Last time — the fundamental group of X at x is

$$\pi_1(X, x) = \{\text{loops in } X \text{ based at } x\}/\sim_p$$

under $[\beta] * [\gamma] := [\beta * \gamma]$

identity elt is $[e_x]$, where e_x is the constant loop

Ex — $\pi_1(\mathbb{R}, x)$ is trivial for any x in $\mathbb{R}$

indeed: for any loop γ at x, have $\gamma \sim_p e_x$ via
$\varphi(s, t) = (1 - t)\gamma(s) + tx$

Prob — in general:
if $X \subseteq \mathbb{R}^n$ is star convex,
then $\pi_1(X, x)$ is trivial

Ex — the path $\omega_n : [0, 1] \to S^1$ def by

$$\omega_n(s) = (\cos 2\pi ns, \sin 2\pi ns)$$

is a loop based at $q := (1, 0)$

what is ω_0? [just e_q]

Thm — the map $\Phi : \mathbb{Z} \to \pi_1(S^1, q)$ def by
$\Phi(n) = [\omega_n]$ is an iso of groups

means:

- $\omega_{m + n} \sim_p \omega_m * \omega_n$ for all m, n
- every loop at q is $\sim_p \omega_n$ for some n
- if $m \neq n$, then $\omega_m \not\sim_p \omega_n$

first, more basic properties of π_1:

<u>Thm</u> if α is a path in X from x to x', and $\bar{\alpha}(s) = \alpha(1 - s)$ is its <u>reverse</u>, then there is an iso

$$\hat{\alpha} : \pi_1(X, x) \to \pi_1(X, x')$$

def by $\hat{\alpha}([\gamma]) = [\bar{\alpha} * \gamma * \alpha]$ [draw]

<u>Cor</u> the iso class of $\pi_1(X, x)$ only depends on the path component of X containing x

<u>Pf of Thm</u> <u>Lem</u> [draw]

1) $\alpha * \bar{\alpha} \sim_p e_x$
2) $\bar{\alpha} * \alpha \sim_p e_{x'}$

thus,

$$\begin{aligned}
\hat{\alpha}([\beta * \gamma]) &= [\bar{\alpha} * \beta * \gamma * \alpha] \\
&= [\bar{\alpha} * \beta * \alpha * \bar{\alpha} * \gamma * \alpha] \\
&= [\bar{\alpha} * \beta * \alpha] * [\bar{\alpha} * \gamma * \alpha] \\
&= \hat{\alpha}([\beta]) * \hat{\alpha}([\gamma])
\end{aligned}$$

so $\hat{\alpha}$ is a homomorphism

also,

$$\hat{\bar{\alpha}} \circ \hat{\alpha} = \mathrm{id}_{\pi_1(X, x)}$$
$$\hat{\alpha} \circ \hat{\bar{\alpha}} = \mathrm{id}_{\pi_1(X, x')}$$

so $\hat{\alpha}$ is bijective

<u>Thm</u> if $f : X \to Y$ is a cts map, then there is a hom

$f_* : \pi_1(X, x)$ to $\pi_1(Y, f(x))$

def by $f_*([\gamma]) = [f \circ \gamma]$ [draw]

[spaces to groups, cts maps to group homs]

Rem must check that f_* is well-defined!

$\gamma \sim_p \gamma'$, implies $f \circ \gamma \sim_p f \circ \gamma'$ [by earlier prob]
so $[\gamma] = [\gamma']$ implies $f_*([\gamma]) = f_*([\gamma'])$

Pf Lem for any paths β, γ in X
s.t. $\beta(1) = \gamma(0)$,
and cts $f : X$ to Y,

$f \circ (\beta * \gamma) = (f \circ \beta) * (f \circ \gamma)$ as paths in Y

thus, $[f \circ (\beta * \gamma)] = [f \circ \beta] * [f \circ \gamma]$

whence $f_*([\beta * \gamma]) = f_*([\beta]) * f_*([\gamma])$

Ex for any m in $\mathbb{Z}$, have a map

$p_m : S^1$ to S^1

def in a polar coord θ by $p_m(\theta) = m\theta$

what is $p_{m, *} : \pi_1(X, q)$ to $\pi_1(X, q)$?

$$\begin{array}{lccc}
 & & \Phi & \\
 & Z & \simeq & \pi_1(S^1, q) \\
p_{m, *} & \text{to} & & \text{to} \qquad\qquad n \mapsto mn \\
 & Z & \simeq & \pi_1(S^1, q)
\end{array}$$

Thm if $f : X \to Y$ and $g : Y \to X$ form a homotopy equivalence

then $f_* : \pi_1(X, x) \to \pi_1(Y, f(x))$
$g_* : \pi_1(Y, y) \to \pi_1(Y, g(y))$

are isomorphisms for all x in X and y in Y

Cor S^1 is not homotopy equivalent to any star convex subset of R^n, for any n

[proof is harder]