MATH 340
Advanced Linear Algebra

mqtrinh.github.io/math/teaching/yale/math-340/

9 psets	36%
1 midterm Wed 2/26	24%
1 final	40%

Axler, Lin Alg Done Right, 4th Ed
Treil, Lin Alg Done Wrong

[late hw policy]
[schedule]
[intros]

(Axler §1B–1C)

F is either
R, the set of real #s
or C, the set of complex #s

addition	identity 0
	inverse –a
multiplication	identity 1
	inverse 1/a when a ≠ 0

F^n, the set of tuples (a_1, a_2, …, a_n)

<u>Idea</u> vectors without choosing coordinates!

Def vector space over F consists of
set V,
addition $+ : V \times V$ to V,
scaling $\bullet : F \times V$ to V,

such that for all u, w, v in V and a, b in F,

$(u + v) + w = u + (v + w)$
$u + v = v + u$
V contains $\mathbf{0}$ s.t. $v + \mathbf{0} = v$
V contains (for each v) some z s.t. $v + z = \mathbf{0}$

$(a \bullet b) \bullet v = a \bullet (b \bullet v)$
$1 \bullet v = v$
$a \bullet (v + w) = a \bullet v + a \bullet w$
$(a + b) \bullet v = a \bullet v + b \bullet v$

Lem $0 \bullet v = \mathbf{0}$ for all v in V

$0 \bullet v = (0 + 0) \bullet v = 0 \bullet v + 0 \bullet v$
add inverse of $0 \bullet v$ to both sides, regroup ( ):
$\mathbf{0} = 0 \bullet v$

Lem $v + z = \mathbf{0}$ if and only if $z = (-1) \bullet v$

if $z = (-1) \bullet v$:
$v + z = 1 \bullet v + (-1) \bullet v = (1 - 1) \bullet v = 0 \bullet v$
which is $\mathbf{0}$ by previous lemma
if $v + z = \mathbf{0}$:
$v + z = v + (-1) \bullet v$ by first half
add z on the left to both sides, regroup ( ):
$z = (-1) \bullet v$

Cor we can write $-v$ without fear

Rem similarly $a \bullet \mathbf{0} = \mathbf{0}$ for all a in F

Ex for any positive integer n
F^n is a vector space over F

Ex $\{\mathbf{0}\}$ (by convention, view as "F^0")

Ex let X be any set:
F^X def by {functions $f : X$ to F}
$f + g$ def by $(f + g)(x) = f(x) + g(x)$
$a \bullet f$ def by $(a \bullet f)(x) = af(x)$

special case: $F^n = F^{\{1, \ldots, n\}}$

Rem $F^{\{0, 1, 2, \ldots\}}$ is bigger than you might think
$F^{[0, 1]}$ is even bigger

Ex F[x] = {polynomials in x with F-coeffs}
$f + g$ def by $(f + g)(x) = f(x) + g(x)$
$a \bullet f$ def by $(a \bullet f)(x) = af(x)$

can multiply polynomials, but it doesn't matter
only scaling matters

[how big is F[x] compared to F^n?]

[is F[x] the same as $F^{\{0, 1, 2, \ldots\}}$?]

Ex fix a, b in F

$V_a = \{(x, ax) \mid x \in F\}$

$V_{\{a, b\}} = \{(x, ax + b) \mid x \in F\}$

[are V_a or $V_{\{a, b\}}$ vector spaces?]

V_a for all a

$V_{\{a, b\}}$ if and only if b = 0

[draw picture]

Ex C is a vector space over R

Ex any vector space over C
is also a vector space over R

(if we can scale by a in C, then certainly by a in R)

[how about the reverse?]

Ex if V is a vector space over R
then its complexification V_C is

$V_C = \{(u, v) \mid u, v \in V\}$

$(u, v) + (u', v')$ def by $(u + u', v + v')$

$(a + ib) \cdot (u, v)$ def by $(au - bv, av + bu)$

think of (u, v) as "u + iv" : see Axler 1B, #8

Ex V = {positive real numbers}

u "+" v def by uv

a "•" v def by $|v^a|$ (for all a in R)

"**0**" def by 1 in V

thus "–v" def by v^{-1}

[is V a vector space over R?]

$(uv)w = u(vw)$, $uv = vu$, $v1 = v$

$v^{ab} = (v^b)^a$, $v^1 = v$

$(uv)^a = (u^a)(v^a)$, $v^{a+b} = (v^a)(v^b)$

[yes!]

Ex define a magic square over F as

a 3 by 3 grid of elements of F s.t.
the sums along rows, cols, main diag's
all match

Mag(F) = {magic squares over F}

via inclusion of Mag(F) into F^9, deduce:
Mag(F) is a vector space over F

[how big is it compared to F^9?]