Skein Relations from Quantum Mechanics

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Themes of this talk:

- Solving problems in quantum mechanics = studying relations among linear operators.
- The algebras generated by these operators can be abstracted to other settings.
- A particular algebra governing quantum angular momentum also shows up in knot theory.
- Not a coincidence: Representation theory predicts a hierarchy of algebras of broad importance.

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1 Quantum Mechanics

classical

An observable is a *function* $f : \mathcal{M} \to \mathbf{R}$ on a state space $\mathcal{M}$.

A measurement in $E \subseteq \mathbf{R}$ lets us infer a state in $f^{-1}(E) \subseteq \mathcal{M}$.

quantum

A state is a line in a Hilbert space $\mathcal{H}$.

An observable is a *projection-valued measure*. It assigns a projection $\pi_E : \mathcal{H} \to \mathcal{H}$ to each $E \subseteq \mathbf{R}$.

The probability of a measurement in E is

$$\langle \varphi, \pi_E(\varphi) \rangle, \quad \text{for a state with unit vector } \varphi \in \mathcal{H}.$$

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The expectation of the observable, given φ, is

$$\langle \varphi, J(\varphi) \rangle, \quad \text{where } J(\varphi) = \int_{\mathbf{R}} \lambda \, d\pi_\lambda(\varphi).$$

We often say that $J : \mathcal{H} \to \mathcal{H}$ "is" the observable.

We'll focus on (total) *quantum angular momentum*:

$$J_x, \quad J_y, \quad J_z.$$

In experiment, the product of the variances of the observables has a strictly positive lower bound.

Heisenberg (∼1925) Can *derive* this mathematically from the identities

$$[J_x, J_y] = i\hbar J_z, \quad [J_y, J_z] = i\hbar J_x, \quad [J_z, J_x] = i\hbar J_y,$$

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Let $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$:

$$[\sigma_x, \sigma_y] = 2i\sigma_z, \quad [\sigma_y, \sigma_z] = 2i\sigma_x, \quad [\sigma_z, \sigma_x] = 2i\sigma_y.$$

The actions $J_x, J_y, J_z \curvearrowright \mathcal{H}$ define a *representation* of the Lie algebra

$$\mathfrak{sl}_2 = \mathbf{C}\sigma_x + \mathbf{C}\sigma_y + \mathbf{C}\sigma_z \quad \subseteq \quad \mathrm{Mat}_2(\mathbf{C}).$$

Classic example where the algebra underlying QM has broader importance.

Our main topic is a fancier, more modern example, arising from *coupled* momenta.

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2 The Algebra of Coupled Momenta

The $\mathfrak{sl}_2$-action puts a lot of structure on $\mathcal{H}$.

The action must respect a direct-sum decomposition

$$\mathcal{H} = \bigoplus_{s=0,\frac{1}{2},1,\frac{3}{2},\dots} V_s^{\oplus m_s}, \quad \text{where } \dim V_s = 2s+1.$$

Above, s is called the *spin number* of V_s.

Elementary particles have fixed spin numbers.

A system of particles with spins $s_1, s_2, \dots$ has a state space given by a tensor product:

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two-body problem

V_c occurs in $V_a \otimes V_b$ if and only if a, b, c form the sides of a triangle and $a + b + c \in \mathbf{Z}$.

In this case, the embedding is unique up to scaling.

$\implies$ $\mathrm{Hom}_{\mathfrak{sl}_2}(V_c, V_a \otimes V_b)$ is one-dimensional.

three-body problem

V_d can occur in $V_a \otimes V_b \otimes V_c$ more than once.

$\mathrm{Hom}(V_d, V_a \otimes V_b \otimes V_c)$ has <u>two</u> bases $(\Phi_e)_e, (\Psi_f)_f$:

$$\mathbf{C}\Phi_e = \mathrm{Hom}(V_d, V_e \otimes V_c) \otimes \mathrm{Hom}(V_e, V_a \otimes V_b),$$
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The *6j symbols* are the entries of the change-of-basis matrix from $(\Phi_e)_e$ to $(\Psi_f)_f$:

$$\begin{Bmatrix} a & b & e \\ c & d & f \end{Bmatrix} \in \mathbf{C}.$$

Using self-duality of V_a, *etc.*, we can show that the symbol is invariant under permutations of a, b, c, d.

Regge (1958) A more surprising symmetry

$$\begin{Bmatrix} a & b & e \\ c & d & f \end{Bmatrix} = \begin{Bmatrix} p-a & p-b & e \\ p-c & p-d & f \end{Bmatrix},$$

where $p = \frac{a+b+c+d}{2}$.

Granovskii–Zhedanov (1988) Conceptual proof of these symmetries via a new operator algebra.

The *$6j$ symbols* are the entries of the change-of-basis matrix from $(\Phi_e)_e$ to $(\Psi_f)_f$:

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On any $\mathcal{H}$, have $J^2 := J_x^2 + J_y^2 + J_z^2$ commuting with J_x, J_y, J_z.

Any nonzero vector in V_s is an eigenvector of J^2 with eigenvalue $\hbar^2 s(s+1)$. Thus, J^2 distinguishes spins.

The $6j$ symbols arise from two ways to parenthesize:

$$\begin{aligned} V_e \otimes V_c &\to (V_a \otimes V_b) \otimes V_c, \\ V_a \otimes V_f &\to V_a \otimes (V_b \otimes V_c). \end{aligned}$$

From $J_{12}^2 \curvearrowright V_a \otimes V_b$ and $J_{23}^2 \curvearrowright V_b \otimes V_c$, we form

$$J_{12}^2 \otimes 1, \quad 1 \otimes J_{23}^2 \quad \curvearrowright \quad V_a \otimes V_b \otimes V_c.$$

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3 Skeins We set $K_{13} = [K_{12}, K_{23}]$, where

$$K_{12} = J_{12}^2 \otimes 1, \quad K_{23} = 1 \otimes J_{23}^2.$$

The other commutation relations look like

$$[K_{23}, K_{13}] = 2(\eta_1 + \theta K_{23} - \{K_{12}, K_{23}\} - K_{23}^2),$$
$$[K_{13}, K_{12}] = 2(\eta_2 + \theta K_{12} - \{K_{12}, K_{23}\} - K_{12}^2),$$

where $\{A, B\} = A \circ B + B \circ A$.

Here, η_1, η_2, θ are polynomial functions of a, b, c.

Now consider these relations on abstract *variables*.

Berest–Samuelson (2018) These relations arise in knot theory, from Kauffman's construction of the *Jones polynomial*.

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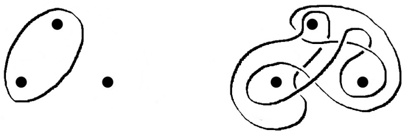

The *Kauffman skein module* of Σ is

$$\mathrm{Sk}_\Sigma(q) = \frac{\mathbf{C}[q^{\pm 1}]\langle\text{unoriented link diagrams in } \Sigma\rangle}{(\text{skein relations})}$$
$$= \mathbf{C}[q^{\pm 1}]\langle \Gamma_1, \Gamma_2, \Gamma_3, \Gamma_{12}, \Gamma_{23}, \Gamma_{13}, \Gamma_{123}\rangle$$

where Γ_I is the loop encircling the punctures in I.

We make $\mathrm{Sk}_\Sigma(q)$ into a ring by declaring: $\Gamma \cdot \Gamma'$ is the diagram where we put Γ' on top of Γ.

$\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_{123}$ then belong to the center.

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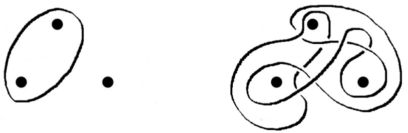

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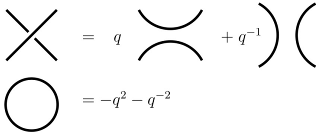

Berest–Samuelson (2018), Fig. 2

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Bullock–Przytycki (1999) $\mathbf{C}(q) \otimes \mathrm{Sk}_\Sigma(q)$ is generated by the elements $\kappa_{ij} := \frac{\Gamma_{ij} - [2]_q}{(q-q^{-1})^2}$ modulo

$$[\kappa_{12}, \kappa_{23}]_q = K_{13},$$
$$[\kappa_{23}, \kappa_{13}]_q = [2]_q(\eta_{1,q} + \theta_q \kappa_{23} - \{\kappa_{12}, \kappa_{23}\} - \kappa_{23}^2),$$
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and one more relation. ($[A, B]_q = qAB - q^{-1}BA$.)

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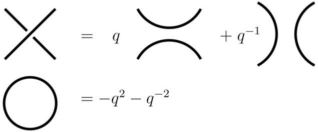

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4 Hecke Algebras

Problem Explain the coincidence of identities.

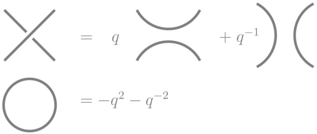
An analogue for *oriented* links should be easier:

$\mathsf{Rep}(\mathfrak{sl}_2)$ has a deformation $\mathsf{Rep}(\mathrm{U}_q(\mathfrak{sl}_2))$ involving a Hopf algebra $\mathrm{U}_q(\mathfrak{sl}_2)$. The swap maps

$$\begin{array}{cc} V \otimes U & v \otimes u \\ \uparrow & \uparrow \\ U \otimes V & u \otimes v \end{array}$$

deform to maps that behave like *braidings*.

Elements in an oriented analogue of $\mathrm{Sk}_\Sigma(q)$ encode diagrams of maps in $\mathsf{Rep}(\mathrm{U}_q(\mathfrak{sl}_2))$.



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An analogue for *oriented* links should be easier:

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Spin in QM is explained by the classification

$$\{\text{finite-dim. irreps of } \mathfrak{sl}_2\} = \{V_s\}_{s=0,\frac{1}{2},1,\frac{3}{2},\ldots}$$

Analogous classifications known for $\mathrm{Sk}(q)$, DAHAs...

Problem <u>Construct</u> the irreps from knot theory, *etc.*

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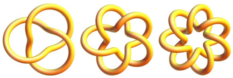
KhR polynomial in $a, q, t \rightsquigarrow$ Jones polynomial in q.



Wolfram, "Torus Knots"

Trinh (2021) Uniform character formula generalizing to torus *links* (and beyond):

$$\sum_{\lambda \vdash n} \mathrm{Deg}_\lambda(e^{2\pi i/n})[\Delta_{m/n}(\chi^\lambda)]_q.$$

Problem Explain it using $\mathrm{Sk}_T(q) \twoheadrightarrow \mathrm{sDAHA}$.

Problem Lift theory from DAHA to $\mathrm{Sk}_T(q), \mathrm{Sk}_\Sigma(q)$.

Thank you for listening.