7.

Notes on Lachowska–Qi and Hemelsoet–Kivinen–Lachowska.

7.1.

Let G be a simply-connected, complex semisimple algebraic group with Weyl group W. Let Q be the coweight = coroot lattice of G, and let $d_1 \leq \cdots \leq d_r$ be the fundamental degrees of the W-action on Q. The largest degree is the Coxeter number $h := d_r$. For any integer $\ell > 0$ coprime to h, we define the rational Catalan number of (W, ℓ) to be

$$\mathrm{Cat}_{W,\ell} = \prod_{1 \leq i \leq r} \frac{\ell + d_i - 1}{d_i}.$$

When $W = S_n$, so that $r = n - 1$ and $d_i = i + 1$, this formula recovers $\frac{(\ell+n-1)!}{\ell! n!}$.

7.2.

Fix odd $\ell > h$ coprime to $h, h+1$, and the determinant of the Cartan matrix. Let ζ be a primitive ℓth root of unity. Let $\mathfrak{u}_\zeta^\vee$ denote Lusztig's small quantum group attached to $\mathfrak{g}^\vee = \mathrm{Lie}(G^\vee)$ and ζ.

The small quantum group is a finite-dimensional unimodular Hopf algebra. For a general Hopf algebra H, the adjective *unimodular* means that there is a nonzero central element $\nu \in Z(H)$ characterized up to scaling by the *left-* and *right-integrality* identities

$$x\nu = \nu x = \varepsilon(x)\nu, \quad \text{where } \varepsilon : H \to \mathbf{C} \text{ is the counit.}$$

Recall that the center $Z(H)$ is precisely the subspace of $\mathfrak{u}_\zeta^\vee$ of elements z invariant under the Hopf adjoint action, in the sense that $\mathrm{ad}(x)z = \varepsilon(x)z$. From this description, we can check that any element of the form $\mathrm{ad}(\nu)h$ for some $h \in H$ is central. The *Higman ideal* is the two-sided ideal of $Z(H)$ formed by

$$Z_{\mathrm{Hig}}(H) := \{\mathrm{ad}(\nu)h \mid h \in \mathfrak{u}_\zeta^\vee\}.$$

Lachowska–Qi show that

$$\dim Z_{\mathrm{Hig}}(\mathfrak{u}_\zeta^\vee) = \mathrm{Cat}_{W,\ell}.$$

The proof relies on a representation-theoretic description of the Higman ideal.

7.3.

Given a general finite-dimensional Hopf algebra H, let $C(H)$, *resp.* $C_l(H)$, be the vector space of trace-like functions on H, *resp.* left-shifted trace-like functions:

$$C(H) = \{f \in H^* \mid f(xy) = f(yx) \text{ for all } x, y\},$$
$$C_l(H) = \{f \in H^* \mid f(xy) = f(yS^2(x)) \text{ for all } x, y\}, \quad \text{where } S : H \to H \text{ is the antipode.}$$

Via the coproduct of H, these vector spaces actually form $\mathbf{C}$-algebras.

If H is quasi-triangular with R-matrix $R = R_{12} \sum R_1 \otimes R_2 \in H \otimes H$, then the element $u = \sum S(R_2)R_1$ satisfies $S^2(h) = uhu^{-1}$ for all $h \in H$. One then finds that there is an isomorphism of $\mathbf{C}$-algebras

$$\mu_l : C(H) \xrightarrow{\sim} C_l(H) \quad \text{defined by } \mu_l(f) = f(u(-))$$

Let $R(H)$ be the Grothendieck ring of finite-dimensional H-modules. Via the map that sends any H-module to its character, we obtain an inclusion $R(H) \subseteq C(H)$. Let $P(H) \subseteq R(H)$ be the ideal generated by (the characters of) the projective modules. Let $P_l(H) \subseteq R_l(H) \subseteq C_l(H)$ correspond to $P(H) \subseteq R(H) \subseteq C(H)$ under μ_l.

7.4.

Let λ_l denote the unique left-integral, nonzero central element of the dual Hopf algebra H^*. Again, left integrality means $f\lambda_l = f(1)\lambda_l$. If H is unimodular, then $\lambda_l \in C_l(H)$. The *Radford isomorphism* is the isomorphism of H-modules

$$\psi_l : H \xrightarrow{\sim} H^* \quad \text{defined by } \psi_l(h) = \lambda_l((-)h),$$

where H acts on H^* through the coadjoint action $\mathrm{ad}^*(x)f = f(\mathrm{ad}(S^{-1}(x))(-))$. Radford proves that ψ_l restricts to an isomorphism of $Z(H)$-modules $\psi_l : Z(H) \xrightarrow{\sim} C_l(H)$.

Observe that $u = m(S \otimes 1)(R_{21})$, where $R_{21} = \sum R_2 \otimes R_1$ and $m : H \otimes H \to H$ is the multiplication of H. Lachowska–Qi check that the map

$$j_l : H^* \to H \quad \text{defined by } j_l(f) = m(((f \circ S^{-1}) \otimes 1)(R_{21}R_{12}))$$

is a map of H-modules. We say that H is *factorizable* if and only if j_l is surjective. In this case, j_l is an isomorphism, which Lachowska–Qi call the *left-shifted Drinfeld isomorphism*, and restricts to an isomorphism of $Z(H)$-modules $j_l : C_l(H) \xrightarrow{\sim} Z(H)$. For factorizable H, their Proposition 2.26 implies that

$$\psi_l(\mathrm{Z}_{\mathrm{Hig}}(H)) = P_l(H) \quad \text{and} \quad \mathrm{Z}_{\mathrm{Hig}}(H) = j_l(P_l(H)). \tag{7.1}$$

The *(left-shifted) Harish-Chandra center* of H is

$$\mathrm{Z}_{\mathrm{HC}}(H) := j_l(R_l(H)).$$

We thus have $\mathrm{Z}_{\mathrm{Hig}}(H) \subseteq \mathrm{Z}_{\mathrm{HC}}(H)$.

The *Fourier transform* on H is the composition $\mathscr{F} = j_l \circ \psi_l : H \to H$. The Higman ideal is stable under the Fourier transform, but the Harish-Chandra center is not. Nonetheless, from a result of Lachowska that $\mathscr{F}^2 \mid_{Z(H)} = S|_{Z(H)}$, one checks that $\psi_l(\mathscr{F}(\mathrm{Z}_{\mathrm{HC}}(H))) = R_l(H)$, or equivalently,

$$\mathscr{F}(\mathrm{Z}_{\mathrm{HC}}(H)) = \psi_l^{-1}(R_l(H)).$$

Using this characterization, one proves that $\mathscr{F}(\mathrm{Z}_{\mathrm{HC}}(H))$ forms an ideal of $Z(H)$, contained inside the subspace annihilated by the action of the nilradical of $Z(H)$.

7.5.

We return to $H = \mathfrak{u}_\zeta^\vee$. Then $\mathfrak{u}_\zeta^\vee$ decomposes into blocks indexed by the orbits of Q under the action of the ℓ-dilated extended affine Weyl group $W \ltimes \ell Q$. We write $\mathfrak{u}_\zeta^{\vee,\lambda}$ for the block corresponding to $\lambda \in Q$. The principal block corresponds to $\lambda = 0$.

Lachowska–Qi Theorem 4.3 asserts that the following numbers are all equal:

(1) The number of blocks of $\mathfrak{u}_\zeta^\vee$: *i.e.*, the number of coweights in the ℓ-dilated fundamental alcove of Q.
(2) The dimension of $Z_{\mathrm{Hig}}(\mathfrak{u}_\zeta^\vee)$.
(3) The dimension of $Z_{\mathrm{HC}}(\mathfrak{u}_\zeta^\vee) \cap \mathscr{F}(Z_{\mathrm{HC}}(\mathfrak{u}_\zeta^\vee))$.

Work of Haiman and Sommers shows that (1) equals $\mathrm{Cat}_{W,\ell}$.

The proof of Theorem 4.3 goes like this. First, each block of $\mathfrak{u}_\zeta^\vee$ has a nonzero Cartan matrix, so the total number of blocks is bounded above by the decomposition matrix expressing the multiplicities of simple $\mathfrak{u}_\zeta^\vee$-modules in projective $\mathfrak{u}_\zeta^\vee$-modules. From (7.1), it follows that (1) $\leq$ (2). Next, (2) $\leq$ (3) by the inclusion of $Z_{\mathrm{Hig}}(\mathfrak{u}_\zeta^\vee)$ into $Z_{\mathrm{HC}}(\mathfrak{u}_\zeta^\vee) \cap \mathscr{F}(Z_{\mathrm{HC}}(\mathfrak{u}_\zeta^\vee))$. Finally, to get (3) $\leq$ (1): Note that $Z_{\mathrm{HC}}(\mathfrak{u}_\zeta^\vee) \cap \mathscr{F}(Z_{\mathrm{HC}}(\mathfrak{u}_\zeta^\vee))$ consists of elements of $Z_{\mathrm{HC}}(\mathfrak{u}_\zeta^\vee)$ annihilated by the nilradical of $Z(\mathfrak{u}_\zeta^\vee)$. By a result of Brown–Gordon, each block of the Harish-Chandra center is isomorphic to some ring of invariants in the coinvariant algebra of W:

$$Z_{\mathrm{HC}}(\mathfrak{u}_\zeta^\vee) \cap \mathfrak{u}_\zeta^{\vee,\lambda} \simeq \mathbf{C}[Q]^{W_\lambda}/\mathbf{C}[Q]_+^W, \quad \text{where } W_\lambda = \{w \in W \mid w \cdot \lambda = \lambda\}.$$

These are local Frobenius algebras. It follows that in each block, the subspace annihilated by the nilradical of $Z(H)$ is one-dimensional. So the dimension of the total subspace of $Z_{\mathrm{HC}}(\mathfrak{u}_\zeta^\vee)$ annihilated this way is equal to the number of blocks.

As a byproduct, the inclusion $Z_{\mathrm{Hig}}(\mathfrak{u}_\zeta^\vee) \subseteq Z_{\mathrm{HC}}(\mathfrak{u}_\zeta^\vee) \cap \mathscr{F}(Z_{\mathrm{HC}}(\mathfrak{u}_\zeta^\vee))$ is an equality. Furthermore, the one-dimensional subspace of $Z_{\mathrm{HC}}(\mathfrak{u}_\zeta^\vee) \cap \mathfrak{u}_\zeta^{\vee,\lambda}$ mentioned above is precisely $Z_{\mathrm{Hig}}(\mathfrak{u}_\zeta^\vee) \cap \mathfrak{u}_\zeta^{\vee,\lambda}$.

7.6.

For each parabolic subgroup $W' \subseteq W$, let $[W']$ denote its conjugacy class. Let $d_{\ell,[W']}$ be the number of coweights λ in the ℓ-dilated fundamental alcove such that $[W_\lambda] = [W']$. Sommers observes that there is an isomorphism of W-representations:

$$\mathbf{C}[Q/\ell Q] \simeq \bigoplus_{[W']} d_{\ell,[W']} \operatorname{Ind}_{W'}^W(1).$$

Taking W-invariants, we get

$$\dim \mathbf{C}[Q/\ell Q]^W = \sum_{[W']} d_{\ell,[W']} = \mathrm{Cat}_{W,\ell}.$$

The right-hand equality generalizes the decomposition of the Catalan numbers into so-called Kreweras numbers.

Note that $\operatorname{Ind}_{W'}^W(1)$ is just the underlying W-representation of $\mathbf{C}[Q]^{W'}/\mathbf{C}[Q]_+^W$. So by the Brown–Gordon result, $Z_{\mathrm{HC}}(\mathfrak{u}_\zeta^\vee)$ is isomorphic as a W-representation to $\mathbf{C}[Q/\ell Q]$, and under this isomorphism $Z_{\mathrm{Hig}}(\mathfrak{u}_\zeta^\vee)$ corresponds to the subspace of W-invariants.

7.7.

There is a $G^\vee$-action on $Z(\mathfrak{u}_\zeta^\vee)$. The work of Bezrukavnikov–Boixeda-Alvarez–Shan–Vasserot gives an embedding of $\mathbf{C}$-algebras of the form

$$\mathrm{H}^*(\mathrm{Gr}^{\zeta, t^{\ell-1}s})^{(\widetilde{W}, \cdot)} \subseteq Z(\mathfrak{u}_\zeta^\vee)^{G^\vee}.$$

Let us explain the notation above. First, set $\mathrm{Gr} = G((t))/G[[t]]$, the affine Grassmannian of G. For general $\gamma \in \mathfrak{g}((t))$, set

$$\mathrm{Gr}^{\zeta,\gamma} = \mathrm{Gr}^\zeta \cap \mathrm{Gr}^\gamma \subseteq \mathrm{Gr},$$

where Gr^ζ is the μ_ℓ-fixed locus of Gr under loop rotation and Gr^γ is the affine Springer fiber for γ. The choice in the theorem is $\gamma = t^{\ell-1}s$, where s is a regular semisimple element of $\mathfrak{g}$. The underlying ind-variety of $\mathrm{Gr}^{\zeta,\gamma}$ is equipped with an action of the extended affine Weyl group $\widetilde{W} := W \ltimes Q$ called the *dot* or *centralizer-monodromy action*. Essentially, W acts via the monodromy of a family of affine Springer fibers into which Gr^γ embeds, and $\lambda \in Q$ acts via translation by t^λ.

Let $\overline{Z} = \mathrm{H}^*(\mathrm{Gr}^{\zeta, t^{\ell-1}s})^{(\widetilde{W}, \cdot)}$. It is conjectured that the inclusion $\overline{Z} \subseteq Z(\mathfrak{u}_\zeta^\vee)^{G^\vee}$ is an equality. Helmsoet–Kivinen–Lachowska prove that

$$\dim \overline{Z} = \mathrm{Cat}_{W, \ell(h+1)-h},$$

another Catalan number.

We sketch the proof. First, by Riche–Williamson, “Smith–Treumann Theory…”, Proposition 4.7, there is a decomposition of the form

$$\mathrm{Gr}^\zeta \simeq \coprod_{[\lambda] \in \Lambda/(W \ltimes \ell Q)} \mathrm{Fl}_\lambda^{(\ell)},$$

where $\mathrm{Fl}_\lambda^{(\ell)} = G((t^\ell))/\mathbf{P}_\lambda$ and $\mathbf{P}_\lambda$ is the parahoric of $G((t^\ell))$ associated with λ. One then shows that the contribution from $\mathrm{Fl}_\lambda^{(\ell)}$ to the cohomology of $\mathrm{Gr}^{\zeta, t^{\ell-1}s}$ takes the form

$$(\mathrm{sgn} \otimes \mathbf{C}[Q/(h+1)Q])^{W_\lambda} \simeq \mathrm{Hom}_W(\mathrm{Ind}_{W_\lambda}^W(1), \mathrm{sgn} \otimes \mathbf{C}[Q/(h+1)Q]).$$

(Probably, the appearance of sgn comes from using $\gamma = t^{\ell-1}s$ as opposed to using $\gamma = t^\ell s$.) Summing over λ, we get isomorphisms of vector spaces:

$$\begin{aligned}
\overline{Z} &\simeq \bigoplus_\lambda \mathrm{Hom}_W(\mathrm{Ind}_{W_\lambda}^W(1), \mathrm{sgn} \otimes \mathbf{C}[Q/(h+1)Q]) \\
&\simeq \bigoplus_{[W']} d_{\ell,[W']} \mathrm{Hom}_W(\mathrm{Ind}_{W'}^W(1), \mathrm{sgn} \otimes \mathbf{C}[Q/(h+1)Q]) \\
&\simeq \mathrm{Hom}_W\left(\mathrm{sgn}, \bigoplus_{[W']} d_{\ell,[W']} \mathrm{Ind}_{W'}^W(1) \otimes \mathbf{C}[Q/(h+1)Q]\right) \\
&\simeq \mathrm{Hom}_W(\mathrm{sgn}, \mathbf{C}[Q/\ell Q] \otimes \mathbf{C}[Q/(h+1)Q]) \\
&\simeq \mathrm{Hom}_W(\mathrm{sgn}, \mathbf{C}[Q/(\ell(h+1))Q]).
\end{aligned}$$

Finally, a trick from, *e.g.*, Springer theory (see Helmsoet–Kivinen–Lachowska Corollary 2.4) shows that $\mathrm{Hom}_W(\mathrm{sgn}, \mathbf{C}[Q/mQ]) \simeq \mathrm{Hom}_W(1, \mathbf{C}[Q/(m-h)Q])$.