MATH 250: TOPOLOGY I PROBLEM SET #4

FALL 2025

Due Friday, October 31. Please attempt all of the problems. Six of them will be graded. You may consult books, papers, and websites as long as you cite all sources and write up your solutions in your own words.

Problem 1 (Munkres 157, #1(a))**.** Show that no two of the spaces

$$(0,1), \quad (0,1], \quad [0,1]$$

are homeomorphic. *Hint:* What happens if you remove any point from $(0,1)$?

Problem 2 (Munkres 158, #3)**.** Let $f : X \to X$ be continuous. Show that:

(1) If $X = [0,1]$, then f has a *fixed point*: that is, a point $x \in X$ such that $f(x) = x$. *Hint:* Intermediate Value Theorem.
(2) If $X = [0,1)$, then the analogue of (1) fails.

Problem 3 (Munkres 162, #4)**.** Show that if X is locally path connected, then every connected open subset of X is path connected. *Hint:* Munkres Theorem 25.5.

Problem 4 (Munkres 171, #5)**.** Let X be Hausdorff, and let A, B be disjoint compact subspaces of X. Show that there exist disjoint open $U, V \subseteq X$ such that $A \subseteq U$ and $B \subseteq V$. *Hint:* Munkres Lemma 26.4.

Problem 5 (Munkres 171, #7)**.** Show that if Y is compact, then for any space X, the projection $\mathrm{pr}_X : X \times Y \to X$ defined by

$$\mathrm{pr}_X(x, y) = x$$

is a *closed map*, meaning it takes closed sets to closed sets.

Problem 6. Read the definition of the *T_1 axiom* in Munkres §17, and the definitions of *regular* and *normal* spaces in Munkres §31. (The Hausdorff axiom is sometimes called the T_2 axiom.)

(1) Put the four conditions above in order from most to least restrictive.
(2) Show that $\mathbf{R}$ is not Hausdorff in the finite complement topology.
(3) Show directly, without using tools from Munkres §32 onwards, that $\mathbf{R}$ is normal in the analytic topology.

Problem 7 (Munkres 330, #2)**.** For any spaces X, Y, let $[X, Y]$ be the set of homotopy classes of maps of X into Y. For clarity, let $I = [0,1]$. Show that:

(1) If X is nonempty, then $[X, I]$ is a singleton.
(2) If Y is nonempty and path-connected, then $[I, Y]$ is a singleton.

Problem 8 (Munkres 330, #3)**.** Keep the notation of Problem 7. We say that a nonempty space X is *contractible* if and only if its identity map is nulhomotopic: *i.e.*, homotopic to some constant-valued map. Show that:

(1) I and $\mathbf{R}$ are contractible.
(2) Any contractible (nonempty) space is path-connected.
(3) If X, Y are nonempty and Y is contractible, then $[X, Y]$ is a singleton.
(4) If X, Y are nonempty, X is contractible, and Y is path-connected, then $[X, Y]$ is a singleton.

In Problems 9–12, we study inverse limits. For any poset I with partial order $\preceq$ (`\preceq`), we define an *inverse system* over I to consist of:

- A collection of sets $\{X_i\}_{i \in I}$.
- A collection of maps $\{\phi_{i,j} : X_j \to X_i\}_{i \preceq j}$, such that for all $i, j, k \in I$ with $i \preceq j \preceq k$, we have $\phi_{i,k} = \phi_{i,j} \circ \phi_{j,k}$.

We define the *inverse limit* $\varprojlim_i X_i$ to be the set

$$\varprojlim_i X_i = \left\{ (x_i)_i \in \prod_{i \in I} X_i \;\middle|\; \phi_{i,j}(x_j) = x_i \text{ for all } i, j \in I \text{ with } i \preceq j \right\}.$$

Problem 9. Fix a positive integer p. Consider the following inverse system:

- The poset is $\mathbf{Z}_{>0}$ under $\leq$.
- The sets are $\mathbf{Z}/p^i\mathbf{Z}$ for $i \in \mathbf{Z}_{>0}$.
- The map $\phi_{i,j} : \mathbf{Z}/p^j\mathbf{Z} \to \mathbf{Z}/p^i\mathbf{Z}$ is reduction mod p^i for all $i, j \in \mathbf{Z}_{>0}$ with $i \leq j$.

The inverse limit is denoted $\mathbf{Z}_p$. For prime p, it is the set of *p-adic integers*.

Let $\mathbf{T}_p$ be the collection of formal power series $\alpha(t) = \sum_{i \geq 0} \alpha_i t^i$ such that $\alpha_i \in \{0, 1, \ldots, p-1\}$ for all i. Show that for all $\alpha \in \mathbf{T}_p$, the sequence $(x_{i,\alpha})_i \in \prod_i \mathbf{Z}/p^i\mathbf{Z}$ defined by

$$x_i = \sum_{0 \leq l < i} \alpha_l p^l \bmod p^i \quad \text{for all } i > 0$$

is an element of $\mathbf{Z}_p$.

(In fact, the map from $\mathbf{T}_p$ to $\mathbf{Z}_p$ that sends $\alpha \mapsto (x_{i,\alpha})_i$ is bijective. Hence, $\mathbf{Z}_p$ is infinite.)

Problem 10. Consider the following inverse system.

- The poset is the collection $\mathcal{B}$ of bounded open subsets of $\mathbf{R}$ under $\subseteq$.
- The sets are $X_U = \{\text{continuous functions from } U \text{ to } \mathbf{R}\}$ for $U \in \mathcal{B}$.
- The maps $\phi_{U,V} : X_V \to X_U$ are given by $\phi_{U,V}(f) = f|_U$, where $|_U$ means restriction of domain, for all $U, V \in \mathcal{B}$ with $U \subseteq V$.

Let $X_{\mathbf{R}} = \{\text{continuous functions from } \mathbf{R} \text{ to } \mathbf{R}\}$.

(1) Show that the map $X_{\mathbf{R}} \to \varprojlim_U X_U$ defined by $f \mapsto (f|_U)_U$ is a bijection.
(2) If we replace the word "continuous" with the word "bounded" throughout this problem, does the analogue of (1) still hold?

Problem 11. Let $(\{X_i\}_{i\in I}, \{\phi_{i,j}\}_{i\preceq j})$ be an inverse system. Suppose that each set X_i is endowed with a topology, such that each map $\phi_{i,j}$ is continuous. View $\varprojlim_i X_i$ as a subspace of $\prod_i X_i$ in the product topology. Show that:

(1) If X_i is Hausdorff for all i, then $\varprojlim_i X_i$ is Hausdorff.
(2) If X_i is Hausdorff for all i, then $\varprojlim_i X_i$ is closed in $\prod_i X_i$. *Hint:* Observe that the composition

$$\prod_i X_i \xrightarrow{\mathrm{pr}_j \times \mathrm{pr}_i} X_j \times X_i \xrightarrow{\phi_{i,j} \times \mathrm{id}} X_i \times X_i$$

is continuous for all $i, j \in I$ with $i \preceq j$. Use Problem Set 3, #7(3).
(3) If X_i is compact for all i, then $\varprojlim_i X_i$ is compact. *Hint:* Combine part (2) above with Tychonoff's theorem.

Problem 12. We keep the setup of Problem 9, but now, endow $\mathbf{Z}/p^i\mathbf{Z}$ with the discrete topology for all i.

(1) Show that the maps $\phi_{i,j}$ are all continuous, and that $\mathbf{Z}_p$ is compact and Hausdorff.
(2) For all $j \in \mathbf{Z}_{>0}$ and $a \in \mathbf{Z}$, we define $a + p^j\mathbf{Z}_p$ to be the preimage of the residue a mod p^j under the composition

$$\mathbf{Z}_p \to \prod_{i>0} \mathbf{Z}/p^i\mathbf{Z} \xrightarrow{\mathrm{pr}_j} \mathbf{Z}/p^j\mathbf{Z}.$$

Show that $a + p^j\mathbf{Z}_p$ is always clopen.

(Using (2), one can show that $\mathbf{Z}_p$ is totally disconnected. However, $\mathbf{Z}_p$ is not discrete.)