Warmup suppose $T : F^2$ to F^2 sends

$v1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ to $\begin{pmatrix} 2 \\ 2 \end{pmatrix}$ $v2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ to $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

matrix of T wrt std basis (e1, e2) of F^2?

1) matrix wrt (v1, v2): $M = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}$

2) v1 = e1 + e2 $B = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$
v2 = –e1 + e2

3) det(B) = 2 [$B^{-1} = \begin{pmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{pmatrix}$]
$\mathrm{adj}(B) = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$

$$
\begin{aligned}
BMB^{-1} &= (1/2)\, B\, M\, \mathrm{adj}(B) \\
&= (1/2) \begin{pmatrix} 2 & 1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \\
&= (1/2) \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1/2 & 3/2 \\ 3/2 & 1/2 \end{pmatrix} \quad \text{[works]}
\end{aligned}
$$

(Axler §5A) given a linear op T : V to V:

Df a subspace W sub V is T-stable,
aka T-invariant,
iff w in W implies Tw in W

Ex above: Fv1, Fv2 are T-stable
Fe1, Fe2 are not

Ex for any linear op T : V to V,
{**0**} and V are are [trivially] T-stable

Ex suppose T : F^3 to F^3 has the matrix

*	*	*	wrt some basis
*	*	*	
0	0	*	

then {(x, y, 0) | x, y} is a nontrivial T-stable subsp.

how about	*	*	0	?	*	*	0	?
	*	*	0		*	*	0	
	*	*	*		0	0	*	

[only {(0, 0, z) | z} for LHS, both for RHS]

a block-diagonal matrix for T with k blocks
corresponds to
a T-stable direct sum V = W_1 + … + W_k

[in particular:]
a diagonal matrix for T
corresponds to
a decomposition of V into T-stable lines

Df for any linear op T : V to V

an <u>eigenline</u> of T is a T-stable line [dim-1 subsp.]

an <u>eigenvector</u> of T is v in V s.t. Fv is an eigenline
(which forces v ≠ **0**)
here, if Tv = λv, then λ is the <u>eigenvalue</u> of T on v

Ex P : F^2 to F^2 given by the matrix wrt the std basis

$$\begin{pmatrix} 0 & 1 \\ 0 & 3 \end{pmatrix}$$

[does P have any eigenvectors?]

$$\begin{pmatrix} \lambda x \\ \lambda y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 3 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ 3y \end{pmatrix}$$

[take $\lambda = 0$:] {(x, 0)} eigenline with eigenvalue 0

[take $\lambda \neq 0$:] {(y/3, y)} eigenline with eigenvalue 3

Ex N : F^2 to F^2 given by the matrix wrt the std basis

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} \lambda x \\ \lambda y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ 0 \end{pmatrix}$$

[regardless of λ:] y = 0

{(x, 0)} is the only eigenline with eigenvalue 0

[same regardless of whether F = R or F = C]

Ex H : R^2 to R^2 given by the matrix wrt the std basis

$$\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} \lambda x \\ \lambda y \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x - y \\ x + y \end{pmatrix}$$

messy to solve…

notice: $(1/\sqrt{2})\, H = \begin{matrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{matrix}$

$= \begin{matrix} \cos(\pi/4) & -\sin(\pi/4) \\ \sin(\pi/4) & \cos(\pi/4) \end{matrix}$

so H is the composition of: rotate by $\pi/4$
scale by $\sqrt{2}$

no H-stable lines through **0**

compare to H' : C^2 to C^2 given by

$\begin{matrix} 1 & -1 \\ 1 & 1 \end{matrix}$

claim: $1 + i$ and $1 - i$ are eigenvalues

$(1 + i)x = x - y$ imply $ix = -y$ and $iy = x$
$(1 + i)y = x + y$

so $\{(x \quad ix)\}$ eigenline with eigenvalue $1 + i$
$\{(x \quad -ix)\}$ eigenline with eigenvalue $1 - i$

<u>Rem</u> C^2 is iso to the complex'n of R^2

$$\begin{matrix} (R^2)_C & \xrightarrow{H_C} & (R^2)_C \\ = & & \\ C^2 & \xrightarrow[H']{} & C^2 \end{matrix}$$

Moral choice of $\mathbb{R}$ vs $\mathbb{C}$ affects eigenstuff

Thm suppose $F = \mathbb{C}$

V is fin. dim. and not $\{\mathbf{0}\}$

then any linear op on V has an eigenline

key idea: plug linear op T into polynomials

$p(z) = \sum_k a_k z^k$ gives

$p(T) = \sum_k a_k T^k$ where T^k = kth iterate of T

note: constant term a_0 treated as $a_0 \, \mathrm{id}_V$

Lem for any F and V fin. dim. and v in V:

some nonzero $p(z)$ gives $p(T) v = \mathbf{0}$

Pf set $n = \dim V$

$v, Tv, \ldots, T^n v$ must be lin. dep.

so there are $a_0, \ldots, a_n$ in F s.t.

$a_i \neq 0$ for some i,

$(a_0 + a_1 T + \ldots + a_n T^n) v = \mathbf{0}$

Pf of Thm since $V \neq \{\mathbf{0}\}$, can pick $v \neq \mathbf{0}$

using lemma, pick $f(z)$ of minimal deg s.t.

f is nonzero and $f(T) v = \mathbf{0}$

since $v \neq \mathbf{0}$, know f is nonconst

by the fund. thm of algebra, f has a root λ: i.e.,

$f(z) = (z - \lambda)\, g(z)$ for (nonzero) $g(z)$ in $\mathbb{C}[z]$

now $(T - \lambda\, id\_V)\, (g(T)\, v) = \mathbf{0}$
notice $T(g(T)v) = \lambda(g(T)v)$
so just need $g(T)\, v \neq \mathbf{0}$
if g nonconst, then done bc $\deg(g) < \deg(f)$
if g const, then done bc g nonzero and $v \neq \mathbf{0}$ □

[where did we use F = C? fund. thm of algebra]

Summary

if W sub V is stable under T : V to V
then T restricts to a lin op T|_W, easier to study

nicer when	V is a sum of T-stable subspaces
nicest when	V is a sum of eigenlines
	i.e., T is diagonalizable

over R	T may have no eigenlines
over C	T will have some eigenline,
	but V need not be sum(eigenlines)

Rem the sum of all eigenlines with eigenval λ is called the λ-eigenspace of T