__Thm__ R^2 and S^1 are not homeomorphic

__Pf__ R^2 is not compact [why?], but

__Lem__ S^1 is compact

__Pf__ $p(t) = (\cos 2\pi t, \sin 2\pi t)$
is a surjective cts map
from $[0, 1]$ onto S^1

cpt image of compact is
compact

__Thm__ R is not homeomorphic to
either S^1 or R^2

$R - \{t\}$ is disconnected for any t in R [why?], but

__Lem__ $S^1 - \{q\}$ is connected
for any q in S^1

__Pf__ WLOG $q = (1, 0)$
p is a homeo from an open
interval onto $S^1 - \{q\}$

__Lem__ $R^2 - \{q\}$ is connected
for any q in R^2

__Pf__ $R^2 - \{q\}$ is path connected

even though R and R^2 are not homeomorphic,
they are still homotopy equivalent [weaker]

(Munkres §51)

<u>Df</u> suppose f, g : X to Y are cts maps

a <u>homotopy</u> from f to g is a cts map

$\varphi : X \times [0, 1]$ to Y

s.t. $\varphi(x, 0) = f(x)$ and $\varphi(x, 1) = g(x)$ for all x in X

[a cts "movie" in time, showing f at time t = 0 and g at time t = 1]

<u>Ex</u> let f, g : R to R be def by
f(x) = x and g(x) = –x

$\varphi(x, t) = (1 - 2t)x$ is a homotopy from f to g [draw]

<u>Ex</u> if f = g, then $\varphi(x, t) = f(x)$ is a homotopy

<u>Ex</u> if $\varphi : X \times [0, 1]$ to Y is a homotopy, then
$\psi(x, t) = \varphi(x, 1 - t)$ is a homotopy

<u>Prob</u> suppose f_1, f_2, f_3 : X to Y are all cts

given homotopies φ from f_1 to f_2,
ψ from f_2 to f_3,

find homotopy from f_1 to f_3 in terms of φ, ψ

<u>Df</u> f, g are <u>homotopic</u> iff there is
some homotopy from f to g

in this case we write $f \sim g$

Cor ~ is an equiv rel on maps from X to Y

Thm any two cts maps from R to R are homotopic

Pf since ~ is an equivalence relation,

enough to show that any $f : R$ to R is nulhomotopic = homotopic to some constant map

$\varphi(x, t) = (1 - t)f(x)$ works

Df a [nonempty] space is contractible iff its identity map is nulhomotopic

so we've shown that R is contractible

Prob let f, g : X to Y and F, G : Y to Z be cts

if $f \sim g$ and $F \sim G$, then $F \circ f \sim G \circ g$

Prob use the f = g case above to show:

if X, Y are nonempty and Y is contractible, then any cts map from X into Y is nulhomotopic

(Munkres §58)

homotopy	: equiv rel on maps
homotopy equivalence	: equiv rel on spaces

Df a homotopy equivalence from X to Y is a pair of cts maps ($f : X$ to Y, $g : Y$ to X)

s.t. $g \circ f \sim id_X$ and $f \circ g \sim id_Y$

in this case, write $X \sim Y$

Ex if $f : X$ to Y is a homeomorphism then (f, f^{-1}) is a homotopy equiv

Ex ($f : R$ to R^2, $g : R^2$ to R) def by $f(x) = (x, 0)$ and $g(x, y) = x$ is a homotopy equivalence

$(g \circ f)(x) = x = id_R(x)$, so $g \circ f = id_R$

$(f \circ g)(x, y) = (x, 0)$, so $\varphi((x, y), t) = (x, ty)$ is a homotopy from $f \circ g$ to id_{R^2}

[draw]

Thm if X is contractible, say with $id_X \sim$ const map at c in X, then $X \sim \{c\}$

Pf take $f : X$ to $\{c\}$ def by $f(x) = c$ for all x
take $g : \{c\}$ to X def by $g(c) = c$

then $g \circ f =$ const map at $c \sim id_X$
$f \circ g = id_{\{c\}}$