# What Gauss Knew about Knots & Braids

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This talk is inspired by the paper:

M. Epple. Orbits of Asteroids, a Braid, and the First Link Invariant. *Math. Intelligencer*, **20**(1) (1998), 45–52.

It discusses:

**1** Gauss's integral to compute the *linking number* of two closed curves.

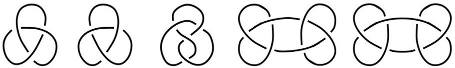
**2** Gauss's sketch of a 4-strand braid.

We'll go further by discussing how these ideas evolved in the 20th and 21st centuries.

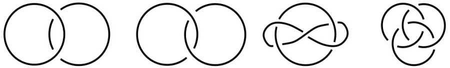
We'll also use many pictures from *Introduction to Vassiliev Knot Invariants* by Chmutov–Duzhin–Mostovoy.

In modern language, we'll be interested in KNOTS and their more general cousins, LINKS.

Some knot (diagram)s:



Some link (diagram)s:



(What mathematicians call a knot is what sailors and climbers would call a *grommet*.)

A *knot* is the image of a (smooth) injective map from the circle $S^1$ into 3-space $\mathbf{R}^3$.

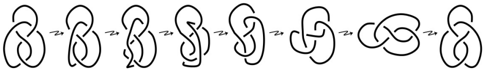
Two such maps $u, v : S^1 \to \mathbf{R}^3$ are *isotopic* iff there exists a (smooth) map

$$\phi : S^1 \times [0,1] \to \mathbf{R}^3$$

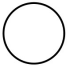
such that:

- $\phi_t = \phi(-, t) : S^1 \to \mathbf{R}^3$ defines a knot for all $t \in [0,1]$.
- $\phi_0 = u$ and $\phi_1 = v$.

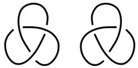
Then we say that the associated knots are *isotopic* as well.



Perhaps the simplest kind of knot is an *unknot*:

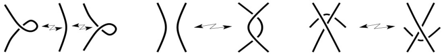


Below are left- and right-handed *trefoils*, respectively:



Are the trefoils isotopic to the unknot? To each other?

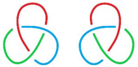
**Thm (Reidemeister)** Two knots are isotopic iff they differ by some sequence of the following local "moves":



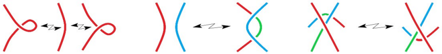
Let's say that a knot is *tricolor* iff, in some diagram, each arc can be colored so that

- Globally, at least *two* colors are used.
- At each crossing, either the three arcs are all the *same* color, or they each have a *different* color.

An unknot is not tricolor, while a trefoil is.



Tricolorability is preserved by Reidemeister's moves.



This proves that the trefoils are not isotopic to the unknot.

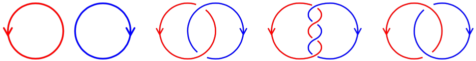
A *link of $n$ components* is the image of an injective map

$$\underbrace{S^1 \sqcup \cdots \sqcup S^1}_{n \text{ copies}} \to \mathbf{R}^3.$$

Again, we'd like a way to show that various links are not isotopic.

Let's restrict attention to links of 2 components.

If the components are *oriented*, then we have an intuitive idea of the number of times they intertwine:



should have *linking numbers* $0, 1, 2$, and $-1$, respectively.

Gauss (1777-1855) lived a century before Reidemeister (1893-1971).

Back then, people didn't bother to define isotopy rigorously. Topology was still called *geometria situs*.

Yet in 1833, Gauss found a precise definition for the linking number $\langle K_1, K_2 \rangle$ of an oriented link $K_1 \sqcup K_2$.

**Def-Thm (Gauss)** If $K_1, K_2$ admit parametrizations $v_1, v_2 : S^1 \to \mathbf{R}^3$, then

$$\langle K_1, K_2 \rangle = \frac{1}{4\pi} \oint_{K_1} \oint_{K_2} \frac{v_2 - v_1}{|v_2 - v_1|^3} \cdot (dv_1 \times dv_2).$$

Apparently, Gauss was led to this integral by his study of celestial mechanics (and later, electromagnetic induction).

But his key insight is purely mathematical. We'll describe it with modern machinery.

As setup, define *(ordered) 2-point configuration space* to be

$$\mathit{Conf}_2 = \{(v_1, v_2) \in \mathbf{R}^3 \times \mathbf{R}^3 : v_1 \neq v_2\}.$$

Define the *Gauss map* $\Gamma : \mathit{Conf}_2 \to S^2 \subseteq \mathbf{R}^3$ by

$$\Gamma(v_1, v_2) = \frac{v_2 - v_1}{|v_2 - v_1|}.$$

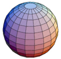
That is, $\Gamma(v_1, v_2)$ is the unit vector pointing from $v_1$ to $v_2$.

Now, a 2-component link

$$v_1 \sqcup v_2 : S^1 \sqcup S^1 \to \mathbf{R}^3$$

gives rise to a composite map

$$\Gamma_{v_1,v_2} \quad : \quad S^1 \times S^1 \xrightarrow{v_1 \times v_2} Conf_2 \xrightarrow{\Gamma} S^2$$



Both the torus $S^1 \times S^1$ and the unit sphere $S^2$ are *closed oriented surfaces*.

Any map between such surfaces has an integer *degree*.

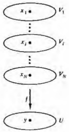
What Gauss's integral really computes is $\deg \Gamma_{v_1,v_2} \in \mathbf{Z}$.
We'll only give a vague sketch.

Gauss's integrand is the Jacobian of $\Gamma_{v_1,v_2}$. His integral

$$\frac{1}{4\pi} \iint_{S^1 \times S^1} \mathrm{Jac}(\Gamma_{v_1,v_2})$$

is a ratio of signed surface areas—roughly, the number of times that $S^1 \times S^1$ wraps over its image in $S^2$.

The preimage of a *generic* small circle in $S^2$ is a disjoint union of circles in $S^1 \times S^1$:



Each has a signed *local degree*.

$\deg \Gamma_{v_1,v_2}$ is their sum, which matches the number above.

In knot theory, there's a tension between:

- "Geometric" definitions that don't rely on a choice of diagram, *e.g.*, Gauss's linking integral.
- "Diagrammatic" definitions, *e.g.*, tricolorability.

In 1928, Alexander found an invariant

$$\Delta : \{links\}/isotopy \longrightarrow \mathbf{Z}[q]$$

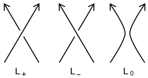
that had both geometric and diagrammatic definitions.

In 1985, Jones found a different invariant

$$\mathbf{V} : \{links\}/isotopy \longrightarrow \mathbf{Z}[q^{\pm 1}]$$

that only had a diagrammatic definition. Unlike $\Delta$, it could distinguish left- and right-handed trefoils.

Consider the local crossings




$\mathbf{V}$ is defined inductively by the local rule

$$q^{-2}\mathbf{V}(L_+) - q^2\mathbf{V}(L_-) = (q - q^{-1})\mathbf{V}(L_0)$$

and the base case $\mathbf{V}(unknot) = 1$.

In 1989, Witten gave a surprising *geometric* definition of $\mathbf{V}$, in terms of path integrals in a quantum field theory.

Reshetikhin–Turaev showed his formula to be rigorous, but only by returning to diagrammatics.

Here's a more recent integral in knot theory.

How can we detect *three* components that are linked without being *pairwise* linked?

Let $Conf_3 = \{(v_1, v_2, v_3) \in (\mathbf{R}^3)^3 : v_1 \neq v_2 \neq v_3 \neq v_1\}$.

We want a map $\Gamma : Conf_3 \to S^2$, so that

$$v_1 \sqcup v_2 \sqcup v_3 : S^1 \sqcup S^1 \sqcup S^1 \to \mathbf{R}^3$$

gives rise to

$$\Gamma_{v_1,v_2,v_3} : S^1 \times S^1 \times S^1 \xrightarrow{v_1 \times v_2 \times v_3} Conf_3 \xrightarrow{\Gamma} S^2.$$

**Thm (D–G–K–M–N–S–V*)** Given $v_1, v_2, v_3 \in \mathbf{R}^3$, let

$$\delta_1 = v_3 - v_2, \quad \delta_2 = v_1 - v_3, \quad \delta_3 = v_2 - v_1,$$

and let $\tilde{\Gamma}(v_1, v_2, v_3)$ be the sum

$$\frac{\delta_1}{|\delta_1|} + \frac{\delta_2}{|\delta_2|} + \frac{\delta_3}{|\delta_3|} + \frac{\delta_1 \times \delta_2}{|\delta_1||\delta_2|} + \frac{\delta_2 \times \delta_3}{|\delta_2||\delta_3|} + \frac{\delta_3 \times \delta_1}{|\delta_3||\delta_1|}.$$

Then:

**1** $\Gamma = \tilde{\Gamma}/|\tilde{\Gamma}|$ is a map $Conf_3 \to S^2$.

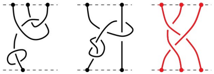
**2** The "Pontryagin $\nu$-invariant" of $\Gamma_{v_1,v_2,v_3}$ detects the Borromean rings.

The blue part only vanishes when $\triangle(v_1, v_2, v_3)$ is equilateral; the green part only vanishes when $v_1, v_2, v_3$ are collinear. Moreover, they are orthogonal.

* DeTurck, Gluck, Komendarczyk, Melvin, Nuchi, Shonkwiler, Vela-Vick (2013)

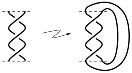
In 1923, before discovering $\Delta$, Alexander had shown that every link can be "combed" into the closure of a braid.

A *braid* is like a link, but it connects $n$ inputs at one end of a box to $n$ outputs at the other without trackbacks.



Only the red diagram depicts a braid (for $n = 3$).

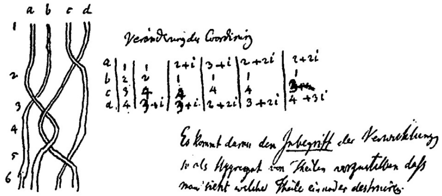
A braid $\beta$ can be closed up into a link $\hat{\beta}$:



Different braids can give the *same* link!

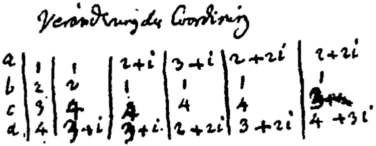
As we'll explain, the braids on $n$ strands form a *group*, which Emil Artin introduced in 1947.

Yet the following sketch appears in Gauss's notebooks between 1815 and 1830.




Did Gauss hope to construct an isotopy invariant of braids?

A close-up of the table of numbers:


Veränderung der Coordinaten

| | | | | | | |
|---|---|---|---|---|---|---|
| a | 1 | 1 | 2+i | 3+i | 2+2i | 2+2i |
| b | 2 | 2 | 1 | 1 | 1 | 1 |
| c | 3 | 4 | 4 | 4 | 4 | ~~3~~ |
| d | 4 | 3+i | 3+i | 2+2i | 3+2i | 4+3i |

How do the changes between columns of numbers correspond to crossings in the braid?

Epple's article offers a precise interpretation, after remarking:

> *"While Gauss was looking for a notation. . . to decide whether or not two braids were equivalent, he came close to defining a nontrivial invariant for braids, namely[,] the last row of the table he set up. . . "*

Groups lurk implicitly throughout much of Gauss's work. Strangely, he never took interest in developing an abstract definition.

A *group* is a set $G$ equipped with a binary operation

$$(-) \circ (-) : G \times G \longrightarrow G$$

and an *identity* $\mathbf{1} \in G$ such that:

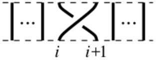
- For all $x, y, z \in G$, we have $(x \circ y) \circ z = x \circ (y \circ z)$.
- For all $x \in G$, we have $x \circ \mathbf{1} = \mathbf{1} \circ x = x$.
- Every $x \in G$ has an *inverse* $x^{-1} \in G$ such that $x \circ x^{-1} = x^{-1} \circ x = \mathbf{1}$.

The *symmetric group* $G = S_n$ is the set of self-bijections of $\{1, \ldots, n\}$, where $\circ$ is composition of maps and $\mathbf{1} = \mathrm{id}$.

The set $Br_n$ of isotopy classes of braids on $n$ strands gives rise to a group, where:

- The operation $\circ$ is concatenating braids end-to-end.
- The identity **1** is the braid that connects inputs to outputs by straight lines, without any crossings.

We can *generate* $Br_n$ by the elements $\sigma_1, \ldots, \sigma_{n-1}$, where $\sigma_i$ is (the isotopy class of)



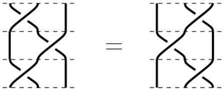

That is, every braid is formed by repeated concatenation of the $\sigma_i$ and $\sigma_i^{-1}$ in some order.

Once you've found a set of generators for a group, it remains to find the *relations* among the "words" they generate.

Artin showed that $Br_n$ only has two types of relations:

**1** $\quad \sigma_i \circ \sigma_j = \sigma_j \circ \sigma_i$ when $|i - j| \geq 2$:

**2** $\quad \sigma_i \circ \sigma_{i+1} \circ \sigma_i = \sigma_{i+1} \circ \sigma_i \circ \sigma_{i+1}$ for all $i$:



Note that (2) is an analogue of a Reidemeister move.

Recall that a braid $\beta$ gives rise to a link closure $\hat{\beta}$.

Notably, if $\alpha, \beta \in Br_n$, then it turns out $\widehat{\alpha \circ \beta} = \widehat{\beta \circ \alpha}$.

So, to study isotopy invariants of links, we might study functions $f$ on $Br_n$ such that

$$f(\alpha \circ \beta) = f(\beta \circ \alpha)$$

always holds. These are called *class functions*.

This is how Jones (with input of Birman) discovered

$$\mathbf{V} : \{\mathit{links}\} \to \mathbf{Z}[q^{\pm 1}].$$

Ocneanu refined his method to get a sharper invariant

$$\mathbf{P} : \{\mathit{links}\} \to \mathbf{Z}[a^{\pm 1}, z^{\pm 1}].$$

We conclude this talk by describing $\mathbf{P}$.

Using the $\sigma_i$, fix inclusions $Br_1 \subseteq Br_2 \subseteq Br_3 \subseteq \ldots$

**Thm (Ocneanu)** The following rules uniquely determine $\mathbf{Z}[z^{\pm 1}]$-linear functions

$$\mathrm{tr}_n : \mathbf{Z}[z^{\pm 1}][Br_n] \to \mathbf{Z}[a^{\pm 1}, z^{\pm 1}]$$

such that $\mathrm{tr}_n(\alpha \circ \beta) = \mathrm{tr}_n(\beta \circ \alpha)$ for all $n$ and $\alpha, \beta \in Br_n$:

**1** $\mathrm{tr}_1(\mathbf{1}) = 1$.

**2** $\mathrm{tr}_n(\beta) = z^{-1}(a - a^{-1})\,\mathrm{tr}_{n-1}(\beta)$ for all $\beta \in Br_{n-1}$.

**3** $\mathrm{tr}_n(\sigma_{n-1}^{\pm 1} \circ \beta) = -a^{\mp 1}\,\mathrm{tr}_{n-1}(\beta)$ for all $\beta \in Br_{n-1}$.

**4** $\sigma_i \circ \sigma_i = \mathbf{1} + z\sigma_i$ for all $i$.

Rule (4) is called the *Iwahori–Hecke relation*.

**Thm (Jones–Ocneanu)** If $\beta \in Br_n$ has length $\ell$ in the generators $\sigma_1, \ldots, \sigma_{n-1}$, then

$$\mathbf{P}(\hat{\beta}) = (-a)^{\ell}\,\mathrm{tr}_n(\beta)$$

only depends on the link closure $\hat{\beta}$, not on $\beta$ itself(!).

$\mathbf{P}$ was discovered independently by several teams of mathematicians.

It's named the *(reduced) HOMFLYPT invariant* after their initials.

**Ex** If $\beta = \sigma_1 \circ \sigma_2 \circ \sigma_1 \circ \sigma_2 \in Br_3$, then $\hat{\beta}$ is a trefoil, and

$$\mathrm{tr}_3(\beta) = (z^2 + 2)a^{-2} - 1,$$

so $\mathbf{P}(\hat{\beta}) = (z^2 + 2)a^2 - a^4$. By comparison, $\mathbf{P}(\mathit{unknot}) = 1$.

*Thank you for listening.*