## 1. Lattices and Rectifiability

1.1. Recall that if $V$ is a $k(\!(t)\!)$-vector space, then a $k[\![t]\!]$-submodule $M \subseteq V$ is a **lattice** in $V$ iff:

(1) It is free over $k[\![t]\!]$.
(2) The map $M[\frac{1}{t}] = M \otimes_{k[\![t]\!]} k(\!(t)\!) \to V$ is an isomorphism.

If hypothesis (1) holds and $V$ is finite-dimensional, then hypothesis (2) holds if and only if the $k[\![t]\!]$-rank of $M$ equals the dimension of $V$.

Now, let $A$ be a $k$-algebra and $V$ a free module over $A \hat{\otimes} F$ of finite rank. Let $M$ be an arbitrary $(A \hat{\otimes} \mathcal{O})$-submodule of $V$.

**Definition 1.2.** We say that $M$ is an **$\mathcal{O}$-lattice** in $V$ iff:

(1) It is *locally* free over $A \hat{\otimes} \mathcal{O}$ in the Zariski topology.
(2) The map $M \otimes_{A \hat{\otimes} \mathcal{O}} (A \hat{\otimes} F) \to V$ is an isomorphism.

Note that if hypothesis (1) holds, then hypothesis (2) holds if and only if the $(A \hat{\otimes} \mathcal{O})$-rank of $M$ equals the $(A \hat{\otimes} F)$-rank of $V$.

By [SP, Tag 00NX], a finitely-generated module over an arbitrary ring is locally free in the Zariski topology if and only if it is projective. By [SP, Tag 0593], any projective module over a local ring is free. Therefore, Definition 1.2 recovers the definition in §1.1 when $\mathcal{O} = k[\![t]\!]$ and $A = k$.

We say that an $\mathcal{O}$-lattice is **free** iff it is free over $A \hat{\otimes} \mathcal{O}$, not just locally free. Henceforth, we fix a free $\mathcal{O}$-lattice

$$V_{\mathcal{O}} \subseteq V.$$

For instance, if we have chosen an isomorphism $V \simeq (A \hat{\otimes} F)^{\oplus n}$, then we can take $V_{\mathcal{O}} = (A \hat{\otimes} \mathcal{O})^{\oplus n}$.

**Definition 1.3.** Let $i \geq 0$ be an integer. We say that $M$ is **$i$-rectifiable** with respect to $V_{\mathcal{O}}$ iff:

(1) $\mathfrak{m}^i V_{\mathcal{O}} \subseteq M \subseteq \mathfrak{m}^{-i} V_{\mathcal{O}}$.
(2) $(\mathfrak{m}^{-i} V_{\mathcal{O}})/M$ is locally free over $A$ of finite rank.

We say that $M$ is **rectifiable** with respect to $V_{\mathcal{O}}$ iff it is $i$-rectifiable for some $i$.

Our terminology differs from Görtz's terminology in [G, §2.3]. What he calls a lattice is what we call a rectifiable module. However, he always assumes that $\mathcal{O} = k[\![t]\!]$. The results below show that $\mathcal{O}$-lattices are always rectifiable, and in the case where $\mathcal{O} = k[\![t]\!]$, the converse holds.

**Proposition 1.4.** *Every $\mathcal{O}$-lattice in $V$ is rectifiable with respect to $V_{\mathcal{O}}$.*

**Lemma 1.5.** *If $M$ is a free $\mathcal{O}$-lattice in $V$, then it admits a free complement as an $A$-submodule of $V$.*

*Proof.* We can assume $V = (A \hat{\otimes} F)^{\oplus n}$ for some $n$. Since $M$ is a free $\mathcal{O}$-lattice, it is the image of $(A \hat{\otimes} \mathcal{O})^{\oplus n}$ under some invertible $(A \hat{\otimes} \mathcal{O})$-linear map $\Phi : V \to V$. Pick a splitting of $k$-vector spaces $F \simeq \mathcal{O} \oplus E$. Then we have a splitting of $A$-modules $A \hat{\otimes} F \simeq (A \hat{\otimes} \mathcal{O}) \oplus (A \otimes E)$, so the image of $(A \otimes E)^{\oplus n}$ under $\Phi$ is the desired complement of $M$. □

**Lemma 1.6.** *If $M, M'$ are free $\mathcal{O}$-lattices in $V$ and $M' \subseteq M$, then $M/M'$ is locally free over $A$ of finite rank.*

*Proof.* By Lemma 1.5, we have $A$-module isomorphisms $V \simeq M \oplus N \simeq M' \oplus N'$, where $N, N'$ are free over $A$. The short exact sequence

$$0 \to M/M' \to N' \to N \to 0$$

splits, so $M/M'$ is projective over $A$. Moreover, $M/M'$ is finitely generated over $A$ because $M$ is. By [SP, 00NX], we deduce that $M$ is locally free over $A$. $\square$

*Proof of Proposition 1.4.* First, we will construct an $A$-algebra $A'$ such that, if

$$M' = M \otimes_{A \hat{\otimes} \mathcal{O}} (A' \hat{\otimes} \mathcal{O}), \tag{1}$$

then $M$ is free over $A' \hat{\otimes} \mathcal{O}$. Pick a *finite* Zariski open cover of $\operatorname{Spec}(A \hat{\otimes} \mathcal{O})$ that trivializes $M$, say, $A \hat{\otimes} \mathcal{O} \to \prod_i (A \hat{\otimes} \mathcal{O})_{f_i}$. For each $i$, let $a_i \in A$ be the image of $f_i$ under $A \hat{\otimes} \mathcal{O} \to A \hat{\otimes} k = A$. Since the $f_i$ generate $A \hat{\otimes} \mathcal{O}$, the $a_i$ generate $A$. Taking $A' = \prod_i A_{a_i}$, we see that the map $A \hat{\otimes} \mathcal{O} \to A' \hat{\otimes} \mathcal{O}$ factors as

$$A \hat{\otimes} \mathcal{O} \to \prod_i (A \hat{\otimes} \mathcal{O})_{f_i} \to \prod_i (A_{a_i} \hat{\otimes} \mathcal{O})_{f_i} \xrightarrow{\sim} \prod_i (A_{a_i} \hat{\otimes} \mathcal{O}) \xrightarrow{\sim} A' \hat{\otimes} \mathcal{O},$$

where the last isomorphism uses the finiteness of the product. Thus, the module $M'$ defined by (1) is free over $A' \hat{\otimes} \mathcal{O}$.

In what follows, let $V'$, *resp.* $V'_{\mathcal{O}}$ denote the base change of $V$, *resp.* $V_{\mathcal{O}}$ along $A \hat{\otimes} \mathcal{O} \to A' \hat{\otimes} \mathcal{O}$. Since $M'$ is free of finite rank over $A' \hat{\otimes} \mathcal{O}$, we can pick $i$ large enough that we get inclusions

$$\mathfrak{m}^i V'_{\mathcal{O}} \subseteq M' \subseteq \mathfrak{m}^{-i} V'_{\mathcal{O}}$$

of $(A' \hat{\otimes} \mathcal{O})$-submodules of $V'$. We claim that they restrict to inclusions

$$\mathfrak{m}^i V_{\mathcal{O}} \subseteq M \subseteq \mathfrak{m}^{-i} V_{\mathcal{O}}.$$

Indeed, the map $A \hat{\otimes} \mathcal{O} \to A' \hat{\otimes} \mathcal{O}$ is injective because the map $A \to A'$, being faithfully flat, is injective. Since $V_{\mathcal{O}}$ and $M$ are flat over $A \hat{\otimes} \mathcal{O}$, the induced maps $V_{\mathcal{O}} \to V'_{\mathcal{O}}$ and $M \to M'$ are also injective, and the claim follows. It remains to check that $(\mathfrak{m}^{-i} V_{\mathcal{O}})/M$ is locally free over $A$ of finite rank. Observe that

$$\begin{aligned}
(\mathfrak{m}^{-i} V'_{\mathcal{O}})/M' &\simeq \frac{(\mathfrak{m}^{-i} V'_{\mathcal{O}})/(\mathfrak{m}^i V'_{\mathcal{O}})}{M'/(\mathfrak{m}^i V'_{\mathcal{O}})} \\
&\simeq \frac{(\mathfrak{m}^{-i} V_{\mathcal{O}})/(\mathfrak{m}^i V_{\mathcal{O}}) \otimes_{A \hat{\otimes} \mathcal{O}} (A' \hat{\otimes} \mathcal{O})}{M/(\mathfrak{m}^i V_{\mathcal{O}}) \otimes_{A \hat{\otimes} \mathcal{O}} (A' \hat{\otimes} \mathcal{O})} \\
&\simeq \frac{(\mathfrak{m}^{-i} V_{\mathcal{O}})/(\mathfrak{m}^i V_{\mathcal{O}}) \otimes_A A'}{M/(\mathfrak{m}^i V_{\mathcal{O}}) \otimes_A A'} \\
&\simeq (\mathfrak{m}^{-i} V_{\mathcal{O}}/M) \otimes_A A'.
\end{aligned}$$

By Lemma 1.6, $(\mathfrak{m}^{-i} V'_{\mathcal{O}})/N$ is locally free over $A'$ of finite rank, so we're done by Zariski descent. $\square$

*Remark* 1.7. Suppose that $A \to B$ is a faithfully-flat morphism of $k$-algebras. It is *not* necessarily true that $A \mathbin{\hat{\otimes}} \mathcal{O} \to B \mathbin{\hat{\otimes}} \mathcal{O}$ is faithfully flat: See

https://mathoverflow.net/a/152538

for a counterexample. Consequently, it is also not necessarily true that $B \mathbin{\hat{\otimes}} \mathcal{O} \simeq B \otimes_A (A \mathbin{\hat{\otimes}} \mathcal{O})$. So in the above proof, the difference between base change along $A \to A'$ and along $A \mathbin{\hat{\otimes}} \mathcal{O} \to A' \mathbin{\hat{\otimes}} \mathcal{O}$ is essential where it appears.

**Proposition 1.8.** *If $\mathcal{O}$ is normal, then every rectifiable $(A \mathbin{\hat{\otimes}} \mathcal{O})$-submodule of $V$ is an $\mathcal{O}$-lattice.*

*Proof.* This is the (1) $\implies$ (2) direction in [G, Lem. 2.11]. □

**Example 1.9.** We give two examples that show how Proposition 1.8 fails when $\mathcal{O}$ is not normal.

(1) Let $\mathcal{O} = k[\![x, y]\!]/(xy)$, so that $F = k(\!(x)\!) \times k(\!(y)\!)$. Let $M = k[\![x]\!] \times k[\![y]\!]$.
(2) Let $\mathcal{O} = k[\![t^2, t^3]\!]$, so that $F = k(\!(t)\!)$. Let $M = k[\![t]\!]$.

In both cases, $M \subseteq F$ is 1-rectifiable with respect to $\mathcal{O}$, but not locally free over $\mathcal{O}$.