Last time suppose $X = \prod_{i \in I} X_i$

the box topology on X is gen by
$\{\prod_i U_i \mid U_i \text{ open in } X_i \text{ for all } i\}$

the product topology on X is gen by
$\{\prod_i U_i \mid U_i \text{ open in } X_i \text{ for all } i,$
$U_i \neq X_i \text{ for only fin many } i\}$

if $I = \{1, \ldots, n\}$ and $X_i = \mathbb{R}$ for all i, then $X = \mathbb{R}^n$

here, box = product

Q1 is it the same as the analytic top? [yes]
[square balls are open in box top]
[prod's of anlytc opens are anlytc open]

Q2 any open set in $\mathbb{R}^n$ not of the form
$U_1 \times U_2 \times \ldots \times U_n$?

[draw]

if $I = \mathbb{Z}_+$ and $X_i = \mathbb{R}$ for all i, then $X = \mathbb{R}^\omega$

Q3 let $V = (-1, 1) \times (-1, 1) \times \ldots$ in $\mathbb{R}^\omega$

open in box topology? [yes]
open in product topology? [no]

why not? [$V \neq \varnothing$, but no product basis elt sub V]

in general: prod top can be coarser than box top

Df — for all i' in I
let $pr_{i'} : \prod_i X_i \to X_{i'}$ be

the *projection* map $pr_{i'}((x_i)_i) = x_{i'}$

Thm — the product topology on $\prod_i X_i$ is
the coarsest s.t. $pr_{i'}$ is cts for all i'

Pf — suppose that T is a top on $\prod_i X_i$
in which $pr_{i'}$ is cts for all i'

then $pr_{i'}^{-1}(U_{i'})$ in T for all i' and open $U_{i'}$

so for any finite $J \subset I$,

$\bigcap_{i' \in J} pr_{i'}^{-1}(U_{i'})$ is open

but $\bigcap_{i' \in J} pr_{i'}^{-1}(U_{i'})$
$= \{(x_i) \mid x_{i'} \in U_{i'} \text{ for all } i' \in J\}$
$= \{\prod_i U_i \mid U_i = X_i \text{ for all } i \notin J\}$

so T contains the basis for the product top
so T contains the product top

similarly,

Thm — the subspace top on $A \subset X$ is
the coarsest s.t. the inclusion of A is cts

	subspace topology	product topology
makes	inclusion cts	projections cts

(Munkres §17, 21)   suppose A sub X

interior Int_X(A)
= {a in X | have U open in X s.t. a in U sub A}
= union of open sets of X that are subsets of A

closure Cl_X(A)
= X – Int_X(X – A)
= X – bigcup_{V sub X – A and open in X} V
= bigcap_{V sub X – A and open in X} (X – V)
= bigcap_{Z sup A and closed in X} Z
= intersection of closed sets of X containing A

[draw]

alternatively: Cl_X(A)
= X – {x | have V open in X s.t. x in V sub X – A}
= {x | no V open in X s.t. x in V sub X – A}
= {x | if V is open in X and x in V,
then V intersects A}

<u>Ex</u>   X = R^ω and A = R^∞

<u>Q</u>   what is the closure of R^∞ in R^ω
in the box top? in the product top?

consider x = (1, 1/2, 1/3, 1/4, …)

in Cl_{R^ω}(R^∞) for box? [draw]

no: x in (0, 2) × (0, 1) × (0, 2/3) × …

for product?

yes: suppose V open in R^ω and x in V

pick basis elt B s.t. x in B sub V
$B = \prod_i B_i$, where $B_i \neq R$ for only fin many i
so B contains elts of R^∞
so V intersects R^∞

<u>Df</u> a sequence $x_1, x_2, \ldots$ of points in X
<u>converges</u> to x
iff, for all open V containing x
have N s.t. $x_N, x_{N + 1}, \ldots$ in V

<u>Q</u> can a sequence converge to
more than one pt?

<u>Ex</u> give X the indiscrete topology:
then every sequence of pts converges to
every pt of X at once!

Df X is <u>Hausdorff</u> iff, for all $x \neq y$ in X,
there are disjoint open U and V
s.t. x in U and y in V

<u>Thm</u> if X is Hausdorff
then any sequence in X converges to
at most one pt

<u>Pf</u> suppose $(x_n)_n$ converges to x and y
suppose $x \neq y$: then have disj open U, V
s.t. x in U and y in V
if $x_N, x_{N + 1}, \ldots$ in U, then notin V