MATH 430
Introduction to Topology

mqtrinh.github.io/math/teaching/yale/math-430/

9 psets	36%
1 midterm Mon 2/24	24%
1 final	40%

Munkres, Topology, 2nd Ed.

[late hw policy]
[schedule]
[intros]

I) algebraic topology:

how to tell shapes apart?

Euler: Königsberg (1736), “V – E + F = 2” (1750)
Poincaré: Analysis situs (1890s–1900s)

II) point-set topology:

what IS shape?

Weyl: rigorous defn of “surface” (1913)
Hausdorff: topological spaces (1914)

[we first do (II), then do (I)]

(Munkres §12) fix a set X

Def a topology on X is
a collection T of subsets of X

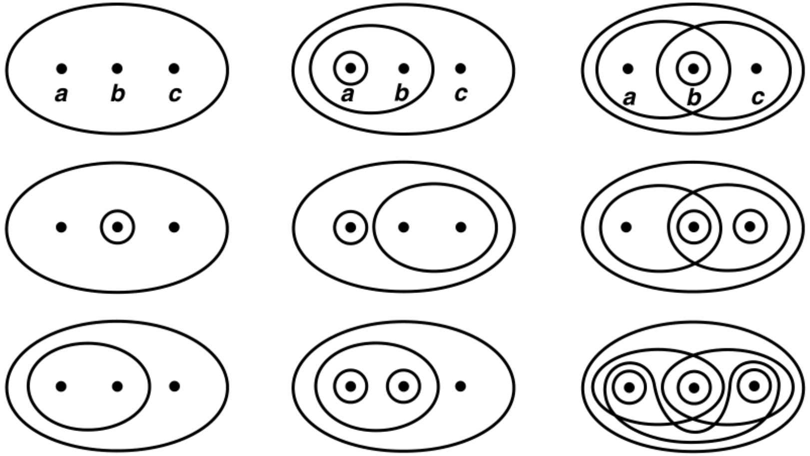
s.t. 1) Ø and X are in T

2) if {U_i}_i is a subcollection of T, then the union of the U_i in X is an element of T

3) if {U_i}_i is a finite subcollection of T, then the intersection of the U_i in X is an element of T

we say that T is a topology on X
the elements of T are its open sets

Ex X = {a, b, c}



[in each case, Ø is not depicted, of course]

[give a collection of subsets that isn’t a topology?]

Ex X = R^n with |x| = sqrt(sum_i x_i^2)

write B(x, δ) = {y in R^n | |y – x| < δ}

we say U sub R^n is analytically open iff
for all x in U,
there is some δ > 0 s.t. B(x, δ) sub U

Thm {analytically open sets} is a topology on R^n

1) easy
2) suppose U_i anlytc opens, U = bigcup_i U_i
pick x in U
x belongs to x in U_j for some j
pick δ > 0 s.t. B(x, δ) sub U_j sub U

3) suppose finitely many i,
U_i analytic opens, V = bigcap_i U_i
pick x in V
for all i, pick δ_i > 0 s.t. B(x, δ_i) sub U_i

[what next?] let δ = min_i δ_i
then B(x, δ) sub B(x, δ_i) sub U_i for all i
therefore B(x, δ) sub V

[observe: 3) wouldn't work for infinite δ_i → 0]

we call this the analytic topology T_{an} on R^n

Rem far from the only topology possible!
always have discrete and indiscrete topologies

Ex X arbitrary

$T_f = \{\emptyset\} \cup \{U \subset X \text{ s.t. } X - U \text{ is finite}\}$

Prop T_f is a topology on X

1) easy
2) suppose $X - U_i$ finite for all i
   $X - \bigcup_i U_i = \bigcap_i U_i$
   [what if instead $U_i = \emptyset$ for some i?]
3) suppose $X - U_i$ finite for all i
   $X - \bigcap_i U_i = \bigcup_i X - U_i$
   unions of finite sets are finite
   [what if instead $U_i = \emptyset$ for some i?]

we call this the finite complement topology

Df given topologies T, T′ on the same X

when T is a subcollection of T′, we say that
T is coarser than T′
and T′ is finer than T

[T′ is more refined: it sees more open sets]

Ex [how do the topologies on R^n compare?
analytic, discrete, indiscrete, finite-comp]

$T_{indisc} \subset T_f \subset T_{an} \subset T_{disc}$

Rem topologies can be incomparable:
think about X = {a, b, c}

Ex X = Z, the set of integers

we say U sub Z is evenly spaced iff
U is a union of sets of the form aZ + b
with a, b in Z and a ≠ 0

Prop {evenly spaced sets} is a topology on Z

[assume this for now]

Cor there are infinitely many prime numbers

[deduction observed by Furstenberg in 1955]

Proof assume finitely many primes p

then Z – bigcup_p pZ = bigcap_p (Z – pZ) is open
because Z – pZ is open for all p

but Z – bigcup_p pZ = {±1}
because if |a| > 1, then some prime divides a

so {±1} is open, but not evenly spaced □