ERRATA TO "THE HILB-VS-QUOT CONJECTURE"

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The errata described in this document do not affect the results of the paper.

#1 is an omission in the acknowledgments. #2 is an error in a comment not used elsewhere. #3 is a typo in notation. #4, the most extensive error, is a misstatement of the closed formulas for a, q, t-Catalan numbers established in [GMV20], which affects the discussion about the relationship between [GN15, Mel22, HM19, GMV20], but not the results depending on these formulas.

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1.

The acknowledgments should mention that the two figures in the paper (both in Section 4, "Torus Knots") were produced using the Penpa+ applet at this link: https://swaroopg92.github.io/penpa-edit/.

2.

In §3.5, the first paragraph and accompanying commutative diagram are incorrect. The paragraph should be changed to the following:

> For each integer $m \geq 0$, let $P_{n-m,m} \subseteq \mathrm{GL}_n$ be the parabolic subgroup of block-upper-triangular matrices with an upper block of size $n-m$ and a lower block of size m. In the Lie algebra of $P_{n-m,m}$, let $\mathcal{M}_{n-m,m}$ be the $P_{n-m,m}$-stable subvariety of nilpotent elements whose lower block is identically zero. Let $X_{m\text{-}nest}$ and $\rho_X = \rho_{X,m}$ be defined by the cartesian square:
>
> $$\begin{tikzcd} X_{m\text{-}nest} \arrow[r] \arrow[d, "\rho_X"'] & {[\mathcal{M}_{n-m,m}/P_{n-m,m}]} \arrow[d] \\ X \arrow[r, "p"] & {[\mathcal{N}/\mathrm{GL}_n]} \end{tikzcd}$$
>
> Explicitly, if we regard GL_n as acting on column vectors from the left, then the A-points of $[\mathcal{M}_{n-m,m}/P_{n-m,m}]$ form the groupoid of triples (V, θ, V'), where V is a locally free rank-n A-module, θ is a

nilpotent endomorphism of V, and V' is a locally free submodule of V of codimension m that contains the image of θ.

3.

In §4.4, in Lemma 4.3, cut the phrase "Writing $\Gamma_{E,>0} = \Gamma(E) \setminus \{0\}$". Change the text surrounding the above display to the following:

> Writing $\Gamma(R)_{>0} = \Gamma(R) \setminus \{0\}$, let
>
> $$I^{\ell}_{m\text{-}nest}(E) = \{(\Delta, \Delta') \in I^{\ell}(E) \times I^{\ell+m}(E) \mid \Delta \supseteq \Delta' \supseteq \Delta + \Gamma(R)_{>0}\}.$$
>
> The following lemma is proved in...

4.

4.1. Let $n, d > 0$ be integers. Consider an $n \times d$ lattice rectangle with a chosen diagonal. Let π be an $n \times d$ Dyck path on one side of the diagonal. We write $v_*(\pi)$ for the set of lattice points on π where the path moves *toward* the diagonal along two edges. We write $v^*(\pi)$ for the set of lattice points on π where it moves *away from* the diagonal along two edges.

These notations match [Mel22, Footnote 5] and the paragraph below Remark 4.5 in our paper. But the definitions of $v_*(\pi), v^*(\pi)$ above have the advantage that they stay the same regardless of how the diagonal or the Dyck path are oriented.

In particular, suppose that the $n \times d$ Dyck path π corresponds to the n, d-invarant set $\Delta \in D_{n,d}$. Then, as explained in our paper, $v_*(\pi)$ is in bijection with the set of *generators* $\mathsf{Gen}(\Delta) \setminus \{0\}$, where

$$\mathsf{Gen}(\Delta) = \{k \in \Delta \mid k - n,\ k - d \notin \Delta\}.$$

Meanwhile, $v^*(\pi)$ is in bijection with the set of *cogenerators*

$$\mathsf{Cog}(\Delta) = \{k \in \mathbf{Z} \setminus \Delta \mid k + n,\ k + d \in \Delta\}.$$

In our paper, the subset of *nonnegative cogenerators* $\mathsf{Cog}(\Delta) \cap \mathbf{Z}_{\geq 0}$ was denoted $\mathsf{Cogen}(\Delta)$. Here, we will call it $\mathsf{Cog}_{\geq 0}(\Delta)$.

4.2. Recall that for any n, d, we write $\bar{\mathsf{X}}_{n,d}(\mathsf{a}, \mathsf{q}, \mathsf{t})$ to denote the graded dimension of the *unreduced* Khovanov–Rozansky link homology of the (n, d) torus link, in the normalization described in our appendix.

In [GMV20], Gorsky–Mazin–Vazirani describe $\bar{\mathsf{X}}_{n,d}(\mathsf{a}, \mathsf{q}, \mathsf{t})$ in terms of the $\boldsymbol{u} = 0^{n+d}$ case of a series $\hat{P}_{\boldsymbol{u}}(q, t, a)$, itself derived from Hogancamp–Mellit's recursion in [HM19]. The definition of $\hat{P}_{\boldsymbol{u}}(q, t, a)$ involves a sum over a collection of n, d-invariant sets $I_{\boldsymbol{u}}$. The Δth term of the sum involves a product over $\mathsf{Cog}_{\geq 0}(\Delta)$.

It is pointed out in [GMV20, Remark 5.2] that if $\boldsymbol{u} = 0^{n+d}$ and $\Delta \in I_{\boldsymbol{u}}$, then $\mathsf{Cog}_{\geq 0}(\Delta) = \mathsf{Cog}(\Delta)$. By contrast, if n, d are coprime and $\Delta \in D_{n,d}$, then $\mathsf{Cog}_{\geq 0}(\Delta)$ is often smaller than $\mathsf{Cog}(\Delta)$. So Remark 4.10 of our paper is incorrect.

Instead, here is the correct translation of $\hat{P}_{0^{n+d}}(q,t,a)$ into Dyck paths, in the case where n, d are coprime. First, for any lattice point p, let $\kappa_\pi(p)$ denote the set of unit steps of π that are parallel to the side of the rectangle of length n, and intersect (in their interiors) the line through p parallel to the chosen $n \times d$ diagonal. (This is the definition of $\kappa_\pi(p)$ above Lemma 4.6 of our paper.) After telescoping a geometric series, we see that [GMV20, Definition 5.5] gives

$$\hat{P}_{0^{n+d}}(q,t,a) = \frac{q^{n+d}}{1-q} \sum_{\substack{n \times d \\ \text{Dyck paths } \pi}} q^{\mathsf{area}(\pi)} t^{\mathsf{codinv}(\pi)} \prod_{p \in v^*(\pi)} (1 + at^{|\kappa_\pi(p)|}).$$

Note that the product over $\mathsf{Cog}_{\geq 0}(\Delta) = \mathsf{Cog}(\Delta)$ for $\Delta \in I_{0^{n+d}}$ in the GMV formula becomes the product over $v^*(\pi)$ above.

For general n, d, the correct conversion between $\hat{P}_{0^{n+d}}(q,t,a)$ and $\bar{\mathsf{X}}_{n,d}(\mathsf{a},\mathsf{q},\mathsf{t})$ is

$$\frac{1}{1+\mathsf{a}} \bar{\mathsf{X}}_{n,d}(\mathsf{a},\mathsf{q},\mathsf{t}) = \mathsf{q}^{-d-n} \hat{P}_{0^{n+d}}(\mathsf{q},\mathsf{t},\mathsf{a}).$$

Note that there is no $\mathsf{a} \mapsto \mathsf{a}\mathsf{q}^{-1}$ substitution on the right-hand side, unlike what was claimed at the end of our appendix.

4.3. The error in Remark 4.10 does not affect any results in our paper. The only other passages that require correction are the following.

- In §1.5, below items (A)–(B), it is inaccurate to say that the Gorsky–Neguţ formula involves the generators of the semigroup modules $\Delta \in D_{n,d}$: Rather, the formulas in [GN15, Mel22, GMV20] all involve cogenerators.

  It seems that generators only enter the picture through the ring $\mathbf{C}[\![t^n, t^d]\!]$: that is, through the plane curve $y^n = x^d$.
- Remark 4.11 is correct if "Cogen formula" is redefined as the closed formula for $\bar{\mathsf{X}}_{n,d}(\mathsf{a},\mathsf{q},\mathsf{t})$ arising from [GMV20].
- In §4.8, identity (4.6) is now merely a conjecture. Remarkably, it still seems to be true. We will investigate it in future work. The discussion in §4.8 is also no longer related to [HM19, GMV20].
- At the very end of our appendix, the last display should be changed to:

$$\begin{aligned}
\frac{1}{1+\mathsf{a}} \bar{\mathsf{X}}_{n,d}(\mathsf{a},\mathsf{q},\mathsf{t}) &= \frac{1}{1-\mathsf{q}} \mathsf{X}_{n,d}(\mathsf{a},\mathsf{q},\mathsf{t}^{-1}) \\
&= R_{0^n,0^d}(\mathsf{q},\mathsf{t}^{-1},\mathsf{a}) \\
&= \hat{Q}_{0^n,0^d}(\mathsf{q},\mathsf{t}^{-1},\mathsf{a}) && \text{by [GMV20, Cor. 5.10]} \\
&= \mathsf{q}^{-d-n} \hat{P}_{0^n,0^d}(\mathsf{q},\mathsf{t},\mathsf{a}) && \text{by [GMV20, (11)].}
\end{aligned}$$

4.4. For the sake of completeness, we mention a further typo in one of the references. Conjecture 6.2 in [GN15] is stated correctly, whereas Conjecture 1.2 is not. The latter claims that the product in the Dyck-path formula runs over all vertices of the Dyck path, rather than just internal vertices.

References

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