

## Last time

Thm if  $f : X \rightarrow Y$  and  $g : Y \rightarrow X$  form  
a homotopy equivalence

then  $f^* : \pi_1(X, x) \rightarrow \pi_1(Y, f(x))$   
 $g^* : \pi_1(Y, y) \rightarrow \pi_1(X, g(y))$

are isomorphisms for all  $x$  in  $X$  and  $y$  in  $Y$

easier special cases:

Ex for any  $X$  and  $x$ ,

$(\text{id}_X)^* = \text{id}_{\{\pi_1(X, x)\}}$   
as maps  $\pi_1(X, x)$  to  $\pi_1(X, x)$

## Ex

if  $f : X \rightarrow Y$  is a homeo  
then  $f^* : \pi_1(X, x) \rightarrow \pi_1(Y, f(x))$  is  
an isomorphism

Lem 1 given  $f : X \rightarrow Y$ ,  
 $g : Y \rightarrow Z$ ,

$(g \circ f)^* = g^* \circ f^*$   
as maps  $\pi_1(X, x)$  to  $\pi_1(Z, g(f(x)))$

so if  $f$  and  $g$  are two-sided inverses of each other,

$g^* \circ f^* = (\text{id}_X)^* = \text{id}_{\{\pi_1(X, x)\}}$   
 $f^* \circ g^* = (\text{id}_Y)^* = \text{id}_{\{\pi_1(Y, f(x))\}}$

[ Pf of Thm

will show that  $f_*$  is an iso  
as argument for  $g$  is similar

by Lem 1,

$$\begin{aligned} g_* \circ f_* &= (g \circ f)_* \\ f_* \circ g_* &= (f \circ g)_* \end{aligned}$$

Lem 2 if  $s : G$  to  $H$  and  $r : H$  to  $K$

s.t.  $r \circ s$  is bijective,  
then  $s$  is injective and  $r$  is surjective

so to show  $f_*$ ,  $g_*$  bijective,  
enough to show  $(g \circ f)_*$  and  $(f \circ g)_*$  bijective

will show that  $(g \circ f)_*$  is bijective  
as argument for  $f \circ g$  is similar

pick homotopy  $\varphi$  from  $g \circ f$  to  $\text{id}_X$

Lem 3

if  $\alpha$  is a path in  $X$  from  $x$  to  $x'$   
then iso  $\hat{\alpha} : \pi_1(X, x)$  to  $\pi_1(X, x')$   
def by  $\hat{\alpha}([y]) = [\bar{\alpha} * y * \alpha]$

further: if  $F, G : A$  to  $X$  are cts,  
 $\varphi$  is a homotopy from  $F$  to  $G$ ,  
 $a$  in  $A$ ,

then  $\alpha_\varphi = \varphi(a, -)$  is a path from  $F(a)$  to  $G(a)$  s.t.

$$G_* = \hat{\alpha_\varphi} \circ F_*$$

now:  $(g \circ f)_* = \hat{\alpha_\varphi} \circ (\text{id}_X)_* = \hat{\alpha_\varphi}$   
so by Lem 3,  $(g \circ f)_*$  is an iso ]

(Munkres §59) [more examples of  $\pi_1$ 's:]

Thm  $\pi_1(S^n)$  is trivial for all  $n \geq 2$

Df in general, a space is simply-connected iff it is path-connected with trivial  $\pi_1$

"Pf of Thm" given  $x$  in  $S^n$  and a loop  $y$  at  $x$ , pick  $p$  not in the image of  $y$   $S^n - \{p\}$  is homeomorphic to  $R^n$

$R^n$  is simply-connected

i.e., any loop in  $R^n$  is  $\sim_p$  constant loop

so  $y \sim_p$  constant loop at  $x$  within  $S^n - \{p\}$

so  $y \sim_p$  constant loop at  $x$  within  $S^n$

but the map  $y : [0, 1]$  to  $X$  could be surjective!

to fix: show that  $y \sim_p$  some non-surjective loop [somewhat hard]

Thm for any  $x$  in  $X$  and  $y$  in  $Y$ ,

$$\pi_1(X \times Y, (x, y)) \simeq \pi_1(X, x) \times \pi_1(Y, y)$$

Cor  $\pi_1(S^1 \times S^1) \simeq Z \times Z$  under +

(Munkres §70) by contrast:

$$\pi_1(\text{figure-eight}) \simeq Z * Z \text{ [free group]}$$

special cases of the Seifert–van Kampen Thm