$\frac{1}{2\pi i} \int \frac{\zeta^n \, d\zeta}{\zeta - \lambda} = \frac{1}{2} \lambda^n.$$

So we obtain

$$\operatorname{Tr} T_n^+(g) + \operatorname{Tr} T_n^-(g) = -\frac{\lambda_g^n - \lambda_g^{-n}}{\lambda_g - \lambda_g^{-1}}, \tag{6}$$

when λ_g and λ_g^{-1} are complex numbers.

Now we have explicit formulae for $\operatorname{Tr} T_n^+(g) - \operatorname{Tr} T_n^-(g)$ and for $\operatorname{Tr} T_n^+(g) + \operatorname{Tr} T_n^-(g)$. We write down the formulae for $\operatorname{Tr} T_n^+(g)$ and $\operatorname{Tr} T_n^-(g)$, which follow immediately from them.

On the set of matrices g with real eigenvalues we have

$$\operatorname{Tr} T_n^+(g) = \operatorname{Tr} T_n^-(g) = \frac{\lambda_g^{-n}}{\lambda_g - \lambda_g^{-1}}, \tag{7}$$

where λ_g is the eigenvalue of greater absolute value.

On the set of matrices g with complex eigenvalues we have

$$\operatorname{Tr} T_n^+(g) = \frac{e^{-in\varphi}}{e^{i\varphi} - e^{-i\varphi}} \tag{8}$$

$$\operatorname{Tr} T_n^-(g) = \frac{e^{-in\varphi}}{e^{-i\varphi} - e^{i\varphi}} \tag{9}$$

where φ *is determined from the condition that the matrix*

$$\begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix}$$

is conjugate to g.

§ 6. THE INVERSION FORMULA AND THE PLANCHEREL FORMULA ON G

1. Statement of the Problem. Let $f(g)$ be a finite function† on G. With every unitary representation $T_\pi(g)$ of the continuous or discrete series of G we associate the operator

$$T_\pi(f) = \int f(g)\, T_\pi(g)\, dg. \tag{1}$$

The operator function $T_\pi(f)$, which is defined on the set of representations $T_\pi(g)$ of the continuous and the discrete series of a group, is called the Fourier transform of $f(g)$. Our task is to find an inversion formula for (1), that is, to express $f(g)$ in terms of its Fourier transform.

It is more convenient to state this problem in terms of generalized functions:‡ *to expand the Delta-function* $\delta(g)$ *on* G *by the traces of the representations of the continuous and the discrete series.* In other words, we have to find a function $\mu(\pi)$ on the set of representations such that

$$\delta(g) = \int \mu(\pi) \operatorname{Tr} T_\pi(g)\, d\pi. \tag{2}$$

The integral is taken over the set of representations of the continuous and the discrete series.

Note that the representations $T_\pi(g)$ and $T_{\pi^{-1}}(g)$ are equivalent, so that $\operatorname{Tr} T_\pi(g) = \operatorname{Tr} T_{\pi^{-1}}(g)$. By virtue of this fact, the function

† For a connected field we assume that $f(g)$ is infinitely differentiable; for a disconnected field, that $f(g)$ is constant in sufficiently small domains on G.

‡ The generalized function $\delta(g)$ is defined as follows: $(\delta(g), f(g)) = f(e)$, where e is the unit element of the group.

$\mu(\pi)$ in (2) is not uniquely determined. It is natural to impose on the required function $\mu(\pi)$ the additional condition:

$$\mu(\pi) = \mu(\pi^{-1}).$$

The inversion formula for the function $f(g)$ on G and the Plancherel formula, which we are looking for, are immediate consequences of (2). For, *let $f(g)$ be a finite function on the group belonging to the space of basic functions. Then* (2) *leads to the inversion formula*

$$f(g_0) = \int \mu(\pi) \operatorname{Tr} (T_\pi(f) T_\pi^{-1}(g_0)) \, d\pi \tag{3}$$

and to the Plancherel formula

$$\int |f(g)|^2 = \int \mu(\pi) \operatorname{Tr} (T_\pi(f) T_\pi^*(f)) \, d\pi, \tag{4}$$

where T^* denotes the adjoint operator.

For by multiplying both sides of (2) by $f(gg_0)$ and integrating over g we find

$$f(g_0) = \int \mu(\pi) \operatorname{Tr}\left(\int f(gg_0) T_\pi(g) \, dg\right) d\pi.$$

After the change of variable $gg_0 = g_1$ we arrive precisely at (3).

To obtain the Plancherel formula we apply (3) to the function

$$F(g) = \int f(g_1)\overline{f(g_1 g^{-1})} \, dg_1.$$

For $g = e$ we find

$$F(e) = \int \mu(\pi) \operatorname{Tr} T_\pi(F) \, d\pi. \tag{5}$$

It is easy to check that

$$T_\pi(F) = T_\pi(f) \cdot T_\pi^*(f) \tag{6}$$

On the other hand, we have

$$F(e) = \int |f(g)|^2 \, dg. \tag{7}$$

Substituting (6) and (7) in (5) we obtain the required Plancherel formula.

Thus, our main task is to find the expansion of $\delta(g)$ by the traces of the irreducible representations

$$\delta(g) = \int \mu(\pi) \operatorname{Tr} T_\pi(g) \, d\pi. \tag{8}$$

This problem will be solved in **§ 6.2** for a disconnected field, and in **§ 6.5** for a connected field.

We give another expression for (8), by going over from the functions $\mu(\pi)$ and $\operatorname{Tr} T_\pi(g)$ to their Mellin transforms. For representations $T_\pi(g)$ of the continuous series we set

$$S(g;t) = \int \operatorname{Tr} T_\pi(g)\pi(t)\, d\pi, \tag{9}$$

$$\varphi(t) = \int \mu(\pi)\pi(t)\, d\pi, \tag{10}$$

where $t \in \mathbf{K}$ and the integral is taken over the group of multiplicative characters on $\mathbf{K}$.

For representations $T_{\pi_\tau}(g)$ of the discrete series corresponding to the extension $\mathbf{K}(\sqrt{\tau})$ of $\mathbf{K}$ we set

$$S_\tau(g;t) = \int \operatorname{Tr} T_{\pi_\tau}(g)\pi(t)\, d\pi_\tau\,, \tag{9'}$$

$$\varphi_\tau(t) = \int \mu(\pi_\tau)\pi_\tau(t), \tag{10'}$$

where t is a point on the circle $t\bar{t} = 1$ in the plane $\mathbf{K}(\sqrt{\tau})$ and the integral is taken over the group of characters π_τ on $t\bar{t} = 1$.

Then (8) takes the form

$$\delta(g) = \int \varphi(t)S(g;t)\, d^*t + \sum_\tau \int_{t\bar{t}=1} \varphi_\tau(t)S_\tau(g;t)\, d^*t + a \operatorname{Tr} T_0(g). \tag{11}$$

Here the sum is over the set of discrete series of G (hence, for a connected field it consists of three terms: $\tau = \mathfrak{p}$, $\varepsilon\mathfrak{p}$, and ε).

The last term in (11) is the trace of the singular representation of G (see **§ 3.7**); it occurs only for a disconnected field $\mathbf{K}$.

The traces of the representations of the continuous and the discrete series, and also their Mellin transforms, were found in **§ 5.** Substituting the expressions for $S(g;t)$ and $S_\tau(g;t)$ (see **§ 5.4**) we obtain the inversion formula in the following form:

$$\delta(g) = \theta(g)\,\frac{\varphi(\lambda_g) + \varphi(\lambda_g^{-1})}{|\lambda_g - \lambda_g^{-1}|} + a\left[\theta(g)\,\frac{|\lambda_g| + |\lambda_g^{-1}|}{\lambda_g - \lambda_g^{-1}} - 1\right]$$
$$+ 2\sum_\tau \int_{t\bar{t}=1} \varphi(t)\,\frac{\operatorname{sign}_\tau(\lambda_g + \lambda_g^{-1} - t - t^{-1})}{|\lambda_g + \lambda_g^{-1} - t - t^{-1}|}\, d^*t, \tag{12}$$

where λ_g and λ_g^{-1} are the eigenvalues of g; $\theta(g) = 1$ when λ_g, $\lambda_g^{-1} \in \mathbf{K}$, and otherwise $\theta(g) = 0$.

The functions $\varphi(t)$ and $\varphi_\tau(t)$ and the coefficient a are so far not determined; we have to find them.

2. The Inversion Formula for a Disconnected Field. Suppose that the elements of the matrices of G belong to a disconnected field $\mathbf{K}$. We denote by $T_\pi(g)$ the representations of the continuous series of G, by $T_0(g)$ its special representation (**§ 3.1** and **§ 3.8**), and by

$T_{\pi_\tau}(g)$ the representations of the discrete series corresponding to the extension $\mathbf{K}(\sqrt{\tau})$ of $\mathbf{K}$ ($\tau = \mathfrak{p}$, $\varepsilon\mathfrak{p}$, or ε). Here we derive the following *inversion formula*:

$$c\delta(g) = \int \mu(\pi) \operatorname{Tr} T_\pi(g)\, d\pi + 2 \operatorname{Tr} T_0(g) + \sum_\tau \int \mu(\pi_\tau) \operatorname{Tr} T_{\pi_\tau}(g)\, d\pi_\tau, \quad (1)$$

where†

$$\mu(\pi) = -\int_K \pi(t)\, |1 - t|^{-2}\, dt, \quad (2)$$

$$\mu(\pi_\varepsilon) = -\int_{t\bar{t}=1} \pi(t)\, |1 - t|^{-2}\, d^*t, \quad (2')$$

$$\mu(\pi_\tau) = -\int_{t\bar{t}=1,|1-t|<1} \pi(t)[|1 - t|^{-2} + 1]\, d^*t, \quad \tau = \mathfrak{p},\ \varepsilon\mathfrak{p} \quad (2'')$$

$c = \dfrac{2(q + 1)}{q^2}$. (The value of the constant c will be computed in **§ 6.4**). The integrals (2′) and (2″) are taken over the circle $t\bar{t} = 1$ in $K(\sqrt{\varepsilon})$ and $K(\sqrt{\tau})$, $\tau = \mathfrak{p}$ or $\varepsilon\mathfrak{p}$, respectively.

Note that all the integrals (2)–(2″) diverge so that they have to be understood in the sense of the regularizing value. For example, $\mu(\pi)$ is the value of the analytic function in ν,

$$\varphi(\nu) = -\int \pi(t)\, |1 - t|^\nu\, dt,$$

for $\nu = -2$ (see **§ 2.6**).

First of all we substitute in (1) the expressions for the traces of the representations and pass to the Mellin transforms with respect to π (see **§ 6.1**). As a result the formula assumes the form

$$\begin{aligned} c\delta(g) = &-\theta(g) \frac{2\,|\lambda_g|}{|\lambda_g - \lambda_g^{-1}|\,|1 - \lambda_g|^2} \\ &+ 2\left(\theta(g) \frac{|\lambda_g| + |\lambda_g^{-1}|}{|\lambda_g - \lambda_g^{-1}|} - 1\right) \\ &-2 \int_{t\bar{t}=1} \frac{\operatorname{sign}_\varepsilon (\lambda_g + \lambda_g^{-1} - t - t^{-1})}{|\lambda_g + \lambda_g^{-1} - t - t^{-1}|\,|1 - t|^2}\, d^*t \\ &-2 \sum_{\tau=\mathfrak{p},\varepsilon\mathfrak{p}} \int_{\substack{|1-t|<1\\ t\bar{t}=1}} \frac{\operatorname{sign}_\tau (\lambda_g + \lambda_g^{-1} - t - t^{-1})}{|\lambda_g + \lambda_g^{-1} - t - t^{-1}|\,|1 - t|^2}\, d^*t \\ &-2 \sum_{\tau=\mathfrak{p},\varepsilon\mathfrak{p}} \int_{t\bar{t}=1,|1-t|<1} \frac{\operatorname{sign}_\tau (\lambda_g + \lambda_g^{-1} - t - t^{-1})}{|\lambda_g + \lambda_g^{-1} - t - t^{-1}|}\, d^*t. \quad (3) \end{aligned}$$

† The norm $|t|$ in an expression of $\mathbf{K}$ is defined by $|t| = |t\bar{t}|^{1/2}$.

Here λ_g and λ_g^{-1} are the eigenvalues of g; $\theta(g) = 1$, when λ_g, $\lambda_g^{-1} \in \mathbf{K}$, $\theta(g) = 0$ otherwise.

The derivation of (3) is given in two stages. First, we check that the expression on the right-hand side of (3), which we denote briefly by $I(g)$, is zero for $\lambda_g = \pm 1$. Then we show that $I(g)$ is zero for all $g \neq e$ so that $I(g)$ is concentrated at the point $g = e$. It follows immediately that $I(g) = c\delta(g)$. The coefficient c will be computed in **§ 6.4.**

The fact that $I(g) = 0$ for $\lambda_g \neq \pm 1$ can be verified directly, by computing the integrals in (3). Here we have to discuss separately the following possible cases:

1. $\lambda_g \in \mathbf{K}, \quad |\lambda_g| \neq 1,$
2. $\lambda_g \in \mathbf{K}, \quad |\lambda_g| = 1, \quad |\lambda_g - 1| = |\lambda_g + 1| = 1,$
3. $\lambda_g \in \mathbf{K}, \quad |\lambda_g| = 1, \quad |\lambda_g - 1| < 1,$
4. $\lambda_g \in \mathbf{K}, \quad |\lambda_g| = 1, \quad |\lambda_g + 1| < 1,$
5. $\lambda_g \in \mathbf{K}(\sqrt{\tau}), \quad \tau = \mathfrak{p}, \varepsilon\mathfrak{p}, \quad |\lambda_g - 1| < 1,$
6. $\lambda_g \in \mathbf{K}(\sqrt{\tau}), \quad \tau = \mathfrak{p}, \varepsilon\mathfrak{p}, \quad |\lambda_g + 1| < 1,$
7. $\lambda_g \in \mathbf{K}(\sqrt{\varepsilon}), \quad |\lambda_g - 1| < 1,$
8. $\lambda_g \in \mathbf{K}(\sqrt{\varepsilon}), \quad |\lambda_g + 1| < 1,$
9. $\lambda_g \in \mathbf{K}(\sqrt{\varepsilon}), \quad |\lambda_g - 1| = |\lambda_g + 1| = 1,$

Below we give a table of the values of the integrals occurring in (3). The evaluation of some of these integrals will be given in **3.** The detailed verification of the fact that $I(g) = 0$ for $\lambda_g \neq \pm 1$ is left to the reader.

Derivation of the formulae. Notation: λ and λ^{-1} are the eigenvalues of g; $\nu = \lambda + \lambda^{-1} - 2$; q is the order of the finite residue class field O/P associated with $\mathbf{K}$ (see **§ 1.3**); $\left(\frac{a}{q}\right)$ is the Legendre symbol ($a \neq 0$ is an element of the finite field F of order q): $\left(\frac{a}{q}\right) = 1$ if a is the square of an element of F; $\left(\frac{a}{q}\right) = -1$ if a is not a square. $\left(\frac{-1}{q}\right) = 1$ when $q \equiv 1 \bmod 4$; $\left(\frac{-1}{q}\right) = -1$ when $q \equiv 3 \bmod 4$.

1. Value of the integral

$$I_\tau^{(1)}(\lambda) = \int\limits_{t\bar{t}=1, |1-t|<1} \frac{\operatorname{sign}_\tau(\lambda + \lambda^{-1} - t - t^{-1})}{|\lambda + \lambda^{-1} - t - t^{-1}|\,|1 - t|^2} d^*t$$

If $|\nu| \geq 1$, then

$$I_\tau^{(1)}(\lambda) = c_\tau' \frac{\operatorname{sign}_\tau \nu}{|\nu|}$$

where $c_\tau' = -\frac{1}{2}$ for $\tau = \mathfrak{p}$ or $\varepsilon\mathfrak{p}$; $c_\varepsilon' = -\dfrac{q}{q+1}$.

If $|\nu| < 1$, then the values $I_\tau^{(1)}(\lambda)$ are given below:

a. $\lambda \in \mathbf{K}$

$$I_{\mathfrak{p}}^{(1)}(\lambda) = I_{\varepsilon\mathfrak{p}}^{(1)}(\lambda) = -\frac{q^2+1}{2(q^2+q+1)}|\nu|^{-3/2} - \frac{q}{2(q^2+q+1)},$$

$$I_{\varepsilon}^{(1)}(\lambda) = -\frac{q}{q^2+q+1}|\nu|^{-3/2} - \frac{q^3}{(q+1)(q^2+q+1)}.$$

b. λ a point on the circle $t\bar{t} = 1$ in $\mathbf{K}(\sqrt{\mathfrak{p}})$

$$I_{\mathfrak{p}}^{(1)}(\lambda) = -\frac{1}{2\sqrt{q}}\left(\frac{q^2+q}{q^2+q+1} + \left(\frac{-1}{q}\right)(q+1)\right)|\nu|^{-3/2} - \frac{q}{2(q^2+q+1)},$$

$$I_{\varepsilon\mathfrak{p}}^{(1)}(\lambda) = -\frac{1}{2\sqrt{q}}\left(\frac{q^2+q}{q^2+q+1} - \left(\frac{-1}{q}\right)(q+1)\right)|\nu|^{-3/2} - \frac{q}{2(q^2+q+1)},$$

$$I_{\varepsilon}^{(1)}(\lambda) = \frac{q^2+q}{\sqrt{q}\,(q^2+q+1)}|\nu|^{-3/2} - \frac{q^3}{(q+1)(q^2+q+1)}.$$

c. λ a point on the circle $t\bar{t} = 1$ in $\mathbf{K}(\sqrt{\varepsilon\mathfrak{p}})$

$$I_{\mathfrak{p}}^{(1)}(\lambda) = -\frac{1}{2\sqrt{q}}\left(\frac{q^2+q}{q^2+q+1} - \left(\frac{-1}{q}\right)(q+1)\right)|\nu|^{-3/2} - \frac{q}{2(q^2+q+1)},$$

$$I_{\varepsilon\mathfrak{p}}^{(1)}(\lambda) = -\frac{1}{2\sqrt{q}}\left(\frac{q^2+q}{q^2+q+1} + \left(\frac{-1}{q}\right)(q+1)\right)|\nu|^{-3/2} - \frac{q}{2(q^2+q+1)},$$

$$I_{\varepsilon}^{(1)}(\lambda) = \frac{q^2+q}{\sqrt{q}\,(q^2+q+1)}|\nu|^{-3/2} - \frac{q^3}{(q+1)(q^2+q+1)}.$$

d. λ a point on the circle $t\bar{t} = 1$ in $\mathbf{K}(\sqrt{\varepsilon})$

$$I_{\mathfrak{p}}^{(1)}(\lambda) = I_{\varepsilon\mathfrak{p}}^{(1)}(\lambda) = \frac{(q+1)^2}{2(q^2+q+1)}|\nu|^{-3/2} - \frac{q}{2(q^2+q+1)},$$

$$I_{\varepsilon}^{(1)}(\lambda) = -\frac{(q+1)^2}{q^2+q+1}|\nu|^{-3/2} - \frac{q^3}{(q+1)(q^2+q+1)}.$$

2. Value of the integral

$$I_\tau^{(2)}(\lambda) = \int\limits_{t\bar{t}=1,\ |1-t|<1} \frac{\operatorname{sign}_\tau(\lambda + \lambda^{-1} - t - t^{-1})}{|\lambda + \lambda^{-1} - t - t^{-1}|}\, d^*t.$$

If $|\nu| \geq 1$, then

$$I_\tau^{(2)}(\lambda) = c_\tau'' \frac{\operatorname{sign}_\tau \nu}{|\nu|},$$

where $c_\tau'' = 1/2$ for $\tau = \mathfrak{p}$ or $\varepsilon\mathfrak{p}$; $c_\varepsilon'' = \dfrac{1}{q+1}$.

If $|\nu| < 1$, then the value $I_\tau^{(2)}(\lambda)$ is given below:

a. $\lambda \in \mathbf{K}$

$$I_{\mathfrak{p}}^{(2)}(\lambda) = I_{\varepsilon\mathfrak{p}}^{(2)}(\lambda) = |\nu|^{-1/2} - 1/2, \qquad I_{\varepsilon}^{(2)}(\lambda) = |\nu|^{-1/2} - \frac{q}{q+1}.$$

b. λ a point on the circle $t\bar{t} = 1$ in $\mathbf{K}(\sqrt{\tau})$, $\tau = \mathfrak{p}$, $\varepsilon\mathfrak{p}$, or ε

$$I_{\mathfrak{p}}^{(2)}(\lambda) = I_{\varepsilon\mathfrak{p}}^{(2)}(\lambda) = -\tfrac{1}{2}, \qquad I_{\varepsilon}^{(2)}(\lambda) = -\frac{q}{q+1}.$$

3. Value of the integral

$$I_{\varepsilon}^{\beta}(\lambda) = \int_{A_\beta} \frac{\operatorname{sign}_{\varepsilon}(\lambda + \lambda^{-1} - t - t^{-1})}{|\lambda + \lambda^{-1} - t - t^{-1}|}\, d^*t,$$

where the integral is taken over the component A_β of the circle $t\bar{t} = 1$ in $\mathbf{K}(\sqrt{\varepsilon})$ defined by the condition $|t - \beta| < 1$. Here β is a point of the circle for which $|\beta + 1| = |\beta - 1| = 1$:

$$I_{\varepsilon}^{\beta}(\lambda) = \frac{1}{q-1}\frac{\operatorname{sign}_{\varepsilon} \nu}{|\nu|} \qquad \text{when} \quad |\lambda| \neq 1,$$

$I_{\varepsilon}^{\beta}(\lambda) = -\dfrac{q-1}{2(q+1)}$, when λ is a point on the circle $t\bar{t} = 1$ in $\mathbf{K}(\sqrt{\varepsilon})$ and either $|\lambda - \beta| < 1$ or $|\lambda - \bar{\beta}| < 1$, $I_{\varepsilon}^{\beta}(\lambda) = \dfrac{1}{q+1}$ in all the remaining cases.

Now we have to verify that $I(g) = 0$ for all the matrices $g \neq e$.

Observe that the integrals in (3) reduce to one of the following forms $a\,|\nu|^{-3/2} + b$, $a\,|\nu|^{-1/2} + b$, $a\,|\nu'|^{-1/2} + b$, where

$$\nu = \lambda_g + \lambda_g^{-1} - 2, \qquad \nu' = \lambda_g + \lambda_g^{-1} + 2$$

(see the derivation of the formulae). In fact, they can all be simplified. The cases $\lambda_g = 1$ and $\lambda_g = -1$ are special, because then $\nu = 0$ and $\nu' = 0$, respectively. Therefore they require a separate investigation. Let us show that the functions $|\nu|^{-3/2}$, $|\nu|^{-1/2}$ and $|\nu'|^{-1/2}$, regarded as generalized functions on the group, have no singularity for $g \neq e$. In other words, $(|\nu|^{-3/2}, f)$, $(|\nu|^{-1/2}, f)$ and $(|\nu'|^{-1/2}, f)$ are continuous functionals in the subspace of finite functions f on G that are equal to zero in a neighborhood of e.† Hence, it follows easily that the generalized function $I(g)$ is concentrated at e.

It is not hard to check that the integrals

$$(|\nu|^{-1/2}, f) = \int |\nu|^{-1/2} f(g)\, dg$$

and

$$(|\nu'|^{-1/2}, f) = \int |\nu'|^{-1/2} f(g)\, dg$$

converge in the usual sense for every finite function $f(g)$. Therefore it is sufficient to treat the integral

$$(|\nu|^{-3/2}, f) = \int |\nu|^{-3/2} f(g)\, dg.$$

† We recall that when we speak of finite functions, we assume in addition that the functions are "piecewise constant," that is, that they are constant in a sufficiently small neighborhood of every point g.

This integral must be understood in the sense of the regularizing value: $(|\nu|^{-3/2}, f)$ is the value of the analytic function of s

$$\varphi(s) = \int |\nu|^s f(g)\, dg \tag{4}$$

for $s = -3/2$. Our aim is to show that if $f(g) = 0$ in a neighborhood of $g = e$, then the function $\varphi(s)$ has no singularity for $s = -3/2$.

We may assume, without loss of generality, that $f(g)$ is concentrated in a sufficiently small neighborhood of a matrix $g_0 \neq e$ with the eigenvalues $\lambda_g = \lambda_g^{-1} = 1$.

We introduce a coordinate system in this neighborhood. The matrix $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ belonging to this neighborhood has at least one of the elements β or γ different from zero, say $\gamma \neq 0$. Then we can take as coordinates in a neighborhood of g_0 the values γ, α, and $\nu = \alpha + \delta - 2$. In these coordinates the formula (4) for $\varphi(s)$ takes the form

$$\varphi(s) = \int |\nu|^s f(\nu, \alpha, \gamma) \frac{d\alpha\, d\gamma}{|\gamma|}\, d\nu.$$

But we know from **§ 1.3** that the only singularity of the generalized function $|\nu|^s$ is a pole at $s = -1$. Consequently, $\varphi(s)$ has no singularity at $s = -3/2$.

So we have shown that the generalized function $I(g)$ on the right side of (3) is concentrated at $g = e$. Hence it follows that $I(g) = c\delta(g)$, where c is a constant (see **§ 2.2**), and the inversion formula (1) is proved.

3. Computation of Certain Integrals. We show how to compute the integrals that occur in the derivation of the formulae in **§ 6.2**. We take as an example

$$I_{\mathfrak{p}}^{(1)}(\lambda) = \int\limits_{t\bar{t}=1,\ |t-1|<1} \frac{\operatorname{sign}_{\mathfrak{p}}(\lambda + \lambda^{-1} - t - t^{-1})}{|\lambda + \lambda^{-1} - t - t^{-1}|\, |1 - t|^2}\, d^*t \tag{1}$$

If $|\lambda + \lambda^{-1} - 2| > 1$, then for every t, $t\bar{t} = 1$, $|1 - t| < 1$, we have

$\operatorname{sign}_\tau (\lambda + \lambda^{-1} - t - t^{-1}) = \operatorname{sign}_\tau (\lambda + \lambda^{-1} - 2)$,

$|\lambda + \lambda^{-1} - t - t^{-1}| = |\lambda + \lambda^{-1} - 2|$, so that the integral (1) simplifies considerably:

$$I_{\mathfrak{p}}^{(1)}(\lambda) = \frac{\operatorname{sign}_{\mathfrak{p}}(\lambda + \lambda^{-1} - 2)}{|\lambda + \lambda^{-1} - 2|} \int \frac{d^*t}{|1 - t|^2}.$$

We give the details of the most complicated case when $|\lambda + \lambda^{-1} - 2| < 1$; then we have $|\lambda| = 1$ and $|\lambda - 1| < 1$. Suppose, for the sake of definiteness, that λ lies on the circle $t\bar{t} = 1$ in $\mathbf{K}(\sqrt{\mathfrak{p}})$. In the other possible cases the integral is computed similarly. According to **§ 1.8** the elements of the circle $t\bar{t} = 1$,

$|1 - t| < 1$ in $\mathbf{K}(\sqrt{\mathrm{p}})$ have the following parametric representations:†

$$t = \frac{1 + \sqrt{\mathrm{p}}x}{1 - \sqrt{\mathrm{p}}x} = \frac{1 + \mathrm{p}x^2}{1 - \mathrm{p}x^2} + \sqrt{\mathrm{p}}\,\frac{2x}{1 - \mathrm{p}x^2},$$

where x ranges over all the integers in $\mathbf{K}$ (that is, $|x| \leq 1$). It is easy to check that $d^*t = \frac{1}{2}dx$, where dx is the invariant measure on $\mathbf{K}^+$.

We can represent λ in the same form:

$$\lambda = \frac{1 + \sqrt{\mathrm{p}}x_0}{1 - \sqrt{\mathrm{p}}x_0}$$

Substituting these expressions in (1) and transmitting the variable t to x, we find

$$I_{\mathrm{p}}^{(1)}(\lambda) = \frac{1}{2} \int\limits_{|x| \leq 1} \frac{\operatorname{sign}_{\mathrm{p}}\left(2\,\dfrac{1 + \mathrm{p}x_0^2}{1 - \mathrm{p}x_0^2} - 2\,\dfrac{1 + \mathrm{p}x^2}{1 - \mathrm{p}x^2}\right)}{\left|2\,\dfrac{1 + \mathrm{p}x_0^2}{1 - \mathrm{p}x_0^2} - 2\,\dfrac{1 + \mathrm{p}x^2}{1 - \mathrm{p}x^2}\right| \cdot \left|\dfrac{2\sqrt{\mathrm{p}}x}{1 - \sqrt{\mathrm{p}}x}\right|^2}\, dx.$$

This expression can be simplified considerably, because the functions $\operatorname{sign}_p x$ and $|x|$ depend only on the first terms of the expansion of x. We obtain

$$I_{\mathrm{p}}^{(1)}(\lambda) = \frac{1}{2} \int\limits_{|x| \leq 1} \frac{\operatorname{sign}_{\mathrm{p}}(\mathrm{p}x_0^2 - \mathrm{p}x^2)}{|\mathrm{p}x_0^2 - \mathrm{p}x^2|\,|\mathrm{p}x^2|}\, dx. \tag{2}$$

Now let us compute this integral. First we add to, and subtract from, $I_{\mathrm{p}}^{(1)}(\lambda)$ the integral

$$\frac{1}{2} \int\limits_{|x| > q^k} \frac{\operatorname{sign}_{\mathrm{p}}(\mathrm{p}x_0^2 - \mathrm{p}x^2)}{|\mathrm{p}x_0^2 - \mathrm{p}x^2|\,|\mathrm{p}x^2|}\, dx = \frac{1}{2}\int \frac{dx}{|\mathrm{p}x^2|^2}.$$

After elementary transformations we obtain

$$I_{\mathrm{p}}^{(1)}(\lambda) = \frac{q^2}{2}\, |x_0|^{-3} \int\limits_{K} \frac{\operatorname{sign}_{\mathrm{p}}(\mathrm{p} - \mathrm{p}x^2)}{|1 - x^2|\,|x^2|}\, dx - \frac{q^2}{2} \int\limits_{|x| > 1} \frac{dx}{|x|^4}. \tag{3}$$

The second integral is easily computed:

$$\int\limits_{|x| > 1} \frac{dx}{|x|^4} = \sum_{k=1}^{\infty} \int\limits_{|x| = q^k} \frac{dx}{|x|^4} = \left(1 - \frac{1}{q}\right) \sum_{k=1}^{\infty} q^{-3k} = \frac{1}{q(q^2 + q + 1)}$$

Now we turn to the evaluation of the first integral in (3).

---

† The transformation $t = \dfrac{1 + \sqrt{\mathrm{p}}x}{1 - \sqrt{\mathrm{p}}x}$ is an analogue of the Cayley transformation for the field of real numbers. Note that when x ranges over the domain $|x| > 1$, then t ranges over another component of the circle $t\bar{t} = 1 : |1 + t| < 1$.

The normalizing factor $\frac{1}{2}$ in the formula for the measure is explained by the fact that $\displaystyle\int\limits_{|x| \leq 1} dx = 1$, whereas $\displaystyle\int\limits_{t\bar{t}=1, |1-t|<1} d^*t = \frac{1}{2}$.

First, we split it into three terms:

$$I = \int_K \frac{\operatorname{sign}_\mathfrak{p}(\mathfrak{p} - \mathfrak{p}x^2)}{|1 - x^2|\,|x^2|}\,dx = \int_{|x|>1} \frac{\operatorname{sign}_\mathfrak{p}(\mathfrak{p} - \mathfrak{p}x^2)}{|1 - x^2|\,|x^2|}\,dx$$
$$+ \int_{|x|<1} \frac{\operatorname{sign}_\mathfrak{p}(\mathfrak{p} - \mathfrak{p}x^2)}{|1 - x^2|\,|x^2|}\,dx + \int_{|x|=1} \frac{\operatorname{sign}_\mathfrak{p}(\mathfrak{p} - \mathfrak{p}x^2)}{|1 - x^2|\,|x^2|}\,dx. \tag{4}$$

We get†

$$I = \int_{|x>1|} \frac{dx}{|x|^4} + \operatorname{sign}_\mathfrak{p}\mathfrak{p} \int_{|x|<1} \frac{dx}{|x|^2} + \operatorname{sign}_\mathfrak{p}\mathfrak{p} \int_{|x|=1} \frac{\operatorname{sign}_\mathfrak{p}(1 - x^2)}{|1 - x^2|}\,dx$$
$$= \frac{1}{q(q^2 + q + 1)} - \left(\frac{-1}{q}\right) + \left(\frac{-1}{q}\right) \int_{|x|=1} \frac{\operatorname{sign}_\mathfrak{p}(1 - x^2)}{|1 - x^2|}\,dx. \tag{5}$$

The last integral can be computed by splitting the set of elements x, $|x| = 1$, into residue classes modulo $\mathfrak{p}$. We find

$$\int_{|x|=1} \frac{\operatorname{sign}_\mathfrak{p}(1 - x^2)}{|1 - x^2|}\,dx = \frac{1}{q} \sum_{a \neq 0, \pm 1} \left(\frac{1 - a^2}{q}\right)$$
$$+ \int_{|x|=1, |1-x|<1} \frac{\operatorname{sign}_\mathfrak{p}(1 - x^2)}{|1 - x^2|}\,dx + \int_{|x|=1, |1+x|<1} \frac{\operatorname{sign}_\mathfrak{p}(1 - x^2)}{|1 - x^2|}\,dx. \tag{6}$$

Here the sum is taken over the elements a of the finite field O/P of order q, other than 0 and ± 1. A straightforward calculation shows that

$$\frac{1}{q} \sum_{a \neq 0, \pm 1} \left(\frac{1 - a^2}{q}\right) = -\frac{1}{q}\left[1 + \left(\frac{-1}{q}\right)\right].$$

On the other hand, it is easy to show that each of the integrals in (6) is zero. So we find

$$I = \frac{1}{q(q^2 + q + 1)} - \left(\frac{-1}{q}\right) - \frac{1}{q}\left[1 + \left(\frac{-1}{q}\right)\right].$$

Substituting this expression in formula (3) for $I_\mathfrak{p}^{(1)}(\lambda)$ we obtain finally

$$I_\mathfrak{p}^{(1)}(\lambda) = -\frac{q}{2}\left(\frac{q^2 + q}{q^2 + q + 1} + \left(\frac{-1}{q}\right)(q + 1)\right)|x_0|^{-3} - \frac{q}{2(q^2 + q + 1)}.$$

To reach precise agreement of this formula with that of the table (p. 213, case b) we observe that

$$x_0 = \frac{1}{\sqrt{\mathfrak{p}}} \frac{\lambda - 1}{\lambda + 1}, \quad \text{and therefore} \quad |x_0| = q^{1/2}\,|\lambda - 1| = q^{1/2}\,|\lambda + \lambda^{-1} - 2|^{1/2}.$$

† The integral $\int_{|x|<1} |x|^{-2}\,dx$ is to be understood here in the sense of the regularizing value, as the value of the analytic function of s, $\varphi(s) = \int_{|x|<1} |x|^{-s}\,dx$ for $s = 2$; $\left(\frac{a}{q}\right)$ is the Legendre symbol (see p. 213).

4. Computation of the Constant c in the Inversion Formula. To obtain the value of the constant c in the inversion formula of **§ 6.2** we apply this formula to an arbitrary fixed function $f(g)$ on G.

Let U be the subgroup of matrices of G whose elements are integers of **K**. Obviously U is compact and is an open set in G.

We take the function $f(g)$ that is equal to unity on U and to zero outside U, and apply the inversion formula to it.

It can be shown that $\operatorname{Tr} T_\pi(f) \neq 0$ only for the representations of the continuous series that correspond to the character $\pi(t) = |t|^{i\rho}$. Consequently, in the inversion formula for $f(g)$ only the terms corresponding to these representations occur. As a result we find

$$c = \int_U \int \mu(\pi_\rho) \operatorname{Tr} T_{\pi_\rho}(g)\, d\pi_\rho\, dg, \tag{1}$$

where $\pi_\rho(g) = |t|^{i\rho}$,

$$\mu(\pi_\rho) = -\int_K |t|^{i\rho}\, |1 - t|^{-2}\, dt, \tag{2}$$

Now we evaluate the integral (1). We recall that

$$\operatorname{Tr} T_{\pi_\rho}(g) = \theta(g)\, \frac{|\lambda_g|^{i\rho} + |\lambda_g|^{-i\rho}}{|\lambda_g - \lambda_g^{-1}|},$$

where λ_g and λ_g^{-1} are the eigenvalues of g; $\theta(g) = 1$ when $\lambda_g, \lambda_g^{-1} \in K$, $\theta(g) = 0$ otherwise. Since $|\lambda_g| = 1$ for matrices g belonging to the compact subgroup U, we have

$$\int_U \operatorname{Tr} T_{\pi_\rho}(g)\, dg = 2 \int_U \theta(g)\, |\lambda_g - \lambda_g^{-1}|^{-1}\, dg.$$

So we see that this integral does not depend on π_ρ. Therefore,

$$c = -2 \int_U \theta(g)\, |\lambda_g - \lambda_g^{-1}|^{-1}\, dg \int_K |t|^{i\rho}\, |1 - t|^{-2}\, dt\, d\pi_\rho\,. \tag{3}$$

The second integral in (3) is easily computed:†

$$\int_K |t|^{i\rho}\, |1 - t|^{-2}\, dt\, d\pi_\rho = \frac{q}{q-1} \int_{|t|=1} |1 - t|^{-2}\, dt = -\frac{2}{q-1}\, *.$$

We do not evaluate here the first integral, but give only the final result:‡

$$\int_U \theta(g)\, |\lambda_g - \lambda_g^{-1}|^{-1}\, dg = \frac{(q+1)(q-1)}{2q^2}\,.$$

---

† The factor $q(q-1)^{-1}$ arises as a consequence of the chosen normalization of $d\pi_\rho$. For we postulate that

$$\int \pi_\rho(t)\, |t|^{-1}\, dt\, d\pi_\rho = 1$$

(see **§ 2.9**).

‡ This integral may be computed by representing the matrix g of U in the form $g = z^{-1}\delta\zeta z$, where $z = \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix}$, $\delta = \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & \lambda \end{pmatrix}$, $\zeta = \begin{pmatrix} 1 & \zeta \\ 0 & 1 \end{pmatrix}$, and taking the elements z, ζ, and λ as parameters of g. It then turns out that $dg = |\lambda_g - \lambda_g^{-1}|\, d\lambda\, d\zeta\, dz$.

And so we have the final value of the constant c:

$$c = \frac{2(q+1)}{q^2}.$$

5. The Inversion Formula for Connected Fields. We now consider the case of a connected field **K**, that is, the field of complex or of real numbers. It can be shown that then the inversion formula is similar to that for a disconnected field.

If G is the group of complex matrices, then the inversion formula takes the form

$$\delta(g) = \int \mu(\pi) \operatorname{Tr} T_\pi(g)\, d\pi, \tag{1}$$

where

$$\mu(\pi) = c \int \pi(t)\, |1 - t|^{-2}\, dt. \tag{2}$$

The integration in (2) is taken over the complex plane of t; the integral (2) must be understood in the sense of the regularizing value (see **§ 2.9**).†

If G is the group of real matrices, then the inversion formula takes the form

$$\delta(g) = \int \mu(\pi) \operatorname{Tr} T_\pi(g)\, d\pi + \sum_n \mu(\pi_n) \operatorname{Tr} T_{\pi_n}(g), \tag{3}$$

where

$$\mu(\pi) = c \int_{-\infty}^{\infty} \pi(t)\, |1 - t|^{-2}\, dt. \tag{4}$$

$$\mu(\pi_n) = c \int_{t\bar{t}=1} \pi_n(t)\, |1 - t|^{-2}\, d^*t. \tag{5}$$

Here $\pi(t)$ are the multiplicative characters of the group of real numbers, $T_\pi(g)$ are the corresponding representations of the continuous series; $\pi_n(t)$ are the characters of the group of rotations of a circle, $T_{\pi_n}(g)$ the corresponding representations of the discrete series; d^*t is the measure on the circle $t\bar{t} = 1$, normalized by the condition $\int d^*t = 1$.

The inversion formulae (1) and (3) can be derived just as in the case of a disconnected field because the integrals in the formulae can be evaluated explicitly. We have to verify that the expression $I(g)$ on the right-hand side of (1) or (3), respectively, is a function concentrated at $g = e$. After that it is not hard to show that $I(g) = c\delta(g)$. We omit the detailed derivation of (1) and (3).

† We recall that in our notation $|z|$ denotes the *square* of the modulus of the complex number z.

Note that the computation of the integral for $\mu(\pi)$ leads to essentially distinct expressions for the field of complex and the field of real numbers. In fact, for the field of complex numbers, every character $\pi(t)$ is of the form

$$\pi(t) = t^{n+i\rho/2}\bar{t}^{-n+i\rho/2},$$

where n is an integer and ρ a real number. Evaluating the integral (2) we find

$$\mu(\pi) = c(\rho^2 + n^2).$$

Now we take the field of real numbers. There are two types of multiplicative characters on the real line: the characters

$$\pi(t) = |t|^{i\rho},$$

where ρ is a real number, and the characters

$$\pi(t) = |t|^{i\rho} \operatorname{sign} t.$$

Evaluating the integral (4) we find that

$$\mu(\pi) = c\pi\rho \tanh \frac{\pi\rho}{2}$$

for a character of the first type, and

$$\mu(\pi) = c\pi\rho \coth \frac{\pi\rho}{2}$$

for a character of the second type.

On the circle $t\bar{t} = 1$ the characters $\pi_n(t)$ have the form

$$\pi_n(t) = e^{in \arg t}.$$

Computing the integral (5) we find that

$$\mu(\pi_n) = c\,|n|.$$

APPENDIX TO CHAPTER 2

1. Some Facts from the Theory of Operator Rings in Hilbert Space. Here we confine ourselves to the statement of results. Their proofs can be found, for example, in Dixmier [14] or Naimark, [52].

A von Neumann algebra is a ring R of operators in Hilbert space satisfying the following conditions:

1. R contains the identity operator;

2. If $A \in R$, then $A^* \in R$, where A^* is the operator adjoint to A;

3. R is closed in the weak operator topology.

For every set S of operators in Hilbert space we denote the collection of all operators that commute with the operators from S by S'. It is easy to verify that when S contains with each operator its conjugate operator, then S' is a von Neumann algebra. If the original set S is a von Neumann algebra, then $(S')' = S$.

A von Neumann algebra R is called a *factor* if $R \cap R'$ consists only of scalar operators. Every von Neumann algebra can be realized canonically as a direct sum (possibly continuous) of factors.

If H is a finite-dimensional Hilbert space, then all factors may be obtained by the following construction. We represent H as a tensor product of two spaces H_1 and H_2: $H = H_1 \otimes H_2$. For R we take the set of all operators of the form $A \otimes 1$. Then R' consists of the operators of the form $1 \otimes B$, and the intersection $R \cap R'$ obviously contains only scalar operators. Of course, this construction is also applicable to infinite-dimensional spaces. But in an infinite-dimensional space not all the factors can be obtained in this way. Those that can be obtained are called factors of type I.

It is customary to classify factors by the structure of the set of projection operators in the factor. Factors of type I are characterized by the property that they are all minimal projections in this set (corresponding to operators of the form $P \otimes 1$, where P is a projection operator of rank 1).

In factors of type II there are no minimal projections, but there are so-called finite projections, that is, projections that are not adjoint to their regular part.

In factors of type III there are neither minimal, nor finite projections.

A representation T of a group G is called a factor-representation if the ring generated by all the operators $T(g)$, $g \in G$, is a factor. We say that a group G belongs to type I if each factor representation of it is generated by a factor of type I and is, therefore, a multiple of an irreducible representation.

Let G_1 and G_2 be two groups and T an irreducible representation of their direct product $G = G_1 \cdot G_2$. We denote by R_i the ring generated by the operators $T(g), g \in G_i \subset G$. Then $R_1' \cap R_2' = \{\lambda E\}$ by the irreducibility of T. Moreover, $R_1 \subset R_2'$, because the elements of G_1 and G_2 commute. Hence, it follows that

$$R_1 \cap R_1' \subset R_2' \cap R_1' = \{\lambda E\}$$

so that R_1 is a factor. The same is true for R_2.

If at least one of the groups G_1 and G_2 is of type I, then the restriction of T to this group is a multiple of an irreducible representation. In this case it is easy to show that the representation T is of the form $T_1 \otimes T_2$, where T_i are representations of G_i.

In the general case this is not true. One of the simplest examples can be constructed as follows. Let G be a countable discrete group in which each class of conjugate elements, except the unit class, is infinite. (An example of such a group is the group of rational matrices of the form $\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$, or the group of permutations of a countable set that shift only a finite number of points.) We consider the representation T of $G \cdot G$ in the space $L^2(G)$ given by the formula: $T(g_1, g_2)f(g)) = f(g_1^{-1}gg_2)$. This representation is irreducible, but cannot be written in the form $T_1 \otimes T_2$, where the T_i are representations of G. The restriction of T to G is not a multiple of an irreducible representation and is a factor of type II.

We now assume that the irreducible representation T of $G = G_1 \cdot G_2$ has the following property.

There exists a function $\varphi \in L_1(G_1 \cdot G_2)$ of the form

$$\varphi(g_1, g_2) = \varphi_1(g_1)\,\varphi_2(g_2)$$

for which

$$T(\varphi) = \int \varphi(g_1, g_2)\, T(g_1, g_2)\, dg_1\, dg_2$$

is a nonzero completely continuous operator. We show that then T is a tensor product of irreducible representations of G_1 and G_2.

First we observe that when φ satisfies condition (A), so does the function† $\psi = \varphi * \varphi^*$. It also is of the form $\psi_1(g_1)\psi_2(g_2)$, where $\psi_i = \varphi * \varphi_i^*$. The operators $A_i = \int \psi_i(g)\,T(g)\,dg$ are nonnegative, commute with each other, and their product is completely continuous. Hence, it is easy to deduce that each operator A_i has a pure point spectrum. Furthermore, if H_i is an eigenspace for A_i corresponding to a nonzero eigenvalue, then the intersection $H_1 \cap H_2$ is finite-dimensional, because all the vectors in this intersection are eigenvectors for A_1A_2 with nonzero eigenvalues. The projection operator P_i onto H_i belongs to the factor R_i generated by the operators $T(g)$, $g \in G_i$. It is well known (see, for example, Naimark [52], Chapter 2, **§ 3**) that every factor R has the following property: if the operators X and Y lie in R and R', respectively, then the product XY is zero if and only if one of the factors is zero.

Among all the nonzero projection operators in R_2 we now consider an operator P for which the rank of the product P_1P takes the smallest value. (The existence of such an operator P is guaranteed by the fact that the rank of P_1P_2 is finite.) We show that P is a minimal projector in R_2. For if P can be represented in the form

† We use the standard notation for the operators of multiplication and involution in the group ring of G. See, for example, Naimark [52].

$P' + P''$, where P' and P'' are orthogonal projectors in R_2, then at least one of the operators P_1P' or P_1P'' has a rank less than P_1P, which is impossible. Hence, R_2 is a factor of type I. As we have seen above, this implies that T is of the form $T_1 \otimes T_2$, where T_i are irreducible representations of G_i.

2. The Connection Between the Unitary Representations of the Group $\tilde{G}$ of all Nonsingular Matrices of Order 2 and the Subgroup of Matrices of the Form $\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$. In this and the next subsections we establish some properties of irreducible unitary representations of the group of matrices of order 2 with elements from a disconnected topological field **K**. It is convenient to consider instead of the group G of unimodular matrices the group $\tilde{G}$ of all nonsingular matrices. The transition from $\tilde{G}$ to G proceeds without difficulty (see **5**).

In $\tilde{G}$ we consider the subgroup G_0 of matrices of the form $g_{a,b} = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$. Our object is to prove the following proposition.

THEOREM 1. *Every unitary irreducible representation $T(g)$ of G remains irreducible upon restriction to G_0.*

We enumerate the unitary irreducible representations of G_0. All but one are one-dimensional and are of the form $V(g_{a,b}) = \pi(a)$, where π is a multiplicative character of K. The only infinite-dimensional irreducible representation is realized in the space $L^2(\mathbf{K}^*, d^*x)$ and is of the form

$$U(g_{a,b})\,\varphi(x) = \chi(bx)\,\varphi(ax),$$

where χ is a fixed nontrivial additive character of K.

The proof of the fact that there are no irreducible unitary representations of G_0 except the ones listed above proceeds by a standard device of the theory of induced representations, and we omit it.

LEMMA. *The restriction of T to G_0 is a multiple of an irreducible representation.*

Proof. The restriction of T to G_0, like every unitary representation, may be realized in the form of a direct integral of irreducible representations.

First we assume that in this expansion the one-dimensional representations form a set of positive measure. Since the elements of the subgroup $N = \{g_{1,b}\}$ map into the unit operator under one-dimensional representations, the space H of T contains a vector ξ that is invariant relative to $T(g)$, $g \in N$. We assume that $\|\xi\| = 1$.

We consider the function $F_\xi(g) = (T(g)\,\xi, \xi)$. Clearly, this is a continuous positive definite function on G, and constant on the double cosets of N. Since the matrices $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ and $\begin{pmatrix} \alpha\delta - \beta\gamma & 0 \\ \gamma & 1 \end{pmatrix}$ for $\gamma \neq 0$, lie in the same coset, we find

$$F_\xi\left(\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\right) = F_\xi\left(\begin{pmatrix} \alpha\delta - \beta\gamma & 0 \\ \gamma & 1 \end{pmatrix}\right).$$

By passing to the limit, as $\gamma \to 0$, we obtain

$$F_\xi\left(\begin{pmatrix} \alpha & \beta \\ 0 & \delta \end{pmatrix}\right) = F_\xi\left(\begin{pmatrix} \alpha\delta & 0 \\ 0 & 1 \end{pmatrix}\right).$$

In particular, for $\delta = \alpha^{-1}$ we have $F_\xi\left(\begin{pmatrix} \alpha & \beta \\ 0 & \alpha^{-1} \end{pmatrix}\right) = 1$. Hence, it follows that the vector ξ is invariant relative to the subgroup K of matrices of the form $g = \begin{pmatrix} \alpha & \beta \\ 0 & \alpha^{-1} \end{pmatrix}$. In fact, for such matrices g we can write

$$\| T(g)\xi - \xi \|^2 = (T(g)\xi = \xi,\ T(g)\xi - \xi) = 2 - 2 \operatorname{Re} F_\xi(g) = 0$$

But then the function $F_\xi(g)$ must be constant on the double cosets of K. It is easy to check that for $\gamma \neq 0$, $x \neq 0$, the matrices $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ and $\begin{pmatrix} \alpha\delta - \beta\gamma & 0 \\ \gamma x^2 & 1 \end{pmatrix}$ lie in the same coset. Therefore,

$$F_\xi\left(\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\right) = F_\xi\left(\begin{pmatrix} \alpha\delta - \beta\gamma & 0 \\ \gamma x^2 & 1 \end{pmatrix}\right).$$

Passing to the limit, as $x \to 0$, we find

$$F_\xi\left(\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\right) = F_\xi\left(\begin{pmatrix} \alpha\delta - \beta\gamma & 0 \\ 0 & 1 \end{pmatrix}\right).$$

In particular, for every unimodular matrix g we have $F_\xi(g) = 1$. As above, it now follows that $T(g)\xi = \xi$ for every unimodular matrix g.

We denote by H_0 the subspace of H that consists of the vectors invariant under the unimodular subgroup. A simple computation shows that H_0 is invariant under all the operators $T(g)$, $g \in \tilde{G}$. Since H is irreducible, we must have $H_0 = H$. So we see that T is trivial on the subgroup G of unimodular matrices and can therefore be regarded as a representation of the factor group $\tilde{G}/G$. Since this is a commutative group, T must be one-dimensional. But for one-dimensional representations our lemma and Theorem 1 are trivially true.

Now we consider the second case, when there are no one-dimensional representations of G in the expansion of T. Since G_0 has only a single representation U that is not one-dimensional, in this case the restriction of T to G_0 is a multiple of U, and the lemma is proved.

From these arguments we can derive the following more general proposition:

If T is a factor representation of $\tilde{G}$, then its restriction to G_0 is a multiple of an irreducible representation.

For the only place in our arguments where we have used the irreducibility of T is the proof of the equation $H_0 = H$. For a factor representation this equation can be proved as follows. We denote by P the projection operator onto H_0. As we have mentioned above, P commutes with all the operators $T(g)$ and hence with all the operators from the weakly closed ring R generated by $T(g)$. It is not hard to verify that P also commutes with all the operators of the ring R' consisting of the operators that commute with the elements from R. Therefore $P \in R \cap R'$. But by definition of a factor representation the rings R and R' are factors, that is, the intersection $R \cap R'$ consists only of scalar operators. Therefore $P = E$ and $H_0 = H$.

It is advantageous to go over to another realization of the representation U, by considering instead of functions on $\mathbf{K}^*$ their Fourier transforms on the dual group Π.

In this realization the representation operators take the form

$$U(g_{a,b})\,\varphi(\pi_1) = \int \pi_1\pi_2^{-1}(b)\,\Gamma(\pi_2\pi_1^{-1})\,\pi_2(a)\,\varphi(\pi_2)\,d\pi_2$$

for $b \neq 0$, and

$$U(g_{a,0})\,\varphi(\pi) = \pi(a)\,\varphi(\pi). \tag{1}$$

From the lemma it follows that the restriction of T to G_0 is given by the same formulae, only instead of ordinary functions we have to consider vector functions with values in a certain Hilbert space L.

Furthermore, when g lies in the center of $\tilde{G}$, the operators $T(g)$ commute with all the representation operators and are therefore multiples of the unit operator. Hence, it follows that if $d_\lambda = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$, then $T(d_\lambda) = \pi_0(\lambda)E$, where π_0 is a fixed character on $\mathbf{K}^*$. We denote by s the matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. From the identity $sg_{a,0}s^{-1} = g_{a^{-1},0}\,d_a$ it follows that $T(s)$ is of the form

$$T(s)\,\varphi(\pi) = s(\pi)\,\varphi(\pi_0\pi^{-1}),$$

where $s(\pi)$ is a function on Π whose values are unitary operators in L.

Now we observe that the irreducibility of $T(g)$ implies that of the set of operators $s(\pi)$ in L. For if L_1 is a subspace of L invariant under all (or even almost all) $s(\pi)$, then the subspace $H_1 \subset H$

consisting of the vector functions with values in L_1 is invariant under the $T(g)$, $g \in G_0$, and under $T(s)$. But the subgroup G_0 and the element s generate the whole group $\tilde{G}$. Therefore H_1 is invariant under all the $T(g)$, and this contradicts the irreducibility of H.

To prove Theorem 1 it is now sufficient to show that the operators $s(\pi)$ commute with each other. Then the set $s(\pi)$ is irreducible only when L is one-dimensional, and so the restriction of $T(g)$ to G_0 coincides with U. The identity $sg_{1,1}s = g_{1,-1}sg_{1,-1}$ reduces to the following condition on $s(\pi)$:

$$s(\pi_1)\,\Gamma(\pi_1\pi_2\pi_0^{-1})s(\pi_2) = \pi_1(-1)\pi_2(-1)\int \Gamma(\pi\pi_1^{-1})s(\pi)\,\Gamma(\pi\pi_2^{-1})\,d\pi, \quad (2)$$

from which it follows immediately that $s(\pi_1)$ and $s(\pi_2)$ commute for almost all pairs (π_1, π_2). The proof of Theorem 1 is now complete.

3. Theorem on the Complete Continuity of the Operator T_φ. Here we show that the group $\tilde{G}$ is of type I so that all unitary factor representations of $\tilde{G}$ are multiples of an irreducible representation.

For this purpose we prove the following stronger proposition.

THEOREM 2. *If $T(g)$ is an irreducible unitary representation of $\tilde{G}$, and φ a summable function on $\tilde{G}$, then*

$$T\varphi = \int \varphi(g)\,T(g)\,dg$$

is a completely continuous operator.†

Proof. Let $\varphi'_{k,\theta}$ be the generalized function on $\tilde{G}$ given by the formula $(\varphi'_{k,\theta}, f) = q^k \int f(g_{a,b})\theta^{-1}(a)\,d^*a\,db$, where θ is a multiplicative character and the integral is taken over the set

$$g_{a,b} = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}, \quad |a| = 1, \quad |b| \le q^{-k}.$$

We set $\varphi_{k,\theta} = \varphi'_{k,\theta} - \varphi'_{k-1,\theta}$. It is not hard to check that $U_{\varphi_{k,\theta}}$ is a projection operator on the one-dimensional subspace of $L^2(\mathbf{K}^*)$ generated by the functions

$$e_{k,\theta}(x) = \begin{cases} \theta(y) & \text{for } |x| = q^{-k}, \\ 0 & \text{for } |x| \ne q^{-k}. \end{cases}$$

Obviously, the set of functions $e_{k,\theta}$ forms an orthogonal basis of $L^2(\mathbf{K}^*)$.

Now let M be the set of all functions $\varphi \in L^1(\tilde{G})$ for which the

† Groups for which this statement is true are called CCR-groups, following Kaplansky who first singled out this class of groups and proved that they are of type I.

operator $T\varphi$ has finite rank. Clearly, M is a two-sided ideal in $L^1(\tilde{G})$ and contains all functions of the form $\varphi_{k,\theta} * f$, $f \in L^1(\tilde{G})$. If $u \in L^\infty(\tilde{G})$ is a functional on $L^1(\tilde{G})$ that is zero on M, then the function u and all its translates have the property $u * \varphi_{k,\theta} = 0$. Hence, $u = \text{const}$ and the closure of M contains all the functions on $L^1(\tilde{G})$ whose integral is zero.

On the other hand, there are functions on M with a nonzero integral, for example the characteristic function of U (see the next subsection). Therefore, $\bar{M} = L^1(\tilde{G})$. So we have shown that every function $\varphi \in L^1(\tilde{G})$ may be approximated in the norm of $L^1(\tilde{G})$ by functions from M. Hence, the operator $T\varphi$ may be approximated (in the sense of the topology defined by the operator norm) by operators of finite rank and is, therefore, completely continuous. The proof of the theorem is now complete.

4. The Decomposition of an Irreducible Representation of $\tilde{G}$ Relative to Representations of its Maximal Compact Subgroup. The Theorem on the Existence of a Trace. Our object is to prove the following proposition.

THEOREM 3. *Let $T(g)$ be a unitary irreducible representation of $\tilde{G}$. In the decomposition of the restriction of $T(g)$ to a maximal compact subgroup $U \subset \tilde{G}$ every irreducible component has finite multiplicity.*

Proof. We are going to use results obtained in the proof of Theorem 1. We have seen that the representation operator $T(s)$ corresponding to the matrix $s = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ has the following form in the π-realization:

$$T(s)\,\varphi(\pi) = s(\pi)\,\varphi(\pi_0\pi^{-1}), \tag{1}$$

where π ranges over the character group Π dual to $\mathbf{K}^*$. Here the function $s(\pi)$ satisfies the following relation:

$$s(\pi_1)\,\Gamma(\pi_1\pi_2\pi_0^{-1})s(\pi_2) = \pi_1\pi_2(-1)\int \Gamma(\pi\pi_1^{-1})s(\pi)\,\Gamma(\pi\pi_2^{-1})\,d\pi. \tag{2}$$

We expand $s(\pi)$ in a Laurent series and find the relations for the coefficients of the expansions.

We recall that according to **§ 2.6** every character is given by a complex number λ, $|\lambda| = 1$, and a character $\theta(y)$ defined on the group O^* of elements of norm 1. It is expressed by the following formula: if $x = \mathfrak{p}^k y$, $|y| = 1$, then

$$\pi(x) = \lambda^k\theta(y).$$

We substitute in (2) the expansions of $s(\pi) = s(\lambda, \theta)$ and of $\Gamma(\pi)$ in Laurent series

$$s(\pi) = \sum_{k=-\infty}^{+\infty} \lambda^k s_k(\theta), \qquad \Gamma(\pi) = \sum_{k=-\infty}^{+\infty} \lambda^k \Gamma_k(\theta).$$

Then we obtain the following relation for the coefficients $s_k(\theta)$:†

$$(1 - q^{-1}) \sum_m s_{k+m}(\theta_1)\, \Gamma_{-m}(\theta_1\theta_2\theta_0^{-1})\, \lambda_0^m s_{l+m}(\theta_2)$$
$$= \theta_1\theta_2(-1) \sum_\theta \Gamma_{-k}(\theta\theta_1^{-1}) s_{k+l}(\theta)\, \Gamma_{-l}(\theta\theta_2^{-1}). \quad (3)$$

We investigate this condition by taking account of the following formulae for the coefficients of $\Gamma_k(\theta)$ obtained in **§ 2.6**.

If the rank of θ is $m > 0$, then $\Gamma_k(\theta) = 0$ for all $k \neq -m$;

$$|\Gamma_{-m}(\theta)| = q^{-m/2}.$$

If the rank of θ is 0, that is, $\theta(x) \equiv 1$, then

$$\Gamma_k(\theta) = \begin{cases} q^{-1} & \text{for } k < -1, \\ -q^{-1} & \text{for } k = -1, \\ 1 - q^{-1} & \text{for } k > -1. \end{cases}$$

First of all, by taking (3) for fixed θ_1, θ_2, l and sufficiently large positive k, we see that the right-hand side is 0, and for $\theta_1, \theta_2 \neq \theta_0$ the sum on the left-hand side reduces to a single term in which m is the rank of $\theta_1\theta_2\theta_0^{-1}$.‡ Hence, it is easy to derive that for each θ the coefficients $s_k(\theta)$ vanish for sufficiently large positive k.

Secondly, if $k \leq 0$, $l \leq 0$, $\theta_1 \neq \theta_2$, $\theta_1\theta_2 \neq \theta_0$, then it follows from (3) that $s_{k+m}(\theta_1)s_{l+m}(\theta_2) = 0$, where m is the rank of the character $\theta_1\theta_2\theta_0^{-1}$. We assume that for some θ_1 and some $n \leq 0$ the coefficient $s_n(\theta_1)$ is different from zero. Setting $k = n - m$ we then find that for all θ_2, other than θ_1 and $\theta_0\theta_1^{-1}$, the coefficients $s_{l+m}(\theta_2)$ are zero for $l \leq 0$. So we have shown that for all θ, except possibly the one pair θ_1, $\theta_0\theta_1^{-1}$, among the coefficients $s_k(\theta)$ there are only finitely many different from zero. Finally, for the excluded characters θ_1 and $\theta_0\theta_1^{-1}$ we easily obtain from (3) a recurrence relation between the $s_k(\theta)$ from which it follows that $|s_k(\theta)|$ decreases like a geometric progression as $k \to -\infty$ (see **6**).

From all we have shown it follows that *the function $s(\pi) = s(\lambda, \theta)$ is infinitely differentiable with respect to λ for every θ.*

Now we are in a position to prove Theorem 3. We note first that all the maximal compact subgroups of $\tilde{G}$ are conjugate to the group U consisting of those matrices g for which the matrix elements of g and of g^{-1} belong to O. U has a family of normal subgroups U_n, consisting of the matrices that are congruent to the unit matrix modulo p^n.

Obviously, U itself as well as the subgroups U_n are open subsets of $\tilde{G}$ and form a complete system of neighborhoods of the unit element of $\tilde{G}$.

---

† Throughout we denote by (λ_0, θ_0) the components of the character π_0.

‡ For the definition of rank see **§ 2.6**.

It is easy to check that every irreducible representation of U is trivial on U_n for sufficiently large n.

We denote by H_n the subspace of H consisting of the vectors that are invariant under the operators $T(g)$, $g \in U_n$. Theorem 3 is equivalent to the statement that all these spaces H_n are finite-dimensional.

First, we find the space $H_n^0 \subset H_n$ consisting of the vectors that are invariant under the $T(g)$, $g \in U_n \cap G_0$. This is very easy if we use the original realization of the representation of U. We only state the final result.

The space H_n^0 consists of the functions $\varphi(\pi) = \sum \varphi_k(\theta)\lambda^k$, satisfying the condition $\varphi_k(\theta) = 0$ if (rank θ) $> n$ or $k < -n$.

Since $sU_ns^{-1} = U_n$, H_n is invariant under $T(s)$. Therefore, if $\varphi(\pi) \in H_n$, the functions $\varphi(\pi)$ and $T(s)\varphi(\pi) = s(\pi)\varphi(\pi_0\pi^{-1})$ both satisfy the conditions stated above, or both fail to do so. The fact that H_n is finite-dimensional now follows from the infinite differentiability with respect to λ of the function $s(\pi) = s(\lambda, \theta)$ and from the following easily verified proposition.

Let $s(\lambda)$ be an infinitely differentiable function on the unit circle Λ and $|s(\lambda)| \equiv 1$. Then the space $L^2(\Lambda)$ contains only a finite number of linearly independent functions $\varphi(\lambda)$ satisfying the conditions:

1. *The function $\varphi(\lambda)$ is orthogonal to λ^k for $k < -n$.*
2. *The function $s(\lambda)\varphi(\lambda)$ is orthogonal to λ^k for $k > n$.*

The proof of the theorem is now complete.

Observe that in the proof of Theorem 3 we have nowhere used the maximality of the compact subgroup. In fact, it is sufficient to postulate that the compact subgroup U contains all the subgroups U_n for sufficiently large n, in other words, that it is an open compact subgroup of $\tilde{G}$. Thus, Theorem 3 remains valid for every open compact subgroup U of $\tilde{G}$, in particular, for each of the subgroups U_n.

COROLLARY. *If φ belongs to the space S of finite piecewise constant functions on $\tilde{G}$, then the operator $T\varphi = \int \varphi(g)T(g)\,dg$ has finite rank.*

Proof. For each function φ in S we can find an n such that φ is constant on the double cosets of U_n. Hence, it follows that for every x in the representation space H and every $g \in U_n$ we have

$$T(g)T_\varphi = T_\varphi.$$

So we see that the domain of values of the operator $T\varphi$ lies in the space H_n consisting of the vectors that are invariant under U_n. The fact that H_n is finite-dimensional was established in the proof of Theorem 3.

From what we have shown it follows that the operator T_φ has a trace, and that this trace is a linear functional in S. This fact can also be stated in the following way:

For every irreducible unitary representation $T(g)$ of $\tilde{G}$ whatever the trace Tr $T(g)$ *of the operator $T(g)$ exists as a generalized function in the space S.*

5. Representations of the Unimodular Group. Let us show how we can carry over the Theorems 2 and 3, which we have previously proved for the full matrix group $\tilde{G}$, to the group G of matrices with determinant 1. Here we confine ourselves to the case when the group $W = \mathbf{K}^*/(\mathbf{K}^*)^2$ is finite. In **§ 1** it was shown that if the finite field $L = O/P$ has characteristic other than 2, then W is of order 4.

If the field L has characteristic 2 and $\mathbf{K}$ characteristic 0 (as, for example, in the important case of the field of two-adic numbers), then G is also finite. For the series

$$(1 - x)^{1/2} = 1 - \sum_{n=1}^{\infty} \frac{(2n-1)!!}{(2n)!!} x^n$$

converges for $\left|\frac{x}{4}\right| < 1$. Therefore $(\mathbf{K}^*)^2$ contains O_n^* for sufficiently large n. Furthermore, all the even powers of the generator p occur in $(\mathbf{K}^*)^2$. Hence, the order of W does not exceed the order of the finite group $Z_2 \cdot O^*/O_n^*$.

But if $\mathbf{K}$ has characteristic 2, then W is an infinite group (it is isomorphic to the product of a countable number of groups Z_2). We exclude this case from our discussion, although even here one can prove that G is of type I.

Let $\hat{G}$ be the set of all irreducible representations, identified up to equivalence, of G. The group $\tilde{G}$ acts in $\hat{G}$ in the following way. If $T \in \hat{G}$, $g \in \tilde{G}$, then we set $T^{(g)}(g_1) = T(gg_1g^{-1})$. Clearly, $T^{(g)}$ is also an irreducible unitary representation of G. We consider the stability subgroup of the point $T \in G$. Clearly, this subgroup contains G, because if $g \in G$, then $T^{(g)}(g_1) = T(g)T(g_1)T^{-1}(g)$, from which it is obvious that T and $T^{(g)}$ are equivalent. Furthermore, all the matrices of the form $g = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$ belong to the stability group, because for such g the representation $T^{(g)}$ simply coincides with T.

Therefore the stability subgroup contains all the matrices of $\tilde{G}$ whose determinant belongs to $(\mathbf{K}^*)^2$. Since we have assumed that $\mathbf{K}^*/(\mathbf{K}^*)^2$ is finite, the orbit of T under the action of $\tilde{G}$ consists of a finite number of points $T_1, T_2, \ldots, T_k$. Let $H_1, H_2, \ldots, H_k$ be the spaces in which these representations act. Then we can give in the direct sum $H_1 \oplus H_2 \oplus \cdots \oplus H_k$ an irreducible representation T' of $\tilde{G}$ whose restriction to G leaves every H_i invariant and coincides in this subspace with T_i.

Theorems 2 and 3 for T now follow easily from the same theorems for T'.

6. Classification of all Irreducible Representations of G and $\tilde{G}$. Condition (3) in **4** enables us to give a complete classification of all irreducible unitary representations of $\tilde{G}$ and G.

THEOREM 4. *These are no irreducible unitary representations of G other than representations of the principal, the supplementary, and the discrete series, together with the special and the unit representation.*

We find it useful to prove an analogous theorem for $\tilde{G}$. We give a list of the representations of this group, where g denotes the matrix $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \tilde{G}$.

1. The continuous series consists of the representations T_{π_1,π_2} where π_1 and π_2 are unitary multiplicative characters of $\mathbf{K}$. The representation space is $L^2(\mathbf{K}, dx)$ (see **§ 3.1**), and

$$T_{\pi_1,\pi_2}(g)f(x) = \pi_1(\beta x + \delta)\pi_2\left(\frac{\alpha\delta - \beta\gamma}{\beta x + \delta}\right)\frac{|\alpha\delta - \beta\gamma|^{1/2}}{|\beta x + \delta|}f\left(\frac{\alpha x + \gamma}{\beta x + \delta}\right).$$

2. The supplementary series consists of the representations $V_{\pi_0,\rho}$, where π_0 is a unitary multiplicative character of $\mathbf{K}$, and ρ a real number from the interval $(0, 1)$. The representation space consists of the functions on K with the scalar product (see **§ 3.7**)

$$(f_1, f_2) = \int f_1(x)\overline{f_2(y)}\,|x - y|^{-2\rho}\,dx\,dy.$$

The representation operators act according to the formula

$$V_{\pi_0,\rho}(g)f(x) = \pi_0(\alpha\delta - \beta\gamma)\frac{|\alpha\delta - \beta\gamma|^{1-\rho}}{|\beta x + \delta|^{2-2\rho}}f\left(\frac{\alpha x + \gamma}{\beta x + \delta}\right).$$

3. The singular series consists of the representations S_{π_0}, where π_0 is a unitary multiplicative character of $\mathbf{K}$. The representations of this series act in the space of functions on $\mathbf{K}$ for which $\int_K f(x)\,dx = 0$, and the scalar product is given by the formula (see **§ 3.8**)

$$(f_1, f_2) = \int f_1(x)\overline{f_2(y)}\,\ln|x - y|\,dx\,dy.$$

The representation operators are of the form

$$S_{\pi_0}f(x) = \pi_0(\alpha\delta - \beta\gamma)\frac{|\alpha\delta - \beta\gamma|}{|\beta x + \delta|^2}f\left(\frac{\alpha x + \gamma}{\beta x + \delta}\right).$$

4. The discrete series consists of the representations $U_{\pi_0,\Pi}$, where π_0, Π are unitary multiplicative characters of $\mathbf{K}$ and $\mathbf{K}(\sqrt{\tau})$, respectively. The representation acts in the space $L^2(\mathbf{K}, dx)$ according to the formula (see **§ 4.1**)

$$U_{\pi_0,\Pi}(g)f(x)$$

$$= \begin{cases} \int K(g \mid x, y) f(y)\, dy, & \text{when } \beta \neq 0 \\ \pi_0(\alpha\delta)\ \mathrm{sign}_\tau\,(\alpha)\,\Pi(\alpha^{-1}) \left|\dfrac{\delta}{\alpha}\right|^{1/2} \chi\!\left(\dfrac{\gamma x}{\alpha}\right) \varphi\!\left(\dfrac{\delta x}{\alpha}\right), & \text{when } \beta = 0. \end{cases}$$

The kernel $K(g \mid x, y)$ is different from zero only for $\mathrm{sign}_\tau\,(xy\Delta) = 1$ and has the form

$$K(g \mid x, y)$$

$$= a_\tau c_\tau \pi_0(\Delta)\, |\Delta|^{1/2} \frac{\mathrm{sign}_\tau\, \beta x \Delta}{|\beta|} \chi\!\left(\frac{\delta x + \alpha y}{\beta}\right) \int_{\bar{t}=y/\Delta x} \chi\!\left(\frac{x\,\Delta t + y t^{-1}}{-\beta}\right) \Pi(t)\, d^*t,$$

where we have set $\Delta = \alpha\delta - \beta\gamma$.

5. The degenerate series consists of the one-dimensional representations

$$W_{\pi_0}(g) = \pi_0(\alpha\delta - \beta\gamma),$$

where π_0 is a unitary multiplicative character of K.

THEOREM 4′. *There are no irreducible unitary representations of $\tilde{G}$ other than the representations listed above.*

Proof. First we consider a finite-dimensional representation T of $\tilde{G}$. The operators $P_n = \int_{U_n} T(g)\, dg$ are obviously self-adjoint projection operators in the representation space H of T. Furthermore, since the subgroups U_n form a complete system of neighborhoods of the unit element of $\tilde{G}$, the sequence $\{P_n\}$ strongly converges to the unit operator.†

In a finite-dimensional space this is possible only if for all n after a certain n, we have $P_n = E$. But this means that all the vectors in H are invariant under U_n. The kernel of T is a normal subgroup of $\tilde{G}$ containing U_n. Hence, it follows that the kernel of T contains the whole subgroup G. Therefore, T is, in fact, a representation of the factor group $\tilde{G}/G$.

The latter group is commutative and isomorphic to the multiplicative group of **K**. Hence, the only finite-dimensional unitary irreducible representations of $\tilde{G}$ are the representations W_{π_0}, which form the degenerate series.

Now let T be an infinite-dimensional irreducible representation. As we have seen in **2**, the restriction of T to G_0 is also irreducible and coincides with a certain fixed representation of G_0 given by the formulae (1).

---

† We recall that U_n consists of all matrices that are congruent to E modulo $\mathfrak{p}^n O$.

Since $\tilde{G}$ is generated by G_0 and the matrix $s = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, T is given when we know the operator $T(s)$. This operator, as was shown in **2**, has the form

$$T(s)\,\varphi(\pi) = s(\pi)\,\varphi(\pi_0\pi^{-1}),$$

where $s(\pi)$ is a function on the set Π of unitary multiplicative characters of $\mathbf{K}$ that assumes complex values of modulus 1. The character π_0 in this formula is determined by the equation $T(d_\lambda) = \pi_0(\lambda)E$, where $d_\lambda = \begin{pmatrix} \delta & 0 \\ 0 & \delta \end{pmatrix}$. Now we give explicit expressions for the functions $s(\pi)$ corresponding to the series of representations listed above.† The symbol $\pi_{(\rho)}$ denotes the non-unitary character $\pi_{(\rho)}(x) = |x|^\rho$.

1. $T = T_{\pi_1,\pi_2}$, a representation of the principal series

$$s(\pi) = \pi_1\pi_2(-1)\,\Gamma(\pi^{-1}\pi_1\pi_{(1/2)})\,\Gamma(\pi^{-1}\pi_2\pi_{(1/2)}).$$

2. $T = V_{\pi_0,\rho}$, a representation of the supplementary series

$$s(\pi) = \Gamma(\pi^{-1}\pi_0\pi_{(\rho)})\,\Gamma(\pi^{-1}\pi_0\pi_{(1-\rho)}).$$

3. $T = S_{\pi_0}$, a representation of the special series

$$s(\pi) = \Gamma(\pi^{-1}\pi_0)\,\Gamma(\pi^{-1}\pi^0\pi_{(1)}).$$

4. $T = U_{\pi,0\Pi}$, a representation of the discrete series

$$s(\pi) = c\,\Gamma_\tau(\pi_\tau),$$

where Γ_τ is the Gamma-function of $\mathbf{K}(\sqrt{\tau})$ and π_τ the nonunitary multiplicative character of this field given by the formula

$$\pi(z) = \Pi^{-1}(z)\,\pi_0\pi^{-1}\pi_{(1/2)}(z\bar{z}).$$

To prove Theorem 4′ it is enough to verify that the functional equation (2) derived in **2** has no solutions other than the ones listed above. We shall do this in the following way.

First, we explicitly find all the solutions of this functional equation that have in their expansion as a Laurent series

$$s(\pi) = s(\lambda, \theta) = \sum_k s_k(\theta)\,\lambda^k \tag{1}$$

at least one nonzero coefficient for a nonpositive power of λ. It turns out that all such solutions are connected with the representations of the principal, the supplementary, and the singular series. The remaining solutions cannot be given in explicit form, but it can

† Observe that it follows immediately from these formulae that the representations T_{π_2,π_1}, and T_{π_2,π_1} are equivalent, and also the representations $V_{\pi_0,\rho}$ and $V_{\pi_0,1-\rho}$.

be shown that in the representations corresponding to these solutions the matrix elements are square-summable on G. Hence, it follows that the associated representations occur as discrete terms in the decomposition of $L^2(G)$ into irreducible components. The Plancherel formula derived in **§ 6**, which gives the decomposition of $L^2(G)$ into representations of the fundamental and the discrete series, then shows that the remaining solutions of our functional equation are connected with the representations of the discrete series.

Now we carry out this program.

We recall some results of **4**. The fundamental functional equation, in terms of the coefficients of the Laurent series, has the form

$$(1 - |\mathfrak{p}|) \sum_m s_{k+m}(\theta_1) \lambda_0^m \Gamma_{-m}\left(\frac{\theta_1\theta_2}{\theta_0}\right) s_{l+m}(\theta_2)$$
$$= \theta_1\theta_2(-1) \sum_\theta \Gamma_{-k}\left(\frac{\theta}{\theta_1}\right) s_{k+l} \Gamma_{-l}\left(\frac{\theta}{\theta_2}\right). \quad (2)$$

The coefficients $s_k(\theta)$ have the following properties. For each θ there exists a number $\rho(\theta)$ such that $s_k(\theta) = 0$ for $k > \rho(\theta)$. If only a finite number of the coefficients $s_k(\theta)$ is different from zero, then $s(\lambda, \theta) = s_{\rho(\theta)}(\theta) \cdot \lambda^{\rho(\theta)}$ (that is, only one coefficient is different from zero; this follows easily from the fact that $|s(\lambda, \theta)| = 1$ for $|\lambda| = 1$).

We use the notation $\theta^* = \theta_0\theta^{-1}$. Then $s_k(\theta^*) = s_k(\theta)$.

We call a character θ exceptional if among the coefficients $s_k(\theta)$, $k \leq 0$, there is at least one different from zero. We have shown that these are not more than two exceptional characters. We treat three cases separately.

1. There are two distinct exceptional characters θ_1 and θ_1^*. Let $s_{-n}(\theta_1) \neq 0$, $n \geq 0$. Let θ_2 be any character other than θ_1^*. We denote by r^* the rank of the character $\dfrac{\theta_1\theta_2}{\theta_0} = \dfrac{\theta_1}{\theta_1^*}$. Then the sum on the left-hand side of (2) reduces to the single term

$$s_{k+r^*}(\theta_1) \Gamma_{-r^*}\left(\frac{\theta_2}{\theta_1^*}\right) \lambda_0^{r^*} s_{l+r^*}(\theta_2).$$

We set $k = -n - r^*$. Then the sum on the right also reduces to a single term, because $\Gamma_{n+r^*}\left(\dfrac{\theta}{\theta_1}\right)$ is different from zero only for $\theta = \theta_1$. So we obtain the equation

$$(1 - |\mathfrak{p}|) s_{-n}(\theta_1) \Gamma_{-r^*}\left(\frac{\theta_2}{\theta_1}\right) \lambda_0^{r^*} s_{l+r}(\theta_2)$$
$$= \theta_1\theta_2(-1) \Gamma_{n+r^*}(1) s_{l-n-r^*}(\theta_1) \Gamma_{-l}\left(\frac{\theta_1}{\theta_2}\right). \quad (3)$$

Since $\Gamma_{n+r^*}(1) = 1 - |\mathfrak{p}|$, we have

$$s_{l+r^*}(\theta_2) = \frac{\theta_1\theta_2(-1)s_{l-n-r^*}(\theta_1)\,\Gamma_{-l}\left(\dfrac{\theta_1}{\theta_2}\right)}{s_{-n}(\theta_1)\,\Gamma_{-r^*}\left(\dfrac{\theta_2}{\theta_1}\right)\lambda_0^{r^*}}. \tag{4}$$

In particular, if $\theta_2 \neq \theta_{1_\theta}$, then $s_{l+r^*}(\theta_2)$ differs from zero only when l is the rank of $\dfrac{1}{\theta_2}$, which we denote by r.

Finally, if θ is a character other than θ_1 and θ_1^*, then

$$s(\lambda, \theta) = \theta\theta_1(-1)\frac{s_{r-r^*-n}(\theta_1)\,\Gamma_{-r}\left(\dfrac{\theta_1}{\theta}\right)}{s_{-n}(\theta_1)\,\Gamma_{-r^*}\left(\dfrac{\theta}{\theta_1^*}\right)\lambda_0^{r^*}}\lambda^{r+r^*}. \tag{5}$$

Now, we set $\theta_2 = \theta_1$ in (2) and denote by r_0 the rank of $\dfrac{\theta_1}{\theta_1^*}$. If at least one of the numbers k or l is nonpositive, then the sum on the right-hand side reduces to the single term with $\theta = \theta_1$, and we obtain the equation

$$(1 - |\mathfrak{p}|)s_{k+r_0}(\theta_1)\,\lambda_0^{r_0}\,\Gamma_{-r_0}\left(\frac{\theta_1}{\theta_1^*}\right)s_{l+r_0}(\theta_1) = \Gamma_{-k}(1)s_{k+l}(\theta_1)\,\Gamma_{-l}(1).$$

After the substitution $s_k(\theta_1) = \sigma_{2r_0-k}\dfrac{1 - |\mathfrak{p}|}{\Gamma_{-r_0}\left(\dfrac{\theta_1}{\theta_1^*}\right)\lambda_0^{r_0}}$ this becomes the relation

$$\sigma_k\sigma_l = \begin{cases} \sigma_{k+l} & \text{for } k \geq r_0, \quad l \geq r_0, \\ \dfrac{|\mathfrak{p}|}{1 - |\mathfrak{p}|} & \text{for } k = r_0 - 1, \quad l \geq r_0, \\ 0 & \text{for } k < r_0 - 1, \quad l \geq r_0. \end{cases}$$

From this the quantities σ_k can easily be found. We only state the final result. There exists a complex number τ such that $\sigma_k = \dfrac{\Gamma_{k-r_0}(1)\,\tau^k}{1 - |\mathfrak{p}|}$. For $s_k(\theta_1)$ we now derive the expressions†

$$s_k(\theta_1) = \frac{\Gamma_{r_0-k}(1)\,\tau^{2r_0-k}}{\lambda_0^{r_0}\,\Gamma_{-r_0}\left(\dfrac{\theta_1}{\theta_1^*}\right)} = \theta_0(-1)\frac{\Gamma_{r_0-k}(1)\,\tau^{2r_0-k}\,\Gamma_{-r_0}\left(\dfrac{\theta_1^*}{\theta_1}\right)}{\lambda_0^{r_0}\,|\mathfrak{p}|^{r_0}}, \tag{6}$$

† Here, and subsequently, we use the identity $\overline{\Gamma_k(\theta)} = \theta(-1)\Gamma_k(\theta^{-1})$ (see **§ 2.6**).

whence

$$s(\lambda, \theta_1) = \sum_k \theta_0(-1) \frac{\Gamma_{r_0-k}(1)\,\lambda^k \tau^{2r_0-k}\,\Gamma_{-r_0}\left(\dfrac{\theta_1^*}{\theta_1}\right)}{\lambda_0^r\,|\mathfrak{p}|^{r_0}}$$

$$= \sum_k \theta_0(-1)\,\Gamma_{r_0-k}(1)\left(\frac{\tau}{\lambda}\right)^{r_0-k}\Gamma_{-r_0}\left(\frac{\lambda_0\,|\mathfrak{p}|^{1/2}}{\tau\lambda}\right)^{-r_0}. \qquad (7)$$

Substituting the value of $s_k(1)$ from (6) in (5) we find that for θ, other than θ_1 and θ_1^*,

$$s(\lambda, \theta) = \theta\theta_1(-1) \frac{\tau^{r^*-r}\,\Gamma_{-r}\left(\dfrac{\theta_1}{\theta}\right)\lambda^{r+r^*}}{\Gamma_{-r^*}\left(\dfrac{\theta}{\theta_1^*}\right)\lambda_0^{r^*}}$$

$$= \theta_0(-1)\,\Gamma_{-r}\left(\frac{\theta_1}{\theta}\right)\left(\frac{\tau}{\lambda}\right)^{-r}\Gamma_{-r^*}\left(\frac{\theta_1^*}{\theta}\right)\left(\frac{\lambda_0\,|\mathfrak{p}|}{\tau\lambda}\right)^{-r^*}. \qquad (8)$$

Note that $|s(\lambda, \theta)| = 1$ is equivalent to $|\tau| = |\mathfrak{p}|^{1/2}$.

Now we compare these formulae (7) and (8) with the previously derived expression

$$s(\pi) = \pi_1\pi_2(-1)\,\Gamma(\pi^{-1}\pi_1\pi_{(1/2)})\,\Gamma(\pi^{-1}\pi_2\pi_{(1/2)}) \qquad (9)$$

for the function $s(\pi)$ corresponding to the representation T_{π_1,π_2}.

We denote by π_1 and π_2 the characters with the coordinates $(\tau\,|\mathfrak{p}|^{-1/2}, \theta_1)$ and $\left(\dfrac{\lambda_0\,|\mathfrak{p}|^{1/2}}{\tau}, \theta_1^*\right)$, respectively. It is not hard to check that for these π_1 and π_2 the formula (9), rewritten in the coordinates (λ, θ), turns into (7) and (8).

So we have shown that all the solutions of the functional equation (2) having two exceptional characters are connected with the representations of the principal series.

2. There exists only one exceptional character $\theta_1 = \theta_1^*$. We may assume that $\theta_1 \equiv 1$. For it is easy to verify that the function $\tilde{s}(\pi) = s(\pi\tilde{\pi})$ satisfies a functional equation of the form (2) in which π_0 is replaced by $\pi_0\tilde{\pi}^{-2}$. If the excluded character for $s(\pi)$ is θ, then for $\tilde{s}(\pi)$ it is $\theta\tilde{\theta}^{-1}$. The same argument shows that we may confine our discussion to the case $\lambda_0 = 1$.

After the substitution $\theta_1 \equiv 1$ the fundamental equation (2) takes the form

$$(1 - |\mathfrak{p}|) \sum_r s_{k+r}(1)\,\Gamma_{-r}(\theta_2)s_{l+r}(\theta_2) = \theta_2(-1) \sum_\theta \Gamma_{-k}(\theta)s_{k+l}(\theta)\,\Gamma_{-l}\left(\frac{\theta}{\theta_2}\right). \qquad (10)$$

In particular, if $k \leq 0$ and the rank of θ_2 is $r_2 \neq 0$, then we obtain

$$s_{k+r_2}(1)\,\Gamma_{-r_2}(\theta_2)s_{l+r_2}(\theta_2) = \theta_2(-1)s_{k+l}(1)\,\Gamma_{-l}(\theta_2^{-1}).$$

The right-hand side of this equation can be different from zero only when l = rank of $\theta_2^{-1} = r_2$. Choosing a nonpositive k such that $s_{k+r_2}(1) \neq 0$ and setting $l = r_2$ we find

$$s_{2r_2}(\theta_2) = \frac{\theta_2(-1)\,\Gamma_{-r_2}(\theta_2^{-1})}{\Gamma_{-r_2}(\theta_2)}.$$

So we have found the coefficients $s_k(\theta)$ for $\theta \neq 1$. Now we set $k > 0$, $l = r_2$ in (10) and separate on the left-hand side the term corresponding to $\theta = 1$:

$$\begin{aligned}(1 - |\mathfrak{p}|)s_{k+r_2}(1)\,\Gamma_{-r_2}(\theta_2)s_{2r_2}(\theta_2) &\\ = \theta_2(-1)\,\Gamma_{-k}(1)s_{k+r_2}(1)\,\Gamma_{-r_2}(\theta_2^{-1}) &\\ + \theta_2(-1)\sum_{\theta\neq 1}\Gamma_{-k}(\theta)s_{k+r_2}(\theta)\,\Gamma_{-r_2}\left(\frac{\theta}{\theta_2}\right)&.\end{aligned}$$

Substituting here the value of $s_{2r_2}(\theta_2)$ found above, we obtain

$$s_{k+r_2}(1)\,\Gamma_{-r_2}(\theta_2^{-1})(1 - |\mathfrak{p}| - \Gamma_{-k}(1)) = \sum_{\theta\neq 1}\Gamma_{-k}(\theta)s_{k+r_2}(\theta)\,\Gamma_{-r_2}\left(\frac{\theta}{\theta_2}\right).$$

Since $k > 0$, the coefficient of $s_{k+r_2}(1)$ on the left-hand side is different from zero. We write the conditions under which at least one term on the right-hand side may be different from zero:

$$k = \text{rank } \theta; \qquad k + r_2 = 2(\text{rank } \theta); \qquad r_2 = \text{rank } \left(\frac{\theta}{\theta_2}\right).$$

Hence, it follows that $k = \text{rank } \theta = r_2 = \text{rank}\left(\frac{\theta}{\theta_2}\right)$, and that $s_{k+r_2}(1)$ can be different from zero only for $k = r_2$. Consequently, $s_n(1) = 0$ for $n > 2$, because every $n > 2$ can be represented in the form $n = k + r_2$, $k \neq r_2$. For $n = 2$ the expression on the right may be summed explicitly, and we find

$$s_2(1) = \sum_{\theta\neq 1}\theta(-1)\,\frac{\Gamma_{-1}(\theta)\,\Gamma_{-1}\left(\frac{\theta}{\theta_2}\right)}{\Gamma_{-1}(\theta_2^{-1})} = |\mathfrak{p}|.$$

Note that all the coefficients we have found so far are uniquely determined and do not depend on the representation in question.

To determine the remaining coefficients $s_k(\theta)$, $k < 2$, we set $\theta_1 = \theta_2 = 1$ in the fundamental equation (2) and assume that at least one of the numbers k or l is nonpositive. Then we obtain the equation

$$(1 - |\mathfrak{p}|)\sum_r s_{k+r}(1)\,\Gamma_{-r}(1)s_{l+r}(1) = \Gamma_{-k}(1)s_{k+l}(1)\,\Gamma_{-l}(1)$$

or

$$-\frac{|\mathfrak{p}|}{1 - |\mathfrak{p}|}s_{k+1}^{(1)}s_{l+1}^{(1)} + \sum_{r\leqslant 0}s_{k+r}(1)s_{l+r}(1) = \frac{\Gamma_{-k}(1)\,\Gamma_{-l}(1)}{1 - |\mathfrak{p}|}s_{k+l}(1).$$

But by virtue of the condition $|s(\lambda, \theta)| = 1$ the coefficients $s_k(\theta)$ satisfy the relations: $\sum_r s_{k+r}(\theta) s_{l+r}(\theta) = \delta_{kl}$. Hence,

$$\frac{|\mathfrak{p}|}{-1 + |\mathfrak{p}|} s_{k+1}(1) s_{l-1}(1) - \sum_{r>0} s_{k+r}(1) s_{l+r}(1) + \delta_{kl}$$
$$= \frac{\Gamma_{-k}(1)\,\Gamma_{-l}(1)}{1 - |\mathfrak{p}|} s_{k+l}(1).$$

Here we set $l = 0$ and take into account that $s_r(1) = 0$ for $r > 2$:

$$\frac{|\mathfrak{p}|}{-1 + |\mathfrak{p}|} s_{k+1}(1) s_1(1) - s_{k+1}(1) s_1(1)$$
$$- s_{k+2}(1) s_2(1) + \delta_{k0} = \frac{\Gamma_{-k}(1)}{1 - |\mathfrak{p}|} s_k(1).$$

In particular, for $k = 0$ we find

$$s_0(1) + \frac{s_1^2(1)}{1 - |\mathfrak{p}|} = 1 - |\mathfrak{p}|^2,$$

and for $k < 0$:

$$s_k(1) + s_{k+1}(1) \cdot \frac{s_1(1)}{1 - |\mathfrak{p}|} + s_{k-2}(1)\,|\mathfrak{p}| = 0.$$

So we have obtained a recurrence relation for $s_k(1)$. It has the general solution

$$s_k(1) = A\tau_1^k + B\tau_2^k \qquad \text{for } k \leq 1,$$

where τ_1 and τ_2 are two complex numbers linked by the condition $\tau_1\tau_2 = |\mathfrak{p}|^{-1}$.

Hence,

$$s(\lambda, 1) = \sum_k s_k(1)\lambda^k = |\mathfrak{p}|\,\lambda^2 + \sum_{k\leqslant 1} (A\tau_1^k\lambda^k + B\tau_2^k\lambda^k)$$
$$= |\mathfrak{p}|\,\lambda^2 + \frac{A\lambda\tau_1}{1 - \lambda^{-1}\tau_1^{-1}} + \frac{B\lambda\tau_2}{1 - \lambda^{-1}\tau_2^{-1}}$$
$$= \frac{(1 - \lambda\alpha)(1 - \lambda\beta)}{(1 - \lambda^{-1}\tau_1^{-1})(1 - \lambda^{-1}\tau_2^{-1})},$$

where α and β are two complex numbers depending on A, B, τ_1, and τ_2. The condition $|s(\lambda, 1)| = 1$ is satisfied only when $\alpha = \bar{\tau}_1^{-1}$, $\beta = \bar{\tau}_2^{-1}$ or $\alpha = \bar{\tau}_2^{-1}$, $\beta = \bar{\tau}_1^{-1}$. In this case we find

$$s(\lambda, 1) = \frac{(1 - \lambda\bar{\tau}_1^{-1})(1 - \lambda\bar{\tau}_2^{-1})}{(1 - \lambda^{-1}\tau_1^{-1})(1 - \lambda^{-1}\tau_2^{-1})}.$$

Let $|\tau_1| = |\mathfrak{p}|^\rho$; by π_0 we denote the character with the coordinates $\left(\frac{\tau_1}{|\tau_1|}, 1\right)$. Then our function $s(\pi)$ may be written in the form

$$s(\pi) = \Gamma(\pi^{-1}\pi_0\pi_{(\rho)})\,\Gamma(\pi^{-1}\pi_0\pi_{(1-\rho)}).$$

We see that a representation corresponding to this function belongs to the supplementary or the singular series.

3. There are no exceptional characters.

We show that then the matrix elements of the representation are square-summable on G. In the representation space of T (in the π-realization) we choose a basis consisting of the function $e_{k,\theta}$:

$$e_{k,\theta}(\lambda, \theta_1) = \begin{cases} \lambda^k & \text{for } \theta_1 = \theta, \\ 0 & \text{for } \theta_1 \neq \theta. \end{cases}$$

In this basis the representation operators have the form

$$\begin{aligned} T(g_{a,0})e_{k,\theta} &= \theta(\alpha)e_{k-n,\theta} \qquad \text{for } a = \alpha\, |\mathfrak{p}|^n,\ \alpha \in O^*; \\ T(g_{1,b})e_{k,\theta} &= (1 - |\mathfrak{p}|)^{-1} \sum_{\theta'} \Gamma_{m+k}\left(\frac{\theta}{\theta'}\right) \frac{\theta}{\theta'}(\beta) e_{k,\theta'} \qquad (11) \\ &\qquad \text{for } b = \beta\, |\mathfrak{p}|^n,\ \beta \in O^*. \\ T(s)e_{k,\theta} &= \sum_i s_t(\theta^*)\, \lambda_0^k e_{t-k,\theta^*} \end{aligned}$$

Almost every element of G can be written in the form

$$g = g_{a,0}^{-1} g_{1,b_1}^{-1} s g_{1,b_2} g_{a,0}.$$

The invariant measure on G with parameters a, b_1, b_2 is $d\mu(g) = d^*a\, db_1\, db_2$.

We examine the domain $D(n, m_1, m_2)$ on G given by the conditions

$$a = \mathfrak{p}^n\alpha,\ b_1 = \mathfrak{p}^{m_1}\beta_1,\ b_2 = \mathfrak{p}^{m_2}\beta_2 \qquad \text{where } \alpha,\ \beta_1,\ \beta_2 \in O^*.$$

Using (11) we can write down the following expression for the matrix elements of $T(g)$, with $g \in D(n, m_1, m_2)$;

$$\begin{aligned} (T(g)e_{k_1,\theta_1}, e_{k_2,\theta_2}) &= \varphi(a, b_1, b_2) \\ &= (1 - |\mathfrak{p}|)^{-2} \sum_{\theta} \theta_0(\beta_2)\theta_2(\alpha\beta_2^{-1})\theta_1(-\alpha^{-1}\beta_1)\, \lambda_0^{k_2-n} \\ &\qquad \Gamma_{m_1+k_1-n}\left(\frac{\theta}{\theta_1}\right) \Gamma_{m_2+k_2-n}\left(\frac{\theta}{\theta_2^*}\right) \theta^{-1}(-\beta_1\beta_2) s_{k_1+k_2-2n}(\theta). \qquad (12) \end{aligned}$$

We investigate under what conditions on n, m_1, m_2 this expression may be different from zero.

First let n be fixed. There are only finitely many characters θ for which $s_{k_1+k_2-2n}(\theta) \neq 0$.

Indeed, the arguments we have already used in the first and second case show that for all θ, except possibly finitely many,

$$s(\lambda, \theta) = s_{\rho(\theta)}(\theta) \cdot \lambda^{\rho(\theta)}, \qquad \text{where } \rho(\theta) = 2(\text{rank } \theta) \qquad (13)$$

Our assertion now follows from the fact that the number of characters of a given rank is finite.

Next, since the rank of θ is bounded and the ranks of θ_1 and

θ_2 are fixed, for negative m_1 or m_2 of sufficiently large absolute value the coefficients of the Gamma-function on the right-hand side of (12) are zero. This means that for every fixed n the domain of the group on which our matrix element is different from zero has finite volume. For all sufficiently large positive n we may use (13) to obtain a more accurate estimate, which shows that this volume is bounded by a constant independent of n.

So far we have nowhere used the fact that there are no exceptional characters.

Now let us take this into account. First of all, it implies (by the definition of exceptional characters) that $s_k(\theta) = 0$ for $k \leq 0$.

This means that for $n \leq \dfrac{k_1 + k_2}{2}$ our matrix element φ vanishes.

In investigating the summability of φ we need therefore consider only the domain $n > N$, where N is a sufficiently large positive number. Here we may use (13) and assume that the rank of θ is greater than that of θ_1 and of θ_2^*. Hence it follows that the right-hand side of (12) is different from zero only for $m_1 = \dfrac{k_2 - k_1}{2}$, $m_2 = \dfrac{k_1 - k_2}{2}$ and has the form

$$\varphi_n(\alpha, \beta_1, \beta_2) = (1 - |\mathfrak{p}|)^{-2}\theta_0(\beta_1)\theta_1(-\alpha^{-1}\beta_1)\theta_2(\alpha\beta_2)\,\lambda_0^{k_2-n}$$
$$\sum_{\theta} \Gamma_{-r}\left(\frac{\theta}{\theta_1}\right)\Gamma_{-r}\left(\frac{\theta}{\theta_2^*}\right)\theta(-\beta_1^{-1}\beta_2^{-1})s_{2r}(\theta),$$

where the sum is taken over all the characters of rank

$$r = \frac{k_1 + k_2}{2} + n.$$

Then,

$$|\varphi_n(\alpha, \beta_1, \beta_2)|^2 = (1 - |\mathfrak{p}|)^{-4}$$
$$\sum_{\theta,\theta'} \Gamma_{-r}\left(\frac{\theta}{\theta_1}\right)\Gamma_{-r}\left(\frac{\theta}{\theta_2^*}\right)s_{2r}(\theta)\,\overline{\Gamma_{-r}\left(\frac{\theta'}{\theta_1}\right)\Gamma_{-r}\left(\frac{\theta'}{\theta_2^*}\right)s_{2r}(\theta')^{-1}}\theta\theta'(-\beta_1\beta_2).$$

Therefore, the integral of $|\varphi(g)|^2$ over the domain $\bigcup_{m_1, m_2} D(n, m_1, m_2)$ is equal to

$$\int |\varphi_n(\alpha, \beta_1, \beta_2)|^2\, d\beta_1\, d\beta_2 = (1 - |\mathfrak{p}|)^{-4} \sum_{\theta}\left| \Gamma_{-r}\left(\frac{\theta}{\theta_1}\right)\Gamma_{-r}\left(\frac{\theta}{\theta_2^*}\right)s_{2r}(\theta)\right|^2.$$

Bearing in mind that $|s_{2r}(\theta)| = 1$, $|\Gamma_{-r}(\theta)| = |\mathfrak{p}|^{r/2}$ and that the number of characters of rank r is $|\mathfrak{p}|^{-r}(1 - |\mathfrak{p}|)$ we obtain the required estimate.

This concludes the proof of Theorem 4′.

We mention that by other straightforward arguments it can be shown that $\varphi(g) \in L^p(G)$ for every $p \geq 1$.