UNOFFICIAL SYLLABUS FOR MATH 7313

MATH 7313. Representation theory: modern introduction.

Lecturer: Ivan Loseu.
Contact info: Office 519 LA, Phone 617-373-5512, e-mail i.loseu@neu.edu.
Meeting times and locations: MW, 4.10-5.40pm, 509 LA.
Office hours: MW, 2.50-3.50.

What is this about? It's about introducing students to some modern aspects of representation theory, in other words, what representation theorists cared about chronologically around 90's on average. It's also about a bunch of surprising connections between seemingly unrelated topics in Representation theory.

Topics.
1) The representation theory of the symmetric groups in zero characteristic (following Okounkov and Vershik).
2) The representation theory of semisimple Lie algebras and semisimple algebraic groups over algebraically closed fields of zero and positive characteristic.
3) The complex representation theory of reductive algebraic groups over finite fields. Hecke algebras.
4) Quantum groups and applications to link invariants.
5) Representations of quivers. Deformed preprojective algebras and applications to linear algebra.
6) The representation theory of the symmetric groups in positive characteristic. Introduction to Lie algebra actions on categories.

Grading: There will be 5 homework problem sets on the first five topics. A problem set is posted after the topic is covered and is due two weeks after that. The grade is based entirely on homework. The final grade is determined as follows: A $\geq$ 90%, A- $\geq$ 87%, B+ $\geq$ 85%, B $\geq$ 75%, B- $\geq$ 72%, C+ $\geq$ 70%, C $\geq$ 60%, C- $\geq$ 55%, D+ $\geq$ 50%, D $\geq$ 45%, D- $\geq$ 40%.

Prerequisites and corequisites. A definite prerequisite is a graduate "abstract algebra" class. Representation theoretic prerequisites will be covered in the first lecture. Some familiarity with C^∞-manifolds/ algebraic varieties will be useful.
As an additional reading, a text by Etingof et. al. available at
http://math.mit.edu/∼etingof/replect.pdf
is highly recommended. Comparing to our course, this should be viewed as a "classical introduction".

Literature/textbooks: Not really. Lecture notes will be posted. They will contain references for further reading.

REPRESENTATION THEORY, LECTURE 0. BASICS

IVAN LOSEV

Introduction

The aim of this lecture is to recall some standard basic things about the representation theory of finite dimensional algebras and finite groups. First, we recall restriction, induction and coinduction functors. Then we recall the Schur lemma and deduce consequences about the action of the center and the structure of completely reducible representations. Then we explain the structure and representation theory of simple finite dimensional algebras over algebraically closed fields. Next, we proceed to semisimple algebras. Finally, we use the latter to recall basics about the representation theory of finite groups.

1. Restriction, induction and coinduction

Let A, B be associative unital algebras over a field $\mathbb{F}$ with a homomorphism $B \to A$. Let M be an A-module. Of course, we can view M as a B-module. On the other hand, A is a left A-module and a right B-module. These two operations commute, one says in this case that A is an A-B-bimodule. It follows that, for a B-module N, the space $A \otimes_B N$ carries a natural structure of a left A-module (induced module). Also A is a B-A-bimodule. So $\operatorname{Hom}_B(A, N)$ is a left A-module via $(a\varphi)(a') := \varphi(a'a), \varphi \in \operatorname{Hom}_B(A, N), a, a' \in A$. This is a *coinduced* module.

Lemma 1.1. *For an A-module M and a B-module N we have natural isomorphisms*

$$\operatorname{Hom}_B(N, M) \cong \operatorname{Hom}_A(A \otimes_B N, M), \quad \operatorname{Hom}_B(M, N) \cong \operatorname{Hom}_A(M, \operatorname{Hom}_B(A, N)).$$

Proof. Consider the map $\operatorname{Hom}_B(N, M) \to \operatorname{Hom}_A(A \otimes_B N, M)$ that sends η to ψ_η given $\psi_\eta(a \otimes n) = a\eta(n)$ and the map in the opposite direction that sends ψ to η_ψ given by $\eta_\psi(n) := \psi(1 \otimes n)$. It is left as an exercise to check that the maps are well-defined (i.e., land in the required Hom spaces) and are mutually inverse. Establishing a natural isomorphism $\operatorname{Hom}_B(M, N) \cong \operatorname{Hom}_A(M, \operatorname{Hom}_B(A, N))$ is also left as an exercise. □

2. Schur lemma and its consequences

2.1. **Schur lemma.** The following important result is known as the Schur lemma.

Proposition 2.1. *Let $\mathbb{F}$ be algebraically closed, A be an associative unital $\mathbb{F}$-algebra and let U, V be finite dimensional irreducible A-modules. Then the following is true.*

1. *If U, V are non-isomorphic, then* $\operatorname{Hom}_A(U, V) = 0$.
2. $\operatorname{End}_A(U)$ *consists of constant maps. In particular,* $\dim \operatorname{End}_A(U) = 1$.

Under some assumptions, this can be generalized to infinite dimensional irreducible modules.

2.2. **The action of the center.** Consider the center of A, $Z(A) := \{z \in A | za = az, \forall a \in A\}$. This is a commutative algebra. The following claim is a corollary of the Schur lemma.

Corollary 2.2. *Let $z \in Z(A)$, and let U be a finite dimensional irreducible A-module. Then z acts on U by a scalar.*

It follows that there is an algebra homomorphism $\chi_U : Z(A) \to \mathbb{F}$ (called the central character of U) such that z acts on U as the multiplication by $\chi_U(z)$.

2.3. **Multiplicities in completely reducible modules.** Let A be an associative unital algebra over an algebraically closed field $\mathbb{F}$. Let $V_i, i \in I$, be the finite dimensional irreducible A-modules, where I is an indexing set. Now let V be a completely reducible finite dimensional A-module. Since V is completely reducible, there are non-negative integers $m_i, i \in I$, with only finitely many nonzero such that $V \cong \bigoplus_{i\in I} V_i^{\oplus m_i}$.

The following lemma is a consequence of the Schur lemma and the additivity of Hom's: $\mathrm{Hom}_A(V_i, V' \oplus V'') \cong \mathrm{Hom}_A(V_i, V') \oplus \mathrm{Hom}_A(V_i, V'')$.

Lemma 2.3. *The number m_i coincides with* $\dim \mathrm{Hom}_A(V_i, V)$.

We call $\mathrm{Hom}_A(V_i, V)$ the *multiplicity space* for V_i in V. The name is justified by the observation that the natural homomorphism

$$\bigoplus_{i\in I} V_i \otimes \mathrm{Hom}_A(V_i, V) \to V, \quad \sum_{i\in I} v_i \otimes \varphi_i \mapsto \sum_{i\in I} \varphi_i(v_i) \tag{2.1}$$

is an isomorphism of A-modules.

2.4. **Endomorphisms of completely reducible modules.** The isomorphism (2.1) together with the Schur lemma imply the following description of the endomorphism algebra $\mathrm{End}_A(V)$:

$$\mathrm{End}_A(V) = \bigoplus_{i\in I} \mathrm{End}(\mathrm{Hom}_A(V_i, V)).$$

Here we assume that the endomorphisms of the zero space are zero (and so we sum over all i such that $\mathrm{Hom}_A(V_i, V) \neq 0$, in particular, the sum is finite). The $\mathrm{End}_A(V)$-module structure on $\mathrm{Hom}_A(V_i, V)$ is given by the composition:

$$\varphi \cdot \psi := \varphi \circ \psi, \varphi \in \mathrm{End}_A(V), \psi \in \mathrm{Hom}_A(V_i, V).$$

3. Simple algebras

3.1. **Burnside theorem.** Let A be an associative algebra over an algebraically closed field $\mathbb{F}$ and let V be a finite dimensional A-module. So we have an algebra homomorphism $A \to \mathrm{End}(V)$.

Proposition 3.1. *If V is irreducible, then the homomorphism $A \to \mathrm{End}(V)$ is surjective.*

Proof. The proof is in several steps.

Step 1. Consider the A-module $V \otimes M$, where M is a finite dimensional vector space (and A acts on the first factor). We claim that every A-submodule $U \subset V \otimes M$ has the form $V \otimes M_0$, where M_0 is a subspace in M. Indeed, $\mathrm{Hom}_A(V, U) \hookrightarrow \mathrm{Hom}_A(V, V \otimes M)$. By the Schur lemma, the target space is naturally identified with M. The subspace $\mathrm{Hom}_A(V, U) \subset M$ is M_0 we need: by complete reducibility, if $U \neq V \otimes M_0$, there is a homomorphism $\varphi : V \to U$ that does not lie in M_0.

Step 2. The space V^* is a right A-module via $(\varphi \cdot a)(v) := \varphi(a \cdot v), \varphi \in V^*, v \in V, a \in A$. Note that V^* is irreducible (if $U' \subset V^*$ is a proper submodule, then the annihilator of U' is a proper A-submodule in V).

Step 3. Recall that $\mathrm{End}(V)$ is naturally identified with $V \otimes V^*$. Both $\mathrm{End}(V)$ and $V \otimes V^*$ are A-bimodules and the isomorphism $\mathrm{End}(V) \cong V \otimes V^*$ is that of A-bimodules. Replacing A with its image in $\mathrm{End}(V)$, we may assume that $A \subset \mathrm{End}(V)$. Clearly, $A \subset V \otimes V^*$ is a subbimodule. Apply Step 1 to A viewed as a left A-module. We get that $A = V \otimes M'$, where $M' \subset V^*$. Applying (the obvious analog of) Step 1 to the right A-module $V \otimes V^*$, we get $A = M \otimes V^*$, where $M \subset V$. Since $V \otimes M' = M \otimes V^*$, we see that $M = V, M' = V^*$. □

3.2. **Simple algebras over algebraically closed fields.** Let A be an associative unital algebra over $\mathbb{F}$. We say that A is simple if it has no proper two-sided ideals. For example, $\mathrm{Mat}_n(\mathbb{F})$ is a simple algebra, this can be deduced similarly to Step 3 of the proof of the Burnside theorem.

Proposition 3.2. *Let $\mathbb{F}$ be algebraically closed and A be a finite dimensional simple A-algebra. Then $A \cong \mathrm{Mat}_n(\mathbb{F})$ for some n.*

Proof. Consider a minimal (w.r.t. inclusion) left ideal $I \subset A$ (just take a left ideal of minimal dimension). It is an irreducible left A-module. So we get a homomorphism $A \to \mathrm{End}(I)$. Its injective because A is simple. It is surjective by the Burnside theorem. So $A \xrightarrow{\sim} \mathrm{End}(I)$. □

3.3. **Representations of the matrix algebra.** Let V be a finite dimensional vector space over $\mathbb{F}$. We are going to understand the representation theory of the algebra $A = \mathrm{End}(V)$.

Proposition 3.3. *Every finite dimensional A-module U is completely reducible and the only irreducible module is V itself.*

Proof. There are $u_1, \ldots, u_k \in U$ such that $U = Au_1 + \ldots + Au_k$. This gives an A-module epimorphism $A^{\oplus k} \twoheadrightarrow U, (a_1, \ldots, a_k) \mapsto a_1u_1 + \ldots + a_ku_k$. Moreover, $A \cong V^{\oplus n}$, where $n = \dim V$ and so $V^{\oplus nk} \twoheadrightarrow U$. Being a quotient of a completely reducible module, U is completely reducible itself. If U is irreducible, $\mathrm{Hom}_A(V^{\oplus nk}, U) \neq 0$. Therefore $\mathrm{Hom}_A(V, U) \neq 0$. By the Schur lemma, $V \cong U$. □

4. Semisimple algebras

Let A be a finite dimensional algebra over $\mathbb{F}$ (still assumed to be algebraically closed). We say that A is *semisimple*, if it is a direct sum of simple algebras.

4.1. **Criteria for semisimplicity.** We are going to explain some criteria for semisimplicity.

Lemma 4.1. *Let A be a finite dimensional associative unital algebra. Let I, J be two-sided ideals in A consisting of nilpotent elements (we say that $a \in A$ is nilpotent if $a^n = 0$ for some $n > 0$). Then $I + J$ is a two-sided ideal consisting of nilpotent elements.*

The proof is left as an exercise.

So A has the unique maximal two-sided ideal consisting of nilpotent elements, it is called the *radical* of A and is denoted by $\mathrm{Rad}(A)$.

Also we can define a distinguished element $\mathrm{tr}_A \in A^*$. It sends $a \in A$ to the the trace of m_a, the operator $A \to A$ given by $m_a(b) = ab$. Note that $\mathrm{tr}_A(ab) = \mathrm{tr}_A(ba)$. So $(a, b)_A := \mathrm{tr}_A(ab)$ is a symmetric bilinear form.

Proposition 4.2. *Let A be a finite dimensional algebra over an algebraically closed field $\mathbb{F}$. The following conditions are equivalent.*

(i) *The algebra A is semisimple.*
(ii) $\mathrm{Rad}(A) = \{0\}$.
(iii) *A is completely reducible as a left A-module.*
(iv) *Every finite dimensional representation of A is completely reducible.*

If the characteristic of $\mathbb{F}$ is zero, then (i)-(iv) are equivalent to the following condition.

(v) *The form $(\cdot,\cdot)_A$ is non-degenerate.*

Proof. Proof of (i)⇒(ii). Note that the radical of a simple algebra is zero. Also the radical of the direct sum is the direct sum of radicals. This proves the required implication.

Proof of (ii)⇒(iii). We have an A-module filtration $A = A_0 \supsetneq A_1 \ldots \supsetneq A_n \supsetneq A_{n+1} = 0$ such that A_i/A_{i+1} is irreducible for all $i = 0,\ldots,n$. Consider the corresponding algebra homomorphism $\varphi : A \to \bigoplus_{i=0}^n \mathrm{End}(A_i/A_{i+1})$. The inclusion $a \in \ker\varphi$ is equivalent to $aA_i \subset A_{i+1}$ for all i. So for any $a \in \ker\varphi$ we have $a^{n+1} = 0$ and hence $\ker\varphi \subset \mathrm{Rad}(A)$. Therefore $\ker\varphi = \{0\}$ and we have an embedding $A \hookrightarrow \bigoplus_{i=0}^n \mathrm{End}(A_i/A_{i+1})$ of algebras, so, in particular, of left A-modules. The A-module $\bigoplus_{i=0}^n \mathrm{End}(A_i/A_{i+1}) = \bigoplus_{i=0}^n (A_i/A_{i+1}) \otimes (A_i/A_{i+1})^*$ is completely reducible. Being a submodule in a completely reducible module, A is completely reducible.

Proof of (iii)⇒(iv) repeats (a part of) the proof of Proposition 3.3.

Proof of (iv)⇒(i). We just need that A is completely reducible. Let $A = \bigoplus_{i=1}^k V_i^{\oplus m_i}$, where all V_i are irreducible and all m_i are positive. Since A is a faithful A-module (only zero acts by zero), the same is true for $\bigoplus_{i=1}^k V_i$. So we get an algebra embedding $A \hookrightarrow \bigoplus_{i=1}^k \mathrm{End}(V_i)$. In particular, this is a left A-module embedding. It follows that $m_i \leqslant \dim V_i$, as the right hand side is the multiplicity of V_i in the left A-module $\bigoplus_{i=1}^k \mathrm{End}(V_i)$.

On the other hand, by the Burnside theorem, the composition of the embedding $A \hookrightarrow \bigoplus_{i=1}^k \mathrm{End}(V_i)$ with the projection to $\mathrm{End}(V_i)$ is surjective. It follows that $m_i \geqslant \dim V_i$. We conclude that $m_i = \dim V_i$ and $A \xrightarrow{\sim} \bigoplus_{i=1}^k \mathrm{End}(V_i)$.

Proof of (ii)⇔(v). It is enough to show that the radical of A coincides with $\ker(\cdot,\cdot)_A$ when $\mathrm{char}\,\mathbb{F} = 0$. Indeed, if a is in the radical, then ab is nilpotent for all $b \in A$, and $(a,b)_A = 0$. So $\mathrm{Rad}(A) = \ker(\cdot,\cdot)_A$. On the other hand, $\ker(\cdot,\cdot)_A$ is a two-sided ideal because $(ab,c)_A = (a,bc)_A$ for all $a,b,c \in A$. Also if $a \in \ker(\cdot,\cdot)_A$, then $\mathrm{tr}_A(a^n) = 0$ for all $n > 0$. It follows that a is nilpotent (here we use that $\mathrm{char}\,\mathbb{F} = 0$). We see that $\ker(\cdot,\cdot)_A$ is contained in the radical. This finishes the proof of (ii)⇔(v). □

4.2. **Representations of semisimple algebras.**

Lemma 4.3. *Let $A = \bigoplus_{i=1}^k \mathrm{End}(V_i)$. Then the set of irreducible A-modules, to be denoted by $\mathrm{Irr}(A)$, coincides with $\{V_1,\ldots,V_k\}$.*

Proof. Let A_1, A_2 be associative algebras. Then any $A_1 \oplus A_2$-module V canonically decomposes as $V_1 \oplus V_2$, where V_i is an A_i-module. Namely, if e_i is the unit in A_i, then $V_i = e_i V$. In particular, $\mathrm{Irr}(A_1 \oplus A_2) = \mathrm{Irr}(A_1) \sqcup \mathrm{Irr}(A_2)$. The claim of the lemma follows from here (and trivial induction). □

Often one wants to compute the number of irreducible representations of A without referring to the decomposition $A = \bigoplus_{i=1}^k \mathrm{End}(V_i)$.

Lemma 4.4. *Let A be a semisimple algebra. Then $|\mathrm{Irr}(A)| = \dim Z(A)$.*

Proof. This follows from the observation that the center of $\mathrm{End}(V_i)$ consists of scalar operators combined with $Z(A_1 \oplus A_2) = Z(A_1) \oplus Z(A_2)$. $\square$

4.3. **Irreducible representations of arbitrary finite dimensional algebras.** Now let A be a finite dimensional $\mathbb{F}$-algebra. By (ii) of Proposition 4.2, the algebra $A/\operatorname{Rad}(A)$ is semisimple. Besides $\operatorname{Rad}(A)$ acts by 0 on all irreducible representations. We conclude that pulling back a representation from $A/\operatorname{Rad}(A)$ to A gives rise to a bijection between $\operatorname{Irr}(A/\operatorname{Rad}(A))$ and $\operatorname{Irr}(A)$.

5. Finite groups

5.1. **Group algebra and its semisimplicity.** Let G be a finite group and $\mathbb{F}$ be a field. We can form the group algebra $\mathbb{F}G$ of G, a vector space with basis G, where the basis elements multiply as in G. A representation of G is the same thing as a representation of $\mathbb{F}G$.

Proposition 5.1. *Let $\mathbb{F}$ be algebraically closed and of characteristic 0. Then $\mathbb{F}G$ is a semisimple algebra. In particular, any finite dimensional representation of G over $\mathbb{F}$ is completely reducible.*

Proof. We will check (v) of Proposition 4.2. On the basis elements, we have $\mathrm{tr}_{\mathbb{F}G}(g) = \delta_{g1}|G|$, where δ_{g1} is the Kroneker symbol. So $(g,h)_{\mathbb{F}G} = \delta_{g,h^{-1}}|G|$. Clearly, this form is nondegenerate. $\square$

We remark that this proposition is no longer true when the characteristic of $\mathbb{F}$ is positive.

5.2. **The number of irreducible representations.** By Lemma 4.4, the number of the irreducible representations of G coincides with the dimension of the center. So let us investigate the structure of $Z(\mathbb{F}G)$ as a vector space.

Proposition 5.2. *There is a basis $b_C \in Z(\mathbb{F}G)$, where C runs over the set of conjugacy classes in G. It is given by $b_C := \sum_{g\in C} g$.*

Proof. The inclusion $\sum_{g\in G} c_g g \in Z(\mathbb{F}G)$ is equivalent to $h\sum_{g\in G} c_g g = \sum_{g\in G} c_g gh$, which in its turn is equivalent to $\sum_{g\in G} c_g(hgh^{-1}) = \sum_{g\in G} c_g g$. In other words, $\sum_{g\in G} c_g g \in Z(\mathbb{F}G)$ if and only if the function $g \mapsto c_g$ is constant on conjugacy classes. This implies the claim of the proposition. $\square$

6. What happens when $\mathbb{F}$ is not algebraically closed

First, we need to modify the second part of the Schur lemma: the endomorphism algebra $\mathrm{End}_A(V)$ is a skew-field. An analog of the Burnside theorem still works: the image of A in $\mathrm{End}(V)$ is $\mathrm{End}_{\mathbb{S}}(V)$, where $\mathbb{S}$ is the skew-field $\mathrm{End}_A(V)$. The proof is somewhat more involved. The simple algebras are precisely $\mathrm{Mat}_n(\mathbb{S})$, where $\mathbb{S}$ is a finite dimensional skew-field over $\mathbb{F}$, the proof repeats that of Proposition 3.2. An analog of Proposition 3.3 holds for $\mathrm{Mat}_n(\mathbb{S})$. An analog of Proposition 4.2 holds too. The details are left to the reader.

LECTURE 1: REPRESENTATIONS OF SYMMETRIC GROUPS, I

IVAN LOSEV

1. Introduction

In this lecture we start to study the representation theory of the symmetric groups S_n over $\mathbb{C}$. Let us summarize a few things that we already know.

0) A representation of S_n is the same thing as a representation of the group algebra $\mathbb{C}S_n$.

1) As with all finite groups, any representation of S_n over $\mathbb{C}$ is completely reducible.

2) The number of irreducible representations of S_n coincides with the number of conjugacy classes.

The conjugacy classes are in a natural bijection with partitions of n. Namely, we take an element $\sigma \in S_n$ and decompose it into the product of disjoint cycles. The lengthes of cycles form a partition of n that is independent of the choice of σ in the conjugacy class. We assign this partition to the conjugacy class of σ.

We would like to emphasize that 2) does not establish any preferred bijection between the irreducible representations of S_n and the partitions of n. To establish such a bijection is our goal in this part. We will follow a "new" approach to the representation theory of the groups S_n due to Okounkov and Vershik, [OV]. Our exposition follows [K, Section 2]. For a "traditional" approach based on Young symmetrizers, the reader is welcome to consult [E] or [F].

2. Inductive approach

A key observation is that symmetric groups for different n are embedded into one another: $\{1\} = S_1 \subset S_2 \subset S_3 \ldots \subset S_{n-1} \subset S_n \subset \ldots$, where we view S_{n-1} as the subgroup of S_n fixing $n \in \{1, \ldots, n\}$. We could try to use "induction", i.e., to study the irreducible representations of S_n by restricting them to S_{n-1}. In fact, this naive idea does not quite work, but this is our starting point.

2.1. **Centralizer $Z_B(A)$ and restrictions of representations.** We start with the following question: given a finite dimensional semisimple associative algebra A and its semisimple subalgebra B, understand the restriction of $V \in \mathrm{Irr}(A)$ (the set of isomorphism classes of finite dimensional irreducible A-modules) to B. The answer to this question is controlled by the subalgebra $Z_B(A) \subset A$ (the centralizer of B in A) defined by $Z_B(A) = \{a \in A | ba = ab, \forall b \in B\}$. More precisely, we have the following fact, where, recall that $\mathrm{Hom}_B(U, V)$ stands for the space of B-module homomorphisms $U \to V$.

Lemma 2.1. *We have an isomorphism* $Z_B(A) = \bigoplus_{U,V} \mathrm{End}(\mathrm{Hom}_B(U, V))$, *where the sum is taken over all pairs* $(U, V) \in \mathrm{Irr}(B) \times \mathrm{Irr}(A)$ *satisfying* $\mathrm{Hom}_B(U, V) \neq \{0\}$.

In other words, the algebra $Z_B(A)$ is semisimple and the irreducible $Z_B(A)$-modules are precisely the nonzero multiplicity spaces $\mathrm{Hom}_B(U, V)$.

Proof. Since A is semisimple, it can be identified with $\bigoplus_{V\in \mathrm{Irr}(A)} \mathrm{End}(V)$, see Proposition 4.2 from Lecture 0. In this realization, we have $Z_B(A) = \bigoplus_{V\in \mathrm{Irr}(A)} \mathrm{End}_B(V)$. Recall from Lecture 0, Section 3.4, that

$$\mathrm{End}_B(V) = \bigoplus_U \mathrm{End}(\mathrm{Hom}_B(U, V)),$$

where the summation is taken over all $U \in \mathrm{Irr}(B)$ such that $\mathrm{Hom}_B(U, V) \neq \{0\}$. $\square$

Here is a corollary of this lemma that will be very useful for us in what follows.

Corollary 2.2. *The following two conditions are equivalent:*

(1) *For any* $U \in \mathrm{Irr}(B), V \in \mathrm{Irr}(A)$, *we have* $\dim \mathrm{Hom}_B(U, V) \leqslant 1$.
(2) $Z_B(A)$ *is commutative.*

Proof. The algebra $Z_B(A) = \bigoplus_{U,V} \mathrm{End}(\mathrm{Hom}_B(U, V))$ is commutative if and only if the summands are. For a nonzero vector space W, $\mathrm{End}(W) = \mathrm{Mat}_{\dim W}(\mathbb{C})$ is commutative if and only if $\dim W = 1$. This implies that (1) $\Leftrightarrow$ (2). $\square$

What this corollary gives us is that if (2) is satisfied, then any irreducible A-module uniquely decomposes into the sum of irreducible B-modules, meaning that the summands are uniquely determined as subspaces.

Remark 2.3. It is instructive to describe the structure of a $Z_B(A)$-module on $\mathrm{Hom}_B(U, V)$ without referring to the decomposition $A = \bigoplus \mathrm{End}(V)$. Let $z \in Z_B(A)$ and $\varphi \in \mathrm{Hom}_B(U, V)$. We define $z \cdot \varphi \in \mathrm{Hom}_B(U, V)$ by $[z \cdot \varphi](u) = z \cdot \varphi(u)$, for all $u \in U$. To check that this is well defined and is compatible with the previous module structure (see Section 3.4 of Lecture 0) is left to the reader.

2.2. **The structure of $Z_m(n)$.** We take $A = \mathbb{C}S_n$ and $B = \mathbb{C}S_m$, where $m < n$. We write $Z_m(n)$ for $Z_B(A)$. We are interested in finding generators for the algebra $Z_m(n)$.

Theorem 2.4. *The algebra $Z_m(n)$ is generated by the subalgebra $Z_m(m)$, the subgroup $S_{[m+1,n]} \subset S_n$ (fixing $1, \ldots, m$ and permuting $m+1, \ldots, n$) and the* Jucys-Murphy elements $L_k := \sum_{i=1}^{k-1} (ik)$, *where k ranges from $m + 1$ to n.*

Note that $L_1 = 0$.

Proof. The proof is in several steps.

Step 1. Let $H \subset G$ be finite groups. Then we have a basis in $Z_{\mathbb{C}H}(\mathbb{C}G)$ indexed by the H-conjugacy classes in G. Namely, to a class c we assign an element $b_c \in Z_{\mathbb{C}H}(\mathbb{C}G)$ given by $z_c := \sum_{g\in c} g$. This is a straightforward generalization of Proposition 5.2 in Lecture 0.

Step 2. Let $G = S_n, H = S_m$. Recall that the G-conjugacy classes in G are parameterized by cycle types, e.g., in S_6 we have a conjugacy class $(***)(**)$. The H-conjugacy classes are parameterized by class types with marked elements $m + 1, \ldots, n$, e.g., we have the following S_4-conjugacy classes in S_6: $(56*)(**), (65*)(**), (6**)(5*), (***)(56)$, etc. Note that $b_{(*k)} = L_k - \sum_{j=m+1}^{k-1} (jk)$ for $k > m$. In particular, $L_k \in Z_m(n)$. To a class c we assign its *degree* $\deg c$ that, by definition, is equal to the number of elements in $\{1, \ldots, n\}$ moved by an element in c (this is independent of the choice of the element).

Step 3. Let A be the subalgebra in $\mathbb{C}S_n$ generated by $Z_m(m), S_{[m+1,n]}, L_{m+1}, \ldots, L_n$. It is easy to see that $A \subset Z_m(n)$. To prove the opposite inclusion, assume that $b_c \notin A$ for some c. We pick c of minimal degree with this property. In the next four steps we will arrive at a contradiction.

Step 4. Assume, first, that c has more than one cycle of length at least 2. Break c into the union of two cycle types c', c'', e.g., if $c = (6**)(5*)$, then we can take $c' = (6**), c'' = (5*)$. Note that

$$b_{c'}b_{c''} = \alpha b_c + \sum_{c_0, \deg c_0 < \deg c} \alpha_{c_0} b_{c_0},$$

where $\alpha > 0$. By our inductive assumption, $b_{c_0} \in A$ and $b_{c'}b_{c''} \in A$. So $b_c \in A$, which contradicts the choice of c.

Step 5. Now let us pick a cycle $(i_1, i_2, \ldots, i_k) \in S_n$ and consider the product $(i_1, \ldots, i_k)(i_s, j)$. If $j \notin \{i_1, \ldots, i_k\}$, then we get $(i_1, \ldots, i_s, j, i_{s+1}, \ldots, i_k)$. If $j \in \{i_1, \ldots, i_k\}$, then $(i_1, \ldots, i_k)(i_k j)$ either splits into the product of two cycles of total degree k or is a cycle of degree $k-1$.

Step 6. Now suppose that the cycle in c has both an element from $\{1, \ldots, m\}$ (does not matter which, we denote it by $*$) and $k \in \{m+1, \ldots, n\}$. We may assume that k is right after $*$ in the cycle. Let c' denote the cycle obtained from c by deleting k. Then $b_{c'}L_k = \alpha b_c + \sum_{c_0} \alpha_{c_0} b_{c_0}$, where the summation is over c_0 that are products of two disjoint cycles with $\deg c_0 = \deg c$ or have $\deg c_0 < \deg c$. This is a consequence of Step 5, as the left hand side is the sum of products of pairs of cycles that share a common element, k. Similarly to Step 4, we arrive at a contradiction with the choice of c.

Step 7. So either the elements in the only cycle of c are all from $\{1, \ldots, m\}$, in which case $b_c \in Z_m(m)$, or are all from $\{m+1, \ldots, n\}$, in which case $b_c \in S_{[m+1,n]}$. Contradiction. □

Corollary 2.5. *The following is true.*

1. *$Z_m(m)$ lies in the center of $Z_m(n)$.*
2. *The algebra $Z_{n-1}(n)$ is commutative.*

Proof. The algebra $Z_m(n)$ commutes with $\mathbb{C}S_m$ and $Z_m(m) \subset Z_m(n) \cap \mathbb{C}S_m$. So $Z_m(m)$ is in the center of $Z_m(n)$.

The algebra $Z_{n-1}(n)$ is generated by $Z_{n-1}(n-1)$ and L_n. Since the former is central, the algebra $Z_{n-1}(n)$ is commutative. □

3. Basis and weights

We will use Corollary 2.5 to construct a basis in $\bigoplus_{V \in \mathrm{Irr}(S_n)} V$ and encode elements of this basis with n-tuples of complex numbers to be called *weights*.

3.1. **Branching graph.**

Basis elements will be labelled by paths in a graph that is called the branching graph for the symmetric groups. The vertices of this graph will be $\bigsqcup_{n \geqslant 1} \mathrm{Irr}(S_n)$. We draw a single arrow between $V^{n-1} \in \mathrm{Irr}(S_{n-1}), V^n \in \mathrm{Irr}(S_n)$ if $\mathrm{Hom}_{S_{n-1}}(V^{n-1}, V^n)$ has dimension 1 (by Corollary 2.5 the only other option is 0). There are no other edges.

Paths in the branching graph label bases in Hom spaces. For vertices $V^m \in \mathrm{Irr}(S_m), V^n \in \mathrm{Irr}(S_n)$ with $m < n$, denote by $\mathsf{Path}(V^m, V^n)$ the set of paths from V^m to V^n.

Lemma 3.1. *There is a basis in* $\mathrm{Hom}_{S_m}(V^m, V^n)$ *indexed by* $\mathsf{Path}(V^m, V^n)$.

Proof. We have $V^n = \bigoplus_{V^{n-1}, V^{n-1} \to V^n} V^{n-1}$. Now decompose V^{n-1} into the sum of irreducible representations of S_{n-2}. Plugging this decomposition into the sum above, we get

$$V^n = \bigoplus_{V^{n-2}, P \in \mathsf{Path}(V^{n-2}, V^n)} V_P^{n-2},$$

where V_P^{n-2} denotes the copy of V_n embedded into V^n via $V^{n-2} \hookrightarrow V^{n-1} \hookrightarrow V^n$, where $P = V^{n-2} \to V^{n-1} \to V^n$. We continue in this manner and get

$$V^n = \bigoplus_{V^m, P \in \mathsf{Path}(V^m, V^n)} V_P^m.$$

Let $\varphi_P \in \mathrm{Hom}_{S_m}(V^m, V^n)$ be the embedding of $V^m \xrightarrow{\sim} V_P^m \subset V^n$. We see that $\varphi_P, P \in \mathsf{Path}(V^m, V^n)$, is a basis in $\mathrm{Hom}_{S_m}(V^m, V^n)$. □

Remark 3.2. Note that the element φ_P is defined uniquely up to proportionality. Also note that if $P_2 \in \mathsf{Path}(V^k, V^m), P_1 \in \mathsf{Path}(V^m, V^n)$, then $\varphi_{P_1} \circ \varphi_{P_2}$ is proportional to $\varphi_{P_1 P_2}$, where $P_1 P_2 \in \mathsf{Path}(V^k, V^n)$ is the concatenation of P_1 and P_2.

3.2. **Basis and weights.** If in Lemma 3.1 we take $m = 1$, we will get a basis in $\mathrm{Hom}_{S_1}(V^1, V^n) = \mathrm{Hom}(\mathbb{C}, V^n) = V^n$, we will write v_P for φ_P in this case. By the construction, if $P = V^1 \to V^2 \to \ldots \to V^n$, then v_P lies in V^1 uniquely embedded into V^2 that is uniquely embedded into V^3, etc.

Lemma 3.3. *The following is true.*

1. *The vector v_P is an eigenvector for all Jucys-Murphy elements $L_k, k = 1, \ldots, n$.*
2. *The eigenvalue of L_k on v_P depends only on the V^{k-1} and V^k components in $P = V^1 \to V^2 \to \ldots \to V^n$.*

We postpone the proof a little bit, to give a definition and an example.

Definition 3.4. Define the *weight* $w_P = (w_1, \ldots, w_n) \in \mathbb{C}^n$ of the path P (or of the basis vector v_P) by $L_k v_P = w_k v_P, k = 1, \ldots, n$.

Example 3.5. Consider the reflection representation R^n of S_n. It can be realized as the submodule $\{(x_1, \ldots, x_n) | x_1 + \ldots + x_n = 0\}$ in the permutation representation $\mathbb{C}^n$ of S_n. The restriction of R^n to S_{n-1} decomposes as $R^{n-1} \oplus T^{n-1}$, where we write T^{n-1} for the trivial representation of S_{n-1}. The copy of R^{n-1} is realized as $\{(x_1, \ldots, x_n) | x_1 + \ldots + x_{n-1} = 0, x_n = 0\}$, while the copy of T^{n-1} is spanned by $(1, \ldots, 1, 1-n)$. The paths indexing the basis in R^n are

$$P_m := T^1 \to T^2 \to T^{m-1} \to R^m \to R^{m+1} \to \ldots \to R^n, m = 2, \ldots, n.$$

The corresponding basis vector is $v_{P_m} = (1, \ldots, 1, 1-m, 0, \ldots, 0)$. The weight w_{P_m} equals $(0, 1, \ldots, m-2, -1, m-1, \ldots, n-2)$.

Proof of Lemma 3.3. Note that v_P lies in the unique copy of V^{k-1} in V^k. It is enough to check that L_k acts on that copy of V^{k-1} by a scalar (that depends only on V^{k-1}, V^k because $L_k \in \mathbb{C}S_k$). But L_k commutes with $\mathbb{C}S_{k-1}$ and so the operator of multiplication by L_k gives an element in $\mathrm{Hom}_{S_{k-1}}(V^{k-1}, V^k)$. Since the dimension of the latter space is 1, the multiplication by L_{k-1} gives an endomorphism of the S_{k-1}-module V^{k-1}. This endomorphism is scalar by the Schur lemma. □

3.3. **Maximal commutative subalgebra.** A natural question to ask at this point is: can two different paths $P \in \mathsf{Path}(V^1, V^n), P' \in \mathsf{Path}(V^1, V'^n)$ give the same weight? Here we will see that the answer is "no": a weight determines a path uniquely.

Consider the subalgebra $A \subset \mathbb{C}S_n$ consisting of all elements a such that all v_P are eigenvectors for a. In other words, if we identify $\mathrm{End}(V^n)$ with $\mathrm{Mat}_{\dim V^n}(\mathbb{C})$ using the basis $v_P, P \in \mathsf{Path}(V^1, V^n)$, and $\mathbb{C}S_n$ with $\bigoplus_{V^n \in \mathrm{Irr}(S_n)} \mathrm{End}(V^n)$, then A is the direct sum of the subalgebras of diagonal matrices in $\mathrm{Mat}_{\dim V^n}(\mathbb{C})$. Note that A is a maximal commutative subalgebra in $\mathbb{C}S_n$.

There are two alternative descriptions of A.

Proposition 3.6. *The following subalgebras of $\mathbb{C}S_n$ coincide.*

(i) *A introduced above.*
(ii) *A' generated by $Z_k(k), k = 1, \dots, n$.*
(iii) *A'' generated by $L_1, \dots, L_n$.*

Proof. We will prove that $A \subset A', A' \subset A''$, and $A'' \subset A$.

Proof of $A \subset A'$. We have a basis in A labelled by the paths $P \in \mathsf{Path}(V^1, V^n)$, where V^n runs over $\mathrm{Irr}(S_n)$. Namely, define e_P by $e_P v_{P'} = \delta_{PP'} v_{P'}$ (i.e., e_P is the diagonal matrix element corresponding to P). Let $P = V^1 \to V^2 \to \dots \to V^n$.

Define $e_{V^m} \in \mathbb{C}S_m$ as the identity in the summand $\mathrm{End}(V^m)$ of $\mathbb{C}S_m = \bigoplus_{U \in \mathrm{Irr}(S_m)} \mathrm{End}(U)$ and zero in all other summands. This element is central, in other words, $e_{V^m} \in Z_m(m)$. Now consider the product $e_{V^1} e_{V^2} \dots e_{V^n}$ and its action on $\bigoplus_{U^n \in \mathrm{Irr}(S_n)} U^n$. Applying e_{V^n} we project to the summand V^n. Applying $e_{V^{n-1}}$ next, we project to the summand V^{n-1} inside V^n. And so on. From the construction of the element v_P, we conclude that $e_{V^1} \dots e_{V^n}$ coincides with e_P. Since $e_{V^1} \dots e_{V^n} \in A'$, we see that $e_P \in A'$, and we are done.

Proof of $A' \subset A$. We prove this by induction on n: suppose that $Z_1(1) = \mathbb{C}, \dots, Z_{n-1}(n-1)$ lie in the subalgebra generated by $L_1, \dots, L_{n-1}$. Note that $Z_n(n) \subset Z_{n-1}(n)$. By Theorem 2.4, $Z_{n-1}(n)$ is generated by $Z_{n-1}(n-1)$ and L_n. So $Z_n(n) \subset A''$ and hence $A' \subset A''$.

Proof of $A'' \subset A$. By Lemma 3.3, every v_P is an eigenvector for L_k. So $L_k \in A$ for any k. The inclusion $A'' \subset A$ follows. □

Corollary 3.7. *If $P \neq P'$, then $w_P \neq w_{P'}$.*

Proof. If $w_P = w_{P'}$, then every element $a \in A''$ acts on $v_P, v_{P'}$ with the same eigenvalue. But $e_P \in A''$ obviously does not have this property. □

3.4. **Road map.** Let $\mathsf{Wt}(n)$ denote the set of all possible weights, this is a subset of $\mathbb{C}^n$. On $\mathsf{Wt}(n)$ we have an equivalence relation: we say that $w_P \sim w_{P'}$, if P, P' lead to the same irreducible V^n. What we need to do to classify $\mathrm{Irr}(S_n)$ is to solve the following two problems:

a) Describe $\mathsf{Wt}(n)$.
b) Determine the equivalence relation $\sim$ on $\mathsf{Wt}(n)$.

This will be done in the next lecture.

References

[E] P. Etingof et al. *Introduction to representation theory.* Student Mathematical Library, 59. American Mathematical Society, Providence, RI, 2011. arXiv:0901.0827.

[F] W. Fulton. *Young tableaux. With applications to representation theory and geometry.* London Mathematical Society Student Texts, 35. Cambridge University Press, Cambridge, 1997.

[K] A. Kleshchev. *Linear and projective representations of symmetric groups.* Cambridge Tracts in Mathematics, 163. Cambridge University Press, Cambridge, 2005.

[OV] A. Okounkov, A. Vershik. *A new approach to representation theory of symmetric groups.* Selecta Math. (N.S.) 2 (1996), no. 4, 581605.

LECTURE 2: REPRESENTATIONS OF SYMMETRIC GROUPS, II

IVAN LOSEV

1. Introduction

Recall that last time we have introduced the set of weights $\mathsf{Wt}(n) \subset \mathbb{C}^n$ and an equivalence relation $\sim$ on $\mathsf{Wt}(n)$. Our goal is to describe this set and this equivalence relation – the equivalence classes are in a natural bijection with $\mathrm{Irr}(S_n)$. The first step here is as follows. Pick a path $P = V^1 \to V^2 \to \ldots \to V^n$ in the branching graph and an integer k with $1 < k < n$. We fix all vertices but V^k and vary V^k. The questions that we are going to answer: how many paths do we get? What is the relation between their weights? Let $\mathsf{Path}(P,k)$ be the resulting set of paths and $V^n_{P,k} := \mathrm{Span}(v_{P'} | P' \in \mathsf{Path}(P,k))$. We will see that $V^n_{P,k}$ is an irreducible $Z_{k-1}(k+1)$-submodule in V^n, where, recall, $Z_{k-1}(k+1)$ stands for the centralizer of $\mathbb{C}S_{k-1}$ inside $\mathbb{C}S_{k+1}$. Next, we will construct the degenerate affine Hecke algebra $\mathcal{H}(2)$ and its homomorphism to $Z_{k-1}(k+1)$ so that $V^n_{P,k}$ becomes an irreducible $\mathcal{H}(2)$-module. We will obtain a complete classification of the finite dimensional irreducible $\mathcal{H}(2)$-modules. This will ultimately allow us to describe the set $\mathsf{Wt}(n)$ and $\sim$.

2. Degenerate affine Hecke algebras

2.1. Comparing $V^n_{P,k}$ to $\mathrm{Hom}_{S_{k-1}}(V^{k-1}, V^{k+1})$.

Recall that the space $\mathrm{Hom}_{S_{k-1}}(V^{k-1}, V^{k+1})$ has a basis $\varphi_{\underline{P}}$, where $\underline{P}$ runs over $\mathsf{Path}(V^{k-1}, V^{k+1})$. The space $V^n_{P,k}$ has a basis indexed by the same set, $v_{P_1\underline{P}P_2}$, where $P_1 = V^1 \to \ldots \to V^{k-1}, P_2 = V^{k+1} \to \ldots \to V^n$ are the fixed parts of the paths in $\mathsf{Path}(P,k)$.

Lemma 2.1. *The map $\psi : \mathrm{Hom}_{S_{k-1}}(V^{k-1}, V^{k+1}) \to V^n$ given by $\varphi \mapsto \varphi_{P_2} \circ \varphi(v_{P_1})$ is a $Z_{k-1}(k+1)$-equivariant embedding whose image coincides with $V^n_{P,k}$.*

Proof. By Remark 3.2 in Lecture 1 (concatenation of paths gives the composition of homomorphisms), $v_{P_1\underline{P}P_2}$ is proportional to $\varphi_{P_2}(\varphi_{\underline{P}}(v_{P^1}))$. This shows that ψ is an embedding whose image coincides with $V^n_{P,k}$. It remains to check that it is $Z_{k-1}(k+1)$-equivariant. Note that φ_{P_2} is $\mathbb{C}S_{k+1}$-equivariant and hence is $Z_{k-1}(k+1)$-equivariant. By Remark 2.3 in Lecture 1, we have $[z \cdot \varphi](u) = z \cdot \varphi(u)$ for $z \in Z_{k-1}(k+1), \varphi \in \mathrm{Hom}_{S_{k-1}}(V^{k-1}, V^{k+1})$ and $u \in V^{k-1}$. This implies the $Z_{k-1}(k+1)$-equivariance of ψ. $\square$

Recall the Jucys-Murphy elements $L_m = \sum_{i=1}^{m-1}(im)$. We have $L_k, L_{k+1}, (k, k+1) \in Z_{k-1}(k+1)$.

Corollary 2.2. *The subspace $V^n_{P,k}$ is stable under $L_k, L_{k+1}, (k, k+1)$ (the latter stands for the transposition of $k, k+1$) and has no proper stable subspaces.*

Proof. Recall, Theorem 2.4 from Lecture 1, that $Z_{k-1}(k+1)$ is generated by $Z_{k-1}(k-1), L_k, L_{k+1}, (k, k+1)$. By (1) of Corollary 2.5 of Lecture 1, $Z_{k-1}(k-1)$ is in the center of $Z_{k-1}(k+1)$. So elements of $Z_{k-1}(k-1)$ act by scalars on any irreducible $Z_{k-1}(k+1)$-module U. So U is irreducible with respect to $L_k, L_{k+1}, (k, k+1)$. Since $V^n_{P,k}$ is an irreducible $Z_{k-1}(k+1)$-module, we are done. $\square$

2.2. **Degenerate affine Hecke algebra $\mathcal{H}(2)$.** Corollary 2.2 motivates us to find relations between $L_k, L_{k+1}, (k, k+1)$. Then we can form an associative algebra with three generators corresponding to $L_k, L_{k+1}, (k, k+1)$ and relations we have found, the space $V^n_{P,k}$ will be an irreducible module over this algebra.

Lemma 2.3. *We have the following relations*

$$L_k L_{k+1} = L_{k+1} L_k, \quad (k, k+1)^2 = 1, \quad (k, k+1)L_k = L_{k+1}(k, k+1) - 1. \tag{2.1}$$

Proof. The element L_{k+1} commutes with $\mathbb{C}S_k$ and hence with $L_k \in \mathbb{C}S_k$. This gives the first relation. The second relation is obvious. The left hand side of the third relation is

$$(k, k+1)\sum_{i=1}^{k-1}(ik) = \sum_{i=1}^{k-1}(k, k+1)(ik) = \sum_{i=1}^{k-1}(i, k, k+1).$$

The right hand side is

$$\left(\sum_{i=1}^{k}(i, k+1)\right)(k, k+1) - 1 = \sum_{i=1}^{k+1}(i, k+1)(k, k+1) - 1 =$$
$$= \sum_{i=1}^{k-1}(i, k, k+1) + (k, k+1)^2 - 1 = \sum_{i=1}^{k-1}(i, k, k+1).$$

These computations prove the third relation. $\square$

Define the *degenerate affine Hecke algebra* $\mathcal{H}(2)$ by generators X_1, X_2, T and relations that mirror those found in Lemma 2.1.

$$X_1X_2 = X_2X_1, T^2 = 1, TX_1 = X_2T - 1.$$

There is a consequence of these relations: $X_1T = TX_2 - 1$ (multiply the third relation by T both from the left and from the right).

Our conclusion is that we have a unique homomorphism $\mathcal{H}(2) \to Z_{k-1}(k+1)$ given on generators by $X_1 \mapsto L_k, X_2 \mapsto L_{k+1}, T \mapsto (k, k+1)$.

Corollary 2.4. *The space $V^n_{P,k}$ is an irreducible $\mathcal{H}(2)$-module.*

2.3. **Classification of irreducible $\mathcal{H}(2)$-modules.** Let us classify the finite dimensional irreducible $\mathcal{H}(2)$-modules M.

Since X_1, X_2 commute, they have a common eigenvector $m \in M$. Let $X_1m = am, X_2m = bm$, where $a, b \in \mathbb{C}$.

Let us consider two cases:

1) Tm is proportional to m. Since $T^2 = 1$, we have two options:

1.1) $Tm = m$. Let us apply the third relation to m. The left hand side gives $TX_1m = am$, while the right hand side gives $(X_2T - 1)m = (b - 1)m$, so here $b = a + 1$.

1.2) $Tm = -m$. Similarly to the previous case, we get $b = a - 1$.

2) m and Tm are linearly independent. Let us see how X_1, X_2 act on Tm:

$$X_1(Tm) = [X_1T = TX_2 - 1] = TX_2m - m = b(Tm) - m,$$
$$X_2(Tm) = [X_2T = TX_1 + 1] = TX_1m + m = a(Tm) + m.$$

In particular, we see that $\mathrm{Span}(m, Tm)$ is stable under $\mathcal{H}(2)$. Since M is irreducible, we see that m and Tm form a basis in M. In this basis, we have

$$T \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, X_1 \mapsto \begin{pmatrix} a & 0 \\ -1 & b \end{pmatrix}, X_2 \mapsto \begin{pmatrix} b & 0 \\ 1 & a \end{pmatrix}. \tag{2.2}$$

In particular, we see that, in case 2), two modules M, M' that give the same pair of eigenvalues a, b are isomorphic. Note that if $b = a \pm 1$, then (2.2) give a reducible module. Indeed, assume in the sake of being definite, that $b = a+1$. Then the line $\mathbb{C}(m+Tm)$ is a submodule, and our 2-dimensional module is a non-split extension of the 1-dimensional modules in 1.1) and 1.2). In particular, the pair (a, b) of eigenvalues for X_1, X_2 determines any irreducible $\mathcal{H}(2)$-module uniquely up to an isomorphism. Let us denote the corresponding module by $M(a, b)$.

Note that if $a \neq b, b \pm 1$, then $\dim M(a, b) = 2$ and the action of X_1, X_2 on $M(a, b)$ is diagonalizable, as X_1 has distinct eigenvalues and X_2 commutes with X_1. The pairs of eigenvalues that appear are (a, b) and (b, a). It follows that $M(a, b) \cong M(b, a)$. Moreover, if $M(a, b) = M(a', b')$ and $(a, b) \neq (a', b')$, then $b \neq a \pm 1$, and $a' = b, b' = a$.

We arrive at the following classification result.

Proposition 2.5. *The finite dimensional irreducible $\mathcal{H}(2)$-modules are classified by pairs of complex numbers, $(a, b) \mapsto M(a, b)$, with $M(a, b) \cong M(b, a)$ if $b \neq a, a \pm 1$. The pair (a, b) is a pair of simultaneous eigenvalues of X_1, X_2 in $M(a, b)$. Moreover, the following is true.*

1. *If $b = a + 1$, then $M(a, b) = \mathbb{C}$ with $T \mapsto 1, X_1 \mapsto a, X_2 \mapsto b$.*
2. *If $b = a - 1$, then $M(a, b) = \mathbb{C}$ with $T \mapsto -1, X_1 \mapsto a, X_2 \mapsto b$.*
3. *If $b \neq a \pm 1$, then formulas (2.2) define an irreducible representation, and this is $M(a, b)$.*
4. *The action of X_1, X_2 on $M(a, b)$ is diagonalizable if and only if $a \neq b$.*

2.4. **Algebras $\mathcal{H}(d)$.** One can define the algebra $\mathcal{H}(d)$ for all $d \geqslant 1$. It is generated by generators $X_1, \ldots, X_d, T_1, \ldots, T_{d-1}$ with relations:

$$\begin{aligned}
&X_i X_j = X_j X_i, \\
&T_i^2 = 1, \quad T_i T_j = T_j T_i, \text{ for } |i - j| > 1, \quad T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}, \\
&X_i T_j = T_j X_i, \text{ for } i - j \neq 0, 1, \quad T_i X_i = X_{i+1} T_i - 1.
\end{aligned}$$

Note that in the second line we have precisely the relations for the transpositions $(i, i+1) \in \mathbb{C}S_n, i = 1, \ldots, n-1$. So the map $(i, i+1) \mapsto T_i$ extends to an algebra homomorphism $\mathbb{C}S_d \to \mathcal{H}(d)$. We also have an algebra homomorphism $\mathbb{C}[X_1, \ldots, X_d] \to \mathcal{H}(d)$ given in an obvious way.

Also note that we have a homomorphism $\mathcal{H}(d) \to Z_{n-d}(n)$ with $X_i \mapsto L_{n-d+i}, T_i \mapsto (n - d + i, n - d + i + 1)$. In particular, $\mathbb{C}S_n$ is a quotient of $\mathcal{H}(n)$ by the two-sided ideal generated by X_1, the element X_i gets mapped to L_i. The algebra $\mathcal{H}(d)$ first appeared in connection with Yangians, [D], and then was used to study the representations of the usual affine Hecke algebra, [L], that, in its turn, arises in the study of the representations of $\mathrm{GL}(\mathbb{Q}_p)$.

A classification of the finite dimensional irreducible $\mathcal{H}(d)$-modules and a computation of their dimensions is known but isn't easy, formulas will be in terms of so called Kazhdan-Lusztig polynomials, see [BK].

3. Completion of classification

3.1. **Consequences for weights.** Here we are going to get some restrictions for $w_{P'}$, where $P' \in \mathsf{Path}(P,k)$. Let $w_P = (w_1, \ldots, w_n)$ and $w_{P'} = (w'_1, \ldots, w'_n)$. Recall (Lemma 3.3 of Lecture 1) that w'_i is completely determined by V'_i and V'_{i-1}. Since $V'_i = V_i$ when $i \neq k$, we see that $w'_i = w_i$ if $i \neq k, k+1$.

Below we are going to use the following notation. We write s_i for $(i, i+1) \in S_n$. For $x := (x_1, \ldots, x_n) \in \mathbb{C}^n$, we write $s_i x$ for $(x_1, \ldots, x_{i+1}, x_i, \ldots, x_n)$.

Proposition 3.1. *Let P be a path of length n. The following is true:*

(a) *If $w_k \neq w_{k+1} \pm 1$, then $s_k w_P$ is a weight equivalent to w_P.*
(b1) $w_1 = 0$.
(b2) $w_k \neq w_{k+1}$ *for any k.*
(b3) *If $w_k = w_{k+2}$, then $w_{k+1} \neq w_k \pm 1$.*

Proof. The proof is based on the observation that if $P' \in \mathsf{Path}(P,k)$, then $V^n_{P,k} \cong M(w'_k, w'_{k+1})$, an isomorphism of $\mathcal{H}(2)$-modules, where on the left hand side the structure of the module is given by $X_1 \mapsto L_k, X_2 \mapsto L_{k+1}, T \mapsto (k, k+1)$. So we can use Proposition 2.5.

In the situation of (a), the module $M(w_k, w_{k+1})$ is two-dimensional. So $\mathsf{Path}(P,k)$ consists of two elements, P and $P' \neq P$. Since $M(w'_k, w'_{k+1}) \cong V^n_{P,k} \cong M(w_k, w_{k+1})$, we use (3) of Proposition 2.5 to see that $w'_k = w_{k+1}, w'_{k+1} = w_k$. This proves (a).

(b1) follows from $L_1 = 0$.

To prove (b2), we note that the action of X_1, X_2 on $V^n_{P,k} \cong M(w_k, w_{k+1})$ is diagonalizable. So $w_k \neq w_{k+1}$ by (4) of Proposition 2.5.

Let us prove (b3). Assume the converse, say, $w_k = w_{k+2} = w_{k+1} - 1$. Then $\mathbb{C}v_P \cong M(w_k, w_k + 1)$ via $X_1 \mapsto L_k, X_2 \mapsto L_{k+1}, T \mapsto (k, k+1)$. In particular, $(k, k+1)v_P = v_P$. Similarly, $(k+1, k+2)v_P = -v_P$. But $(k+1, k)(k+1, k+2)(k, k+1) = (k+1, k+2)(k, k+1)(k+1, k+2)$ in S_n. Applying the two sides to v_P, we arrive at $-v_P = v_P$. Contradiction. $\square$

3.2. **Combinatorial weights and combinatorial equivalence.** Motivated by (a), define an equivalence relation $\sim_c$ (combinatorial equivalence) on $\mathbb{C}^n$ by $x \sim y$ if y is obtained from x by a sequence of permutations of adjacent elements whose difference is not ± 1 (we call such a permutation *admissible*). We define the subset $\mathsf{CWt}(n) \subset \mathbb{C}^n$ (combinatorial weights) as the set of all x such that any y with $y \sim_c x$ satisfies (b1)-(b3). The next lemma follows from Proposition 2.5.

Lemma 3.2. $\mathsf{Wt}(n)$ *is the union of some equivalence classes for $\sim_c$ in $\mathsf{CWt}(n)$. Moreover, for $w, w' \in \mathsf{Wt}(n)$, we have $w \sim_c w' \Rightarrow w \sim w'$.*

Note that $\mathsf{CWt}(n) \subset \mathbb{Z}^n$.

We are going to embed the set of equivalence classes $\mathsf{CWt}(n)/\sim_c$ into $\mathcal{P}(n)$, the set of partitions on n. This is done by reducing an element of $\mathsf{CWt}(n)$ to a "normal form". We say that an element $x \in \mathsf{CWt}(n)$ is *normal* if it has the form

$$(0, 1, \ldots, n_1, -1, 0, \ldots, n_2, -2, \ldots, n_3, \ldots),$$

where $n_1 + 1 \geqslant n_2 + 2 \geqslant \ldots \geqslant 0$.

Proposition 3.3. *For any $y \in \mathsf{CWt}(n)$, there exists a unique normal $x \in \mathsf{CWt}(n)$ with $y \sim_c x$.*

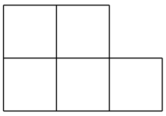
Proof. Let n_1 be the maximal entry in y. We can find $y' \sim_c y$ such that $y'_k = n_1$ and $y'_{k+1}, \ldots, y'_n < n_1$, while there is no other $z \sim_c y$ with $z_k, \ldots, z_n < n_1$ (the maximal entries in y' are as far to the left as possible). We claim that $k = n_1 + 1$ and $y_i = i - 1$ for $i \leqslant k$. Indeed, if $y'_{k-1} \neq n_1 - 1$, then we can switch y'_{k-1} and y'_k (here we use (b2)), etc.

Then we freeze the first $n_1 + 1$ entries in y' (we no longer permute them) and pick the maximal unfrozen entry n_2 in y'. Again, consider y'', where the entries n_2 are as far to the left as possible (with the first $n_1 + 1$ entries frozen). Similarly to the previous paragraph, y'' starts with $0, 1, \ldots, n_1, a, a + 1, \ldots, n_2$. If $a < -1$, then we can move a all the way to the left and get a contradiction with (b1). If $a > -1$, then we can move it the left until we get a segment $a, a + 1, a$, which contradicts (b3). So $a = -1$.

Then we freeze the first $(n_1 + 1) + (n_2 + 2)$ entries and repeat the argument. This shows the existence of x.

Uniqueness follows from the observation that n_1 is the maximal element of $y_1, \ldots, y_n$, n_2 is maximal after removing $0, 1, \ldots, n_1$, etc. $\square$

3.3. **Young diagrams and tableaux, finally.** Partitions of n that are often depicted as Young diagrams, the following diagram corresponds to the partition $5 = 3 + 2$.



Define a map $\mathsf{CWt}(n)/\sim_c \to \mathcal{P}(n)$ by sending x to the partition with parts $(n_1+1), (n_2+2)$, where n_1, n_2, etc., are as in Proposition 3.3. By the uniqueness part of that proposition, our map is an embedding. The following theorem completes the classification of $\mathrm{Irr}(S_n) = \mathsf{Wt}(n)/\sim$.

Theorem 3.4. *We have* $\mathsf{Wt}(n) = \mathsf{CWt}(n), \sim = \sim_c$ *and* $\mathsf{CWt}(n)/\sim_c \xrightarrow{\sim} \mathcal{P}(n)$.

Proof. We have a surjection $\mathsf{Wt}(n)/\sim_c \twoheadrightarrow \mathsf{Wt}(n)/\sim$ and embeddings

$$(\mathsf{Wt}(n)/\sim_c) \hookrightarrow (\mathsf{CWt}(n)/\sim_c) \hookrightarrow \mathcal{P}(n).$$

Therefore we get the following chain of inequalities:

$$|\mathsf{Wt}(n)/\sim| \leqslant |\mathsf{Wt}(n)/\sim_c| \leqslant |\mathsf{CWt}(n)/\sim_c| \leqslant |\mathcal{P}(n)|$$

By Introduction to Lecture 1, $|\mathrm{Irr}(S_n)| = |\mathsf{Wt}(n)/\sim| = |\mathcal{P}(n)|$. So all these embeddings and a surjection are bijections. $\square$

Now let us relate $\mathsf{Wt}(n)$ to the set of *standard Young tableaux* $\mathsf{SYT}(n)$. Recall that a standard Young tableaux on a Young diagram λ with n boxes is a filling of λ with numbers from 1 to n that strictly increase bottom to top and left to right. For example, these two fillings are examples of Young tableaux of shape $(3, 2)$.

4	5	
1	2	3

3	5	
1	2	4

To a Young tableaux T we assign its *content* as follows. Let (x_i, y_i) be the coordinate of the box numbered by i. Then the content $c(T)$ of T is, by definition, $(x_1 - y_1, x_2 - y_2, \ldots, x_n - y_n)$. The following two collections are contents of the tableaux in the previous example: $(0, 1, 2, -1, 0)$ and $(0, 1, -1, 2, 0)$.

Proposition 3.5. *The map $T \mapsto c(T)$ is a biijection $\mathsf{SYT}(n) \to \mathsf{Wt}(n)$ that intertwines the surjections $\mathsf{SYT}(n) \twoheadrightarrow \mathcal{P}(n)$ (taking the shape) and $\mathsf{Wt}(n) \twoheadrightarrow \mathcal{P}(n)$.*

Proof. We can define an admissible permutation of k and $k+1$ in a tableaux T: it permutes k and $k+1$ if the result is still a tableau. For example, the two tableaux above are obtained from one another by permuting 3 and 4.

The admissible permutations give rise to an equivalence relation $\sim_c$ on $\mathsf{SYT}(n)$. Clearly, an admissible permutation of $k, k+1$ in T corresponds to the admissible permutation s_k of $c(T)$. It is not hard to show that $c(T)$ satisfies conditions (b1)-(b3). So $c(T)$ is indeed an element of $\mathsf{CWt}(n)$ and the image of c is the union of equivalence classes for $\sim_c$.

We can define normal Young tableaux, where we fill the first row by numbers from 1 to some n_1, then the second row by the numbers from n_1+1 to n_1+n_2, etc., for example, the first tableau above is normal. Clearly, if T is normal, then so is $c(T)$. From Proposition 3.3, it follows that c is surjective. On the other hand, it is easy to check that c is injective. $\square$

References

[BK] J. Brundan, A. Kleshchev. *The degenerate analogue of Ariki's categorification theorem.* Math. Z. 266 (2010), no. 4, 877-919. arXiv:0901.0057.

[D] V. Drinfeld. *Degenerate affine Hecke algebra and Yangians.* Funktsional. Anal. i Prilozh., 20 (1986), 69-70.

[L] G. Lusztig. *Affine Hecke algebras and their graded version.* J. Amer. Math. Soc. 2 (1989), no. 3, 599635.

LECTURE 3: REPRESENTATION THEORY OF $\mathrm{SL}_2(\mathbb{C})$ AND $\mathfrak{sl}_2(\mathbb{C})$

IVAN LOSEV

Introduction

We proceed to studying the representation theory of algebraic groups and Lie algebras. Algebraic groups are the groups defined inside $\mathrm{GL}_n(\mathbb{F})$ by polynomial equations, such as $\mathrm{SL}_n(\mathbb{F}), \mathrm{O}_n(\mathbb{F}), \mathrm{Sp}_n(\mathbb{F})$. We are interested in their representations with polynomial matrix coefficients. Often, this problem is reduced to studying representations of Lie algebras (some sort of linearization). The connection between representations of algebraic groups and their Lie algebras is very tight in characteristic 0 but is much more loose in characteristic p, as we will see in the next lecture.

This lecture consists of two parts. First, we define algebraic groups, their Lie algebras, representations of both and connections between those. In the second part we fully classify the representations of $\mathrm{SL}_2(\mathbb{C})$ and its Lie algebra $\mathfrak{sl}_2(\mathbb{C})$.

1. Algebraic groups and their Lie algebras

1.1. **Algebraic groups.** Let $\mathbb{F}$ be an algebraically closed field. By a (linear) *algebraic group* G we mean a subgroup of $\mathrm{GL}_n(\mathbb{F}) = \{A \in \mathrm{Mat}_n(\mathbb{F}) | \det A \neq 0\}$ defined by polynomial equations[1]. Examples of algebraic groups include $\mathrm{GL}_n(\mathbb{F})$ itself or $\mathrm{SL}_n(\mathbb{F}) = \{A \in \mathrm{GL}_n(\mathbb{F}) | \det A = 1\}$. To get further examples, pick a non-degenerate symmetric or skew-symmetric form B on $\mathbb{F}^n$ and consider the subgroup $G_B = \{A \in \mathrm{GL}_n(\mathbb{F}) | B(Au, Av) = B(u, v), \forall u, v \in \mathbb{F}^n\}$. If J is the matrix of this form in some fixed basis, then $G = \{A \in \mathrm{GL}_n(\mathbb{F}) | A^t J A = J\}$. This group is denoted by $\mathrm{O}_n(\mathbb{F})$ if B is symmetric, and by $\mathrm{Sp}_n(\mathbb{F})$ if B is skew-symmetric (note that in this case n is even, because we assume B is non-degenerate).

By a polynomial function on an algebraic group $G \subset \mathrm{GL}_n(\mathbb{F})$ we mean a polynomial in matrix coefficients and $\det^{-1}$. The polynomial functions on G form an algebra to be denoted by $\mathbb{F}[G]$. By a homomorphism of algebraic groups $G \to G' (\subset \mathrm{GL}_n(\mathbb{F}))$, we mean a group homomorphism whose matrix coefficients are polynomial functions. By an isomorphism of algebraic groups, we mean a homomorphism that has an inverse that is also a homomorphism. This allows to consider algebraic groups regardless their embeddings to $\mathrm{GL}_n(\mathbb{F})$ (there is also an internal definition: a linear algebraic group is an affine algebraic variety that is a group such that the group operations are morphisms of algebraic varieties).

We want to study representations $G \to \mathrm{GL}_N(\mathbb{F})$ that are homomorphisms of algebraic groups (they are traditionally called *rational*). Examples are provided by the representation of G in $\mathbb{F}^n$, its dual, their tensor products, subs and quotients of those.

1.2. **Lie algebras.** Let $G \subset \mathrm{GL}_n(\mathbb{F})$ be an algebraic group. Consider $\mathfrak{g} := T_1 G$, the tangent space to G at $1 \in G$, this is a subspace in $T_1 \mathrm{GL}_n(\mathbb{F}) = \mathrm{Mat}_n(\mathbb{F})$. It consists of the tangent vectors to curves $\gamma(t)$ in G with $\gamma(0) = 1$. When $\mathbb{F} = \mathbb{C}$ we can take γ to be a map from a small interval in $\mathbb{R}$ containing 0. In general, we can take some "formal curve", a formal power series $1 + A_1 t + A_2 t^2 + \ldots$ with $A_i \in \mathrm{Mat}_n(\mathbb{F})$ satisfying the defining equations of G.

[1]We consider algebraic groups with the reduced scheme structure

Such a curve exists for every $A_1 \in \mathfrak{g}$ because every algebraic group is a smooth algebraic variety.

Let us compute the tangent spaces for the groups $\mathrm{SL}_n(\mathbb{F}), \mathrm{O}_n(\mathbb{F}), \mathrm{Sp}_n(\mathbb{F})$. In the case of $\mathrm{SL}_n(\mathbb{F})$, we get $\mathfrak{g} = \{x \in \mathrm{Mat}_n(\mathbb{F}) | \operatorname{tr} x = 0\}$, this space is usually denoted by $\mathfrak{sl}_n(\mathbb{F})$. For $G_B = \mathrm{O}_n(\mathbb{F})$ or $\mathrm{Sp}_n(\mathbb{F})$, we get $\mathfrak{g}_B = \{x \in \mathrm{Mat}_n(\mathbb{F}) | B(xu, v) + B(u, xv) = 0, \forall u, v \in \mathbb{F}^n\}$. When B is orthogonal (resp., symplectic), this space is denoted by $\mathfrak{so}_n(\mathbb{F})$ (resp., $\mathfrak{sp}_n(\mathbb{F})$).

All these spaces have an interesting feature, they are closed with respect to the commutator of matrices, $[x, y] := xy - yx$. This is a general phenomenon: if G is an algebraic group, then $\mathfrak{g}$ is closed with respect to $[\cdot, \cdot]$. The reason is that if $\gamma(t), \eta(s)$ are two curves with $\gamma(t) = 1 + tx + \ldots, \eta(s) = 1 + sy + \ldots$, then the group commutator $\gamma(t)\eta(s)\gamma(t)^{-1}\eta(s)^{-1}$ expands as $1 + ts[x, y] + \ldots$, where "$\ldots$" denotes the terms of order 3 and higher (note that all of them include both t and s).

The commutator on $\mathrm{Mat}_n(\mathbb{F})$ is bilinear and satisfies the following two important identities

$$[x, x] = 0 \tag{1.1}$$

(this implies that the commutator is skew-symmetric) and the *Jacobi identity*

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0. \tag{1.2}$$

This motivates the following definition.

Definition 1.1. A Lie algebra is a vector space $\mathfrak{g}$ equipped with a bilinear operation $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ satisfying (1.1) and (1.2).

As we have seen, for any algebraic group $G \subset \mathrm{GL}_n(\mathbb{F})$, the tangent space $T_1 G$ is a Lie algebra.

The notions of a Lie algebra homomorphism, product of Lie algebras, subalgebras, quotient algebras, ideals are introduced in a standard way.

1.3. **Correspondence between representations.** Now let $\Phi : G \to G'$ be an algebraic group homomorphism. The description of the bracket using the expansion of the commutator in the group shows that the tangent map $\varphi : \mathfrak{g} \to \mathfrak{g}'$ is a homomorphism of Lie algebras.

We write $\mathfrak{gl}(V)$ for $\mathrm{End}(V)$ when view it as a Lie algebra. By a representation of an arbitrary Lie algebra $\mathfrak{g}$ one means a Lie algebra homomorphism $\mathfrak{g} \to \mathfrak{gl}(V)$. We can also view a representation as a bilinear map $\mathfrak{g} \times V \to V, (x, v) \mapsto x \cdot v$ satisfying $[x, y] \cdot v = x \cdot y \cdot v - y \cdot x \cdot v$. In this case, we call V a $\mathfrak{g}$-module.

Example 1.2. Consider the group $\mathrm{GL}_1(\mathbb{F})$ that coincides with the multiplicative group $\mathbb{F}^\times$ of $\mathbb{F}$. The corresponding Lie algebra is just $\mathbb{F}$ with zero bracket (the one-dimensional abelian Lie algebra). Consider the one-dimensional representations of $\mathbb{F}$ and of $\mathbb{F}^\times$. In the former case, a representation is given by multiplying by an arbitrary $z \in \mathbb{F}$. In the group case, a representation is given by sending $z \in \mathbb{F}^\times$ to $f(z) \in \mathbb{F}^\times$ such that $f(zz') = f(z)f(z'), f(1) = 1$. A function $f(z)$ is a polynomial on $\mathbb{F}^\times$, i.e., a Laurent polynomial. It has no nonzero roots, so it has to be $\alpha z^n, n \in \mathbb{Z}, \alpha \in \mathbb{F}^\times$. Clearly, $\alpha = 1$. The corresponding representation of $\mathbb{F}$ is given by the multiplication by n.

Example 1.3. Let $G \subset \mathrm{GL}_n(\mathbb{F})$. Then $\mathfrak{gl}_n(\mathbb{F})$ carries a representation of G given by $g.x = gxg^{-1}$. The subalgebra $\mathfrak{g} \subset \mathfrak{gl}_n(\mathbb{F})$ is a subrepresentation. The representation of G in $\mathfrak{g}$ is called the *adjoint representation*. Note that $g.[x, y] = [g.x, g.y]$, in other words, G acts by automorphisms of the Lie algebra $\mathfrak{g}$. The corresponding representation of $\mathfrak{g}$ (also called adjoint and denoted by ad) is given by $\mathrm{ad}(y)x = [y, x]$.

Recall that, for rational representations of G, we can take tensor products and dual representations. This corresponds to taking tensor products and duals of Lie algebra representations. For example, if U, V are representations of G, then the representation of G in $U \otimes V$ is defined by $g \cdot (u \otimes v) = (g \cdot u) \otimes (g \cdot v)$. We plug $\gamma(t) = 1 + tx + \ldots$ for g, differentiate, and set $t = 0$ to get $x \cdot (u \otimes v) = (x \cdot u) \otimes v + u \otimes (x \cdot v)$. We take this formula for the definition of a $\mathfrak{g}$-action on $U \otimes V$. For similar reasons, we define a $\mathfrak{g}$-module structure on V^* by $(x \cdot \alpha)(v) := -\alpha(x \cdot v)$. Finally, if U, V are $\mathfrak{g}$-modules, then we define a $\mathfrak{g}$-module structure on $\mathrm{Hom}(U, V)$ by $(x \cdot \varphi)(u) = x \cdot \varphi(u) - \varphi(x \cdot u)$. Using these constructions we can produce new representations of Lie algebras from existing ones.

Now let us consider the following questions. Given a Lie algebra homomorphism $\varphi : \mathfrak{g} \to \mathfrak{g}'$ (e.g., a representation), is there a homomorphism $\Phi : G \to G'$, whose tangent map is φ? In this case we say that Φ integrates φ. If so, is Φ unique? Here is a place, where the answer crucially depends on $\operatorname{char} \mathbb{F}$. In the next lecture, we will see that if $\operatorname{char} \mathbb{F} > 0$, then the answers to both questions are "no" (we already have seen some of this in Example 1.2).

Proposition 1.4. *Let* $\operatorname{char} \mathbb{F} = 0$ *and let* G *be connected. Then the following is true.*

(1) *If* Φ *exists, then it is unique.*
(2) *If* G *is simply connected and* $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$ *(the right had side denotes the linear span of all elements of the form* $[x, y]$*, where* $x, y \in \mathfrak{g}$*), then* Φ *exists.*

Proof/discussion. We will explain how (1) is proved (for $\mathbb{F} = \mathbb{C}$) and give a short discussion of (2) (explaining what "connected" and "simply connected" mean for general $\mathbb{F}$).

Let $\mathbb{F} = \mathbb{C}$. We have a distinguished map $\gamma : \mathbb{R} \to G$ with $\gamma(t) = 1 + tx + \ldots$, it is given by $\gamma(t) = \exp(tx)$. In fact, it is the only differentiable group homomorphism with given tangent vector x. By this uniqueness, $\Phi(\exp(tx)) = \exp(t\varphi(x))$. The group G is connected, so elements of the form $\exp(tx)$ generate G. This shows (1).

Now let us discuss (2) for $\mathbb{F} = \mathbb{C}$. Every algebraic group is also a complex Lie group, i.e., a group, which is a complex manifold, so that the group operations are complex analytic. A general result from the theory of Lie groups (based on the existence/uniqueness of solutions of ODE's) shows that there is a complex Lie group homomorphism $\Phi : G \to G'$ integrating φ. If $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$, then Φ is a morphism of algebraic varieties, see [OV, Section 3.4].

For an arbitrary linear algebraic group G over an algebraically closed field $\mathbb{F}$, we say that G is connected if it is irreducible as an algebraic variety. We say that G is simply connected if there is no surjective non-bijective algebraic group homomorphism $\tilde{G} \to G$. In this setting (2) is proved using the structure theory of algebraic groups. $\square$

Now let us explain why conditions in (2) are necessary producing counter-examples when either of the two conditions in (2) fails. Consider $G = \mathrm{SO}(3)$, the special orthogonal group $\mathrm{SO}(n)$, by definition, is $\mathrm{O}(n) \cap \mathrm{SL}(n)$. Then $\mathfrak{g} = \mathfrak{so}_3(\mathbb{C}) \cong \mathfrak{sl}_2(\mathbb{C})$. The 2-dimensional representation of $\mathfrak{sl}_2(\mathbb{C})$ given by the natural inclusion $\mathfrak{sl}_2(\mathbb{C}) \subset \mathfrak{gl}_2(\mathbb{C})$ does not integrate to $\mathrm{SO}_3(\mathbb{C})$.

On the other hand, one can consider $G = \mathbb{C}$, the additive group of $\mathbb{C}$. The only one dimensional representation that integrates to a rational representation of G is the zero one.

Remark 1.5. Let $\operatorname{char} \mathbb{F} = 0$. Let V, V' be two representations of G and $\psi : V \to V'$ a homomorphism of representations of $\mathfrak{g}$. Then ψ is a homomorphism of representations of G provided G is connected. This is proved similarly to part (1) of the previous proposition provided $\mathbb{F} = \mathbb{C}$. In particular, if V is completely reducible over $\mathfrak{g}$, then it is completely reducible over G.

2. Representation theory of $\mathfrak{sl}_2(\mathbb{C})$

2.1. **Universal enveloping algebras.** Let $\mathfrak{g}$ be a Lie algebra over a field $\mathbb{F}$. As with finite groups, we would like to have an associative algebra whose representation theory is the same as that of $\mathfrak{g}$. Such an algebra is called the *universal enveloping algebra*, it is defined as

$$U(\mathfrak{g}) := T(\mathfrak{g})/(x \otimes y - y \otimes x - [x, y]|x, y \in \mathfrak{g}),$$

where we write $T(\mathfrak{g})$ for the tensor algebra of $\mathfrak{g}$ viewed as a vector space. There is a natural Lie algebra homomorphism $\iota : \mathfrak{g} \to U(\mathfrak{g})$ that has the following universal property: if A is an associative unital algebra and $\varphi : \mathfrak{g} \to A$ is a Lie algebra homomorphism, then there is unique associative unital algebra homomorphism $\psi : U(\mathfrak{g}) \to A$ with $\varphi = \psi \circ \iota$. In particular, a $U(\mathfrak{g})$-module is the same thing as a $\mathfrak{g}$-module.

The algebra $U(\mathfrak{g})$ is infinite dimensional if $\mathfrak{g} \neq \{0\}$. We can describe a basis in $U(\mathfrak{g})$ as follows. Choose a basis $x_1, \ldots, x_n$ in $\mathfrak{g}$ (we assume that $\mathfrak{g}$ is finite dimensional, using infinite indexing sets we can cover the case when $\mathfrak{g}$ is infinite dimensional). It is easy to see that the ordered monomials $x_1^{d_1} x_2^{d_2} \ldots x_n^{d_n} \in U(\mathfrak{g})$ span $U(\mathfrak{g})$ as a vector space. The following claim is known as the Poincare-Birkhoff-Witt (shortly, PBW) theorem.

Proposition 2.1. *The elements $x_1^{d_1} x_2^{d_2} \ldots x_n^{d_n}$, where $d_1, \ldots, d_n \in \mathbb{Z}_{\geqslant 0}$, form a basis in $U(\mathfrak{g})$.*

For example, when $\mathfrak{g} = \mathfrak{sl}_2$, we have a basis $e^k h^\ell f^m \in U(\mathfrak{g})$.

There is no easy multiplication rule for the basis elements. An easy general observation is that $(x_1^{d_1} \ldots x_n^{d_n}) \cdot (x_1^{e_1} \ldots x_n^{e_n}) = x_1^{d_1+e_1} \ldots x_n^{d_n+e_n} + \ldots$, where "$\ldots$" means a linear combination of monomials of total degree less then $\sum_{i=1}^n (d_i + e_i)$.

Now let $\mathfrak{h} \subset \mathfrak{g}$ is a Lie subalgebra. The universal property of $U(\mathfrak{h})$ and a Lie algebra homomorphism $\mathfrak{h} \to \mathfrak{g} \to U(\mathfrak{g})$ give rise to an associative algebra homomorphism $U(\mathfrak{h}) \to U(\mathfrak{g})$. By the PBW theorem, this homomorphism is injective. So we can view $U(\mathfrak{h})$ as a subalgebra in $U(\mathfrak{g})$.

2.2. **Classification of irreducible representations of $\mathfrak{sl}_2(\mathbb{C})$.** First, we will need a lemma from linear algebra.

Lemma 2.2. *Let V be a finite dimensional vector space over an algebraically closed field $\mathbb{F}$. Let $A, B \in \mathrm{End}(V)$ be such that $[A, B] = zB$ for $z \neq 0$. Further, for $a \in \mathbb{F}$, let V_a denote the generalized eigenspace for A in V with eigenvalue a. Then the following is true.*

(1) *$B(V_a) \subset V_{a+z}$ for all a.*
(2) *If $\mathsf{char}(\mathbb{F}) = 0$, then B is nilpotent and there is an eigenvector v for A such that $Bv = 0$.*

Proof. We have $(A - (a + z)\,\mathrm{id})(Bv) = B(A - a\,\mathrm{id})v$ that implies (1). If $\mathsf{char}(\mathbb{F}) = 0$, then all numbers $a + nz, n \in \mathbb{Z}_{\geqslant 0}$, are different. Since we have only finitely many eigenvalues for A in V, (2) follows. □

Let V be a finite dimensional representation of $\mathfrak{g} := \mathfrak{sl}_2(\mathbb{C})$. Taking $A = h$ and $B = e$ so that $[h, e] = 2e$ we get the following corollary.

Corollary 2.3. *There is a nonzero vector $v \in V$ such that $hv = zv, ev = 0$ for some $z \in \mathbb{C}$.*

Now let us introduce Verma modules. Pick $z \in \mathbb{C}$. Let $\mathfrak{b} \subset \mathfrak{g}$ be the subspace with basis h, e, this is a subalgebra. Consider the $\mathfrak{b}$-module $\mathbb{C}_z$ that is $\mathbb{C}$ as a vector space, h acts by

z and e acts by 0. We view $\mathbb{C}_z$ as a $U(\mathfrak{b})$-module. As was mentioned before, we can view $U(\mathfrak{b})$ as a subalgebra in $U(\mathfrak{g})$. Then set

$$\Delta(z) := U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_z = U(\mathfrak{g})/U(\mathfrak{g})\,\mathrm{Span}_{\mathbb{C}}(h-z, e).$$

This is a left $U(\mathfrak{sl}_2)$-module. The reason why we need this is that, for any $U(\mathfrak{g})$-module V, we have

$$\begin{aligned}\mathrm{Hom}_{U(\mathfrak{g})}(\Delta(z), V) &= \mathrm{Hom}_{U(\mathfrak{g})}(U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_z, V) = \mathrm{Hom}_{U(\mathfrak{b})}(\mathbb{C}_z, V) = \\ &= \{v \in V | hv = zv, ev = 0\}.\end{aligned} \tag{2.1}$$

So if V is irreducible and there is $v \in V, v \neq 0$ with $hv = zv, ev = 0$, then there is a nonzero homomorphism $\Delta(z) \to V$ and hence V is a quotient of $\Delta(z)$. So we need to understand the structure of $\Delta(z)$.

As a right $U(\mathfrak{b})$-module, $U(\mathfrak{g})$ has basis $f^n, n \in \mathbb{Z}_{\geqslant 0}$. It follows that $\Delta(z)$ has basis $f^n v_z$, where v_z is the image of $1 \in U(\mathfrak{g})$ in $\Delta(z)$. The action of f, h, e on this basis is given as follows

$$f \cdot f^n v_z = f^{n+1} v_z, \quad h f^n v_z = (z-2n) f^n v_z, \quad e f^n v_z = (z-n+1) n f^{n-1} v_z. \tag{2.2}$$

The first equality is clear, the second is easy and the last one follows from the identity

$$e f^n = f^{n-1}(h+1-n) + f^n e. \tag{2.3}$$

Lemma 2.4. *The module $\Delta(z)$ is irreducible if and only if $z \notin \mathbb{Z}_{\geqslant 0}$. If $z \in \mathbb{Z}_{\geqslant 0}$, then $\Delta(z)$ has a unique proper submodule and the quotient by this submodule is finite dimensional.*

Proof. Let U be a submodule of $\Delta(z)$. Since $\Delta(z)$ splits into the direct sum of eigenspaces for h, so does U. So $U = \mathrm{Span}(f^n v_z | n \geqslant m)$ for some $m > 0$. From the third equation in (2.2), we conclude that $z = m-1$. All claims of the lemma follow easily from here. □

Remark 2.5. Note that the kernel $\Delta(n) \twoheadrightarrow L(n)$ is $\Delta(-2-n)$.

Proposition 2.6. *There is a bijection $\mathbb{Z}_{\geqslant 0} \xrightarrow{\sim} \mathrm{Irr}_{fin}(\mathfrak{sl}_2)$ (the set of finite dimensional irreducible $\mathfrak{sl}_2$-modules). It sends $n \in \mathbb{Z}_{\geqslant 0}$ to the proper quotient $L(n)$ of $\Delta(n)$. There is a basis $u_0, u_1, \ldots, u_n$ in $L(n)$, where the action of e, h, f is given by*

$$f u_i = (n-i) u_{i+1}, h u_i = (n-2i) u_i, e u_i = i u_{i-1}. \tag{2.4}$$

Proof. We already established the existence of $L(n)$ with required properties. Clearly, these modules are pairwise non-isomorphic. Now apply (2.1) to $V \in \mathrm{Irr}_{fin}(\mathfrak{sl}_2)$ and z found in Lemma 2.3. We see that V is a quotient of $\Delta(z)$. By Lemma 2.4, $z \in \mathbb{Z}_{\geqslant 0}$ and $V \cong L(z)$.

We set $u_i := \frac{f^i}{i!} v_n$. Clearly, (2.4) is satisfied. □

2.3. **Complete reducibility.** Consider an element $C := ef + fe + h^2/2 = 2fe + h^2/2 + h \in U(\mathfrak{g})$ (the *Casimir element*). It is straightforward to check that it is central in $U(\mathfrak{g})$. So C acts by scalar on every irreducible finite dimensional module. To determine this scalar for $L(n)$ compute $Cv_n = (h^2/2 + h)v_n = (n^2/2 + n)v_n$. In particular, we see that these scalars distinguish the irreducible modules $L(n)$.

Lemma 2.7. *Any finite dimensional $\mathfrak{sl}_2$-module V is completely reducible.*

Proof. Let us decompose V into the direct sum of generalized eigenspaces for C. We may assume that only one eigenvalue occurs, say $n^2/2 + n$. It remains to show that $L(n)$ has no self-extensions. Indeed, suppose we have an exact sequence $0 \to V_1 \to V \to V_2 \to 0$ with $V_i \cong L(n)$ that does not split. Pick a vector $v_0' \notin V_1$ that lies in the generalized eigenspace for h with eigenvalue n. We see that $(h-n)v$ lies in the eigenspace for h in V with eigenvalue n. Note that $\pm(n+2)$ are not eigenvalues of h in V. Therefore $ev = f^{n+1}v = 0$. By (2.3) applied to $n+1$ rather than to n, $f^n(h-n)v = 0$. Therefore $(h-n)v = 0$. So we get a homomorphism $\Delta(n) \to V$ mapping v_z to v_0'. It must factor through $L(n) \to V$. It follows that our exact sequence splits. $\square$

2.4. **Representations of** $\mathrm{SL}_2(\mathbb{C})$**.** We have the following classification result.

Proposition 2.8. *Every rational representation of* $\mathrm{SL}_2(\mathbb{C})$ *is completely reducible. For every* $n \in \mathbb{Z}_{\geqslant 0}$*, there is a unique (up to isomorphism) representation* $L(n)$ *of dimension* $n+1$*.*

Proof. The first claim follows from Remark 1.5. To prove the second one, it only remains to check that every $\mathfrak{sl}_2(\mathbb{C})$-module $L(n)$ integrates to $\mathrm{SL}_2(\mathbb{C})$. Consider the $\mathrm{SL}_2(\mathbb{C})$-module $S^n(\mathbb{C}^2)$. Its basis is $x^n, x^{n-1}y, \ldots, y^n$, where x, y is a natural basis of $\mathbb{C}^2$. The eigenvalues of h on the corresponding $\mathfrak{sl}_2(\mathbb{C})$-module are $n, n-2, \ldots, -n$. It follows that $S^n\mathbb{C}^2 \cong L(n)$ as an $\mathfrak{sl}_2(\mathbb{C})$-module, and so $L(n)$ indeed integrates to $\mathrm{SL}_2(\mathbb{C})$. $\square$

References

[OV] A.L. Onishchik, E.B. Vinberg, *Lie groups and algebraic groups.* Translated from the Russian and with a preface by D. A. Leites. Springer Series in Soviet Mathematics. Springer-Verlag, Berlin, 1990. xx+328 pp.

LECTURE 4: REPRESENTATION THEORY OF $\mathrm{SL}_2(\mathbb{F})$ AND $\mathfrak{sl}_2(\mathbb{F})$

IVAN LOSEV

In this lecture we will discuss the representation theory of the algebraic group $\mathrm{SL}_2(\mathbb{F})$ and of the Lie algebra $\mathfrak{sl}_2(\mathbb{F})$, where $\mathbb{F}$ is an algebraically closed field of positive characteristic, say p.

1. Representations of $\mathfrak{sl}_2(\mathbb{F})$

Here we assume that $p > 2$. The reason is that $\mathfrak{sl}_2(\mathbb{F})$ is not simple, i.e., contains proper ideals when $p = 2$.

1.1. **Center of $U(\mathfrak{sl}_2)$.** The first thing we want to understand is the structure of the center. It is useful because the central elements act by scalars on any finite dimensional irreducible module.

An important feature of the positive characteristic story is that the center becomes bigger. Roughly speaking, in addition to elements that were central in characteristic 0, we get a large central subalgebra called the p-center. A general principle is that one should look at the pth powers of elements.

We start with $\mathfrak{gl}_n(\mathbb{F})$. For $x \in \mathfrak{gl}_n(\mathbb{F})$, we can take $x^p \in U(\mathfrak{gl}_n(\mathbb{F}))$. On the other hand, $\mathfrak{gl}_n(\mathbb{F}) = \mathrm{Mat}_n(\mathbb{F})$ is itself an associative algebra and we can take the pth power of x there (known as the *restricted pth power*). In order to distinguish between these two situations, we write $x^{[p]}$ for the pth power of x in $\mathrm{Mat}_n(\mathbb{F})$.

Proposition 1.1. *The element $x^p - x^{[p]}$ is central in $U(\mathfrak{gl}_n(\mathbb{F}))$.*

In the proof we will use the following lemma.

Lemma 1.2. *Let A be an associative $\mathbb{F}$-algebra and $x, y \in A$. Recall that we write $\mathrm{ad}(x)$ for the map $A \to A, \mathrm{ad}(x)(a) := (xa - ax)$. We have $\mathrm{ad}(x)^p y = [x^p, y]$.*

Proof. We distribute $\mathrm{ad}(x)^p y$. Let l_x, r_x denote the operators $y \mapsto xy$ and $y \mapsto yx$ so that $\mathrm{ad}(x) = l_x - r_x$. Since l_x, r_x commute, we get

$$\mathrm{ad}(x)^p y = \sum_{i=0}^{p} \binom{p}{i} (-1)^i x^i y x^{p-i}.$$

Since $\binom{p}{i} = 0$ for $0 < i < p$ (we are in characteristic p), the right hand side is $x^p y - y x^p$. $\square$

Proof of Proposition 1.1. Pick $x, y \in \mathfrak{gl}_n(\mathbb{F})$. Applying Lemma 1.2 to $A = U(\mathfrak{gl}_n(\mathbb{F}))$, we have $[x^p, y] = \mathrm{ad}(x)^p y$ in $U(\mathfrak{gl}_n(\mathbb{F}))$. Note that the right hand side is in $\mathfrak{g}$. Now apply Lemma 1.2 to $A = \mathrm{Mat}_n(\mathbb{F})$. We get $[x^{[p]}, y] = \mathrm{ad}(x)^p y$. So $[x^p - x^{[p]}, y] = 0$. Since $\mathfrak{gl}_n(\mathbb{F})$ generates $U(\mathfrak{gl}_n(\mathbb{F}))$, we see that $x^p - x^{[p]}$ is central. $\square$

Now let $G \subset \mathrm{GL}_n(\mathbb{F})$ be an algebraic group defined over $\mathbb{F}_p$ (i.e., by polynomials with coefficients in $\mathbb{F}_p$). One can show that $\mathfrak{g} \subset \mathfrak{gl}_n(\mathbb{F})$ is closed with respect to the map $x \mapsto x^{[p]}$, see [J, Section 7] for details. In any case, for the subalgebras $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{F}), \mathfrak{so}_n(\mathbb{F}), \mathfrak{sp}_n(\mathbb{F})$, the claim that $\mathfrak{g}$ is closed with respect to $x \mapsto x^{[p]}$ can be checked by hand. For example, in the case of $\mathfrak{sl}_n(\mathbb{F})$, the condition that $x \in \mathfrak{sl}_n(\mathbb{F})$ is equivalent to $\lambda_1 + \ldots + \lambda_n = 0$, where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of x counted with multiplicities. The eigenvalues of $x^{[p]}$ are $\lambda_1^p, \ldots, \lambda_n^p$, we get $\lambda_1^p + \ldots + \lambda_n^p = (\lambda_1 + \ldots + \lambda_n)^p = 0$ because the map $\lambda \mapsto \lambda^p$ is a ring automorphism of $\mathbb{F}$.

So we get a map $\iota : \mathfrak{g} \to Z(U(\mathfrak{g})), x \mapsto x^p - x^{[p]}$. Recall that the group G acts on $\mathfrak{g}$ (adjoint representation). This action is by Lie algebra automorphisms so it extends to an action of G by algebra automorphisms on $U(\mathfrak{g})$. Therefore, it preserves the center $Z(U(\mathfrak{g}))$.

Proposition 1.3. *The map $\iota : \mathfrak{g} \to Z(U(\mathfrak{g}))$ is additive, semi-linear in the sense that $\iota(\lambda x) = \lambda^p x$ for $\lambda \in \mathbb{F}$, and G-equivariant. Moreover, if $x_1, \ldots, x_n$ form a basis in $\mathfrak{g}$, then the elements $x_i^p - x_i^{[p]} \in U(\mathfrak{g})$ are algebraically independent.*

The claims that ι is semilinear and G-equivariant are straightforward. The claim that ι is additive is more subtle and is left for the homework. The claim about the algebraic independence follows from the PBW theorem.

The subalgebra generated by $\iota(\mathfrak{g})$ inside $U(\mathfrak{g})$ is called the *p-center*. For $\mathfrak{g} = \mathfrak{sl}_2$, it is generated by $e^p, f^p, h^p - h$. It does not coincide with the whole center of $U(\mathfrak{sl}_2)$. For example, the Casimir element $C = ef + fe + h^2/2 \in U(\mathfrak{sl}_2)$ lies in the center but not in the p-center.

1.2. **p-reductions.** Pick an element $\alpha \in \mathfrak{g}^*$. Consider the quotient $U_\alpha(\mathfrak{g})$ of $U(\mathfrak{g})$ by the relations $\iota(x) - \langle \alpha, x\rangle, x \in \mathfrak{g}$. The algebras $U_\alpha(\mathfrak{g})$ below will be called *p-reductions* (short from "p-central reductions"). The element α is called the *p-character*.

Since any element of the p-center acts on $M \in \operatorname{Irr}(\mathfrak{g})$ by a scalar, we see that the $U(\mathfrak{g})$-action on M factors through $U_\alpha(\mathfrak{g})$ for α uniquely determined by M. So it is a natural question to describe the structure of $U_\alpha(\mathfrak{g})$.

Proposition 1.4. *The dimension of $U_\alpha(\mathfrak{g})$ equals $p^{\dim \mathfrak{g}}$. Moreover, if $x_1, \ldots, x_n$ form a basis of $\mathfrak{g}$, then the ordered monomials $x_1^{m_1} \ldots x_n^{m_n}$ with $0 \leqslant m_i \leqslant p-1$ form a basis in $U_\alpha(\mathfrak{g})$.*

Proof. Recall that the monomials

$$x_1^{pd_1+m_1} \ldots x_n^{pd_n+m_n} \tag{1.1}$$

$d_i \in \mathbb{Z}_{\geqslant 0}, m_i \in \{0, \ldots, p-1\}$ form a basis in $U(\mathfrak{g})$ (the PBW theorem). Now consider a new set of element indexed by the same d_i, m_i:

$$x_1^{m_1} \ldots x_n^{m_n}(x_1^p - x_1^{[p]} - \langle x_1, \alpha\rangle)^{d_1} \ldots (x_n^p - x_n^{[p]} - \langle x_n, \alpha\rangle)^{d_n}. \tag{1.2}$$

We claim that the monomials (1.2) form a basis in $U(\mathfrak{g})$. Indeed, the difference between (1.1) and (1.2) has degree that is strictly less than the total degree of any of these monomials, this implies our claim about basis.

Now note that the elements $x_i^p - x_i^{[p]} - \langle \alpha, x_i\rangle$ are central. So the two-sided ideal generated by these elements coincides with the linear span of basis elements (1.2), where at least one of the degrees d_i is positive. This easily implies the proposition. $\square$

So $\operatorname{Irr}_{fin}(U(\mathfrak{g})) = \bigsqcup_{\alpha \in \mathfrak{g}^*} \operatorname{Irr}_{fin}(U_\alpha(\mathfrak{g}))$ (in the left hand side we have the set of finite dimensional irreducible representations, note that there are no others by Problem 1 in Homework 1 because $U(\mathfrak{g})$ is a finitely generated module over its center). For every α, $\operatorname{Irr}(U_\alpha(\mathfrak{g})) = \operatorname{Irr}_{fin}(U_\alpha(\mathfrak{g}))$ is non-empty because $\dim U_\alpha(\mathfrak{g}) < \infty$.

In particular, in the case of $\mathfrak{sl}_2(\mathbb{F})$ the elements e, f no longer need to act nilpotently on a finite dimensional representations.

Note that, if $\alpha, \beta \in \mathfrak{g}^*$ are such that $\alpha = g \cdot \beta$, then the action of $g \in G$ on $U(\mathfrak{g})$ induces an algebra isomorphism $U_\beta(\mathfrak{g}) \xrightarrow{\sim} U_\alpha(\mathfrak{g})$. So it is enough to compute $\mathrm{Irr}(U_\alpha(\mathfrak{g}))$ for one α per every G-orbit.

1.3. **The case of $\mathfrak{sl}_2(\mathbb{F})$.** Since we assume that $p > 2$, the form $(x, y) = \mathrm{tr}(xy)$ on $\mathfrak{g} := \mathfrak{sl}_2(\mathbb{F})$ is non-degenerate. This form is $\mathrm{SL}_2(\mathbb{F})$-invariant. So the G-module $\mathfrak{g}$ and $\mathfrak{g}^*$ are identified. We know how to classify elements of $\mathfrak{sl}_2(\mathbb{F})$ (and, more generally, $\mathfrak{sl}_n(\mathbb{F})$) up to G-conjugacy, the classification is given by the Jordan normal form theorem. So up to G-conjugacy, we have three possibilities for $M \in \mathrm{Irr}_{fin}(\mathfrak{sl}_2)$:

(1) $e^p, f^p, h^p - h$ act by 0. Here $\alpha = 0$.
(2) $e^p, f^p - 1, h^p - h$ act by 0. Here $\alpha = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.
(3) $e^p, f^p, h^p - h - \lambda$ act by 0 for $\lambda \neq 0$. Here $\alpha = \begin{pmatrix} \lambda/2 & 0 \\ 0 & -\lambda/2 \end{pmatrix}$.

In the first case, M is called *restricted* – we will see below that it is obtained from an irreducible representation of $\mathrm{SL}_2(\mathbb{F})$. In all three cases e acts nilpotently, and as in the characteristic 0 case, we can find an h-eigenvector $v \in M$ annihilated by e. Let z be an eigenvalue. Note that $z^p - z = 0$ in cases (1),(2) (equivalently $z \in \mathbb{F}_p$), while, in case (3), z is one of the p solutions of $z^p - z = \lambda$.

Now we are going to proceed as in the characteristic 0 case: we introduce analogs of Verma modules for $U_\alpha(\mathfrak{g})$. Namely, $h, e \in U_\alpha(\mathfrak{g})$ generate a subalgebra in $U_\alpha(\mathfrak{g})$ with basis $h^\ell e^k, 0 \leqslant \ell, k \leqslant p-1$. We denote this subalgebra by $U_\alpha(\mathfrak{b})$. Consider the one-dimensional $U(\mathfrak{b})$-module $\mathbb{F}_z$, where e acts by 0 and h acts by z, where z is above. It factors through $U_\alpha(\mathfrak{b}) = U(\mathfrak{b})/(h^p - h - \alpha(h), e^p)$.

Form the induced module $\Delta_\alpha(z) := U_\alpha(\mathfrak{g}) \otimes_{U_\alpha(\mathfrak{b})} \mathbb{F}_z$ (the baby Verma modules). This module has basis $f^i v_z, i = 0, \ldots, p-1$, where $hv_z = zv_z, ev_z = 0$.

The following is the classification irreducible $U_\alpha(\mathfrak{g})$-modules (as well as the description of the structure of the modules $\Delta_\alpha(z)$).

Theorem 1.5. *The following is true.*

(i) *In case (1), there are p pairwise non-isomorphic irreducible $U_\alpha(\mathfrak{g})$-modules $L(z), z = 0, \ldots, p-1$ of dimension $z+1$, where $L_\alpha(z)$ is a simple quotient of $\Delta_\alpha(z)$. The module $\Delta_\alpha(p-1)$ is irreducible, while for $i \neq p-1$, there is an exact sequence $0 \to L_\alpha(-2-i) \to \Delta_\alpha(i) \to L_\alpha(i) \to 0$.*
(ii) *In case (2), all $\Delta_\alpha(z)$ are irreducible. We have $\Delta_\alpha(z) \cong \Delta_\alpha(z')$ if and only if $z + z' = -2$. We have $(p+1)/2$ irreducible $U_\alpha(\mathfrak{g})$-modules in this case.*
(iii) *In case (3), the $\mathfrak{g}$-modules $\Delta_\alpha(z)$ are non-isomorphic and form a complete collection of the irreducible $U_\alpha(\mathfrak{g})$-modules.*

The proof of this theorem will be in the homework.

2. Representations of $\mathrm{SL}_2(\mathbb{F})$

2.1. **Correspondence between group and Lie algebra.** A connection between the representation theories of an algebraic group G and the corresponding Lie algebra $\mathfrak{g}$ is much more loose than in characteristic 0. In a sentence, the representation theory of G in characteristic p is much closer to characteristic 0, than the representation theory of the corresponding Lie algebra.

Let V be a representation of G. It is also a representation of $\mathfrak{g}$.

Lemma 2.1. *Suppose V is irreducible over G and $\mathfrak{g}^{*G} = \{0\}$. Then the only eigenvalue of $\iota(\mathfrak{g})$ in V is zero.*

In fact, a stronger statement is true: for a rational representation V of G, the $U(\mathfrak{g})$-action on V factors through $U_0(\mathfrak{g})$.

Proof. For a $\mathfrak{g}$-module M and $g \in G$, define a new $\mathfrak{g}$-module M^g that coincides with M as a vector space but the action of $\mathfrak{g}$ is modified: if φ is a representation of $\mathfrak{g}$ in M, then $\varphi \circ \mathrm{Ad}(g)$ is a representation of $\mathfrak{g}$ in M^g. Similarly, we can define the G-module V^g. But $V \cong V^g$ as a G-module, and so also as a $\mathfrak{g}$-module. On the other hand, if α is a p-character of M, then $g^{-1} \cdot \alpha$ is a p-character of M^g. Since $\mathfrak{g}^{*G} = \{0\}$, we see that the only eigenvalue of $\iota(\mathfrak{g})$ in V is zero. □

On the other hand, many nontrivial G-modules give rise to trivial $\mathfrak{g}$-modules, we have seen this already in Example 1.2 of Lecture 3. Consider the Frobenius automorphism, $\mathsf{Fr} : \mathbb{F} \to \mathbb{F}, x \mapsto x^p$. It lifts to an automorphism of $\mathrm{GL}_n(\mathbb{F})$ (entry-wise) also denoted by Fr. For any algebraic subgroup of $\mathrm{GL}_n(\mathbb{F})$ defined over $\mathbb{F}_p$, the automorphism Fr of $\mathrm{GL}_n(\mathbb{F})$ restricts to an automorphism $G \to G$. So, for a rational representation V of G, we can consider its pullback Fr^*V, which is also a rational representation of G (if ρ is a representation of G in V, then $\rho \circ \mathsf{Fr}$ is the representation of G in Fr^*V). Note that the tangent map of Fr is zero at any point (derivatives of the pth powers are equal to 0). So $\mathfrak{g}$ acts on Fr^*V by 0. From here it is easy to that even if a representation of $\mathfrak{g}$ integrates to a representation of G, then it does so in infinitely many non-isomorphic ways.

2.2. **Weight decomposition.** The first thing to notice is that a rational representation of $\mathrm{SL}_2(\mathbb{F})$ still has a weight decomposition, which in this case is a decomposition with respect to the action of the subgroup T of diagonal matrices in $\mathrm{SL}_2(\mathbb{F})$ that is isomorphic to the multiplicative group $\mathbb{F}^\times$.

Lemma 2.2. *Any rational representation V of $\mathbb{F}^\times$ splits into the direct sum of one dimensional rational representations of $\mathbb{F}^\times$.*

Proof. From Linear algebra, we know that any two commuting diagonalizable operators are simultaneously diagonalizable. In fact, this is true for any collection (even infinite) of diagonalizable operators. The collection we take consists of all elements of $\mathbb{F}^\times$ that are of finite order coprime to p hence diagonalizable. Denote this collection by Λ. So the elements from Λ are simultaneously diagonalizable. The non-diagonal matrix coefficients of V vanish on Λ and so are zero. □

Recall (Example 1.2 of Lecture 3) that the one-dimensional rational representations of $\mathbb{F}^\times$ are classified by integers: to $n \in \mathbb{Z}$ we assign the representation given by $z \mapsto z^n$.

Now let V be a rational representation of $\mathrm{SL}_2(\mathbb{F})$. We can decompose it into the direct sum $V = \bigoplus_{n\in\mathbb{Z}} V_n$, where $V_n = \{v \in V | \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix} v = z^n v\}$. By a highest (resp., lowest) weight of V we mean the maximal (resp., minimal) n such that $V_n \neq \{0\}$. Note that $V_n \cong V_{-n}$. An isomorphism between these spaces is provided by the action of $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ because conjugating $\begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix}$ with $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ we get $\begin{pmatrix} z^{-1} & 0 \\ 0 & z \end{pmatrix}$. So the lowest weight is negative the highest weight.

Theorem 2.3. *For each $n \in \mathbb{Z}_{\geqslant 0}$, there is a unique irreducible representation $L(n)$ of $\mathrm{SL}_2(\mathbb{F})$ with highest weight n.*

This theorem will be proved below. An idea of the proof is similar to the case of $\mathfrak{sl}_2(\mathbb{C})$: we will produce an analog of a Verma module called a *Weyl module.*

2.3. **(Dual) Weyl modules.** Consider the representation $W^\vee(n) := S^n(\mathbb{F}^2)$ (a *dual Weyl module*) of $\mathrm{SL}_2(\mathbb{F})$. Let $x, y \in \mathbb{F}^2$ be a basis with weights $1, -1$. Then $x^n, x^{n-1}y, \ldots, y^n$ form a basis in $W^\vee(n)$ with weights $n, n-2, \ldots, -n$. The reason why this representation is useful is that it has the following universality property. Let B denote the group of upper triangular matrices. Consider the one-dimensional representation $\mathbb{F}_{-n}$ of B, where $b := \begin{pmatrix} z & x \\ 0 & z^{-1} \end{pmatrix}$ acts by $\chi(b)^{-n}$, $\chi(b) := z$.

Proposition 2.4. *For a rational representation V of $G = \mathrm{SL}_2(\mathbb{F})$, we have a natural isomorphism* $\mathrm{Hom}_G(V, W^\vee(n)) \cong \mathrm{Hom}_B(V, \mathbb{F}_{-n})$.

Proof. In the proof we will need a geometric realization of $W^\vee(n)$: as the global sections $\Gamma(\mathcal{O}(n))$ of the line bundle $\mathcal{O}(n)$ on $\mathbb{P}^1$. The group G acts on $\mathbb{P}^1 = \mathbb{P}(\mathbb{F}^2)$ and moreover, $\mathbb{P}^1$ is the homogeneous space G/B. The bundle $\mathcal{O}(n)$ carries an action of G that is compatible with the action of G on $\mathbb{P}^1$. In other words, this is a homogeneous vector bundle. To give such a bundle one only needs to specify a fiber at one point that is a representation of the stabilizer. Pick a point $[1:0] \in \mathbb{P}^1$ whose stabilizer is B. The fiber $\mathcal{O}(-1)_{[x:y]}$ of $\mathcal{O}(-1)$ at a point $[x:y] \in \mathbb{P}^1$ is the line passing through this point. It follows that $\mathcal{O}(-1)_{[1:0]} \cong \mathbb{F}_1$ (an isomorphism of B-modules). So $\mathcal{O}(n)_{[1:0]} = (\mathcal{O}(-1)^*)^{\otimes n}_{[1:0]}$ is the B-module $\mathbb{F}_{-n}$.

Now we are ready to produce an isomorphism $\mathrm{Hom}_G(V, W^\vee(n)) \to \mathrm{Hom}_B(V, \mathbb{F}_{-n})$. Note that $W^\vee(n) = \Gamma(\mathcal{O}(n))$ coincides with $\{f \in \mathbb{F}[G] | f(gb) = \chi(b)^n f(g)\}$. It follows that $\mathrm{Hom}(V, W^\vee(n))$ coincides with the space of polynomial maps $\varphi : G \to V^*$ such that $\varphi(gb) = \chi(b)^n \varphi(g)$. So $\mathrm{Hom}_G(V, W^\vee(n)) = (V^* \otimes W^\vee(n))^G$ coincides with $H := \{\varphi : G \to V^* | \varphi(gb) = g\chi^n(b)\varphi(1)\}$ (such a map is automatically polynomial). The map $\varphi \mapsto \varphi(1)$ establishes an isomorphism of H with $(V^* \otimes \mathbb{F}_{-n})^B$. $\square$

We define the *Weyl module* $W(n)$ as $W^\vee(n)^*$. It follows from Proposition 2.4 that

$$\mathrm{Hom}_G(W(n), V) = \mathrm{Hom}_B(\mathbb{F}_n, V) = \{v \in V | bv = \chi(b)^n v, \forall b \in B\}. \tag{2.1}$$

Proof of Theorem 2.3. Let V be an irreducible $\mathrm{SL}_2(\mathbb{F})$-module with highest weight n. Any vector in V_n is invariant under the subgroup $\{\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}\}$ (this is a homework problem). From (2.1) we conclude that V is an irreducible quotient of $W(n)$. Now the proof repeats that of Proposition 2.6, Lecture 3. In particular, $L(n)$ is the unique irreducible quotient of $W(n)$. $\square$

Remark 2.5. Note that $L(n) \cong L(n)^*$ (both sides are irreducibles with highest weight n). So $L(n) \hookrightarrow W^\vee(n)$. The kernel of $W(n) \twoheadrightarrow L(n)$ only has simple composition factors $L(m)$ with $m < n$ because all the weights in the kernel are less than n. The same is true for the cokernel of $L(n) \hookrightarrow W^\vee(n)$.

2.4. **Steinberg decomposition.** Here we are going to explain the structure of $L(n)$. The description we are going to give is "inductive". The base is $n < p$.

Lemma 2.6. *Let $n < p$. Then $L(n) \cong W^\vee(n) \cong W(n)$. As a $\mathfrak{g}$-module, $L(n) \cong L_0(n)$.*

Proof. Since $n < p$, we have $S^n(V)^* \cong S^n(V^*)$ for any finite dimensional $\mathbb{F}$-space V. Further, the $\mathrm{SL}_2(\mathbb{F})$-module $\mathbb{F}^2$ is self-dual. So the module $W^\vee(n) = S^n(\mathbb{F}^2)$ is self-dual. From Remark 2.5, it follows that $W^\vee(n) = L(n)$. The claim about the isomorphism of $\mathfrak{g}$-modules is straightforward. $\square$

Here is an "induction step".

Lemma 2.7. *For $n < p$ and $d \geqslant 0$, we have an isomorphism $L(dp+n) \cong L(n) \otimes \mathsf{Fr}^* L(d)$ of $\mathrm{SL}_2(\mathbb{F})$-modules.*

Proof. The highest weight of the right hand side is $dp+n$ so it is enough to show that the right hand side is irreducible. As a $\mathfrak{g}$-module, $\mathrm{Fr}^* L(d)$ is trivial. If V is a G-submodule of $L(n) \otimes \mathsf{Fr}^* L(d)$, then it is also a $\mathfrak{g}$-submodule. By the proof of the Burnside theorem in Lecture 1, $V = L(n) \otimes V_0$, where $V_0 \subset \mathsf{Fr}^* L(d)$. We have $V_0 = \mathrm{Hom}_{\mathfrak{g}}(L(n), V) \hookrightarrow \mathrm{Hom}_{\mathfrak{g}}(L(n), L(n) \otimes \mathsf{Fr}^* L(d)) = \mathsf{Fr}^* L(d)$. Note that $\mathrm{Hom}_{\mathfrak{g}}(U_1, U_2)$, where U_1, U_2 are G-modules, is a G-module (unlike in the characteristic 0, this module can be nontrivial). The equality $\mathrm{Hom}_{\mathfrak{g}}(L(n), L(n) \otimes \mathsf{Fr}^* L(d)) = \mathsf{Fr}^* L(d)$ is that of G-modules. But the pull-back of an irreducible module under a group homomorphism is itself irreducible. So the G-module $\mathrm{Hom}_{\mathfrak{g}}(L(n), V)$ is included into an irreducible G-module. It follows that $V_0 = \mathsf{Fr}^* L(d)$ that finishes the proof. $\square$

Our conclusion is that any $L(n)$ decomposes (in a unique way) into the tensor product of iterated pullbacks under Fr of $L(m)$'s with $m < p$ (Steinberg decomposition). This allows to determine the weight decomposition of $L(n)$. It also allows to determine the multiplicities of $L(n)$'s in the composition series of $W(n')$.

References

[J] J.C. Jantzen. *Representations of algebraic groups.* Academic Press, 1987.

LECTURE 5: SEMISIMPLE LIE ALGEBRAS OVER $\mathbb{C}$

IVAN LOSEV

Introduction

In this lecture I will explain the classification of finite dimensional semisimple Lie algebras over $\mathbb{C}$. Semisimple Lie algebras are defined similarly to semisimple finite dimensional associative algebras but are far more interesting and rich. The classification reduces to that of simple Lie algebras (i.e., Lie algebras with non-zero bracket and no proper ideals). The classification (initially due to Cartan and Killing) is basically in three steps.

1) Using the structure theory of simple Lie algebras, produce a combinatorial datum, the root system.

2) Study root systems combinatorially arriving at equivalent data (Cartan matrix/ Dynkin diagram).

3) Given a Cartan matrix, produce a simple Lie algebra by generators and relations.

In this lecture, we will cover the first two steps. The third step will be carried in Lecture 6.

1. Semisimple Lie algebras

Our base field is $\mathbb{C}$ (we could use an arbitrary algebraically closed field of characteristic 0).

1.1. **Criteria for semisimplicity.**

We are going to define the notion of a semisimple Lie algebra and give some criteria for semisimplicity. This turns out to be very similar to the case of semisimple associative algebras (although the proofs are much harder).

Let $\mathfrak{g}$ be a finite dimensional Lie algebra.

Definition 1.1. We say that $\mathfrak{g}$ is *simple*, if $\mathfrak{g}$ has no proper ideals and $\dim \mathfrak{g} > 1$ (so we exclude the one-dimensional abelian Lie algebra). We say that $\mathfrak{g}$ is *semisimple* if it is the direct sum of simple algebras.

Any semisimple algebra $\mathfrak{g}$ is the Lie algebra of an algebraic group, we can take the automorphism group $\mathrm{Aut}(\mathfrak{g})$. The connected component of 1 is denoted by $\mathrm{Ad}(\mathfrak{g})$, it should be viewed as the group of "inner" automorphisms of $\mathfrak{g}$. One can show that the algebra $\mathfrak{g}$ is simple if and only if $\mathrm{Ad}(\mathfrak{g})$ is simple as an abstract group.

We define the *Killing form* on a finite dimensional Lie algebra $\mathfrak{g}$ by $(x, y) = \mathrm{tr}(\mathrm{ad}(x)\,\mathrm{ad}(y))$, this is a symmetric bilinear form. It is *invariant* in the sense that

$$([x, y], z) + (y, [x, z]) = 0 \tag{1.1}$$

(equivalently, $(\cdot,\cdot)$ is annihilated by the representation of $\mathfrak{g}$ in the space $S^2(\mathfrak{g}^*)$). If $\mathfrak{g}$ is a Lie algebra of an algebraic group (or complex Lie group) G, then $(\cdot,\cdot)$ is G-invariant in the usual sense $(g\cdot y, g\cdot z) = (y, z)$ (just differentiate the latter equality to get (1.1)).

Theorem 1.2. *The following conditions are equivalent:*

(1) $\mathfrak{g}$ *is semisimple.*

(2) $(\cdot,\cdot)$ *is non-degenerate.*
(3) *Any finite dimensional representation of* $\mathfrak{g}$ *is completely reducible.*

We also have the notion of the radical of $\mathfrak{g}$ (=the maximal solvable ideal). The algebra $\mathfrak{g}$ is semisimple if and only if the radical is zero.

(2) gives a practical way to check semisimplicity, for example, $\mathfrak{sl}_n(\mathbb{C}), \mathfrak{so}_n(\mathbb{C})$ (for $n>2$) and $\mathfrak{sp}_n(\mathbb{C})$ are semisimple, this is left as an exercise.

1.2. **Cartan subalgebras.** Let $\mathfrak{g}$ be a semisimple Lie algebra. We say that an element in $\mathfrak{g}$ is semisimple (resp., nilpotent) if it acts by a semisimple (resp., nilpotent) operator in some faithful finite dimensional representation. Then it acts by a semisimple (resp., nilpotent) operator in any finite dimensional representation.

Proposition 1.3. *The following is true:*

(1) *A Zariski generic element* $x\in\mathfrak{g}$ *is semisimple.*
(2) *The centralizer* $\mathfrak{z}_{\mathfrak{g}}(x) := \{y\in\mathfrak{g} | [x,y]=0\}$ *is an abelian subalgebra in* $\mathfrak{g}$ *consisting of semisimple elements. This centralizer is called a* Cartan subalgebra.
(3) *Any two Cartan subalgebras are conjugate by an element of* $\mathrm{Ad}(\mathfrak{g})$.

For $\mathfrak{g}=\mathfrak{sl}_n(\mathbb{C})$, we take x with all distinct eigenvalues. Any Cartan subalgebra is the subalgebra of all elements diagonal in some basis. In the definition of $\mathfrak{g}=\mathfrak{so}_n(\mathbb{C})$, we use the form $(u,v)=\sum_{i=1}^n u_{n+1-i}v_i$ so that $\mathfrak{so}_n(\mathbb{C})$ consists of all matrices that are skew-symmetric with respect to the main antidiagonal. We again take x with all distinct eigenvalues. For a Cartan subalgebra, we can take the subalgebra of all diagonal matrices contained in $\mathfrak{so}_n(\mathbb{C})$, these matrices have the form $(x_1,\ldots,x_m,-x_m,\ldots,-x_1)$ if $n=2m$ and $(x_1,\ldots,x_m,0,-x_m,\ldots,-x_1)$ if $n=2m+1$. The case $\mathfrak{g}=\mathfrak{sp}_{2m}(\mathbb{C})$ is treated similarly – we take the form $\omega(x,y)=\sum_{i=1}^m (x_i y_{2n+1-i} - x_{2n+1-i}y_i)$.

1.3. **Root systems.** Let $\mathfrak{h}$ denote a Cartan subalgebra. For $\alpha\in\mathfrak{h}^*$, we set $\mathfrak{g}_\alpha := \{x\in\mathfrak{g} | [h,x]=\alpha(h)x, \forall h\in\mathfrak{h}\}$. The set of all $\alpha\in\mathfrak{h}^*\setminus\{0\}$ such that $\mathfrak{g}_\alpha\neq\{0\}$ is called the *root system* of $\mathfrak{g}$. We will write Δ (or $\Delta(\mathfrak{g})$) for the root system so that $\mathfrak{g}=\mathfrak{h}\oplus\bigoplus_{\alpha\in\Delta}\mathfrak{g}_\alpha$. Note that $[\mathfrak{g}_\alpha,\mathfrak{g}_\beta]\subset\mathfrak{g}_{\alpha+\beta}$. Note also that $(\cdot,\cdot)$ restricts to a perfect pairing $\mathfrak{g}_\alpha\times\mathfrak{g}_{-\alpha}\to\mathbb{C}$ (indeed, $\mathfrak{g}_\alpha$ is orthogonal to any $\mathfrak{g}_\beta$ with $\alpha+\beta\neq 0$) and to a non-degenerate form on $\mathfrak{h}$. So we have a non-degenerate symmetric form on $\mathfrak{h}^*$ also denoted by $(\cdot,\cdot)$.

We have the following properties of the subspaces $\mathfrak{g}_\alpha$. (1)-(6) are obtained using the representation theory of $\mathfrak{sl}_2(\mathbb{C})$.

Proposition 1.4. *We have the following properties of* $\mathfrak{g}_\alpha$*'s and* Δ.

(1) *Let* $e\in\mathfrak{g}_\alpha$ *and* $f\in\mathfrak{g}_{-\alpha}$ *be such that* $(e,f)\neq 0$. *Then we can rescale* f *in such a way that* $h:=[e,f]$ *(that is an element of* $\mathfrak{h}$*) satisfies* $\alpha(h)=2$. *We have* $[h,e]=2e, [h,f]=-2f$. *In other words, the map* $\mathfrak{sl}_2\to\mathfrak{g}$ *given by* $\begin{pmatrix}0&1\\0&0\end{pmatrix}\mapsto e, \begin{pmatrix}0&0\\1&0\end{pmatrix}\mapsto f, \begin{pmatrix}1&0\\0&-1\end{pmatrix}\mapsto h$ *is a Lie algebra homomorphism.*
(2) $\dim\mathfrak{g}_\alpha=1$ *for all* $\alpha\in\Delta$. *So we have elements* $e_\alpha\in\mathfrak{g}_\alpha, f\in\mathfrak{g}_{-\alpha}, h_\alpha\in\mathfrak{h}$ *as in (1). Note that* h_α *is uniquely determined.*
(3) $\beta(h_\alpha)\in\mathbb{Z}$ *for any* $\alpha,\beta\in\Delta$.
(4) *If* $\alpha\in\Delta$, *then* $2\alpha\notin\Delta$.
(5) *For* $\beta\in\Delta$, *define a linear map* $s_\beta:\mathfrak{h}^*\to\mathfrak{h}^*$ *by* $\lambda\mapsto\lambda-\lambda(h_\beta)\beta$. *The map* s_β *maps* Δ *to itself.*

(6) *Let $\alpha+\beta\neq 0$. If $\beta+k\alpha, \beta+\ell\alpha\in\Delta$, then $\beta+i\alpha\in\Delta$ for any integer i between k,ℓ.*
(7) *Δ spans $\mathfrak{h}^*$. Moreover, let $\mathfrak{h}_{\mathbb{R}}$ denotes the $\mathbb{R}$-span of Δ. Then the restriction of $(\cdot,\cdot)$ to $\mathfrak{h}_{\mathbb{R}}$ is positive definite.*
(8) *Under the identification of $\mathfrak{h}^*$ and $\mathfrak{h}$ by means of $(\cdot,\cdot)$, we get $h_\alpha = 2\alpha/(\alpha,\alpha)$ (we'll use notation $\alpha^\vee$ for the right hand side).*

Example 1.5. Let $\mathfrak{g}=\mathfrak{sl}_n$. Let ϵ_i denote the linear function on $\mathfrak{h}$ taking the entry (i,i) so that $\sum_{i=1}^n \epsilon_i = 0$. Then the root system Δ consists of the elements $\epsilon_i-\epsilon_j, i\neq j$. The root subspace $\mathfrak{g}_\alpha$ for $\alpha=\epsilon_i-\epsilon_j$ is spanned by the unit matrix E_{ij}. The root system Δ is called the root system of type A_{n-1}.

Note that, for $x\in\mathfrak{h}$, we have $(x,x)=\sum_{\alpha\in\Delta}\alpha(x)^2 = 2\sum_{i<j}(x_i-x_j)^2 = 2(n-1)\sum_{i=1}^n x_i^2 - 4\sum_{i<j}x_ix_j$. Since $\sum_{i=1}^n x_i = 0$, we arrive at $(x,x)=2(n+1)\sum_{i=1}^n x_i^2$.

Example 1.6. Let $\mathfrak{g}=\mathfrak{so}_{2n+1}$. Then $\mathfrak{h}^*$ has a basis of functions $\epsilon_i, i=1,\ldots,n$ defined as in the previous example. The root system Δ consists of the elements $\pm\epsilon_i\pm\epsilon_j, i\neq j, \pm\epsilon_i$. This is the root system of type B_n.

Example 1.7. For $\mathfrak{g}=\mathfrak{sp}_{2n}$, the root system Δ consists of $\pm\epsilon_i\pm\epsilon_j, i\neq j, \pm 2\epsilon_i$ (type C_n).

Example 1.8. For $\mathfrak{g}=\mathfrak{so}_{2n}$, the root system Δ consists of $\pm\epsilon_i\pm\epsilon_j, i\neq j$, (type D_n).

In the last three examples, the form (x,x) on $\mathfrak{h}$ is proportional to $\sum_{i=1}^n x_i^2$.

1.4. **Irreducible root systems.** We say that Δ is *irreducible* if there are no proper subspaces $\Delta_1,\Delta_2\subset\Delta$ such that $\Delta=\Delta_1\cup\Delta_2$ and $(\alpha_1,\alpha_2)=0$ for $\alpha_i\in\Delta_i$.

Lemma 1.9. *The algebra $\mathfrak{g}$ is simple if and only if Δ is irreducible.*

Example 1.10. The algebras $\mathfrak{sl}_n, \mathfrak{so}_{2n+1}, \mathfrak{sp}_{2n}$ are simple for $n>1$. The algebra $\mathfrak{so}_{2n}$ is simple if and only if $n>2$. For $n=2$, the root system Δ equals $\{\pm\epsilon_1\pm\epsilon_2\}$ and we can take $\Delta_1=\{\pm(\epsilon_1-\epsilon_2)\}$ and $\Delta_2=\{\pm(\epsilon_1+\epsilon_2)\}$. And, indeed, we have $\mathfrak{so}_4\cong\mathfrak{sl}_2\times\mathfrak{sl}_2$.

2. Classification of root systems

2.1. **Abstract root systems.** Let E be a finite dimensional Euclidian space and $\Delta\subset E\setminus\{0\}$ be a finite collection of elements. For $\alpha\in\Delta$, we write $\alpha^\vee$ for $\frac{2\alpha}{(\alpha,\alpha)}$. Suppose that the following is true

(R1) $(\alpha^\vee,\beta)$ is an integer for any $\alpha,\beta\in\Delta$.
(R2) Define an automorphism s_β of E given by $s_\beta(v)=v-(\beta^\vee,v)\beta$. This automorphism preserves Δ.
(R3) Δ spans E.

Note that $s_\alpha(\alpha)=-\alpha$ and so Δ is closed under multiplication by -1.

We say that Δ is *reduced* if $\alpha\in\Delta$ implies $2\alpha\notin\Delta$. We see that $\Delta(\mathfrak{g})$ is a reduced root system in $E=\mathfrak{h}^*_{\mathbb{R}}$. Similarly to Section 1.4, we can speak about irreducible root systems.

We say that two root systems $\Delta\subset E, \Delta'\subset E'$ are *isomorphic* if there is a linear isomorphism $\varphi: E\to E'$ such that $\varphi(\Delta)=\Delta'$ and $(\alpha^\vee,\beta)=(\varphi(\alpha)^\vee,\varphi(\beta)), \forall\alpha,\beta\in\Delta$.

2.2. **Weyl group and Weyl chambers.** Note that s_β is the reflection about the hyperplane $\beta^\perp\subset E$. We consider the subgroup $W\subset \mathrm{O}(E)$ generated by the reflections s_β. It preserves Δ. Since Δ is finite and spans E, we see that W is finite. It is called the *Weyl group* of Δ.

Example 2.1. Let Δ be the root system of type A_n. We can take $(x,x) = \sum_{i=1}^{n+1} x_i^2$ to be the scalar product on E. Let $\beta = \epsilon_i - \epsilon_j$. Then $\beta^\vee = \epsilon_i - \epsilon_j$ and $(\beta^\vee, x) = x_i - x_j$, so the reflection s_β is given by $x \mapsto x - (x_i - x_j)(\epsilon_i - \epsilon_j)$, where $x = \sum_{i=1}^{n+1} x_i \epsilon_i$. So it just swaps the ith and jth coordinates of x. It follows that the Weyl group is the symmetric group S_{n+1}. In types B_n, C_n, we get the group $S_n \ltimes (\mathbb{Z}/2\mathbb{Z})^n$, where the elements of S_n permute the basis vectors $\epsilon_1, \ldots, \epsilon_n \in E$ and the elements of $(\mathbb{Z}/2\mathbb{Z})^n$ switch the signs of the basis vectors. In type D_n, the group $W(D_n)$ is the index two normal subgroup of $S_n \times \mathbb{Z}/2\mathbb{Z}$ consisting of all elements that switch an even number of signs.

The hyperplanes $\beta^\perp$ split E into the union of regions called *Weyl chambers*. The following proposition describes the properties of the chambers.

Proposition 2.2. *The following is true.*

(1) *The group W permutes the Weyl chambers simply transitively.*
(2) *For each Weyl chamber C, there are n roots $\alpha_1, \ldots, \alpha_n$ such that $C = \{v \in E | (\alpha_i, v) \geqslant 0, i = 1, \ldots, n\}$.*
(3) *C is a* fundamental domain *for W meaning that for each $v \in E$, there is a unique $u \in C$ with $v \in Wu$.*
(4) *The reflections $s_{\alpha_1}, \ldots, s_{\alpha_n}$ (a.k.a. simple reflections) generate the Weyl group.*

Example 2.3. Consider the root system of type A_n. The hyperplanes $\beta^\perp$ are $x_i = x_j$. So we have $(n+1)!$ Weyl chambers, they are specified by an ordering of $x_1, \ldots, x_{n+1}$. An example of a Weyl chamber is given by $C = \{(x_1, \ldots, x_{n+1}) | x_1 \geqslant x_2 \geqslant \ldots \geqslant x_{n+1}\}$. The corresponding simple reflections are the simple transpositions $(i, i+1), i = 1, \ldots, n$. They clearly generate S_n.

In types B_n, C_n, we can take $C = \{(x_1, \ldots, x_n) | x_1 \geqslant x_2 \geqslant \ldots \geqslant x_n \geqslant 0\}$. In type D_n, we can take $C = \{(x_1, \ldots, x_n) | x_1 \geqslant \ldots \geqslant x_n \geqslant -x_{n-1}\}$.

2.3. **Simple roots, Cartan matrix and Dynkin diagram.** By a system of simple roots, we mean a subset of the form $\alpha_1, \ldots, \alpha_n$ for some Weyl chamber C, see (2) of Proposition 2.2. Note that W acts simply transitively on the set of systems of simple roots. So it does not matter which system we pick.

Proposition 2.4. *Let $\alpha_1, \ldots, \alpha_n$ be a system of simple roots. Then*

(1) *$\alpha_1, \ldots, \alpha_n$ form a basis in E.*
(2) *For any $\beta \in \Delta$, either $\beta = \sum_{i=1}^n n_i \alpha_i$ with all $n_i \geqslant 0$ (positive root), or $\beta = \sum_{i=1}^n n_i \alpha_i$ with all $n_i \leqslant 0$ (negative root).*

Example 2.5. For the chambers chosen in Example 2.3, we have the following systems of simple roots:

(A_n) $\alpha_i := \epsilon_i - \epsilon_{i+1}, i = 1, \ldots, n$.
(B_n) $\alpha_i := \epsilon_i - \epsilon_{i+1}, i = 1, \ldots, n-1, \alpha_n := \epsilon_n$.
(C_n) $\alpha_i := \epsilon_i - \epsilon_{i+1}, i = 1, \ldots, n-1, \alpha_n := 2\epsilon_n$.
(D_n) $\alpha_i := \epsilon_i - \epsilon_{i+1}, i = 1, \ldots, n-1, \alpha_n := \epsilon_{n-1} + \epsilon_n$.

We can encode a simple root system by the *Cartan matrix*, this is an $n \times n$-matrix with entries $n_{ij} = \alpha_i^\vee(\alpha_j)$. This matrix is defined up to a conjugation with a monomial matrix (corresponding to re-ordering $\alpha_1, \ldots, \alpha_n$). We have the following important result.

Proposition 2.6. *Let Δ, Δ' be two reduced root systems with the same Cartan matrix. Then Δ, Δ' are isomorphic.*

We can depict Cartan matrices (or simple root systems) getting so called Dynkin diagrams. Namely, the simple roots are nods. We draw $n_{ij}n_{ji} = \frac{4(\alpha_i,\alpha_j)^2}{(\alpha_i,\alpha_i)(\alpha_j,\alpha_j)}$ un-oriented edges between two nodes. We orient them by putting an arrow (that also can be viewed as an inequality sign) from the longer root to a shorter root. Note that, since $n_{ij}n_{ji} < 4$, we can recover the lengthes of roots from that. The Dynkin diagrams for the root systems A_n-D_n are shown in Picture 1.

2.4. **Classification of Cartan matrices.** By an abstract Cartan matrix we mean a square matrix $A = (a_{ij})_{i,j=1}^n$ such that

(1) $a_{ii} = 2$,
(2) $a_{ij} \leqslant 0$ for $i \neq j$,
(3) $a_{ij} = 0$ if and only if $a_{ji} = 0$.

We say that A is *irreducible* if we cannot partition $\{1, 2, \ldots, n\}$ into $I \sqcup J$ such that $a_{ij} = 0, i \in I, j \in J$. We say that A is *symmetrizable* if there is a diagonal matrix D with positive entries and a symmetric matrix G such that $A = DG$. Note that if A is irreducible, then D is defined up to a positive scalar factor. We say that A is *positive definite* if so is S.

The matrix $A(\Delta)$ is symmetrizable, we can take $D = \mathrm{diag}(2/(\alpha_i, \alpha_i))_{i=1}^n$, then S is the Gram matrix of the basis α_i. So A is positive definite.

One can classify the irreducible symmetrizable positive definite Cartan matrices. Besides the matrices/ Dynkin diagrams A_n-D_n, there are just five more exceptional diagrams, E_6, E_7, E_8, F_4, G_2, see Picture 2.

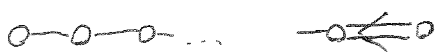
One can explicitly produce the root systems corresponding to these diagrams. We are not going to do that. Instead, we will produce simple Lie algebras with these Dynkin diagrams. This will complete the classification of simple Lie algebras (over $\mathbb{C}$ or over general algebraically closed fields of characteristic 0).

Pic 1:

A_n

B_n

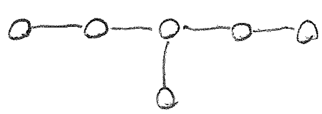
C_n



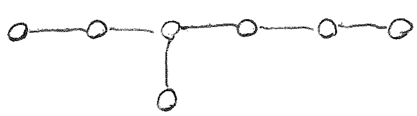
D_n

Pic 2:

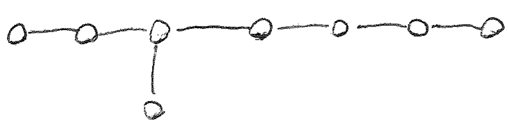
E_6



E_7



E_8



F_4

G_2

LECTURE 6: KAC-MOODY ALGEBRAS, REDUCTIVE GROUPS, AND REPRESENTATIONS

IVAN LOSEV

Introduction

We start by introducing Kac-Moody algebras and completing the classification of finite dimensional semisimple Lie algebras.

We then discuss the classification of finite dimensional representations of semisimple Lie algebras (and, more generally, integrable highest weight representations of Kac-Moody algebras).

We finish by discussing the structure and representation theory of reductive algebraic groups.

1. Generators and relations. Kac-Moody algebras

1.1. **Relations in simple Lie algebras.** Let $\mathfrak{g}$ be a simple Lie algebra, $\mathfrak{h} \subset \mathfrak{g}$ a Cartan subalgebra, $\alpha_1, \ldots, \alpha_n$ a simple root system. We write e_i, h_i, f_i for $e_{\alpha_i}, h_{\alpha_i}, f_{\alpha_i}$. Let $A = (a_{ij})_{i,j=1}^n$ be the Cartan matrix, where, recall, $a_{ij} = \alpha_j(h_i)$. Set $\mathfrak{n} := \bigoplus_{\alpha>0} \mathfrak{g}_\alpha, \mathfrak{n}^- := \bigoplus_{\alpha<0} \mathfrak{g}_\alpha$, these are Lie subalgebras of $\mathfrak{g}$ because $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$. We have $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ (as vector spaces) because every root is either positive or negative.

Lemma 1.1. *The elements e_i (resp., f_i) generate $\mathfrak{n}$ (resp., $\mathfrak{n}^-$). In particular, $e_i, h_i, f_i, i = 1, \ldots, n$, generate $\mathfrak{g}$. Further, we have the following relations (a.k.a. Serre relations):*

(1) $[h_i, h_j] = 0$.
(2) $h_i = [e_i, f_i], [e_i, f_j] = 0$ *for* $i \neq j$.
(3) $[h_i, e_j] = a_{ij} e_j, [h_i, f_j] = -a_{ij} f_j$.
(4) $\mathrm{ad}(e_i)^{1-a_{ij}} e_j = 0, \mathrm{ad}(f_i)^{1-a_{ij}} f_j = 0, i \neq j$.

Example 1.2. Let $\mathfrak{g} = \mathfrak{sl}_n$. Then $\mathfrak{n}$ (resp., $\mathfrak{n}^-$) is the subalgebra of all strictly upper (resp., lower) triangular matrices. We have $h_i := E_{i,i} - E_{i+1,i+1}, e_i = E_{i,i+1}, f_i = E_{i+1,i}$. We have $[E_{ij}, E_{jk}] = E_{ik}$ if $i \neq k$. Lemma 1.1 basically follows from this identity.

Proof. Let us check that the elements e_i generate $\mathfrak{n}$ (the claim about f_i's and $\mathfrak{n}^-$ is analogous). Assume the contrary: there is $\alpha > 0$ such that $\mathfrak{g}_\alpha$ does not lie in the subalgebra $\mathfrak{n}_0$ generated by the e_i's. We have $\alpha = \sum_{i=1}^n m_i \alpha_i$ with $m_i \geqslant 0$. We may assume that $\sum_{i=1}^n m_i$ is minimal possible such that $\mathfrak{g}_\alpha \not\subset \mathfrak{n}_0$. Since all $m_j \geqslant 0$ and $0 < (\alpha, \alpha) = \sum_j m_j (\alpha, \alpha_j)$, we see that there is j with $\alpha(h_j) = \frac{2(\alpha,\alpha_j)}{(\alpha_j,\alpha_j)} > 0$. The elements α and $s_j(\alpha) = \alpha - \alpha(h_j)\alpha_j$ are roots. (6) of Proposition 1.4 of Lecture 5 implies that $\alpha - \alpha_j$ is a root. By the inductive assumption, $\mathfrak{g}_{\alpha-\alpha_j} \subset \mathfrak{n}_0$. Consider the $\mathfrak{sl}_2$-module $\sum_{z\in\mathbb{Z}} \mathfrak{g}_{\alpha+z\alpha_j}$. By the representation theory of $\mathfrak{sl}_2$, we see that $[e_j, \mathfrak{g}_{\alpha-\alpha_j}] = \mathfrak{g}_\alpha$, and we are done.

Let us check the relations. (1) is obvious. The first equality in (2) is the definition of h_i. The second one follows from $[e_i, f_j] \in \mathfrak{g}_{\alpha_i-\alpha_j} = 0$ because $\alpha_i - \alpha_j$ is neither positive nor negative root. (3) follows from $a_{ij} = \alpha_j(h_i)$.

Let us prove that $\mathrm{ad}(f_i)^{1-a_{ij}} f_j = 0$. We know that

$$[e_i, f_j] = 0, [h_i, f_j] = -a_{ij} f_j. \tag{1.1}$$

Consider the $\mathfrak{sl}_2$-subalgebra spanned by e_i, h_i, f_i and its module $\bigoplus_{z\in\mathbb{Z}} \mathfrak{g}_{-\alpha_j+z\alpha_i}$. Considering f_j as an element of this module and using (1.1) and the representation theory of $\mathfrak{sl}_2$, we see that $\mathrm{ad}(f_i)^{1-a_{ij}} f_j = 0$. The other equality in (4) is proved similarly. $\square$

1.2. **Kac-Moody algebras.** Now let A be an arbitrary symmetrizable irreducible Cartan matrix. We can define the Lie algebra $\mathfrak{g}(A)$ by generators h_i, e_i, f_i and relations (1)-(4). This is the Kac-Moody algebra (with Cartan matrix A).

Let $\mathfrak{n}$ (resp., $\mathfrak{n}^-$, $\mathfrak{h}$) be the subalgebra in $\mathfrak{g}(A)$ generated by the elements e_i (resp., f_i, resp., h_i). Then we have the following important result

Proposition 1.3. *We have $\mathfrak{g}(A) = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}$ (as vector spaces). Moreover, $h_1, \ldots, h_n$ form a basis in $\mathfrak{h}$.*

To simplify the notation let us write $\mathfrak{g}$ for $\mathfrak{g}(A)$. Consider the vector space $\mathfrak{h}^\vee$ with basis $\alpha_1, \ldots, \alpha_n$. This space comes with a pairing $\mathfrak{h}^\vee \times \mathfrak{h} \to \mathbb{C}$ given by $(\alpha_i, h_j) \to a_{ij}$ and with a symmetric form defined by S (note that the pairing and the form are degenerate if and only if $\det A = 0$ – which is an interesting case, but it requires some extra care because the elements α_i cannot be viewed as functions on $\mathfrak{h}$).

Let Q^+ (resp., Q) denote the sub-semigroup (resp., subgroup) in $\mathfrak{h}^\vee$ spanned by $\alpha_1, \ldots, \alpha_n$. We assign degrees $\alpha_i, 0, -\alpha_i$ to e_i, h_i, f_i. Since the relations in $\mathfrak{g}$ are Q-homogeneous, we see that $\mathfrak{n}$ is Q^+-graded and $\mathfrak{n}^-$ is $-Q^+$-graded. For $\beta \in Q^+ \sqcup -Q^+$, let $\mathfrak{g}_\beta$ denote the corresponding graded component. We say that β is a *root* if $\mathfrak{g}_\beta \neq \{0\}$. The notions of positive and negative roots are introduced in an obvious way.

We still can define the bijections $s_i : \mathfrak{h}^\vee \to \mathfrak{h}^\vee$ as before (the simple reflections). The subgroup of $\mathrm{GL}(\mathfrak{h}^\vee)$ generated by the elements s_i is called the Weyl group of $\mathfrak{g}$ or of A. The Weyl group elements map roots to roots preserving the dimensions of root spaces, this again follows from the representation theory of $\mathfrak{sl}_2$. Roots obtained from $\alpha_1, \ldots, \alpha_n$ by applying Weyl group elements are called *real*, for a real root α we have $\dim \mathfrak{g}_\alpha = 1$. The other roots are called imaginary.

Lemma 1.4. *If β is an imaginary root, then $(\beta, \beta) \leqslant 0$.*

1.3. **Case of positive definite Cartan matrix.** Now suppose that A is positive definite (and irreducible).

Theorem 1.5. *The algebra $\mathfrak{g}(A)$ is finite dimensional and simple.*

Sketch of proof. The Weyl group W is a subgroup in $\mathrm{O}(\mathfrak{h}^\vee)$ that preserves the lattice generated by $\alpha_1, \ldots, \alpha_n$. Such a group is finite. By Lemma 1.4, every root is real. So the root system of $\mathfrak{g}$ is finite and all $\mathfrak{g}_\alpha$'s have dimension 1. So $\dim \mathfrak{g}(A) < \infty$. Any ideal in $\mathfrak{g}(A)$ can be shown to intersect $\mathfrak{h}$ which contradicts the irreducibility of A. $\square$

When A is not positive definite, the algebra $\mathfrak{g}(A)$ is not finite dimensional. A family of examples will be described in the homework.

2. Finite dimensional representations

2.1. **The case of finite dimensional algebras.** Let $\mathfrak{g}$ be a finite dimensional semisimple Lie algebra over $\mathbb{C}$ and V be its finite dimensional module. Since $\mathfrak{h}$ is commutative and consists of semisimple elements, we have the *weight* decomposition $V = \bigoplus_{\nu\in\mathfrak{h}^*} V_\nu$, where V_ν is the eigenspace for $\mathfrak{h}$ with eigenvalue ν. If $V_\nu \neq \{0\}$, we say that ν is a *weight* of V. The number of $\nu(h_i)$ is a weight for an $\mathfrak{sl}_2$-module and so is integral. Set $P := \{\mu \in \mathfrak{h}^* | \mu(h_i) \in \mathbb{Z}, \forall i\}$. We conclude that any weight ν is in P (and hence P is called the weight lattice).

On $\mathfrak{h}^*$ we introduce a partial order $\leqslant$: $\nu' \leqslant \nu$ if $\nu - \nu'$ is the sum of positive roots. By a *highest weight* of an irreducible module V, we mean a maximal weight λ (existing because the set of weights is finite). Note that any $v \in V_\lambda$ is annihilated by $\mathfrak{n}$.

Lemma 2.1. *Let λ be a highest weight of V. Then*

(1) $\lambda(h_i) \geqslant 0$.
(2) *Further,* $f_i^{\lambda(h_i)+1} v_\lambda = 0$.

Proof. (1) follows from $e_i v = 0$. (2) follows from the representation theory of $\mathfrak{sl}_2$, compare to the proof of Lemma 1.1. □

The elements $\lambda \in \mathfrak{h}^*$ satisfying (1) are called *dominant*. The set of dominant weights is denoted by P^+.

The following theorem generalizes the corresponding result for $\mathfrak{sl}_2$.

Theorem 2.2. *There is a bijection between* $\mathrm{Irr}_{fin}(\mathfrak{g})$ *and* P^+ *that sends a module to its (unique) highest weight.*

The proof is similar to the $\mathfrak{sl}_2$-case, see Lecture 3. Set $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$. Now for an arbitrary $\lambda \in \mathfrak{h}^*$ we can form the one-dimensional $\mathfrak{b}$-module $\mathbb{C}_\lambda$ ($\mathfrak{h}$ acts by λ and $\mathfrak{n}$ acts by 0, compare to $\mathbb{C}_z$ in the case $\mathfrak{g} = \mathfrak{sl}_2$). So we can form the Verma module $\Delta(\lambda) := U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda$ that has the universal property:

$$\mathrm{Hom}_{\mathfrak{g}}(\Delta(\lambda), V) = \{v \in V_\lambda | \mathfrak{n} v = 0\}.$$

Let v_λ denote the image of $1 \in U(\mathfrak{g})$ in $\Delta(\lambda)$.

Lemma 2.3. *The following is true:*

(1) $\Delta(\lambda) = \bigoplus_{\nu\leqslant\lambda} \Delta(\lambda)_\nu$ *and* $\dim \Delta(\lambda)_\nu < \infty$.
(2) *There is a unique simple quotient $L(\lambda)$ of $\Delta(\lambda)$.*
(3) *Moreover, if $v \in L(\lambda)$ is annihilated by $\mathfrak{n}$, then it is proportional to v_λ.*

Proof. By the PBW theorem, $\Delta(\lambda)$ has basis $\prod_{\alpha>0} f_\alpha^{-n_\alpha} v_\lambda$ (for some fixed ordering of positive roots). The weight of this basis vector is $\lambda - \sum_{\alpha>0} n_\alpha \alpha$. This implies (1).

Note that any proper submodule of $\Delta(\lambda)$ is contained in $\bigoplus_{\nu<\lambda} \Delta(\lambda)_\nu$. This implies (2).

Let us prove (3). Assuming it is false, we can find $v \in L(\lambda)_\nu$ with $\mathfrak{n} v = 0$. We have $\nu < \lambda$ and hence the image of $\Delta(\nu)$ in $L(\lambda)$ is proper. Contradiction. □

Corollary 2.4. *Any irreducible finite dimensional $\mathfrak{g}$-module has a single highest weight. Two finite dimensional irreducible modules with the same highest weight are isomorphic.*

To prove Theorem 2.2 it remains to prove the following proposition.

Proposition 2.5. *If λ is dominant, then* $\dim L(\lambda) < \infty$.

Sketch of the proof. The proof is in several steps.

Step 1. Deduce that f_i, e_i act on $L(\lambda)$ locally nilpotently (for e_i this follows from the weight considerations, while for f_i one needs to use the second part of Lemma 2.1).

Step 2. Deduce that $L(\lambda)$ is the sum of its finite dimensional $\mathfrak{sl}_2$-modules for any $\mathfrak{sl}_2 = \mathrm{Span}(e_i, h_i, f_i)$. So the action of this $\mathfrak{sl}_2$ integrates to $\mathrm{SL}_2(\mathbb{C})$.

Step 3. By using the element $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C})$, show that the set of weights of $L(\lambda)$ is closed under s_i.

Step 4. By Step 3, the set of weights is closed under the action of the Weyl group W. On the other hand, any weight μ has the property that $\lambda - \mu$ is the sum of positive roots. From here one can deduce that the set of weights is finite.

Step 5. Since the dimensions of all weight spaces are finite (this is so even in $\Delta(\lambda)$ by (1) of Lemma 2.3), we see that $\dim L(\lambda) < \infty$. $\square$

2.2. **Fundamental weights.** For $i = 1, \ldots, n$, define $\omega_i \in \mathfrak{h}^*$ by $\omega_i(\alpha_j^\vee) = \delta_{ij}$. This elements form bases in the group P and in the monoid P^+.

The irreducible representations corresponding to fundamental weights are important because of the following lemma.

Lemma 2.6. *Let λ, μ be dominant weights. Then $L(\lambda + \mu)$ coincides with the submodule in $L(\lambda) \otimes L(\mu)$ generated by $v_\lambda \otimes v_\mu$.*

Proof. First of all, note that a $\mathfrak{g}$-submodule in any $\mathfrak{g}$-module generated by $v \in V_\nu$ with $\mathfrak{n}v = 0$ is isomorphic to $L(\nu)$. Indeed, it is a finite dimensional image of a homomorphism $\Delta(\nu) \to V$. Any finite dimensional image of $\Delta(\nu)$ is $L(\nu)$ by (2) of Lemma 2.3 and the complete reducibility of finite dimensional modules.

The vector $v_\lambda \otimes v_\mu$ has weight $\lambda + \mu$ and is annihilated by $\mathfrak{n}$. Our claim follows. $\square$

Example 2.7. Let $\mathfrak{g} = \mathfrak{sl}_{n+1}$. We have $\omega_i = \sum_{j=1}^i \epsilon_j$ and $L(\omega_i) = \Lambda^i \mathbb{C}^{n+1}$. Indeed, let $v_1, \ldots, v_{n+1}$ be a natural basis in $\mathbb{C}^{n+1}$. The vector $v_1 \wedge v_2 \wedge \ldots \wedge v_i$ has weight ω_i and is annihilated by $\mathfrak{n}$. It is not difficult to see that ω_i is the only dominant weight in $\Lambda^i \mathbb{C}^{n+1}$ so the latter is irreducible.

Example 2.8. Let $\mathfrak{g} = \mathfrak{so}_{2n+1}$. We have $\omega_i = \sum_{j=1}^i \epsilon_j$ for $i < n$, and $\omega_n = \frac{1}{2}\sum_{j=1}^n \epsilon_j$. One can show that $\bigwedge^i \mathbb{C}^{2n+1} = L(\omega_i)$ for $i < n$. The irreducible representation $L(\omega_n)$ (the spinor representation) is not realized in this way.

Example 2.9. Let $\mathfrak{g} = \mathfrak{sp}_{2n}$. We have $\omega_i = \sum_{j=1}^i \epsilon_i$, and $L(\omega_i)$ is a direct summand in $\Lambda^i \mathbb{C}^{2n}$ generated by $v_1 \wedge v_2 \wedge \ldots \wedge v_i$ for any $i = 1, \ldots, n$.

Example 2.10. Let $\mathfrak{g} = \mathfrak{so}_{2n}$. We have $\omega_i = \sum_{j=1}^i \epsilon_i$ for $i = 1, \ldots, n-2$, $\omega_{n-1} = \omega_{n-2} + \frac{1}{2}(\epsilon_{n-1} - \epsilon_n), \omega_n = \omega_{n-2} + \frac{1}{2}(\epsilon_{n-1} + \epsilon_n)$. We have $L(\omega_i) = \bigwedge^i \mathbb{C}^{2n}$ for $i \leqslant n-2$. The representations $L(\omega_{n-1}), L(\omega_n)$ are the so called half-spinor representations.

2.3. **Integrable highest weight representations of Kac-Moody algebras.** Now let A be a symmetrizable Cartan matrix and $\mathfrak{g} = \mathfrak{g}(A)$ be the corresponding Kac-Moody algebra. Recall the space $\mathfrak{h}^\vee$ spanned by the simple roots $\alpha_1, \ldots, \alpha_n$. We are interested in studying highest weight representations of $\mathfrak{g}$, i.e., representations V equipped with a grading $V = \bigoplus_{\mu \in -Q^+} V(\mu)$, where $V(\mu)$ is a finite dimensional space, where $\mathfrak{h}$ acts diagonalizably and $e_i V(\mu) \subset V(\mu + \alpha_i), f_i V(\mu) = V(\mu - \alpha_i)$. Note that V decomposes into the sum of weight spaces for $\mathfrak{h}$ but this decomposition does not need to agree with $V = \bigoplus_{\mu \in -Q^+} V(\mu)$ when A is degenerate. If $V(0)$ is a single weight space, then the weight of $\mathfrak{h}$ in it, an element of $\mathfrak{h}^*$, is called the highest weight of V.

We say that a weight $\mathfrak{g}$-module V is *integrable* if e_i, f_i act on V locally nilpotently (so that V integrates to the corresponding infinite dimensional group).

Theorem 2.11. *Any integrable highest weight $\mathfrak{g}$-module is completely reducible. The irreducibles are classified by dominant weights (via taking the highest weight).*

3. Reductive algebraic groups

3.1. **Classification.** An algebraic group is called *unipotent* if it is represented by unipotent operators in any rational representation (equivalently, in some faithful rational representation). A typical example is the group of uni-triangular matrices.

We say that an algebraic group is reductive if it does not have normal unipotent subgroups. For example, an *algebraic tori* (or just tori) $(\mathbb{C}^\times)^n$ are reductive. Those are the only connected commutative reductive groups.

We say that an algebraic group is semisimple if its Lie algebra is semisimple. The following theorems classifies (connected) semisimple and reductive algebraic groups.

Theorem 3.1. *The following is true.*

(1) *For any semisimple Lie algebra $\mathfrak{g}$, there is a unique connected and simply connected algebraic group with this Lie algebra.*
(2) *Let G be a connected semisimple algebraic group. Then there is a simply connected semisimple group $\tilde{G}$ and a finite central subgroup $Z \subset \tilde{G}$ such that $G = \tilde{G}/Z$.*
(3) *A connected algebraic group G is reductive if and only if there is a semisimple algebraic group G', a torus T and a finite central subgroup $Z \subset G' \times T$ such that $G = (T \times G')/Z$. For T we can take the connected component of 1 in the center $Z(G)$. The Lie algebra $\mathfrak{g}'$ coincides with $[\mathfrak{g}, \mathfrak{g}]$.*

So basically, to understand connected reductive algebraic groups, we need to compute the centers of simply connected semisimple groups (that are necessarily finite).

Example 3.2. The groups $\mathrm{SL}_n(\mathbb{C})$ and $\mathrm{Sp}_{2n}(\mathbb{C})$ are simply connected. The centers consist of scalar matrices and so have order n for $\mathrm{SL}_n(\mathbb{C})$ and 2 for $\mathrm{Sp}_{2n}(\mathbb{C})$. The group $\mathrm{SO}_n(\mathbb{C})$ is not simply connected, it has a $2:1$ cover, called the *spinor group* $\mathrm{Spin}_n(\mathbb{C})$.

3.2. **Structure theory.** Let us discuss important structural features of a (connected) reductive algebraic group G. By a *Borel subgroup*, one means a maximal (with respect to inclusion) connected solvable subgroup B. An important result is that all such subgroups are G-conjugate. Moreover, B coincides with its normalizer. The Lie algebra of B, a subalgebra in $\mathfrak{g}$, is called a Borel subalgebra. For example, $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n} \subset \mathfrak{g}$ is a Borel subalgebra. The homogeneous space G/B is a smooth projective variety known as the flag variety for G. This terminology is motivated by the following example.

Example 3.3. Let $G = \mathrm{GL}_n(\mathbb{C})$. One can show that every connected solvable subgroup fixes a complete flag of subspaces in $\mathbb{C}^n$. So the Borel subgroups are precisely the subgroups of upper-triangular matrices with respect to some basis. And G/B is the flag variety.

Another important class of subgroups in G are maximal tori (with respect to inclusion). Note that any torus in G is contained in a Borel subgroup.

Theorem 3.4. *The following is true:*

(1) *Any two maximal tori in a Borel subgroup B are conjugate. Hence any two maximal tori in G are conjugate.*
(2) *If G is semisimple, then the Lie algebra of a maximal torus is a Cartan subalgebra.*

(3) *Let T be a maximal torus and $\mathfrak{h} \subset \mathfrak{g}$ be its Lie subalgebra. Note that since T is abelian its adjoint action on $\mathfrak{h}$ is trivial. So $N_G(T)/T$ acts on $\mathfrak{h}$. This action is faithful and the image coincides with W.*
(4) *T contains the center of G.*

Example 3.5. Consider $G = \mathrm{SL}_n(\mathbb{C})$. Any torus acts by diagonal matrices in some basis. So any maximal torus consists of all diagonal matrices in some basis. The normalizer $N_G(T)$ is the subgroup of all monomial matrices, i.e., non-degenerate matrices that have exactly one nonzero entry in every row. We see that $N_G(T)/T \cong S_n$. The action of $N_G(T)/T$ on $\mathfrak{h}$ is by permuting the entries.

LECTURE 7: CATEGORY $\mathcal{O}$ AND REPRESENTATIONS OF ALGEBRAIC GROUPS

IVAN LOSEV

Introduction

We continue our study of the representation theory of a finite dimensional semisimple Lie algebra $\mathfrak{g}$ by introducing and studying the category $\mathcal{O}$ of $\mathfrak{g}$-modules that has appeared in the seminal paper by Bernstein, Israel and Sergei Gelfand, [BGG]. We establish a block decomposition for this category and use this to prove the Weyl character formula for finite dimensional irreducible $\mathfrak{g}$-modules.

Then we proceed to studying the representations of reductive algebraic groups both in zero and positive characteristic. Our main result is the classification of irreducible rational representations.

1. Category $\mathcal{O}$

1.1. **Definition.** By definition, the category $\mathcal{O}$ consists of all finitely generated $U(\mathfrak{g})$-modules M such that $\mathfrak{h}$ acts on M diagonalizably and $\mathfrak{n}$ acts locally nilpotently, meaning that for each $v \in M$ there is $k \in \mathbb{Z}_{\geqslant 0}$ such that $e_{\gamma_1} \dots e_{\gamma_\ell} v = 0$ for any $\ell \geqslant k$ and any positive roots $\gamma_1, \dots, \gamma_\ell$.

Lemma 1.1. $\Delta(\lambda) \in \mathcal{O}$.

Proof. $\Delta(\lambda)$ is generated by a single vector, v_λ. We have the weight decomposition $\Delta(\lambda) = \bigoplus_{\nu \leqslant \lambda} \Delta(\lambda)_\nu$ so $\mathfrak{h}$ acts on $\Delta(\lambda)$ diagonalizably. Also $e_{\gamma_1} \dots e_{\gamma_k} \Delta(\lambda)_\nu = 0$ provided $\sum_i \gamma_i > \lambda - \nu$ (where the order $\leqslant$ is defined by $\nu' \leqslant \nu$ if $\nu - \nu'$ is the sum of positive roots). $\square$

Lemma 1.2. *The irreducible objects in $\mathcal{O}$ are precisely the irreducible quotients $L(\lambda)$ of $\Delta(\lambda), \lambda \in \mathfrak{h}^*$.*

Proof. Any object $M \in \mathcal{O}$ has a vector v annihilated by $\mathfrak{n}$. Indeed, take any $u \in M$ and let $v = e_{\gamma_1} \dots e_{\gamma_k} u \neq 0$ with maximal possible k. It follows that there is a nonzero homomorphism $\Delta(\lambda) \to M$ for some $\lambda \in \mathfrak{h}^*$. Completing the proof is now an exercise. $\square$

To get more examples of objects in $\mathcal{O}$, we note that if $M \in \mathcal{O}$ and V is a finite dimensional $\mathfrak{g}$-module, then $V \otimes M \in \mathcal{O}$.

1.2. **Structure of the center.** In order to proceed further with our study of the category $\mathcal{O}$, we need to understand the structure of the center of $U(\mathfrak{g})$, let us denote this center by Z. It was described by Harish-Chandra. Set $\rho := \sum_{\alpha > 0} \alpha/2$, where the sum is taken over all positive roots.

Theorem 1.3. *We have an isomorphism $z \mapsto f_z : Z \xrightarrow{\sim} \mathbb{C}[\mathfrak{h}^*]^W$ (the algebra of W-invariant polynomials on $\mathfrak{h}^*$) with the property that z acts on $\Delta(\lambda)$ by $f_z(\lambda + \rho)$.*

We will sketch the proof below after providing an example.

Example 1.4. Let $\mathfrak{g} = \mathfrak{sl}_2$. Here $\mathfrak{h}^* = \mathbb{C}$ (with $\epsilon_1 = 1$). So the only positive root α equals 2 and ρ is 1. One can show that the Casimir element $C = 2fe + \frac{1}{2}h^2 + h$ generates the center of $U(\mathfrak{g})$. It acts on $\Delta(z)$ by $\frac{1}{2}z^2 + z = \frac{1}{2}((z+1)^2 - 1)$. The subalgebra $\mathbb{C}[\mathfrak{h}^*]^W \subset \mathbb{C}[\mathfrak{h}^*]$ is $\mathbb{C}[x^2] \subset \mathbb{C}[x]$. We can take $f_C(x) = \frac{1}{2}(x^2+1)$. This actually proves the Harish-Chandra theorem for $\mathfrak{g} = \mathfrak{sl}_2$.

Sketch of proof of Theorem 1.3. Step 1. Let us construct a homomorphism $Z \hookrightarrow S(\mathfrak{h}) = \mathbb{C}[\mathfrak{h}^*]$. Note that $Z \subset U(\mathfrak{g})_0$, the zero weight space for the adjoint action of $\mathfrak{h}$ on $U(\mathfrak{g})$. Note that $U(\mathfrak{g})\mathfrak{n} \cap U(\mathfrak{g})_0$ is a two-sided ideal in $U(\mathfrak{g})_0$ with $U(\mathfrak{g})_0/(U(\mathfrak{g})_0 \cap U(\mathfrak{g})\mathfrak{n}) = U(\mathfrak{h})(= \mathbb{C}[\mathfrak{h}^*])$. The homomorphism we need is $Z \hookrightarrow U(\mathfrak{g})_0 \twoheadrightarrow \mathbb{C}[\mathfrak{h}^*]$. Denote the image of z by $\tilde{f}_z$. By the construction, z acts on $\Delta(\lambda)$ by $\tilde{f}_z(\lambda)$.

Step 2. We need to show that $\tilde{f}_z(\lambda + \rho)$ is W-invariant. This is a consequence of the following fact about Verma modules: suppose that $\lambda \in P$ and α_i be such that $m := \lambda(\alpha_i^\vee) \geqslant 0$. Then $e_j f_i^{m+1} v_\lambda = 0$ for any j (for $j \neq i$ we use $[e_j, f_i] = 0, e_j v_\lambda = 0$ and for $i = j$ we use the representation theory of $\mathfrak{sl}_2$). We get a nonzero homomorphism $\Delta(\lambda - (m+1)\alpha_i) \to \Delta(\lambda)$. It follows that $\tilde{f}_z(\lambda) = \tilde{f}_z(\lambda - (m+1)\alpha_i)$. Note that $\lambda - (m+1)\alpha_i = s_i(\lambda + \rho) - \rho$. Therefore $\tilde{f}_z(\lambda) = \tilde{f}_z(w(\lambda+\rho) - \rho)$. The claim in the beginning of the step follows. We set $f_z(\lambda) := \tilde{f}_z(\lambda - \rho)$.

Step 3. Now let us show that $z \mapsto f_z$ is injective. The algebra $U(\mathfrak{g})$ is filtered, $U(\mathfrak{g}) = \bigcup_{m \geqslant 0} U(\mathfrak{g})^{\leqslant m}$, where $U(\mathfrak{g})^{\leqslant m}$ has basis $x_1^{d_1} \ldots x_n^{d_n}$ with $d_1 + \ldots + d_n \leqslant m$. This is an algebra filtration meaning that $U(\mathfrak{g})^{\leqslant m} U(\mathfrak{g})^{\leqslant m'} \subset U(\mathfrak{g})^{\leqslant m+m'}$. So we can consider the associated graded algebra $\operatorname{gr} U(\mathfrak{g}) := \bigoplus_{m \geqslant 0} U(\mathfrak{g})^{\leqslant m}/U(\mathfrak{g})^{\leqslant m-1}$, where the multiplication is given by $(a + U(\mathfrak{g})^{\leqslant m-1})(b + U(\mathfrak{g})^{\leqslant k-1}) := ab + U(\mathfrak{g})^{\leqslant k+m-1}$. Recall that $(x_1^{d_1} \ldots x_n^{d_n})(x_1^{e_1} \ldots x_n^{e_n})$ equals $x_1^{d_1+e_1} \ldots x_n^{d_n+e_n}$ plus lower degree terms. In other words, $\operatorname{gr} U(\mathfrak{g}) = S(\mathfrak{g})$. We note that Z coincides with $U(\mathfrak{g})^G$. Since the G-module $U(\mathfrak{g})$ is completely reducible, we see that $\operatorname{gr} U(\mathfrak{g})^G = S(\mathfrak{g})^G$. The right hand side is $\mathbb{C}[\mathfrak{g}^*]^G \cong \mathbb{C}[\mathfrak{g}]^G$ (here we use the identification of $\mathfrak{g}$ and $\mathfrak{g}^*$ induced by $(\cdot,\cdot)$) and we have the restriction (from $\mathfrak{g}$ to $\mathfrak{h}$) homomorphism $\mathbb{C}[\mathfrak{g}]^G \to \mathbb{C}[\mathfrak{h}]$. This homomorphism is injective because $G\mathfrak{h}$ is Zariski dense in $\mathfrak{g}$ (see Proposition 1.3 in Lecture 5). On the other hand, the associated graded of the homomorphism $Z \to \mathbb{C}[\mathfrak{h}^*] = \mathbb{C}[\mathfrak{h}]$ coincides with the restriction homomorphism $\mathbb{C}[\mathfrak{g}]^G \to \mathbb{C}[\mathfrak{h}]$. So the homomorphism $z \mapsto f_z : Z \to \mathbb{C}[\mathfrak{h}^*]$ is injective.

Step 4. Note that Step 3 implies that the image of the restriction homomorphism $\mathbb{C}[\mathfrak{g}]^G \to \mathbb{C}[\mathfrak{h}]$ lies in $\mathbb{C}[\mathfrak{h}]^W$. This also can be checked directly using (3) of Theorem 3.4 in Lecture 6.

To show that $Z \hookrightarrow \mathbb{C}[\mathfrak{h}^*]^W$ is surjective, we need to check that $\mathbb{C}[\mathfrak{g}]^G \to \mathbb{C}[\mathfrak{h}]^W$ is surjective. We will do this in the case when $\mathfrak{g} = \mathfrak{sl}_n$, where this is very explicit. In the general case one needs to use some further structure theory of $\mathfrak{g}$ and some Algebraic geometry.

For $\mathfrak{g} = \mathfrak{sl}_n$, the algebra $\mathbb{C}[\mathfrak{h}]^{S_n}$ is generated by the power symmetric functions $\sum_{i=1}^n \epsilon_i^k$, where $k = 2, \ldots, n$. This function is the restriction of $\operatorname{tr}(x^k) \in \mathbb{C}[\mathfrak{g}]^G$. This shows that the restriction homomorphism $\mathbb{C}[\mathfrak{g}]^G \to \mathbb{C}[\mathfrak{h}]^W$ is surjective and completes the proof. $\square$

Corollary 1.5. *An element $z \in Z$ acts on $L(\lambda)$ by $f_z(\lambda + \rho)$.*

1.3. **Infinitesimal blocks.** First of all, let us prove the following lemma.

Lemma 1.6. *Any object of $\mathcal{O}$ has finite length (i.e., has finite composition series).*

Proof. First of all, let us check that $\Delta(\lambda)$ has finite length. Set $w \cdot \lambda := w(\lambda + \rho) - \rho$. If $L(\mu)$ is a composition factor of $\Delta(\lambda)$, then $f_z(\mu + \rho) = f_z(\lambda + \rho)$ for any $z \in Z$. By Theorem

1.3, $\mu = w \cdot \lambda$. So only finitely many different simples can occur in the composition series of $\Delta(\lambda)$. The multiplicity of $L(\mu)$ is bounded by $\dim \Delta(\lambda)_\mu$. So $\Delta(\lambda)$ indeed has finite length.

Now we claim that every module in $\mathcal{O}$ has a finite filtration whose successive quotients are quotients of Verma modules. Indeed, by above, M has a sub of required form. The algebra $U(\mathfrak{g})$ is Noetherian. This is because $S(\mathfrak{g})$ is Noetherian, $U(\mathfrak{g})$ is $\mathbb{Z}_{\geqslant 0}$-filtered, and $\operatorname{gr} U(\mathfrak{g}) = S(\mathfrak{g})$. This establishes the claim in the beginning of the paragraph and completes the proof of the lemma. $\square$

For $\lambda \in \mathfrak{h}^*$, define $\mathcal{O}_\lambda$ as a full subcategory in $\mathcal{O}$ consisting of all modules M such that every $z \in Z$ acts on M with generalized eigenvalue $f_z(\lambda + \rho)$. Note that $\mathcal{O}_\lambda = \mathcal{O}_{w\cdot\lambda}$, by definition and $L(w \cdot \lambda), w \in W$, are precisely the irreducible objects in $\mathcal{O}_\lambda$. In particular, the number of simples equals $|W/W_{\lambda+\rho}|$, where $W_{\lambda+\rho}$ is the stabilizer of $\lambda + \rho$ in W.

Proposition 1.7. *Any $M \in \mathcal{O}$ splits as $\bigoplus_{\lambda \in \mathfrak{h}^*/W} M^\lambda$, where $M^\lambda \in \mathcal{O}_\lambda$.*

Proof. This is the decomposition into the generalized eigenspaces for Z (that exists because M has finite length). $\square$

The point of the previous proposition is that the study of $\mathcal{O}$ reduces to that of $\mathcal{O}_\lambda$'s. These subcategories are called the *infinitesimal blocks*.

1.4. **Characters.** Let $M \in \mathcal{O}$. All weight spaces in the simples $L(\lambda)$ are finite dimensional (this is true even for $\Delta(\lambda)$). Since M has finite length, we have $\dim M_\nu < \infty$ for all ν. So we can consider the formal character $\mathsf{ch} M = \sum_{\nu \in \mathfrak{h}^*} \dim M_\nu e^\nu$, where e^ν is ν viewed as an element of the group algebra of $\mathfrak{h}^*$. The sum $\mathsf{ch} M$ is finite if and only if M is finite dimensional.

Example 1.8. Let us compute $\mathsf{ch}\Delta(\lambda)$. As we have seen in the proof of (1) of Lemma 2.3 in Lecture 6, we have a basis $\prod_{\alpha>0} f_\alpha^{m_\alpha} v_\lambda, m_\alpha \in \mathbb{Z}_{\geqslant 0}$. The latter element has weight $\lambda - \sum_{\alpha>0} m_\alpha \alpha$. It follows that

$$\mathsf{ch}\Delta(\lambda) = e^\lambda \prod_{\alpha>0} \sum_{i=0}^{\infty} e^{-i\alpha} = e^\lambda \prod_{\alpha>0} (1 - e^{-\alpha})^{-1}.$$

Using this example and Section 1.3, we are going to compute the characters of the finite dimensional irreducible modules $L(\lambda)$ (the Weyl character formula). For this we need a notation. Set $F(\lambda) := \sum_{w \in W} \det(w) e^{w\lambda}$ (here we take the determinant of w in $\mathfrak{h}$).

Theorem 1.9. *For $\lambda \in P^+$, we have $\mathsf{ch} L(\lambda) = F(\lambda + \rho)/F(\rho)$.*

Sketch of proof. First of all, we have the following formula:

$$F(\rho) = e^\rho \prod_{\alpha>0} (1 - e^{-\alpha}). \tag{1.1}$$

On the other hand, recall that each $\Delta(w \cdot \lambda)$ admits an epimorphism onto $L(w \cdot \lambda)$ such that the kernel is filtered with $L(w' \cdot \lambda)$, where $w' \cdot \lambda < w \cdot \lambda$. It follows, in particular, that

$$\mathsf{ch} L(\lambda) = \sum_{w \in W} n_w \mathsf{ch}\Delta(w \cdot \lambda), \tag{1.2}$$

where all $n_w \in \mathbb{Z}$ and $n_1 = 1$. Combining (1.1) with (1.2), we get

$$F(\rho) \mathsf{ch} L(\lambda) = \sum_{w \in W} n_w e^{w(\lambda+\rho)}. \tag{1.3}$$

Also note that $\mathsf{ch}L(\lambda)$ is W-invariant because W acts on the set of weights of $L(\lambda)$ preserving the dimensions of weight spaces. So $w(F(\rho)\mathsf{ch}L(\lambda)) = \det(w)F(\rho)\mathsf{ch}L(\lambda)$. So $n_w = \det(w)n_1 = \det(w)$. □

One can ask how to compute $L(\lambda)$ for general $\lambda \in \mathfrak{h}^*$. This computation is, basically, in three steps.

(1) To do the case when λ in integral and $\lambda + \rho$ is regular (meaning that the stabilizer of $\lambda + \rho$ in W is trivial). This is a very nontrivial problem. The answer is expressed in terms of so called Kazhdan-Lusztig bases in the Hecke algebra of W to be covered later in this class. Finding this answer (Kazhdan-Lusztig) and proving it (Brylinski-Kashiwara and Beilinson-Bernstein) is one of the most significant achievements of Representation theory of the second half of the 20th century.
(2) The case when λ is integral but $\lambda + \rho$ is not regular is reduced to the previous case by means of so called translation functors. This is relatively easy.
(3) The case when λ is not integral is reduced to the integral case for a smaller Weyl group. This reduction, due to Soergel, is not so easy but is not nearly as hard as Step 1.

2. Representation theory of reductive groups

2.1. **Structure theory in arbitrary characteristic.** Let G be a connected (i.e., irreducible as an algebraic variety) reductive algebraic group over an algebraically closed field $\mathbb{F}$, $T \subset B \subset G$ be a maximal torus and a Borel subgroup. The subgroups T and B were introduced in Lecture 6 in the case of characteristic 0 field but the results there hold in arbitrary characteristic (when we do not refer to Lie algebras). We can speak about simple (resp., semisimple) groups as well: these are the connected reductive groups without any (resp., solvable) connected normal subgroups.

For an algebraic group H, let $X(H)$ denote the set $\mathrm{Hom}(H, \mathbb{F}^\times)$ of group homomorphisms (a.k.a. characters). The set $X(H)$ carries a natural group structure.

Lemma 2.1. *Let $T \cong (\mathbb{F}^\times)^n$. Then $X(T) = \mathbb{Z}^n$ and all representations of T are completely reducible.*

Proof. This is a direct generalization (together with a proof) of the corresonding claims for $\mathbb{F}^\times$, see Example 1.2 in Lecture 3, Lemma 2.2 in Lecture 4. □

We can develop the theory of roots in arbitrary characteristic using the adjoint T-action on $\mathfrak{g}$. We have the Weyl group $W = N_G(T)/T$ acting on $\mathbb{R} \otimes_{\mathbb{Z}} X(T)$ and it behaves as in characteristic 0. For each root α, we have the corresponding group homomorphism $\mathrm{SL}_2 \to G$.

Example 2.2. For $G = \mathrm{SL}_n(\mathbb{F})$, $\alpha = \epsilon_i - \epsilon_j$, we consider $\mathrm{SL}_2(\mathbb{F}) \subset \mathrm{SL}_n(\mathbb{F})$ "located" in the rows and columns i, j.

We can decompose B into the semidirect product $T \ltimes U$, where U is the maximal normal unipotent subgroup. In the examples we consider ($G = \mathrm{SL}_n(\mathbb{F}), \mathrm{Sp}_{2n}(\mathbb{F}), \mathrm{SO}_n(\mathbb{F})$ – for the latter two groups we assume $\mathrm{char}\,\mathbb{F} > 2$ – for U we take the subgroup of all unitriangular matrices in G). The group U is generated by the images of $\{\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}\} \subset \mathrm{SL}_2(\mathbb{F})$ under all homomorphisms associated to the positive roots (one can restrict to the simple roots). This is easy to see for $\mathrm{SL}_n(\mathbb{F})$ using Example 2.2 and can be checked by hand for other classical groups. The decomposition $B = T \ltimes U$ yields an identification $X(B) \cong X(T)$ because $X(U) = \{1\}$.

Using this and emulating the classification theorem for simple Lie algebras in characteristic 0 one can prove the following result.

Theorem 2.3. *The following is true.*

(1) *The simple simply connected algebraic groups are in bijection with the Dynkin diagrams A_n-G_2. Any semisimple simply connected group is the direct product of simple ones.*
(2) *Any semisimple algebraic group is the quotient of a simple one by a finite central subgroup.*

Recall that we say that G is semisimple if there are no etale covers $\tilde{G} \to G$, where $\tilde{G}$ is an algebraic group.

2.2. **Representation theory of algebraic groups.** Here G is connected and reductive. Let V be a rational representation of G. We have the weight decomposition $V = \bigoplus_{\nu \in X(T)} V_\nu$. We say that $\lambda \in X(T)$ is dominant if $\lambda(\alpha^\vee) \in \mathbb{Z}_{\geqslant 0}$ for any positive root α. The definition of a highest weight of an irreducible G-module V is given in the same way as for Lie algebras in characteristic 0.

Theorem 2.4. *The irreducible rational representations of a connected reductive group G are in bijection with the dominant elements in $X(T)$ (an irreducible representation corresponds to its unique highest weight). In characteristic 0, any rational representation of G is completely reducible.*

In characteristic 0, this can be deduced from the corresponding results about semisimple Lie algebras combined with Lemma 2.1. Below we will explain what to do in characteristic p.

Similarly to Problem 3 in Homework 2, we prove the following result.

Lemma 2.5. *Let λ be a highest weight of an irreducible G-module V and $v \in V_\lambda$. Then $bv_\lambda = \lambda(b)v_\lambda$ for any $b \in B$.*

The next step in the proof of Theorem 2.4 is to produce the *Weyl module* $W(\lambda)$ that has the universal property

$$\mathrm{Hom}_G(W(\lambda), V) = \{v \in V | bv = \lambda(b)v, \forall v \in V\}.$$

We start by producing the dual Weyl module $W^\vee(\lambda)$ with lowest weight λ^{*-1}, where we write λ^* for the highest weight of $L(\lambda)^*$ (in characteristic 0). This is done in the same way as in the SL_2-case: we take the line bundle $\mathcal{O}(\lambda^*)$ on G/B that is the homogeneous vector bundle with fiber $\mathbb{F}_{\lambda^{*-1}}$ over $eB \in G/B$. Then we set $W^\vee(\lambda) := \Gamma(\mathcal{O}(\lambda^*)) = \{f \in \mathbb{F}[G] | f(gb) = \lambda^*(b)f(g)\}$. Then we set $W(\lambda) = W^\vee(\lambda^*)^*$. Similarly to the SL_2-case, this module has the required universal property.

Now we need to establish the following two facts: $W(\lambda)_\lambda = \mathbb{F}$ and if $W(\lambda)_\nu \neq \{0\}$ implies $\nu \leqslant \lambda$. This will imply that there is a unique simple quotient $L(\lambda)$ of $W(\lambda)$ and complete the proof of Theorem 2.4. Both claims above follow from the next theorem.

Theorem 2.6. *We have $\mathsf{ch}W(\lambda) = F(\lambda + \rho)/F(\rho)$. In other words, the character is independent of the characteristic.*

Sketch of proof. Note that if M is a finitely generated $\mathbb{Z}$-module, then $\dim \mathbb{Q}\otimes_{\mathbb{Z}} M = \chi(\mathbb{F}_p \otimes^L_{\mathbb{Z}} M)$, where χ is the Euler characteristic. With some standard manipulations along these lines, we conclude that $\bigoplus_{i=0}^{\infty} \mathsf{ch} H^i(G/B, \mathcal{O}(\lambda^*))$ is independent of the characteristic. But $H^i(G/B, \mathcal{O}(\lambda^*)) = 0$ for $i > 0$, this is the Borel-Weil-Bott theorem (in characteristic p it was proved by Kempf). $\square$

2.3. **Characters of simples.** First, let us explain the general form of the Steinberg decomposition. We say that a dominant weight λ is restricted if $\langle \lambda, \alpha_i^\vee \rangle < p$ for all i. So for any dominant weight λ there is a unique p-adic expansion $\lambda = \lambda_0 + p\lambda_1 + \ldots + p^\ell \lambda_\ell$, where all λ_i are restricted. The following theorem (due to Steinberg) reduces the computation of $L(\lambda)$ to that of $L(\lambda_i)$'s.

Theorem 2.7. *Let λ_0 be restricted and $\lambda = \lambda_0 + p\mu$. Then $L(\lambda) \cong L(\lambda_0) \otimes \mathrm{Fr}^* L(\mu)$.*

The question of computation of the characters of $L(\lambda)$ with restricted λ's is wide open. The answer is known (and complicated) when p is very large, [AJS] (there are actual bounds, but they are huge, see [F]). For quite a long time, there was a conjecture on the multiplicities of $L(\lambda)$'s in $W(\mu)$'s when $p \geqslant h$, where h is the so called Coxeter number (it is equal to n for $\mathrm{SL}_n(\mathbb{F})$). Recently, this conjecture was disproved by Williamson, [W]. The problem of computing the multiplicities is wide open even on the level of conjectures.

References

[AJS] H. H. Andersen, J. C. Jantzen, and W. Soergel. *Representations of quantum groups at a pth root of unity and of semisimple groups in characteristic p: independence of p.* Asterisque, (220):321, 1994.

[BGG] J. Bernstein, I. Gelfand, S. Gelfand, *A certain category of $\mathfrak{g}$-modules.* (Russian) Funkcional. Anal. i Prilozhen. 10 (1976), no. 2, 18.

[F] P. Fiebig. *An upper bound on the exceptional characteristics for Lusztig's character formula.* J. Reine Angew. Math. 673 (2012), 1-31. arXiv:0811.1674.

[W] G. Williamson *Schubert calculus and torsion explosion.* arXiv:1309.5055.

LECTURE 8: REPRESENTATIONS OF $\mathfrak{g}_{\mathbb{F}}$ AND OF $\mathrm{GL}_n(\mathbb{F}_q)$

IVAN LOSEV

Introduction

We start by briefly explaining key results on the representation theory of semisimple Lie algebras in large enough positive characteristic.

After that, we start a new topic: the complex representation theory of finite groups of Lie type and Hecke algebras. Today we consider the most basic group: $G := \mathrm{GL}_n(\mathbb{F}_q)$. We introduce the Hecke algebra as the endomorphism algebra of the G-module $\mathbb{C}[B \setminus G]$. We describe the basis in this algebra and some of the multiplication rules that allow to present the Hecke algebra by generators and relations. Then we use the Tits deformation principle to show that the Hecke algebra is isomorphic to $\mathbb{C}S_n$.

1. Representations of semisimple Lie algebras in positive characteristic

Let $G_{\mathbb{F}}$ be a simple algebraic group over an algebraically closed field $\mathbb{F}$ of positive characteristic and $\mathfrak{g}_{\mathbb{F}}$ be its Lie algebra. In this section we assume that the characteristic of $\mathbb{F}$ is large enough. This will guarantee that $(\cdot,\cdot)$ is non-degenerate, that $\mathfrak{g}_{\mathbb{F}}$ is simple (by root considerations), and several more subtle things. In a sentence, the structure theory of $\mathfrak{g}_{\mathbb{F}}$ will be the same as of $\mathfrak{g}$, while the representation theory will be crucially different.

1.1. **Case of nilpotent p-character.** Recall that any irreducible $\mathfrak{g}_{\mathbb{F}}$-module has the so called p-character, an element of $\mathfrak{g}_{\mathbb{F}}$ to be denoted by α. In this section, we assume that α is nilpotent. As we will see below, the general case can be reduced to this one. Since $p \gg 0$, the nilpotent $G_{\mathbb{F}}$-orbits in $\mathfrak{g}_{\mathbb{F}}$ are in a natural bijection with the nilpotent orbits of $G(=G_{\mathbb{C}})$ in $\mathfrak{g}$ (this should be clear when $\mathfrak{g} = \mathfrak{sl}_n$, and is easy to show when $\mathfrak{g} = \mathfrak{so}_n$ or $\mathfrak{sp}_{2n}$). So we can view α also as an element of $\mathfrak{g}$ (defined up to G-conjugacy).

Consider an irreducible $\mathfrak{g}$-module M with p-character α. We still have the so called *Harish-Chandra* center $U(\mathfrak{g}_{\mathbb{F}})^{G_{\mathbb{F}}}$. As in characteristic 0, $U(\mathfrak{g}_{\mathbb{F}})^{G_{\mathbb{F}}} = \mathbb{F}[\mathfrak{h}^*_{\mathbb{F}}]^W$. For $\lambda \in \mathfrak{h}^*_{\mathbb{F}}$ let $U_{\alpha,\lambda}(\mathfrak{g}_{\mathbb{F}})$ denote the corresponding quotient of $U_\alpha(\mathfrak{g}_{\mathbb{F}})$. One can show that if $U_{\alpha,\lambda}(\mathfrak{g}_{\mathbb{F}}) \neq \{0\}$, then $\lambda \in \mathfrak{h}^*_{\mathbb{F}_p}$.

Below we will consider the case when the stabilizer of $\lambda+\rho$ in W is trivial (the regular case). One reduces the general case to this one using translation functors. Consider the variety $\mathcal{B}$ of all Borel subalgebras in $\mathfrak{g}$. This is nothing else but the flag variety G/B. Inside, we have the closed (generally, singular) subvariety $\mathcal{B}_\alpha$ of all subalgebras containing α.

Example 1.1. Let $\mathfrak{g} = \mathfrak{sl}_n$. Then $\mathcal{B}$ is the variety Fl of full flags, and $\mathcal{B}_\alpha$ consists of all flags $\{0\} \subsetneq V_1 \subsetneq V_2 \subsetneq \ldots \subsetneq V_n$ such that $\alpha V_i \subset V_{i-1}$ for all i (recall that α is nilpotent and so if α preserves V_i, V_{i-1}, then it maps V_i to V_{i-1}).

Theorem 1.2. *Let α be nilpotent, and λ be regular. Then* $|\mathrm{Irr}(U_{\alpha,\lambda}(\mathfrak{g}))| = \dim H_*(\mathcal{B}_e)$.

Example 1.3. Let $\mathfrak{g} = \mathfrak{sl}_2$. If $\alpha = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, then $\mathcal{B}_e$ consists of one point, $\{0\} \subsetneq \operatorname{im}\alpha \subsetneq \mathbb{C}^2$. We have $(p+1)/2$ points in $\mathfrak{h}^*_{\mathbb{F}_p}/\{\pm 1\}$, and $(p+1)/2$ irreducible $U_\alpha(\mathfrak{g})$-modules. Taking the scalar of action of C, we get a bijection between these two sets, as (partially) predicted by Theorem 1.2.

If $\alpha = \{0\}$, then $\mathcal{B}_e = \mathcal{B}$ and the homology has dimension 2. We have p irreducible $U_0(\mathfrak{g})$-modules, $L_0(z), z = 0, \ldots, p-1$. The eigenvalue of C on $L_0(z)$ is $\frac{1}{2}((z+1)^2 - 1)$. Regular λ corresponds to $z \neq -1$. We see that $\mathrm{Irr}(U_{0,\lambda}(\mathfrak{g})) = \{L(\lambda), L(-2-\lambda)\}$.

Example 1.4. Let $\mathfrak{g} = \mathfrak{sl}_n$. Let $n_1, \ldots, n_k$ be the sizes of Jordan blocks in the Jordan decomposition of α. One can show that

$$\dim H_*(\mathcal{B}_e) = \frac{n!}{n_1! \ldots n_k!}.$$

In this case, the number of finite dimensional irreducible modules can be computed by more elementary techniques than those of [BMR].

Let us proceed to explaining what is known about the dimensions of the simple $U_\alpha(\mathfrak{g})$-modules. They are known in principle, [BM], but the answer is involved and quite unexplicit. There is a nice general fact proved by Premet, [P].

Theorem 1.5. *We have $U_\alpha(\mathfrak{g}_{\mathbb{F}}) \cong \mathrm{Mat}_{p^d}(\mathcal{W}_{\alpha,\mathbb{F}})$, where $\mathcal{W}_{\alpha,\mathbb{F}}$ is some associative algebra and $d = \frac{1}{2}\dim \mathfrak{g}\cdot\alpha$. In particular, the dimension of any $U_\alpha(\mathfrak{g}_{\mathbb{F}})$-module is divisible by p^d.*

1.2. **Reduction to a nilpotent p-character.** Let $\alpha \in \mathfrak{g}_{\mathbb{F}}$. We can decompose α into the sum $\alpha_s + \alpha_n$ of commuting diagonalizable and nilpotent elements (Jordan decomposition). Let $\mathfrak{g}_{0,\mathbb{F}}$ stand for the centralizer of α_s in $\mathfrak{g}_{\mathbb{F}}$ (a so called Levi subalgebra), when $\mathfrak{g} = \mathfrak{sl}_n$, then $\mathfrak{g}_0$ is conjugate a subalgebra of block-diagonal matrices.

Then we have the following result.

Proposition 1.6. *$U_\alpha(\mathfrak{g}_{\mathbb{F}}) \cong \mathrm{Mat}_{p^k}(U_\alpha(\mathfrak{g}_{0\mathbb{F}}))$, where $k = \frac{1}{2}\dim\mathfrak{g}\cdot\alpha_s$. In particular, there is a natural bijection $\mathrm{Irr}(U_\alpha(\mathfrak{g}_{\mathbb{F}})) \cong \mathrm{Irr}(U_\alpha(\mathfrak{g}_{0\mathbb{F}}))$.*

Since α_s is central in $\mathfrak{g}_0$, we have an isomorphism $U_\alpha(\mathfrak{g}_{0\mathbb{F}}) \cong U_{\alpha_n}(\mathfrak{g}_{0\mathbb{F}})$.

2. Representations of $\mathrm{GL}_n(\mathbb{F}_q)$

Let $\mathbb{F}_q$ be a finite field with q elements (so that $q = p^\ell$ for some prime p and positive integer ℓ). We are interested in representations of the finite group $G := \mathrm{GL}_n(\mathbb{F}_q)$ over $\mathbb{C}$. In particular, such representations are completely reducible and we only need to classify the irreducible representations. The number of those is the same as the number of conjugacy classes in G. We will explain the classification of conjugacy classes later. In this lecture we will produce the irreducible representations that correspond to unipotent conjugacy classes. Recall that the classification of unipotent matrices up to conjugacy does not depend on the field: the Jordan normal form theorem holds for all operators with eigenvalues in the base field. In particular, we see that the unipotent conjugacy classes are in one-to-one correspondence with the partitions of n.

The idea of construction of the corresponding representations comes from the representation theory of reductive groups. Namely, let B be the subgroup of all upper-triangular matrices in G. We are looking at the irreducible representations of G that have a B-fixed vector. We will see that these irreducible representations are classified by the partitions of n. A crucial tool here is the so called Hecke algebra, a deformation of $\mathbb{C}S_n$.

2.1. $\mathbb{C}[B \setminus G]$ **and its endomorphisms.** We are interested in the irreducible G-modules V such that $V^B \neq \{0\}$, equivalently, such that $(V^*)^B = (V^B)^* \neq 0$. Of course, $(V^*)^B = \mathrm{Hom}_B(V, \mathbb{C})$, where we write $\mathbb{C}$ for the trivial B-module. Recall the coinduced module $\mathrm{Hom}_B(\mathbb{C}G, \mathbb{C}) = \mathbb{C}[B \setminus G]$, where we write $B \setminus G$ for the set of left B-cosets in G and G acts on $\mathbb{C}[B \setminus G]$ by $g.f(g') = f(g'g)$. By its universal property, $\mathrm{Hom}_B(V, \mathbb{C}) = \mathrm{Hom}_G(V, \mathbb{C}[B \setminus G])$. So $V^B \neq \{0\}$ if and only if V is a summand of $\mathbb{C}[B \setminus G]$. Recall that the assignment $V \mapsto \mathrm{Hom}_B(V, \mathbb{C}) = \mathrm{Hom}_G(V, \mathbb{C}[B \setminus G]))$ gives rise to a bijection between the set of $V \in \mathrm{Irr}(G)$ that are summands in $\mathbb{C}[B \setminus G]$ and the $\mathrm{Irr}(\mathrm{End}_G(\mathbb{C}[B \setminus G]))$. So we need to understand the structure of the algebra $\mathrm{End}_G(\mathbb{C}[B \setminus G])$.

First of all, we will give an alternative description of this algebra. Consider the space $\mathbb{C}[B \setminus G]^B$ of B-invariant functions on $B \setminus G$ that is naturally identified with the set of $B \times B$-invariant functions on G. One can define the convolution product $C[B \setminus G]^B \otimes \mathbb{C}[B \setminus G] \to \mathbb{C}[B \setminus G]$ as follows:

$$F * f(g) = |B|^{-1} \sum_{h \in G} F(h) f(h^{-1}g).$$

We have $F * f \in \mathbb{C}[B \setminus G]$ because

$$F * f(bg) = |B|^{-1} \sum_{h \in G} F(h) f(h^{-1}bg) = |B|^{-1} \sum_{h \in G} F(b^{-1}h) f(h^{-1}g) = F * f(g).$$

Also note that $F*? : \mathbb{C}[B \setminus G] \to \mathbb{C}[B \setminus G]$ is a G-equivariant homomorphism. It follows that it restricts to a bilinear map $\mathbb{C}[B \setminus G]^B \otimes \mathbb{C}[B \setminus G]^B \to \mathbb{C}[B \setminus G]^B$. It is straightforward to check that $(F' * F) * f = F' * (F * g)$. So we see that $\mathbb{C}[B \setminus G]^B$ is an associative algebra with respect to convolution that acts on $\mathbb{C}[B \setminus G]$ by G-equivariant endomorphisms. In particular, we have an algebra homomorphism $\mathbb{C}[B \setminus G]^B \to \mathrm{End}_G(\mathbb{C}[B \setminus G])$.

Lemma 2.1. *The homomorphism $\mathbb{C}[B \setminus G]^B \to \mathrm{End}_G(\mathbb{C}[B \setminus G])$ is an isomorphism.*

Proof. We have $\mathrm{Hom}_G(\mathbb{C}[B \setminus G], \mathbb{C}[B \setminus G]) = \mathrm{Hom}_B(\mathbb{C}[B \setminus G], \mathbb{C}) = \mathbb{C}[B \setminus G]^B$. So the two algebras have the same dimension. It remains to check that the homomorphism is injective. Applying $\sum_{h \in G} F(h) f(h^{-1}g) = 0$ to the characteristic functions f of B-orbits in G, we see that $\sum_{h \in B} F(hg) = 0$ for any $g \in G$. We conclude that $F(g) = 0$. □

The realization $\mathrm{End}_G(\mathbb{C}[B \setminus G]) = \mathbb{C}[B \setminus G]^B$ is beneficial for several reasons. First of all, we can find a basis in the right hand side. Embed $W := S_n$ into $G = \mathrm{GL}_n(\mathbb{F}_q)$ as the group of monomial matrices with unit nonzero coefficients. The Gauss elimination algorithm proves the following fact known as the *Bruhat* decomposition.

Lemma 2.2. *We have $G = \bigsqcup_{w \in W} BwB$. In particular, we have the basis $T_w, w \in W$, in $\mathbb{C}[B \setminus G]^B$, where T_w is the characteristic function of BwB.*

Now let us study the product $T_u * T_w$. Let $\mu_{u,w} : BuB \times BwB \to G$ be the multiplication map. Note that B acts freely on $BuB \times BwB, b.(x, y) = (xb^{-1}, by)$ and $\mu_{u,w}$ is B-equivariant so that the fibers are unions of B-orbits. Then

$$T_u * T_w(g) = \frac{1}{|B|} |\mu_{u,w}^{-1}(g)|. \tag{2.1}$$

In particular, we see that T_1 is the unit in $\mathbb{C}[B \setminus G]^B$. Now consider the case when $u = s_i$, the simple transposition $(i, i+1)$. Consider the length function $\ell : W \to \mathbb{Z}_{\geqslant 0}$ that to $w \in W$ assigns the minimal number ℓ such that $w = s_{i_1} \ldots s_{i_\ell}$ for some $i_1, \ldots, i_\ell$ (such decompositions are called *reduced*). It equals to the number of inversions in w. Note that $\ell(s_i w) = \ell(w) \pm 1$.

Proposition 2.3. *We have* $T_sT_w = T_{sw}$ *if* $\ell(sw) = \ell(w)+1$ *and* $T_sT_w = qT_{sw} + (q-1)T_w$ *if* $\ell(sw) = \ell(w)-1$ *(where we write* s *for* s_i*).*

Proof. We have $|BwB|/|B| = q^{\ell(w)}$. This can be deduced from the Gauss elimination algorithm or from the equality $|BwB|/|B| = |B|/|B\cap wBw^{-1}|$ (the intersection can be described explicitly).

Consider the case $\ell(sw) = \ell(w)+1$ so that $|BswB|/|B| = (|BsB|/|B|)(|BwB|/|B|)$. Note that $BswB$ lies in the image of $\mu_{s,w}$ and so we get $|\mu_{s,w}^{-1}(g)| = |B|$ if $g \in BswB$ and $\mu_{s,w}^{-1}(g)$ is empty else. We deduce that $T_sT_w = T_{sw}$.

Now let us consider the case when $\ell(sw) = \ell(w)-1$. Let $u = sw$. By the previous case, $T_w = T_s * T_u$. So we just need to prove that $T_s^2 = q + (q-1)T_s$. We have the inclusion $BsB \subset P_i$, where P_i consists of all matrices (a_{jk}) such that $a_{jk} \neq 0$ implies $j \leqslant k$ or $j = i+1, k = i$. Therefore $BsBBsB \subset BsB \sqcup B$ so the only basis elements that can occur with nonzero multiplicities in T_s^2 are $T_s, 1$. The preimage of 1 under $\mu_{s,s}$ is isomorphic to BsB and so the coefficient of 1 equals $|BsB|/|B| = q$. Since $|BsB|^2 = |\mu^{-1}(1)||B| + |\mu^{-1}(s)||BsB| = q|B| + |\mu^{-1}(s)|q|B|$, we deduce that $|\mu^{-1}(s)|/|B| = q-1$. This proves $T_s^2 = q + (q-1)T_s$. □

Below we will write T_i instead of T_{s_i}.

Corollary 2.4. *We have* $T_i^2 = (q-1)T_i + q, T_iT_j = T_iT_j$ *if* $|i-j| > 1$ *and* $T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}$.

An advantage of looking at $\operatorname{End}_G(\mathbb{C}[B\setminus G])$ is that this algebra is manifestly semisimple.

2.2. **Hecke algebra over** $\mathbb{Z}[v^{\pm 1}]$**.** Let v be an independent variable. We define the $\mathbb{Z}[v^{\pm 1}]$-algebra $H_v(n)$ by the generators $T_i, i = 1, \ldots, n-1$ and the relations as in Corollary 2.4, where q is replaced with v. For $w \in W$, we define an element T_w as follows. Choose a reduced expression $w = s_{i_1}\ldots s_{i_\ell}$, where $\ell = \ell(w)$. It is a classical fact that any two reduced expressions of w are obtained from one another by a sequence of *braid moves*: replacing s_is_j with s_js_i when $|i-j| > 1$, and replacing $s_is_{i+1}s_i$ with $s_{i+1}s_is_{i+1}$ and vice versa. Set $T_w = T_{i_1}\ldots T_{i_\ell}$, this is well-defined.

Theorem 2.5. *The algebra* $H_v(n)$ *is a free* $\mathbb{Z}[v^{\pm 1}]$*-module with basis* $T_w, w \in W$.

Proof. We note that $T_iT_w = T_{s_iw}$ if $\ell(s_iw) = \ell(w)+1$, and $T_iT_w = (v-1)T_w + vT_{s_iw}$ if $\ell(s_iw) = \ell(w)-1$. So the span of T_w's is closed under the multiplication by the generators and hence T_w's span $H_v(n)$. In order to show that the elements T_w are linearly independent over $\mathbb{Z}[v^{\pm 1}]$, consider the free $\mathbb{Z}[v^{\pm 1}]$-module U with basis $u_w, w \in W$. Define an action of the generators T_i on U by

$$T_iu_w = \begin{cases} u_{s_iw}, & \ell(s_iw) = \ell(w)+1, \\ (v-1)u_w + vu_{s_iw}, & \ell(s_iw) = \ell(w)-1. \end{cases} \tag{2.2}$$

It is straightforward (but tedious) to check that this extends to an $H_v(n)$-action. Since $T_wu_1 = u_w$, we see that the elements T_w are linearly independent. □

For $z \in \mathbb{C}^\times$, set $H_{\mathbb{C},z}(n) := \mathbb{C}_z \otimes_{\mathbb{Z}[v^{\pm 1}]} H_v(n)$, where the homomorphism $\mathbb{Z}[v^{\pm 1}] \to \mathbb{C}_z$ is given by $v \mapsto z$. We note that $\mathbb{C}[B\setminus G]^B = H_{\mathbb{C},q}(n)$, while $\mathbb{C}S_n = H_{\mathbb{C},1}(n)$.

2.3. **Structure of Hecke algebras.** The following result is known as the Tits deformation principle.

Theorem 2.6. *Let X be a principal open subset in $\mathbb{C}^n$ and A is a free $\mathbb{C}[X]$-algebra of finite rank. For any two points $x, y \in X$, if the specializations A_x, A_y are semisimple, then they are isomorphic.*

Corollary 2.7. *We have an isomorphism $\mathcal{H}_{\mathbb{C},q}(n) \cong \mathcal{H}_{\mathbb{C},1}(n)$.*

Proof. We apply Theorem 2.6 to $X = \mathbb{C}^\times, A = \mathbb{C}[v^{\pm 1}] \otimes_{\mathbb{Z}[v^{\pm 1}]} H_v(n), x = 1, y = q$. □

Proof of Theorem 2.6. The proof is in several steps.

Step 1. Let r be the rank of A. Pick some basis $v_1, \ldots, v_r$ of the $\mathbb{C}[X]$-module A. The coefficient of v_ℓ in $v_i v_j$ is an element of $\mathbb{C}[X]$. So we get a morphism $\varphi : X \to \mathbb{C}^{r*} \otimes \mathbb{C}^{r*} \otimes \mathbb{C}^r$ of algebraic varieties that sends $x \in X$ to the multiplication of A_x (in basis $v_1, \ldots, v_r$).

Step 2. Recall the form $(\cdot, \cdot)$ on the associative algebra A given by $(a, b) = \mathrm{tr}_A(ab)$. Its entries are again functions on $\mathbb{C}[X]$. The locus where this form is non-degenerate is the locus of $x \in X$ such that A_x is semisimple. So we can replace X with a principal open subset and assume that A_x is semisimple for any $x \in X$.

Step 3. The group $\mathrm{GL}_r(\mathbb{C})$ acts on the space of products $\mathbb{C}^{r*} \otimes \mathbb{C}^{r*} \otimes \mathbb{C}^r$ by base changes. There are finitely many orbits of this group corresponding to semisimple associative algebras. We see that the image of φ lies in the union of these orbits.

Step 4. Pick a point x and let $y_1, \ldots, y_n$ be affine coordinates on X centered at x. Set $R := \mathbb{C}[[y_1, \ldots, y_n]]$. Consider the algebra $\hat{A} = R \otimes_{\mathbb{C}[X]} A$. This is an R-algebra that is a free finite rank module over R such that $\hat{A}/\mathfrak{m}\hat{A} = A_x$, where $\mathfrak{m} \subset R$ is the maximal ideal. We want to prove that $\hat{A} \cong R \otimes A_x$ (in other words, A is a trivial bundle of algebras over the formal neighborhood of x in X).

Step 5. We will use the result about lifting of idempotents: if e is an element in A_x such that $e^2 = e$, then there is an element $\hat{e} \in \hat{A}$ that maps to e under the projection $\hat{A} \twoheadrightarrow A_x$ and satisfies $\hat{e}^2 = \hat{e}$.

Pick primitive idempotents (=diagonal matrix unit) $e_1, \ldots, e_k$, one per each direct summand. Lift them to idempotents $\hat{e}_1, \ldots, \hat{e}_k \in \hat{A}$. So $\hat{V}_i := \hat{A}\hat{e}_i$ is free over R and $\hat{V}_i/\mathfrak{m}\hat{V}_i = V_i (= A_x e_i)$. We have an algebra homomorphism $\hat{A} \to \bigoplus_{i=1}^k \mathrm{End}_R(\hat{V}_i)$ given by the action of $\hat{A}$ on $\hat{V}_1 \oplus \ldots \oplus \hat{V}_k$. It specializes to the homomorphism $A_x \to \bigoplus_{i=1}^k \mathrm{End}(V_i)$ given by the action of A_x on $V_1 \oplus \ldots \oplus V_k$. But the latter homomorphism is an isomorphism. Now we are done by a standard fact: let $\psi : M \to N$ be a homomorphism of free finite rank R-modules that is an isomorphism after specializing to the residue field. Then ψ is an isomorphism.

Step 6. The preimage of a locally closed subvariety under a morphism is a locally closed subvariety. The claim that A is a trivial bundle of algebras over a formal neighborhood of x in X shows that the preimage of any orbit of a semisimple associative algebra under φ is open. Since X is irreducible, we see that only one of these preimages is nonzero. □

One can ask, for which q the algebra $\mathcal{H}_{\mathbb{C},q}(n)$ is semisimple. The answer is: if and only if q is not a root of 1 of order $\leqslant n$. In fact, one can develop the representation theory of $\mathcal{H}_{\mathbb{C},q}(n)$ for q as above in the same fashion as for $\mathbb{C}S_n$ by using the multiplicative versions of the Jucys-Murphi elements to be introduced in Homework 3. Using this construction one can produce a natural bijection between $\mathrm{Irr}(\mathcal{H}_{\mathbb{C},q}(n))$ and $\mathrm{Irr}(S_n)$ (that cannot be deduced from Theorem 2.6).

References

[BM] R. Bezrukavnikov, I. Mirkovic. *Representations of semisimple Lie algebras in prime characteristic and noncommutative Springer resolution.* Ann. Math. 178 (2013), n.3, 835-919.

[BMR] R. Bezrukavnikov, I. Mirkovic, D. Rumynin. *Localization of modules for a semisimple Lie algebra in prime characteristic* (with an appendix by R. Bezrukavnikov and S. Riche), Ann. of Math. (2) 167 (2008), no. 3, 945-991.

[P] A. Premet, *Irreducible representations of Lie algebras of reductive groups and the Kac-Weisfeiler conjecture*, Invent. Math. 121(1995), 79-117.

LECTURE 9: FINITE GROUPS OF LIE TYPE AND HECKE ALGEBRAS

IVAN LOSEV

Introduction

We continue to study the representation theory of finite groups of Lie type and its connection to Hecke algebras, now in a more general setting. We start by defining Hecke algebras of arbitrary Weyl groups. Then we introduce finite groups G of Lie type and relate the co-induced representations $\mathbb{C}[B \setminus G]$ to Hecke algebras, similarly to what was done for $\mathrm{GL}_n(\mathbb{F}_q)$.

In the second part of this lecture we discuss a way to produce representations of finite groups of Lie type that is of crucial importance for the classification of irreducibles and computation of their characters: the Deligne-Lusztig induction. We start by explaining the induction (or, in our conventions, co-induction) of a character of B lifted from a character of T. Then we describe maximal tori of G, since our field is not algebraically closed two maximal tori do not need to be conjugate. Finally, we discuss the Deligne-Lusztig induction, it is constructed using étale cohomology and produces a virtual representation of G starting with a character of a torus.

1. Hecke algebras and finite groups of Lie type

1.1. **Generic Hecke algebras.** Let $G_{\mathbb{C}}$ be a semisimple algebraic group over $\mathbb{C}$ and W be its Weyl group. Recall that each reflection in W is conjugate to a simple reflection. Let S denote the set of simple reflections in W. We define independent variables v_s such that $v_s = v_t$ and $s, t \in S$ are such that s and t are conjugate in W. For example, in types A, D, E (simply laced types) all reflections are conjugate (any two adjacent simple roots are conjugate in W and hence the corresponding reflections are conjugate). So here we have one variable. In types B, C, F, G we have two possible lengths of roots that lead to two conjugacy classes and so will have two variables v_s. For $s, t \in S$ let m_{st} be the number of edges between s, t in the Dynkin diagram plus 2 so that $m = m_{st}$ is the minimal positive integer such that $(st)^m = 1$, equivalently, $sts \ldots = tst \ldots$, where in each side we have m factors.

We define the generic Hecke algebra $\mathcal{H} = \mathcal{H}(W)$ as the algebra over $Z[v_s^{\pm 1}]_{s \in S}$ generated by the elements $T_s, s \in S$, with relations

$$T_s T_t T_s \ldots = T_t T_s T_t \ldots \ (m_{st} \text{ factors}), \quad (T_s - v_s)(T_s + 1) = 0.$$

If $W = S_n$, we get the algebra $\mathcal{H}_v(n)$ introduced in the previous lecture. As another example, consider the Weyl group of type B_n (or C_n, they are the same). We number the simple reflections as follows $s_0 = s_{\epsilon_n}, s_i = s_{\epsilon_{n+1-i} - \epsilon_{n-i}}, i = 1, \ldots, n-1$. We have $m_{ij} = 2$ if $|i - j| > 1, m_{01} = 4, m_{i,i+1} = 3$ if $i > 0$. We write v for $v_i, i > 0$ and V for v_0. The relations become as follows: $T_1, \ldots, T_{n-1}$ have relations as in $\mathcal{H}_v(n)$, while $T_0 T_i = T_i T_0$ for $i > 1$, $T_0 T_1 T_0 T_1 = T_1 T_0 T_1 T_0, (T_0 - V)(T_0 + 1) = 0$.

For $w \in W$, we have an element $T_w = T_{i_1} \ldots T_{i_k}$ for a reduced expression $w = s_{i_1} \ldots s_{i_k}$, again, T_w is well-defined. Similarly to the S_n-case we have the following theorem.

Theorem 1.1. *The algebra $\mathcal{H}(W)$ is a free module over $\mathbb{Z}[v_s^{\pm 1}]$ with basis $T_w, w \in W$.*

For numerical values $q_s \in \mathbb{C}^\times$ of the variables v_s, we can consider the specialization $\mathcal{H}_{\mathbb{C},q_\bullet}(W)$ of $\mathcal{H}(W)$, a $\mathbb{C}$-algebra. The Tits deformation principle implies the following.

Proposition 1.2. *Suppose the algebra $\mathcal{H}_{\mathbb{C},q_\bullet}(W)$ is semisimple. Then it is isomorphic to $\mathbb{C}W$.*

Remark 1.3. All constructions here generalize to real reflection groups (finite groups of isometries of a Euclidian space generated by reflections), e.g. to dihedral groups.

We can also define Hecke algebras for arbitrary symmetrizable Cartan matrices, they will deform the group algebras of the Weyl groups. These Hecke algebras are going to be important for us when we discuss the character formulas for irreducible rational representations of reductive algebraic groups. Note that Proposition 1.2 no longer holds because the Hecke algebra is no longer finite dimensional.

1.2. **Finite groups of Lie type.** Let $\mathbb{F}$ be the algebraic closure of $\mathbb{F}_p$. Consider a connected reductive algebraic group $G_\mathbb{F}$. This group is known to be defined over $\mathbb{F}_p$. Further, we can find a maximal torus and Borel subgroup $T_\mathbb{F} \subset B_\mathbb{F} \subset G_\mathbb{F}$ defined over $\mathbb{F}_p$. For the classical groups $\mathrm{SL}_n(\mathbb{F}), \mathrm{Sp}_{2n}(\mathbb{F}), \mathrm{SO}_n(\mathbb{F})$, this can be checked directly (we take all diagonal matrices for $T_\mathbb{F}$ and all upper triangular matrices for $B_\mathbb{F}$, recall that we take forms given by anti-diagonal matrices to define $\mathrm{Sp}_{2n}(\mathbb{F}), \mathrm{SO}_n(\mathbb{F})$).

Now pick $\ell > 0$ and set $q = p^\ell$. Let Fr denote the automorphism $x \mapsto x^q$ of $\mathbb{F}$ so that $\mathbb{F}_q$ is the fixed point locus of Fr (we will write Fr_q when we want to indicate the dependence on q). Since $G_\mathbb{F}$ is defined over $\mathbb{F}_p$, we get the Frobenius homomorphism $\mathsf{Fr} : G_\mathbb{F} \to G_\mathbb{F}$. We can define the group $G = G_{\mathbb{F}_q}$ as the fixed point set of Fr in $G_\mathbb{F}$. The group G is a special case of a finite group of Lie type. Examples include $\mathrm{SL}_n(\mathbb{F}_q)$ (or $\mathrm{GL}_n(\mathbb{F}_q)$), $\mathrm{Sp}_{2n}(\mathbb{F}_q)$ and $\mathrm{SO}_n(\mathbb{F}_q)$. Let us write $N_\mathbb{F}$ for the normalizer of $T_\mathbb{F}$ in $G_\mathbb{F}$. Clearly, $N_\mathbb{F}$ is Fr-stable and we set $N := N_\mathbb{F}^{\mathsf{Fr}}$. So we have subgroups $T := T_{\mathbb{F}_q} \subset N, B := B_{\mathbb{F}_q} \subset G$ (a maximal torus and a Borel subgroup).

More generally, let Φ be an automorphism of $G_\mathbb{F}$ such that some power of Φ is Fr_{q^k}. Suppose that $T_\mathbb{F}, B_\mathbb{F}$ are Φ-stable (so that $N_\mathbb{F}$ is also Φ-stable). We get the fixed point subgroup $G := G_\mathbb{F}^\Phi \subset G_{\mathbb{F}_{q^k}}$. This is a general case of a *split* finite group of Lie type. We also get subgroups $T := T_\mathbb{F}^\Phi \subset N := N_\mathbb{F}^\Phi, B := B_\mathbb{F}^\Phi$.

We want to give a classical example of $G = G_\mathbb{F}^\Phi, \Phi \neq \mathsf{Fr}_q$: a finite unitary group. Recall that if we have a hermitian form h on a complex vector space V, we can define the unitary group $U(h)$ of all linear transformations of V that are unitary with respect to h (when h is positive definite we get the usual unitary group). If J is the matrix of h, then $U(h) = \{A \in \mathrm{GL}_n(\mathbb{C}) | \bar{A}^t J A = J\}$ (the superscript "t" means the transposed matrix).

Now consider the group $\mathrm{GL}_n(\mathbb{F})$. Let J denote the matrix with 1's on the main anti-diagonal and zeroes elsewhere. We have an automorphism α of $\mathrm{GL}_n(\mathbb{F})$ given by $A \mapsto JA^tJ$. Note that $\alpha^2 = 1$ and $\alpha \circ \mathsf{Fr}_q = \mathsf{Fr}_q \circ \alpha$. Set $\Phi := \alpha \circ \mathsf{Fr}_q$ so that $\Phi = \alpha \circ \Phi, \Phi^2 = \mathsf{Fr}_{q^2}$. Consider the group $\mathrm{GL}_n(\mathbb{F})^\Phi = \{A \in \mathrm{GL}_n(\mathbb{F}_{q^2}) | \bar{A}^t J A = J\}$, where $\bar{A}$ is obtained from A by applying $\mathsf{Fr}_q : \mathbb{F}_{q^2} \to \mathbb{F}_{q^2}$. The group $\mathrm{GL}_n(\mathbb{F})^\Phi$ is called the finite unitary group and is denoted by $\mathrm{GU}_n(\mathbb{F}_q)$.

One reason to care about finite groups of Lie type comes from the theory of finite simple groups. Each finite simple group is either

- the alternating group A_n,
- or a (generally, non-split) finite group of Lie type,
- or one of finitely many sporadic finite simple groups.

1.3. **Hecke algebras and representations of G.** Let $G := G^\Phi_{\mathbb{F}}$, where Φ is as above. Then we have subgroups $T \subset N, B \subset G$. Clearly $T \subset N$ is a normal subgroup and we can form the quotient $W := N/T$. We set $BwB := BnB$, where n is any representative of w in N. The following theorem gives the Bruhat decomposition.

Theorem 1.4. *We have $G = \bigsqcup_{w\in W} BwB$.*

Inside W we can find a subset S of involutions such that (W, S) becomes a Coxeter system (so that W is a real reflection group and S is the set of simple reflections). Furthermore, we have $BswB = BsBBwB$ if $\ell(sw) = \ell(w)+1$ and $BswB \sqcup BwB = BswB$ if $\ell(sw) = \ell(w)-1$. Set $q_s := |BsB|/|B|$, one can show $q_s = q_t$ if s, t are conjugate in W. Moreover, if $\Phi = \mathsf{Fr}_q$, we have $q_s = q$ for all $s \in S$.

Example 1.5. Consider the case $G = \mathrm{GU}_n(\mathbb{F}_q)$. In this case, W is the Weyl group of type $B_{\lfloor n/2\rfloor}$. We have $q_i = q^2$ for $i > 0$. If n is even, we get $q_0 = q$, and if n is odd, then $q_0 = q^3$. This is a part of Homework 3.

Consider the specialization $\mathcal{H}_{\mathbb{C},q_\bullet}(W)$, where the variable v_s goes to q_s.

Theorem 1.6. *The endomorphism algebra $\mathrm{End}_G(\mathbb{C}[B \setminus G])$ is isomorphic to $\mathcal{H}_{\mathbb{C},q_\bullet}(W)$.*

This allows to produce some irreducible representations of G (the number is equal to the number of W-irreps). Of course, this is just a tiny portion of all representations.

2. Deligne-Lusztig induction

2.1. **Induction from Borel.** Let us produce more irreducible representations. Before we co-induced from the trivial B-module. Now let us consider one-dimensional B-modules with trivial action of $U := U^\Phi_{\mathbb{F}}$, the unipotent subgroup (in our examples, this is the group of all strictly upper-triangular matrices). Such a module is given by a character of T, say χ. The coinduced module $\mathrm{Hom}_B(\mathbb{C}G, \mathbb{C}_\chi)$ coincides with $\mathbb{C}[B \setminus_\chi G] = \{f \in \mathbb{C}[G] | f(bg) = \chi(b)f(g)\}$. The endomorphism algebra $\mathrm{End}_G(\mathbb{C}[B \setminus_\chi G])$ is $\mathrm{Hom}_G(\mathbb{C}[B \setminus_\chi G], \mathbb{C}_\chi) = \mathbb{C}[G]^{B\times B,\chi\times\chi^{-1}}$, where, by definition, the right hand side is $\{f \in \mathbb{C}[G] | f(b_1gb_2) = \chi(b_1)f(g)\chi(b_2)\}$. The latter coincides with $\bigoplus_{w\in W} \mathbb{C}[BwB]^{B\times B,\chi\times\chi^{-1}}$. The space $\mathbb{C}[BwB]^{B\times B,\chi\times\chi^{-1}}$ is one dimensional if $\chi \in \mathrm{Hom}(T, \mathbb{C}^\times)$ is fixed by w^{-1} and is zero else. Indeed, take $b_1, b_2 \in T, b_1 = nb_2n^{-1}$, where n is a representative of w. Then we get $f(n)\chi(b_2) = f(nb_2) = f(b_1n) = \chi(nb_2n^{-1})f(n)$ meaning $\chi(b_2) = \chi(nb_2n^{-1}) = (w^{-1}.\chi)(b_2)$. So if $f(n) \neq 0$, then χ is fixed by w^{-1}. A converse is an exercise.

As we have seen, if $\chi = 1$, then each space $\mathbb{C}[BwB]^{B\times B,\chi\times\chi^{-1}}$ is one-dimensional. The other extreme is when χ is generic, i.e., it is not fixed by any non-trivial Weyl group element. In this case $\dim \mathrm{End}_G(\mathbb{C}[B \setminus_\chi G]) = 1$ and so $\mathbb{C}[B \setminus_\chi G]$ is irreducible. The general case interpolates between these two: the endomorphism algebra will be isomorphic to the (suitably understood) Hecke algebra of W_χ, the stabilizer of χ. Let us point out that W_χ does not need to be a Coxeter group. This does not happen for $G = \mathrm{GL}_n(\mathbb{F}_q)$: here χ can be thought as an element of $(\mathbb{F}_q^\times)^n$, and the group W_χ is the product of symmetric groups.

Lemma 2.1. *The modules $\mathbb{C}[B \setminus_\chi G]$ and $\mathbb{C}[B \setminus_{\chi'} G]$ have common direct summands if and only if χ, χ' are W-conjugate. In the latter case, they are isomorphic.*

Proof. Consider the space $\{f \in \mathbb{C}[BwB] | f(b_1gb_2) = \chi(b_1)f(g)\chi'(b_2)\}$. It is one-dimensional if and only if $\chi = w\chi'$ (this follows from an argument above) and is zero else. This implies the first claim.

The second claim for $G = \mathrm{GL}_n(\mathbb{F}_q)$ will be a part of Homework 3. $\square$

2.2. **Conjugacy classes and tori.** Now let us discuss the structure of conjugacy classes in G. We start with $G = \mathrm{GL}_n(\mathbb{F}_q)$. Let α be a primitive element in $\mathbb{F}_{q^m}$. Consider the endomorphism $A_\alpha : \mathbb{F}_q^m \to \mathbb{F}_q^m$ (given by the multiplication by α in $\mathbb{F}_q^m = \mathbb{F}_{q^m}$). After a base change to $\mathbb{F}$, the operator A_α becomes diagonalizable with eigenvalues $\alpha, \mathsf{Fr}_q(\alpha), \ldots, \mathsf{Fr}_q^{m-1}\alpha$. So the operators A_α, A_β are conjugate if and only if $\beta = \mathsf{Fr}_q^k\alpha$ in this case we say that α, β are equivalent.

We say that an element $A \in \mathrm{GL}_n(\mathbb{F}_q)$ is semisimple if it becomes diagonalizable over $\mathbb{F}$. Any semisimple element is conjugate to an operator of the form $\mathrm{diag}(A_{\alpha_1}, \ldots, A_{\alpha_k})$, where $\alpha_1, \ldots, \alpha_k$ are defined uniquely up to a permutation and equivalences. More generally, we have an obvious analog of the Jordan normal form theorem.

Now fix a partition $\nu = (\nu_1, \ldots, \nu_k)$ of n and consider the set T_ν of all elements of the form $\mathrm{diag}(A_{\alpha_1}, \ldots, A_{\alpha_k})$, where $\alpha_i \in \mathbb{F}_{q^{\nu_i}}$ (not necessarily primitive). This is a subgroup. For $\nu = (1, 1, \ldots, 1)$, we get $T_\nu = T$. The subgroups T_ν are the maximal tori in $G = \mathrm{GL}_n(\mathbb{F}_q)$ (up to conjugacy). All of them but T are not included into a Borel subgroup (=do not preserve a complete flag).

Let us extend this construction to a general G (with $\Phi = \mathsf{Fr}_q$) and make it more conceptual. If $S_\mathbb{F} \subset G_\mathbb{F}$ a Fr-stable maximal torus, then $(S_\mathbb{F})^{\mathsf{Fr}}$ is an abelian subgroup in G. But it does not need to be conjugate to T. We have $S_\mathbb{F} = gT_\mathbb{F}g^{-1}$ for some $g \in G_\mathbb{F}$. The equality $\mathsf{Fr}(S_\mathbb{F}) = S_\mathbb{F}$ is equivalent to $g^{-1}\mathsf{Fr}(g) \in N_\mathbb{F}$. Now we have the following theorem of Lang.

Theorem 2.2. *The map $L : G_\mathbb{F} \to G_\mathbb{F}$ given by $g \mapsto g^{-1}\mathsf{Fr}(g)$ is surjective.*

Proof. The proof is based on the observation that the tangent map of Fr is zero at all points. Consider the action of $G_\mathbb{F}$ on itself given by $g.h := gh\mathsf{Fr}(g)^{-1}$. For any fixed h, the map $g \mapsto g.h$ is etale (all tangent maps are iso). So any $G_\mathbb{F}$-orbit on $G_\mathbb{F}$ has dimension $\dim G_\mathbb{F}$ and is open. Since $G_\mathbb{F}$ is connected, we get a single orbit. This shows that the map $g \mapsto g\mathsf{Fr}(g)^{-1}$ is surjective. It follows that L is surjective. $\square$

So pick $n \in N_\mathbb{F}$ and set $S_\mathbb{F} = gT_\mathbb{F}g^{-1}$, where $g \in G_\mathbb{F}$ is such that $g^{-1}\mathsf{Fr}(g) = n$. Up to G-conjugacy, the subgroup $(S_\mathbb{F})^{\mathsf{Fr}}$ depends only on the image w of n in W. So we denote it by T_w. Moreover, it only depends on the conjugacy class of w.

Example 2.3. When $G = \mathrm{GL}_n(\mathbb{F})$, then $T_\nu = T_w$, where ν is the cycle type of w. It is enough to check this when w is a single cycle. Under the identification of $T_\mathbb{F}$ with $S_\mathbb{F} = gT_\mathbb{F}g^{-1}$ given by $t \mapsto gtg^{-1}$, the morphism $\mathsf{Fr} : S_\mathbb{F} \to S_\mathbb{F}$ becomes $t \mapsto w(\mathsf{Fr}(t))$. We can identify $T_\mathbb{F}$ with $(\mathbb{F}^\times)^n$ such that w permutes the coordinates on $T_\mathbb{F}$ (in a cycle). So the fixed points in $T_\mathbb{F}$ become $(z, \mathsf{Fr}(z), \ldots, \mathsf{Fr}^{n-1}(z))$, where $\mathsf{Fr}^n(z) = z$, i.e., $z \in \mathbb{F}_{q^n}$. From here it is easy to see that $(S_\mathbb{F})^{\mathsf{Fr}}$ is conjugate to $T_{(n)}$.

In fact, any semisimple element in G is conjugate to an element in T_w, note that the conjugacy class of w is not uniquely determined by the element, for example, the constant matrices in $\mathrm{GL}_n(\mathbb{F}_q)$ lie in all tori T_w.

2.3. **Deligne-Lusztig induction.** Recall that we can produce a representation $\mathbb{C}[B \setminus_\chi G]$ from a character χ of T. This involves the choice of B but as different B's containing T are conjugate by W, by Lemma 2.1, this choice does not affect $\mathbb{C}[B \setminus_\chi G]$. We denote this representation by $R_T^G(\chi)$.

We want to have a similar construction for an arbitrary maximal torus T_w of G. The problem is that this torus is not included into a Borel subgroup. Deligne and Lusztig solved this problem in [DL] defining what is now called the Deligne-Lusztig induction.

Let $S_{\mathbb{F}} \subset G_{\mathbb{F}}$ be a Fr-stable maximal torus and let $B_{\mathbb{F}} = S_{\mathbb{F}} \ltimes U_{\mathbb{F}}$ be a Borel subgroup, it is not Fr-stable, in general. Consider the subvariety $Y := L^{-1}(U_{\mathbb{F}}) \subset G_{\mathbb{F}}$, i.e., $\{g \in G | g^{-1}\mathsf{Fr}(g) \in U_{\mathbb{F}}\}$. Note that G acts on Y by left multiplications: for $h \in G$, we have $L(hg) = (hg)^{-1}\mathsf{Fr}(hg) = g^{-1}h^{-1}\mathsf{Fr}(h)\mathsf{Fr}(g) = g^{-1}\mathsf{Fr}(g) = L(g)$. The group $T_w = (S_{\mathbb{F}})^{\mathsf{Fr}}$ acts by right multiplications because $S_{\mathbb{F}}$ normalizes $U_{\mathbb{F}}$. Clearly, the actions of G and T_w commute.

We want to get a virtual representation of $G \times T_w$ from its action on Y ("virtual" means that it is a formal linear combination of irreducibles with integral coefficients). This representation will be the Euler characteristic of Y. In order to define the Euler characteristic, we need some cohomology theory. The variety Y is over $\mathbb{F}$ so it is not a topological space in any reasonable sense.

Yet there is a suitable cohomology theory, it is called étale cohomology. More precisely, we pick a prime ℓ that does not divide q. Consider the ℓ-adic field $\mathbb{Q}_\ell$. Then, for an algebraic variety X over $\mathbb{F}$, we can define the ith cohomology group with compact support $H^i_c(X, \mathbb{Q}_\ell)$. This cohomology group is a finite dimensional vector space over $\mathbb{Q}_\ell$, it is zero when $i < 0$ or i is large enough so it makes sense to speak about the Euler characteristic $\chi(X, \mathbb{Q}_\ell)$. When X has an action of a group H, all cohomology groups $H^i_c(X, \mathbb{Q}_\ell)$ carry a representation of H. In particular, $\chi(Y, \mathbb{Q}_\ell)$ is a virtual representation of $G \times T_w$ (over $\mathbb{Q}_\ell$). The algebraic closure $\overline{\mathbb{Q}}_\ell$ is known to be isomorphic to $\mathbb{C}$. So $\chi(Y, \mathbb{C}) := \mathbb{C} \otimes_{\mathbb{Q}_\ell} \chi(Y, \mathbb{Q}_\ell)$ is a complex virtual representation of $G \times T_w$ that can be shown to be independent of the choice of $B_{\mathbb{F}}$.

Example 2.4. Suppose that $U_{\mathbb{F}}$ is Fr-stable. Then the Lang map $L_U : U_{\mathbb{F}} \to U_{\mathbb{F}}$ is surjective. The fiber over 1 is U. It follows that $Y = G \times_U U_{\mathbb{F}}$. The group $U_{\mathbb{F}}$ is an affine space and so $H^i_c(U_{\mathbb{F}}, \mathbb{Q}_\ell)$ vanishes unless $i = 2\dim U_{\mathbb{F}}$, in the latter case, $\dim H^i_c(U_{\mathbb{F}}, \mathbb{Q}_\ell) = 1$. So, $\chi(Y, \mathbb{C}) = \mathbb{C} \otimes_{\mathbb{Q}_\ell} H^{2\dim U_{\mathbb{F}}}_c(Y, \mathbb{Q}_\ell) = \mathbb{C}[U \setminus G]$ (an equality of $G \times T$-modules).

Now let us explain the construction of $R^G_{T_w}(\chi)$, where χ is a character of T_w. We simply take $R^G_{T_w}(\chi) := [\chi(Y, \mathbb{C}) \otimes \chi]^{T_w}$. In the case when $T_w = T$, we recover the co-induced module $\mathbb{C}[B \setminus_\chi G]$.

The virtual representations $R^G_{T_w}(\chi)$ were extensively studied by Deligne and Lusztig, and then by Lusztig, see, e.g. [L], there is also an exposition of these results in [C]. For example, one can show that every irreducible representation of G appears in one of $R^G_{T_w}(\chi)$.

One can define the notion of conjugate characters χ of T_w, and χ' of $T_{w'}$. It is easy to define this notion when w is conjugate to w', while in general one should view χ, χ' as elements of the so called Langlands dual group. For example, the trivial characters of different tori are all conjugate. It was shown in [DL] that if χ, χ' are not conjugate, then $R^G_{T_w}(\chi)$ and $R^G_{T_{w'}}(\chi')$ do not have common summands.

The most interesting and complicated case is when $\chi = 1$. The representations that appear in $R^G_{T_w}(1)$ are called *unipotent*. In the case when $G = \mathrm{GL}_n(\mathbb{F}_q)$ all unipotent representations are already realized for $w = 1$, while outside of type A, this is not so. Morally, the unipotent representations should correspond to unipotent conjugacy classes, but this correspondence is subtle (for example, there are three unipotent conjugacy classes for $\mathrm{SL}_2(\mathbb{F}_q)$ for odd q but only two unipotent representations).

References

[C] R. Carter. *Finite groups of Lie type. Conjugacy classes and complex characters.* 2nd ed. John Wiley and sons, 1993.

[DL] P. Deligne, G. Lusztig. *Representations of reductive groups over finite fields.* Ann. of Math. (2), 103 (1976), 103-161.

[L] G. Lusztig. *Characters of reductive groups over a finite field*, Ann. Math. Studies 107, Princeton University Press (1984).

LECTURE 10: KAZHDAN-LUSZTIG BASIS AND CATEGORIES $\mathcal{O}$

IVAN LOSEV

Introduction

In this and the next lecture we will describe an entirely different application of Hecke algebras, now to the category $\mathcal{O}$. In the first section we will define the Kazhdan-Lusztig basis in the Hecke algebra of W and explain how to read the multiplicities in the category $\mathcal{O}$ from this basis (the Kazhdan-Lusztig conjecture proved independently by Beilinson-Bernstein and Brylinski-Kashiwara).

In the remainder of this lecture and in the next one, we will explain some steps towards a proof of this conjecture based on works of Soergel and of Elias-Williamson. We will start by defining projective functors between different infinitesimal blocks of category $\mathcal{O}$. As an application, we will show how the computation of multiplicities in $\mathcal{O}_\lambda$ for $\lambda \in P$ reduces to $\lambda = 0$.

1. Kazhdan-Lusztig basis and conjecture

1.1. **Recap on category $\mathcal{O}$.** Pick a semisimple Lie algebra $\mathfrak{g}$ over $\mathbb{C}$. We have the triangular decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}$. Let W denote the Weyl group. Let $\rho := \frac{1}{2}\sum_{\alpha>0}\alpha = \sum_i \omega_i$ (where ω_i denote the fundamental weight corresponding to a simple root α_i). Define the shifted action of W on $\mathfrak{h}$ by $w \cdot \lambda := w(\lambda+\rho) - \rho$.

Recall that in Lecture 7 we have introduced the BGG category $\mathcal{O}$ consisting of all finitely generated $U(\mathfrak{g})$-modules with diagonalizable action of $\mathfrak{h}$ and locally nilpotent action of $\mathfrak{n}$. Also we have identified the center Z of $U(\mathfrak{g})$ with $\mathbb{C}[\mathfrak{h}]^{W\cdot} = \{f \in \mathbb{C}[\mathfrak{h}] | f(w\cdot\lambda) = f(\lambda), \forall w \in W, \lambda \in \mathfrak{h}\}$, where we send $z \in Z$ to the polynomial $\tilde{f}_z$ such that z acts by $\tilde{f}_z(\lambda)$ on the Verma module $\Delta(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda$. This allowed to split $\mathcal{O}$ into the direct sum of infinitesimal blocks $\mathcal{O}_\lambda$ consisting of all modules M in $\mathcal{O}$, where z acts with generalized eigenvalue $\tilde{f}_z(\lambda)$. We are going to be interested in $\mathcal{O}_\lambda$, where $\lambda \in P$ (the weight lattice) and mainly in $\mathcal{O}_0$ (we will see that the study of $\mathcal{O}_\lambda$ with $\lambda \in P$ basically reduces to the study of $\mathcal{O}_0$). The simple objects in $\mathcal{O}_0$ are $L(w\cdot 0), w \in W$, all of these objects are different because $W_\rho = \{1\}$. We have seen in Lecture 7 that any $L(w\cdot 0)$ appears in the composition series of $\Delta(w\cdot 0)$ once, and all other composition are $L(w'\cdot 0)$, where $w'\cdot 0 < w\cdot 0$ meaning that $w\cdot 0 - w'\cdot 0$ is a sum of positive roots. In fact, we can take a weaker *Bruhat* order (getting a stronger result).

Definition 1.1. We say that $u \prec w$ (in the Bruhat order) if $w = s_{\beta_k}\ldots s_{\beta_1}u$, where $\beta_k,\ldots,\beta_1$ are roots (not necessarily simple) and $\ell(s_{\beta_i}\ldots s_{\beta_1}u) > \ell(s_{\beta_{i-1}}\ldots s_{\beta_1}u)$ for all i.

In this order, the minimal element in W is 1, while the maximal element is the longest (with respect to the length $\ell(w)$) element $w_0 \in W$. It is uniquely characterized by the property that it maps the positive Weyl chamber C to $-C$. For $W = S_n$, we have $w_0(i) = n+1-i$ for all i.

Here are properties of $\prec$ to be used below.

Lemma 1.2. *The following is true:*

(1) *If* $u \prec w$*, then* $u \cdot 0 > w \cdot 0$.
(2) *If* u *is obtained from* w *by deleting some elements in the reduced expression of* w*, then* $u \prec w$.
(3) $u \preceq w$ *if and only if* $w_0 w \preceq w_0 u$.

The proof is left as an exercise.

We will write L_w for $L(w_0 w \cdot 0)$ and $\Delta_w = \Delta(w_0 w \cdot 0)$. One can show that if L_u is a composition factor of Δ_w, then $u \preceq w$. What we want to compute is the character of L_w. Let m_w^u denote the multiplicity of L_u in Δ_w. Consider the multiplicity matrix M, it is unitriangular and hence invertible. Let $M^{-1} = (n_w^u)$. So $\mathsf{ch} L_w = \sum_{u \preceq w} n_w^u \mathsf{ch}\Delta_u$. So what we need to compute is the numbers n_w^u.

It is convenient to reformulate this problem. The category $\mathcal{O}_0$ is abelian. So we can consider its *Grothendieck group* $K_0(\mathcal{O}_0)$. It is defined as the quotient of the free group generated by the isomorphism classes of the objects $M \in \mathcal{O}_0$ modulo the relation $M = M' \oplus M''$ if there is an exact sequence $0 \to M' \to M \to M'' \to 0$. We denote the image of M in $K_0(\mathcal{O}_0)$ by $[M]$. Since the objects in $\mathcal{O}_0$ have finite length, the classes $[L_w]$ form a basis in $K_0(\mathcal{O}_0)$. Since the matrix M is uni-triangular, the same is true for $[\Delta_w]$. We identify $K_0(\mathcal{O}_0)$ with the group ring $\mathbb{Z}W$ in such a way that $[\Delta_w]$ corresponds to w. So we need to compute the basis $[L_w] = \sum_{u \preceq w} n_w^u u$.

Example 1.3. It is easy to compute two basis elements $[L_w]$. Namely, we have $n_w^w = 1$ and $n_w^u \neq 0 \Rightarrow u \preceq w$. This immediately implies $[L_1] = 1$. The proof of the Weyl character formula in Lecture 7 says $[L_{w_0}] = \sum_{w \in W} \operatorname{sgn}(w_0 w) w$.

In general, however, we cannot even describe the basis $[L_w]$ staying inside $\mathbb{Z}W$. This is where the Hecke algebra comes into play.

1.2. **Kazhdan-Lusztig basis.** First, it will be convenient to modify the Hecke algebra slightly. Let us recall the previous definition (in the specialization $v_s = v$ for all $s \in S$, where S denotes the set of simple reflections in W). The Hecke algebra $\mathcal{H}_v(W)$ is generated by elements that we will now denote by T'_s with relations $T'_s T'_t T'_s \ldots = T'_t T'_s T'_t \ldots$ (m_{st} times) and $(T'_s - v)(T'_s + 1) = 0$. Now let q be another independent variable (that has nothing to do with a prime power). Define the $\mathbb{Z}[q^{\pm 1}]$-algebra $\mathcal{H}_q(W)$ by generators T_s with relations $T_s T_t T_s \ldots = T_t T_s T_t \ldots$ and $(T_s - q)(T_s + q^{-1}) = 0$. Clearly, $\mathcal{H}_q(W) = \mathcal{H}_v(W)[q^{\pm 1}]/(v - q^2)$ with $T'_s \mapsto qT_s$. We see that $\mathcal{H}_q(W)$ has basis T_w such that

$$T_s T_w = \begin{cases} T_{sw}, & \text{if } \ell(sw) = \ell(w) + 1, \\ T_{sw} + (q - q^{-1}) T_w, & \text{if } \ell(sw) = \ell(w) - 1. \end{cases} \tag{1.1}$$

We have a ring involution of $\mathcal{H}_q(W)$ (called the bar involution and denoted by $\bar{\bullet}$), given on generators by $\bar{q} := q^{-1}, \bar{T}_s := T_s^{-1} (= T_s + q^{-1} - q)$. Since $\bar{\bullet}$ preserves the relations, we see that $\bar{\bullet}$ is indeed a well-defined ring involution. Note that $\bar{T}_w = (T_{w^{-1}})^{-1}$.

The following fundamental result is due to Kazhdan and Lusztig, [KL].

Theorem 1.4. *For any* $w \in W$*, there is a unique element* $C_w \in \mathcal{H}_q(W)$ *such that* $C_w = T_w + \sum_{u \prec w} P_w^u(q) T_u$*, where* $P_w^u(q) \in q\mathbb{Z}[q]$*, and* $\bar{C}_w = C_w$.

Since the matrix of expressing C_w's in terms of T_w's is uni-triangular, we see that the elements $C_w, w \in W$, form a basis in $\mathcal{H}_q(W)$. This is a so called *Kazhdan-Lusztig* basis.

Proof. The proof is by induction with respect to the Bruhat order: we assume that C_u exits and is unique for all $u \prec w$. Let $w = s_{i_1} \ldots s_{i_\ell}$ be a reduced expression. We have

$\bar{T}_w = \bar{T}_{i_1} \dots \bar{T}_{i_\ell} = (T_{i_1} + q^{-1} - q) \dots (T_{i_\ell} + q^{-1} - q)$. Decompose $\bar{T}_w$ in the basis T_u. We have $\bar{T}_w = T_w + \sum_{u \prec w} R^u_w(q) T_u$ (all u's are obtained by removing some simple reflections from the reduced decomposition of w and so $u \prec w$ by (2) of Lemma 1.2). By the existence of C_u, what we need to show that there is a unique $\tilde{P}^u_w(q) \in q\mathbb{Z}[q]$ such that $C_w = T_w + \sum_{u \prec w} \tilde{P}^u_w(q) C_u$ and $\bar{C}_w = C_w$. We also have $\bar{T}_w - T_w = \sum_{u \prec w} Q^u_w(q) C_u$. Applying $\bar{\bullet}$ to the equation, we get $T_w - \bar{T}_w = \sum_{u \prec w} Q^u_w(q^{-1}) C_u$ and so $\bar{Q}^u_w(q^{-1}) = -\bar{Q}^u_w(q)$. But we have

$$\bar{C}_w = \bar{T}_w + \sum_{u \prec w} \tilde{P}^u_w(q^{-1}) C_u = T_w + \sum_{u \prec w} Q^u_w(q) C_u + \sum_{u \prec w} \tilde{P}^u_w(q^{-1}) C_u$$

So we need to prove that there is a unique $\tilde{P}^u_w(q) \in q\mathbb{Z}[q]$ such that $\tilde{P}^u_w(q^{-1}) - \tilde{P}^u_w(q) = Q^u_w(q)$. This follows from $Q^u_w(q^{-1}) = -Q^u_w(q)$. □

Example 1.5. We have $C_1 = 1$ and $C_s = T_s - q$, where $s \in S$.

Let us consider a more interesting example: $W = S_3$. Let s, t denote the simple reflections. The Bruhat order is that $1 < s, t < st, ts < sts = tst$ (elements in the same group are not comparable). We have

$$C_{st} = T_{st} - q(T_s + T_t) + q^2, C_{ts} = T_{ts} - q(T_s + T_t) + q^2, C_{sts} = T_{sts} - q(T_{st} + T_{ts}) + q^2(T_s + T_t) - q^3.$$

More generally, $C_{w_0} = \sum_{w \in W} (-q)^{\ell(w_0) - \ell(w)} T_w$. To check these equalities is a part of the homework.

1.3. **Kazhdan-Lusztig conjecture.** We have a surjection $\mathcal{H}_q(W) \twoheadrightarrow \mathbb{Z}W$ given by setting $q = 1$. The following theorem was conjectured by Kazhdan-Lusztig and proved by Beilinson-Bernstein, [BB], and Brylinski-Kashiwara, [BK].

Theorem 1.6. *We have* $[L_w] = C_w|_{q=1}$.

By Example 1.5, this agrees with the Weyl character formula: $[L_{w_0}] = \sum_{w \in W_0} \mathrm{sgn}(w_0 w) w$.

This is a difficult theorem whose proof found in the 80's required a heavy machinery and is one of the greatest achievements of Geometric Representation theory. Recently, a more elementary (but also difficult) proof was found, see [EW]. Starting the next section, we will outline some ideas relevant for that proof.

1.4. **Stronger version.** Now we are going to explain how to recover C_w itself (not just its specialization to 1) from the structure of Verma modules. This description was found in [BGS].

Let M be an object of $\mathcal{O}$. By $\mathsf{head}(M)$ we mean the maximal semisimple quotient of M and by the radical $\mathsf{Rad}(M)$ we mean the kernel $M \twoheadrightarrow \mathsf{head}(M)$. Now define the *radical filtration* $M_0 \supsetneq M_1 \supsetneq M_2 \supsetneq \dots$ by setting $M_i := \mathsf{Rad}(M_{i-1})$. Now take $M := \Delta_w$ and for $u \preceq w$ define $m^u_w(q) := \sum [M_i/M_{i+1} : L_u] q^i$, where the square bracket denotes the multiplicity of L_u in the composition series of M_i/M_{i+1}. For example, $m^w_w(q) = 1$.

Theorem 1.7. *We have* $T_w = \sum_{u \preceq w} m^u_w(q) C_u$.

Example 1.8. For $\mathfrak{g} = \mathfrak{sl}_2$, the module $\Delta(1)$ has simple radical, $\Delta(-2) = L(-2)$. For $s \in S_2 \setminus \{1\}$, we get $m^s_s(q) = 1, m^1_s(q) = q$. Indeed, $T_s = C_s + qC_1$.

2. Projective functors, I

We are going to explain how to reduce the study of $\mathcal{O}_\lambda$ with $\lambda \in P$ to $\lambda = 0$.

2.1. **Tensor products with finite dimensional modules.** Recall that if V is a finite dimensional $\mathfrak{g}$-module and $M \in \mathcal{O}$, then $V \otimes M \in \mathcal{O}$. So we get the functor $V \otimes \bullet : \mathcal{O} \to \mathcal{O}$. This functor is exact (preserves exact sequences), it has both left and right adjoints, both are given by $V^* \otimes \bullet$.

Now we are going to get a partial description of $V \otimes \Delta(\lambda)$. Pick a weight basis $v_1, \ldots, v_m$ of V and let $\nu_1, \ldots, \nu_m \in \mathfrak{h}^*$ be the corresponding weights. We may assume that they are ordered compatibly with the order on $\mathfrak{h}^*$, i.e., if $\nu_i \geqslant \nu_j$, then $i \geqslant j$.

Proposition 2.1. *There is a filtration $V \otimes M = M_0 \supset M_1 \supset \ldots \supset M_m = \{0\}$ such that $M_{i-1}/M_i \cong \Delta(\lambda + \nu_i)$.*

Proof. Recall that $\Delta(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda$. We claim that $V \otimes \Delta(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} (V \otimes \mathbb{C})$. This follows from

$$\begin{aligned}&\operatorname{Hom}_{\mathfrak{g}}(V \otimes \Delta(\lambda), M) = \operatorname{Hom}_{\mathfrak{g}}(\Delta(\lambda), V^* \otimes M) = \operatorname{Hom}_{\mathfrak{b}}(\mathbb{C}_\lambda, V^* \otimes M) = \\ &= \operatorname{Hom}_{\mathfrak{b}}(V \otimes \mathbb{C}_\lambda, M) = \operatorname{Hom}_{\mathfrak{g}}(U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} (V \otimes \mathbb{C}_\lambda), M).\end{aligned}$$

Consider the filtration $V \otimes \mathbb{C}_\lambda = N_0 \supset \ldots \supset N_m = \{0\}$, where $N_i := \operatorname{Span}_{\mathbb{C}}(v_{i+1}, \ldots, v_m)$. This is $\mathfrak{b}$-module filtration (because $\mathfrak{n}$ increases weights) with $N_{i-1}/N_i = \mathbb{C}_{\lambda+\nu_i}$. Set $M_i := U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} N_i$. Recall that $U(\mathfrak{g})$ is a free right $U(\mathfrak{b})$-module. So the functor $U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \bullet$ is exact and we have $M_{i-1}/M_i = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} (N_{i-1}/N_i) = \Delta(\lambda + \nu_i)$. $\square$

Let pr_μ denote the functor $\mathcal{O} \twoheadrightarrow \mathcal{O}_\mu$ that sends $M \in \mathcal{O}$ to the generalized eigenspace of Z in M with eigenvalue μ. The functors of the form $\mathsf{pr}_\mu(V \otimes \bullet) : \mathcal{O}_\lambda \to \mathcal{O}_\mu$ (and their compositions) are known as *projective functors*. They are tremendously useful in the study of $\mathcal{O}$.

Corollary 2.2. *The object $\mathsf{pr}_\mu(V \otimes \Delta(w \cdot \lambda))$ admits a filtration by $\Delta(\lambda + \nu_i)$ with $w \cdot \lambda + \nu_i \in W \cdot \mu$.*

2.2. **Application: translation functors.** We are going to consider a special case of Corollary 2.2, where it is especially easy to describe what weights ν_i appear.

Proposition 2.3. *Assume that $\lambda, \mu \in P$ are such that $\lambda, \lambda - \mu, \mu + \rho$ are dominant. Let V be the irreducible finite dimensional module with highest weight $\lambda - \mu$. Then $\mathsf{pr}_\mu(V^* \otimes \Delta(w \cdot \lambda)) = \Delta(w \cdot \mu)$ and $\mathsf{pr}_\lambda(V \otimes \Delta(w \cdot \mu))$ is filtered with $\Delta(wu \cdot \lambda), u \in W_{\mu+\rho}$.*

Note that $W_{\mu+\rho}$ is generated by the simple reflections s_i such that $\langle \mu + \rho, \alpha_i^\vee \rangle = 0$.

Proof. To prove the claim about $\mathsf{pr}_\mu(V^* \otimes \Delta(w \cdot \lambda))$ we need to find all weights ν of V^* such that $w \cdot \lambda + \nu \in W \cdot \mu$, equivalently, $\lambda + \rho + w^{-1}\nu = u(\mu + \rho)$ for some $u \in W$. We have $u(\mu + \rho) \leqslant \mu + \rho$ for any $u \in W$ and $w^{-1}\nu \geqslant \mu - \lambda$ (the lowest weight of V^*) with equality if and only if $w = w_0$. So $\lambda + \rho + w^{-1}\nu \geqslant \lambda + \rho + \mu - \lambda = \mu + \rho \geqslant u(\mu + \rho)$. The equality $\mathsf{pr}_\mu(V^* \otimes \Delta(w \cdot \lambda)) = \Delta(w \cdot \mu)$ follows from Corollary 2.2.

The claim about $\mathsf{pr}_\lambda(V \otimes \Delta(w \cdot \mu))$ follows similarly using the observation that $\lambda - w \cdot \mu \geqslant \lambda - \mu$ for any $w \in W$, and the equality is equivalent to $w \in W_{\mu+\rho}$. $\square$

This proposition has several important corollaries.

Corollary 2.4. *Let λ_1, λ_2 be dominant. Then there is an equivalence $\mathcal{O}_{\lambda_1} \xrightarrow{\sim} \mathcal{O}_{\lambda_2}$ sending $\Delta(w \cdot \lambda_1)$ to $\Delta(w \cdot \lambda_2)$.*

Proof. We may assume that $\lambda_1 - \lambda_2$ is dominant. Otherwise, we replace λ_1 with $\lambda_1 + \lambda_2$ and take a composed equivalence $\mathcal{O}_{\lambda_1} \xrightarrow{\sim} \mathcal{O}_{\lambda_1+\lambda_2} \xrightarrow{\sim} \mathcal{O}_{\lambda_2}$.

Apply Proposition 2.3 to $\lambda = \lambda_1 + \lambda_2$ and $\mu = \lambda_2$. We get functors $\varphi : \mathcal{O}_\mu \to \mathcal{O}_\lambda, \varphi(M) := \mathsf{pr}_\lambda(V \otimes M)$ and $\varphi^* := \mathsf{pr}_\mu(V^* \otimes \bullet) : \mathcal{O}_\lambda \to \mathcal{O}_\mu$. The notation φ^* is justified by the observation that φ^* is left and right adjoint to φ. We are going to prove that φ^*, φ are mutually inverse (quasi-inverse, if we want to be precise).

Note that $\varphi(\Delta(w \cdot \mu)) = \Delta(w \cdot \lambda)$ and $\varphi^*(\Delta(w \cdot \lambda)) = \Delta(w \cdot \mu)$ (we have $W_{\mu+\rho} = \{1\}$). Also we have an adjointness homomorphism $\varphi^* \circ \varphi(M) \to M$ (induced by $\varphi^* \circ \varphi(M) \hookrightarrow V^* \otimes V \otimes M \to M$). This homomorphism is zero if and only if $\varphi(M)$ is zero. Now apply this to $M = \Delta(w \cdot \lambda)$, we get a nonzero homomorphism $\Delta(w \cdot \mu) = \varphi^* \circ \varphi(\Delta(w \cdot \mu)) \to \Delta(w \cdot \mu)$. But any Verma module is generated by its highest weight vector and any endomorphism maps that vector to its multiple. We deduce that any nonzero endomorphism of a Verma module is an isomorphism. So $\varphi^* \circ \varphi(M) \xrightarrow{\sim} M$ when M is a Verma module. Since any object in $\mathcal{O}_\mu$ is filtered by quotients of Verma modules, we see that $\varphi^* \circ \varphi(M) \xrightarrow{\sim} M$ for any $M \in \mathcal{O}_\lambda$. So φ^* is left inverse of φ. Similarly, we see that φ^* is a right inverse of φ. □

Now let us consider the case when $\mu + \rho$ is dominant, but $W_{\mu+\rho}$ is non-trivial. The simples in $\mathcal{O}_\mu$ are naturally labelled by $W/W_{\mu+\rho}$. There is a distinguished representative in each right coset $wW_{\mu+\rho}$ – it is known that such a coset contains a unique longest element (w.r.t. the length function ℓ; it also contains a unique shortest element, but we do not need this). So it is natural to label the simples in $\mathcal{O}_{\mu+\rho}$ with longest elements of right $W_{\mu+\rho}$-cosets.

Corollary 2.5. *Let λ, μ be such as in Proposition 2.3. Then $\mathsf{pr}_\mu(V^* \otimes L(w \cdot \lambda)) = L(w \cdot \mu)$ if w is longest in its right $W_{\mu+\rho}$-coset $wW_{\mu+\rho}$ and is zero else.*

Sketch of proof. The proof is again based on using adjoint functors $\varphi := \mathsf{pr}_\mu(V^* \otimes \bullet)$ and $\varphi^* := \mathsf{pr}_\lambda(V \otimes \bullet)$.

Step 1. We need to show that $\varphi(L(w \cdot \lambda)) = 0$ when w is not longest in its right $W_{\mu+\rho}$-coset. In other words, we can find a simple reflection $s_i \in W_{\mu+\rho}$ such that $\ell(ws_i) > \ell(w)$. In this case, we have a nonzero homomorphism $\eta : \Delta(ws_i \cdot \lambda) \to \Delta(w \cdot \lambda)$. One can show that $\varphi(\eta) \neq 0$. So $\varphi(\eta)$ is an isomorphism. In particular, $\varphi(\operatorname{coker} \eta) = 0$ and hence $\varphi(L(w \cdot \lambda)) = 0$.

Step 2. Now let w be longest in its right $W_{\mu+\rho}$-coset. The object $\varphi(L(w \cdot \lambda))$ is a quotient of $\varphi(\Delta(w \cdot \lambda)) = \Delta(w \cdot \mu)$. So we need to show that $\varphi(L(w \cdot \lambda)) \neq 0$ and that $\operatorname{Hom}(\Delta(w' \cdot \mu), \varphi(L(w \cdot \lambda))) = 0$ if $w' \cdot \mu < w \cdot \mu$ (this will show that $\varphi(L(w \cdot \lambda))$ is simple). The equality follows because $\operatorname{Hom}(\Delta(w' \cdot \mu), \varphi(L(w \cdot \lambda))) = \operatorname{Hom}(\varphi^*(\Delta(w' \cdot \mu)), L(w \cdot \lambda))$ and $\Delta(w \cdot \lambda)$ does not appear in the filtration of $\varphi^*(\Delta(w' \cdot \mu))$. On the other hand, if $\varphi(L(w \cdot \lambda)) = 0$, then the class $[\varphi(\Delta(w \cdot \lambda))]$ is a linear combination of $[\varphi(L(w' \cdot \lambda))]$ with $w' \cdot \mu < w \cdot \mu$ and hence of $[\varphi(\Delta(w' \cdot \lambda))]$ with $w' \cdot \mu < w \cdot \mu$. But $\varphi(\Delta(w \cdot \lambda)) = \Delta(w \cdot \mu)$ and $\varphi(\Delta(w' \cdot \lambda)) = \Delta(w' \cdot \mu)$, contradiction. □

Let us give a corollary of the previous two corollaries that reduces the question about the multiplicities in the categories $\mathcal{O}_\mu$ for $\lambda \in P$ to $\lambda = 0$.

Corollary 2.6. *Let μ be such that $\mu + \rho$ is dominant. Pick w that is longest in its right $W_{\lambda+\rho}$-coset. Then $[\Delta(u \cdot \mu) : L(w \cdot \mu)] = [\Delta(u \cdot 0) : L(w \cdot 0)]$.*

References

[BB] A. Beilinson, J. Bernstein. Localisation de $\mathfrak{g}$-modules. C. R. Acad. Sci. Paris Sér. I Math. 292 (1981), no. 1, 1518.

[BGS] A. Beilinson, V. Ginzburg, W. Soergel. *Koszul duality patterns in representation theory*. J. Amer. Math. Soc. 9 (1996), no. 2, 473527.

[BK] J.-L. Brylinski, M. Kashiwara. *Kazhdan-Lusztig conjecture and holonomic systems*. Invent. Math. 64 (1981), no. 3, 387410.

[EW] B. Elias, G. Williamson. *The Hodge theory of Soergel bimodules*. Ann. of Math. (2) 180 (2014), no. 3, 10891136. arXiv:1212.0791.

[KL] D. Kazhdan, G. Lusztig. *Representations of Coxeter groups and Hecke algebras*. Invent. Math. 53 (1979), no. 2, 165-184.

LECTURE 11: SOERGEL BIMODULES

IVAN LOSEV

Introduction

In this lecture we continue to study the category $\mathcal{O}_0$ and explain some ideas towards the proof of the Kazhdan-Lusztig conjecture.

We start by introducing projective functors $\mathcal{P}_i : \mathcal{O}_0 \to \mathcal{O}_0$ that act by $w \mapsto w(1+s_i)$ on $K_0(\mathcal{O}_0)$. Using these functors we produce a projective generator of $\mathcal{O}_0$.

In Section 2 we explain some of the work of Soergel that ultimately was used by Elias and Williamson to give a relatively elementary proof of the Kazhdan-Lusztig conjecture. In order to relate the category $\mathcal{O}_0$ to the Hecke algebra $\mathcal{H}_q(W)$ one needs to produce a graded lift of that category. In order to do that, Soergel constructed a functor $\mathcal{O}_0 \to \mathbb{C}[\mathfrak{h}]^{coW}$-mod, where $\mathbb{C}[\mathfrak{h}]^{coW}$ is the so called *coinvariant algebra*. He proved that this functor is fully faithful on the projective objects and has described the image of a projective generator that turns out to be a graded module. This gives rise to a graded lift of $\mathcal{O}_0$. Also these results of Soergel lead to the notion of Soergel (bi)modules that are certain (bi)modules over $\mathbb{C}[\mathfrak{h}]$. They are of great importance for modern Representation theory.

We finish by briefly describing some related constructions: Kazhdan-Lusztig bases for Hecke algebras with unequal parameters and multiplicities for rational representations of semisimple algebraic groups in positive characteristic.

1. Projective functors, II

1.1. **Functors $\mathcal{P}_i$.** Let $\alpha_1, \ldots, \alpha_n$ denote the simple roots. We want to define a projective functor $\mathcal{P}_i : \mathcal{O}_0 \to \mathcal{O}_0$. For this we pick $\lambda, \mu \in P$ such that $\lambda, \mu+\rho, \lambda-\mu$ are dominant and the only positive root vanishing on $\mu+\rho$ is α_i (so $\lambda+\rho$ lies inside the dominant Weyl chamber and $\mu+\rho$ lies on the wall corresponding to α_i). Let V be the irreducible module with highest weight $\lambda-\mu$. So we have functors $\varphi := \mathsf{pr}_\lambda(V \otimes \bullet) : \mathcal{O}_\mu \to \mathcal{O}_\lambda$ and its biadjoint $\varphi^* := \mathsf{pr}_\mu(V^* \otimes \bullet) : \mathcal{O}_\lambda \to \mathcal{O}_\mu$.

Lemma 1.1. *The object $\varphi \circ \varphi^*(\Delta(w\cdot\lambda))$ admits a 2 step filtration by $\Delta(w\cdot\lambda), \Delta(ws_i\cdot\lambda)$, where the Verma with the smaller weight appears as the quotient and Verma with the larger weight appears as a sub.*

This is a special case of Proposition 2.3 in the previous lecture.

Now recall (Corollary 2.4 of Lecture 10) that there is an equivalence $\mathcal{O}_0 \xrightarrow{\sim} \mathcal{O}_\lambda$ with $\Delta(w\cdot 0) \mapsto \Delta(w\cdot\lambda)$. Transferring the functor $\varphi\circ\varphi^*$ to $\mathcal{O}_0$, we get the functor $\mathcal{P}_i$ we need. Note that $\mathcal{P}_i$ is exact (and, moreover, is self-adjoint). In particular, $\mathcal{P}_i$ induces an endomorphism of $K_0(\mathcal{O}_0) = \mathbb{Z}W$ to be denoted by $[\mathcal{P}_i]$.

Corollary 1.2. *We have $[\mathcal{P}_i]w = w(s_i+1)$.*

Remark 1.3. This may be viewed as a reason for the identification $K_0(\mathcal{O}_0) \cong \mathbb{Z}W$ (that is regarded as a right W-module). Indeed, we see that the generators s_i+1 of the group algebra $\mathbb{Z}W$ lift to endofunctors $\mathcal{P}_i$ of $\mathcal{O}_0$. This gives one of the first examples of *categorification.*

Remark 1.4. One can ask whether it is possible to lift the elements s_i to endofunctors of $\mathcal{O}_0$. The answer is yes if we are willing to replace $\mathcal{O}_0$ with the bounded derived category $D^b(\mathcal{O}_0)$. Then we can consider the *reflection functor* R_i given by the cone of the adjunction morphism $1 \to \mathcal{P}_i = \varphi \circ \varphi^*$. The functors R_i are self-equivalences of $D^b(\mathcal{O}_0)$ that give rise to an action of the braid group B_W on $D^b(\mathcal{O}_0)$. Recall that the group B_W is generated by the elements T_i with relations $T_iT_jT_i \ldots = T_jT_iT_j \ldots$ (m_{ij} factors).

1.2. **Projective objects in $\mathcal{O}_0$.** We start by recalling basics on projective objects in an abelian category, say $\mathcal{C}$. An object $P \in \mathcal{C}$ is called *projective* if the functor $\operatorname{Hom}_{\mathcal{C}}(P, \bullet)$ is exact. By $\mathcal{C}$-proj we denote the full subcategory of $\mathcal{C}$ consisting of projective objects ("full" means that the morphisms in $\mathcal{C}$-proj are the same as in $\mathcal{C}$). Note that this is an additive category that is closed under taking direct summands but, in general, has neither kernels nor cokernels.

We say that $\mathcal{C}$ has enough projectives if every object is a quotient of a projective. If all objects in $\mathcal{C}$ have finite length, the condition that $\mathcal{C}$ has enough projectives is equivalent to the condition that every simple $L \in \mathcal{C}$ has a *projective cover* P_L, i.e., an indecomposable projective object surjecting onto L (recall that an object is called *indecomposable* if it cannot be decomposed into a proper direct sum). If a projective cover exists, it is unique up to an isomorphism and has no Hom's to other simple objects. Every projective splits into the direct sum of projective covers.

We can recover $\mathcal{C}$ from $\mathcal{C}$-proj if $\mathcal{C}$ has enough projectives and all objects have finite length. For simplicity, assume, in addition, that $\mathcal{C}$ has finitely many simple objects. Let P be a projective generator of $\mathcal{C}$, i.e., a projective object that surjects onto any simple object. Then the functor $\operatorname{Hom}_{\mathcal{C}}(P, \bullet)$ is an equivalence of $\mathcal{C}$ and the category of right $\operatorname{End}_{\mathcal{C}}(P)$-modules. We note that $\operatorname{End}_{\mathcal{C}}(P)$ is a finite dimensional associative unital algebra. Conversely, if A is a finite dimensional associative unital algebra, then the category A-mod of finite dimensional A-modules has enough projectives, finitely many simples and all objects have finite length.

We are interested in $\mathcal{C} = \mathcal{O}_0$. As we have seen in Lecture 7, it has finitely many simples and all objects have finite length. Let us give an example of a projective.

Lemma 1.5. *The object $\Delta(0)$ is projective in $\mathcal{O}_0$.*

Proof. Recall that $\operatorname{Hom}_{\mathcal{O}_0}(\Delta(0), M) = \{m \in M_0 | \mathfrak{n}m = 0\}$. But 0 is the maximal weight of an object in $\mathcal{O}_0$ (because $w \cdot 0 \leqslant 0$ for all w). Since the action of $\mathfrak{n}$ increases weights, we see that $M_0^{\mathfrak{n}} = M_0$. The functor $M \mapsto M_0$ is exact and so $\Delta(0)$ is projective. $\square$

Now let $w \in W$ have reduced expression $s_{i_1} \ldots s_{i_\ell}$. Let $\underline{w}$ denote the sequence $(s_{i_1}, \ldots, s_{i_\ell})$. Set $P_{\underline{w}} := \mathcal{P}_{i_\ell} \ldots \mathcal{P}_{i_1} \Delta(0)$.

Proposition 1.6. *The category $\mathcal{O}_0$ has enough projectives. Moreover, the object $P_{\underline{w}}$ decomposes into the direct sum of the projective cover of $L(w \cdot 0)$ (it appears with multiplicity 1) and of the projective covers of $L(u \cdot 0)$ with $u \prec w$ (in the Bruhat order).*

Below $P(w \cdot 0)$ denotes the projective cover of $L(w \cdot 0)$.

Proof. The functors $\mathcal{P}_i$ are self-adjoint and so map projectives to projectives. Hence $P_{\underline{w}}$ is projective. So it is enough to show that $\dim \operatorname{Hom}_{\mathcal{O}_0}(P_{\underline{w}}, L(w \cdot 0)) = 1$ and $\operatorname{Hom}_{\mathcal{O}_0}(P_{\underline{w}}, L(w' \cdot 0)) \neq 0$ implies $w' \preceq w$. The object $P_{\underline{w}}$ has a filtration with successive Verma quotients. Set $w_k = s_{i_1} \ldots s_{i_k}$ and $\underline{w}_k := (s_{i_1}, \ldots, s_{i_k})$. By induction on k (where the step follows from Lemma 1.1), we see that

(1) $\Delta(w_k \cdot 0)$ is a quotient of $P_{\underline{w}_k}$.

(2) All other Verma modules in a filtration of $P_{\underline{w}_k}$ have highest weights $u \cdot 0$, where u is obtained from w_k by removing some simple reflections (and hence $u \prec w_k$).

For $k = \ell$, (1) implies $\mathrm{Hom}_{\mathcal{O}_0}(P_{\underline{w}}, L(w \cdot 0)) \neq 0$, while (2) shows that $\mathrm{Hom}_{\mathcal{O}_0}(P_{\underline{w}}, L(w' \cdot 0)) \neq 0$ for $w' \neq w$ implies $w' \prec w$, and $\dim \mathrm{Hom}_{\mathcal{O}_0}(P_{\underline{w}}, L(w \cdot 0)) < 1$. □

Example 1.7. Consider the case of $\mathfrak{sl}_2$. Then $0 \to \Delta(0) \to P_s \to \Delta(-2) \to 0$. This exact sequence does not split (this is a part of the homework). So $P_s = P(-2)$. Applying $\mathcal{P}$ to $P(-2)$, we get $P(-2)^{\oplus 2}$.

2. Soergel (bi)modules

Our goal is to give some description of $\mathcal{O}_0$-proj and see that $\mathcal{O}_0$ is equivalent to A-mod, where A is graded. We will follow an approach by Soergel, [S1]. But first let us explain why we need graded algebras here.

2.1. **Graded lift.** The first problem in proving the Kazhdan-Lusztig conjecture is that we do not know how to relate $\mathcal{O}_0$ to $\mathcal{H}_q(W)$: the Grothendieck group of $\mathcal{O}_0$ is $\mathbb{Z}W$, it does not "see" q, while it is impossible to define the Kazhdan-Lusztig basis without having q. This is remedied by considering "graded lifts".

Namely, let A be a finite dimensional algebra equipped with an algebra grading $A = \bigoplus_{i \in \mathbb{Z}} A_i$. We can consider the category $\mathcal{C} := A$-mod. Or we can consider the category of *graded* A-modules. Its objects are finite dimensional graded A-modules, i.e., A-modules M equipped with an A-module grading $M = \bigoplus_{i \in \mathbb{Z}} M_i$ (this is a part of the structure). The morphisms in this category are the grading preserving homomorphisms of A-modules. Let us write $\tilde{\mathcal{C}}$ for the category of graded A-modules. It is an abelian category and we can consider its Grothendieck group $K_0(\tilde{\mathcal{C}})$.

The point is that $K_0(\tilde{\mathcal{C}})$ is not only an abelian group, but also a $\mathbb{Z}[q^{\pm 1}]$-module. For a graded A-module, we can consider the module $M\{d\}$ with shifted grading $M\{d\}_i := M_{i+d}$. We set $q^d[M] := [M\{d\}]$.

Not all A-modules admit a grading. However, we have the following lemma.

Lemma 2.1. *All indecomposable projective A-modules and all simple A-modules admit a grading. Furthermore, if an indecomposable A-module admits a grading, then the grading is unique up to a shift.*

We do not give a proof (it is actually based on the properties of algebraic groups – a basic observation here is that a grading on M gives rise to a $\mathbb{C}^\times$-action on M).

So fix some gradings on the simple A-modules $L_1, \ldots, L_k$. It follows that the simple graded A-modules are precisely $L_i\{d\}, i = 1, \ldots, k, d \in \mathbb{Z}$. Hence $K_0(\tilde{\mathcal{C}})$ is a free $\mathbb{Z}[q^{\pm 1}]$-module with basis $[L_1], \ldots, [L_k]$. Besides $K_0(\mathcal{C}) = K_0(\tilde{\mathcal{C}})/(q-1)$.

Now let P be a projective generator of $\mathcal{O}_0$. Then $A := \mathrm{End}_{\mathcal{O}_0}(P)$ is a finite dimensional algebra and $\mathcal{O}_0 \cong A^{opp}$-mod (the category of right A-modules). The first step to prove the Kazhdan-Lusztig conjecture is to equip A with a grading (for a suitable choice of P).

2.2. **Structural results on $\mathcal{O}_0$-proj.** The idea of Soergel, [S1], was to study the functor $\mathbb{V} := \mathrm{Hom}_{\mathcal{O}_0}(P(w_0 \cdot 0), \bullet)$. The projective $P(w_0 \cdot 0)$ plays a very special role, for example, it is the only indecomposable projective that is also injective. First, one needs to understand the target category for this functor, i.e., to compute the endomorphisms of $P(w_0 \cdot 0)$. Let $\mathbb{C}[\mathfrak{h}]^W_+$ denote the ideal of all elements in $\mathbb{C}[\mathfrak{h}]^W$ without constant term. We write $\mathbb{C}[\mathfrak{h}]^{coW}$ (the "coinvariant algebra") for the quotient $\mathbb{C}[\mathfrak{h}]/(\mathbb{C}[\mathfrak{h}]^W_+)$, where we write $(\mathbb{C}[\mathfrak{h}]^W_+) = \mathbb{C}[\mathfrak{h}]\mathbb{C}[\mathfrak{h}]^W_+$. This is a graded algebra that is isomorphic to $\mathbb{C}W$ as a W-module.

Theorem 2.2. *We have* $\operatorname{End}_{\mathcal{O}_0}(P(w_0 \cdot 0)) \xrightarrow{\sim} \mathbb{C}[\mathfrak{h}]^{coW}$.

The functor $\mathbb{V}$ is very far from being an equivalence, for example, it kills all simples but $L(w_0 \cdot 0) = \Delta(w_0 \cdot 0)$. However, we have the following important result.

Theorem 2.3. *The functor $\mathbb{V}$ is fully faithful (induces an isomorphism of Hom spaces) on the projective objects.*

In the case of $\mathfrak{sl}_2$, these theorems can be verified directly (this is a part of the homework).

So, for a projective generator P of $\mathcal{O}_0$, we get $\operatorname{End}_{\mathcal{O}_0}(P) = \operatorname{End}_{\mathbb{C}[\mathfrak{h}]^{coW}}(\mathbb{V}(P))$.

Now let us compute $\mathbb{V}(\Delta(0))$ and understand what the functor $\mathcal{P}_i$ looks like on the level of $\mathbb{C}[\mathfrak{h}]^{coW}$-mod. Let $\mathbb{C}[\mathfrak{h}]^{s_i}$ denote the subalgebra of all s_i-invariant polynomials in $\mathbb{C}[\mathfrak{h}]$.

Theorem 2.4. *We have a functorial isomorphism* $\mathbb{V}(\mathcal{P}_i(M)) \cong \mathbb{V}(M) \otimes_{\mathbb{C}[\mathfrak{h}]^{s_i}} \mathbb{C}[\mathfrak{h}]$. *Moreover,* $\mathbb{V}(\Delta(0)) \cong \mathbb{C}(= \mathbb{C}[\mathfrak{h}]/(\mathfrak{h}))$.

Note that $\mathbb{C}[\mathfrak{h}]$ is a graded $\mathbb{C}[\mathfrak{h}]^{s_i}$-module. So if $\mathbb{V}(M)$ is a graded $\mathbb{C}[\mathfrak{h}]^{coW}$-module, then $\mathbb{V}(\mathcal{P}_i M)$ gets graded (as a tensor product of graded modules). So the object $\mathbb{V}(P_{\underline{w}})$ gets graded. Moreover, $P := \bigoplus_{\underline{w}} P_{\underline{w}}$ (we take one reduced expression per w) is a projective generator of $\mathcal{O}_0$ by Proposition 1.6.

Corollary 2.5. *The algebra $A := \operatorname{End}_{\mathcal{O}_0}(P)$ is graded.*

Proof. By Theorem 2.3, we have $A = \operatorname{End}_{\mathbb{C}[\mathfrak{h}]^{coW}}(\mathbb{V}(P))$. The algebra $\mathbb{C}[\mathfrak{h}]^{coW}$ is graded and $\mathbb{V}(P)$ admits a grading. So $\operatorname{End}_{\mathbb{C}[\mathfrak{h}]^{coW}}(\mathbb{V}(P))$ has a natural grading. □

2.3. **Soergel modules and bimodules.** Consider the $\mathbb{C}[\mathfrak{h}]$-bimodule $\mathcal{B}_{s_i} := \mathbb{C}[\mathfrak{h}] \otimes_{\mathbb{C}[\mathfrak{h}]^{s_i}} \mathbb{C}[\mathfrak{h}]$. It is a graded bimodule (by the total degree), where, for convenience, we take $\deg \mathfrak{h} = 2$. For a sequence $\underline{w} := (s_{i_1}, \ldots, s_{i_k})$ of simple reflections, we set $\mathcal{B}_{\underline{w}} := \mathcal{B}_{s_{i_1}} \otimes_{\mathbb{C}[\mathfrak{h}]} \mathcal{B}_{s_{i_2}} \otimes_{\mathbb{C}[\mathfrak{h}]} \ldots \otimes_{\mathbb{C}[\mathfrak{h}]} \mathcal{B}_{s_{i_k}}$. This is a so called *Bott-Samelson* $\mathbb{C}[\mathfrak{h}]$-bimodule. Note that the left and right $\mathbb{C}[\mathfrak{h}]^W$-module structures on $\mathcal{B}_{\underline{w}}$ coincide.

Remark 2.6. The bimodules $\mathcal{B}_{\underline{w}}$ have a geometric meaning. Let $P_i, i = 1, \ldots, n$ denote the minimal parabolic subgroup of G corresponding to the simple root α_i, its Lie algebra $\mathfrak{p}_i$ equals $\mathfrak{b} \oplus \mathbb{C}f_i$. Consider the product $P_{i_1} \times \ldots \times P_{i_k}$. On this product, the group B^k acts: $(b_1, \ldots, b_k).(p_1, \ldots, p_k) = (p_1 b_1^{-1}, b_1 p_2 b_2^{-1}, \ldots, b_{k-1} p_k b_k^{-1})$. The quotient $P_{i_1} \times \ldots \times P_{i_k}/B^k$ is a so called Bott-Samelson variety, to be denoted by $\mathsf{BS}_{\underline{w}}$, since $P_{i_k}/B \cong \mathbb{P}^1$, the variety $\mathsf{BS}_{\underline{w}}$ is an iterated $\mathbb{P}^1$-bundle. Now suppose that $\underline{w}$ is a reduced expression of w. Note that we have a natural morphism $\mathsf{BS}_{\underline{w}} \to G/B$ (taking the product). One can show that the image is the Schubert subvariety $\overline{BwB}/B$ and that the morphism we consider is actually a resolution of singularities.

The group B still acts on $\mathsf{BS}_{\underline{w}}$ on the left and we can consider its equivariant cohomology $H^*_B(\mathsf{BS}_{\underline{w}})$. This is a module over $H^*_B(G/B) = \mathbb{C}[\mathfrak{h}] \otimes_{\mathbb{C}[\mathfrak{h}]^W} \mathbb{C}[\mathfrak{h}]$. In fact, $H^*_B(\mathsf{BS}_{\underline{w}}) = \mathcal{B}_{\underline{w}}$. From here we deduce that $H^*(\mathsf{BS}_{\underline{w}}) = \mathbb{C} \otimes_{\mathbb{C}[\mathfrak{h}]} \mathcal{B}_{\underline{w}}$, the module over $H^*(G/B) = \mathbb{C}[\mathfrak{h}]^{coW}$.

Definition 2.7. By a *Soergel bimodule* we mean any graded $\mathbb{C}[\mathfrak{h}]$-bimodule that appears as a graded direct summand of the bimodules of the form $\bigoplus_{\underline{w}} \mathcal{B}_{\underline{w}}\{d_{\underline{w}}\}$, where $d_{\underline{w}} \in \mathbb{Z}$ is a grading shift. A *Soergel module* is a graded direct summand in $\mathbb{C} \otimes_{\mathbb{C}[\mathfrak{h}]} B$, where B is a Soergel bimodule.

The motivation is as follows. The Soergel modules are precisely the images of the projectives in $\mathcal{O}_0$ under the functor $\mathbb{V}$. Taking the tensor product with $\mathcal{B}_{s_i}$ corresponds to the functor $\mathcal{P}_i$. So we should view Soergel bimodules as graded analogs of the projective functors.

The following result of Soergel classifies the indecomposable Soergel (bi)modules.

Theorem 2.8. *The indecomposable Soergel (bi)modules (up to a grading shift) are parameterized by W. More precisely, let $w \in W$, and $\underline{w} = (s_{i_1}, \ldots, s_{i_\ell})$ give a reduced expression of W. Then there is a unique indecomposable summand of $\mathcal{B}_{\underline{w}}$ depending only on w that does not occur in $\mathcal{B}_{\underline{u}}$, where $\ell(\underline{u}) < \ell$. Further, a Soergel bimodule B is indecomposable if and only if $\mathbb{C} \otimes_{\mathbb{C}[\mathfrak{h}]} B$ is indecomposable.*

The classification of the indecomposable Soergel modules can be deduced from their connection to the projective objects in $\mathcal{O}$. For bimodules, the claim is more complicated.

Example 2.9. Consider the bimodule $\mathcal{B}_s = \mathbb{C}[\mathfrak{h}] \otimes_{\mathbb{C}[\mathfrak{h}]^s} \mathbb{C}[\mathfrak{h}]$. It is indecomposable. Indeed, $\mathbb{C} \otimes_{\mathbb{C}[\mathfrak{h}]} \mathcal{B}_s = \mathbb{C}[\mathfrak{h}]/(\mathbb{C}[\mathfrak{h}]^s_+)$ is indecomposable as a $\mathbb{C}[\mathfrak{h}]$-module and the graded Nakayma lemma implies that $\mathcal{B}_s$ is indecomposable. On the other hand, for $W = S_3$, the bimodule $\mathcal{B}_{(s,t,s)}$ is not indecomposable, it is the direct sum of $\mathcal{B}_s$ and $\mathbb{C}[\mathfrak{h}] \otimes_{\mathbb{C}[\mathfrak{h}]^W} \mathbb{C}[\mathfrak{h}]$ with suitable grading shifts, both summands are indecomposable.

2.4. **Tensor structure and graded K_0.** By definition, $\mathsf{BS}_{\underline{w}} \otimes_{\mathbb{C}[\mathfrak{h}]} \mathsf{BS}_{\underline{u}} = \mathsf{BS}_{\underline{wu}}$, where we write $\underline{wu}$ for the concatenation of $\underline{w}$ and $\underline{u}$. Because of this, Sbim is closed under taking tensor products over $\mathbb{C}[\mathfrak{h}]$. We write $B \cdot B'$ for $B \otimes_{\mathbb{C}[\mathfrak{h}]} B'$.

The category Sbim is not abelian (it does not have kernels and cokernels). But it is additive and is closed under taking direct summand. For such a category $\mathcal{C}$, we can define the *split* Grothendieck group, the quotient of the free group on the isomorphism classes of objects by $M = M' + M''$ if $M \cong M' \oplus M''$. We still denote this group by $K_0(\mathcal{C})$ (for an abelian category, the split Grothendieck group is huge and not useful at all, so this abuse of notation does not harm) and write $[M]$ for a class of M. A basis in $K_0(\mathcal{C})$ is formed by the (graded isomorphism classes of) indecomposable objects. Since we have the grading shifts on Sbim, the group $K_0(\mathsf{Sbim})$ is a $\mathbb{Z}[q^{\pm 1}]$-module. We can define a $\mathbb{Z}[q^{\pm 1}]$-algebra structure on $K_0(\mathsf{Sbim})$ by $[B] \cdot [B'] = [B \cdot B']$.

The following result is due to Soergel (a.k.a. Soergel's categorification theorem).

Theorem 2.10. *We have a $\mathbb{Z}[q^{\pm 1}]$-isomorphism $K_0(\mathsf{Sbim}) \cong \mathcal{H}_q(W)$ that sends $[\mathcal{B}_s]$ to $q^{-1}T_s + q^{-2}$.*

Sketch of proof. Both sides are free $\mathbb{Z}[q^{\pm 1}]$-modules of rank $|W|$. So the only thing that we need to prove is that $q[\mathcal{B}_s] - q^{-1}$ satisfies the relations for the elements T_s. We have relations of two kinds: quadratic relations $T_s^2 = 1 + (q - q^{-1})T_s$ and the braid relations $T_sT_tT_s \ldots = T_tT_sT_t \ldots$. The former are easy and we will check them, the latter are more complicated (the case $m_{st} = 2$ is still easy, the case of $m_{st} = 3$ follows from Example 2.9).

What we need to check is an isomorphism $\mathcal{B}_s \cdot \mathcal{B}_s = \mathcal{B}_s \oplus \mathcal{B}_s\{-2\}$. This is easily reduced to the case of $\mathfrak{sl}_2$. There $\mathcal{B}_s = \mathbb{C}[x] \otimes_{\mathbb{C}[x^2]} \mathbb{C}[x]$ and

$$\mathcal{B}_s \cdot \mathcal{B}_s = (\mathbb{C}[x] \otimes_{\mathbb{C}[x^2]} \mathbb{C}[x]) \otimes_{\mathbb{C}[x]} (\mathbb{C}[x] \otimes_{\mathbb{C}[x^2]} \mathbb{C}[x]) = \mathbb{C}[x] \otimes_{\mathbb{C}[x^2]} \mathbb{C}[x] \otimes_{\mathbb{C}[x^2]} \mathbb{C}[x].$$

As a $\mathbb{C}[x^2]$-bimodule, $\mathbb{C}[x]$ decomposes as $\mathbb{C}[x^2] \oplus \mathbb{C}[x^2]\{-2\}$ (the second summand is $x\mathbb{C}[x^2]$ and, by our convention, $\deg x = 2$). We conclude that $\mathcal{B}_s \cdot \mathcal{B}_s = \mathcal{B}_s \oplus \mathcal{B}_s\{-2\}$. $\square$

2.5. **Kazhdan-Lusztig conjecture via Soergel bimodules.** A reasonable question is what are the classes of the indecomposable Soergel bimodules in $\mathcal{H}_q(W)$. Let $\underline{\mathcal{B}}_w$ denote the indecomposable summand in $\mathsf{BS}_{\underline{w}}\{\ell(w)\}$ that does not appear in $\mathsf{BS}_{\underline{u}}$ with $\ell(\underline{u}) < \ell(w)$. We want to describe the classes of $\underline{\mathcal{B}}_w$ (note that we have normalized the choice of grading).

It turns out that these classes are very closely related to the basis elements C_w. Namely, we have a ring involution $\bullet^*$ of $\mathcal{H}_q(W)$ defined on the generators by $T_s \mapsto T_s, q \mapsto -q^{-1}$ (this clearly preserves the relations).

Theorem 2.11. *We have* $[\underline{\mathcal{B}}_w] = C_w^*$.

This theorem can be shown to imply the Kazhdan-Lusztig conjecture (and determines the classes of $[P(w \cdot 0)] \in \mathbb{Z}W = K_0(\mathcal{O}_0)$). It was first proved by Soergel (using the geometric methods such as perverse sheaves). An alternative proof follows from the work of Elias and Williamson, see [EW1], and also a survey [EW2]. The main new ingredient of that work is a very clever emulation of Hodge theory in the context of Soergel bimodules.

3. Complements

3.1. **Hecke algebras with unequal parameters.** We have used the specialization of all Hecke algebra parameters v_s to q^2. One can consider more general specializations: v_s gets specialized to $q^{2L(s)}$, where $L : S \to \mathbb{Z}_{\geqslant 0}$ is a function that is constant on the conjugacy classes of reflections. The Kazhdan-Lusztig basis in the corresponding specialization is defined as before. However, these bases are much more mysterious. For example, let $P_w^u(q)$ be the coefficient of T_u in C_w. In the equal parameter case it is known that $(-1)^{\ell(w)-\ell(u)} P_w^u(q)$ has nonnegative coefficients. This is not known in general.

3.2. **Multiplicities for algebraic groups.** The Kazhdan-Lusztig basis in a suitable Hecke algebra controls the characters of irreducible $G_{\mathbb{F}}$-modules $L(\lambda)$, where $G_{\mathbb{F}}$ is a semisimple algebraic group over an algebraically closed field $\mathbb{F}$ of characteristic p, where p is very large comparing to the rank of $G_{\mathbb{F}}$ (the dimension of the maximal torus). The Hecke algebra is taken for the affine Weyl group $W^{aff} = W \ltimes Q$, where W is the Weyl group of G and Q is the root lattice, i.e., the group in $\mathfrak{h}^*$ generated by the simple roots, compare to Problem 5 in Homework 2. We refer to Section 2 of [F] for details.

References

[EW1] B. Elias, G. Williamson. *The Hodge theory of Soergel bimodules*, Ann. of Math. (2) 180 (2014), no. 3, 1089-1136. arXiv:1212.0791.

[EW2] B. Elias, G. Williamson. *Kazhdan-Lusztig conjectures and shadows of Hodge theory*, arXiv:1403.1650.

[F] P. Fiebig. *Lusztig's conjecture as a moment graph problem.* Bull. London Math. Soc. 42(6) (2010), 957-972. arXiv:0712.3909.

[S1] W. Soergel. *Kategorie O, perverse Garben und Moduln über den Koinvarianten zur Weylgruppe.* J. Amer. Math. Soc. 3 (1990), no. 2, 421-445.

[S2] W. Soergel. *Kazhdan-Lusztig-Polynome und unzerlegbare Bimoduln über Polynomringen*, J. Inst. Math. Jussieu, 6(3) (2007), 501-525.

LECTURE 12: HOPF ALGEBRA $U_q(\mathfrak{sl}_2)$

IVAN LOSEV

Introduction

In this lecture we start to study quantum groups $U_q(\mathfrak{g})$, certain deformations of the universal enveloping algebras $U(\mathfrak{g})$. The algebras $U_q(\mathfrak{g})$ are *Hopf algebras* that basically means that we can take tensor products and duals of their representations. In Section 1 we define Hopf algebras.

In Section 2 we start discussing quantum groups themselves concentrating mostly on the simplest case, $U_q(\mathfrak{sl}_2)$. An important feature here is that the tensor product is not commutative in a naive sense. This is a feature and not a bug, this is one of the main reasons why the quantum groups were introduced.

1. Hopf algebras

1.1. **Tensor products and duals.** Recall that for a group G and two G-modules V_1, V_2 we can define G-module structures on $V_1 \otimes V_2$ and V_1^* by

$$g.(v_1 \otimes v_2) := gv_1 \otimes gv_2, \langle g.\alpha, v_1\rangle := \langle \alpha, g^{-1}v_1\rangle.$$

We also have the trivial one-dimensional module $\mathbb{C}$, where $g \in G$ acts by 1.

Similarly, for a Lie algebra $\mathfrak{g}$ and two $\mathfrak{g}$-modules V_1, V_2, we can define $\mathfrak{g}$-module structures on $V_1 \otimes V_2$ and V_1^* by

$$x.(v_1 \otimes v_2) = (x.v_1) \otimes v_2 + v_1 \otimes (x.v_2), \langle x.\alpha, v_1\rangle = -\langle \alpha, x.v_1\rangle.$$

And we have the trivial one-dimensional module $\mathbb{C}$, where $x \in \mathfrak{g}$ acts by 0.

Recall also that a G-module (resp., $\mathfrak{g}$-module) is the same thing as a module over the group algebra $\mathbb{C}G$ (resp., over the universal enveloping algebra $U(\mathfrak{g})$). Both $\mathbb{C}G, U(\mathfrak{g})$ are associative algebras. Note, however, that if A is an associative algebra, then we do not have natural A-module structures on $V_1 \otimes V_2, V_1^*, \mathbb{C}$ (where V_1, V_2 are A-modules). Indeed, $V_1 \otimes V_2$ carries a natural structure of $A \otimes A$-module by $(a \otimes b).(v_1 \otimes v_2) = (av_1) \otimes (bv_2)$. The dual space V_1^* is naturally a module over the opposite algebra A^{op}, which is the same vector space as A but with opposite multiplication: $a \cdot b := ba$. An A^{op}-module is the same thing as a right A-module, and we set $(\alpha a)(v_1) := \alpha(av_1)$. Finally, $\mathbb{C}$ is naturally a $\mathbb{C}$-module. We could equip $V_1 \otimes V_2$ with an A-module structure if we have a distinguished algebra homomorphism $\Delta : A \to A \otimes A$ (then we just pull the $A \otimes A$-module structure back to A). This homomorphism Δ is called a *coproduct*. Similarly, to equip V_1^* and $\mathbb{C}$ with A-module structures we need algebra homomorphisms $S : A \to A^{op}$ (antipode) and $\eta : A \to \mathbb{C}$ (counit).

Let us construct these homomorphisms for $A = \mathbb{C}G$ and $A = U(\mathfrak{g})$.

Example 1.1. For $A = \mathbb{C}G$, we have $\Delta(g) := g \otimes g, S(g) = g^{-1}, \eta(g) = 1$ for $g \in G$.

Example 1.2. Let $A = U(\mathfrak{g})$. Since Δ, S, η are supposed to be algebra homomorphisms, it is enough to define them on $\mathfrak{g}$. We set $\Delta(x) = x \otimes 1 + 1 \otimes x, S(x) = -x, \eta(x) = 1$, where $x \in \mathfrak{g}$.

1.2. **Coassociativity.** We need some additional assumptions on Δ, S, ϵ in order to guarantee some natural properties of tensor products such as associativity. Axiomatizing these properties, we arrive at the definition of a *Hopf algebra*.

First, let us examine the associativity of the tensor product. We have a natural isomorphism $(V_1 \otimes V_2) \otimes V_3 \to V_1 \otimes (V_2 \otimes V_3), (v_1 \otimes v_2) \otimes v_3 \mapsto v_1 \otimes (v_2 \otimes v_3)$. We want this isomorphism to be A-linear. We have two homomorphisms $A \to A^{\otimes 3}$ produced from Δ. First, we have $(\Delta \otimes \mathrm{id}) \circ \Delta$. The algebra A acts on $(V_1 \otimes V_2) \otimes V_3$ via this homomorphism $A \to A^{\otimes 3}$. Indeed, if $\Delta(a) = \sum_{i=1}^k a_i^1 \otimes a_i^2$, then $a.((v_1 \otimes v_2) \otimes v_3) = \sum_{i=1}^k a_i^1.(v_1 \otimes v_2) \otimes a_i^2 v_3 = \sum_{i=1}^k \Delta(a_i^1)(v_1 \otimes v_2) \otimes a_i^2 v_3$, and $(\Delta \otimes \mathrm{id}) \circ \Delta(a) = \sum_{i=1}^k \Delta(a_i^1) \otimes a_i^2$. Similarly, A acts on $V_1 \otimes (V_2 \otimes V_3)$ via $(\mathrm{id} \otimes \Delta) \circ \Delta : A \to A^{\otimes 3}$. So, if we want to the isomorphism $(v_1 \otimes v_2) \otimes v_3 \mapsto v_1 \otimes (v_2 \otimes v_3)$ to be A-linear, it is natural to require that $(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta$. In other words, we want the following diagram to be commutative.

$$\begin{tikzcd}
A \otimes A \arrow[r, "\mathrm{id} \otimes \Delta"] & A \otimes A \otimes A \\
A \arrow[u, "\Delta"] \arrow[r, "\Delta"] & A \otimes A \arrow[u, "\Delta \otimes \mathrm{id}"']
\end{tikzcd}$$

If this holds, then we say that Δ is coassociative.

Let us motivate the terminology ("coproduct" and "coassociative"). Let A be a finite dimensional algebra. Let us write $m : A \otimes A \to A$ for the product. Then m is associative (i.e., $m(m(a \otimes b) \otimes c) = m(a \otimes m(b \otimes c))$) if and only if the following diagram is commutative.

$$\begin{tikzcd}
A \otimes A \arrow[d, "m"] & A \otimes A \otimes A \arrow[l, "\mathrm{id} \otimes m"'] \arrow[d, "m \otimes \mathrm{id}"] \\
A & A \otimes A \arrow[l, "m"']
\end{tikzcd}$$

Now let us dualize. We get the space A^* together with the map $m^* : A^* \to A^* \otimes A^*$ that is natural to call a coproduct. Clearly, m is associative if and only if m^* is coassociative.

1.3. **Axioms of Hopf algebras.** We need to axiomatically describe two more maps: the counit $\eta : A \to \mathbb{C}$ and the antipode $S : A \to A^{op}$.

An axiom of a counit should be dual to that of the unit, $e : \mathbb{C} \to A, z \mapsto z \cdot 1$. The element $e(1)$ is a unit if and only if the following diagram is commutative.

$$\begin{tikzcd}
 & A \otimes A \arrow[d, "m"] & \\
\mathbb{C} \otimes A \arrow[ur, "e \otimes \mathrm{id}"] \arrow[r, "\cong"] & A & A \otimes \mathbb{C} \arrow[ul, "\mathrm{id} \otimes e"'] \arrow[l, "\cong"']
\end{tikzcd}$$

Dualizing this diagram we get the counit axiom: the following diagram is commutative.

$$\begin{tikzcd}
& A\otimes A \arrow[dl, "\eta\otimes\mathrm{id}"'] \arrow[dr, "\mathrm{id}\otimes\eta"] & \\
\mathbb{C}\otimes A & A \arrow[l, "\cong"'] \arrow[r, "\cong"] \arrow[u, "\Delta"] & A\otimes\mathbb{C}
\end{tikzcd}$$

Finally, the antipode axiom is the commutativity of the following diagram.

$$\begin{tikzcd}
& A\otimes A \arrow[r, "S\otimes\mathrm{id}"] & A\otimes A \arrow[dr, "m"] & \\
A \arrow[ur, "\Delta"] \arrow[dr, "\Delta"'] \arrow[r, "\eta"] & \mathbb{C} \arrow[rr, "e"] & & A \\
& A\otimes A \arrow[r, "\mathrm{id}\otimes S"] & A\otimes A \arrow[ur, "m"'] &
\end{tikzcd}$$

Let us illustrate this axiom in the example of $A = \mathbb{C}G$, where $S(g) = g^{-1}$. There $\Delta(g) = g\otimes g, S\otimes\mathrm{id}(g\otimes g) = g^{-1}\otimes g$, $m(g^{-1}\otimes g) = 1 = e\circ\eta(g)$.

Definition 1.3. By a Hopf algebra we mean a $\mathbb{C}$-vector space A with five maps (m, e, Δ, η, S), where $m : A\otimes A \to A, e : \mathbb{C}\to A, \Delta : A \to A\otimes A, \eta : A\to\mathbb{C}, S : A\to A$ such that:

(1) (A, m, e) is an associative unital algebra.
(2) $\Delta : A\to A\otimes A, S : A \to A^{op}, \eta : A\to\mathbb{C}$ are algebra homomorphisms.
(3) Δ is coassociative, and η satisfies the counit axiom.
(4) S satisfies the antipode axiom.

Remark 1.4. In fact, once m, e, Δ are specified, S and η are recovered in at most one way.

It is straightforward to check that $\mathbb{C}G$ and $U(\mathfrak{g})$ are Hopf algebras.

1.4. **Duality of Hopf algebras.** Now let $(A, m, e, \Delta, \eta, S)$ be a finite dimensional Hopf algebra. One can show that $(A^*, \Delta^*, \eta^*, m^*, e^*, S^*)$ is a Hopf algebra as well.

Example 1.5. Let us describe $(\mathbb{C}G)^*$. As a vector space, $(\mathbb{C}G)^*$ is the algebra of functions on G, to be denoted by $\mathbb{C}[G]$. The map $\Delta : \mathbb{C}G \to \mathbb{C}G\otimes\mathbb{C}G$ sends g to $g\otimes g$. So $\Delta^*(\alpha\otimes\beta)(g) = \alpha\otimes\beta(g\otimes g) = \alpha(g)\beta(g)$ is the usual multiplication of functions. Similarly, η^* sends 1 to the identity function. The map $m^* : \mathbb{C}[G] \to \mathbb{C}[G]\otimes\mathbb{C}[G] = \mathbb{C}[G\times G]$ sends $\alpha\in\mathbb{C}[G]$ to $m^*(\alpha)(g, h) := \alpha(gh)$. The map $e^* : \mathbb{C}[G]\to\mathbb{C}$ maps α to $\alpha(1)$. Finally, we have $(S^*\alpha)(g) = \alpha(g^{-1})$.

1.5. **Cocommutativity.** In the cases of $A = U(\mathfrak{g}), \mathbb{C}G$ the isomorphism $V_1\otimes V_2 \xrightarrow{\sim} V_2\otimes V_1$ is that of A-modules. The reason for this is that the *opposite coproduct* $\Delta^{op} := \sigma\circ\Delta$, where $\sigma : A^{\otimes 2}\to A^{\otimes 2}, a\otimes b\mapsto b\otimes a$, coincides with Δ. The Hopf algebras with $\Delta = \Delta^{op}$ are called *cocommutative*. However, there are Hopf algebras that are not cocommutative, e.g. $\mathbb{C}[G]$.

The Hopf algebras we have encountered so far are commutative as algebras ($\mathbb{C}[G]$) or cocommutative ($\mathbb{C}G, U(\mathfrak{g})$). Of course, one can cook a Hopf algebra that is neither commutative nor cocommutative: the tensor product of two Hopf algebras carries a natural Hopf algebra structure and we can take the tensor product of a non-commutative Hopf algebra with a non-cocommutative one. But this is very boring. In the next section, we will study a far more interesting example.

2. $U_q(\mathfrak{sl}_2)$

2.1. $U_q(\mathfrak{sl}_2)$ **as a Hopf algebra.** We will define the "quantum $\mathfrak{sl}_2$" by generators and relations (as an algebra) and then define Δ, η, S on the generators.

Let $q \in \mathbb{C} \setminus \{0, \pm 1\}$ (we can also take q to be an independent variable in the field of rational functions $\mathbb{C}(q)$). We define the algebra $U_q(\mathfrak{sl}_2)$ generated by E, F, K, K^{-1} subject to the following relations:

$$KK^{-1} = K^{-1}K = 1, \quad KEK^{-1} = q^2 E, \quad KFK^{-1} = q^{-2}F, \quad EF - FE = \frac{K - K^{-1}}{q - q^{-1}}.$$

Note that the algebra $U := U_q(\mathfrak{sl}_2)$ is spanned by the monomials $F^k K^\ell E^m$, where $k, m \in \mathbb{Z}_{\geqslant 0}$, and $\ell \in \mathbb{Z}$. In fact, these monomials are linearly independent (the PBW theorem).

Now let us define the Hopf algebra structure. We set

$$\begin{aligned} &\Delta(E) = E \otimes 1 + K \otimes E, \quad \Delta(F) = F \otimes K^{-1} + 1 \otimes F, \quad \Delta(K) = K \otimes K, \\ &\eta(E) = \eta(F) = 0, \quad \eta(K) = 1, \\ &S(E) = -K^{-1}E, S(F) = -FK, S(K) = K^{-1}. \end{aligned} \tag{2.1}$$

Proposition 2.1. *Δ, η, S extend to required algebra homomorphisms. Moreover, U becomes a Hopf algebra.*

Proof. This is a mighty tedious check... What we need to verify is that Δ, S, η respect the relations in U and that the axioms (3),(4) in the definition of a Hopf algebra hold on the generators E, K, F. Let us check that $\Delta([E, F]) = [\Delta(E), \Delta(F)]$, which is the hardest relation to check. We have

$$\Delta([E, F]) = \Delta\left(\frac{K - K^{-1}}{q - q^{-1}}\right) = \frac{K \otimes K - K^{-1} \otimes K^{-1}}{q - q^{-1}}$$

On the other hand,

$$\begin{aligned} &[\Delta(E), \Delta(F)] = [E \otimes 1 + K \otimes E, F \otimes K^{-1} + 1 \otimes F] = [E, F] \otimes K^{-1} + K \otimes [E, F] + \\ &+ [K \otimes E, F \otimes K^{-1}] = \frac{(K - K^{-1}) \otimes K^{-1}}{q - q^{-1}} + \frac{K \otimes (K - K^{-1})}{q - q^{-1}} + KF \otimes EK^{-1} - \\ &- FK \otimes K^{-1}E = \frac{K \otimes K - K^{-1} \otimes K^{-1}}{q - q^{-1}} + KF \otimes EK^{-1} - (q^2 KF) \otimes (q^{-2}EK^{-1}) = \\ &= \frac{K \otimes K - K^{-1} \otimes K^{-1}}{q - q^{-1}}. \end{aligned}$$

$\square$

We note that $\Delta \neq \Delta^{op}$. In particular, the map $v_1 \otimes v_2 \mapsto v_2 \otimes v_1$ does not give an isomorphism $V_1 \otimes V_2 \to V_2 \otimes V_1$, in general. However, in the next lecture we will find an element $R \in U_q(\mathfrak{sl}_2) \otimes U_q(\mathfrak{sl}_2)$ (this is a slight lie, we need a certain completion) with $R^{-1}\Delta(u)R = \Delta^{op}(u)$. This element, called the universal R-matrix, is extremely important. In particular, it will allow us to construct link invariants, such as the Jones polynomial.

2.2. $U_q(\mathfrak{sl}_2)$ **vs** $U(\mathfrak{sl}_2)$**.** The algebra $U_q(\mathfrak{sl}_2)$ should be thought as a deformation of $U(\mathfrak{sl}_2)$ (the latter corresponds to $q = 1$). This however requires some care, we cannot put $q = 1$ in the definition of $U_q(\mathfrak{sl}_2)$. In order to make the claim about the deformation more precise, we will need to consider the formal version of $U_q(\mathfrak{sl}_2)$, we will call it $U_\hbar(\mathfrak{sl}_2)$. This will be an algebra over $\mathbb{C}[[\hbar]]$.

By definition, as an algebra, $U_\hbar(\mathfrak{sl}_2)$ is the quotient of $T(\mathfrak{sl}_2)[[\hbar]]$ by the relations

$$[h,e]=2e,[h,f]=-2f,[e,f]=\frac{\exp(\hbar h)-\exp(-\hbar h)}{\exp(\hbar)-\exp(-\hbar)}.$$

Note that $\frac{\exp(\hbar h)-\exp(-\hbar h)}{\exp(\hbar)-\exp(-\hbar)}$ is a formal power series in $\hbar$, modulo $\hbar$ it equals h. It follows that $U_\hbar(\mathfrak{sl}_2)/(\hbar)=U(\mathfrak{sl}_2)$.

One can show that $\hbar$ is not a zero divisor in $U_\hbar(\mathfrak{sl}_2)$. Note that $E=e,F=f,K=\exp(\hbar h),q=\exp(\hbar)$ satisfy the relations of $U_q(\mathfrak{sl}_2)$. Indeed, for example, we get

$$\exp(\hbar h)e\exp(-\hbar h)=\exp(\hbar\,\mathrm{ad}(h))e=\exp(2\hbar)e.$$

One can introduce the Hopf algebra structure on $U_\hbar(\mathfrak{sl}_2)$ but one needs to extend the definition to allow Δ to be a homomorphism $U_\hbar(\mathfrak{sl}_2)\to U_\hbar(\mathfrak{sl}_2)\widehat{\otimes}_{\mathbb{C}[[\hbar]]}U_\hbar(\mathfrak{sl}_2)$. Here $\widehat{\otimes}$ denotes the *completed tensor product*. While the usual tensor product consists of all finite sums of decomposable tensors, the completed product consists of all converging (in the $\hbar$-adic topology) infinite sums.

2.3. **Algebras $U_q(\mathfrak{g})$.** We can define quantum groups $U_q(\mathfrak{g})$ for any semisimple Lie algebra $\mathfrak{g}$ (or, more generally, any Kac-Moody algebra $\mathfrak{g}(A)$ for a symmetrizable Cartan matrix A). Let us start with $\mathfrak{g}=\mathfrak{sl}_{n+1}$.

Recall that the usual universal enveloping algebra $U(\mathfrak{sl}_{n+1})$ is defined by the generators $e_i,h_i,f_i,i=1,\ldots,n$, and the following relations:

(i) $[h_i,e_i]=2e_i,[h_i,f_i]=-2f_i,[e_i,f_i]=h_i$.
(ii) $[h_i,h_j]=0$.
(iii) $[h_i,e_j]=a_{ij}e_j,[h_i,f_j]=-a_{ij}f_j$.
(iv) $e_if_j=f_je_i,i\neq j$.
(v) $e_ie_j=e_je_i$, if $a_{ij}=0$, and $e_i^2e_j-2e_ie_je_i+e_je_i^2=0$, if $a_{ij}=-1$.
(vi) $f_if_j=f_jf_i$, if $a_{ij}=0$, and $f_i^2f_j-2f_if_jf_i+f_jf_i^2=0$, if $a_{ij}=-1$.

Recall that here $a_{ij}=-1$ if $|i-j|=1$ and $a_{ij}=0$ if $|i-j|>1$.

The quantum group $U_q(\mathfrak{sl}_{n+1})$ is defined by the generators $E_i,K_i^{\pm1},F_i,i=1,\ldots,n$, with relations

(i$_q$) $K_iE_iK_i^{-1}=q^2E_i,K_iF_iK_i^{-1}=q^{-2}F_i,[E_i,F_i]=\frac{K_i-K_i^{-1}}{q-q^{-1}}$.
(ii$_q$) $[K_i,K_j]=0$.
(iii$_q$) $K_iE_jK_i^{-1}=q^{a_{ij}}E_j,K_iF_jK_i^{-1}=q^{-a_{ij}}F_j$.
(iv$_q$) $E_iF_j=F_jE_i,i\neq j$.
(v$_q$) $E_iE_j=E_jE_i$ if $a_{ij}=0$ and $E_i^2E_j-[2]_qE_iE_jE_i+E_jE_i^2=0$ if $a_{ij}=-1$.
(vi$_q$) $F_iF_j=F_jF_i$ if $a_{ij}=0$ and $F_i^2F_j-[2]_qF_iF_jF_i+F_jF_i^2=0$ if $a_{ij}=-1$.

Here $[2]_q$ denotes the "quantum 2", i.e., $q+q^{-1}$.

The similar definition will work for any simply laced Cartan matrix A (meaning that $a_{ij}\in\{0,-1\}$ if $i\neq j$). When A is not simply laced (e.g., of type B_n,C_n,F_4,G_2), the definition is more technical, one needs to use different q's for the "$\mathfrak{sl}_2$-subalgebras" of $U_q(\mathfrak{g})$ according the length of the corresponding root. Namely, when $\mathfrak{g}$ is finite dimensional, we define $d_i\in\{1,2,3\}$ as $(\alpha_i,\alpha_i)/2$, where $(\cdot,\cdot)$ is a W-invariant form on $\mathfrak{h}^*$ normalized in such a way that $(\alpha,\alpha)=2$ for the short roots (we have two different root lengthes). This can be generalized to an arbitrary symmetrizable Kac-Moody algebra but we are not going to explain that.

Now set $q_i := q^{d_i}$ (so that $q_1 = q$). We also define the quantum integer $[n]_{q_i} = q_i^{n-1} + q_i^{n-2} + \ldots + q_i^{1-n}$, and the quantum factorial $[n]_{q_i}! = [1]_{q_i} \ldots [n]_{q_i}$. We set

$$\binom{n}{k}_{q_i} = \frac{[n]_{q_i}!}{[k]_{q_i}![n-k]_{q_i}!}.$$

Now we define $U_q(\mathfrak{g})$ as the algebra generated by E_i, K_i, F_i subject to the relations

(i$_q$) $K_iE_iK_i^{-1} = q_i^2 E_i, K_iF_iK_i^{-1} = q_i^{-2}F_i, [E_i, F_i] = \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}$.

(ii$_q$) $[K_i, K_j] = 0$.

(iii$_q$) $K_iE_jK_i^{-1} = q_i^{a_{ij}} E_j, K_iF_jK_i^{-1} = q_i^{-a_{ij}} F_j$.

(iv$_q$) $E_iF_j = F_jE_i, i \neq j$.

(v$_q$) $\sum_{k=0}^{1-a_{ij}} (-1)^k \binom{1-a_{ij}}{k}_{q_i} E_i^{1-a_{ij}-k} E_j E_i^k = 0$.

(vi$_q$) $\sum_{k=0}^{1-a_{ij}} (-1)^k \binom{1-a_{ij}}{k}_{q_i} F_i^{1-a_{ij}-k} F_j F_i^k = 0$.

Note that they are obtained from the relations for $U(\mathfrak{g})$ in the same fashion as the relations for $U_q(\mathfrak{sl}_{n+1})$ are obtained from those for $U(\mathfrak{sl}_{n+1})$.

The Hopf algebra structure on $U_q(\mathfrak{g})$ is introduced as follows: we just define Δ, S, η on E_i, F_i, K_i as in $U_{q_i}(\mathfrak{sl}_2)$.

LECTURE 13: REPRESENTATIONS OF $U_q(\mathfrak{g})$ AND R-MATRICES

IVAN LOSEV

Introduction

In this lecture we study the representation theory of $U_q(\mathfrak{g})$ when q is not a root of 1. In Section 1, we classify the finite dimensional irreducible representations of $U := U_q(\mathfrak{sl}_2)$, sketch the proof of complete reducibility and explain what happens for a general $\mathfrak{g}$.

As we have seen in the previous lecture, the obvious isomorphism $\sigma : v_1 \otimes v_2 \to v_2 \otimes v_1 : V_1 \otimes V_2 \xrightarrow{\sim} V_2 \otimes V_1$ is not U-linear. However, one can find an element R in a suitable completion of $U \otimes U$ such that $R \circ \sigma : V_1 \otimes V_2 \xrightarrow{\sim} V_2 \otimes V_1$ is a U-module isomorphism. This will be done in Section 2. This construction is of importance for knot invariants, as will be explained in the next lecture.

1. Representation theory of $U_q(\mathfrak{g})$, I

Recall that, as an algebra, U is given by generators $E, F, K^{\pm 1}$ and relations

$$KEK^{-1} = q^2 E, \quad KFK^{-1} = q^{-2} F, \quad EF - FE = \frac{K - K^{-1}}{q - q^{-1}}.$$

1.1. **Classification of the irreducibles.** Let us start by producing some examples of the irreducible representations of U.

Example 1.1. Let us classify the one-dimensional representations. We have $KEK^{-1} = q^2E, KFK^{-1} = q^{-2}F$. Since $q \neq \pm 1$, it follows that E, F act by 0. So $K - K^{-1} = (q - q^{-1})(EF - FE)$ acts by 0. We deduce that K acts by ± 1. Both choice give representations. Of course, the representation, where K acts by 1, E, F act by 0, is the trivial representation, one that is given by the counit η.

Example 1.2. The assignment $E \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, F \mapsto \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, K \mapsto \begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix}$ gives rise to a two-dimensional representation of U. This is a so called tautological representation.

Lemma 1.3. *Suppose q is not a root of* 1. *Let V be a finite dimensional U-module. Then the elements E, F act on V nilpotently.*

Proof. For $\alpha \in \mathbb{C}^\times$, let V_α denote the generalized eigenspace for K with eigenvalue α. It is easy to show that $EV_\alpha \subset V_{q^2\alpha}$. Since q is not a root of 1, we see that all numbers $q^{2n}\alpha$ are different. It follows that E acts nilpotently. For the same reasons, F acts nilpotently. $\square$

Now the classification of $\mathrm{Irr}_{fin}(U)$ works in the same way as for $U(\mathfrak{sl}_2)$. Namely, we have the subalgebra $U_q(\mathfrak{b}) \subset U$ spanned by K, E, it has a basis $K^\ell E^m$ for $\ell \in \mathbb{Z}, m \in \mathbb{Z}_{\geqslant 0}$. The algebra U is a free right module over $U_q(\mathfrak{b})$ with basis $F^k, k \in \mathbb{Z}_{\geqslant 0}$. Then, for $\alpha \in \mathbb{C}^\times$, we can form the Verma module $\Delta_q(\alpha) := U \otimes_{U_q(\mathfrak{b})} \mathbb{C}_\alpha$, where K acts on $\mathbb{C}_\alpha$ by α and E acts by

0. The U-module $\Delta_q(\alpha)$ has basis $v_k := F^k v_\alpha, k \in \mathbb{Z}_{\geqslant 0}$. The action of F, K, E in this basis is given by

$$Fv_k = v_{k+1}, \quad Kv_k = q^{-2k}\alpha v_k, \quad Ev_k = [k]_q \frac{\alpha q^{1-k} - \alpha^{-1}q^{k-1}}{q - q^{-1}} v_{k-1}.$$

The third equation follows from

$$[E, F^n] = [n]_q F^{n-1} \frac{Kq^{1-n} - K^{-1}q^{n-1}}{q - q^{-1}}. \tag{1.1}$$

Theorem 1.4. *Suppose q is not a root of 1. Then the finite dimensional irreducible U-modules are in one-to-one correspondence with the set $\{\pm q^n\}_{n \in \mathbb{Z}_{\geqslant 0}}$. The module $L(\pm q^n)$, the irreducible quotient of $\Delta_q(\pm q^n)$, has basis $u_0, \ldots, u_n$, where the action of the generators is given by*

$$Ku_i = \pm q^{n-2i} u_i, Fu_i = [n-i]_q u_{i+1}, Eu_i = \pm [i]_q u_{i-1}.$$

Proof. As in the proof for U, we need to understand when $\Delta_q(\alpha)$ has a proper quotient. This is only possible when $Ev_k = 0$. The number $[k]_q = \frac{q^k - q^{-k}}{q - q^{-1}}$ is never zero thanks to our assumption that q is not a root of 1. So $Ev_k = 0$ if and only if

$$\frac{\alpha q^{1-k} - \alpha^{-1} q^{k-1}}{q - q^{-1}} = 0 \Leftrightarrow \alpha^2 = q^{2(k-1)} \Leftrightarrow \alpha = \pm q^{k-1}.$$

In this case, we have a k-dimensional quotient of $\Delta_q(\alpha)$. Thanks to the standard universal property of $\Delta_q(\alpha)$, every simple module is a quotient of one of $\Delta_q(\alpha)$ (and it is easy to see that α is recovered uniquely from the simple module). We set $u_k := v_k / [n-k]_q!$. □

Example 1.5. The modules from Example 1.1 are $L(\pm 1)$. The module from Example 1.2 is $L(q)$. Note that $L(-q^n) = L(-1) \otimes L(q^n)$. Thanks to this, one usually only studies the modules $L(q^n)$.

1.2. **Complete reducibility.** We can introduce the quantum Casimir element

$$C = FE + \frac{Kq + K^{-1}q^{-1}}{(q - q^{-1})^2} \in U.$$

One can show (this is a part of the homework) that this element is central in U.

Theorem 1.6. *Let q be not a root of 1. Then any finite dimensional U-module is completely reducible.*

Proof. For $v_\alpha \in \Delta_q(\alpha)$, we get $Cv_\alpha = \frac{\alpha q + \alpha^{-1} q^{-1}}{(q - q^{-1})^2} v_\alpha$. In particular, we see that all scalars of the action of C on $L(\pm q^n)$ are distinct. As in the case of $U(\mathfrak{sl}_2)$, we only need to prove that $L(\pm q^n)$ has no self-extensions. This is done similarly to that case. □

1.3. **General $U_q(\mathfrak{g})$.** We assume that q is not a root of 1. Let $\mathfrak{g}$ be a semisimple Lie algebra with generators $e_i, h_i, f_i, i = 1, \ldots, n$. Let us explain the classification of finite dimensional irreducible $U_q(\mathfrak{g})$-modules.

Theorem 1.7. *Any finite dimensional representation of $U_q(\mathfrak{g})$ is completely reducible.*

Let us explain how to classify the finite dimensional irreducible representations. First, let us consider the one-dimensional representations. On such a representation, all elements E_i, F_i act by 0, and the elements K_i act by ± 1. So for $\kappa \in \{\pm 1\}^n$, we have the one-dimensional module $L(\kappa)$, where K_i acts by κ_i.

Modulo these 2^n modules, the finite dimensional representation theory of $U_q(\mathfrak{g})$ looks just like the representation theory of $\mathfrak{g}$. Recall that P^+ denotes the set of dominant weights.

Theorem 1.8. *There is a bijection* $\{\pm 1\}^n \times P^+ \xrightarrow{\sim} \mathrm{Irr}_{fin}(U_q(\mathfrak{g}))$ *that sends* (κ, λ) *to a unique finite dimensional irreducible module* $L(\kappa q^\lambda)$ *that has a highest vector* v_λ *such that* $E_i v_\lambda = 0, K_i v_\lambda = \kappa_i q^{\lambda(\alpha_i^\vee)} v_\lambda$.

We do not provide a proof. We note that this description implies $L(\kappa q^\lambda) = L(\kappa) \otimes L(q^\lambda)$.

To finish let us point out that one can define the notion of a character of $L(q^\lambda)$ in a natural way. The character is given by the Weyl character formula.

2. Universal R-matrix

2.1. **Three coproducts.** Recall the coproduct $\Delta : U \to U^{\otimes 2}$ given on the generators by

$$\Delta(K) = K \otimes K, \Delta(E) = E \otimes 1 + K \otimes E, \Delta(F) = F \otimes K^{-1} + 1 \otimes F.$$

The opposite coproduct $\Delta^{op} := \sigma \circ \Delta$, where σ denotes the permutation of the tensor factors is then given by

$$\Delta^{op}(K) = K \otimes K, \Delta^{op}(E) = E \otimes K + 1 \otimes E, \Delta^{op}(F) = F \otimes 1 + K^{-1} \otimes F.$$

We want to find an element $R \in U^{\otimes 2}$ (in fact, we will have to use a completion) such that $R\Delta^{op}(u) = \Delta(u)R$. If V_1, V_2 are U-modules, then the map $R_{V_1,V_2} \circ \sigma : V_1 \otimes V_2 \to V_2 \otimes V_1, v_1 \otimes v_2 \mapsto R_{V_1,V_2}(v_2 \otimes v_1)$ is an isomorphism of U-modules.

In order to produce R, we will need the third coproduct, Δ', given by

$$\Delta'(K) = K \otimes K, \Delta'(E) = E \otimes 1 + K^{-1} \otimes E, \Delta'(F) = F \otimes K + 1 \otimes F.$$

This coproduct is obtained from Δ by a twist with an anti-involution τ of U_q given by $\tau(K) = K^{-1}, \tau(E) = E, \tau(F) = F$. Then $\Delta'(u) = \tau \otimes \tau(\Delta(\tau(u)))$. This equality implies that Δ' also gives a coassociative coproduct on U (or we can check this directly). We will produce R as a product $\Theta\Psi$ with $\Theta^{-1}\Delta(u)\Theta = \Delta'(u)$ and $\Psi^{-1}\Delta'(u)\Psi = \Delta^{op}(u)$.

2.2. **Construction of Θ.** Let us construct Θ. This will be an infinite sum of the form $\sum_{n=0}^{\infty} a_n F^n \otimes E^n$, where we will find the coefficients a_n from $\Theta\Delta'(E) = \Delta(E)\Theta$.

$$\begin{aligned}
&\Theta\Delta'(E) = \Delta(E)\Theta \\
\Leftrightarrow &\Big(\sum_{n=0}^{\infty} a_n F^n \otimes E^n\Big)(E \otimes 1 + K^{-1} \otimes E) = (E \otimes 1 + K \otimes E)\Big(\sum_{n=0}^{\infty} a_n F^n \otimes E^n\Big) \\
\Leftrightarrow &\sum_{n=0}^{\infty} a_n (F^n E \otimes E^n + F^n K^{-1} \otimes E^{n+1}) = \sum_{n=0}^{\infty} a_n (EF^n \otimes E^n + KF^n \otimes E^{n+1}) \\
\Leftrightarrow &\sum_{n=0}^{\infty} a_n [E, F^n] \otimes E^n = \sum_{n=0}^{\infty} a_n (F^n K^{-1} - KF^n) \otimes E^{n+1} \\
\Leftrightarrow &a_{n+1}[E, F^{n+1}] = a_n (F^n K^{-1} - KF^n).
\end{aligned}$$

By (1.1), we get $[E, F^{n+1}] = [n+1]_q F^n \frac{Kq^{-n} - K^{-1}q^n}{q - q^{-1}}$. On the other hand $F^n K^{-1} - KF^n = F^n(K^{-1} - q^{-2n}K)$. Therefore

$$a_{n+1}[n+1]_q F^n \frac{Kq^{-n} - K^{-1}q^n}{q - q^{-1}} = a_n F^n (K^{-1} - q^{-2n}K) \Leftrightarrow a_{n+1} = \frac{(q^{-1} - q)q^{-n}}{[n+1]_q} a_n.$$

We conclude that

$$\Theta = \sum_{n=0}^{\infty} \frac{(q^{-1}-q)^n q^{-n(n-1)/2}}{[n]_q!} F^n \otimes E^n. \tag{2.1}$$

One can show that $\Theta\Delta'(K) = \Delta(K)\Theta$ (this is almost immediate) and that $\Theta\Delta'(F) = \Delta(F)\Theta$ (this is a computation very similar to what was done above).

Example 2.1. Let us compute $\Theta_{V\otimes V}$, where $V = L(q)$. We have $\Theta_{V\otimes V} = 1+(q^{-1}-q)F\otimes E$. Let us write this operator as a matrix. Let v_1, v_2 (resp., v_1', v_2') be the natural basis in the first and in the second factor. Then in the basis $v_1\otimes v_1', v_1\otimes v_2', v_2\otimes v_1', v_2\otimes v_2'$ we get the following matrix of Θ: $\begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & q^{-1}-q & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}$.

2.3. **Construction of Ψ.** An element Ψ satisfying $\Psi\Delta^{op}(u) = \Delta'(u)\Psi$ will depend only on K. But it is not expressed in terms of algebraic functions in q, K (we have to use log's). So we will just define $\Psi_{V_2\otimes V_1} : V_2 \otimes V_1 \to V_2\otimes V_1$, where K acts on V_2, V_1 with powers of q (we can also define Ψ as an element of the *idempotent completion* of $U\otimes U$, where we add infinite sums $\sum_{\lambda,\mu} a_{\lambda\mu}\pi_\lambda \otimes \pi_\mu$, where $\lambda, \mu \in \{\pm q^n\}$ and π_λ acts on a U-module V as the projection to V_λ). Let $\Psi_{V_2\otimes V_1}(v_n\otimes u_m) = \psi(n,m)v_n\otimes u_m$, where $v_n \in V_2, u_m\in V_1$ are K-eigenvectors with eigenvalues q^n, q^m, and ψ is a function we need to determine. We have

$$\Psi\Delta^{op}(E)(v_n\otimes u_m) = \Psi(E\otimes K + 1\otimes E)(v_n\otimes u_m) = \Psi(q^m Ev_n\otimes u_m + v_n\otimes Eu_m) =$$
$$= q^m\psi(n+2,m)Ev_n\otimes u_m + \psi(n,m+2)v_n\otimes Eu_m.$$
$$\Delta'(E)\Psi(v_n\otimes u_m) = \psi(n,m)(K^{-1}\otimes E + E\otimes 1)(v_n\otimes u_m) =$$
$$= \psi(n,m)(Ev_n\otimes u_m + q^{-n}v_n\otimes Eu_m).$$
$$\Psi\Delta^{op}(E) = \Delta'(E)\Psi \Leftrightarrow \psi(n+2,m) = q^{-m}\psi(n,m) \text{ and } \psi(n,m+2) = q^{-n}\psi(n,m).$$

Conversely, for any ψ satisfying the conditions above, we have $\Psi\Delta^{op}(u) = \Delta'(u)\Psi$. Indeed, for $u = K$ this holds for any ψ and for $u = F$ the conditions on ψ are equivalent to what we had above. Note that to recover ψ we just need to specify the values $\psi(\alpha,\beta)$ when $\alpha,\beta\in\{-1,0\}$.

Example 2.2. Let us consider the case when $V_1 = V_2 = V = L(q)$. We set $\psi(-1,-1) = q$. Then $\psi(1,-1) = \psi(-1,1) = 1, \psi(1,1) = q^{-1}$. In the same basis as in Example 2.1, we get $\Psi_{V\otimes V} = \mathrm{diag}(q^{-1})$. So

$$R_{V\otimes V} = \Theta_{V\otimes V}\Psi_{V\otimes V} = \begin{pmatrix} q^{-1} & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & q^{-1}-q & 1 & 0\\ 0 & 0 & 0 & q^{-1}\end{pmatrix}, \quad R_{V\otimes V}\circ\sigma = \begin{pmatrix} q^{-1} & 0 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 1 & q^{-1}-q & 0\\ 0 & 0 & 0 & q^{-1}\end{pmatrix}.$$

Of course, it is easy to check directly that $R\circ\sigma : V\otimes V \to V\otimes V$ is U-linear. This is the case that we will mostly need. But for higher dimensional V_1, V_2 constructing an isomorphism by hand is very hard.

2.4. **Yang-Baxter equation.** We have constructed an isomorphism $R_{V_2\otimes V_1}\circ\sigma : V_1\otimes V_2 \to V_2\otimes V_1$. Now pick three U-modules, V_1, V_2, V_3. We can produce two isomorphisms $V_1\otimes V_2\otimes V_3 \to V_3\otimes V_2\otimes V_1$ by using the isomorphisms of the form $R_{?\otimes ?}\circ\sigma$ (note that applying this to V_1 and V_3 and inserting id_{V_2} in the middle does not give a U-linear map). Let us write $\tau_{?,?}$ for $R_{?\otimes ?}\circ\sigma$.

$$\begin{tikzcd}
 & V_2\otimes V_1\otimes V_3 \arrow[r, "\mathrm{id}_{V_2}\otimes\tau_{V_1,V_3}"] & V_2\otimes V_3\otimes V_1 \arrow[dr, "\tau_{V_2,V_3}\otimes\mathrm{id}_{V_1}"] & \\
V_1\otimes V_2\otimes V_3 \arrow[ur, "\tau_{V_1,V_2}\otimes\mathrm{id}_{V_3}"] \arrow[dr, "\mathrm{id}_{V_1}\otimes\tau_{V_2,V_3}"'] & & & V_3\otimes V_2\otimes V_1 \\
 & V_1\otimes V_3\otimes V_2 \arrow[r, "\tau_{v_1,V_3}\otimes\mathrm{id}_{V_2}"] & V_3\otimes V_1\otimes V_2 \arrow[ur, "\mathrm{id}_{V_3}\otimes\tau_{V_1,V_2}"'] &
\end{tikzcd}$$

We want this diagram to commute (the hexagon axiom). Note that the top isomorphism is $(R_{23}R_{13}R_{12})_{V_3\otimes V_2\otimes V_1}\circ\sigma_{13}$, while the bottom isomorphism is $(R_{12}R_{13}R_{23})_{V_3\otimes V_2\otimes V_1}\circ\sigma_{13}$. Here the notation is as follows. We write R_{12} for $R\otimes 1\in U^{\tilde{\otimes}3}$ (where we put $\tilde{\otimes}$ to indicate that we take a completion) and R^{23} for $1\otimes R$. The notation R_{13} means $\sum_i R_i^1\otimes 1\otimes R_i^2$, where $R=\sum_i R_i^1\otimes R_i^2$. We write σ_{13} for the permutation $u\otimes v\otimes w\mapsto w\otimes v\otimes u$. So the hexagon diagram is commutative provided

$$R_{12}R_{13}R_{23}=R_{23}R_{13}R_{12}. \tag{2.2}$$

This is the quantum Yang-Baxter equation (shortly, QYBE) that first appeared in Statistical Physics and was an initial motivation for introducing quantum groups.

Theorem 2.3. *QYBE holds for $R\in U\tilde{\otimes}U$.*

We omit the proof.

2.5. **Braid group representation.** There is an alternative way to view QYBE when we are dealing with the n-fold tensor product of a single U-module V. Define the U-module automorphism $\tau_{i,i+1}$ of $V^{\otimes n}$ as $\mathrm{id}_V^{\otimes i-1}\otimes\tau_{V,V}\otimes\mathrm{id}^{n-1-i}$ (we permute the ith and $i+1$th copies). The hexagon axiom gives $\tau_{i,i+1}\tau_{i+1,i+2}\tau_{i,i+1}=\tau_{i+1,i+2}\tau_{i,i+1}\tau_{i+1,i+2}$ for all $i\in\{1,\dots,n-1\}$. Clearly, $\tau_{i,i+1}\tau_{j,j+1}=\tau_{j,j+1}\tau_{i,i+1}$ if $|i-j|>1$.

Definition 2.4. The braid group B_n is the group with generators $T_1,\dots,T_{n-1}$ and relations $T_iT_j=T_jT_i$ when $|i-j|>1$ and $T_iT_{i+1}T_i=T_{i+1}T_iT_{i+1}$.

So we have a representation of B_n in $V^{\otimes n}$ given by $T_i\mapsto\tau_{i,i+1}$. When $V=L(q)$, one can verify this directly without referring to Theorem 2.3. Now note that $\tau=\tau_{1,2}\in\mathrm{End}(V\otimes V)$ satisfies $\tau^2=1+(q^{-1}-q)\tau$. In other words, the action of $\mathbb{C}B_n$ on $V^{\otimes n}$ factors through the Hecke algebra $\mathcal{H}_{q^{-1}}(n)$.

Remark 2.5. Let $\mathfrak{g}$ be a finite dimensional semisimple Lie algebra so that we can form the quantum group $U_q(\mathfrak{g})$. We still have the universal R-matrix $R\in U_q(\mathfrak{g})\tilde{\otimes}U_q(\mathfrak{g})$ satisfying QYBE and such that $R_{V_2\otimes V_1}\circ\sigma: V_1\otimes V_2\to V_2\otimes V_1$ is an isomorphism of $U_q(\mathfrak{g})$-modules. It was constructed by Drinfeld.

LECTURE 14: LINK INVARIANTS FROM QUANTUM GROUPS

IVAN LOSEV

Introduction

In this lecture we explain how to construct invariants of links from representations of quantum groups. We use the representation $V = L(q)$ of $U_q(\mathfrak{sl}_2)$ to produce the invariant known as the Jones polynomial.

We start in Section 1 by recalling the basic notions of knot theory and introducing the Jones polynomial. The definition can be used to show its uniqueness but not existence.

One way to prove the existence of the Jones polynomial is to relate links to braids. Any link can be obtained as a braid closure. A link invariant then corresponds to a *Markov trace* on the braid groups, a collection of maps $B_n \to X$ (where X is some set) satisfying certain compatibility relations. We produce such a trace from the B_n-action on $V^{\otimes n}$.

In the last section we explain another way to produce link invariants from representations of quantum groups due to Reshetikhin and Turaev. It is better computationally, one can compute the invariant directly from the diagram. More generally, a construction produces a homomorphism of suitable quantum group modules from a tangle.

1. Background from knot theory

1.1. **Links and their diagrams.** By a *link* we mean a continuous embedding of $\mathbb{S}^1 \sqcup \ldots \sqcup \mathbb{S}^1$ (the disjoint union of k circles) into $\mathbb{R}^3$. A link with a single component is called a knot. We view links up to isotopy (a continuous family of diffeomorphisms of $\mathbb{R}^3$). We can also consider oriented knots and links.

Usually, knots and links are presented by their two dimensional diagrams by picking a suitable projection $\mathbb{R}^3 \to \mathbb{R}^2$. Namely, we consider projections that have simple transverse intersections, i.e. we do not allow tangent strands or three strands intersecting in a single point. See examples in Picture 1.1.

We can speak about isotopic diagrams – we use continuous families of diffeomorphisms of $\mathbb{R}^2$. But isotopic links may have a non-isotopic diagrams. One can consider so called *Reidemeister moves* (see Picture 1.2), they take a piece of a diagram and transform it in such a way that change an isotopy class of a diagram but not of a link.

Theorem 1.1. *Two diagrams correspond to isotopic (oriented) links if and only if they can be obtained from one another by diagram isotopies and (oriented, just put various orientations on the fragments) Reidemeister moves.*

There is no algorithm however to test whether two diagrams can be obtained from one another as described in the theorem. So one tries to produce invariants of (oriented) diagrams that are preserved by diagram isotopies and Reidemeister moves and that are algorithmically computable.

1.2. **Jones polynomial.** Let us take a small circle in a diagram that contains precisely one intersection of strands. Then the diagram inside the circle looks like one of two fragments L_+ or L_-, see Picture 1.3, that are not isotopic (inside the circle). Another fragment we can have inside the circle is L_0. Now consider the ambient links that are the same outside the circle and are equal to L_+, L_0, L_- inside it. Abusing the notation we still denoted these links by L_+, L_-, L_0.

Theorem 1.2. *There is a unique oriented link invariant $L \mapsto P(L) \in \mathbb{Z}[q^{\pm 1}]$ such that $q^{-2}P(L_+) - q^2 P(L_-) = (q^{-1} - q)P(L_0)$ (skein relation) whose value on the trivial link with n components (unlink) is $(q + q^{-1})^{n-1}$.*

Theorem implies that $P(L)|_{q=1} = 2^{k-1}$, where k is the number of components.

It is possible to compute this invariant algorithmically. Namely, pick a point on a diagram and move this point according to the orientation. When we reach a crossing we put the strand we are on on top if it was on the bottom. If the diagram has changed, we write the skein relation expressing the Jones polynomial of the previous diagram as the sum of two. When we return to the starting point we will get the expression for the original polynomial in terms of a bunch of summands with one less crossing and a summand, where the link component we are on became untangled (meaning that it gives a trivial embedding $\mathbb{S}^1 \hookrightarrow \mathbb{R}^3$ that is not linked to other components).

Example 1.3. We compute the Jones polynomial of the Hopf link oriented as in Picture 1.1 (two different orientations may – and will – give different Jones polynomials). Let us consider the upper crossing point, see Picture 1.4. Then our initial Hopf link gives L_- so we will write $\tilde{L}_-$ for that link. Switching the crossing to L_+, we'll get the link $\tilde{L}_+$ that is two unlinked circles. Switching the crossing to L_0, we'll get $\tilde{L}_0$ that is the unknot. So $P(\tilde{L}_+) = q + q^{-1}$ and $P(\tilde{L}_0) = 1$. From the skein relation, we find

$$q^{-2}P(\tilde{L}_+) - q^2 P(\tilde{L}_-) = (q^{-1} - q)P(\tilde{L}_0) \Rightarrow P(\tilde{L}_-) = q^{-4}(q + q^{-1}) - q^{-2}(q^{-1} - q) = q^{-5} + q^{-1}.$$

Example 1.4. For the trefoil K in Picture 1.1 we have $P(K) = q^2 + q^6 - q^8$, see Picture 1.5 for some explanation.

2. Jones polynomial as Markov trace

2.1. **Braids, geometrically.** Recall the braid group B_n introduced in the previous lecture. It admits a geometric presentation similar (and closely related) to links. We will write B_n^g for this realization. As a set B_n^g consists of the configurations of n strands in $\mathbb{R}^2 \times [0, 1]$ connecting points $(i, 0, 0)$ to points $(j, 0, 1)$ (one-to-one), where $i, j = 1, \ldots, n$, in some order in such a way that

(a) each strand projects isomorphically to $[0, 1]$
(b) and the strands do not intersect.

We identify two braids that are obtained by an isotopy (fixing the $2n$ points and preserving the conditions above). We can present braids by braid diagrams, see Picture 2.1.

Proposition 2.1. *Two braid diagrams give isotopic braids if one is obtained from the other by a sequence of diagram isotopies and Reidemeister moves (R2) and (R3) (condition (a) prohibits the situation in (R1)). Let B_n^g denote the set of all these geometric braids.*

The set B_n^g admits an associative product (concatenation, Picture 2.2). This product has a unit given by the trivial braid (straight strands connecting $(i, 0, 0)$ to $(i, 0, 1)$ for each i).

As a monoid B_n^g is generated by the braids T_i, T_i^{-1} presented on Picture 2.3. That these elements generate B_n^g should be clear from Picture 2.4 (just perturb the diagram so that the projections of all crossings to $[0,1]$ are distinct). (R2) precisely says that T_i and T_i^{-1} are inverse to one another, so our notation is justified. In particular, B_n^g is a group rather than just a monoid. Note that $T_iT_j = T_jT_i$ when $|i-j|>1$ (via a diagram isotopy). Also $T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}$, this is precisely (R3). So we get a group epimorphism $B_n \twoheadrightarrow B_n^g$. The following result is a consequence of Proposition 2.1.

Theorem 2.2. *The epimorphism $B_n \twoheadrightarrow B_n^g$ is an isomorphism.*

2.2. **Braids vs links.** Given a braid b, we orient it from right to left. Then we can take the so called *braid closure*, see Picture 2.5, and get an oriented link. The following result is due to Alexander.

Theorem 2.3. *Any oriented link is the closure of some braid.*

Now let us figure out when two braids $b \in B_n, b' \in B_{n'}$ give the same link. Note that $\overline{ab} = \overline{ba}$, Pic 2.6. Now let us take $b \in B_n$. We can embed B_n into B_{n+1} (just put a strand from $(n+1,0,0)$ to $(n+1,0,0)$ that is below all other strands). Pick $b \in B_n$. We can view b as an element of B_{n+1} and form the product $bT_n^{\pm 1} \in B_{n+1}$. Then $\overline{bT_n^{\pm 1}} = \bar{b}$.

The following important result is due to Markov.

Theorem 2.4. *Braids $b_1 \in B_{n_1}, b_2 \in B_{n_2}$ have the same closure if and only if b_1 can be obtained from b_2 by a sequence of Markov moves*

(M1) *$ab \leftrightarrow ba$, for a, b in same B_n.*
(M2) *$b \leftrightarrow bT_n^{\pm 1}$, for $b \in B_n \hookrightarrow B_{n+1}$.*

By a *Markov trace*, we mean a collection of maps $\varphi_n : B_n \to \mathbb{C}$ (or some other target) that do not change under the Markov moves. By Theorem 2.4, this is the same thing as an oriented link invariant. The reason why we call it a trace is that $\varphi_n(ab) = \varphi_n(ba)$ is satisfied as soon as $\varphi_n(b) = \operatorname{tr}(\Phi_n(b))$ for some representation Φ_n of B_n.

2.3. **Markov trace from $L(q)$.** Let V be the U-module $L(q)$, where we write $U := U_q(\mathfrak{sl}_2)$. We have a homomorphism $B_n \to \mathbb{Z}$ called the degree (and denoted by deg). It is defined on the generators $\deg(T_i) = 1$ (and extends to B_n because all relations preserve the degrees). Now recall from the previous lecture that B_n acts on $V^{\otimes n}$ by U-linear automorphisms: T_i maps to $\tau_{i,i+1} = \mathrm{id}^{\otimes(i-1)} \otimes (R_{V\otimes V} \circ \sigma) \otimes \mathrm{id}^{\otimes n-i-1}$. Denote this representation by Φ'_n. The action of B_n commutes with the action of K that is given by an iterated Δ of K, i.e., by $K^{\otimes n}$. The trace of Φ'_n "almost" give a Markov trace but not quite.

Theorem 2.5. *The maps φ_n given by $\varphi_n(b) = q^{2\deg(b)} \operatorname{tr}(K^{\otimes n}\Phi'_n(b))$ form a Markov trace. Moreover, $\varphi_n(b) = (q+q^{-1})P(\bar{b})$.*

The proof of this theorem (in a more general setting, where we replace $U_q(\mathfrak{sl}_2)$ by $U_q(\mathfrak{sl}_n)$ is a part of the homework).

Example 2.6. Let us compute $\varphi_n(b)$ for $b = 1 \in B_n$. We get $\varphi_n(b) = q^0 \operatorname{tr}(K^{\otimes n}) = \operatorname{tr}(K)^n = (q+q^{-1})^n$.

Now let us compute $\varphi_2(T_1^2)$. We have $\Phi'_2(T_1^2) = 1 + (q^{-1}-q)\Phi_2(T_1)$. So $\varphi_2(T_1^2) = q^4 \operatorname{tr}(K^{\otimes}2) + q^4(q^{-1}-q)\operatorname{tr}(K^{\otimes 2}T_1)$. But we know that φ_n form the Markov trace, so $\varphi_2(T_1) = q^2 \operatorname{tr}(K^{\otimes 2}T_1) = \varphi_1(1) = (q+q^{-1})$. So $\varphi_2(T_1^2) = q^4(q+q^{-1})^2 + q^2(q^{-1}-q)(q+q^{-1}) = (q+q^{-1})(q^5+q)$. Note that the closure of T_1^2 is a Hopf link.

The theorem above proves the existence of the Jones polynomial but is not very useful for computations. In the next section we will consider another construction of the Jones polynomial, which also proves the existence and is better for computations.

3. Tangles and representations of quantum groups

3.1. **Tangles.** Tangles generalize both braids and links. A tangle is the following configuration: it consists of oriented links in $\mathbb{R}^2 \times [0,1]$ and oriented strands that connect some n fixed points on $\mathbb{R}^2 \times \{0\}$ and m fixed points on $\mathbb{R}^2 \times \{1\}$ (we can connect two points on $\mathbb{R}^2 \times \{0\}$ or two points on $\mathbb{R}^2 \times \{1\}$ with an oriented arc), points are connected one-to-one, in particular, $n+m$ has to be even. We consider tangles up to isotopy that fixes the $n+m$ points. We get the set $T(n,m)$ of isotopy classes. Note that $T(0,0)$ consists precisely of the oriented links.

By a *signed set* we mean a set together with a map to $\{\pm\}$. A tangle gives structures of signed sets on $\{1,\ldots,n\}$ and $\{1,\ldots,m\}$: sinks on $\mathbb{R}^2 \times \{0\}$ and sources on $\mathbb{R}^2 \times \{1\}$ are sent to a $+$, all other points are sent to a $-$. So, for two signed sets, M,N with $|M|=m, |N=n|$, we can define the subset $T(N,M) \subset T(n,m)$ corresponding to given signed sets.

We can still represent tangles by tangle diagrams, see Picture 3.1. Two tangles T_1, T_2 are isotopic if and only if the diagram of T_2 is obtained from that of T_1 by a sequence of diagram isotopies and the Reidemeister moves (R1),(R2),(R3).

We can compose tangles getting a partial composition map $T(K,N) \times T(N,M) \to T(K,M)$ similarly to the braids. Generating tangles are the crossings $X_+, X_- \in T(2,2)$, Picture 3.2, and also caps and cups in $T(2,0)$ and $T(0,2)$ (usually tangles are drawn vertically, hence the names), each with 2 possible orientations. Note that all other crossings are obtained as compositions of $X_\pm$ with cups and caps, see Picture 3.3 (we can rotate the crossing using cups and caps). Now the argument to show that $X_\pm$, cups and caps are generators is the same as for the braids.

We also have the tensor product $T(n_1,m_1) \times T(n_2,m_2) \to T(n_1+n_2, m_1+m_2)$, by definition, the diagram of $T_1 \otimes T_2$ is obtained by putting the diagram of T_2 above the diagram of T_1, see Picture 3.4.

3.2. **Functor.** Let $V = L(q)$. We assign V, V^* to the $n+m$ points: V goes to the point labeled by a $+$ and V^* to a point labeled by a $-$. To a signed set M we assign the module to be denoted by $V^{\otimes M}$, which is the tensor product of modules assigned to points in M.

Our goal is, for $T \in T(N,M)$, construct a U-linear homomorphism $\varphi_T : V^{\otimes M} \to V^{\otimes N}$ in such a way that $\varphi_{T_1 \circ T_2} = \varphi_{T_1} \circ \varphi_{T_2}$ and $\varphi_{T_1 \otimes T_2} = \varphi_{T_1} \otimes \varphi_{T_2}$.

This is done as follows: we need to define φ_T for generating tangles, extend it to arbitrary tangles so that diagrams corresponding to the same tangle give the same homomorphisms. In other words, we need to check that the homomorphism is preserved under a diagram isotopy and respects the three Reidemeister moves. We are not going to discuss this check, it requires a much more careful examination of how tangle isotopies work.

The generating tangles are the cups in $T(0,2)$, caps in $T(2,0)$ and the crossings in $T(2,2)$ (lines should clearly give the identity isomorphism). The homomorphism corresponding to X_+ (and to its rotations) is $q^2 \tau_{?\otimes ?}$, while the homomorphism corresponding to X_- is $q^{-2}\tau_{?\otimes ?}^{-1}$. The homomorphisms corresponding to caps and cups are between $V \otimes V^*$ (or $V^* \otimes V$) and $\mathbb{C}$. This is discussed in the next section.

Let us note that once $T \mapsto \varphi_T$ is constructed, it gives a link invariant. The invariant produced from $V = L(q)$ is the Jones polynomial. An advantage of the present construction is that it is much easier to compute the Jones polynomial from a link diagram (we just need to decompose the diagram into the composition of the generating tangles and write the corresponding composition of homomorphisms, see Picture 3.5).

3.3. **Duality.** Let V be a finite dimensional representation of U. We are going to define natural homomorphisms between $V\otimes V^*$ (and $V^*\otimes V$) and the trivial module $\mathbb{C}$. Recall that U acts on V^* via $\langle u\alpha, v\rangle = \langle \alpha, S(u)v\rangle$. Recall that S is given by $S(E) = -K^{-1}E, S(F) = -FK, S(K) = K^{-1}$.

First of all, note that the natural isomorphism $V \cong V^{**}$ is not U-linear. Indeed, U acts on $V^{**} = V$ via $u \cdot v = S^2(u)v$, where in the right hand side we have the usual action on V. We get $S^2(u) = K^{-1}uK$ (it is enough to check this on the generators, where this straightforward). So $v \mapsto K^{-1}v$ is a U-module isomorphism $V \to V^{**}$.

The natural map $p : V^* \otimes V \to \mathbb{C}, \alpha\otimes v \mapsto \alpha(v)$ is U-linear. For example, let us check that p intertwines E, i.e., $p\circ E = 0$. We have

$$E(\alpha\otimes v) = \Delta(E)(\alpha\otimes v) = (E\otimes 1 + K\otimes E)(\alpha\otimes v) = E\alpha\otimes v + K\alpha\otimes Ev,$$

$$p(E(\alpha\otimes v)) = \langle E\alpha, v\rangle + \langle K\alpha, Ev\rangle = \langle \alpha, -K^{-1}Ev\rangle + \langle \alpha, K^{-1}Ev\rangle = 0.$$

The map $V^{**}\otimes V^* \to \mathbb{C}$ is U-linear hence $V\otimes V^* \to \mathbb{C}, v\otimes\alpha \mapsto \langle K^{-1}v, \alpha\rangle$ is U-linear.

Now let us get U-linear isomorphisms $\mathbb{C} \to V\otimes V^*, V^*\otimes V$. The former is the naive map: we can identify $V\otimes V^* \cong \mathrm{End}(V)$ via $(v\otimes\alpha).v' = \langle\alpha, v'\rangle v$ and the image of 1 in $V\otimes V^*$ is the identity map. This map is U-linear because the map $V\otimes V^*\otimes V \to V$ is U-linear. Similarly, we define $\mathbb{C} \to V^*\otimes V$ in such a way that the map $V^*\otimes V\otimes V^* \to V^*$ is U-linear: $1 \in \mathbb{C}$ goes to $K^{-1}_{V^*}$ under the natural identification $V^*\otimes V \cong \mathrm{End}(V^*)$.

We will use the notations ev_V for $V^*\otimes V \to \mathbb{C}$, ev^*_V for $V\otimes V^* \to \mathbb{C}$, coev_V for $\mathbb{C} \to V^*\otimes V$ and $\mathsf{coev}^*_V : \mathbb{C} \to V\otimes V^*$.

Example 3.1. Let us compute the four maps above for $V = L(q)$. Let v_1, v_2 be the natural basis of V and α_1, α_2 be the dual basis in V^*. Then $\mathsf{ev}_V(\sum_{i,j=1}^2 a_{ij}\alpha_i \otimes v_j) = a_{11}+a_{22}, \mathsf{ev}^*_V(\sum_{i,j=1}^2 b_{ij}v_i\otimes\alpha_j) = q^{-1}b_{11}+qb_{22}$. Further, $\mathsf{coev}^*_V(1) = v_1\otimes\alpha_1+v_2\otimes\alpha_2, \mathsf{coev}_V = q\alpha_1\otimes v_1 + q^{-1}\alpha_2\otimes v_2$. Note that $\mathsf{ev}_V\circ\mathsf{coev}_V = \mathsf{ev}^*_V\circ\mathsf{coev}^*_V = q+q^{-1}$.

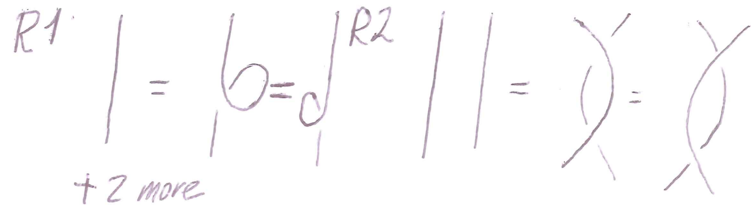
These maps are assigned to cups and caps as shown in Picture 3.6.

Pic 1

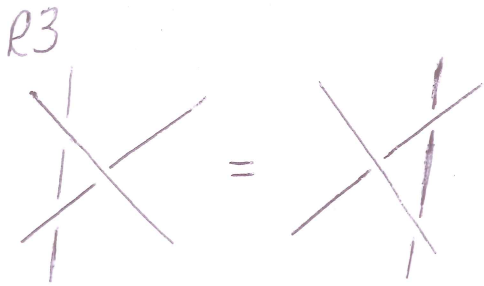
Pic 1.1. Link diagrams

-Hopf link   -Trefoil

Pic 1.2 Reidemeister moves

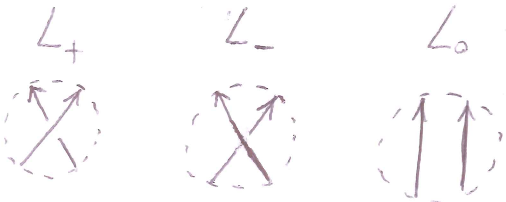


+2 more



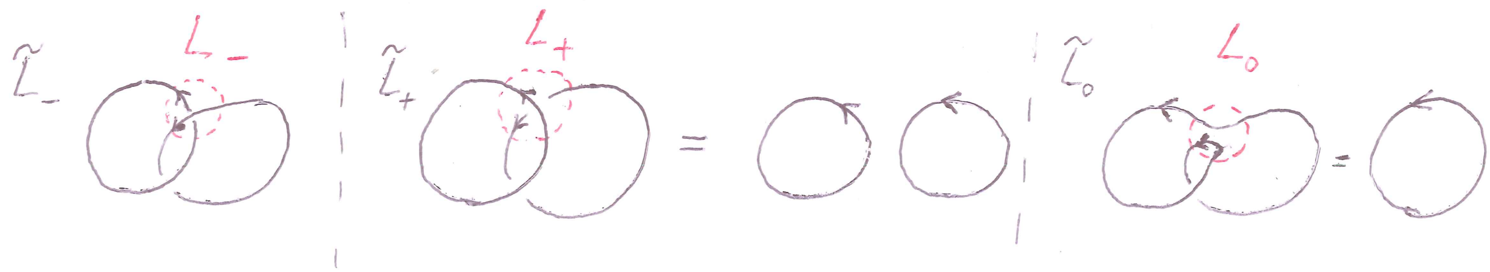
+ other variants (moved strand on top or in between

Pic 1.3: L_+, L_-, L_0:



Pic 1.4. Jones polyn. for Hopf link

$$P = q^{-5} + q^{-1}$$



Pic 1.5.: Jones polynomial for trefoil $P = q^2 + q^6 - q^8$

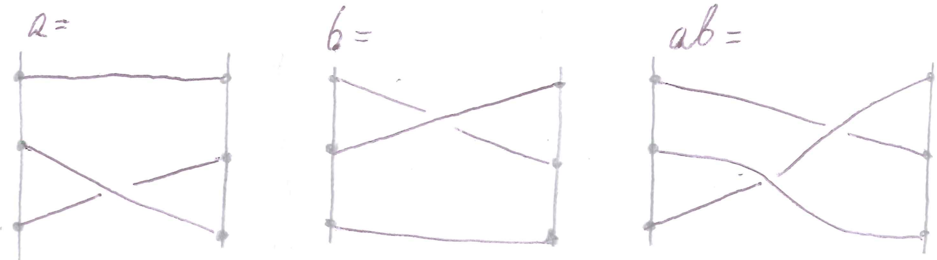
-unknot

-Hopf link w. orient. diff. from Pic 1.1 w. $P = q^5 + q$

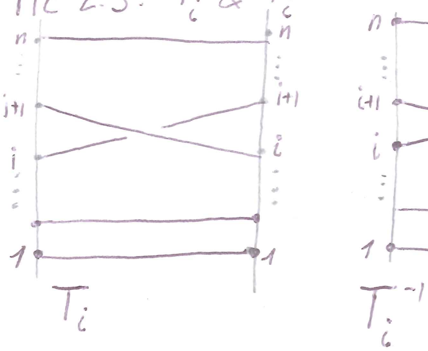
Picture 2

Pic 2.1: braid diagrams

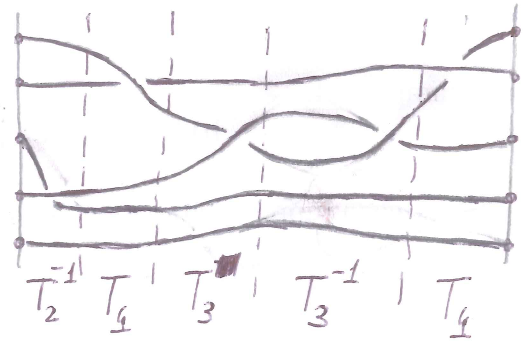
Pic 2.2: product of braids



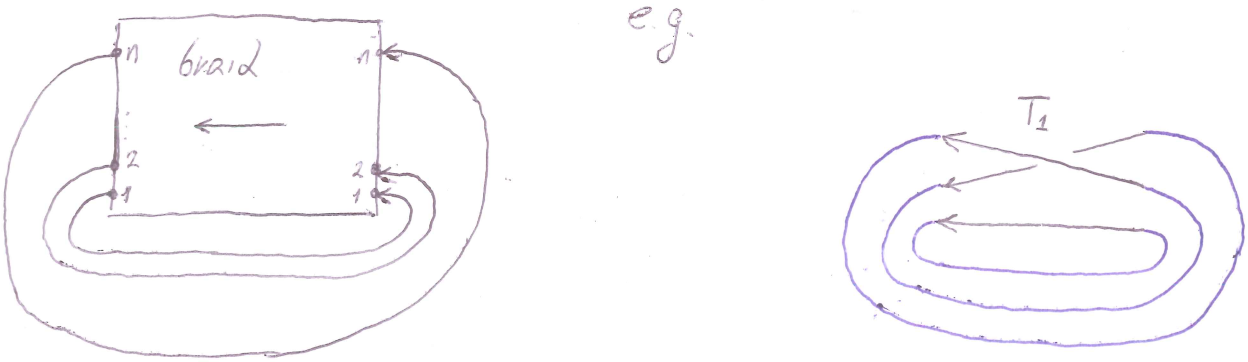
Pic 2.3: T_i & T_i^{-1}



Pic 2.4

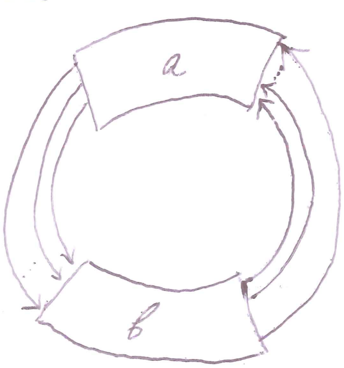


Pic 2.5: braid closure



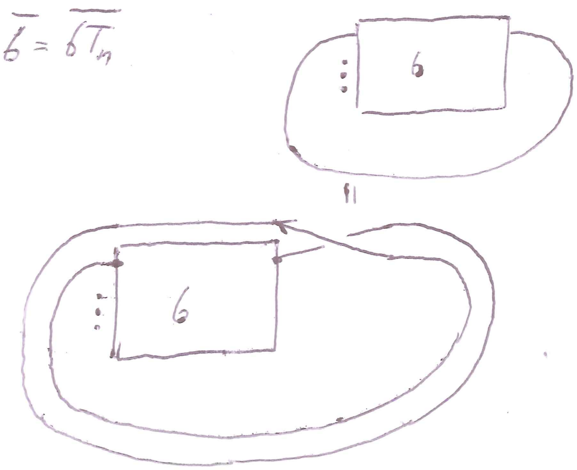
Pic 2.6 Markov move I

$\overline{ab} = \overline{ba}$



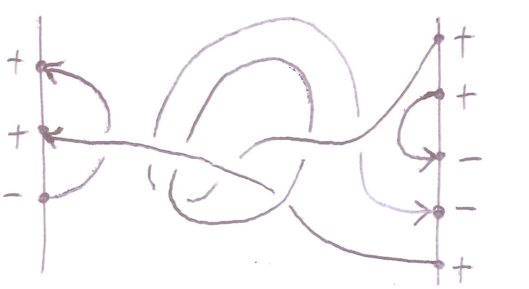
Markov move II

$\overline{b} = \overline{bT_n}$

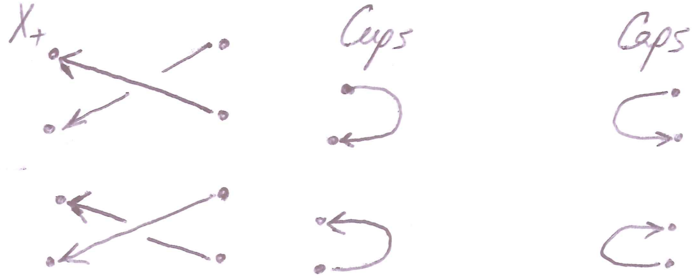


Picture 3: tangles

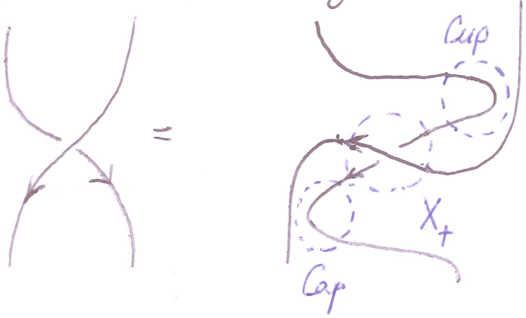
Pic 3.1: tangle diagram



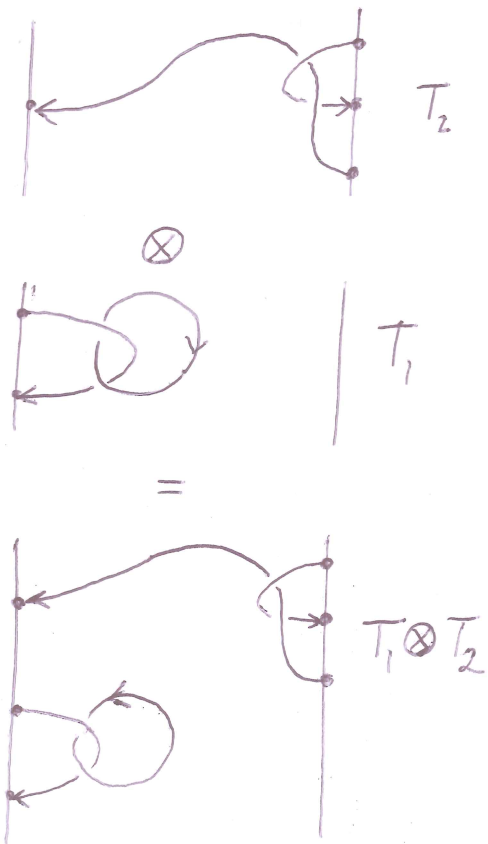
Tangles $X_\pm$, caps and cups (Pic 3.2)



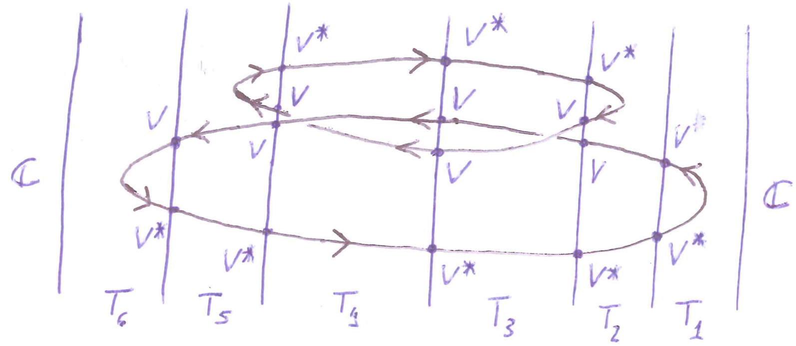
Pic 3.3: Other crossings



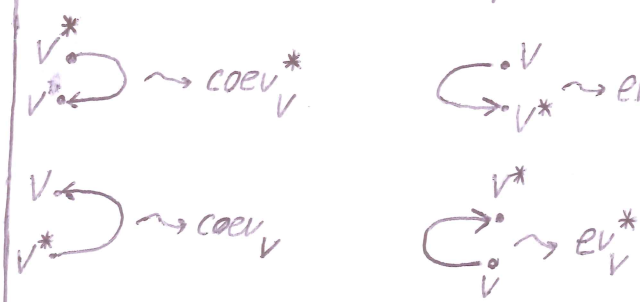
Pic 3.4: Tensor product



Pic 3.5



Pic 3.6: Homomorphisms φ for cups and caps



LECTURE 15: REPRESENTATIONS OF $U_q(\mathfrak{g})$ AT ROOTS OF 1

IVAN LOSEV

Introduction

In Lecture 13 we have studied the representation theory of $U_q(\mathfrak{g})$ when q is not a root of 1. We have seen that the representation theory basically looks like the representation theory of $\mathfrak{g}$ (over $\mathbb{C}$).

In this lecture we are going to study a more complicated case: when q is a root of 1 (we still need to exclude some small roots of 1 to make the algebra $U_q(\mathfrak{g})$ defined). When we deal with the usual definition of $U_q(\mathfrak{g})$ we see features of the algebra $U(\mathfrak{g}_{\mathbb{F}})$, where $\mathbb{F}$ is a field of positive characteristic. In particular, $U_q(\mathfrak{g})$ has an analog of the p-center and is finite over its center.

Or we can modify our definition of $U_q(\mathfrak{g})$ including divided powers, this is the case we are going to mostly care about. The corresponding algebra has many features of the semisimple algebraic groups over field of positive characteristic (such as the Frobenius homomorphism). In fact, it is the connection with quantum groups that allowed to compute the characters of irreducible representations of $G_{\mathbb{F}}$ when $\operatorname{char}\mathbb{F} \gg 0$.

For the most part of this lecture we consider the case of $U_q(\mathfrak{sl}_2)$ that can be treated by hand. In the second part we consider a far more complicated case of $U_q(\mathfrak{sl}_n)$. We mention the connection between the representation theory of $U_q(\mathfrak{sl}_n)$ and that of the affine Lie algebra $\hat{\mathfrak{sl}}_n$ discovered by Kazhdan-Lusztig. This is the main tool to study the representation theory of $U_q(\mathfrak{sl}_n)$ (and of other $U_q(\mathfrak{g})$, there we need the affine Lie algebra $\hat{\mathfrak{g}}$).

1. Case of $U_q(\mathfrak{sl}_2)$

Recall that $U_q(\mathfrak{sl}_2)$ is defined by generators $E, K^{\pm 1}, F$ and relations

$$KEK^{-1} = q^2E, KFK^{-1} = q^{-2}F, EF - FE = \frac{K - K^{-1}}{q - q^{-1}}$$

These relations give us some freedom of choosing the base ring. In this lecture, q will be viewed as an element of $R := \mathbb{C}[q^{\pm 1}][(q - q^{-1})^{-1}]$, it obviously makes sense to consider $U_q(\mathfrak{sl}_2)$ as an R-algebra. This algebra will be denoted by U_R. For $\epsilon \in \mathbb{C} \setminus \{0, \pm 1\}$, we set $U_\epsilon := \mathbb{C}_\epsilon \otimes_R U_R$. By $\mathbb{K}$ we denote the fraction field $\mathbb{C}(q)$ of R and we set $U_{\mathbb{K}} = \mathbb{K} \otimes_R U_R$.

For the remainder of this section, we write $\mathfrak{g}$ for $\mathfrak{sl}_2$ and G for SL_2 (over $\mathbb{C}$ by default).

1.1. **Usual $U_\epsilon(\mathfrak{sl}_2)$.** Let ϵ denote a primitive root of 1 of order d. We set $d_0 = d/2$ if d is even and $d_0 = d$ if d is odd.

Proposition 1.1. *The elements $E^{d_0}, K^{d_0}, F^{d_0} \in U_\epsilon$ are central.*

The proof is a part of the homework.

Let Z_{d_0} denote the subalgebra of U_ϵ generated by $F^{d_0}, K^{d_0}, E^{d_0}$. It is an analog of the p-center of $U(\mathfrak{sl}_2(\mathbb{F}))$.

Proposition 1.2. *Elements $F^{kd_0}K^{\ell d_0}E^{md_0}$, where $k,m\in\mathbb{Z}_{\geqslant 0}, \ell\in\mathbb{Z}$, form a basis in Z_{d_0}. The algebra U_ϵ is a free Z_{d_0}-module with basis $F^{k_1}K^{\ell_1}E^{m_1}$, where $k_1,\ell_1,m_1\in\{0,\ldots,d_0-1\}$.*

The proof of the first claim is based on the formula for E^rF^s that will be provided below. All other claims are easy.

The representation theory of U_ϵ is very similar to that of $U(\mathfrak{g}_{\mathbb{F}})$, where $\mathfrak{g}=\mathfrak{sl}_2$, but we are not going to provide details here. One thing that we will need is that the modules $L_0(\kappa\epsilon^s)$ over U_ϵ are irreducible when $s\in\{0,\ldots,d_0-1\}$ (and the central elements $F^{d_0}, K^{d_0}-\kappa^{d_0}, E^{d_0}$ act by 0).

1.2. **Divided power algebra.** Now we are going to define a different algebra to be a denoted by $\dot{U}_R$ and its specializations. The algebra $\dot{U}_R$ is defined as the subalgebra inside $U_{\mathbb{K}}$ generated $K^{\pm 1}$ by the divided powers $E^{(k)}:=E^k/[k]_q!, F^{(k)}:=F^k/[k]_q!$. By the very definition, $\mathbb{K}\otimes_R\dot{U}_R:=U_{\mathbb{K}}$. We set $\dot{U}_\epsilon:=\mathbb{C}_\epsilon\otimes_R\dot{U}_R$. Note that if ϵ is not a root of 1, the algebra $\dot{U}_\epsilon$ coincides with U_ϵ, while for a root of 1 we get something different. This is because still have elements $E^{(k)},F^{(k)}$ in $\dot{U}$ but they are no longer polynomials in E,F. Since $[d_0]_\epsilon!=\frac{\epsilon^{d_0}-\epsilon^{-d_0}}{\epsilon-\epsilon^{-1}}=0$, we get $E^{d_0}=F^{d_0}=0$ in $\dot{U}_\epsilon$. Below we will only (for simplicity) consider the case when d is odd so that $d=d_0$. Note that $[k]_\epsilon=0$ if and only if k is divisible by d.

Remark 1.3. This construction is strongly motivated by the representation theory of algebraic groups in characteristic p. There one defines the *hyperalgebra* $\dot{U}_{\mathbb{Z}}(\mathfrak{g})\subset U(\mathfrak{g}_{\mathbb{Q}})$ and its specialization $\dot{U}_{\mathbb{F}}(\mathfrak{g})$. The point is that the category of finite dimensional $\dot{U}_{\mathbb{F}}(\mathfrak{g})$-modules coincides with that of the rational representations of $G_{\mathbb{F}}$.

Lemma 1.4. *The algebra $\dot{U}_\epsilon$ is generated by $E,K^{\pm1},F,E^{(d)},F^{(d)}$.*

Proof. Note that

$$[kd]_q=\frac{q^{kd}-q^{-kd}}{q-q^{-1}}=\frac{q^{kd}-q^{-kd}}{q^d-q^{-d}}\frac{q^d-q^{-d}}{q-q^{-1}}=[k]_{q^d}[d]_q.$$

So

$$([kd]_q/[d]_q)\,|_{q=\epsilon}=k. \tag{1.1}$$

It follows that any $E^{(k)}$ is a polynomial in E and $E^{(d)}$ and the similar statement holds for $F^{(k)}$. ☐

Our next goal is to establish the triangular decompositions

$$\dot{U}_R=\dot{U}_R^-\otimes_R\dot{U}_R^0\otimes\dot{U}_R^+,\quad \dot{U}_\epsilon=\dot{U}_\epsilon^-\otimes\dot{U}_\epsilon^0\otimes\dot{U}_\epsilon^+$$

(as R-modules/vector spaces). In order to do this we will need some identities in $U_{\mathbb{K}}$. First, some notation. For $a\in\mathbb{Z}$, we set

$$[K;a]:=\frac{Kq^a-K^{-1}q^{-a}}{q-q^{-1}}$$

so that $[E,F]=[K;0]$. Consider the "binomial coefficient"

$$\binom{K;a}{i}=\left(\prod_{j=0}^{i-1}[K;a-j]\right)/[i]_q!.$$

Lemma 1.5. *We have the following equalities in $U_{\mathbb{K}}$:*

$$E^{(r)}F^{(s)} = \sum_{i=0}^{\min(r,s)} F^{(s-i)} \binom{K; 2i-r-s}{i} E^{(r-i)}, \tag{1.2}$$

$$\binom{K; a+1}{i} = q^i \binom{K;a}{i} + q^{i-a-1}K^{-1}\binom{K;a}{i-1}, \tag{1.3}$$

$$\binom{K;a}{i} E^{(k)} = E^{(k)} \binom{K; a+2k}{i}, \tag{1.4}$$

$$\binom{K;a}{i} F^{(k)} = F^{(k)} \binom{K; a-2k}{i}. \tag{1.5}$$

Proof. (1.2) is proved by the double induction. (1.3), a q-analog of a classical binomial identity, is obtained by a direct calculation. (1.4),(1.5) are straightforward corollaries of $KEK^{-1} = q^2E, KFK^{-1} = q^{-2}F$, respectively. $\square$

(1.2) and (1.3) imply that all $\binom{K;a}{i} \in \dot{U}_R$. We will denote the specialization of $\binom{K;a}{i}$ to $\dot{U}_\epsilon$ by the same symbol.

Now let us proceed to the triangular decompositions. Set $\dot{U}_R^? := \dot{U}_R \cap U_{\mathbb{K}}^?$, where $? = +, 0, -$, and $U_{\mathbb{K}}^+, U_{\mathbb{K}}^0, U_{\mathbb{K}}^-$ are the subalgebras of $U_{\mathbb{K}}$ generated by E, K, F, respectively so that we have $U_{\mathbb{K}} = U_{\mathbb{K}}^- \otimes_{\mathbb{K}} U_{\mathbb{K}}^0 \otimes U_{\mathbb{K}}^+$. We write $\dot{U}_\epsilon^? = \mathbb{C}_\epsilon \otimes_R \dot{U}_R^?$.

Proposition 1.6. *We have the following triangular decompositions.*

1. $\dot{U}_R = \dot{U}_R^- \otimes_R \dot{U}_R^0 \otimes_R \dot{U}_R^+$. *The algebra $\dot{U}_R^\pm$ is a free R-module with basis $E^{(k)}$ (or $F^{(k)}$), where $k \in \mathbb{Z}_{\geqslant 0}$. The R-module $\dot{U}_R^0$ is spanned by the elements of the form $K^\ell\binom{K;0}{i}$, where $\ell, i \in \mathbb{Z}$.*
2. $\dot{U}_\epsilon = \dot{U}_\epsilon^- \otimes \dot{U}_\epsilon^0 \otimes \dot{U}_\epsilon^+$. *The space $\dot{U}_\epsilon^\pm$ has basis $E^k(E^{(d)})^\ell$ (or $F^k(F^{(d)})^\ell$) with $k \in \{0,1,\ldots,\ell-1\}$ or $\ell \in \mathbb{Z}_{\geqslant 0}$. The space $\dot{U}_\epsilon^0$ has basis $K^m\binom{K;0}{d}^\ell$, where $0 \leqslant d \leqslant 2m-1$.*

Proof. Let us prove (1). The algebra $\dot{U}_R$ is spanned by $K^sE^{(\ell_1)}F^{(m_1)}\ldots E^{(\ell_n)}F^{(m_n)}$. Then we use relations of Lemma 1.5 to show that $\dot{U}_R^- \otimes_R \dot{U}_R^0 \otimes_R \dot{U}_R^+ \twoheadrightarrow U_R$. This map is an isomorphism after base change to K. All factors are torsion free over R and so is their product. We deduce that $\dot{U}_R^- \otimes_R \dot{U}_R^0 \otimes_R \dot{U}_R^+ = U_R$. The claims about bases/spanning sets are left as exercises.

Let us prove (2). The triangular decomposition is a straightforward corollary of the claim for $\dot{U}_R$. The claim about the bases in $\dot{U}_\epsilon^\pm$ follows from the corresponding claims for $\dot{U}_R^\pm$ and (1.1). The claim for $\dot{U}_\epsilon^0$ is more subtle and is left as an exercise (note for example that we have $\prod_{i=0}^{d-1}[K;-i] = 0$ in $\dot{U}_\epsilon^0$ hence we can only use $K^m, m = 0,\ldots,2d-1$, in generators). $\square$

Now let us present $\dot{U}_\epsilon$ by generators and relations.

Proposition 1.7. *The algebra $\dot{U}_\epsilon$ is generated by $E, F, K^{\pm 1}, E^{(d)}, F^{(d)}, \binom{K;0}{d}$ with the following relations:*

$$EE^{(d)} = E^{(d)}E, \quad FF^{(d)} = F^{(d)}F, \quad K\binom{K;0}{d} = \binom{K;0}{d}K, \tag{1.6}$$

$$KEK^{-1} = \epsilon^2 E, \quad KFK^{-1} = \epsilon^{-2}F, \quad KE^{(d)}K^{-1} = E^{(d)}, \quad KF^{(d)}K^{-1} = F^{(d)}, \tag{1.7}$$

$$E\binom{K;0}{d} = \binom{K;2}{d}E, \tag{1.8}$$

$$\binom{K;0}{d}F = F\binom{K;2}{d}, \tag{1.9}$$

$$\binom{K;0}{d}E^{(d)} = E^{(d)}\binom{K;0}{d} + 2E^{(d)}, \quad \binom{K;0}{d}F^{(d)} = F^{(d)}\binom{K;0}{d} - 2F^{(d)}. \tag{1.10}$$

$$[E,F] = [K;0], \quad [E, F^{(d)}] = F^{(d-1)}[K;1-d], \quad [E^{(d)}, F] = [K;1-d]E^{(d-1)}, \tag{1.11}$$

$$[E^{(d)}, F^{(d)}] = \binom{K;0}{d} + \sum_{i=1}^{d-1} F^{(d-i)}\binom{K;2i-2d}{i}E^{(d-i)}. \tag{1.12}$$

In (1.8,1.9) one recovers $\binom{K;2}{d}$ from $\binom{K;0}{d}$ and polynomials in K using (1.3). In the summation in the right hand side of (1.12) and in the right hand side of (1.11) we have an expression of F, K, E.

Proof. Let $\tilde{U}_\epsilon$ be the algebra generated by the generators and relations above. All the relations above hold in $\dot{U}_\epsilon$ and so we get $\tilde{U}_\epsilon \twoheadrightarrow \dot{U}_\epsilon$. Using the relations for $\tilde{U}_\epsilon$ we can see that elements of the form $F^{k_1}(F^{(d)})^{k_2}K^{\ell_1}\binom{K;0}{d}^{\ell_2}E^{m_1}(E^{(d)})^{m_2}$ with $k_1, m_1 \in \{0, \ldots, d-1\}, \ell_1 \in \{0, \ldots, 2d-1\}, k_2, \ell_2, m_2 \in \mathbb{Z}_{\geqslant 0}$, span $\tilde{U}_\epsilon$. Hence $\tilde{U}_\epsilon \xrightarrow{\sim} \dot{U}_\epsilon$. □

One can check that the subalgebra $\dot{U}_R \subset U_{\mathbb{K}}$ is a Hopf subalgebra. It follows that $\dot{U}_\epsilon$ is a Hopf algebra.

Now let us investigate the Hopf algebra structures on $\dot{U}_R, \dot{U}_\epsilon$.

Lemma 1.8. *$\dot{U}_R \subset U_{\mathbb{K}}$ is a Hopf subalgebra. So $\dot{U}_\epsilon$ becomes a Hopf algebra.*

Proof. This boils down to show that $\Delta(\dot{U}_R) \subset \dot{U}_R \otimes_R \dot{U}_R$ and $S(\dot{U}_R) \subset \dot{U}_R$. We will do the first check. It will follow if we check that $\Delta(E^{(k)}), \Delta(F^{(k)}) \subset \dot{U}_R \otimes_R \dot{U}_R$. We have

$$\Delta(E^{(k)}) = ([k]_q!)^{-1}(E \otimes 1 + K \otimes E)^k = \sum_{i=0}^{k} q^{i(k-i)}E^{(k-i)}K^i \otimes E^{(i)},$$

$$\Delta(F^{(k)}) = ([k]_q!)^{-1}\sum_{i=0}^{k} q^{i(k-i)}F^{(k-i)} \otimes F^{(i)}K^{-i}.$$

Both right hand sides are in $\dot{U}_R \otimes_R \dot{U}_R$. □

1.3. **Small quantum group and quantum Frobenius.** By the small quantum group u_ϵ we mean the subalgebra of $\dot{U}_\epsilon$ generated by $E, K^{\pm 1}, F$. Another way to define it is as the image of the natural homomorphism $U_\epsilon \to \dot{U}_\epsilon$. Note that u_ϵ is a Hopf quotient of U_ϵ and a Hopf subalgebra of $\dot{U}_\epsilon$.

Proposition 1.9. *The algebra u_ϵ is the quotient of U_ϵ by $(E^d, F^d, K^{2d} - 1)$. It has basis $F^kK^\ell E^m$, where $k, m \in \{0, \ldots, d-1\}$ and $\ell \in \{0, \ldots, 2d-1\}$.*

Now we want to describe the the quotient of $\dot{U}_\epsilon$ by $(E, F, K-1)$.

Proposition 1.10. *We have* $\dot{U}_\epsilon/(E,F,K-1) = U(\mathfrak{sl}_2)$ *with* $E^{(d)} \mapsto e, F^{(d)} \mapsto f$ *and* $\binom{K;0}{d} \mapsto h$. *This is an isomorphism of Hopf algebras.*

Proof. By Proposition 1.7, $\dot{U}_\epsilon/(E,F,K-1)$ is generated by the elements $E^{(d)}, F^{(d)}, \binom{K;0}{d}$ that satisfy the defining relations of $U(\mathfrak{sl}_2)$. This gives the required isomorphism. It preserves the Hopf algebra structure because it intertwines $\Delta(E^{(d)})$ with $\Delta(e)$ and $\Delta(F^{(d)})$ with $\Delta(f)$. The latter is a consequence of the computation in the proof of Lemma 1.8. $\square$

The epimorphism $\dot{U}_\epsilon \twoheadrightarrow U(\mathfrak{sl}_2)$ is called the *quantum Frobenius* epimorphism and is denoted by Fr.

1.4. **Classification of the irreducibles and characters.** The classification of finite dimensional irreducible $\dot{U}_\epsilon$-modules works similarly to to that of $\mathrm{SL}_2(\mathbb{F})$). Besides, we have an analog of the Steinberg decomposition.

Theorem 1.11. *There is a bijection between* $\mathrm{Irr}_{f.d.}(\dot{U}_\epsilon)$ *and the set* $\{\pm 1\} \times \mathbb{Z}_{\geqslant 0}$. *An element* (κ, n) *in the latter set goes to the unique finite dimensional irreducible module* $L(\kappa, n)$ *that has a vector* $v_{\kappa,n}$ *with (here* $n = dm + r$, *where* $0 \leqslant r < d$,*)*

$$K v_{\kappa,n} = \kappa \epsilon^n v_{\kappa,n}, \quad \binom{K;0}{d} v_{\kappa,n} = m v_{\kappa,n}, \quad E^{(i)} v_{\kappa,n} = 0, i > 0. \tag{1.13}$$

then $L(\kappa, n) = L(\kappa, r) \otimes \mathsf{Fr}^* L(m)$, *where* $E^{(d)}, F^{(d)}$ *act by* 0 *on* $L(\kappa, r)$ *and* $L(m)$ *stands for the irreducible* $\mathfrak{sl}_2(\mathbb{C})$*-module with highest weight* m.

To prove this theorem is a part of the homework.

Moreover, we note that we still have the universal R-matrix for $\dot{U}_\epsilon$ (but not for U_ϵ!). Indeed, Θ lies in a completion of $\dot{U}_R$ (and becomes a finite sum in $\dot{U}_\epsilon$). Moreover, one can make sense for Ψ for $\dot{U}_\epsilon$. The resulting R-matrix enjoys the same properties as for $q \neq \sqrt[\bullet]{1}$.

2. Case of $U_q(\mathfrak{sl}_n)$

In this section we deal with $\mathfrak{g} = \mathfrak{sl}_n$. The case of a more general semisimple Lie algebra is treated similarly but we want to keep the exposition simpler.

We can define the subalgebra $\dot{U}_R(\mathfrak{sl}_n) \subset U_{\mathbb{K}}(\mathfrak{sl}_n)$ similarly to the above. We then consider the specialization $\dot{U}_\epsilon(\mathfrak{sl}_n)$, where ϵ is a primitive dth root of 1 (we still assume that d is odd). This specialization admits a triangular decomposition $\dot{U}_\epsilon(\mathfrak{sl}_n) = \dot{U}_\epsilon^- \otimes \dot{U}_\epsilon^0 \otimes \dot{U}_\epsilon^+$, where $\dot{U}_\epsilon^-$ is generated by the elements $F_i^{(k)}$, $\dot{U}_\epsilon^0$ is generated by the elements $K_i^{\pm 1}$ and $\binom{\{K_i;a}{j\}}$, $\dot{U}_\epsilon^+$ is generated by the elements $E_i^{(k)}$.

The finite dimensional irreducible representations are still parameterized by $\{\pm 1\}^{n-1} \times P^+$, where we write P^+ for the monoid of dominant weights. We will be interested in the representations parameterized by P^+. Every such module V admits a weight decomposition $V = \bigoplus_\mu V_\mu$, where V_μ is determined as follows. Let $\mu = (\mu_1, \ldots, \mu_n)$. Then V_μ consists of all elements $v \in V$ such that $K_i v = \epsilon^{r_i} v$ and $\binom{K;0}{d} v = d m_i v$. Here we write $\mu_i = d m_i + r_i$ for the division with remainder. For $\lambda \in P^+$, we write $L(\lambda)$ for the corresponding irreducible representation.

The module $L(\lambda)$ can be realized as the unique irreducible quotient of the Weyl module $W(\lambda)$. The latter is defined by a single vector v_λ and the following relations: $v \in W(\lambda)_\lambda$,

$E_i^{(k)}v_\lambda = 0$ for all $k > 0$ and $F_i^{(k)}v_\lambda = 0$ for $k > \mu_i$. Note that those are precisely the relations defining the irreducible module with highest weight λ over $U(\mathfrak{sl}_n)$.

The module $W(\lambda)$ enjoys the same universal property as the Weyl module for the representations of $\mathrm{SL}_n(\mathbb{F})$:

$$\mathrm{Hom}_{\dot{U}_\epsilon}(W(\lambda), V) = \{v \in V_\lambda | E_i^{(k)}v = 0, \forall i = 1, \ldots, n, \forall k > 0\}.$$

As usual, an important problem is to compute the characters of the modules $L(\lambda)$. This boils down to determining the multiplicities of $L(\mu)$'s in $W(\lambda)$'s. The solution follows from the work of Kazhdan and Lusztig, [KL] (for an arbitrary finite dimensional semisimple Lie algebra $\mathfrak{g}$). They established an equivalence of $\dot{U}_\epsilon$-$\mathrm{mod}_{f.d.}$ and a certain category of representations of the affine Lie algebra $\hat{\mathfrak{sl}}_n$ (that was defined in Homework 5). The multiplicities in the latter are given by values at 1 of suitable Kazhdan-Lusztig polynomials for the affine symmetric group $\hat{S}_n$.

2.1. **Connection to representation theory of $\hat{\mathfrak{sl}}_n$.** As usual, an important problem is to compute the characters of the modules $L(\lambda)$. This boils down to determining the multiplicities of $L(\mu)$'s in $W(\lambda)$'s. The solution follows from the work of Kazhdan and Lusztig, [KL] (for an arbitrary finite dimensional semisimple Lie algebra $\mathfrak{g}$). They established an equivalence of $\dot{U}_\epsilon$-$\mathrm{mod}^1_{f.d.}$ (the category of modules, where K acts with eigenvalues that are powers of ϵ) and a certain category of representations of the affine Lie algebra $\hat{\mathfrak{sl}}_n$ (that was defined in Homework 5). The multiplicities in the latter are given by values at 1 of suitable Kazhdan-Lusztig polynomials for the affine symmetric group $\hat{S}_n$.

We are going to finish by describing the Kazhdan-Lusztig category $\mathcal{C}$ on the side for $\mathfrak{g} = \hat{\mathfrak{sl}}_n$. Pick $\kappa \notin \mathbb{Q}_{>0}$ such that $\exp(\pi\sqrt{-1}\kappa^{-1}) = \epsilon$. By definition, $\mathcal{C}$ consists of the modules with the following properties:

- $\mathfrak{m}_+ := t\mathfrak{g}[t]$ acts locally nilpotently.
- $\mathfrak{g} \subset \hat{\mathfrak{g}}$ acts locally finitely (any vector lies in a finite dimensional $\mathfrak{g}$-submodule).
- $c \in \hat{\mathfrak{g}}$ acts by the scalar $\kappa - m$.

Let us produce an example of a module in $\mathcal{C}$, the Weyl module (a.k.a. a parabolic Verma module) $\Delta(\lambda)$. Let $V(\lambda)$ denote the irreducible $\mathfrak{g}$-module with highest weight λ. We turn it into a $\mathfrak{g}[t]$-module by making $t\mathfrak{g}[t]$ act by 0. We make c to act by $\kappa - m$. Then set $\Delta(\lambda) = U(\hat{\mathfrak{g}}) \otimes_{U(\mathfrak{g}[t]\oplus\mathbb{C}c)} V(\lambda)$.

The following is the main result of [KL] (for an arbitrary semisimple Lie algebra $\mathfrak{g}$).

Theorem 2.1. *There is a category equivalence between $U_q(\mathfrak{g})$-$\mathrm{mod}^1_{f.d.}$ and the category $\mathcal{C}$ that maps $W(\lambda)$ to $\Delta(\lambda)$.*

The main idea of the proof is to recover the tensor product on the level of $\mathcal{C}$. This is done using a construction called *conformal blocks* that comes from Math Physics.

References

[KL] D. Kazhdan, G. Lusztig. *Tensor structures arising from affine Lie algebras*, I,II: J. Amer. Math. Soc., 6 (1993), no. 4, 905-947, 949-1011, III,IV: J. Amer. Math. Soc. 7 (1994), no. 2, 335-381, 383-453.

LECTURE 16: REPRESENTATIONS OF QUIVERS

IVAN LOSEV

Introduction

Now we proceed to study representations of quivers. We start by recalling some basic definitions and constructions such as the path algebra and indecomposable representations. Then we state a theorem of Kac that describes the dimensions, where the indecomposable representations occur as well as the number of parameters needed to describe their isomorphism classes. We will prove the Kac theorem only partially using Crawley-Boevey's approach based on deformed preprojective algebras. This approach does not allow to prove Kac's theorem completely but it is more elementary than Kac's original approach.

1. Representations of quivers

1.1. **Quivers and their representations.** By a *quiver* we mean an oriented graph, possibly with multiple edges and loops. Formally, it is a quadruple $Q := (Q_0, Q_1, t, h)$, where Q_0, Q_1 are finite sets of vertices and arrows and $t, h : Q_1 \to Q_0$ are maps (taking an arrow a to its tail and head), see Picture 1.1. See Pictures 1.2, 1.3 for some examples of quivers.

A representation of a quiver Q is an assignment that takes every vertex $i \in Q_0$ to a vector space V_i and every arrow $a \in Q_1$ to a map $x_a : V_{t(a)} \to V_{h(a)}$. In particular, a representation of the quiver in Picture 1.2(a) is a pair V_1, V_2 of vector spaces and maps $x_a : V_1 \to V_2, x_b : V_2 \to V_1$.

As with groups and algebras, a representation of a quiver Q is the same thing as a module over a suitable associative algebra, the *path algebra* $\mathbb{C}Q$ of Q. This algebra is constructed as follows. A basis in this algebra is formed by all paths in Q. By a path in Q we mean either the empty path $p = \epsilon_i, i \in Q_0$, or a sequence $p = (a_1, \ldots, a_k)$ of arrows with $t(a_i) = h(a_{i+1})$. We set $t(p) = t(a_k), h(p) = h(a_1)$ (and $h(\epsilon_i) = t(\epsilon_i) = i$). The multiplication is introduced as follows: we have $p_1 p_2 = 0$ if $h(p_2) \neq t(p_1)$, and $p_1 p_2$ is the concatenation of p_1, p_2 else. Note that $\mathbb{C}Q$ becomes an associative algebra with unit $1 = \sum_{i \in Q_0} \epsilon_i$.

Let us produce a natural bijection between the representations of Q and the $\mathbb{C}Q$-modules. Given a representation (V_i, x_a) we define the $\mathbb{C}Q$-module $V := \bigoplus_{i \in Q_0} V_i$, where the action is introduced by $\epsilon_i u_j = \delta_{ij} u_j, a u_j = x_a(\delta_{jt(a)} u_j)$, where $u_j \in V_j$. We extend the multiplication to an arbitrary path p in an obvious way. Conversely, given a $\mathbb{C}Q$-module V, we get a representation (V_i, x_a) by $V_i := \epsilon_i V, x_a = a\epsilon_{t(a)}$.

In particular, we see that the representations of Q form an abelian category. A morphism $(V_i, x_a) \to (V_i', x_a')$ can be interpreted as a collection of maps $y_i : V_i \to V_i'$ with $y_{h(a)} \circ x_a = x_a' \circ y_{t(a)}$.

We are interested in the case when $\dim V_i < \infty$ for all i. Then we can define the dimension vector $v := (\dim V_i)_{i \in Q_0}$.

1.2. **Indecomposable representations and equivalence.** As usual, the main goal in our study of the representations of Q (equivalently, of $\mathbb{C}Q$) is their classification up to an isomorphism. The Krull-Schmidt theorem reduces this task to the classification of the indecomposable representations. Recall that by an indecomposable representation of an algebra A we mean a representation U that does not split into the direct sum of two nonzero representations of A. Clearly, any finite dimensional representation decomposes into the sum of indecomposable ones. The Krull-Schmidt theorem says that such a decomposition is unique. More precisely, we have the following.

Theorem 1.1. *Let A be an algebra, U be a finite dimensional A-module and $U = \bigoplus_{i\in I} U_i = \bigoplus_{j\in J} U'_j$ be two decompositions into the indecomposable representations. Then there is a bijection $\sigma : I \xrightarrow{\sim} J$ such that $U_i \cong U'_{\sigma(i)}$.*

So what we want to study is the indecomposable representations of $\mathbb{C}Q$ up to an isomorphism. What makes this problem especially nice is that it reduces to describing orbits of a reductive group action on a vector space. Namely, fix vector spaces V_i, let v be the dimension vector. Then the linear maps (x_a) naturally form a vector space $\mathrm{Rep}(Q, v) = \bigoplus_{a\in Q_1} \mathrm{Hom}(V_{t(a)}, V_{h(a)})$. On this space we have an action of the group $G_v := \prod_{i\in Q_0} \mathrm{GL}(V_i)$ given by "change of bases": $(g_i).(x_a) = (g_{h(a)}x_a g^{-1}_{t(a)})$. Clearly, the representations of $\mathbb{C}Q$ on V given by $(x_a), (x'_a)$ are isomorphic if and only if (x_a) and (x'_a) lie in the same G_v-orbit. So what we need to do is to describe the G_v-orbits in $\mathrm{Rep}(Q, v)$.

To finish this section let us provide two classical linear algebraic examples.

Example 1.2. Let us consider the quiver of type A_2 (two vertices and one arrow between them). Then $\mathrm{Rep}(Q, v) = \mathrm{Hom}(V_1, V_2)$ and $G_v = \mathrm{GL}(V_1) \times \mathrm{GL}(V_2)$ acts on this space $(g_1, g_2).x = g_2 x g^{-1}$. The maps x, x' lie in the same orbit if and only if $\mathrm{rk}\, x = \mathrm{rk}\, x'$. Such a map is indecomposable if and only if $\dim V_1 = 1, \dim V_2 = 0$ or vice versa (and x_a is zero) or $\dim V_1 = \dim V_2 = 1$ and x_a is an isomorphism.

Example 1.3. Consider the case when Q is the Jordan quiver. Here V is a single vector space and $G_v = \mathrm{GL}(V)$ acts on V by conjugations. The orbits are classified by the Jordan normal forms (up to permutation of blocks). The indecomposable representations are the single Jordan blocks.

1.3. **Questions.** In general, the complete description of the G_v-orbits in $\mathrm{Rep}(Q, v)$ is a wild problem that can only be solved completely when Q is a type A, D, E or a type $\tilde{A}, \tilde{D}, \tilde{E}$ type diagram (see Picture 1.3. for the latter).

One question that we will answer completely is about the possible dimensions of the indecomposable representations. We will also describe the "number of parameters" that is needed to describe the equivalence classes of the indecomposable representations in $\mathrm{Rep}(Q, v)$. Let us make this formal.

Let X be an algebraic variety equipped with an action of an algebraic group G. Define the subset $X_{\leqslant i} := \{x \in X \mid \dim Gx \leqslant i\}$ and $X_i := X_{\leqslant i} \setminus X_{\leqslant i-1}$. Note that $X_{\leqslant i}$ is a closed subvariety of X and hence X_i is open in $X_{\leqslant i}$. We define $p_G(X)$, the number of parameters for the G-orbits in X, to be $p_G(X) := \max_i(\dim X_i - i)$. Note that $p_G(X) = 0$ if and only if X consists of finitely many orbits.

To talk about the number of parameters of the indecomposable representations we need to extend the function p_G to G-stable constructible subsets of X. Recall that a constructible subset, by definition, is a union of locally closed subvarieties. By the Chevalley theorem, the image of a morphism of algebraic varieties is constructible.

Lemma 1.4. *The following is true.*

(1) *A G-stable constructible subset Y is a union of G-stable locally closed subvarieties.*

(2) *The subset of indecomposable representations in* $\mathrm{Rep}(Q,v)$ *is constructible.*

Proof. There is an open subset $Y^0 \subset Y$ such that $\dim Y^0 > \dim \overline{Y \setminus Y^0}$. We can replace Y^0 with GY^0 (still open) and assume that Y^0 is G-stable. Then we induct on $\dim \overline{Y}$ to prove (1).

Let us prove (2). For any decomposition $v = v' \oplus v''$, we have a G_v-equivariant morphism $\psi_{v',v''} : G_v \times \mathrm{Rep}(Q,v') \times \mathrm{Rep}(Q,v'') \to \mathrm{Rep}(Q,v)$, $(g,(x'_a),(x''_a)) \mapsto g.(x'_a \oplus x''_a)$. The set of the indecomposable representations is the complement to $\bigcup \operatorname{im} \psi_{v',v''}$, where the union is taken over all proper decompositions $v = v' \oplus v''$. Hence it is constructible. $\square$

This lemma allows to define p_v, the number of parameters needed to describe the orbits of indecomposable representations in $\mathrm{Rep}(Q,v)$.

Example 1.5. For the quiver of Dynkin type A_2, the number of orbits is finite. So the number of parameters is 0.

Example 1.6. For the Jordan quiver, $p_n = 1$, for all $n \in \mathbb{Z}_{>0}$.

2. Kac's theorems

2.1. **Root system and Weyl group.** It turns out that the answers to our questions about $\mathrm{Rep}(Q,v)$ are stated in terms of the root system of the Kac-Moody algebra $\mathfrak{g}(A)$, where A is the Cartan matrix of Q, given by $a_{ij} = 2\delta_{ij} - n_{ij}$, where n_{ij} stands for the number of edges between i,j. We are going to recall the corresponding definitions in the form that we need (we need to work in a more general setting, where we have loops, in which case the algebra $\mathfrak{g}(A)$ was not defined).

Define two spaces $\mathfrak{h}, \mathfrak{h}^*$ with bases $\alpha_i^\vee, \alpha_i$, where $i \in Q_0$. Define the *Tits form* on $\mathfrak{h}^*$ with matrix A. In other words, if we identify $\mathfrak{h}^*$ with $\mathbb{C}^{Q_0}$ by $\sum_{i\in Q_0} x_i\alpha_i \mapsto x = (x_i)_{i\in Q_0}$, then the form is given by $(x,y) = 2\sum_{i\in Q_0} x_iy_i - \sum_{a\in Q_1}(x_{t(a)}y_{h(a)} - y_{t(a)}x_{h(a)})$. For $i \in Q_0$ without loops, we define a map $s_i : \mathfrak{h}^* \to \mathfrak{h}^*$ by $x \mapsto x - (x,\alpha_i)\alpha_i$. This map is a reflection with respect to the hyperplane $(\alpha,\cdot) = 0$. It maps $\sum_{i\in Q_0} x_i\alpha_i$ to $\sum_{i\in Q_0} x'_i\alpha_i$, where $x_j = x'_j$ if $j \neq i$ and $x'_i = \sum_j n_{ij}x_j - x_i$. The subgroup of $\mathrm{GL}(\mathfrak{h}^*)$ generated by the reflections s_i is denoted by W (or $W(Q)$) and is called the Weyl group of Q.

Now let us define the real and imaginary roots of Q. A real root is a $W(Q)$-conjugate of some α_i, where i has no loops. Note that $(\alpha,\alpha) = 2$ for any real root α. To define imaginary roots without referring to the corresponding Kac-Moody algebra is more complicated. We define the *support* $\operatorname{Supp}\alpha$ of an element $\alpha = \sum_i x_i\alpha_i \in \mathfrak{h}^*$ as the set of all i such that $x_i \neq 0$. So we can speak about connected and disconnected (as subgraphs of Q w/o orientation) supports. By an imaginary root we mean a nonzero element $\alpha \in \mathrm{Span}_{\mathbb{Z}_{\geqslant 0}}(\alpha_i)_{i\in Q_0}$ with connected support such that $(\alpha,\alpha) \leqslant 0$.

Lemma 2.1. *Let α be an imaginary root. Then $W(Q)\alpha \subset \mathrm{Span}_{\mathbb{Z}_{\geqslant 0}}(\alpha_i)_{i\in Q_0}$.*

The proof is a part of the homework.

2.2. **Kac theorem.** Here's one of the main result about indecomposable representations of quivers.

Theorem 2.2. *The following is true.*

(a) *There is an indecomposable representation with dimension vector $v \in \mathbb{C}^{Q_0}$ if and only if $\sum_i v_i\alpha_i$ is a root.*

(b) *If v is a real root, then there is a unique indecomposable representation with dimension vector v.*
(c) *In general, if v is a root, then $p_v = 1 - \frac{1}{2}(v, v)$.*

Example 2.3. If Q is of type A_2, then there are three roots: $(1,1),(0,1),(1,0)$, and Kac's theorem clearly holds.

Example 2.4. Let Q be the Jordan quiver (type $\tilde{A}_0$). Then the positive root system coincides with $\mathbb{Z}_{>0}$. We have $(n,n) = 0$ for any n. The Kac theorem predicts that the number of parameters describing the orbits of indecomposable representations equals $1 - 0 = 1$. We have seen that this is indeed the case.

We are not going to prove all statements of Kac's theorem. We will only check that the dimension of any indecomposable representation is a root, we will prove part (b), and we will check (c) assuming (a) holds and v is primitive (i.e. $\mathrm{GCD}(v_i)_{i\in Q_0} = 1$).

We also note that the answer in Kac's theorem does not depend on an orientation of Q. As the homework shows, in many examples some orientations are easier than the others.

3. Deformed preprojective algebras

3.1. **Definition.** We fix an element $\lambda \in \mathbb{C}^{Q_0}$ and define an algebra $\Pi^\lambda(Q)$ (due to Crawley-Boevey and Holland) depending on λ. This algebra is known as the deformed preprojective algebra.

First, let us define the double quiver $\overline{Q}$. We have $\overline{Q}_0 = Q_0$ and $\overline{Q}_1 = Q_1 \sqcup Q_1^{op}$, where Q_1^{op} is identified with Q_1 as a set (we denote the corresponding bijection by $a \mapsto a^*$) with $t(a^*) = h(a), h(a^*) = t(a)$ (in other words, for every arrow in Q, we add the opposite arrow).

We set

$$\Pi^\lambda(Q) = \mathbb{C}\overline{Q}/(\sum_{a\in Q_1}[a,a^*] - \sum_{i\in Q_0}\lambda_i\epsilon_i).$$

This relation can be expanded as the Q_0-tuple of relations $\sum_{a,h(a)=i} aa^* - \sum_{a,t(a)=i} a^*a = \lambda_i\epsilon_i$.

Example 3.1. Consider the A_2-quiver. Then $\mathbb{C}\overline{Q}$ is generated by the elements $\epsilon_1, \epsilon_2, a, b$ with relations $\epsilon_2 a = a\epsilon_1 = a, \epsilon_1 b = b\epsilon_2 = b$. In $\Pi^\lambda(Q)$, we have two more relations $ba = \lambda_1\epsilon_1, ab = \lambda_2\epsilon_2$. Here the algebra $\Pi^\lambda(Q)$ is finite dimensional but its dimension depends on λ_1, λ_2.

Example 3.2. Consider the Jordan quiver. Here $\Pi^\lambda(Q) = \mathbb{C}\langle a, a^*\rangle/([a,a^*] = \lambda)$, the first Weyl algebra (for $\lambda \neq 0$) and the polynomial algebra for $\lambda = 0$.

An important property of $\Pi^\lambda(Q)$ is that it is independent of the orientation of Q (up to a natural isomorphism). Namely, suppose that we have changed an orientation of one arrow, say a, getting a new quiver Q'. Then the map $b \mapsto b, b^* \mapsto b, (b \neq a), a \mapsto a^*, a^* \mapsto -a$ defines an isomorphism $\mathbb{C}\overline{Q} \cong \mathbb{C}\overline{Q}'$ that induces an isomorphism $\Pi^\lambda(Q) \cong \Pi^\lambda(Q')$.

3.2. **Moment maps.** The subvariety $\mathrm{Rep}(\Pi^\lambda(Q), v) \subset \mathrm{Rep}(\overline{Q}, v)$ is given by the equations $\sum_{a,h(a)=i} x_a x_{a^*} - \sum_{a,t(a)=i} x_{a^*}x_a = \lambda_i \,\mathrm{id}_{V_i}, i \in Q_0$. Let us denote the left hand side by $\mu_i(x_a, x_{a^*})$.

Set $\mu := (\mu_i)_{i\in Q_0} : \mathrm{Rep}(\overline{Q}, v) \to \mathfrak{g}_v$. It turns out that $\mu : \mathrm{Rep}(\overline{Q}, v) \to \mathfrak{g}_v$ is the so called moment map for the action of $G_v = \prod_{i\in Q_0}\mathrm{GL}(V_i)$. Let us explain what this means.

First of all, let us notice that $\mathrm{Rep}(\overline{Q}, v)$ is a symplectic vector space. Indeed, for any finite dimensional vector spaces U, U', we can identify $\mathrm{Hom}(U, U')^*$ with $\mathrm{Hom}(U', U)$ via the trace form. So $\mathrm{Rep}(\overline{Q}, v) = \mathrm{Rep}(Q, v) \oplus \mathrm{Rep}(Q^{op}, v) = \mathrm{Rep}(Q, v) \oplus \mathrm{Rep}(Q, v)^*$. It follows that $\mathrm{Rep}(\overline{Q}, v)$ comes equipped with a natural non-degenerate skew-symmetric form. Using our identifications, it can be written as follows $\omega((x_a, x_{a^*}), (y_a, y_{a^*})) = \sum_{a \in Q_1} \mathrm{tr}(x_a y_{a^*} - y_a x_{a^*})$. Clearly, this form is G_v-invariant.

Now let U be a symplectic vector space with form ω together with a homomorphism $G \to \mathrm{Sp}(U)$ of algebraic groups. By a moment map for this action we mean a G-equivariant map $\mu : U \to \mathfrak{g}^*$ with the property that

$$\langle d_v \mu(u), \xi \rangle = \omega_v(\xi v, u) \tag{3.1}$$

(this condition prescribes $d\mu$ completely). There is a simple formula for μ: $\langle \mu(v), \xi \rangle = \frac{1}{2}\omega(\xi v, v)$, this formula immediately implies both properties.

Lemma 3.3. *The map $\mu = (\mu_i)_{i \in Q_0}$ is a moment map for the G_v-action on the symplectic vector space $\mathrm{Rep}(\overline{Q}, v)$ under the identification of $\mathfrak{g}_v$ with $\mathfrak{g}_v^*$ by means of the trace form.*

Proof. What we need to prove is that

$$\sum_{i \in Q_0} \mathrm{tr} \left(\sum_{h(a)=i} x_a x_{a^*} y_i - \sum_{t(a)=i} x_{a^*} x_a y_i \right) =$$
$$\frac{1}{2}\omega((y_{h(a)} x_a - x_a y_{t(a)}, y_{t(a)} x_{a^*} - x_{a^*} y_{h(a)}), (x_a, x_{a^*}), (x_a, x_{a^*}))$$

By definition, the right hand side is

$$\frac{1}{2} \sum_{a \in Q_1} \left(\mathrm{tr}([y_{h(a)} x_a - x_a y_{t(a)}) x_{a^*}] - \mathrm{tr}([y_{t(a)} x_{a^*} - x_{a^*} y_{h(a)}] x_a) \right).$$

Rearranging the summation and using the cyclicity of trace, we see that the previous expression coincides with the l.h.s. $\square$

Let us explain why moment maps are important. Let $H_\xi = \langle \mu, \xi \rangle$, this is a function on U. To this function we can assign the skew-gradient $v(H_\xi)$ with respect to ω. Condition (3.1) means precisely that $v(H_\xi)$ coincides with the vector field $u \mapsto \xi u$ on U.

LECTURE 17: DEFORMED PREPROJECTIVE ALGEBRAS

IVAN LOSEV

1. Introduction/recap

In the previous lecture we have stated the Kac theorem and introduced the deformed preprojective algebras. In this lecture we will prove a weaker version of the theorem by studying the representation theory of those algebras.

Theorem 1.1. *Let Q be a quiver and v be a dimension vector. Then the following is true.*

(1) *If there is an indecomposable representation of dimension v, then v is a root.*
(2) *If v is a real root, then there is a unique (up to an isomorphism) indecomposable representation of dimension v.*
(3) *If v is primitive (meaning that $\mathrm{GCD}(v_i) = 1$) and there is an indecomposable representation of dimension v, then p_v, the number of parameters for the isomorphism classes of indecomposable representations, equals*

$$1 - (v, v)/2(= 1 - \dim G_v + \dim \mathrm{Rep}(Q, v)).$$

Remark 1.2. Suppose that there is i such that $v_j = 0$ for all $j \neq i$ and there is no loop at i. Then $\mathrm{Rep}(Q, v) = \{0\}$. The zero representation is indecomposable if and only if $v_i = 1$ (i.e., v corresponds to a simple root).

Also note that if there is an indecomposable representation of dimension v, then the support of v is connected.

Now recall that a deformed preprojective algebra is defined by

$$\Pi^\lambda(Q) = \mathbb{C}\overline{Q}/(\sum_{a\in Q_1}[a, a^*] - \sum_{i\in Q_0}\lambda_i\epsilon_i).$$

The set $\mathrm{Rep}(\Pi^\lambda(Q), v) \subset \mathrm{Rep}(\overline{Q}, v)$ coincides with $\mu^{-1}(\sum_i \lambda_i \mathrm{id}_{V_i})$, where $\mu : \mathrm{Rep}(\overline{Q}, v) \to \mathfrak{g}_v$ is the moment map, $\mu_i(x_a, x_{a^*}) = \sum_{a,h(a)=i} x_a x_{a^*} - \sum_{a,t(a)=i} x_{a^*} x_a$. Being a moment map means that μ is G_v-equivariant and

$$\langle d_x\mu(v), \xi\rangle = \omega(\xi x, v). \tag{1.1}$$

Note that $\sum_i \mathrm{tr}\,\mu_i(x_a, x_{a^*}) = 0$. By $\bar{\mathfrak{g}}_v$, we denote the subalgebra of $\mathfrak{g}_v$ consisting of all elements (y_i) with $\sum_i \mathrm{tr}(y_i) = 0$. So $\mu : \mathrm{Rep}(\bar{Q}, v) \to \bar{\mathfrak{g}}_v$.

2. Connection to indecomposable representations of Q

Let $\pi : \mathrm{Rep}(\Pi^\lambda(Q), v) \to \mathrm{Rep}(Q, v)$ denote the projection, it sends $(x_a, x_{a^*})_{a\in Q_1}$ to $(x_a)_{a\in Q_1}$. Our goal is to describe the pre-image of (x_a).

2.1. **Exact sequence.** A key tool for this is the following lemma. We define a map $c : \mathrm{Rep}(Q^{op}, v) \to \mathfrak{g}_v$ by $c(x_{a^*}) := \mu(x_a, x_{a^*})$ and a map $t : \mathfrak{g}_v \to \mathrm{End}(x_a)^*$ by $\langle t(y_i), (z_i)\rangle = \sum_i \mathrm{tr}(y_i z_i)$. Recall that $\mathrm{End}(x_a)$ denote the endomorphism algebra of the representation x_a, it consists of all Q_0-tuples (z_i) with $z_{h(a)}x_a = x_a z_{t(a)}$.

Lemma 2.1. *The sequence*

$$\mathrm{Rep}(Q^{op}, v) \xrightarrow{c} \mathfrak{g}_v \xrightarrow{t} \mathrm{End}(x_a)^* \to 0$$

is exact.

Proof. The map t is the composition of the identification $\mathfrak{g}_v \cong \mathfrak{g}_v^*$ and the projection $\mathfrak{g}_v^* \twoheadrightarrow \mathrm{End}(x_a)^*$, so t is surjective.

Let us prove that $t \circ c = 0$. This is equivalent to $\sum_{i\in Q_0} \mathrm{tr}(\mu_i(x_a, x_{a^*}), z_i) = 0$. But

$$\sum_{i\in Q_0} \mathrm{tr}(\mu_i(x_a, x_{a^*}), z_i) = \sum_{a\in Q_1} (\mathrm{tr}(x_a x_{a^*} z_{h(a)}) - \mathrm{tr}(x_{a^*} x_a z_{t(a)})) =$$
$$\sum_{a\in Q_1} (\mathrm{tr}(x_{a^*}(z_{h(a)}x_a - x_a z_{t(a)}))) = 0.$$

In order to check that $\ker t = \mathrm{im}\, c$, we will compare the dimensions. We have

$$\ker c = \{(x_{a^*}) | d_{(x_a),0}\mu((0, x_{a^*})) = 0\} = [(1.1)] = \{(x_{a^*}) | \langle (x_{a^*}), \mathfrak{g}_v.(x_a)\rangle = 0\}.$$

The dimension of $\mathfrak{g}_v.(x_a)$ is $\dim \mathfrak{g} - \dim \mathrm{End}(x_a)$ and so $\dim \ker c = \dim \mathrm{Rep}(Q^{op}, v) - \dim \mathfrak{g}_v + \dim \mathrm{End}(x_a)$. We conclude that $\dim \mathrm{im}\, c = \dim \mathfrak{g}_v - \dim \mathrm{End}(x_a) = \dim \ker t$. The second equality holds because t is surjective. $\square$

2.2. **Consequences.** Now let us deduce some corollaries on $\pi^{-1}(x_a)$.

Corollary 2.2. *The following is true.*

1. *We have $\pi^{-1}(x_a) \neq \varnothing$ if and only if $\sum_{i\in Q_0} \lambda_i \mathrm{tr}(z_i) = 0$ for any $(z_i) \in \mathrm{End}(x_a)$.*
2. *If $\pi^{-1}(x_a)$ is non-empty, then it is an affine space of dimension $\dim \mathrm{Rep}(Q, v) - \dim G_v.(x_a)$.*
3. *Suppose that v is generic with $\lambda \cdot v = 0$ meaning that the equality $\lambda \cdot v' = 0$ with $v' \leqslant v$ (component-wise) implies $v = kv'$ for some $k \in \mathbb{Q}$ (here we write $\lambda \cdot v = \sum_{i\in Q_0} \lambda_i v_i$). Then $\pi^{-1}(x_a) \neq \varnothing$ if and only if the dimensions of all direct summands of (x_a) are proportional to v.*
4. *In addition, suppose v is primitive. Then $\pi^{-1}(x_a) \neq \varnothing$ if and only if (x_a) is indecomposable. Moreover, all representations of $\Pi^\lambda(Q)$ of dimension v are irreducible.*

Proof. (1) is a direct corollary of Lemma 2.1. (2) follows from the proof because $\dim \pi^{-1}(x_a) = \dim \ker c = \dim \mathrm{Rep}(Q^{op}, v) - \dim \mathfrak{g}_v + \dim \mathrm{End}(x_a) = \dim \mathrm{Rep}(Q, v) - \dim G_v.(x_a)$.

Let us prove (3). Let (x'_a) be a direct summand of (x_a) of dimension v'. Let $(z_i) \in \bigoplus_i \mathrm{End}(V_i)$ denote the corresponding projection. Then it is an element of $\mathrm{End}(x_a)$. So $\sum_{i\in Q_0} \lambda_i \mathrm{tr}(z_i) = \lambda \cdot v' = 0$. Since λ is generic, we see that v' is proportional to v.

Conversely, let $(x_a) = \bigoplus_j (x_a^j)$ be the decomposition into indecomposables. Assume that the dimensions v^i are proportional to v. Let us write an endomorphism (z_i) of (x_a) as a matrix (z^{jk}), with $z^{jk} \in \mathrm{Hom}_Q((x_a^j), (x_a^k))$. Note that since (x_a^j) is indecomposable, the endomorphism z^{jj} acts on the corresponding representation space $V^j = \bigoplus_i V_i^j$ with a single eigenvalue. It follows that the vector $(\mathrm{tr}(z_i^{jj}))_{i\in Q_0}$ is proportional to v^j. We see that $\sum_{i\in Q_0} \lambda_i \mathrm{tr}(z_i^{jj}) = 0$ and so $\sum_{i\in Q_0} \lambda_i \mathrm{tr}(z_i) = 0$. (3) is fully proved.

Now let us prove (4). The first claim is a direct corollary of (3). To prove the second statement, let (x'_a, x'_{a^*}) be a nonzero sub of $(x_a, x_{a^*}) \in \operatorname{Rep}(\Pi^\lambda(Q), v)$. Then $\pi^{-1}(x'_a) \neq 0$ and hence, by (4), we need to have $\lambda \cdot v' = 0$. Since v is primitive, this is only possible if $v = v'$. □

2.3. **Application to Kac's theorem.** Let us compute d_v under some additional assumptions.

Corollary 2.3. *Assume that v is primitive and λ is generic with $\lambda \cdot v = 0$. If there is an indecomposable representation in* $\operatorname{Rep}(Q, v)$ *or* $\operatorname{Rep}(\Pi^\lambda(Q), v) \neq \varnothing$*, then $p_v = 1 - (v, v)/2$.*

Proof. Let $\operatorname{Rep}^{ind}(Q, v) \subset \operatorname{Rep}(Q, v)$ denote the subset of the indecomposable representations. Then $\operatorname{Rep}(\Pi^\lambda(Q), v)$ admits a morphism π with image $\operatorname{Rep}^{ind}(Q, v)$ whose fiber over (x_a) is an affine space of dimension $\dim \operatorname{Rep}(Q, v) - \dim G_v.(x_a)$. By (4) of the previous corollary all representations in $\operatorname{Rep}(\Pi^\lambda(Q), v)$ are irreducible. By the Schur lemma, all their endomorphisms are constant.

Let $\bar{G}_v$ denote the quotient of G_v by the one-dimensional subgroup of constant matrices. The kernel of $G_v \twoheadrightarrow \bar{G}_v$ acts on $\operatorname{Rep}(Q, v)$ trivially so we get an action of $\bar{G}_v$ on $\operatorname{Rep}(Q, v)$. The Lie algebra of $\bar{G}_v$ is naturally identified with $\bar{\mathfrak{g}}_v$ and $\mu : \operatorname{Rep}(\bar{Q}, v) \to \bar{\mathfrak{g}}_v$ is the moment map. Also note that the action of $\bar{G}_v$ on $\operatorname{Rep}(\Pi^\lambda(Q), v) = \mu^{-1}(\lambda)$ is free. From here and (1.1) one deduces that μ is a submersion at all points of $\operatorname{Rep}(\Pi^\lambda(Q), v)$ and hence $\dim \operatorname{Rep}(\Pi^\lambda(Q), v) = \dim \operatorname{Rep}(\overline{Q}, v) - \dim \mathfrak{g}_v$.

Let us cover $\operatorname{Rep}^{ind}(Q, v)$ with locally closed G_v-stable subvarieties with constant dimensions of orbits, $\operatorname{Rep}^{ind}(Q, v) = \bigsqcup_i X_i$, let d_i denote the dimension of a G_v-orbit in X_i. Let Y_i denote the preimage of X_i in $\mu^{-1}(\lambda)$, it is an affine bundle with rank $\dim \operatorname{Rep}(Q, v) - d_i$ over X_i. So we see that $2 \dim \operatorname{Rep}(Q, v) - \dim \overline{G}_v = \max_i(\dim Y_i) = \max_i(\dim X_i + \dim \operatorname{Rep}(Q, v) - d_i) = \dim \operatorname{Rep}(Q, v) + \max_i(p(X_i))$. It follows that $p_v = \max_i(p(X_i)) = \dim \operatorname{Rep}(Q, v) - \dim G_v + 1 = 1 - (v, v)/2$. □

3. Reflection functors

We will view $\lambda = (\lambda_j)_{j \in Q_0}$ as an element of $\mathfrak{h}$ and a dimension vector v as an element of $\mathfrak{h}^*$ (the pairing is by $\langle \lambda, v \rangle = \lambda \cdot v$). Recall that $W(Q)$ acts on $\mathfrak{h}^*$ as follows: $(s_i v)_j = v_j$ for $j \neq i$ and $(s_i v)_i = \sum_j n_{ij} v_j - v_i$, where n_{ij} is the number of edges between i and j. So $W(Q)$ acts on $\mathfrak{h}$ as follows: $(s_i \lambda)_i = -\lambda_i, (s_i \lambda)_j = \lambda_j + n_{ij} \lambda_i$.

The main result of this section is as follows.

Theorem 3.1. *Pick $i \in Q_0$ such that there are no loops at i. Suppose $\lambda_i \neq 0$. Then is an equivalence* $\Pi^\lambda(Q)\text{-mod} \xrightarrow{\sim} \Pi^{s_i \lambda}(Q)\text{-mod}$ *that maps a representation of dimension v to a representation of dimension $s_i v$.*

Before proving this theorem we will explain how it applies to the Kac theorem.

3.1. **Application to Kac's theorem.**

Corollary 3.2. *Suppose there is an indecomposable representation of dimension vector v. Then v is a root.*

Proof. We can assume that for all $v' \leqslant v, v' \neq v$ (componentwise), the claim is true. We can also assume $(v, v) > 0$, otherwise we are done by Remark 1.2. If $(v, \epsilon_i) \leqslant 0$ for all i, then $(v, v) = \sum_i v_i (v, \epsilon_i) \leqslant 0$. Note that if there is a loop at i, then $(v, \epsilon_i) \leqslant 0$. So it's enough to consider the case when there is i such that there is no loop at i and $(v, \epsilon_i) > 0$ so that $s_i v = v - (v, \epsilon_i)\epsilon_i < v$.

Let us prove that if $\mathrm{Rep}(\Pi^\lambda(Q), v)$ for a Zariski generic λ with $\lambda \cdot v = 0$ contains an indecomposable representation, then v is a real root. We prove it by induction. By Theorem 3.1, $\mathrm{Rep}(\Pi^{s_i\lambda}(Q), s_i v)$ contains an indecomposable representation. This provides an inductive step. The base is given by $v = m\epsilon_i$: there the representation is zero and so $m = 1$.

By (3) of Corollary 2.2, if $\mathrm{Rep}(Q, v)$ contains an indecomposable representation, then so does $\mathrm{Rep}(\Pi^\lambda(Q), v)$. This completes the proof. □

Corollary 3.3. *Let v be a real root. Then there is a unique (up to isomorphism) indecomposable representation of Q with dimension vector v.*

Proof. Let λ be generic with $\lambda \cdot v = 0$. Let us check that there is a unique (up to an isomorphism) representation of $\Pi^\lambda(Q)$ of dimension vector v. If $v = \epsilon_i$, then there is only the zero representation and so we are done. Theorem 3.1 gives the induction step.

Note that, as any real root, v is indecomposable. Now the claim of this corollary follows from (4) of Corollary 2.2. □

3.2. **Construction of equivalence.** Now let us construct the required equivalence. Pick a representation (x_a, x_{a^*}) with dimension vector v. Recall that $\Pi^\lambda(Q)$ does not depend on the orientation of Q up to an isomorphism. So we may assume that i is a sink in Q. Let $W_i := \bigoplus_{a, t(a)=i} V_{h(a)}$. We can write (x_a, x_{a^*}) as $(A, B, \underline{x})$, where $A := \bigoplus_{a,t(a)=i} : V_i \to W_i, B := \bigoplus_{a,t(a)=i} x_{a^*} : W_i \to V_i$ and $\underline{x}$ includes all x_b, x_{b^*} with $t(b) \neq i$. Multiplying the relation of $\Pi^\lambda(Q)$ by ϵ_i, we see that $BA = -\lambda_i \,\mathrm{id}_{V_i}$. Since $\lambda_i \neq 0$, we see that A is injective, B is surjective. Also, we see that $W_i = \operatorname{im} A \oplus \ker B$. Identifying V_i with $\operatorname{im} A$, we can assume that A is the inclusion $V_i \hookrightarrow W$, and $B = -\lambda_i \pi$, where π is the projection along $\ker B$.

Now let us proceed to defining a representation of $\Pi^{s_i\lambda}(Q)$ with dimension vector $s_i v$. The space $V' := \bigoplus V_i'$ is determined as follows: $V_j' := V_j$ if $j \neq i$, and $V_i' := \ker B$. In particular, $v' = s_i v$. The representation is given by $(A', B', \underline{x})$, where A' is the inclusion $V_i' \hookrightarrow W_i$ and B' is $\lambda_i \pi'$, where $\pi' : W_i \twoheadrightarrow V_i'$ is the projection along $\operatorname{im} A$. Note that we have

$$A'B' - AB = \lambda_i \,\mathrm{id}_{W_i}\,. \tag{3.1}$$

Now let us check that the resulting representation $(A', B', \underline{x})$ factors through $\Pi^{s_i\lambda}(Q)$. For $a \in Q_1$ with $t(a) = i$, let ρ_a, ι_a denote the projection $W_i = \bigoplus_{a,t(a)=i} V_{h(a)} \twoheadrightarrow V_{h(a)}$ and the inclusion $V_{h(a)} \hookrightarrow W_i$ corresponding to this arrow. So we have $x_a = \rho_a \circ A, x_{a^*} = B \circ \iota_a, x_a' = \rho_a \circ A', x_{a^*}' = B' \circ \iota_a$. We have $-\sum_{t(a)=i} x_{a^*}' x_a' = -B'A' = -\lambda_i \,\mathrm{id}_{V_i'}$. So what we need to check is that for $j \neq i$, we have

$$\sum_{a, h(a)=j} x_a' x_{a^*}' - \sum_{a, t(a)=j} x_{a^*}' x_a' = (s_i\lambda)_j \,\mathrm{id}_{V_j} = (\lambda_j + n_{ij}\lambda_i)\,\mathrm{id}_{V_i}\,.$$

This will follow if we check that

$$\sum_{a, h(a)=j} (x_a' x_{a^*}' - x_a x_{a^*}) - \sum_{a, t(a)=j} (x_{a^*}' x_a' - x_{a^*} x_a) = n_{ij}\lambda_i \,\mathrm{id}_{V_j}\,.$$

If $t(a) \neq i$, then $x_a = x'_a, x_{a^*} = x'_{a^*}$. So the left hand side is

$$\sum_{t(a)=i,h(a)=j} (x'_a x'_{a^*} - x_a x_{a^*}) = \sum_{t(a)=i,h(a)=j} \rho_a \circ (A'B' - AB) \circ \iota_a =$$

$$= [(3.1)] = \sum_{t(a)=i,h(a)=j} \rho_a \circ (\lambda_i \operatorname{id}_{W_i}) \circ \iota_a = n_{ij}\lambda_i \operatorname{id}_{V_j},$$

as required. So we indeed get a a representation of $\Pi^{s_i\lambda}(Q)$.

Our construction is functorial. Indeed, let $(y_i) : (V_i, x_a, x_{a^*}) \to (\bar{V}_i, \bar{x}_a, \bar{x}_{a^*})$ be a homomorphism of representations. This induces a homomorphism $y : W_i \to \bar{W}_i$ that intertwines A, B with $\bar{A}, \bar{B}$. In particular, y restricts to $\ker B \to \ker \overline{B}$. So it induces a homomorphism $(y'_i) : (V'_i, x'_a, x'_{a^*}) \to (\bar{V}'_i, \bar{x}'_a, \bar{x}'_{a^*})$. We indeed get a functor $\Pi^\lambda(Q)$-mod $\to \Pi^{s_i\lambda}(Q)$-mod that behaves as s_i on the dimension vectors.

We also have a similarly defined functor $\psi : \Pi^{s_i\lambda}(Q)$-mod $\to \Pi^\lambda(Q)$-mod. It sends a representation $(A', B', \underline{x})$ back to $(A, B, \underline{x})$. It is easy to see that $\psi \circ \varphi$ is isomorphic to the identity functor of $\Pi^\lambda(Q)$-mod. Similarly, $\varphi \circ \psi$ is isomorphic to the identity functor. This shows that φ is an equivalence (with quasi-inverse ψ).

4. Further results and applications

4.1. Irreducible representations. A basic question about the representation theory of $\Pi^\lambda(Q)$ is to describe its irreducible representations. Let us state the corresponding result of Crawley-Boevey, we are not going to provide a proof.

For $v \in \mathbb{Z}_{\geqslant 0}^{Q_0}$, set $p(v) = 1 - \frac{1}{2}(v, v)$. Define the set Σ_λ of all positive roots such that $\lambda \cdot v = 0$ $p(v) > \sum_{i=1}^k p(v_i)$ for all proper decompositions of v into the sum $\sum_{i=1}^k v_i$, where all $v_i \in (\mathbb{Z}_{\geqslant 0})^{Q_0} \setminus \{0\}$ such that $\lambda \cdot v_i = 0$. It is not so easy to describe Σ_λ, but this is a combinatorial object.

Theorem 4.1. *The algebra* $\Pi^\lambda(Q)$ *has an irreducible representation of dimension* v *if and only if* $v \in \Sigma_\lambda$. *Moreover,* $\operatorname{Rep}(\Pi^\lambda(Q), v) \subset \operatorname{Rep}(\mathbb{C}\overline{Q}, v)$ *is an irreducible subvariety of dimension* $\dim \operatorname{Rep}(Q, v) + p(v)$ *and a Zariski generic point in* $\operatorname{Rep}(\Pi^\lambda(Q), v)$ *gives an irreducible representation.*

4.2. Application to additive Deligne-Simpson problem. The additive Deligne-Simpson problem (we'll abbreviate this as DS problem) asks about the conditions on the conjugacy classes $C_1, \ldots, C_k$ in $\operatorname{Mat}_n(\mathbb{C})$ such that there are matrices $Y_i \in \operatorname{Mat}_n(\mathbb{C})$ satisfying the following two conditions:

(1) $\sum_{i=1}^k Y_i = 0$,
(2) and there are no proper subspaces in $\mathbb{C}^n$ stable under all Y_i.

From $C_1, \ldots, C_k$, Crawley-Boevey have constructed a quiver Q, a dimension vector v, and $\lambda \in \mathbb{C}^{Q_0}$ such that there is a bijection between

(a) solutions $(Y_1, \ldots, Y_k)$ of the DS problem (up to $\mathrm{GL}_n(\mathbb{C})$-conjugacy),
(b) irreducible dimension v representations of $\Pi^\lambda(Q)$ (up to an isomorphism).

Then the solution of the DS problem follows from Theorem 4.1 (one needs to use some complicated combinatorics to get the answer explicitly).

LECTURE 18: DELIGNE-SIMPSON PROBLEM

IVAN LOSEV

Introduction

The Deligne-Simpson problem asks to find a condition on conjugacy classes $C_1, \ldots, C_k \subset \mathrm{Mat}_n(\mathbb{C})$ such that there are matrices $Y_i \in C_i$ with

(1) $\sum_{i=1}^k Y_i = 0$,
(2) and there are no proper subspaces in $\mathbb{C}^n$ stable under all Y_i.

Crawley-Boevey reduced this problem to checking if there is an irreducible representation in $\mathrm{Rep}(\Pi^\lambda(Q), v)$ for suitable Q, λ, v produced from $C_1, \ldots, C_k$. Recall, Section 4.1 of Lecture 17, that this is equivalent to $v \in \Sigma_\lambda$, where $\Sigma_\lambda \subset \mathbb{Z}^{Q_0}$ is a combinatorially defined set.

Crawley-Boevey's approach was strongly motivated the Kraft-Procesi construction who proved that the closures of conjugacy classes of matrices are normal. The proof easily reduces to the case of nilpotent orbits. Kraft and Procesi realized their closures as certain quotients that are special cases of Nakajima quiver varieties. This allowed them to prove the normality.

In the first section we will recall the necessary background from Invariant theory, a field that studies quotients under group actions. Then we will explain the Kraft-Procesi construction. Finally, we will explain Crawley-Boevey's approach to the DS problem.

1. Invariant theory

Let X be an affine algebraic variety and let G be a reductive algebraic group (such as $\mathrm{GL}(n)$ or, more generally, G_v). We assume that G acts on X in such a way that the action map $G \times X \to X$ is a morphism of algebraic varieties. If $X = V$ is a vector space, then a rational representation of G in V provides an example of such an action. In general, we can G-equivariantly embed X into V (as a closed subvariety). Our goal is to study the algebra $\mathbb{C}[X]^G = \{f \in \mathbb{C}[X] | f(g.x) = f(x), \forall x \in X, g \in G\}$.

The first general result here is due to Hilbert.

Theorem 1.1. *The algebra $\mathbb{C}[X]^G$ is finitely generated.*

So we can consider the variety $X//G$ with algebra $\mathbb{C}[X]^G$ of polynomial functions. The inclusion $\mathbb{C}[X]^G \hookrightarrow \mathbb{C}[X]$ gives rise to a dominant morphism $\pi : X \to X//G$. It turns out that this morphism has very nice properties (that follow because G is reductive).

Theorem 1.2. *The following is true.*

(1) *The morphism π is surjective.*
(2) *Let $Y_1, Y_2 \subset X$ be closed G-stable subvarieties of X with $Y_1 \cap Y_2 = \varnothing$. Then $\pi(Y_1) \cap \pi(Y_2) = \varnothing$. In particular, every fiber of π contains a unique closed orbit.*
(3) *Let Z be an affine algebraic variety with a G-invariant morphism $\varphi : X \to Z$. Then there is a unique morphism $\psi : X//G \to Z$ such that $\varphi = \psi \circ \pi$.*
(4) *In particular, if $X' \subset X$ is a closed G-stable subvariety, then the induced morphism $X'//G \hookrightarrow X//G$ is a closed embedding with image $\pi(X')$.*

Here is an important example of a computation of $X//G$ and π. Let V, V' be finite dimensional vector spaces, $X = \mathrm{Hom}(V,V') \times \mathrm{Hom}(V',V), G = \mathrm{GL}(V')$ acts on X by $g.(A,B) = (gA, Bg^{-1})$. The following result is traditionally known as the (first and second) "main theorem of invariant theory for $\mathrm{GL}(V')$".

Theorem 1.3. *The quotient $X//G$ is the subvariety of all operators of rank $\leqslant \dim V'$ in $\mathrm{End}(V)$. The quotient morphism $\pi : X \to X//G$ is given by $(A,B) \mapsto BA$.*

We will need the following corollary of Theorem 1.3 combined with (4) of Theorem 1.2.

Corollary 1.4. *Let $X_0 \subset \mathrm{Hom}(V,V') \times \mathrm{Hom}(V',V)$ be a $\mathrm{GL}(V')$-stable subvariety. Then $X_0//\mathrm{GL}(V') = \{BA | (A,B) \in X_0\}$.*

Note that $X_0//\mathrm{GL}(V')$ is a closed $\mathrm{GL}(V)$-stable subvariety of $\mathrm{End}(V)$.

2. Orbit closures

2.1. **Induction lemma.** We are going to investigate the following question. Let $\lambda = (\lambda_1, \ldots, \lambda_k)$ be a Young diagram with n boxes (we assume that $\lambda_k > 0$). To this diagram we can assign the nilpotent orbit $O_\lambda \subset \mathrm{Mat}_n(\mathbb{C})$ consisting of all matrices whose Jordan normal form has blocks of sizes $\lambda_1, \ldots, \lambda_k$. Now set $\lambda' = (\lambda_1 - 1, \ldots, \lambda_k - 1)$, this is the Young diagram obtained from λ by removing the first column. Let $n' := |\lambda'|$, the number of boxes in λ'. Further, let $\lambda^{(i)}$ denote the diagram obtained from λ by removing the first i columns. Note that $O_\lambda = \{Y \in \mathrm{Mat}_n(\mathbb{C}) | \operatorname{rk} Y^i = |\lambda^{(i)}|\}$, while $\overline{O}_\lambda = \{Y \in \mathrm{Mat}_n(\mathbb{C}) | \operatorname{rk} Y^i \leqslant |\lambda^{(i)}|\}$.

Set $V := \mathbb{C}^n, V' := \mathbb{C}^{n'}$.

Lemma 2.1. *The set $\{BA | A \in \mathrm{Hom}(V,V'), B \in \mathrm{Hom}(V',V), AB \in \overline{O}_{\lambda'}\}$ coincides with $\overline{O}_\lambda$. Moreover, we have $BA \in O_\lambda$ if and only if A is surjective, B is injective, and $AB \in O_{\lambda'}$.*

Proof. Note that $(BA)^{i+1} = B(AB)^i A$. We have $\mathrm{rk}((BA)^{i+1}) \leqslant \mathrm{rk}((AB)^i) \leqslant |\lambda^{(i+1)}|$. It follows that $BA \in \overline{O}_\lambda$. The condition that $BA \in O_\lambda$ is equivalent to $\mathrm{rk}((BA)^{i+1}) = |\lambda^{(i+1)}|$ for all i. The condition that $\mathrm{rk}(BA) = |\lambda^{(1)}| = \dim V'$ is equivalent to A being surjective and B being injective. If that holds, then $\mathrm{rk}((BA)^{i+1}) = \mathrm{rk}((AB)^i)$. This completes the proof. □

2.2. **$\overline{O}_\lambda$ as a quiver variety.** Let $m := \lambda_1$. Consider the Dynkin quiver of type A_m, we number vertices by numbers from 0 to $m-1$. We orient it left to right. Set $v_i := |\lambda^{(i)}|$ and let $V_0 = V, V_1, \ldots, V_{m-1}$ be the spaces of these dimensions. The corresponding representation space $\mathrm{Rep}(\overline{Q}, v)$ consists of the $2m-2$-tuples $(A_1, B_1, A_2, B_2, \ldots, A_{m-1}, B_{m-1})$, where $A_i \in \mathrm{Hom}(V_{i-1}, V_i)$ and $B_i \in \mathrm{Hom}(V_i, V_{i-1})$:

$$V_0 \underset{B_1}{\overset{A_1}{\rightleftarrows}} V_1 \underset{B_2}{\overset{A_2}{\rightleftarrows}} V_2 \underset{B_3}{\overset{A_3}{\rightleftarrows}} \cdots \underset{B_{m-1}}{\overset{A_{m-1}}{\rightleftarrows}} V_{m-1}$$

Consider the group $G = \mathrm{GL}(V_1) \times \ldots \times \mathrm{GL}(V_{m-1})$. The moment map $\mu : \mathrm{Rep}(\overline{Q}, v) \to \mathfrak{g}$ is given by $\mu = (\mu_1, \ldots, \mu_{m-1})$, where μ_i sends $(A_j, B_j)_{j=1}^m$ to $A_i B_i - B_{i+1} A_{i+1}$ (where we set $A_m = B_m = 0$).

The following result is due to Kraft and Procesi.

Proposition 2.2. *The image of the map $\mu^{-1}(0) \to \mathrm{End}(V)$ sending $(A_i, B_i)_{i=1}^{m-1}$ to $B_1 A_1$ has image $\overline{O}_\lambda$ and induces an isomorphism $\mu^{-1}(0)//G \xrightarrow{\sim} \overline{O}_\lambda$.*

Proof. Our proof is by induction on i. The base is the empty quiver. To do the induction step, assume that the analog of our claim is established for the subquiver Q' with vertices $1, \ldots, m-1$ and the group $G' = \mathrm{GL}(V_2) \times \ldots \mathrm{GL}(V_{m-1})$. Let $\mu' : \mathrm{Rep}(\overline{Q}', v') \to \mathfrak{g}'$ be the corresponding moment map so that $\mathrm{Rep}(\overline{Q}, v) = \mathrm{Hom}(V, V_1) \times \mathrm{Hom}(V_1, V) \times \mathrm{Rep}(\overline{Q}', v'), G = \mathrm{GL}(V_1) \times G', \mu((A_i, B_i)) = (A_1B_1 - B_2A_2, \mu')$. We can produce the quotient $\mu^{-1}(0)//G$ in two steps: $(\mu^{-1}(0)//G')// \mathrm{GL}(V_1)$. Note that A_1, B_1 are G'-invariant. By the inductive step, $\mu'^{-1}(0)//G' \xrightarrow{\sim} \overline{O}_{\lambda'}$ via $(A_2, B_2, \ldots, A_{m-1}, B_{m-1}) \to B_2A_2$. It follows that $\mu^{-1}(0)//G' = \{(A_1, B_1) | A_1B_1 \in \overline{O}_{\lambda'}\}$. Combining Corollary 1.4 with Lemma 2.1, we see that the map $(A_1, B_1) \mapsto B_1A_1$ gives rise to an isomorphism of $\mu^{-1}(0)//G$ and $\overline{O}_\lambda$. $\square$

Remark 2.3. Let us explain how this helps to prove normality. First, if an affine variety X is normal, then so is $X//G$, this is an exercise. Now to prove that $\mu^{-1}(0)$ is normal we need to do two things: to show that it is a complete intersection (its codimension equals $\dim G$) and apply the Serre normality criterium saying that $\mu^{-1}(0)$ is normal if the complement of the locus where μ is not submersive has codimension bigger than 1. In order to do this one observes that μ is a submersion at a point v precisely when G_v is discrete (in our case, this is equivalent for $G_v = \{1\}$). Both claims are proved inductively based on Lemma 2.1.

2.3. **Closure of an arbitrary orbit.** Lemma 2.1 generalizes to arbitrary orbits as follows. Take an orbit $C \subset \mathrm{Mat}_n(\mathbb{C})$. Let λ be the Young diagram corresponding to the nilpotent component of C. Set $n' := n - |\lambda| + |\lambda'|$ and let C' be the orbit in $\mathrm{Mat}_{n'}(\mathbb{C})$ obtained from C by replacing the nilpotent part O_λ with $O_{\lambda'}$. The proof of the following lemma is left to the reader.

Lemma 2.4. *The set $\{BA | A \in \mathrm{Hom}(V, V'), B \in \mathrm{Hom}(V', V), AB \in \overline{C}'\}$ coincides with $\overline{C}$. Moreover, we have $BA \in C$ if and only if A is surjective, B is injective, and $AB \in C'$.*

When C consists of non-degenerate operators we can replace C with $C - \chi$, where χ is an eigenvalue of C (and we write $C - \chi = \{Y - \chi \,\mathrm{id}_{\mathbb{C}^n} | Y \in C\}$) and apply Lemma 2.4 to that orbit. This motivates the following construction.

Let $\chi_1, \ldots, \chi_m$ be all roots of the minimal polynomial of C counted with multiplicities so that, for $Y \in C$, we have $\prod_{i=1}^m (Y - \chi_i) = 0$. Check the Dynkin diagram of type A_m oriented as before. Set $v_i := \mathrm{rk} \prod_{j=1}^i (Y - \chi_i)$ and $\xi_i = \chi_{i+1} - \chi_i, i = 1, \ldots, m-1$. Consider G, μ as before and set $\xi := \sum_{i=1}^m \xi_i \,\mathrm{id}_{V_i}$.

The proof of the following proposition repeats that of Theorem 1.3

Proposition 2.5. *We have an isomorphism $\mu^{-1}(\xi)//G \xrightarrow{\sim} \overline{C}$ induced by $(A_i, B_i)_{i=1}^{m-1} \mapsto B_1A_1 + \chi_1$.*

Example 2.6. Let us consider an example of this. Let C consist of the projectors in $\mathrm{Mat}_n(\mathbb{C})$ of rank $r < n$. The quiver Q will have type A_2. Then we can set $\chi_1 = 0$ and $\chi_2 = 1$. We have $\xi_2 = 1, v_1 = r$, the moment map equation is $AB = 1$ and the map $\mu^{-1}(0) \to \overline{C} = C$ is $(A, B) \mapsto BA$. Or we can set $\chi_1 = 1, \chi_2 = 0$. In this case, $v_1 = n - r, \xi_1 = -1$. The moment map equation is $AB = -1$ and the map $\mu^{-1}(0) \to C$ is $(A, B) \mapsto BA + 1$.

Remark 2.7. Let $(A_i, B_i)_{i=1}^{m-1} \in \mu^{-1}(\xi)$ be such that $Y := B_1A_1 + \chi_1 \in C$. We can recover $(A_i, B_i)_{i=1}^{m-1}$ up to G-conjugacy from Y as follows. Since $Y \in C$, we see that all A_i are surjective and all B_i are injective. We can assume that $V = V_0 \supset V_1 \supset V_2 \supset \ldots$ and all B_i's are inclusions. Then $A_1 = Y - \chi_1 : V_0 \to V_1 = \mathrm{im}(Y - \chi_1)$. We have $A_2|_{V_1} + \chi_2 = B_2A_2 + \chi_2 = A_1B_1 + \chi_1 = (Y - \chi_1)|_{V_1} + \chi_1 = Y|_{V_1}$. So $A_2 = (Y - \chi_2)|_{V_1}$. Continuing in this way we see that $V_i = \mathrm{im} \prod_{j=1}^i (Y - \chi_j)$ and $A_i = (Y - \chi_i)|_{V_{i-1}}$ for all i.

3. Solution to Deligne-Simpson problem

Recall that we are interested in solutions of the DS problem: describe $\mathrm{GL}_n(\mathbb{C})$-orbits $C_1,\ldots,C_k$ in $\mathrm{Mat}_n(\mathbb{C})$ such that there are $Y_i\in C_i$ with

(1) $\sum_{i=1}^k Y_i=0$,
(2) and there are no proper subspaces in $\mathbb{C}^n$ stable under all Y_i.

We are going to produce a quiver Q, a dimension vector v and a parameter λ out of $C_1,\ldots,C_k$.

The quiver Q and the dimension vector v are as follows. Produce the type A quivers from $C_1,\ldots,C_k$, let $[0,i],[1,i],\ldots,[m_i,i]$ denote the vertices of the quiver corresponding to C_i. Identify the vertices $[0,i]$ with a single vertex 0 getting a star-shaped quiver. Denote the dimension vector v so that $(v_{[0,i]},\ldots,v_{[m_i,i]})$ is the dimension vector produced from C_i (so that $v_0=n$). Define $\xi_{[j,i]}$ as above for $j>0$, set $\xi_0=\sum_{i=1}^k\chi_{[1,i]}$. Note that ξ_0 is chosen in such a way that the equalities $\sum_{i=1}^k\mathrm{tr}(Y_i)=0$ (a necessary condition for the DS problem to have a solution) and $v_0\xi_0+\sum_{i=1}^k\sum_{j=1}^{m_i}v_{[j,i]}\xi_{[j,i]}=0$ (a necessary condition for $\Pi^\xi(Q)$ to have a representation of dimension v) are equivalent.

Example 3.1. Assume that $n=6$ and $k=3$. We take the following conjugacy classes C_1,C_2,C_3: C_1 contains $\mathrm{diag}(-2,-1,0,1,2,3)$, C_2 contains $\mathrm{diag}(-1,-1,-1,0,0,0)$, and $C_3=O_\lambda$, for $\lambda=(3,3)$. Then we get $m_1=5,m_2=1,m_3=2$ with $v_{[?,1]}=(5,4,3,2,1),v_{[1,2]}=3,v_{[?,3]}=(4,2)$. Further, $\xi_{[?,1]}=(1,1,1,1,1),\xi_{[1,2]}=1,\xi_{[?,3]}=(0,0),\xi_0=-3$. We note that the resulting quiver is of type $\tilde{E}_8$ and v is the indecomposable imaginary root δ.

Theorem 3.2. *There is a bijection between the solutions $Y_1,\ldots,Y_k$ of the DS problem (up to conjugacy) and the irreducible representations in $\mathrm{Rep}(\Pi^\xi(Q),v)$ (up to isomorphism).*

First, let us get a necessary and sufficient condition for a representation of $\Pi^\xi(Q)$ to be irreducible.

Lemma 3.3. *The following two conditions are equivalent:*

(1) *A representation $(A_{[j,i]},B_{[j,i]})\in\mathrm{Rep}(\Pi^\xi(Q),v)$ is irreducible.*
(2) *All maps $A_{[j,i]}$ are surjective, all maps $B_{[j,i]}$ are injective, and the space $\mathbb{C}^n$ is irreducible with respect to the operators $B_{[1,i]}A_{[1,i]}$.*

Proof. Suppose (1) holds and let us establish (2). Assume the map $A_{[j,i]}$ is not surjective. Set $U_{[j',i']}:=V_{[j',i']}$ if $i'\neq i$ or $j'<j$, $U_{[k,i]}:=A_{[k,i]}U_{[k-1,i]}$ for $k\geqslant j$. The moment map equation implies that $B_{[k,i]}U_{[k+1,i]}\subset U_{[k,j]}$. Indeed, we have $B_{[k,i]}U_{[k+1,i]}=B_{[k,j]}A_{[k,j]}U_{[k,j]}=(A_{[k-1,j]}B_{[k-1,j]}+\xi_{[k-1,i]})U_{[k,i]}\subset U_{[k,i]}$. We deduce that $(U_{[k,i]})\subset(V_{[k,i]})$ defines a proper subrepresentation, a contradiction. The remaining parts of (2) are left as exercises.

Conversely, suppose that (2) holds. Let $(U_{[j,i]})\subset(V_{[j,i]})$ be a proper subrepresentation. The condition that V_0 is irreducible w.r.t. $B_{[1,i]}A_{[1,i]}$ implies that $U_0=\{0\}$ or $U_0=V_0$. In the former case, $U_{[j,i]}=\{0\}$ because all $B_{[j,i]}$ are injective, in the latter case, $U_{[j,i]}=V_{[j,i]}$ because all $A_{[j,i]}$ are surjective. □

Proof of Theorem 3.2. Let $(A_{[j,i]},B_{[j,i]})$ be an irreducible representation in $\mathrm{Rep}(\Pi^\xi(Q),v)$. Then $Y_i:=B_{[1,i]}A_{[1,i]}+\chi_{[1,i]},i=1,\ldots,k$ is a solution to the Deligne-Simpson problem (the claim that $\sum_i Y_i=0$ is the moment map condition at 0, while the claim that $\mathbb{C}^n$ is irreducible w.r.t. the Y_i's follows from Lemma 3.3). Conversely, let $(Y_i)_{i=1}^k$ be the solution to the DS problem. We can form the maps $A_{[i,j]},B_{[i,j]}$ as in Remark 2.7, and this will give an irreducible representation in $\mathrm{Rep}(\Pi^\xi(Q), v)$. On the level of isomorphism classes, these two maps are inverse to each other. $\square$

Example 3.4. In the example above, the DS problem has a solution. Moreover, the variety of conjugacy classes of (Y_1, Y_2, Y_3) can be shown to be 2-dimensional.

Lecture 18 $\frac{1}{2}$

Deformations of Kleinian singularities

1) Kleinian groups & McKay correspondence
2) Kleinian singularities & their deformations
3) Construction via DPA.

1) Kleinian gr-p = finite subgroup in $SL_2(\mathbb{C})$. ~~Those~~ These subgroups up to $SL_2(\mathbb{C})$-conjugacy are parameterized by affine Dynkin diagrams $\tilde{A}_n, \tilde{D}_n, \tilde{E}_\ell$, $\ell = 6,7,8$

Ex 1: $\Gamma \simeq \mathbb{Z}/_{n+1} = \left\{ \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon^{-1} \end{pmatrix}, \epsilon^{n+1} = 1 \right\} \rightsquigarrow \tilde{A}_n$: 

Γ = [dihedral gr-p] $= \left\{ \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon^{-1} \end{pmatrix}, \begin{pmatrix} 0 & \epsilon \\ -\epsilon^{-1} & 0 \end{pmatrix} \middle| \epsilon^{2n} = 1 \right\} \rightsquigarrow$  $(\tilde{D}_{n+2})$, $n \geq 2$

Exceptional groups $\rightsquigarrow \tilde{E}_6, \tilde{E}_7, \tilde{E}_8$.

Construction of diagram from the group (McKay): $V = \mathbb{C}^2$

$\Gamma \subset SL_2(\mathbb{C}) \rightsquigarrow N_0, \dots, N_r$ - the irreps of Γ, N_0 = triv

Quiver $\overline{Q}$: $\overline{Q}_0 = \{0, 1, \dots, r\}$

$\#\{a : i \to j\} = \dim \operatorname{Hom}_\Gamma(V \otimes N_i, N_j)$ $\left(= \dim \operatorname{Hom}_\Gamma(N_j \otimes V, N_i) \text{ b/c } V \text{ is self-dual}\right)$

$\rightsquigarrow \overline{Q}$ = double of Q b/c have no loops (check). if $\Gamma \neq \{1\}$

Total # edges between i, j in Q is $\dim \operatorname{Hom}_\Gamma(N_i \otimes V, N_j)$

Ex: $\Gamma \simeq \mathbb{Z}/_{n+1} \rightsquigarrow$ irred char-s $\chi_k : z \mapsto \exp\left(\frac{2\pi\sqrt{-1}}{n+1} z\right)$, have edges from k to $k \pm 1$ b/c $V = \chi_1 \oplus \chi_{-1}$

Thm: Q is an affine Dynkin quiver. Moreover, 0 is the extending vertex (one added to the Dynkin quiver). Finally, $(\dim N_i)_{i=0}^{r}$ is the indecomposable imaginary root for Q (to be denoted by δ)

Rem: The last claim can be proved conceptually: note that $V \otimes \mathbb{C}\Gamma \simeq \mathbb{C}\Gamma \oplus \mathbb{C}\Gamma$

2) Kleinian singularity: $\mathbb{C}^2/\Gamma$ - affine alg-c variety w. algebra of functions $\mathbb{C}[x,y]^\Gamma$. This algebra has 3 generators, say a, b, c, and one relation, $F(a,b,c)$

Ex: $\Gamma = \mathbb{Z}_{n+1}$: $a = x^{n+1}, b = y^{n+1}, c = xy$. $F = ab - c^{n+1}$

Others: $\tilde{D}_r$: $a^{r-1} + ab^2 + c^2 = 0$

$\tilde{E}_6$: $a^4 + b^3 + c^2 = 0$

$\tilde{E}_7$: $a^3b + b^3 + c^2 = 0$

$\tilde{E}_8$: $a^5 + b^3 + c^2 = 0$

$A = \mathbb{C}[x,y]^\Gamma$ is graded (as a subalgebra of $\mathbb{C}[x,y]$). Our goal is to produce filtered deformations, i.e filtered associative algebras $\mathcal{A}$ s.t. $\operatorname{gr}\mathcal{A} = A$ (isomorphism of graded algebras)

Now $\Pi^\lambda(Q)$ has a filtration induced from $\mathbb{C}\bar{Q}$ (by length of the path) and $\Pi^0(Q)$ is actually graded (the relation is homogeneous of degree 2). Consider the subspace $\epsilon_0\Pi^\lambda(Q)\epsilon_0 \subset \Pi^\lambda(Q)$. It's closed under the product and ϵ_0 is the unit. So $\epsilon_0\Pi^\lambda(Q)\epsilon_0$ is a filtered associative algebra w. 1 (the filtration is restricted from $\Pi^\lambda(Q)$). We note that $\Pi^0(Q) \twoheadrightarrow \operatorname{gr}\Pi^\lambda(Q)$ (the relations for $\Pi^0(Q)$ ~~are~~ is top degree component of that for $\Pi^\lambda(Q)$)

Thm (Crawley-Boevey & Holland)

- $\operatorname{gr}\Pi^\lambda(Q) \simeq \Pi^0(Q)$ ($\Rightarrow \operatorname{gr}\epsilon_0\Pi^\lambda(Q)\epsilon_0 = \epsilon_0\Pi^0(Q)\epsilon_0$)
- $\epsilon_0\Pi^0(Q)\epsilon_0 = \mathbb{C}[x,y]^\Gamma$

So the algebras $\epsilon_0\Pi^\lambda(Q)\epsilon_0$ are filtered deformations of $\mathbb{C}[x,y]^\Gamma$. In fact, all filtered deformations can be obtained in this way.

3) We need an alternative description of $\Pi^\lambda(Q)$. 1st step is as follows.

3.1) Semi-direct tensor products.

Let A be an associative unital algebra with an action of a finite group Γ (e.g $A = \mathbb{C}[x,y]$, $\Gamma \subset SL_2(\mathbb{C})$). We define an algebra $A \otimes \mathbb{C}\Gamma$ as follows: it is $A \otimes \mathbb{C}\Gamma$ as a vector space and the product of monomials is defined as follows:

$a_1 \otimes \gamma_1 \cdot a_2 \otimes \gamma_2 = a_1\gamma_1(a_2) \otimes \gamma_1\gamma_2$ (where $\gamma_1(a_2)$ stands for the image of a_2 under the action of γ_1)

We get an associative algebra with unit $1\otimes 1$.

Let us explain a connection between $A\otimes\mathbb{C}\Gamma$ and the invariant subalgebra A^Γ. Consider the trivial idempotent $e=\frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma}\gamma\in\mathbb{C}\Gamma\subset A\otimes\mathbb{C}\Gamma$

Consider the subalgebra $e(A\otimes\mathbb{C}\Gamma)e$ (with unit e)

Lem: The map $a\mapsto ea(=ae)$, $A^\Gamma\to e(A\otimes\mathbb{C}\Gamma)e$ is an algebra isomorphism

Proof is an exercise

Main examples of A: $A=\mathbb{C}[x,y]$, $A'=\mathbb{C}\langle x,y\rangle$ - free algebra in 2 generators ($\Gamma\subset SL_2(\mathbb{C})$)

3.2) Deformations of $A\otimes\mathbb{C}\Gamma$ & connection to DPA

$c:\Gamma\to\mathbb{C}$ - function constant on conjugacy classes

$\rightsquigarrow C=\sum_{\gamma\in\Gamma}c(\gamma)\gamma$ - central element

$\rightsquigarrow H_c=A'\otimes\mathbb{C}\Gamma/([x,y]=C)$ ($H_0=A\otimes\mathbb{C}\Gamma$)

Thm 1 (CB & H) $\operatorname{gr}H_c=H_0$

The algebra H_c is related to $\Pi^\lambda(Q)$ as follows. Recall that $N_0\ldots N_r$ denote the irreducible representations of Γ. Pick a primitive idempotent $e_i\in\operatorname{End}(N_i)\subset\mathbb{C}\Gamma\subset H_c$. Set $\tilde{e}=\sum_{i=0}^r e_i$, $\lambda_i=\operatorname{tr}_{N_i}C$

Thm 2 (CB & H) $\Pi^\lambda(Q)\simeq\tilde{e}H_c\tilde{e}$ (w. $e_i\leftrightarrow\epsilon_i$)

Cor: $\epsilon_0\Pi^\lambda(Q)\epsilon_0=eH_ce$ ($e=e_0$)

The proof is in 2 steps. First, we'll check that $\tilde{e}(A'\otimes\mathbb{C}\Gamma)\tilde{e}\simeq\mathbb{C}\overline{Q}$ (easy and pretty formal part). A more difficult part is that $\tilde{e}(xy-yx-C)_{A'\otimes\mathbb{C}\Gamma}\tilde{e}=(\sum_a[a,a^*]-\sum_i\lambda_i\epsilon_i)_{\mathbb{C}\overline{Q}}$

This will prove Thm 2

3.3) Tensor algebras

We are going to show that $\tilde{e}(A'\otimes\mathbb{C}\Gamma)\tilde{e}\simeq\mathbb{C}\overline{Q}$. The main point is that both $A'\otimes\mathbb{C}\Gamma$, $\mathbb{C}\overline{Q}$ are tensor algebras (of bimodules over finite dimensional algebras.

The general construction is as follows. Let A_0 be an algebra and A_1 is its bimodule. We can form the tensor power $A_1^{\otimes k} = A_1 \otimes_{A_0} A_1 \otimes_{A_0} \dots \otimes_{A_0} A_1$

Example: 1) $A_0 = \mathbb{C}Q_0$, $A_1 = \mathbb{C}\bar{Q}_1 \rightsquigarrow T_{A_0}(A_1) = \mathbb{C}\bar{Q}$.

2) $A_0 = \mathbb{C}\Gamma$, $A_1 = \mathbb{C}^2 \otimes \mathbb{C}\Gamma$, $\gamma_1(v \otimes a)\gamma_2 = \gamma_1 v \otimes \gamma_1 a \gamma_2$

$\rightsquigarrow T_{A_0}(A_1) = \mathbb{C}\langle x,y\rangle \otimes \mathbb{C}\Gamma$

Now assume that A_0 is s/simple, $A_0 = \bigoplus_i \mathrm{End}_{\mathbb{C}}(N_i)$, $e_i \in \mathrm{End}_{\mathbb{C}}(N_i)$ primitive idempotent, $\tilde{e} = \sum_i e_i$.

Lem: $\tilde{e}\, T_{A_0}(A_1)\tilde{e} = T_{\tilde{e}A_0\tilde{e}}(\tilde{e}A_1\tilde{e})$

Proof: The algebra $T_{\tilde{e}A_0\tilde{e}}(\tilde{e}A_1\tilde{e})$ has a univ. property: let A be an algebra with an embedding $\tilde{e}A_0\tilde{e} \hookrightarrow A$ and $\tilde{e}A_0\tilde{e}$-bimodule map $\tilde{e}A_1\tilde{e} \to A$. Then they extend to a unique algebra homomorphism $T_{\tilde{e}A_0\tilde{e}}(\tilde{e}A_1\tilde{e}) \to \tilde{e}\,T_{A_0}(A_1)\tilde{e}$. Now $\tilde{e}$ gives a Morita equivalence A_0-bimod $\xrightarrow{\sim} \tilde{e}A_0\tilde{e}$-bimod, $B \to \tilde{e}B\tilde{e}$. This shows that the homomorphism is an iso □

Cor: $\tilde{e}(\mathbb{C}\langle x,y\rangle \# \Gamma)\tilde{e} \xrightarrow{\sim} \mathbb{C}\bar{Q}$

Proof: $\tilde{e}\,\mathbb{C}\Gamma\tilde{e} \simeq \mathbb{C}Q_0$ $(e_i \leftrightarrow \epsilon_i)$. Now we need to check that $\tilde{e}(\mathbb{C}^2 \otimes \mathbb{C}\Gamma)\tilde{e} = \mathbb{C}\bar{Q}_1$. But $\bar{e}_i(\mathbb{C}^2 \otimes \mathbb{C}\Gamma)\bar{e}_j = \bigoplus_k e_i(\mathbb{C}^2 \otimes N_k) \otimes N_k^* e_j$ $= e_i(\mathbb{C}^2 \otimes N_j) = \mathrm{Hom}(N_i, \mathbb{C}^2 \otimes N_j) = \epsilon_i \mathbb{C}\bar{Q}_1 \epsilon_j$, this implies $\tilde{e}(\mathbb{C}^2 \otimes \mathbb{C}\Gamma)\tilde{e}$ $= \mathbb{C}\bar{Q}_1$ □

3.4) Equality $\tilde{e}(xy - yx - c)_{A' \otimes \mathbb{C}\Gamma}\tilde{e} = \left(\sum_{a \in Q_1}[a, a^*] - \sum_{i \in Q_0} \lambda_i \epsilon_i\right)_{\mathbb{C}\bar{Q}}$

First note that $\tilde{e}$ commutes w. $xy - yx - c$ b/c $xy - yx - c$ is Γ-invariant. So l.h.s. is gen-d by $(xy - yx - c)e_i$.

Prop: $(xy - yx - c)e_i = \sum_{h(a)=i} aa^* - \sum_{t(a)=i} a^*a - \lambda_i \epsilon_i$ after a suitable choice of generators a, a^* (under the isomorphism of 3.3)

This is quite technical. (see Lemma 4.2 in Lec 5 of the SRA class)

Let's do an example instead of proof.

Ex: $\Gamma = \mathbb{Z}/(n+1)\mathbb{Z}$ in which case $\tilde{e} = 1$ so we have $A' \otimes \mathbb{C}\Gamma =$

$\mathbb{C}\bar{Q}$. Then we have $C = \sum_i \lambda_i \epsilon_i$. ~~We take the counter clockwise arrows for~~ Let $a_i : i \to i+1$ and $a_i^* : i+1 \to i$. We set $a_i = x\epsilon_i$, $a_i^* = \epsilon_i y$.

The DPA relation becomes $[x,y] = C$.

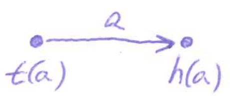
35) Further results

A natural questions is when eH_ce is commutative.

Thm: eH_ce is commutative $\iff \lambda \cdot \delta = 0$

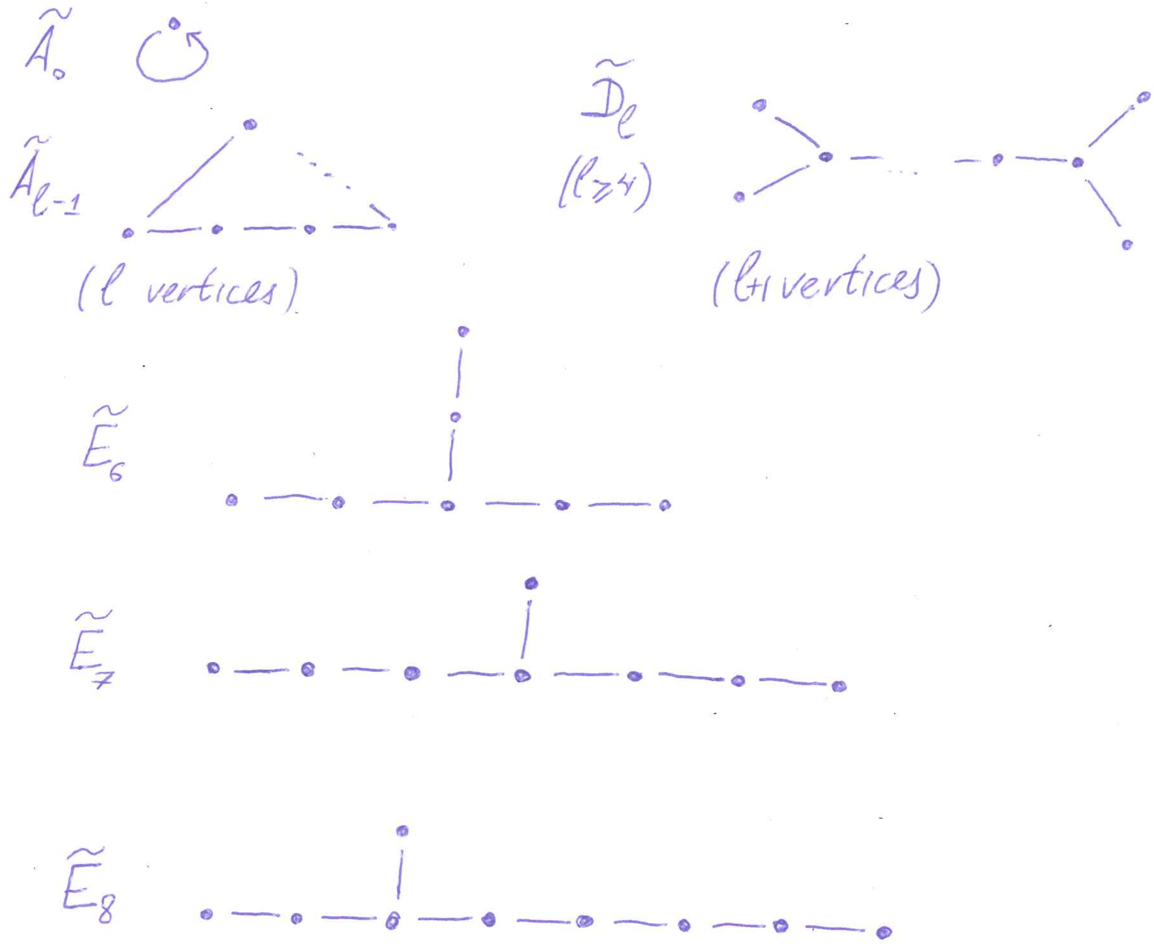
Quivers.

Pic 1.1: head and tail



Examples (Pic 1.2)

Pic 1.3: affine Dynkin quivers:



LECTURE 19: KAC-MOODY ALGEBRA ACTIONS ON CATEGORIES, I

IVAN LOSEV

1. Introduction

We have started this class by studying the representation theory of the symmetric group S_n over the complex numbers. We finish by giving a brief introduction to the representation theory of S_n over a field $\mathbb{F}$ of positive characteristic p. We will also establish a connection between the representations of $\hat{\mathfrak{sl}}_p$ and those of $\mathbb{F}S_n$. This connection was one of motivations to consider Kac-Moody algebra actions on categories. We would like to point out that while the representation theory of S_n in characteristic 0 is a classical and very well understood subject (all representations are completely reducible, the irreducible ones are classified by the Young diagrams, and character formulas are known in some way, at least), the representation theory in characteristic p is very complicated (representations are no longer completely reducible, and, although the classification of the irreducible representations is known, currently, there is not even a conjecture on their characters).

1.1. **Kac-Moody algebras and their representations.** Let us give a reminder regarding Kac-Moody algebras. We are interested only in symmetric Kac-Moody algebras. Those are associated to unoriented graphs I without loops. From I we can define its Cartan matrix $A = (a_{ij})_{i,j\in I}$ by $a_{ii} = 2$ and $a_{ij} = -n_{ij}$, where n_{ij} is the number of edges between i and j. Then we can define the Kac-Moody algebra $\mathfrak{g}(I)$ by generators $e_i, h_i, f_i, i \in I$, and the following relations:

(R1) $[h_i, h_j] = 0, [h_i, e_j] = a_{ij}e_j, [h_i, f_j] = -a_{ij}f_j$.
(R2) $[e_i, f_j] = \delta_{ij}h_j$.
(R3) $\mathrm{ad}(e_i)^{1-a_{ij}}e_j = \mathrm{ad}(f_i)^{1-a_{ij}}f_j = 0$.

Example 1.1. First, let us recall that, if I is a Dynkin diagram of type A_ℓ, then $\mathfrak{g}(I) = \mathfrak{sl}_{\ell+1}$. We will need an infinite version of this, the algebra $\mathfrak{sl}_\infty$ that consists of infinite (in both directions) matrices with finitely many nonzero entries and trace 0. It corresponds to the graph I, where the vertices are the integers and we connect vertices whose difference is ± 1.

Now let I be a cycle with ℓ vertices. We can view vertices as elements of $\mathbb{Z}/\ell\mathbb{Z}$ connected if the difference is ± 1 (for $\ell = 2$ we have two edges). The corresponding Kac-Moody algebra is $\hat{\mathfrak{sl}}_\ell$. It can be defined as $\hat{\mathfrak{sl}}_\ell = \mathfrak{sl}_\ell \otimes \mathbb{C}[t^{\pm 1}] \oplus \mathbb{C}c$ with commutation relations $[x\otimes t^p, y\otimes t^q] = [x, y]\otimes t^{p+q} + p\delta_{p+q,0}\,\mathrm{tr}(xy)c, [c, x\otimes t^p] = 0$. Note that $h_i = E_{ii} - E_{i+1,i+1}, i = 1, \ldots, \ell-1, h_0 = c + E_{\ell\ell} - E_{11}$. We have a Cartan subalgebra $\mathfrak{h} := \mathrm{Span}(h_i | i \in I) \subset \mathfrak{g}(I)$, the elements h_i form a basis in $\mathfrak{h}$.

Often one considers a slightly bigger algebra, $\tilde{\mathfrak{sl}}_\ell = \hat{\mathfrak{sl}}_\ell \oplus \mathbb{C}d$, where the additional commutation relations are $[d, c] = 0, [d, x\otimes t^k] = kx\otimes t^k$. In this case, one includes d as one more basis element in $\mathfrak{h}$.

To give a representation of $\mathfrak{g}(I)$ in a vector space V, we need to equip V with operators e_i, f_i, h_i satisfying the relations above. We only care about so called weight representations. We say that V is a weight representation if $V = \bigoplus_{\mu\in\mathfrak{h}^*} V_\mu$, where $V_\mu = \{v \in V | xv =$

$\langle \mu, x\rangle v, \forall x \in \mathfrak{h}\}$. The relations (R1) are equivalent to $e_i V_\mu \subset V_{\mu+\alpha_i}, f_i V_\mu \subset V_{\mu-\alpha_i}$, where $\alpha_i \in \mathfrak{h}^*$ are the simple roots defined by $[x, e_i] = \langle \alpha_i, x\rangle e_i, \forall x \in \mathfrak{h}$.

1.2. **Actions on categories.** Now let us discuss what one should mean by an action of $\mathfrak{g}(I)$ on a category. We are going to work with abelian categories $\mathcal{C}$ that are linear over some field $\mathbb{F}$ and where all objects have finite length. $\mathbb{F}S_n$-mod is an example of such a category. To $\mathcal{C}$ we can assign its complexified Grothendieck group $[\mathcal{C}] := \mathbb{C} \otimes_{\mathbb{Z}} K_0(\mathcal{C})$. For an exact functor $F : \mathcal{C} \to \mathcal{C}$, we get a linear map $[\mathcal{F}] : [\mathcal{C}] \to [\mathcal{C}], [M] \mapsto [\mathcal{F}M]$.

So here is a rough idea of what an action of $\mathfrak{g}(I)$ on $\mathcal{C}$ should mean. We want a collection of exact functors, $E_i, F_i : \mathcal{C} \to \mathcal{C}$ and a weight decomposition $\mathcal{C} = \bigoplus_{\mu\in\mathfrak{h}^*} \mathcal{C}_\mu$ such that $e_i := [E_i], f_i := [F_i]$ and $[\mathcal{C}] = \bigoplus_\mu [\mathcal{C}_\mu]$ gives a weight representation of $\mathfrak{g}(I)$. This occurs in many examples but is not powerful enough to produce an interesting theory. A crucial idea due to Chuang and Rouquier was to additionally include some functor morphisms that are also present in examples.

The category $\mathcal{C}$ we are interested in is $\mathcal{C} = \bigoplus_{n\geqslant 0} \mathbb{F}S_n$-mod. We will see that if $\operatorname{char}\mathbb{F} = 0$, then $\mathcal{C}$ carries a categorical action of $\mathfrak{sl}_\infty$ (relatively boring case), while, for $\operatorname{char}\mathbb{F} = p$, $\mathcal{C}$ carries a categorical action of $\hat{\mathfrak{sl}}_p$ that makes $[\mathcal{C}]$ into the irreducible module $V(\omega_0)$ (and, in particular, computes the number of the irreducible $\mathbb{F}S_n$-modules for any n). The functors E_i come from restrictions (from S_n to S_{n-1}), while functors F_i come from inductions (from S_{n-1} to S_n).

We'll proceed as follows. First, we produce the functors E_i. Then we decompose $\mathcal{C}$ into the direct sum of subcategories (that later will be shown to be weight subcategories). Next, we will define the functors F_i. Finally, we will discuss functor morphisms we need. In the next lecture, we will start by showing that $[E_i], [F_i]$ define a weight Kac-Moody representation on $[\mathcal{C}]$.

2. Restriction and induction functors

Let $\mathbb{Z}_{\mathbb{F}}$ denote the ring of integers inside $\mathbb{F}$ (i.e., $\mathbb{Z}$ if $\operatorname{char}\mathbb{F} = 0$ and $\mathbb{Z}/p\mathbb{Z}$ if $\operatorname{char}\mathbb{F} = p$). We write $\mathcal{C}$ (or $\mathcal{C}_{\mathbb{F}}$ if we want to indicate the dependence on the base field $\mathbb{F}$) for $\bigoplus_{n\geqslant 0} \mathbb{F}S_n$-mod. Here $S_0 = S_1 = \{1\}$.

2.1. **Functors E_i.** As in Lectures 1,2, to study the representations of $\mathbb{F}S_n$ we use "induction" based on the chain of inclusions $S_0 \subset S_1 \subset \ldots \subset S_{n-1} \subset S_n \subset \ldots$ (where, recall, S_{n-1} consists of all permutations in S_n that fix n). We can restrict an $\mathbb{F}S_n$-module to $\mathbb{F}S_{n-1}$ getting the *restriction functor* $\operatorname{Res}^n_{n-1} : \mathbb{F}S_n\text{-mod} \to \mathbb{F}S_{n-1}\text{-mod}$. Our goal now is to decompose $\operatorname{Res}^n_{n-1}(M)$ into a direct sum in a functorial (in M) way.

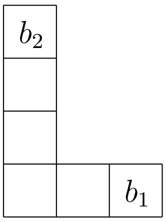
For this, recall the Jucys-Murphy element $L_n := \sum_{i=1}^{n-1}(in) \in \mathbb{F}S_n$. It commutes with S_{n-1} and hence the map $X_M : \operatorname{Res}^n_{n-1}(M) \to \operatorname{Res}^n_{n-1}(M)$ given by $X_M(m) = L_n m$ is an endomorphism of an $\mathbb{F}S_{n-1}$-module. This endomorphism is functorial in M because any S_n-linear homomorphism $M \to M'$ commutes with L_n. So the endomorphisms X_M form an endomorphism of the functor $\operatorname{Res}^n_{n-1}$.

The decomposition of $\operatorname{Res}^n_{n-1}(M)$ we need is that into the generalized eigenspaces for X_M. Let us recall how this works when $\operatorname{char}\mathbb{F} = 0$. Independently of $\mathbb{F}$, the irreducible S_n-modules are parameterized by the Young diagrams λ with n boxes, we write M_λ for the irreducible module corresponding to λ. We have

$$\operatorname{Res}^n_{n-1}(M_\lambda) = \bigoplus_\mu M_\mu,$$

where the sum is taken over all diagrams μ obtained from λ by removing a single box. Moreover, the summands M_μ are precisely the eigenspaces for L_n. More precisely, let b be the box in $\lambda \setminus \mu$. We define its *content* $c(b)$ as $x - y$, where x, y are the coordinates of b, see the example below.

Example 2.1. Let $\lambda = (3, 1, 1, 1)$, the diagram is as follows



We have two removable boxes in λ denoted by b_1, b_2. Note that $c(b_1) = 3 - 1 = 2$ and $c(b_2) = 1 - 4 = -3$. So $\mathrm{Res}_5^6(M) = M_{\mu_1} \oplus M_{\mu_2}$, where $\mu_1 = (2, 1, 1, 1)$ and $\mu_2 = (3, 1, 1)$. The eigenvalues of L_6 on M_{μ_1}, M_{μ_2} are 2 and -3, respectively.

Lemma 2.2. *Let* $\mathrm{char}\,\mathbb{F} = p$. *Then the eigenvalues of* X_M *are in* $\mathbb{Z}/p\mathbb{Z}$.

Proof. For $\mathbb{F} = \mathbb{C}$, the eigenvalues of L_n in any module, in particular, in $\mathbb{C}S_n$ are integral. So there are integers $a_1, \ldots, a_m$ such that $\prod_{i=1}^m (L_n - a_i) = 0$. This equality holds in $\mathbb{C}S_m$ and hence also in $\mathbb{Z}S_n$. Therefore it also holds in $\mathbb{F}S_n$ and we are done. $\square$

For $i \in \mathbb{Z}_\mathbb{F}$, let $\mathrm{Res}_{n-1}^n(M)_i$ denote the generalized eigenspace for X_M in M with eigenvalue i. The assignment $M \mapsto \mathrm{Res}_{n-1}^n(M)_i$ is a functor $\mathbb{F}S_n$-mod $\to \mathbb{F}S_{n-1}$-mod (again, for the reason that any S_n-linear homomorphism is also L_n-linear). We have $\mathrm{Res}_{n-1}^n = \bigoplus_{i \in \mathbb{Z}_\mathbb{F}} \mathrm{Res}_{n-1}^n(\bullet)_i$.

We write E for $\bigoplus_{i=0}^\infty \mathrm{Res}_{n-1}^n(\bullet)$ so that $EM = \mathrm{Res}_{n-1}^n(M)$ for $M \in \mathbb{F}S_n$-mod. Then E is an endofunctor of $\mathcal{C}$. Here we set $\mathrm{Res}_{-1}^0 = 0$. Further, we set $E_i := \bigoplus_{n=0}^\infty \mathrm{Res}_{n-1}^n(\bullet)_i$. In other words, E_i is the generalized eigen-subfunctor of E for the endomorphism X with eigenvalue i. We have $E = \bigoplus_{i \in \mathbb{Z}_\mathbb{F}} E_i$.

2.2. **Direct sum decomposition for $\mathcal{C}$.** Let $\mathcal{H}(n)$ denote the degenerate affine Hecke algebra with generators $X_1, \ldots, X_n, T_1, \ldots, T_{n-1}$ to be recalled later. Recall that we have an algebra isomorphism $\mathcal{H}(n)/(X_1 = 0) \xrightarrow{\sim} \mathbb{F}S_n$ sending X_i to the Jucys-Murphy element $L_i = \sum_{j=1}^{i-1} (ji)$. Recall, Problem 3 in Homework 1, that the symmetric polynomials in the elements X_i are central in $\mathcal{H}(n)$. We get the following corollary.

Lemma 2.3. *Symmetric polynomials in* $L_1, \ldots, L_n$ *are central in* $\mathbb{F}S_n$.

Let us take an unordered n-tuple A of elements of $\mathbb{Z}_\mathbb{F}$. For $M \in \mathbb{F}S_n$-mod, define the generalized eigenspace for $\mathbb{Z}[L_1, \ldots, L_n]^{S_n}$ with eigenvalue A by

$$M_A := \{m \in M | (P(L_1, \ldots, L_n) - P(A))^k m = 0, \forall k \gg 0, P \in \mathbb{Z}[x_1, \ldots, x_n]^{S_n}\}.$$

Then $M = \bigoplus_A M_A$. Define $\mathbb{F}S_n$-mod_A as the full subcategory of $\mathbb{F}S_n$-mod consisting of all modules M that coincide with M_A. We have $\mathbb{F}S_n$-mod $= \bigoplus_A \mathbb{F}S_n$-$\mathrm{mod}_A$ (there are no homomorphisms/extensions between modules belonging to different categories $\mathbb{F}S_n$-mod_A).

Example 2.4. Let $\mathrm{char}\,\mathbb{F} = 0$. Recall that M_λ has a basis v_T labelled by the standard Young tableaux T of shape λ. We have $L_i v_T = c(b_i) v_T$, where b_i is the box labelled by i in T. Hence $M_\lambda = (M_\lambda)_{c(\lambda)}$. So if A coincides with $c(\lambda)$ (the unordered collection of the contents of the boxes in λ), then $\mathbb{F}S_n\text{-mod}_A$ is spanned by M_λ and, otherwise, $\mathbb{F}S_n\text{-mod}_A$ is zero.

We view A as a multiset. We will write $A\setminus\{i\}$ (resp., $A\cup\{i\}$) for the multiset, where the multiplicity of i is decreased (resp., increased) by 1.

Lemma 2.5. *Let $M\in \mathbb{F}S_n\text{-mod}_A$. If $i\not\in A$, then $E_iM=0$. Otherwise $E_iM\in \mathbb{F}S_n\text{-mod}_{A\setminus\{i\}}$.*

Proof. A is a collection of simultaneous eigenvalues of $(L_1,\ldots,L_n)$. So if $i\not\in A$, then $E_iM=0$. If $i\in A$, and B is a collection of simultaneous eigenvalues of $(L_1,\ldots,L_{n-1})$ in E_iM, then $B\cup\{i\}$ is a collection of simultaneous eigenvalues of $L_1,\ldots,L_n$ in M. So $A=B\cup\{i\}$ and $B=A\setminus\{i\}$. $\square$

Let π_A denote the projection functor $\mathbb{F}S_n\text{-mod}\twoheadrightarrow \mathbb{F}S_n\text{-mod}_A$, $\pi_A(M):=M_A$. Then, for $M\in\mathbb{F}S_n\text{-mod}_A$, we get $E_iM=\pi_{A\setminus\{i\}}\circ EM$, where we assume that $\pi_{A\setminus\{i\}}:=0$ if $i\not\in A$. Note that π_A is both left and right adjoint for the inclusion functor $\mathbb{F}S_n\text{-mod}_A\hookrightarrow\mathbb{F}S_n\text{-mod}$.

2.3. **Functors F_i.** We have a left adjoint Ind_n^{n-1} and a right adjoint Coind_n^{n-1} functors to $\mathrm{Res}^n_{n-1}:\mathbb{F}S_n\text{-mod}\to\mathbb{F}S_{n-1}\text{-mod}$. The former is given by $N\mapsto \mathbb{F}S_n\otimes_{\mathbb{F}S_{n-1}}N$, while the latter is given by $N\mapsto \mathrm{Hom}_{S_{n-1}}(\mathbb{F}S_n,N)$.

Lemma 2.6. *We have a functor isomorphism* $\mathrm{Ind}_n^{n-1}\cong\mathrm{Coind}_n^{n-1}$.

Proof. Note that $\mathrm{Coind}_n^{n-1}(N)=(\mathbb{F}S_n)^*\otimes_{\mathbb{F}S_{n-1}}N$, where $(\mathbb{F}S_n)^*$ is equipped with a bimodule structure given by $\langle a\alpha b,c\rangle:=\langle\alpha,bca\rangle$. The claim about the isomorphism of functors will follow if we check that $\mathbb{F}S_n\cong(\mathbb{F}S_n)^*$ as an $\mathbb{F}S_n$-$\mathbb{F}S_{n-1}$-bimodule. In fact, we have an isomorphism of $\mathbb{F}S_n$-bimodules. Namely, consider the bilinear form $(\cdot,\cdot)$ on $\mathbb{F}S_n$ given by $(g,h)=\delta_{gh,1}$. It is a direct check that the identification $\mathbb{F}S_n\cong(\mathbb{F}S_n)^*$ with respect to this form is an isomorphism of $\mathbb{F}S_n$-bimodules. $\square$

Our goal now is to produce a left adjoint of the functor $E_i, i\in\mathbb{Z}_{\mathbb{F}}$, to be denoted by F_i. This can be done in two equivalent ways. We can define the functors F_i on all categories $\mathbb{F}S_{n-1}\text{-mod}_B$ and then extend them to $\mathbb{F}S_{n-1}\text{-mod}$ by additivity. On $\mathbb{F}S_{n-1}\text{-mod}_B$, the functor F_i is defined by $\pi_{B\cup\{i\}}\circ F$ so that $F=\bigoplus_{i\in\mathbb{Z}_{\mathbb{F}}}F_i$.

Lemma 2.7. *The functor F_i is biadjoint to E_i.*

Proof. Let us show that F_i is left adjoint to E_i, the other adjunction is similar. It is enough to establish a bi-functorial isomorphism $\mathrm{Hom}_{S_n}(F_iN,M)\cong\mathrm{Hom}_{S_{n-1}}(N,E_iM)$ for $N\in\mathbb{F}S_{n-1}\text{-mod}_B, M\in\mathbb{F}S_n\text{-mod}_A$. Both sides are zero if $A\neq B\cup\{i\}$. If $A=B\cup\{i\}$, then the l.h.s. is $\mathrm{Hom}_{S_n}(FN,M)$ (because $\mathrm{Hom}_{S_{n-1}}(F_jN,M)=0$ for $j\neq i$) and similarly the r.h.s. is $\mathrm{Hom}_{S_{n-1}}(N,EM)$. Since F is left adjoint to E, we are done. $\square$

Here is an equivalent way to produce F_i. Since F is left adjoint to E, the algebra $\mathrm{End}(E)$ gets identified with $\mathrm{End}(F)^{opp}$ (where the superscript means that the multiplication is taken in the opposite order). This is a consequence of the Yoneda lemma and the adjointness. So we get an endomorphism $X\in\mathrm{End}(F)^{opp}$. Then F_i is the generalized eigenfunctor for X with eigenvalue i.

2.4. **Functor morphisms.** As we have mentioned before, we also need to consider some functor morphisms. We have already seen some of those: we had an endomorphism X of the functor E. We also had morphisms $1\to EF, FE\to 1$, where 1's denote the identity functors (from the adjointness: F is left adjoint to E). But we actually need more morphisms. Those will be endomorphisms of $E^d=\bigoplus\mathrm{Res}^n_{n-d}$.

A recipe to construct these endomorphisms is similar to what was done for X: we will get them from elements of $(\mathbb{F}S_n)^{S_{n-d}}$. The elements that we are going to use are $L_{n-d+i} = \sum_{j=1}^{n-d+i-1}(j, n+d-i), i = 1, \ldots, d$, and $(n-d+i, n-d+i+1), i = 1, \ldots, i-1$. Recall from Lecture 2 that these elements satisfy the relations of the degenerate affine Hecke algebra $\mathcal{H}(d)$ that is generated by the elements $X_1, \ldots, X_d, T_1, \ldots, T_{d-1}$ subject to the relations:

$$X_iX_j = X_jX_i,$$
$$T_iT_j = T_jT_i, \text{ for } |i-j| > 1, \quad T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}, \quad T_i^2 = 1,$$
$$T_iX_j = X_jT_i, \text{ for } j-i \neq 0, 1, \quad T_iX_i = X_{i+1}T_i - 1.$$

As we have seen, there is an algebra homomorphism $\mathcal{H}(d) \to (\mathbb{F}S_n)^{S_{n-d}}$ mapping X_i to L_{n-d+i} and T_i to $(n-d+i, n-d+i+1)$. This yields an algebra homomorphism $\mathcal{H}(d) \to \mathrm{End}(E^d)$.

Let us make a remark that will become useful later. We can recover the images of X_i in $\mathrm{End}(E^d)$ from $X \in \mathrm{End}(E)$. For this we need the following general construction. Let $\mathcal{F}, \mathcal{G}$ be endofunctors of a category $\mathcal{C}$ and let $X \in \mathrm{End}(\mathcal{F}), Y \in \mathrm{End}(\mathcal{G})$. Then we get the endomorphisms of $\mathcal{F}\mathcal{G}$ given by $(X1)_M := X_{\mathcal{G}(M)}, (1Y)_M := \mathcal{F}(Y_M)$. With this notation, X_i goes to the endomorphism $1^{i-1}X1^{d-i}$.

Similarly, we can recover the image of T_i from $T \in \mathrm{End}(E^2)$ that is the image of T_1, i.e., $T_M(m) = (n-1, n)m$ for $M \in \mathbb{F}S_n\text{-mod}, m \in M$. Namely, T_i maps to $1^{i-1}T1^{d-i-1}$.

Note that this description makes some of the relations between the images of X_i, T_i in $\mathrm{End}(E^d)$ automatic (e.g., the relation that $X_iX_j = X_jX_i$), while others only need to be checked for small d (for example, it is enough to check that $T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}$ for $d = 3$).

LECTURE 20: KAC-MOODY ALGEBRA ACTIONS ON CATEGORIES, II

IVAN LOSEV

1. Introduction

1.1. **Recap.** In the previous lecture we have considered the category $\mathcal{C}_{\mathbb{F}} := \bigoplus_{n\geqslant 0} \mathbb{F}S_n$-mod. We have equipped it with two endofunctors, $E = \bigoplus_n \operatorname{Res}^n_{n-1}$ and $F = \bigoplus_n \operatorname{Ind}^n_{n+1}$ that are biadjoint. We have decomposed E into the direct sum of eigenfunctors, $E = \bigoplus_{i\in\mathbb{Z}_{\mathbb{F}}} E_i$, for the endomorphism X that is given by $X_M m = L_n m$ for $M \in \mathbb{F}S_n$-mod, where L_n is the Jucys-Murphy element $\sum_{i=1}^{n-1}(in)$. We have also considered the corresponding decomposition $F = \bigoplus_{i\in\mathbb{Z}_{\mathbb{F}}} F_i$.

Besides, we have introduced the decomposition $\mathbb{F}S_n$-mod $= \bigoplus_A \mathbb{F}S_n$-mod$_A$, where the summation is taken over all cardinality n multi-subsets in $\mathbb{Z}_{\mathbb{F}}$, and $\mathbb{F}S_n$-mod$_A$ consists of all $M \in \mathbb{F}S_n$-mod such that $P(L_1,\ldots,L_n)$ acts on M with a single eigenvalue $P(A)$, for every $P \in \mathbb{Z}[x_1,\ldots,x_n]^{S_n}$. This decomposition is related to the functors E_i, F_i as follows. Let π_A denote the projection $\mathbb{F}S_n$-mod $\twoheadrightarrow \mathbb{F}S_n$-mod$_A$. Then, for $M \in \mathbb{F}S_n$-mod$_A$, we have $E_iM = \pi_{A\setminus\{i\}}(EM), F_iM = \pi_{A\cup\{i\}}(FM)$. Below, we will write $\mathcal{C}_{\mathbb{F},A} = \mathbb{F}S_{|A|}$-mod$_A$. So we get the direct sum decomposition $\mathcal{C}_{\mathbb{F}} = \bigoplus_A \mathcal{C}_{\mathbb{F},A}$, where the sum is taken over all multi-subsets A of $\mathbb{Z}_{\mathbb{F}}$.

Finally, we have also introduced an endomorphism T of E^2: $T_M m = (n-1,n)m$ for $m \in M, M \in \mathbb{F}S_n$-mod. We have seen that the assignment $X_i \mapsto 1^{i-1}X1^{d-i}, T_i \mapsto 1^{i-1}T1^{d-i-1}$ extends to an algebra homomorphism $\mathcal{H}(d) \to \operatorname{End}(E^d)$.

1.2. **Goals.** First of all, we will show that $[E_i],[F_i], i\in\mathbb{Z}_{\mathbb{F}}$, together with the decomposition $[\mathcal{C}_{\mathbb{F}}] = \bigoplus_A[\mathcal{C}_{\mathbb{F},A}]$ define the structure of a weight representation of $\hat{\mathfrak{sl}}_p$ (if $\operatorname{char}\mathbb{F} = p$) or of $\mathfrak{sl}_\infty$ (if $\operatorname{char}\mathbb{F} = p$). The characteristic 0 case is easy as we can determine $[\mathcal{C}_{\mathbb{F}}],[E_i],[F_i]$ very explicitly. The case when $\operatorname{char}\mathbb{F} = p$ is more tricky because we do not understand the structure of $[\mathcal{C}_{\mathbb{F}}]$ at this point. We will treat this case by reducing to characteristic 0.

After this is done we will give an abstract definition of an action of $\hat{\mathfrak{sl}}_p$ on a category ($\mathcal{C}_{\mathbb{F}}$ for $\operatorname{char}\mathbb{F} = p$ will be the main example). Then we will give an application: modulo some results of Chuang and Rouquier, we will show that $[\mathcal{C}_{\mathbb{F}}]$ is an irreducible $\hat{\mathfrak{sl}}_p$-module.

2. $\hat{\mathfrak{sl}}_p$-action on K_0

Let $\mathbb{F}$ be a characteristic p field. In this section, we will show that the operators $[E_i],[F_i]$ on $[\mathcal{C}_{\mathbb{F}}]$ give rise to a $\hat{\mathfrak{sl}}_p$-action. Moreover, we will check that $[\mathcal{C}_{\mathbb{F}}] = \bigoplus_A[\mathcal{C}_{\mathbb{F},A}]$, where we write $\mathcal{C}_A$ for $\mathbb{F}S_{|A|}$-mod$_A$, is a weight decomposition for $\tilde{\mathfrak{sl}}_p$.

2.1. **Comparison of K_0's in characteristics 0 and p.** Consider the following situation. Let R be a local Dedekind domain containing $\mathbb{Z}$. Let $\mathbb{K}$ denote the fraction field of R and let $\mathbb{F}$ be the residue field, we will assume that it has characteristic p. E.g., we can take $R = \mathbb{Z}_p$, then $\mathbb{K} = \mathbb{Q}_p, \mathbb{F} := \mathbb{F}_p$. Let A_R be an associative unital R-algebra that is a free finite rank R-module. An example is provided by RS_n. Set $A_{\mathbb{K}} = \mathbb{K}\otimes_R A_R, A_{\mathbb{F}} = \mathbb{F}\otimes_R A_R$.

Consider the categories $A_{\mathbb{K}}$-mod and $A_{\mathbb{F}}$-mod of finite dimensional $A_{\mathbb{K}}$- and $A_{\mathbb{F}}$-modules. We are going to produce a group map $K_0(A_{\mathbb{K}}\text{-mod}) \to K_0(A_{\mathbb{F}}\text{-mod})$. Take $M \in A_{\mathbb{K}}$-mod. We can pick an *R-lattice* $M_R \subset M$ meaning a finitely generated R-submodule M_R with $\mathbb{K} \otimes_R M_R \xrightarrow{\sim} M$ that is automatically free over R. Then we get $M_{\mathbb{F}} := \mathbb{F} \otimes_R M_R \in A_{\mathbb{F}}$-mod. There are different lattices $M_R \subset M$ leading to non-isomorphic modules $M_{\mathbb{F}}$. However, a standard fact (left as an exercise) is that the class of $M_{\mathbb{F}}$ in K_0 does not depend on the choice of M_R. So we do get a well-defined map $K_0(A_{\mathbb{K}}\text{-mod}) \to K_0(A_{\mathbb{F}}\text{-mod})$.

Lemma 2.1. *This map is additive.*

Proof. Let $M' \subset M$ be an $A_{\mathbb{K}}$-submodule with the projection $\pi : M \to M/M'$. Then $M'_R := M' \cap M_R$ is a lattice in M', while $\pi(M_R)$ is a lattice in M/M' so that we have an exact sequence $0 \to M'_R \to M_R \to \pi(M_R) \to 0$. Since $\pi(M_R)$ is free over R, we see that the sequence $0 \to M'_{\mathbb{F}} \to M_{\mathbb{F}} \to \mathbb{F} \otimes_R \pi(M_R) \to 0$ is exact. This completes the proof. □

The following result is much more interesting.

Proposition 2.2. *The map $K_0(\mathbb{K}S_n\text{-mod}) \to K_0(\mathbb{F}S_n\text{-mod})$ is surjective.*

We will discuss why this is true in the next lecture.

2.2. **Fock space.** Let $\mathcal{C}_{\mathbb{K}} := \bigoplus_{n \geqslant 0} \mathbb{K}S_n\text{-mod}$. The $\mathbb{C}$-vector space $[\mathcal{C}_{\mathbb{K}}]$ has basis $[M_\lambda]$ labeled by all partitions λ. It is customary to write $|\lambda\rangle$ for $[M_\lambda]$. The space $\mathcal{C}_{\mathbb{K}}$ is known as the Fock space. We will denote it by $\mathcal{F}$.

Let us produce an action of $\mathfrak{sl}_\infty$ on $\mathcal{F}$. We set $e_i^\infty|\lambda\rangle = |\mu\rangle$, where μ is obtained from λ by deleting a box of content i if such μ exists, and $e_i^\infty|\lambda\rangle = 0$, else. Similarly, set $f_i^\infty|\lambda\rangle = |\nu\rangle$ if ν is obtained from λ by adding a box of content i if such ν exists, and $f_i^\infty|\lambda\rangle = 0$, else. Finally, set $h_i^\infty|\lambda\rangle = (a_i^\infty(\lambda) - r_i^\infty(\lambda))|\lambda\rangle$, where $a_i^\infty(\lambda)$ is the number of addable boxes of content i in λ and $r_i^\infty(\lambda)$ is the number of removable boxes of content i in λ.

Lemma 2.3. *The operators e_i^∞, f_i^∞ give rise to a weight representation of $\mathfrak{sl}_\infty$ in $\mathcal{F}$ (with h_i^∞ as specified above).*

The proof is left as an exercise.

We have seen in Section 2.1 of Lecture 19 that $e_i^\infty = [E_i^{\mathbb{K}}]$ (we write $E_i^{\mathbb{K}}$ for the functor E_i for $\mathcal{C}_{\mathbb{K}}$). From the adjointness of $E_i^{\mathbb{K}}, F_i^{\mathbb{K}}$, we conclude that $\operatorname{Hom}_{\mathcal{C}_{\mathbb{K}}}(F_i^{\mathbb{K}} M_\lambda, M_\nu) = \operatorname{Hom}_{\mathcal{C}_{\mathbb{K}}}(M_\lambda, E_i^{\mathbb{K}} M_\nu)$ and therefore $F_i^{\mathbb{K}} M_\lambda = M_\nu$ if ν is obtained from λ by adding a box of content i if such ν exists, and $F_i^{\mathbb{K}} M_\lambda = 0$, else. So $[F_i^{\mathbb{K}}] = f_i^\infty$.

Let us proceed to an action of $\tilde{\mathfrak{sl}}_p$ on $\mathcal{F}$. For $j \in \mathbb{Z}/p\mathbb{Z}$, by a j-box we mean a box whose content is congruent to j modulo i. Let $a_j(\lambda), r_j(\lambda)$ denote the number of addable and removable j-boxes in λ. We set

$$e_j = \sum_{i \equiv j \mod p} e_i^\infty, f_j = \sum_{i \equiv j \mod p} e_i^\infty, h_j|\lambda\rangle = (a_j(\lambda) - r_j(\lambda))|\lambda\rangle, d|\lambda\rangle = |\lambda||\lambda\rangle.$$

The next lemma follows mostly from Lemma 2.3.

Lemma 2.4. *The operators e_j, f_j define a weight representation of $\tilde{\mathfrak{sl}}_p$ in $\mathcal{F}$ (with h_j, d acting as specified).*

Example 2.5. Take the diagram $\lambda = (3, 1, 1, 1)$ and assume $p = 3$. This diagram has two removable boxes: $(3, 1), (1, 4)$ and three addable boxes: $(4, 1), (2, 2), (1, 5)$. The boxes $(4, 1), (2, 2), (1, 4)$ are 0-boxes, while $(3, 1), (1, 5)$ are 2-boxes and there are no 1-boxes. So we have $e_0|\lambda\rangle = |\mu_2\rangle, e_1|\lambda\rangle = 0, e_2|\lambda\rangle = |\mu_1\rangle$, where $\mu_1 = (2, 1, 1, 1), \mu_2 = (3, 1, 1)$. Further, we have $f_0|\lambda\rangle = |\nu_1\rangle + |\nu_2\rangle, f_1|\lambda\rangle = 0, f_2|\lambda\rangle = |\nu_3\rangle$, where $\nu_1 = (4,1,1,1), \nu_2 = (3,2,1,1), \nu_3 = (3,1,1,1,1)$. So $h_0|\lambda\rangle = |\lambda\rangle, h_1|\lambda\rangle = h_2|\lambda\rangle = 0$ and $d|\lambda\rangle = 6|\lambda\rangle$.

Now let us discuss the weight spaces for $\tilde{\mathfrak{sl}}_p$ in $\mathcal{F}$.

Lemma 2.6. *For diagrams λ, λ' the following are equivalent.*

(1) $c(\lambda) \mod p = c(\lambda') \mod p$ *(the equality of multisubsets of $\mathbb{Z}_{\mathbb{F}}$).*
(2) $a_j(\lambda) - r_j(\lambda) = a_j(\lambda') - r_j(\lambda')$ *for all j and $|\lambda| = |\lambda'|$.*

Proof. Let n_j denote the number of j-boxes in λ so that (1) means $n_j(\lambda) = n_j(\lambda')$ for all j. Adding a j-box, we increase $a_{j\pm 1} - r_{j\pm 1}$ by 1 (if $p > 2$; for $p = 2$ we increase it by 2) and decrease $a_j - r_j$ by 2. We also increase $|\lambda|$ by 1. It follows that $a_j(\lambda) - r_j(\lambda) = n_{j+1}(\lambda) + n_{j-1}(\lambda) - 2n_j(\lambda) + \delta_{j0}$. Clearly, $|\lambda| = \sum_j n_j(\lambda)$. These equalities easily imply that (1) and (2) are equivalent. $\square$

For a multisubset $A \subset \mathbb{Z}_{\mathbb{F}}$ define the subspace $\mathcal{F}_A$ as the span of all $|\lambda\rangle$ with $c(\lambda) = A$. So $\mathcal{F} = \bigoplus_A \mathcal{F}_A$ is the weight decomposition for the action of $\tilde{\mathfrak{sl}}_p$.

2.3. **Action of $\tilde{\mathfrak{sl}}_p$ on $[\mathcal{C}_{\mathbb{F}}]$.** Now we are ready to prove the following theorem.

Theorem 2.7. *The surjection $[\mathcal{C}_{\mathbb{K}}] \twoheadrightarrow [\mathcal{C}_{\mathbb{F}}]$ intertwines the operator e_j with $[E_j^{\mathbb{F}}]$, the operator f_j with $[F_j^{\mathbb{F}}]$, and maps $\mathcal{F}_A$ onto $[\mathbb{F}S_n\text{-mod}_A]$, where $n = |A|$. In particular, $[\mathcal{C}_{\mathbb{F}}] = \bigoplus_A [\mathcal{C}_{\mathbb{F},A}]$ is a weight representation of $\tilde{\mathfrak{sl}}_p$.*

Proof. The proof is in several steps. Let $\rho : \mathcal{F} \twoheadrightarrow [\mathcal{C}_{\mathbb{F}}]$ denote the surjection.

Step 1. Let us show that $\rho(\mathcal{F}_A) = [\mathcal{C}_{\mathbb{F},A}]$. Since ρ is a surjection, it is enough to show that $\rho(\mathcal{F}_A) \subset [\mathcal{C}_{\mathbb{F},A}]$. Pick λ with $c(\lambda) = \tilde{A}$, where $\tilde{A} \mod p = A$. Then $\rho(|\lambda\rangle) = [M_{\lambda,\mathbb{F}}]$, where $M_{\lambda,R} \subset M_{\lambda,\mathbb{K}}$ is an R-form. For $P \in \mathbb{Z}[x_1, \ldots, x_n]^{S_n}$, the polynomial $P(L_1, \ldots, L_n)$ acts on $M_{\lambda,\mathbb{K}}$ with the single eigenvalue $P(\tilde{A})$. So the same is true for $M_{\lambda,R}$ and hence for $M_{\lambda,\mathbb{F}}$. It follows that $M_{\lambda,\mathbb{F}} \in \mathbb{F}S_n\text{-mod}_A$.

Step 2. Set $f = \sum_j f_j, e = \sum_j e_j$ and let us show that $\rho \circ e = [E] \circ \rho, \rho \circ f = [F] \circ \rho$. To prove the former, note that, tautologically, $\operatorname{Res}^n_{n-1} M_R$ is an R-lattice in $\operatorname{Res}^n_{n-1} M_{\mathbb{K}}$ and hence $(\operatorname{Res}^n_{n-1} M)_{\mathbb{F}} = \operatorname{Res}^n_{n-1}(M_{\mathbb{F}})$. To prove $\rho \circ f = [F] \circ \rho$ note that $\operatorname{Ind}^{n-1}_n M_R$ is an R-lattice in $\operatorname{Ind}^{n-1}_n M_{\mathbb{K}}$.

Step 3. Let us prove that $\rho \circ e_i = [E_i] \circ \rho$. It is enough to prove that $\rho(e_i|\lambda\rangle) = [E_i](\rho|\lambda\rangle)$. Note that $e_i|\lambda\rangle$ coincides with the projection of $e|\lambda\rangle$ to $\mathcal{F}_{c(\lambda)\setminus\{i\}}$ (here we consider $c(\lambda)$ modulo p). From Step 1, it follows that $\rho(e_i|\lambda\rangle)$ coincides with the projection to $[\mathbb{F}S_n\text{-mod}_A]$ of $\rho(e|\lambda\rangle)$. By Step 2, $\rho(e|\lambda\rangle)$ equals the projection to $[\mathbb{F}S_n\text{-mod}_A]$ of $[E] \circ \rho(|\lambda\rangle)$. As we have seen above, the last projection coincides with $[E_i](\rho(|\lambda\rangle))$.

The proof of $\rho \circ f_i = [F_i] \circ \rho$ is similar. $\square$

3. Action of $\hat{\mathfrak{sl}}_p$ on a category

Let $\mathbb{F}$ be a characteristic p field and let $\mathcal{C}$ be an $\mathbb{F}$-linear abelian category. We suppose that all objects in $\mathcal{C}$ have finite length. The category $\mathcal{C} = \bigoplus_n \mathbb{F}S_n\text{-mod}$ is of this kind.

An action of $\hat{\mathfrak{sl}}_p$ on $\mathcal{C}$ is a collection of data together with four axioms. For us, the data is a pair of functors E, F with fixed adjointness – F is left adjoint to E – as well as endomorphisms $X \in \operatorname{End}(E), T \in \operatorname{End}(E^2)$. The axioms are as follows:

(1) F is isomorphic to a right adjoint of E (and hence both E, F are exact).

(2) $E = \bigoplus_{i\in\mathbb{Z}_{\mathbb{F}}} E_i$, where E_i is the generalized eigenfunctor with eigenvalue i for the action of X on E. By the fixed adjointness, we get the decomposition $F = \bigoplus_{i\in\mathbb{Z}_{\mathbb{F}}} F_i$ so that F_i is left adjoint to E_i.
(3) We have a weight decomposition $\mathcal{C} = \bigoplus_\nu \mathcal{C}_\nu$ such that the decomposition $[\mathcal{C}] = \bigoplus_\nu[\mathcal{C}_\nu]$ and the maps $[E_i], [F_i]$ define an integrable representation of $\hat{\mathfrak{sl}}_p$ on $[\mathcal{C}]$. Recall that a representation of $\hat{\mathfrak{sl}}_p$ is called *integrable* if the operators e_i, f_i are locally nilpotent. Also note that, thanks to the weight decomposition of $\mathcal{C}$, F_i is isomorphic to the right adjoint of E_i.
(4) The assignment $X_i \mapsto 1^{i-1}X1^{d-i}, T_i \mapsto 1^{i-1}T1^{d-1-i}$ lifts to an algebra homomorphism $\mathcal{H}(d) \to \mathrm{End}(E^d)$, where we write $\mathcal{H}(d)$ for the degenerate affine Hecke algebra.

We have already seen that we have a categorical $\hat{\mathfrak{sl}}_p$-action on $\bigoplus_{n\geqslant 0} \mathbb{F}S_n$-mod.

Let us make a couple of remark regarding this definition. First, it has a multiplicative version that will work for the categories of modules over the type A Hecke algebra (an interesting case is when q is a root of 1). Second, we can extend this definition to other Lie algebras of type A. For example, to get a categorical action of $\mathfrak{sl}_2$ we need to require that X acts on E with a single eigenvalue and to modify (3) in an obvious way. In this way, a categorical action of $\hat{\mathfrak{sl}}_p$ gives rise to p categorical actions of $\mathfrak{sl}_2$. It is possible to define categorical actions of Kac-Moody algebras outside of type A but this requires essentially new ideas. Finally, let us note that the functors E, F are symmetric, i.e., we can have another categorical action with these functors swapped (for this we need, in particular, that the algebra $\mathcal{H}(d)$ is naturally identified with its opposite, which is left as an exercise).

4. Application: crystals

4.1. **E_i and F_i on irreducible objects.** We would like to understand the structure of E_iL, F_iL, where L is a simple object in $\mathcal{C}$. Here we have the following result due to Chuang and Rouquier (who have introduced the notion of a categorical $\mathfrak{sl}_2$-action).

Proposition 4.1. *The following is true.*
(1) *Suppose $E_iL \neq \{0\}$. Then the head (the maximal semisimple quotient) and the socle (the maximal semisimple sub) of E_iL are simple and isomorphic (let's denote this simple object by $\tilde{e}_iL$).*
(2) *Let d be the maximal number such $E_i^dL \neq 0$. Then $e_i[L] = d[\tilde{e}_iL] + \sum_{L_0}[L_0]$, where the sum is taken over simples L_0 with $E_i^{d-1}L_0 = 0$.*

The similar results also hold for F_iL (in particular, we get the simple/head socle of F_iL to be denoted by $\tilde{f}_iL$).

If $E_iL = 0$ (resp., $F_iL = 0$), then we set $\tilde{e}_iL = 0$ (resp., $\tilde{f}_iL$). So we get a collection of maps $\tilde{e}_i, \tilde{f}_i : \mathrm{Irr}(\mathcal{C}) \to \mathrm{Irr}(\mathcal{C}) \sqcup \{0\}$. A nice and very useful exercise is to check that if $\tilde{e}_iL \neq 0$, then $\tilde{f}_i\tilde{e}_iL = L$.

Proposition 4.1 implies, in particular, that the classes $[L], L \in \mathrm{Irr}(\mathcal{C})$, form a so called *perfect basis* (as defined by Berenstein and Kazhdan). This implies that the maps $\tilde{e}_i, \tilde{f}_i$ endow $\mathrm{Irr}(\mathcal{C})$ with a crystal structure (a crystal is a combinatorial shadow of a Lie algebra action first constructed by Kashiwara using quantum groups).

4.2. **Irreducibility of $\bigoplus_n[\mathbb{F}S_n$-mod].** Using Proposition 4.1, we will show that the $\hat{\mathfrak{sl}}_p$-module $\bigoplus_n[\mathbb{F}S_n$-mod] is irreducible (and hence it is the irreducible highest weight module of weight ω_0, a.k.a., the basic representation of $\hat{\mathfrak{sl}}_p$).

Theorem 4.2. *The $\hat{\mathfrak{sl}}_p$-module $V := \bigoplus_n [\mathbb{F}S_n\text{-mod}]$ is irreducible.*

Proof. The module V is a quotient of $\mathcal{F}$ and so is an integrable highest weight representation. Such a representation is irreducible if and only if it has a unique *singular* (=annihilated by all e_i) vector v. One such vector is $[\mathbb{C}] \in [\mathbb{F}S_0\text{-mod}]$. Moreover, the space V^0 of singular vectors does not contain any other vector of the form $[L]$. Indeed, $\sum_i e_i[L] = [EL]$, but $EL = \mathrm{Res}^n_{n-1} L$ is nonzero if $L \notin [\mathbb{F}S_0\text{-mod}]$. The following lemma combined with Proposition 4.1 implies that V^0 is spanned by vectors of the form $[L], L \in \mathrm{Irr}(\mathcal{C})$. This completes the proof. □

Lemma 4.3. *Let V be an $\hat{\mathfrak{sl}}_p$-module and let $\mathcal{B}$ be a basis with the following property. Let $d_i(b)$ denote the maximal number d such that $e_i^d b \neq 0$. For any $b \in \mathcal{B}, i \in \mathbb{Z}/p\mathbb{Z}$, we have that either $e_i b = 0$ or $e_i b = \alpha \tilde{e}_i b + \sum_{b_0} n_{b_0} b_0$, where $\alpha \neq 0$, if $n_{b_0} \neq 0$, then $e_i^{d_i(b)-1} b_0 = 0$, and $\tilde{e}_i b \in \mathcal{B}$. Assume further that $\tilde{e}_i b_1 = \tilde{e}_i b_2 \neq 0$ implies $b_1 = b_2$. Then the space of singular vectors V^0 is spanned by $V^0 \cap \mathcal{B}$.*

Proof. Pick $v \in V^0$ and expand it in the basis $\mathcal{B}$, $v = \sum_{b \in \mathcal{B}'} n_b b$, where $n_b \neq 0$ for all $b \in \mathcal{B}'$. Let $d := \max\{d_i(b) | b \in \mathcal{B}'\}$. Let $\mathcal{B}'_i := \{b \in \mathcal{B}' | d_i(b) = d\}$. Assume $d > 0$. Then

$$0 = e_i^d v = \sum_{b \in \mathcal{B}'_i} m_b \tilde{e}_i^d b,$$

where all m_b's are nonzero and all $\tilde{e}_i^d b$ are distinct. We get a contradiction that finishes the proof. □

Categorical actions, III.

1) Reminder: cat-y $\mathcal{O}$ for $\mathfrak{gl}_n$.

2) Categorical action

3) Parabolic versions

1) $\mathfrak{g}=\mathfrak{gl}_n(\mathbb{C})$. Consider "integral part" of category $\mathcal{O}$. (before we were dealing w. s/simple $\mathfrak{g}$ but $\mathfrak{g}=\mathfrak{gl}_n$ is not very different)

$\mathcal{O}=\{M\in U(\mathfrak{g})\text{-mod}:$ fin gen

$\mathfrak{h}$ acts diagon. w. integral eigenvalues

$\mathfrak{n}$ acts locally nilpotently$\}$

Ex: $\lambda=(\lambda_1,\dots \lambda_n)\in\mathbb{Z}^n \rightsquigarrow \mathbb{C}_\lambda\in U(\mathfrak{b})\text{-mod}$, $\mathfrak{b}=\left\{\begin{pmatrix}* & *\\ 0 & *\end{pmatrix}\right\}$

$\rightsquigarrow \Delta(\lambda)=U(\mathfrak{g})\otimes_{U(\mathfrak{b})}\mathbb{C}_\lambda$

$\exists!$ irred $L(\lambda)\twoheadleftarrow\Delta(\lambda)$ & $\mathbb{Z}^n\xrightarrow{\sim}\mathrm{Irr}(\mathcal{O})$, $\lambda\mapsto L(\lambda)$

$[\mathcal{O}]=(\mathbb{C}^{\mathbb{Z}})^{\otimes n}$, where $\mathbb{C}^{\mathbb{Z}}$ is a vector space w. basis $v_i, i\in\mathbb{Z}$

$[\Delta(\lambda)]\longmapsto v_{\lambda_1}\otimes v_{\lambda_2-1}\otimes\dots\otimes v_{\lambda_n+1-n}$

Reason for shift: $\rho=\frac{1}{2}\sum_{\alpha>0}\alpha=\frac{1}{2}\sum_{i<j}(\epsilon_i-\epsilon_j)=\left(\frac{n-1}{2},\frac{n-3}{2},\dots,\frac{1-n}{2}\right)$ - up to $-\frac{n-1}{2}(\epsilon_1+\dots+\epsilon_n)$ get $(0,-1,\dots,1-n)$ - same values on roots as ρ. So we redefine ρ as $(0,-1,\dots,1-n)$

Blocks: Define equiv. on $\mathbb{Z}^n$ by $\lambda\sim\mu$ if $\lambda+\rho\sim_{S_n}\mu+\rho$. Then $\mathcal{O}=\bigoplus_{\alpha\in\mathbb{Z}^n/\sim}\mathcal{O}_\alpha$, where $\mathcal{O}_\alpha$ is the Serre span of $L(\lambda)$, $\lambda\in\alpha$

$e_i v_j=\delta_{ij}v_{j+1}$, $f_i v_j=\delta_{i,j+1}v_j$

Observation: $\mathfrak{sl}_\infty\curvearrowright\mathbb{C}^{\mathbb{Z}}$ (tautological representation) $\rightsquigarrow\mathfrak{sl}_\infty\curvearrowright(\mathbb{C}^{\mathbb{Z}})^{\otimes n}$

Want to show that $\mathcal{O}$ carries a categorical $\mathfrak{sl}_\infty$-action categorifying that on $(\mathbb{C}^{\mathbb{Z}})^{\otimes n}$.

Rmk: All objects in $\mathcal{O}$ have finite length.

2) Recall that we need: functors $E,F\curvearrowright\mathcal{O}$ w. fixed one-sided adjunction and endomorphisms $X\in\mathrm{End}(E)$, $T\in\mathrm{End}(E^2)$ subject to:

(1) E,F are biadjoint

(2) E-decomp-n of E w.r.t. X looks like $E=\bigoplus_{i\in\mathbb{Z}}E_i$

1

by fixed adjointness, have $F = \bigoplus_{i\in\mathbb{Z}} F_i$

(3) There's decomp-n $\mathcal{O} = \bigoplus_\nu \mathcal{O}_\nu$ s.t. $[\mathcal{O}] = \bigoplus_\nu [\mathcal{O}_\nu]$ is the weight decomp-n of $(\mathbb{C}^{\mathbb{Z}})^{\otimes n}$ for the $\mathfrak{sl}_\infty$-action, and $[E_i], [F_i]$ coincide w. e_i, f_i (defined by $e_i \cdot v_{i_1}\otimes \ldots \otimes v_{i_n} = \sum_{j=1}^n \delta_{i i_j} v_{i_1}\otimes \ldots \otimes v_{i+1} \otimes \ldots \otimes v_{i_n}$, $f_i \cdot v_{i_1}\otimes \ldots \otimes v_{i_n} = \sum_{j=1}^n \delta_{i, i_j+1} v_{i_1}\otimes \ldots \otimes v_{i-1} \otimes \ldots \otimes v_{i_n}$).

(4) The assignment $X_i \mapsto 1^{i-1}X1^{d-i}$, $T_i \mapsto 1^{i-1}T1^{d-i-1}$ extends to an algebra homomorphism $\mathcal{H}(d) \longrightarrow \mathrm{End}(E^d)$

2.1) Data: $E(M) = \mathbb{C}^n \otimes M$ ($\mathbb{C}^n$ - tautol. $\mathfrak{g}$-module), $F(M) = (\mathbb{C}^n)^* \otimes M$

$X_M(v\otimes m) = \sum_{i,j=1}^n E_{ij} v \otimes E_{ji} m$, $T_M(v\otimes v' \otimes m) = v' \otimes v \otimes m$

tensor Casimir

Axiom 1 is clear, Axiom 4 is Problem 4 in HW2

2.2) Objects $E\Delta(\lambda)$, $F\Delta(\lambda)$

To check axioms 2, 3 we compute $E\Delta(\lambda)$, $F\Delta(\lambda)$ and how X acts on these objects.

Let V be a finite dimensional GL_n-module w. weight basis $v_1 \ldots v_m$ w. weights $\nu_1 \ldots \nu_m$ ordered in non-decreasing way. Then recall (Prop 2.1 in Lecture 10) that $V\otimes\Delta(\lambda)$ is filtered w. successive quotients $\Delta(\lambda+\nu_i)$ (ordered bottom to top, e.g. $\Delta(\lambda+\nu_1)$ is a sub and $\Delta(\lambda+\nu_m)$ is a quotient)

Ex: $V = \mathbb{C}^n$, then get weights $\epsilon_1 > \epsilon_2 > \ldots > \epsilon_n$ and filtr-n is by $\Delta(\lambda+\epsilon_1), \Delta(\lambda+\epsilon_2), \ldots \Delta(\lambda+\epsilon_n)$ $(\epsilon_i = (\delta_{1i}, \ldots \delta_{ni}))$

$V = (\mathbb{C}^n)^* \rightsquigarrow -\epsilon_n > -\epsilon_{n-1} > \ldots > -\epsilon_1$, filtr-n by $\Delta(\lambda-\epsilon_n), \ldots \Delta(\lambda-\epsilon_1)$

Prop: X_M preserves the filtr-n on $E\Delta(\lambda)$ and acts on the subquotient $\Delta(\lambda+\epsilon_i)$ by $\lambda_i + 1 - i$

Proof: The first claim follows from $\mathrm{Hom}(\Delta(\lambda+\epsilon_i), \Delta(\lambda+\epsilon_j)) = 0$ (this is because $\lambda+\epsilon_i > \lambda+\epsilon_j$). To deduce it is left as an exercise

Any endomorphism of a Verma module acts on it by scalar. To determine the scalar, we need to compare the tensor Casimir $\sum_{i,j=1}^{n} E_{ij}\otimes E_{ji} \in U(\mathfrak{g})\otimes U(\mathfrak{g})$ with the usual Casimir $\sum_{i,j=1}^{n} E_{ij}E_{ji}\ (=C) \in U(\mathfrak{g})$. Let $\delta: U(\mathfrak{g}) \to U(\mathfrak{g})\otimes U(\mathfrak{g})$ be the coproduct. Then $\sum_{i,j=1}^{n} E_{ij}\otimes E_{ji} = \frac{1}{2}(\delta(C) - C\otimes 1 - 1\otimes C)$. ~~The element Σ acts on~~ We can rewrite C as $\sum_{j<i} 2E_{ij}E_{ji} + \sum_{j<i}[E_{ji}, E_{ij}] + \sum_{i=1}^{n} E_{ii}^2$

So $C|_{\Delta(\mu)} = \sum_{i=1}^{n}(n+1-2i)\mu_i + \sum_{i=1}^{n}\mu_i^2$

The element $\sum_{i,j=1}^{n} E_{ij}\otimes E_{ji} = \frac{1}{2}(\delta(C) - C\otimes 1 - 1\otimes C)$ acts on the subquotient $\Delta(\lambda+\epsilon_i)$ by $\frac{1}{2}(C|_{\Delta(\lambda+\epsilon_i)} - C|_{\Delta(\epsilon_1)} - C|_{\Delta(\lambda)}) =$

$= \frac{1}{2}(n+1-2i + 2\lambda_i + 1 - n) = \lambda_i + i - 1$ $\square$

Cor: Axiom 2 holds

Proof: We only need to check that eigenvalues of X_M on EM are integral for every simple M. Since every simple is a quotient of some $E\Delta(\lambda)$ $\square$

Cor: Axiom 3 holds

Proof: Proposition shows that $[E_j\Delta(\lambda)] = e_j[\Delta(\lambda)]$. Let's show the claim about F_j's. Let α be the equivalence class of λ and α' be the class where one entry j is replaced with $j+1$. For $M\in\mathcal{O}_\alpha$ we have $E_jM = \pi_{\alpha'}\circ EM$. Using adjointness we deduce that, for $N\in\mathcal{O}_{\alpha'}$, we have $F_jM = \pi_\alpha\circ FM$. The claim that $[F_j\Delta(\lambda)] = f_j\Delta(\lambda)$ easily follows from here $\square$

2.3)* Crystals. Here we determine operators $\tilde{e}_j, \tilde{f}_j$ for $Irr(\mathcal{O})$. Take $\lambda\in\mathbb{Z}^n$ and write $\lambda+\rho = (\lambda_1, \lambda_2 - 1, \ldots, \lambda_n + 1 - n)$. To each entry $j+1$ we assign bracket $($, and to each j we assign $)$. For example, $j=3$, $\lambda+\rho = (3,4,4,5,3,2,4)$. Then we cancel all brackets that

$)\ (\ \cancel{(\quad)}\ ($ $\longleftarrow$ are connect or $()(()))(\rightsquigarrow)($. It's the standard fact that the result doesn't depend on the order of cancellations. We end up with a sequence like $)\ldots)(\ldots($. To define $\tilde{e}_j L(\lambda)$ we switch the rightmost $)$ to $($ and set $\tilde{e}_j L(\lambda)=L(\lambda')$, where $\lambda' = \lambda+\epsilon_k$, k is the position where the switch occurred. If there are no "$)$", then we set $\tilde{e}_j L(\lambda)=0$. To compute $\tilde{f}_j L(\lambda)$ we switch the left-most $($ to $)$ and set $\tilde{f}_j L(\lambda)=L(\lambda'')$, $\lambda''=\lambda-\epsilon_\ell$ (or $\tilde{f}_j L(\lambda)=0$ if there is no $($. In the example above, $\lambda'+\rho=(4,4,4,5,3,2,4)$, $\lambda''+\rho=(3,3,4,5,3,2,4)$

3) Parabolic categories

This is a generalization of $\mathcal{O}$. Pick positive integers $n_1\ldots n_k$ w. $n_1+\ldots+n_k=n$ and denote $(n_1\ldots n_k)=\underline{n}$. We introduce some notations: Let L denote the subgroup of GL_n consisting of all block diagonal matrices, where blocks have sizes $n_1\ldots n_k$. Let $\mathfrak{m}$ ~~denote~~ (resp $\mathfrak{m}^-$) denote the subalgebra of all strictly upper triangular block (resp. lower triangular) block matrices. For example, take $\underline{n}=(1,2,1)$.

Then $L=\left\{\begin{pmatrix} * & 0 & 0 & 0 \\ 0 & * & * & 0 \\ 0 & * & * & 0 \\ 0 & 0 & 0 & * \end{pmatrix}\right\}$, $\mathfrak{m}=\left\{\begin{pmatrix} 0 & * & * & * \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 \end{pmatrix}\right\}$ $\mathfrak{m}^-=\left\{\begin{pmatrix} 0 & 0 & 0 & 0 \\ * & 0 & 0 & 0 \\ * & 0 & 0 & 0 \\ * & * & * & 0 \end{pmatrix}\right\}$.

~~Consider~~ Let $\mathfrak{l}$ denote the Lie algebra of L and $\mathfrak{p}=\mathfrak{l}\oplus\mathfrak{m}$, this is a Lie subalgebra.

Now consider the category $\mathcal{O}^{\underline{n}}=\{M\in U(\mathfrak{g})\text{-mod} \mid$ fin. gen.

$\mathfrak{l}$-action is loc. finite and integrates to L

$\mathfrak{m}$ acts locally nilpotently$\}$

An example of an object in $\mathcal{O}^{\underline{n}}$ is provided by a parabolic Verma module. Namely pick an irreducible representation of L, say V and set $\Delta^{\underline{n}}(V)=U(\mathfrak{g})\otimes_{U(\mathfrak{p})}V$. Note that the irreducibles L are labeled by highest weights $(\lambda_1\ldots\lambda_n)$ subject to $\lambda_1\geq\ldots\geq\lambda_{n_1}$, $\lambda_{n_1+1}\geq\ldots\geq\lambda_{n_2},\ldots$, $\lambda_{n_1+\ldots+n_{k-1}+1}\geq\ldots\geq\lambda_n$. For such λ we write $\Delta^{\underline{n}}(\lambda)$ instead of $\Delta^{\underline{n}}(V)$.

Note that $\mathcal{O}^{\underline{n}} \subset \mathcal{O}$. Let's determine $[\mathcal{O}^{\underline{n}}] \subset [\mathcal{O}] = (\mathbb{C}^{\mathbb{Z}})^{\otimes n}$. It's easy to see that the classes $[\Delta^{\underline{n}}(\lambda)]$ constitute a basis in $[\mathcal{O}^{\underline{n}}]$ (compare to the analogous claim for $\mathcal{O}$). The ~~Weyl character formula for the an irreducible module $V(\lambda)$ of L together shows that~~ ~~$[V(\lambda)] =$~~ The Weyl character formula (for L) implies the following (exercise)

$$[\Delta^{\underline{n}}(\lambda)] = (v_{\lambda_1} \wedge \ldots \wedge v_{\lambda_{n_1}+1-n_1}) \otimes (v_{\lambda_{n_1+1}-n_1} \wedge \ldots \wedge v_{\lambda_{n_2}+1-n_2}) \otimes \ldots$$

Thm: $\mathcal{O}^{\underline{n}} \subset \mathcal{O}$ is a categorical $\mathfrak{sl}_\infty$-representation with $[\mathcal{O}^{\underline{n}}] = \Lambda^{n_1}\mathbb{C}^{\mathbb{Z}} \otimes \Lambda^{n_2}\mathbb{C}^{\mathbb{Z}} \otimes \ldots \otimes \Lambda^{n_k}\mathbb{C}^{\mathbb{Z}}$.

We don't provide the proof.