LEVEL-RANK DUALITIES FOR FINITE REDUCTIVE GROUPS

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Abstract. In this survey, we present our work "Level-Rank Dualities from Φ-Cuspidal Pairs and Affine Springer Fibers" with an emphasis on motivations from combinatorics, modular representation theory, and classical level-rank duality, while omitting any detailed discussion of affine Springer fibers.

This is an extended abstract of [TX23]. We will explain the evidence for a family of surprising coincidences within the representation theory of a finite reductive group G: more precisely, dualities between blocks of cyclotomic Hecke algebras that Broué–Malle attach to Φ-cuspidal pairs of G, where the Hecke parameters are specialized not to the order of the finite field $\mathbf{F}_q$, but to roots of unity. For the groups $G = \mathrm{GL}_n(\mathbf{F}_q)$, these coincidences can be expressed very concretely, in terms of the combinatorics of partitions, and the whole story recovers an avatar of Frenkel's level-rank duality studied by Uglov, Chuang–Miyachi, and others.

1. Partitions

1.1. Let Π be the set of all integer partitions. We will regard a partition as a weakly decreasing sequence $\pi = (\pi_1, \pi_2, \ldots)$ such that $\pi_i = 0$ for all i large enough; in practice, we write a partition like $\pi = (3, 3, 1, 0, \ldots)$ using a shorthand like $(3^2, 1)$. We define the size of π to be $|\pi| := \pi_1 + \pi_2 + \ldots$ and the length of π to be the number of nonzero entries.

For any positive integer m, we define an *m-partition* to be an m-tuple of partitions $\vec{\pi} = (\pi^{(0)}, \pi^{(1)}, \ldots, \pi^{(m-1)})$, indexed from 0 through $m-1$. We will occasionally regard these indices as elements of $\mathbf{Z}_m := \mathbf{Z}/m\mathbf{Z}$. We define the size of $\vec{\pi}$ to be $|\vec{\pi}| := |\pi^{(0)}| + \cdots + |\pi^{(m-1)}|$.

1.2. The combinatorial part of our story starts with an analogue, for partitions, of the notion of long division by m. Recall that a partition is an *m-core* if and only if it contains no hook lengths divisible by m. For instance, the only 1-core is the empty partition; the 2-cores are the "staircase" partitions 1, $(2, 1)$, $(3, 2, 1)$, *etc.*; but m-cores for $m \geq 3$ are more complicated. Let $\Pi_{m\text{-}cor} \subseteq \Pi$ be the subset of m-cores. Then the analogue of the map sending an integer to its quotient and remainder modulo m is a bijection $\Pi \xrightarrow{\sim} \Pi^m \times \Pi_{m\text{-}cor}$, usually known as the *core-quotient bijection*.

Following Uglov [Ugl00], we define it this way: Let $\mathbf{B}$ be the collection of subsets $\beta \subseteq \mathbf{Z}$ with the property that $n \in \beta$ when n is a sufficiently negative integer, but $n \notin \beta$ when n is sufficiently positive. There is a bijection $\Pi \times \mathbf{Z} \xrightarrow{\sim} \mathbf{B}$ that takes a pair (π, s) to the set

$$\beta_{\pi,s} = \{\pi_i - i + s \mid i = 1, 2, \ldots\}.$$

There is also a bijection $\upsilon_m = (\upsilon_m^{(0)}, \upsilon_m^{(1)}, \ldots, \upsilon_m^{(m-1)}) : \mathbf{B} \xrightarrow{\sim} \mathbf{B}^m$, defined by setting

$$\upsilon_m^{(r)}(\beta) = \{q \in \mathbf{Z} \mid mq + r \in \beta\} \quad \text{for } 0 \leq r < m.$$

The composition

$$\Upsilon_m : \Pi \times \mathbf{Z} \xrightarrow{\sim} \mathbf{B} \xrightarrow{\upsilon_m} \mathbf{B}^m \xrightarrow{\sim} (\Pi \times \mathbf{Z})^m = \Pi^m \times \mathbf{Z}^m$$

recovers the core-quotient map as follows: The m-quotient of π is the m-partition formed by the Π^m-component of $\Upsilon_m(\pi, 0)$. The m-core of π is the unique partition μ such that $\Upsilon_m(\mu, s)$ and $\Upsilon_m(\pi, s)$ have the same $\mathbf{Z}^m$-component for any s.

Note that μ is an m-core if and only if the Π^m-component of $\Upsilon_m(\mu, s)$ is the empty m-partition for some (or any) s. Also, note that there is a purely graphical way to calculate the core-quotient map, using Young diagrams [JK81, §2.7].

In the literature, elements of $\Pi^m \times \mathbf{Z}^m$ are called *charged m-partitions* and written with the ket notation $|\vec{\pi}, \vec{s}\rangle$ of quantum mechanics. The vector $\vec{s}$ is called the *m-charge*. Elements of $\mathbf{B}^m$ are sometimes called *m-runner abacus configurations*, or *m-abaci*. They can be pictured as subsets of $\mathbf{Z} \times \mathbf{Z}_m$, or more vividly, as configurations of beads on a horizontal abacus with m runners. Under the identification $\Pi^m \times \mathbf{Z}^m \simeq \mathbf{B}^m$, the operation $|\vec{\pi}, \vec{s}\rangle \mapsto |\vec{\emptyset}, \vec{s}\rangle$ corresponds to sliding all beads as far left as possible.

Example 1.1. Let $\pi = (8, 6, 1)$ and $s = 2$. Then $\beta_{\pi,s} \in \mathbf{B}$ is represented by the following 1-abacus:

$$\begin{matrix} \cdots & -4 & -3 & -2 & - & 0 & - & - & - & - & - & 6 & - & - & 9 \end{matrix}$$

So $\Upsilon_3(\pi, s) \in \mathbf{B}^3$ is represented by the following 3-abacus, where we have labeled each position (q, r) with the value of $3q + r$. Note that r increases *downward.*

$$\begin{matrix} \cdots & -12 & -9 & -6 & -3 & 0 & - & 6 & 9 \\ \cdots & -11 & -8 & -5 & -2 & - & - & - & \\ \cdots & -10 & -7 & -4 & - & - & - & - & \end{matrix}$$

Sliding the beads at 6 and 9 as far left as possible gives the 3-abacus of $\Upsilon_3(\mu, s)$, where $\mu = (5, 3, 1)$. So the 3-core of π is $(5, 3, 1)$.

1.3. We will discuss generalizations of υ_m and Υ_m that involve fixing another integer $l > 0$. Let $\upsilon_m^l = (\upsilon_m^{l,(0)}, \upsilon_m^{l,(1)}, \ldots, \upsilon_m^{l,(m-1)}) : \mathbf{B}^l \xrightarrow{\sim} \mathbf{B}^m$ be defined by

$$lq + r_l \in \upsilon_m^{l,(r_m)}(\vec{\beta}) \iff mq + r_m \in \beta^{(r_l)} \quad \begin{array}{l}\text{for all } q, r_l, r_m \in \mathbf{Z} \text{ such that} \\ 0 \le r_l < l \text{ and } 0 \le r_m < m.\end{array}$$

It is also possible to rewrite υ_m^l as a composition $\mathbf{B}^l \xrightarrow{\upsilon_l^{-1}} \mathbf{B} \xrightarrow{\upsilon_m^*} \mathbf{B}^m$, where υ_m^* is some modified version of υ_m. We can now form the composition

$$\Upsilon_m^l : \Pi^l \times \mathbf{Z}^l = \mathbf{B}^l \xrightarrow{\upsilon_m^l} \mathbf{B}^m = \Pi^m \times \mathbf{Z}^m.$$

Note that $\upsilon_m^1 = \upsilon_m$ and $\Upsilon_m^1 = \Upsilon_m$.

Example 1.2. The bijection Υ_4^3 takes the 4-abacus

$$\begin{array}{ccccc} \cdots & -12 & -8 & -4 & 0 \\ \cdots & -11 & -7 & -3 & - \\ \cdots & -10 & -6 & -2 & 2 \\ \cdots & -9 & -5 & - & 3 \end{array}$$

to the 3-abacus in Example 1.1.

Here is a graphical interpretation of Υ_4^3: If we subdivide the 4-abacus above into 3×4 arrays of points (q_4, r_4), according to the quotient of q_4 modulo 3, and subdivide the 3-abacus in Example 1.1 into 4×3 arrays of points (q_3, r_3), according to the quotient of q_3 modulo 4, then Υ_4^3 amounts to reflecting and rotating each 3×4 array onto a corresponding 4×3 array. In our example:

$$\left.\begin{array}{ccc|ccc} \bullet & \bullet & \bullet & \bullet & - & - \\ \bullet & \bullet & \bullet & - & - & - \\ \bullet & \bullet & \bullet & \bullet & - & - \\ \bullet & \bullet & - & \bullet & - & - \end{array}\right. \quad \mapsto \quad \begin{array}{cccc|cccc} \bullet & \bullet & \bullet & \bullet & \bullet & - & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & - & - & - & - \\ \bullet & \bullet & \bullet & - & - & - & - & - \end{array}$$

In the next section, we will explain how Υ_m^l holds meaning in the representation theory of the wreath products

$$S_{N,m} := S_N \ltimes \mathbf{Z}_m^N,$$

or rather, the modular representation theory of associated Hecke algebras. Note that the $m = 1$ and $m = 2$ cases recover the Weyl groups of types A and BC, respectively.

2. Representations

2.1. Recall that S_N is a reflection group generated by the system of simple reflections $s_1, s_2, \ldots, s_{N-1}$, where s_i transposes letters i and $i + 1$. Writing Br_N for the braid group on N strands, we have a quotient map $Br_N \to S_N$, sending the ith positive simple twist σ_i to the transposition s_i. It descends to a quotient map of rings $H_N(x) \to \mathbf{Z}S_N$, where the domain is the *Hecke algebra*

$$H_N(x) := \frac{\mathbf{Z}[x^{\pm 1}]Br_N}{\langle (\sigma_i - 1)(\sigma_i + x) \mid i = 1, \dots, N-1 \rangle}.$$

For any nonzero complex number ξ, we set

$$H_N(\xi) = \mathbf{C} \otimes_{\mathbf{Z}[x^{\pm 1}]} H_N(x), \quad \text{where } \mathbf{Z}[x^{\pm 1}] \to \mathbf{C} \text{ sends } x \mapsto \xi.$$

For sufficiently generic $\xi \in \mathbf{C}^\times$, there is an isomorphism $H_N(\xi) \simeq \mathbf{C}S_N$, and hence, a bijection between simple $H_N(\xi)$-modules up to isomorphism and irreducible characters of S_N. By work of Frobenius, Schur, and Young, the latter are also in bijection with integer partitions of size N. But $H_N(\xi)$ may not even be semisimple at special ξ.

2.2. Now fix a positive integer m. The group $S_{N,m}$ is a *complex* reflection group, generated by S_N and a pseudo-reflection of order m. The map $Br_N \to S_N = S_{N,1}$ generalizes to a map $Br_{N,m} \to S_{N,m}$, where

$$Br_{N,m} = \begin{cases} Br_N & m = 1, \\ \langle Br_N, \tau \mid \tau\sigma_1\tau\sigma_1 = \sigma_1\tau\sigma_1\tau \text{ and } \tau\sigma_j = \sigma_j\tau \text{ for } j \neq 1 \rangle. & m > 1. \end{cases}$$

The Hecke algebra generalizes to the so-called *Ariki–Koike algebra*

$$H_{N,m}(\vec{x}) = \frac{\mathbf{Z}[\vec{x}^{\pm 1}]Br_{N,m}}{\left\langle \begin{array}{l} (\sigma_i - 1)(\sigma_i + x_\sigma) \text{ for } i = 1, \dots, N-1, \\ (\tau - x_{\tau,0})(\tau - x_{\tau,1}) \cdots (\tau - x_{\tau,m-1}) \end{array} \right\rangle},$$

where $\vec{x}$ denotes a collection of indeterminates $(x_\sigma, x_{\tau,0}, \dots, x_{\tau,m-1})$, and $\vec{x}^{\pm 1}$ means we also adjoin their inverses. For any $\vec{\xi} \in (\mathbf{C}^\times)^m$, we define $H_{N,m}(\vec{\xi})$ similarly to how we defined $H_N(\xi)$.

Like before, there is an isomorphism $H_{N,m}(\vec{\xi}) \simeq \mathbf{C}S_{N,m}$ at generic $\vec{\xi}$, inducing a bijection between simple $H_{N,m}(\vec{\xi})$-modules up to isomorphism and irreducible characters of $S_{N,m}$. By work of Clifford, the latter are indexed by m-partitions of size N. Again like before, $H_{N,m}(\vec{\xi})$ can degenerate at special $\vec{\xi}$.

We are especially interested in parameters of the form

$$\vec{\xi} = (\zeta, \zeta^{\vec{s}}) := (\zeta, \zeta^{s^{(0)}}, \zeta^{s^{(1)}}, \dots, \zeta^{s^{(m-1)}}), \tag{2.1}$$

where ζ is a root of unity and $\vec{s} = (s^{(0)}, s^{(1)}, \dots, s^{(m-1)})$ is an m-charge. It turns out that the representation theory of $H_{N,m}(\zeta, \zeta^{\vec{s}})$ encodes and is encoded by the combinatorics of m-cores and m-quotients.

2.3. Given an associative algebra H, we write $\mathsf{K}_0(H)$ for the Grothendieck group of the category of finitely-generated H-modules, and $\mathsf{K}_0^+(H) \subseteq \mathsf{K}_0(H)$ for its non-negative part. Let $H_{N,m}^\circ(\vec{x}) = \mathbf{K} \otimes_{\mathbf{Z}[\vec{x}^{\pm 1}]} H_{N,m}(\vec{x})$ for some field $\mathbf{K} \supseteq \mathbf{Z}[\vec{x}^{\pm 1}]$ such that $H_{N,m}^\circ(\vec{x}) \simeq \mathbf{K}S_{N,m}$, and inside which the subring $\mathbf{Z}[\vec{x}^{\pm 1}]$ is integrally closed.

As explained in [GP00, Ch. 7] and [GJ11, Ch. 3], the specialization map from $H_{N,m}(\vec{x})$ to $H_{N,m}(\vec{\xi})$ induces a *decomposition map*

$$\mathrm{d}_{\vec{\xi}} : \mathsf{K}_0^+(H_{N,m}^\circ(\vec{x})) \to \mathsf{K}_0^+(H_{N,m}(\vec{\xi})). \tag{2.2}$$

At the same time, every character of $S_{N,m}$ corresponds to a module over $H_{N,m}^\circ(\vec{x})$ up to isomorphism. Altogether, we get maps

$$\{\vec{\pi} \in \Pi^m \mid |\vec{\pi}| = N\} \xrightarrow{\sim} \mathrm{Irr}(S_{N,m}) \to \mathsf{K}_0^+(H_{N,m}^\circ(\vec{x})) \xrightarrow{\mathrm{d}_{\vec{\xi}}} \mathsf{K}_0^+(H_{N,m}(\vec{\xi})).$$

The elements of the image of the composition are the classes of the so-called *Specht modules*. We say that two m-partitions of size N, or irreducible characters of $S_{N,m}$, belong to the same *$\vec{\xi}$-block* if and only if the corresponding Specht modules fit into a finite sequence of $H_{N,m}(\vec{\xi})$-modules in which consecutive modules always share a Jordan–Hölder factor. Note that $H_{N,m}(\vec{\xi})$ is semisimple if and only if each block is a singleton. In general, $\vec{\xi}$-blocks can be more complicated.

2.4. Fix positive integers l, m. Fix a primitive lth root of unity η_l and a primitive mth root of unity η_m.

In a slogan, Υ_m^l relates $(\eta_m, \eta_m^{\vec{r}})$-blocks of l-partitions, as we run over l-charges $\vec{r}$, and $(\eta_l, \eta_l^{\vec{s}})$-blocks of m-partitions, as we run over m-charges $\vec{s}$. To make this precise, we define a *rank-m, level-l Uglov datum* to be a triple $(K, \vec{r}, \mathbf{a})$, where $K \geq 0$ is any integer, $\vec{r}$ is an l-charge, and $\mathbf{a}$ is an $(\eta_m, \eta_m^{\vec{r}})$-block of the l-partitions of size K. There is a natural map from the set of charged l-partitions to the set of rank-m, level-l Uglov data: To $|\vec{\pi}, \vec{r}\rangle$, we associate the Uglov datum in which $K = |\vec{\pi}|$ and $\mathbf{a}$ is the $(\eta_m, \eta_m^{\vec{r}})$-block containing $\vec{\pi}$. The following result is implicit in [Ugl00], via a theorem of Ariki [GJ11, Thm. 6.2.21]:

Theorem 2.1 (Uglov)**.** *The bijection Υ_m^l descends to a bijection*

$$\{\textit{rank-}m\textit{, level-}l\textit{ Uglov data}\} \xrightarrow{\sim} \{\textit{rank-}l\textit{, level-}m\textit{ Uglov data}\}.$$

In particular, if the latter sends $(K, \vec{r}, \mathbf{a})$ to $(N, \vec{s}, \mathbf{b})$, then Υ_m^l restricts to a bijection

$$\{|\vec{\pi}, \vec{r}\rangle \mid \vec{\pi} \in \mathbf{a}\} \xrightarrow{\sim} \{|\vec{\varpi}, \vec{s}\rangle \mid \vec{\varpi} \in \mathbf{b}\}.$$

Theorem 2.1 is remarkable because there is no direct relationship between the algebras $H_{M,l}(\eta_m, \eta_m^{\vec{r}})$ and $H_{N,m}(\eta_l, \eta_l^{\vec{s}})$ beyond the numerics of their parameters.

2.5. We now sketch how Theorem 2.1 is related to Frenkel's *level-rank duality* between two affine Lie algebras, or rather, quantized versions $\mathrm{U}_v(\widehat{\mathfrak{sl}}_l)_{\mathbf{Q}}$ and $\mathrm{U}_v(\widehat{\mathfrak{sl}}_m)_{\mathbf{Q}}$ over the field $\mathbf{Q}(v)$ [Fre82].

For any integer $l > 0$ and m-charge $\vec{s}$, there is a module over the quantum affine algebra $\mathrm{U}_v(\widehat{\mathfrak{sl}}_m)_{\mathbf{Q}}$ called the *Fock space of level l and charge $\vec{s}$*. Its underlying vector space is a formal span of charged m-partitions:

$$\Lambda_v^{\vec{s}} = \bigoplus_{\vec{\pi} \in \Pi^m} \mathbf{Q}(v)|\vec{\pi}, \vec{s}\rangle.$$

It controls the representation theory of the algebras $H_{N,m}(\eta_l, \eta_l^{\vec{s}})$ through the theorem of Ariki mentioned earlier. Namely, Ariki identifies the sum of the Grothendieck groups $\mathsf{K}_0(H_{N,m}(\eta_l, \eta_l^{\vec{s}}))$ for $N \geq 0$ with an irreducible, highest-weight submodule of $\Lambda_v^{\vec{s}}$. If we set $|\vec{s}| = s^{(0)} + \cdots + s^{(m-1)}$, then for any fixed integer s, Uglov's commuting actions arise from the linear isomorphisms

$$\bigoplus_{|\vec{r}|=s} \Lambda_v^{\vec{r}} \xleftarrow{\upsilon_l} \Lambda_v^{s} \xrightarrow{\upsilon_m^*} \bigoplus_{|\vec{s}|=s} \Lambda_v^{\vec{s}}$$

induced by the maps υ_l and υ_m^* such that $\upsilon_m^l = \upsilon_m^* \circ \upsilon_l^{-1}$. (See the end of §1.)

3. Categories

3.1. After Uglov, several teams of authors worked to categorify Theorem 2.1(3) to a statement about highest-weight covers of the module categories of the Hecke algebras $H_{N,m}(\vec{x})$. To explain this, it is convenient to generalize from the groups $S_{N,m}$ to a finite complex reflection group C with reflection representation V. Generalizing $Br_{N,m}$, the *braid group* of C is the fundamental group $Br_C := \pi_1(V^\circ/C)$, where $V^\circ \subseteq V$ is the open locus where C acts freely. Generalizing $H_{N,m}(\vec{x})$, the *Hecke algebra* $H_C(\vec{x})$ is a certain quotient of $\mathbf{Z}[\vec{x}^{\pm 1}]Br_C$, where $\vec{x}$ is now a collection of indeterminates indexed in terms of the reflection hyperplanes in V and the orders of certain corresponding complex reflections [BMR98]. Extending scalars to $\mathbf{C}$, and specializing $\vec{x}$ to a vector of nonzero complex numbers $\vec{\xi}$, we obtain an algebra $H_C(\vec{\xi})$ a decomposition map $\mathrm{d}_{\vec{\xi}}$ generalizing the map in (2.2), and a notion of $\vec{\xi}$-blocks. The passage from $H_C(\vec{x})$ to $H_C(\vec{\xi})$ may be viewed as fixing conditions on monodromy over V°/C.

The *rational double affine Hecke algebra* or *rational Cherednik algebra* of C, at a vector of complex numbers $\vec{\nu}$ with the same indices as $\vec{\xi}$, is an algebra of deformed polynomial *differential operators* over $V \mathbin{/\!\!/} C$. It takes the form

$$D_C^{\mathrm{rat}}(\vec{\nu}) = (\mathbf{C}C \ltimes (\mathrm{Sym}(V) \otimes \mathrm{Sym}(V^*)))/I(\vec{\nu})$$

for some ideal $I(\vec{\nu})$ deforming the Heisenberg–Weyl relations $xy - yx = \langle x, y\rangle$ for $x \in V$ and $y \in V^*$. As shown in [GGOR03], the rational Cherednik algebra shares many features with the universal enveloping algebras of semisimple Lie algebras: It has a triangular decomposition, where $\mathbf{C}C$ plays the role of the Cartan subalgebra, and an analogue of the Bernstein–Gelfand–Gelfand category O, which we will denote $\mathsf{O}_C(\vec{\nu})$. In particular, $\mathsf{O}_C(\vec{\nu})$ is a highest-weight category whose simple objects are indexed by $\mathrm{Irr}(C)$. For any $\chi \in \mathrm{Irr}(C)$, we write $\Delta_{\vec{\nu}}(\chi)$ to denote the *Verma* or *standard module* with the corresponding simple quotient.

When $\vec{\xi}$ is related to $\vec{\nu}$ by a certain exponential formula, modules over $H_C(\vec{\xi})$ and $D_C^{\mathrm{rat}}(\vec{\nu})$ are related through a Riemann–Hilbert correspondence. Namely, localizing a $D_C^{\mathrm{rat}}(\vec{\nu})$-module to V°/W, then taking monodromy along a Knizhnik–Zamolodchikov-type connection, defines a functor

$$\mathsf{KZ} : \mathsf{O}_C(\vec{\nu}) \to \mathsf{Mod}(H_C(\vec{\xi})). \tag{3.1}$$

When C is a *real* reflection group, the formula is just $\vec{\xi} = \exp(2\pi i\vec{\nu})$.

Ginzburg–Guay–Opdam–Rouquier show that the KZ functor induces a bijection between the block *subcategories* of $\mathsf{O}_C(\vec{\nu})$ and those of $\mathsf{Mod}(H_C(\vec{\xi}))$ [GGOR03]. In fact, KZ is representable by a certain projective object P, and the pair $(\mathsf{O}_C(\vec{\nu}), P)$ forms a *highest-weight cover* of $\mathsf{Mod}(H_C(\vec{\xi}))$ in the sense of Rouquier [Rou08].

3.2. Let $(K, \vec{r}, \mathbf{a})$ and $(N, \vec{s}, \mathbf{b})$ be Uglov data corresponding to each other in the setup of Theorem 2.1. The following result was essentially conjectured by Chuang–Miyachi [CM12], and proved by Rouquier–Shan–Varagnolo–Vasserot [RSVV16, Thm. 7.4]. (See also [SVV14, Cor. 6.5].)

Theorem 3.1 (Categorical Level-Rank Duality). *Let $(K, \vec{r}, \mathbf{a})$ and $(N, \vec{s}, \mathbf{b})$ be Uglov data corresponding to each other in the setup of Theorem 2.1. Then the bijection $\mathsf{a} \xrightarrow{\sim} \mathsf{b}$ is categorified by an equivalence of bounded derived categories*

$$\mathsf{D}^b(\mathsf{O}_{S_{K,l}}(\vec{\nu}_l)_{\mathbf{a}}) \simeq \mathsf{D}^b(\mathsf{O}_{S_{N,m}}(\vec{\nu}_m)_{\mathbf{b}})$$

that combines Koszul and Ringel dualities. Above, the Cherednik parameters $\vec{\nu}_l \in \mathbf{C}^l$ and $\vec{\nu}_m \in \mathbf{C}^m$ respectively map to the Hecke parameters $(\eta_m, \eta_m^{\vec{r}})$ and $(\eta_l, \eta_l^{\vec{s}})$ under (3.1); the subscripts $\mathbf{a}, \mathbf{b}$ indicate the blocks of the Cherednik categories O lifting the appropriate blocks of the Hecke module categories.

The proof uses further equivalences between the Cherednik categories and truncated parabolic categories O of the Lie algebras $\widehat{\mathfrak{sl}}_m$, proved independently by Losev [Los16] and Rouquier–Shan–Varagnolo–Vasserot [RSVV16].

4. Finite Reductive Groups

4.1. We will propose a generalization of Theorem 2.1, replacing the groups $S_{N,m}$ with complex reflection groups arising from the representation theory of finite groups of Lie type. As we will explain, the *general linear* case of our story recovers the case of Theorem 2.1 where l, m are coprime. We expect that the other classical linear cases also recover Theorem 2.1—in type D, possibly after replacing $S_{N,m}$ by an index-2 subgroup.

Fix a prime power q and a (connected, smooth) reductive algebraic group $\mathbf{G}$ over $\bar{\mathbf{F}}_q$, equipped with an $\mathbf{F}_q$-structure corresponding to a Frobenius map $F : \mathbf{G} \to \mathbf{G}$. We say that $G = \mathbf{G}^F$ is a *finite reductive group*. For instance, if $\mathbf{G} = \mathrm{GL}_n$, then different choices of F produce either $G = \mathrm{GL}_n(\mathbf{F}_q)$ or $G = \mathrm{GU}_n(\mathbf{F}_q)$.

Throughout, we use **boldface** uppercase letters for spaces over $\bar{\mathbf{F}}_q$, and ordinary *italics* for their loci of F-fixed points.

4.2. By the work of Deligne and Lusztig, the irreducible representations of G can all be obtained from certain induction maps

$$R_{\mathbf{M}}^{\mathbf{G}} : \mathsf{K}_0(M) \to \mathsf{K}_0(G), \quad \text{where } \mathbf{M} \text{ runs over } F\text{-stable Levi subgroups of } \mathbf{G},$$

and $M = \mathbf{M}^F$. In more detail, $R_{\mathbf{M}}^{\mathbf{G}}$ arises from commuting actions of G and M on the compactly-supported étale cohomology of algebraic varieties $\mathbf{Y}_{\mathbf{P}}^{\mathbf{G}}$ over $\bar{\mathbf{F}}_q$, now called *Deligne–Lusztig varieties*. The precise definition depends on a choice of parabolic subgroup $\mathbf{P} \subseteq \mathbf{G}$ containing $\mathbf{M}$. For $\mathbf{F}_q$ of sufficiently large characteristic, the map $R_{\mathbf{M}}^{\mathbf{G}}$ is independent of $\mathbf{P}$.

In fact, to construct the irreducibles of G, it suffices to run over maximal tori, not Levis. An irreducible character is *unipotent* if and only if it occurs in $R_{\mathbf{T}}^{\mathbf{G}}(1)$ for some maximal torus $\mathbf{T}$, where 1 is the trivial character. Following Lusztig, let

$$\mathrm{Uch}(G) = \{\text{unipotent irreducible characters of } G\}.$$

Lusztig shows that $\mathrm{Uch}(G)$ can be indexed in a way independent of q, depending only on the Weyl group W of $\mathbf{G}$ itself [Lus78, Lus84].

4.3. An irreducible character $\mu \in \mathrm{Irr}(M)$ is *cuspidal* if and only if it does not occur in $R_{\mathbf{L}}^{\mathbf{M}}(\lambda)$ for some strictly smaller F-stable Levi $\mathbf{L} \subseteq \mathbf{M}$ and irreducible $\lambda \in \mathrm{Irr}(L)$. For a unipotent cuspidal $\mu \in \mathrm{Uch}(M)$, we define the *Harish-Chandra series* associated with (M, μ) to be the set

$$\mathrm{Uch}(G)_{M,\mu} = \{\rho \in \mathrm{Uch}(G) \mid (\rho, R_{\mathbf{M}}^{\mathbf{G}}(\mu))_G \neq 0\}.$$

Harish-Chandra observed that the sets $\mathrm{Uch}(G)_{M,\mu}$ are pairwise disjoint and partition $\mathrm{Uch}(G)$ as we run over $\mathbf{G}$-conjugacy classes of such *cuspidal pairs* (M, μ) for which the underlying Levi $\mathbf{M} \subseteq \mathbf{G}$ is F-maximally split. This last condition means that $\mathbf{M} = Z_{\mathbf{G}}(\mathbf{T})^\circ$ for some F-stable torus $\mathbf{T}$, maximally split over $\mathbf{F}_q$.

Motivated by ideas from the ℓ-modular representation theory of G at large primes ℓ, Broué–Malle–Michel discovered that Harish–Chandra's theory is the $m = 1$ case of a theory that exists for any positive integer m [BMM93]. (To obtain a relation between ℓ-modular representation theory and what follows, one takes m to be the multiplicative order of q modulo ℓ.) To state their results cleanly, they introduce the notion of a *generic finite reductive group* that interpolates the groups G as we keep the root datum and its Frobenius automorphism fixed, but vary the prime power q. For simplicity, we will avoid this formalism in our discussion below.

Let Φ_m denote the mth cyclotomic polynomial. It defines a class of F-stable tori $\mathbf{T} \subseteq \mathbf{G}$, not necessarily maximal, called *Φ_m-tori*: The corresponding groups $T \subseteq G$ are characterized by the property that $|T|$ is a power of $\Phi_m(q)$.

An F-stable Levi subgroup $\mathbf{M} \subseteq \mathbf{G}$ is called *Φ_m-split* if and only if it takes the form $\mathbf{M} = Z_{\mathbf{G}}(\mathbf{T})^\circ$ for some Φ_m-torus $\mathbf{T}$. For such $\mathbf{M}$, we say that $\mu \in \mathrm{Uch}(M)$ is *Φ_m-cuspidal* if and only if it does not occur in $R_{\mathbf{L}}^{\mathbf{M}}(\lambda)$ for any smaller Φ_m-split Levi $\mathbf{L}$ and $\lambda \in \mathrm{Uch}(L)$. In this case, we say that (M, μ) is a *Φ_m-cuspidal pair*. By taking $m = 1$, we recover the usual notions of maximally split tori, maximally split Levis, and cuspidal pairs.

Note that G acts by conjugation on the set of Φ_m-cuspidal pairs. The following result is essentially part (1) of Theorem 3.2 in [BMM93]:

Theorem 4.1 (Broué–Malle–Michel)**.** *For any fixed integer $m > 0$, the Harish-Chandra series $\mathrm{Uch}(G)_{M,\mu}$ are disjoint and partition $\mathrm{Uch}(G)$, as (M, μ) runs over a full set of representatives for the conjugacy classes of Φ_m-cuspidal pairs.*

4.4. In the classical case where $m = 1$, Lusztig observed [Lus77, Lus78] that each Harish-Chandra series $\mathrm{Uch}(G)_{M,\mu}$ has a parametrization of the form

$$\chi_{M,\mu} : \mathrm{Uch}(G)_{M,\mu} \xrightarrow{\sim} \mathrm{Irr}(W_{M,\mu}^G),$$

where $W_{M,\mu}^G$ is the stabilizer of μ under the action of $W_M^G := N_G(M)/M$. We say that W_M^G, *resp.* $W_{M,\mu}^G$, is the *relative Weyl group* of M, *resp.* (M, μ), in G. These form real reflection groups; for most choices of μ, they coincide.

The bijection $\chi_{M,\mu}$ arises from comparing the G-commutant of the cohomology of $Y_{\mathbf{P}}^{\mathbf{G}}$ to the group algebra of $W_{M,\mu}^G$. More precisely, Lusztig shows that there is a $\bar{\mathbf{Q}}[x^{\pm 1/\infty}]$-algebra $H_{M,\mu}^G(x)$, obtained from the algebra $H_{W_{M,\mu}^G}(\vec{x})$ discussed earlier by extending scalars and specializing $\vec{x}$, such that for generic $\xi \in \mathbf{C}^\times$, we have

$$\mathbf{C} \otimes_{\bar{\mathbf{Q}}[\vec{x}^{\pm 1/\infty}]} H_{M,\mu}^G(\xi) \simeq \mathbf{C}W_{M,\mu}^G, \quad \text{where } x \mapsto \xi, \tag{4.1}$$

but at the same time,

$$\bar{\mathbf{Q}}_\ell \otimes_{\bar{\mathbf{Q}}[\vec{x}^{\pm 1/\infty}]} H_{M,\mu}^G(x) \simeq \mathrm{End}_G(\mathrm{H}_c^*(Y_{\mathbf{P}}^{\mathbf{G}}, \bar{\mathbf{Q}}_\ell)), \quad \text{where } x \mapsto q. \tag{4.2}$$

Then $\chi_{M,\mu}$ arises from a composition of isomorphisms

$$\mathsf{K}_0(G) \xrightarrow{\sim} \mathsf{K}_0(H_{M,\mu}^G(q)) \xleftarrow{\sim} \mathsf{K}_0(H_{M,\mu}^G(x)) \xrightarrow{\sim} \mathsf{K}_0(W_{M,\mu}^G),$$

where the first arrow arises from the double-centralizer theorem and the latter two arise from Tits deformation. In fact, these isomorphisms are isometries with respect to natural multiplicity pairings.

Lusztig observed similar results in the cases where the underlying Levi is not F-maximally split, but a maximal torus of Coxeter type [Lus76]. This condition implies that it is Φ_h-split for some integer h, called the (twisted) Coxeter number of G. In these cases, the relative Weyl group is cyclic of order h.

Broué–Malle observed that for general m, the relative Weyl groups arising from Φ_m-split Levis and Φ_m-cuspidal pairs are always *complex* reflection groups [BM93]. Generalizing Lusztig's results, they define an algebra $H^G_{M,\mu}(x)$ for any G, m, M, μ, using case-by-case formulas, such that (4.1) still holds, and various numerical properties of $H^G_{M,\mu}(x)$ are consistent with (4.2).

Conjecture 4.2 (Broué–Malle)**.** (4.2) *holds for all* G, m, M, μ*.*

Although few cases of this conjecture are known beyond the ones that Lusztig established, Broué–Malle–Michel were still able to generalize his parametrizations $\chi_{M,\mu}$ to all G, m, M, μ, by direct and case-by-case constructions. The following result is part (2) of Theorem 3.2 in [BMM93].

Theorem 4.3 (Broué–Malle–Michel)**.** *For any fixed integer* $m > 0$ *and* m*-cuspidal pair* (M, μ)*, there is a map*

$$(\varepsilon_{M,\mu}, \chi_{M,\mu}) : \mathrm{Uch}(G)_{M,\mu} \to \{\pm 1\} \times \mathrm{Irr}(W^G_{M,\mu})$$

such that $\varepsilon_{M,\mu}\chi_{M,\mu} : \mathbf{Z}\mathrm{Uch}(G)_{M,\mu} \to \mathsf{K}_0(W^G_{M,\mu})$ *defines an isometry with respect to the natural multiplicity pairings. In particular,* $\chi_{M,\mu}$ *is bijective. Moreover:*

(1) *These maps are compatible with inclusions of* m*-split Levis. They intertwine the maps* R^M_L *with the ordinary induction maps between relative Weyl groups.*
(2) *The collection of maps* $(\varepsilon_{M,\mu}, \chi_{M,\mu})$ *is stable under the conjugation action of* G *on* Φ_m*-cuspidal pairs.*

Remark 4.4. Broué–Malle–Michel show that if $\rho \in \mathrm{Uch}(G)$ satisfies $\mathrm{Deg}_\rho(\zeta_m) \neq 0$, then $\rho \in \mathrm{Uch}(G)_{T,1}$, where T represents the conjugacy class of Φ_m-split Levis that are maximal tori, and 1 denotes the trivial character of T. In this case, $\mathrm{Deg}_\rho(\zeta_m) = \varepsilon_{T,1}(\rho) \deg \chi_{T,1}(\rho)$ [Bro01, 76].

5. Finite General Linear Groups

5.1. Take $\mathbf{G} = \mathrm{GL}_n$ under the standard Frobenius map, so that $G = \mathrm{GL}_n(\mathbf{F}_q)$. Then there is the following dictionary between the notions from §1 and the notions we have just introduced.

First, recall that every unipotent irreducible character of $\mathrm{GL}_n(\mathbf{F}_q)$ arises from a principal series attached to an irreducible character of its Weyl group S_n. Restated in the language of Harish-Chandra series: For any maximally split maximal torus $\mathbf{A} \subseteq \mathbf{G}$, we have $\mathrm{Uch}(G)_{A,1} = \mathrm{Uch}(G)$ and $W_{A,1}^G \simeq S_n$. At the same time, $\mathrm{Irr}(S_n)$ is indexed by partitions of size n. Thus, we may regard $\chi_{A,1}$ as a bijection:

$$\mathrm{Uch}(G) \simeq \{\rho \in \Pi \mid |\rho| = n\}. \tag{5.1}$$

Henceforth, we abuse notation by writing ρ to denote both characters of G and the corresponding partitions.

Next, for any $m \geq 1$, it turns out that every Φ_m-split Levi subgroup of $\mathbf{G}$ takes the form $\mathbf{M} \simeq \mathrm{GL}_{n-mN} \times \mathbf{S}_m^N$, where $S \simeq \mathbf{F}_{q^m}^\times$. For such $\mathbf{M}$, the analogue of (5.1) restricts to a bijection

$$\{m\text{-cuspidals } \mu \in \mathrm{Uch}(M)\} \simeq \{\mu \in \Pi_{m\text{-}cor} \mid |\mu| = n - mN\}.$$

Henceforth, we abuse notation by writing μ to denote both cuspidal characters of M and the corresponding m-core partitions. For a fixed μ, (5.1) restricts to a bijection

$$\mathrm{Uch}(G)_{M,\mu} \simeq \{\rho \in \Pi \mid |\rho| = n \text{ and } m\text{-core}(\rho) = \mu\}. \tag{5.2}$$

At the same time, it turns out that $W_{M,\mu}^G \simeq W_M^G \simeq S_{N,m}$, giving a bijection

$$\mathrm{Irr}(W_{M,\mu}^G) \simeq \{\vec{\pi} \in \Pi^m \mid |\vec{\pi}| = N\}.$$

Under these identifications, $\chi_{M,\mu} : \mathrm{Uch}(G)_{M,\mu} \xrightarrow{\sim} \mathrm{Irr}(W_{M,\mu}^G)$ is essentially the map sending a partition to its m-quotient. To be precise: $\chi_{M,\mu}(\rho)$ is the "shifted" m-quotient $\Upsilon_m(\rho, \ell_\mu)$, where ℓ_μ denotes the length of μ as a partition.

5.2. We can now provide the explicit definition of the Broué–Malle algebra $H_{M,\mu}^G(x)$ for $G = \mathrm{GL}_n(\mathbf{F}_q)$. Recall that if $W_{M,\mu}^G \simeq S_{N,m}$ for some N, then $H_{M,\mu}^G(x)$ is obtained from $H_{W_{M,\mu}^G}(\vec{x})$ by extending scalars to $\bar{\mathbf{Q}}[x^{\pm 1/\infty}]$ and specializing $\vec{x}$.

By the core-quotient bijection, the $\mathbf{Z}^m$-component of $\Upsilon_m(\rho, \ell_\mu)$ is the same vector $\vec{b}_m(\mu)$ for all $\rho \in \Pi^m$. In the notation of §2, the map $H_{S_{N,m}}(\vec{x}) \to H_{M,\mu}^G(x)$ sends

$$\begin{cases} x_\sigma \mapsto x^m, \\ x_{\tau,j} \mapsto x^{mb_m^{(j)}(\mu)+j} \text{ for all } j. \end{cases} \tag{5.3}$$

Note that we always have $b_m^{(0)}(\mu) = 0$.

6. Conjectures and Results

6.1. We return to the general setting of Section 4. We define *$H_{M,\mu}^G(\xi)$-blocks* of $\mathrm{Irr}(W_{M,\mu}^G)$ similarly to the $\vec{\xi}$-blocks in §2, but through a composition of the form

$$\mathrm{Irr}(W_{M,\mu}^G) \to \mathsf{K}_0^+(H_{W_{M,\mu}^G}^\circ(\vec{x})) \xrightarrow{\mathrm{d}_\xi} \mathsf{K}_0^+(H_{M,\mu}^G(\xi)),$$

where d_ξ is induced by $H_{W^G_{M,\mu}}(\vec{x}) \to H^G_{M,\mu}(x) \to H^G_{M,\mu}(\xi)$.

Keeping $q, \mathbf{G}, F$, we now consider a Φ_l-cuspidal pair (L, λ) and a Φ_m-cuspidal pair (M, μ), for separate positive integers l and m. Fix a primitive lth root of unity ζ_l and a primitive mth root of unity ζ_m. Our main conjecture is:

Conjecture 6.1. *Let* $\mathrm{Irr}(W^G_{L,\lambda})_{M,\mu}$ *and* $\mathrm{Irr}(W^G_{M,\mu})_{L,\lambda}$ *be the images of the maps*

$$\mathrm{Irr}(W^G_{L,\lambda}) \xleftarrow{\chi_{L,\lambda}} \mathrm{Uch}(G)_{L,\lambda} \cap \mathrm{Uch}(G)_{M,\mu} \xrightarrow{\chi_{M,\mu}} \mathrm{Irr}(W^G_{M,\mu}).$$

Then:

(1) $\mathrm{Irr}(W^G_{L,\lambda})_{M,\mu}$ *and* $\mathrm{Irr}(W^G_{M,\mu})_{L,\lambda}$ *are unions of* $H^G_{L,\lambda}(\zeta_m)$*- and* $H^G_{M,\mu}(\zeta_l)$*-blocks, respectively.*

(2) *The bijection* $\mathrm{Irr}(W^G_{L,\lambda})_{M,\mu} \xrightarrow{\sim} \mathrm{Irr}(W^G_{M,\mu})_{L,\lambda}$ *induced by* $\chi_{L,\lambda}$ *and* $\chi_{M,\mu}$ *respects blocks. That is, it descends to a bijection* $\chi^{L,\lambda}_{M,\mu}$ *of the form*

$$\{H^G_{L,\lambda}(\zeta_m)\text{-blocks } \mathbf{a} \subseteq \mathrm{Irr}(W^G_{L,\lambda})_{M,\mu}\} \xrightarrow{\sim} \{H^G_{M,\mu}(\zeta_l)\text{-blocks } \mathbf{b} \subseteq \mathrm{Irr}(W^G_{M,\mu})_{L,\lambda}\}.$$

(3) *If* $\chi^{L,\lambda}_{M,\mu}(\mathbf{a}) = \mathbf{b}$, *then the bijection* $\mathbf{a} \xrightarrow{\sim} \mathbf{b}$ *is categorified by a bounded derived equivalence between appropriate highest-weight covers: that is, an equivalence*

$$\mathsf{D}^b(\mathsf{O}_{W^G_{L,\lambda}}(\vec{\nu}_l)_{\mathbf{a}}) \simeq \mathsf{D}^b(\mathsf{O}_{W^G_{M,\mu}}(\vec{\nu}_m)_{\mathbf{b}})$$

for any vectors $\vec{\nu}_l, \vec{\nu}_m$ *related to* $\mathbf{a}, \mathbf{b}$ *by certain numerical conditions.*

6.2. Our main theorem proves Conjecture 6.1 for the groups $G = \mathrm{GL}(\mathbf{F}_q)$ and coprime l, m, via Theorem 2.1. To bridge the two settings, we need two further observations.

First, from comparing (2.1) to (5.3), we see that $H_{N,m}(\eta_l, \eta_l^{\vec{s}})$ takes the form $H^G_{M,\mu}(\zeta_l)$ if and only if

$$\begin{aligned} \eta_l &= \zeta_l^m, \\ m\vec{s} &\equiv m\vec{b}_m(\mu) + (0, 1, \ldots, m-1) \pmod{l}. \end{aligned}$$

Henceforth, we assume that l, m are coprime, so that ζ_l^m remains a primitive lth root of unity and ζ_m^l remains a primitive mth root of unity.

Second, observe that the affine symmetric group $\mathbf{Z}^m \rtimes S_m$ acts on the set of m-charges. There is a corresponding action on the set of possible parameters $\vec{\nu}$ for the rational Cherednik algebra, such that if $\vec{\nu}$ maps to $(\eta_l, \eta_l^{\vec{s}}) = (\zeta_l^m, \zeta_l^{m\vec{s}})$ under (3.1), then $w \cdot \vec{\nu}$ maps to $(\eta_l, \eta_l^{\bar{w}\cdot\vec{s}}) = (\zeta_l^m, \zeta_l^{m(\bar{w}\cdot\vec{s})})$ under (3.1), for any $w \in \mathbf{Z}^m \rtimes S_m$ with finite part $\bar{w} \in S_m$. Rouquier conjectured, and Gordon–Losev [GL14, §5.13–5.14] and Webster [Web17, Thm. B(3)] independently proved, equivalences of the form $\mathsf{D}^b(\mathsf{O}_{S_{N,m}}(\vec{\nu})) \xrightarrow{\sim} \mathsf{D}^b(\mathsf{O}_{S_{N,m}}(w \cdot \vec{\nu}))$. As we run over w, these functors define a strong categorical action of the affine braid group on the set of such categories.

The proof of the following theorem relies on the abacus combinatorics discussed in §1, as well as the classification of $\vec{\xi}$-blocks due to Lyle–Mathas.

Theorem 6.2. *Suppose that $\mathbf{G} = \mathrm{GL}_n$ (under the standard Frobenius) and ℓ, m are coprime. Then all three parts of Conjecture 6.1 hold. Moreover:*

(1) *$\mathrm{Irr}(W_{L,\lambda}^G)_{M,\mu}$ and $\mathrm{Irr}(W_{M,\mu}^G)_{L,\lambda}$ are single $H_{L,\lambda}^G(\zeta_m)$- and $H_{M,\mu}^G(\zeta_l)$-blocks, respectively.*

(2) *The bijections in part (2) of Conjecture 6.1 are essentially the bijections Υ_m^l, in that there are affine permutations $w_l \in S_l \ltimes \mathbf{Z}^l$ and $w_m \in S_m \ltimes \mathbf{Z}^m$ giving a commutative diagram:*

$$\begin{tikzcd}
\mathrm{Irr}(W_{L,\lambda}^G) \arrow[d] & \mathrm{Uch}(G)_{L,\lambda} \cap \mathrm{Uch}(G)_{M,\mu} \arrow[l, "\chi_{L,\lambda}"'] \arrow[r, "\chi_{M,\mu}"] \arrow[d] & \mathrm{Irr}(W_{M,\mu}^G) \arrow[d] \\
\Pi^l \times \mathbf{Z}^l \arrow[d, "w_l"'] & \Pi \arrow[l, "\Upsilon_l(\rho,\ell_\lambda) \mapsfrom \rho"'] \arrow[r, "\rho \mapsto \Upsilon_m(\rho,\ell_\mu)"] & \Pi^m \times \mathbf{Z}^m \arrow[d, "w_m"] \\
\Pi^l \times \mathbf{Z}^l \arrow[rr, "\Upsilon_m^l"] & & \Pi^m \times \mathbf{Z}^m
\end{tikzcd}$$

Above, ℓ_λ, ℓ_μ refer to the lengths of λ and μ as partitions.

(3) *We can choose w_m so that*

$$m(w_m \cdot \vec{b}_m(\mu)) \equiv m\vec{b}_m(\mu) + (0, 1, \ldots, m-1) \pmod{l},$$

and similarly with l, m, λ, w_l in place of m, l, μ, w_m.

Corollary 6.3. *Conjecture 6.1 holds when $G = \mathrm{GL}_n(\mathbf{F}_q)$ and ℓ, m are coprime.*

We expect that, up to modifiying both parts of Theorem 6.2, we can remove the coprimality hypothesis from the corollary.

Moreover, we have checked in almost all cases where $\mathbf{G}$ has exceptional type that Conjecture 6.1(1) is compatible with the sizes of the sets $\mathrm{Uch}(G)_{L,\lambda} \cap \mathrm{Uch}(G)_{M,\mu}$ and the sizes of the blocks contained in $\mathrm{Irr}(W_{L,\lambda}^G)_{M,\mu}$ and $\mathrm{Irr}(W_{M,\mu}^G)_{L,\lambda}$, where these are known or can be deduced.

Example 6.4. Let $G = \mathrm{GL}_8(\mathbf{F}_q)$ and $l = 4$ and $m = 3$. Let $\lambda = (2^2)$, a 4-core, and $\mu = (2)$, a 3-core. Then (5.2) gives

$$\begin{aligned}
\mathrm{Uch}(G)_{L,\lambda} &\simeq \{(6,2), (5,3), (2^3,1^2), (2^2,1^4)\},\\
\mathrm{Uch}(G)_{M,\mu} &\simeq \{(8), (5,3), (5,2,1), (5,1^3), (4,3,1), (3^2,1^2), (2^4), (2^2,1^4), (2,1^6)\},
\end{aligned}$$

from which $\mathrm{Uch}(G)_{L,\lambda} \cap \mathrm{Uch}(G)_{M,\mu} \simeq \{(5,3), (2^2,1^4)\}$.

Let $\pi = (5,3)$. The abaci that represent $\Upsilon_l(\pi, \ell_\lambda) = \Upsilon_4((5,3),2)$ and $\Upsilon_m(\pi, \ell_\mu) = \Upsilon_3((5,3),1)$ are below.

$$\begin{array}{cccccc} \cdots & -12 & -8 & -4 & - & - \\ \cdots & -11 & -7 & -3 & - & - \\ \cdots & -10 & -6 & -2 & - & 6 \\ \cdots & -9 & -5 & -1 & 3 & \end{array} \qquad \begin{array}{ccccccc} \cdots & -12 & -9 & -6 & -3 & - & - \\ \cdots & -11 & -8 & -5 & -2 & - & - \\ \cdots & -10 & -7 & -4 & - & 2 & 5 \end{array}$$

Let $w_4 = (1,0,0,-1) \circ \bar{w}_4$, where $\bar{w}_4 = [1032] \in S_4$. Let $w_3 = (2,0,-1) \circ \bar{w}_3$, where $\bar{w}_3 = [201] \in S_3$. Then w_3 satisfies the congruence in Theorem 6.2(3), and similarly with w_4. Under these affine permutations, the abaci above become the abaci in Examples 1.1–1.2.

$$\begin{array}{ccccc} \cdots & -12 & -8 & -4 & 0 \\ \cdots & -11 & -7 & -3 & - \\ \cdots & -10 & -6 & -2 & 2 \\ \cdots & -9 & -5 & - & 3 \end{array} \qquad \begin{array}{ccccccccc} \cdots & -12 & -9 & -6 & -3 & 0 & - & 6 & 9 \\ \cdots & -11 & -8 & -5 & -2 & - & - & - & \\ \cdots & -10 & -7 & -4 & - & - & - & - & \end{array}$$

We saw in Example 1.2 that Υ_4^3 transforms the left abacus into the right one.

6.3. **Postscript.** An important theme that our discussion has omitted: When $\mathbf{L}, \mathbf{M}$ are maximal tori, we expect novel *geometric* incarnations of the duality in Conjecture 6.1, via bimodules constructed from the cohomology of algebraic varieties. They suggest analogies between these varieties and certain F-twisted Steinberg varieties related to Deligne–Lusztig theory, but with roots of unity replacing the prime power q.

One family of incarnations is of Dolbeault nature: The (ind-)varieties are *homogeneous affine Springer fibers* that first appeared in work of Lusztig–Smelt, and more recently, in work of Varagnolo–Vasserot [VV09], Oblomkov–Yun [OY16], and Boixeda-Alvarez–Losev–Kivinen [BAL24] on the geometric representation theory of (degenerate) double affine Hecke algebras. Another family is of Betti nature: The varieties are *braid Steinberg varieties* that first appeared in work of Trinh on triply-graded link homology [Tri21].

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