Thm if V is fin dim and T : V to W is linear,
then dim V = dim ker(T) + dim im(T)

Pf Outline

pick basis {w_1, …, w_r} for im(T)
pick u_1, …, u_r s.t. T(v_i) = w_i for all i
let U = span(u_1, …, u_r)
pick basis v_1, …, v_k for ker(T)
I) {u_1, …, u_r} is a basis for U
II) ker(T) + U = V
II) ker(T) + U is a direct sum
then dim V = dim ker(T) + dim im(T) □

Pf of I) if sum_i a_iu_i = 0, then
sum_i a_iT(u_i) = 0 so a_i = 0 for all i

Pf of II) pick v in V

know T(v) = sum_i b_iw_i for some b_i
so T(v) = sum_i b_iT(u_i) = T(sum_i b_iu_i)
so T(v – sum_i b_iu_i) = **0_W**
so v – sum_i b_iu_i in ker(T)
so v in ker(T) + U

Pf of III) recall: W + U is a direct sum iff
W cap U = {**0**}
so want ker(T) cap U = {**0_V**}

pick v in ker(T) cap U
since v in U, have v = sum_i a_iu_i for some a_i
since v in ker(T), have sum_i a_iw_i = T(v) = **0_W**
but {w_i}_i lin. indep. so a_i = 0 for all i

assuming V, W are both finite-dimensional:

Cor if T : V to W is linear and dim V > dim W,
then [?] T is not injective

Pf dim im(T) ≤ dim W < dim V,
so
dim ker(T) = dim V – dim im(T) > 0

Cor if T : V to W is linear and dim V < dim W,
then T is not surjective

Pf exercise

Cor if T : V to W is linear and bijective,
then [?] dim V = dim W [converse false!]

(Axler §3D)

Thm TFAE for a linear map T : V to W:

1) T is bijective
2) T takes any basis for V onto a basis for W
3) T takes some basis for V onto one for W
4) there is a linear map S : V to W s.t.
S(T(v)) = v and T(S(w)) = w

Df in the situation above, we say that T is
a linear isomorphism from V onto W

Pf of Thm not hard to show that
2) implies 3) implies 4) implies 1)
remains to show 1) implies 2)

so suppose $\{e_i\}_i$ is a basis for V

claim that $\{T(e_i)\}_i$ spans W:

for all w in W, have $w = T(v)$ for some v in V

by surjectivity of T

$v = \sum_i a_i e_i$ for some a_i

so $w = \sum_i a_i T(e_i)$

claim that $\{T(e_i)\}_i$ is lin. indep.

suppose $\sum_i b_i T(e_i) = \mathbf{0}_W$

then $T(\sum_i b_i e_i) = \mathbf{0}_W$

so $\sum_i b_i e_i$ in $\ker(T)$

so $\sum_i b_i e_i = \mathbf{0}_V$

by injectivity of T

so $b_i = 0$ for all i

by lin. independence of $\{e_i\}_i$ □

(Axler §3C) [recap:]

Slogan #1 [if we know a basis for V, then]

a linear map V to W is det by

where it sends [the] a basis

and any choices will do

Slogan #2 a linear isomorphism is

a linear map taking bases to bases

<u>Ex</u> there's a linear map from F^4 to $F[x]$

sending the ith std basis vec to x^{i-1}

it <u>restricts</u> to a linear iso from F^4 to P_3

where $P_3 = \{p \mid p = 0 \text{ or } \deg p \leq 3\}$

[in this sense, linear maps out of F^n are easy]
[what else can we do with linear maps?]
a composition of linear maps is linear: given linear

$A : V$ to W,
$B : W$ to U,

$B(A(v + v')) = B(A(v) + A(v')) = B(A(v)) + B(A(v'))$,
$B(A(c \bullet v)) = B(c \bullet A(v)) = c \bullet B(A(v))$

so $B \circ A : V$ to U is also linear

suppose V with <u>ordered</u> basis $(v_1, \ldots, v_n)$,
W … $(w_1, \ldots, w_m)$,
U … $(u_1, \ldots, u_\ell)$
$A(v_i) = \sum_j a_{j,i} w_j$,
$B(w_j) = \sum_k b_{k,j} u_k$

$$
\begin{aligned}
B(A(v_i)) &= B(\textstyle\sum_j a_{j,i} w_j) \\
&= \textstyle\sum_j a_{j,i} B(w_j) \\
&= \textstyle\sum_j a_{j,i} \sum_k b_{k,j} u_k \\
&= \textstyle\sum_{k,j} a_{j,i} b_{k,j} u_k \\
&= \textstyle\sum_k c_{k,i} u_k \\
&\qquad \text{where } c_{k,i} = \textstyle\sum_j a_{j,i} b_{k,j}
\end{aligned}
$$

matrix multiplication = shorthand for these calc's

<u>Df</u> matrix of A wrt the ordered bases $(v_i)_{i = 1}^n, (w_j)_{j = 1}^m$:

$$
\begin{matrix}
a11 & a12 & \ldots & a1n \\
a21 & a22 & \ldots & a2n \\
\ldots & \ldots & & \ldots \\
am1 & am2 & \ldots & amn
\end{matrix}
$$

rows is n (dim V),
cols is m (dim W)

henceforth write F^n, F^m, etc in <u>column notation</u>:

$$v = \sum_i c_i v_i: \quad \begin{matrix} c_1 \\ c_2 \\ \dots \\ c_n \end{matrix}$$

matrix × vector rule for Av:

$$\begin{matrix} a_{11} & \dots & a_{1n} \\ a_{21} & \dots & a_{2n} \\ \dots & & \dots \\ a_{m1} & \dots & a_{mn} \end{matrix} \quad \begin{matrix} c_1 \\ c_2 \\ \dots \\ c_n \end{matrix} \quad = \quad \begin{matrix} \dots \\ a_{j1} c_1 + \dots + a_{jn} c_n \\ \dots \\ \dots \end{matrix}$$

e.g., Av_i: $\begin{matrix} a_{1i} \\ a_{2i} \\ \dots \\ a_{mi} \end{matrix}$ (ith matrix col)

matrix × matrix rule for $B \circ A$:

$$\begin{matrix} b_{11} & \dots & b_{1m} \\ b_{21} & \dots & b_{2m} \\ \dots & & \dots \\ b_{\ell 1} & \dots & b_{\ell m} \end{matrix} \quad \begin{matrix} a_{11} & \dots & a_{1n} \\ a_{21} & \dots & a_{2n} \\ \dots & & \dots \\ a_{m1} & \dots & a_{mn} \end{matrix}$$

$$= \quad \begin{matrix} \dots & \dots & \dots \\ \dots & \sum_j a_{j,i} b_{kj} & \dots \\ \dots & \dots & \dots \\ \dots & \dots & \dots \end{matrix}$$

so	vectors	become	columns,
	linear maps		matrices,
	$\circ$		matrix multiplication

Warning

if dim V = dim W, then matrix is square

e.g., if V = W

but even if V = W,
the ordered bases $(v_i)_i$, $(w_j)_j$ can still differ!
i.e., rows and cols not indexed the same

Convention

if $V = F^n$ and $W = F^m$, then take the std bases

e.g., if V = W, then $(v_i)_i$ = std = $(w_j)_j$