

# MATH 340: ADVANCED LINEAR ALGEBRA

## PROBLEM SET #9

SPRING 2025

**Due Friday, April 25.** You may consult books, papers, and websites as long as you cite all sources and write up your solutions in your own words. **Updated Problem 8 on 4/16 at 12:30 am.**

**Problem 1.** Recall that in the two-element field  $\mathbf{F}$  introduced on Problem Set 2, #8, we have  $-1 = 1$ : that is, 1 is its own additive inverse.

- (1) Show that if  $F \in \{\mathbf{R}, \mathbf{C}\}$ , then a bilinear form on a vector space over  $F$  is alternating if and only if it is antisymmetric.
- (2) What goes wrong in (1) when  $F = \mathbf{F}$ ?

*In the remainder of this problem set, we exclude this field, returning to our usual assumption that all vector spaces are defined over  $\mathbf{R}$  or  $\mathbf{C}$ .*

**Problem 2.** Let  $\mathbf{E} \subseteq \mathfrak{sl}(2, \mathbf{C})$  be the real vector space of trace-zero, Hermitian  $2 \times 2$  complex matrices. (See Problem Set 3, #6, and Problem Set 7, #2.)

- (1) Using part (2) of Problem Set 3, #6, check that  $E_1, E_2, E_3$  form a basis for  $\mathbf{E}$ , where

$$E_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad E_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

- (2) Let  $S : \mathbf{E} \xrightarrow{\sim} \mathbf{R}^3$  be the linear isomorphism that sends  $(E_1, E_2, E_3)$  onto the standard basis. Show that

$$S(X) \cdot S(Y) = \frac{1}{2} \operatorname{tr}(XY)$$

for all  $X, Y \in V$ , where  $- \cdot -$  is the dot product. Thus, the right-hand side defines an inner product on  $\mathbf{E}$ .

**Problem 3.** On a complex inner product space, operators  $T$  such that  $\langle Tu, Tv \rangle = \langle u, v \rangle$  for all vectors  $u, v$  are usually called *unitary*, rather than orthogonal. Let

$$\operatorname{SO}(n) = \{n \times n \text{ real matrices } M \mid \det(M) = 1 \text{ and } M^t M = I\},$$

$$\operatorname{SU}(n) = \{n \times n \text{ complex matrices } M \mid \det(M) = 1 \text{ and } M^* M = I\},$$

where  $M_{j,i}^* = \bar{M}_{i,j}$ . The notations SO and SU stand for *special orthogonal* and *special unitary*. Using Problem Set 3, #8 and Problem Set 5, #8, give bijections

$$M : \{z \in \mathbf{C} \mid |z| = 1\} \xrightarrow{\sim} \operatorname{SO}(2), \quad M : \{z \in \mathbf{H} \mid |z| = 1\} \xrightarrow{\sim} \operatorname{SU}(2)$$

such that  $M(z_1 z_2) = M(z_1) \cdot M(z_2)$  in both cases. Above, the absolute value of a quaternion is given by  $|a1 + bi + cj + dk| = \sqrt{a^2 + b^2 + c^2 + d^2}$ .

**Problem 4.** Keep the notation of Problems 2–3. Show that if  $M \in \text{SU}(2)$ , then

$$T_M : \mathbf{E} \rightarrow \mathbf{E} \quad \text{defined by } T_M(X) = MXM^{-1}$$

is a well-defined, linear, and orthogonal with respect to  $\langle X, Y \rangle := \frac{1}{2} \text{tr}(XY)$ .

Together, Problem 3 and this problem encapsulate W. R. Hamilton's formalism for describing 3-dimensional rotations via quaternions.

**Problem 5.** Recall, from Problem Set 8, #4, the inner product on  $\mathbf{C}[t]$  defined by

$$\langle f, g \rangle = \int_{-1}^1 f(t) \overline{g(t)} dt.$$

- (1) Show explicitly that the linear operator  $D(p(t)) = \frac{d}{dt}p(t)$  is not self-adjoint with respect to  $\langle -, - \rangle$ .
- (2) Is the linear operator  $T(p(t)) = tp(t)$  self-adjoint?

**Problem 6.** Let  $T$  be an orthogonal linear operator on an inner product space  $V$  of dimension  $n$ . In the language of Problem Set 8, #7, the problem below proves the *Cartan–Dieudonné theorem* that  $T$  is a composition of  $\leq n$  reflections.

- (1) Show that for all  $v, w \in V$  such that  $\|v\| = \|w\|$ , the reflection  $S_{v-w} : V \rightarrow V$  swaps  $v$  and  $w$ .
- (2) Let  $(e_1, \dots, e_n)$  be a basis for  $V$ , and let  $f_i = Te_i$  for all  $i$ . Suppose that for some  $k$ , there is an orthogonal linear operator  $T_k : V \rightarrow V$  such that  $T_k e_i = f_i$  for all  $i \leq k$ . Using Problem Set 8, #8(1), show that

$$\begin{aligned} \|T_k e_{k+1}\| &= \|f_{k+1}\|, \\ \|T_k e_{k+1} - f_i\| &= \|f_{k+1} - f_i\| \quad \text{for all } i < k+1. \end{aligned}$$

*Hint:* Orthogonal operators preserve the norms of vectors.

- (3) Let  $T_0 = \text{Id}_V$  and  $T_{k+1} = S_{T_k e_{k+1} - f_{k+1}} \circ T_k$  for all  $k \geq 0$ . Use (1)–(2) to show that  $T_{k+1} e_i = f_i$  for all  $i \leq k+1$ . Thus,  $T_n = T$ .

*Hint:* In the case where  $i < k+1$ , expand the second display from (2), then apply the first display to arrive at  $\langle T_k e_{k+1}, f_i \rangle = \langle f_{k+1}, f_i \rangle$ . This shows that  $f_i$  is orthogonal to  $T_k e_{k+1} - f_{k+1}$ .

**Problem 7.** Show that if  $M \in \text{Mat}_n(\mathbf{C})$  is invertible, then  $M = QR$  for some unitary  $Q$  and invertible upper-triangular  $R$ . This is called the *QR decomposition* of  $M$ . *Hint:* Interpret the Gram–Schmidt process for the columns of  $M$  in terms of right multiplication by another matrix.

**Problem 8.** Let  $A \in \text{Mat}_n(\mathbf{C})$ .

- (1) Show that if the pairing  $\langle u, v \rangle := u^t A \bar{v}$  is an inner product, then  $A = B^* B$  for some invertible  $B$ . *Hint:* The spectral theorem.
- (2) Use Problem 7 to deduce that in this case,  $A = R^* R$  for some invertible and upper-triangular  $R$ . This is called the *Cholesky decomposition* of  $A$ . It is a special case of *singular value decomposition*. (The square roots of the eigenvalues of  $A$  are the *singular values* of  $R$ .)