last two weeks: bilinear forms β

[what does β alternating mean?]

β is alternating iff, for all v, we have $\beta(v, v) = 0$

[what does β nondegenerate mean?]

β is nondegenerate iff, for all v,

($\beta(v, -)$ zero or $\beta(-, v)$ zero) implies $v = \mathbf{0}$

<u>Df</u> a bilinear form β is definite iff, for all v in V,

$\beta(v, v) = 0$ implies $v = \mathbf{0}$

[which implies which?]

if β is definite, then β is nondegenerate

if β is alternating, then $V = \{\mathbf{0}\}$ or β is not definite

Ex take $\beta((a, c), (b, d)) = ad - bc$

is it definite? no

is it nondegenerate? yes [why?]

over R: $\beta((a, c), (-c, a)) = a^2 + c^2$

over C: $\beta((a, c), (-c^*, a^*)) = |a|^2 + |c|^2$

[takeaways:]

- nondegenerate does not imply definite
- R versus C sometimes matters

(Axler §6A) let V be a real vector space

<u>Df</u> a bilinear form β on V is positive iff

$\beta(v, v) \geq 0$ for all v in V

Df — an inner product on V is a positive, definite, symmetric bilinear form

Rem — for inner products, we usually write $\langle\ ,\ \rangle$ in place of β

but this is NOT the evaluation pairing

[how to check that $\langle\ ,\ \rangle$ is an inner product?]

- positivity, definiteness, symmetry
- $\langle a \cdot u + u', v\rangle = a\langle u, v\rangle + \langle u', v\rangle$

  [as symmetry takes care of the 2nd coord]

[easiest ex?]

Ex — the dot product on $\mathbb{R}^n$ is always an inner product

[why care? let us measure length:]

Df — a norm on a (real or complex) vec. sp. V is a map $N : V \to \mathbb{R}_{\geq 0}$ s.t.

1) $N(v) = 0$ implies $v = \mathbf{0}$
2) $N(a \cdot v) = |a|\, N(v)$
3) $N(u + v) \leq N(u) + N(v)$ [triangle ≤]

Thm — if $\langle\ ,\ \rangle$ is an inner product on (real) V, then $\|v\| := \sqrt{\langle v, v\rangle}$ is a norm on V

Pf of Thm — $\|\ \|$ is valued in $\mathbb{R}_{\geq 0}$ by positivity

1) $\|v\| = 0$ implies $\langle v, v\rangle = \|v\|^2 = 0$

   so $v = \mathbf{0}$ by definiteness

2) $\langle a \cdot v, a \cdot v\rangle = a^2\langle v, v\rangle$

   so taking sqrts, $\|a \cdot v\| = |a|\, \|v\|$

3) [harder:] want $||u + v|| \le ||u|| + ||v||$

since $|| \ || \ge 0$, this is equivalent to

$$||u + v||^2 \le ||u||^2 + 2\,||u||\,||v|| + ||v||^2$$

LHS is $||u||^2 + 2\langle u, v\rangle + ||v||^2$ so need:

Lem (Cauchy–Schwarz) $|\langle u, v\rangle| \le ||u||\,||v||$
[what's the intuition?]

Ex if $V = R^n$ and $\langle \ , \ \rangle$ is the dot product
then $||v||$ is the Euclidean length of v

check: if $u, v \ne \mathbf{0}$ and α is the angle btw them
then $u \bullet v = ||u||\,||v|| \cos \alpha$

[draw picture]

Pf of Lem if $v = \mathbf{0}$ then done
else $v \ne \mathbf{0}$

let $u' = (\langle u, v\rangle/\langle v, v\rangle)\, v$, the projection of u onto v
then $\langle u', v\rangle = \langle u, v\rangle$
so $\langle u - u', v\rangle = 0$
now $u = u' + (u - u')$ where
- u' is a multiple of v (projection)
- $u - u'$ is orthogonal to v (complement)

so $u - u'$ is also orthogonal to u'

[Pythagoras:]
$$\begin{aligned} ||u||^2 &= ||u'||^2 + ||u - u'||^2 \\ &\ge ||u'||^2 \\ &= \langle u, v\rangle^2/||v||^2 \end{aligned}$$

<u>Summary</u> inner product $\langle\ ,\ \rangle$ on V gives us:

- notion of orthogonality
- orthogonal projections + complements
- Pythagorean identity [for orthogonal vec's]
- Cauchy–Schwarz inequality
- triangle inequality
- norm $\|\ \|$

the pair $(V, \langle\ ,\ \rangle)$ is called an <u>inner product space</u>
but often use this term for V itself

<u>Ex</u> let $V = \mathbb{R}^2$ and $\langle u, v\rangle = u^t M v$

where $M = \begin{matrix} 1 & -1/2 \\ -1/2 & 1 \end{matrix}$

it turns out that $\langle\ ,\ \rangle$ is another inner product
hence $\|\ \| : \mathbb{R}^2$ to $\mathbb{R}_{\geq 0}$ def by

$$\|(x, y)\| = \sqrt{x^2 - xy + y^2}$$

is another norm

<u>Q</u> how to classify inner product spaces?

<u>A</u> [next time] can always reduce general case to $\mathbb{R}^n$ under the dot product, after finding a new basis

<u>Q</u> how to generalize from $\mathbb{R}$ to $\mathbb{C}$?
the dot product on $\mathbb{C}^n$ is not positive
[same issue with the norm]

let V now be a complex vector space

<u>Df</u> a skew-linear or sesquilinear functional on V is a map $\eta : V$ to C s.t.

$$\eta(v + v') = \eta(v) + \eta(v')$$
$$\eta(a \bullet v) = a^* \eta(v)$$

for all v, v' in V and a in C

a skew-linear form on V is a map $\langle\ ,\ \rangle : V \times V$ to C s.t.

$\langle -, v\rangle$ is linear for all v
$\langle u, -\rangle$ is skew-linear for all u

define positivity and definiteness like before
define conjugate-symmetry to be

$$\langle u, v\rangle = \langle v, u\rangle^*$$

<u>Df</u> an inner product on V (over C) is a positive, definite, conjugate-symmetric skew-linear form

<u>Ex</u> the skew dot product $\langle u, v\rangle = u \bullet v^*$

with this defn, everything before still works:

- orthogonality
- Pythagoras
- Cauchy–Schwarz and triangle inequalities
- $||v|| := \langle v, v\rangle$ is a norm