Last time — (X, x) gives $\pi_1(X, x)$; f gives f_*

if f is a homeo, then f_* is an iso

Ex — if $f = id_X$, then $f_* = id_{\pi_1(X, x)}$

Q — is the converse true?

Ex — R and {0} are not homeomorphic [why?]
but $\pi_1(R, 0) = \pi_1(\{0\}, 0)$

Thm — if X is a convex subspace of R^n
then for any x and loop γ based at x,
have $\gamma \sim_p e_x$
thus $\pi_1(X, x) = \{[e_x]\}$

Pf — recall the homotopy
$h(s, t) = (1 - t)*x + t*\gamma(s)$

is it a path homotopy?

$h(0, t) = (1 - t)*x + t*\gamma(0) = (1 - t)*x + t*x = x$
$h(1, t) = (1 - t)*x + t*\gamma(1) = (1 - t)*x + t*x = x$

Moral — convex subspaces of R^n need not be homeomorphic
but they are all simply connected:
their π_1's are all trivial

but recall:
any nonempty convex X is contractible:
there is x_0 in X s.t. $id_X \sim$ (constant map at x_0)

(Munkres §58)

<u>Df</u> a <u>homotopy equivalence</u> btw X and Y is a pair of cts $f : X$ to Y and $g : Y$ to X

s.t. $g \circ f \sim id\_X$ and $f \circ g \sim id\_Y$

here we say X and Y are <u>homotopy equivalent</u>

<u>Thm</u> if $f : X$ to Y and $g : Y$ to X form a homotopy equivalence

then $f\_* : \pi\_1(X, x)$ to $\pi\_1(Y, f(x))$, $g\_* : \pi\_1(Y, y)$ to $\pi\_1(X, g(y))$

are isomorphisms for any x in X and y in Y

<u>Ex</u> suppose X is contractible
pick x_0 in X s.t. $id\_X \sim$ (const map at x)

$Y = \{x\_0\}$
$f : X$ to Y $f(x) = x\_0$, the constant map
$g : Y$ to X $g(x\_0) = x\_0$, the inclusion

then $(g \circ f)(x) = x\_0$, so $g \circ f \sim id\_X$
while $f \circ g = id\_Y \sim id\_Y$

so X is homotopy equivalent to $\{x\_0\}$

<u>Ex</u> $X = R^2 – \{(0, 0)\}$ and $Y = S^1$

$r : X$ to Y $r(x, y) = (x, y)/|(x, y)|$
$i : Y$ to X $i(x, y) = (x, y)$

then $(i \circ r)(x, y) = (x, y)/|(x, y)|$, so $i \circ r \sim id_X$
via $h((x, y), t) = ((1 - t)/|(x, y)| + t)*(x, y)$
while $(r \circ i) = id_Y$

so $R^2 - \{(0, 0)\}$ is homotopy equivalent to S^1

Rem these examples are a specific kind of homotopy equivalence called a deformation retract

Pf of Thm suppose $f : X$ to Y and $g : Y$ to X
s.t. $g \circ f \sim id_X$ and $f \circ g \sim id_Y$

will show that $f_* : \pi_1(X, x)$ to $\pi_1(Y, f(x))$
is an iso for any x in X [argument for g_* is similar]

three lemmas:

1) if $f : X$ to Y and $g : Y$ to Z are cts maps
then $(g \circ f)_* = g_* \circ f_*$ [from last time]

2) if $\varphi : G$ to H and $\psi : H$ to K are maps
s.t. $\psi \circ \varphi$ is bijective
then φ is injective and ψ is surjective

3) if α is a path in X from x_0 to x_1
then $\breve{\alpha} : \pi_1(X, x_0)$ to $\pi_1(X, x_1)$ def by

$$\breve{\alpha}([\gamma]) = [\bar{\alpha} * \gamma * \alpha]$$

is an isomorphism

by 1), $g_* \circ f_* = (g \circ f)_*$,
$f_* \circ g_* = (f \circ g)_*$

so by 2), just need $(g \circ f)_*$ and $(f \circ g)_*$ to be isos

pick homotopies j from $g \circ f$ to id_X,
k from $f \circ g$ to id_Y

can use 3) to show: if $f, f' : A$ to X are cts,
h a homotopy from f to f',
a in A,

then $f'_* = \breve{\alpha}_h \circ f_* : \pi_1(A, a)$ to $\pi_1(X, f'(a))$

where $\alpha_h(s) = h(a, s)$, a path from $f(a)$ to $f'(a)$

apply to j and k:

$$(g \circ f)_* = \breve{\alpha}_j \circ id_{X, *} = \breve{\alpha}_j$$
$$(f \circ g)_* = \breve{\alpha}_k \circ id_{Y, *} = \breve{\alpha}_k$$

but by 3), $\breve{\alpha}_j$ and $\breve{\alpha}_k$ are isos