

(Munkres §24)

Q is \mathbb{R} connected in the analytic topology?
why is this tricky?

last time, showed:

\mathbb{Q} as a subspace of \mathbb{R} is disconnected,
even though it is dense in \mathbb{R} (i.e. $\text{Cl}_{\mathbb{R}}(\mathbb{Q}) = \mathbb{R}$)

any proof that \mathbb{R} is connected must use
some fact about \mathbb{R} that fails for \mathbb{Q}

Df given a set X with a total order $<$
it has the least upper bound property iff
every subset of X bounded above
has a least upper bound (= sup)

Ex in an intro analysis course,
we define \mathbb{R} from scratch

the defn shows that \mathbb{R} has the LUBP
but \mathbb{Q} does not: e.g. $\{x \in \mathbb{Q} \mid x^2 < 2\}$ has no LUB

the LUBP is not enough to prove \mathbb{R} is connected:
why?

Ex if $X \subset \mathbb{R}$ finite and $|X| \geq 2$
then X disconnected but has the LUBP

Df a set X with a total order $<$ is called
a linear continuum iff

- 1) it has the LUBP
- 2) for all $x < y$ in X , there is z s.t. $x < z < y$

thus R is a linear continuum

Thm R is connected in the analytic topology

Pf suppose U, V form a separation of R
pick a in U and b in V
WLOG we can assume $a < b$

let $A = [a, b] \cap U$ and $B = [a, b] \cap V$

then A, B form a separation of $[a, b]$

in its subspace top

but A is bounded above by b

so by the LUBP, $\sup A$ exists and is in $[a, b]$

remains to show 1) $\sup A \notin B$

2) $\sup A \notin A$

1) assume $\sup A \in B$

then $\sup A > a$: else $A = \emptyset$

but B is open in $[a, b]$, so $(\sup A - \delta, \sup A] \subset B$
for some $\delta > 0$

but then, $\sup A - \varepsilon$ is a smaller upper bound for A
whenever $0 < \varepsilon \leq \delta$

2) assume $\sup A \in A$

then $\sup A < b$: else, $B = \emptyset$

but A is open in $[a, b]$, so $[\sup A, \sup A + \delta) \subset A$
for some $\delta > 0$

but then $\sup A + \varepsilon$ is an elt of A above $\sup A$
whenever $0 < \varepsilon < \delta$ \square

Rem thm generalizes to any linear continuum
in its order topology (we did not define)

for this course, just note:

the proof still works if we replace \mathbb{R} with

(a, b) , $(a, b]$, $[a, b)$, $[a, b]$

$(-\infty, b)$, $(-\infty, b]$, (a, ∞) , $[a, \infty)$

Thm (Intermediate Value)

suppose X connected and $f : X$ to \mathbb{R} cts

wrt the analytic top on \mathbb{R}

if a, b in X and $f(a) \leq \alpha \leq f(b)$

then there is some c in X s.t. $f(c) = \alpha$

Lem if X is connected, $f : X$ to Y cts,
then $f(X)$ sub Y is connected

Pf preimage of separation is a separation

Pf of Thm suppose no c in X s.t. $f(c) = \alpha$

then $f(X)$ has a separation: $\{f(x) < \alpha \mid x \text{ in } X\}$,
 $\{f(x) > \alpha \mid x \text{ in } X\}$

but X is connected, so contradiction with lemma

Moral the connectedness of intervals in \mathbb{R}
is very well-understood

Idea study connectedness in other spaces
by comparing them to intervals in \mathbb{R}

Df for any X and x, y in X
a path from x to y is a cts map
 $\gamma : [a, b]$ to X s.t. $\gamma(a) = x$ and $\gamma(b) = y$

we say X is path-connected iff
there is a path between every pair of pts in X

Rem we require $a \leq b$
but otherwise a, b can be any numbers

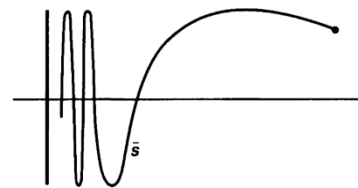
Lem if X is path-connected
then X is connected

Pf if $X = \emptyset$, then done
else, can fix x in X

for all y in X , pick a path γ_y from x to y
then $X = \bigcup \gamma_y([0, 1])$
each $\gamma_y([0, 1])$ is connected and contains x
so X is connected

[famous non-example:]

Ex in analytic \mathbb{R}^2 : consider
 $S = \{(x, \sin(1/x)) \mid 0 < x \leq 1\}$
 $A = \{(0, y) \mid -1 \leq y \leq 1\}$
 $\check{S} = S \cup A = \text{the closure of } S \text{ in } \mathbb{R}^2$



the topologist's sine curve: \check{S} in its subspace top.

not hard: S, A are path-connected
claim: \check{S} is not

sketch: fix a in A and s in S
 suppose $\gamma : [0, 1]$ to \check{S} a path from a to s

claim there is a largest t_0 in $[0, 1]$ s.t. $\gamma(t_0)$ in A :

A is closed in R , hence in \check{S}
 so $\gamma^{-1}(A)$ is closed in $[0, 1]$
 so $\gamma^{-1}(A)$ is its own closure
 so it contains its sup

so for all t in $(t_0, 1]$, have $\gamma(t)$ in S

let $(t_n)_n$ be a decreasing seq converging to t_0

for all n : pick $0 < x_n < x\text{-coord}(\gamma(t_n))$ s.t.

$$\sin(1/x_n) = (-1)^n$$

by IVT: $x\text{-coord}(\gamma(t_0)) < x_n < x\text{-coord}(\gamma(t_n))$

means there is t'_n in $[t_0, t_n]$ s.t.

$$\gamma(t'_n) = x_n$$

(Munkres §25)

Lem for any X , the following are equiv. rel's:

- 1) $x \sim y$ iff there is a connected subspace of X that contains both x and y
- 2) $x \leftrightarrow y$ iff there is a path between x and y in X

Pf transitivity for 1): if x, y in A & y, z in B ,
 then $A \cap B \ni y$

transitivity for 2): "pasting lem" M. 18.3

Df the connected components of X are
 the equiv. classes under \sim
 the path components of X are
 the equiv. classes under \leftrightarrow

Ex with some work, we can show:

S and A are the conn. components of \check{S} ,
and also, its path components