Thm if $p : E \to X$ is a covering, then:

I) for any path $\gamma : [0, 1] \to X$ and $e \in E$ s.t. $p(e) = \gamma(0)$,

a unique $\Gamma : [0, 1] \to \mathbb{R}$ s.t. $\Gamma(0) = e$ and $\gamma = p \circ \Gamma$, called a lift of γ

II) for any path homotopy $h : [0, 1]^2 \to X$ s.t. $h(–, 0) = \gamma$, and lift Γ of γ,

a unique path-homotopy $H : [0, 1]^2 \to \mathbb{R}$ s.t. $H(–, 0) = \Gamma$ and $h = p \circ H$, called a lift of h

Pf Lemmas 54.1 and 54.2 in Munkres

recall $o = (1, 0) \in S^1$

$\omega_n(s) = (\cos(2\pi ns), \sin(2\pi ns))$ ($n \in \mathbb{Z}$)

Cor $\Phi : \mathbb{Z} \to \pi_1(S^1, o)$ def by $\Phi(n) = [\omega_n]$ is bijective

[previously: Φ is a homomorphism]

Φ surjective for any loop γ at o want n s.t. $[\gamma] = [\omega_n]$

observe: $p^{-1}(o) = \mathbb{Z}$

so for any $a \in \mathbb{Z}$, I) gives Γ s.t.

$\Gamma(0) = a$ and $\gamma = p \circ \Gamma$

let $b = \Gamma(1)$

then $b \in p^{-1}(o) = \mathbb{Z}$

now, by PS6, #2, have $[\Gamma] = [\acute{\omega}_{a, b}]$
so by PS5, #1, have $[\gamma] = [\omega_{b - a}]$

<u>Φ injective</u> suppose that $[\omega_m] = [\omega_n]$

observe: $\acute{\omega}_{0, m}$ lifts ω_m
so for any h from ω_m to ω_n, II) gives H s.t.
$H(–, 0) = \acute{\omega}_{0, m}$ and $h = p \circ H$
let $\Gamma = H(–, 1)$
then Γ lifts $\omega_n = h(–, 1)$

but by I), there is a <u>unique</u> lift of ω_n starting at 0
so $\Gamma = \acute{\omega}_{0, n}$
so H goes from $\acute{\omega}_{0, m}$ to $\acute{\omega}_{0, n}$
so $m = n$

proof that Φ is injective shows in general:

<u>Cor</u> if $p : E$ to X is a covering,
$\gamma_0 \sim_p \gamma_1$ in X,
e in E s.t. $p(e) = \gamma_0(0) = \gamma_1(0)$,

then the unique lifts Γ_0, Γ_1 starting at e satisfy
$\Gamma_0 \sim_p \Gamma_1$

<u>Cor</u> if $p : E$ to X is a covering,
$p(e) = x$,

then $p_* : \pi_1(E, e)$ to $\pi_1(X, x)$ is injective

<u>Pf</u> let Γ_0, Γ_1 be loops in E based at e s.t.
$p_*([\Gamma_0]) = p_*([\Gamma_1])$

since $p_*(\Gamma_i) = [p \circ \Gamma_i]$ and Γ_i lifts $p \circ \Gamma_i$,
we require $[\Gamma_0] = [\Gamma_1]$

<u>Ex</u> recall the covering $p_n : S^1$ to S^1

under $\pi_1(S^1) \simeq \mathbb{Z}$, we have $\mathrm{im}((p_n)_*) \simeq n\mathbb{Z}$

(Munkres §79)

<u>Df</u> a <u>pointed covering</u> of X is a pair (p, e)
s.t. $p : E$ to X is a path-conn. covering,
e in E

a <u>pointed equivalence</u> from (p, e) to (p', e') is
a homeo $f : (E, e)$ to (E', e') s.t. $p = p' \circ f$ [draw]

<u>Thm</u> (Galois Correspondence)

if X is conn. & locally simply-conn., then

$$(p : E \to X, e) \mapsto p_*(\pi_1(E, e))$$

is a bijection

{pointed coverings of X}/~
to
{subgroups of $\pi_1(X, x)$}

leads to topological analogue of Galois theory

<u>Rem</u> if $E = E'$ but $e \neq e'$
then (p, e), (p', e') may not be equivalent