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Preface

This issue of Pure and Applied Mathematics Quarterly is dedicated to Professor George Lusztig.

Professor George Lusztig was born in 1946 in Timisoara, Romania. He graduated from the University of Bucharest in 1968, and received both the M.A. and Ph.D. from Princeton University in 1971 under the direction of Michael Atiyah and William Browder. He held faculty positions at the University of Warwick before joining MIT as a Full Professor in 1978. He was appointed Norbert Wiener Professor at MIT during 1999–2009. He is currently the Abdun-Nur Professor of Mathematics at MIT.

Professor Lusztig has received numerous research distinctions, including the Berwick Prize of the London Mathematical Society (1977), the AMS Cole Prize in Algebra (1985), and the Brouwer Medal of the Dutch Mathematical Society (1999). In 2008, he received the AMS Leroy P. Steele Prize for Lifetime Achievement "for entirely reshaping representation theory, and in the process changing much of mathematics". In 2014, he received the Shaw Prize in Mathematical Sciences "for his fundamental contributions to algebra, algebraic geometry, and representation theory, and for weaving these subjects together to solve old problems and reveal beautiful new connections". In 2022, he was awarded the Wolf Prize in Mathematics "for groundbreaking contributions to representation theory and related areas".

Professor Lusztig is a Fellow of the Royal Society (1983), a Fellow of the American Academy of Arts and Sciences (1991), and a Member of the National Academy of Sciences (1992).

As of 2023, Professor Lusztig has written close to 300 papers. Amazingly, his productivity (measured in terms of number of papers per year) has been almost monotonically increasing against his age. For instance, about half of his papers were written after year 2000.

It is beyond the scope of this Preface to give a serious account of Lusztig's work. We refer the readers to the following excellent surveys for more details:

- R. W. Carter, A survey of the work of George Lusztig.
  Nagoya Math. J. 182 (2006), 1–45.
  Errata, Nagoya Math. J. 183 (2006), (i)–(ii).
- G. Lusztig, Algebraic and geometric methods in representation theory. arXiv:1409.8003. (This is an expanded version of a lecture given by Lusztig in Hong Kong in 2014 when he was awarded the Shaw prize.)

Professor Lusztig's work addresses central questions in almost all aspects of representation theory and Lie theory: complex representations of finite Chevalley groups and p-adic groups, modular representations of algebraic groups and Lie algebras, representations of (finite and affine) Lie algebras, quantum groups and canonical basis, structure and representations of Weyl groups and Hecke algebras, total positivity, etc. In each of these subjects, he either solves the fundamental questions completely (such as the complete classification of irreducible complex representations of finite Chevalley groups, the complete classification of simple modules of affine Hecke algebras, and complete classification of unipotent representations of p-adic groups), or makes deep conjectures that becomes the guiding principles for subsequent research (such as the Kazhdan-Lusztig conjecture, and the conjectural character formula for modular representations). His work has completely reshaped representation theory since the last quarter of the 20th century.

The tools he uses to prove deep results in algebra are often from algebraic geometry and topology, including étale cohomology, intersection cohomology and K-theory. Here are some famous examples from Lusztig's work: the Deligne-Lusztig construction of virtual representations of finite Chevalley groups uses étale cohomology of varieties defined over finite fields; the Kazhdan-Lusztig polynomials are interpreted as the graded dimension of local intersection cohomology groups of Schubert varieties; the characters of irreducible representations of the finite Chevalley groups are computed from Lusztig's character sheaves; and the irreducible modules of the affine Hecke algebra are classified using equivariant K-theory or equivariant homology of Springer fibers (Kazhdan-Lusztig; Lusztig). Because of his pioneering use of geometric methods in representation theory, he is universally recognized as one of the founders of the subject nowadays known as Geometric Representation Theory.

On the other hand, simple clean statements and *tour de force* calculations are also important characteristics of Professor Lusztig's work. He makes considerable effort to formulate his results in as elementary terms as possible. He is not satisfied with defining an abstract concept but always looks for algorithms for computing various quantities that are amenable to computer calculations. Many of his deep uniform results were first discovered by thorough calculations (usually by hand) in a case-by-case manner.

Professor Lusztig has supervised 20 PhD students. Many mathematicians of the younger generation have been greatly influenced by Lusztig through his mentoring, communications and by reading his books and papers.

Professor Lusztig has also contributed tremendously to the mathematical community. He donated a part of his Shaw Prize money to establish the

AMS Chevalley Prize in Lie theory, awarded every other year since 2016. He also donated money to establish the George Lusztig PRIMES mentorships to award PhD students and postdocs for being excellent mentors for high school research projects.

On behalf of the editors, we would like to thank all the authors who have contributed to the special volume. Professor Lusztig's mathematical impact will be clearly seen from these papers. Finally, we would like to thank Professor Lusztig for making representation theory and Lie theory such rich and beautiful subjects.

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A binary operation on irreducible components of Lusztig's nilpotent varieties I: definition and properties

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To George Lusztig, with admiration

Abstract: We define a binary operation on the set of irreducible components of Lusztig's nilpotent varieties of a quiver. We study commutativity, cancellativity and associativity of this operation. We focus on rigid irreducible components and discuss inductive ways to construct them.

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1. Introduction

In this paper we define and study a binary operation on the set Comp of irreducible components of Lusztig's nilpotent varieties (of all graded dimensions) pertaining to a finite quiver Q without loops. Recall that Comp parameterizes Lusztig's canonical basis of the negative part U^- of the quantum enveloping algebra pertaining to Q, as well as the vertices of the corresponding crystal graph $B(\infty)$ [Lus91, Lus90, Kas91, KS97]. In fact, the binary operation extends the basic crystal operators. In the case where the underlying graph of Q is a simply laced Dynkin diagram, Comp is in bijection with the equivalence classes of finite-dimensional representations of Q. By Gabriel's theorem, the indecomposable representations of Q determine the positive roots in the root system defined by the Dynkin diagram [Gab72, BGfP73]. Hence, in this case, Comp is in bijection with multisets of positive roots.

The nilpotent varieties classify nilpotent modules of a given graded dimension of the preprojective algebra Π of Q. (The nilpotency condition is redundant in the Dynkin case.) A special role is played by the rigid irreducible components, i.e., those containing an open orbit under the natural action of the group of grading preserving linear isomorphisms. Their analogues, the so-called "real" simple modules of either the quiver Hecke algebras (aka. KLR algebras) or quantum groups, were studied by many people, including Geiss, Hernandez, Kang, Kashiwara, Kim, Leclerc, Oh and Schröer, in the context of monoidal categorification of cluster algebras [HL10, GLS06, KKKO18]. We refer the reader to [Kas18] and [HL21] for recent surveys on this topic.

The binary operation on Comp is modeled after a construction of Lusztig. Given $C_1, C_2 \in$ Comp we define $C_1 * C_2$ to be the Zariski closure of the constructible set formed by all possible extensions of x_1 by x_2 for all (x_1, x_2) in a suitable open subset of $C_1 \times C_2$. Our first main result (Theorem 3.1) is that $C_1 * C_2$ is indeed an irreducible component. In the case where C_1 or C_2 corresponds to a simple nilpotent module (i.e., the graded dimension is the indicator function of a vertex i of Q) this gives rise to the crystal operator $\tilde{f}_i$ of $B(\infty)$. Crawley-Boevey and Schröer studied this construction for the module varieties of a general finitely generated algebra, and showed that it gives rise to an irreducible subvariety, which under certain circumstances is an irreducible component [CBS02]. More recently, in the case of nilpotent varieties, Baumann, Kamnitzer and Tingley treated cases of "torsion pairs" [BKT14].[1] However, the fact that for nilpotent varieties the result is always an irreducible component seems to have gone unnoticed until now. This operation

[1] We are grateful to the referee for pointing out this reference to us.

is neither commutative nor associative in general. However, we can analyze the lack of commutativity and associativity. We also study the rigidity of $C_1 * C_2$ in terms of conditions on C_1 and C_2.

It is known that U^- is categorified by the KLR algebra R. (See e.g. [KKKO18, Theorem 2.1.2] and the references therein for a precise statement. We will freely use the terminology of [ibid.] in the following discussion.) Moreover, if R is symmetric, then in this categorification, at least in characteristic 0, the dual canonical basis corresponds to the self-dual simple R-modules [VV11] (see also [KKKO18, Theorem 2.1.4]). Denote by $C \mapsto \pi(C)$ the resulting bijection between Comp and the set of equivalence classes of simple self-dual R-modules. It is conjectured that if C is rigid, then $\pi(C)$ is real. (See [GLS11, Conjecture 18.1] for a related conjecture.) Assuming this, it follows from [KKKO15] that if C_1 or C_2 is rigid, then the socle of $\pi(C_1) \circ \pi(C_2)$ is simple. Here, $\circ$ denotes the convolution product defined in [KKKO18, §2.1]. It is then natural to conjecture that this socle is $\pi(C_1 * C_2)$.

In the case $Q = A_n$, one can reformulate the conjecture in terms of representation theory of the general linear groups over a local, non-archimedean field. This will be done in the second part of the paper [LM24], together with a verification of the conjecture in a special case.

We now describe the contents of the paper in more detail.

In §2 we recall the well known algebraic/geometric objects pertaining to quivers and their representations, namely preprojective algebras and the nilpotent varieties defined by Lusztig. As already mentioned, the irreducible components of the nilpotent varieties are especially important. We recall the formula of Crawley-Boevey for the dimension of Ext^1 of two modules of the preprojective algebra [CB00] and the switch-duality of Ext^1 [GLS07]. The case where the underlying graph of the quiver is a simply laced Dynkin diagrams is discussed in detail.

In §3 we introduce the binary operation $*$ on Comp, which is the main object of the paper. The fact that it is well defined relies on a simple dimension counting argument using Crawley-Boevey's formula and the fact that the nilpotent varieties are of pure dimension.

The central notion of *rigid* modules and irreducible components is recalled in §4.

Cancellativity of rigid irreducible components with respect to $*$ and a divisibility criterion are the subject matter of §5.

Commutativity and associativity of $*$ (or lack thereof) are discussed in §6 and §7, respectively.

Further technical results are presented in §8. They will be used in [LM24].

Finally, in §9 we go back to the description of modules of the preprojective algebra. Roughly speaking, a Π-module can be viewed as a representation of Q with supplementary data. Following Ringel [Rin98], in the case where the quiver does not admit cycles (which includes the Dynkin case) this data can be formulated in terms of the Coxeter functors of Bernstein–Gel'fand–Ponomarev [BGfP73] using their cohomological properties [Gab80]. As a complement to [GLS07, §8] we give a switch-dual short exact sequence for Ext^1_Π which is useful in computations.

2. Quivers, preprojective algebras and nilpotent varieties

Throughout this paper we fix an algebraically closed field K. All pertinent objects (vector spaces, algebras, tensor products, varieties, morphisms, dimensions, etc.) are implicitly over K. By convention, unless indicated otherwise, all modules are left-modules and all ideals are two-sided. We denote the dual of a vector space V by V^*. For vector spaces U, V we denote by $\mathrm{Lin}(U, V)$ the vector space of linear maps from U to V. If $U = V$, we write for brevity $\mathrm{Lin}(V) = \mathrm{Lin}(V, V)$.

For any finite set A, let $\mathbb{N}A$ be the free abelian monoid generated by A. We think of an element in $\mathbb{N}A$ as either a set of elements of A with multiplicities (i.e., a multiset), or a function from A to $\mathbb{N}$.

In this section we recall some standard definitions and facts about quivers, their representations and related cohomological and algebraic geometric aspects. We mainly follow Lusztig [Lus91] and Ringel [Rin98].

Let $Q = (I, \Omega)$ be a finite quiver without loops with vertex set I. The data is encoded by a function $\Omega \to I \times I$, $h \mapsto (h', h'')$ (representing arrows $h' \xrightarrow{h} h''$) such that $h' \neq h''$ for all $h \in \Omega$. Let KQ be the path algebra of Q. Recall that it is hereditary. For each $i \in I$, let e_i be the idempotent of KQ corresponding to the trivial path with vertex i. A representation of Q is an I-graded vector space $V = \oplus_{i\in I} V_i$ together with linear maps $A_h : V_{h'} \to V_{h''}$ for each $h \in \Omega$. This is the same as a KQ-module such that $e_i V = V_i$ for all $i \in I$. Henceforth, unless otherwise mentioned, we will only consider finite-dimensional representations.

We write $\mathcal{V}$ for the category of finite-dimensional I-graded vector spaces and for simplicity write $V \in \mathcal{V}$ to mean that V is an object in $\mathcal{V}$. If V has graded dimension $\operatorname{grdim} V = \boldsymbol{d} \in \mathbb{N}I$, then we write $V \in \mathcal{V}(\boldsymbol{d})$.

Let $V \in \mathcal{V}(\boldsymbol{d})$. The representations of Q on V form an affine space $R_Q(V)$ of dimension

$$\sum_{h\in\Omega} \boldsymbol{d}(h')\boldsymbol{d}(h'').$$

The group $G_V = \prod_{i\in I} \mathrm{GL}(V_i)$ of grading-preserving automorphisms of V acts linearly on $R_Q(V)$ by conjugation. The orbits are the isomorphism classes of representations of Q of graded dimension $\boldsymbol{d}$.

It is easy to write a projective resolution of KQ as a bi-module over itself. Consequently, for any Q-representations M and N with data $\mu_h : M_{h'} \to M_{h''}$ and $\nu_h : N_{h'} \to N_{h''}$, $h \in \Omega$, we have an exact sequence

$$(2.1)\quad 0 \to \mathrm{Hom}_Q(M,N) \to \oplus_{i\in I} \mathrm{Lin}(M_i,N_i) \xrightarrow{\alpha_{\mu,\nu}} \oplus_{h\in\Omega} \mathrm{Lin}(M_{h'},N_{h''}) \to \mathrm{Ext}^1_Q(M,N) \to 0$$

where the middle morphism is defined by

$$(2.2)\qquad \alpha_{\mu,\nu}((A_i)_{i\in I}) = (\nu_h A_{h'} - A_{h''}\mu_h)_{h\in\Omega}.$$

In particular, if M and N are of graded dimension $\boldsymbol{d}$ and $\boldsymbol{e}$ respectively, then

$$(2.3)\quad \dim \mathrm{Hom}_Q(M,N) - \dim \mathrm{Ext}^1_Q(M,N) = \langle \boldsymbol{d}, \boldsymbol{e}\rangle_Q := \sum_{i\in I} \boldsymbol{d}(i)\boldsymbol{e}(i) - \sum_{h\in\Omega} \boldsymbol{d}(h')\boldsymbol{e}(h'').$$

Note that if $\operatorname{grdim} V = \boldsymbol{d}$, then $\langle \boldsymbol{d}, \boldsymbol{d}\rangle_Q = \dim G_V - \dim R_Q(V)$.

For every $i \in I$ denote by $S_Q(i)$ the (simple) representation whose graded dimension is the indicator function of i. It is unique up to isomorphism. Moreover, if $\boldsymbol{d}$ is concentrated at i, i.e., if $\boldsymbol{d}(j) \neq 0$ for all $j \neq i$, then up to isomorphism, $\overbrace{S_Q(i) \oplus \cdots \oplus S_Q(i)}^{\boldsymbol{d}(i)}$ is the unique representation of graded dimension $\boldsymbol{d}$.

By definition, a representation M of Q is nilpotent if it satisfies the following equivalent conditions.

1. The augmentation ideal of KQ generated by the arrows acts nilpotently on M.
2. Every path of sufficiently large length acts trivially on M.
3. The closure of the G_V-orbit of M contains 0.
4. Every simple subquotient of M is of the form $S_Q(i)$ for some $i \in I$.

These conditions are automatic if Q does not admit oriented cycles, i.e., if KQ is finite-dimensional. (Henceforth, by a cycle we always mean an oriented cycle.) Note that an extension of a nilpotent representation by a nilpotent representation is nilpotent. The subset $R^{\mathrm{nilp}}_Q(V) \subseteq R_Q(V)$ of nilpotent representations is connected, Zariski closed and G_V-stable.

For any arrow h let h° be the opposite arrow (i.e., $(h^\circ)' = h''$ and $(h^\circ)'' = h'$). Let $Q^\circ = (I, \Omega^\circ)$ be the opposite quiver where $\Omega^\circ = \{h^\circ \mid h \in \Omega\}$ and let $\overline{Q}$ be the double quiver (I, H) where $H = \Omega \cup \Omega^\circ$ (disjoint union). The map $V \mapsto V^*$ is a duality between the category of representations of Q and those of Q°. We can also identify canonically $R_{Q^\circ}(V)$ with the dual vector space of $R_Q(V)$ (by taking the trace of the product) and $R_{\overline{Q}}(V) = R_Q(V) \times R_{Q^\circ}(V)$ with the cotangent bundle $T^*(R_Q(V))$ of $R_Q(V)$.

By definition, the preprojective algebra $\Pi = \Pi_Q$ is the quotient of the path algebra of $\overline{Q}$ by the ideal generated by $\sum_{h\in\Omega}[h, h^\circ]$. Let $R_\Pi(V)$ be the space of Π_Q-module structures on V such that $e_i V = V_i$ for all i. This is a closed subvariety of $R_{\overline{Q}}(V)$. It is the fiber over 0 of the moment map

$$T^*(R_Q(V)) \to \oplus_{i\in I} \operatorname{Lin}(V_i). \tag{2.4}$$

The projection

$$\pi_Q : R_\Pi(V) \to R_Q(V)$$

is the pullback with respect to the homomorphism (in fact, embedding) $KQ \to \Pi_Q$.

Let $\Lambda(V) = \Lambda_Q(V) = R_\Pi(V) \cap R_{\overline{Q}}^{\mathrm{nilp}}(V)$ be the Zariski closed, G_V-stable subvariety of nilpotent Π-modules. This is Lusztig's nilpotent variety studied in [Lus91, §12]. It is of pure dimension $\dim R_Q(V)$ and in fact a Lagrangian subvariety of $T^*(R_Q(V))$.

We denote by $\operatorname{Comp}(V)$ the (finite) set of irreducible components of $\Lambda(V)$. Since the group G_V is connected, each irreducible component of $\Lambda(V)$ is G_V-stable. Thus, $\operatorname{Comp}(V)$ depends only on the graded dimension of V. We will therefore also use the notation $\operatorname{Comp}(\boldsymbol{d})$ for any $\boldsymbol{d} \in \mathbb{N}I$.

For instance, if $\boldsymbol{d}$ is concentrated at a single vertex, then $R_\Pi(V) = \Lambda(V) = \{0\}$. In particular, $\operatorname{Comp}(\boldsymbol{d})$ is a singleton in this case and we call the element of $\operatorname{Comp}(\boldsymbol{d})$ *unicolor*. We denote by $S(i)$ the (simple) Π-module whose graded dimension is the indicator function of $i \in I$. By abuse of notation, we also denote by $S(i)$ the corresponding irreducible component.

If Q does not have cycles, then for any $V \in \mathcal{V}$, $R_Q(V) \times \{0\}$ and $\{0\} \times R_{Q^\circ}(V)$ are irreducible components of $\Lambda(V)$.

Set

$$\operatorname{Comp} = \bigcup_{\boldsymbol{d}\in\mathbb{N}I} \operatorname{Comp}(\boldsymbol{d}) \quad \text{(disjoint union)}.$$

Let $(\cdot,\cdot)$ be the bi-additive symmetric form on $\mathbb{N}I$ given by

$$(\boldsymbol{d}, \boldsymbol{e}) = \langle \boldsymbol{d}, \boldsymbol{e}\rangle_Q + \langle \boldsymbol{d}, \boldsymbol{e}\rangle_{Q^\circ} = \langle \boldsymbol{d}, \boldsymbol{e}\rangle_Q + \langle \boldsymbol{e}, \boldsymbol{d}\rangle_Q\,. \tag{2.5}$$

Note that if $\operatorname{grdim} V = \boldsymbol{d}$, then

$$(\boldsymbol{d}, \boldsymbol{d}) = 2(\dim G_V - \dim \Lambda(V)). \tag{2.6}$$

By a result of Crawley-Boevey [CB00, Lemma 1] for any Π-modules M and N of graded dimensions $\boldsymbol{d}$ and $\boldsymbol{e}$ respectively we have

$$\dim \operatorname{Hom}_\Pi(M, N) - \dim \operatorname{Ext}^1_\Pi(M, N) + \dim \operatorname{Hom}_\Pi(N, M) = (\boldsymbol{d}, \boldsymbol{e}). \tag{2.7}$$

In particular,

$$\dim \operatorname{Ext}^1_\Pi(M, M) = 2 \dim \operatorname{End}_\Pi(M) - (\boldsymbol{d}, \boldsymbol{d}). \tag{2.8}$$

In fact, (see [BKT14, §4.2] or [GLS07, §8] and the references therein) we have a functorial isomorphism

$$\operatorname{Ext}^1_\Pi(N, M) \simeq \operatorname{Ext}^1_\Pi(M, N)^* \tag{2.9}$$

and an injection[2]

$$\operatorname{Ext}^2_\Pi(N, M) \hookrightarrow \operatorname{Hom}_\Pi(M, N)^*. \tag{2.10}$$

Note that we can rewrite (2.7) as

$$\begin{aligned}\dim \operatorname{Ext}^1_\Pi(M, N) =&(\dim \Lambda(V \oplus V') - \dim G_{V \oplus V} \cdot (M \oplus N)) - \\ &(\dim \Lambda(V) - \dim G_V \cdot M) - (\dim \Lambda(V') - \dim G_V \cdot N)\end{aligned}$$

where $M \in R_\Pi(V)$ and $N \in R_\Pi(V')$.

For any $C_1, C_2 \in \operatorname{Comp}$ we set

$$\begin{aligned}\hom_\Pi(C_1, C_2) &= \min\{\dim \operatorname{Hom}_\Pi(x_1, x_2) \mid x_1 \in C_1, x_2 \in C_2\}, \\ \operatorname{ext}^1_\Pi(C_1, C_2) &= \min\{\dim \operatorname{Ext}^1_\Pi(x_1, x_2) \mid x_1 \in C_1, x_2 \in C_2\}.\end{aligned}$$

We recall that $\dim \operatorname{Hom}_\Pi(x_1, x_2)$ and $\dim \operatorname{Ext}^1_\Pi(x_1, x_2)$ are upper semicontinuous functions, so that the minimal loci $\operatorname{argmin}_{C_1 \times C_2} \dim \operatorname{Hom}_\Pi(x_1, x_2)$ and $\operatorname{argmin}_{C_1 \times C_2} \dim \operatorname{Ext}^1_\Pi(x_1, x_2)$ are open in $C_1 \times C_2$. By (2.7) we have

$$\hom_\Pi(C_1, C_2) - \operatorname{ext}^1_\Pi(C_1, C_2) + \hom_\Pi(C_2, C_1) = (\boldsymbol{d}, \boldsymbol{e})$$

for any $C_1 \in \operatorname{Comp}(\boldsymbol{d})$, $C_2 \in \operatorname{Comp}(\boldsymbol{e})$.

[2] If no connected component of Q is of Dynkin type, then in fact $\operatorname{Ext}^2_\Pi(N, M) \simeq \operatorname{Hom}_\Pi(M, N)^*$.

The preprojective algebra is isomorphic to its opposite algebra. (More precisely, $\Pi_Q^\circ \simeq \Pi_{Q^\circ} = \Pi_Q$, the first isomorphism takes a path to its opposite.) This gives a self-duality $M \mapsto M^*$ on finite-dimensional Π-modules. It yields an isomorphism $\Lambda(V) \to \Lambda(V^*)$, and a bijection $\mathrm{Comp}(V) \to \mathrm{Comp}(V^*)$, which we also denote by $C \mapsto C^*$. We therefore get an involution on $\mathrm{Comp}(\boldsymbol{d})$ for any $\boldsymbol{d} \in \mathbb{N}I$.

Up to isomorphism, the preprojective algebra, and hence its nilpotent varieties, depend only on $\overline{Q}$ (or in other words, on the underlying graph of Q, without orientation) [Lus91, §12.15]. These isomorphisms depend on certain choices of signs and square-roots of -1, but the induced bijection on the set of irreducible components does not depend on this choice.

The Dynkin case

(See [Gab72] [BGfP73] [GP79] [Lus91, §14])

In this subsection we assume that Q is of Dynkin type, i.e., it satisfies the following equivalent conditions.

1. The underlying graph of Q is a simply laced Dynkin diagram.
2. The quadratic form $\langle \boldsymbol{d}, \boldsymbol{d} \rangle_Q$ is positive-definite.
3. $R_Q(V)$ admits an open G_V-orbit for all $V \in \mathcal{V}$.
4. $R_Q(V)$ admits finitely many G_V-orbits for all $V \in \mathcal{V}$.
5. Up to isomorphism, there are only finitely many indecomposable representations of Q, i.e., KQ is representation finite.
6. Π_Q is finite-dimensional.

The map $M \mapsto \operatorname{grdim} M$ is a bijection between the equivalence classes of indecomposable representations of Q and the set

$$\Psi = \{\boldsymbol{d} \in \mathbb{N}I \mid \langle \boldsymbol{d}, \boldsymbol{d} \rangle_Q = 1\}.$$

We can identify Ψ with the set of positive roots in the root system corresponding to the Dynkin diagram. The set of simple roots is I itself.

For $\beta \in \Psi$ denote by $M_Q(\beta)$ the indecomposable representation with graded dimension β. It is simple if and only if $\beta \in I$. Every indecomposable representation M is a brick (i.e., $\mathrm{End}_Q(M) = K$) and is rigid (i.e., $\mathrm{Ext}^1_Q(M, M) = 0$).

We write a typical element of $\mathfrak{M} := \mathbb{N}\Psi$ as a formal sum $\mathfrak{m} = \beta_1 + \cdots + \beta_k$, $\beta_i \in \Psi$. Then,

$$\mathfrak{m} = \beta_1 + \cdots + \beta_k \mapsto M_Q(\mathfrak{m}) = M_Q(\beta_1) \oplus \cdots \oplus M_Q(\beta_k)$$

is a bijection between $\mathfrak{M}$ and the isomorphism classes of representations of Q. Extend the inclusion $\Psi \to \mathbb{N}I$ to an additive map

$$\operatorname{grdim} : \mathfrak{M} \to \mathbb{N}I.$$

Then, $\operatorname{grdim} M_Q(\mathfrak{m}) = \operatorname{grdim} \mathfrak{m}$. Moreover, $\Lambda(V) = R_\Pi(V)$ (i.e., every Π-module is nilpotent) and $\Lambda(V)$ is the disjoint union of the conormal bundles to the (finitely many) G_V-orbits in $R_Q(V)$. Thus, the irreducible components of $\Lambda(V)$ are the closure of these conormal bundles. Therefore, $\operatorname{Comp}(\boldsymbol{d})$ is in bijection with the isomorphism classes of Q-representations of graded dimension $\boldsymbol{d}$. This gives rise to a bijection

$$\lambda_Q : \mathfrak{M} \to \operatorname{Comp}$$

such that $\operatorname{Comp}(\boldsymbol{d}) = \lambda_Q(\mathfrak{M}(\boldsymbol{d}))$ where $\mathfrak{M}(\boldsymbol{d}) = \{\mathfrak{m} \in \mathfrak{M} \mid \operatorname{grdim} \mathfrak{m} = \boldsymbol{d}\}$. We denote by $\mu_Q(\mathfrak{m})$ the conormal bundle of the orbit of $M_Q(\mathfrak{m})$. Thus, if $\operatorname{grdim} V = \operatorname{grdim} \mathfrak{m}$, then $\mu_Q(\mathfrak{m})$ is the inverse image of the G_V-orbit of $M_Q(\mathfrak{m})$ under π_Q and $\lambda_Q(\mathfrak{m})$ is the Zariski closure of $\mu_Q(\mathfrak{m})$ in $\Lambda(V)$. Of course, we also have a bijection

$$\lambda_{Q^\circ} : \mathfrak{M} \to \operatorname{Comp}$$

since the preprojective algebra pertaining to Q° coincides with Π.

In particular, $\mathfrak{m} \in \mathbb{N}I$ (i.e., all the roots in $\mathfrak{m}$ are simple) if and only if $\lambda_Q(\mathfrak{m}) = \pi_Q^{-1}(\{0\}) = \{0\} \times R_{Q^\circ}(V)$ if and only if $\lambda_{Q^\circ}(\mathfrak{m}) = R_Q(V) \times \{0\}$ where $V \in \mathcal{V}(\operatorname{grdim} \mathfrak{m})$.

Also, for any $\beta \in \Psi$ we have $\lambda_Q(\beta) = R_Q(V) \times \{0\}$ (for $V \in \mathcal{V}(\beta)$), i.e., the G_V-orbit of $M_Q(\beta)$ is open in $R_Q(V)$.

We remark that in the context of a Lie group G acting linearly, with finitely many orbits on a vector space V, the fiber of 0 of the moment map $T^*(V) \to \operatorname{Lie}(G)^*$ was considered in [Pja75], where it was shown to be of pure dimension $\dim V$.

We have

$$M_Q(\mathfrak{m})^* \simeq M_{Q^\circ}(\mathfrak{m}) \tag{2.11}$$

and consequently,

$$\lambda_Q(\mathfrak{m})^* = \lambda_{Q^\circ}(\mathfrak{m})$$

for any $\mathfrak{m} \in \mathfrak{M}$.

3. Extensions

For the rest of the paper we fix a quiver Q.

Let $V^i \in \mathcal{V}(\boldsymbol{d}_i)$, $i = 1,2$ and let $V \in \mathcal{V}(\boldsymbol{d})$ with $\boldsymbol{d} = \boldsymbol{d}_1 + \boldsymbol{d}_2$. For any $G_{V^1} \times G_{V^2}$-stable constructible subset S of $\Lambda(V^1) \times \Lambda(V^2)$ let

$$\mathcal{E}_2(S) = \{x \in R_\Pi(V) \mid \exists \text{ a short exact sequence } \\ 0 \to x_2 \to x \to x_1 \to 0 \text{ with } (x_1, x_2) \in S\}.$$

It is a G_V-stable, constructible subset of $\Lambda(V)$.

Let $C_i \in \mathrm{Comp}(V^i)$, $i = 1,2$ and take

$$S = \operatorname{argmin}_{C_1 \times C_2} \dim \operatorname{Ext}^1_\Pi(x_1, x_2). \tag{3.1}$$

Then, S is a nonempty open subset of $C_1 \times C_2$ (hence irreducible) and $\mathcal{E}_2(S)$ is irreducible ([CBS02, Theorem 1.3(ii)]). We denote its Zariski closure by $C_1 * C_2$.

Clearly,

$$(C_1 * C_2)^* = C_2^* * C_1^*. \tag{3.2}$$

Theorem 3.1. *$C_1 * C_2$ is an irreducible component of $\Lambda(V)$. Moreover,*

$$\text{if } \emptyset \neq S' \subseteq S \text{ is open and } G_{V^1} \times G_{V^2}\text{-stable,} \\ \text{then } \mathcal{E}_2(S') \text{ is dense in } C_1 * C_2. \tag{3.3}$$

Proof. It will be more convenient to "forget" about the grading of V. Thus, instead of $R_\Pi(V)$ we consider the larger variety $\tilde{R}_\Pi(V)$ of Π-module structure on V (i.e., algebra homomorphisms $\Pi \to \mathrm{Lin}(V)$) such that for all $i \in I$, e_i acts as a projection onto a $\boldsymbol{d}(i)$-dimensional, but otherwise arbitrary, subspace of V. Fixing a generating set of Π of size g, we can identify $\tilde{R}_\Pi(V)$ with a closed subvariety of $\mathrm{Lin}(V)^g$ which is stable under $\mathrm{GL}(V)$ (acting diagonally by conjugation). Let $\tilde{\Lambda}(V) \subseteq \tilde{R}_\Pi(V)$ be the closed, connected, $\mathrm{GL}(V)$-stable subvariety of nilpotent modules. Clearly,[3]

$$\tilde{R}_\Pi(V) = \mathrm{GL}(V) \overset{G_V}{\times} R_\Pi(V) \text{ and } \tilde{\Lambda}(V) = \mathrm{GL}(V) \overset{G_V}{\times} \Lambda(V).$$

[3]If H is a subgroup of G and X is an H-set, then we write $G \overset{H}{\times} X$ for the G-set $(G \times X)/H$ where H acts freely on $G \times X$ by $(g,x)h = (gh, h^{-1}x)$.

Thus, $\tilde\Lambda(V)$ is of pure dimension $\delta(\boldsymbol{d}) = \dim\Lambda(V) + \dim \mathrm{GL}(V) - \dim G_V$ and the irreducible components of $\tilde\Lambda(V)$ and $\Lambda(V)$ are in natural bijection. Note that by (2.6),

$$(\boldsymbol{d}, \boldsymbol{d}) = 2(\dim \mathrm{GL}(V) - \delta(\boldsymbol{d})). \tag{3.4}$$

Let $\tilde C_i = \mathrm{GL}(V^i)\cdot C_i$ be the irreducible component of $\tilde\Lambda(V^i)$ corresponding to C_i, $i = 1, 2$. Write

$$\tilde S = (\mathrm{GL}(V^1)\times \mathrm{GL}(V^2))\cdot S \subseteq \tilde C_1 \times \tilde C_2, \quad \widetilde{S'} = (\mathrm{GL}(V^1)\times \mathrm{GL}(V^2))\cdot S' \subseteq \tilde S$$

and $\tilde{\mathcal{E}} = \tilde{\mathcal{E}}_2(\widetilde{S'}) = \mathrm{GL}(V)\cdot \mathcal{E}_2(S')$. It is enough to show that the dimension of $\tilde{\mathcal{E}}$ is $\delta(\boldsymbol{d})$.

For simplicity, take $V = V^1 \oplus V^2$. Let Z be the subset of $\tilde R_\Pi(V)$ (and in fact, of $\tilde\Lambda(V)$) consisting of the module structures on V for which V^2 is a submodule and the pair of induced structures on $V^1 = V/V^2$ and V^2 is an element of $\widetilde{S'}$. (With respect to the decomposition $V = V^1 \oplus V^2$, any element of Z, viewed in $\mathrm{Lin}(V)^g$, is block lower triangular in each coordinate.) Thus, $\tilde{\mathcal{E}} = \mathrm{GL}(V)\cdot Z$.

As explained in [CBS02, §4 and §5], the dimensions of $\mathrm{Ext}^1_\Pi(x_1, x_2)$ and $\mathrm{Hom}_\Pi(x_1, x_2)$ are constant (say N_1 and N_0) on $\tilde S$ and via the canonical map $p : Z \to \widetilde{S'}$, Z becomes a vector bundle over $\widetilde{S'}$ of rank $N_1 - N_0 + \dim V^1 \cdot \dim V^2$. In particular, Z is irreducible of dimension

$$\dim \widetilde{S'} + N_1 - N_0 + \dim V^1 \cdot \dim V^2 = \delta(\boldsymbol{d}_1) + \delta(\boldsymbol{d}_2) + N_1 - N_0 + \dim V^1 \cdot \dim V^2.$$

In the case at hand, we have (by (2.7) and (3.4))

$$\begin{aligned}\dim \mathrm{Ext}^1_\Pi(x_1, x_2) = \delta(\boldsymbol{d}) - \delta(\boldsymbol{d}_1) - \delta(\boldsymbol{d}_2) - 2\dim V^1 \cdot \dim V^2 + \\ \dim \mathrm{Hom}_\Pi(x_1, x_2) + \dim \mathrm{Hom}_\Pi(x_2, x_1).\end{aligned} \tag{3.5}$$

It follows that $\tilde S \subseteq \operatorname{argmin}_{C_1\times C_2} \mathrm{Hom}_\Pi(x_2, x_1)$ and

$$\delta(\boldsymbol{d}) - \dim Z = \dim V^1 \cdot \dim V^2 - \hom_\Pi(C_2, C_1). \tag{3.6}$$

Let R be the unipotent radical of the parabolic subgroup of $\mathrm{GL}(V)$ preserving V^1. Let Y be the Zariski closure of $R\cdot Z$. Clearly, Y is irreducible. We show that $\dim Y = \delta(\boldsymbol{d})$, which will clearly imply the theorem. Let $X = R\times Z$ and consider the action map

$$f : X \to Y.$$

Identify R with $\mathrm{Lin}(V^2, V^1)$ via $T \mapsto \left(\begin{smallmatrix} 1 & T \\ & 1 \end{smallmatrix}\right)$. We claim that for any $(x_1, x_2) \in \widetilde{S}'$, the fiber X_x of f at $x = x_1 \oplus x_2 \in Z$ is $\mathrm{Hom}_\Pi(x_2, x_1) \times \{x\}$. Indeed, the lower left corner is preserved under the action of R. Therefore, if $T \cdot z = x$ with $T \in R$ and $z \in Z$, then z is diagonal, which clearly implies that $z = x$ and $T \in \mathrm{Hom}_\Pi(x_2, x_1)$.

In particular, by (3.6)

$$\dim X_x = \dim R + \dim Z - \delta(\boldsymbol{d}) = \dim X - \delta(\boldsymbol{d}).$$

It follows from [Sha13, Theorem 1.25] that

$$\dim Y \geq \dim X - \dim X_x = \delta(\boldsymbol{d}).$$

Since evidently, $\dim Y \leq \delta(\boldsymbol{d})$, we conclude that $\dim Y = \delta(\boldsymbol{d})$, as required. $\square$

The theorem gives rise to a binary operation (also denoted by $*$) on Comp such that $\mathrm{Comp}(\boldsymbol{d}_1) * \mathrm{Comp}(\boldsymbol{d}_2) \subseteq \mathrm{Comp}(\boldsymbol{d}_1 + \boldsymbol{d}_2)$.

Remark 3.2. In the case where C_1 (or C_2) is unicolor, Theorem 3.1, together with Proposition 5.1 below, was proved by Lusztig (see [Lus91, §12]). It was conjectured by him [Lus90] that this gives rise to the crystal graph defined by Kashiwara [Kas91]. This was subsequently proved by Kashiwara and Saito [KS97].

A broader situation, from which the previous case can be deduced, was considered by Baumann–Kamnitzer–Tingley in [BKT14]. It is pertaining to cases where $\hom_\Pi(C_2, C_1) = 0$. More precisely, let $(\mathscr{T}, \mathscr{F})$ be a torsion pair (see [ibid., §3.1]) in the category of nilpotent Π-modules. Assume that $\mathscr{T}(\boldsymbol{d}) = \Lambda(\boldsymbol{d}) \cap \mathscr{T}$ and $\mathscr{F}(\boldsymbol{d}) = \Lambda(\boldsymbol{d}) \cap \mathscr{F}$ are open in $\Lambda(\boldsymbol{d})$ for all $\boldsymbol{d} \in \mathbb{N}I$. Let $\mathfrak{T}(\boldsymbol{d})$ (resp., $\mathfrak{F}(\boldsymbol{d})$) be the irreducible components of $\mathscr{T}(\boldsymbol{d})$ (resp., $\mathscr{F}(\boldsymbol{d})$), viewed as subsets of $\mathrm{Comp}(\boldsymbol{d})$ (by taking the Zariski closure). Let

$$\mathfrak{T} = \bigcup_{\boldsymbol{d}} \mathfrak{T}(\boldsymbol{d}), \ \ \mathfrak{F} = \bigcup_{\boldsymbol{d}} \mathfrak{F}(\boldsymbol{d}).$$

Then, $(C_1, C_2) \mapsto C_1 * C_2$ defines a bijection $\mathfrak{F} \times \mathfrak{T} \to \mathrm{Comp}$ ([ibid., Theorem 4.4]).[4] Many important examples of such torsion pairs are given in [ibid., §5].

We say that C_1 and C_2 *strongly commute* if there exist $x_i \in C_i$, $i = 1, 2$ such that $\mathrm{Ext}^1_\Pi(x_1, x_2) = 0$. (This is an open condition in $(x_1, x_2) \in \Lambda(V^1) \times \Lambda(V^2)$.) We caution that in general, an irreducible component does not necessarily strongly commute with itself.

[4]Note that the convention for the product in [ibid.] is opposite to ours.

Let $V = V^1 \oplus V^2$ and denote by $C_1 \oplus C_2$ the Zariski closure of the set

$$G_V \cdot \{x_1 \oplus x_2 \mid x_1 \in C_1 \text{ and } x_2 \in C_2\}.$$

This is an irreducible subset of $\Lambda(V)$. Clearly,

$$C_1 * C_2 \supseteq C_1 \oplus C_2. \tag{3.7}$$

Corollary 3.3 (cf. [CBS02]). *C_1 and C_2 strongly commute if and only if $C_1 \oplus C_2$ is an irreducible component of $\Lambda(V)$, in which case $C_1 \oplus C_2 = C_1 * C_2$.*

Indeed, if C_1 and C_2 strongly commute then $\mathcal{E}_2(S) \subseteq C_1 \oplus C_2$ and therefore $C_1 \oplus C_2 = C_1 * C_2$ is an irreducible component. Conversely, if $C_1 \oplus C_2$ is an irreducible component, then the argument of [CBS02, p. 216] shows that C_1 and C_2 strongly commute.

Remark 3.4. Corollary 3.3 is a special case of a general result of Crawley-Boevey and Schröer. Namely, let A be a finitely generated algebra and for any finite-dimensional vector space V let $R_A(V)$ be the variety of A-module structures on V. Let $V = V^1 \oplus V^2$ and let C_i be irreducible components of $R_A(V^i)$, $i = 1, 2$. Define $C_1 \oplus C_2$ as above. Then, by [CBS02], $C_1 \oplus C_2$ is an irreducible component of $R_A(V)$ if and only if there exist $x_i \in C_i$, $i = 1, 2$ such that $\mathrm{Ext}^1_A(x_1, x_2) = \mathrm{Ext}^1_A(x_2, x_1) = 0$. (When A admits orthogonal idempotents e_i, $i \in I$ such that $\sum e_i = 1$, we immediately deduce a version with graded dimensions.) To obtain Corollary 3.3 we take A to be the quotient of Π by a suitable power of the augmentation ideal. (In the Dynkin case, A is Π itself.) Of course, in the preprojective case, the condition $\mathrm{Ext}^1_\Pi(x_1, x_2) = 0$ is symmetric by (2.9).

4. Rigid modules

Recall that by definition, a Π-module x is *rigid* if $\mathrm{Ext}^1_\Pi(x, x) = 0$.

Clearly, x is rigid if and only if x^* is rigid.

If $x = x_1 \oplus \cdots \oplus x_k$, then

$$x \text{ is rigid} \iff \mathrm{Ext}^1_\Pi(x_i, x_j) = 0 \text{ for all } i, j. \tag{4.1}$$

For instance, if the graded dimension of x is concentrated at $i \in I$, then x is rigid. Also, every projective module is rigid.

Let V be a finite-dimensional I-graded vector space. Let $\mathbf{R}_\Pi(V)$ be the *scheme* of Π-modules on V such that $e_i V = V_i$ for all $i \in I$. By Voigt's lemma (which is valid for any finitely generated algebra – see [Gab74, §1.1] for a standard reference) for any $x \in R_\Pi(V)$ with G_V-orbit $\mathcal{O}(x)$ we have

$$\operatorname{Ext}^1_\Pi(x,x) \simeq N_x^{\mathbf{R}_\Pi(V)}(\mathcal{O}(x)) = T_x\mathbf{R}_\Pi(V)/T_x\mathcal{O}(x) \tag{4.2}$$

where $T_x\mathbf{R}_\Pi(V)$ is the Zariski tangent space of the scheme $\mathbf{R}_\Pi(V)$ at x. In particular, x is rigid if and only if $\mathcal{O}(x)$ is an open subscheme of $\mathbf{R}_\Pi(V)$. (Of course, we can forgo the condition $e_iV = V_i$ and consider the $\mathrm{GL}(V)$-orbit instead.)

On the other hand, by (2.8), x is rigid if and only if

$$2\dim \operatorname{End}_\Pi(x) = (\boldsymbol{d}, \boldsymbol{d}).$$

Since

$$\dim \mathcal{O}(x) = \dim G_V - \dim \operatorname{End}_\Pi(x),$$

we also have

$$\dim \operatorname{Ext}^1_\Pi(x,x) = 2(\dim \Lambda(V) - \dim \mathcal{O}(x))$$

(by (2.6)). Comparing with (4.2) we get

$$\tfrac{1}{2}\dim \operatorname{Ext}^1_\Pi(x,x) = \dim \Lambda(V) - \dim \mathcal{O}(x) = \dim T_x\mathbf{R}_\Pi(V) - \dim \Lambda(V).$$

In particular, if $x \in \Lambda(V)$ then

$$\dim \operatorname{Ext}^1_\Pi(x,x) = 2\operatorname{codim} \mathcal{O}(x) \quad \text{(codimension in } \Lambda(V)).$$

Therefore, the following conditions are equivalent for $x \in \Lambda(V)$.

1. x is rigid.
2. $\mathcal{O}(x)$ is open in $\Lambda(V)$.
3. The Zariski closure $\overline{\mathcal{O}(x)}$ is an irreducible component of $\Lambda(V)$.
4. $\dim \operatorname{End}_\Pi(x) = \dim G_V - \dim \Lambda(V)$.
5. $\dim \operatorname{End}_\Pi(x) \le \dim G_V - \dim \Lambda(V)$.
6. x is a smooth point in the scheme $\mathbf{R}_\Pi(V)$ (in particular, it lies in a unique irreducible component C of $R_\Pi(V)$) and $\dim C = \dim \Lambda(V)$.[5]

Definition 4.1. *An irreducible component C of $\Lambda(V)$ is called rigid if it contains a rigid module, or equivalently it contains a (necessarily unique) open G_V-orbit.*

[5] The condition $\dim C = \dim \Lambda(V)$ is redundant in the case Q is of Dynkin type.

In this case, the open orbit in C consists of the rigid modules in C.

For instance, any unicolor $C \in \operatorname{Comp}$ is rigid.

If C is rigid, then C^* is rigid.

Remark 4.2. Suppose that Q is of Dynkin type. Then, an irreducible component C of $\Lambda(V)$ is rigid if and only if the scheme $\mathbf{R}_\Pi(V)$ is generically smooth (or equivalently, generically reduced) at C.[6]

If $C = \lambda_Q(\mathfrak{m})$, then C is rigid if and only if $\mu_Q(\mathfrak{m})$ contains an open G_V-orbit. In this case, $\lambda_{Q^\circ}(\mathfrak{m}) = \lambda_Q(\mathfrak{m})^*$ is also rigid.

For any $V \in \mathcal{V}$, $R_Q(V) \times \{0\}$ and $\{0\} \times R_{Q^\circ}(V)$ are rigid. In particular, $\lambda_Q(\beta)$ is rigid for any $\beta \in \Psi$.

In general, we have a bijection between the set of rigid irreducible components of $\Lambda(V)$ and the set of G_V-orbits of rigid modules in $\Lambda(V)$.

The first examples of non-rigid irreducible components were given by Leclerc in [Lec03]. (We will recall it [LM24].)

The role of rigid modules and irreducible components was highlighted in the work of Geiss, Leclerc and Schröer (e.g., [GLS06] and also below).

Remark 4.3. Let $C_1, C_2, C \in \operatorname{Comp}$.

1. By (4.1), if C_1 and C_2 strongly commute, then $C_1 * C_2 = C_1 \oplus C_2$ is rigid if and only if C_1 and C_2 are rigid.
2. If C is rigid, then C strongly commutes with itself. The converse, however is not true as Leclerc's example shows.
3. It is possible for $C * C$ to be rigid even if C itself is not. We will give an example in [LM24].

5. Cancellation

We show that rigid irreducible components are cancellative. (We do not know whether non-rigid irreducible components are cancellative as well.)

Fix a rigid $C_1 \in \operatorname{Comp}$ and a rigid element x_1 in C_1. For any $V \in \mathcal{V}(\boldsymbol{d})$ and an irreducible component $C \in \operatorname{Comp}(V)$, denote by C' the (possibly empty) constructible, G_V-stable subset of C consisting of the elements x that admit x_1 as a quotient of Π-modules. Denote by $\operatorname{Comp}_{C_1\text{-cvr}}(V)$ or $\operatorname{Comp}_{C_1\text{-cvr}}(\boldsymbol{d})$ the set of irreducible components $C \in \operatorname{Comp}(V)$ for which C' is dense in C. Finally, set

$$\operatorname{Comp}_{C_1\text{-cvr}} = \cup_{\boldsymbol{d} \in \mathbb{N}I} \operatorname{Comp}_{C_1\text{-cvr}}(\boldsymbol{d}) \subseteq \operatorname{Comp}.$$

[6] The latter equivalence holds for any irreducible component of a scheme of finite type over K.

The following proposition generalizes a result of Lusztig in the case where C_1 is unicolor.

Proposition 5.1. *The map* $C_2 \in \mathrm{Comp} \mapsto C_1 * C_2 \in \mathrm{Comp}$ *defines a bijection* $\mathrm{Comp} \to \mathrm{Comp}_{C_1\text{-cvr}}$. *Its inverse takes* $C \in \mathrm{Comp}_{C_1\text{-cvr}}(V)$ *to the closure of the set of* Π*-modules* $\{\operatorname{Ker}\phi \mid x \in S', \phi : x \twoheadrightarrow x_1\}$, *for any open, nonempty* G_V*-stable subset* S' *of*

$$S = \{x \in C' \mid \dim \operatorname{Hom}_\Pi(x, x_1) = \hom_\Pi(C, C_1)\}.$$

Similarly, $C_2 \mapsto C_2 * C_1$ *is one-to-one and its image consists of the set of irreducible components* C *for which the subset* C'' *consisting of the* x*'s that admit* x_1 *as a submodule is dense in* C. *The inverse map is given by taking the closure of the set of* Π*-modules* $\{\operatorname{Coker}\phi \mid x \in S', \phi : x_1 \hookrightarrow x\}$ *for any open, nonempty* G_V*-stable subset* S' *of* $\{x \in C'' \mid \dim \operatorname{Hom}_\Pi(x_1, x) = \hom_\Pi(C_1, C)\}$.

Proof. We only need to prove the first statement, since the second one can then be deduced by passing to C^*.

Clearly, $C_1 * C_2 \in \mathrm{Comp}_{C_1\text{-cvr}}$ for any $C_2 \in \mathrm{Comp}$. Conversely, suppose that $C \in \mathrm{Comp}_{C_1\text{-cvr}}(V)$ with $\operatorname{grdim} V = \boldsymbol{d}$. Write $\boldsymbol{d} = \boldsymbol{d}_1 + \boldsymbol{d}_2$ where $C_1 \in \mathrm{Comp}(\boldsymbol{d}_1)$. Let d be the total dimension of $\boldsymbol{d}$. Similarly for d_1 and d_2.

As in the proof of Theorem 3.1 we prove an analogous (but equivalent) statement for $\tilde{\Lambda}(V) = \mathrm{GL}(V) \overset{G_V}{\times} \Lambda(V)$. Thus, we replace C, S and S' by their image under the action of $\mathrm{GL}(V)$ (so that C is an irreducible component of $\tilde{\Lambda}(V)$) and similarly for C_1. Fix a d_2-dimensional subspace $V_2 \subseteq V$ and let $V_1 = V/V_2$. We view C_1 as an irreducible component of $\tilde{\Lambda}(V_1)$.

Let

$$X = \{(x, \varphi) \mid x \in S', \varphi \in \operatorname{Hom}_\Pi(x, x_1), \operatorname{Ker}\varphi = V_2\},$$

and consider the morphism

$$f : X \to \tilde{\Lambda}(V_1) \times \tilde{\Lambda}(V_2)$$

that takes $(x, \varphi) \in X$ to the quotient and submodule structures induced by x on V_1 and V_2.

Let Y be the closure of the image of f. We claim that $Y = C_1 \times C_2$ where C_2 is the unique irreducible component of $\tilde{\Lambda}(V_2)$ such that $C = C_1 * C_2$.

We first show that X is irreducible of dimension $\delta(\boldsymbol{d}) + \hom_\Pi(C, C_1) - d_1 d_2$ where $\delta(\boldsymbol{d}) = \dim \tilde{\Lambda}(V)$. Let

$$C^\circ = \operatorname{argmin}_{x \in C} \dim \operatorname{Hom}_\Pi(x, x_1),$$

which is a nonempty open subset of C. Consider

$$E = \{(x, \varphi) \mid x \in C^\circ, \varphi \in \mathrm{Hom}_\Pi(x, x_1)\}.$$

Thus, E is the fiberwise kernel of the morphism

$$\xi : C^\circ \times \mathrm{Lin}(V, V_1) \to C^\circ \times \mathrm{Lin}(V, V_1)^N$$

of trivial vector bundles over C° given by

$$\xi(x, \varphi) = (x, (\varphi \circ x(a_i) - x_1(a_i) \circ \varphi)_{i=1}^N)$$

where $a_1, \dots, a_N$ are fixed generators of Π. Since ξ is of constant rank (by the definition of C°), E is a vector bundle over C° by a standard result ([LP97, Proposition 1.7.2]).[7]

We may consider V and V_1 as constant sheaves of vector spaces $\mathcal{F}$ and $\mathcal{G}$ over E. The map

$$\Phi : \mathcal{F} \to \mathcal{G}$$

given by φ is a morphism of sheaves over E. Let $\mathcal{B} = \operatorname{Coker} \Phi$. Note that the fiber $\mathcal{B}_{(x,\varphi)}$ is $\operatorname{Coker} \varphi$. Since $C \in \mathrm{Comp}_{C_1\text{-cvr}}$, the complement E° of the support of $\mathcal{B}$ in E is nonempty (and open). Note that

$$E^\circ = \{(x, \varphi) \mid x \in C^\circ, \varphi \in \mathrm{Hom}_\Pi(x, x_1), \varphi \text{ is surjective}\}.$$

Thus, S, which is the image of E° under the canonical map $E \to C^\circ$, is open. Let

$$E' = \{(x, \varphi) \in E^\circ \mid x \in S'\},$$

which is open in E. Let Z be the Grassmannian variety of d_2-dimensional subspaces of V. Let $\alpha : E' \to Z$ and $\beta : \mathrm{GL}(V) \to Z$ be the morphisms

$$\alpha(x, \varphi) = \operatorname{Ker} \varphi, \quad \beta(g) = g^{-1}(V_2)$$

and let F be the pull back $E' \times_Z \mathrm{GL}(V)$. Then, F is a principal H-bundle over E', where H is the parabolic subgroup of $\mathrm{GL}(V)$ stabilizing V_2. In particular, F is irreducible of dimension

$$\begin{aligned}\dim E' + d^2 - \dim Z &= \dim C + \hom_\Pi(C, C_1) + d^2 - d_1 d_2 \\ &= \delta(\boldsymbol{d}) + \hom_\Pi(C, C_1) + d^2 - d_1 d_2.\end{aligned}$$

[7]We thank Yakov Varshavsky for providing us this reference.

Note that X is equal to the fiber of the map $F \to \mathrm{GL}(V)$ over the identity. Now, the maps α and β are $\mathrm{GL}(V)$-equivariant with respect to the $\mathrm{GL}(V)$-action on E' given by $g(x, \varphi) = (g \cdot x, \varphi \circ g^{-1})$, the right regular action of $\mathrm{GL}(V)$ on itself, and the usual action of $\mathrm{GL}(V)$ on Z. Thus, $\mathrm{GL}(V)$ acts freely on F and the action map gives rise to an isomorphism

$$\mathrm{GL}(V) \times X \simeq F.$$

It follows that X is irreducible of dimension $\delta(\boldsymbol{d}) + \hom_\Pi(C, C_1) - d_1 d_2$, as claimed.

Next, we analyze the fibers of f as in [CBS02, §4 and §5], except that we also have to take φ into account. Suppose that $y = (y_1, y_2)$ is in the image of f. Then, y_1 lies in the orbit of x_1 and the fiber X_y is given by the product of a vector space of dimension $\dim \mathrm{Ext}^1_\Pi(y_1, y_2) - \dim \mathrm{Hom}_\Pi(y_1, y_2) + d_1 d_2$ with the $\mathrm{Aut}_\Pi(x_1)$-torsor of isomorphisms of Π-modules between y_1 and x_1. In particular, X_y is irreducible of dimension

$$\begin{aligned}&\dim \mathrm{Ext}^1_\Pi(x_1, y_2) - \dim \mathrm{Hom}_\Pi(x_1, y_2) + d_1 d_2 + \dim \mathrm{Aut}_\Pi(x_1)\\ &= \delta(\boldsymbol{d}) - \delta(\boldsymbol{d}_1) - \delta(\boldsymbol{d}_2) - d_1 d_2 + \dim \mathrm{Hom}_\Pi(y_2, x_1) + \dim \mathrm{Aut}_\Pi(x_1)\end{aligned}$$

(by (3.5)). Note that since x_1 is rigid, for any $(x, \varphi) \in X_y$ we have a short exact sequence

$$0 \to \mathrm{Hom}_\Pi(y_1, x_1) \to \mathrm{Hom}_\Pi(x, x_1) \to \mathrm{Hom}_\Pi(y_2, x_1) \to 0$$

and hence

$$\hom_\Pi(C, C_1) = \dim \mathrm{Hom}_\Pi(x, x_1) = \dim \mathrm{End}_\Pi(x_1) + \dim \mathrm{Hom}_\Pi(y_2, x_1).$$

Thus,

$$\dim X_y = \delta(\boldsymbol{d}) - \delta(\boldsymbol{d}_1) - \delta(\boldsymbol{d}_2) - d_1 d_2 + \hom_\Pi(C, C_1),$$

which is independent of y.

It follows that

$$\dim Y = \dim X - \dim X_y = \delta(\boldsymbol{d}_1) + \delta(\boldsymbol{d}_2),$$

and hence Y is an irreducible component of $\tilde\Lambda(V_1) \times \tilde\Lambda(V_2)$. Since Y is contained in $C_1 \times \tilde\Lambda(V_2)$, it is necessarily of the form $Y = C_1 \times C_2$ for some irreducible component C_2 of $\tilde\Lambda(V_2)$, as claimed.

Clearly, for any $\mathrm{GL}(V_1) \times \mathrm{GL}(V_2)$-stable subset U of $\tilde\Lambda(V_1) \times \tilde\Lambda(V_2)$ we have $\mathcal{E}_2(U) \supseteq p(f^{-1}(U))$ where $p : F \to C$ is the composition of the canonical maps

$F \to E' \hookrightarrow E \to C^\circ \hookrightarrow C$. (Recall that X is contained in F and $F = \mathrm{GL}(V) \cdot X$, and note that the restriction of p to X is the first projection $X \to S'$.) Since these maps are open and $\mathrm{GL}(V)$-equivariant, if U is open, then $\mathcal{E}_2(U)$ contains the open set $p(\mathrm{GL}(V) \cdot f^{-1}(U))$ of C. Taking $U = \operatorname{argmin}_{C_1 \times C_2} \operatorname{Ext}^1_\Pi(x_1, x_2)$ we infer that $C_1 * C_2 \supseteq C$ and hence $C_1 * C_2 = C$.

Finally, suppose that $C = C_1 * C_2'$ for some irreducible component C_2' of $\tilde{\Lambda}(V_2)$. Let $U = \operatorname{argmin}_{x_2 \in C_2'} \dim \operatorname{Ext}^1_\Pi(x_1, x_2)$, which is a nonempty open subset of C_2'. Let W be the subset of $\tilde{\Lambda}(V)$ consisting of the Π-module structures on V for which V_2 is a submodule in U and the induced structure on V_1 is x_1. Then, W is a vector bundle over U and by Theorem 3.1, $\mathrm{GL}(V) \cdot W$ is a constructible dense subset of C. Hence, $W^\circ = W \cap S'$ is nonempty and open in W (since S' is $\mathrm{GL}(V)$-stable). Let $p : W \to U$ be the canonical map. Then, $p(W^\circ)$ is open in U (hence in C_2') and $\{x_1\} \times p(W^\circ)$ is contained in the image of f. Therefore, $C_2' = C_2$.

The proposition follows. $\square$

6. Commutativity

Next, we discuss the lack of commutativity of the $*$ operation.

This transpires already in the simplest example of $Q = A_2$, $C_i = S(i)$, $i = 1, 2$. Writing $I = \{1, 2\}$ with $1 \to 2$ and $\Psi = \{\alpha_1, \alpha_2, \beta = \alpha_1 + \alpha_2\}$, we have

$$S(1) * S(2) = \lambda_Q(\{\beta\}) \text{ but } S(2) * S(1) = \lambda_{Q^\circ}(\{\beta\}) = \lambda_Q(\{\alpha_1\} + \{\alpha_2\}).$$

We say that two irreducible components $C_1, C_2 \in \mathrm{Comp}$ *weakly commute* if

$$C_1 * C_2 = C_2 * C_1.$$

Recall that, as the name suggests, strong commutativity implies weak commutativity by Corollary 3.3. (See Corollary 9.4 below for a more precise statement in the case where Q is of Dynkin type.) By Remark 4.3, the converse is not true in general, even if $C_1 = C_2$. However, the converse holds in the case where C_1 or C_2 is rigid.

Proposition 6.1. *Let C_1, C_2 be irreducible components in* Comp. *Suppose that C_1 or C_2 is rigid. Then,*

C_1 and C_2 weakly commute if and only if C_1 and C_2 strongly commute.

If $x_1 \in C_1$ (say) is rigid, then the commutativity is equivalent to

$$\mathrm{Ext}^1_{\Pi}(x_1, x_2) = 0 \text{ for some } x_2 \in C_2. \tag{6.1}$$

(This condition is open in x_2.)

The equivalence of strong commutativity with (6.1) (in the case where C_1 is rigid) is clear. In order to show that weak commutativity implies strong one in the case at hand, we prove the following general result.

Lemma 6.2. *Let A be any algebra. Suppose that we have two short exact sequences of finite-dimensional A-modules*

$$\begin{gathered} 0 \to x_2 \to x \to x_1 \to 0 \\ 0 \to x_1 \to x \to x_3 \to 0 \end{gathered} \tag{6.2}$$

Assume that

1. *x_1 is rigid.*
2. *$\dim \mathrm{Hom}_A(x_1, x_2) = \dim \mathrm{Hom}_A(x_1, x_3)$.*
3. *$\dim \mathrm{Hom}_A(x_2, x_1) = \dim \mathrm{Hom}_A(x_3, x_1)$.*

Then, the short exact sequences (6.2) split (and hence $x_3 \simeq x_2$).

Proof. The short exact sequences (6.2) and the rigidity of x_1 yield two exact sequences

$$0 \to \mathrm{Hom}_A(x_1, x_2) \to \mathrm{Hom}_A(x_1, x) \xrightarrow{\alpha} \mathrm{End}_A(x_1) \xrightarrow{\beta} \mathrm{Ext}^1_A(x_1, x_2) \xrightarrow{\gamma} \mathrm{Ext}^1_A(x_1, x) \to 0$$

and

$$0 \to \mathrm{End}_A(x_1) \to \mathrm{Hom}_A(x_1, x) \to \mathrm{Hom}_A(x_1, x_3) \to 0.$$

Comparing dimensions and taking into account our hypothesis, we get

$$\dim \mathrm{Ext}^1_A(x_1, x_2) = \dim \mathrm{Ext}^1_A(x_1, x).$$

Hence, γ is an isomorphism. Therefore $\beta = 0$, so that α is surjective. Thus, the first sequence in (6.2) splits. By a dual argument, the second one also splits. □

Proposition 6.1 follows from Lemma 6.2. Indeed, we may assume without loss of generality that C_1 is rigid. Let x_1 be a rigid element in C_1 and suppose that $C = C_1 * C_2 = C_2 * C_1$. Let

$$S_2' = \operatorname{argmin}_{x_2 \in C_2} \dim \operatorname{Ext}^1_\Pi(x_1, x_2) = \operatorname{argmin}_{x_2 \in C_2} \dim \operatorname{Ext}^1_\Pi(x_2, x_1),$$

an open, nonempty subset of C_2, and let $S' = \mathcal{O}(x_1) \times S_2'$ and $S'' = S_2' \times \mathcal{O}(x_1)$. Then, $\mathcal{E}_2(S')$ and $\mathcal{E}_2(S'')$ are open in C and by [CBS02, Lemma 4.4] the conditions of Lemma 6.2 are satisfied for every $x \in \mathcal{E}_2(S') \cap \mathcal{E}_2(S'')$. Therefore, $C \subseteq C_1 \oplus C_2$ and hence, $C_1 \oplus C_2 = C$, as required.

Henceforth, if C_1 or C_2 is rigid, then we simply say that C_1 and C_2 commute if they weakly (or equivalently, strongly) commute.

7. Associativity

The operation $*$ is not associative, already for $Q = A_2$.

Example 7.1. Let $C_i = S(i)$, $i = 1, 2$. Then,

$$(C_1 * C_2) * C_1 = (C_1 * C_2) \oplus C_1, \quad C_1 * (C_2 * C_1) = C_1 \oplus (C_2 * C_1).$$

However, $C_1 * C_2 \neq C_2 * C_1$. Hence,

$$(C_1 * C_2) * C_1 \neq C_1 * (C_2 * C_1).$$

Nevertheless, it turns out that there is a useful cohomological criterion which guarantees associativity, as will be explained below.

For the next result, which is in the spirit of [CBS02, Theorem 1.3(ii)], let A be any finitely generated algebra. We consider the module varieties $\operatorname{mod}_A(d)$ of d-dimensional A-modules, $d \geq 0$. Fix $d_1, d_2, d_3 \geq 0$ and let S be a $\mathrm{GL}_{d_1} \times \mathrm{GL}_{d_2} \times \mathrm{GL}_{d_3}$-stable, constructible subset of $\operatorname{mod}_A(d_1) \times \operatorname{mod}_A(d_2) \times \operatorname{mod}_A(d_3)$. Let $d = d_1 + d_2 + d_3$ and define

$$\begin{aligned}\mathcal{E}_3(S) = \{M \in \operatorname{mod}_A(d) \mid \exists \text{ submodules } N_1 \subseteq N_2 \subseteq M \text{ such that} \\ \text{the isomorphism classes of } (M/N_2, N_2/N_1, N_1) \text{ belongs to } S\}.\end{aligned}$$

Proposition 7.2. *Let $\smile$ be the cup product*

$$\smile\colon \operatorname{Ext}^1_A(M_1, M_2) \otimes \operatorname{Ext}^1_A(M_2, M_3) \to \operatorname{Ext}^2_A(M_1, M_3). \tag{7.1}$$

Suppose that S is irreducible and that

1. *For each* $1 \le i < j \le 3$*, the dimensions of* $\operatorname{Ext}^1_A(M_i, M_j)$ *are constant on* S*.*
2. *For* $(M_1, M_2, M_3) \in S$*, the varieties*

$$\mathcal{Q}_{(M_1,M_2,M_3)} = \{(\phi, \psi) \in \operatorname{Ext}^1_A(M_1, M_2) \times \operatorname{Ext}^1_A(M_2, M_3) \mid \phi \smile \psi = 0\}$$

defined by quadratic equations are irreducible of constant dimension. (For instance, this is satisfied if $\smile$ *is trivial.)*

Then, $\mathcal{E}_3(S)$ *is irreducible.*

Proof. We recall some standard facts. For any A-modules M_1 and M_2, the space $M = \operatorname{Lin}(M_1, M_2)$ is an A-A bimodule and

$$\operatorname{Ext}^*_A(M_1, M_2) = H^*(A, M)$$

where the right-hand side is Hochschild cohomology. In particular,

$$\operatorname{Ext}^1_A(M_1, M_2) = \operatorname{Der}(A, M)/\operatorname{Der}^0(A, M)$$

where $\operatorname{Der}(A, M)$ is the space of derivatives, namely,

$$\operatorname{Der}(A, M) = \{d \in \operatorname{Lin}(A, M) \mid d(ab) = ad(b) + d(a)b \ \forall a, b \in A\}$$

and $\operatorname{Der}^0(A, M)$ is the subspace of inner derivatives, i.e., those of the form $a \mapsto am - ma$ for some $m \in M$. Also,

$$\operatorname{Ext}^2_A(M_1, M_2) = \operatorname{Fac}(A \otimes A, M)/\operatorname{Fac}^0(A \otimes A, M)$$

where $\operatorname{Fac}(A \otimes A, M)$ denotes the space of factor sets, i.e., bilinear maps $f : A \times A \to M$ satisfying

$$a_1 f(a_2, a_3) + f(a_1, a_2 a_3) = f(a_1 a_2, a_3) + f(a_1, a_2)a_3 \quad \forall a_1, a_2, a_3 \in A$$

and $\operatorname{Fac}^0(A \otimes A, M)$ is the image of the coboundary map

$$g \in \operatorname{Lin}(A, M) \mapsto \partial g(a, b) = g(ab) - ag(b) - g(a)b$$

whose kernel is $\operatorname{Der}(A, M)$.

Now let M_1, M_2, M_3 be three A-modules. For brevity we write $\mathcal{L}_{i,j} = \operatorname{Lin}(M_i, M_j)$, $1 \le i < j \le 3$. The bilinear map

$$\operatorname{Der}(A, \mathcal{L}_{1,2}) \times \operatorname{Der}(A, \mathcal{L}_{2,3}) \to \operatorname{Fac}(A \otimes A, \mathcal{L}_{1,3})$$

given by

$$(f_1, f_2) \mapsto (f_2 \otimes f_1)(a, b) = f_2(a) f_1(b)$$

induces the cap product (7.1).

Now, consider the closed subset Z of $\mathrm{mod}_A(d)$ consisting of homomorphisms $A \to \mathrm{Mat}_d$ that are block lower triangular with respect to the decomposition $d = d_1 + d_2 + d_3$. The diagonal blocks induce a morphism

$$\pi : Z \to \mathrm{mod}_A(d_1) \times \mathrm{mod}_A(d_2) \times \mathrm{mod}_A(d_3). \tag{7.2}$$

The fiber at (M_1, M_2, M_3) is given by the triples

$$(f_1, f_2, g) \in \mathrm{Der}(A, \mathcal{L}_{1,2}) \times \mathrm{Der}(A, \mathcal{L}_{2,3}) \times \mathrm{Lin}(A, \mathcal{L}_{1,3})$$

satisfying

$$f_2 \otimes f_1 = \partial g.$$

This is an affine bundle (with fiber $\mathrm{Der}(A, \mathcal{L}_{1,3})$) over

$$\{(f_1, f_2) \in \mathrm{Der}(A, \mathcal{L}_{1,2}) \times \mathrm{Der}(A, \mathcal{L}_{2,3}) \mid f_2 \otimes f_1 \in \mathrm{Fac}^0(A \otimes A, \mathcal{L}_{1,3})\}$$

which in turn is an affine bundle over $\mathcal{Q}_{(M_1,M_2,M_3)}$ with fiber $\mathrm{Der}^0(A, \mathcal{L}_{1,2}) \times \mathrm{Der}^0(A, \mathcal{L}_{2,3})$.

By [CBS02, Lemma 4.4], the first condition on S guarantees that for all $1 \leq i < j \leq 3$, the dimensions of $\mathrm{Der}(A, \mathcal{L}_{i,j})$ and $\mathrm{Der}^0(A, \mathcal{L}_{i,j})$ are constant for $(M_1, M_2, M_3) \in S$.

Together with the second condition on S, this ensures that the fiber of π over any point $(M_1, M_2, M_3) \in S$ is irreducible of constant dimension. Let V be the vector space

$$\mathrm{Lin}(A, \mathcal{L}_{1,2}) \times \mathrm{Lin}(A, \mathcal{L}_{2,3}) \times \mathrm{Lin}(A, \mathcal{L}_{1,3})$$

(which depends only on d_1, d_2, d_3) with $\mathbb{G}_m$ acting by multiplication by t, t and t^2 respectively on the three factors. Then, we can identify $\pi^{-1}(S)$ with a closed $\mathbb{G}_m$-stable subset of

$$S \times V.$$

Since S is irreducible, we infer that $\pi^{-1}(S)$ is irreducible by Lemma 7.3 below.

Hence, $\mathcal{E}_3(S)$ is irreducible since it is the image of $\pi^{-1}(S)$ under the action of GL_d. $\square$

Lemma 7.3. *Let Y be an irreducible variety and V a vector space. Suppose that the multiplicative group $\mathbb{G}_m$ acts trivially on Y and with positive exponents on V. Let X be a closed, $\mathbb{G}_m$-stable subvariety of $Y \times V$ and let $p : X \to Y$ be the canonical projection. Suppose that the fibers of p are irreducible and of constant dimension. Then, X is irreducible.*

Proof. We may assume that $X \neq Y \times \{0\}$. It is enough to prove that $X' = X \setminus (Y \times \{0\})$ is irreducible, since it is dense in X. The weighted projective space $\mathbb{P}(V) = (V \setminus \{0\})/\mathbb{G}_m$ is a projective variety, and hence, the map $Y \times \mathbb{P}(V) \to Y$ is closed. Since $Z := X'/\mathbb{G}_m$ is a closed subvariety of $Y \times \mathbb{P}(V)$, we obtain a closed map $f : Z \to Y$ whose fibers are irreducible of constant dimension. By a standard result (cf. proof of [Sha13, Theorem 1.26]), Z is irreducible. Therefore, X' is irreducible. ☐

As always, in the case where A admits a set of orthogonal idempotents such that $e_1 + \cdots + e_n = 1$, Proposition 7.2 immediately extends to the module varieties of given graded dimensions.

We will apply this to the case $A = \Pi$.

Harking back to Example 7.1, let $M_1 = M_3 = S(1)$ and $M_2 = S(2)$. Then, the spaces $\mathrm{Ext}^1_\Pi(M_1, M_2)$, $\mathrm{Ext}^1_\Pi(M_2, M_3)$ and $\mathrm{Ext}^2_\Pi(M_1, M_3)$ are one-dimensional and the cup product (7.1) is non-trivial. Therefore, the quadric $\mathcal{Q}_{(M_1,M_2,M_3)}$ is the cross $xy = 0$, which is reducible. This explains the failure of associativity in this case.

In general, we have the following.

Corollary 7.4. *Let $C_i \in \mathrm{Comp}(V^i)$, $i = 1, 2, 3$. Suppose that generically in $C_1 \times C_2 \times C_3$, the variety $\mathcal{Q}_{(M_1,M_2,M_3)}$ is irreducible. Then,*

$$(C_1 * C_2) * C_3 = C_1 * (C_2 * C_3).$$

In particular, this holds if one of the following conditions hold.

1. $\mathrm{Ext}^1_\Pi(M_1, M_2) \smile \mathrm{Ext}^1_\Pi(M_2, M_3) = 0$ *generically in $C_1 \times C_2 \times C_3$. For instance, this holds if at least one of the following conditions hold.*
   (a) C_1 and C_2 strongly commute.
   (b) C_2 and C_3 strongly commute.
   (c) $\hom_\Pi(C_3, C_1) = 0$ *(by (2.10)).*
2. *On an open, nonempty subset of $C_1 \times C_2 \times C_3$,* $\dim \mathrm{Ext}^2_\Pi(M_1, M_3) = 1$ *and the rank of the cup product bilinear form (7.1) is bigger than one.*

Proof. Let $S \subseteq C_1 \times C_2 \times C_3$ be an open, nonempty (hence irreducible) subset for which

$$\begin{gathered}\dim \operatorname{Ext}^1_\Pi(M_i, M_j) = \operatorname{ext}^1_\Pi(C_i, C_j) \text{ for all } 1 \le i < j \le 3, \text{ and} \\ \text{the varieties } \mathcal{Q}_{(M_1,M_2,M_3)} \text{ are irreducible of constant dimension}\end{gathered}$$

for every $(M_1, M_2, M_3) \in S$. By Proposition 7.2, $\mathcal{E}_3(S)$ is irreducible. However, it follows from (3.3) that $\overline{\mathcal{E}_3(S)}$ contains both $(C_1 * C_2) * C_3$ and $C_1 * (C_2 * C_3)$, which are irreducible components of $\Lambda(V^1 \oplus V^2 \oplus V^3)$. The corollary follows. ☐

Remark 7.5. Corollary 7.4 generalizes [BKT14, Proposition 4.5].

Remark 7.6. Let U, V, W be linear spaces and $B : U \times V \to W$ a bilinear map. Let

$$X = \{(u, v) \in U \times V \mid B(u, v) = 0\}.$$

The varieties $\mathcal{Q}_{(M_1,M_2,M_3)}$ are of this type. In general, we do not know a simple exact criterion for the irreducibility of X.

For any $u \in U$ let $u^\perp$ be the annihilator of u in V with respect to B. Then, $U^\circ = \operatorname{argmin}_U \dim u^\perp \neq \emptyset$ is open in U [CBS02, Lemma 4.2]. The inverse image of U° under the projection $p_1 : X \to U$ is a vector bundle over U° whose closure X_1 is an irreducible component of X. Define X_2 similarly by interchanging U and V. Obviously, if X is irreducible, then $X_2 = X_1$. We do not know whether the converse holds in general. Note that in general, X may admit irreducible components other than X_1 and X_2. In the case where $X = \mathcal{Q}_{(M_1,M_2,M_3)}$ one can (ostensibly) weaken the assumption of Corollary 7.4 to assume that $X_1 = X_2$.

8. Further results

The following result gives a useful way to construct new rigid modules from old ones.

Lemma 8.1. *Let*

$$0 \to x_2 \to x \to x_1 \to 0 \tag{8.1}$$

be a short exact sequence of Π*-modules. Suppose that* $\operatorname{Ext}^1_\Pi(x, x_1) = 0$ *and* x_2 *is rigid. Then,* x_1 *and* x *are rigid.*

Proof. Consider the commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_\Pi(x_2,x) \arrow[r] \arrow[d, "f"] & \mathrm{Ext}^1_\Pi(x_1,x) \arrow[d] & \\
\mathrm{Hom}_\Pi(x_2,x_1) \arrow[r, "g"] \arrow[d] & \mathrm{Ext}^1_\Pi(x_1,x_1) \arrow[r] & \mathrm{Ext}^1_\Pi(x,x_1) \\
\mathrm{Ext}^1_\Pi(x_2,x_2) & &
\end{tikzcd}$$

where the middle row and left column are exact. Since $\mathrm{Ext}^1_\Pi(x_1,x) = 0$, $g\circ f = 0$. On the other hand, g (resp., f) is onto since $\mathrm{Ext}^1_\Pi(x,x_1) = 0$ (resp., $\mathrm{Ext}^1_\Pi(x_2,x_2) = 0$). Hence, $\mathrm{Ext}^1_\Pi(x_1,x_1) = 0$, so that x_1 is rigid. Similarly, the rigidity of x is argued using the diagram

$$\begin{tikzcd}
\mathrm{Ext}^1_\Pi(x_1,x_2) \arrow[r] \arrow[d] & \mathrm{Ext}^1_\Pi(x,x_2) \arrow[r] \arrow[d] & \mathrm{Ext}^1_\Pi(x_2,x_2) \\
\mathrm{Ext}^1_\Pi(x_1,x) \arrow[r] & \mathrm{Ext}^1_\Pi(x,x) \arrow[d] & \\
& \mathrm{Ext}^1_\Pi(x,x_1) &
\end{tikzcd}$$

The lemma follows. $\square$

Together with Proposition 5.1 we conclude

Corollary 8.2. *Let $C = C_1 * C_2$ with $C_1, C_2 \in \mathrm{Comp}$. Suppose that C_2 is rigid and C strongly commutes with C_1. Then C is rigid if and only if C_1 is rigid.*

Remark 8.3. Let x_1 and x_2 be any Π-modules. Suppose that x is a rigid Π-module which is an extension of x_1 by x_2. Then, up to isomorphism, x is determined by x_1 and x_2. This follows from [CBS02, Theorem 1.3(ii)].

Remark 8.4. A special case of Lemma 8.1, which was proved in [GLS06, Proposition 5.7], is when (8.1) does not split and $\dim \mathrm{Ext}^1_\Pi(x_1,x_2) = 1$. Indeed, by [GLS06, Lemma 5.11] (which is valid for modules over any algebra A) under these conditions, if x_1 is rigid, then $\mathrm{Ext}^1_A(x_1,x) = 0$. For convenience, we recall the argument. By the rigidity of x_1, we have an exact sequence

$$0 \to \mathrm{Hom}_A(x_1,x_2) \to \mathrm{Hom}_A(x_1,x) \xrightarrow{\gamma} \mathrm{Hom}_A(x_1,x_1) \xrightarrow{\delta} \mathrm{Ext}^1_A(x_1,x_2) \to \mathrm{Ext}^1_A(x_1,x) \to 0.$$

The assumption that (8.1) does not split means that δ is nonzero. Since $\mathrm{Ext}^1_A(x_1, x_2)$ is one-dimensional, we infer that δ is surjective. Consequently, $\mathrm{Ext}^1_A(x_1, x) = 0$.

A dual argument shows that if x_2 is rigid, then $\mathrm{Ext}^1_A(x, x_2) = 0$. Note however that Lemma 8.1 is more general since we may have x_1 and x_2 rigid and $\mathrm{Ext}^1_\Pi(x, x_1) = 0$ without $\mathrm{Ext}^1_\Pi(x, x_2) = 0$.

Remark 8.5. Suppose that x is a rigid, non-simple Π-module. Then, one may hope that there exists a short exact sequence (8.1) such that x_1 and x_2 are rigid and non-trivial and $\mathrm{Ext}^1_\Pi(x, x_1) = 0$. If true, this would give an inductive approach to understand rigid modules. We will prove a special case in Lemma 9.5 below.

We finish this section with another useful result.

Lemma 8.6. *Suppose that* $\mathrm{Hom}_\Pi(x_2, x_1) = 0$ *and* x *is an extension of* x_1 *by* x_2*. Let* $\gamma \in \mathrm{Ext}^1_\Pi(x_1, x_2)$ *be the class of* x *and let*

$$\omega : \mathrm{Aut}_\Pi(x_1) \times \mathrm{Aut}_\Pi(x_2) \to \mathrm{Ext}^1_\Pi(x_1, x_2)$$

be the orbit map pertaining to γ*. Then,*

$$\dim \mathrm{Ext}^1_\Pi(x, x) = \dim \mathrm{Ext}^1_\Pi(x_1, x_1) + \dim \mathrm{Ext}^1_\Pi(x_2, x_2) + 2 \dim \mathrm{Coker}\, d\omega_{e,e}.$$

In particular, x *is rigid if and only if* x_1 *and* x_2 *are rigid and the* $\mathrm{Aut}_\Pi(x_1) \times \mathrm{Aut}_\Pi(x_2)$*-orbit of* γ *in* $\mathrm{Ext}^1_\Pi(x_1, x_2)$ *is open and separable.*

Finally, if x *is rigid and* x_2 *is a brick (that is,* $\mathrm{End}_\Pi(x_2) = K$*), then* $\mathrm{Ext}^1_\Pi(x_1, x) = 0$*.*

Proof. Consider the commutative diagram

$$\begin{tikzcd}
 & \mathrm{End}_\Pi(x_1) \arrow[r, equal] \arrow[d, "\partial_1"] & \mathrm{Hom}_\Pi(x, x_1) \arrow[d] & \\
\mathrm{End}_\Pi(x_2) \arrow[r, "\partial_2"] \arrow[d, equal] & \mathrm{Ext}^1_\Pi(x_1, x_2) \arrow[r] \arrow[d, "\iota^1"] & \mathrm{Ext}^1_\Pi(x, x_2) \arrow[r, two heads] \arrow[d, "\beta"] & \mathrm{Ext}^1_\Pi(x_2, x_2) \arrow[d, hook] \\
\mathrm{Hom}_\Pi(x_2, x) \arrow[r, "\partial_x"] & \mathrm{Ext}^1_\Pi(x_1, x) \arrow[r, "\alpha"] \arrow[d, two heads] & \mathrm{Ext}^1_\Pi(x, x) \arrow[r, "\beta^*"] \arrow[d, "\alpha^*"] & \mathrm{Ext}^1_\Pi(x_2, x) \arrow[d] \\
 & \mathrm{Ext}^1_\Pi(x_1, x_1) \arrow[r, hook] & \mathrm{Ext}^1_\Pi(x, x_1) \arrow[r] & \mathrm{Ext}^1_\Pi(x_2, x_1)
\end{tikzcd}$$

with exact rows and columns. Here we used the relation (2.9) as well as the assumption $\mathrm{Hom}_\Pi(x_2, x_1) = 0$, which implies that $\mathrm{Ext}^2_\Pi(x_1, x_2) = 0$ by (2.10). Observe that

$$\dim \mathrm{Im}\,\alpha = \dim \mathrm{Coker}\,\partial_x = \dim \mathrm{Coker}(\iota^1 \partial_2)$$
$$= \dim \mathrm{Coker}\,\iota^1 + \dim \mathrm{Coker}\,\partial_{12} = \dim \mathrm{Ext}^1_\Pi(x_1, x_1) + \dim \mathrm{Coker}\,\partial_{12}$$

where

$$\partial_{12} = (-\partial_1) \oplus \partial_2 : \mathrm{End}_\Pi(x_1) \oplus \mathrm{End}_\Pi(x_2) \to \mathrm{Ext}^1_\Pi(x_1, x_2).$$

Similarly,

$$\dim \mathrm{Im}\,\beta = \dim \mathrm{Ext}^1_\Pi(x_2, x_2) + \dim \mathrm{Coker}\,\partial_{12}.$$

Thus,

$$\begin{aligned} \dim \mathrm{Ext}^1_\Pi(x, x) &= \dim \mathrm{Im}\,\alpha + \dim \mathrm{Coker}\,\alpha = \dim \mathrm{Im}\,\alpha + \dim \mathrm{Ker}\,\alpha^* \\ &= \dim \mathrm{Im}\,\alpha + \dim \mathrm{Im}\,\beta \\ &= \dim \mathrm{Ext}^1_\Pi(x_1, x_1) + \dim \mathrm{Ext}^1_\Pi(x_2, x_2) + 2 \dim \mathrm{Coker}\,\partial_{12}. \end{aligned}$$

It remains to observe that ∂_{12} is the differential of ω at the identity.

For the last part, the assumption that x_2 is a brick implies that $\mathrm{Im}\,\partial_2 \subseteq \mathrm{Im}\,\partial_1$. Hence, $\partial_x = \iota^1 \partial_2 = 0$ and therefore $\mathrm{Ext}^1_\Pi(x_1, x) = 0$ since by assumption, $\alpha = 0$. ☐

Corollary 8.7. *Let $C_1, C_2 \in \mathrm{Comp}$ and $C = C_1 * C_2$. Suppose that C is rigid and $\mathrm{hom}_\Pi(C_2, C_1) = 0$. Then, C_1 and C_2 are rigid.*

9. Data for Π-modules[8]

Let M be a Q-representation with defining data $\mu_h : M_{h'} \to M_{h''}$, $h \in \Omega$. In order to extend M to a representation of Π we need to endow M with a structure of a Q°-representation (on the same underlying I-graded vector space) such the resulting $K\overline{Q}$-module structure descends to Π. This data is the same as an element of the kernel of the linear transformation

$$\oplus_{h \in \Omega} \mathrm{Lin}(M_{h''}, M_{h'}) \to \oplus_{i \in I} \mathrm{Lin}(M_i, M_i)$$

[8]We take the opportunity to thank the referee once again for simplifying the discussion of this section.

given by

$$(A_h)_{h\in\Omega} \mapsto \Big(\sum_{h\in\Omega|h'=i} A_h\mu_h - \sum_{h\in\Omega|h''=i} \mu_h A_h \Big)_{i\in I}.$$

We can identify this transformation with $\alpha^*_{\mu,\mu}$ (see (2.2)). By (2.1) we have $\mathrm{Ext}^1_Q(M,M)^* = \mathrm{Coker}(\alpha_{\mu,\mu})^* = \mathrm{Ker}(\alpha^*_{\mu,\mu})$, so we can view a Π-module as a Q-representation M together with an element of $\mathrm{Ext}^1_Q(M,M)^*$. Alternatively, by Voigt's Lemma this is the same as an element of the conormal of the G_V-orbit of M in $R_Q(V)$. Geometrically, this reflects the fact that the fiber over 0 of the moment map (2.4) is the union of the conormal bundles of the G_V-orbits in $R_Q(V)$. This discussion is closely related to [Rin98] except that we do not explicitly use the Auslander–Reiten translation functor. We will comment further about it at the end of the section.

Next, we describe Hom_Π and Ext^1_Π in this picture. Since $\mathrm{Ext}(\cdot,\cdot)$ is a bifunctor, for any Q-representations M and N we have a canonical bilinear map

$$\mathrm{Hom}_Q(M,N) \times \mathrm{Ext}^1_Q(N,M) \to \mathrm{Ext}^1_Q(M,M) \oplus \mathrm{Ext}^1_Q(N,N).$$

Dually, for any $f \in \mathrm{Hom}_Q(M,N)$ we get a linear map

$$f^* : \mathrm{Ext}^1_Q(M,M)^* \oplus \mathrm{Ext}^1_Q(N,N)^* \to \mathrm{Ext}^1_Q(N,M)^*.$$

Now, if M and N are Π-modules with data

$$\mathfrak{d} \in \mathrm{Ext}^1_Q(M,M)^*, \;\; \mathfrak{d}' \in \mathrm{Ext}^1_Q(N,N)^*,$$

then we get a linear map

$$\mathcal{T}_{M;N} : \mathrm{Hom}_Q(M,N) \longrightarrow \mathrm{Ext}^1_Q(N,M)^* \tag{9.1}$$

by

$$\mathcal{T}_{M;N} f = f^*(\mathfrak{d}, -\mathfrak{d}').$$

The following is a slight elaboration on [GLS07, §8] which is probably well-known to experts. For convenience, we include some details.

Proposition 9.1. *Let M and N be two Π-modules. Then, we have a functorial long exact sequence*

$$\begin{aligned} &0 \to \mathrm{Hom}_\Pi(M,N) \to \mathrm{Hom}_Q(M,N) \xrightarrow{\mathcal{T}_{M;N}} \mathrm{Ext}^1_Q(N,M)^* \to \\ &\mathrm{Ext}^1_\Pi(M,N) \to \mathrm{Ext}^1_Q(M,N) \xrightarrow{\mathcal{T}^*_{N;M}} \mathrm{Hom}_Q(N,M)^* \to \\ &\mathrm{Hom}_\Pi(N,M)^* \to 0. \end{aligned} \tag{9.2}$$

In particular,

$$\mathrm{Hom}_\Pi(M,N) = \mathrm{Ker}\,\mathcal{T}_{M;N} \tag{9.3}$$

and we have a functorial short exact sequence

$$0 \to \mathrm{Coker}(\mathcal{T}_{M;N}) \to \mathrm{Ext}^1_\Pi(M,N) \to \mathrm{Ker}(\mathcal{T}^*_{N;M}) = \mathrm{Coker}(\mathcal{T}_{N;M})^* \to 0.$$

Proof. Let $\mu_h : M_{h'} \to M_{h''}$, $\mu'_h : M_{h''} \to M_{h'}$ (resp., $\nu_h : N_{h'} \to N_{h''}$, $\nu'_h : N_{h''} \to N_{h'}$), $h \in \Omega$ be the data defining M (resp. N). Set

$$X = Z = \oplus_{i\in I} \mathrm{Lin}(M_i, N_i) = \oplus_{i\in I} \mathrm{Lin}(N_i, M_i)^*$$

and $Y = Y_1 \oplus Y_2$ where

$$\begin{aligned} Y_1 &= \oplus_{h\in\Omega} \mathrm{Lin}(M_{h'}, N_{h''}) = \oplus_{h\in\Omega} \mathrm{Lin}(N_{h''}, M_{h'})^*, \\ Y_2 &= \oplus_{h\in\Omega} \mathrm{Lin}(M_{h''}, N_{h'}) = \oplus_{h\in\Omega} \mathrm{Lin}(N_{h'}, M_{h''})^*. \end{aligned}$$

By the description of [GLS07, §8], the cohomologies of the complex

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

are $\mathrm{Hom}_\Pi(M,N)$, $\mathrm{Ext}^1_\Pi(M,N)$ and $\mathrm{Hom}_\Pi(N,M)^*$ where in the notation of (2.2) $f = \alpha_{\mu,\nu} \oplus \alpha_{\mu',\nu'}$ and $g = \alpha^*_{\nu',\mu'} \oplus \alpha^*_{\nu,\mu}$. The exactness of (9.2) follows from the lemma below together with (2.1). ☐

Lemma 9.2. *Suppose that in an Abelian category we have a complex*

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

and a short exact sequence

$$0 \to Y_2 \to Y \to Y_1 \to 0.$$

Let f_1 be the projection of f to Y_1 and let g_2 be restriction of g to Y_2. Then, we have an exact sequence

$$\begin{aligned} 0 \to \mathrm{Ker}\, f \to &\mathrm{Ker}\, f_1 \xrightarrow{f} \mathrm{Ker}\, g_2 \to \mathrm{Ker}\, g / \mathrm{Im}\, f \to \\ &\mathrm{Coker}\, f_1 \xrightarrow{g} \mathrm{Coker}\, g_2 \to \mathrm{Coker}\, g \to 0. \end{aligned}$$

Indeed, this is just the long exact sequence in cohomology pertaining to the short exact sequence of complexes

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & X \arrow[r, equal] \arrow[d, "f"] & X \arrow[r] \arrow[d, "f_1"] & 0 \arrow[d] \\
0 \arrow[r] \arrow[d] & Y_2 \arrow[r] \arrow[d, "g_2"] & Y \arrow[r] \arrow[d, "g"] & Y_1 \arrow[r] \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & Z \arrow[r, equal] & Z \arrow[r] & 0 \arrow[r] & 0
\end{tikzcd}$$

Corollary 9.3. *In the notation above we have*

$$\dim \operatorname{Ext}^1_\Pi(M,N) = \dim \operatorname{Coker} \mathcal{T}_{M;N} + \dim \operatorname{Coker} \mathcal{T}_{N;M}.$$

In particular, $\operatorname{Ext}^1_\Pi(M,N) = 0$ *if and only if both* $\mathcal{T}_{M;N}$ *and* $\mathcal{T}_{N;M}$ *are surjective.*

Corollary 9.4. *Suppose that Q is of Dynkin type. Then, for any* $\mathfrak{m}, \mathfrak{n} \in \mathfrak{M}$*,*

(9.4) $\lambda_Q(\mathfrak{m}) * \lambda_Q(\mathfrak{n}) = \lambda_Q(\mathfrak{m}+\mathfrak{n}) \iff$
$\mathcal{T}_{N;M}$ *is surjective for some* $(M,N) \in \mu_Q(\mathfrak{m}) \times \mu_Q(\mathfrak{n})$.

(The latter is an open condition in (M,N)*.) In particular,*

(9.5) $\lambda_Q(\mathfrak{m})$ *and* $\lambda_Q(\mathfrak{n})$ *strongly commute* $\iff$
$\lambda_Q(\mathfrak{m}) * \lambda_Q(\mathfrak{n}) = \lambda_Q(\mathfrak{m}+\mathfrak{n}) = \lambda_Q(\mathfrak{n}) * \lambda_Q(\mathfrak{m})$.

Proof. Let M and N be the Π-modules with underlying Q-representations $M_Q(\mathfrak{m})$ and $M_Q(\mathfrak{n})$ and data

$$\mathfrak{d} \in \operatorname{Ext}^1_Q(M_Q(\mathfrak{m}), M_Q(\mathfrak{m}))^* \text{ and } \mathfrak{d}' \in \operatorname{Ext}^1_Q(M_Q(\mathfrak{n}), M_Q(\mathfrak{n}))^*.$$

Then, by Proposition 9.1, $\mathcal{T}_{N;M}$ is surjective if and only if

(9.6) the map $\operatorname{Ext}^1_\Pi(M,N) \longrightarrow \operatorname{Ext}^1_Q(M_Q(\mathfrak{m}), M_Q(\mathfrak{n}))$ is identically zero.

If the latter condition is satisfied for some data $\mathfrak{d}$ and $\mathfrak{d}'$, then since this condition is open, $\mathcal{E}_2(S) \subseteq \mu_Q(\mathfrak{m}+\mathfrak{n})$ for a nonempty open subset S of $\mu_Q(\mathfrak{m}) \times \mu_Q(\mathfrak{n})$. Thus, $\lambda_Q(\mathfrak{m}) * \lambda_Q(\mathfrak{n}) \subseteq \overline{\mu_Q(\mathfrak{m}+\mathfrak{n})} = \lambda_Q(\mathfrak{m}+\mathfrak{n})$, and hence $\lambda_Q(\mathfrak{m}) * \lambda_Q(\mathfrak{n}) = \lambda_Q(\mathfrak{m}+\mathfrak{n})$.

Conversely, if $\lambda_Q(\mathfrak{m}) * \lambda_Q(\mathfrak{n}) = \lambda_Q(\mathfrak{m}+\mathfrak{n})$, then by (3.3), since $\mu_Q(\mathfrak{m}+\mathfrak{n})$ is open in $\lambda_Q(\mathfrak{m}+\mathfrak{n})$, there exists $M \in \mu_Q(\mathfrak{m})$ and $N \in \mu_Q(\mathfrak{n})$ such that the set E of extensions of M by N is contained in $\lambda_Q(\mathfrak{m}+\mathfrak{n})$ and intersects $\mu_Q(\mathfrak{m}+\mathfrak{n})$. Recall that we can write E as $G_V \cdot U$ where U is isomorphic to a vector space and under this isomorphism, the surjection $\kappa : U \to \mathrm{Ext}^1_\Pi(M,N)$ is a quotient map of vector spaces. Hence, $U \cap \mu_Q(\mathfrak{m}+\mathfrak{n})$ is non-empty and open in U. On the other hand, the composition $\tilde{\kappa}$ of κ with $\mathrm{Ext}^1_\Pi(M,N) \to \mathrm{Ext}^1_Q(M_Q(\mathfrak{m}), M_Q(\mathfrak{n}))$ is zero on $U \cap \mu_Q(\mathfrak{m}+\mathfrak{n})$. (Here was used the standard fact that a non-trivial extension of two finite-dimensional modules cannot be isomorphic as a module to the direct sum.) Hence $\tilde{\kappa} \equiv 0$ and therefore (9.6) holds. Our assertion follows. □

We also remark that in the Dynkin case we have

$$\lambda_Q(\mathfrak{m}+\mathfrak{n}) \supseteq \lambda_Q(\mathfrak{m}) \oplus \lambda_Q(\mathfrak{n}) \tag{9.7}$$

for any $\mathfrak{m}, \mathfrak{n} \in \mathfrak{M}$.

Next, we mention a partial converse to Lemma 8.1 (cf. Remark 8.5).

Lemma 9.5. *Let x be a Π-module with underlying Q-representation M and data $\mathfrak{d} \in \mathrm{Ext}^1_Q(M,M)^*$. Suppose that as a Q-representation we have*

$$M = M_1 \oplus M_2 \text{ with } \mathrm{Ext}^1_Q(M_1, M_2) = 0.$$

Write $\mathfrak{d} = \left(\begin{smallmatrix} \mathfrak{d}_1 & \\ * & \mathfrak{d}_2 \end{smallmatrix}\right)$ where $\mathfrak{d}_i \in \mathrm{Ext}^1_Q(M_i, M_i)^*$, $i = 1, 2$ and let x_i be the Π-module with underlying Q-representation M_i and data $\mathfrak{d}_i$. Then, x is an extension of x_1 by x_2. Suppose moreover that $\mathrm{Hom}_Q(M_2, M_1) = 0$ and x is rigid. Then x_1 and x_2 are rigid. If in addition $\mathrm{End}_Q(M_2) = K$, then $\mathrm{Ext}^1_\Pi(x, x_1) = 0$.

Indeed, by the vanishing of $\mathrm{Ext}^1_Q(M_1, M_2)$, the projection $M \to M_1$ lies in the kernel of $\mathcal{T}_{x;x_1}$. The first part follows. The other parts follow from Lemma 8.6.

Relation with Coxeter functor

Finally, let us assume that Q is acyclic, i.e., KQ is finite-dimensional. Although the maps $\mathcal{T}_{M;N}$ defined above admit a concrete linear algebra realization (taking into account (2.1)), at least from a computational aspect it is advantageous to consider a different point of view using Ringel's formalism [Rin98].

Let $\Phi^\pm$ be the Coxeter functors defined by Bernstein–Gel'fand–Ponomarev on the category $\mathcal{C}_Q$ of representations of Q [BGfP73]. Then, Φ^- is the left adjoint of Φ^+. Let $\epsilon = \epsilon_Q$ be the involution of KQ given by multiplying each arrow by -1. It gives rise to an involutive auto-equivalence on $\mathcal{C}_Q$ also denoted by ϵ. Let $\tau = \epsilon\Phi^+$. (Of course, τ is defined for any representation although we will only consider finite-dimensional ones.)

By the Brenner–Butler–Gabriel theorem (see [Gab80, Proposition 5.3]), τ coincides with the Auslander–Reiten translation functor. Thus, since KQ is hereditary,

$$\tau M = \mathrm{Ext}^1_Q(M, KQ)^*.$$

Here KQ is considered as a left-module over itself; the Ext space is a right KQ-module, and its dual is a left KQ-module. Also, we have functorial isomorphisms

$$\mathrm{Ext}^1_Q(M, N) \simeq \mathrm{Hom}_Q(N, \tau M)^*.$$

Under this isomorphism, a data for a Π-representation is given by an element $\mathfrak{d} \in \mathrm{Hom}_Q(M, \tau M)$. If M and N are Π-modules with data $\mathfrak{d} \in \mathrm{Hom}_Q(M, \tau M)$ and $\mathfrak{d}' \in \mathrm{Hom}_Q(N, \tau N)$, then

$$\mathcal{T}_{M;N} : \mathrm{Hom}_Q(M, N) \longrightarrow \mathrm{Hom}_Q(M, \tau N) \tag{9.8}$$

is given by

$$f \mapsto \tau(f) \circ \mathfrak{d} - \mathfrak{d}' \circ f.$$

In practice, this gives a convenient realization of $\mathcal{T}_{M;N}$ (for instance, in translating the rigidity condition of a Π-module into linear algebra using Corollary 9.3).

Remark 9.6. ([GP79]) As a KQ-module, Π is isomorphic to $\oplus_{m\geq 0}(\tau^-)^m(KQ)$ where $\tau^- = \Phi^-\epsilon$ is the left adjoint of τ. Decomposing

$$KQ = \oplus_{i\in I} KQe_i$$

as the direct sum of the indecomposable projective KQ-modules, we obtain

$$\Pi = \oplus_{i\in I} P(i) \text{ where } P(i) = \Pi e_i \simeq \oplus_{m\geq 0}(\tau^-)^m(KQe_i). \tag{9.9}$$

If Q is of Dynkin type, then only finitely many summands on the right-hand side are nonzero, Π is finite-dimensional, and the $P(i)$'s are precisely the indecomposable projective Π-modules.

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Positivity of $\imath$canonical bases of type AIII/AIV

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To George Lusztig with admiration

Abstract: For the quasi-split quantum symmetric pair $(\mathbf{U}, \mathbf{U}^\imath)$ of type AIII/AIV, we show various positivity properties of the $\imath$canonical bases on finite-dimensional simple $\mathbf{U}$-modules, as well as their tensor products. This answers affirmatively a conjecture in [BW13].

1. Introduction

1.1.

Let $\mathbf{U} = \mathbf{U}_q(\mathfrak{sl}_{n+1})$ be the quantum group associated with the Lie algebra $\mathfrak{sl}_{n+1}$ over the field $\mathbb{Q}(q)$ with a generic parameter q through out this paper. We denote by $\mathbb{N}$ the set of non-negative integers. Let $(\mathbf{U}, \mathbf{U}^\imath)$ be the quasi-split (that is, without black dots in the Satake diagram) quantum symmetric pair of type AIII/AIV. This is the quantum analog of the symmetric pair either $(\mathfrak{sl}_{n+1}, \mathfrak{s}(\mathfrak{gl}_{\frac{n+1}{2}} \times \mathfrak{gl}_{\frac{n+1}{2}}))$ or $(\mathfrak{sl}_{n+1}, \mathfrak{s}(\mathfrak{gl}_{\frac{n+2}{2}} \times \mathfrak{gl}_{\frac{n}{2}})$ depending on the parity of n. Here $\mathbf{U}^\imath$ is a right coideal subalgebra of $\mathbf{U}$, that is, the coproduct of $\mathbf{U}$ maps $\mathbf{U}^\imath$ to $\mathbf{U}^\imath \otimes \mathbf{U}$. We call the subalgebra $\mathbf{U}^\imath$ the $\imath$ quantum group. The structure theory of quantum symmetric pairs was established by Letzter ([Le02]).

In a program initiated with Wang in [BW13, BW18], we established systematically a theory of $\imath$canonical bases arising from quantum symmetric pairs. This generalizes Lusztig's theory [Lu94] of canonical bases arising from quantum groups. This paper considers the positivity properties of $\imath$canonical bases in quasi-split type AIII/AIV, generalizing positivity properties of Lusztig's canonical bases. Note that the definition of $\imath$ quantum groups contains parameters. We consider two sets of parameters that are consist with either [BW13] or [Bao17]; see Definition 4.1 and 4.7. We shall treat the two cases tegother in the introduction. We remark that such positivity does not exist for general parameters.

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1.2.

Let us explain the main result of this paper in more detail. We denote by X the integral weight lattice of $\mathbf{U}$, and denote by X^+ the set of dominant weights. For any $\lambda \in X^+$, we denote by $L(\lambda)$ the simple $\mathbf{U}$-module of highest weight λ. For any $\lambda_1, \dots, \lambda_k \in X^+$, the tensor product $L(\lambda_1) \otimes \cdots \otimes L(\lambda_k)$ admits the canonical basis $B(\lambda_1, \dots, \lambda_k)$ by [Lu94], as well as the $\imath$canonical basis $B^\imath(\lambda_1, \dots, \lambda_k)$ following [BW13, Bao17].

Theorem 1.1 (Theorems 4.6 and 4.10)**.** *Let $\lambda_1, \dots, \lambda_k \in X^+$ and $0 \leq l \leq k$. We have*

$$b^\imath = \sum_{b_1^\imath, b_2} t_{b;b_1,b_2} b_1^\imath \otimes b_2 \in L(\lambda_1) \otimes \cdots \otimes L(\lambda_k), \qquad \textit{with } t_{b;b_1,b_2} \in \mathbb{N}[q],$$

where $b^\imath \in B^\imath(\lambda_1, \dots, \lambda_k)$, $b_1^\imath \in B^\imath(\lambda_1, \dots, \lambda_l)$, and $b_2 \in B(\lambda_{l+1}, \dots, \lambda_k)$.

We identify these coefficients $t_{b;b_1,b_2}$ with certain polynomials arising from (variations of) the Kazhdan-Lusztig bases of Hecke algebras. In extreme cases, they are exactly the (parabolic) Kazhdan-Lusztig polynomials. To establish such identification, we use Lusztig's theory of based modules ([Lu94, Chap. 27]), as well as its $\imath$ counterpart for quantum symmetric pairs. The positivity of $t_{b;b_1,b_2}$ essentially arises from the hyperbolic localization on flag varieties after such identification; see [Bra03, GH]. Special case of the theorem has been obtained in [FL15], where they considered the case all $L(\lambda_i)$ being the natural representation $\mathbb{V}$ and $l = 0$.

We have the following corollary concerning finite-dimensional simple $\mathbf{U}$-modules, which answers affirmatively [BW13, Conjecture 4.24].

Corollary 1.2. *Let $\lambda \in X^+$. Let $B^\imath(\lambda)$ and $B(\lambda)$ be the sets of $\imath$-canonical basis and canonical basis on $L(\lambda)$, respectively. We have*

$$b^\imath = \sum_{b_1 \in B} t_{b;b_1} b_1, \qquad \textit{with } t_{b;b_1} \in \mathbb{N}[q].$$

Our method can easily be adapted to give a simple proof (see [We15] for a proof via categorification) of a similar positivity statement for Lusztig's canonical bases arising from $\mathbf{U}$, that is, for $b \in B(\lambda_1, \dots, \lambda_k)$, $b_1 \in B(\lambda_1, \dots, \lambda_l)$, and $b_2 \in B(\lambda_{l+1}, \dots, \lambda_k)$, we have

$$b = \sum_{b_1^\imath, b_2} t'_{b;b_1,b_2} b_1 \otimes b_2 \in L(\lambda_1) \otimes \cdots \otimes L(\lambda_k), \qquad \text{with } t'_{b;b_1,b_2} \in \mathbb{N}[q].$$

1.3.

This paper is organised as follows. We recall various results on Hecke algebras in Section 2. We study based modules of quantum groups in Section 3. We prove the main theorem in Section 4.

2. The Hecke algebras

2.1.

Let (W, S) be a finite Weyl group. We denote the length function by $\ell(\cdot)$. For any $J \subset S$, we denote by W_J the corresponding parabolic subgroup, and denote by w_J the longest element of W_J.

Let W^J and ${}^JW$ be the set of minimal length coset representatives of W/W_J and $W_J\backslash W$, respectively. For $I, J \subset S$, we denote by ${}^IW^J$ the set of minimal length double coset representatives of $W_I\backslash W/W_J$. For any (left, right, or double) coset p, we denote by p_- the minimal length representative of this coset. We shall abuse the notation and denote the Bruhat orders on W, as well as on any cosets, by $\leq$. The following lemma can be found in [Ca93, Theorems 2.7.4 and 2.7.5].

Lemma 2.1. *Let $I, J \subset S$ and $p \in W_I\backslash W/W_J$. Let $K = I \cap p_- J p_-^{-1}$.*

(1) We have

$$W_I \cap p_- W_J p_-^{-1} = W_{I \cap p_- J p_-^{-1}}.$$

(2) The map

$$\begin{aligned}(W^K \cap W_I) \times W_J &\longrightarrow p \\ (u, v) &\mapsto u p_- v\end{aligned}$$

is a bijection and $\ell(u p_- v) = \ell(u) + \ell(p_-) + \ell(v)$.

2.2.

Let $\mathcal{H} = \mathcal{H}_W$ be the Hecke algebra of (W, S). This is a $\mathbb{Q}(q)$-algebra with the standard basis $\{H_w | w \in W\}$ satisfying the relations:

$$\begin{aligned} H_v H_w &= H_{vw}, && \text{if } \ell(vw) = \ell(v) + \ell(w); \\ (H_s + q)(H_s - q^{-1}) &= 0, && \text{for } s \in S. \end{aligned}$$

Let $^{-}: \mathcal{H} \to \mathcal{H}$ be the $\mathbb{Q}(q)$-semilinear bar involution such that $\overline{H_s} = H_s^{-1}$ and $\overline{q} = q^{-1}$.

Thanks to [Lu03, Chap 5], for any $w \in W$, there is a unique element $\underline{H_w}$ such that

(1) $\overline{\underline{H_w}} = \underline{H_w}$;
(2) $\underline{H_w} = \sum_{y \in W} p_{y,w} H_y$ where
  - $p_{y,w} = 0$ unless $y \leq w$;
  - $p_{w,w} = 1$;
  - $p_{y,w} \in q\mathbb{Z}[q]$ if $y < w$.

The set $\{\underline{H_w} | w \in W\}$ forms a $\mathbb{Q}(q)$-basis of $\mathcal{H}$. This is the canonical (or Kazhdan-Lusztig) basis of $\mathcal{H}$.

2.3.

We then recall the parabolic canonical basis, which was originally defined in [Deo87]. Modules of Hecke algebras are usually right modules.

Let $J \subset S$. Let e_J^+ be the 1-dimensional trivial representation of $\mathcal{H}_J$, where $H_w \in \mathcal{H}_J$ acts via multiplication by $q^{-\ell(w)}$. We define the induced module $M_J = \mathrm{Ind}_{\mathcal{H}_J}^{\mathcal{H}} e_J^+ = M_e \cdot \mathcal{H}$, where the generator M_e satisfies $M_e \cdot H_w = q^{-\ell(w)} M_e$ for $w \in W_J$. The module M_J admits a standard basis $\{M_w = M_e \cdot H_w | w \in {}^J W\}$. This module admits a $\mathbb{Q}(q)$-semilinear bar involution compatible with the bar involution on $\mathcal{H}$, such that

$$\overline{M_e} = M_e, \qquad \overline{m \cdot h} = \overline{m} \cdot \overline{h}, \qquad \text{for } h \in \mathcal{H}, m \in M_J.$$

Hence similar to [Lu03, Chap. 5], for any $w \in {}^J W$, there is a unique element $\underline{M_w}$ such that

(1) $\overline{\underline{M_w}} = \underline{M_w}$;
(2) $\underline{M_w} = \sum_{y \in {}^J W} p^+_{y,w} M_y$ where
  - $p^+_{y,w} = 0$ unless $y \leq w$;
  - $p^+_{w,w} = 1$;
  - $p^+_{y,w} \in q\mathbb{Z}[q]$ if $y < w$.

The set $\{\underline{M_w} | w \in {}^J W\}$ forms the canonical basis of M_J. We actually have the following embedding of $\mathcal{H}$-modules

$$\begin{aligned} p_J^+ : M_J &\to \mathcal{H}, \\ M_w &\mapsto \underline{H_{w_J}} \cdot H_w, \quad w \in {}^J W, \\ \underline{M_w} &\mapsto \underline{H_{w_J w}}, \quad w \in {}^J W. \end{aligned} \tag{2.1}$$

In a more concrete way, we have

$$\sum_{y\in {}^JW} p^+_{y;w}\underline{H_{w_J}}\cdot H_y = \underline{H_{w_Jw}}. \tag{2.2}$$

2.4.

Let $I\subset S$. We are interested in the restriction of $\mathcal{H}$ and M_J as (right) $\mathcal{H}_I$-modules. Let us start with the module $\mathcal{H}$. We have

$$\mathcal{H}\cong\bigoplus_{p\in W^I} H_{p_-}\cdot\mathcal{H}_I,\qquad \text{as } \mathcal{H}_I\text{-modules.} \tag{2.3}$$

For any $w=p_-w'$ with $p\in W^I$ and $w'\in W_I$, we write $\underline{H^I_w}=H_{p_-}\cdot\underline{H_{w'}}$. It is easy to see that $\{\underline{H^I_w}|w\in W\}$ forms a basis of $\mathcal{H}_W$. This is the so-called hybrid basis in [GH].

We then turn to the module M_J. We have

$$M_J\cong\bigoplus_{p\in {}^JW^I} M_e\cdot H_{p_-}\cdot\mathcal{H}_I=\bigoplus_{p\in {}^JW^I} M_{p_-}\cdot\mathcal{H}_I,\ \text{as } \mathcal{H}_I\text{-modules.} \tag{2.4}$$

Now thanks to Lemma 2.1, we have

$$M_{p_-}\cdot\mathcal{H}_I\cong \operatorname{ind}^{\text{-}\mathcal{H}_I}_{\text{-}\mathcal{H}_K} e^+_K,\qquad \text{where } K=K_p=p_-^{-1}Jp_-\cap I.$$

Therefore we have the standard basis $\{M_{p_-w}|w\in W_I\cap {}^{K_p}W\}$, and can also define the canonical basis $\{\underline{M^I_{p_-w}}|w\in W_I\cap {}^{K_p}W\}$ of each summand $M_{p_-}\cdot\mathcal{H}_I$ follwing §2.3. Therefore we obtain the (parabolic) hybrid basis $\{\underline{M^I_{p_-w}}|w\in W_I\cap {}^{K_p}W, K_p=p_-^{-1}Jp_-\cap I, p\in {}^JW^I\}$ of M_J as an $\mathcal{H}_I$-module. Recall the embedding $p^+_J: M_J\to\mathcal{H}$ in (2.1).

Lemma 2.2. *We fix a double coset $p\in W_J\backslash W/W_I$. Let $K=p_-^{-1}Jp_-\cap I$, $K'=p_-Kp_-^{-1}=J\cap p_-Ip_-^{-1}$, and $w\in W_I\cap {}^KW$. We have*

$$p^+_J(\underline{M^I_{p_-w}})=\sum_{r\in W_J\cap W^{K'}} q^{\ell(w_J)-\ell(rw_{K'}p_-w)+\ell(p_-w)}\underline{H^I_{rw_{K'}p_-w}}.$$

Proof. Recall that we have

$$\underline{H_{w_J}}=\sum_{w\in W_J} q^{\ell(w_J)-\ell(w)}H_w=\sum_{r\in W_J\cap W^{K'}} q^{\ell(w_J)-\ell(w_{K'})-\ell(r)}H_r\cdot\underline{H_{w_{K'}}}.$$

On the other hand, thanks to Lemma 2.1 we have

$$\underline{H_{w_{K'}}} \cdot H_{p_-} = H_{p_-} \cdot \underline{H_{w_K}}.$$

Hence we have

$$\underline{H_{w_J}} \cdot H_{p_-} = \sum_{r \in W_J \cap W^{K'}} q^{\ell(w_J)-\ell(w_K)-\ell(r)} H_r \cdot H_{p_-} \cdot \underline{H_{w_K}}.$$

Now we can write

$$\begin{aligned}
p_J^+(\underline{M^I_{p_-w}}) &= p_J^+\Big(\sum_{y \in W_I \cap {}^K W} p^+_{y;w} M_{p_-} \cdot H_y\Big) \\
&= \sum_{y \in W_I \cap {}^K W} p^+_{y;w} \underline{H_{w_J}} \cdot H_{p_-} \cdot H_y \\
&= \sum_{y \in W_I \cap {}^K W} p^+_{y;w} \Big(\sum_{r \in W_J \cap W^{K'}} q^{\ell(w_J)-\ell(w_K)-\ell(r)} H_r \cdot H_{p_-} \cdot \underline{H_{w_K}} \cdot H_y\Big) \\
&= \sum_{r \in W_J \cap W^{K'}} q^{\ell(w_J)-\ell(w_K)-\ell(r)} H_r \cdot H_{p_-} \cdot \Big(\sum_{y \in W_I \cap {}^K W} p^+_{y;w} \cdot \underline{H_{w_K}} \cdot H_y\Big) \\
&\overset{\heartsuit}{=} \sum_{r \in W_J \cap W^{K'}} q^{\ell(w_J)-\ell(w_K)-\ell(r)} H_{rp_-} \cdot \underline{H_{w_K w}} \\
&= \sum_{r \in W_J \cap W^{K'}} q^{\ell(w_J)-\ell(w_{K'})-\ell(r)} \underline{H^I_{rw_{K'}p_-w}} \\
&= \sum_{r \in W_J \cap W^{K'}} q^{\ell(w_J)-\ell(rw_{K'}p_-w)+\ell(p_-w)} \underline{H^I_{rw_{K'}p_-w}}.
\end{aligned}$$

The identity ($\heartsuit$) follows from (2.2). This finishes the proof. □

2.5.

The following proposition is the key for the positivity results in this paper.

Proposition 2.3. *[GH, Bra03] We have, as an equation in $\mathcal{H}$,*

$$\underline{H_w} = \sum_{y \le w} p^I_{y;w} \underline{H^I_y}, \quad p^I_{y;w} \in \mathbb{N}[q].$$

Proposition 2.4. *Let $w \in {}^J W$. We have the following positivity result:*

$$\underline{M_w} = \sum_{y \in {}^J W} p^{I,+}_{y,w} \underline{M^I_y} \in M_J, \qquad \textit{with } p^{I,+}_{y,w} \in \mathbb{N}[q].$$

Remark 2.5. The proposition can also be proved similar to Proposition 2.3 via hyperbolic localization on partial flag varieties following [GH, Bra03].

We give another proof identifying $p^{I,+}_{y,w}$ with $p^{I}_{y',w'}$ up to a "positive" scalar for certain y', $w' \in W$. This should be considered as an analog of [Deo87, Proposition 3.4].

Proof. We compare the images of both sides under the embedding (2.1).

For the left hand side, we have

$$p^{+}_{J}(\underline{M_w}) = \underline{H_{w_J w}} = \sum_{y\in W} p^{I}_{y,w_J w}\underline{H^{I}_{y}}.$$

Thanks to Lemma 2.2, we have

$$\begin{aligned}
&p^{+}_{J}\Big(\sum_{y\in {}^J W} p^{I,+}_{y,w}\underline{M^{I}_{y}}\Big)\\
=&p^{+}_{J}\Big(\sum_{p\in W_J\backslash W/W_I}\Big(\sum_{y\in {}^J W, y\in p} p^{I,+}_{y,w}\underline{M^{I}_{y}}\Big)\Big)\\
=&\sum_{p\in W_J\backslash W/W_I}\Big(\sum_{y\in {}^J W, y\in p} p^{I,+}_{y,w}\Big(\sum_{r\in W_J\cap W^{K'_p}} q^{\ell(w_J)-\ell(rw_{K'_p}y)+\ell(y)}\underline{H^{I}_{rw_{K'_p}y}}\Big)\Big),
\end{aligned}$$

where $K'_p = J\cap p_- I p_-^{-1}$.

Then by comparing the coefficients, we have

$$q^{\ell(rw_{K'_p}y)} p^{I}_{rw_{K'_p}y, w_J w} = q^{\ell(w_J)+\ell(y)} p^{I,+}_{y,w}, \qquad \text{for } y\in p\in W_J\backslash W/W_I.$$

The proposition then follows from Proposition 2.3. □

3. Based modules of quantum groups

In this section, we review based modules of quantum groups. The main reference is [Lu94, Chap. 27]. We denote the standard generators of $\mathbf{U} = \mathbf{U}_q(\mathfrak{sl}_{n+1})$ by $E_i, F_i, K_i^{\pm 1}$ for $i = 1, 2, \dots, n$, and denote the anti-linear ($q \mapsto q^{-1}$) bar involution on $\mathbf{U}$ by ψ (denoted by $^-$ in [Lu94, §3.1.12]).

We use the following convention on the coproduct of $\mathbf{U}$:

$$\begin{aligned}
\Delta :&\mathbf{U} \longrightarrow \mathbf{U}\otimes\mathbf{U},\\
\Delta(E_i) &= 1\otimes E_i + E_i\otimes K_i^{-1},\\
\Delta(F_i) &= F_i\otimes 1 + K_i\otimes F_i,\\
\Delta(K_i) &= K_i\otimes K_i.
\end{aligned}$$

This is the same as the one used in [BW13], but is different from the one used in [Lu94].

3.1.

Let (M, B) be a finite-dimensional based $\mathbf{U}$-module ([Lu94, §27.1.2]). We shall denote the associated bar involution on M by ψ. We shall abuse the notation and denote by ψ the bar involution on any based $\mathbf{U}$-module. Elements of B are ψ-invariant by definition. We have the following compatibility of bar involutions:

$$\psi(ux) = \psi(u)\psi(x), \qquad u \in \mathbf{U}, x \in M.$$

Let (M', B') be another based $\mathbf{U}$-module. A based morphism from a based module (M, B) to a based module (M', B') is by definition a morphism $f : M \to M'$ of $\mathbf{U}$-modules such that

(1) for any $b \in B$, we have $f(b) \in B' \cup \{0\}$;
(2) $B \cap \ker f$ is a basis of $\ker f$.

As a consequence, we see that $f \circ \psi = \psi \circ f$. We also have the obvious notions of based submodules and based quotients.

Lemma 3.1. *[Lu94, Theorem 27.3.2] The tensor product $M \otimes M'$ is a based $\mathbf{U}$-module with the basis $B \diamondsuit B' = \{b \diamondsuit b' | (b, b') \in B \times B'\}$, where $b \diamondsuit b'$ is the unique ψ-invariant element of the form*

$$b' \otimes b' + \sum_{(b_1, b_1') \in B \times B'} q\mathbb{Z}[q] b_1 \otimes b_1'. \tag{3.1}$$

Here the bar involution ψ on $M_1 \otimes M_2$ is defined as $\psi = \Theta \circ (\psi \otimes \psi)$, where Θ denotes the quasi-$\mathcal{R}$ matrix ([Lu94, §4.1]).

Remark 3.2. Note that while it is crucial to consider the integral lattice $\mathbb{Z}[q, q^{-1}]$ in the theory of based modules, it plays little role (after the establishment of canonical bases) for the question we are interested in in this paper. So we shall not make this explicit.

Lemma 3.3. *Let (M_i, B_i) be finite-dimensional based $\mathbf{U}$-modules for $i = 1, 2, 3, 4$. Let $f : M_1 \to M_3$ and $g : M_2 \to M_4$ be morphisms of based $\mathbf{U}$-modules. Then the morphism $f \otimes g : M_1 \otimes M_2 \to M_3 \otimes M_4$ is a based morphism of $\mathbf{U}$-modules.*

Proof. It is clear that $f \otimes g$ is a morphism of $\mathbf{U}$-modules. We prove it is based.

Since f and g are both based morphisms, we know that $(f\otimes g)\circ(\psi\otimes\psi) = (\psi \otimes \psi) \circ (f \otimes g)$. Since Θ is in the completion of $\mathbf{U} \otimes \mathbf{U}$ ([Lu94, §4.1]), we see that $f \otimes g$ commutes with Θ automatically. Hence $f \otimes g$ commutes with the bar involutions $\psi = \Theta \circ (\psi \otimes \psi)$.

Let $b_1 \diamondsuit b_2 \in B_1 \diamondsuit B_2$. We have

$$\psi\Big((f \otimes g)(b_1 \diamondsuit b_2)\Big) = (f \otimes g)\Big(\psi(b_1 \diamondsuit b_2)\Big) = (f \otimes g)(b_1 \diamondsuit b_2).$$

On the other hand, $(f \otimes g)(b_1 \diamondsuit b_2)$ is of the form (thanks to (3.1))

$$f(b_1) \otimes g(b_2) + \sum_{(b_1',b_2') \in B_1 \times B_2} q\mathbb{Z}[q] f(b_1') \otimes g(b_2'),$$

where $f(b_1), f(b_1') \in B_3 \cup \{0\}$ and $g(b_2), g(b_2') \in B_4 \cup \{0\}$, since f and g are based.

Now if neither $f(b_1)$ nor $g(b_2)$ are zero, thanks to Lemma 3.1, we see that $(f \otimes g)(b_1 \diamondsuit b_2) = f(b_1) \diamondsuit g(b_2) \in B_3 \diamondsuit B_4$.

if $f(b_1)$ or $g(b_2)$ is zero, $(f \otimes g)(b_1 \diamondsuit b_2)$ is a ψ-invariant elment of the form

$$\sum_{(b_1',b_2') \in B_1 \times B_2} q\mathbb{Z}[q] f(b_1') \otimes g(b_2'),$$

hence has to be zero. Therefore we have

$$(f \otimes g)(b_1 \diamondsuit b_2) = \begin{cases} f(b_1) \diamondsuit g(b_2), & \text{if } (f(b_1), g(b_2)) \in B_3 \times B_4; \\ 0, & \text{otherwise.} \end{cases}$$

It follows immediately that $f \otimes g$ is based. ☐

3.2.

Let $\mathbb{I} = \{1, 2, \dots, n+1\}$ be the index set. Let $\mathbb{V}$ be the natural representation of $\mathbf{U}$. Then $\mathbb{V}$ is a based $\mathbf{U}$-module with the basis $B = \{v_i | i \in \mathbb{I}\}$ being the standard basis as well as the canonical basis. The tensor product $\mathbb{V}^{\otimes m}$ admits the standard basis $\{v_f = v_{f(1)} \otimes \cdots \otimes v_{f(m)} | f \in \mathbb{I}^m\}$. It is also a based module with basis $B^{\diamondsuit m}$ thanks to Lemma 3.1. We call an element $f \in \mathbb{I}^m$ anti-dominant, if $f(1) \leq f(2) \leq \cdots \leq f(m)$.

Let $1 \leq m \leq n$. Let $\mathcal{H}_{A_{m-1}}$ be the Hecke algebra of the Weyl group $W_{A_{m-1}}$, where we denote the set of simple reflections of $W_{A_{m-1}}$ by S. The Hecke algebra $\mathcal{H}_{A_{m-1}}$ and the Weyl group $W_{A_{m-1}}$ both act (from the right) naturally on $\mathbb{V}^{\otimes m}$ (c.f. [BW13, §5.1]). We have the following identification of $\mathcal{H}_{A_{m-1}}$-modules,

$$\begin{aligned}\mathbb{V}^{\otimes m} &= \bigoplus_{f\in\mathbb{I}^m} v_f \cdot \mathcal{H}_{A_{m-1}} \\ &\cong \bigoplus_{f\in\mathbb{I}^m} \underline{H_{w_{J(f)}}} \cdot \mathcal{H}_{A_{m-1}} \cong \bigoplus_{f\in\mathbb{I}^m} M_{J(f)}, \text{ for anti-dominant } f \in \mathbb{I}^m,\end{aligned}$$

where $J(f) \subset S$ such that $W_{J(f)} \subset W_{A_{m-1}}$ is the stablizer of v_f.

Proposition 3.4. *[FKK98] The basis $B^{\diamondsuit m}$ of $\mathbb{V}^{\otimes m}$ can be identified with the (parabolic) Kazhdan-Lusztig basis associated with the Hecke algebra $\mathcal{H}_{A_{m-1}}$ via the q-Schur duality.*

We define $R = \sum_{w\neq e} \mathbb{V}^{\otimes m} \cdot \underline{H_w}$, for $\underline{H_w} \in \mathcal{H}_{A_{m-1}}$.

Lemma 3.5. *The quotient space $\wedge^m\mathbb{V} = \mathbb{V}^{\otimes m}/R$ is a based* **U***-module. Moreover, the quotient map $\pi : \mathbb{V}^{\otimes m} \to \wedge^m\mathbb{V}$ is a morphism of based* **U***-modules.*

Proof. It suffices to prove that R is a based **U**-submodule of $\mathbb{V}^{\otimes m}$. It follows from the q-Schur duality that R is a **U**-submodule. We prove it is based.

Recall $\mathbb{V}^{\otimes m} = \bigoplus_{\text{anti-dominant } f\in\mathbb{I}^m} v_f \cdot \mathcal{H}_{A_{m-1}}$. For any anti-dominant f with $J(f) \neq \emptyset$, we have

$$\begin{aligned}&\sum_{e\neq w\in W} v_f \cdot \mathcal{H}_{A_{m-1}} \cdot \underline{H_w} \\ \cong &\sum_{e\neq w\in W} \underline{H_{w_{J(f)}}} \cdot \mathcal{H}_{A_{m-1}} \cdot \underline{H_w} \\ = &\sum_{e\neq w\in W} \underline{H_{w_{J(f)}}} \cdot \mathcal{H}_{A_{m-1}} \\ \cong &\, v_f \cdot \mathcal{H}_{A_{m-1}}.\end{aligned}$$

So in this case, the subspace $\sum_{e\neq w\in W} v_f \cdot \mathcal{H}_{A_{m-1}} \cdot \underline{H_w}$ admits basis $\{v_f\underline{H_w} | w \in {}^{J(f)}W\}$. On the other hand, if f is anti-dominant with $J(f) = \emptyset$, the subspace $\sum_{e\neq w\in W} v_f \cdot \mathcal{H}_{A_{m-1}} \cdot \underline{H_w}$ admits the basis $\{v_f\underline{H_w} | e \neq w \in W\}$. It follows that R is a based submodule. □

Corollary 3.6. *Let $m_1, m_2, \dots, m_k \leq n$. The quotient map $\pi : \mathbb{V}^{\otimes(m_1+m_2+\cdots+m_k)} \to \wedge^{m_1}\mathbb{V} \otimes \wedge^{m_2}\mathbb{V} \otimes \cdots \wedge^{m_k} \mathbb{V}$ is a morphism of based* **U***-modules.*

Proof. The corollary follows from Lemma 3.3 and Lemma 3.5. □

Lemma 3.7. *Let $L(\lambda_i)$ be finite-dimensional simple* $\mathbf{U}$*-modules for $i = 1, \dots, l$. We have the following based embedding of* $\mathbf{U}$*-modules for suitable $m_1, \dots, m_k \le n$:*

$$L(\lambda_1) \otimes \cdots \otimes L(\lambda_l) \longrightarrow \wedge^{m_1}\mathbb{V} \otimes \wedge^{m_2}\mathbb{V} \otimes \cdots \wedge^{m_k}\mathbb{V}.$$

Proof. Thanks to Lemma 3.3, it suffices to prove the statement for $l = 1$. We write $\lambda_1 = \sum_{i=1}^{n-1} a_i\omega_i \in X^+$, where ω_i is the ith fundamental weight. Recall we have $\wedge^i\mathbb{V} \cong L(\omega_i)$ for $1 \le i \le n-1$.

Therefore we have the natural embedding

$$L(\lambda_1) \longrightarrow \mathbb{V}^{\otimes a_1} \otimes (\wedge^2\mathbb{V})^{a_2} \otimes \cdots \otimes (\wedge^{n-1}\mathbb{V})^{a_{n-1}}.$$

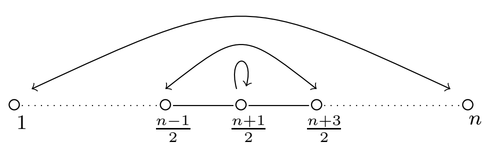
The fact that this is a based embedding follows from [Lu94, §27.2], since $L(\lambda_1)$ is exactly the submodule $\Big(\mathbb{V}^{\otimes a_1} \otimes (\wedge^2\mathbb{V})^{a_2} \otimes \cdots \otimes (\wedge^{n-1}\mathbb{V})^{a_{n-1}}\Big)[\ge \lambda_1]$. □

4. Positivity of ıcanonical bases

4.1.

Let us recall the theory of canonical bases arising from quantum symmetric pairs. We first recall the following two families of quantum symmetric pairs considered in [BW13].

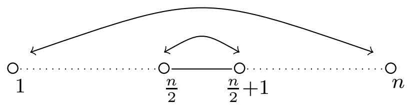
Definition 4.1. (1) If $n+1 = 2r+2$ is even, then we consider the quantum symmetric pair $(\mathbf{U}^\imath, \mathbf{U})$ whose Satake diagram is of the form



The coideal subalgebra $\mathbf{U}^\imath$ of $\mathbf{U}$ is defined to be the $\mathbb{Q}(q)$-subalgebra of $\mathbf{U}$ generated by

$$B_i = E_i + q^{\delta_{i,\frac{n+1}{2}}} F_{n+1-i}K_i^{-1} + \delta_{i,\frac{n+1}{2}}K_i^{-1}, \quad K_iK_{n+1-i}^{-1}, \quad i = 1, 2, \dots, n.$$

(2) If $n+1 = 2r+1$ is odd, then we consider the quantum symmetric pair $(\mathbf{U}^\imath, \mathbf{U})$ whose Satake diagram is of the form



The coideal subalgebra $\mathbf{U}^\imath$ of $\mathbf{U}$ is defined to be the $\mathbb{Q}(q)$-subalgebra of $\mathbf{U}$ generated by

$$B_i = E_i + q^{-\delta_{i,\frac{n}{2}+1}} F_{n+1-i} K_i^{-1}, \quad K_i K_{n+1-i}^{-1}, \quad i = 1, 2, \dots, n.$$

We shall treat both cases simultaneously. The algebra $\mathbf{U}^\imath$ admits an anti-linear involution $\psi_\imath$ such that

$$\psi_\imath(B_i) = B_i, \quad \psi_\imath(K_i K_{n+1-i}^{-1}) = K_i^{-1} K_{n+1-i}.$$

There also exists a unique (up to a scalar) intertwiner Υ in the completion of $\mathbf{U}$, such that (as an identity in the completion)

$$\psi_\imath(u)\Upsilon = \Upsilon\psi(u), \quad \text{for } u \in \mathbf{U}^\imath.$$

The element Υ becomes a well-defined operator for any finite-dimensional $\mathbf{U}$-module. For any based $\mathbf{U}$-module (M, B) with the associated involution ψ, we let

$$\psi_\imath = \Upsilon \circ \psi : M \longrightarrow M. \tag{4.1}$$

Proposition 4.2. *[BW13, Theorems 4.20 and 6.22] Let (M, B) be a based $\mathbf{U}$-module. Then there exists a unique $\imath$-canonical basis $B^\imath = \{b^\imath | b \in B\}$ of the $\mathbf{U}^\imath$-module M, such that*

$$\psi_\imath(b^\imath) = b^\imath, \quad \textit{and} \quad b^\imath = b + \sum_{b \in B} t_{b;b'} b', \quad \textit{with } t_{b;b'} \in q\mathbb{Z}[q]. \tag{4.2}$$

Lemma 4.3. *Let $f : M_1 \to M_2$ be a based morphism of based $\mathbf{U}$-modules (M_1, B_1) and (M_2, B_2). Then $f(B_1^\imath) \subset B_2^\imath \cup \{0\}$.*

Proof. Recall Υ is in certain completion of $\mathbf{U}$, and acts (well-definedly) on any finite-dimensional $\mathbf{U}$-module. Hence f commutes with the map $\psi_\imath = \Upsilon \circ \psi$ on M_1 and M_2. The rest of the argument (involving the partial orders and the integral lattices) is entirely similar to that of Lemma 3.3. $\square$

4.2.

Let W_{B_m} be the Weyl group of type B_m with simple reflections $\{s_0, s_1, \dots, s_{m-1}\}$, where we have

$$\begin{aligned} s_i^2 &= 1, \quad \text{for all } i, \\ s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, &\quad \text{and} \quad s_i s_j = s_j s_i, \quad \text{for } |i-j| > 1, \\ s_0 s_1 s_0 s_1 = s_1 s_0 s_1 s_0, &\quad \text{and} \quad s_0 s_i = s_i s_0, \quad \text{for } i > 1. \end{aligned}$$

Let $\mathcal{H}_{B_m}$ be the associated Hecke algebra of type B_m.

Both the Hecke algebra $\mathcal{H}_{B_m}$ and the coideal subalgebra $\mathbf{U}^\imath$ act naturally on the tensor space $\mathbb{V}^{\otimes m}$.

Proposition 4.4. *[BW13, §5 and §6]*

(1) The actions of $\mathbf{U}^\imath$ and $\mathcal{H}_{B_m}$ on $\mathbb{V}^{\otimes m}$ commute.
(2) The ı-canonical basis on $\mathbb{V}^{\otimes m}$ can be identified with the (parabolic) Kazhdan-Lusztig basis of $\mathcal{H}_{B_m}$.

Proposition 4.5. *Let $\alpha, \beta \geq 0$ with $\alpha+\beta = m$. Let $(\mathbb{V}^{\otimes \alpha}, B_\alpha)$, $(\mathbb{V}^{\otimes \beta}, B_\beta)$ and $(\mathbb{V}^{\otimes m}, B)$ be based $\mathbf{U}$-modules of the tensor product of natural representations. For any $b \in B$, we have*

$$b^\imath = \sum_{(b_\alpha, b_\beta) \in B_\alpha \times B_\beta,} t_{b;b_\alpha,b_\beta}(b_\alpha^\imath \otimes b_\beta), \qquad \text{with } t_{b;b_\alpha,b_\beta} \in \mathbb{N}[q].$$

Proof. We consider the parabolic subalgebra $\mathcal{H}_{B_\alpha} \times \mathcal{H}_{A_{\beta-1}}$ of $\mathcal{H}_{B_m}$ generated by

$$\{H_{s_0}, \dots, H_{s_{\alpha-1}}, H_{s_{\alpha+1}}, \dots, H_{s_{m-1}}\}.$$

It acts naturally on $\mathbb{V}^{\otimes \alpha} \otimes \mathbb{V}^{\otimes \beta} = \mathbb{V}^{\otimes m}$. Then by Proposition 3.4 and Proposition 4.4, we see that the basis $\{b_\alpha^\imath \otimes b_\beta | b_\alpha^\imath \in B_\alpha, b_\beta \in B_\beta\}$ can be identified with the (parabolic) Kazhdan-Lusztig basis of $\mathcal{H}_{B_\alpha} \times \mathcal{H}_{A_{\beta-1}}$. Therefore the proposition follows from Proposition 2.4 and Proposition 2.3. □

Theorem 4.6. *Let $L(\lambda_i)$ be finite-dimensional simple $\mathbf{U}$-modules for $i = 1, \dots, k$. For any $0 \leq l \leq k$, we consider the based modules $(L(\lambda_1) \otimes \cdots \otimes L(\lambda_l), B_\alpha)$, $(L(\lambda_{l+1}) \otimes \cdots \otimes L(\lambda_k), B_\beta)$, and $(L(\lambda_1) \otimes \cdots \otimes L(\lambda_k), B)$. For any $b \in B$, we have*

$$b^\imath = \sum_{(b_\alpha, b_\beta) \in B_\alpha \times B_\beta,} t_{b;b_\alpha,b_\beta}(b_\alpha^\imath \otimes b_\beta), \qquad \text{with } t_{b;b_\alpha,b_\beta} \in \mathbb{N}[q].$$

Proof. We first consider the case where $\lambda_i = \omega_{m_i}$ for $1 \leq m_i \leq n$. Recall that we have

$$L(\omega_{m_i}) \cong \wedge^{m_i}\mathbb{V}.$$

Let $m_1 + m_2 + \cdots + m_l = \alpha$ and $m_{l+1} + m_{l+2} + \cdots + m_k = \beta$. Recall the based surjective morphism

$$\pi = \pi_\alpha \otimes \pi_\beta : \mathbb{V}^{\otimes \alpha+\beta} \cong \mathbb{V}^{\otimes\alpha} \otimes \mathbb{V}^{\otimes\beta} \twoheadrightarrow \wedge^{m_1}\mathbb{V} \otimes \wedge^{m_2}\mathbb{V} \otimes \cdots \otimes \wedge^{m_k}\mathbb{V}.$$

Thanks to Lemma 4.3, all three morphisms π, π_α, and π_β preserve both canonical bases and $\imath$canonical bases. Therefore applying π to the identity in Proposition 4.5, we have

$$\pi(b^\imath) = \sum_{(b_\alpha, b_\beta)\in B_\alpha \times B_\beta,} t_{b;b_\alpha,b_\beta}\Big(\pi_\alpha(b^\imath_\alpha) \otimes \pi_\beta(b_\beta)\Big), \qquad \text{with } t_{b;b_\alpha,b_\beta} \in \mathbb{N}[q].$$

The proposition follows in this case.

For general $\lambda_i \in X^+$, the theorem follows from the previous case and the following based embedding thanks to Lemma 3.7 (for some suitable $m_1, m_2, \ldots, m_s$):

$$L(\lambda_1) \otimes \cdots \otimes L(\lambda_k) \longrightarrow \wedge^{m_1}\mathbb{V} \otimes \wedge^{m_2}\mathbb{V} \otimes \cdots \otimes \wedge^{m_s}\mathbb{V}. \qquad \square$$

4.3.

In this section, we consider the quantum symmetric pair with another set of parameters. We refer to [BW18] for more details regarding the parameters. The connection between such quantum symmetric pairs and the BGG category $\mathcal{O}$ of type D was studied in [ES13]. We developed the relevant theory of $\imath$canonical bases in [Bao17], which were then used to reformulate the Kazhdan-Lusztig theory of type D.

Definition 4.7. (1) If $n+1 = 2r+2$ is even, then the coideal subalgebra $\mathbf{U}^\imath_1$ of $\mathbf{U}$ is defined to be the $\mathbb{Q}(q)$-subalgebra generated by

$$B_i = E_i + q^{\delta_{i,\frac{n+1}{2}}} F_{n+1-i}K_i^{-1}, \quad K_iK^{-1}_{n+1-i}, \quad i = 1, 2, \ldots, n.$$

(2) If $n+1 = 2r+1$ is odd, then the coideal subalgebra $\mathbf{U}^\imath_1$ of $\mathbf{U}$ is defined to be the $\mathbb{Q}(q)$-subalgebra generated by

$$B_i = E_i + q^{-\delta_{i,\frac{n}{2}}} F_{n+1-i}K_i^{-1}, \quad K_iK^{-1}_{n+1-i}, \quad i = 1, 2, \ldots, n.$$

We shall again treat both cases simultaneously. The algebra $\mathbf{U}^\imath_1$ admits an anti-linear involution $\psi_\imath$ such that

$$\psi_\imath(B_i) = B_i, \quad \psi_\imath(K_i K^{-1}_{n+1-i}) = K_i^{-1} K_{n+1-i}.$$

For any based $\mathbf{U}$-module (M, B), we can again define the anti-linear involution (entirely similar to (4.1))

$$\psi_\imath : M \longrightarrow M.$$

Proposition 4.8 ([Bao17, Theorem 2.15])**.** *Let (M, B) be a based $\mathbf{U}$-module. Then there exists a unique ı-canonical basis $B^\imath = \{b^\imath | b \in B\}$ of the $\mathbf{U}^\imath_1$-module M, such that*

$$\psi_\imath(b^\imath) = b^\imath, \quad \text{and} \quad b^\imath = b + \sum_{b \in B} t_{b;b'} b', \quad \text{with } t_{b;b'} \in q\mathbb{Z}[q]. \tag{4.3}$$

Remark 4.9. Note that the basis we obtained here is generally different from that of Proposition 4.2, since we have taken a different subalgebra $\mathbf{U}^\imath_1$.

Theorem 4.10. *Let $L(\lambda_i)$ be finite-dimensional simple $\mathbf{U}$-modules for $i = 1, \dots, k$. For any $0 \leq l \leq k$, we consider the based modules $(L(\lambda_1) \otimes \cdots \otimes L(\lambda_l), B_\alpha)$, $(L(\lambda_{l+1}) \otimes \cdots \otimes L(\lambda_k), B_\beta)$, and $(L(\lambda_1) \otimes \cdots \otimes L(\lambda_k), B)$. For any $b \in B$, we have*

$$b^\imath = \sum_{(b_\alpha, b_\beta) \in B_\alpha \times B_\beta,} t_{b;b_\alpha,b_\beta}(b^\imath_\alpha \otimes b_\beta), \qquad \text{with } t_{b;b_\alpha,b_\beta} \in \mathbb{N}[q].$$

Proof. The proof is entirely similar to that in Theorem 4.6. In this case, we can either consider the duality between the coideal subalgebra $\mathbf{U}^\imath_1$ with the Hecke algebra of type D, or with the Hecke algebra of type B of unequal parameters. More details of the dualities can be found in [Bao17, ES13]. ☐

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Non-abelian Hodge moduli spaces and homogeneous affine Springer fibers

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To George Lusztig with admiration

Abstract: Starting from a homogeneous affine Springer fiber Fl_ψ, we construct three moduli spaces that correspond to the Dolbeault, de Rham and Betti aspects of a hypothetical Simpson correspondence with wild ramifications. We show that Fl_ψ is homeomorphic to the central Lagrangian fiber in the Dolbeault space, prove that the Dolbeaut and de Rham spaces both have the same cohomology as Fl_ψ, and construct a map from the de Rham space to the Betti space which we conjecture to be an analytic isomorphism.

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1. Introduction

1.1. Springer fiber and Slodowy slice

Let G be a reductive group over an algebraically closed characteristic zero field k with Lie algebra $\mathfrak{g}$. For a nilpotent element $e \in \mathcal{N} \subset \mathfrak{g}$, the usual Springer fiber $\mathcal{B}_e = \{gB \in G/B | \mathrm{Ad}(g^{-1})e \in \mathrm{Lie}\ B\}$ is the fiber over e of the Springer resolution $\pi : T^*(G/B) = \widetilde{\mathcal{N}} \to \mathcal{N}$. Moreover, $\mathcal{B}_e$ can be realized as a Lagrangian subscheme in a symplectic smooth variety, via the construction of the *Slodowy slice* through e. More precisely, let $\{e, h, f\}$ be an $\mathfrak{sl}_2$-triple containing e. Let $\mathcal{S}_e = (e + \mathfrak{g}^f) \cap \mathcal{N}$ be the nilpotent part of the Slodowy slice through e. Let $\widetilde{\mathcal{S}}_e = \mathcal{S}_e \times_{\mathcal{N}} \widetilde{\mathcal{N}}$ be the preimage of $\mathcal{S}_e$ under the Springer resolution. The following well known properties of these varieties play an important role in geometric representation theory.

(1) $\widetilde{\mathcal{S}}_e$ is a smooth variety over k with a canonical symplectic form.
(2) The map $\pi_e : \widetilde{\mathcal{S}}_e \to \mathcal{S}_e$ is a symplectic resolution.
(3) There are compatible $\mathbb{G}_m$-actions on $\widetilde{\mathcal{S}}_e$ and $\mathcal{S}_e$ such that the symplectic form on $\widetilde{\mathcal{S}}_e$ has weight 2, and $\mathcal{S}_e$ contracts to the point e under the $\mathbb{G}_m$-action. In particular, if $k = \mathbb{C}$ the embedding $\mathcal{B}_e \hookrightarrow \widetilde{\mathcal{S}}_e$ induces a homotopy equivalence between the corresponding complex varieties. The action is the product of the dilation action on $T^*(G/B)$ and conjugation by a cocharacter coming from the homomorphism $SL(2) \to G$ provided by the Jacobson-Morozov Theorem.
(4) The subvariety $\mathcal{B}_e \hookrightarrow \widetilde{\mathcal{S}}_e$ (the fiber over e) is Lagrangian in $\widetilde{\mathcal{S}}_e$.
(5) The symplectic variety $\widetilde{\mathcal{S}}_e$ can be obtained from $T^*(G/B)$ by Hamiltonian reduction for a unipotent subgroup $U_e \subset G$ equipped with an additive character.

The main goal of this paper is to construct and study an analogue of the resolved Slodowy slice $\widetilde{\mathcal{S}}_e$ when the Springer fiber $\mathcal{B}_e$ is replaced by an affine Springer fiber Fl_ψ for a homogeneous element ψ. Roughly speaking, a homogeneous element ψ is a regular semisimple element in the loop Lie algebra $\mathfrak{g}((t))$ for which there exists an analogue of the Jacobson-Morozov cocharacter, i.e. a cocharacter of the Kac-Moody group for which ψ is an eigenvector.

1.2. Summary of main results

Starting from a homogeneous element ψ in $\mathfrak{g}((t))$, we construct three moduli spaces that serve as the Dolbeault, de Rham and Betti aspects in the terminology of the non-abelian Hodge theory.

(1) The Dolbeault space $\mathcal{M}_\psi$ is a moduli space of G-Higgs bundles on $\mathbb{P}^1$ with level structures at 0 and ∞, and prescribed irregular part at ∞. This is the space we propose as a loop group analogue of the resolved Slodowy slice. It shares some of its properties although there are also some important differences. In particular, it is a symplectic algebraic space with a $\mathbb{G}_m$ action contracting the space to a Lagrangian subspace homeomorphic to Fl_ψ (see Theorem 2.8.1); it can be obtained from $T^*\mathrm{Fl}$ by Hamiltonian reduction for a subgroup $\mathbf{J}_\infty$ in the loop group equipped with an additive character (see §2.11). However, the analogue of the Springer map is the Hitchin map $f : \mathcal{M}_\psi \to \mathcal{A}_\psi$ which is a completely integrable system rather than a symplectic resolution.
Also, the spaces Fl_ψ and $\mathcal{M}_\psi$ are finite dimensional but in general have infinite type: in the simplest example Fl_ψ is an infinite chain of projective lines. They are both of finite type if ψ is elliptic.

(2) The de Rham space $\mathcal{M}_{\mathrm{dR},\psi}$ is a moduli space of G-connections on $\mathbb{P}^1$ with level structures and prescribed irregular part at ∞.

(3) The Betti space $\mathcal{M}_{\mathrm{Bet},\psi}$ is a complex analytic stack that depends only on the positive braid determined by ψ. The essential part of it is, up to a quotient by G, defined as an explicit subvariety in a product $G\times(G/B)^n$. The key property justifying its name is the interpretation of $\mathcal{M}_{\mathrm{Bet},\psi}$ as the moduli space of Stokes data (See §4.3).

The three spaces $\mathcal{M}_\psi$, $\mathcal{M}_{\mathrm{dR},\psi}$, $\mathcal{M}_{\mathrm{Bet},\psi}$ are related as follows.

There is a one-parameter deformation $\mathcal{M}_{\mathrm{Hod},\psi}$ of $\mathcal{M}_\psi$ with general fiber isomorphic to $\mathcal{M}_{\mathrm{dR},\psi}$. This family carries a $\mathbb{G}_m$-action contracting it to the Lagrangian subspace Fl_ψ in the special fiber, which allows one to show that the restriction maps induce isomorphisms on cohomology (see Theorem 2.8.4 and Corollary 3.2.3):

$$\mathrm{H}^*(\mathrm{Fl}_\psi) \xleftarrow{\sim} \mathrm{H}^*(\mathcal{M}_\psi) \xleftarrow{\sim} \mathrm{H}^*(\mathcal{M}_{\mathrm{Hod},\psi}) \xrightarrow{\sim} \mathrm{H}^*(\mathcal{M}_{\mathrm{dR},\psi}). \tag{1.1}$$

Recall that for other types of connections (for example, for non-singular connections on a projective curve over $\mathbb{C}$) the non-abelian Hodge theory of Corlette, Donaldson, Hitchin and Simpson defines a hyper-Kähler structure on the Hitchin moduli space and an isomorphism between the Dolbeault and the De Rham moduli spaces as real manifolds, and realizes the Hodge moduli space as an open subvariety (preimage of $\mathbb{C}$ under the projection to $\mathbb{CP}^1$) in the twistor moduli space. We do not know if that theory can be extended to our setting.

Another, more elementary comparison available for nonsingular connections over a complete complex curve is provided by the Riemann-Hilbert correspondence: it yields an isomorphism between the de Rham and the Betti moduli spaces as complex analytic (but not as algebraic) varieties. In §4.4, using Stokes theory for G-connections (see §4.3), we defined an enhanced Riemann-Hilbert map

$$\widetilde{\mathrm{RH}} : \mathcal{M}_{\mathrm{dR},\psi} \to \mathcal{M}_{\mathrm{Bet},\psi}. \tag{1.2}$$

We conjecture that this map is an analytic isomorphism.

In the main body of the paper, we also include a variant of the above spaces $\mathcal{M}_\psi, \mathcal{M}_{\mathrm{dR},\psi}$ and $\mathcal{M}_{\mathrm{Bet},\psi}$ with an arbitrary semisimple part of the residue/monodromy at 0.

1.3. Relation to earlier results

Many of the constructions presented here are related to ones found in the literature.

Precursors of the Dolbeault moduli space of this type, besides of course the original construction of Hitchin [23], appeared in the work of Beauville [3], Biquard-Garcia Prada-Mundet i Riera [6], Boalch [7], Bottacin [9], Markman [30], Oblomkov-Yun [35] and Simpson [40]. The paper [18] by Fredrickson and Neitzke studies a moduli space of Higgs bundles closely related to the case $G = \mathrm{GL}_n, \nu = d/n$ in this paper.

Biquard and Boalch [5] have developed non-abelian Hodge theory for irregular connections whose polar part (excepting the residue) is semisimple. When the underlying curve is $\mathbb{P}^1$ and the polar part is homogeneous, these are a special case of connections considered in the present work. However, the moduli spaces we consider are different in that we endow the connection with a framing at infinity: the moduli space of Biquard and Boalch can be obtained from a special case of our moduli space by Hamiltonian reduction with respect to a torus action. Boalch and Yamakawa [8] construct moduli spaces of Stokes data for G-local systems, and endow them with Poisson structures. Our Betti spaces, although presented differently, should be special cases of their construction. The work of Bremer-Sage [12] studies moduli of flat connections with level structure on the bundles, although for them, the underlying vector bundle is always trivial.

Mochizuki [31] has extended the irregular non-abelian Hodge correspondence to (unframed) wild Higgs bundles whose polar parts are not semisimple. To our knowledge it is not known if this induces an isomorphism of real manifolds in the general case.

The Betti space is a variant of a special case of the moduli spaces defined by Minh-Tam Trinh [43]. Its definition is purely Lie-theoretic and it uses only the braid group element determined by ψ (or rather its slope). A version of this space already appeared in Lusztig's definition of character sheaves [28]. In the work of Shende-Treumann-Williams-Zaslow [39], when $G = \mathrm{GL}(n)$, similar moduli spaces were interpreted as a moduli space of local systems on $\mathbb{P}^1 \backslash \{0, \infty\}$ with Stokes data at ∞.

In a research proposal, Vivek Shende speculated an irregular non-abelian Hodge correspondence for curves and a P=W conjecture in that setting. In his thesis, Minh-Tam Trinh [42] formulated a precise P=W conjecture connecting cohomology of Hitchin fibers and cohomology of Steinberg-like varieties attached to braids (which are closely related to $\mathcal{M}_{\mathrm{Bet},\psi}$). Our work gives supporting evidence for these conjectures in the case of homogeneous Hitchin fibers: assuming (1.2) is an analytic isomorphism and combing it with the isomorphisms (1.1), one would conclude that $\mathcal{M}_{\mathrm{Bet},\psi}$ has the same cohomology as the affine Springer fiber Fl_ψ.

1.4. Microlocal sheaves on $\mathbf{Fl}_\psi$ and wildly ramified geometric Langlands

While we believe the above results to be of independent interest, our motivation for considering those spaces came from our study of the categories of sheaves on the affine flag manifold and related categories of microlocal sheaves. We now briefly explain the setting and the motivating conjectures, our results in this direction will be presented elsewhere.

Denote by $\mu\mathrm{Sh}_\Lambda(X)$ the category of microlocal sheaves on a polarised conical X supported on a conical Lagrangian Λ as constructed from work of [25], [38] and [33].

Consider $\mathcal{M}^{G^\vee}_{0,\mathrm{Bet},\psi}$ the Betti space for $G^\vee$, the Langlands dual group, using the positive braid defined by ψ and the definition in §4.4.3.

1.4.1 Conjecture. *There is a fully faithful functor*

$$\mu\mathrm{Sh}_{\mathrm{Fl}_\psi}(\mathcal{M}_\psi) \hookrightarrow \mathrm{IndCoh}(\mathcal{M}^{0,G^\vee}_{\mathrm{Bet},\psi}).$$

This conjecture can be viewed as a geometric Langlands correspondence for deeper level structures/wild ramifications. At the same time, it can be viewed as an instance of homological mirror symmetry between $\mathcal{M}_\psi$ and $\mathcal{M}^{G^\vee}_{\mathrm{Bet},\psi}$. A version of this conjecture in the case ψ is homogeneous of slope 1 is proved in [4].

2. The Dolbeault moduli space

Let G be a reductive group over an algebraically closed field k such that the adjoint G_{ad} is simple. Let $\mathfrak{g}$ be the Lie algebra of G. Fix a G-invariant non-degenerate symmetric bilinear form $\langle\cdot,\cdot\rangle$ on $\mathfrak{g}$ to identify $\mathfrak{g}$ with $\mathfrak{g}^*$. Let $T_0 \subset G$ be a maximal torus and $W_0 = W(G, T_0)$ be the corresponding Weyl group of G. We assume $|W_0|$ is invertible in k.

In this section, we will construct analogues of the resolved Slodowy slice $\widetilde{\mathcal{S}}_e$ when the Springer fiber $\mathcal{B}_e$ is replaced with affine Springer fibers Fl_ψ attached to certain elements $\psi \in \mathfrak{g}(\!(t)\!)$ called *homogeneous* (see §2.1). We will construct a moduli space $\mathcal{M}_\psi$ of Higgs bundles with the following features:

(1) $\mathcal{M}_\psi$ is a smooth algebraic space over k with a canonical symplectic form.
(2) There is a Hitchin map $f : \mathcal{M}_\psi \to \mathcal{A}_\psi$ that is a completely integrable system. It is proper when ψ is elliptic.
(3) There are compatible $\mathbb{G}_m$-actions on $\mathcal{M}_\psi$ and $\mathcal{A}_\psi$, (compute the weight of the symplectic form), contracting $\mathcal{A}_\psi$ to the point a_ψ. The central fiber $f^{-1}(a_\psi) = \mathcal{M}_{a_\psi}$ is homeomorphic to the affine Springer fiber Fl_ψ.

More precise statements will be given in Theorem 2.8.1. We will give three constructions of $\mathcal{M}_\psi$ each having its own advantages in proving certain geometric properties.

2.1. Homogeneous affine Springer fibers

Let $\mathfrak{g}(\!(t)\!) = \mathfrak{g} \otimes k(\!(t)\!)$ be the loop Lie algebra. Let $G(\!(t)\!)$ be the loop group of G and $G[\![t]\!]$ be the positive loop group so that $(G(\!(t)\!))(k) = G(k(\!(t)\!))$ and $(G[\![t]\!])(k) = G(k[\![t]\!])$. Then $G(\!(t)\!)$ acts on $\mathfrak{g}(\!(t)\!)$ by the adjoint action. Let $\mathfrak{a} = \mathfrak{g} /\!\!/ G$ be the adjoint quotient, and $\mathfrak{a}(\!(t)\!) = \mathfrak{a}(k(\!(t)\!))$.

Let $\mathbb{G}_m^{\mathrm{rot}}$ be the one-dimensional torus acting on $\mathfrak{g}(\!(t)\!)$ by loop rotation $\mathbb{G}_m^{\mathrm{rot}} \ni s : A(t) \mapsto A(st)$. Let $\mathbb{G}_m^{\mathrm{dil}}$ be the one-dimensional torus acting on $\mathfrak{g}(\!(t)\!)$ by dilation $\mathbb{G}_m^{\mathrm{dil}} \ni s : A(t) \mapsto sA(t)$. The action of $\mathbb{G}_m^{\mathrm{rot}} \times \mathbb{G}_m^{\mathrm{dil}}$ on $\mathfrak{g}(\!(t)\!)$ induces an action on $\mathfrak{a}(\!(t)\!)$.

Homogeneous elements in the loop Lie algebra $\mathfrak{g}(\!(t)\!)$ are defined by Varagnolo and Vasserot [44, Section 1.3]. Recall from *loc. cit.* and [35, Definition 3.1.2] that a regular semisimple element $a \in \mathfrak{a}(\!(t)\!)$ is called *homogeneous of* *slope* $\nu = d/m \in \mathbb{Q}$ (where $d \in \mathbb{Z}$, $m \in \mathbb{N}$ in lowest terms) if it is fixed by the action of the subtorus $\mathbb{G}_m(\nu) \subset \mathbb{G}_m^{\mathrm{rot}} \times \mathbb{G}_m^{\mathrm{dil}}$ given by $\{(s^{-m}, s^d) | s \in \mathbb{G}_m\}$. A regular semisimple element $\psi \in \mathfrak{g}((t))$ is called *homogeneous of slope* $\nu = d/m$ if its image in $\mathfrak{a}((t))$ is. In other words, $\psi(t) \in \mathfrak{g}((t))$ is homogeneous of slope $\nu = d/m$ if and only if it is regular semisimple, and that $s^{-d}\psi(s^m t)$ is in the same $G((t))$-orbit of $\psi(t)$ for all $s \in k^\times$.

By [35, Theorem 3.2.5], a homogeneous element $\psi \in \mathfrak{g}((t))$ of slope $\nu = d/m$ exists if and only if m is a regular number for the Weyl group W_0, i.e., m is the order of a regular element in W_0 in the sense of Springer [41] (we will describe the conjugacy class of this regular element in Remark 2.4.1).

Fix a Borel subgroup $B_0 \subset G$ containing T_0, and let $\mathbf{I}_0 \subset G[\![t]\!]$ be the corresponding Iwahori subgroup. Let $\mathrm{Fl} = G((t))/\mathbf{I}_0$ be the affine flag variety of G. For $\psi \in \mathfrak{g}((t))$ homogeneous of slope $\nu = d/m \geq 0$, following [26], we may consider its affine Springer fiber Fl_ψ

$$\mathrm{Fl}_\psi = \{g\mathbf{I}_0 \in \mathrm{Fl} | \mathrm{Ad}(g^{-1})\psi \in \mathrm{Lie}\ \mathbf{I}_0\}.$$

This is an ind-scheme that is a union of projective schemes over k. It is an affine analogue of Springer fibers.

2.1.1 Example. In the case $G = \mathrm{SL}_n$, consider the slope $\nu = d/n$, where $d \geq 1$ is coprime to n. The element

$$\psi = \begin{pmatrix} & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \\ t & & & & \end{pmatrix}^d$$

is homogeneous of slope $\nu = d/n$. The affine Springer fiber Fl_ψ has been studied by Lusztig and Smelt [29]. It classifies chains of lattices in $k((t))^n$ stable under ψ. We may reinterpret Fl_ψ as classifying chains of fractional ideals of the ring $k[\![t, y]\!]/(y^n - t^d)$.

2.2. Moy-Prasad filtration

The point $x \in \mathbb{X}_*(T_0)_\mathbb{Q}$ (the standard apartment of the building of $G((t))$) defines a Moy-Prasad grading on $\mathfrak{g}[t, t^{-1}]$

$$\mathfrak{g}[t, t^{-1}] = \bigoplus_{s \in \mathbb{Q}} \mathfrak{g}[t, t^{-1}]_{x,s}.$$

Here $\mathfrak{g}[t,t^{-1}]_{x,s}$ is spanned by the affine root spaces of $\mathfrak{g}[t,t^{-1}]$ for the affine roots $\alpha+n$ such that $\alpha(x)+n=s$.

Fix $\nu=d/m>0$ in lowest terms, where m is a regular number for W_0. We shall choose x such that the following hold:

- m is the smallest positive integer such that $\mathfrak{g}[t,t^{-1}]_{x,s}\neq 0$ implies $s\in\frac{1}{m}\mathbb{Z}$.
- $\mathfrak{g}[t,t^{-1}]_{x,\nu}$ contains a regular semisimple element (as an element in $\mathfrak{g}(\!(t)\!)$).

For such an x, a regular semisimple element $\psi'\in\mathfrak{g}[t,t^{-1}]_{x,\nu}$ is homogeneous of slope ν. Conversely, for any homogeneous $\psi\in\mathfrak{g}(\!(t)\!)$ of slope ν, there exists x satisfying the above conditions, and a regular semisimple $\psi'\in\mathfrak{g}[t,t^{-1}]_{x,\nu}$ such that ψ and ψ' are in the $G(\!(t)\!)$-adjoint orbit. Indeed, by [35, 3.3], we can take $x=\rho^\vee/m$. There many be more than one choice of x for a homogeneous ψ.

In the following we shall fix an $x\in\mathbb{X}_*(T_0)_{\mathbb{Q}}$ satisfying the above conditions, and fix a regular semisimple $\psi\in\mathfrak{g}[t,t^{-1}]_{x,\nu}$, which is homogeneous of slope ν. We will simply write $\mathfrak{g}[t,t^{-1}]_{x,s}$ as $\mathfrak{g}[t,t^{-1}]_s$, so that

$$\mathfrak{g}[t,t^{-1}]=\bigoplus_{i\in\mathbb{Z}}\mathfrak{g}[t,t^{-1}]_{i/m}. \tag{2.1}$$

2.3. The curve and the parahoric subgroup

Let $X=\mathbb{P}^1_k$ with affine coordinate t. Then t is a uniformizer at $0\in\mathbb{P}^1$, and $\tau=t^{-1}$ is a uniformizer at $\infty\in\mathbb{P}^1$. We identify the loop group $G(\!(t)\!)$ with the loop group of G at $0\in X$.

We also have the loop group $G(\!(\tau)\!)$ at $\infty\in X$. For $i\in\mathbb{Z}$, let

$$\mathfrak{g}(\!(\tau)\!)_{\le i/m}=\widehat{\bigoplus}_{j\le i}\mathfrak{g}[t,t^{-1}]_{j/m}$$

where $\widehat{\oplus}$ denotes the τ-adic completion of the direct sum. Then $\mathfrak{g}(\!(\tau)\!)_{\le i/m}$ is a $k(\!(\tau)\!)$-lattice in $\mathfrak{g}(\!(\tau)\!)$.

We will use the three notations $\mathfrak{g}(\!(\tau)\!)_{j/m}=\mathfrak{g}(\!(t)\!)_{j/m}=\mathfrak{g}[t,t^{-1}]_{j/m}$ interchangeably depending on the context.

Let $\mathbf{P}_\infty\subset G(\!(\tau)\!)$ be the parahoric subgroup whose Lie algebra is $\mathfrak{g}(\!(\tau)\!)_{\le 0}$. Let $\mathbf{P}_\infty(\frac{i}{m})\subset\mathbf{P}_\infty$ be the Moy-Prasad subgroups of $\mathbf{P}_\infty$: its Lie algebra is $\mathfrak{g}(\!(\tau)\!)_{\le -i/m}$. Let $\mathbf{P}_\infty^+=\mathbf{P}_\infty(\frac{1}{m})$ be the pro-unipotent radical of $\mathbf{P}_\infty$.

In particular, $\psi\in\mathfrak{g}[t,t^{-1}]_\nu$ is viewed as a linear character of $\mathfrak{g}(\!(\tau)\!)_{\le-\nu}$.

2.4. The centralizer of ψ

Let C be the torus over $X\backslash\{0,\infty\}$ that is the centralizer of ψ (which is a regular semisimple section of $\mathfrak{g}$ over $X\backslash\{0,\infty\}$) under G. Note that C is not necessarily split; it becomes split over the μ_m-cover of $X\backslash\{0,\infty\} = \mathbb{G}_m$, with monodromy given by a regular element in W of order m. Let $\mathbf{C}_\infty \subset C(\!(\tau)\!)$ be the unique parahoric subgroup, and $\mathbf{C}_\infty^+$ be the pro-unipotent radical of $\mathbf{C}_\infty$.

The grading (2.1) on $\mathfrak{g}[t,t^{-1}]$ restricts to a grading on $\mathfrak{c}[t,t^{-1}]$, the global sections of the sheaf of Lie algebras Lie C over $X\backslash\{0,\infty\}$

$$\mathfrak{c}[t,t^{-1}] = \bigoplus_{i\in\mathbb{Z}} \mathfrak{c}[t,t^{-1}]_{i/m}$$

where $\mathfrak{c}[t,t^{-1}]_{i/m} \subset \mathfrak{g}[t,t^{-1}]_{i/m}$ are elements commuting with ψ.

Let $\mathfrak{c}(\!(\tau)\!)$ be the $\tau = t^{-1}$-adic completion of $\mathfrak{c}[t,t^{-1}]$. Let $\mathfrak{c}(\!(\tau)\!)_{\le i/m} = \mathfrak{g}(\!(\tau)\!)_{\le i/m} \cap \mathfrak{c}(\!(\tau)\!)$. Then Lie $\mathbf{C}_\infty = \mathfrak{c}(\!(\tau)\!)_{\le 0}$, and Lie $\mathbf{C}_\infty^+ = \mathfrak{c}(\!(\tau)\!)_{\le -1/m}$.

2.4.1 Remark. Specializing both sides of (2.1) at $t = 1$, each $\mathfrak{g}[t,t^{-1}]_{i/m}$ can be identified with a subspace of $\mathfrak{g}$. Thus we get a $\mathbb{Z}/m\mathbb{Z}$ grading of $\mathfrak{g}$

$$\mathfrak{g} = \bigoplus_{i\in\mathbb{Z}/m\mathbb{Z}} \mathfrak{g}_{i/m}. \tag{2.2}$$

Recall $x \in \mathbb{X}_*(T)_\mathbb{Q}$ defines the Moy-Prasad grading on $\mathfrak{g}[t,t^{-1}]$. Since the graded pieces $\mathfrak{g}[t,t^{-1}]_{x,s}$ are nonzero only for $s \in \frac{1}{m}\mathbb{Z}$, we can write $x = \xi/m$ where $\xi \in \mathbb{X}_*(T_0^{\mathrm{ad}})$. View ξ as a homomorphism $\xi : \mathbb{G}_m \to T_0^{\mathrm{ad}}$. Then (2.2) is the grading obtained by the adjoint action of μ_m via $\xi|_{\mu_m}$. Evaluating at $t = 1$, $\mathfrak{c}[t,t^{-1}]_{i/m}$ is identified with a subspace $\mathfrak{c}_{i/m} \subset \mathfrak{g}_{i/m}$ depending only on $i/m \mod \mathbb{Z}$. Their sum

$$\mathfrak{t} := \oplus_{i\in\mathbb{Z}/m\mathbb{Z}} \mathfrak{c}_{i/m} \subset \mathfrak{g}$$

is a Cartan subalgebra of $\mathfrak{g}$ stable under the $\mathbb{Z}/m\mathbb{Z}$-grading (2.2). Indeed, let $\overline{\psi} \in \mathfrak{g}_{d/m}$ be the image of ψ, then $\mathfrak{t}$ is the Lie algebra of the maximal torus $T := C_G(\overline{\psi}) \subset G$, the fiber of C over $1 \in X$. Let $\zeta_m \in \mu_m$ be a primitive m-th root of unity, then $\mathrm{Ad}(\xi(\zeta_m))$ (whose eigenvalues on $\mathfrak{g}$ gives the $\mathbb{Z}/m\mathbb{Z}$-grading) normalizes T, hence determines an element $w = w(\zeta_m)$ in the Weyl group $W = W(G,T)$ that is regular of order m. Different choices of ζ_m yields conjugate $w(\zeta_m)$. This defines a regular conjugacy class in W of order m.

2.5. Construction of $\mathcal{M}_\psi$

Let $D_0 = \text{Spec } k[\![t]\!], D_0^\times = \text{Spec } k(\!(t)\!)$ and $D_\infty = \text{Spec } k[\![\tau]\!], D_\infty^\times = \text{Spec } k(\!(\tau)\!)$.

Let $\mathcal{M}_\psi$ be the moduli stack parametrizing pairs $(\mathcal{E}, \varphi)$ where

- $\mathcal{E}$ is a G-bundle over X with $\mathbf{K}_\infty := \mathbf{P}_\infty(\frac{d}{m})\mathbf{C}_\infty^+$-level structure at ∞ and $\mathbf{I}_0$-level structure at 0. We denote by $\text{Ad}(\mathcal{E})$ the vector bundle over $X \backslash \{0, \infty\}$ associated to the adjoint representation $\mathfrak{g}$ of G.
- φ is a section of $\text{Ad}(\mathcal{E}) \otimes \omega_{X\backslash\{0,\infty\}}$ satisfying the following conditions:
  (i) After choosing a trivialization of $\mathcal{E}|_{D_\infty}$ together with its $\mathbf{K}_\infty$-level structure, we require
  $$\varphi|_{D_\infty^\times} \in (\psi + \mathfrak{g}(\!(\tau)\!)_{\leq 0})d\tau/\tau.$$
  Note that the right side is invariant under the adjoint action by $\mathbf{K}_\infty$, therefore this condition is independent of the trivialization of $\mathcal{E}|_{D_\infty}$.
  (ii) After choosing a trivialization of $\mathcal{E}|_{D_0}$ together with its $\mathbf{I}_0$-level structure, we require
  $$\varphi|_{D_0^\times} \in \text{Lie } (\mathbf{I}_0^+)dt/t.$$

2.6. Hitchin base

The Hitchin base $\mathcal{A}_\psi$ is defined as follows. Fix homogeneous generators $f_i \in k[\mathfrak{g}]^G$ of degree d_i, $1 \leq i \leq r$, which give an isomorphism

$$(f_1, \cdots, f_r) : \mathfrak{a} = \mathfrak{g} /\!\!/ G \xrightarrow{\sim} \mathbb{A}^r.$$

We identify $\omega_X(0 + \infty)$ with $\mathcal{O}_X$ using $\frac{dt}{t}$. In particular, $f_i(\psi)$ is a global section of $\mathcal{O}([\frac{dd_i}{m}] \cdot \infty)$ (the twisting means pole order at ∞). Let $\mathcal{A}_\psi \subset \prod_{i=1}^r \Gamma(\mathbb{P}^1, \mathcal{O}([\frac{dd_i}{m}] \cdot \infty))$ be the subspace of sections $a = (a_i)_{1 \leq i \leq r}$ such that for each $i = 1, \cdots, r$

- $a_i(0) = 0$;
- $a_i \equiv f_i(\psi) \mod \tau^{-[\frac{d(d_i-1)}{m}]}$ near ∞.

In other words, if we identify $\Gamma(\mathbb{P}^1, \mathcal{O}([\frac{dd_i}{m}] \cdot \infty))$ with polynomials in t, then $f_i(\psi)$ is a monomial of degree dd_i/m if $dd_i/m \in \mathbb{Z}$ and zero otherwise, and a_i is a polynomial of the form

$$a_i(t) = f_i(\psi) + \sum_{j=1}^{[d(d_i-1)/m]} a_{i,j} t^j.$$

2.6.1 Lemma. *Taking a Higgs bundle* $(\mathcal{E},\varphi)\in\mathcal{M}_\psi$ *to* $(f_i(\varphi))_{1\le i\le r}$ *defines a map*

$$f:\mathcal{M}_\psi\to\mathcal{A}_\psi.$$

Proof. Since the residue of φ at 0 is nilpotent, $f_i(\varphi)$ vanishes at 0. To compute the pole order of $f_i(\varphi)-f_i(\psi)$, we choose a trivialization of $\mathcal{E}|_{D_\infty}$ so that $\varphi\in(\psi+\mathfrak{g}(\!(\tau)\!)_{\le 0})d\tau/\tau$. Write $\varphi=(\psi+\theta)d\tau/\tau$ for some $\theta\in\mathfrak{g}(\!(\tau)\!)_{\le 0}$. Inside $\mathfrak{g}(\!(\tau^{1/m})\!)$ we can $G(\!(\tau^{1/m})\!)$-conjugate the grading $\mathfrak{g}(\!(\tau)\!)_{i/m}$ (extended $k(\!(\tau^{1/m})\!)$-linearly to $\mathfrak{g}(\!(\tau^{1/m})\!)$) to the standard one given by powers of $\tau^{1/m}$, i.e., $\mathfrak{g}\otimes\tau^{i/m}$. After this conjugation, φ becomes $\varphi'=(x\tau^{-d/m}+\sum_{j\ge 0}\varphi_j\tau^{j/m})d\tau/\tau$, for some regular semisimple $x\in\mathfrak{g}$ such that $x\tau^{-d/m}$ is in the same $G(\!(\tau^{1/m})\!)$-orbit of ψ, and $\varphi_j\in\mathfrak{g}$ for $j\ge 0$. Then it is clear that $f_i(\varphi)=f_i(\varphi')$ has leading term $f_i(x)\tau^{-dd_i/m}=f_i(\psi)$, with other terms starting with $\tau^{-[\frac{d(d_i-1)}{m}]}$. □

2.6.2 Lemma. *The Hitchin base* $\mathcal{A}_\psi$ *is an affine space of dimension* $\dim\mathcal{A}_\psi=\frac{1}{2}(\frac{d}{m}|\Phi|-r+\dim\mathfrak{t}^w)$ *(where* w *is a regular element in* W *of order* m *defined in §2.2 and* Φ *the set of roots of* $\mathfrak{g}$*).*

Proof. From the definition we have $\dim\mathcal{A}_\psi$ is the same as the space of $(a_1,\cdots,a_r)$ where a_i is a section of $\mathcal{O}([\frac{d(d_i-1)}{m}]\cdot\infty)$ vanishing at 0. Therefore

$$\dim\mathcal{A}_\psi=\sum_{i=1}^r\left[\frac{d(d_i-1)}{m}\right].$$

We have

$$\sum\frac{d(d_i-1)}{m}=\frac{d}{m}\sum_i(d_i-1)=\frac{d}{m}\frac{|\Phi|}{2}.$$

Therefore we reduce to showing

$$\sum_i\{(d_i-1)/m\}=(r-\dim\mathfrak{t}^w)/2. \tag{2.3}$$

Here $\{\cdots\}$ denotes the fractional part. Let $\zeta\in\mu_m$ be a primitive mth root of unity. We claim that

(2.4) The eigenvalues of w on $\mathfrak{t}$ are $\{\zeta^{d_i-1}\}_{1\le i\le r}$ as a multi-set.

Indeed, by Remark 2.4.1, we may take $\mathfrak{t}=\oplus_{i\in\mathbb{Z}/m\mathbb{Z}}\mathfrak{c}_{i/m}$ to be the centralizer of a regular semisimple element $y\in\mathfrak{g}_{1/m}$ (the grading defined in (2.2) using $x=\xi/m\in\mathbb{X}_*(T)_\mathbb{Q}$), and w acts on $\mathfrak{t}$ by $\mathrm{Ad}(\xi(\zeta))$ for a primitive

$\zeta \in \mu_m$. Since the claim (2.4) is independent of the choice of x satisfying the conditions in §2.2, we may take $x = \rho^\vee/m$ so that $\xi = \rho^\vee$. Therefore the grading (2.2) is given by eigenspaces of $\mathrm{Ad}(\rho^\vee(\zeta))$. Consider the open dense subset $\mathfrak{g}_{1/m}^{\mathrm{reg}} \subset \mathfrak{g}_{1/m}$ consisting of regular (but not necessarily semisimple) elements. It suffices to show that for all $y \in \mathfrak{g}_{1/m}^{\mathrm{reg}}$, the action of $\rho^\vee(\zeta)$ on $\mathfrak{z}_y$ (the centralizer of y in $\mathfrak{g}$) has eigenvalues $\{\zeta^{d_i-1}\}_{1\le i\le r}$ as a multi-set, because applying this statement to a regular semisimple y gives (2.4). Since $\mathfrak{g}_{1/m}^{\mathrm{reg}}$ is open dense in $\mathfrak{g}_{1/m}$, hence it is connected. The Lie algebra universal centralizer $\mathfrak{z} \to \mathfrak{g}_{1/m}^{\mathrm{reg}}$ is a vector bundle with an action of $\mathrm{Ad}(\rho^\vee(\zeta))$, hence all fibers have the same multi-set of eigenvalues under $\mathrm{Ad}(\rho^\vee(\zeta))$. Now we take y to be the regular nilpotent element $y_0 = \sum e_{\alpha_i}$ (for simple roots α_i, noting that $\mathfrak{g}_{\alpha_i} \subset \mathfrak{g}_{1/m}$). Since the weights of $\mathrm{Ad}(\rho^\vee)$ on $\mathfrak{z}_{y_0}$ are the exponents of $\mathfrak{g}$ by definition, i.e., $\{d_i - 1\}$, we see that the eigenvalues of $\mathrm{Ad}(\rho^\vee(\zeta))$ on $\mathfrak{z}_{y_0}$ are $\{\zeta^{d_i-1}\}$. This proves (2.4).

Therefore, when summing up $\{(d_i-1)/m\}$, each pair of eigenvalues λ, λ^{-1} ($\lambda \neq \pm 1$) of w on $\mathfrak{t}$ contributes 1; $\lambda = -1$ contributes 1/2 and $\lambda = 1$ contributes zero. Hence (2.3). □

2.7. The $\mathbb{G}_m(\nu)$-actions

2.7.1. Action on Fl_ψ The one-dimensional torus $\mathbb{G}_m^{\mathrm{rot}}$ acts on $k(\!(t)\!)$ by scaling the parameter t. We denote the action of $s \in \mathbb{G}_m^{\mathrm{rot}}$ by $\mathrm{rot}(s)$.

Recall $x = \xi/m$ (where $\xi \in \mathbb{X}_*(T_0^{\mathrm{ad}})$) defines the Moy-Prasad grading on $\mathfrak{g}[t,t^{-1}]$ such that $\psi \in \mathfrak{g}[t,t^{-1}]_\nu$.

We have a $\mathbb{G}_m(\nu)$-action on Fl_ψ, where $(s^{-m}, s^d) \in \mathbb{G}_m(\nu)$ acts by

$$s : g\mathbf{I}_0 \mapsto \mathrm{rot}(s^{-m})\mathrm{Ad}(\xi(s^{-1}))g\mathbf{I}_0.$$

2.7.2. Action on $\mathcal{M}_\psi$ Let $\mathbb{G}_m^{\mathrm{rot}}$ act on $X = \mathbb{P}^1$ by scaling the coordinate t. We denote the action of $s \in \mathbb{G}_m^{\mathrm{rot}}$ on X by $\mathrm{rot}(s)$. Note that for any $s \in \mathbb{G}_m$,

$$s^d \cdot \mathrm{rot}(s^{-m})(\mathrm{Ad}(\xi(s^{-1}))\psi) = \psi.$$

Since $\mathrm{rot}(s^{-m})\mathrm{Ad}(\xi(s^{-1}))$ fixes the line $k\psi$, it normalizes $\mathbf{C}_\infty$ and $\mathbf{C}_\infty^+$. Clearly $\mathbb{G}_m^{\mathrm{rot}} \times T_0^{\mathrm{ad}}$ normalizes $\mathbf{P}_\infty(\frac{i}{m})$ for all i, therefore $\mathrm{rot}(s^{-m})\mathrm{Ad}(\xi(s^{-1}))$ normalizes $\mathbf{K}_\infty = \mathbf{P}_\infty(\frac{d}{m})\mathbf{C}_\infty^+$. Similarly, the action of $s^d \cdot \mathrm{rot}(s^{-m})\mathrm{Ad}(\xi(s^{-1}))$ stabilizes $\psi + \mathfrak{g}(\!(\tau)\!)_{\le 0} \subset \mathfrak{g}(\!(\tau)\!)$. Therefore we get a $\mathbb{G}_m(\nu)$-action on $\mathcal{M}_\psi$ with $(s^{-m}, s^d) \in \mathbb{G}_m(\nu)$ sending $(\mathcal{E}, \varphi) \in \mathcal{M}_\psi$ to $(\mathcal{E}', \varphi')$ defined as follows. First let $\mathcal{E}''$ be the G-bundle $\mathrm{rot}(s^{-m})^*\mathcal{E}$ with $\mathbf{K}''_\infty = \mathrm{rot}(s^{-m})\mathbf{K}_\infty$-level at ∞ and $\mathbf{I}_0$-level at 0. Since $\mathrm{Ad}(\xi(s^{-1}))\mathbf{K}''_\infty = \mathbf{K}_\infty$, the action of $\mathrm{Ad}(\xi(s^{-1}))$ on $G(\!(\tau)\!)$

induces an equivalence between the groupoids of G-bundles with $\mathbf{K}''_\infty$-level and with $\mathbf{K}_\infty$-level at ∞. This turns $\mathcal{E}''$ into a G-bundle $\mathcal{E}'$ with $\mathbf{K}_\infty$-level at ∞ and still $\mathbf{I}_0$-level at 0. Finally $\varphi' = s^d \mathrm{rot}(s^{-m})^*\varphi$.

2.7.3. Action on $\boldsymbol{\mathcal{A}_\psi}$ The torus $\mathbb{G}_m(\nu)$ also acts on $\mathcal{A}_\psi$ so that (s^{-m}, s^d) sends $(a_i(t))_i$ to $(s^{dd_i}a_i(s^{-m}t))_i$. This action contracts $\mathcal{A}_\psi$ to the unique fixed point $a_\psi = (f_i(\psi))_{1\le i\le r} \in \mathcal{A}_\psi$.

2.8. Main results on $\boldsymbol{\mathcal{M}_\psi}$

The main geometric results on $\mathcal{M}_\psi$ are:

2.8.1 Theorem. *For a homogeneous element $\psi \in \mathfrak{g}((t))$ of slope $\nu = d/m$, the following hold.*

1. *The stack $\mathcal{M}_\psi$ is a smooth algebraic space over k of dimension $\frac{d}{m}|\Phi| - r + \dim \mathfrak{t}^w$ (where w is a regular element in W of order m defined in §2.2 and Φ the set of roots of $\mathfrak{g}$).*
2. *$\mathcal{M}_\psi$ carries a canonical symplectic structure of weight d under the $\mathbb{G}_m(\nu)$-action.*
3. *The map $f : \mathcal{M}_\psi \to \mathcal{A}_\psi$ is a $\mathbb{G}_m(\nu)$-equivariant completely integrable system (i.e., fibers of f are Lagrangians).*
4. *There is a natural map $\mathrm{Fl}_\psi \to \mathcal{M}_{a_\psi} = f^{-1}(a_\psi)$ which is a universal homeomorphism.*
5. *When ψ is elliptic (equivalently, w is elliptic), f is proper.*

2.8.2 Remark. Constructions similar to $\mathcal{M}_\psi$ have appeared before.

1. When $G = \mathrm{SL}_n$, Markman [30] has constructed a Poisson moduli of meromorphic Higgs bundles for arbitrary curve X and showed that the Hitchin fibration is a completely integrable system (namely generic fibers are Lagrangian in symplectic leaves of maximal rank). If we take $\psi = t^d A$ where $d \ge 1$ and $A \in \mathfrak{g}$ is regular semisimple, our $\mathcal{M}_\psi$ is a symplectic leaf in Markman's Poisson moduli space.
2. For any G and homogeneous ψ, Oblomkov and one of the authors [35] constructed a Poisson moduli of Higgs bundles (on a weighted projective line) with a contracting $\mathbb{G}_m$-action whose central fiber is closely related to Fl_ψ.

Both of the constructions above are more closely related to the Poisson moduli space $\mathcal{M}^\dagger_\psi$ in §2.13.

2.8.3 Remark. Consider the case $k = \mathbb{C}$. With the realization of Fl_ψ as a conical Lagrangian in the symplectic ambient space $\mathcal{M}_\psi$, it makes sense to consider the category $\mu\mathrm{Sh}_{\mathrm{Fl}_\psi}(\mathcal{M}_\psi)$ of microlocal sheaves on $\mathcal{M}_\psi$ supported on Fl_ψ. More precisely, using the realization of $\mathcal{M}_\psi$ as a Hamiltonian reduction of the cotangent bundle of $\mathrm{Bun}_G(\mathbf{J}^1_\infty, \mathbf{I}_0)$ in §2.11, one may define $\mu\mathrm{Sh}_{\mathrm{Fl}_\psi}(\mathcal{M}_\psi)$ to be a full subcategory of sheaves on $\mathrm{Bun}_G(\mathbf{J}^1_\infty, \mathbf{I}_0)/\mathbb{G}_m(\nu)$ with singular support in Fl_ψ. This can be thought of a quantization of $\mathcal{M}_\psi$, or as an affine analogue of modules over W-algebras. In [4] we study this category in the special case where ψ is homogeneous of slope 1.

We also prove the following cohomological result.

2.8.4 Theorem. *The canonical map* $\gamma : \mathrm{Fl}_\psi \to \mathcal{M}_{a_\psi} \hookrightarrow \mathcal{M}_\psi$ *induces an isomorphism on cohomology* $\gamma^* : \mathrm{H}^*(\mathcal{M}_\psi) \xrightarrow{\sim} \mathrm{H}^*(\mathrm{Fl}_\psi)$.

Here, $\mathrm{H}^*(\mathcal{M}_\psi)$ is defined to be the projective limit of cohomology of finite-type open subspaces; and $\mathrm{H}^*(\mathrm{Fl}_\psi)$ is defined to be the projective limit of cohomology of finite-type closed subschemes. When ψ is elliptic, one can use the fact that f is proper and the contraction principle to deduce Theorem 2.8.4 immediately. In general, the proof is more involved and uses hyperbolic localization.

2.9. Proof of Theorem 2.8.1(1)

From the well-known properties of moduli stack of G-bundles we know that $\mathcal{M}_\psi$ is an algebraic stack locally of finite type over k.

2.9.1. We show that $\mathrm{Aut}(\mathcal{E}, \varphi)$ is trivial as an algebraic group for any k-point $(\mathcal{E}, \varphi) \in \mathcal{M}_\psi$. By [34, Cor 4.11.3], $\mathrm{Aut}(\mathcal{E}, \varphi)$ is isomorphic to a subgroup of a maximal torus of G, hence diagonalizable (this is proved in the case without level structure but the argument works with level structure). On the other hand, restricting to D_∞, $\mathrm{Aut}(\mathcal{E}, \varphi)$ is a subgroup of the pro-unipotent group $\mathbf{K}_\infty$, hence itself unipotent. Therefore, $\mathrm{Aut}(\mathcal{E}, \varphi)$ is the trivial algebraic group over k. This implies that $\mathcal{M}_\psi$ is an algebraic space locally of finite type over k.

2.9.2. We show that $\mathcal{M}_\psi$ is a smooth Deligne-Mumford stack over k.

Let $\mathcal{E}$ be a G-bundle over X with $\mathbf{I}_0$ and $\mathbf{K}_\infty = \mathbf{P}_\infty(\frac{d}{m}) \cdot \mathbf{C}^+_\infty$ level structures at 0 and ∞ respectively. For a $\mathbf{I}_0$-invariant lattice $\Lambda_0 \subset \mathfrak{g}((t))$ and a $\mathbf{K}_\infty$-invariant lattice $\Lambda_\infty \subset \mathfrak{g}((\tau))$, we define $\mathrm{Ad}(\mathcal{E}; \Lambda_0, \Lambda_\infty)$ to be the subsheaf of $j_*\mathrm{Ad}(\mathcal{E})$ (where $j : X\backslash\{0, \infty\} \hookrightarrow X$) that is equal to $\mathrm{Ad}(\mathcal{E})$ over $\mathbb{P}^1\backslash\{0, \infty\}$ and its local sections near 0 (resp. ∞) lies in Λ_0 (resp. Λ_∞) after trivializing $\mathcal{E}|_{D_0}$ (resp. $\mathcal{E}|_{D_\infty}$).

The tangent complex of $\mathcal{M}_\psi$ at $(\mathcal{E},\varphi)\in\mathcal{M}_\psi$ is $\mathrm{H}^*(X,\mathcal{K})$ where $\mathcal{K}$ is the two step complex of vector bundles on X placed in degrees -1 and 0:

$$\mathcal{K}=\mathcal{K}_{(\mathcal{E},\varphi)}:=[\mathrm{Ad}(\mathcal{E};\mathrm{Lie}\ \mathbf{I}_0,\mathfrak{k}_\infty)\xrightarrow{[-,\varphi]}\mathrm{Ad}(\mathcal{E};\mathrm{Lie}\ \mathbf{I}_0^+,\mathfrak{g}(\!(\tau)\!)_{\le 0})]. \tag{2.5}$$

Here $\mathfrak{k}_\infty=\mathrm{Lie}\ \mathbf{K}_\infty$. The obstruction to the infinitesimal deformations of $(\mathcal{E},\varphi)$ lies in $\mathrm{H}^1(X,\mathcal{K})$ while the Lie algebra of $\mathrm{Aut}(\mathcal{E},\varphi)$ is $\mathrm{H}^{-1}(X,\mathcal{K})$. To show $\mathcal{M}_\psi$ is smooth Deligne-Mumford, we need to show that $\mathrm{H}^1(X,\mathcal{K})$ and $\mathrm{H}^{-1}(X,\mathcal{K})$ vanish.

We consider the complex $\mathcal{K}^\vee=\underline{\mathrm{Hom}}(\mathcal{K},\omega_X[1])$, obtained by taking the Serre dual of $\mathcal{K}$ termwise but still placed in degrees -1 and 0:

$$\mathcal{K}^\vee=[\mathrm{Ad}(\mathcal{E};\mathrm{Lie}\ \mathbf{I}_0,\mathfrak{g}(\!(\tau)\!)_{\le -1/m})\xrightarrow{[-,\varphi]}\mathrm{Ad}(\mathcal{E};\mathrm{Lie}\ \mathbf{I}_0^+,\mathfrak{k}_\infty^\vee)].$$

Here we are using the pairing on $\mathrm{Ad}(\mathcal{E})$ induced from $\langle\cdot,\cdot\rangle$ on $\mathfrak{g}$; $\mathfrak{k}_\infty^\vee\subset\mathfrak{g}(\!(\tau)\!)$ is the dual lattice of $\mathfrak{k}_\infty=\mathfrak{g}(\!(\tau)\!)_{\le -d/m}+\mathfrak{c}(\!(\tau)\!)_{\le -1/m}$, i.e. $\mathfrak{k}_\infty^\vee=\{v\in\mathfrak{g}(\!(\tau)\!)|\langle v,\mathfrak{k}_\infty d\tau/\tau\rangle\subset k[\![\tau]\!]d\tau\}$. Let $\mathfrak{c}(\!(\tau)\!)^\perp\subset\mathfrak{g}(\!(\tau)\!)$ be the orthogonal complement of $\mathfrak{c}(\!(\tau)\!)\subset\mathfrak{g}(\!(\tau)\!)$ under $\langle\cdot,\cdot\rangle$. Let $\mathfrak{c}(\!(\tau)\!)^\perp_{\le i/m}=\mathfrak{c}(\!(\tau)\!)^\perp\cap\mathfrak{g}(\!(\tau)\!)_{\le i/m}$. Then

$$\mathfrak{k}_\infty^\vee=\mathfrak{c}(\!(\tau)\!)^\perp_{\le (d-1)/m}\oplus\mathfrak{c}(\!(\tau)\!)_{\le 0}.$$

In particular, $\mathfrak{g}(\!(\tau)\!)_{\le 0}\subset\mathfrak{k}_\infty^\vee$. Hence there is a natural inclusion $\iota:\mathcal{K}\hookrightarrow\mathcal{K}^\vee$. We claim that ι induces a quasi-isomorphism in $D^b\mathrm{Coh}(X)$. Indeed, after trivializing $\mathcal{E}|_{D_\infty}$, $\mathrm{coker}(\iota)$ is the two-step complex

$$\mathfrak{g}(\!(\tau)\!)_{\le -1/m}/\mathfrak{k}_\infty\xrightarrow{[-,\varphi]}\mathfrak{k}_\infty^\vee/\mathfrak{g}(\!(\tau)\!)_{\le 0}.$$

Moy-Prasad filtration induces filtrations $(\mathfrak{k}_\infty^\vee/\mathfrak{g}(\!(\tau)\!)_{\le 0})_{\le j/m}$ and $(\mathfrak{g}(\!(\tau)\!)_{\le -1/m}/\mathfrak{k}_\infty)_{\le j/m}$ on both sides, with associated graded

$$(\mathfrak{g}(\!(\tau)\!)_{\le -1/m}/\mathfrak{k}_\infty)_{j/m}\cong\begin{cases}\mathfrak{c}(\!(\tau)\!)^\perp_{j/m}\cong\mathfrak{g}_{j/m}/\mathfrak{c}_{j/m}, & -d+1\le j\le -1\\ 0 & \text{otherwise.}\end{cases}$$

$$(\mathfrak{k}_\infty^\vee/\mathfrak{g}(\!(\tau)\!)_{\le 0})_{j/m}\cong\begin{cases}\mathfrak{c}(\!(\tau)\!)^\perp_{j/m}\cong\mathfrak{g}_{j/m}/\mathfrak{c}_{j/m}, & 1\le j\le d-1\\ 0 & \text{otherwise.}\end{cases}$$

The map $[-,\varphi]$ sends $(\mathfrak{g}(\!(\tau)\!)_{\le -1/m}/\mathfrak{k}_\infty)_{\le j/m}$ to $(\mathfrak{k}_\infty^\vee/\mathfrak{g}(\!(\tau)\!)_{\le 0})_{\le (j+d)/m}$, and the induced map on the associated graded is $\mathrm{ad}(\psi)_{j/m}:\mathfrak{g}_{j/m}/\mathfrak{c}_{j/m}\to\mathfrak{g}_{(j+d)/m}/\mathfrak{c}_{(j+d)/m}$ for $-d+1\le j\le -1$. Since ψ is regular semisimple, $\mathrm{ad}(\psi)_{j/m}$ is an isomorphism. Therefore $\mathrm{coker}(\iota)$ is acyclic, hence $\iota:\mathcal{K}\hookrightarrow\mathcal{K}^\vee$ is a quasi-isomorphism. Therefore ι induces an isomorphism $\mathrm{H}^*(X,\mathcal{K})\xrightarrow{\sim}\mathrm{H}^*(X,\mathcal{K}^\vee)$.

On the other hand, the complexes $\mathrm{H}^*(X,\mathcal{K})$ and $\mathrm{H}^*(X,\mathcal{K}^\vee)$ are linearly dual to each other. Therefore we conclude that there is a perfect pairing between $\mathrm{H}^1(X,\mathcal{K})$ and $\mathrm{H}^{-1}(X,\mathcal{K})$, and a perfect pairing on $\mathrm{H}^0(X,\mathcal{K})$ which is easily seen to be symplectic.

By §2.9.1, $\mathrm{H}^{-1}(X,\mathcal{K}) = \mathrm{Lie}\,\mathrm{Aut}(\mathcal{E},\varphi) = 0$. Therefore $\mathrm{H}^1(X,\mathcal{K}) = 0$ as well, hence $\mathcal{M}_\psi$ is a smooth algebraic space. Moreover, the tangent space $\mathrm{H}^0(X,\mathcal{K})$ at every point $(\mathcal{E},\varphi)$ carries a canonical symplectic form, hence a globally defined non-degenerate 2-form Ω on $\mathcal{M}_\psi$. The fact that $d\Omega = 0$ will be shown in §2.11 where we identify $\mathcal{M}_\psi$ as a Hamiltonian reduction from a cotangent bundle, which then carries a canonical symplectic form, and it is easy to check that the form coincides with Ω.

2.9.3. We compute $\dim \mathcal{M}_\psi$, or $\dim \mathrm{H}^0(X,\mathcal{K})$. By the vanishing of $\mathrm{H}^{\neq 0}(X,\mathcal{K})$, we have

$$\begin{aligned}\dim \mathrm{H}^0(X,\mathcal{K}) &= \deg \mathrm{Ad}(\mathcal{E};\mathrm{Lie}\,\mathbf{I}_0^+, \mathfrak{g}(\!(\tau)\!)_{\le 0}) - \deg \mathrm{Ad}(\mathcal{E};\mathrm{Lie}\,\mathbf{I}_0, \mathfrak{k}_\infty)\\ &= \dim_k \mathfrak{g}(\!(\tau)\!)_{\le 0}/\mathfrak{k}_\infty - \dim_k \mathrm{Lie}\,\mathbf{I}_0/\mathrm{Lie}\,\mathbf{I}_0^+\\ &= \dim_k \mathfrak{g}(\!(\tau)\!)_{\le 0}/\mathfrak{k}_\infty - r.\end{aligned} \tag{2.6}$$

By construction, we have

$$\mathfrak{g}(\!(\tau)\!)_{\le 0}/\mathfrak{k}_\infty \cong \mathfrak{g}_0 \oplus \bigoplus_{i=1}^{d-1} \mathfrak{g}_{i/m}/\mathfrak{c}_{i/m}.$$

Therefore

$$\dim \mathfrak{g}(\!(\tau)\!)_{\le 0}/\mathfrak{k}_\infty = \dim \mathfrak{c}_0 + \sum_{i=0}^{d-1} \mathfrak{g}_{i/m}/\mathfrak{c}_{i/m}. \tag{2.7}$$

We consider the roots $\Phi(\mathfrak{g},\mathfrak{c})$ for the Cartan $\mathfrak{c}$. The $\mathbb{Z}/m\mathbb{Z}$-grading on $\mathfrak{g}$ is induced by $w \in W(\mathfrak{g},\mathfrak{c})$ regular of order m. Now w permutes $\Phi(\mathfrak{g},\mathfrak{c})$ freely with $|\Phi|/m$ orbits. Each orbit contributes 1-dimension to each $\mathfrak{g}_{i/m}/\mathfrak{c}_{i/m}$ for all $i \in \mathbb{Z}/m\mathbb{Z}$. Therefore $\dim \mathfrak{g}_{i/m}/\mathfrak{c}_{i/m} = |\Phi|/m$ for all $i \in \mathbb{Z}$. Using (2.7), we see that

$$\dim \mathfrak{g}(\!(\tau)\!)_{\le 0}/\mathfrak{k}_\infty = \dim \mathfrak{c}_0 + \frac{d}{m}|\Phi| = \dim \mathfrak{t}^w + \frac{d}{m}|\Phi|.$$

Combined with (2.6) we get

$$\dim_{(\mathcal{E},\varphi)} \mathcal{M}_\psi = \dim \mathrm{H}^0(X, \mathcal{K}_{(\mathcal{E},\varphi)}) = \dim \mathfrak{t}^w + \frac{d}{m}|\Phi| - r.$$

□

2.10. Proof of Thereom 2.8.1(4)

Ngô's product formula [34, Prop. 4.15.1] and its extension by Bouthier and Cesnavicius [10, Theorem 4.3.8] has an analogue for our level structures which we spell out.

Define a reduced sub-ind-scheme of $G(\!(\tau)\!)/\mathbf{K}_\infty$:

$$\mathcal{X}_{\psi,\infty} = \{g\mathbf{K}_\infty \in G(\!(\tau)\!)/\mathbf{K}_\infty | \mathrm{Ad}(g^{-1})\psi \in \psi + \mathfrak{g}(\!(\tau)\!)_{\leq 0}\}.$$

This is an analog of an affine Springer fiber.

Recall the torus C over $\mathbb{P}^1\backslash\{0,\infty\}$ defined in §2.4 as the centralizer of ψ. Extend C to a group scheme $\mathcal{C}$ on $\mathbb{P}^1$ with parahoric level structures at 0 and ∞. Let $\mathrm{Pic}_C(\widehat{0};\widehat{\infty})$ be the moduli space of $\mathcal{C}$ torsors over $\mathbb{P}^1$ with trivializations on D_0 and D_∞. Then $C[\![t]\!]$ and $C[\![\tau]\!]$ act on $\mathrm{Pic}_C(\widehat{0};\widehat{\infty})$ by changing the trivializations, and these actions extend to actions of the loop tori $C(\!(t)\!)$ and $C(\!(\tau)\!)$. On the other hand, $C(\!(t)\!)$ and $C(\!(\tau)\!)$ act on Fl_ψ and $\mathcal{X}_{\psi,\infty}$ respectively by left translations.

There is a canonical morphism

$$\alpha : \mathrm{Pic}_C(\widehat{0};\widehat{\infty}) \overset{C(\!(t)\!)\times C(\!(\tau)\!)}{\times} (\mathrm{Fl}_\psi \times \mathcal{X}_{\psi,\infty}) \to \mathcal{M}_{a_\psi} \tag{2.8}$$

defined as follows. Given a $\mathcal{C}$-torsor $\mathcal{Q}$ over $\mathbb{P}^1$ with trivializations on D_0 and D_∞, we get a G-torsor $\mathcal{E}^\circ = \mathcal{Q} \overset{C}{\times} G$ over $\mathbb{P}^1\backslash\{0,\infty\}$ with a Higgs field given by ψ and trivializations on D_0 and D_∞. A point $g_0\mathbf{I}_0 \in \mathrm{Fl}_\psi$ gives a G-torsor $\mathcal{E}_0$ over D_0 with $\mathbf{I}_0$-level structure together with a trivialization over $D_0^\times$. We can glue $\mathcal{E}_0$ with $\mathcal{E}^\circ$ along $D_0^\times$ using the trivializations. Similarly, a point $g_\infty\mathbf{K}_\infty \in \mathcal{X}_{\psi,\infty}$ gives a G-torsor $\mathcal{E}_\infty$ over D_∞ with $\mathbf{K}_\infty$-level structure together with a trivialization over $D_\infty^\times$, which we can glue with $\mathcal{E}^\circ$ along $D_0^\times$. This way we have extended $\mathcal{E}^\circ$ to a G-torsor $\mathcal{E}$ on $\mathbb{P}^1$ with $\mathbf{I}_0$ and $\mathbf{K}_\infty$-level structures. The Higgs field ψ on $\mathcal{E}^\circ$ extends to $\mathcal{E}$ because of the conditions defining Fl_ψ and $\mathcal{X}_{\psi,\infty}$. This gives the map α. The same argument of [10, Theorem 4.3.8] shows that α is a universal homeomorphism: the reason is that for any $(\mathcal{E},\varphi) \in \mathcal{M}_{a_\psi}(R)$ where R is a seminormal strictly Henselian local k-algebra, $(\mathcal{E},\varphi)|_{\mathbb{P}^1_R\backslash\{0,\infty\}}$ reduces to a C-torsor, and the restriction of any C-torsor over Spec $R(\!(t)\!)$ and Spec $R(\!(\tau)\!)$ must be trivial, as shown in [10, Theorem 3.2.4] (using that m is invertible in k, hence C splits over a tamely ramified cover of $\mathbb{G}_m$).

By Lemma 2.10.1 below, the action of $C(\!(\tau)\!)$ on $\mathcal{X}_{\psi,\infty}$ is transitive. The stabilizer of $C(\!(\tau)\!)$ at the base point $1 \in \mathcal{X}_{\psi,\infty}$ is $C(\!(\tau)\!) \cap \mathbf{K}_\infty = \mathbf{C}_\infty^+$. Therefore the action map $C(\!(\tau)\!)/\mathbf{C}_\infty^+ \to \mathcal{X}_{\psi,\infty}$ is an isomorphism on the reduced structures. This allows us to simplify the left side of (2.8) to

$$\mathrm{Pic}_{\mathcal{C}}(\widehat{0};\mathbf{C}^+_\infty)\overset{C((t))}{\times}\mathrm{Fl}_\psi\to\mathcal{M}_{a_\psi}. \tag{2.9}$$

Here $\mathrm{Pic}_{\mathcal{C}}(\widehat{0};\mathbf{C}^+_\infty)$ is the moduli space of $\mathcal{C}$-torsors over $\mathbb{P}^1$ with a trivialization on D_0 and a $\mathbf{C}^+_\infty$-level structure at ∞. Let $\mathbf{C}_0\subset C((t))$ be the parahoric subgroup. Let $\mathrm{Pic}_{\mathcal{C}}(\mathbf{C}_0;\mathbf{C}^+_\infty)$ be the moduli space of $\mathcal{C}$-torsors over $\mathbb{P}^1$ with a $\mathbf{C}_0$-level structure at 0 and a $\mathbf{C}^+_\infty$-level structure at ∞. Then $\mathrm{Pic}_{\mathcal{C}}(\mathbf{C}_0;\mathbf{C}^+_\infty)$ is the discrete space $\mathbb{X}_*(T)_{\langle w\rangle}$ of coinvariants under the action of $\langle w\rangle$. Indeed, a computation of tangent space shows that $\mathrm{Pic}_{\mathcal{C}}(\mathbf{C}_0;\mathbf{C}^+_\infty)$ is discrete; the automorphism group of the identity point is trivial hence the automorphism group of all points are trivial since $\mathrm{Pic}_{\mathcal{C}}(\mathbf{C}_0;\mathbf{C}^+_\infty)$ is a Picard groupoid. By [22, Lemma 16] the connected components of $\mathrm{Pic}_{\mathcal{C}}(\mathbf{C}_0;\mathbf{C}_\infty)$ are canonically indexed by $\mathbb{X}_*(T)_{\langle w\rangle}$, hence the same is true for $\mathrm{Pic}_{\mathcal{C}}(\mathbf{C}_0;\mathbf{C}^+_\infty)$. On the other hand, the Kottwitz map gives an isomorphism $(C((t))/\mathbf{C}_0)^{\mathrm{red}}\xrightarrow{\sim}\mathbb{X}_*(T)_{\langle w\rangle}$ (see [36, Theorem 5.1, step A]), and $\mathrm{Pic}_{\mathcal{C}}(\mathbf{C}_0;\mathbf{C}^+_\infty)$ is a trivial torsor under $(C((t))/\mathbf{C}_0)^{\mathrm{red}}$. Therefore the action of $C((t))$ on $\mathrm{Pic}_{\mathcal{C}}(\widehat{0};\mathbf{C}^+_\infty)$ is transitive, and the reduced stabilizer is trivial. Hence the natural map $\mathrm{Fl}_\psi\to\mathrm{Pic}_{\mathcal{C}}(\widehat{0};\mathbf{C}^+_\infty)\overset{C((t))}{\times}\mathrm{Fl}_\psi$ is an isomorphism on the reduced structure. Since (2.9) is a universal homeomorphism, we conclude that the composition

$$\mathrm{Fl}_\psi\to\mathrm{Pic}_{\mathcal{C}}(\widehat{0};\mathbf{C}^+_\infty)\overset{C((t))}{\times}\mathrm{Fl}_\psi\to\mathcal{M}_{a_\psi}$$

is a universal homeomorphism. $\square$

2.10.1 Lemma. *Let $\psi'\in\psi+\mathfrak{g}((\tau))_{\geq 0}$ be in the same $G((\tau))$-orbit of ψ, then there exists $g\in\mathbf{P}_\infty(\frac{d}{m})$ such that $\mathrm{Ad}(g)\psi'=\psi$.*

Proof. We construct inductively a sequence of elements $g_j\in\mathbf{P}_\infty(\frac{d}{m})$ (for $j\leq 1$) such that

(1) $g_1=1$;
(2) For $j\leq 0$, $g_j\in g_{j+1}\mathbf{P}_\infty(\frac{-j+d}{m})$;
(3) For $j\leq 0$, $\mathrm{Ad}(g_j)\psi'\equiv\psi\mod\mathfrak{g}((\tau))_{\leq j-1}$.

Then the limit $g=\lim_{j\to-\infty}g_j$ exists in $\mathbf{P}_\infty(\frac{d}{m})$, and it satisfies $\mathrm{Ad}(g)\psi'=\psi$.

Take $g_1=1$. Suppose g_{j+1} has been constructed. Then we have

$$\mathrm{Ad}(g_{j+1})\psi'=\psi+X_j+X_{j-1}+\cdots,\quad X_k\in\mathfrak{g}((\tau))_{j/m}.$$

We look for $Y\in\mathfrak{g}((\tau))_{(j-d)/m}$ such that $[Y,\psi]=X_j$; then $g_j=\exp(-Y)g_{j+1}$ satisfies all the requirements.

Now solve the equation $[Y,\psi]=X_j$ for $Y\in\mathfrak{g}((\tau))_{(j-d)/m}$. Recall $f_1,\cdots,f_r \in k[\mathfrak{g}]^G$ is a set of homogeneous generators with degrees $d_1,\cdots,d_r$. By assumption $f_i(\psi')=f_i(\psi)$, hence

$$f_i(\psi)=f_i(\mathrm{Ad}(g_{j+1})\psi')=f_i(\psi+X_j+X_{j-1}+\cdots).$$

Taking Taylor expansion of f_i at ψ with respect to the Moy-Prasad grading, we get

$$\langle df_i(\psi),X_j\rangle=0,\quad \forall i=1,\cdots,r.$$

Here $df_i(\psi)\in\mathfrak{g}^*((\tau))$, and the pairing $\langle,\rangle$ is the $k((\tau))$-bilinear between $\mathfrak{g}^*((\tau))$ and $\mathfrak{g}((\tau))$. Since ψ is regular semisimple as an element in $\mathfrak{g}((\tau))$, the differentials $\{df_i(\psi)\}_{1\le i\le r}$ span a subspace of $\mathfrak{g}^*((\tau))$ which under the Killing form can be identified with $\mathfrak{c}((\tau))$ (the centralizer of ψ). Therefore, the annihilator of the span of $\{df_i(\psi)\}_{1\le i\le r}$ is $[\mathfrak{g}((\tau)),\psi]$. The above equations imply that $X_j\in[\mathfrak{g}((\tau)),\psi]$, so $X_j\in[Z,\psi]$ for some $Z\in\mathfrak{g}((\tau))$. Let Y be the $\mathfrak{g}((\tau))_{(j-d)/m}$ homogeneous component of Z, then $[Y,\psi]=X_j$. This completes the inductive construction of g_j. □

2.11. Construction of $\mathcal{M}_\psi$ as a Hamiltonian reduction

The symplectic structure on $\mathcal{M}_\psi$ mentioned in Theorem 2.8.1 comes from a realization of $\mathcal{M}_\psi$ as a Hamiltonian reduction of a certain cotangent space.

We define a subgroup $\mathbf{J}_\infty\subset G((\tau))$ as follows:

(1) If d is odd, then $\mathbf{J}_\infty=\mathbf{P}_\infty(\frac{(d+1)/2}{m})\cdot\mathbf{C}_\infty^+$.
(2) If d is even, then $\mathfrak{g}((\tau))_{-d/2m}$ carries an alternating form $(x,y)\mapsto\langle\psi,[x,y]\rangle$. Let $\mathfrak{m}\subset\mathfrak{g}((\tau))_{-d/2m}$ be a maximal isotropic subspace, and let $\mathbf{P}_\infty(\frac{d/2}{m})_\mathfrak{m}$ be its preimage in $\mathbf{P}_\infty(\frac{d/2}{m})$ under the projection $\mathbf{P}_\infty(\frac{d/2}{m})\to\mathfrak{g}((\tau))_{-d/(2m)}$. Let $\mathbf{J}_\infty=\mathbf{P}_\infty(\frac{d/2}{m})_\mathfrak{m}\cdot\mathbf{C}_\infty^+$.

Then ψ has a unique extension to a linear character $\widetilde{\psi}:\mathbf{J}_\infty\to\mathbb{G}_a$ such that, on the level of Lie algebras, $\widetilde{\psi}$ is trivial on $(\mathrm{Lie}\,\mathbf{J}_\infty)\cap\mathfrak{g}((\tau))_{i/m}$ for $i<-d$.

To construct this note that $\mathbf{J}_\infty$ is pro-unipotent so we just need check that the linear map $\widetilde{\psi}$ given by

$$\mathrm{Lie}\,\mathbf{J}_\infty\subset\mathrm{Lie}\,\mathbf{P}_\infty(\frac{1}{m})\to\mathfrak{g}((\tau))_{-d/2m}\xrightarrow{<\psi,->}k$$

is a Lie algebra homomorphism. That is to check that $[\mathrm{Lie}\,\mathbf{J}_\infty,\mathrm{Lie}\,\mathbf{J}_\infty]$ is in the kernel. This follows from the definition of $\mathbf{P}_\infty(\frac{d/2}{m})_\mathfrak{m}$ and the fact that $\mathbf{C}_\infty^+$ is a commutative group centralizing ψ.

Recall the notation of Hamiltonian (or Marsden-Weinstein) reduction: let $\mathfrak{X}$ be a smooth stack with the action of an algebraic group H. Let $\zeta : \mathfrak{h} \to k$ be a character of the Lie algebra $\mathfrak{h} = \mathrm{Lie}\, H$, viewed as an element in $\mathfrak{h}^*$. Let $\mu_H : T^*\mathfrak{X} \to \mathfrak{h}$ be the moment map. Then define the stack

$$T^*\mathfrak{X}//_\zeta H = \mu_H^{-1}(\zeta)/H.$$

When ζ is a regular value of μ_H, $T^*\mathfrak{X}//_\zeta H$ is a smooth stack that inherits an exact symplectic structure from that of $T^*\mathfrak{X}$.

We apply this construction to the case $\mathfrak{X} = \mathrm{Bun}_G(\mathbf{J}^1_\infty; \mathbf{I}_0)$, the moduli stack of G-bundles on $X = \mathbb{P}^1$ with $\mathbf{J}^1_\infty := \ker(\widetilde{\psi}) \subset \mathbf{J}_\infty$-level structure at ∞ and $\mathbf{I}_0$-level structure at 0, with the action of $H = \mathbb{G}_a = \mathbf{J}_\infty/\mathbf{J}^1_\infty$ and the character $\zeta = \widetilde{\psi} \in \mathfrak{h}^*$. Here $\mathbf{J}_\infty$ acts on $\mathrm{Bun}_G(\mathbf{J}^1_\infty; \mathbf{I}_0)$ by changing the $\mathbf{J}^1_\infty$-level structure, where $\mathbf{J}^1_\infty$- acts trivially and thus this descends to an action of $H = \mathbb{G}_a = \mathbf{J}_\infty/\mathbf{J}^1_\infty$.

2.11.1 Proposition. *There is a canonical isomorphism between $\mathcal{M}_\psi$ and the Hamiltonian reduction of $T^*\mathrm{Bun}_G(\mathbf{J}^1_\infty; \mathbf{I}_0)//_{\widetilde{\psi}}\mathbb{G}_a$. In particular, $\mathcal{M}_\psi$ carries a canonical symplectic structure which coincides with the 2-form Ω defined in §2.9.*

Proof. We describe $\mathcal{N}_\psi := T^*\mathrm{Bun}_G(\mathbf{J}^1_\infty; \mathbf{I}_0)//_{\widetilde{\psi}}\mathbb{G}_a$ as a moduli space of Higgs bundles as follows. Let $\mathfrak{j}_\infty = \mathrm{Lie}\, \mathbf{J}_\infty$ and $\mathfrak{j}^\vee_\infty \subset \mathfrak{g}(\!(\tau)\!)$ be the dual lattice under the form $\langle \cdot, \cdot \rangle$ extended $k(\!(\tau)\!)$-linearly, i.e., $\mathfrak{j}^\vee_\infty = \{v \in \mathfrak{g}(\!(\tau)\!) | \langle v, \mathfrak{j}_\infty d\tau/\tau \rangle \subset k[\![\tau]\!]d\tau\}$.

Recall the notations $\mathfrak{c}(\!(\tau)\!)^\perp, \mathfrak{c}(\!(\tau)\!)^\perp_{\leq j/m}$ from §2.9.2. Then

$$\mathfrak{j}^\vee_\infty = \begin{cases} \mathfrak{c}(\!(\tau)\!)^\perp_{\leq (d-1)/(2m)} \oplus \mathfrak{c}(\!(\tau)\!)_{\leq 0} & d \text{ odd;} \\ \mathfrak{m}^\perp \oplus \mathfrak{c}(\!(\tau)\!)^\perp_{\leq (d/2-1)/m} \oplus \mathfrak{c}(\!(\tau)\!)_{\leq 0} & d \text{ even.} \end{cases} \tag{2.10}$$

Here $\mathfrak{m}^\perp \subset \mathfrak{g}(\!(\tau)\!)_{d/(2m)}$ is the orthogonal complement of $\mathfrak{m} \subset \mathfrak{g}(\!(\tau)\!)_{-d/(2m)}$ under the pairing $\langle \cdot, \cdot \rangle$.

Then $\mathcal{N}_\psi$ classifies pairs $(\mathcal{E}, \varphi)$ where

- $\mathcal{E}$ is a G-bundle over X with $\mathbf{J}_\infty$-level structure at ∞ and $\mathbf{I}_0$-level structure at 0. We denote by $\mathrm{Ad}(\mathcal{E})$ the vector bundle over $X \backslash \{0, \infty\}$ associated to the adjoint representation $\mathfrak{g}$ of G.
- φ is a section of $\mathrm{Ad}(\mathcal{E}) \otimes \omega_{X\backslash\{0,\infty\}}$ satisfying the following conditions:
  (i) Under some (equivalently, any) trivialization of $\mathcal{E}|_{D_\infty}$ together with its $\mathbf{J}_\infty$-level structure, we require

$$\varphi|_{D^\times_\infty} \in (\psi + \mathfrak{j}^\vee_\infty)d\tau/\tau.$$

(ii) Under some (equivalently, any) trivialization of $\mathcal{E}|_{D_0}$ together with its $\mathbf{I}_0$-level structure, we require

$$\varphi|_{D_0^\times} \in \mathrm{Lie}\,(\mathbf{I}_0^+)dt/t.$$

Since $\mathbf{K}_\infty \subset \mathbf{J}_\infty$ and $\psi + \mathfrak{g}(\!(\tau)\!)_{\leq 0} \subset \psi + \mathfrak{j}_\infty^\vee$, we have a natural map

$$F : \mathcal{M}_\psi \to \mathcal{N}_\psi.$$

We need to show that F is an isomorphism of algebraic stacks.

Let $U_\psi = (\psi + \mathfrak{j}_\infty^\vee)/\mathfrak{g}(\!(\tau)\!)_{\leq 0}$ (an affine space). Let $\overline{J} = \mathbf{J}_\infty/\mathbf{P}(\frac{d}{m})$. Then $\overline{J}$ acts on U_ψ by the adjoint action. Let $\overline{C} \subset \overline{J}$ be the image of $\mathbf{C}_\infty^+$, then $\overline{C}$ stabilizes the point $\psi \in U_\psi$. We thus get a morphism of stacks

$$\iota : [\{\psi\}/\overline{C}] \to [U_\psi/\overline{J}].$$

On the other hand, we have an evaluation map

$$\epsilon : \mathcal{N}_\psi \to [U_\psi/\overline{J}]$$

by taking the Laurent expansion of $\varphi|_{D_\infty^\times}$ modulo $\mathfrak{g}(\!(\tau)\!)_{\leq 0}d\tau/\tau$. From the definitions we have a Cartesian diagram

$$\begin{tikzcd} \mathcal{M}_\psi \arrow[r, "F"] \arrow[d] & \mathcal{N}_\psi \arrow[d, "\epsilon"] \\ {[\{\psi\}/\overline{C}]} \arrow[r, "\iota"] & {[U_\psi/\overline{J}]} \end{tikzcd}$$

To show F is an isomorphism, it suffices to show that ι is an isomorphism.

Consider the action map $\alpha : \overline{J} \to U_\psi$ sending $g \in \overline{J}$ to $\mathrm{Ad}(g)\psi \in U_\psi$. It passes to the quotient

$$\overline{\alpha} : \overline{J}/\overline{C} \cong \mathbf{J}_\infty/\mathbf{K}_\infty \to U_\psi$$

Then ι is an isomorphism if and only if $\overline{\alpha}$ is an isomorphism.

For $d/2 \leq j \leq d$ and $d \in \mathbb{Z}$, let

$$\mathbf{Q}_j = \mathbf{P}_\infty(\frac{j}{m})\mathbf{C}_\infty^+,$$
$$\Lambda_j = \mathfrak{c}(\!(\tau)\!)^\perp_{\leq (d-j)/m} + \mathfrak{g}(\!(\tau)\!)_{\leq 0}.$$

We have

$$\mathbf{K}_\infty = \mathbf{Q}_d \subset \cdots \subset \mathbf{Q}_j \subset \mathbf{Q}_{j-1} \subset \cdots \mathbf{Q}_{\lceil d/2 \rceil} \subset \mathbf{J}_\infty,$$
$$\mathfrak{g}((\tau))_{\le 0} = \Lambda_d \subset \cdots \subset \Lambda_j \subset \Lambda_{j-1} \subset \cdots \subset \Lambda_{\lceil d/2 \rceil} \subset \mathfrak{j}_\infty^\vee.$$

Moreover, $\mathrm{Ad}(\mathbf{Q}_j)\psi \subset \Lambda_j$ for $j \ge d/2$.

We show inductively that for $j \ge d/2$, the map

$$\alpha_j : \mathbf{J}_\infty/\mathbf{Q}_j \to (\psi + \mathfrak{j}_\infty^\vee)/\Lambda_j$$

defined by $g \mapsto \mathrm{Ad}(g)\psi \mod \Lambda_j$ is an isomorphism. Since $\alpha_d = \overline{\alpha}$, this would finish the proof.

When d is odd, the initial step $j = (d+1)/2$ is trivial since both sides of α_j are singletons. When d is even, we have $\mathbf{J}_\infty/\mathbf{Q}_{d/2} \cong \mathfrak{m}/\mathfrak{c}_{-d/(2m)}$, and $(\psi + \mathfrak{j}_\infty^\vee)/\Lambda_{d/2} \cong \psi + \mathfrak{m}^\perp$, and the map $\alpha_{d/2}$ is $[-,\psi]$. By definition $\mathfrak{m}$ is a maximal isotropic subspace of $\mathfrak{g}_{-d/(2m)}$ under the form $(x,y) \mapsto \langle x, [y,\psi]\rangle$. This form has kernel $\mathfrak{c}_{-d/(2m)}$, hence $[-,\psi]$ maps $\mathfrak{m}/\mathfrak{c}_{-d/(2m)}$ isomorphically to $\mathfrak{m}^\perp$.

Now assume α_j is an isomorphism (where $d/2 \le j < d$). We have a commutative diagram

$$\begin{tikzcd} \mathbf{J}_\infty/\mathbf{Q}_{j+1} \arrow[r, "\alpha_{j+1}"] \arrow[d, "q_j"] & (\psi + \mathfrak{j}_\infty^\vee)/\Lambda_{j+1} \arrow[d, "p_j"] \\ \mathbf{J}_\infty/\mathbf{Q}_j \arrow[r, "\alpha_j"] & (\psi + \mathfrak{j}_\infty^\vee)/\Lambda_j \end{tikzcd}$$

Since the above diagram is $\mathbf{J}_\infty$-equivariant and the map α_j is assumed to be an isomorphism, to show α_{j+1} is an isomorphism it suffices to show that

$$\alpha_{j+1}|_{q_j^{-1}(1)} : q_j^{-1}(1) = \mathbf{Q}_j/\mathbf{Q}_{j+1} \to p_j^{-1}(\psi) = (\psi + \Lambda_j)/\Lambda_{j+1}$$

is an isomorphism. This map can be identified with

$$[-,\psi] : \mathfrak{c}((\tau))^\perp_{-j/m} \to \mathfrak{c}((\tau))^\perp_{(d-j)/m}.$$

Since ψ is regular semisimple with centralizer $\mathfrak{c}((\tau))$ under $\mathfrak{g}((\tau))$, the above map is an isomorphism. This completes the inductive step.

To see the 2-form coincides we note that the tangent complex of $\mathcal{N}_\psi$ is

$$\mathcal{K} = \mathcal{K}_{(\mathcal{E},\varphi)} := [\mathrm{Ad}(\mathcal{E}; \mathrm{Lie}\ \mathbf{I}_0, \mathfrak{j}_\infty) \xrightarrow{[-,\varphi]} \mathrm{Ad}(\mathcal{E}; \mathrm{Lie}\ \mathbf{I}_0^+, \mathfrak{g}((\tau))_{\le 0})].$$

similarly defined to the tangent complex in equation 2.5. The result of the 2-forms now follows by noting that the above argument shows both tangent complexes agree under the map ι. $\square$

2.12. Proof of Theorem 2.8.1(3)

By Theorem 2.8.1(1) and Lemma 2.6.2, we see that $\dim \mathcal{M}_\psi = 2\dim \mathcal{A}_\psi$. Therefore all fibers of f have dimension $\geq \dim \mathcal{A}_\psi$. Also, by Theorem 2.8.1(4), the central fiber of f has dimension $\dim \mathrm{Fl}_\psi = \dim \mathcal{A}_\psi = \dim \mathcal{M}_\psi - \dim \mathcal{A}_\psi$. Since all points in $\mathcal{A}_\psi$ contract to a_ψ under the $\mathbb{G}_m$-action, all fibers of f have dimension $\leq \dim \mathrm{Fl}_\psi$. Combine both inequalities, we conclude that all fibers of f have dimension equal to $\dim \mathcal{A}_\psi$. Since $\mathcal{M}_\psi$ and $\mathcal{A}_\psi$ are both smooth, f is flat of relative dimension equal to $\dim \mathcal{A}_\psi$.

It remains to check that the functions given by coordinates of $\mathcal{A}_\psi$ are Poisson commuting. Let $(\mathcal{E},\varphi) \in \mathcal{M}_\psi$ with image $a \in \mathcal{A}_\psi$. We need to show that the image of the cotangent map $f^* : T^*_a\mathcal{A}_\psi \to T^*_{(\mathcal{E},\varphi)}\mathcal{M}_\psi$ is isotropic.

Let $\mathcal{F} = \bigoplus_{i=1}^r \mathcal{O}([d(d_i-1)/m]\cdot\infty - \underline{0})$, where $\underline{0}$ denotes the point $0 \in X$. Then $T_a\mathcal{A}_\psi$ is identified with $\mathrm{H}^0(X,\mathcal{F})$. Recall from the proof of Theorem 2.8.1 that the tangent space $T_{(\mathcal{E},\varphi)}\mathcal{M}_\psi \cong \mathrm{H}^0(X,\mathcal{K})$. The tangent map $f_* : T_{(\mathcal{E},\varphi)}\mathcal{M}_\psi \to T_a\mathcal{A}_\psi$ is induced from the following map of coherent complexes on X by taking global sections

$$\begin{tikzcd} \mathrm{Ad}(\mathcal{E};\mathrm{Lie}\,\mathbf{I}_0,\mathrm{Lie}\,\mathbf{K}_\infty) \arrow[r, "{[-,\varphi]}"] & \mathrm{Ad}(\mathcal{E};\mathrm{Lie}\,\mathbf{I}_0^+,\mathfrak{g}(\!(\tau)\!)_{\leq 0}) \arrow[d, "(df_i)_{1\leq i\leq r}"] \\ & \mathcal{F} \end{tikzcd}$$

Here $df_i : \mathrm{Ad}(\mathcal{E};\mathrm{Lie}\,\mathbf{I}_0^+,\mathfrak{g}(\!(\tau)\!)_{\leq 0}) \to \mathcal{O}([d(d_i-1)/m]\cdot\infty - \underline{0})$ is the $\mathcal{O}_X$-linear map given fiberwise by the differential of f_i at φ. The map $(df_i)_{1\leq i\leq r}$ above factors through $p : \mathcal{H}^0\mathcal{K} \to \mathcal{F}$. The tangent map f_* at $(\mathcal{E},\varphi)$ is thus given by

$$T_{(\mathcal{E},\varphi)}\mathcal{M}_\psi = \mathrm{H}^0(X,\mathcal{K}) \twoheadrightarrow \mathrm{H}^0(X,\mathcal{H}^0\mathcal{K}) \xrightarrow{p} \mathrm{H}^0(X,\mathcal{F}) = T_a\mathcal{A}_\psi.$$

Dually, the cotangent map f^* at $(\mathcal{E},\varphi)$ is given by

$$T^*_a\mathcal{A}_\psi = \mathrm{H}^1(X,\mathcal{F}^*\otimes\omega_X) \xrightarrow{p^*} \mathrm{H}^1(X,\mathcal{H}^{-1}(\mathcal{K}^\vee)) \hookrightarrow \mathrm{H}^0(X,\mathcal{K}^\vee) = T^*_{(\mathcal{E},\varphi)}\mathcal{M}_\psi.$$

Here $\mathcal{K}^\vee = \underline{\mathrm{Hom}}(\mathcal{K},\omega_X[1])$ is the Serre dual of $\mathcal{K}$, and $(-)^*$ denotes linear dual $\underline{\mathrm{Hom}}(-,\mathcal{O}_X)$. In the proof of Theorem 2.8.1, we showed that the obvious

map $\iota : \mathcal{K} \to \mathcal{K}^\vee$ is a quasi-isomorphism, which then induces an isomorphism of short exact sequences
(2.11)

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{H}^1(X, \mathcal{H}^{-1}\mathcal{K}) \arrow[r] \arrow[d, "\cong"] & T_{(\mathcal{E},\varphi)}\mathcal{M}_\psi \arrow[r] \arrow[d, "\iota"', "\cong"] & \mathrm{H}^0(X, \mathcal{H}^0\mathcal{K}) \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & \mathrm{H}^1(X, \mathcal{H}^{-1}(\mathcal{K}^\vee)) \arrow[r] & T^*_{(\mathcal{E},\varphi)}\mathcal{M}_\psi \arrow[r] & \mathrm{H}^0(X, \mathcal{H}^0(\mathcal{K}^\vee)) \arrow[r] & 0
\end{tikzcd}$$

By construction, the middle vertical map gives the symplectic form on $T_{(\mathcal{E},\varphi)}$ $\mathcal{M}_\psi$. The composition

$$T_a^*\mathcal{A}_\psi \xrightarrow{f^*} T^*_{(\mathcal{E},\varphi)}\mathcal{M}_\psi \xrightarrow{\iota^{-1}} T_{(\mathcal{E},\varphi)}\mathcal{M}_\psi \xrightarrow{f_*} T_a\mathcal{A}_\psi$$

factors through the composition of either row in (2.11), hence is zero. This shows that the image of $T_a^*\mathcal{A}_\psi$ in $T^*_{(\mathcal{E},\varphi)}\mathcal{M}_\psi$ is isotropic. □

2.13. Construction of $\mathcal{M}_\psi$ as a symplectic leaf

Let $\mathcal{M}_\psi^\dagger$ be the moduli stack parametrizing pairs $(\mathcal{E}, \varphi)$ where

- $\mathcal{E}$ is a G-bundle over X with $\mathbf{P}_\infty^+$-level structure at ∞ and $\mathbf{I}_0$-level structure at 0.
- φ is a section of $\mathrm{Ad}(\mathcal{E}) \otimes \omega_{X\setminus\{0,\infty\}}$ satisfying the following conditions:
  - (i) Under some (equivalently, any) trivialization of $\mathcal{E}|_{D_\infty}$ together with its $\mathbf{P}_\infty^+$-level structure, we require

$$\varphi|_{D_\infty^\times} \in (\psi + \mathfrak{g}(\!(\tau)\!)_{\leq(d-1)/m})d\tau/\tau.$$

  - (ii) Under some (equivalently, any) trivialization of $\mathcal{E}|_{D_0}$ together with its $\mathbf{I}_0$-level structure, we require

$$\varphi|_{D_0^\times} \in \mathrm{Lie}\,(\mathbf{I}_0^+)dt/t.$$

The Hitchin base $\mathcal{A}_\psi^\dagger$ for $\mathcal{M}_\psi^\dagger$ is the closed subscheme $\mathcal{A}_\psi^\dagger \subset \prod_{i=1}^r \Gamma(\mathbb{P}^1,$ $\mathcal{O}([\frac{dd_i}{m}] \cdot \infty))$ of sections $a = (a_i)_{1\leq i\leq r}$ such that for each $i = 1, \cdots, r$

- $a_i(0) = 0$;
- $a_i \equiv f_i(\psi) \mod \tau^{-\frac{dd_i-1}{m}}$ near ∞ (i.e., the leading coefficient of degree $\tau^{-dd_i/m}$, if $m|d_i$, of a_i at ∞ is the same as that of $f_i(\psi)$).

We have the Hitchin map

$$f^\dagger : \mathcal{M}^\dagger_\psi \to \mathcal{A}^\dagger_\psi.$$

It is also clear from the construction that there is a canonical map $\mathcal{M}_\psi \to \mathcal{M}^\dagger_\psi$ and an inclusion $\mathcal{A}_\psi \subset \mathcal{A}^\dagger_\psi$.

2.13.1 Proposition. *The canonical maps give a Cartesian diagram*

$$\begin{tikzcd} \mathcal{M}_\psi \arrow[r] \arrow[d, "f"] & \mathcal{M}^\dagger_\psi \arrow[d, "f^\dagger"] \\ \mathcal{A}_\psi \arrow[r, hook] & \mathcal{A}^\dagger_\psi \end{tikzcd}$$

In particular, $\mathcal{M}_\psi \cong \mathcal{M}^\dagger_\psi \times_{\mathcal{A}^\dagger_\psi} \mathcal{A}_\psi$.

Proof. Let $Q = \mathbf{P}^+_\infty/\mathbf{P}_\infty(\frac{d}{m})$. Let $V_\psi = (\psi + \mathfrak{g}(\!(\tau)\!)_{\leq(d-1)/m})/\mathfrak{g}(\!(\tau)\!)_{\leq 0}$. Then Q acts on the affine space V_ψ by the adjoint action. Each $(\mathcal{E}, \varphi) \in \mathcal{M}^\dagger_\psi$ gives an Q-torsor $\mathcal{Q}$ induced from the $\mathbf{P}^+_\infty$-level structure of $\mathcal{E}$, and the polar terms of $\varphi(d\tau/\tau)^{-1}$ give a section of the associated affine space bundle $\mathcal{Q} \overset{Q}{\times} V_\psi$. This construction gives a map

$$\epsilon : \mathcal{M}^\dagger_\psi \to [V_\psi/Q].$$

The stabilizer Q_ψ of $\psi \in V_\psi$ is the image of $\mathbf{C}^+_\infty$ in Q. By the definition of $\mathcal{M}_\psi$, we have a Cartesian diagram

$$\begin{tikzcd} \mathcal{M}_\psi \arrow[r] \arrow[d] & \mathcal{M}^\dagger_\psi \arrow[d, "\epsilon"] \\ {[\{\psi\}/Q_\psi]} \arrow[r, hook] & {[V_\psi/Q]} \end{tikzcd} \tag{2.12}$$

Let $\mathfrak{a}_\psi$ be the affine space of $(\overline{a}_i)_{1\leq i\leq r}$ where $\overline{a}_i \in \tau^{-[dd_i/m]}k[\![\tau]\!]/k[\![\tau]\!]$ with leading term $f_i(\psi)$ in degree $\tau^{-dd_i/m}$ if $m|d_i$. Then $(f_i)_{1\leq i\leq r}$ gives a map

$$\overline{f} : V_\psi \to \mathfrak{a}_\psi.$$

We have a map $\pi : \mathcal{A}^\dagger_\psi \to \mathfrak{a}_\psi$ by taking the first few terms of the Laurent expansion a_i of at ∞. Let $\overline{a}_\psi = \pi(a_\psi) \in \mathfrak{a}_\psi$. Then $\mathcal{A}_\psi = \pi^{-1}(\overline{a}_\psi)$. Let

$V_\psi^0 := \overline{f}^{-1}(\overline{a}_\psi)$. Therefore

$$\mathcal{M}_\psi^\dagger \times_{\mathcal{A}_\psi^\dagger} \mathcal{A}_\psi = \epsilon^{-1}([V_\psi^0/Q]).$$

In view of (2.12), to show that $\mathcal{M}_\psi$ is equal to the left side above, it suffices to show that $[\{\psi\}/Q_\psi] \cong [V_\psi^0/Q]$, or equivalently,

(2.13)
The action map $\alpha : Q \to V_\psi^0$, $q \mapsto \mathrm{Ad}(q)\psi$, is smooth and surjective.

We first show that α is surjective on k-points. Let $\psi' \in \psi + \mathfrak{g}(\!(\tau)\!)_{\le (d-1)/m}$ be such that $f_i(\psi') = f_i(\psi)$ for all $1 \le i \le r$. We want to construct $q \in \mathbf{P}_\infty^+$ such that $\mathrm{Ad}(q)\psi - \psi' \in \mathfrak{g}(\!(\tau)\!)_{\le 0}$. The same argument as in the proof of Lemma 2.10.1 works to construct h_j inductively modulo $\mathbf{P}_\infty(\frac{j}{m})$ for $j = 1, 2, \cdots, d$ such that $\mathrm{Ad}(q)\psi - \psi' \in \mathfrak{g}(\!(\tau)\!)_{\le (d-j)/m}$. We omit details.

We then show that α is smooth. Since $(V_\psi^0)(k)$ is a single orbit of $Q(k)$, it suffices to show that the tangent map of α is surjective at 1. The tangent map of α at 1 is

$$[-, \psi] : \mathrm{Lie}\ Q = \bigoplus_{j=1}^{d-1} \mathfrak{g}(\!(\tau)\!)_{(j-d)/m} \to \bigoplus_{j=1}^{d-1} \mathfrak{c}(\!(\tau)\!)^\perp_{j/m},$$

which is surjective since ψ is regular semisimple. This finishes the proof of (2.13). The proposition is proved. □

2.13.2 Remark. One can show that $\mathcal{M}_\psi^\dagger$ is a smooth Poisson algebraic space, and $\mathcal{M}_\psi$ is a symplectic leaf in $\mathcal{M}_\psi^\dagger$. We omit the proof.

2.14. Proof of Theorem 2.8.1(5)

Assuming w is elliptic, we show that f is proper. By Proposition 2.13.1, it suffices to show that $f^\dagger$ is proper. We introduce a variant $\mathcal{M}^\ddagger$ of $\mathcal{M}_\psi^\dagger$: it classifies $(\mathcal{E}, \varphi)$ where $\mathcal{E}$ has $\mathbf{P}_\infty$ and $\mathbf{I}_0$-level structures, and the Higgs field is required to lie in $\mathfrak{g}(\!(\tau)\!)_{\le d/m} d\tau/\tau$ near ∞ such that its projection to $\mathfrak{g}(\!(\tau)\!)_{d/m}$ is regular semisimple, and in $\mathrm{Lie}\ \mathbf{I}_0^+ dt/t$ near 0 (after trivializations). Let $\widetilde{\mathcal{A}}^\ddagger$ be the affine space of $(a_i \in \Gamma(X, \mathcal{O}([dd_i/m]) \cdot \infty)_{1 \le i \le r}$ with the condition that $a_i(0) = 0$. Evaluating the leading coefficient at ∞ gives a map $\widetilde{\mathcal{A}}^\ddagger \to \mathfrak{g}_{d/m} /\!\!/ L_{\mathbf{P}_\infty}$, and let $\mathcal{A}^\ddagger \subset \widetilde{\mathcal{A}}^\ddagger$ be the preimage of $\mathfrak{g}_{d/m}^{\mathrm{rs}} /\!\!/ L_{\mathbf{P}_\infty}$ (where $L_{\mathbf{P}_\infty}$ is the Levi quotient of $\mathbf{P}_\infty$; it is identified the connected subgroup of G with Lie algebra $\mathfrak{g}_0$). We have the Hitchin fibration $f^\ddagger : \mathcal{M}^\ddagger \to \mathcal{A}^\ddagger$. Then the fiber product $\mathcal{M}^\ddagger \times_{\mathcal{A}^\ddagger} \mathcal{A}_\psi^\dagger$ admits a description that is almost identical to

$\mathcal{M}_\psi^\dagger$, except that the $\mathbf{P}_\infty^+$-level structure is replaced with the slightly larger level group $\mathbf{P}_\infty^+\mathbf{C}'_\infty$, where $\mathbf{C}'_\infty \subset \mathbf{P}_\infty$ is the centralizer of ψ in $\mathbf{P}_\infty$. Since $(\mathbf{P}_\infty^+\mathbf{C}'_\infty)/\mathbf{P}_\infty^+ \cong C_{L_{\mathbf{P}_\infty}}(\psi)$ is finite over k for w elliptic, $\mathcal{M}_\psi^\dagger \to \mathcal{M}^\ddagger \times_{\mathcal{A}^\ddagger} \mathcal{A}_\psi^\dagger$ is finite. Therefore, to show $f^\dagger$ is proper it suffices to show that $f^\ddagger$ is proper. Now $f^\ddagger$ is a parahoric Hitchin fibration, and [35, Proposition 6.3.7(2)] implies that $f^\ddagger$ is proper over the elliptic locus. However, we shall argue that the whole $\mathcal{A}^\ddagger$ consists of elliptic points. Indeed, the non-elliptic locus $Z \subset \mathcal{A}^\ddagger$ is closed and $\mathbb{G}_m(\nu)$-stable, so must contain a $\mathbb{G}_m(\nu)$-fixed point if non-empty. But $\mathcal{A}^{\ddagger,\mathbb{G}_m(\nu)}$ consists of images of ψ' for $\psi' \in \mathfrak{g}[t,t^{-1}]_{d/m}$, and they are all elliptic. □

2.15. Comparison of cohomology

The goal of this subsection is to prove Theorem 2.8.4 about the cohomology of $\mathcal{M}_\psi$ and Fl_ψ.

2.15.1. The situation We consider the following general situation. Let $\mathfrak{X}$ be an algebraic space locally of finite type over k, equipped with a $\mathbb{G}_m$-action. Let $\mathfrak{X}^{\mathbb{G}_m} = \coprod_{\alpha\in I} Z_\alpha$ be an open-closed decomposition of the fixed point locus.

In [14] Drinfeld introduces the attractor $\mathfrak{X}^+ := \mathrm{Map}_{\mathbb{G}_m}(\mathbb{A}^1, \mathfrak{X})$. It is equipped with two maps

$$\mathfrak{X}^{\mathbb{G}_m} \xleftarrow{\mathfrak{q}^+} \mathfrak{X}^+ \xrightarrow{\mathfrak{p}^+} \mathfrak{X}$$

Here $\mathfrak{p}^+$ (resp. $\mathfrak{q}^+$) is evaluation at $1 \in \mathbb{A}^1$ (resp. $0 \in \mathbb{A}^1$). It is shown in [14, Theorem 1.4.2] that $\mathfrak{X}^+$ is represented by an algebraic space of finite type over k, and $\mathfrak{q}^+$ is an affine morphism.

One defines the repeller $\mathfrak{X}^-$ to be the attractor for the inverted $\mathbb{G}_m$-action, and we have maps $\mathfrak{p}^- : \mathfrak{X}^- \to \mathfrak{X}$ and $\mathfrak{q}^- : \mathfrak{X}^- \to \mathfrak{X}^{\mathbb{G}_m}$.

For each $\alpha \in I$, let $\mathfrak{X}_\alpha^\pm = \mathfrak{q}^{\pm,-1}(Z_\alpha)$. Let $\mathfrak{p}_\alpha^\pm = \mathfrak{p}^\pm|_{\mathfrak{X}_\alpha^\pm}$; $\mathfrak{q}_\alpha^\pm = \mathfrak{q}^\pm|_{\mathfrak{X}_\alpha^\pm}$.

We make the following assumptions:

(1) For each $\alpha \in I$, $\mathfrak{q}_\alpha^+ : \mathfrak{X}_\alpha^+ \to \mathfrak{X}$ is a locally closed embedding.
(2) For each $\alpha \in I$, the reduced image of $\mathfrak{q}_\alpha^- : \mathfrak{X}_\alpha^- \to \mathfrak{X}$ is a locally closed subspace X_α^- of $\mathfrak{X}$, and the induced map $\mathfrak{X}_\alpha^- \to X_\alpha^-$ is a homeomorphism.
(3) $\cup_{\alpha\in I}\mathfrak{X}_\alpha^+ = \mathfrak{X}$, i.e., for any $x \in \mathfrak{X}$, the limit $\lim_{t\to 0} t\cdot x$ exists.
(4) There exists a partial order $\le$ on I such that for each $\alpha \in I$
   - The set $\{\alpha' \in I; \alpha' \le \alpha\}$ is finite.
   - $\mathfrak{X}_{\le\alpha}^+ := \cup_{\alpha'\le\alpha}\mathfrak{X}_{\alpha'}^+$ is open in $\mathfrak{X}$, and is of finite type over k.

- $X^-_{\leq\alpha} := \cup_{\alpha'\leq\alpha} X^-_\alpha$ is proper over k. In particular, each Z_α is proper over k.

The cohomology of a locally finite algebraic space is the projective limit of the cohomology of finite type open subspaces. Therefore

$$\mathrm{H}^*(\mathfrak{X}, \overline{\mathbb{Q}}_\ell) = \varprojlim_{\alpha\in I} \mathrm{H}^*(\mathfrak{X}^+_{\leq\alpha}, \overline{\mathbb{Q}}_\ell).$$

On the other hand, we define an ind-space

$$\mathfrak{Y} := \varinjlim_{\alpha\in I} X^-_{\leq\alpha}$$

as a union of finite type closed subspaces. We define the cohomology of $\mathfrak{Y}$ to be

$$\mathrm{H}^*(\mathfrak{Y}, \overline{\mathbb{Q}}_\ell) := \varprojlim_{\alpha\in I} \mathrm{H}^*(X^-_{\leq\alpha}, \overline{\mathbb{Q}}_\ell).$$

2.15.2 Proposition. *Under the above assumptions, the restriction maps* $\mathrm{H}^*(\mathfrak{X}, \overline{\mathbb{Q}}_\ell) \to \mathrm{H}^*(\mathfrak{Y}, \overline{\mathbb{Q}}_\ell)$ *and* $\mathrm{H}^*_{\mathbb{G}_m}(\mathfrak{X}, \overline{\mathbb{Q}}_\ell) \to \mathrm{H}^*_{\mathbb{G}_m}(\mathfrak{Y}, \overline{\mathbb{Q}}_\ell)$ *are isomorphisms.*

Proof. First note that $X^-_{\leq\alpha} \subset \mathfrak{X}^+_{\leq\alpha}$. Indeed, if $x \in X^-_{\leq\alpha}$, the limit $\lim_{t\to 0} t\cdot x$ exists in $X^-_{\leq\alpha}$ since $X^-_{\leq\alpha}$ is proper. But $(X^-_{\leq\alpha})^{\mathbb{G}_m} = \coprod_{\alpha'\leq\alpha} Z_{\alpha'}$, hence $\lim_{t\to 0} t\cdot x \in Z_{\alpha'}$ for some $\alpha' \leq \alpha$, therefore $x \in \mathfrak{X}^+_{\leq\alpha}$.

We extend $\leq$ to a total ordering on I and add the minimal element 0 to I. Let $\mathfrak{X}^+_0 = \mathfrak{X}^-_0 = X^-_0 = \varnothing$. We denote by $i^\pm_\alpha : Z_\alpha \hookrightarrow \mathfrak{X}^\pm_\alpha$ the inclusions. Let $s_\alpha : X^-_{\leq\alpha} \hookrightarrow \mathfrak{X}^+_{\leq\alpha}$ be the inclusion. If α' is the predecessor of α, let $\mathfrak{X}^\pm_{<\alpha} := \mathfrak{X}^\pm_{\leq\alpha'}$, $X^-_{<\alpha} := X^-_{\leq\alpha'}$ and let $s_{<\alpha} = s_{\alpha'}$.

We prove by induction on α that the restriction map $s^*_\alpha : \mathrm{H}^*(\mathfrak{X}^+_{\leq\alpha}) \to \mathrm{H}^*(X^-_{\leq\alpha})$ is an isomorphism, and the same is true for $\mathbb{G}_m$-equivariant cohomology. This would imply the proposition by taking projective limits. The case $\alpha = 0$ is clear.

Suppose the $s^*_{<\alpha}$ is an isomorphism, we show s^*_α is also an isomorphism. Consider the commutative diagram (for simplicity we have omitted most of the subscripts α)

$$\begin{tikzcd}
Z_\alpha \arrow[rr, "i^+"] \arrow[d, "i^-"'] & & \mathfrak{X}^+_\alpha \arrow[d, "k^+"] \\
\mathfrak{X}^-_\alpha \arrow[r, "j^-"] \arrow[rr, bend left, "k^-"] & X^-_{\leq\alpha} \arrow[r, "s_\alpha"] & \mathfrak{X}^+_{\leq\alpha} \\
& X^-_{<\alpha} \arrow[r] \arrow[u, "v"] & \mathfrak{X}^+_{<\alpha} \arrow[u, "u"']
\end{tikzcd}$$

Here, $i^+, i^-, k^+, s_\alpha, s_{<\alpha}, v$ are closed embeddings, j^- is the composition $\mathfrak{X}_\alpha^- \to X_\alpha^- \hookrightarrow X_{\le\alpha}^-$ hence a topological open embedding; u is an open embedding.

We consider the following diagram of distinguished triangles in $D^b_{\mathbb{G}_m}(X^+_{\le\alpha})$:

(2.14)
$$\begin{tikzcd} k_!^+ k^{+!}\overline{\mathbb{Q}}_\ell \arrow[r] \arrow[d, "\zeta"] & \overline{\mathbb{Q}}_\ell \arrow[r] \arrow[d] & u_* u^* \overline{\mathbb{Q}}_\ell \arrow[d] \\ k_!^- k^{-*}\overline{\mathbb{Q}}_\ell \arrow[r] & s_{\alpha *} s_\alpha^* \overline{\mathbb{Q}}_\ell \arrow[r] & s_{<\alpha *} s_{<\alpha}^* \overline{\mathbb{Q}}_\ell \end{tikzcd}$$

The rows are given by the open-closed decompositions $\mathfrak{X}^+_{\le\alpha} = \mathfrak{X}^+_{<\alpha} \cup \mathfrak{X}^+_\alpha$ and $X^-_{\le\alpha} = X^-_\alpha \cup X^-_{<\alpha}$ (by assumption $\mathfrak{X}^-_\alpha \to X^-_\alpha$ is a homeomorphism). The only map that requires explanation is ζ, which is the composition natural transformations

$$\begin{aligned} \zeta : k_*^+ k^{+!} &\to s_{\alpha *} s_\alpha^* k_*^+ k^{+,!} \xrightarrow{\xi} s_{\alpha *} h_*^- i^{+*} k^{+!} \xrightarrow{\rho} s_{\alpha *} h_*^- i^{-!} k^{-*} \\ &= s_{\alpha *} j_!^- i_*^- i^{-!} j^{-*} s_\alpha^* \to s_{\alpha *} j_!^- j^{-*} s_\alpha^* = k_!^- k^{-*}. \end{aligned}$$

Here $\xi : s_\alpha^* k_*^+ \to h_*^- i^{+*}$ is given by $k^+ \circ i^+ = s_\alpha \circ h^-$ and adjunction; $\rho : i^{+*} k^{+!} \to i^{-!} k^{-*}$ is the comparison map of hyperbolic localization functors, see Braden [11, top of page 212]. The commutativity of (2.14) is a diagram chase that we omit.

Taking global sections of (2.14) we get a map between long exact sequences of cohomology groups

$$\begin{tikzcd} \cdots \arrow[r] & \mathrm{H}^*(\mathfrak{X}^+_\alpha, k^{+!}\overline{\mathbb{Q}}_\ell) \arrow[r] \arrow[d, "\mathrm{H}^*(\zeta)"] & \mathrm{H}^*(\mathfrak{X}^+_{\le\alpha}) \arrow[r] \arrow[d, "s_\alpha^*"] & \mathrm{H}^*(\mathfrak{X}^+_{<\alpha}) \arrow[r] \arrow[d, "s_{<\alpha}^*"] & \cdots \\ \cdots \arrow[r] & \mathrm{H}^*_c(\mathfrak{X}^-_\alpha, k^{-*}\overline{\mathbb{Q}}_\ell) \arrow[r] & \mathrm{H}^*(X^-_{\le\alpha}) \arrow[r] & \mathrm{H}^*(X^-_{<\alpha}) \arrow[r] & \cdots \end{tikzcd}$$

By induction hypothesis, $s_{<\alpha}^*$ is an isomorphism, therefore to show s_α^* is an isomorphism, it suffices to show $\mathrm{H}^*(\zeta)$ is. Since $\mathfrak{X}^+_\alpha$ contracts to Z_α under $\mathfrak{q}^+_\alpha : \mathfrak{X}^+_\alpha \to Z_\alpha$, the contraction principle gives an isomorphism $\mathfrak{q}^+_{\alpha *} k^{+!}\overline{\mathbb{Q}}_\ell \cong i^{+*} k^{+!}\overline{\mathbb{Q}}_\ell$. Taking global sections we get

$$\mathrm{H}^*(\mathfrak{X}^+_\alpha, k^{+!}\overline{\mathbb{Q}}_\ell) \cong \mathrm{H}^*(Z_\alpha, i^{+*} k^{+!}\overline{\mathbb{Q}}_\ell).$$

Similarly, using the contraction $\mathfrak{q}^-_\alpha : \mathfrak{X}^-_\alpha \to Z_\alpha$ we have an isomorphism $\mathfrak{q}^-_{\alpha !} k^{-*}\overline{\mathbb{Q}}_\ell \cong i^{-!} k^{-*}\overline{\mathbb{Q}}_\ell$; taking global sections with compact support and using that Z_α is proper, we get

$$\mathrm{H}^*_c(\mathfrak{X}^-_\alpha, k^{-*}\overline{\mathbb{Q}}_\ell) \cong \mathrm{H}^*(Z_\alpha, i^{-!}k^{-*}\overline{\mathbb{Q}}_\ell).$$

Under these isomorphisms, the induced map on cohomology by ζ is the comparison map of hyperbolic localizations to Z_α:

$$\mathrm{H}^*(\zeta) = \mathrm{H}^*(\rho) : \mathrm{H}^*(Z_\alpha, i^{+*}k^{+!}\overline{\mathbb{Q}}_\ell) \to \mathrm{H}^*(Z_\alpha, i^{-!}k^{-*}\overline{\mathbb{Q}}_\ell).$$

By Braden's theorem [11, Theorem 1] and its extension to algebraic spaces by Drinfeld-Gaitsgory [15, Theorem 3.1.6], $\mathrm{H}^*(\rho)$ is an isomorphism. This shows $\mathrm{H}^*(\zeta)$ is an isomorphism, hence s^*_α is an isomorphism.

Taking $\mathbb{G}_m$-equivariant global sections of (2.14), the same argument proves that s^*_α is an isomorphism on $\mathbb{G}_m$-equivariant cohomology. □

2.15.3. Attractors and repellers for $\boldsymbol{\mathcal{M}_\psi}$ We would like to apply the above discussions to $\mathcal{M}_\psi$ with the $\mathbb{G}_m(\nu)$-action. For this we collect some facts about its attractors and repellers.

Since the action of $\mathbb{G}_m(\nu)$ on $\mathcal{A}_\psi$ contracts to the point a_ψ, the $\mathbb{G}_m(\nu)$-fixed points $\mathcal{M}_\psi^{\mathbb{G}_m(\nu)}$ necessarily lie in the central fiber $\mathcal{M}_{a_\psi}$, hence homeomorphic to $\mathrm{Fl}_\psi^{\mathbb{G}_m(\nu)}$ by Theorem 2.8.1(4).

Recall $\mathbf{P}_0 \subset G(\!(t)\!)$ is the parahoric subgroup whose Lie algebra is $\mathfrak{g}(\!(t)\!)_{\geq 0}$. By [27, §5.1], the $\mathbf{P}_0$-orbits on Fl are parametrized by $W_{\mathbf{P}}\backslash\widetilde{W}$, where $W_{\mathbf{P}}$ is the Weyl group of the Levi $L_{\mathbf{P}}$ of $\mathbf{P}_\infty$. For $w \in W_{\mathbf{P}}\backslash\widetilde{W}$, let $\mathrm{Fl}_w \subset \mathrm{Fl}$ be the $\mathbf{P}_0$-orbit containing any lifting of w. Then $W_{\mathbf{P}}\backslash\widetilde{W}$ is equipped with the partial order $\leq$ such that $w \leq w'$ if and only if $\mathrm{Fl}_w \subset \overline{\mathrm{Fl}_{w'}}$. This is the partial order induced from the Bruhat order on $\widetilde{W}$, if we identify $W_{\mathbf{P}}\backslash\widetilde{W}$ as a subset of $\widetilde{W}$ using longest representatives. Let $\mathrm{Fl}_{w,\psi} = \mathrm{Fl}_w \cap \mathrm{Fl}_\psi$. Denote by ${}^w\mathbf{I}_0$ (resp. ${}^w(\mathrm{Lie}\ \mathbf{I}_0^+)$) the conjugate of $\mathbf{I}_0$ (resp. $\mathrm{Lie}\ \mathbf{I}_0^+$) by any lift of $w \in \widetilde{W}$.

Consider the action of $\mathbb{G}_m(\nu)$ on Fl given in §2.7.1 that stabilizes Fl_ψ. Since the affine roots $\alpha + n\delta$ in $\mathbf{P}_0$ are those with $\alpha(\xi/m) + n \geq 0$, and $\alpha(\xi/m)+n = 0$ if and only if $\alpha+n\delta$ is a root of the Levi $L_{\mathbf{P}}$, the fixed points $\mathrm{Fl}^{\mathbb{G}_m(\nu)}$ is the disjoint union of $L_{\mathbf{P}}w\mathbf{I}_0/\mathbf{I}_0$ for $w \in W_{\mathbf{P}}\backslash\widetilde{W}$. Therefore $\mathrm{Fl}_\psi^{\mathbb{G}_m(\nu)}$ admits a decomposition

$$\mathrm{Fl}_\psi^{\mathbb{G}_m(\nu)} = \coprod_{w \in W_{\mathbf{P}}\backslash\widetilde{W}} \mathcal{H}_\psi(w), \quad \mathcal{H}_\psi(w) = L_{\mathbf{P}}w\mathbf{I}_0/\mathbf{I}_0 \cap \mathrm{Fl}_\psi. \tag{2.15}$$

If we choose a representative $\widetilde{w} \in \widetilde{W}$ of w, then $\mathcal{H}_\psi(w)$ is isomorphic to a Hessenberg variety for $L_{\mathbf{P}}$ defined using the $L_{\mathbf{P}}$-module $\mathfrak{g}(\!(t)\!)_{d/m}$ and its subspace ${}^{\widetilde{w}}(\mathrm{Lie}\ \mathbf{I}_0^+) \cap \mathfrak{g}(\!(t)\!)_{d/m}$:

$$\mathcal{H}_\psi(w) \cong \{h \in L_{\mathbf{P}}/(L_{\mathbf{P}} \cap {}^{\widetilde{w}}\mathbf{I}_0) | \mathrm{Ad}(h^{-1})(\psi) \in {}^{\widetilde{w}}(\mathrm{Lie}\ \mathbf{I}_0^+) \cap \mathfrak{g}(\!(t)\!)_{d/m}\}.$$

Let $\gamma : \mathrm{Fl}_\psi \to \mathcal{M}_\psi$ be the canonical map.

We shall use the notations from §2.15.1. For $w \in W_{\mathbf{P}} \backslash \widetilde{W}$, let $Z_w = \gamma(\mathcal{H}_\psi(w))$ (it is open-closed in $\mathcal{M}_\psi^{\mathbb{G}_m}$). Let $\mathfrak{X}_w^+$ and $\mathfrak{X}_w^-$ be the attractor and repeller of Z_w, and we have maps $\mathfrak{p}_w^\pm : \mathfrak{X}_w^\pm \to \mathcal{M}_\psi$ and $\mathfrak{q}_w^\pm : \mathfrak{X}_w^\pm \to Z_w$.

2.15.4 Lemma. *The map $\gamma_w = \gamma|_{\mathcal{H}_\psi(w)} : \mathcal{H}_\psi(w) \to Z_w$ is an isomorphism.*

Proof. Since $\mathcal{M}_\psi$ is smooth, so is Z_w by [14, Prop. 1.4.20]. It is well-known that $\mathcal{H}_\psi(w)$ is smooth. The map γ_w is a homeomorphism between smooth spaces, hence an isomorphism. □

2.15.5 Lemma. *The image of the map $\mathfrak{p}_w^- : \mathfrak{X}_w^- \to \mathcal{M}_\psi$ is the locally closed subspace $X_w^- := \gamma(\mathrm{Fl}_{w,\psi})$. There is a unique isomorphism $\mathfrak{X}_w^- \cong \mathrm{Fl}_{w,\psi}$ compatible with the maps $\mathfrak{p}_w^-$ and γ. In particular, $\mathfrak{p}_w^-$ is a homeomorphism onto its image $X_w^- = \gamma(\mathrm{Fl}_{w,\psi})$.*

Proof. We first show that the image of $\mathfrak{p}_w^-$ is $X_w^- = \gamma(\mathrm{Fl}_{w,\psi})$ (with the reduced structure). Since the action of $\mathbb{G}_m(\nu)$ is contracting to a_ψ, if $\lim_{s\to\infty} s \cdot (\mathcal{E}, \varphi)$ exists, $f(\mathcal{E}, \varphi)$ must be equal to a_ψ, i.e., $(\mathcal{E}, \varphi) \in \mathcal{M}_{a_\psi}$. Now we can identify $(\mathcal{E}, \varphi)$ with a geometric point $g\mathbf{I}_0 \in \mathrm{Fl}_\psi$ under γ. Since the inverted $\mathbb{G}_m(\nu)$-action $s \cdot g = \mathrm{rot}(s^m)\mathrm{Ad}(\xi(s))g$ contracts $\mathbf{P}_0$ to $L_{\mathbf{P}}$, $\mathrm{Fl}_w = \mathbf{P}_0 w \mathbf{I}_0/\mathbf{I}_0$ is the repelling subscheme of $L_{\mathbf{P}} w \mathbf{I}_0/\mathbf{I}_0$ in Fl. Therefore, $\lim_{s\to\infty} s \cdot g\mathbf{I}_0 \in \mathcal{H}_\psi(w)$ if and only if $g\mathbf{I}_0 \in \mathrm{Fl}_{w,\psi}$.

From the above we also see that each geometric point of $\mathcal{M}_{a_\psi}$ has a unique limit point under the action of $s \in \mathbb{G}_m(\nu), s \to \infty$. By [14, Prop. 1.4.11(i)], $\mathfrak{p}_w^- : \mathfrak{X}_w^- \to \mathcal{M}_\psi$ is unramified with image X_w^-. The uniqueness of limit points as $s \to \infty$ implies that $\mathfrak{p}_w^-$ is a monomorphism (geometric fibers are a reduced singleton).

Since $\mathcal{M}_\psi$ is smooth, by [14, Prop. 1.4.20], the map $\mathfrak{q}_w^- : \mathfrak{X}_w^- \to Z_w$ is smooth. By [20, §4.5], the contraction map $q : \mathrm{Fl}_{w,\psi} \to \mathcal{H}_\psi(w)$ is an iterated affine space bundle, hence also smooth. Therefore the homeomorphism $\mathrm{Fl}_{w,\psi} \to X_w^-$ is the normalization map. The map $\mathfrak{p}_w^- : \mathfrak{X}_w^- \to X_w^-$ thus uniquely lifts to $\widetilde{p} : \mathfrak{X}_w^- \to \mathrm{Fl}_{w,\psi}$. We have a commutative diagram

$$\begin{tikzcd} \mathfrak{X}_w^- \arrow[r, "\widetilde{p}"] \arrow[d, "\mathfrak{q}_w^-"] & \mathrm{Fl}_{w,\psi} \arrow[d, "q"] \\ Z_w \arrow[r, "\sim"] & \mathcal{H}_\psi(w) \end{tikzcd}$$

The bottom map is an isomorphism by Lemma 2.15.4. We claim that $\widetilde{p}$ is an isomorphism.

Note that $\widetilde{p}$ is a monomorphism because $\mathfrak{q}_w^-$ is; it is also surjective. Replacing Z_w by its connected components, and taking preimages in $\mathfrak{X}_w^-$ and $\mathrm{Fl}_{w,\psi}$, we may assume Z_w is connected. In this case, $\mathfrak{X}_w^-$ is connected because every point of it contracts to Z_w under $s \to \infty$. Now $\widetilde{p}$ is a monomorphic surjection between two smooth connected spaces. By considering differentials we conclude that $\widetilde{p}$ is an isomorphism. □

On the other hand, by [16, Cor. 16] and [27, §5.2] the isomorphism classes of $\mathrm{Bun}_G(\mathbf{P}_\infty, \mathbf{I}_0)$ are also indexed by $W_{\mathbf{P}}\backslash\widetilde{W}$. Denote the locally closed substack with isomorphism class w by $\mathrm{Bun}_G^w(\mathbf{P}_\infty, \mathbf{I}_0)$. By [16, §3] $\mathrm{Bun}_G^w(\mathbf{P}_\infty, \mathbf{I}_0) \subset \overline{\mathrm{Bun}_G^{w'}(\mathbf{P}_\infty, \mathbf{I}_0)}$ if and only if $w \geq w'$ under the partial order already defined on $W_{\mathbf{P}}\backslash\widetilde{W}$.

Let $\omega : \mathcal{M}_\psi \to \mathrm{Bun}_G(\mathbf{P}_\infty, \mathbf{I}_0)$ be the forgetful map.

2.15.6 Lemma. *For any $(\mathcal{E}, \varphi) \in \mathcal{M}_\psi$, the limit $\lim_{s\to 0} s \cdot (\mathcal{E}, \varphi)$ exists in $\mathcal{M}_\psi^{\mathbb{G}_m(\nu)}$. Moreover, the map $\mathfrak{p}_w^+ : \mathfrak{X}_w^+ \to \mathcal{M}_\psi$ is a locally closed embedding whose image is $\omega^{-1}(\mathrm{Bun}_G^w(\mathbf{P}_\infty, \mathbf{I}_0))$.*

Proof. We first show the limit $\lim_{s\to 0} s \cdot (\mathcal{E}, \varphi)$ always exists. Consider the uniformization map $u : \mathrm{Fl} = G((t))/\mathbf{I}_0 \to \mathrm{Bun}_G(\mathbf{K}_\infty, \mathbf{I}_0)$. We think of Fl as classifying a $\mathcal{E}$-torsor on X with $\mathbf{K}_\infty$-level at ∞ and $\mathbf{I}_0$-level structure at 0 and a trivialization on $X\backslash\{0\}$. Let $\mathcal{N}$ be the base change of $\mathcal{M}_\psi$ along u. Then $\mathcal{N}$ can be described as the moduli space of pairs $(g\mathbf{I}_0, \theta)$ where $g\mathbf{I}_0 \in \mathrm{Fl}$ and $\theta \in \mathfrak{g}[t, t^{-1}]_{\leq 0}$ such that

$$\mathrm{Ad}(g^{-1})(\psi + \theta)dt/t \in \mathrm{Lie}\ \mathbf{I}_0^+ dt/t.$$

Indeed, given $(g\mathbf{I}_0, \theta)$, we define $\mathcal{E}$ as the image of $g\mathbf{I}_0$ under u, equipped with the Higgs field $\varphi = (\psi + \theta)dt/t$ over $X\backslash\{0\}$.

The action of $\mathbb{G}_m(\nu)$ on $\mathcal{M}_\psi$ lifts to $\mathcal{N}$, and is given by

$$s \cdot (g\mathbf{I}_0, \theta) = (\mathrm{rot}(s^{-m})\mathrm{Ad}(\xi(s^{-1}))g\mathbf{I}_0, s^d \mathrm{rot}(s^{-m})\mathrm{Ad}(\xi(s^{-1}))\theta).$$

Since $\theta \in \mathfrak{g}[t, t^{-1}]_{\leq 0}$, $\lim_{s\to 0} s^d \mathrm{rot}(s^{-m})\mathrm{Ad}(\xi(s^{-1}))\theta = 0$. On the other hand, since Fl is ind-proper, $\lim_{s\to 0}(\mathrm{rot}(s^{-m})\mathrm{Ad}(\xi(s^{-1}))g)\mathbf{I}_0 \in \mathrm{Fl}^{\mathbb{G}_m(\nu)}$ exists. This shows that $\lim_{s\to 0} s\cdot(g\mathbf{I}_0, \theta)$ exists for any point $(g\mathbf{I}_0, \theta) \in \mathcal{N}$. Since $\mathcal{N} \to \mathcal{M}_\psi$ is surjective and $\mathbb{G}_m(\nu)$-equivariant, the same is true for $\mathcal{M}_\psi$.

We next show that the image of $\mathfrak{p}_w^+$ is $\omega^{-1}(\mathrm{Bun}_G^w(\mathbf{P}_\infty, \mathbf{I}_0))$. In other words, for a geometric point $(\mathcal{E}, \varphi)$ of $\mathcal{M}_\psi$, $\lim_{s\to 0} s \cdot (\mathcal{E}, \varphi) \in \gamma(\mathcal{H}_\psi(w))$ if and only if the image of $\mathcal{E}$ in $\mathrm{Bun}_G(\mathbf{P}_\infty, \mathbf{I}_0)$ is the point w. Let $(g\mathbf{I}_0, \theta) \in \mathcal{N}$

be a preimage of $(\mathcal{E},\varphi)$. Then $\lim_{s\to 0} s\cdot(\mathcal{E},\varphi)\in\gamma(\mathcal{H}_\psi(w))$ if and only if $\lim_{s\to 0}\mathrm{rot}(s^{-m})\mathrm{Ad}(\xi(s^{-1}))g\mathbf{I}_0\in L_{\mathbf{P}}w\mathbf{I}_0/\mathbf{I}_0$. Now $\mathrm{Fl}^{\mathbb{G}_m(\nu)}=\coprod_{w\in W_{\mathbf{P}}\backslash\widetilde{W}} L_{\mathbf{P}}w\mathbf{I}_0/\mathbf{I}_0$. Note that $\mathrm{Bun}_G(\mathbf{P}_\infty,\mathbf{I}_0)=\Gamma_\infty\backslash\mathrm{Fl}$ where $\Gamma_\infty=\mathbf{P}_\infty\cap G[t,t^{-1}]$, and the action of $\mathrm{rot}(s^{-m})\mathrm{Ad}(\xi(s^{-1}))$ contracts the group Γ_∞ to the Levi $L_{\mathbf{P}}$. Therefore, the limit point $\lim_{s\to 0}\mathrm{rot}(s^{-m})\mathrm{Ad}(\xi(s^{-1}))g\mathbf{I}_0\in L_{\mathbf{P}}w\mathbf{I}_0/\mathbf{I}_0$ if and only if $g\mathbf{I}_0$ lies in the Γ_∞-orbit of w, i.e., the image of $g\mathbf{I}_0$ (or $\mathcal{E}$) in $\mathrm{Bun}_G(\mathbf{P}_\infty,\mathbf{I}_0)$ is in $\mathrm{Bun}_G^w(\mathbf{P}_\infty,\mathbf{I}_0)$.

So far we have shown that $\mathfrak{p}_w^+$ induces a map $\widetilde{p}:\mathfrak{X}_w^+\to\Omega_w:=\omega^{-1}(\mathrm{Bun}_G^w(\mathbf{P}_\infty,\mathbf{I}_0))$. Finally we show that $\widetilde{p}$ is an isomorphism, therefore $\mathfrak{p}_w^+$ is a locally closed embedding. In Lemma 2.15.7 below, we construct a smooth map $\gamma:\Omega_w\to\mathcal{H}_\psi(w)$ such that the following diagram is commutative

$$\begin{tikzcd} \mathfrak{X}_w^+ \arrow[r, "\widetilde{p}"] \arrow[d, "\mathfrak{q}_w^+"'] & \Omega_w \arrow[d, "\gamma"] \\ Z_w \arrow[r, "\sim"] & \mathcal{H}_\psi(w) \end{tikzcd} \tag{2.16}$$

By [14, Proposition 1.4.20], $\mathfrak{q}_w^+$ is smooth with connected fibers (since its contracting to Z_w). On the other hand γ is also smooth. Moreover, by [14, Proposition 1.4.11(i)], $\widetilde{p}$ is unramified. In our situation $\widetilde{p}$ is a monomorphism and a surjection, therefore the. By comparing relative differentials of $\mathfrak{q}_w^+$ and γ, we conclude that $\widetilde{p}$ is an isomorphism. $\square$

2.15.7 Lemma. *There is a smooth map $\gamma:\Omega_w\to\mathcal{H}_\psi(w)$ making (2.16) commutative.*

Proof. To see this we first construct a smooth map $\gamma:\Omega_w\to\mathcal{H}_\psi(w)$. Let $H=\mathbf{P}_\infty/\mathbf{P}_\infty(d/m)$, $H^+=\mathbf{P}_\infty^+/\mathbf{P}_\infty(d/m)$ which acts on $V=\oplus_{i=1}^d\mathfrak{g}(\!(\tau)\!)_{i/m}$. Let $V_w=V\cap\mathrm{Ad}(w)(\mathrm{Lie}\,(G[t]\cap\mathbf{I}_0^+))$ and $\widetilde{V}_w=\mathfrak{g}(\!(\tau)\!)_{\leq d/m}\cap\mathrm{Ad}(w)(\mathrm{Lie}\,(G[t]\cap\mathbf{I}_0^+))$. Let

$$\widehat{\Omega}_w=\{(h,v)\in H\times\widetilde{V}_w|\mathrm{Ad}(h)v\in\psi+\mathfrak{g}(\!(\tau)\!)_{\leq 0}\}.$$

Let $A_w=\mathbf{P}_\infty\cap\mathrm{Ad}(w)(G[t]\cap\mathbf{I}_0)$, which is the automorphism group of the point w in $\mathrm{Bun}_G(\mathbf{P}_\infty,\mathbf{I}_0)$. It acts on $\widehat{\Omega}_w$ on the right by $(h,v)\cdot a=(ha,\mathrm{Ad}(a^{-1})v)$. Let ψ_V be ψ viewed as an element in V, and $C_{H^+}(\psi)$ be its stabilizer under H^+. Then $C_{H^+}(\psi)$ acts on $\widehat{\Omega}_w$ by left translation on $h\in H$. Then there is an isomorphism

$$C_{H^+}(\psi)\backslash\widehat{\Omega}_w/A_w\cong\Omega_w.$$

This identification follows using [16, Cor. 16] restricted over $\mathrm{Bun}_G^w(\mathbf{P}_\infty,\mathbf{I}_0)$.

Let $\widetilde{\Omega}_w = \{h \in H | \mathrm{Ad}(h^{-1})\psi_V \in V_w\}$. There is a natural map $\beta : \widehat{\Omega}_w \to \widetilde{\Omega}_w$ (sending (h, v) to h) is an affine space bundle with fibers isomorphic to $\mathfrak{g}(\!(\tau)\!)_{\leq 0} \cap \mathrm{Ad}(w)(\mathrm{Lie}\ (G[t] \cap \mathbf{I}_0^+))$. In particular, β is smooth.

Let $V_{w,i/m} = \mathfrak{g}(\!(\tau)\!)_{i/m} \cap \mathrm{Ad}(w)(\mathrm{Lie}\ (G[t] \cap \mathbf{I}_0^+))$. Let $\widetilde{\mathcal{H}}_w = \{\ell \in L_{\mathbf{P}} | \mathrm{Ad}(\ell^{-1})\psi \in V_{w,d/m}\}$. Then $\widetilde{\mathcal{H}}_w \to \mathcal{H}_\psi(w)$ is a torsor under $B_w = L_{\mathbf{P}} \cap \mathrm{Ad}(w)(G[t] \cap \mathbf{I}_0)$ (a Borel subgroup of $L_{\mathbf{P}}$). We have a map $\alpha : \widetilde{\Omega}_w \to \widetilde{\mathcal{H}}_w$ sending $h \in H$ to its image in $L_{\mathbf{P}}$. Note that B_w is a quotient of A_w, and the composition $\alpha\beta : \widehat{\Omega}_w \to \widetilde{\mathcal{H}}_w$ is $C_{H^+}(\psi)$ invariant and A_w-equivariant (via the quotient B_w on $\widetilde{\mathcal{H}}_w$). Therefore $\alpha\beta$ induces a map $\gamma : \Omega_w \to \mathcal{H}_\psi(w)$ by passing to the quotients. To summarize, we have a commutative diagram

$$\begin{tikzcd}
\widehat{\Omega}_w \arrow[r, "\beta"] \arrow[d, "\pi"] & \widetilde{\Omega}_w \arrow[r, "\alpha"] & \widetilde{\mathcal{H}}_w \arrow[d, "\pi'"] \\
\Omega_w \arrow[rr, "\gamma"] & & \mathcal{H}_\psi(w)
\end{tikzcd}$$

where π, π' and β are smooth. To show γ is smooth, it suffices to show that α is smooth.

We have a commutative diagram

$$\begin{tikzcd}
H \arrow[r, "\theta"] \arrow[d, "\widetilde{\alpha}"] & V/V_w \arrow[d, "p"] \\
L_{\mathbf{P}} \arrow[r, "\eta"] & \mathfrak{g}(\!(\tau)\!)_{d/m}/V_{w,d/m}
\end{tikzcd} \tag{2.17}$$

Here $\widetilde{\alpha}$ and p are the projections, $\theta(h) = \mathrm{Ad}(h^{-1})\psi_V \mod V_w$; $\eta(\ell) = \mathrm{Ad}(\ell^{-1})\psi \mod V_{w,d/m}$. By definition, $\widetilde{\Omega}_w = \theta^{-1}(0)$, $\widetilde{\mathcal{H}}_w = \eta^{-1}(0)$ and α is the restriction of $\widetilde{\alpha}$. Let $h \in \widetilde{\Omega}_w$ with image $\overline{h} = \widetilde{\alpha}(h) \in L_{\mathbf{P}}$. The tangent maps of (2.17) at h are given by

$$\begin{tikzcd}
\mathfrak{h} = \oplus_{i=0}^{d-1}\mathfrak{g}(\!(\tau)\!)_{-i/m} \arrow[r, "T_h\theta = {[-,\mathrm{Ad}(h^{-1})\psi_V]}"] \arrow[d] & V/V_w \arrow[d] \\
\mathfrak{l} = \mathfrak{g}(\!(\tau)\!)_0 \arrow[r, "T_{\overline{h}}\eta = {[-,\mathrm{Ad}(\overline{h}^{-1})\psi]}"] & \mathfrak{g}(\!(\tau)\!)_{d/m}/V_{w,d/m}
\end{tikzcd}$$

We need to show that $T_h\theta$ is surjective (which implies that $\widetilde{\Omega}_w$ is smooth at h) and that the induced map $T_h\Omega_w = \ker(T_h\theta) \to \ker(T_{\overline{h}}\eta) = T_{\overline{h}}\widetilde{\mathcal{H}}_w$ is surjective. Consider the filtrations $F_{-i}\mathfrak{h} = \oplus_{i \leq i' \leq (d-1)}\mathfrak{g}(\!(\tau)\!)_{-i'/m}$, $F_i(V/V_w) =$

$\oplus_{1\le i'\le i}\mathfrak{g}(\!(\tau)\!)_{i'/m}/V_{w,i'/m}$. Then $T_h\theta$ sends $F_{-i}\mathfrak{h}$ to $F_{d-i}(V/V_w)$, and the induced map on the associated graded is

$$\mathrm{Gr}_{-i}(T_h\theta) : F_{-i}\mathfrak{h} = \mathfrak{g}(\!(\tau)\!)_{-i/m} \xrightarrow{[-,\psi']} F_{d-i}(V/V_w) = \mathfrak{g}(\!(\tau)\!)_{(d-i)/m}/V_{w,(d-i)/m}$$

where $\psi' = \mathrm{Ad}(\overline{h}^{-1})\psi \in \mathfrak{g}(\!(\tau)\!)_{d/m}$. In particular, $T_{\overline{h}}\eta = \mathrm{Gr}_0(T_h\theta)$. It remains to show that each $\mathrm{Gr}_{-i}(T_h\theta)$ is surjective, for $0 \le i \le d-1$.

Passing to the dual spaces and using the Killing form to identify $\mathfrak{g}(\!(\tau)\!)_{i/m}$ with the dual of $\mathfrak{g}(\!(\tau)\!)_{-i/m}$, we reduce to showing that

$$(\mathrm{Gr}_{-i}(T_h\theta))^* : \mathfrak{g}(\!(\tau)\!)_{(i-d)/m} \cap \mathrm{Ad}(w)(\mathfrak{n} + t\mathfrak{g}[t]) \xrightarrow{[-,\psi']} \mathfrak{g}(\!(\tau)\!)_{i/m}$$

is injective for $0 \le i \le d-1$. Let $\mathfrak{c}'(\!(\tau)\!) \subset \mathfrak{g}(\!(\tau)\!)$ be the centralizer of ψ'. This is a Cartan subalgebra of $\mathfrak{g}(\!(\tau)\!)$ with the induced grading $\mathfrak{c}'(\!(\tau)\!)_{j/m} \subset \mathfrak{g}(\!(\tau)\!)_{j/m}$. Then the kernel of $(\mathrm{Gr}_{-i}(T_h\theta))^*$ is $\mathfrak{c}'(\!(\tau)\!)_{(i-d)/m} \cap \mathrm{Ad}(w)(\mathfrak{n} + \tau^{-1}\mathfrak{g}[\tau^{-1}])$. Let $x \in \mathfrak{c}'(\!(\tau)\!)_{(i-d)/m} \cap \mathrm{Ad}(w)(\mathfrak{n} + t\mathfrak{g}[t])$ and let $f \in k[\mathfrak{g}]^G$ be a homogeneous invariant polynomial of positive degree. Since $i < d$, $x \in \mathfrak{g}(\!(\tau)\!)_{<0}$, $f(x) \in \tau k[\![\tau]\!]$; since $x \in \mathrm{Ad}(w)(\mathrm{Lie}\ (G[t] \cap \mathbf{I}_0^+))$, $f(x) \in tk[t]$. Therefore $f(x) = 0$ for all any non-constant homogeneous $f \in k[\mathfrak{g}]^G$, hence x is nilpotent. Since $\mathfrak{c}'(\!(\tau)\!)$ consists of semisimple elements only, x must be 0. This shows $(\mathrm{Gr}_{-i}(T_h\theta))^*$ is injective and $\mathrm{Gr}_{-i}(T_h\theta)$ is surjective, for all $i < d$. This concludes the proof that α is smooth. By previous discussion, it follows that γ is also smooth.

The commutativity of (2.16) is clear by checking geometric points (at which level $\widetilde{p}$ is a bijection). □

2.15.8. Proof of Theorem 2.8.4 We need to verify that the $\mathbb{G}_m(\nu)$-action on $\mathfrak{X} = \mathcal{M}_\psi$ satisfies the conditions in §2.15.1. We use the decomposition (2.15) for the fixed point subspace, and the partial order $W_{\mathbf{P}}\backslash\widetilde{W}$ defined in §2.15.3. We use notations $\mathfrak{X}^{\pm}_w$ for attractors and repellers.

Condition (1) and (3) follow from Lemma 2.15.6.

Condition (2) follows from Lemma 2.15.5.

Condition (4). Since $\le$ on $W_{\mathbf{P}}\backslash\widetilde{W}$ is defined to be the closure order of $\mathbf{P}_0$-orbits on Fl, we have $X^-_{\le w} = \gamma(\overline{\mathrm{Fl}_w} \cap \mathrm{Fl}_\psi)$ is proper since $\overline{\mathrm{Fl}_w}$ is. This also shows that $\{w'; w' \le w\}$ is finite. On the other hand, $\mathrm{Bun}_G^{\le w}(\mathbf{P}_\infty, \mathbf{I}_0) = \cup_{w'\le w}\mathrm{Bun}_G^{w'}(\mathbf{P}_\infty, \mathbf{I}_0)$ is open in $\mathrm{Bun}_G(\mathbf{P}_\infty, \mathbf{I}_0)$, therefore its preimage $\mathfrak{X}^+_{\le w}$ in $\mathcal{M}_\psi$ is open. This verifies all conditions so Proposition 2.15.2 applies. □

2.16. A further Hamiltonian reduction

We introduce a variant $\mathcal{M}^{\flat}_{\psi}$ of $\mathcal{M}_{\psi}$ by making a Hamiltonian reduction with respect to the torus $C_0 := \mathbf{C}_\infty/\mathbf{C}^+_\infty \cong T^{w,\circ}$. We define $\mathcal{M}^{\flat}_{\psi}$ to be the moduli stack of pairs $(\mathcal{E}, \varphi)$ where

- $\mathcal{E}$ is a G-bundle over X with $\mathbf{P}_\infty(\frac{d}{m})\mathbf{C}_\infty$-level structure at ∞ and $\mathbf{I}_0$-level structure at 0.
- φ is a section of $\mathrm{Ad}(\mathcal{E}) \otimes \omega_{X\backslash\{0,\infty\}}$ such that under some (equivalently any) trivialization of $\mathcal{E}|_{D_\infty}$ together with its $\mathbf{P}_\infty(\frac{d}{m})\mathbf{C}_\infty$-level structure, we have
$$\varphi|_{D^\times_\infty} \in (\psi + \mathfrak{c}(\!(\tau)\!)^{\perp}_{\leq 0} + \mathfrak{c}(\!(\tau)\!)_{<0})d\tau/\tau.$$
and, under some (equivalently, any) trivialization of $\mathcal{E}|_{D_0}$ together with its $\mathbf{I}_0$-level structure, $\varphi|_{D^\times_0} \in \mathrm{Lie}\ (\mathbf{I}^+_0)dt/t$.

There is a Hamiltonian action of $C_0 = \mathbf{C}_\infty/\mathbf{C}^+_\infty$ on $\mathcal{M}_\psi$ since $\mathbf{P}_\infty(\frac{d}{m})\mathbf{C}_\infty$ normalizes $\mathbf{K}_\infty$. The moment map for this action is taking residue at ∞

$$\mathrm{res}_\infty : \mathcal{M}_\psi \to \mathfrak{c}_0 = \mathfrak{c}(\!(\tau)\!)_0 = \mathrm{Lie}\ C_0$$

defined as follows: for $(\mathcal{E}, \varphi) \in \mathcal{M}_\psi$ such that under some local trivialization $\varphi|_{D^\times_\infty} = (\psi + x_0 + \mathfrak{g}(\!(\tau)\!)_{\leq -1/m})d\tau/\tau$, where $x_0 \in \mathfrak{g}_0$, we let $\mathrm{res}_\infty(\mathcal{E}, \varphi)$ be the projection of x_0 to $\mathfrak{c}_0 = \mathfrak{g}_0/\mathfrak{c}^{\perp}_0$. This projection is independent of the trivialization of $\mathcal{E}|_{D_\infty}$. By definition, we have

$$\mathcal{M}^{\flat}_{\psi} = \mathrm{res}^{-1}_\infty(0)/C_0. \tag{2.18}$$

This realizes $\mathcal{M}^{\flat}_{\psi}$ as the Hamiltonian reduction of $\mathcal{M}_\psi$ by C_0.

On the other hand, we have a map

$$\mathrm{res}_{\mathcal{A}} : \mathcal{A}_\psi \to \prod_{1\leq i\leq r, m|(d_i-1)} \mathbb{A}^1$$

sending $(a_i)_{1\leq i\leq r}$ to the coefficient of $\tau^{-\frac{d(d_i-1)}{m}}$ of a_i, for those $1 \leq i \leq r$ such that $m|(d_i - 1)$. There is a commutative diagram

$$\begin{tikzcd} \mathcal{M}_\psi \arrow[r, "\mathrm{res}_\infty"] \arrow[d, "f"] & \mathfrak{c}_0 \arrow[d, "\phi"] \\ \mathcal{A}_\psi \arrow[r, "\mathrm{res}_{\mathcal{A}}"] & \prod_{1\leq i\leq r, m|(d_i-1)} \mathbb{A}^1 \end{tikzcd}$$

Here ϕ sends $x_0 \in \mathfrak{c}_0$ to $((\partial_{x_0} f_i)(\psi))_{1\le i\le d, m|(d_i-1)}$. It is easy to see that ϕ is a linear isomorphism. We define

$$\mathcal{A}^\flat_\psi = \mathrm{res}_{\mathcal{A}}^{-1}(0) \subset \mathcal{A}_\psi.$$

In other words, in addition to the conditions defining $\mathcal{A}_\psi$, we require the coefficient of $\tau^{-\frac{d(d_i-1)}{m}}$ of a_i (when $m|(d_i-1)$) to be zero. Then f induces a map

$$f^\flat : \mathcal{M}^\flat_\psi \to \mathcal{A}^\flat_\psi.$$

One can prove the following properties of $\mathcal{M}^\flat_\psi$, analogous to those of $\mathcal{M}_\psi$, using either the same idea of proof or formal deduction from Hamiltonian reduction (2.18).

2.16.1 Theorem. *For a homogeneous element $\psi \in \mathfrak{g}((t))$ of slope $\nu = d/m$, the following hold.*

(1) The stack $\mathcal{M}^\flat_\psi$ is a smooth algebraic stack over k of dimension $\frac{d}{m}|\Phi| - r - \dim \mathfrak{t}^w$.
(2) $\mathcal{M}^\flat_\psi$ carries a canonical symplectic structure of weight d under the $\mathbb{G}_m(\nu)$-action.
(3) The map $f^\flat : \mathcal{M}_\psi \to \mathcal{A}_\psi$ is a $\mathbb{G}_m(\nu)$-equivariant completely integrable system.
(4) There is a natural homeomorphism $[\mathrm{Fl}_\psi/C_0] \to \mathcal{M}^\flat_{a_\psi} = f^{\flat,-1}(a_\psi)$.
(5) When ψ is elliptic (equivalently, w is elliptic), $\mathcal{M}^\flat_\psi = \mathcal{M}_\psi, \mathcal{A}^\flat_\psi = \mathcal{A}_\psi$, and in particular $f^\flat = f$ is proper.

2.16.2 Remark. One could also consider the variant of $\mathcal{M}^\flat_\psi$ by specifying an arbitrary residue in $\mathfrak{c}_0$ at ∞ and an arbitrary residue in $\mathfrak{t}$ at 0. They do not have $\mathbb{G}_m(\nu)$-action in general but their Hitchin fibrations are still completely integrable systems.

3. The de Rham moduli space

Similar to the usual Hitchin moduli space, $\mathcal{M}_\psi$ admits a one-parameter deformation into the moduli space of certain λ-connections. We denote the $\lambda = 1$ fiber by $\mathcal{M}_{\mathrm{dR},\psi}$. The main result in this section gives a canonical isomorphism between the cohomologies of $\mathcal{M}_{\mathrm{dR},\psi}$ and $\mathcal{M}_\psi$.

3.1. Moduli of $\boldsymbol{\lambda}$-connections

3.1.1. $\boldsymbol{\mathcal{M}_{\mathrm{Hod},\psi}}$ and $\boldsymbol{\mathcal{M}_{\mathrm{dR},\psi}}$ Let $\mathcal{M}_{\mathrm{Hod},\psi}$ be the moduli stack of triples $(\lambda, \mathcal{E}, \nabla)$ where

- $\lambda \in \mathbb{A}^1$.
- $\mathcal{E}$ is a G-bundle over X with $\mathbf{K}_\infty := \mathbf{P}_\infty(d/m)\mathbf{C}^+_\infty$-level structure at ∞ and $\mathbf{I}_0$-level structure at 0.
- ∇ is a λ-connection on $\mathcal{E}|_{X\backslash\{0,\infty\}}$ satisfying the following conditions:
  - (i) Under any (equivalently, some) trivialization of $\mathcal{E}|_{D_\infty}$ together with its $\mathbf{K}_\infty$-level structure, $\nabla|_{D^\times_\infty}$ takes the form

$$\nabla|_{D^\times_\infty} \in \lambda d + (\psi + \mathfrak{g}(\!(\tau)\!)_{\leq 0})d\tau/\tau.$$

  - (ii) Under any (equivalently, some) trivialization of $\mathcal{E}|_{D_0}$ together with its $\mathbf{I}_0$-level structure, $\nabla|_{D^\times_0}$ takes the form

$$\nabla|_{D^\times_0} \in \lambda d + \mathrm{Lie}\ (\mathbf{I}^+_0)dt/t.$$

We have the projection $\lambda : \mathcal{M}_{\mathrm{Hod},\psi} \to \mathbb{A}^1$ recording λ. The fiber $\lambda^{-1}(0)$ is identified with $\mathcal{M}_\psi$. Define

$$\mathcal{M}_{\mathrm{dR},\psi} := \lambda^{-1}(1).$$

3.1.2. $\mathbb{G}_{\boldsymbol{m}}(\boldsymbol{\nu})$-action The $\mathbb{G}_m(\nu)$-action on $\mathcal{M}_\psi$ extends to an action on $\mathcal{M}_{\mathrm{Hod},\psi}$: we interpret the scaling by s^d as multiplying ∇ by s^d, so that the function λ has weight d under the $\mathbb{G}_m(\nu)$-action. We denote this action by

$$s : (\lambda, \mathcal{E}, \nabla) \mapsto s \cdot (\lambda, \mathcal{E}, \nabla), \quad s \in \mathbb{G}_m(\nu), (\lambda, \mathcal{E}, \nabla) \in \mathcal{M}_{\mathrm{Hod},\psi}.$$

Since the function λ has weight $d > 0$, the $\mathbb{G}_m(\nu)$-fixed points $\mathcal{M}^{\mathbb{G}_m(\nu)}_{\mathrm{Hod},\psi}$ necessarily lie in the central fiber of $\mathcal{M}_\psi$, hence $\mathcal{M}^{\mathbb{G}_m(\nu)}_{\mathrm{Hod},\psi} = \mathcal{M}^{\mathbb{G}_m(\nu)}_\psi$, which isomorphic to $\mathrm{Fl}^{\mathbb{G}_m(\nu)}_\psi = \coprod \mathcal{H}_\psi(w)$ by Lemma 2.15.4.

Similar to Lemma 2.15.6, we have a version for $\mathcal{M}_{\mathrm{Hod},\psi}$.

3.1.3 Lemma. *For any $(\lambda, \mathcal{E}, \nabla) \in \mathcal{M}_{\mathrm{Hod},\psi}$, the limit $\lim_{s\to 0} s \cdot (\lambda, \mathcal{E}, \nabla)$ exists in $\mathcal{M}^{\mathbb{G}_m(\nu)}_\psi$. Moreover, $\lim_{s\to 0} s \cdot (\lambda, \mathcal{E}, \nabla) \in Z_w$ if and only if the image of $\mathcal{E}$ in $\mathrm{Bun}_G(\mathbf{P}_\infty, \mathbf{I}_0)$ is w.*

Proof. The proof is almost identical to that of Lemma 2.15.6. We only indicate the modifications. Let $\mathcal{N}_{\mathrm{Hod}} = \mathcal{M}_{\mathrm{Hod},\psi} \times_{\mathrm{Bun}_G(\mathbf{K}_\infty,\mathbf{I}_0)} \mathrm{Fl}$, where $u : \mathrm{Fl} \to \mathrm{Bun}_G(\mathbf{K}_\infty, \mathbf{I}_0)$ is the the uniformization map. Then $\mathcal{N}_{\mathrm{Hod}}$ is the moduli space of triples $(\lambda, g\mathbf{I}_0, \theta)$ where $\lambda \in \mathbb{A}^1$, $g\mathbf{I}_0 \in \mathrm{Fl}$ and $\theta \in \mathfrak{g}[t, t^{-1}]_{\leq 0}$ such that

$$\lambda g^{-1}dg - \mathrm{Ad}(g^{-1})(\psi + \theta)dt/t \in \mathrm{Lie}\ \mathbf{I}_0^+ dt/t.$$

Indeed, given $(\lambda, g\mathbf{I}_0, \theta)$, we define $\mathcal{E}$ as the image of $g\mathbf{I}_0$ under u, equipped with the λ-connection $\nabla = \lambda d - (\psi + \theta)dt/t$ over $X \backslash \{0\}$.

The action of $\mathbb{G}_m(\nu)$ on $\mathcal{M}_{\mathrm{Hod},\psi}$ lifts to $\mathcal{N}$, and is given by

$$s \cdot (\lambda, g\mathbf{I}_0, \theta) = (s^d\lambda, (\mathrm{rot}(s^{-m})\mathrm{Ad}(\xi(s^{-1}))g)\mathbf{I}_0, s^d\mathrm{rot}(s^{-m})\mathrm{Ad}(\xi(s^{-1}))\theta).$$

The rest of the argument can be copied verbatim from that of Lemma 2.15.6. □

3.1.4 Theorem. *The stack $\mathcal{M}_{\mathrm{Hod},\psi}$ is an algebraic space smooth over $\mathbb{A}^1$ with pure relative dimension equal to* $\dim \mathcal{M}_\psi$.

Proof. We first remark that $\mathcal{M}_{\mathrm{Hod},\psi}$ is an algebraic stack locally of finite type over $\mathrm{Bun}_G(\mathbf{K}_\infty, \mathbf{I}_0) \times \mathbb{A}^1$. Indeed, letting $\mathcal{E}^{\mathrm{univ}}$ be the universal G-bundle over $\mathrm{Bun}_G(\mathbf{K}_\infty, \mathbf{I}_0) \times X$, there is an extension of vector bundles over $\mathrm{Bun}_G(\mathbf{K}_\infty, \mathbf{I}_0) \times X \times \mathbb{A}^1$ of the form

$$(3.1)\quad 0 \to p^*_{\mathrm{Bun}\times X}\mathrm{Ad}(\mathcal{E}^{\mathrm{univ}}; \mathrm{Lie}\ \mathbf{I}_0^+, \mathfrak{g}(\!(\tau)\!)_{\leq d/m}) \to \mathcal{A} \to p_X^* T_X(-0-\infty) \to 0$$

whose splittings over $\lambda \in \mathbb{A}^1$ classify λ-connections on $\mathcal{E}$ which locally looks like $\lambda d + \mathrm{Lie}\ \mathbf{I}_0^+ dt/t$ near 0 and looks like $\lambda d + \mathfrak{g}(\!(\tau)\!)_{\leq d/m} d\tau/\tau$ near ∞. The moduli stack $\widetilde{\mathcal{M}}$ of splittings of (3.1) is an algebraic stack locally of finite type over $\mathrm{Bun}_G(\mathbf{K}_\infty, \mathbf{I}_0) \times \mathbb{A}^1$. Our $\mathcal{M}_{\mathrm{Hod},\psi}$ is a closed substack of $\widetilde{\mathcal{M}}$, hence locally of finite type over k.

Now we show $\lambda : \mathcal{M}_{\mathrm{Hod},\psi} \to \mathbb{A}^1$ is smooth. Let $Z \subset \mathcal{M}_{\mathrm{Hod},\psi}$ be the closed substack where λ fails to be smooth. Suppose $Z \neq \varnothing$. Since λ is $\mathbb{G}_m(\nu)$-equivariant, Z is stable under the $\mathbb{G}_m(\nu)$-action. By Lemma 3.1.3, Z contains a fixed point under $\mathbb{G}_m(\nu)$, hence in particular, $Z \cap \mathcal{M}_\psi \neq \varnothing$. At a geometric point $(\lambda, \mathcal{E}, \nabla) \in \mathcal{M}_{\mathrm{Hod},\psi}$, the relative tangent complex of $\lambda : \mathcal{M}_{\mathrm{Hod},\psi} \to \mathbb{A}^1$ is the de Rham cohomology $\mathrm{H}^*(X, \mathcal{K}_{(\mathcal{E},\nabla)})$, where $\mathcal{K}_{(\mathcal{E},\nabla)}$ is the complex in degrees -1 and 0:

$$\mathrm{Ad}(\mathcal{E}; \mathrm{Lie}\ \mathbf{I}_0, \mathfrak{k}_\infty) \xrightarrow{\nabla_{\mathrm{Ad}}} \mathrm{Ad}(\mathcal{E}; \mathrm{Lie}\ \mathbf{I}_0^+, \mathfrak{g}(\!(\tau)\!)_{\leq 0})$$

and ∇_{Ad} is the λ-connection on $\mathrm{Ad}(\mathcal{E})$ induced from ∇. In particular, at a point $(\mathcal{E},\varphi)\in\mathcal{M}_\psi$, the relative tangent complex of λ is the same as the tangent complex of $\mathcal{M}_\psi$ at $(\mathcal{E},\varphi)$, whose obstruction group vanishes by Theorem 2.8.1(1). Therefore λ is smooth at any point of $\mathcal{M}_\psi$, i.e., $Z\cap\mathcal{M}_\psi=\varnothing$. Contradiction! This shows $Z=\varnothing$ hence λ is smooth.

To see $\mathcal{M}_{\mathrm{Hod},\psi}$ has trivial automorphism groups, let $Z'\subset\mathcal{M}_{\mathrm{Hod},\psi}$ be the closed substack where the automorphism group is nontrivial. Then Z is $\mathbb{G}_m(\nu)$-stable but $Z\cap\mathcal{M}_\psi=\varnothing$ since $\mathcal{M}_\psi$ is known to be an algebraic space. Therefore the same argument as above using Lemma 3.1.3 shows that $Z=\varnothing$.

Finally, we compute the dimension of $\mathcal{M}_{\mathrm{Hod},\psi}$ at any geometric point $(\lambda,\mathcal{E},\nabla)$. Since $\mathcal{M}_{\mathrm{Hod},\psi}$ is smooth over $\mathbb{A}^1$ with trivial automorphism group at $(\lambda,\mathcal{E},\nabla)$, the complex $R\Gamma(X,\mathcal{K}_{(\mathcal{E},\nabla)})$ is concentrated in degree 0, and $\mathrm{H}^0(X,\mathcal{K}_{(\mathcal{E},\nabla)})$ is the relative tangent space of λ at $(\lambda,\mathcal{E},\nabla)$. Therefore the relative dimension of λ at $(\lambda,\mathcal{E},\nabla)$ is

$$\chi(X,\mathcal{K}_{(\mathcal{E},\nabla)})=\chi(X,\mathrm{Ad}(\mathcal{E};\mathrm{Lie}\ \mathbf{I}_0^+,\mathfrak{g}(\!(\tau)\!)_{\le 0}))-\chi(X,\mathrm{Ad}(\mathcal{E};\mathrm{Lie}\ \mathbf{I}_0,\mathfrak{k}_\infty)),$$

which is the same as $\dim\mathcal{M}_\psi$. □

3.1.5 Remark. The smooth map $\lambda:\mathcal{M}_{\mathrm{Hod},\psi}\to\mathbb{A}^1$ carries a canonical symplectic structure constructed as follows. Consider the "Serre dual" complex $\mathcal{K}^\vee_{(\mathcal{E},\nabla)}$ given by

$$\mathrm{Ad}(\mathcal{E};\mathrm{Lie}\ \mathbf{I}_0,\mathfrak{g}(\!(\tau)\!)_{\le -1/m})\xrightarrow{\nabla_{\mathrm{Ad}}}\mathrm{Ad}(\mathcal{E};\mathrm{Lie}\ \mathbf{I}_0^+,\mathfrak{k}_\infty^\vee).$$

Here Serre dual is in quotation marks because the differential in $\mathcal{K}_{(\mathcal{E},\nabla)}$ is not $\mathcal{O}_X$-linear. Here we take the termwise Serre dual (and the Killing form to identify $\mathrm{Ad}(\mathcal{E})|_{\mathbb{G}_m}$ with $\mathrm{Ad}(\mathcal{E})^*|_{\mathbb{G}_m}$), and the differential in $\mathcal{K}^\vee_{(\mathcal{E},\nabla)}$ is still given by the adjoint connection.

The same argument as in §2.9.2 shows that the natural map $\mathcal{K}_{(\mathcal{E},\nabla)}\to\mathcal{K}^\vee_{(\mathcal{E},\nabla)}$ is a quasi-isomorphism in the derived category of sheaves of abelian groups on X, hence a canonical isomorphism $\mathrm{H}^*(X,\mathcal{K}_{(\mathcal{E},\nabla)})\xrightarrow{\sim}\mathrm{H}^*(X,\mathcal{K}^\vee_{(\mathcal{E},\nabla)})$. We claim that there is a perfect pairing between $\mathrm{H}^*(X,\mathcal{K}_{(\mathcal{E},\nabla)})$ and $\mathrm{H}^*(X,\mathcal{K}^\vee_{(\mathcal{E},\nabla)})$ even though $\mathcal{K}_{(\mathcal{E},\nabla)}$ is not an $\mathcal{O}_X$-linear complex. Indeed, by construction there is a k-linear pairing of complexes of sheaves

$$\mathcal{K}_{(\mathcal{E},\nabla)}\otimes_k\mathcal{K}^\vee_{(\mathcal{E},\nabla)}\to\omega_X[1]. \tag{3.2}$$

Taking cohomology induces a pairing between $\mathrm{H}^i(X,\mathcal{K}_{(\mathcal{E},\nabla)})$ and $\mathrm{H}^{-i}(X,\mathcal{K}^\vee_{(\mathcal{E},\nabla)})$ valued in $\mathrm{H}^0(X,\omega_X[1])\cong k$. Writing $\mathcal{K}=\mathcal{K}_{(\mathcal{E},\nabla)}$ as $[\mathcal{K}^{-1}\to$

$\mathcal{K}^0]$, $\mathrm{H}^*(X, \mathcal{K}_{(\mathcal{E},\nabla)})$ fits into a long exact sequence exact at the two ends

$$\mathrm{H}^{-1}(\mathcal{K}) \to \mathrm{H}^0(\mathcal{K}^{-1}) \to \mathrm{H}^0(\mathcal{K}^0) \to \mathrm{H}^0(\mathcal{K}) \to \mathrm{H}^1(\mathcal{K}^{-1}) \to \mathrm{H}^1(\mathcal{K}^0) \to \mathrm{H}^1(\mathcal{K}) \tag{3.3}$$

Similarly for $\mathrm{H}^*(X, \mathcal{K}^\vee_{(\mathcal{E},\nabla)})$, or its dual

$$\begin{aligned} \mathrm{H}^1(\mathcal{K}^\vee)^* &\to \mathrm{H}^1(\mathcal{K}^{-1,\vee})^* \to \mathrm{H}^1(\mathcal{K}^{0,\vee})^* \\ &\to \mathrm{H}^0(\mathcal{K}^\vee)^* \to \mathrm{H}^0(\mathcal{K}^{-1,\vee})^* \to \mathrm{H}^0(\mathcal{K}^{0,\vee})^* \to \mathrm{H}^{-1}(\mathcal{K}^\vee)^* \end{aligned} \tag{3.4}$$

The pairing gives a map of long exact sequences from (3.3) to (3.4). One checks that the maps $\mathrm{H}^i(\mathcal{K}^j) \to \mathrm{H}^{1-i}(\mathcal{K}^{j,\vee})^*$ are the usual Serre duality ($i = 0, 1$, $j = -1, 0$), hence are isomorphisms. Therefore the maps $\mathrm{H}^i(\mathcal{K}) \to \mathrm{H}^{-i}(\mathcal{K}^\vee)^*$ given by the pairing (3.2) is also an isomorphism. It is easy to check that this is gives a symplectic form on $\mathrm{H}^0(X, \mathcal{K}_{(\mathcal{E},\nabla)})$, and a perfect pairing between $\mathrm{H}^{-1}(X, \mathcal{K}_{(\mathcal{E},\nabla)})$ and $\mathrm{H}^1(X, \mathcal{K}_{(\mathcal{E},\nabla)})$. It can be shown by calculations similar to that done by R.Fedorov [17, §6.3] that this 2-form is closed. Therefore the map λ has a relative symplectic structure. In particular, $\mathcal{M}_{\mathrm{dR},\psi}$ is a symplectic algebraic space.

3.2. Comparison of cohomology

The non-abelian Hodge theory suggests that $\mathcal{M}_{\mathrm{dR},\psi}$ should be diffeomorphic to $\mathcal{M}_\psi$. In particular they should have isomorphic cohomology. In this subsection we prove the cohomological isomorphism without showing they are diffeomorphic.

3.2.1. The situation Consider the following general situation. Let $f : \mathfrak{X} \to \mathbb{A}^1$ be a regular function on an algebraic space $\mathfrak{X}$ locally of finite type over an algebraically closed field k. Let $\mathfrak{X}_\lambda = f^{-1}(\lambda)$ for $\lambda \in k$. Let $\mathbb{G}_m$ act on $\mathfrak{X}$ such that f has weight $d > 0$. Let $\mathfrak{X}^{\mathbb{G}_m} = \coprod_{\alpha\in I} Z_\alpha$ be an open-closed decomposition. We use notation $\mathfrak{X}^+_\alpha$ and $\mathfrak{q}^+_\alpha : \mathfrak{X}^+_\alpha \to Z_\alpha$ from §2.15.1 for attractors.

We make the following assumptions:

(1) The function f is a smooth morphism $f : \mathfrak{X} \to \mathbb{A}^1$. In particular, $\mathfrak{X}$ is smooth over k.
(2) The map $\mathfrak{q}^+_\alpha : \mathfrak{X}^+_\alpha \to \mathfrak{X}$ is a locally closed embedding.
(3) $\cup_{\alpha\in I}\mathfrak{X}^+_\alpha = \mathfrak{X}$, i.e., for any $x \in \mathfrak{X}$, the limit $\lim_{t\to 0} t \cdot x$ exists.
(4) There exists a partial order $\leq$ on I such that for each $\alpha \in I$
    - The set $\{\alpha' \in I; \alpha' \leq \alpha\}$ is finite.

- $\mathfrak{X}^+_{\leq\alpha} := \cup_{\alpha'\leq\alpha}\mathfrak{X}^+_{\alpha}$ is open in $\mathfrak{X}$, and is of finite type over k.

3.2.2 Lemma. *Under the above assumptions, the restriction map* $\mathrm{H}^*(\mathfrak{X}) \to \mathrm{H}^*(\mathfrak{X}_1)$ *is an isomorphism.*

Proof. Let $i_1 : \mathfrak{X}_1 \hookrightarrow \mathfrak{X}$ be the inclusion. The canonical map $i_{1*}\overline{\mathbb{Q}}_\ell[-2](-1) \cong i_{1*}i_1^!\overline{\mathbb{Q}}_\ell \to \overline{\mathbb{Q}}_\ell$ induces a map

$$r : \mathrm{H}^*_c(\mathfrak{X}_1, i_1^!\overline{\mathbb{Q}}_\ell) \cong \mathrm{H}^{*-2}_c(\mathfrak{X}_1, \overline{\mathbb{Q}}_\ell(-1)) \to \mathrm{H}^*_c(\mathfrak{X}, \overline{\mathbb{Q}}_\ell).$$

It suffices to prove r is an isomorphism, and the statement on cohomology follows by Poincaré duality.

Extend $\leq$ to a total order on I. Let $\mathfrak{X}^+_{1,\leq\alpha} = \mathfrak{X}^+_{\leq\alpha} \cap \mathfrak{X}_1$, and similarly define $\mathfrak{X}^+_{1,\alpha}$ and $\mathfrak{X}^+_{1,<\alpha}$. We have an analogue $r_{\leq\alpha}$ of r for the inclusion $\mathfrak{X}^+_{1,\leq\alpha} \hookrightarrow \mathfrak{X}^+_{\leq\alpha}$, and similarly we have r_α and $r_{<\alpha}$. We have a map of long exact sequences

$$\begin{tikzcd}
\cdots \arrow[r] & \mathrm{H}^{*-2}_c(\mathfrak{X}^+_{1,<\alpha})(-1) \arrow[r] \arrow[d, "r_{<\alpha}"] & \mathrm{H}^{*-2}_c(\mathfrak{X}_{1,\leq\alpha})(-1) \arrow[r] \arrow[d, "r_{\leq\alpha}"] & \mathrm{H}^{*-2}_c(\mathfrak{X}^+_{1,\alpha})(-1) \arrow[r] \arrow[d, "r_\alpha"] & \cdots \\
\cdots \arrow[r] & \mathrm{H}^*_c(\mathfrak{X}^+_{<\alpha}) \arrow[r] & \mathrm{H}^*_c(\mathfrak{X}_{\leq\alpha}) \arrow[r] & \mathrm{H}^*_c(\mathfrak{X}^+_\alpha) \arrow[r] & \cdots
\end{tikzcd}$$

By induction on α, it suffices to show that r_α is an isomorphism for each α.

Let $f_\alpha = f|_{\mathfrak{X}^+_\alpha} : \mathfrak{X}^+_\alpha \to \mathbb{A}^1$, and $\pi_\alpha = (f_\alpha, \mathfrak{q}^+_\alpha) : \mathfrak{X}^+_\alpha \to \mathbb{A}^1 \times Z_\alpha$. We first claim that π_α is smooth. Since the critical locus of π_α is closed and stable under the $\mathbb{G}_m$-action, it must intersect $\{0\} \times Z_\alpha$ if not empty. Therefore it suffices to show that π_α is smooth along Z_α. Let $z \in Z_\alpha$ be a geometric point. By [14, Prop.1.4.11(vi)], $(T_z\mathfrak{X}^+_\alpha) = (T_z\mathfrak{X})^+$ is the summand where $\mathbb{G}_m$ acts with positive weights; $T_zZ_\alpha = (T_z\mathfrak{X})^0$ is the zero weight space. The tangent map of π_α at $z \in Z_\alpha$ is (T_zf, pr_0), where pr_0 is the natural projection. By assumption, f has weight $d > 0$, therefore $T_zf : T_z\mathfrak{X} \to T_0\mathbb{A}^1 = \mathbb{A}^1$ factors through the weight d summand $T_z\mathfrak{X} \twoheadrightarrow (T_z\mathfrak{X})^+ \twoheadrightarrow (T_z\mathfrak{X})^d$. Since f is a smooth morphism, $df(z)$ is nonzero. Therefore $T_z\pi_\alpha = (T_zf, \mathrm{pr}_0)$ is surjective, and π_α is smooth.

We then show that the geometric fibers of π_α are isomorphic to affine spaces. By [14, Theorem 1.4.2], $\mathfrak{q}^+_\alpha : \mathfrak{X}^+_\alpha \to Z_\alpha$ is an affine morphism of finite type. Let $z \in Z_\alpha$ be a geometric point valued in K, then $\mathfrak{X}^+_z = \mathfrak{q}^{+,-1}_\alpha(z)$ is a smooth affine scheme over K with a contracting $\mathbb{G}_m$-action. We have $\mathfrak{X}^+_z = \mathrm{Spec}\ A$ where $A = \oplus_{n\geq 0}A_n$ is a finitely generated graded K-algebra with $A_0 = K$. Since Spec A is smooth at the cone point z, one can choose liftings $t_1, \cdots, t_m \in A_+ = \oplus_{n>0}A_n$ of a homogeneous basis of the cotangent space A_+/A_+^2, so that $K[t_1, \cdots, t_m] \xrightarrow{\sim} A$ as graded algebras. Now $df_\alpha(z)$ is a nonzero homogeneous element in $T^*_z(\mathrm{Spec}\ A)$ by the smoothness of π_α, we may assume $t_1 = f_\alpha$ so that it lifts $df_\alpha(z)$. This way, we have an isomorphism of the pair $(\mathfrak{X}^+_z, f_\alpha)$ with $(\mathrm{Spec}\ K[t_1, \cdots, t_m], t_1)$, so the geometric fibers of $f_\alpha|\mathfrak{X}^+_z$ are affine spaces of dimension $m-1$.

If needed we may decompose Z_α further to ensure π_α is equidimensional. Since π_α is smooth with geometric fibers isomorphic to $\mathbb{A}^{d_\alpha}$, we have a canonical isomorphism

$$K_\alpha := R\pi_{\alpha!}\overline{\mathbb{Q}}_\ell \cong \overline{\mathbb{Q}}_\ell[-2d_\alpha](-d_\alpha) \in D^b_c(\mathbb{A}^1 \times Z_\alpha).$$

Let $\iota_1 : \{1\} \times Z_\alpha \to \mathbb{A}^1 \times Z_\alpha$ be the inclusion. The map r_α is induced from the canonical map $\iota_{1*}\iota_1^! K_\alpha \to K_\alpha$ by taking $\mathrm{H}^*_c(-)$. Now K_α is constant, it is clear that $\iota_{1*}\iota_1^! K_\alpha \to K_\alpha$ induces an isomorphism on compactly supported cohomology on $\mathbb{A}^1 \times Z_\alpha$. Therefore r_α is an isomorphism. □

3.2.3 Corollary. *Both restriction maps*

$$\mathrm{H}^*(\mathcal{M}_\psi) \xleftarrow{i_0^*} \mathrm{H}^*(\mathcal{M}_{\mathrm{Hod},\psi}) \xrightarrow{i_1^*} \mathrm{H}^*(\mathcal{M}_{\mathrm{dR},\psi})$$

are isomorphisms.

Proof. To show i_1^* is an isomorphism, we would like to apply Lemma 3.2.2. We need to check that $\lambda : \mathcal{M}_{\mathrm{Hod},\psi} \to \mathbb{A}^1$ satisfies the conditions in §3.2.1. Condition (1) follows from Theorem 3.1.4; (2) can be proved in the same way as Lemma 2.15.6; (3) follows from Lemma 3.1.3; for (4), the partial order on $W_{\mathbf{P}}\backslash\widetilde{W}$ is the same one used §2.15.3.

Now we show that i_0^* is an isomorphism. Consider the further restriction map along $\gamma_{\mathrm{Hod}} : \mathrm{Fl}_\psi \to \mathcal{M}_{\mathrm{Hod},\psi}$. We have a factorization

$$\gamma^*_{\mathrm{Hod}} : \mathrm{H}^*(\mathcal{M}_{\mathrm{Hod},\psi}) \xrightarrow{i_0^*} \mathrm{H}^*(\mathcal{M}_\psi) \xrightarrow{\gamma^*} \mathrm{H}^*(\mathrm{Fl}_\psi).$$

By Theorem 2.8.4, γ^* is an isomorphism. Observe that Prop. 2.15.2 also applies to $\mathcal{M}_{\mathrm{Hod},\psi}$, which proves that γ^*_{Hod} is an isomorphism. Therefore i_0^* is also an isomorphism. □

3.3. Variants

3.3.1. Changing the level group We have a one-parameter deformation $\mathcal{M}^\dagger_{\mathrm{Hod},\psi}$ of the Poisson moduli space $\mathcal{M}^\dagger_\psi$ introduced in §2.13: it classifies $(\lambda, \mathcal{E}, \nabla)$ where

- $\lambda \in \mathbb{A}^1$.
- $\mathcal{E}$ is a G-bundle over X with $\mathbf{P}^+_\infty$-level structure at ∞ and $\mathbf{I}_0$-level structure at 0.
- ∇ is a λ-connection on $\mathcal{E}|_{X\backslash\{0,\infty\}}$ satisfying the following conditions:
  - (i) Under some (equivalently, any) trivialization of $\mathcal{E}|_{D_\infty}$ together with its $\mathbf{P}^+_\infty$-level structure, we require

$$\nabla|_{D^\times_\infty} \in \lambda d + (\psi + \mathfrak{g}((\tau))_{\leq (d-1)/m})d\tau/\tau. \tag{3.5}$$

  - (ii) Under some (equivalently, any) trivialization of $\mathcal{E}|_{D_0}$ together with its $\mathbf{I}_0$-level structure, we require

$$\nabla|_{D^\times_0} \in \lambda d + \text{Lie }(\mathbf{I}^+_0)dt/t.$$

A small part of the Hitchin map $f^\dagger$ for $\mathcal{M}^\dagger_\psi$ continues to make sense for $\mathcal{M}^\dagger_{\text{Hod},\psi}$. Recall in the proof of Proposition 2.13.1 we have introduced an affine space $\mathfrak{a}_\psi$ with a surjection $\mathcal{A}^\dagger_\psi \to \mathfrak{a}_\psi$ that records a Laurent tail of a_i at ∞. Let $\overline{a}_\psi \in \mathfrak{a}_\psi$ be the image of a_ψ. We also introduced in the proof of Proposition 2.13.1 an affine space $V_\psi = (\psi + \mathfrak{g}((\tau))_{\leq (d-1)/m})/\mathfrak{g}((\tau))_{\leq 0}$ with the action of $Q = \mathbf{P}^+_\infty/\mathbf{P}_\infty(\frac{m}{d})$. The invariant polynomials $(f_1, \cdots, f_r)$ give a map $[V_\psi/Q] \to \mathfrak{a}_\psi$. Taking the irregular part of the connection ∇ at ∞ yields a map

$$\overline{f}^\dagger_\psi : \mathcal{M}^\dagger_{\text{Hod},\psi} \to [V_\psi/Q] \to \mathfrak{a}_\psi.$$

We have an analogue of Proposition 2.13.1 with the same proof.

3.3.2 Proposition. *The natural map $\mathcal{M}_{\text{Hod},\psi} \to \mathcal{M}^\dagger_{\text{Hod},\psi}$ identifies $\mathcal{M}_{\text{Hod},\psi}$ with the fiber of $\overline{f}^{\dagger,-1}_\psi(\overline{a}_\psi)$. Equivalently, $\mathcal{M}_{\text{Hod},\psi}$ can be identified with the closed subspace of $\mathcal{M}^\dagger_{\text{Hod},\psi}$ obtained by replacing the condition (3.5) with: under* some *trivialization of $\mathcal{E}|_{D_\infty}$, $\nabla|_{D^\times_\infty} \in \lambda d + (\psi + \mathfrak{g}((\tau))_{\leq 0})d\tau/\tau$.*

3.3.3. Hamiltonian reduction by C_0 As in §2.16, C_0 acts on $\mathcal{M}_{\text{Hod},\psi}$, and we have the map of taking formal residue at ∞

$$\text{res}_{\text{Hod},\infty} : \mathcal{M}_{\text{Hod},\psi} \to \mathfrak{c}_0.$$

For $(\mathcal{E}, \nabla) \in \mathcal{M}_{\text{Hod},\psi}$, such that $\nabla|_{D^\times_\infty} = \lambda d + (\psi + x_0 + \mathfrak{g}((\tau))_{\leq -1/m})d\tau/\tau$ under some local trivialization, where $x_0 \in \mathfrak{g}((\tau))_0 = \mathfrak{g}_0$, $\text{res}_{\text{Hod},\infty}(\mathcal{E}, \nabla)$ is the projection of x_0 to $\mathfrak{c}_0 = \mathfrak{g}_0/\mathfrak{c}^\perp_0$.

We define

$$\mathcal{M}^\flat_{\mathrm{Hod},\psi} = \mathrm{res}^{-1}_{\mathrm{Hod},\infty}(0)/C_0, \quad \mathcal{M}^\flat_{\mathrm{dR},\psi} = (\mathrm{res}^{-1}_{\mathrm{Hod},\infty}(0) \cap \mathcal{M}_{\mathrm{dR},\psi})/C_0.$$

Then $\mathcal{M}^\flat_{\mathrm{dR},\psi}$ is a smooth algebraic stack of the same dimension as $\mathcal{M}^\flat_\psi$. The analog of Corollary 3.2.3 holds, giving a canonical isomorphism $\mathrm{H}^*(\mathcal{M}^\flat_{\mathrm{dR},\psi}) \cong \mathrm{H}^*(\mathcal{M}^\flat_{\mathrm{Hod},\psi}) \cong \mathrm{H}^*(\mathcal{M}^\flat_\psi)$.

3.3.4. Varying semisimple monodromy at 0 The space $\mathcal{M}_{\mathrm{Hod},\psi}$ admits a deformation over the universal Cartan $\mathfrak{h}$ as follows. Consider the moduli stack ${}_{\mathfrak{h}}\mathcal{M}_{\mathrm{Hod},\psi}$ of triples $(\lambda, \mathcal{E}, \nabla)$ as in the definition of $\mathcal{M}_{\mathrm{Hod},\psi}$, except that we relax the condition near 0 to be: under any (equivalently, some) trivialization of $\mathcal{E}|_{D_0}$ together with its $\mathbf{I}_0$-level structure, $\nabla|_{D_0^\times}$ takes the form

$$\nabla|_{D_0^\times} \in \lambda d + \mathrm{Lie}\ (\mathbf{I}_0) dt/t.$$

We have a map

$$\rho : {}_{\mathfrak{h}}\mathcal{M}_{\mathrm{Hod},\psi} \to \mathbb{A}^1 \times \mathfrak{h}$$

where the $\mathfrak{h}$-factor sends $(\lambda, \mathcal{E}, \nabla)$ to the image of $\mathrm{Res}_0 \nabla$ in the universal Cartan $\mathfrak{h} = \mathrm{Lie}\ \mathbf{I}_0/\mathrm{Lie}\ \mathbf{I}_0^+$.

We define

$$\begin{aligned} {}_{\mathfrak{h}}\mathcal{M}_{\mathrm{dR},\psi} &= \rho^{-1}(\{1\} \times \mathfrak{h}) \subset {}_{\mathfrak{h}}\mathcal{M}_{\mathrm{Hod},\psi}, \\ {}_{\mathfrak{h}}\mathcal{M}_{\psi} &= \rho^{-1}(\{0\} \times \mathfrak{h}) \subset {}_{\mathfrak{h}}\mathcal{M}_{\mathrm{Hod},\psi}. \end{aligned}$$

The map ρ is equivariant with respect to the $\mathbb{G}_m(\nu)$-action on ${}_{\mathfrak{h}}\mathcal{M}_{\mathrm{Hod},\psi}$ and the scaling action on $\mathbb{A}^1 \times \mathfrak{h}$ by the d-th power. If we fix $s \in \mathfrak{h}$, then we get a $\mathbb{G}_m(\nu)$-equivariant one-parameter family by restricting ${}_{\mathfrak{h}}\mathcal{M}_{\mathrm{Hod},\psi}$ along the line of $\mathbb{A}^1 \times \mathfrak{h}$ through $(1, s)$:

$$\lambda_s : {}_s\mathcal{M}_{\mathrm{Hod},\psi} \to \mathbb{A}^1_s := \{(\lambda, \lambda s) | \lambda \in \mathbb{A}^1\} \subset \mathbb{A}^1 \times \mathfrak{h}$$

whose fiber over $\lambda = 1$ we denote by ${}_s\mathcal{M}_{\mathrm{dR},\psi}$. Note its fiber over $\lambda = 0$ is $\mathcal{M}_\psi$.

The formal residue construction extends to ${}_{\mathfrak{h}}\mathcal{M}_{\mathrm{Hod},\psi}$. In particular it restricts to a formal residue map on the de Rham space

$$\mathrm{res}_{\mathrm{dR},\infty} : {}_{\mathfrak{h}}\mathcal{M}_{\mathrm{dR},\psi} \to \mathfrak{c}_0.$$

For $\theta \in \mathfrak{c}_0$, let

$$ {}_{\mathfrak{h}}\mathcal{M}_{\mathrm{dR},\psi,\theta} = \mathrm{res}^{-1}_{\mathrm{dR},\infty}(\theta).$$

3.3.5 Theorem. *Both restriction maps*

$$\mathrm{H}^*(\mathcal{M}_\psi) \xleftarrow{i_0^*} \mathrm{H}^*({}_s\mathcal{M}_{\mathrm{Hod},\psi}) \xrightarrow{i_1^*} \mathrm{H}^*({}_s\mathcal{M}_{\mathrm{dR},\psi})$$

are isomorphisms. In particular, there is a canonical isomorphism

$$\mathrm{H}^*({}_s\mathcal{M}_{\mathrm{dR},\psi}) \cong \mathrm{H}^*(\mathcal{M}_\psi)$$

for any $s \in \mathfrak{h}$.

The proof is along the same lines as Corollary 3.2.3, using the general results about $\mathbb{G}_m$-contracting families in §2.15 and §3.2.

3.3.6 Remark. For $s \in \mathfrak{h}$, let ${}_s\mathcal{M}_\psi = \rho^{-1}(0, s) \subset {}_{\mathfrak{h}}\mathcal{M}_\psi$. This is the Higgs moduli space analogous to $\mathcal{M}_\psi$ but with residue at 0 mapping to $s \in \mathfrak{h}$ under $\mathfrak{b} \to \mathfrak{h}$. By scaling s, there is an $\mathbb{A}^1$-family connecting with general fiber ${}_s\mathcal{M}_\psi$ and 0-fiber $\mathcal{M}_\psi$. The same argument as Corollary 3.2.3 gives a canonical isomorphism $\mathrm{H}^*({}_s\mathcal{M}_\psi) \cong \mathrm{H}^*(\mathcal{M}_\psi)$.

3.4. Symmetry of $\mathcal{M}_{\mathrm{Hod},\psi}$ coming from ∞

We shall construct an action of $C((\tau))/\mathbf{C}_\infty^+$ on $\mathcal{M}_{\mathrm{Hod},\psi}$.

3.4.1. First we describe a filtration on the loop torus $C((\tau))$.

Recall from Remark 2.4.1 we defined a maximal torus $T \subset G$ to be the fiber of C at $t = 1$. Also T is the centralizer of $\overline{\psi} \in \mathfrak{c}_{d/m}$. Fix an mth root of τ and denote it by $\tau^{1/m}$. Now ψ and $\overline{\psi}\tau^{-d/m}$ are in the same $G_{\mathrm{ad}}((\tau^{1/m}))$-orbit, we get a canonical isomorphism between their centralizers (which are abelian) inside $G((\tau^{1/m}))$:

$$\mathrm{can} : C((\tau^{1/m})) \cong T((\tau^{1/m})).$$

This allows us to identify $C((\tau))$ with the fixed points under the diagonal μ_m-action on $T((\tau^{1/m}))$

$$C((\tau)) \cong (T((\tau^{1/m})))^{\mu_m} \tag{3.6}$$

where $\zeta \in \mu_m$ acts on $\tau^{1/m}$ via the Galois action $\tau^{1/m} \mapsto \zeta\tau^{1/m}$, and μ_m acts on T via an injective homomorphism

$$\omega : \mu_m \to W = W(G, T)$$

that sends a primitive element $\zeta \in \mu_m$ to a regular element of order m.

Later when we work with $k = \mathbb{C}$, we shall let w be the image of $\zeta_m := \exp(2\pi i/m)$ under ω. In general, we use $\langle w\rangle$ to denote the image of ω, keeping in mind that w is a regular element of m up to taking prime to m powers.

Recall $\mathbf{C}_\infty \subset C(\!(\tau)\!)$ is the parahoric subgroup, with pro-unipotent radical $\mathbf{C}_\infty^+$. Then the isomorphism (3.6) gives a canonical isomorphism $\mathbf{C}_\infty/\mathbf{C}_\infty^+ \cong T^{\langle w\rangle,\circ}$, the neutral component of the $\langle w\rangle$-fixed points on T. We have a Kottwitz isomorphism between $\pi_0(C(\!(\tau)\!)) = (C(\!(\tau)\!)/\mathbf{C}_\infty)^{\mathrm{red}}$ and the $\langle w\rangle$-coinvariants on $\mathbb{X}_*(T)$

$$\kappa_C : (C(\!(\tau)\!)/\mathbf{C}_\infty)^{\mathrm{red}} \cong \mathbb{X}_*(T)_{\langle w\rangle}. \tag{3.7}$$

See [36, §2.a.2, Theorem 5.1 step A]. This isomorphism makes the following diagram commutative

$$\begin{tikzcd}
(T(\!(\tau^{1/m})\!)/\mathbf{T}_\infty)^{\mathrm{red}} \arrow[r, "\kappa_T"] \arrow[d, "\mathrm{Nm}"] & \mathbb{X}_*(T) \arrow[d, "p"] \\
(C(\!(\tau)\!)/\mathbf{C}_\infty)^{\mathrm{red}} \arrow[r, "\kappa_C"] & \mathbb{X}_*(T)_{\langle w\rangle}
\end{tikzcd}$$

Here $\mathbf{T}_\infty = T[\![\tau^{1/m}]\!]$, κ_T is given by the $\tau^{1/m}$-adic valuation, Nm is the norm map $x \mapsto x\zeta(x)\cdots\zeta^{m-1}(x)$ (for $\zeta \in \mu_m$ acting on $T(\!(\tau^{1/m})\!)$ diagonally), and p is the canonical quotient map.

Let $\mathbf{C}_\infty^\natural \subset C(\!(\tau)\!)$ be the maximal bounded subgroup that corresponds to $(T[\![\tau^{1/m}]\!]^{\langle w\rangle}$ under (3.6). Then under the isomorphism (3.7), $\mathbf{C}_\infty^\natural/\mathbf{C}_\infty$ corresponds to the torsion subgroup of $\mathbb{X}_*(T)_{\langle w\rangle}$. On the other hand, $\mathbf{C}_\infty^\natural/\mathbf{C}_\infty^+ \cong T^{\langle w\rangle}$, and $\mathbf{C}_\infty^\natural/\mathbf{C}_\infty$ can also be identified with $\pi_0(T^{\langle w\rangle})$. To summarize, we have a filtration of $C(\!(\tau)\!)$ with reduced associated graded as follows:

$$\mathbf{C}_\infty^+ \underbrace{\subset}_{T^{\langle w\rangle,\circ}} \mathbf{C}_\infty \underbrace{\subset}_{\mathbb{X}_*(T)_{\langle w\rangle,\mathrm{tors}}} \mathbf{C}_\infty^\natural \underbrace{\subset}_{\mathbb{X}_*(T)_{\langle w\rangle}/\mathrm{tors}} C(\!(\tau)\!) \tag{3.8}$$

3.4.2. Residue map We construct a residue map

$$\mathrm{res}_{C,\infty} : C(\!(\tau)\!)/\mathbf{C}_\infty^\natural \cong \mathbb{X}_*(T)_{\langle w\rangle}/\mathrm{tors} \to \mathfrak{c}_0 \tag{3.9}$$

as follows. First, for the split torus $T(\!(\tau^{1/m})\!)$ we have the usual residue map

$$\mathrm{res}_{T,\infty} : T(\!(\tau^{1/m})\!)/\mathbf{T}_\infty = \frac{1}{m}\mathbb{X}_*(T) \hookrightarrow \mathfrak{t}.$$

defined by $x \mapsto \mathrm{res}_{\tau=0}(x^{-1}dx)$ (taking the coefficient of $d\tau/\tau$). Then $\mathrm{res}_{C,\infty}$ is obtained from $\mathrm{res}_{T,\infty}$ by restricting to μ_m-fixed points (noting that $\mathfrak{c}_0 = \mathfrak{t}^{\langle w\rangle}$).

We use the norm map to identify $\mathfrak{t}_{\langle w\rangle} \xrightarrow{\sim} \mathfrak{t}^{\langle w\rangle}$ ($x \mapsto x + wx + \cdots + w^{m-1}x$). Then the residue map fits into a commutative diagram

$$\begin{tikzcd}
m\mathrm{res}_{T,\infty}: & T((\tau^{1/m}))/\mathbf{T}_\infty \arrow[r, "\kappa_T"] \arrow[d, "\mathrm{Nm}"] & \mathbb{X}_*(T) \arrow[r] \arrow[d, "p"] & \mathfrak{t} \arrow[d, "p"] \\
\mathrm{res}_{C,\infty}: & C((\tau))/\mathbf{C}^\natural \arrow[r, "\kappa_C"] & \mathbb{X}_*(T)_{\langle w\rangle}/\mathrm{tors} \arrow[r] & \mathfrak{c}_0 \cong \mathfrak{t}_{\langle w\rangle}
\end{tikzcd}$$

Here the maps indexed by p are natural projections.

When $k = \mathbb{C}$, we may take the cokernel of the horizontal maps as complex tori, and get a canonical isomorphism

$$\mathfrak{c}_0/\mathrm{Im}(\mathrm{res}_{C,\infty}) \xrightarrow{\sim} T_{\langle w\rangle} \quad (\langle w\rangle\text{-covariants on } T) \tag{3.10}$$

3.4.3. Let $\widehat{\mathcal{M}}_{\mathrm{Hod},\psi}$ be the moduli space of $(\lambda, \mathcal{E}, \alpha_\infty, \nabla)$ where $(\lambda, \mathcal{E}, \nabla)$ is as in the definition of $\mathcal{M}_{\mathrm{Hod},\psi}$, and α_∞ is a trivialization of $\mathcal{E}|_{D_\infty}$ (together with its $\mathbf{K}_\infty$-level structure) under which $\nabla|_{D_\infty^\times}$ takes the form $\lambda d + (\psi + \mathfrak{g}((\tau))_{\le 0})d\tau/\tau$. Note that $\mathcal{M}_{\mathrm{Hod},\psi} = \widehat{\mathcal{M}}_{\mathrm{Hod},\psi}/\mathbf{K}_\infty$ where $\mathbf{K}_\infty$ acts by changing the trivialization α_∞.

Sending $(\lambda, \mathcal{E}, \alpha_\infty, \nabla)$ to the connection one-form of $\nabla|_{D_\infty^\times}$ under the trivialization α_∞ gives a map

$$\widehat{\mathcal{M}}_{\mathrm{Hod},\psi} \to \psi + \mathfrak{g}((\tau))_{\le 0}.$$

For each $\varphi \in \psi + \mathfrak{g}((\tau))_{\le 0}$, we have the centralizer group scheme C_φ over $D_\infty^\times$, and its maximal bounded subgroup $\mathbf{C}_\varphi^\natural$, parahoric subgroup $\mathbf{C}_\varphi$ and pro-unipotent radical $\mathbf{C}_\varphi^+$. As φ varies, these groups form families over $\varphi \in \psi + \mathfrak{g}((\tau))_{\le 0}$. For example, the C_φ form a torus J over $D_R^\times = \mathrm{Spec}\ R((\tau))$, where $R = \Gamma(\psi + \mathfrak{g}((\tau))_{\le 0}, \mathcal{O})$; we have integral models $\mathbf{J}^\natural, \mathbf{J}$ and $\mathbf{J}^+$ of J over $D_R = \mathrm{Spec}\ R[\![\tau]\!]$ whose fibers over $\varphi \in \psi + \mathfrak{g}((\tau))_{\le 0}$ are $\mathbf{C}_\varphi^\natural, \mathbf{C}_\varphi$ and $\mathbf{C}_\varphi^+$ respectively.

Let $J((\tau))$ be the loop group of J, which is a group ind-scheme over the infinite-dimensional affine space $\psi + \mathfrak{g}((\tau))_{\le 0} = \mathrm{Spec}\ R$. This is a subgroup of $G((\tau)) \times (\psi + \mathfrak{g}((\tau))_{\le 0})$. We have an action of $J((\tau))$ on $\widehat{\mathcal{M}}_{\mathrm{Hod},\psi}$ over $\psi + \mathfrak{g}((\tau))_{\le 0}$ by changing on the trivialization α_∞. Here we are using that, for $\varphi \in \psi + \mathfrak{g}((\tau))_{\le 0}$ and $g \in C_\varphi((\tau))$, we have $g^{-1}dg \in \mathfrak{g}((\tau))_{\le 0}d\tau/\tau$.

3.4.4 Lemma. *The group scheme* $\mathbf{J}^\natural$ *admits a canonical trivialization:* $\mathbf{J}^\natural \cong \mathbf{C}^\natural_\infty \widehat{\times}_k \mathrm{Spec}\ R$ *over* D_R *(here we abuse the notation to view* $\mathbf{C}^\natural_\infty$ *as a group scheme over* D_∞*) whose restriction to* D_∞ *(corresponding to* ψ*) is the identity.*

Proof. We base change to the cyclic cover $D_R^{(m)} := \mathrm{Spec}\ R[\![\tau^{1/m}]\!] \to D_R = \mathrm{Spec}\ R[\![\tau]\!]$. We choose $g \in G(\!(\tau^{1/m})\!)$ such that the adjoint action by g sends $\psi + \mathfrak{g}(\!(\tau)\!)_{\leq 0} = \mathrm{Spec}\ R$ into a closed subscheme of $\mathrm{Spec}\ R' = \psi_0\tau^{-d/m} + \mathfrak{g}[\![\tau^{1/m}]\!]$ for some regular semisimple $\psi_0 \in \mathfrak{t}^{\mathrm{rs}}$. Consider the centralizer group scheme J' over $D_{R'}^{(m)}$, and its integral model $\mathbf{J}' = \mathbf{J}'^\natural$ (both maximally bounded and parahoric since J' is split when specialized to each point of R'). Moreover, $\mathbf{J}'|_{D_R^{(m)}}$ carries a μ_m-equivariant structure since it is the parahoric subgroup of $J'|_{D_R^{(m)}} = J \times_{D_R^\times} D_R^{(m),\times}$. Then we have

$$\mathbf{J}^\natural = (\mathrm{Res}_{D_R^{(m)}/D_R}(\mathbf{J}'|_{D_R^{(m)}}))^{\mu_m}. \tag{3.11}$$

Let $\mathbf{C}'$ be the restriction of $\mathbf{J}'$ to $D_\infty^{(m)}$ (corresponding to $\psi_0\tau^{-d/m} \in \mathrm{Spec}\ R'$). We first give a canonical isomorphism of tori over $D_{R'}^{(m)}$

$$\gamma'_m : \mathbf{J}' \cong \mathbf{C}' \widehat{\times}_k \mathrm{Spec}\ R'.$$

By the rigidity of homomorphisms between tori, it suffices to give a trivialization of the restriction $J'|_{\mathrm{Spec}\ R'}$ (where $\mathrm{Spec}\ R' \hookrightarrow D_{R'}^{(m)}$ is defined by $\tau^{1/m} = 0$). However, J' is the group scheme of centralizers of $\psi_0 + \tau^{d/m}\mathfrak{g}[\![\tau^{1/m}]\!]$, hence its reduction modulo $\tau^{1/m}$ is canonically trivialized.

Restricting both sides of γ' to $D_R^{(m)}$ we get an isomorphism of tori

$$\gamma_m : \mathbf{J}'|_{D_R^{(m)}} \cong \mathbf{C}' \widehat{\times}_k \mathrm{Spec}\ R$$

whose restriction to $D_\infty^{(m)}$ is the identity. It can be checked that the isomorphism γ_m is independent of how one conjugates $\psi + \mathfrak{g}(\!(\tau)\!)_{\leq 0}$ into $\psi_0\tau^{-d/m} + \mathfrak{g}[\![\tau^{1/m}]\!]$ inside $G(\!(\tau^{1/m})\!)$ (since J is commutative).

Both sides of γ_m admit μ_m-equivariant structures. It is easy to show that the isomorphism γ is compatible with the μ_m-equivariant structures (again it suffices to check it over the center of the disk $\mathrm{Spec}\ R$). Taking restriction of scalars from $D_R^{(m)}$ to D_R and taking μ_m-fixed points (see (3.11)), we get the desired isomorphism

$$\gamma_m : \mathbf{J}^\natural|_{D_R^{(m)}} \cong \mathbf{C}^\natural_\infty \widehat{\times}_k \mathrm{Spec}\ R. \qquad \square$$

By this lemma, $J, \mathbf{J}$ and $\mathbf{J}^+$ all admit canonical trivializations over $\psi + \mathfrak{g}(\!(\tau)\!)_{\le 0}$. Using the trivializations, the action of the group ind-scheme $J(\!(\tau)\!)$ on $\widehat{\mathcal{M}}_{\mathrm{Hod},\psi}$ over $\psi + \mathfrak{g}(\!(\tau)\!)_{\le 0}$ becomes an action of the constant group $C(\!(\tau)\!)$ on $\widehat{\mathcal{M}}_{\mathrm{Hod},\psi}$. On the quotient $\mathcal{M}_{\mathrm{Hod},\psi} = \widehat{\mathcal{M}}_{\mathrm{Hod},\psi}/\mathbf{K}_\infty$, the action of $\mathbf{C}^+_\infty \subset C(\!(\tau)\!)$ is trivial, hence we get an action of $C(\!(\tau)\!)/\mathbf{C}^+_\infty$ on $\mathcal{M}_{\mathrm{Hod},\psi}$.

Summarizing, we get:

3.4.5 Proposition. *There is a canonical action of $C(\!(\tau)\!)/\mathbf{C}^+_\infty$ on $\mathcal{M}_{\mathrm{Hod},\psi}$. The formal residue map $\mathrm{res}_{\mathrm{Hod},\infty}$ is equivariant under the $C(\!(\tau)\!)/\mathbf{C}^+_\infty$-action. Here, the action on $\mathfrak{c}_0$ factors through the lattice quotient $C(\!(\tau)\!)/\mathbf{C}^\natural_\infty \cong \mathbb{X}_*(T)_{\langle w\rangle}/\mathrm{tors}$, and is given by translation under the map $\mathrm{res}_{C,\infty}$ in (3.9).*

The same statements hold for ${}_{\mathfrak{h}}\mathcal{M}_{\mathrm{Hod},\psi}$.

4. The Betti moduli space

In this section, we first recall the moduli space $\mathcal{M}(\beta)$ defined using a positive braid β. When $k = \mathbb{C}$ and $G = \mathrm{GL}(n)$, we recall the interpretation of $\mathcal{M}(\beta)$ as a moduli space of Stokes filtered local systems. For arbitrary G, we define $\mathcal{M}_{\mathrm{Bet},\psi}$ as an enhanced version of $\mathcal{M}(\beta)$, and a set-theoretic map $\mathcal{M}_{\mathrm{dR},\psi} \to \mathcal{M}_{\mathrm{Bet},\psi}$ which is conjectured to be biholomorphic.

4.1. The stack $\mathcal{M}(\beta)$

In this subsection, G and k are in the same generality as §2.

Let $(\mathbf{W}, S)$ be the abstract Weyl group of G with simple reflections S, i.e., $\mathbf{W}$ is the set of G-orbits on $\mathcal{B}^2$, where $\mathcal{B}$ is the flag variety of G. For $w \in \mathbf{W}$, let $BS(w) \subset \mathcal{B}^2$ be the corresponding G-orbit.

Let H be the universal Cartan of G, i.e., the reductive quotient of any Borel subgroup of G. The abstract Weyl group $\mathbf{W}$ acts on H. It is conceptually important to distinguish between the maximal torus T attached to ψ and the universal Cartan H, and between the Weyl group $W = W(G, T)$ and the abstract Weyl group $\mathbf{W}$.

Let $Br_{\mathbf{W}}$ be the braid group of $\mathbf{W}$ and $Br^+_{\mathbf{W}}$ be the monoid of positive braids. For $w \in W$, let $\widetilde{w}$ be its canonical lifting to $Br^+_{\mathbf{W}}$ as a reduced word in S.

Let $\beta \in Br^+_{\mathbf{W}}$ and write

$$\beta = \widetilde{w}_1 \cdots \widetilde{w}_n \tag{4.1}$$

for a sequence of elements $w_1, \cdots, w_n \in \mathbf{W}$. Let $w \in \mathbf{W}$ be the image of β in $\mathbf{W}$, i.e., $w = w_1 \cdots w_n$.

4.1.1 Definition. Let $\mathcal{M}(\beta)$ be the moduli stack parametrizing:

(1) An $n+1$-tuple $(E_0, ..., E_n)$ of B-torsors over a point (or a test scheme).
(2) For $0 \leq i \leq n-1$, an isomorphisms of G-torsors $\iota_i : E_i \times^B G \to E_{i+1} \times^B G$, such that the two B-reductions of the identified G-torsor are in relative position w_i.
(3) An isomorphism of B-torsors $\tau : E_n \xrightarrow{\sim} E_0$.

By [13, Application 2], $\mathcal{M}(\beta)$ depends only on the positive braid β and not on the decomposition (4.1), up to a canonical isomorphism.

The composition of the isomorphisms $\iota_0, \cdots, \iota_{n-1}$ together with τ defines an automorphism of the G-torsor $E_0 \times^B G$, therefore a map

$$\mu_{\beta,G} : \mathcal{M}(\beta) \to [G/\mathrm{Ad}(G)]. \tag{4.2}$$

On the other hand, each E_i induces a T-torsor K_i via the surjection $B \twoheadrightarrow T$. The map $E_i \times^B G \to E_{i+1} \times^B G$ induces an isomorphism between K_i and $w_i(K_{i+1}) = K_{i+1} \times^{T,w_i} T$. Taking the composition of all these maps we get an isomorphism between K_0 and $w(K_0)$. This gives a map

$$\mu_{\beta,H} : \mathcal{M}(\beta) \to [H/\mathrm{Ad}_w(H)] \tag{4.3}$$

where $\mathrm{Ad}_w(H)$ means $t \in H$ is acting on H by $x \mapsto txw(t^{-1})$.

We give an alternative description of $\mathcal{M}(\beta)$, following the construction in [39] and [43]. Let $\mathcal{M}^\sharp(\beta)$ be the moduli of $(E_0, \cdots, E_n, \iota_i, \tau)$ as in $\mathcal{M}(\beta)$ together with a trivialization of $E_0 \times^B G$. The monodromy map 4.2 lifts to $\mathcal{M}^\sharp(\beta) \to G$. Via the isomorphisms of G-torsors, $(E_0, \cdots, E_n)$ give B-reductions of $E_0 \times^B G$, which via the trivialization give a tuple of Borel subgroups of G. We are led to the following description

$$\begin{aligned} \mathcal{M}^\sharp(\beta) \cong \{B_0, ..., B_n, g) \in \mathcal{B}^{n+1} \times G | (B_i, B_{i+1}) \in BS(w_i) \\ \text{for } 0 \leq i \leq n-1, \text{ and } B_n = {}^g B_0\}. \end{aligned}$$

The G-action on $\mathcal{M}^\sharp(\beta)$ by changing the trivialization of $E_0 \times^B G$ corresponds to the diagonal action on $B_i \in \mathcal{B}$ and the conjugation action on $g \in G$. Then

$$\mathcal{M}(\beta) = [G \backslash \mathcal{M}^\sharp(\beta)].$$

From this description we easily see that

4.1.2 Lemma. *For any $\beta \in Br^+_{\mathbf{W}}$, $\mathcal{M}(\beta)$ is a smooth algebraic stack over k of dimension $\ell(\beta)$, the length of β.*

4.1.3 Example. Consider the case $\beta = \widetilde{w}_0^2$ is the "full twist", where w_0 is the longest element in $\mathbf{W}$. This case will come up when we consider connections from $\mathcal{M}_{\mathrm{dR},\psi}$ when ψ is homogeneous of slope $\nu = 1$.

In this case, $\mathcal{M}^\sharp(\beta)$ classifies (B_0, B_1, B_2, g) where both pairs of Borel subgroups (B_0, B_1) and (B_1, B_2) are opposite, and ${}^gB_0 = B_2$. Fix a pair of opposite Borel subgroups B^+ and B^- with $T = B^+ \cap B^-$. Up to G-action we may assume $B_0 = B^+$ and $B_1 = B^-$. Then $(B^-, {}^gB^+)$ is in general position if and only if $g \in B^-B^+$. We get

$$\mathcal{M}(\widetilde{w}_0^2) \cong [B^-B^+/\mathrm{Ad}(T)].$$

The map $\mu_{\widetilde{w}_0^2,G} : \mathcal{M}(\widetilde{w}_0^2) \to [G/\mathrm{Ad}(G)]$ is induced from the inclusion $B^-B^+ \subset G$; the map $\mu_{\widetilde{w}_0^2,T} : \mathcal{M}(\widetilde{w}_0^2) \to [T/\mathrm{Ad}(T)]$ is induced from the projection $B^-B^+ = N^-TN^+ \to T$.

4.2. Stokes filtered local systems

From now on we set $k = \mathbb{C}$. In this subsection, we consider the case $G = \mathrm{GL}(n)$.

Given a meromorphic connection $(\mathcal{E}, \nabla)$ on a punctured disk $\Delta^\times$ with coordinate τ, taking analytic flat sections defines a local system on $\Delta^\times$ endowed with a Stokes structure at $\tau = 0$. We now recall the definition of this Stokes structure, following Sabbah's presentation in [37, Chapter 2]. The material that follows, up to and including Section 4.2.7, is well-known, and we include it for clarity.

Let $\widetilde{\Delta}(0)$ denote the real blow-up of the disk Δ at $\tau = 0$, with boundary circle $\mathbf{S}$. Consider the constant sheaf $\mathcal{J}_1$ on $\mathbf{S}$ with fiber

$$\mathcal{P} := \mathbb{C}(\!(\tau)\!)/\mathbb{C}[\![\tau]\!].$$

Sections are given by finite sums

$$\phi = \sum_{i<0} a_i \tau^i. \tag{4.4}$$

The stalk of $\mathcal{J}_1$ over $\theta \in \mathbf{S}$ is partially ordered by the rate of growth of a section as τ approaches 0 along the ray $\arg(\tau) = \theta$. We denote this order by $\leq_\theta$. Write $\mathbf{S}_{\phi\leq\chi}$ for the open subset of $\mathbf{S}$ on which $\phi \leq_\theta \chi$.

Let $\mathcal{L}$ be a local system on $\mathbf{S}$. An *unramified pre-Stokes filtration* on $\mathcal{L}$ is a collection of subsheaves $\mathcal{L}_{\leq\phi} \subset \mathcal{L}$ for $\phi \in \mathcal{P}$ such that for all $\nu \in \mathbf{S}$

$$\phi \leq_\theta \chi \implies \mathcal{L}_{\leq\phi,\theta} \subset \mathcal{L}_{\leq\chi,\theta}.$$

Let $\mathcal{L}_{<\phi}$ be the subsheaf of $\mathcal{L}_{\leq\phi}$ such that $\mathcal{L}_{<\phi,\theta} = \sum_{\chi<_\theta\phi} \mathcal{L}_{\leq\chi,\theta}$. Let $\mathrm{gr}_\phi\mathcal{L} = \mathcal{L}_{\leq\phi}/\mathcal{L}_{<\phi}$. We can associate to this filtration a $\mathcal{P}$-graded sheaf $\mathrm{gr}\mathcal{L} = \bigoplus_{\phi\in\Psi} \mathrm{gr}_\phi\mathcal{L}$, where $\Psi \subset \mathcal{P}$ is the finite subset consisting of ϕ such that $\mathrm{gr}_\phi\mathcal{L} \neq 0$. The subset Ψ is called the *exponential factors* of $(\mathcal{L}, \mathcal{L}_{\leq\bullet})$. The graded pieces are in general not locally constant on $\mathbf{S}$. When they are, the result is a $\mathcal{P}$-graded local system on $\mathbf{S}$, which in this context is called an *unramified Stokes-graded local system*.

Conversely, to an unramified Stokes-graded local system $\mathcal{L}_\bullet$, we can assign an unramified pre-Stokes filtration as follows.

4.2.1 Definition. Let $\Psi \subset \mathcal{P}$ be a finite subset, and let $\mathcal{L}_\bullet = \bigoplus_{\phi\in\Psi} \mathcal{L}_\phi$ be an unramified Stokes-graded local system. The graded unramified Stokes filtration on $\mathcal{L}$ is

$$\mathcal{L}_{\leq\phi} = \bigoplus_{\phi\in\Psi} \beta_{\chi\leq\phi}\mathcal{L}_\chi$$

where $\beta_{\chi\leq\phi}$ indicates restriction to the open $\mathbf{S}_{\chi\leq\phi}$ followed by extension by zero.

4.2.2 Definition. An unramified Stokes filtered local system $(\mathcal{L}, \mathcal{L}_{\leq\bullet})$ on $\mathbf{S}$ is a pre-Stokes filtration which is locally isomorphic to a graded unramified Stokes filtration.

The map $\tau \to \tau^m$ induces a map of real blow ups at $\tau = 0$. Restricting to the boundary circles, we obtain an m-fold cover $\rho_m : \mathbf{S}' \to \mathbf{S}$. Denote by $\sigma : \mathbf{S}' \to \mathbf{S}'$ the generator of the automorphism group of this cover given by $\sigma(\tau^{1/m}) = e^{2\pi i/m}\tau^{1/m}$.

Define $\mathcal{P}_m = \mathbb{C}(\!(\tau^{1/m})\!)/\mathbb{C}[\![\tau^{1/m}]\!]$. It carries a natural action of the Galois group $\langle\sigma\rangle \cong \mathbb{Z}/m\mathbb{Z}$.

4.2.3 Definition. A Stokes-filtered local system is a triple $(\mathcal{L}, \Psi, \mathcal{L}'_{\leq\bullet})$ where

(1) $\mathcal{L}$ is a local system on $\mathbf{S}$.
(2) $\Psi \subset \mathcal{P}_m$ is a finite subset stable under the action of the Galois group $\langle\sigma\rangle$.
(3) $\mathcal{L}'_{\leq\bullet}$ is a pre-Stokes filtration on $\mathcal{L}' := \rho_m^*\mathcal{L}$ such that $(\mathcal{L}', \mathcal{L}'_{\leq\bullet})$ is an unramified Stokes-filtered local system on $\mathbf{S}'$ with exponents Ψ. Moreover, we require the canonical isomorphism $\sigma^*\mathcal{L}' \cong \mathcal{L}'$ to identify the subsheaves $\mathcal{L}'_{\leq\phi}$ and $\mathcal{L}'_{\leq\sigma(\phi)}$, for all $\phi \in \mathcal{P}_m$.

From now on, we fix $\Psi \subset \mathbb{C}(\tau^{1/m})$ to be the set of eigenvalues of ψ. It consists of n distinct monomials of degree $-d/m$. We will be concerned with Stokes-filtered local systems $(\mathcal{L}, \Psi, \mathcal{L}'_{\leq\bullet})$ for which $\dim \mathrm{gr}_\phi\mathcal{L}' = 1$ for each

$\phi \in \Psi$. We fix a degree m cover $\mathbf{S}' \to \mathbf{S}$. Monodromy around $\tau = 0$ defines a permutation w of Ψ of order m.

Choose a base point $\theta \in \mathbf{S}$. Let $T \cong (\mathbb{G}_m)^\Psi$ denote the subgroup of graded automorphisms of the fiber of $\mathrm{gr}\mathcal{L}'$ at ν. Then w acts on T by permuting factors.

4.2.4 Definition. Let $T_{\langle w \rangle}$ be the coinvariant torus of T under the action of the cyclic group $\langle w \rangle$. The formal monodromy of $(\mathcal{L}, \Psi, \mathcal{L}'_{\leq \bullet})$ is the element in $T_{\langle w \rangle}$ defined by parallel transport in $\mathrm{gr}\mathcal{L}'$ from θ to $\sigma(\theta)$.

4.2.5. Moduli of Stokes filtered local systems We describe the moduli of Stokes local systems in our setting, when $G = \mathrm{GL}(n)$. We will give a more general construction, which makes sense for arbitrary G, in Section 4.3.

The exponents Ψ define a braid as follows. Each $\phi \in \Psi$ defines a function $\Re(\phi) : \mathbf{S}' \to \mathbb{R}$. We say $\theta \in \mathbf{S}$ is a Stokes direction if $\Re(\phi) = \Re(\chi)$ on a preimage of θ under $\mathbf{S}' \to \mathbf{S}$. To simplify the exposition, we assume that the Stokes directions of distinct pairs (ϕ, χ) are distinct. This holds for a dense set of ψ. This assumption will be lifted in Section 4.3.

The Stokes directions divide $\mathbf{S}$ into k Stokes sectors. Fix θ_0 in the interior of such a sector. As $G = GL(n)$, the braid β_ν arises from a loop $S^1 \to \mathrm{Config}_n(\mathbb{C})$, whose basepoint we take to be θ_0. The real projection of this loop is given by the union of graphs of $\Re(\phi), \phi \in \Phi$, viewed as multivalued functions on $\mathbf{S}$. In fact, β_ν can be reconstructed from these graphs. A Stokes sector determines a complete ordering on Ψ. A Stokes direction for the pair (ϕ, χ) determines a positive half-twist interchanging (ϕ, χ). The braid β_ν is the product of these half-twists.

4.2.6 Proposition. *Recall $G = \mathrm{GL}(n)$. The moduli stack of Stokes-filtered local systems with exponential factors Ψ is isomorphic to $\mathcal{M}(\beta)$. This isomorphism identifies the maps 4.2 and 4.3 with the monodromy and formal monodromy respectively.*

Proof. For any given Stokes sector, the set of framings of the fiber $\mathcal{L}$ compatible with the Stokes filtration is a B-torsor. This defines the $k+1$-tuple $(E_0, ..., E_k)$, where E_0 and E_k are both associated to the initial Stokes sector. The isomorphisms $E_i \times_B G \to E_{i+1} \times_B G$ are furnished by parallel transport. The resulting pair of B-reductions of the G-torsor are related by the reflection s_i associated to that Stokes direction.

The identification of the monodromy and formal monodromy is a direct consequence of the definitions. □

4.2.7. Riemann-Hilbert map Let Conn be the category of meromorphic connections $(\mathcal{E}, \nabla)$ on the punctured disk. Let Stokes be the category of Stokes-filtered local systems on $\mathbf{S}$. There is an 'irregular Riemann-Hilbert' functor

$$\mathrm{RH} : \mathrm{Conn} \to \mathrm{Stokes}, \quad (\mathcal{E}, \nabla) \to (\mathcal{L}, \mathcal{L}_{\leq \bullet}),$$

where $\mathcal{L}$ is the local system of ∇-flat sections of $\mathcal{E}$ away from ∞, and the filtration $\mathcal{L}_\bullet$ is by order of growth near ∞, which in this formulation is due to Deligne. It is an equivalence of categories.

The Stokes filtrations depend holomorphically on the connection ∇ in the following sense. Let $\nabla_u = d + A(\tau, u)\frac{d\tau}{\tau}$ be a family of connections on the trivial bundle $\mathcal{E} = \mathcal{O}^n$ over the analytic punctured disk $\Delta_\infty^\times$, depending holomorphically on an auxiliary variable u which varies in a domain $U \subset \mathbb{C}^N$. Suppose that the irregular part of $A(\tau, u)$ is constant in u, with regular semisimple leading term, and only finitely many coefficients of $A(\tau, u)$ are non-constant.

Each such connection determines the same set of Stokes directions $\mathrm{St}(\mathcal{E}, \nabla) \subset \mathbf{S}$. For each $u \in U$ and $\theta \in \mathbf{S} \setminus \mathrm{St}(\mathcal{E}, \nabla)$, we obtain a flag $B_{\theta,u}$ in the fiber $\mathcal{E}_\theta = \mathbb{C}^n$.

4.2.8 Lemma. *The map $U \to \mathcal{B}$ defined by the above construction is complex analytic.*

Proof. This follows from [2, Remark 1.8], where it is explained that the sectorial flat sections of $(\mathcal{E}, \nabla)$, out of which the Stokes filtrations are constructed, vary holomorphically with ∇. $\square$

4.3. Riemann-Hilbert map for *G*-connections

Now we are back in the setting of general reductive group G, and $k = \mathbb{C}$.

4.3.1. Stokes directions for *G*-connections Let $(\mathcal{E}, \nabla)$ be a meromorphic G-connection on the punctured disk $\Delta^\times$ with coordinate τ. The restriction of $(\mathcal{E}, \nabla)$ to the formal punctured disk $D_\infty^\times$ (around $\tau = 0$), after passing to a ramified cover with parameter $\tau^{1/m}$, can be formally gauge transformed to $\nabla \in d + (B(\tau^{1/m}) + \mathfrak{g}[\![\tau^{1/m}]\!]) d\tau/\tau$, for some irregular part $B(\tau^{1/m}) \in \mathfrak{t}[\tau^{-1/m}]$ (where $\mathfrak{t} \subset \mathfrak{g}$ is a Cartan subalgebra). Recall that $\mathfrak{a} = \mathfrak{t} /\!\!/ W$. The image of $B(\tau^{1/m})$ under the projection $\mathfrak{t}[\tau^{-1/m}] \to \mathfrak{a}[\tau^{-1/m}]$ lies in $\mathfrak{a}[\tau^{-1}]$, which we denote by $A(\tau) \in \mathfrak{a}[\tau^{-1}]$.

Below we assume that $B(\tau^{1/m})$ is regular semisimple, i.e., for any root α of $\mathfrak{g}$ with respect to $\mathfrak{t}$, $\alpha(B(\tau^{1/m})) \neq 0$.

For any finite-dimensional representation V of G, we have the associated local system $\mathcal{E}(V)^\nabla$ of flat sections of $\mathcal{E}(V) = \mathcal{E} \times^G V$ on $\mathbb{P}^1 \backslash \{0\}$. For $\theta \in \mathbf{S}$, we have a subspace $M(V)_\theta \subset (\mathcal{E}(V)^\nabla)_\theta$ (meaning the stalk of $\mathcal{E}(V)^\nabla$ along any point in the ray θ) consisting of solutions defined in a small sector containing the ray of θ, that are of maximal decay along the ray θ. Since $B(\tau^{1/m})$ is regular semisimple, for V irreducible $\dim M(V)_\theta = 1$ for all but finitely many $\theta \in \mathbf{S}$. Note that

$$M(V_1 \otimes V_2)_\theta = M(V_1)_\theta \otimes M(V_2)_\theta \subset \mathcal{E}(V_1 \otimes V_2)^\nabla_\theta. \tag{4.5}$$

A point $\theta \in \mathbf{S}$ is called a *Stokes direction* for $(\mathcal{E}, \nabla)$ if for some irreducible representation V of G, $\dim M(V)_\theta > 1$. We claim that there are only finitely many Stokes directions. Indeed, $V_\lambda \in \mathrm{Rep}(G)$ is the irreducible representation with highest weight λ, and for some finite subset $\{\lambda_1, \cdots, \lambda_N\}$ that generate the monoid of dominant weights of G, $\dim M(V_{\lambda_i})_\theta = 1$ for $1 \leq i \leq N$ implies $\dim M(V_\lambda)_\theta = 1$ for all λ (using (4.5)).

Let $\mathrm{St}(\mathcal{E}, \nabla) \subset \mathbf{S}$ be the set of Stokes directions. A connected component of $\mathbf{S} \backslash \mathrm{St}(\mathcal{E}, \nabla)$ is called a *Stokes sector*.

Below, we fix a base point $\theta_0 \in \mathbf{S}$. We label elements in $\mathrm{St}(\mathcal{E}, \nabla)$ counterclockwisely as $\{\sigma_1, \sigma_2, \cdots, \sigma_n\}$, starting with the one immediately next to θ_0 in the counterclockwise direction. Let $I_i = (\sigma_i, \sigma_{i+1})$ be the Stokes sectors (for $0 \leq i \leq n-1$, and $\sigma_0 := \sigma_n$).

4.3.2 Construction (B-torsors for each sector)**.** For $\theta \in \mathbf{S} \backslash \mathrm{St}(\mathcal{E}, \nabla)$, we define a B-reduction of $\mathcal{E}_\theta$ as follows. Indeed, the assignment $V \mapsto M(V)_\theta \subset \mathcal{E}(V)^\nabla_\theta$ satisfies the relation (4.5) and $M(V)_\theta$ is one-dimensional for V irreducible. Such data determines a B-reduction of the G-torsor $\mathcal{E}^\nabla_\theta$ along the ray θ. Clearly, this B-reduction is locally constant as θ moves in a Stokes sector. Therefore, on each Stokes sector $I_i \subset \mathbf{S} \backslash \mathrm{St}(\mathcal{E}, \nabla)$, we have a canonical B-torsor E_i constructed from $(\mathcal{E}, \nabla)$, for $0 \leq i \leq n-1$. We let $E_n = E_0$.

4.3.3 Construction (The braid)**.** Let S^1_ϵ be the circle of radius $\epsilon > 0$ around $\tau = 0$. There is a canonical isomorphism $S^1_\epsilon \cong \mathbf{S}$. Restricting the map $A : \mathbb{A}^1_{\tau^{-1}} \to \mathfrak{a} = \mathfrak{h} /\!/ \mathbf{W}$ to S^1_ϵ the image lands in $\mathfrak{a}^{\mathrm{rs}}$.

Fix a base point $\theta_0 \in \mathbf{S}$, which gives a corresponding base point $\epsilon e^{i\theta_0} \in S^1_\epsilon$. If $\widetilde{\theta}_0$ is a lifting of θ_0 to $\mathbb{R}$, we abuse the notation to denote the interval $[\widetilde{\theta}_0, \widetilde{\theta}_0 + 2\pi]$ by $[\theta_0, \theta_0 + 2\pi]$. Let $a_0 = A(\epsilon e^{i\theta_0}) \in \mathfrak{a}^{\mathrm{rs}}$. Choose a lifting $\widetilde{a}_0 \in \mathfrak{h}^{\mathrm{rs}}$ of a_0. Then $A|_{S^1_\epsilon} : S^1_\epsilon \to \mathfrak{a}^{\mathrm{rs}}$ lifts uniquely to $\widetilde{A}_\epsilon : [\theta_0, \theta_0 + 2\pi] \to \mathfrak{h}^{\mathrm{rs}}$ with $\widetilde{A}_\epsilon(\theta_0) = a_0$.

Let $\widetilde{\sigma}_i$ be the preimage of σ_i in $[\theta_0, \theta_0 + 2\pi]$, and similarly let $\widetilde{I}_i \subset [\theta_0, \theta_0 + 2\pi]$ be the preimage of I_i. Note $\widetilde{I}_0 = [\theta_0, \widetilde{\sigma}_1] \sqcup [\widetilde{\sigma}_n, \theta_0 + 2\pi]$. We denote $J_i = [\theta_0, \widetilde{\sigma}_1]$ if $i = 0$, $J_i = \widetilde{I}_i$ for $1 \leq i \leq n-1$, and $J_n = [\widetilde{\sigma}_n, \theta_0 + 2\pi]$.

Consider the projection $\widetilde{A}_{\mathbb{R},\epsilon} : [\theta_0, \theta_0 + 2\pi] \xrightarrow{\widetilde{A}_\epsilon} \mathfrak{h}^{\mathrm{rs}} \to \mathfrak{h}_{\mathbb{R}}$ (projection to the real part). Then $\{\widetilde{\sigma}_1, \cdots, \widetilde{\sigma}_n\}$ is precisely the preimage of the root hyperplanes in $\mathfrak{h}_{\mathbb{R}}$ under $\widetilde{A}_\epsilon$. The image $\widetilde{A}_\epsilon(J_i)$ is contained in a unique Weyl chamber $C_i \subset \mathfrak{h}^{\mathrm{rs}}_{\mathbb{R}}$. For ϵ sufficiently small, C_i are independent of ϵ. The relative positions of two Weyl chambers in $\mathfrak{h}_{\mathbb{R}}$ are indexed by $\mathbf{W}$. Let $w_i \in \mathbf{W}$ be the relative position of the Weyl chambers (C_{i-1}, C_i) for $1 \le i \le n$. Then define

$$\beta_{\theta_0} = \widetilde{w}_1 \widetilde{w}_2 \cdots \widetilde{w}_n \in Br^+_{\mathbf{W}}.$$

Recall for $w \in \mathbf{W}$, we write $\widetilde{w} \in Br^+_{\mathbf{W}}$ its canonical lift. The braid β_{θ_0} does not depend on the lifting $\widetilde{a}_0$ of a_0. Changing the base point θ_0, β_{θ_0} changes by a cyclic shift of words.

4.3.4 Remark. There is another natural way to get a conjugacy class in $Br_{\mathbf{W}}$ from $A(\tau)$. The map $A|_{S^1_\epsilon}$ gives an element in $\pi_1(\mathfrak{a}^{\mathrm{rs}}, A(\epsilon e^{i\theta_0}))$, which is isomorphic to $Br_{\mathbf{W}}$ (and the isomorphism is unique up to conjugacy). We thus get a conjugacy class $[\beta]$ in $Br_{\mathbf{W}}$. One can show that $[\beta]$ is independent of ϵ and θ_0 as long as ϵ is sufficiently small, and that β_{θ_0} defined above belongs to the conjugacy class $[\beta]$.

Recall the irregularity of the adjoint connection $(\mathrm{Ad}(\mathcal{E}), \nabla)$ associated with $(\mathcal{E}, \nabla)$ is defined to be

$$\mathrm{Irr}(\mathrm{Ad}(\mathcal{E}), \nabla) = \sum_{\alpha \in \Phi} -\operatorname{ord}_{\tau=0} \alpha(B(\tau^{1/m})).$$

4.3.5 Lemma. *We have* $\ell(\beta) = \mathrm{Irr}(\mathrm{Ad}(\mathcal{E}), \nabla)$.

Proof. We first consider the case $m = 1$, i.e., $B(\tau) \in \mathfrak{h}[\tau^{-1}]$. On the one hand, $\ell(\beta) = \sum_{i=1}^n \ell(w_i)$, and $\ell(w_i)$ is the number of root hyperplanes separating C_{i-1} and C_i. Therefore $\ell(\beta)$ is the number of times the image of $B_{\mathbb{R}}|_{S^1_\epsilon}$ crosses the root hyperplanes (where $B_{\mathbb{R}}$ is the real projection $\mathbb{A}^1_{\tau^{-1}} \to \mathfrak{h} \to \mathfrak{h}_{\mathbb{R}}$). For the root hyperplane H_α defined by $\alpha = 0$, $B_{\mathbb{R}}|_{S^1_\epsilon}$ intersects H_α exactly when $B_\alpha(\tau) = \alpha(B(\tau))$ takes values in $i\mathbb{R}$ for $|\tau| = \epsilon$. For $\epsilon \ll 1$, the map $B_\alpha : S^1_\epsilon \to \mathbb{C}^\times$ has mapping degree $-\operatorname{ord}_{\tau=0} \alpha(B(\tau))$, and $B_\alpha|_{S^1_\epsilon}$ intersects $i\mathbb{R}$ transversely (with the same sign of intersection) in $-2\operatorname{ord}_{\tau=0} \alpha(B(\tau))$ times. The total number of times $B_{\mathbb{R}}|_{S^1_\epsilon}$ crosses the root hyperplanes is

$$\sum_{\alpha \in \Phi^+} -2\operatorname{ord}_{\tau=0} \alpha(B(\tau)) = \sum_{\alpha \in \Phi} -\operatorname{ord}_{\tau=0} \alpha(B(\tau))$$

which is $\mathrm{Irr}(\mathrm{Ad}(\mathcal{E}), \nabla)$.

In general, let $\pi : \Delta^\times_{\tau^{1/m}} \to \Delta^\times_\tau$ be the projection. Then the irregular part of $\pi^*(\mathcal{E},\nabla)$ lies in $\mathfrak{h}[\tau^{-1/m}]$. Let $\widetilde{\beta}$ be the braid attached to $\pi^*(\mathcal{E},\nabla)$. On the one hand we have $\mathrm{Irr}(\mathrm{Ad}(\mathcal{E}),\nabla) = \frac{1}{m}\mathrm{Irr}(\pi^*\mathrm{Ad}(\mathcal{E}),\nabla)$, which is equal to $\frac{1}{m}\ell(\widetilde{\beta})$ by the previous paragraph. On the other hand, from the construction of $\widetilde{\beta}$ one sees that $\widetilde{\beta} = \beta_1\beta_2\cdots\beta_m$ where each β_i is a positive braid with the same length as β. Therefore $\ell(\widetilde{\beta}) = m\ell(\beta)$. Combining these facts we get $\mathrm{Irr}(\mathrm{Ad}(\mathcal{E}),\nabla) = \frac{1}{m}\ell(\widetilde{\beta}) = \ell(\beta)$. □

4.3.6 Construction (A point in $\mathcal{M}(\beta_{\theta_0})$)**.** We shall construct a point in $\mathcal{M}(\beta_{\theta_0})$ from $(\mathcal{E},\nabla)$. In Construction 4.3.2 we have constructed a B-torsor E_i for $0 \le i \le n$, with $E_n = E_0$. Let $\tau = \mathrm{id} : E_n \xrightarrow{\sim} E_0$ be the identity isomorphism. For each σ_i, let $\sigma_i^- \in I_{i-1}$ and $\sigma_i^+ \in I_i$, then parallel transport along the path from σ_i^- to σ_i^+ that passes through σ_i identifies the stalks $\mathcal{E}^\nabla_{\sigma_i^-}$ and $\mathcal{E}^\nabla_{\sigma_i^+}$. This gives a canonical isomorphism of G-torsors

$$\iota_{i-1} : E_{i-1} \times^B G \cong \mathcal{E}^\nabla_{\sigma_i^-} \cong \mathcal{E}^\nabla_{\sigma_i} \cong \mathcal{E}^\nabla_{\sigma_i^+} \cong E_i \times^B G. \tag{4.6}$$

By the lemma below, the data $(E_0,\cdots,E_n,\iota_0,\cdots,\iota_{n-1},\mathrm{id})$ defines a point in $\mathcal{M}(\beta_{\theta_0})$.

4.3.7 Lemma. *The relative position of the two B-reductions E_{i-1} and E_i of $\mathcal{E}^\nabla_{\sigma_i}$ (see (4.6)) is equal to $w_i \in \mathbf{W}$ defined in Construction 4.3.3.*

Proof. We treat the case $m = 1$ below; the general case is proved by the same argument after pullback to the cover $\Delta^\times_{\tau^{1/m}} \to \Delta^\times_\tau$. Also, without loss of generality we may assume $i = 1$. Let $w \in \mathbf{W}$ be the relative position of E_0 and E_1.

Let $\sigma^- \in I_0$ and $\sigma^+ \in I_1$. Let $x^-_\epsilon = \Re B(\epsilon e^{i\sigma^-}) \in C_0$ (the dominant Weyl chamber, for $\epsilon \ll 1$), and $x^+_\epsilon = \Re B(\epsilon e^{i\sigma^+}) \in C_1$ for $\epsilon \ll 1$.

Fix a regular anti-dominant weight λ, and denote by V_λ the irreducible representation of G with lowest weight λ. Let Ω_λ be the set of weights of V_λ. The B-reduction E_- gives an increasing filtration $F_\mu\mathcal{E}(V_\lambda)^\nabla_{\sigma^-}$ on $\mathcal{E}(V_\lambda)^\nabla_{\sigma^-}$ indexed by the poset Ω_λ (where $\mu_1 \ge \mu_2$ if and only if $\mu_1 - \mu_2$ is a sum of positive coroots). Note that the smallest sub $F_\lambda\mathcal{E}(V_\lambda)^\nabla_{\sigma^-}$ is the maximal decay line $M(V_\lambda)_{\sigma^-}$. On the other hand, the maximal decay line $M(V_\lambda)_{\sigma^+} \subset \mathcal{E}(V_\lambda)^\nabla_{\sigma^+}$, after parallel transport to σ^- via the path through σ_1, gives a line $\mathcal{L}_+ \subset \mathcal{E}(V_\lambda)^\nabla_{\sigma^-}$. Since λ is regular, the relative position w can be characterized as follows: it is the unique element $w \in \mathbf{W}$ such that $\mathcal{L}_+ \subset F_{w\lambda}\mathcal{E}(V_\lambda)^\nabla_{\sigma^-}$ and $\mathcal{L}_+$ maps injectively to the associated graded $\mathrm{Gr}^F_{w\lambda}\mathcal{E}(V_\lambda)^\nabla_{\sigma^-}$. We would like to show that this property of w implies $w = w_1$.

Let $\Psi_\lambda \subset \mathcal{P}$ be the set of exponential factors of the connection $(\mathcal{E}(V_\lambda), \nabla)$. Then Ψ_λ is the image of $\Omega_\lambda \to \mathcal{P}$ given by $\mu \mapsto \langle \mu, B(\tau) \rangle$. As a Stokes filtered local system, $\mathcal{E}(V_\lambda)^\nabla$ is locally (near σ_1) isomorphic to a Stokes graded local system $\mathcal{L} \cong \oplus_{\chi \in \Psi_\lambda} \mathcal{L}_\chi$. We may assume $\sigma^\pm$ to be close enough to σ_1 so that this isomorphism is defined along the arc $[\sigma^-, \sigma^+]$. In particular, there is a unique $\chi^+ \in \Psi_\lambda$ such that $\mathcal{L}_{\chi^+, \sigma^+} = M(V_\lambda)_{\sigma^+}$.

We claim that $\chi^+ = \langle w_1 \lambda, B(\tau) \rangle$. Indeed, write $\chi^+ = \langle \mu, B(\tau) \rangle$ for some $\mu \in \Omega_\lambda$. Since $\mathcal{L}_{\chi^+, \sigma^+}$ is the maximal decay line at σ^+, for $\epsilon \ll 1$, $\langle \mu, x^+_\epsilon \rangle \leq \langle \mu', x^+_\epsilon \rangle$ for all $\mu' \in \Omega_\lambda$. Equivalently, $\langle w_1^{-1} \mu, w_1^{-1} x^+_\epsilon \rangle \leq \langle \mu', w_1^{-1} x^+_\epsilon \rangle$ for all $\mu' \in \Omega_\lambda$. Since $w_1^{-1} x^+_\epsilon \in w_1^{-1} C_1 = C_0$, which is the dominant Weyl chamber, the above inequality implies $w_1^{-1} \mu = \lambda$ (the minimal element in Ω_λ), i.e., $\mu = w_1 \lambda$, hence $\chi^+ = \langle w_1 \lambda, B(\tau) \rangle$.

Now $\mathcal{L}_+ = \mathcal{L}_{\chi^+, \sigma^-}$. By the definition of w, $\mathcal{L}_+ = \mathcal{L}_{\chi^+, \sigma^-}$ lies in $F_{w\lambda} \mathcal{E}(V_\lambda)^\nabla_{\sigma^-}$ and maps injectively to the associated graded $\mathrm{Gr}_{w\lambda} \mathcal{E}(V_\lambda)^\nabla_{\sigma^-}$. This means χ^+ has the same decay rate as $\langle w\lambda, B(\tau) \rangle$ along the ray σ^-. Using $\chi^+ = \langle w_1 \lambda, B(\tau) \rangle$, this implies $\langle w_1 \lambda, x^-_\epsilon \rangle = \langle w\lambda, x^-_\epsilon \rangle$ for $\epsilon \ll 1$. Since x^-_ϵ is in the interior of C_0, this forces $w_1 \lambda = w\lambda$. Since λ is regular, we conclude that $w_1 = w$. $\square$

4.4. Enhanced Riemann-Hilbert map

4.4.1. Specialization to our setting Consider the context of a homogeneous element $\psi \in \mathfrak{g}(\!(t)\!)$ of slope $\nu = d/m$. For any connection $(\mathcal{E}, \nabla) \in \mathcal{M}_{\mathrm{dR},\psi}$ and the choice of a base point $\theta_0 \in \mathbf{S}$ that is not a Stokes direction, Construction 4.3.3 gives a positive braid β_{θ_0} that depends on ψ but not otherwise on $(\mathcal{E}, \nabla)$. By Remark 4.3.4, β_{θ_0} is a positive braid representing the loop $S^1 \to \mathfrak{a} = \mathfrak{h} /\!\!/ \mathbf{W}$ given by restricting the map $\chi(\psi) : \mathbb{C}^\times \to \mathfrak{g} \xrightarrow{\chi} \mathfrak{a}$. One can show that β_{θ_0} depends only on the slope ν and not otherwise on ψ. We henceforth denote β_{θ_0} by β_{ν,θ_0}, or simply β_ν is the base point θ_0 is fixed. The image of β in $\mathbf{W}$ is conjugate to w^d, which is in turn conjugate to w because both w and w^d are regular of order m.

When ψ is elliptic, we can compute β_ν as follows. We have the regular element $w \in \mathbf{W}$ of order m, unique up to conjugacy. Let us assume w has minimal length in its conjugacy class (then $\ell(w) = |\Phi|/m$). One can take $\beta_\nu = \widetilde{w}^d$, where $\widetilde{w} \in Br^+_{\mathbf{W}}$ is the lifting of w to a positive braid by any reduced expression. We have $\ell(\beta_\nu) = \nu|\Phi|$. Note that w (hence $\widetilde{w}$) is not unique, but different choices of minimal length w differ by cyclic shift of words, as shown by He-Nie in [21, Corollary 4.4]. Therefore, the resulting β_ν also differ by a cyclic shift, which then give isomorphic $\mathcal{M}(\beta_\nu)$.

Constructions 4.3.2 and 4.3.6 give a set theoretic map

$$\mathrm{RH} : \mathcal{M}_{\mathrm{dR},\psi} \to \mathcal{M}(\beta_\nu).$$

When $G = GL(n)$, Lemma 4.2.8 shows it is a holomorphic map.

4.4.2. The enhanced RH map Recall that $\mathcal{M}(\beta_\nu)$ is equipped with two monodromy maps: $\mu_{\beta_\nu,G}$ to $[G/\mathrm{Ad}(G)]$ (which we simply write $[G/G]$ below) and $\mu_{\beta_\nu,H}$ to $[H/\mathrm{Ad}_w(H)]$. Since the connections classified by the global moduli $\mathcal{M}_{\mathrm{dR},\psi}$ also has regular singularities at 0 with nilpotent monodromy lying in a Borel reduction at 0. We have a map

$$\mathrm{res}_{\mathrm{dR},0} : \mathcal{M}_{\mathrm{dR},\psi} \to [\widetilde{\mathcal{N}}/G] = [\mathfrak{n}/B]$$

where $\widetilde{\mathcal{N}} \to \mathcal{N}$ is the Springer resolution of the nilpotent cone $\mathcal{N} \subset \mathfrak{g}$. Let $\widetilde{\mathcal{U}} \to \mathcal{U}$ be the Springer resolution of the unipotent variety $\mathcal{U} \subset G$. The following diagram is commutative by construction

$$\begin{tikzcd}
\mathcal{M}_{\mathrm{dR},\psi} \arrow[rr, "\mathrm{RH}"] \arrow[d, "\mathrm{res}_{\mathrm{dR},0}"] & & \mathcal{M}(\beta_\nu) \arrow[d, "\mu_{\beta_\nu,G}"] \\
{[\widetilde{\mathcal{N}}/G]} \arrow[r, "\exp", "\sim"'] & {[\widetilde{\mathcal{U}}/G]} \arrow[r] & {[G/G]}
\end{tikzcd} \tag{4.7}$$

On the other hand, looking at the formal residue and formal monodromy at ∞, the following diagram should also be commutative

$$\begin{tikzcd}
\mathcal{M}_{\mathrm{dR},\psi} \arrow[rr, "\mathrm{RH}"] \arrow[d, "\mathrm{res}_{\mathrm{dR},\infty}"] & & \mathcal{M}(\beta_\nu) \arrow[d, "\mu_{\beta_\nu,H}"] \\
\mathfrak{c}_0 \arrow[r, "\exp"] & T^{w,\circ} \arrow[r, "\xi_{\theta_0}"] & {[H/\mathrm{Ad}_w(H)]}
\end{tikzcd}$$

Here ξ_{θ_0} is defined as follows. Let $\theta_0' \in \mathbf{S}'$ be a preimage of θ_0 under the degree m cyclic covering $\rho_m : \mathbf{S}' \to \mathbf{S}$. The adjoint Cartan $\mathfrak{t}^{\mathrm{ad}}$ has a canonical real form $\mathfrak{t}^{\mathrm{ad}}_{\mathbb{R}}$ consisting of elements whose value under any root is real. Recall $\overline{\psi} \in \mathfrak{t}$. For $\epsilon > 0$ consider the image of $\overline{\psi}(\epsilon e^{i\widetilde{\theta_0}})^d$ under $\mathfrak{t} \twoheadrightarrow \mathfrak{t}^{\mathrm{ad}} \twoheadrightarrow \mathfrak{t}^{\mathrm{ad}}_{\mathbb{R}}$. Then for $\epsilon \ll 1$ the image lies in a Weyl chamber $C_{\theta_0'} \subset \mathfrak{t}^{\mathrm{ad}}_{\mathbb{R}}$ independent of ϵ. Let $B_{\theta_0'}$ be the Borel subgroup of G containing T corresponding to the chamber $-C_{\theta_0'}$. Thus we have an isomorphism of tori $\iota_{\theta_0'} : T \hookrightarrow B_{\theta_0'} \twoheadrightarrow H$. Changing the choice of θ_0' changes $C_{\theta_0'}$ by the action of the cyclic group $\langle w \rangle$, therefore $\iota_{\theta_0'}|_{T^w}$ is independent of the choice of θ_0', which we denote by ι_{T^w}. The map

ξ_{θ_0} is the composition $T^{w,\circ} \xrightarrow{\iota_{T^w}} H \to [H/\mathrm{Ad}_w(H)]$ where the last map is the composition.

More generally, we consider the variant ${}_s\mathcal{M}_{\mathrm{dR},\psi}$ defined in §3.3.4 for an arbitrary residue $s \in \mathfrak{h}$ at 0. Let $\kappa = \exp(s) \in H$. Let $\widetilde{\mathfrak{g}} \to \mathfrak{g}$ and $\widetilde{G} \to G$ be the Grothendieck alterations. Let $\mathrm{pr}_{\mathfrak{h}} : \widetilde{\mathfrak{g}} \to \mathfrak{h}$ and $\mathrm{pr}_H : \widetilde{G} \to H$ be the projections. Let $\widetilde{\mathfrak{g}}[s] = \mathrm{pr}_{\mathfrak{h}}^{-1}(s) \subset \widetilde{\mathfrak{g}}$ and $\widetilde{G}[\kappa] = \mathrm{pr}_H^{-1}(\kappa) \subset \widetilde{G}$. Then the diagram (4.7) becomes

$$\begin{tikzcd} {}_s\mathcal{M}_{\mathrm{dR},\psi} \arrow[rr, "\mathrm{RH}"] \arrow[d, "\mathrm{res}_{\mathrm{dR},0}"] & & \mathcal{M}(\beta_\nu) \arrow[d, "\mu_{\beta_\nu,G}"] \\ {[\widetilde{\mathfrak{g}}[s]/G]} \arrow[r, "\exp", "\sim"'] & {[\widetilde{G}[\kappa]/G]} \arrow[r] & {[G/G]} \end{tikzcd}$$

These diagrams motivate the following definition.

4.4.3 Definition. Let ψ be a homogeneous element of slope ν.

(1) (Trinh [43]) Form the *derived fiber product*

$$\mathcal{M}^0_{\mathrm{Bet},\psi} := \mathcal{M}(\beta_\nu) \times^{\mathbf{R}}_{[G/G]} [\widetilde{\mathcal{U}}/G],$$

(2) We define the Betti moduli space attached to ψ to be the derived complex analytic stack

$$\mathcal{M}_{\mathrm{Bet},\psi} := \mathfrak{c}_0 \times_{[H/\mathrm{Ad}_w(H)]} \mathcal{M}^0_{\mathrm{Bet},\psi}.$$

Here, the map $\mathfrak{c}_0 \to [H/\mathrm{Ad}_w(H)]$ is given by the exponential map $\mathfrak{c}_0 \cong \mathfrak{t}^w \xrightarrow{\exp} H^{w,\circ} \subset H$ followed by the quotient map.

(3) More generally, for $\kappa \in H$, we define

$$\begin{aligned} {}_\kappa\mathcal{M}^0_{\mathrm{Bet},\psi} &:= \mathcal{M}(\beta_\nu) \times^{\mathbf{R}}_{[G/G]} [\widetilde{G}[\kappa]/G] \\ {}_\kappa\mathcal{M}_{\mathrm{Bet},\psi} &:= \mathfrak{c}_0 \times_{[H/\mathrm{Ad}_w(H)]} {}_\kappa\mathcal{M}^0_{\mathrm{Bet},\psi}. \end{aligned}$$

Note when w is elliptic, ${}_s\mathcal{M}_{\mathrm{Bet},\psi} \to {}_\kappa\mathcal{M}^0_{\mathrm{Bet},\psi}$ is a H^w-torsor, and ${}_\kappa\mathcal{M}_{\mathrm{Bet},\psi}$ is a derived algebraic stack.

4.4.4 Conjecture. *Let $\kappa \in H$.*

(1) For $\nu > 0$, the derived structure on ${}_\kappa\mathcal{M}^0_{\mathrm{Bet},\psi}$ is trivial, and ${}_\kappa\mathcal{M}^0_{\mathrm{Bet},\psi}$ is an algebraic stack smooth over $\mathbb{C}$.

(2) The analytic stack ${}_\kappa\mathcal{M}_{\mathrm{Bet},\psi}$ is a complex analytic manifold with a canonical symplectic structure.

By the above diagrams and Definition 4.4.3, the map RH lifts to

$$\widetilde{\mathrm{RH}} : {}_s\mathcal{M}_{\mathrm{dR},\psi} \to \mathfrak{c}_0 \times_{[H/\mathrm{Ad}_w(H)]} \mathcal{M}(\beta_\nu) \times^{\mathbf{R}}_{[G/G]} [\widetilde{G}[\kappa]/G] = {}_\kappa\mathcal{M}_{\mathrm{Bet},\psi}. \tag{4.8}$$

4.4.5 Conjecture. *For any reductive G over $\mathbb{C}$ and homogeneous element ψ in $\mathfrak{g}(\!(t)\!)$ of slope $\nu > 0$, and any $s \in \mathfrak{h}$ with $\kappa = \exp(s)$, the map $\widetilde{\mathrm{RH}}$ in (4.8) is an analytic isomorphism.*

We plan to return to this conjecture in a future work. As a consistency check, we compare the dimensions of $\mathcal{M}_{\mathrm{dR},\psi}$ and $\mathcal{M}_{\mathrm{Bet},\psi}$. On one hand, we have by Theorem 3.1.4 and Theorem 2.8.1(1)

$$\dim \mathcal{M}_{\mathrm{dR},\psi} = \dim \mathcal{M}_\psi = \nu|\Phi| - r + \dim \mathfrak{t}^w.$$

On the other hand, by Lemma 4.1.2 we have

$$\dim \mathcal{M}(\beta_\nu) = \ell(\beta_\nu).$$

By Lemma 4.3.5, we have for any $(\mathcal{E}, \nabla) \in \mathcal{M}_{\mathrm{dR},\psi}$

$$\ell(\beta_\nu) = \mathrm{Irr}(\mathrm{Ad}(\mathcal{E}), \nabla) = \nu|\Phi|.$$

Combining the two equations we get

$$\dim \mathcal{M}(\beta_\nu) = \nu|\Phi|.$$

Therefore the *derived dimension* of $\mathcal{M}_{\mathrm{Bet},\psi}$ is

$$\begin{aligned}\dim \mathcal{M}(\beta_\nu) + (\dim[\widetilde{\mathcal{U}}/G] - \dim[G/G]) + (\dim \mathfrak{c}_0 - \dim[H/\mathrm{Ad}_w(H)])\\ = \nu|\Phi| - r + \dim \mathfrak{t}^w.\end{aligned}$$

Therefore $\mathcal{M}_{\mathrm{dR},\psi}$ has the same dimension as the derived version of $\mathcal{M}_{\mathrm{Bet},\psi}$.

4.4.6 Remark. By Theorem 3.3.5, the cohomology $\mathrm{H}^*({}_s\mathcal{M}_{\mathrm{dR},\psi})$ is canonically independent of $s \in \mathfrak{h}$. Combined with Conjecture 4.4.5, it then implies that the cohomology $\mathrm{H}^*({}_\kappa\mathcal{M}_{\mathrm{Bet},\psi})$ is independent of $\kappa \in H$ (canonically upon choosing a logarithm of κ). Keeping track of the symmetry by $(C(\!(\tau)\!)/\mathbf{C}^+_\infty)^{\mathrm{red}}$ on ${}_s\mathcal{M}_{\mathrm{dR},\psi}$, this also implies the statement that the cohomology of ${}_\kappa\mathcal{M}^0_{\mathrm{Bet},\psi}$ is independent of κ. In other words, the direct image complex of the map

$$\mathcal{M}(\beta_\nu) \times_{[G/G]} [\widetilde{G}/G] \to [\widetilde{G}/G] \xrightarrow{\mathrm{pr}_H} H$$

should be locally constant. This direct image complex can be interpreted as the parabolic restriction of character sheaves. It would be interesting to prove the local constancy statement directly for a wider class of positive braids β and not just those coming from homogeneous ψ.

4.4.7 Remark. Instead of making a base change of $\mathcal{M}(\beta_\nu)$ along $\mathfrak{c}_0 \to [H/\mathrm{Ad}_w(H)]$, one can reformulate Conjecture 4.4.5 by taking a quotient of $\mathcal{M}_{\mathrm{dR},\psi}$.

It is clear that the map RH is invariant under the action of $C(\!(\tau)\!)/\mathbf{C}^+_\infty$ on the domain, because that action does not change the isomorphism class of the connection on $X\backslash\{\infty\}$.

Consider the action of the reduced part $(C(\!(\tau)\!)/\mathbf{C}^+_\infty)^{\mathrm{red}}$ on $\mathcal{M}_{\mathrm{dR},\psi}$. By (3.8), this group is an extension of the lattice $(C(\!(\tau)\!)/\mathbf{C}^\natural_\infty)^{\mathrm{red}} \cong \mathbb{X}_*(T)_{\langle w\rangle}/\mathrm{tors}$ by $\mathbf{C}^\natural_\infty/\mathbf{C}^+_\infty \cong T^{\langle w\rangle}$. By Proposition 3.4.5, the action of $(C(\!(\tau)\!)/\mathbf{C}^+_\infty)^{\mathrm{red}}$ translates the formal residue via the lattice quotient $(C(\!(\tau)\!)/\mathbf{C}^\natural_\infty)^{\mathrm{red}}$, therefore the quotient of $\mathcal{M}_{\mathrm{dR},\psi}$ by $(C(\!(\tau)\!)/\mathbf{C}^+_\infty)^{\mathrm{red}}$ makes sense as an analytic stack over the quotient torus $\mathfrak{c}_0/\mathrm{Im}(\mathrm{res}_{C,\infty})$, which is identified with $T_{\langle w\rangle}$ in (3.10).

One can reformulate Conjecture 4.4.5 by saying that the map

$$\mathcal{M}_{\mathrm{dR},\psi}/(C(\!(\tau)\!)/\mathbf{C}^+_\infty)^{\mathrm{red}} \to \mathcal{M}(\beta_\nu) \times_{[G/G]} [\widetilde{\mathcal{U}}/G]$$

is an analytic isomorphism. This is consistent with Conjecture 4.4.5, as one can check that $(C(\!(\tau)\!)/\mathbf{C}^+_\infty)^{\mathrm{red}}$ is naturally isomorphic to the fiber of $\mathfrak{c}_0 \to [H/\mathrm{Ad}_w(H)]$.

5. Microlocal sheaves on $\mathbf{Fl}_\psi$ and wildly ramified geometric Langlands

In this section we will expand on the conjectural equivalence in §1.4.

From work of [25] and [32], we can construct a sheaf of categories $\mu\mathrm{Sh}$ on conical open subsets $U \subset T^*X$, such that the global sections category $\mu\mathrm{Sh}(T^*X) \cong D(X)$ is the derived category of constructible sheaves on X.

For a conical Lagrangian $\Lambda \subset T^*X$ we can define the full subcategory $\mu\mathrm{Sh}_\Lambda$ of objects with singular support contained in Λ.

If G is a group acting on X, we can also use this to construct a sheaf of categories on the Hamiltonian reduction $T^*X//G := \mu_G^{-1}(0)/G$.

Let X an algebraic space with a $\mathbb{G}_m \ltimes \mathbb{G}_a$-action, where $\mathbb{G}_m$ acts linearly on $\mathbb{G}_a$. Let $\mu_{\mathbb{G}_m\ltimes\mathbb{G}_a} : T^*X \to \mathbb{A}^1_{\mathbb{G}_m} \times \mathbb{A}^1_{\mathbb{G}_a}$ be the moment map of the action.

We have the following relations of Hamiltonian reductions

$$\begin{aligned}T^*X//_1\mathbb{G}_a &= \mu^{-1}_{\mathbb{G}_m\ltimes\mathbb{G}_a}(\mathbb{A}^1_{\mathbb{G}_m}\times\{1\})/\mathbb{G}_a\\ &\cong \mu^{-1}_{\mathbb{G}_m\ltimes\mathbb{G}_a}(\mathbb{A}^1_{\mathbb{G}_m}\times(\mathbb{A}^1_{\mathbb{G}_a}\setminus\{0\}))/\mathbb{G}_m\ltimes\mathbb{G}_a\\ &\cong \mu^{-1}_{\mathbb{G}_m\ltimes\mathbb{G}_a}(\{0\}\times(\mathbb{A}^1_{\mathbb{G}_a}\setminus\{0\}))/\mathbb{G}_m\subset T^*X//\mathbb{G}_m,\end{aligned}$$

where the last is an open subset of the Hamiltonian reduction. We can thus identify the shifted $\mathbb{G}_a$ Hamiltonian reduction as an open subset of the $\mathbb{G}_m$ Hamiltonian reduction and use this to define the sheaf of categories $\mu\mathrm{Sh}$ on $T^*X//_1\mathbb{G}_a$, and in particular the category of global sections $\mu\mathrm{Sh}(T^*X//_1\mathbb{G}_a)$.

Let $\mathbf{J}_\infty$ be the group defined in §2.11, equipped with a homomorphism $\widetilde{\psi}:\mathbf{J}_\infty\to\mathbb{G}_a$ induced by ψ. In Proposition 2.11.1 we prove the identification $\mathcal{M}_\psi\cong T^*\mathrm{Bun}_G(\mathbf{J}^1_\infty;\mathbf{I}_0)//_1\mathbb{G}_a$, where $\mathbf{J}^1_\infty=\ker(\widetilde{\psi})$. We can thus apply the formalism above to the construction to get a category $\mu\mathrm{Sh}(\mathcal{M}_\psi)$.

5.0.1 Remark. The definition of $\mu\mathrm{Sh}(T^*X//_1\mathbb{G}_a)$ is inspired by a construction of Gaitsgory [19, §1.6] called *Kirillov category*. It allows to define a version of $D_{(\mathbb{G}_a,\psi)}(X)$ (where ψ is understood to be an Artin-Schreier sheaf on $\mathbb{G}_a$, which makes sense in characteristic $p>0$) for X over an arbitrary base field, as long as the $\mathbb{G}_a$-action on X extends to a $\mathbb{G}_m\ltimes\mathbb{G}_a$-action. Denote this Kirillov category by $\mathrm{Kir}(X)$. Then one can construct a functor

$$\mathrm{Kir}(X)\to\mu\mathrm{Sh}(T^*X//_1\mathbb{G}_a).$$

More details will be explained in [4]. In particular, we have a functor

$$\mathrm{Kir}(\mathrm{Bun}_G(\mathbf{J}^1_\infty;\mathbf{I}_0))\to\mu\mathrm{Sh}(\mathcal{M}_\psi).$$

We can consider the full subcategory with fixed singular support along the image of $\mathrm{Fl}_\psi\to\mathcal{M}_\psi$ constructed in §2.8.1(4) and denote this by $\mu\mathrm{Sh}_{\mathrm{Fl}_\psi}(\mathcal{M}_\psi)$.

Consider $\mathcal{M}^{0,G^\vee}_{\mathrm{Bet},\psi}$ the Betti space for $G^\vee$, the Langlands dual group, using the positive braid defined by ψ and the definition in §4.4.3.

Note that the definitions in §4 begin with a homogeneous element $\psi\in\mathfrak{g}((t))$. We can construct a homogeneous element $\psi^\vee\in\mathfrak{g}^\vee((t))$ inducing the same map $\chi(\psi)=\chi(\psi^\vee):\mathbb{C}\to\mathfrak{a}=\mathfrak{h}/W\cong\mathfrak{h}^\vee/W$ constructed in §4.4.1 under the identification $\mathfrak{h}/W\cong\mathfrak{h}^\vee/W$ given by choosing a W-invariant symmetric bilinear form on $\mathfrak{h}$. It therefore gives the same positive braid as defined by ψ. We can thus think of $\mathcal{M}^{0,G^\vee}_{\mathrm{Bet},\psi}$ as the Betti space associated to the homogeneous element $\psi^\vee$.

5.0.2 Conjecture. *There is a fully faithful functor*

$$\mu\mathrm{Sh}_{\mathrm{Fl}_\psi}(\mathcal{M}_\psi) \to \mathrm{IndCoh}(\mathcal{M}^{0,G^\vee}_{\mathrm{Bet},\psi}).$$

5.0.3 Remark. (1) This conjecture can be viewed as a geometric Langlands correspondence for deeper level structures/wild ramifications. At the same time, it can be viewed as an instance of homological mirror symmetry between $\mathcal{M}_\psi$ and $\mathcal{M}^{0,G^\vee}_{\mathrm{Bet},\psi}$.

(2) We expect the image of this functor on compact objects to consist of coherent sheaves that are supported over proper subschemes of $\mathcal{M}^{0,G^\vee}_{\mathrm{Bet},\psi}$. We further expect that $\mathcal{M}^{0,G^\vee}_{\mathrm{Bet},\psi}$ has a unique minimal unipotent orbit and that objects in $\mu\mathrm{Sh}_{\mathrm{Fl}_\psi}(\mathcal{M}_\psi)$ that have finitely many components of Fl_ψ in their singular support should be sent to sheaves living over this smallest unipotent orbit appearing in $\mathcal{M}^{0,G^\vee}_{\mathrm{Bet},\psi}$.

(3) A possible way to upgrade the above conjecture is to use wrapped microlocal sheaves as defined in [32]. These should include objects living over non-proper subschemes of $\mathcal{M}^{0,G^\vee}_{\mathrm{Bet},\psi}$.

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Calogero-Moser spaces vs unipotent representations

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To George Lusztig, with admiration

Abstract: Lusztig's classification of unipotent representations of finite reductive groups depends only on the associated Weyl group W (and the automorphism that the Frobenius automorphism induces on W). All the structural questions (families, Harish-Chandra series, partition into blocks...) have an answer in a combinatorics that can be entirely built directly from W.

Over the years, we have noticed that the same combinatorics seems to be encoded in the Poisson geometry of a Calogero-Moser space associated with W (families should correspond to $\mathbb{C}^\times$-fixed points, Harish-Chandra series should correspond to symplectic leaves, blocks should correspond to symplectic leaves in the fixed point subvariety under the action of a root of unity).

The aim of this survey is to gather all these observations, state precise conjectures and provide general facts and examples supporting these conjectures.

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For this introduction, let us focus on the case where $\mathbf{G}$ is a *split* reductive group over a finite field with q elements $\mathbb{F}_q$ and let $G = \mathbf{G}(\mathbb{F}_q)$ be the finite group consisting of $\mathbb{F}_q$-rational points. Let W denote the Weyl group of $\mathbf{G}$ and let $\mathcal{X}$ denote the Calogero-Moser space associated with W *at equal parameters* (recall that it is an affine Poisson variety endowed with a $\mathbb{C}^\times$-action [EtGi, §4]). Let $\mathrm{Unip}(G)$ denote the set of irreducible unipotent characters of G (as defined by Lusztig). A consequence of a conjecture of Gordon-Martino (2007, [GoMa]) is that the fixed point set $\mathcal{X}^{\mathbb{C}^\times}$ should be in bijection with the set of *Lusztig families* of $\mathrm{Unip}(G)$. This first link was the starting point of our interest in the geometry of Calogero-Moser spaces.

In 2008, Gordon, following works of Haiman, obtained in type A a parametrization of the irreducible components of the fixed point subvariety $\mathscr{Z}^{\boldsymbol{\mu}_d}$ by d-cores of partitions. This fits perfectly with the partition of irreducible unipotent representations of $\mathbf{GL}_n(\mathbb{F}_q)$ into d-Harish-Chandra series (defined by Broué-Malle-Michel [BMM1]). In 2011, Bellamy and Losev (independently) obtained a parametrization *à la Harish-Chandra* of symplectic leaves of $\mathscr{Z}$. In 2013, Rouquier and the author [BoRo1] constructed partitions of W into left, right and two-sided Calogero-Moser cells and conjectured they coincide with Kazhdan-Lusztig cells.

From then, the author has worked (with many different authors) on representations of Cherednik algebras at $t = 0$ and the geometry of Calogero-Moser spaces (see [BoRo1, BoRo2, Bon3, BoMa, BoSh]), for, as main motivation, understanding these strange analogies between the geometry of Calogero-Moser spaces and the representation theory of finite reductive groups. Over the years, the author has enriched these coincidences with several examples but has never exposed them in a paper. This is the aim of this survey to present them, state precise conjectures, and provide a list of examples that support these conjectures. A main reason for waiting for such a long time is that we needed to establish some theoretical background on Calogero-Moser space to state precise conjectures: this is done in [Bon4], where we generalize some results of Bellamy [Bel3] and Losev [Los] on symplectic leaves. We also needed some general results (cohomology, fixed points, regular automorphisms) in accordance with these conjectures [BoSh, BoMa, Bon5].

Let us explain one of the strangest (and most convincing) coincidences. Let ℓ be a prime number not dividing q and assume for simplicity that ℓ does not divide $|W|$. We denote by d the order of q modulo ℓ. Then each ℓ-block B of $\mathrm{Unip}(G)$ should correspond to a *symplectic leaf* $\mathscr{S}_B$ of the fixed point subvariety $\mathscr{Z}^{\boldsymbol{\mu}_d}$ of $\mathscr{Z}$ in such a way that:

- On one hand, the *d-Harish-Chandra theory* of Broué-Malle-Michel [BMM1] associates to B a complex reflection group $\mathscr{W}_B$ whose irreducible characters are in bijection with B. Moreover, Broué-Malle-Michel also associate to B a Deligne-Lusztig variety $\mathfrak{X}_B$ and a parameter k_B and conjecture that the endomorphism algebra of the ℓ-adic cohomology of $\mathfrak{X}_B$ is isomorphic to a Hecke algebra of $\mathscr{W}_B$ with parameter k_B. This association is motivated by Broué's abelian defect conjecture, and its geometric version for finite reductive groups [Bro2, §6] (see also [BMM1, BrMa2]).
- On the other hand, an analogue of a *d-Harish-Chandra theory for symplectic leaves* of $\mathscr{Z}^{\boldsymbol{\mu}_d}$ developed by the author [Bon4] (extending earlier

works of Bellamy [Bel3] and Losev [Los] which deal with the case where $d = 1$) associates to $\mathcal{S}_B$ a finite linear group $\mathcal{W}'_B$ and a parameter k'_B. We conjecture [Bon4, Conj. B] that the normalization $\overline{\mathcal{S}}_B^{\mathrm{nor}}$ of the closure of the symplectic leaf $\mathcal{S}_B$ is the Calogero-Moser space for the pair $(\mathcal{W}'_B, k'_B)$.
- The main intriguing observation is that, in the cases where computations can be done, $\mathcal{W}_B$ is a subgroup of $\mathcal{W}'_B$ (in fact, in most cases, $\mathcal{W}_B = \mathcal{W}'_B$) and the parameter k_B is the restriction of the parameter k'_B. Our main conjecture is that this holds in general.

So, important features of the ℓ-modular representation theory of G seem to be encoded in the (Poisson) geometry of the affine variety $\mathcal{Z}^{\boldsymbol{\mu}_d}$ (where d and ℓ are linked by the fact that q is a primitive d-th root of unity modulo ℓ). Moreover, this correspondence seems to carry more properties, as explained in Section 12. To support our conjectures, we have the following examples available:

- We are able to prove most of them if W is of type A (see Section 15).
- They hold if W is of type B_2 or G_2 and d is the Coxeter number (see Section 13).
- They hold if W is of classical type and $d = 1$ (classical Harish-Chandra theory); see Section 16.
- In the *regular* case (see §7.C for the definition), we have a general result on Calogero-Moser spaces (see Theorem 7.4) which fits with observations made on the unipotent representations side (see Example 12.9).
- Our conjectures are compatible with Ennola duality.

The text is organized as follows. An introductory part presents the set-up and the notation involved all along the text. We summarize in the first part some general questions on the geometry of Calogero-Moser spaces (cohomology, geometry of symplectic leaves...), already contained in [BoRo2, BoSh, Bon4, Bon5]. The second part is a crash-course on unipotent representations of finite reductive groups (we hope it is understandable for non-specialists). The third part contains an explanation of the notion of *genericity* and also a detailed exposition of the different coincidences (stated as conjectures) we expect between the Poisson geometry of $\mathcal{Z}$ and the representation theory of G: this is the heart of this survey. The fourth part contains several very explicit examples confirming the conjectures. The last (short) part is an invitation to the *Spetses* theory of Broué-Malle-Michel [BMM2, BMM3], which have connections with the theme of this paper.

Commentary. Recently, Riche-Williamson [RiWi] provided a geometric proof of the *linkage principle* [Ver, Hum, Jan, And]: in their construction, blocks of the category of rational representations of $\mathbf{G}(\overline{\mathbb{F}}_q)$ are in bijection with the irreducible components of $\mathscr{G}r^{\boldsymbol{\mu}_p}$, where p is the prime number dividing q and $\mathscr{G}r$ is the (complex) affine Grassmannian of the (complex) Langlands dual group to $\mathbf{G}$. So our observation has the same flavor as Riche-Williamson result: the blocks of some category of representations are controlled by the geometry of fixed points under the action of a group of roots of unity on some variety. Of course, the main difference is that Riche-Williamson proved a true theorem, built on the geometric Satake equivalence [Lus5, Gin, BeDr, MiVi] between representations of $\mathbf{G}(\overline{\mathbb{F}}_q)$ and some category of perverse sheaves on $\mathscr{G}r$. Our observations are conjectural, and are only concerned with numerical/combinatorial coincidences. We lack of a *geometric Calogero-Moser equivalence*[1]...

Set-up

1. General notation

Throughout this paper, we will abbreviate $\otimes_{\mathbb{C}}$ as $\otimes$.

If $\mathfrak{X}$ is a quasi-projective scheme of finite type over an algebraically closed field, we denote by $\mathfrak{X}_{\mathrm{red}}$ its reduced subscheme. By an algebraic variety, we mean a quasi-projective reduced scheme of finite type over an algebraically closed field. If $\mathfrak{X}$ is an algebraic variety, we denote by $\mathfrak{X}^{\mathrm{nor}}$ its normalization. If $\mathfrak{X}$ is complex and affine we denote by $\mathbb{C}[\mathfrak{X}]$ its coordinate ring.

If $\mathfrak{X}$ is a complex algebraic variety, we denote by $\mathrm{H}^j(\mathfrak{X})$ its j-th singular cohomology group with coefficients in $\mathbb{C}$. If $\mathfrak{X}$ carries a regular action of a torus $\mathbf{T}$, we denote by $\mathrm{H}^j_{\mathbf{T}}(\mathfrak{X})$ its j-th $\mathbf{T}$-equivariant cohomology group (still with coefficients in $\mathbb{C}$). Note that $\mathrm{H}^{2\bullet}(\mathfrak{X}) = \bigoplus_{j \geqslant 0} \mathrm{H}^{2j}(\mathfrak{X})$ is a graded $\mathbb{C}$-algebra and $\mathrm{H}^{2\bullet}_{\mathbf{T}}(\mathfrak{X}) = \bigoplus_{j \geqslant 0} \mathrm{H}^{2j}_{\mathbf{T}}(\mathfrak{X})$ is a graded $\mathrm{H}^{2\bullet}_{\mathbf{T}}(\mathbf{pt})$-algebra, where $\mathbf{pt} = \mathrm{Spec}(\mathbb{C})$. We identify $\mathrm{H}^{2\bullet}_{\mathbb{C}^\times}(\mathbf{pt})$ with $\mathbb{C}[\hbar]$ in the usual way (note that

[1] Recent works of Dudas-Rouquier relate the category of coherent sheaves on the Hilbert scheme of points in the plane (which is diffeomorphic to the Calogero-Moser space associated with the symmetric group) and representations of finite general linear or unitary groups. This work is still unpublished, but the interested reader might look at numerous videos of some of their talks:
https://www.birs.ca/events/2017/5-day-workshops/17w5003/videos/watch/201710181031-Rouquier.html
https://www.msri.org/workshops/820/schedules/23934
https://www.youtube.com/watch?v=CMBVSJC6EX0

$\mathrm{H}^{2j+1}_{\mathbf{T}}(\mathbf{pt}) = 0$ for all j). If $\mathscr{Y}$ is another complex variety endowed with a regular $\mathbf{T}$-action and if $\varphi : \mathscr{Y} \to \mathscr{X}$ is a $\mathbf{T}$-equivariant morphism of varieties, we denote by $\varphi^* : \mathrm{H}^\bullet_{\mathbf{T}}(\mathscr{X}) \longrightarrow \mathrm{H}^\bullet_{\mathbf{T}}(\mathscr{Y})$ the induced morphism in equivariant cohomology.

2. Finite linear group, reflections

Notation. *We fix in this paper a finite dimensional* $\mathbb{C}$*-vector space* V *and a finite subgroup* W *of* $\mathrm{GL}_{\mathbb{C}}(V)$*.*

2.A. Reflections, hyperplanes

We set $\varepsilon : W \to \mathbb{C}^\times$, $w \mapsto \det(w)$ and

$$\mathrm{Ref}(W) = \{s \in W \mid \dim_{\mathbb{C}} V^s = n - 1\}.$$

Note that, for the moment, we do not assume that $W = \langle \mathrm{Ref}(W) \rangle$. We identify $\mathbb{C}[V]$ (resp. $\mathbb{C}[V^*]$) with the symmetric algebra $\mathrm{S}(V^*)$ (resp. $\mathrm{S}(V)$).

We denote by $\mathscr{A}$ the set of *reflecting hyperplanes* of W, namely

$$\mathscr{A} = \{V^s \mid s \in \mathrm{Ref}(W)\}.$$

If $H \in \mathscr{A}$, we denote by W_H its inertia group, i.e. the group consisting of elements $w \in W$ such that $w(v) = v$ for all $v \in H$. We denote by α_H an element of V^* such that $H = \mathrm{Ker}(\alpha_H)$ and by $\alpha_H^\vee$ an element of V such that $V = H \oplus \mathbb{C}\alpha_H^\vee$ and the line $\mathbb{C}\alpha_H^\vee$ is W_H-stable. We set $e_H = |W_H|$. Note that W_H is cyclic of order e_H and that $\mathrm{Irr}(W_H) = \{\mathrm{Res}^W_{W_H} \varepsilon^j \mid 0 \leqslant j \leqslant e - 1\}$. We denote by $\varepsilon_{H,j}$ the (central) primitive idempotent of $\mathbb{C}W_H$ associated with the character $\mathrm{Res}^W_{W_H} \varepsilon^{-j}$, namely

$$\varepsilon_{H,j} = \frac{1}{e_H} \sum_{w \in W_H} \varepsilon(w)^j w \in \mathbb{C}W_H.$$

If Ω is a W-orbit of reflecting hyperplanes, we write e_Ω for the common value of all the e_H, where $H \in \Omega$. We denote by $\aleph$ the set of pairs (Ω, j) where $\Omega \in \mathscr{A}/W$ and $0 \leqslant j \leqslant e_\Omega - 1$. The vector space of families of complex numbers indexed by $\aleph$ will be denoted by $\mathbb{C}^\aleph$, elements of $\mathbb{C}^\aleph$ will be called *parameters*. If $k = (k_{\Omega,j})_{(\Omega,j)\in\aleph} \in \mathbb{C}^\aleph$, we define $k_{H,j}$ for all $H \in \Omega$ and $j \in \mathbb{Z}$ by $k_{H,j} = k_{\Omega,j_0}$ where Ω is the W-orbit of H and j_0 is the unique element of $\{0, 1, \ldots, e_H - 1\}$ such that $j \equiv j_0 \mod e_H$.

2.B. Filtration

Let $\mathrm{cod} : W \to \mathbb{Z}_{\geqslant 0}$ be defined by

$$\mathrm{cod}(w) = \mathrm{codim}_{\mathbb{C}}(V^w)$$

(note that $\mathrm{Ref}(W) = \mathrm{cod}^{-1}(1)$) and we define a filtration $\mathscr{F}_\bullet(\mathbb{C}W)$ of the group algebra of W as follows: let

$$\mathscr{F}_j(\mathbb{C}W) = \bigoplus_{\mathrm{cod}(w) \,\leqslant\, j} \mathbb{C}w.$$

Then

$$\mathbb{C}\,\mathrm{Id}_V = \mathscr{F}_0(\mathbb{C}W) \subset \mathscr{F}_1(\mathbb{C}W) \subset \cdots \subset \mathscr{F}_n(\mathbb{C}W) = \mathbb{C}W = \mathscr{F}_{n+1}(\mathbb{C}W) = \cdots$$

is a filtration of $\mathbb{C}W$. For any subalgebra A of $\mathbb{C}W$, we set $\mathscr{F}_j(A) = A \cap \mathscr{F}_j(\mathbb{C}W)$, so that

$$\mathbb{C}\,\mathrm{Id}_V = \mathbb{C}\mathscr{F}_0(A) \subset \mathscr{F}_1(A) \subset \cdots \subset \mathscr{F}_n(A) = A = \mathscr{F}_{n+1}(A) = \cdots$$

is also a filtration of A. Write

$$\mathrm{Rees}^\bullet_{\mathscr{F}}(A) = \bigoplus_{j\geqslant 0} \hbar^j \mathscr{F}_j(A) \subset \mathbb{C}[\hbar] \otimes A \qquad \text{(the } \textit{Rees algebra}\text{)},$$

$$\mathrm{gr}^\bullet_{\mathscr{F}}(A) = \bigoplus_{j\geqslant 0} \mathscr{F}_j(A)/\mathscr{F}_{j-1}(A).$$

Recall that $\mathrm{gr}^\bullet_{\mathscr{F}}(A) \simeq \mathrm{Rees}^\bullet_{\mathscr{F}}(A)/\hbar\,\mathrm{Rees}^\bullet_{\mathscr{F}}(A)$.

3. Rational Cherednik algebra at $t = 0$

Notation. *Throughout this paper, we fix a parameter $k \in \mathbb{C}^{\aleph}$.*

3.A. Definition

We define the *rational Cherednik algebra* $\mathbf{H}_k$ to be the quotient of the algebra $\mathrm{T}(V \oplus V^*) \rtimes W$ (the semi-direct product of the tensor algebra $\mathrm{T}(V \oplus V^*)$

with the group W) by the relations

$$(3.1)\qquad \begin{cases} [x,x'] = [y,y'] = 0, \\ [y,x] = \displaystyle\sum_{H\in\mathcal{A}} \sum_{j=0}^{e_H-1} e_H(k_{H,j} - k_{H,j+1}) \frac{\langle y,\alpha_H\rangle \cdot \langle \alpha_H^\vee, x\rangle}{\langle \alpha_H^\vee, \alpha_H\rangle} \varepsilon_{H,j}, \end{cases}$$

for all x, $x' \in V^*$, y, $y' \in V$. Here $\langle\ ,\ \rangle : V \times V^* \to \mathbb{C}$ is the standard pairing. The first commutation relations imply that we have morphisms of algebras $\mathbb{C}[V] \to \mathbf{H}_k$ and $\mathbb{C}[V^*] \to \mathbf{H}_k$. Recall [EtGi, Theo. 1.3] that we have an isomorphism of $\mathbb{C}$-vector spaces

$$(3.2)\qquad \mathbb{C}[V] \otimes \mathbb{C}W \otimes \mathbb{C}[V^*] \xrightarrow{\sim} \mathbf{H}_k$$

induced by multiplication (this is the so-called *PBW-decomposition*).

Remark 3.3. Let $(l_\Omega)_{\Omega\in\mathcal{A}/W}$ be a family of complex numbers and let $k' \in \mathbb{C}^{\aleph}$ be defined by $k'_{\Omega,j} = k_{\Omega,j} + l_\Omega$. Then $\mathbf{H}_k = \mathbf{H}_{k'}$. This means that there is no restriction to generality if we consider for instance only parameters k such that $k_{\Omega,0} = 0$ for all Ω, or only parameters k such that $k_{\Omega,0}+k_{\Omega,1}+\cdots+k_{\Omega,e_\Omega-1} = 0$ for all Ω (as in [BoRo2]). ■

3.B. Calogero-Moser space

We denote by $\mathbf{Z}_k$ the center of the algebra $\mathbf{H}_k$: it is well-known [EtGi, Theo 3.3 and Lem. 3.5] that $\mathbf{Z}_k$ is an integral domain, which is integrally closed. Moreover, it contains $\mathbb{C}[V]^W$ and $\mathbb{C}[V^*]^W$ as subalgebras [Gor1, Prop. 3.6] (so it contains $\mathbf{P} = \mathbb{C}[V]^W \otimes \mathbb{C}[V^*]^W$), and it is a free $\mathbf{P}$-module of rank $|W|$. We denote by $\mathcal{Z}_k$ the affine algebraic variety whose ring of regular functions $\mathbb{C}[\mathcal{Z}_k]$ is $\mathbf{Z}_k$: this is the *Calogero-Moser space* associated with the datum (V, W, k). It is irreducible and integrally closed.

We set $\mathcal{P} = V/W \times V^*/W$, so that $\mathbb{C}[\mathcal{P}] = \mathbf{P}$ and the inclusion $\mathbf{P} \hookrightarrow \mathbf{Z}_k$ induces a finite and flat morphism of varieties

$$\Upsilon_k : \mathcal{Z}_k \longrightarrow \mathcal{P}.$$

Using the PBW-decomposition, we define a $\mathbb{C}$-linear map $\mathbf{\Omega}^{\mathbf{H}_k} : \mathbf{H}_k \longrightarrow \mathbb{C}W$ by

$$\mathbf{\Omega}^{\mathbf{H}_k}(fwg) = f(0)g(0)w$$

for all $f \in \mathbb{C}[V]$, $g \in \mathbb{C}[V^*]$ and $w \in \mathbb{C}W$. This map is W-equivariant with respect to the action on both sides by conjugation, so it induces a well-defined $\mathbb{C}$-linear map

$$\mathbf{\Omega}^k : \mathbf{Z}_k \longrightarrow \mathrm{Z}(\mathbb{C}W).$$

Recall from [BoRo2, Cor. 4.2.11] that $\boldsymbol{\Omega}^k$ is a morphism of algebras, and that

(3.4) $\mathscr{Z}_k$ *is smooth if and only if* $\boldsymbol{\Omega}^k$ *is surjective.*

The "only if" part is essentially due to Gordon [Gor1, Cor. 5.8] (but the reader must take [BoRo2, Prop. 9.6.6 and (16.1.2)] into account for translating Gordon's result in terms of $\boldsymbol{\Omega}^k$) while the "if" part follows from the work of Bellamy, Schedler and Thiel [BeScTh, Cor. 1.4].

3.C. Other parameters

Let $\mathscr{C}$ denote the space of maps $\mathrm{Ref}(W) \to \mathbb{C}$ which are constant on conjugacy classes of reflections. The element

$$\sum_{(\Omega,j)\in\aleph}\sum_{H\in\Omega}(k_{H,j}-k_{H,j+1})e_H\varepsilon_{H,j}$$

of $\mathrm{Z}(\mathbb{C}W)$ is supported only by reflections, so there exists a unique map $c_k \in \mathscr{C}$ such that

$$\sum_{(\Omega,j)\in\aleph}\sum_{H\in\Omega}(k_{H,j}-k_{H,j+1})e_H\varepsilon_{H,j} = \sum_{s\in\mathrm{Ref}(W)}(\varepsilon(s)-1)c_k(s)s.$$

Then the map $\mathbb{C}^\aleph \to \mathscr{C}$, $k \mapsto c_k$, is linear and surjective. With this notation, we have

$$[y,x] = \sum_{s\in\mathrm{Ref}(W)}(\varepsilon(s)-1)\,c_k(s)\,\frac{\langle y,\alpha_s\rangle\cdot\langle\alpha_s^\vee,x\rangle}{\langle\alpha_s^\vee,\alpha_s\rangle}\,s, \tag{3.5}$$

for all $y \in V$ and $x \in V^*$. Here, $\alpha_s = \alpha_{V^s}$ and $\alpha_s^\vee = \alpha_{V^s}^\vee$.

3.D. Extra-structures on the Calogero-Moser space

The Calogero-Moser space $\mathscr{Z}_k$ is endowed with a $\mathbb{C}^\times$-action, a Poisson bracket and an Euler element which are described below.

3.D.1. Grading, $\mathbb{C}^\times$-action The algebra $\mathrm{T}(V\oplus V^*)\rtimes W$ can be $\mathbb{Z}$-graded in such a way that the generators have the following degrees

$$\begin{cases}\deg(y)=-1 & \text{if } y\in V,\\ \deg(x)=1 & \text{if } x\in V^*,\\ \deg(w)=0 & \text{if } w\in W.\end{cases}$$

This descends to a $\mathbb{Z}$-grading on $\mathbf{H}_k$ because the defining relations (3.1) are homogeneous. Since the center of a graded algebra is always graded, the subalgebra $\mathbf{Z}_k$ is also $\mathbb{Z}$-graded. So the Calogero-Moser space $\mathscr{Z}_k$ inherits a regular $\mathbb{C}^\times$-action. Note also that by definition $\mathbf{P} = \mathbb{C}[V]^W \otimes \mathbb{C}[V^*]^W$ is clearly a graded subalgebra of $\mathbf{Z}_k$.

3.D.2. Poisson structure Let $t \in \mathbb{C}$. One can define a deformation $\mathbf{H}_{t,k}$ of $\mathbf{H}_k$ as follows: $\mathbf{H}_{t,k}$ is the quotient of the algebra $\mathrm{T}(V \oplus V^*) \rtimes W$ by the relations

$$\begin{cases} [x, x'] = [y, y'] = 0, \\ [y, x] = t\langle y, x\rangle + \displaystyle\sum_{H \in \mathscr{A}} \sum_{j=0}^{e_H - 1} e_H (k_{H,j} - k_{H,j+1}) \frac{\langle y, \alpha_H\rangle \cdot \langle \alpha_H^\vee, x\rangle}{\langle \alpha_H^\vee, \alpha_H\rangle} \varepsilon_{H,j}, \end{cases} \tag{3.6}$$

for all x, $x' \in V^*$, y, $y' \in V$. It is well-known [EtGi, Theo 1.3] that the PBW decomposition (as in (3.2)) still holds so that the family $(\mathbf{H}_{t,k})_{t\in\mathbb{C}}$ is a flat deformation of $\mathbf{H}_k = \mathbf{H}_{0,k}$. This allows to define a Poisson bracket $\{\ ,\ \}$ on $\mathbf{Z}_k$ as follows: if z_1, $z_2 \in \mathbf{Z}_k$, we denote by z_1^t, z_2^t the corresponding element of $\mathbf{H}_{t,k}$ through the PBW decomposition and we define

$$\{z_1, z_2\} = \lim_{t \to 0} \frac{[z_1^t, z_2^t]}{t}.$$

Finally, note that

(3.7) *The Poisson bracket is $\mathbb{C}^\times$-equivariant.*

3.D.3. Euler element Let $(y_1, \dots, y_m)$ be a basis of V and let $(x_1, \dots, x_m)$ denote its dual basis. As in [BoRo2, §3.3], we set

$$\mathbf{eu} = \sum_{j=1}^m x_j y_j + \sum_{s \in \mathrm{Ref}(W)} \varepsilon(s) c_k(s) s = \sum_{j=1}^m x_j y_j + \sum_{H \in \mathscr{A}} \sum_{j=0}^{e_H - 1} e_H\ k_{H,j} \varepsilon_{H,j}.$$

Recall that $\mathbf{eu}$ does not depend on the choice of the basis of V. Also

$$\mathbf{eu} \in \mathbf{Z}_k, \qquad \mathrm{Frac}(\mathbf{Z}_k) = \mathrm{Frac}(\mathbf{P})[\mathbf{eu}] \tag{3.8}$$

and

$$\{\mathbf{eu}, z\} = dz \tag{3.9}$$

if $z \in \mathbf{Z}_k$ is homogeneous of degree d (see for instance [BoRo2, Prop. 3.3.3]).

Notation. *If ? is one of the above objects defined in this section ($\mathbf{H}_k$, $\mathfrak{X}_k$, $\aleph$, $\mathscr{A}$, $\mathscr{H}_k$ ...), we will sometimes denote it by ?(W) or ?(V,W) if we need to emphasize the context.*

4. Reflection groups

Recall that, for the moment, we did not assume that $W = \langle \mathrm{Ref}(W)\rangle$ (this will be assumed only after this section). Let $W_{\mathrm{ref}} = \langle \mathrm{Ref}(W)\rangle$ be the maximal subgroup of W generated by reflections. Then the set $\mathscr{A}$ depends only on W_{ref} and the finite group W/W_{ref} acts on $\aleph(W_{\mathrm{ref}})$ and $\aleph = \aleph(W_{\mathrm{ref}})/(W/W_{\mathrm{ref}})$. In other words, giving an element $k \in \aleph$ is equivalent to giving an element $k \in \aleph(W_{\mathrm{ref}})$ which is W/W_{ref}-invariant. In this case, the relations ($\mathscr{H}_k$) only depend on W_{ref}. If we denote by $\mathbf{H}_k(W_{\mathrm{ref}})$ the Cherednik algebra defined with W_{ref} instead of W, then $\mathbf{H}_k(W_{\mathrm{ref}})$ is naturally a subalgebra of $\mathbf{H}_k$ and, as a $\mathbb{C}W$-module, $\mathbf{H}_k = \mathbb{C}W \otimes_{\mathbb{C}W_{\mathrm{ref}}} \mathbf{H}_k(W_{\mathrm{ref}})$. Note also that the finite group W/W_{ref} acts on $\mathbf{Z}_k(W_{\mathrm{ref}})$ and on $\mathfrak{X}_k(W_{\mathrm{ref}})$, and that

$$\mathbf{Z}_k = \mathbf{Z}_k(W_{\mathrm{ref}})^{W/W_{\mathrm{ref}}} \qquad \text{and} \qquad \mathfrak{X}_k = \mathfrak{X}_k(W_{\mathrm{ref}})/(W/W_{\mathrm{ref}}). \tag{4.1}$$

We deduce from this the following facts:

Proposition 4.2. *Let $q : \mathfrak{X}_k(W_{\mathrm{ref}}) \longrightarrow \mathfrak{X}_k$ denote the quotient map. Then:*

(a) *We have $\mathfrak{X}_k^{\mathbb{C}^\times} = q(\mathfrak{X}_k(W_{\mathrm{ref}})^{\mathbb{C}^\times})$ and $q^{-1}(\mathfrak{X}_k(W_{\mathrm{ref}})^{\mathbb{C}^\times}) = \mathfrak{X}_k^{\mathbb{C}^\times}$.*
(b) *The morphism q induces isomorphisms*

$$q_* : \mathrm{H}^\bullet(\mathfrak{X}_k) \xrightarrow{\sim} \mathrm{H}^\bullet(\mathfrak{X}_k(W_{\mathrm{ref}}))^{W/W_{\mathrm{ref}}}$$

and

$$q_* : \mathrm{H}^\bullet_{\mathbb{C}^\times}(\mathfrak{X}_k) \xrightarrow{\sim} \mathrm{H}^\bullet_{\mathbb{C}^\times}(\mathfrak{X}_k(W_{\mathrm{ref}}))^{W/W_{\mathrm{ref}}}.$$

Proof. (a) follows since q is a finite morphism and since an action of $\mathbb{C}^\times$ on a finite set is necessarily trivial. (b) is a classical property of cohomology [Bre, Theo. III.2.4]. □

Continuing this reduction, we denote by $W(k)$ the subgroup of W_{ref} generated by the reflections $s \in \mathrm{Ref}(W)$ such that $c_k(s) \neq 0$. It is a normal subgroup of W and W_{ref}. Also, the formula (3.5) shows that, as a $\mathbb{C}W$-module, $\mathbf{H}_k = \mathbb{C}W \otimes_{\mathbb{C}W_{\mathrm{ref}}} \mathbf{H}_{k^\flat}(W_{\mathrm{ref}})$. Here, $k^\flat \in \mathbb{C}^{\aleph(W(k))}$ is such that $c_{k^\flat}^{W(k)} \in \mathscr{C}(W(k))$ is the restriction of c_k to $\mathrm{Ref}(W(k))$. Therefore, as above, we have

$$(4.3)\qquad \mathbf{Z}_k = \mathbf{Z}_{k^\flat}(W(k))^{W/W(k)} \qquad \text{and} \qquad \mathfrak{X}_k = \mathfrak{X}_{k^\flat}(W(k))/(W/W(k)).$$

We deduce from this the following facts:

Proposition 4.4. *Let $q^\flat : \mathfrak{X}_{k^\flat}(W(k)) \longrightarrow \mathfrak{X}_k$ denote the quotient map. Then:*

(a) *We have $\mathfrak{X}_k^{\mathbb{C}^\times} = q^\flat(\mathfrak{X}_{k^\flat}(W(k))^{\mathbb{C}^\times})$ and $(q^\flat)^{-1}(\mathfrak{X}_{k^\flat}(W(k))^{\mathbb{C}^\times}) = \mathfrak{X}_k^{\mathbb{C}^\times}$.*
(b) *The morphism $q^\flat$ induces isomorphisms*

$$q^\flat_* : \mathrm{H}^\bullet(\mathfrak{X}_k) \xrightarrow{\sim} \mathrm{H}^\bullet(\mathfrak{X}_{k^\flat}(W(k)))^{W/W(k)}$$

and

$$q^\flat_* : \mathrm{H}^\bullet_{\mathbb{C}^\times}(\mathfrak{X}_k) \xrightarrow{\sim} \mathrm{H}^\bullet_{\mathbb{C}^\times}(\mathfrak{X}_k(W(k)))^{W/W(k)}.$$

Even though the case where $k = 0$ serves as a base of our conjectures/questions, the really interesting case is when $W(k) = W$: equations (4.3) and Proposition 4.4 help us to recover properties of $\mathfrak{X}_k(W)$ from those of $\mathfrak{X}_k(W(k))$. For instance, Etingof-Ginzburg proved that, if $\mathfrak{X}_k$ is smooth, then $W = W(k)$ (see [EtGi, Prop. 3.10]).

5. Braid group, Hecke algebra

Hypothesis and notation. *From now on, and until the end of this paper, we assume that*

$$W = \langle \mathrm{Ref}(W) \rangle$$

and we fix $k \in \mathbb{C}^{\aleph}$. We set

$$V_{\mathrm{reg}} = V \setminus \bigcup_{H \in \mathcal{A}} H$$

and we recall that V_{reg} is the set of elements of V whose stabilizer in W is trivial (this is Steinberg's Theorem: see for instance [Bro1, Theo. 4.7]).

We fix $v_0 \in V_{\mathrm{reg}}$ and we denote by $\bar{v}_0$ its image in V_{reg}/W. We set

$$\mathbb{B} = \pi_1(V_{\mathrm{reg}}/W, \bar{v}_0) \qquad \textit{and} \qquad \mathbb{P} = \pi_1(V_{\mathrm{reg}}, v_0).$$

Then the group $\mathbb{B}$ (resp. $\mathbb{P}$) is called the ***braid group*** *(resp. the* ***pure braid group****) of W.*

5.A. Generators of $\mathbb{B}$ and $\mathbb{P}$

If $H \in \mathscr{A}$, we denote by s_H the generator of W_H of determinant $\zeta_{e_H} = \exp(2i\pi/e_H)$ and by $\mathbf{s}_H$ a *braid reflection* around H (as defined in [Bro1, Def. 4.13]: it is called a *generator of the monodromy* around H in [BrMaRo]). Through the exact sequence

$$1 \longrightarrow \mathbb{P} \longrightarrow \mathbb{B} \longrightarrow W \longrightarrow 1 \tag{5.1}$$

induced by the unramified covering $V_{\mathrm{reg}} \to V_{\mathrm{reg}}/W$, the image of $\mathbf{s}_H$ is s_H and so $\mathbf{s}_H^{e_H} \in \mathbb{P}$. Moreover,

$$\mathbb{B} = \langle (\mathbf{s}_H)_{H\in\mathscr{A}} \rangle \qquad \text{and} \qquad \mathbb{P} = \langle (\mathbf{s}_H^{e_H})_{H\in\mathscr{A}} \rangle. \tag{5.2}$$

5.B. Hecke algebra

Let F denote the number field generated by the traces of the elements of W (it is generally called the *character field* of W). It is known [Ben, Bes] that the algebra FW is split. We denote by $\mathscr{O}$ the ring of algebraic integers in F and let $R = \mathscr{O}[\mathbf{q}^{\mathbb{C}}]$ be the group algebra of $(\mathbb{C}, +)$ over $\mathscr{O}$, denoted with an exponential notation: namely, we have $\mathbf{q}^a\mathbf{q}^{a'} = \mathbf{q}^{a+a'}$ for all a, $a' \in \mathbb{C}$. We set $\mathbf{q} = \mathbf{q}^1$. The *Hecke algebra* with parameter k, denoted by $\mathscr{H}_k(W)$, is the quotient of the group algebra $R\mathbb{B}$ of $\mathbb{B}$ over R by the ideal generated by the elements

$$\prod_{j=0}^{e_H-1} (\mathbf{s}_H - \zeta_{e_H}^j \mathbf{q}^{k_{H,j}}),$$

where H runs over $\mathscr{A}$.

We denote by T_H the image of $\mathbf{s}_H$ in $\mathscr{H}_k(W)$. We have

$$\prod_{j=0}^{e_H-1} (T_H - \zeta_{e_H}^j \mathbf{q}^{k_{H,j}}) = 0. \tag{5.3}$$

If q is a non-zero complex number, let $\mathscr{H}_k(W,q)$ denote a *specialization of* $\mathscr{H}_k(W)$ *at* q. Namely, we choose a complex logarithm v of q and we denote by $\mathrm{ev}_v : R \to \mathbb{C}$ the morphism of $\mathbb{C}$-algebras such that $\mathbf{q}^a \mapsto q^a = \exp(av)$ for all $a \in \mathbb{C}$. Then $\mathscr{H}_k(W,q)$ is the $\mathbb{C}$-algebra obtained by specialisation through ev_v. This is clearly an abuse of notation, as the specialization might depend on the choice of the logarithm v of q (for instance whenever the parameter k has some non-integer values). But it turns out that, in this survey, this notation will occur only whenever the specialization does not depend on this choice.

Recall that $\mathcal{H}_k(W)$ is a free R-module of rank $|W|$ (see [Ari], [ArKo], [BrMaRo], [Cha1], [Cha2], [Cha3], [Mar1], [Mar2], [Mar3], [MaPf] and [Tsu]) such that its specialization $\mathcal{H}_k(W,1)$ is just the group algebra $\mathbb{C}W$ of W over $\mathbb{C}$.

5.C. Hecke families

Whenever $k_{\Omega,j} \in \mathbb{Z}$ for all $(\Omega, j) \in \aleph$, Broué and Kim [BrKi] defined a partition of $\mathrm{Irr}(W)$ into families, which they call *Rouquier k-families*. In [BoRo2, §6.5], Rouquier and the author extended (easily) the definition of these families to general parameters k, and decided to call them *Hecke k-families*. We will stick to this last terminology in this paper. Let us explain this definition.

Let K denote the fraction field of R. The K-algebra $K\mathcal{H}_k(W)$ is split semisimple [Mal3, Theo. 5.2] so, by Tits deformation Theorem [GePf, Theo. 7.4.6], it is isomorphic to the group algebra KW. Therefore, its irreducible characters are in bijection with $\mathrm{Irr}(W)$. If $\chi \in \mathrm{Irr}(W)$, we denote by χ_k the corresponding irreducible character of $K\mathcal{H}_k(W)$. Now, let R_{cyc} denote the localization of R defined by

$$R_{\mathrm{cyc}} = R[((1-\mathbf{q}^a)^{-1})_{a \in \mathbb{C}\setminus\{0\}}].$$

We say the χ and χ' are *in the same Hecke k-family* if there is a primitive central idempotent b of $R_{\mathrm{cyc}}\mathcal{H}_k(W)$ such that $\chi_k(b) = \chi'_k(b) \neq 0$.

We denote by $\mathrm{Fam}_k^{\mathrm{Hec}}(W)$ the set of Hecke k-families.

5.D. Calogero-Moser families

Calogero-Moser families were defined by Gordon using *baby Verma modules* [Gor1, §4.2 and §5.4]. We explain here an equivalent definition given in [BoRo2, §7.2]. If $\chi \in \mathrm{Irr}(W)$, we denote by $\omega_\chi : \mathrm{Z}(\mathbb{C}W) \to \mathbb{C}$ its central character (i.e., $\omega_\chi(z) = \chi(z)/\chi(1)$ is the scalar by which z acts on an irreducible representation affording the character χ). We denote by e_χ (or e_χ^W if necessary) the primitive central idempotent such that $\omega_\chi(e_\chi) = 1$. We say that two irreducible characters χ and χ' of W belong to the same *Calogero-Moser k-family* if $\omega_\chi \circ \mathbf{\Omega}^k = \omega_{\chi'} \circ \mathbf{\Omega}^k$. If $\mathfrak{F}$ is a subset of $\mathrm{Irr}(W)$, we set

$$e_{\mathfrak{F}} = \sum_{\chi \in \mathfrak{F}} e_\chi \in \mathrm{Z}(\mathbb{C}W).$$

Finally, we denote by $\mathrm{Fam}_k^{\mathrm{CM}}(W)$ the set of Calogero-Moser k-families. Then [BoRo2, (16.1.2)]

$$\mathrm{Im}(\boldsymbol{\Omega}^k) = \bigoplus_{\mathfrak{F} \in \mathrm{Fam}_k^{\mathrm{CM}}(W)} \mathbb{C} e_{\mathfrak{F}} \tag{5.4}$$

and $\mathrm{Im}(\boldsymbol{\Omega}^k)$ can be identified with the algebra of functions on $\mathfrak{X}_k^{\mathbb{C}^\times}$.

In other words, this defines a surjective map

$$\mathfrak{z}_k : \mathrm{Irr}(W) \longrightarrow \mathfrak{X}_k^{\mathbb{C}^\times}$$

whose fibers are the Calogero-Moser k-families. If $p \in \mathfrak{X}_k^{\mathbb{C}^\times}$, we denote by $\mathfrak{F}_p$ (or $\mathfrak{F}_p^k$ if we need to emphasize the parameter) the corresponding Calogero-Moser k-family. The next conjecture can be found in [Mart1]:

> **Conjecture 5.5** (Martino). *Let $k^\sharp$ be the parameter $(k_{\Omega,-j})_{(\Omega,j)\in\aleph} \in \mathbb{C}^\aleph$, where the index j is viewed modulo e_Ω. Then each Calogero-Moser k-family is a union of Hecke $k^\sharp$-families.*

Theorem 5.6. *Conjecture 5.5 is known to hold in the following cases*[2]*:*

1. *If W is of type $G(de, e, r)$, with d, e, $r \geqslant 1$ and e odd whenever $r = 2$.*
2. *If W is of type G_4, G_{12}, G_{13}, G_{20}, G_{22}, $G_{23} = \mathrm{W}(H_3)$ or $G_{28} = \mathrm{W}(F_4)$.*
3. *If W is of type G_5, G_6, G_7, G_8, G_9, G_{10}, G_{14}, G_{15}, G_{16} or G_{24} for generic values of k.*

Proof. For (1), see [Mart1, Bel2, Mart2]. For (2) and (3), see [Thi] (except for $G_{28} = \mathrm{W}(F_4)$: for this one, see [BoTh]). □

Part I. Questions about Calogero-Moser spaces

6. Cohomology

6.A. Localization

We denote by $i_k : \mathfrak{X}_k^{\mathbb{C}^\times} \hookrightarrow \mathfrak{X}_k$ the closed immersion (here, $\mathfrak{X}_k^{\mathbb{C}^\times}$ denotes the reduced zero-dimensional variety of $\mathbb{C}^\times$-fixed points). As explained in §5.D, we have a natural isomorphism of algebras

$$\mathrm{H}_{\mathbb{C}^\times}(\mathfrak{X}_k^{\mathbb{C}^\times}) \simeq \mathbb{C}[\hbar] \otimes \mathrm{Im}(\boldsymbol{\Omega}^k). \tag{6.1}$$

[2] We refer to Shephard-Todd notation for irreducible complex reflection groups [ShTo].

So we view the map i_k^* as a morphism of algebras

$$i_k^* : \mathrm{H}_{\mathbb{C}^\times}(\mathscr{Z}_k) \longrightarrow \mathbb{C}[\hbar] \otimes \mathrm{Im}(\boldsymbol{\Omega}^k).$$

We can now state the following conjecture (see [BoRo2, §16.1] and [BoSh, Conj. 3.3]).

Conjecture 6.2. *With the above notation, we have:*

(a) *If $i \geqslant 0$, then $\mathrm{H}^{2i+1}(\mathscr{Z}_k) = 0$.*

(b) $\mathrm{Im}(i_k^*) = \mathrm{Rees}_{\mathscr{F}}(\mathrm{Im}(\boldsymbol{\Omega}^k))$.

Recall from standard arguments [BoSh, Prop. 2.4] that this conjecture would imply a description of both the cohomology and the equivariant cohomology of $\mathscr{Z}_k$:

Proposition 6.3. *Assume that Conjecture 6.2 holds. Then:*

(a) *If $i \geqslant 0$, then $\mathrm{H}^{2i+1}_{\mathbb{C}^\times}(\mathscr{Z}_k) = 0$.*
(b) $\mathrm{H}^{2\bullet}_{\mathbb{C}^\times}(\mathscr{Z}_k) \simeq \mathrm{Rees}_{\mathscr{F}}(\mathrm{Im}(\boldsymbol{\Omega}^k))$ *as $\mathbb{C}[\hbar]$-algebras.*
(c) $\mathrm{H}^{2\bullet}(\mathscr{Z}_k) \simeq \mathrm{gr}_{\mathscr{F}}(\mathrm{Im}(\boldsymbol{\Omega}^k))$ *as $\mathbb{C}$-algebras.*

Theorem 6.4. *Conjecture 6.2 is known to hold in the following cases:*

(a) *If $k = 0$.*
(b) *If $\dim V = 1$.*
(c) *If $\mathscr{Z}_k$ is smooth.*

Proof. (a) follows from the fact that $\mathscr{Z}_0 = (V \times V^*)/W$ is contractible and $\mathrm{Im}\,\boldsymbol{\Omega}^0 = \mathbb{C}$. For (b), see [BoRo2, Theo. 18.5.8] and [BoSh, Prop. 1.6]. For (c), see [BoSh, Theo. A]. □

Example 6.5. It might be tempting to conjecture that the Calogero-Moser space $\mathscr{Z}_k$ is rationally smooth and p-smooth if p is a prime number not dividing $|W|$. Indeed, $\mathscr{Z}_k$ is a deformation of $\mathscr{Z}_0 = (V \times V^*)/W$ which tends to be smoother and smoother as k becomes more and more generic. However, both statements are false in general:

(1) If $\dim V = 1$ and $m = |W| \geqslant 2$ (so that $\aleph = ((0,j))_{0 \leqslant j \leqslant m-1}$ and we write $k_j = k_{0,j}$ for simplicity), then it follows from [BoRo2, Theo. 18.2.4] that

$$\mathscr{Z}_k = \{(x,y,z) \in \mathbb{C}^3 \mid \prod_{j=0}^{m-1} (z - mk_j) = xy\}.$$

Now, if p is a prime number not dividing m and smaller than m (this always exists if $m \geqslant 3$), and if we choose k such that $k_0 = k_1 = \cdots = k_{p-1} = 0$ and $k_p = k_{p+1} = \cdots = k_{m-1} = 1$, then

$$\mathscr{Z}_k = \{(x,y,z) \in \mathbb{C}^3 \mid z^p(z-m)^{m-p} = xy\}$$

contains a simple singularity of type A_{p-1} and so $\mathscr{Z}_k$ is rationally smooth but not p-smooth while $\mathscr{Z}_0 = \mathbb{C}^2/\boldsymbol{\mu}_m$ is p-smooth because p does not divide m.

(2) If W is of type B_2 and $(k_{\Omega,0}, k_{\Omega,1}) = (k_{\Omega',0}, k_{\Omega',1})$ and $k_{\Omega,0} \neq k_{\Omega,1}$ (where Ω and Ω' are the two orbits of reflecting hyperplanes), then $\mathscr{Z}_k$ admits a unique singular point and the singularity is equivalent to the singularity at 0 of the orbit closure of the minimal nilpotent orbit of the Lie algebra $\mathfrak{sl}_3(\mathbb{C})$ (see [BBFJLS, Theo. 1.3 (b)]): it is well-known that this orbit closure is not rationally smooth. ■

6.B. Morphisms between Calogero-Moser spaces

Let (V', W') be another pair consisting of a finite dimensional complex vector space and a finite subgroup $W' \subset \mathbf{GL}_{\mathbb{C}}(V')$. We fix a parameter $k' \in \mathbb{C}^{\aleph(V',W')}$ and, in this subsection, we will denote by a prime $?'$ the object $?$ defined using (V', W') instead of (V, W), i.e. the object $?(V', W')$. For instance, $\mathscr{Z}'_{k'} = \mathscr{Z}_{k'}(V', W')$ and $\aleph' = \aleph(V', W')$.

Hypothesis. *We assume in this subsection that we are given a $\mathbb{C}^\times$-equivariant morphism of varieties $\varphi : \mathscr{Z}'_{k'} \to \mathscr{Z}_k$.*

We denote by $\varphi_{\mathrm{fix}} : \mathscr{Z}'^{\mathbb{C}^\times}_{k'} \to \mathscr{Z}^{\mathbb{C}^\times}_k$ the induced map. Then φ_{fix} induces a morphism of algebras

$$\varphi^{\#}_{\mathrm{fix}} : \operatorname{Im} \boldsymbol{\Omega}^k \longrightarrow \operatorname{Im} \boldsymbol{\Omega}'^{k'}$$

through the formula

$$\varphi^{\#}_{\mathrm{fix}}(e_{\mathfrak{F}_p}) = \sum_{p' \in \varphi^{-1}_{\mathrm{fix}}(p)} e'_{\mathfrak{F}'_{p'}}.$$

The following proposition should be compared with [BoMa, Cor. 1.5]:

Proposition 6.6. *Assume that Conjecture 6.2 holds for both $\mathscr{Z}_k$ and $\mathscr{Z}'_{k'}$. Then*

$$\varphi^{\#}_{\mathrm{fix}}(\mathscr{F}_j \operatorname{Im} \boldsymbol{\Omega}^k) \subset \mathscr{F}'_j \operatorname{Im} \boldsymbol{\Omega}'^{k'}$$

for all j.

Proof. The maps φ and φ_{fix} induce maps between equivariant cohomology groups which we denote by

$$\varphi^* : \mathrm{H}^\bullet_{\mathbb{C}^\times}(\mathfrak{X}_k) \longrightarrow \mathrm{H}^\bullet_{\mathbb{C}^\times}(\mathfrak{X}'_{k'})$$

and

$$\varphi^*_{\text{fix}} : \mathrm{H}^\bullet_{\mathbb{C}^\times}(\mathfrak{X}_k^{\mathbb{C}^\times}) \longrightarrow \mathrm{H}^\bullet_{\mathbb{C}^\times}(\mathfrak{X}'^{\mathbb{C}^\times}_{k'}).$$

The functoriality properties of cohomology imply that the diagram

$$\begin{tikzcd}
\mathrm{H}^\bullet_{\mathbb{C}^\times}(\mathfrak{X}_k) \arrow[r, "\varphi^*"] \arrow[d, "i_k^*"'] & \mathrm{H}^\bullet_{\mathbb{C}^\times}(\mathfrak{X}'_{k'}) \arrow[d, "i'^*_{k'}"] \\
\mathrm{H}^\bullet_{\mathbb{C}^\times}(\mathfrak{X}_k^{\mathbb{C}^\times}) \arrow[r, "\varphi^*_{\text{fix}}"] & \mathrm{H}^\bullet_{\mathbb{C}^\times}(\mathfrak{X}'^{\mathbb{C}^\times}_{k'})
\end{tikzcd}$$

is commutative. Recall from (6.1) that we identify $\mathrm{H}^\bullet_{\mathbb{C}^\times}(\mathfrak{X}_k^{\mathbb{C}^\times})$ (resp. $\mathrm{H}^\bullet_{\mathbb{C}^\times}(\mathfrak{X}'^{\mathbb{C}^\times}_{k'})$) with $\mathbb{C}[\hbar] \otimes \operatorname{Im} \boldsymbol{\Omega}^k$ (resp. $\mathbb{C}[\hbar] \otimes \operatorname{Im} \boldsymbol{\Omega}'^{k'}$). Through this identification, the map φ^*_{fix} becomes $\mathrm{Id}_{\mathbb{C}[\hbar]} \otimes \varphi^\#_{\text{fix}}$ by construction. Therefore, the above commutative diagram yields a commutative diagram

$$\begin{tikzcd}
\mathrm{H}^\bullet_{\mathbb{C}^\times}(\mathfrak{X}_k) \arrow[r, "\varphi^*"] \arrow[d, "i_k^*"'] & \mathrm{H}^\bullet_{\mathbb{C}^\times}(\mathfrak{X}'_{k'}) \arrow[d, "i'^*_{k'}"] \\
\mathbb{C}[\hbar] \otimes \operatorname{Im} \boldsymbol{\Omega}^k \arrow[r, "\mathrm{Id}_{\mathbb{C}[\hbar]} \otimes \varphi^\#_{\text{fix}}"] & \mathbb{C}[\hbar] \otimes \operatorname{Im} \boldsymbol{\Omega}'^{k'}.
\end{tikzcd}$$

So $\mathrm{Id}_{\mathbb{C}[\hbar]} \otimes \varphi^\#_{\text{fix}}(\operatorname{Im} i_k^*) \subset \operatorname{Im} i'^*_{k'}$. As we assume that Conjecture 6.2 holds for both $\mathfrak{X}_k$ and $\mathfrak{X}'_{k'}$, this is exactly the statement of the proposition. $\square$

Remark 6.7. In the next section, we propose some conjecture which would give many examples of morphisms between Calogero-Moser spaces. In all the cases where these conjectures are proved, the above Proposition 6.6 gives a highly non-trivial link between the character tables of W and W' (see for instance [BoMa, Cor. 4.22] for the case where $W = G(l, 1, n)$). ■

7. Symplectic leaves and fixed points

If $\tau \in \mathrm{N}_{\mathbf{GL}_{\mathbb{C}}(V)}(W)$ has finite order and satisfies $\tau(k) = k$, then τ acts on the Calogero-Moser space $\mathscr{Z}_k$. We are interested in this section in the variety of fixed points $\mathscr{Z}_k^\tau$ of τ (endowed with its reduced structure) and its symplectic leaves. Since W acts trivially on $\mathscr{Z}_k$, the action of τ on $\mathscr{Z}_k$ depends only on its coset $W\tau$.

We say that τ is *W-full* if $\dim(V^\tau) = \max_{w \in W} \dim(V^{w\tau})$ (see [Bon4, §1.F.4]). Recall that τ is W-full if and only if the natural map $V^\tau \longrightarrow (V/W)^\tau$ is onto [Bon4, (3.2)], (the argument is due to Springer [Spr]). Since W acts trivially on $\mathscr{Z}_k$, we may replace τ by any $w\tau$ and assume that τ is W-full. Therefore, we will work in this section under the following hypothesis:

> **Hypothesis and notation.** *We fix in this section, and only in this section, a W-**full** element τ of finite order in $\mathrm{N}_{\mathbf{GL}_{\mathbb{C}}(V)}(W)$ and we assume that $\tau(k) = k$.*

As in [Bon4], let W_τ denote the quotient Σ/Π, where Σ (resp. Π) is the setwise (resp. pointwise) stabilizer of V^τ. A parabolic subgroup P of W is called *τ-split* if P is the stabilizer in W of a vector belonging to V^τ. Note that a τ-split parabolic subgroup is τ-stable. If P is a τ-split parabolic subgroup of W, we set

$$\overline{\mathrm{N}}_{W_\tau}(P_\tau) = \mathrm{N}_{W_\tau}(P_\tau)/P_\tau.$$

Then $\overline{\mathrm{N}}_{W_\tau}(P_\tau)$ acts faithfully on the vector space $(V^P)^\tau$. So one can define Calogero-Moser spaces associated with the pair $((V^P)^\tau, \overline{\mathrm{N}}_{W_\tau}(P_\tau))$, even though $\overline{\mathrm{N}}_{W_\tau}(P_\tau)$ is not necessarily a reflection group for its action on $(V^P)^\tau$.

7.A. Symplectic leaves

Brown-Gordon [BrGo, §3.5] defined a stratification of any complex affine Poisson variety into *symplectic leaves*. They also proved that the Calogero-Moser space $\mathscr{Z}_k$ has only finitely many symplectic leaves [BrGo, Theo. 7.8]. As explained in [Bon4, §4.A], this implies that the variety $\mathscr{Z}_k^\tau$ admits a stratification into symplectic leaves and that there are only finitely many of them. We denote by $\mathcal{S}ymp(\mathscr{Z}_k^\tau)$ the set of its symplectic leaves. Such a symplectic leaf is called *τ-cuspidal* if it has dimension 0 (we also talk about *τ-cuspidal* points[3]). Note that this notion can be defined even if τ is not W-full. Let $\mathrm{Cus}_k^\tau(V, W)$

[3] We can also say that a Calogero-Moser k-family is *τ-cuspidal* if it is associated to a τ-cuspidal point. Of course, a τ-cuspidal family is τ-stable.

denote the set of pairs (P,p) where P is a τ-split parabolic subgroup of W and p is a τ-cuspidal point of $\mathscr{Z}_{k_P}(V/V^P,P)$ (here k_P is the restriction of k to $\aleph(V/V^P,P)$).

Then W_τ acts on $\mathrm{Cus}_k^\tau(V,W)$ and the next result is proved in [Bon4, Theo. A] (whenever $\tau = \mathrm{Id}_V$, it is independently due to Bellamy [Bel3] and Losev [Los]).

Theorem 7.1. *Recall that τ is W-full. Then there is a natural bijection*

$$\mathcal{S}ymp(\mathscr{Z}_k^\tau) \xrightarrow{\sim} \mathrm{Cus}_k^\tau(V,W)/W_\tau.$$

Moreover, the dimension of the symplectic leaf associated with the W_τ-orbit of (P,p) through this bijection is equal to $2\dim(V^P)^\tau$.

We refer to [Bon4, Lem. 8.4] for the explicit description of the bijection: if $(P,p) \in \mathrm{Cus}_k^\tau(V,W)$, we denote by $\mathcal{S}_{P,p}$ its associated symplectic leaf. Recall also [Bon4, Rem. 4.2] that all symplectic leaves are $\mathbb{C}^\times$-stable.

7.B. Normalization

Let $(P,p) \in \mathrm{Cus}_k^\tau(V,W)$. Then $\overline{\mathcal{S}_{P,p}}$ carries a Poisson structure and so does its normalization $\overline{\mathcal{S}_{P,p}}^{\mathrm{nor}}$ (see [Kal]). We proposed in [Bon4, Conj. B] the following conjecture:

> **Conjecture 7.2.** *There exists a parameter $k_{P,p} \in \mathbb{C}^{\aleph((V^P)^\tau, \overline{\mathrm{N}}_{W_\tau}(P_\tau))}$ such that the varieties $\overline{\mathcal{S}_{P,p}}^{\mathrm{nor}}$ and $\mathscr{Z}_{k_{P,p}}((V^P)^\tau, \overline{\mathrm{N}}_{W_\tau}(P_\tau))$ are isomorphic as Poisson varieties endowed with a $\mathbb{C}^\times$-action.*

Theorem 7.3. *Conjecture 7.2 is known to hold in the following cases:*

(a) *If $k = 0$.*
(b) *If $\mathscr{Z}_k$ is smooth.*
(c) *If W is a Weyl group of type B (i.e. C) and $\tau = \mathrm{Id}_V$.*
(d) *If W is of type D and τ is a diagram automorphism.*
(e) *If W is dihedral and τ is the non-trivial diagram automorphism.*
(f) *If W is of type G_4.*

Proof. See [Bon4, Prop. 6.7 and §9] for more details: this relies on works of Bellamy-Maksimau-Schedler [BeMaSc], Maksimau and the author [BoMa] and Thiel and the author [BoTh]. ☐

7.C. Regular case

We say that the element τ of $\mathrm{N}_{\mathbf{GL}_{\mathbb{C}}(V)}(W)$ is *regular* if $V_{\mathrm{reg}}^{\tau} \neq \varnothing$. In this case, $\mathfrak{X}_k^{\tau}$ admits a unique irreducible component of maximal dimension [Bon5, Prop. 2.4]: we denote by by $(\mathfrak{X}_k^{\tau})_{\max}$. It has dimension $2\dim(V^{\tau})$. The following result has been proved in [Bon5, Theo. 2.8] (here, if χ is a τ-stable character of W, we choose an extension $\tilde{\chi}$ of χ to $W\langle\tau\rangle$):

Theorem 7.4. *Assume that τ is a regular element. Let $p \in \mathfrak{X}_k^{\mathbb{C}^\times}$ be such that $\tau(p) = p$ and $\sum_{\chi \in \mathfrak{F}_p^{\tau}} |\tilde{\chi}(\tau)|^2 \neq 0$. Then p belongs to $(\mathfrak{X}_k^{\tau})_{\max}$.*

Moreover, it is conjectured in [Bon5, Conj. 2.6] that the converse holds:

Conjecture 7.5. *Assume that τ is a regular element. Let $p \in \mathfrak{X}_k^{\mathbb{C}^\times}$ be such that $\tau(p) = p$. Then p belongs to $(\mathfrak{X}_k^{\tau})_{\max}$ if and only if $\sum_{\chi \in \mathfrak{F}_p^{\tau}} |\tilde{\chi}(\tau)|^2 \neq 0$.*

Note that Conjecture 7.5 holds for $W = \mathfrak{S}_n$ by [Bon5, Exam. 5.7].

8. Special features of Coxeter groups

Hypothesis and notation. *We assume in this section, and only in this section, that there exists a W-stable $\mathbb{R}$-vector subspace $V_{\mathbb{R}}$ of V such that $V = \mathbb{C} \otimes_{\mathbb{R}} V_{\mathbb{R}}$ as a W-module, that k takes* ***real values*** *and that $c_k(s) \geqslant 0$ for all $s \in \mathrm{Ref}(W)$.*

First, note that the reflections of W have order 2 (so that $e_H = 2$ for all $H \in \mathcal{A}$) and that we have a bijection between $\mathrm{Ref}(W)$ and $\mathcal{A}$. This implies that

$$c_k(s) = k_{H,1} - k_{H,0},$$

where H is the reflecting hyperplane of s. Note that $k = k^{\sharp}$.

8.A. Lusztig families

If $\chi \in \mathrm{Irr}(W)$, we denote by $\mathbf{sch}_{\chi}^{(k)} \in \mathcal{O}[\mathbf{q}^{\mathbb{R}}]$ the *Schur element* associated with the irreducible character χ_k of the Hecke algebra $\mathcal{H}_k(W)$ (see [GePf, §7.2]): since $\mathbb{R}$ is an ordered group, we can set $a_{\chi}^{(k)} = \mathrm{val}\,\mathbf{sch}_{\chi}^{(k)}$ and $A_{\chi}^{(k)} = \deg \mathbf{sch}_{\chi}^{(k)}$: this defines two maps $a^{(k)}$, $A^{(k)} : \mathrm{Irr}(W) \to \mathbb{R}$. Using the map $a^{(k)}$ and the notion of *J-induction*, Lusztig [Lus8, §22] defined the notion of *k-constructible character* (or *c_k-constructible character*) of W. Let $\mathcal{I}rr_k(W)$ denote the graph defined as follows:

- The set of vertices of $\mathcal{I}rr_k(W)$ is $\operatorname{Irr}(W)$.
- There is an edge between two vertices if they both occur in a same k-constructible character.

A *Lusztig k-family* is a subset of $\operatorname{Irr}(W)$ consisting of the vertices of a connected component of $\mathcal{I}rr_k(W)$. We denote by $\operatorname{Fam}_k^{\mathrm{Lus}}(W)$ the set of Lusztig k-families of W. It turns out that

$$\operatorname{Fam}_k^{\mathrm{Hec}}(W) = \operatorname{Fam}_k^{\mathrm{Lus}}(W) \tag{8.1}$$

(see [Chl] and [Lus8]). Martino's Conjecture 5.5 has a more precise version in the Coxeter case:

Conjecture 8.2 (Gordon-Martino)**.** *If W is a Coxeter group, then*

$$\operatorname{Fam}_k^{\mathrm{CM}}(W) = \operatorname{Fam}_k^{\mathrm{Lus}}(W).$$

It follows from the definition [Lus8] that

(8.3) *the maps $a^{(k)}$, $A^{(k)} : \operatorname{Irr}(W) \to \mathbb{R}$ are constant on Lusztig k-families.*

So the next proposition is a strong argument in favor of Conjecture 8.2:

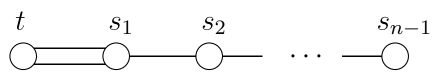
Proposition 8.4. *The map $a^{(k)}+A^{(k)} : \operatorname{Irr}(W) \to \mathbb{R}$ is constant on Calogero-Moser k-families.*

Proof. If χ and χ' belong to the same Calogero-Moser family, then $\omega_\chi(\mathbf{\Omega}^k(\mathbf{eu})) = \omega_{\chi'}(\mathbf{\Omega}^k(\mathbf{eu}))$. But the scalar $\omega_\chi(\mathbf{\Omega}^k(\mathbf{eu}))$ is, up to a suitable renormalization by a fixed affine transformation of $\mathbb{R}$, equal to $a_\chi^{(k)} + A_\chi^{(k)}$ (see [BoRo2, Lem. 7.2.1] and [BrMi, 4.21]). So the result follows. ☐

Remark 8.5. Using the Kazhdan-Lusztig basis of the Hecke algebra $\mathcal{H}_k(W)$ and the associated partition of W into two-sided cells [Lus8, §8] (see also [Bon2, Def. 6.1.4]), Lusztig defined a partition of $\operatorname{Irr}(W)$ into *Kazhdan-Lusztig k-families* as follows: two irreducible characters χ and χ' of W belong to the same Kazhdan-Lusztig k-family if χ_k and χ'_k both occur in the left module associated to a same two-sided cell. Let $\operatorname{Fam}_k^{\mathrm{KL}}(W)$ denote the set of Kazhdan-Lusztig k-families of W.

Lusztig conjectured [Lus8, §23] that $\mathrm{Fam}_k^{\mathrm{Lus}}(W) = \mathrm{Fam}_k^{\mathrm{KL}}(W)$. This conjecture is known to hold in the following cases:

- If c_k is constant [Lus8, Prop. 23.3] (note that this covers the case where W has only one conjugacy class of reflections, i.e. if W is of type ADE or $I_2(2m+1)$).
- If W is dihedral [Lus8].
- If W is of type F_4 [Gec].
- If W is of type B_n and $c_k(t) > (n-1)c_k(s_1)$, where the Coxeter graph is given by



  (see [BoIa] and [Bon1]).

Note that this conjecture involves only the Hecke algebra and is not related to the geometry of the Calogero-Moser space and the theme of this paper. ■

8.B. Cuspidal families

Lusztig also introduced the important notion of τ-*cuspidal* Lusztig k-family [Lus6, §8.1] (note that the definition in [Lus6, §8.1] is for the *equal parameter* case, but the definition can easily be extended to general parameters, as explained in [BeTh, §2.5]). As there is also a notion of τ-cuspidal Calogero-Moser family (see §7.A), it is natural to expect that the equality predicted by Gordon-Martino conjecture preserves this feature:

> **Conjecture 8.6.** *If W is a finite Coxeter group, and if $\tau \in \mathrm{N}_{\mathbf{GL}_{\mathbb{R}}(V_{\mathbb{R}})}(W)$ is W-full and satisfies $\tau(k) = k$, then the τ-cuspidal Lusztig k-families coincide with the τ-cuspidal Calogero-Moser k-families.*

If $\tau = \mathrm{Id}_V$, this conjecture has been proposed by Bellamy and Thiel [BeTh, Conj. B]. The τ-cuspidal Lusztig k-families have been classified (see [Lus6, §8.1] for the equal parameter case and [BeTh, §6, §7] for the unequal parameter case) and it turns out that there is at most one τ-cuspidal Lusztig k-family. So Conjecture 8.6 would imply that there is at most one τ-cuspidal point in $\mathscr{Z}_k$ whenever W is a Coxeter group [BeTh, Conj. D].

Theorem 8.7. *Conjectures 8.2 and 8.6 are known to hold for W of type A, $B = C$, D, $I_2(m)$, H_3 or F_4 (with the restriction that $\tau = \mathrm{Id}_V$ if W is of type F_4).*

Proof. For Conjecture 8.2, see [Gor1, Theo. 5.6] for type A, see [GoMa] for types $B = C$ and D, see [Bel2] for type $I_2(m)$ and [BoTh] for types H_3 and F_4.

For Conjecture 8.6, see [BeTh, Theo. A] for types A, $B = C$, D and $I_2(m)$, and [BoTh] for types H_3 and F_4. $\square$

Part II. Unipotent representations of finite reductive groups

Throughout this part, we will only consider algebraic varieties and algebraic groups defined over an algebraic closure of a finite field. If $\mathbf{G}$ is an algebraic group, we denote by $\mathrm{Z}(\mathbf{G})$ its center. If $\mathbf{S}$ is a torus, we denote by $Y(\mathbf{S})$ its lattice of one-parameter subgroups.

Let $\mathscr{G}roups$ denote the class of triples $(q, \mathbf{G}, F)$ where:

- q is a power of some prime number p.
- $\mathbf{G}$ is a connected reductive group defined over an algebraic closure $\mathbb{F}$ of the finite field $\mathbb{F}_p$ with p elements.
- $F : \mathbf{G} \to \mathbf{G}$ is a Frobenius endomorphism of $\mathbf{G}$ endowing $\mathbf{G}$ with a rational structure over the finite subfield $\mathbb{F}_q$ of $\mathbb{F}$ with q elements.

This part provides a quick survey on unipotent representations of the finite reductive group $\mathbf{G}^F$ (where $(q, \mathbf{G}, F) \in \mathscr{G}roups$) and their associated structures (cuspidal representations, Harish-Chandra theory, d-Harish-Chandra theory...)[4].

Hypothesis and notation. *We fix in this part, and only in this part, a triple $\mathscr{G} = (q, \mathbf{G}, F) \in \mathscr{G}roups$. We denote by p the unique prime number dividing q and by $\mathbb{F}$ the algebraic closure of $\mathbb{F}_p$ over which $\mathbf{G}$ is defined. We fix a prime number ℓ different from p and we denote by $\overline{\mathbb{Q}}_\ell$ an algebraic closure of $\mathbb{Q}_\ell$. Note that $\mathbf{G}$ is not necessarily split over $\mathbb{F}_q$.*

If $\mathbf{X}$ is an algebraic variety over $\mathbb{F}$, we denote by $\mathrm{H}_c^j(\mathbf{X})$ its j-th ℓ-adic cohomology group with compact support with coefficients in $\overline{\mathbb{Q}}_\ell$: it is a finite dimensional $\overline{\mathbb{Q}}_\ell$-vector space. We set $\mathrm{H}_c^\bullet(\mathbf{X}) = \bigoplus_{j \geqslant 0} \mathrm{H}_c^j(\mathbf{X})$.

We fix an F-stable Borel subgroup $\mathbf{B}$ of $\mathbf{G}$ and an F-stable maximal torus $\mathbf{T}$ of $\mathbf{B}$. Let $\mathscr{B} = \mathbf{G}/\mathbf{B}$ denote the flag variety. Let $\mathscr{W} = \mathrm{N}_{\mathbf{G}}(\mathbf{T})/\mathbf{T}$ denote the

[4] Our formalism here does not include the Suzuki and Ree groups.

Weyl group of $\mathbf{G}$. It is acted on by F and we denote by τ the automorphism of $\mathscr{W}$ induced by F. If $\mathbf{\mathfrak{O}}$ is a $\mathbf{G}$-orbit in $\mathscr{B} \times \mathscr{B}$, we denote by

$$\mathbf{X}_{\mathfrak{O}} = \{g\mathbf{B} \in \mathscr{B} \mid (g\mathbf{B}, F(g)\mathbf{B}) \in \mathfrak{O}\}.$$

Then $\mathbf{X}_{\mathfrak{O}}$ is called a *Deligne-Lusztig variety*: it is acted on the left by the finite group $\mathbf{G}^F$. Hence, the vector spaces $\mathrm{H}_c^j(\mathbf{X}_{\mathfrak{O}})$ inherits a structure of $\overline{\mathbb{Q}}_\ell\mathbf{G}^F$-module. An irreducible representation of $\mathbf{G}^F$ (over $\overline{\mathbb{Q}}_\ell$) is called *unipotent* if it appears in such a $\overline{\mathbb{Q}}_\ell\mathbf{G}^F$-module, for some $\mathfrak{O}$ and some j. The set of isomorphism classes of irreducible unipotent representations of $\mathbf{G}^F$ will be denoted by $\mathscr{U}\mathit{nip}(\mathscr{G})$. We define a *unipotent* representation of $\mathbf{G}^F$ to be a direct sum of irreducible unipotent representations.

We define a *Levi subgroup* of $\mathscr{G}$ to be a triple $\mathscr{L} = (q, \mathbf{L}, F)$ where $\mathbf{L}$ is an F-stable Levi complement of a parabolic subgroup of $\mathbf{G}$. In this case, we set $W_{\mathscr{G}}(\mathscr{L}) = \mathrm{N}_{\mathbf{G}^F}(\mathbf{L})/\mathbf{L}^F$: this group acts on $\mathscr{U}\mathit{nip}(\mathscr{L})$ and, if $\lambda \in \mathscr{U}\mathit{nip}(\mathscr{L})$, we denote by $W_{\mathscr{G}}(\mathscr{L}, \lambda)$ its stabilizer.

The set of unipotent representations admits several interesting partitions, which are related to the different problems one may consider: Harish-Chandra series for an algebraic parametrization, d-Harish-Chandra series for blocks in transverse characteristic, families for computing character values through the theory of character sheaves... All these partitions interconnect in a very subtle way.

9. Harish-Chandra theories

9.A. Classical Harish-Chandra theory

If $\mathbf{L}$ is an F-stable Levi complement of an F-stable parabolic subgroup $\mathbf{P}$ of $\mathbf{G}$, we denote by

$$\begin{array}{rccc} \mathscr{R}_{\mathbf{L}\subset\mathbf{P}}^{\mathbf{G}} : & \overline{\mathbb{Q}}_\ell\mathbf{L}^F\text{-mod} & \longrightarrow & \overline{\mathbb{Q}}_\ell\mathbf{G}^F\text{-mod} \\ & M & \longmapsto & \mathrm{Ind}_{\mathbf{P}^F}^{\mathbf{G}^F} \tilde{M} \end{array}$$

the *Harish-Chandra induction* functor [DiMi1, Chap. 4]. Here, $\tilde{M}$ denotes the inflation of M through the surjective morphism $\mathbf{P}^F \twoheadrightarrow \mathbf{L}^F$. If we denote by $\mathbf{U}_\mathbf{P}$ the unipotent radical of $\mathbf{P}$, then $\mathscr{R}_{\mathbf{L}\subset\mathbf{P}}^{\mathbf{G}}$ admits a left and right adjoint functor ${}^*\mathscr{R}_{\mathbf{L}\subset\mathbf{P}}^{\mathbf{G}}$ given by

$$\begin{array}{rccc} {}^*\mathscr{R}_{\mathbf{L}\subset\mathbf{P}}^{\mathbf{G}} : & \overline{\mathbb{Q}}_\ell\mathbf{G}^F\text{-mod} & \longrightarrow & \overline{\mathbb{Q}}_\ell\mathbf{L}^F\text{-mod} \\ & N & \longmapsto & N^{\mathbf{U}_\mathbf{P}^F}. \end{array}$$

The functor ${}^*\mathcal{R}_{\mathbf{L}\subset\mathbf{P}}^{\mathbf{G}}$ is called the *Harish-Chandra restriction*. Note that both functors send a unipotent representation to a unipotent representation. An irreducible unipotent representation N is called *cuspidal* if ${}^*\mathcal{R}_{\mathbf{L}\subset\mathbf{P}}^{\mathbf{G}} N = 0$ for any F-stable Levi complement of a proper F-stable parabolic subgroup $\mathbf{P}$ of $\mathbf{G}$. We denote by $\mathcal{U}nip_{\mathrm{cus}}(\mathcal{G})$ the set of (isomorphism classes of) cuspidal unipotent irreducible representations of $\mathbf{G}^F$.

It turns out that both Harish-Chandra functors do not depend on the choice of the F-stable parabolic subgroup $\mathbf{P}$ admitting $\mathbf{L}$ as a Levi complement. We denote by $\mathcal{C}us(\mathcal{G})$ the set of pairs $(\mathcal{L}, \lambda)$, where $\mathcal{L} = (q, \mathbf{L}, F)$ and $\mathbf{L}$ is an F-stable Levi complement of an F-stable parabolic subgroup of $\mathbf{G}$, and $\lambda \in \mathcal{U}nip_{\mathrm{cus}}(\mathcal{L})$. We denote by $\mathcal{U}nip(\mathcal{G}, \mathcal{L}, \lambda)$ the set of unipotent irreducible representations occuring in $\mathcal{R}_{\mathbf{L}\subset\mathbf{P}}^{\mathbf{G}}\lambda$, where $\mathbf{P}$ is any F-stable parabolic subgroup admitting $\mathbf{L}$ as a Levi complement: this set is called the *Harish-Chandra series* associated with the pair $(\mathcal{L}, \lambda)$. This set depends on the pair $(\mathcal{L}, \lambda)$ up to $\mathbf{G}^F$-conjugacy. We denote by $\mathcal{C}us(\mathcal{G})/\sim$ the set of $\mathbf{G}^F$-conjugacy classes of elements of $\mathcal{C}us(\mathcal{G})$. The *Harish-Chandra theory* can then be summarized as follows [Lus1, §5]:

Theorem 9.1 (Lusztig). *We have*

$$\mathcal{U}nip(\mathcal{G}) = \dot{\bigcup_{(\mathcal{L},\lambda)\in\mathcal{C}us(\mathcal{G})/\sim}} \mathcal{U}nip(\mathcal{G}, \mathcal{L}, \lambda),$$

where $\dot{\cup}$ *means a disjoint union.*

Moreover, if $(\mathcal{L}, \lambda) \in \mathcal{C}us(\mathcal{G})$, *with* $\mathcal{L} = (q, \mathbf{L}, F)$ *and if* $\mathbf{P}$ *is any* F*-stable parabolic subgroup admitting* $\mathbf{L}$ *as a Levi complement, then:*

(a) *The group* $W_{\mathcal{G}}(\mathcal{L})$ *is a Weyl group for its action on the lattice*

$$\{y \in Y(\mathrm{Z}(\mathbf{L})) \mid F(y) = qy\}.$$

Moreover, $W_{\mathcal{G}}(\mathcal{L}, \lambda) = W_{\mathcal{G}}(\mathcal{L})$.

(b) *The endomorphism algebra* $\mathrm{End}_{\mathbf{G}^F} \mathcal{R}_{\mathbf{L}\subset\mathbf{P}}^{\mathbf{G}}\lambda$ *is isomorphic to a Hecke algebra of the form* $\mathcal{H}_{k_{\mathcal{L}}}(W_{\mathcal{G}}(\mathcal{L}), q)$, *where* $k_{\mathcal{L}} \in \aleph(W_{\mathcal{G}}(\mathcal{L}))$ *does not depend on* λ *(and is integer valued).*

(c) *The statement* (a) *induces a bijection*

$$\mathrm{HC}^{\mathcal{G},\mathcal{L},\lambda} : \mathrm{Irr}\, W_{\mathcal{G}}(\mathcal{L}) \xrightarrow{\sim} \mathcal{U}nip(\mathcal{G}, \mathcal{L}, \lambda).$$

In other words,

$$\mathcal{U}nip(\mathcal{G}) = \dot{\bigcup_{(\mathcal{L},\lambda)\in\mathcal{C}us(\mathcal{G})/\sim}} \mathrm{HC}^{\mathcal{G},\mathcal{L},\lambda}(\mathrm{Irr}\, W_{\mathcal{G}}(\mathcal{L})).$$

Example 9.2 (Principal series)**.** As $\mathbf{T}$ is an F-stable maximal torus of the F-stable Borel subgroup $\mathbf{B}$, we may set $\mathscr{T} = (q, \mathbf{T}, F)$ and so there is a Harish-Chandra series associated with the trivial character (denoted by 1) of $\mathbf{T}^F$ (indeed, as there is no proper parabolic subgroup of $\mathbf{T}$, 1 is a cuspidal unipotent representation). Then $W_{\mathscr{G}}(\mathscr{T}) = \mathscr{W}^\tau$ and the injective map $\mathrm{HC}^{\mathscr{G},\mathscr{T},1} : \mathrm{Irr}(\mathscr{W}^\tau) \longrightarrow \mathscr{Unip}(\mathscr{G})$ will be simply denoted by $\mathrm{HC}^{\mathscr{G}}$. Its image is called the *principal* series of unipotent representations of $\mathscr{G}$.

If $\tau = \mathrm{Id}_{\mathscr{W}}$ (i.e., if $\mathbf{G}/\mathrm{Z}(\mathbf{G})$ is split), then the parameter $k_{\mathscr{T}}$ of Theorem 9.1(b) is given by $(k_{\mathscr{T}})_{H,0} = 1$ and $(k_{\mathscr{T}})_{H,1} = 0$ for any reflecting hyperplane H of $\mathscr{W}$. This parameter will be denoted by k_{sp} in the next part. ■

9.B. Deligne-Lusztig induction

For going further, we need to recall the construction of Deligne-Lusztig induction. If $\mathbf{P}$ is a (not necessarily F-stable) parabolic subgroup of $\mathbf{G}$ admitting an F-stable Levi complement $\mathbf{L}$, then we define the variety $\mathbf{Y_P}$ (also called a *Deligne-Lusztig variety*) by

$$\mathbf{Y_P} = \{g\mathbf{U_P} \in \mathbf{G}/\mathbf{U_P} \mid g^{-1}F(g) \in \mathbf{U_P} \cdot F(\mathbf{U_P})\},$$

where $\mathbf{U_P}$ denotes the unipotent radical of $\mathbf{P}$. Then $\mathbf{Y_P}$ inherits a left action of $\mathbf{G}^F$ and a right action of $\mathbf{L}^F$ which commute, endowing the ℓ-adic cohomology groups with compact support $\mathrm{H}_c^j(\mathbf{Y_P})$ with a structure of a $(\overline{\mathbb{Q}}_\ell\mathbf{G}^F, \overline{\mathbb{Q}}_\ell\mathbf{L}^F)$-bimodule. This allows us to define a map $R_{\mathbf{L}\subset\mathbf{P}}^{\mathbf{G}} : \mathbb{Z}\,\mathrm{Irr}(\mathbf{L}^F) \longrightarrow \mathbb{Z}\,\mathrm{Irr}(\mathbf{G}^F)$ between Grothendieck groups by the formula

$$R_{\mathbf{L}\subset\mathbf{P}}^{\mathbf{G}}([M]) = \sum_{k\,\geqslant\,0} (-1)^j [\mathrm{H}_c^j(\mathbf{Y_P}) \otimes_{\overline{\mathbb{Q}}_\ell\mathbf{L}^F} M].$$

This map is called the *Deligne-Lusztig induction* and is the shadow of a functor $\mathscr{R}_{\mathbf{L}\subset\mathbf{P}}^{\mathbf{G}} : \mathrm{D}^b(\overline{\mathbb{Q}}_\ell\mathbf{L}^F\text{-mod}) \longrightarrow \mathrm{D}^b(\overline{\mathbb{Q}}_\ell\mathbf{G}^F\text{-mod})$ between bounded derived categories, which is defined by

$$\mathscr{R}_{\mathbf{L}\subset\mathbf{P}}^{\mathbf{G}}(M) = \mathrm{H}_c^\bullet(\mathbf{Y_P}) \otimes_{\overline{\mathbb{Q}}_\ell\mathbf{L}^F} M.$$

Note that we work in a semisimple world, so that any complex of $\overline{\mathbb{Q}}_\ell\Gamma$-modules (where Γ is a finite group) is quasi-isomorphic to its cohomology: here, $\mathrm{H}_c^\bullet(\mathbf{Y_P})$ is viewed as a complex of $(\overline{\mathbb{Q}}_\ell\mathbf{G}^F, \overline{\mathbb{Q}}_\ell\mathbf{L}^F)$-bimodules whose j-th term is $\mathrm{H}_c^j(\mathbf{Y_P})$ and whose differential is zero.

If the parabolic subgroup $\mathbf{P}$ is F-stable, then the Deligne-Lusztig functor $\mathscr{R}_{\mathbf{L}\subset\mathbf{P}}^{\mathbf{G}}$ is just the functor induced by the Harish-Chandra functor at the level of derived categories (this justifies the use of the same notation). Both $R_{\mathbf{L}\subset\mathbf{P}}^{\mathbf{G}}$ and $\mathscr{R}_{\mathbf{L}\subset\mathbf{P}}^{\mathbf{G}}$ admits adjoints (in two different significations of *adjoint*) which we denote by ${}^*R_{\mathbf{L}\subset\mathbf{P}}^{\mathbf{G}}$ and ${}^*\mathscr{R}_{\mathbf{L}\subset\mathbf{P}}^{\mathbf{G}}$ and which are defined by

$$ {}^*\mathscr{R}_{\mathbf{L}\subset\mathbf{P}}^{\mathbf{G}}(N) = \mathrm{H}_c^\bullet(\mathbf{Y_P})^* \otimes_{\overline{\mathbb{Q}}_\ell\mathbf{G}^F} N $$

and

$$ {}^*R_{\mathbf{L}\subset\mathbf{P}}^{\mathbf{G}}([N]) = \sum_{k\geqslant 0} (-1)^j [\mathrm{H}_c^j(\mathbf{Y_P})^* \otimes_{\overline{\mathbb{Q}}_\ell\mathbf{G}^F} N]. $$

9.C. *d*-Harish-Chandra theory

We fix a natural number $d \geqslant 1$ and we denote by ζ_d a primitive d-th root of unity. The *d-Harish-Chandra theory* of Broué-Malle-Michel [BMM1] is an analogue of Harish-Chandra theory where the F-stable Levi subgroups of F-stable parabolic subgroups are replaced by a larger class of F-stable Levi subgroups, using Deligne-Lusztig induction instead of Harish-Chandra induction. Whenever $d = 1$, we retrieve the usual Harish-Chandra theory. Let us summarize it.

Let $\mathbf{S}$ be a torus over $\mathbb{F}$, endowed with a Frobenius endomorphism $F : \mathbf{S} \to \mathbf{S}$ associated with an $\mathbb{F}_q$-structure on $\mathbf{S}$. Let Φ_d denote the d-th cyclotomic polynomial. Then $\mathbf{S}$ is called a *Φ_d-torus* if the following two conditions are satisfied:

- $\mathbf{S}$ is split over $\mathbb{F}_{q^d}$.
- If $\mathbf{S}'$ is an F-stable subtorus of $\mathbf{S}$ different from 1, and if e divides d and is different from d, then $\mathbf{S}'$ is not split over $\mathbb{F}_{q^e}$.

A Levi subgroup $\mathscr{L} = (q, \mathbf{L}, F)$ of $\mathscr{G}$ is called *d-split* if $\mathbf{L}$ is the centralizer of an F-stable Φ_d-torus of $\mathbf{G}$.

Remark 9.3. Let $\mathbf{L}$ be an F-stable Levi subgroup of $\mathbf{G}$. Then $(q, \mathbf{L}, F)$ is 1-split if and only if $\mathbf{L}$ is the Levi complement of an F-stable parabolic subgroup of $\mathbf{G}$. ■

An irreducible unipotent representation N of $\mathbf{G}^F$ is called *d-cuspidal* if ${}^*R_{\mathbf{L}\subset\mathbf{P}}^{\mathbf{G}}[N] = 0$ for any pair $(\mathbf{L}, \mathbf{P})$ where $\mathbf{P}$ is a parabolic subgroup of $\mathbf{G}$ and $\mathbf{L}$ is an F-stable Levi complement of $\mathbf{P}$ which is d-split as a Levi subgroup of $\mathbf{G}$. We denote by $\mathscr{U}\!\mathit{nip}_{\mathrm{cus}}^d(\mathscr{G})$ the set of (isomorphism classes of) d-cuspidal irreducible unipotent representations of $\mathbf{G}^F$.

We denote by $\mathscr{C}\!\mathit{us}^d(\mathscr{G})$ the set of pairs $(\mathscr{L}, \lambda)$, where $\mathscr{L} = (q, \mathbf{L}, F)$ is a d-split Levi subgroup of $\mathbf{G}$ and λ is a d-cuspidal irreducible unipotent representation of $\mathbf{L}^F$. We denote by $\mathscr{U}\!\mathit{nip}(\mathscr{G}, \mathscr{L}, \lambda)$ the set of unipotent irreducible representations occurring in $R_{\mathbf{L}\subset\mathbf{P}}^{\mathbf{G}}[\lambda]$, where $\mathbf{P}$ is a parabolic subgroup admitting $\mathbf{L}$ as a Levi complement: this set is called the *d-Harish-Chandra series* associated with the pair $(\mathscr{L},\lambda)$. This set depends on the pair $(\mathscr{L},\lambda)$ up to $\mathbf{G}^F$-conjugacy. We denote by $\mathscr{C}\!\mathit{us}^d(\mathscr{G})/\!\sim$ the set of $\mathbf{G}^F$-conjugacy classes of elements of $\mathscr{C}\!\mathit{us}^d(\mathscr{G})$. The *d-Harish-Chandra theory* can then be summarized as follows [BMM1, Theo. 3.2]:

Theorem 9.4 (Broué-Malle-Michel)**.** *We have*

$$\mathscr{U}\!\mathit{nip}(\mathscr{G}) = \dot{\bigcup_{(\mathscr{L},\lambda)\in\mathscr{C}\!\mathit{us}^d(\mathscr{G})/\sim}} \mathscr{U}\!\mathit{nip}(\mathscr{G},\mathscr{L},\lambda),$$

where $\dot{\cup}$ means a disjoint union.

Moreover, if $(\mathscr{L},\lambda)\in\mathscr{C}\!\mathit{us}^d(\mathscr{G})$ with $\mathscr{L}=(q,\mathbf{L},F)$ and if $\mathbf{P}$ is a parabolic subgroup admitting $\mathbf{L}$ as a Levi complement, then:

(a) *The group $W_{\mathscr{G}}(\mathscr{L},\lambda)$ is a complex reflection group for its action on*

$$\{y\in\mathbb{C}\otimes_{\mathbb{Z}} Y(\mathrm{Z}(\mathbf{L}))\mid F(y)=\zeta_d q y\}.$$

(b) *There exists a bijection $\mathrm{HC}_d^{\mathscr{G},\mathscr{L},\lambda}:\mathrm{Irr}\,W_{\mathscr{G}}(\mathscr{L},\lambda)\xrightarrow{\sim}\mathscr{U}\!\mathit{nip}(\mathscr{G},\mathscr{L},\lambda)$ and a sign function $\mathrm{Irr}\,W_{\mathscr{G}}(\mathscr{L},\lambda)\longrightarrow\{1,-1\}$ intertwining ordinary induction and Lusztig induction [BMM1][5]. In other words,*

$$\mathscr{U}\!\mathit{nip}(\mathscr{G}) = \dot{\bigcup_{(\mathscr{L},\lambda)\in\mathscr{C}\!\mathit{us}^d(\mathscr{G})/\sim}} \mathrm{HC}_d^{\mathscr{G},\mathscr{L},\lambda}(\mathrm{Irr}\,W_{\mathscr{G}}(\mathscr{L},\lambda)).$$

Note that this theorem sounds like a miracle and is proved through a case-by-case analysis: it would be better explained by the following conjecture (which is true for $d=1$ by Theorem 9.1).

Conjecture 9.5 (Broué-Malle-Michel)**.** *Let $(\mathscr{L},\lambda)\in\mathscr{C}\!\mathit{us}^d(\mathscr{G})$ with $\mathscr{L}=(q,\mathbf{L},F)$. Then there exists a parabolic subgroup $\mathbf{P}$ of $\mathbf{G}$, admitting $\mathbf{L}$ as a Levi complement, and such that:*

(a) *The $\overline{\mathbb{Q}}_\ell\mathbf{G}^F$-modules $\mathrm{H}_c^j(\mathbf{Y}_\mathbf{P})\otimes_{\overline{\mathbb{Q}}_\ell\mathbf{L}^F}\lambda$ and $\mathrm{H}_c^{j'}(\mathbf{Y}_\mathbf{P})\otimes_{\overline{\mathbb{Q}}_\ell\mathbf{L}^F}\lambda$ have no common irreducible constituent if $j\neq j'$.*

(b) *The endomorphism algebra of the complex of $\overline{\mathbb{Q}}_\ell\mathbf{G}^F$-modules $\mathscr{R}_{\mathbf{L}\subset\mathbf{P}}^{\mathbf{G}}(\lambda)$ is canonically isomorphic to some Hecke algebra of the form $\mathscr{H}_{k_{\mathscr{L},\lambda}}(W_{\mathscr{G}}(\mathscr{L},\lambda),\zeta_d^{-1}q)$ for some parameter $k_{\mathscr{L},\lambda}\in\aleph(W_{\mathscr{G}}(\mathscr{L},\lambda))$.*

[5] Sorry for being somewhat vague in this survey.

Of course, Conjecture 9.5 can easily be reduced to the case where $\mathbf{G}$ is quasi-simple. In this case, and for $d > 1$, the full Conjecture 9.5 is known only whenever d is the Coxeter number (see Lusztig's fundamental paper [Lus1], which served as an inspiration for the conjecture) or whenever $\mathbf{G}$ is of type A_d (see [DiMi2]). Part (a) of Conjecture 9.5 is known if $\mathbf{G}$ is of type A and $\mathscr{L}$ is a torus (see [BDR], which relies in an essential way on the work of Dudas [Dud]). For this part (a), some other cases have been solved by Digne-Michel-Rouquier [DiMiRo] and Digne-Michel [DiMi2].

For part (b), note also that, in many important cases, a map from the group algebra of the braid group of $W_{\mathscr{G}}(\mathscr{L}, \lambda)$ to the endomorphism algebra of the complex of $\overline{\mathbb{Q}}_\ell \mathbf{G}^F$-modules $\mathscr{R}_{\mathbf{L}\subset\mathbf{P}}^{\mathbf{G}}(\lambda)$ has been constructed [BrMi, BrMa2, DiMi2], but it is generally not known whether it is onto and if it factorizes (exactly) through the Hecke algebra $\mathscr{H}_{k_{\mathscr{L},\lambda}}(W_{\mathscr{G}}(\mathscr{L}, \lambda), \zeta_d^{-1} q)$. There are, however, partial results in this direction [BrMi, BrMa2, DiMi2].

Also, extra-properties that should be satisfied by the Hecke algebra involved in Conjecture 9.5 imply that, if $\chi \in \operatorname{Irr}(W_{\mathscr{G}}(\mathscr{L}, \lambda))$, then

$$\deg \mathrm{H}^{\mathscr{G},\mathscr{L},\lambda}(\chi) = \pm \frac{R_{\mathbf{L}\subset\mathbf{P}}^{\mathbf{G}}(\lambda)(1)}{\mathbf{sch}_\chi^{k_{\mathscr{L},\lambda}}}. \tag{9.6}$$

This imposes huge constraints on the parameter $k_{\mathscr{L},\lambda}$ and allows to determine it explicitly in almost all cases [BMM1] (including the classical groups).

Remark 9.7. Let ℓ be a prime number different from p and assume for simplicity that $\ell \geqslant 5$ and ℓ is *very good* for $\mathbf{G}$. Let d denote the smallest integer such that ℓ divides $q^d - 1$. Then two irreducible unipotent representations of $\mathbf{G}^F$ belong to the same ℓ-block if and only if they belong to the same d-Harish-Chandra series (see [BMM1, Theo. 5.24] for the case where ℓ does not divide the order of the Weyl group and [CaEn, Theo. 22.4] for the general case). ■

10. Families

10.A. Almost characters

If χ is a τ-stable irreducible character of $\mathscr{W}$, we fix once and for all an extension $\tilde{\chi}$ of χ to $\mathscr{W} \rtimes \langle\tau\rangle$. If $w \in \mathscr{W}$, we set $\mathbf{T}_w = g_w \mathbf{T} g_w^{-1}$, where $g_w \in \mathbf{G}$ is chosen so that $g_w^{-1} F(g_w) \in \mathrm{N}_{\mathbf{G}}(\mathbf{T})$ is a representative of w. Then $\mathbf{T}_w$ is an F-stable maximal torus so, using Deligne-Lusztig induction, we can define

$$R_\chi^{\mathscr{G}} = \frac{1}{|\mathscr{W}|} \sum_{w\in\mathscr{W}} \tilde{\chi}(w\tau) R_{\mathbf{T}_w}^{\mathbf{G}}(1_{\mathbf{T}_w^F}) \quad \in \mathbb{C}\mathscr{U}nip(\mathscr{G}). \tag{10.1}$$

Then $R_\chi^{\mathscr{G}}$ is called an *almost character* of $\mathbf{G}^F$. Note that $R_\chi^{\mathscr{G}}$ depends on the choice of $\tilde{\chi}$, but only up to multiplication by a root of unity.

10.B. Families

Lusztig [Lus6, Chap. 4] has defined a partition of $\mathscr{U}\!\mathit{nip}(\mathscr{G})$ into *families*: let us recall his construction. Define a graph $\mathfrak{Gr}_{\mathscr{G}}$ on $\operatorname{Irr}(\mathscr{W})^\tau$ as follows:

(G1) The set of vertices of $\mathfrak{Gr}_{\mathscr{G}}$ is $\operatorname{Irr}(\mathscr{W})^\tau$.
(G2) Two τ-stable irreducible characters χ and χ' are linked by an edge in the graph $\mathfrak{Gr}_{\mathscr{G}}$ if $R_\chi^{\mathscr{G}}$ and $R_{\chi'}^{\mathscr{G}}$ have a common irreducible constituent.

If $\mathfrak{C}$ is a connected component of $\mathfrak{Gr}_{\mathscr{G}}$, we denote by $\mathfrak{F}_{\mathfrak{C}}^{\mathscr{G}}$ the set of irreducible unipotent representations $\gamma \in \mathscr{U}\!\mathit{nip}(\mathscr{G})$ such that $\langle R_\chi^{\mathscr{G}}, \gamma\rangle_{\mathbf{G}^F} \neq 0$ for some $\chi \in \mathfrak{C}$. The subset $\mathfrak{F}_{\mathfrak{C}}^{\mathscr{G}}$ of $\mathscr{U}\!\mathit{nip}(\mathscr{G})$ is called a *unipotent Lusztig family* of $\mathscr{G}$. We denote by $\mathscr{F}\!\mathit{am}_{\mathrm{un}}(\mathscr{G})$ the set of such families. By construction, the unipotent Lusztig families form a partition of $\mathscr{U}\!\mathit{nip}(\mathscr{G})$.

One of the main results in Lusztig's work on unipotent representations is the list of following compatibilities between this partition and Harish-Chandra series [Lus6] (some of them are proved by a case-by-case analysis):

Theorem 10.2 (Lusztig)**.** *With the above notation, we have:*

(a) *If* $(\mathscr{L}, \lambda) \in \mathscr{C}\!\mathit{us}(\mathscr{G})$ *and* $\mathfrak{F} \in \mathscr{F}\!\mathit{am}_{\mathrm{un}}(\mathscr{G})$*, then* $(\mathrm{HC}^{\mathscr{G},\mathscr{L},\lambda})^{-1}(\mathfrak{F})$ *is empty or belongs to* $\operatorname{Fam}_{k_{\mathscr{L}}}^{\mathrm{Lus}}(W_{\mathscr{G}}(\mathscr{L}))$ *(recall that* $W_{\mathscr{G}}(\mathscr{L}) = W_{\mathscr{G}}(\mathscr{L}, \lambda)$ *and that* $k_{\mathscr{L}} = k_{\mathscr{L},\lambda}$ *does not depend on* λ*).*
(b) *If* $\mathscr{U}\!\mathit{nip}_{\mathrm{cus}}(\mathscr{G})$ *is non-empty, then it is contained in a single family, which will be denoted by* $\mathfrak{F}_{\mathrm{cus}}^{\mathscr{G}}$.
(c) *If* $\tau = \mathrm{Id}_V$*, then the principal series (see Example 9.2) satisfies the following property: for any* $\mathfrak{F} \in \mathscr{F}\!\mathit{am}_{\mathrm{un}}(\mathscr{G})$ *and any* $\chi \in \operatorname{Irr}(\mathscr{W})$*, then* $R_\chi^{\mathscr{G}} \in \mathbb{C}\mathfrak{F}$ *if and only if* $\mathrm{HC}^{\mathscr{G}}(\chi) \in \mathfrak{F}$*. In particular, every family meets the principal series.*

Note that the analogue of statement (a) for d-Harish-Chandra theory (instead of classical Harish-Chandra theory) is probably true (by replacing Lusztig families Calogero-Moser families) but it is still not known up to now. The analogue of statement (b) for d-Harish-Chandra theory is false in general (for instance, if d is large enough, then all irreducible unipotent representations are d-cuspidal). The analogue of statement (c) for $\tau \neq \mathrm{Id}_V$ is false in general (for instance, in twisted type A_{n-1} with $n \geqslant 3$, every family is a singleton but there are irreducible unipotent representations not belonging to the principal series).

Part III. Genericity vs Calogero-Moser spaces

Hypothesis. *We assume in this third part that there exists a rational structure $V_{\mathbb{Q}}$ on V which is stable under the action of W (i.e., W is a Weyl group). We also assume that $V^W = 0$. We denote by $k_{\mathrm{sp}} \in \mathbb{C}^{\aleph}$ the* ***spetsial*** *parameter, that is, the parameter such that $(k_{\mathrm{sp}})_{H,0} = 1$ and $(k_{\mathrm{sp}})_{H,1} = 0$ for all $H \in \mathcal{A}$.*

We also fix an element $\tau \in \mathrm{N}_{\mathbf{GL}_{\mathbb{Q}}(V_{\mathbb{Q}})}(W)$ of finite order.

We denote by $\mathscr{Groups}(W\tau)$ the class of triples $(q, \mathbf{G}, F) \in \mathscr{Groups}$ such that, if $\mathbf{T}$ is any F-stable maximal torus of the group $\mathbf{G}$, then the pair $(\mathbb{Q} \otimes Y(\mathbf{T}/\mathrm{Z}(\mathbf{G})), N_{\mathbf{G}}(\mathbf{T})/\mathbf{T})$ is isomorphic to $(V_{\mathbb{Q}}, W)$ and, moreover, τ stabilizes the lattice $Y(\mathbf{T}/\mathrm{Z}(\mathbf{G}))$ of $V_{\mathbb{Q}}$ and there exists $w \in W$ such that $F(y) = qw\tau(y)$ for all $y \in Y(\mathbf{T}/\mathrm{Z}(\mathbf{G}))$.

This part may be viewed as the aim of this survey article, where we propose several conjectures which compare the geometry (fixed points, symplectic leaves) of the Calogero-Moser space $\mathscr{Z}_{k_{\mathrm{sp}}}$ with the different partitions (families, d-Harish-Chandra series) of unipotent characters of triples belonging to $\mathscr{Groups}(W\tau)$. A first general remark (due to Lusztig) is that most of these partitions do not depend that much on the triple $\mathscr{G} \in \mathscr{Groups}(W\tau)$: they mainly depend only on the coset $W\tau$. This phenomenon, called *genericity*, was developed and formalized by Broué-Malle-Michel [BMM1] and will be explained in Section 11.

The conjectures will be stated precisely in Section 12. If $\mathscr{G} \in \mathscr{Groups}(W\tau)$, they propose conjectural links between:

- the partition of $\mathscr{Unip}(\mathscr{G})$ into families and the $\mathbb{C}^\times$-fixed points in $\mathscr{Z}_{k_{\mathrm{sp}}}^{\tau}$;
- the partition into d-Harish-Chandra series and symplectic leaves of $\mathscr{Z}_{k_{\mathrm{sp}}}^{\zeta_d\tau}$, where ζ_d is a primitive d-th root of unity.

In the second point, the most spectacular conjecture relates the parameter involved in the description of the normalization of the closure of a symplectic leaf as a Calogero-Moser space (i.e. the parameter $k_{P,p}$ of Conjecture 7.2) and the parameter of the Hecke algebra which conjecturally describes the endomorphism algebra of the cohomology of some Deligne-Lusztig variety.

11. Genericity

11.A. Rough definition

The notions of *generic groups*, *generic unipotent representations*... have been defined rigorously in [BMM1]. In this survey, we will not recall this precise definition, which would require to introduce again much more notation. We will use throughout this part a rather vague definition: when some structure associated with any $\mathscr{G} = (q, \mathbf{G}, F) \in \mathscr{G}roups(W\tau)$ depends only on $W\tau$ and not on the triple $\mathscr{G}$, we will say that this structure behaves *generically*.

A first example is the order of $\mathbf{G}'^F$, where $\mathbf{G}'$ is the derived subgroup of $\mathbf{G}$. Indeed, if $m = \dim_{\mathbb{C}} V$, there exists a choice of algebraically independent homogeneous generators $f_1, \ldots, f_m$ of $\mathbb{C}[V]^W$ which are eigenvectors for the action of τ (and we denote by d_j the degree of f_j and by ξ_j the eigenvalue corresponding to f_j). We can then define the following polynomial $\mathrm{Ord}_{W\tau}(\mathbf{q}) \in \mathbb{Q}[\mathbf{q}]$:

$$\mathrm{Ord}_{W\tau}(\mathbf{q}) = q^{|\mathscr{A}|} \prod_{j=1}^{m} (\mathbf{q}^{d_j} - \xi_j).$$

Then this polynomial does not depend on the precise choice of the f_j's, and

$$|\mathbf{G}'^F| = \mathrm{Ord}_{W\tau}(q) \tag{11.1}$$

for all $(q, \mathbf{G}, F) \in \mathscr{G}roups(W\tau)$.

Remark 11.2 (***d***-splitness)**.** Let $d \geqslant 1$ and let ζ_d denote a primitive d-th root of unity. We set

$$\delta(d) = \max_{w \in W} \dim V^{\zeta_d w\tau}$$

and we denote by w_d an element of W such that $\dim V^{\zeta_d w_d \tau} = \delta(d)$. Recall [Spr] that $\delta(d)$ is the number of $j \in \{1, 2, \ldots, m\}$ such that $\zeta_d^{d_j} = \xi_j$.

Then $w_d\tau$ is well-defined up to W-conjugacy [Spr] and any subspace of the form $V^{\zeta_d w\tau}$ for some $w \in W$ is contained in a subspace of the form $x(V^{\zeta_d w_d \tau})$ for some $x \in W$. We set $\tau_d = \zeta_d w_d \tau \in \mathrm{N}_{\mathbf{GL}_{\mathbb{C}}(V)}(W)$. Note that this choice of w_d implies that the element τ_d is W-full [Spr]. Note also that $\mathscr{X}_{k_{\mathrm{sp}}}^{\tau_d} = \mathscr{X}_{k_{\mathrm{sp}}}^{\zeta_d \tau}$.

Then, for any $\mathscr{G} \in \mathscr{G}roups(W\tau)$, the conjugacy classes of d-split Levi subgroups of $\mathscr{G}$ are in bijection with the W_{τ_d}-orbits of τ_d-split parabolic subgroups of W: the correspondence assigns to the conjugacy class of the d-split Levi subgroup its Weyl group, suitably embedded in W (see [BrMa1] for details). ■

11.B. Genericity of unipotent representations

For our purpose, the most important result about genericity is the following theorem, which says that the unipotent representations of $\mathscr{G} \in \mathscr{G}roups(W\tau)$ and their degree behave generically.

Theorem 11.3 (Lusztig)**.** *There exists a finite set* $\mathrm{Unip}(W\tau)$ *endowed with a map* $\mathbf{deg}_{W\tau} : \mathrm{Unip}(W\tau) \longrightarrow \mathbb{Q}[\mathbf{q}]$, *both depending only on the coset* $W\tau$ *and such that, for each triple* $\mathcal{G} \in \mathcal{G}roups(W)$, *there exists a well-defined bijection*

$$\boldsymbol{\rho}^{\mathcal{G}} : \mathrm{Unip}(W\tau) \xrightarrow{\sim} \mathcal{U}nip(\mathcal{G})$$

satisfying

$$\deg\ \boldsymbol{\rho}^{\mathcal{G}}_{\gamma} = (\mathbf{deg}_{W\tau}\,\gamma)(q)$$

for all $\gamma \in \mathrm{Unip}(W)$.

This Theorem 11.3 follows from the classification of unipotent representations obtained by Lusztig [Lus6] and a case-by-case analysis. More recent works of Lusztig [Lus9] provide general explanations for the existence of the finite set $\mathrm{Unip}(W\tau)$ and the bijection $\boldsymbol{\rho}^{\mathcal{G}}$ but do not explain the polynomial behaviour of the degree of the unipotent representations.

It turns out that the different structures on $\mathcal{U}nip(\mathcal{G})$ (families, Harish-Chandra series, partitions into ℓ-blocks...) can also be read only from the finite set $\mathrm{Unip}(W\tau)$, as it will be explained below. In other words, they behave generically. Our aim here is to provide numerical evidences that these extra-structures can be read from the geometry of the Calogero-Moser space $\mathcal{Z}_{k_{\mathrm{sp}}}$ ($\mathbb{C}^{\times}$-fixed points, symplectic leaves, fixed points under the action of $\boldsymbol{\mu}_d$...).

11.C. Almost characters

We denote by $\mathbb{C}\mathrm{Unip}(W\tau)$ the set of formal $\mathbb{C}$-linear combinations of elements of $\mathrm{Unip}(W\tau)$ and we still denote by $\boldsymbol{\rho}^{\mathcal{G}} : \mathbb{C}\mathrm{Unip}(W\tau) \longrightarrow \mathbb{C}\mathcal{U}nip(\mathcal{G})$ the $\mathbb{C}$-linear extension of the bijection $\boldsymbol{\rho}^{\mathcal{G}}$.

With this notation, the almost characters behave generically. In other words, there exists a (necessarily unique) family $(R_\chi)_{\chi \in \mathrm{Irr}(W)^\tau}$ of elements of $\mathbb{C}\mathrm{Unip}(W\tau)$ such that

$$\boldsymbol{\rho}^{\mathcal{G}}(R_\chi) = R^{\mathcal{G}}_\chi \tag{11.4}$$

for any $\mathcal{G} \in \mathcal{G}roups(W\tau)$ (see [Lus6, Chap. 4]). In particular, the graph $\mathfrak{Gr}_{\mathcal{G}}$ constructed in §10.B is generic: it will also be denoted by $\mathfrak{Gr}_{W\tau}$.

11.D. Families

Families of unipotent characters behave generically. Indeed, it follows from (11.4) that there exists a partition of $\mathrm{Unip}(W\tau)$ into *unipotent Lusztig families* (we denote by $\mathrm{Fam}_{\mathrm{un}}(W\tau)$ the set of such families) such that,

$$\boldsymbol{\rho}^{\mathcal{G}}(\mathrm{Fam}_{\mathrm{un}}(W\tau)) = \mathcal{F}am_{\mathrm{un}}(\mathcal{G}) \tag{11.5}$$

for any $\mathscr{G} \in \mathscr{G}roups(W\tau)$. If $\mathfrak{C}$ is a connected component of the graph $\mathfrak{Gr}_{W\tau}$, we denote by $\mathfrak{F}^{\mathrm{un}}_{\mathfrak{C}} \subset \mathrm{Unip}(W\tau)$ the associated generic family. Lusztig proved the following important result [Lus6]:

Theorem 11.6 (Lusztig)**.** *The map*

$$\begin{array}{ccc} \mathrm{Fam}^{\mathrm{Lus}}_{k_{\mathrm{sp}}}(W)^{\tau} & \longrightarrow & \mathrm{Fam}_{\mathrm{un}}(W\tau) \\ \mathfrak{C} & \longmapsto & \mathfrak{F}^{\mathrm{un}}_{\mathfrak{C}^{\tau}} \end{array}$$

is well-defined and bijective.

This Theorem contains in particular the fact that, if $\mathfrak{C}$ is a τ-stable Lusztig k_{sp}-family of characters of W, then $\mathfrak{C}^{\tau} \neq \varnothing$. But this follows directly from the fact that every Lusztig k_{sp}-family contains a unique character with minimal b-invariant (the *b-invariant* of an irreducible character χ is the minimal value of j such that χ occurs in the j-th symmetric power $\mathrm{S}^j(V)$), called the *special* character of the family [Lus4, §12]: if the family is τ-stable, then its special character is necessarily τ-stable.

11.E. Lusztig's a-function

If $\gamma \in \mathrm{Unip}(W\tau)$, we denote by a_γ (resp. A_γ) the valuation (resp. the degree) of the polynomial $\mathbf{deg}_{W\tau}\,\mathbf{q}$. It follows from Lusztig's work [Lus6] that

(11.7) *the functions a, $A : \mathrm{Unip}(W\tau) \longrightarrow \mathbb{Z}_{\geqslant 0}$ are constant on families.*

11.F. d-Harish-Chandra series

It turns out that d-Harish-Chandra theory behaves also generically [BMM1, Theo. 3.2]. More precisely, if $\mathscr{G} = (q, \mathbf{G}, F) \in \mathscr{G}roups(W\tau)$, then:

- Let $\mathrm{Unip}^d_{\mathrm{cus}}(W\tau)$ denote the set of $\gamma \in \mathrm{Unip}(W\tau)$ such that $\mathbf{deg}_{W\tau}\,\gamma$ is divisible by $(\mathbf{q} - \zeta_d)^{\dim V^{\tau_d}}$. Then $(\boldsymbol{\rho}^{\mathscr{G}})(\mathrm{Unip}^d_{\mathrm{cus}}(W\tau)) = \mathscr{U}nip^d_{\mathrm{cus}}(\mathscr{G})$ (see [BMM1, Prop. 2.9]).
- The $\mathbf{G}^F$-conjugacy classes of d-split Levi subgroups of $\mathscr{G}$ are in bijection with the W_{τ_d}-conjugacy classes of d-split parabolic subgroups of W (see Remark 11.2): if the d-split Levi subgroup $\mathscr{L}$ of $\mathscr{G}$ corresponds to the d-split parabolic subgroup P under this bijection, then $\mathscr{L} \in \mathscr{G}roups(P\tau_d)$ and $W_{\mathscr{G}}(\mathscr{L}) \simeq \overline{\mathrm{N}}_{W_{\tau_d}}(P_{\tau_d})$ (see [BrMa1, Prop. 3.12]).
  Through this isomorphism and the bijection $\boldsymbol{\rho}^{\mathscr{L}}$, the group $\overline{\mathrm{N}}_{W_{\tau_d}}(P_{\tau_d})$ acts on $\mathrm{Unip}(P\tau_d)$ and stabilizes the subset $\mathrm{Unip}^d_{\mathrm{cus}}(\mathscr{L})$. Moreover, the action of $\overline{\mathrm{N}}_{W_{\tau_d}}(P_{\tau_d})$ on $\mathrm{Unip}(P\tau_d)$ is generic. If $\lambda \in \mathrm{Unip}(P\tau_d)$, we denote by $\overline{\mathrm{N}}_{W_{\tau_d}}(P_{\tau_d}, \lambda)$ its stabilizer in $\overline{\mathrm{N}}_{W_{\tau_d}}(P_{\tau_d})$.

- If we denote by $\mathrm{Cus}^d(W\tau)$ the set of pairs (P,λ) where P is a d-split parabolic subgroup of W and $\lambda \in \mathrm{Unip}^d_{\mathrm{cus}}(P\tau_d)$, then the previous point defines a natural bijection between $\mathrm{Cus}^d(W\tau)/\sim$ and $\mathscr{C}\!\mathit{us}^d(\mathscr{G})/\sim$.
- If $(P,\lambda) \in \mathrm{Cus}^d(W\tau)$ corresponds to $(\mathscr{L},\boldsymbol{\rho}^{\mathscr{L}}_\lambda) \in \mathscr{C}\!\mathit{us}^d(\mathscr{G})$, then the maps $\mathrm{HC}_d^{\mathscr{G},\mathscr{L},\boldsymbol{\rho}^{\mathscr{L}}_\lambda}$ and $\boldsymbol{\rho}^{\mathscr{G}}$ define an injection $\mathrm{HC}_d^{W,P,\lambda} : \mathrm{Irr}(\overline{\mathrm{N}}_{W_{\tau_d}}(P_{\tau_d},\lambda)) \hookrightarrow \mathrm{Unip}(W\tau)$ which behaves generically. Its image is denoted by $\mathrm{Unip}_d(W\tau,P,\lambda)$. Then

$$\text{(11.8)}\qquad \mathrm{Unip}(W\tau) = \dot{\bigcup_{(P,\lambda)\in\mathscr{C}\!\mathit{us}^d(W)}} \mathrm{HC}_d^{W,P,\lambda}(\mathrm{Irr}(\overline{\mathrm{N}}_{W_{\tau_d}}(P_{\tau_d},\lambda))).$$

  Moreover, the parameter $k_{\mathscr{L},\boldsymbol{\rho}^{\mathscr{L}}_\lambda}$ in Conjecture 9.5 is generic, i.e. depends only on (P,λ). It will be denoted by $k_{P,\lambda}$.

Here, all the statements stated without reference can be found in [BMM1, Theo. 3.2].

Remark 11.9. Let $(P,\lambda) \in \mathrm{Cus}^d(W\tau)$. It follows from the classification of such pairs (see [BMM1]) that:

- $\overline{\mathrm{N}}_{W_{\tau_d}}(P_{\tau_d},\lambda)$ is always a reflection group for its action on $(V^P)^{\tau_d}$.
- Examples where $\overline{\mathrm{N}}_{W_{\tau_d}}(P_{\tau_d},\lambda) \neq \overline{\mathrm{N}}_{W_{\tau_d}}(P_{\tau_d})$ are very rare. For instance, this never happens if $d=1$ (see Theorem 9.1(a)) or if W is of type A (see [BMM1, §3.A]). ■

Example 11.10 (Principal series)**.** Let us describe the generic version of Example 9.2. First, $\mathrm{Unip}(1)$ consists of a single element that we may (and will) denote by 1. Then the cuspidal pair $(1,1) \in \mathrm{Cus}(W\tau)$ corresponds to the pair $(\mathscr{T},1) \in \mathscr{C}\!\mathit{us}(\mathscr{G})$ associated with an F-stable maximal torus of an F-stable Borel subgroup and the map $\mathrm{HC}^{\mathscr{G}} = \mathrm{HC}^{\mathscr{G},\mathscr{T},1}$ will be simply denoted by $\mathrm{HC}^W : \mathrm{Irr}(W^\tau) \longrightarrow \mathrm{Unip}(W\tau)$, instead of $\mathrm{HC}^{W,1,1}$.

If $\tau = \mathrm{Id}_V$, then the parameter $k_{1,1}$ is equal to k_{sp} (see Example 9.2). ■

11.G. *d*-Harish-Chandra theory and filtration

Assume in this subsection, and only in this subsection, that $\tau = \mathrm{Id}_V$ (on the reductive group side, this means that we work in the split case). Let $\mathrm{Z}(\mathbb{C}W)^{\mathrm{Lus}}$ denote the subalgebra of $\mathrm{Z}(\mathbb{C}W)$ whose basis is given by $(e^W_{\mathfrak{C}})_{\mathfrak{C}\in\mathrm{Fam}^{\mathrm{Lus}}_{k_{\mathrm{sp}}}(W)}$. Let $(P,\lambda) \in \mathrm{Cus}^d(W)$. We define a morphism of algebras

$$(\mathrm{HC}_d^{W,P,\lambda})^\# : \mathrm{Z}(\mathbb{C}W)^{\mathrm{Lus}} \longrightarrow \mathrm{Z}(\mathbb{C}\overline{\mathrm{N}}_{W_{\tau_d}}(P_{\tau_d},\lambda))$$

by

$$(\mathrm{HC}_d^{W,P,\lambda})^\#(e_{\mathfrak{C}}^W) = \sum_{\substack{\chi \in \mathrm{Irr}\,\overline{\mathrm{N}}_{W_{\tau_d}}(P_{\tau_d},\lambda) \\ \text{such that } \mathrm{HC}_d^{W,P,\lambda}(\chi) \in \mathfrak{F}_{\mathfrak{C}}^{\mathrm{un}}}} e_\chi^{\overline{\mathrm{N}}_{W_{\tau_d}}(P_{\tau_d},\lambda)}.$$

Conjecture 11.11. *Assume that* $\tau = \mathrm{Id}_V$ *and fix* $(P,\lambda) \in \mathrm{Cus}^d(W)$. *Then*

$$(\mathrm{HC}_d^{W,P,\lambda})^\#(\mathscr{F}_j \mathrm{Z}(\mathbb{C}W)^{\mathrm{Lus}}) \subset \mathscr{F}_j \mathrm{Z}(\mathbb{C}\overline{\mathrm{N}}_{W_{\tau_d}}(P_{\tau_d},\lambda))$$

for all j.

Remark 11.12. This conjecture seems to come from nowhere and provides a strange link between the character tables of W and $\overline{\mathrm{N}}_{W_{\tau_d}}(P_{\tau_d},\lambda)$. However, if we believe in the links between representation theory of finite reductive groups and geometry of Calogero-Moser spaces (as developed in the next section), then this Conjecture 11.11 is just a consequence of this philosophy (see the upcoming Proposition 12.6) and of Conjecture 6.2. This is an example where the geometry of Calogero-Moser spaces suggests unexpected properties of the representation theory of finite reductive groups.

Note that Conjecture 11.11 holds in the following cases:

- If W is of type A (see [BoMa, Cor. 4.22] and the explanations given in Section 15).
- A pretty convincing result is that it holds if $\dim V \leqslant 8$ (so this includes the type E_8): this has been checked through computer calculations based on all the functions implemented by Jean Michel [Mic][6]. ■

Example 11.13. Let $z = \sum_{s \in \mathrm{Ref}(W)} s \in \mathrm{Z}(\mathbb{C}W)$. Then $z = \sum_{\chi \in \mathrm{Irr}(W)} (|\mathscr{A}| - (a_\chi^{(k_{\mathrm{sp}})} + A_\chi^{(k_{\mathrm{sp}})}) e_\chi^W$ (see [BMM2, Cor.6.9] and [BoRo2, Lem. 7.2.1]). So $z \in \mathscr{F}_1 \mathrm{Z}(\mathbb{C}W)^{\mathrm{Lus}}$ by (8.3). Now, if $\psi \in \mathrm{Irr}\,\overline{\mathrm{N}}_{W_{\tau_d}}(P_{\tau_d},\lambda)$ is such that $\mathrm{HC}_d^{W,P,\lambda}(\psi)$ belongs to the same family as $\mathrm{HC}^W(\chi)$, then it follows from (9.6) that $|\mathscr{A}| - (a_\chi^{(k_{\mathrm{sp}})} - A_\chi^{(k_{\mathrm{sp}})}) = M - (a_\psi^{(k_{P,\lambda})} + A_\psi^{(k_{P,\lambda})})$ for some M which does not depend on ψ or χ (and only on (W,P,λ)). Therefore,

$$(\mathrm{HC}_d^{W,P,\lambda})^\#(z) = \sum_{\chi \in \mathrm{Irr}\,\overline{\mathrm{N}}_{W_{\tau_d}}(P_{\tau_d},\lambda)} (M - (a_\psi^{(k_{P,\lambda})} + A_\psi^{(k_{P,\lambda})}))e_\psi^{\overline{\mathrm{N}}_{W_{\tau_d}}(P_{\tau_d},\lambda)}.$$

[6] We wish to thank again warmly Jean Michel for writing the programs for performing these calculations.

In other words, it follows from [BMM2, Cor.6.9] and [BoRo2, Lem. 7.2.1] that there exists $M' \in \mathbb{C}$ such that

$$(\mathrm{HC}_d^{W,P,\lambda})^\#(z) = M' + \sum_{s \in \mathrm{Ref}(\overline{\mathrm{N}}_{W_{\tau_d}}(P_{\tau_d},\lambda))} c_{k_{P,\lambda}}(s)s \in \mathscr{F}_1 \mathrm{Z}(\mathbb{C}\overline{\mathrm{N}}_{W_{\tau_d}}(P_{\tau_d},\lambda)),$$

as desired. ■

12. Coincidences, conjectures

12.A. Families

By Theorem 10.2(b), the set $\mathrm{Fam}_{\mathrm{un}}(W\tau)$ is in bijection with the set $\mathrm{Fam}_{k_{\mathrm{sp}}}^{\mathrm{Lus}}(W)^\tau$. We conjecture the first link between the geometry of $\mathscr{Z}_{k_{\mathrm{sp}}}$ and the representation theory of finite reductive groups:

Conjecture 12.1. *There exists a unique bijection*

$$\Phi : (\mathscr{Z}_{k_{\mathrm{sp}}}^{\mathbb{C}^\times})^\tau \xrightarrow{\sim} \mathrm{Fam}_{\mathrm{un}}(W\tau)$$

such that, for any $p \in (\mathscr{Z}_{k_{\mathrm{sp}}}^{\mathbb{C}^\times})^\tau$ and for any $\chi \in (\mathfrak{F}_p^{k_{\mathrm{sp}}})^\tau$, the almost character R_χ belongs to $\mathbb{C}\Phi(p)$.

Whenever $\tau = \mathrm{Id}_V$, this Conjecture 12.1 is equivalent to Gordon-Martino Conjecture 8.2 (see Theorem 11.6). For the rest of this section, we assume that Conjecture 12.1 holds, and we keep the notation $\Phi : (\mathscr{Z}_{k_{\mathrm{sp}}}^{\mathbb{C}^\times})^\tau \xrightarrow{\sim} \mathrm{Fam}_{\mathrm{un}}(W\tau)$.

12.B. Fixed points and *d*-cuspidality

We expect that d-cuspidality of unipotent representations and τ_d-cuspidality of points in $\mathscr{Z}_{k_{\mathrm{sp}}}^{\tau_d} = \mathscr{Z}_{k_{\mathrm{sp}}}^{\zeta_d \tau}$ are linked as follows:

Conjecture 12.2. *Assume here that Conjecture 12.1 holds. Let $p \in (\mathscr{Z}_{k_{\mathrm{sp}}}^{\mathbb{C}^\times})^\tau$ be such that there exists $\lambda \in \Phi(p) \subset \mathrm{Unip}(W\tau)$ which is d-cuspidal. Then p is τ_d-cuspidal.*

Note that the converse to Conjecture 12.2 does not hold in general, even for $d = 1$ (for instance for W of type D, as it will be explained in §16.C).

12.C. d-Harish-Chandra theory and symplectic leaves

Assume in this subsection that Conjectures 12.1 and 12.2 hold for W and all its parabolic subgroups. Fix $(P,\lambda) \in \mathrm{Cus}^d(W\tau)$ and let p be the point of $\mathfrak{Z}_{k_{\mathrm{sp}}}(V/V^P, P)$ corresponding to the Lusztig family of λ through Conjecture 12.1. Then p is τ_d-cuspidal by Conjecture 12.2. Therefore, one can associate to the pair (P,p) a symplectic leaf $\mathcal{S}_{P,p}$ of $\mathfrak{Z}_{k_{\mathrm{sp}}}^{\tau_d}$.

Conjecture 12.3. *Recall that we assume that Conjectures 12.1 and 12.2 hold for W and all its parabolic subgroups. Then, with the above notation:*

(a) *Let $p' \in (\mathfrak{Z}_{k_{\mathrm{sp}}}^{\mathbb{C}^\times})^\tau$. Then $p' \in \overline{\mathcal{S}}_{P,p}$ if and only if the d-Harish-Chandra series $\mathrm{HC}^{W,P,\lambda}(\mathrm{Irr}(\overline{\mathrm{N}}_{W_{\tau_d}}(P_{\tau_d},\lambda)))$ meets the family $\Phi(p')$.*

(b) *There exists a parameter $k_{P,p} \in \aleph((V^P)^{\tau_d}, \overline{\mathrm{N}}_{W_{\tau_d}}(P_{\tau_d}))$ such that:*

  (b1) *$\overline{\mathcal{S}}_{P,p}^{\mathrm{nor}} \simeq \mathfrak{Z}_{k_{P,p}}((V^P)^{\tau_d}, \overline{\mathrm{N}}_{W_{\tau_d}}(P_{\tau_d}))$ as Poisson varieties endowed with a $\mathbb{C}^\times$-action.*

  (b2) *The parameter $k_{P,\lambda} \in \aleph((V^P)^{\tau_d}, \overline{\mathrm{N}}_{W_{\tau_d}}(P_{\tau_d},\lambda))$ involved in Conjecture 9.5(b) (and which is generic by the previous section) is the restriction of $k_{P,p}$ to $\overline{\mathrm{N}}_{W_{\tau_d}}(P_{\tau_d},\lambda)$.*

Commentary 12.4. In the previous conjecture, the existence of a parameter $k_{P,p}$ satisfying (b1) is just a restatement of Conjecture 7.2: the main point of the above conjecture is that its restriction should coincide with the parameter of Conjecture 9.5(b), which has to do with a completely different context (ℓ-adic cohomology of Deligne-Lusztig varieties). In some sense, this is a justification of this long paper. ■

The correspondence outlined in Conjecture 12.3 should also be compatible in a more precise way with Harish-Chandra theory. For this survey, keep the notation of the above conjecture and assume moreover that $\overline{\mathrm{N}}_{W_{\tau_d}}(P_{\tau_d},\lambda) = \overline{\mathrm{N}}_{W_{\tau_d}}(P_{\tau_d})$ (recall from Remark 11.9 that this is the most probable situation). Assume also that Conjecture 12.3 holds. Then, by (b1), we get a $\mathbb{C}^\times$-equivariant morphism of varieties

$$\psi : \mathfrak{Z}_{k_{P,p}}((V^P)^{\tau_d}, \overline{\mathrm{N}}_{W_{\tau_d}}(P_{\tau_d})) \longrightarrow \mathfrak{Z}_{k_{\mathrm{sp}}}$$

whose image is the closure $\overline{\mathcal{S}}_{P,p}$ of the symplectic leaf $\mathcal{S}_{P,p}$ of $\mathfrak{Z}_{k_{\mathrm{sp}}}^{\tau_d}$. By restriction to the $\mathbb{C}^\times$-fixed points, we get a map

$$\psi_{\mathrm{fix}} : \mathfrak{Z}_{k_{P,p}}((V^P)^{\tau_d}, \overline{\mathrm{N}}_{W_{\tau_d}}(P_{\tau_d}))^{\mathbb{C}^\times} \longrightarrow \mathfrak{Z}_{k_{\mathrm{sp}}}^{\mathbb{C}^\times}$$

whose image is contained in $(\mathfrak{X}_{k_{\mathrm{sp}}}^{\mathbb{C}^\times})^\tau$. On the other hand, Conjecture 12.1 provides a surjective map $\Phi^* : \mathrm{Unip}(W\tau) \twoheadrightarrow (\mathfrak{X}_{k_{\mathrm{sp}}}^{\mathbb{C}^\times})^\tau$ (whose fibers are the unipotent Lusztig families) and the definition of Calogero-Moser families provides a surjective map

$$\mathfrak{z}_{k_{P,p}} : \mathrm{Irr}(\overline{\mathrm{N}}_{W_{\tau_d}}(P_{\tau_d})) \twoheadrightarrow \mathfrak{X}_{k_{P,p}}((V^P)^{\tau_d}, \overline{\mathrm{N}}_{W_{\tau_d}}(P_{\tau_d}))^{\mathbb{C}^\times}.$$

Finally, recall that d-Harish-Chandra theory (see Theorem 9.4 and its generic version) provides an injective map

$$\mathrm{HC}^{W,P,\lambda} : \mathrm{Irr}(\overline{\mathrm{N}}_{W_{\tau_d}}(P_{\tau_d})) \hookrightarrow \mathrm{Unip}(W\tau).$$

We expect all these maps to be compatible in the following sense:

Conjecture 12.5. *Assume that Conjectures 12.1, 12.2 and 12.3 hold and that $\overline{\mathrm{N}}_{W_{\tau_d}}(P_{\tau_d}, \lambda) = \overline{\mathrm{N}}_{W_{\tau_d}}(P_{\tau_d})$. Then the diagram*

$$\begin{tikzcd}
\mathrm{Irr}(\overline{\mathrm{N}}_{W_{\tau_d}}(P_{\tau_d})) \arrow[r, two heads, "\mathfrak{z}_{k_{P,p}}"] \arrow[d, hook, "\mathrm{HC}^{W,P,\lambda}"'] & \mathfrak{X}_{k_{P,p}}((V^P)^{\tau_d}, \overline{\mathrm{N}}_{W_{\tau_d}}(P_{\tau_d}))^{\mathbb{C}^\times} \arrow[d, "\psi_{\mathrm{fix}}"] \\
\mathrm{Unip}(W\tau) \arrow[r, two heads, "\Phi^*"] & (\mathfrak{X}_{k_{\mathrm{sp}}}^{\mathbb{C}^\times})^\tau
\end{tikzcd}$$

is commutative.

The conjectures stated in this section, together with Conjecture 6.2 on the cohomology of Calogero-Moser spaces, imply the Conjecture 11.11. Let us give some details. First, as in §6.B, the morphism ψ induces a morphism of algebras

$$\psi_{\mathrm{fix}}^{\#} : \mathrm{Im}\,\mathbf{\Omega}_W^{k_{\mathrm{sp}}} \longrightarrow \mathrm{Im}\,\mathbf{\Omega}_{\overline{\mathrm{N}}_{W_{\tau_d}}(P_{\tau_d})}^{k_{P,p}}.$$

Then, if we assume that $\tau = \mathrm{Id}_V$ and that Conjectures 8.2 and 12.1 hold, the map $\psi_{\mathrm{fix}}^{\#}$ is just the map $(\mathrm{HC}_d^{W,P,\lambda})^{\#}$ of Conjecture 11.11. So Proposition 6.6 has the following consequence:

Proposition 12.6. *With the above notation, assume that $\overline{\mathrm{N}}_{W_{\tau_d}}(P_{\tau_d}, \lambda) = \overline{\mathrm{N}}_{W_{\tau_d}}(P_{\tau_d})$ and that Conjectures 6.2, 8.2 12.1, 12.2, 12.3 and 12.5 hold. Then Conjecture 11.11 holds for the d-cuspidal triple (W, P, λ).*

Remark 12.7. The consequence of Proposition 12.6 does not involve anymore the geometry of Calogero-Moser spaces but only the representation theory of finite reductive groups. Therefore, the validity of Conjecture 11.11 in many cases (see Remark 11.12 and Example 11.13) is a good indication that the general philosophy of this paper has some reasonable foundation. ■

Example 12.8 (Principal series)**.** Assume in this example, and only in this example, that $\tau = \mathrm{Id}_V$, that Conjecture 12.1 (i.e. Gordon-Martino's Conjecture 8.2) holds and that $(P, \lambda) = (1, 1) \in \mathrm{Cus}(W)$. Then Conjecture 12.3(a) is a restatement of the fact that every family meets the principal series while Conjecture 12.3(b) and Conjecture 12.5 are vacuous. ■

Example 12.9 (Regular element)**.** Assume in this example, and only in this example, that $\tau = \mathrm{Id}_V$ and d is chosen such that τ_d is regular (see §7.C for the definition: from this definition, the trivial subgroup of W is τ-split). Then, $(1, 1) \in \mathrm{Cus}^d(W)$ and $W_{\tau_d} = C_W(w_d)$.

On the unipotent representation side, if $\mathfrak{C}$ is a Lusztig k_{sp}-family, then it has been checked by J. Michel (unpublished) that the unipotent Lusztig family $\mathfrak{F}^{\mathrm{un}}_{\mathfrak{C}}$ (see Theorem 11.6) meets the d-Harish-Chandra series $\mathrm{Unip}_d(W, 1, 1)$ if and only if $\sum_{\chi \in \mathfrak{C}} |\chi(w_d)|^2 \neq 0$. Moreover, he also checked that

$$\sum_{\chi \in \mathfrak{C}} |\chi(w_d)|^2 = \sum_{\substack{\psi \in \mathrm{Irr}\, C_W(w_d) \\ \text{such that } \mathrm{HC}_d^{W,1,1}(\psi) \in \mathfrak{F}^{\mathrm{un}}_{\mathfrak{C}}}} \psi(1)^2.$$

On the Calogero-Moser space side, the closure of the symplectic leaf associated with $(1, 1)$ is just the irreducible component of maximal dimension $(\mathcal{X}^{\tau}_{k_{\mathrm{sp}}})_{\max}$ defined in §7.C. So the above facts about unipotent representations justify, through the philosophy of this section, Conjecture 7.5 and [Bon5, Conj 5.2]. ■

Part IV. Examples

Hypothesis. *As in the third part, we assume that there exists a rational structure $V_{\mathbb{Q}}$ on V which is stable under the action of W (i.e., W is a Weyl group) and that $V^W = 0$. We also fix an element $\tau \in \mathrm{N}_{\mathbf{GL}_{\mathbb{Q}}(V_{\mathbb{Q}})}(W)$ of finite order, an integer $d \geqslant 1$ and a primitive d-th root of unity ζ_d.*

We aim to illustrate the Conjectures stated in Section 12 by several examples:

(a) We prove that Conjectures 12.1, 12.2, 12.3 and 12.5 hold in rank 2 for d equal to the Coxeter number.
(b) We also prove that, assuming Broué-Malle-Michel Conjecture 9.5 (and particularly the conjectural value of $k_{P,\lambda}$), they hold in type A.
(c) For classical types, we only prove Conjectures 12.1, as well as Conjecture 12.2 whenever $d = 1$ (classical Harish-Chandra theory).

As explained in Commentary 12.4, the most intriguing question is Conjecture 12.3(b), which predicts the equality of parameters coming from two extremely different contexts (cohomology of some Deligne-Lusztig variety vs symplectic leaves of Calogero-Moser spaces). Even for classical Harish-Chandra theory (i.e. whenever the Deligne-Lusztig variety is zero-dimensional), this is somewhat unexpected and certainly reflects some deep connections. In the examples treated in this part, we will mainly focus on this question.

13. Rank 2

The case of type A being treated in the upcoming Section 15, we will just consider here the types B_2 and G_2. We will not fill the details for proving all the Conjectures: indeed, the groups are small enough so that the remaining details can be filled by the reader. So, as explained in the introduction to this part, we only give the details for Conjecture 12.3(b).

Theorem 13.1. *Assume that W is of type B_2 or G_2 and that d is the Coxeter number. Then Conjectures 12.1, 12.2, 12.3 and 12.5 hold.*

Proof. Let s and t be the two simple reflections of W. Let $c = st$ be a standard Coxeter element of W and let $\mathfrak{O}_c$ denote its corresponding $\mathbf{G}$-orbit in $\mathcal{B} \times \mathcal{B}$. Then $\zeta_d c$ is W-full (so we may take $\tau_d = \zeta_d c$) and $W_{\tau_d} = C_W(\zeta_d c) = \langle c \rangle$ is the cyclic group of order d. As there is only one reflecting hyperplane for W_{τ_d}, the parameters for W_{τ_d} will be denoted by $k = (k_0, k_1, \ldots, k_{d-1})$. We denote by k^{cox} the parameter given by:

$$k^{\mathrm{cox}} = \begin{cases} (0,1,2,1) & \text{if } d = 4, \\ (0,1,2,1,1,1) & \text{if } d = 6. \end{cases}$$

Also, there is (up to $\mathbf{G}^F$-conjugacy), only one proper d-split Levi subgroup, namely the Coxeter torus $\mathbf{T}_c$. Computing the Deligne-Lusztig induction of the trivial character of $\mathbf{T}_c^F$ amounts to determining the cohomology of the Deligne-Lusztig variety $\mathbf{X}_{\mathfrak{O}_c}$. This has been done by Lusztig [Lus1], and it follows from his work that Conjecture 9.5 holds in this case.

Let us give more details. First, he proved Conjecture 9.5(a) about the disjointness of the cohomology groups [Lus1, Theo. 6.1] and that the endomorphism algebra of the $\mathbf{G}^F$-module $\mathrm{H}_c^\bullet(\mathfrak{X}_{\mathfrak{C}_c})$ is generated by the Frobenius endomorphism F and he computed the eigenvalues of F in all cases [Lus1, Table 7.3]. This leads to the following presentation for this endomorphism algebra:

$$\begin{cases} \text{Generator: } F \text{ (the Frobenius endomorphism)},\\ \text{Relation: } \prod_{j=0}^{d-1}(F-\zeta_d^j(\zeta_d^{-1}q)^{k_j^{\mathrm{cox}}})=0.\end{cases}$$

In other words,

$$\mathrm{End}_{\mathbf{G}^F}\,\mathrm{H}_c^\bullet(\mathfrak{X}_{\mathfrak{C}_c}) \simeq \mathscr{H}_{k^{\mathrm{cox}}}(W_{\tau_d},\zeta_d^{-1}q). \tag{13.2}$$

On the other hand, the computation of the fixed point subvariety $\mathcal{Z}_{k_{\mathrm{sp}}}^{\boldsymbol{\mu}_d}$ has been done in [Bon3, Theo. 7.1] and the result is given by:

$$\mathcal{Z}_{k_{\mathrm{sp}}}^{\boldsymbol{\mu}_d} \simeq \{(x,y,z)\in\mathbb{C}^3 \mid (z^2-d^2)z^{d-2}=xy\}.$$

Setting $z'=z+d$, we get

$$\mathcal{Z}_{k_{\mathrm{sp}}}^{\boldsymbol{\mu}_d} \simeq \{(x,y,z')\in\mathbb{C}^3 \mid z'(z'-2d)(z'-d)^{d-2}=xy\}.$$

In other words,

$$\mathcal{Z}_{k_{\mathrm{sp}}}^{\boldsymbol{\mu}_d} \simeq \mathcal{Z}_{k^{\mathrm{cox}}}(V^{\tau_d},W_{\tau_d}) \tag{13.3}$$

(see Example 6.5(a)).

We see that the same parameter occurs in (13.2) and (13.3): this shows that Conjecture 12.3(b) holds in this case, as desired. □

14. Some combinatorics

We refer to [JaKe, §2.7] for facts about abaci, d-cores, d-quotients of partitions that will be used here.

14.A. Notation

A *partition* is a sequence $\lambda=(\lambda_k)_{k\geqslant 1}$ of non-negative integers such that $\lambda_k\geqslant\lambda_{k+1}$ for all k and $\lambda_k=0$ for $k\gg 0$. Let Part denote the set of all partitions. If $\lambda\in\mathrm{Part}$, we set $|\lambda|=\sum_{k\geqslant 1}\lambda_k$ and $a_\lambda=\sum_{k\geqslant 1}(k-1)\lambda_k$, and we denote by $Y(\lambda)$ the *Young diagram* of λ, that is, the set of pairs of natural numbers (i,j) such that $j \geqslant 1$ and $1 \leqslant i \leqslant \lambda_j$. If $y \in Y(\lambda)$, we denote by $\mathrm{hk}_\lambda(y)$ the *hook length* of λ based at y, i.e. the number of $(i',j') \in Y(\lambda)$ such that $i' \geqslant i$, $j' \geqslant j$ and $(i-i')(j-j') = 0$. Let

$$\mathbf{deg}\,\lambda = \mathbf{q}^{a_\lambda} \frac{\displaystyle\prod_{k=1}^{|\lambda|}(\mathbf{q}^k - 1)}{\displaystyle\prod_{y \in Y(\lambda)} (\mathbf{q}^{\mathrm{hk}_\lambda(y)} - 1)}.$$

It turns out that $\mathbf{deg}\,\lambda \in \mathbb{Z}[\mathbf{q}]$.

Let $d \geqslant 1$. A partition λ is called a *d-core* if $\mathrm{hk}_\lambda(y) \neq d$ for all $y \in Y(\lambda)$. The subset of Part consisting of d-cores is denoted by Cor_d. An element $\lambda = (\lambda^{(1)}, \dots, \lambda^{(d)})$ of the set Part^d of d-uples of partitions is called a *d-partition*: we set $|\lambda| = |\lambda^{(1)}| + \cdots + |\lambda^{(d)}|$. If $\lambda \in \mathrm{Part}$, we denote by $\mathrm{cor}_d(\lambda) \in \mathrm{Cor}_d$ its *d-core* and by $\mathrm{quo}_d(\lambda) \in \mathrm{Part}^d$ its d-quotient. The map

$$\begin{array}{rccc} \mathrm{cor}_d \times \mathrm{quo}_d : & \mathrm{Part} & \longrightarrow & \mathrm{Cor}_d \times \mathrm{Part}^d \\ & \lambda & \longmapsto & (\mathrm{cor}_d(\lambda), \mathrm{quo}_d(\lambda)) \end{array} \tag{14.1}$$

is bijective. Its inverse will be denoted by

$$\mathrm{par}_d : \mathrm{Cor}_d \times \mathrm{Part}^d \xrightarrow{\sim} \mathrm{Part}.$$

It follows from the definition of both maps that

$$|\lambda| = |\mathrm{cor}_d(\lambda)| + d\,|\mathrm{quo}_d(\lambda)|. \tag{14.2}$$

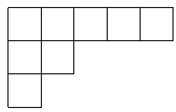
If $r \geqslant 0$, we denote by $\mathrm{Part}(r)$ (resp. $\mathrm{Part}^d(r)$, resp. $\mathrm{Cor}_d(r)$) the set of $\lambda \in \mathrm{Part}$ (resp. $\lambda \in \mathrm{Part}^d$, resp. $\lambda \in \mathrm{Cor}_d$) such that $|\lambda| = r$. We also set $\mathrm{Cor}_d(\equiv r)$ for the set of $\lambda \in \mathrm{Cor}_d$ such that $|\lambda| \leqslant r$ and $|\lambda| \equiv r \mod d$. In other words, $\mathrm{Cor}_d(\equiv r) = \mathrm{cor}_d(\mathrm{Part}(r))$.

If $\lambda \in \mathrm{Part}^d(r)$, we denote by χ_λ the associated irreducible character of the complex reflection group $G(d,1,r)$, following the convention in [GeJa]. If $k \in \mathbb{C}^{\aleph(G(d,1,r))}$ and $\lambda \in \mathrm{Part}^d(r)$, we denote by z_λ^k the element of $\mathfrak{z}_k(\chi_\lambda) \in \mathfrak{X}_k(d,1,r)^{\mathbb{C}^\times}$ defined in §5.D. Note that we do not need to emphasize d or r, as they are determined by λ.

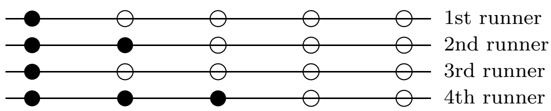
14.B. Abaci

A *d-abacus* is an abacus with d runners. If $\gamma \in \mathrm{Cor}_d$, we denote by $\mathfrak{A}(\gamma)$ its d-abacus, with the convention that the first runner contains the first empty box. Let $b(\gamma) = (b_0(\gamma), b_1(\gamma), \dots, b_{d-1}(\gamma))$ denote the sequence defined as follows: $b_j(\gamma)$ is the number of beads on the $(j+1)$-th runner of $\mathfrak{A}(\gamma)$ minus the number of beads on the first runner. Let $\mathrm{Res}_d(\gamma) = (\rho_0(\gamma), \dots, \rho_{d-1}(\gamma))$ denote the d-residue of γ. It is defined as follows: $\rho_k(\gamma)$ is the number of pairs $(i, j) \in Y(\gamma)$ such that $i - j \equiv k \mod d$.

Example 14.3. Let $\gamma = (5, 2, 1) \in \mathrm{Part}(8)$. Its Young diagram is

$Y(\gamma)$ 

It is easily seen that γ is a 4-core, and its 4-abacus $\mathfrak{A}(\gamma)$ is given by

$\mathfrak{A}(\gamma)$ 

Then $b(\gamma) = (0, 1, 0, 2)$ while $\mathrm{Res}_d(\gamma) = (3, 2, 2, 1)$. ■

Let us define two sequences $k^\gamma = (k_j^\gamma)_{0 \leqslant j \leqslant d-1}$ and $l^\gamma = (l_j^\gamma)_{0 \leqslant j \leqslant d-1}$ associated with a d-core γ:

- $k_j^\gamma = d b_j(\gamma) + j$.
- $l_j^\gamma = b_0(\gamma) + b_1(\gamma) + \cdots + b_{d-1}(\gamma)$
$$+ \begin{cases} d(\rho_{1-j}(\gamma) - \rho_{-j}(\gamma)) + j - 1 & \text{if } 1 \leqslant j \leqslant d-1, \\ d(\rho_1(\gamma) - \rho_0(\gamma)) + d - 1 & \text{if } j = 0. \end{cases}$$

Here, the index in $\rho_{1-j}(\gamma)$ or $\rho_{-j}(\gamma)$ must be understood modulo d. The next result will be useful in the next section:

Proposition 14.4. *Let γ be a d-core and let m denote its length (i.e., the number of non-zero parts). Then*

$$k_j^\gamma = l_{j+1-m}^\gamma$$

for all $j \in \mathbb{Z}/d\mathbb{Z}$.

Proof. If $j \in \mathbb{Z}/d\mathbb{Z}$, we denote by $\bar{j}$ its unique representative in $\{0, 1, \dots, d-1\}$. Note that $m = b_0(\gamma) + b_1(\gamma) + \cdots + b_{d-1}(\gamma)$. We argue by induction on the length m of γ.

If $\gamma = \varnothing$, then $k^\gamma = (0, 1, \dots, d-1)$ and $l^\gamma = (d-1, 0, 1, \dots, d-2)$. This shows the result whenever $m = 0$.

Assume now that $m \geqslant 1$ and write $\gamma = (\gamma_1, \gamma_2, \gamma_3, \cdots)$ with $\gamma_1 > 0$. Let $\gamma' = (\gamma_2, \gamma_3, \cdots)$. Then γ' is a d-core so, by the induction hypothesis, Proposition 14.4 holds for γ'. In other words, $k_j^{\gamma'} = l_{j+2-m}^{\gamma'}$ for all $j \in \mathbb{Z}/d\mathbb{Z}$.

Let us first compare the sequences k^γ and $k^{\gamma'}$. For this, let y denote the unique element of $\{0, 1, \ldots, d-1\}$ such that $b_y(\gamma) = b_y(\gamma')+1$. Then, if $j \neq y$, we get $b_j(\gamma) = b_j(\gamma')$. Therefore,

$$(\#) \qquad k_j^\gamma = \begin{cases} k_j^{\gamma'} & \text{if } j \neq y, \\ k_j^{\gamma'} + d & \text{if } j = y. \end{cases}$$

Let us now compare the sequences l^γ and $l^{\gamma'}$. For this, let x denote the unique element of $\{0, 1, \ldots, d-1\}$ such that $\gamma_1 \equiv x \mod d$. Then, for $j \in \mathbb{Z}/d\mathbb{Z}$, we get

$$\rho_j(\gamma) = \rho_{j-1}(\gamma') + (\gamma_1 - x)/d + \delta_x(j)$$

where δ_x is identically 0 if $x = 0$ and

$$\delta_x(j) = \begin{cases} 1 & \text{if } \bar{j} = 0 \text{ or } 1 \leqslant d - \bar{j} \leqslant x - 1, \\ 0 & \text{if } x \leqslant d - \bar{j} \leqslant d - 1, \end{cases}$$

if $x \geqslant 1$. Also, note that $\gamma_1 = d(b_y(\gamma) - 1) + y + 1 - m$, so $y + 1 - m \equiv x \mod d$. Let us also write l_j^γ, for $j \in \mathbb{Z}/d\mathbb{Z}$, as follows:

$$l_j^\gamma = m + d(\rho_{1-j}(\gamma) - \rho_{-j}(\gamma)) + \bar{j} - 1 + d\delta_{j,0}$$

where $\delta_{j,0}$ is the Kronecker symbol. Putting things together, one gets:

$$\begin{aligned} l_j^\gamma &= m + d(\rho_{-j}(\gamma') - \rho_{-1-j}(\gamma') + \delta_x(1-j) - \delta_x(-j)) + \bar{j} - 1 + d\delta_{j,0} \\ &= 1 + l_{j+1}^{\gamma'} - \overline{j+1} + 1 - d\delta_{j+1,0} + d(\delta_x(1-j) - \delta_x(-j)) + \bar{j} - 1 + d\delta_{j,0}. \end{aligned}$$

But $\overline{j+1} = \bar{j} + 1 - d\delta_{j+1,0}$, so

$$l_j^\gamma = l_{j+1}^{\gamma'} + d(\delta_x(1-j) - \delta_x(-j)) + d\delta_{j,0}.$$

Two cases may occur:

- If $x = 0$, then δ_x is identically 0 and so $d(\delta_x(1-j) - \delta_x(-j)) + d\delta_{j,0} = d\delta_{j,0} = d\delta_{j,-x}$.

- If $x > 0$, then there are only two values of $j \in \mathbb{Z}/d\mathbb{Z}$ for which $\delta_x(1-j) - \delta_x(-j)$ is non-zero, namely $j = 0$ and $j = x$. If $j = 0$, this returns -1 while, if $j = x$, this returns 1. Therefore, we have again $d(\delta_x(1-j) - \delta_x(-j)) + d\delta_{j,0} = d\delta_{j,x}$.

Finally, $l_j^\gamma = l_{j+1}^{\gamma'} + d\delta_{j,-x}$ and $x \equiv y + 1 - m \mod d$ so, by the induction hypothesis and (#),

$$l_j^\gamma = k_{j+m-1}^{\gamma'} + d\delta_{j,m-y} = k_{j+m-1}^{\gamma} - d\delta_{j+m-1,y} + d\delta_{j,x} = k_{j+m-1}^{\gamma},$$

as expected. □

15. The smooth example: type A

The Calogero-Moser space $\mathcal{Z}_{k_{\mathrm{sp}}}$ is smooth if and only if W is a Weyl group of type A. This simplifies drastically its geometry ($\mathbb{C}^\times$-fixed points, symplectic leaves,...) and all the conjectures proposed in Part I are true in this case [BoSh, BoMa].

On the other hand, the almost characters of some $\mathscr{G} \in \mathscr{G}roups$ are all irreducible characters if and only if $\mathscr{G}$ is of type A. This also simplifies drastically its representation theory (unipotent Lusztig families, d-Harish-Chandra theory, blocks,...).

In the spirit of this paper, these two facts should be the shadow of a common phenomenon. We do not propose an explanation for it, but we give details about how the combinatorics on both sides fit perfectly in type A, so that all the conjectures stated in Part III hold in this case (provided that Conjecture 9.5 holds).

Hypothesis. *From now on, and until the end of this section, we fix a natural number $n \geqslant 2$ and we assume that*

$$V = \{(\xi_1, \dots, \xi_n) \in \mathbb{C}^n \mid \xi_1 + \cdots + \xi_n = 0\}$$

and that $W = \mathfrak{S}_n$ acting on V by permutation of the coordinates. The Calogero-Moser space $\mathcal{Z}_{k_{\mathrm{sp}}}(V, \mathfrak{S}_n)$ will be simply denoted by $\mathcal{Z}(n)$.

We set $r = \lfloor n/d \rfloor$ and we denote by w_d a product of r disjoint cycles of length d. Then $\zeta_d w_d$ is $\mathfrak{S}_n$-split.

The $\zeta_d w_d$-split parabolic subgroups of $\mathfrak{S}_n$ are those of the form $\mathfrak{S}_m$, where $m \leqslant n$ and $m \equiv n \mod d$. In this case, $(\mathfrak{S}_n)_{\zeta_d w_d} \simeq G(d,1,r)$ and

$$\overline{\mathrm{N}}_{(\mathfrak{S}_n)_{\zeta_d w_d}}((\mathfrak{S}_m)_{\zeta_d w_d}) \simeq G(d,1,(n-m)/d). \tag{15.1}$$

If $\gamma \in \mathrm{cor}_d(\equiv n)$, we set $r_\gamma = r_\gamma(n) = (n - |\gamma|)/d$ and we denote by $k_\gamma \in \mathbb{C}^{\aleph(G(d,1,r_\gamma))}$ the parameter defined by:

$$\begin{cases} ((k_\gamma)_{\mathrm{Ker}(x_1-x_2),0}, (k_\gamma)_{\mathrm{Ker}(x_1-x_2),1}) = (d,0), & \text{if } r_g \geqslant 2, \\ ((k_\gamma)_{\mathrm{Ker}(x_1),0}, (k_\gamma)_{\mathrm{Ker}(x_1),1}, \dots, (k_\gamma)_{\mathrm{Ker}(x_1),d-1}) = k^\gamma, & \text{if } r_g \geqslant 1 \text{ and } d \geqslant 2, \end{cases}$$

where k^γ is the sequence defined in §14.B. If $r_\gamma = 0$, then k_γ is the zero parameter of the trivial group.

If $\lambda \in \mathrm{Part}(n)$, we denote by z_λ the image of χ_λ through the map $\mathfrak{z}_{k_{\mathrm{sp}}} : \mathrm{Irr}(\mathfrak{S}_n) \longrightarrow \mathcal{X}(n)^{\mathbb{C}^\times}$. Similarly, if $\mu \in \mathrm{Part}^d(m)$, and if $k \in \mathbb{C}^{\aleph(G(d,1,m))}$, we denote by z_μ^k the image of χ_μ through the map $\mathfrak{z}_k : \mathrm{Irr}\, G(d,1,m) \longrightarrow \mathcal{X}_k(G(d,1,m))^{\mathbb{C}^\times}$.

15.A. Geometry of $\mathcal{X}(n)$

Recall that $\mathcal{X}(n)$ is smooth [EtGi, Cor. 16.2]. This has several consequences:

- First, the map $\mathfrak{z}_{k_{\mathrm{sp}}} : \mathrm{Irr}(\mathfrak{S}_n) \longrightarrow \mathcal{X}(n)^{\mathbb{C}^\times}$ defined in §5.D is bijective [Gor1, Cor. 5.8]. This means that

  (15.2) *the map* $\mathrm{Part}(n) \longrightarrow \mathcal{X}(n)^{\mathbb{C}^\times}$, $\lambda \longmapsto z_\lambda$ *is bijective.*

- The variety $\mathcal{X}(n)$ has only one symplectic leaf (it is a symplectic variety). Therefore, if $d \geqslant 1$, then $\mathcal{X}(n)^{\boldsymbol{\mu}_d}$ is also smooth and symplectic, so its symplectic leaves coincide with its irreducible components: in particular, they are closed normal subvarieties of $\mathcal{X}(n)$, so coincide with the normalization of their closure (which is involved in Conjecture 7.2).

The next theorem, which describes these symplectic leaves, has been obtained by Maksimau and the author [BoMa, Theo. 4.21]:

Theorem 15.3. *With the above notation, we have:*

(a) *Let* $\lambda \in \mathrm{Part}(n)$. *Then* z_λ *is* ζ_d*-cuspidal if and only if* λ *is a* d*-core.*
(b) *The map*

$$\begin{array}{ccc} \mathrm{Cor}_d(\equiv n) & \longrightarrow & \mathrm{Cus}_{k_{\mathrm{sp}}}^{\zeta_d w_d}(\mathcal{X}(n)) \\ \gamma & \longmapsto & (\mathfrak{S}_{|\gamma|}, z_\gamma) \end{array}$$

is bijective. We denote by $\mathcal{S}_\gamma(n)$ *the symplectic leaf* $\mathcal{S}_{\mathfrak{S}_{|\gamma|}, z_\gamma}$ *of* $\mathcal{X}(n)^{\boldsymbol{\mu}_d}$.

(c) *If* $\gamma \in \mathrm{Cor}_d(\equiv n)$*, then there exists a* $\mathbb{C}^\times$*-equivariant isomorphism of varieties*

$$i_\gamma : \mathcal{Z}_{k_\gamma}(G(d,1,r_\gamma)) \xrightarrow{\sim} \mathcal{S}_\gamma(n)$$

such that

$$i_\gamma(z_\mu) = z_{\mathrm{par}_d(\gamma,\mu)}$$

for all $\mu \in \mathrm{Part}^d(r_\gamma)$ *(in particular,* $\dim \mathcal{S}_\gamma(n) = 2r_\gamma$*).*

Proof. All the results have been proved in [BoMa, Theo. 4.21], except that we need to make some comments about the parameter. So let $\gamma \in \mathrm{Cor}_d(\equiv n)$ and let $l_\gamma \in \mathbb{C}^{\aleph(G(d,1,r_\gamma))}$ be the parameter defined by:

$$\begin{cases} ((l_\gamma)_{\mathrm{Ker}(x_1-x_2),0}, (l_\gamma)_{\mathrm{Ker}(x_1-x_2),1}) = (d,0), & \text{if } r_g \geqslant 2, \\ ((l_\gamma)_{\mathrm{Ker}(x_1),0}, (l_\gamma)_{\mathrm{Ker}(x_1),1}, \dots, (l_\gamma)_{\mathrm{Ker}(x_1),d-1}) = l^\gamma, & \text{if } r_g \geqslant 1 \text{ and } d \geqslant 2, \end{cases}$$

where l^γ is the sequence defined in §14.B. Then the result of [BoMa, Theo. 4.21] says that $\mathcal{S}_{\mathfrak{S}_{|\gamma|},z_\gamma} \simeq \mathcal{Z}_{l_\gamma}(G(d,1,r_\gamma))$. However, Proposition 14.4 says that l^γ is obtained from k^γ by a cyclic permutation, and so $\mathcal{Z}_{k_\gamma}(G(d,1,r_\gamma)) \simeq \mathcal{Z}_{l_\gamma}(G(d,1,r_\gamma))$ by [BoRo2, (3.5.4)]. $\square$

15.B. Unipotent representations: the split case

We fix in this subsection a triple $\mathcal{G} = (q, \mathbf{G}, F) \in \mathcal{G}roups(\mathfrak{S}_n)$. In other words, $\mathbf{G}'^F$ is a split group of type A_{n-1} (recall that the definition of $\mathcal{G}roups(\mathfrak{S}_n)$ implies no restriction on the rational structure of the center of $\mathbf{G}$). Then it is well-known that

$$(15.4)\quad \mathcal{U}nip(\mathcal{G}) = \{R_\chi^{\mathcal{G}} \mid \chi \in \mathrm{Irr}(\mathfrak{S}_n)\} \qquad \text{and} \qquad \deg R_{\chi_\lambda}^{\mathcal{G}} = (\mathbf{deg}\,\lambda)(q).$$

In other words, we may define the set $\mathrm{Unip}(\mathfrak{S}_n)$, the bijection $\boldsymbol{\rho}^{\mathcal{G}}$ and the map $\mathbf{deg}_{\mathfrak{S}_n}$ as follows:

$$\begin{cases} \mathrm{Unip}(\mathfrak{S}_n) = \mathrm{Part}(n), \\ \boldsymbol{\rho}_\lambda^{\mathcal{G}} = R_{\chi_\lambda}^{\mathcal{G}} \quad \text{for any } \lambda \in \mathrm{Part}(n), \\ \mathbf{deg}_{\mathfrak{S}_n} = \mathbf{deg}. \end{cases}$$

The partition into families is pretty easy in this case:

(15.5) *All the unipotent Lusztig families are singletons.*

A generic translation is given by:

(15.6) *the map* $\mathrm{Part}(n) \longrightarrow \mathrm{Fam}_{\mathrm{un}}(\mathfrak{S}_n)$, $\lambda \longmapsto \{\lambda\}$ *is bijective.*

The following result has been proved in [BMM1, Pages 45-47]:

Theorem 15.7 (Broué-Malle-Michel)**.** *With the above notation, we have:*

(a) *Let* $\lambda \in \mathrm{Part}(n)$*. Then the unipotent character* R_{χ_λ} *is* d*-cuspidal if and only if* λ *is a* d*-core.*

(b) *The map*

$$\begin{array}{ccc} \mathrm{Cor}_d(\equiv n) & \longrightarrow & \mathrm{Cus}_d(\mathfrak{S}_n) \\ \gamma & \longmapsto & (\mathfrak{S}_{|\gamma|}, R_{\chi_\gamma}) \end{array}$$

is bijective. If $\gamma \in \mathrm{Cor}_d(\equiv n)$*, then*

$$\overline{\mathrm{N}}_{(\mathfrak{S}_n)_{\zeta_d w_d}}((\mathfrak{S}_{|\gamma|})_{\zeta_d w_d}) \simeq G(d,1,r_\gamma).$$

(c) *If* $\gamma \in \mathrm{Cor}_d(\equiv n)$*, let* HC_d^γ *denote the bijection*

$$\mathrm{HC}_d^{\mathfrak{S}_n,\mathfrak{S}_m,R_{\chi_\gamma}} : \mathrm{Irr}(G(d,1,r_\gamma)) \longrightarrow \mathrm{Unip}(\mathfrak{S}_n, \mathfrak{S}_{|\gamma|}, R_{\chi_\gamma})$$

defined by the d*-Harish-Chandra theory. Then*

$$\mathrm{HC}_d^\gamma(\chi_\lambda) = R_{\chi_{\mathrm{par}_d(\gamma,\lambda)}}$$

for all $\lambda \in \mathrm{Part}^d(r_\gamma)$*.*

Now, fix a d-core $\gamma \in \mathrm{Cor}_d(\equiv n)$ and let $\mathscr{L}_\gamma = (g, \mathbf{L}_\gamma, F) \in \mathscr{G}roups(\mathfrak{S}_{|\gamma|})$ be such that $\mathbf{L}_\gamma$ is a d-split Levi subgroup of $\mathbf{G}$ (if $\mathbf{G}^F = \mathbf{GL}_n(\mathbb{F}_q)$, then $\mathbf{L}_\gamma^F \simeq \mathbf{GL}_{|\gamma|}(\mathbb{F}_q) \times (\mathbb{F}_{q^d}^\times)^{r_\gamma}$). Conjecture 9.5 predicts the existence of a parabolic subgroup $\mathbf{P}_\gamma$ such that the Deligne-Lusztig variety $\mathbf{Y}_{\mathbf{U}_{\mathbf{P}_\gamma}}$ satisfies the following two properties:

(a) The $\overline{\mathbb{Q}}_\ell \mathbf{G}^F$-modules $\mathrm{H}_c^j(\mathbf{Y}_{\mathbf{P}_\gamma}) \otimes_{\overline{\mathbb{Q}}_\ell \mathbf{L}_\gamma^F} R_{\chi_\gamma}^{\mathscr{L}_\gamma}$ and $\mathrm{H}_c^{j'}(\mathbf{Y}_{\mathbf{P}}) \otimes_{\overline{\mathbb{Q}}_\ell \mathbf{L}^F} R_{\chi_\gamma}^{\mathscr{L}_\gamma}$ have no common irreducible constituent if $j \neq j'$.

(b) $\mathrm{End}_{\mathbf{G}^F} \mathscr{R}_{\mathbf{L}_\gamma \subset \mathbf{P}_\gamma}^{\mathbf{G}}(R_{\chi_\gamma}^{\mathscr{L}_\gamma}) \simeq \mathscr{H}_{k_\gamma^\#}(G(d,1,r_\gamma), \zeta_d^{-1} q)$ for some parameter $k_\gamma^\#$.

This conjecture is far from being proved (see the next remark) but Broué-Malle [BrMa2, Prop. 2.10] proposed a conjectural value for $k_\gamma^\#$:

Conjecture 15.8 (Broué-Malle)**.** *If* $\gamma \in \mathrm{Cor}_d(\equiv n)$*, then Conjecture 9.5 holds for the pair* $(\mathscr{L}_\gamma, R_{\chi_\gamma}^{\mathscr{L}_\gamma})$ *with parameter* $k_\gamma^\# = k_\gamma$*.*

We find remarkable that the parameter k_γ predicted, in this particular case, by Broué-Malle in 1993 in the context of Deligne-Lusztig varieties coincides (up to a non-relevant cyclic permutation) with the parameter l_γ found in 2018 by Maksimau and the author when studying Calogero-Moser spaces.

Remark 15.9. The disjunction of the cohomology (see the above statement (a)) has been proved in [BDR, Theo. 4.3] whenever $\mathbf{L}_\gamma$ is a torus (i.e. $|\gamma| = 0$ or 1), based on earlier works of Dudas [Dud, Cor. 3.2]. The statement (b) is only known if $d = n$ (Lusztig [Lus1, §7.3]) or $d = n - 1$ (Digne-Michel [DiMi2, Theo. 10.1]). ■

The comparison between Theorems 15.3 and 15.7 yields:

Theorem 15.10. *If $W\tau = \mathfrak{S}_n$, then:*

(a) *Conjectures 12.1, 12.2, 12.3(a) and 12.5 hold.*
(b) *If moreover Conjecture 15.8 holds, then Conjecture 12.2(b) holds.*

15.C. The non-split case

In type A, the non-split case corresponds to the case where F acts on the Weyl group $\mathfrak{S}_n$ by the diagram automorphism (i.e. conjugation by the longest element). In our generic description, this corresponds to the coset $-\mathfrak{S}_n$ and objects in $\mathscr{Groups}(-\mathfrak{S}_n)$ (for instance, we may take for $(q, \mathbf{G}, F) \in \mathscr{Groups}(-\mathfrak{S}_n)$ the triple where $\mathbf{G} = \mathbf{GL}_n(\overline{\mathbb{F}}_q)$ and $\mathbf{G}^F$ is the general unitary group). Ennola duality establishes a bijection between $\mathrm{Unip}(\mathfrak{S}_n)$ and $\mathrm{Unip}(-\mathfrak{S}_n)$ and this bijection transforms d-Harish-Chandra series into d'-Harish-Chandra series, where

$$d' = \begin{cases} 2d & \text{if } d \equiv 1 \bmod 2, \\ d/2 & \text{if } d \equiv 2 \bmod 4, \\ d & \text{if } d \equiv 0 \bmod 4. \end{cases}$$

Therefore, the non-split case follows directly from the split one by applying Ennola duality (see [BrMa2, Rem. 2.11]). More precisely:

Theorem 15.11. *If $W\tau = -\mathfrak{S}_n$, then:*

(a) *Conjectures 12.1, 12.2, 12.3(a) and 12.5 hold.*
(b) *If moreover Conjecture 15.8 holds, then Conjecture 12.2(b) holds.*

16. Classical groups and Harish-Chandra theory

We aim to check the following result:

Theorem 16.1. *Assume that W is a Weyl group of classical type and that $d = 1$. Then Conjectures 12.1, 12.2 and 12.3 hold.*

The rest of this section is devoted to the proof of this Theorem. Note that the most difficult part of the work has been previously done by Gordon-Martino [GoMa], Bellamy-Thiel [BeTh] and Bellamy-Maksimau-Schedler [BeMaSc] on the Caloger-Moser space side (as well as an application of Bellamy-Maksimau-Schedler result by the author [Bon4, Cor. 9.8]), and by Lusztig [Lus2, Lus4, Lus5] on the unipotent representations side. The main interest of this section is to relate all these results together following the philosophy of this survey.

Notation. *If $n \geqslant 0$, we set $W_n = G(2,1,n)$ and $W'_n = G(2,2,n)$, so that W_n is a Weyl group of type B_n (i.e. C_n) and W'_n is a Weyl group of type D_n. We denote by $\tau = \mathrm{diag}(-1,1,\dots,1) \in W_n$: it induces the non-trivial involutive diagram automorphism of W'_n. The canonical basis of $V = \mathbb{C}^n$ is denoted by $(y_1,\dots,y_n)$ and its dual basis is denoted by $(x_1,\dots,x_n)$.*

Note that $W_n = \langle\tau\rangle \ltimes W'_n$.

16.A. Families

First, Conjectures 12.1 and 12.2 for $d = 1$ follow immediately from Theorem 8.7 and [BeTh, Theo. A].

16.B. Type B or C

We denote by $\mathbf{BC}(n)$ the set of $r \in \mathbb{Z}_{\geqslant 0}$ such that $r^2 + r \leqslant n$. If r, $m \geqslant 0$, we denote by $k[r]$ the element of $\mathbb{C}^{\aleph(W_m)}$ defined by

$$\begin{cases} k[r]_{\Omega,0} = r, & k[r]_{\Omega,1} = 0, \\ k[r]_{\Omega',1} = 1, & k[r]_{\Omega',1} = 0, \end{cases}$$

where Ω (resp. Ω') is the orbit of the reflecting hyperplane $\mathrm{Ker}(x_1)$ (resp. $\mathrm{Ker}(x_1-x_2)$). Note that $k_{\mathrm{sp}} = k[1]$. The notation $k[r]$ is somewhat ambiguous, as it does not refer to the natural number m, but it will be used only whenever m is clear from the context. The generic version of Harish-Chandra theory can be summarized as follows [Lus2], [Lus3, Tab. II] (note that $\overline{\mathrm{N}}_{W_n}(W_m) \simeq W_{n-m}$ for all $m \leqslant n$):

Theorem 16.2 (Lusztig)**.** *Let $n \geqslant 2$. Then:*

(a) $\mathrm{Unip}_{\mathrm{cus}}(W_n)$ *is non-empty if and only if $n = r^2 + r$ for some $r \in \mathbb{Z}_{\geqslant 0}$. In this case, it contains only one element, which will be denoted by* cus_n.

(b) *The map*

$$\begin{array}{ccc} \mathbf{BC}(n) & \longrightarrow & \mathrm{Cus}(W_n)/W_n \\ r & \longmapsto & (W_{r^2+r}, \mathrm{cus}_{r^2+r}) \end{array}$$

is bijective.

(c) *Let $r \in \mathbf{BC}(n)$. If $\mathscr{G} = (q, \mathbf{G}, F) \in \mathscr{G}roups(W_n)$ and if $\mathscr{L} = (q, \mathbf{L}, F) \in \mathscr{G}roups(W_{r^2+r})$ is such that $\mathbf{L}$ is a 1-split Levi subgroup of $\mathbf{G}$, then*

$$\mathrm{End}_{\mathbf{G}^F} \mathscr{R}_{\mathbf{L}}^{\mathbf{G}} \boldsymbol{\rho}_{\mathrm{cus}_{r^2+r}}^{\mathscr{G}} \simeq \mathscr{H}_{k[2r+1]}(W_{n-(r^2+r)}).$$

On the Calogero-Moser space side, Bellamy-Thiel [BeTh, Theo. 6.24] and Bellamy-Maksimau-Schedler [BeMaSc] proved the following result (note that the proof of (c) by Bellamy-Maksimau-Schedler relies on the description of $\mathscr{Z}_k(W_n)$ as quiver varieties):

Theorem 16.3 (Bellamy-Thiel, Bellamy-Maksimau-Schedler)**.** *Let $n \geqslant 2$. Then:*

(a) $\mathscr{Z}_{k_{\mathrm{sp}}}(W_n)$ *contains a cuspidal point if and only if $n = r^2 + r$ for some $r \in \mathbb{Z}_{\geqslant 0}$. In this case, it contains only one cuspidal point, which will be denoted by z_n^{cus}. It is equal to $z_{(r^{r+1}, \varnothing)}^{k_{\mathrm{sp}}}$.*

(b) *The map*

$$\begin{array}{ccc} \mathbf{BC}(n) & \longrightarrow & \mathrm{Cus}_{\mathrm{Id}_V}(\mathscr{Z}_{k_{\mathrm{sp}}}(W_n))/W_n \\ r & \longmapsto & (W_{r^2+r}, z_{r^2+r}^{\mathrm{cus}}) \end{array}$$

is bijective. We denote by $\mathscr{S}_r(n)$ the symplectic leaf of $\mathscr{Z}_{k_{\mathrm{sp}}}(W_n)$ indexed by $(W_{r^2+r}, z_{r^2+r}^{\mathrm{cus}})$ through Theorem 7.1.

(c) *Let $r \in \mathbf{BC}(n)$. Then*

$$\overline{\mathscr{S}_r(n)}^{\mathrm{nor}} \simeq \mathscr{Z}_{k[2r+1]}(W_{n-(r^2+r)}).$$

The comparison between the above two theorems proves Conjecture 12.3 in type B or C, up to the verification that the cuspidal unipotent representation belongs the same unipotent Lusztig family as $\mathrm{HC}^W(\chi_{(r^{r+1}, \varnothing)})$. We need for this the combinatorics of symbols [Lus2, §3] and its link with unipotent representations [Lus2, Theo. 8.2]. Whenever $n = r(r+1)$, then the cuspidal unipotent representation cus_n is parametrized by the symbol

$\begin{pmatrix} 1 & 2 & \cdots & 2r & 2r+1 \\ & & \varnothing & & \end{pmatrix}$ (with defect $2r+1$) while $\mathrm{HC}^W(\chi_{(r^{r+1},\varnothing)})$ is parametrized by the symbol $\begin{pmatrix} r+1 & r+2 & \cdots & 2r & 2r+1 \\ 1 & 2 & \cdots & r & \end{pmatrix}$ (with defect 1). Since both symbols have the same entries, $\mathrm{cus}_{r(r+1)}$ and $\mathrm{HC}^W(\chi_{(r^{r+1},\varnothing)})$ belong to the same unipotent Lusztig family [Lus4, Theo. 5.8], as desired.

16.C. Type D

We denote by $\mathbf{D}(n)$ the set of $r \in \mathbb{Z}_{\geqslant 0}$ such that $r^2 \leqslant n$ and $r \neq 1$. For $j \in \{0,1\}$, we set

$$\mathbf{D}_j(n) = \{0\} \cup \{r \in \mathbf{D}(n) \mid r \geqslant 2 \text{ and } r \equiv j \mod 2\}.$$

In type D, there are two kinds of possible rational structures (and a third one, if $n = 4$, inducing an order 3 automorphism of W_4': it will not be considered here), a split one and a non-split one. They correspond respectively to the elements $\mathrm{Id}_V = \tau^0$ and τ of the normalizer of W_n'. Note that

$$\overline{\mathrm{N}}_{(W_n')_{\tau^j}}((W_m')_{\tau^j}) \simeq \begin{cases} W_n' & \text{if } (m,j) = (0,0), \\ W_{n-1} & \text{if } (m,j) = (0,1), \\ W_{n-m} & \text{if } m \geqslant 2. \end{cases}$$

We summarize the generic version of Harish-Chandra theory in both cases [Lus2], [Lus3, Tab. II]:

Theorem 16.4 (Lusztig)**.** *We have:*

(a) $\mathrm{Unip}_{\mathrm{cus}}(W_n')$ *(resp.* $\mathrm{Unip}_{\mathrm{cus}}(W_n'\tau_n)$*) is non-empty if and only if* $n = r^2$ *for some* $r \in \mathbb{Z}_{\geqslant 0}$*,* r *even (resp.* r *odd or* $r = 0$*). In this case, it contains only one element, which will be denoted by* cus_n'*.*

(b) *If* $j \in \{0,1\}$*, the map*

$$\begin{array}{ccc} \mathbf{D}_j(n) & \longrightarrow & \mathrm{Cus}(W_n'\tau_n^j)/W_n' \\ r & \longmapsto & (W_{r^2}', \mathrm{cus}_{r^2}') \end{array}$$

is bijective.

(c) *Let* $j \in \{0,1\}$ *and let* $r \in \mathbf{D}_j(n)$*. If* $\mathscr{G} = (q, \mathbf{G}, F) \in \mathscr{G}roups(W_n'\tau_n^j)$ *and if* $\mathscr{L} = (q, \mathbf{L}, F) \in \mathscr{G}roups(W_{r^2}')$ *is such that* $\mathbf{L}$ *is a* 1*-split Levi* *subgroup of* $\mathbf{G}$*, then*

$$\operatorname{End}_{\mathbf{G}^F} \mathscr{R}_{\mathbf{L}}^{\mathbf{G}} \boldsymbol{\rho}^{\mathscr{G}}_{\mathrm{cus}'_{r^2}} \simeq \begin{cases} \mathscr{H}_{k_{\mathrm{sp}}}(W'_n) & \text{if } (r,j) = (0,0), \\ \mathscr{H}_{k[2]}(W_{n-1}) & \text{if } (r,j) = (0,1), \\ \mathscr{H}_{k[2r]}(W_{n-r^2}) & \text{otherwise.} \end{cases}$$

In [Bon4, Cor. 9.8], the author determined the partition into symplectic leaves (as well as their structure) of both $\mathscr{Z}_{k_{\mathrm{sp}}}(W'_n)$ and $\mathscr{Z}_{k_{\mathrm{sp}}}(W'_n)^{\tau_n}$, but it must be said that the essential part of the work was done by Bellamy-Thiel [BeTh, Prop. 4.17] and Bellamy-Maksimau-Schedler [BeMaSc]:

Theorem 16.5 (Bellamy-Thiel, Bellamy-Maksimau-Schedler)**.** *Let* $n \geqslant 4$*. Then:*

(a) $\mathscr{Z}_{k_{\mathrm{sp}}}(W'_n)$ *(resp.* $\mathscr{Z}_{k_{\mathrm{sp}}}(W'_n)^{\tau_n}$*) admits a cuspidal point if and only if there exists* $r \geqslant 2$ *such that* $n = r^2$*. In this case, it contains only one element, which will be denoted by* y_n^{cus}*. By extension, we set* y_0^{cus} *for the unique cuspidal point of* $\mathscr{Z}_{k_{\mathrm{sp}}}(0,1)$*.*

(b) *If* $j \in \{0,1\}$*, the map*

$$\begin{array}{ccc} \mathbf{D}_j(n) & \longrightarrow & \operatorname{Cus}^{\tau_n^j}(\mathscr{Z}_{k_{\mathrm{sp}}}(W'_n)/W'_n \\ r & \longmapsto & (W'_{r^2}, y_{r^2}^{\mathrm{cus}}) \end{array}$$

is bijective. We denote by $\mathscr{S}'_r(n)$ *the cuspidal leaf of* $\mathscr{Z}_{k_{\mathrm{sp}}}(W'_n)^{\tau_n^j}$ *indexed by* $(W'_{r^2}, y_{r^2}^{\mathrm{cus}})$*.*

(c) *Let* $j \in \{0,1\}$ *and let* $r \in \mathbf{D}_j(n)$*. Then*

$$\overline{\mathscr{S}'_r(n)}^{\mathrm{nor}} \simeq \begin{cases} \mathscr{Z}_{k_{\mathrm{sp}}}(W'_n) & \text{if } (r,j) = (0,0), \\ \mathscr{Z}_{k[2]}(W_{n-1}) & \text{if } (r,j) = (0,1), \\ \mathscr{Z}_{k[2r]}(W_{n-r^2}) & \text{otherwise.} \end{cases}$$

The comparison between the above two theorems proves Conjecture 12.3 in type D, in both the untwisted and the twisted case, up to the verification that the cuspidal families correspond on both side: this is done thanks to the combinatoric of symbols, in the same way as for type B or C.

Part V. Spetses

17. What is a spets?

As explained in Section 11, and as noticed in many papers on the subject, essential features of the representation theory of a finite reductive group are controlled by its Weyl group and can be recovered from structures built from it. The *Spetses*[7] program initiated by Broué-Malle-Michel [Mal1, Mal2, BMM2, BMM3], which takes its origin in their work on genericity [BMM1, BrMa2], proposes to attach to some finite complex reflection groups (called *spetsial*, see below) some numerical data ("unipotent representations", "degrees") which admits partitions into "families", "d-Harish-Chandra series" satisfying the same kind of properties as in the case of Weyl groups. This was soon corroborated by computations done by Lusztig [Lus7] and Malle (unpublished) for finite Coxeter groups that are not Weyl groups. This suggests that there should be a mysterious object (the *spets*) admitting some kind of representation theory similar to the representation theory of finite reductive groups.

We try to summarize it (very) quickly in this section, and see what are the possible links with the material of this paper. We come back to the general situation where W is a complex reflection group. The *spetsial* parameter of W, denoted by k_{sp}, is defined by $(k_{\mathrm{sp}})_{\Omega,0} = 1$ and $(k_{\mathrm{sp}})_{\Omega,j} = 0$ for all $\Omega \in \mathcal{A}/W$ and $1 \leqslant j \leqslant e_\Omega - 1$. Broué-Malle-Michel asked whether one can associate with any reflection group W several combinatorial data:

(S1) A set $\mathrm{Unip}(W)$, whose elements are called *irreducible unipotent representations* of the *spets* attached to W, even though there is no group and no representation attached to them.
(S2) A map $\mathbf{deg} : \mathrm{Unip}(W) \longrightarrow \mathbb{C}[\mathbf{q}]$. For ζ a root of unity, an irreducible unipotent representation $\rho \in \mathrm{Unip}(W)$ is called ζ*-cuspidal* if $\mathbf{deg}\rho$ is divisible by $(\mathbf{q}-\zeta)^{\dim(V/W)^\zeta}$.
(S3) A ζ-Harish-Chandra theory: in other words, a partition of $\mathrm{Unip}(W)$ into ζ-Harish-Chandra series built on the same model as in Theorem 9.4. In particular, to each Harish-Chandra series is associated a Hecke algebra of the stabilizer of the corresponding ζ-cuspidal pair (P,λ), with a well-defined parameter $k_{P,\lambda}$.
(S4) Almost characters: these are formal complex linear combinations of irreducible unipotent representations that can be used to define a partition of $\mathrm{Unip}(W)$ into unipotent Lusztig families in the same way as in §10.B.

All these data should satisfy compatibility conditions (axioms) which mimic what is known or conjectured for finite reductive groups. The group W is said to be *spetsial* if some divisibility property of its Schur elements $\mathbf{sch}_\chi^{k_{\mathrm{sp}}}$

[7] Spetses is a Greek island where a conference on finite groups was organized in 1993: this program started there, during a coffee break...

holds [Mal2, §3]. It turns out that many complex reflection groups are not spetsial, but some of them are. The list of irreducible spetsial groups is as follows:

- The groups $G(e,1,n)$ and $G(e,e,n)$ for any $e \geqslant 1$;
- The primitive groups G_j, for

$$j \in \{4, 6, 8, 14, 23, 24, 25, 26, 27, 28, 29, 30, 32, 33, 34, 35, 36, 37\}.$$

Being spetsial is easily seen to be a necessary condition for admitting combinatorial data as in (S1), (S2), (S3) and (S4) satisfying the list of axioms, but it is somewhat astonishing that it is also a sufficient condition [Mal1, BMM2, BMM3]. The natural remaining question is to figure out if there is a category (the spets?) lying above all these combinatorial data. Note the first attempts in this direction using fusion systems and ℓ-compact groups by Kessar-Malle-Semeraro [Sem, KMS1, KMS2].

Of course, if W is a Weyl group (i.e. a finite rational reflection group), then W is spetsial and one recovers the generic theory of unipotent representations of finite groups of the form $\mathbf{G}^F$ where $(q, \mathbf{G}, F) \in \mathscr{Groups}(W)$[8]. In the late 80's, Lusztig associated to each finite Coxeter group which is not a Weyl group a combinatorial datum as in (S1) and (S2) satisfying a few axioms (this was finally published in 1993; see [Lus7]). For H_2, H_3 and H_4, this was rediscovered by Malle in 1992 (unpublished). In this case, the almost characters as in (S4) were obtained for dihedral groups by Lusztig and for H_4 by Malle [LuMa]. About the same period, Malle [Mal1] proved that the imprimitive complex reflection groups $G(e,1,n)$ and $G(e,e,n)$ can be endowed with data satisfying (S1), (S2), (S3), (S4). The case of the primitive complex reflection groups has been done by Broué-Malle-Michel [BMM2, BMM3].

We expect that, for spetsial groups, all the above Broué-Malle-Michel constructions are compatible with the geometry of the Calogero-Moser space associated with W at *spetsial* parameter, and that all the conjectures stated in Section 12 remain valid in this context. In other words, is the spets attached to W hidden in the (Poisson) geometry of $\mathscr{Z}_{k_{\mathrm{sp}}}(W)$?

In the upcoming section, we illustrate again these coincidences in the smallest non-cyclic primitive complex reflection group, namely the group G_4.

[8]Note that, in this case, the ζ-Harish-Chandra series depend only on the order d of ζ and coincide with d-Harish-Chandra series

18. A primitive example

Hypothesis. *We assume in this section, and only in this section, that W is of type G_4. In other words, we set*

$$s = \begin{pmatrix} 1 & 0 \\ 0 & \zeta_3 \end{pmatrix} \qquad \textit{and} \qquad t = \frac{1}{3}\begin{pmatrix} 2\zeta_3+1 & 2(\zeta_3-1) \\ \zeta_3-1 & \zeta_3+2 \end{pmatrix},$$

and we assume that $W = \langle s,t\rangle = G_4$. Here, ζ_3 is a primitive third root of unity.

If $(\delta,\beta) \in \{(1,0),(1,4),(1,8),(2,1),(2,3),(2,5),(3,2)\}$, there is a unique irreducible character of G_4 of degree δ and b-invariant β: it will be denoted by $\phi_{\delta,\beta}$. We have

$$\mathrm{Irr}(G_4) = \{\phi_{\delta,\beta} \mid (\delta,\beta) \in \{(1,0),(1,4),(1,8),(2,1),(2,3),(2,5),(3,2)\}\}.$$

Note that $\phi_{1,0} = 1$ is the trivial character, that $\phi_{1,4} = \varepsilon$, $\phi_{1,8} = \varepsilon^2$, that $\phi_{2,1}$ and $\phi_{2,3}$ are the characters afforded by the representations V and V^* respectively, that $\phi_{2,5}$ is the character afforded by $V \otimes \varepsilon \simeq V^* \otimes \varepsilon^2$ and that $\phi_{3,2}$ corresponds to the second symmetric power $\mathrm{S}^2(V) \simeq \mathrm{S}^2(V^*)$. We denote by C_3 the parabolic subgroup $\langle s\rangle$ of W: it is a cyclic group of order 3.

18.A. Unipotent representations

All the facts stated without proof in this paragraph are taken from [BMM3, §A.4] (see also [Mic]). The set $\mathrm{Unip}_{\mathrm{cus}}(C_3)$ contains a single element (which will be denoted by cus_{C_3}) and the set $\mathrm{Unip}_{\mathrm{cus}}(G_4)$ contains also a single element (which will be denoted by cus_{G_4}). The (classical) 1-Harish-Chandra theory of the spets G_4 may be summarized as follows:

- There are three Harish-Chandra series, namely $\mathrm{Unip}(G_4,1,1)$, $\mathrm{Unip}(G_4,C_3,\mathrm{cus}_{C_3})$ and $\{\mathrm{cus}_{G_4}\}$.
- $\mathrm{Unip}(G_4,1,1)$ is the principal series, and we set $\rho_{\delta,\beta} = \mathrm{HC}^{G_4}(\phi_{\delta,\beta})$.
- We have $\overline{\mathrm{N}}_{G_4}(C_3,\mathrm{cus}_{C_3}) = \overline{\mathrm{N}}_{G_4}(C_3) \simeq \boldsymbol{\mu}_2$ and we set $\rho_{C_3,+} = \mathrm{HC}^{G_4,C_3,\mathrm{cus}_{C_3}}(1)$ and $\rho_{C_3,-} = \mathrm{HC}^{G_4,C_3,\mathrm{cus}_{C_3}}(\sigma)$, where σ is the inclusion $\boldsymbol{\mu}_2 \hookrightarrow \mathbb{C}^\times$.

Therefore,

$$\mathrm{Unip}(G_4) = \{\rho_{1,0},\rho_{1,4},\rho_{1,8},\rho_{2,1},\rho_{2,3},\rho_{2,5},\rho_{3,2},\rho_{C_3,+},\rho_{C_3,-},\mathrm{cus}_{G_4}\}. \tag{18.1}$$

The unipotent Lusztig families are the following four subsets $\mathfrak{F}_{\clubsuit}$, $\mathfrak{F}_{\diamondsuit}$, $\mathfrak{F}_{\heartsuit}$, $\mathfrak{F}_{\spadesuit}$ of $\mathrm{Unip}(G_4)$:

$$\begin{cases} \mathfrak{F}^{\mathrm{un}}_{\clubsuit} = \{\rho_{1,0}\}, \\ \mathfrak{F}^{\mathrm{un}}_{\diamondsuit} = \{\rho_{3,2}\}, \\ \mathfrak{F}^{\mathrm{un}}_{\heartsuit} = \{\rho_{2,1}, \rho_{2,3}, \rho_{C_3,+}\}, \\ \mathfrak{F}^{\mathrm{un}}_{\spadesuit} = \{\rho_{1,4}, \rho_{1,8}, \rho_{2,5}, \rho_{C_3,-}, \mathrm{cus}_{G_4}\}. \end{cases} \tag{18.2}$$

The next table summarizes the numerical data attached to the spets G_4. Let us explain the information given in this table:

- The first column contains the list of irreducible unipotent representations ρ of the spets G_4.
- The second column contains the degree of ρ, where Φ_e denotes the e-th cyclotomic polynomial, $\Phi'_3 = \mathbf{q} - \zeta_3$, $\Phi''_3 = \mathbf{q} - \zeta_3^{-1}$, $\Phi'_6 = \mathbf{q} - \zeta_6$, $\Phi''_6 = \mathbf{q} - \zeta_6^{-1}$ and $\sqrt{-3} = 2\zeta_3 + 1$.
- The third column gives the family of ρ.
- The fourth (resp. fifth) column gives the cuspidal pair (P, λ) parametrizing the ζ_4-Harish-Chandra series (resp. ζ_6-Harish-Chandra series) to which ρ belongs. Note that an empty box means that ρ is ζ_4-cuspidal (resp. ζ_6-cuspidal) and that $(1,1)_d$ denotes the ζ_d-cuspidal pair associated with the trivial parabolic subgroup, which if τ_d-split[9].

ρ	$\mathbf{deg}(\rho)$	Family	ζ_4-series	ζ_6-series
$\rho_{1,0}$	1	$\clubsuit$	$(1,1)_4$	$(1,1)_6$
$\rho_{1,4}$	$-\sqrt{-3}/6\mathbf{q}^4\Phi''_3\Phi_4\Phi''_6$	$\spadesuit$		$(1,1)_6$
$\rho_{1,8}$	$\sqrt{-3}/6\mathbf{q}^4\Phi'_3\Phi_4\Phi'_6$	$\spadesuit$		
$\rho_{2,1}$	$(3+\sqrt{-3})/6\mathbf{q}\Phi'_3\Phi_4\Phi''_6$	$\heartsuit$		$(1,1)_6$
$\rho_{2,3}$	$(3-\sqrt{-3})/6\mathbf{q}\Phi''_3\Phi_4\Phi'_6$	$\heartsuit$		
$\rho_{2,5}$	$1/2\mathbf{q}^4\Phi_2^2\Phi_6$	$\spadesuit$	$(1,1)_4$	
$\rho_{3,2}$	$\mathbf{q}^2\Phi_3\Phi_6$	$\diamondsuit$	$(1,1)_4$	
$\rho_{C_3,+}$	$-\sqrt{-3}/3\mathbf{q}\Phi_1\Phi_2\Phi_4$	$\heartsuit$		$(1,1)_6$
$\rho_{C_3,-}$	$-\sqrt{-3}/3\mathbf{q}^4\Phi_1\Phi_2\Phi_4$	$\spadesuit$		$(1,1)_6$
cus_{G_4}	$-1/2\mathbf{q}^4\Phi_1^2\Phi_3$	$\spadesuit$	$(1,1)_4$	$(1,1)_6$

[9]The interesting ζ-Harish-Chandra series are those attached to a root of unity ζ of order equal to 1, 2, 3, 4 or 6; the $\zeta = -1$ (resp. the $\zeta = \zeta_3$) case can be deduced from the $\zeta = 1$ (resp. $\zeta = \zeta_6$) case thanks to Ennola duality [BMM3, Axiom 5.13], which essentially amounts to replacing $\mathbf{q}$ by $-\mathbf{q}$ in this case.

We conclude this subsection by giving the parameters $k_{P,\lambda}$ for all $(P,\lambda) \in \mathrm{Cus}^d(G_4)$ and $d \in \{1,4,6\}$. Whenever the relative Weyl group $\overline{\mathrm{N}}_{G_4}(P,\lambda)$ is cyclic and isomorphic to $\boldsymbol{\mu}_d$, then the parameter will be given as a list $(k_0,k_1,\dots,k_{d-1})$ of complex numbers:

- $k_{1,1} = k_{\mathrm{sp}}$.
- $k_{C_3,\mathrm{cus}_{C_3}} = (3,0)$.
- $k_{(1,1)_4} = (3,0,1,0)$.
- $k_{(1,1)_6} = (2,0,0,1,0,1)$.

18.B. Calogero-Moser space

As there is only one orbit of reflecting hyperplanes (call it Ω), we will simply denote parameters $k \in \mathbb{C}^{\aleph(G_4)}$ by a triple $(k_0,k_1,k_2) \in \mathbb{C}^3$ where $k_j = k_{\Omega,j}$. For instance, $k_{\mathrm{sp}} = (1,0,0)$. Descriptions of the Calogero-Moser space $\mathscr{Z}_k(G_4)$ have been given in [BoMa] and [BoTh]. Note that the descriptions given in both cases are for parameters $k = (k_0,k_1,k_2) \in \mathbb{C}^3$ satisfying $k_0+k_1+k_2 = 0$: this is not restrictive, thanks to Remark 3.3. So, we set $k_{\mathrm{sp}}^\circ = (2/3,-1/3,-1/3)$, and then $\mathscr{Z}_{k_{\mathrm{sp}}}(G_4) = \mathscr{Z}_{k_{\mathrm{sp}}^\circ}(G_4)$. Specializing the presentation [BoTh] at k_{sp}°, we get that $\mathscr{Z}_{k_{\mathrm{sp}}^\circ}$ is the closed subvariety of $\mathbb{C}^8$ consisting of points $(x_1,x_2,y_1,y_2,a,b,c,e) \in \mathbb{C}^8$ such that

$$\begin{cases}
ab + 12ce + 2x_1y_1 - 15e^4 + 234e^2 + 192e = 0,\\
3ay_1e + 4bc - 9be^3 + 126be + 2x_1y_2 = 0,\\
3a^2e - 2bx_2 + 8cx_1 - 9x_1e^3 - 108x_1e = 0,\\
4ac - 9ae^3 + 126ae + 3bx_1e + 2x_2y_1 = 0,\\
2ay_2 - 3b^2e - 8cy_1 + 9y_1e^3 - 108y_1e = 0,\\
-a^3 - 3ax_1e^2 + 48ax_1 + 2ay_1^2 - b^3 + 2bx_1^2\\
\qquad - 3by_1e^2 + 48by_1 - 8cx_2 - 8cy_2 + 10x_2e^3\\
\qquad - 156x_2e + 128x_2 + 10y_2e^3 - 156y_2e - 128y_2 = 0,\\
16c^2 + 720ce + 9x_1y_1e^2 + 2x_2y_2 - 27e^6 + 864e^3 + 6804e^2 = 0,\\
-2ay_1^2 + b^3 + 3by_1e^2 - 48by_1 + 8cy_2 - 10y_2e^3 + 156y_2e + 128y_2 = 0,\\
5a^2y_1 + 444ab + 5b^2x_1 + 280ce^3 + 4848ce - 1280c + 60x_1y_1e^2\\
\qquad + 648x_1y_1 + 10x_2y_2 - 360e^6 + 7200e^3 + 88776e^2 + 44928e = 0.
\end{cases}$$

The action of $\mathbb{C}^\times$ is given by

$$\xi \cdot (x_1,x_2,y_1,y_2,a,b,c,e) = (\xi^4x_1,\xi^6x_2,\xi^{-4}y_1,\xi^{-6}y_2,\xi^2a,\xi^{-2}b,c,e). \tag{18.3}$$

An immediate computation shows that $\mathcal{Z}_{k_{\mathrm{sp}}^\circ}^{\mathbb{C}^\times}$ contains 4 points, given by

$$\begin{aligned} z_\clubsuit &= (0,0,0,0,0,0,468,8), \qquad & z_\diamondsuit &= (0,0,0,0,0,0,0,0),\\ z_\heartsuit &:= (0,0,0,0,0,0,-45,2) \qquad \text{and} \qquad & z_\spadesuit &= (0,0,0,0,0,0,-18,-4). \end{aligned}$$

We denote by $\mathfrak{F}_\bigstar^{\mathrm{CM}}$ the Calogero-Moser k_{sp}°-family associated with $z_\bigstar$. Then

$$\begin{cases} \mathfrak{F}_\clubsuit^{\mathrm{CM}} = \{\phi_{1,0}\},\\ \mathfrak{F}_\diamondsuit^{\mathrm{CM}} = \{\phi_{3,2}\},\\ \mathfrak{F}_\heartsuit^{\mathrm{CM}} = \{\phi_{2,1},\phi_{2,3}\}\\ \mathfrak{F}_\spadesuit^{\mathrm{CM}} = \{\phi_{1,4},\phi_{1,8},\phi_{2,5}\}. \end{cases} \tag{18.4}$$

The comparison of (18.2) and (18.4) proves the *spetsial* analogue of Conjecture 12.1.

18.B.1. Symplectic leaves of $\boldsymbol{\mathcal{Z}_{k_{\mathrm{sp}}^\circ}}$. Let $\mathcal{S}$ denote the singular locus of $\mathcal{Z}_{k_{\mathrm{sp}}^\circ}$. It has been computed in [BoTh] and it is proved there that it is irreducible of dimension 2 and that

$$z_\clubsuit, z_\diamondsuit \notin \mathcal{S} \qquad \text{and} \qquad z_\heartsuit, z_\spadesuit \in \mathcal{S}. \tag{18.5}$$

Moreover, $z_\spadesuit$ is the only singular point of $\mathcal{S}$. Therefore, there are three symplectic leaves:

- The smooth locus: through the parametrization of Theorem 7.1, it corresponds to the pair $(1,p)$, where p is the unique point of the Calogero-Moser space $\mathcal{Z}_{k_{\mathrm{sp}}^\circ}(0,1)$.
- $\mathcal{S}^\circ = \mathcal{S} \setminus \{z_\spadesuit\}$: through the parametrization of Theorem 7.1, it corresponds to the pair (C_3,q), where q is the unique cuspidal point of the Calogero-Moser space $\mathcal{Z}_{k_{\mathrm{sp}}^\circ}(V/V^{C_3},C_3)$.
- $\{z_\spadesuit\}$: it is cuspidal.

This parametrization fits perfectly with the partition of $\mathrm{Unip}(G_4)$ into Harish-Chandra series, so this proves the spetsial analogue of Conjecture 12.2 and Conjecture 12.3(a) for $d=1$.

Concerning Conjecture 12.3(b), the only interesting case is the second one. Recall that $\overline{\mathrm{N}}_{G_4}(C_3) = \boldsymbol{\mu}_2$. It is proved in [BoTh] that

$$\mathcal{S}^{\mathrm{nor}} \simeq \{(x,y,e) \in \mathbb{C}^3 \mid (e-2)(e+4) = xy\} \simeq \mathcal{Z}_{k_{C_3,\mathrm{cus}_{C_3}}}(\boldsymbol{\mu}_2) \tag{18.6}$$

as Poisson varieties. Note that the computation in [BoTh] is done for the parameter $-3k_{\mathrm{sp}}^{\circ}$, so the equation given here is just obtained after a rescaling. This proves that Conjecture 12.3(b) holds for $d = 1$.

18.B.2. Symplectic leaves of $\boldsymbol{\mathcal{Z}_{k_{\mathrm{sp}}^{\circ}}^{\mu_4}}$ The action of $\mathbb{C}^{\times}$ being given by (18.3), the variety $\mathcal{Z}_{k_{\mathrm{sp}}^{\circ}}^{\boldsymbol{\mu}_4}$ is defined, inside $\mathcal{Z}_{k_{\mathrm{sp}}^{\circ}}$, by the equations $a = b = x_2 = y_2 = 0$. This yields

$$\mathcal{Z}_{k_{\mathrm{sp}}^{\circ}}^{\boldsymbol{\mu}_4} = \{z_{\heartsuit}\} \ \dot{\cup} \ \mathcal{S}_4, \tag{18.7}$$

where

$$\mathcal{S}_4 \simeq \{(x_1, y_1, e) \in \mathbb{C}^3 \mid 4/3x_1y_1 = e(e-8)(e+4)^2\} \tag{18.8}$$

So $\mathcal{S}_4$ admits only one singular point (namely, $z_{\spadesuit}$) and so $\mathcal{Z}_{k_{\mathrm{sp}}^{\circ}}^{\boldsymbol{\mu}_4}$ admits three symplectic leaves:

- There are two τ_4-cuspidal points, namely $z_{\heartsuit}$ and $z_{\spadesuit}$.
- There is one 2-dimensional symplectic leaf, which is the smooth locus of $\mathcal{S}_4$ (i.e. $\mathcal{S}_4 \setminus \{z_{\spadesuit}\}$). Through the parametrization of Theorem 7.1, it corresponds to the pair $(1, p)$, where p is the unique point of the Calogero-Moser space $\mathcal{Z}_{k_{\mathrm{sp}}^{\circ}}(0, 1)$.

This parametrization fits perfectly with the partition of $\mathrm{Unip}(G_4)$ into 4-Harish-Chandra series, so this proves the spetsial analogues of Conjectures 12.2 and 12.3(a) for $d = 4$.

Concerning Conjecture 12.3(b), the only interesting case is the second one. Recall that $W_{\tau_4} \simeq \boldsymbol{\mu}_4$. The above description proves that $\mathcal{S}_4$ (which is the closure of the symplectic leaf $\mathcal{S}_4 \setminus \{z_{\spadesuit}\}$) is normal and that

$$\mathcal{S}_4 \simeq \mathcal{Z}_{k_{(1,1)_4}}(\boldsymbol{\mu}_4) \tag{18.9}$$

as Poisson varieties (for the Poisson bracket, see [BoTh]). So Conjecture 12.3(b) holds for $d = 4$.

18.B.3. Symplectic leaves of $\boldsymbol{\mathcal{Z}_{k_{\mathrm{sp}}^{\circ}}^{\mu_6}}$ The action of $\mathbb{C}^{\times}$ being given by (18.3), the variety $\mathcal{Z}_{k_{\mathrm{sp}}^{\circ}}^{\boldsymbol{\mu}_6}$ is defined, inside $\mathcal{Z}_{k_{\mathrm{sp}}^{\circ}}$, by the equations $a = b = x_1 = y_1 = 0$. This yields

$$\mathcal{Z}_{k_{\mathrm{sp}}^{\circ}}^{\boldsymbol{\mu}_6} = \{z_{\diamondsuit}\} \ \dot{\cup} \ \mathcal{S}_6, \tag{18.10}$$

where

$$\mathcal{S}_6 \simeq \{(x_2, y_2, e) \in \mathbb{C}^3 \mid x_2 y_2 = (e-8)(e-2)^2(e+4)^3\} \tag{18.11}$$

So $\mathcal{S}_6$ admits two singular points (namely, $z_\heartsuit$ and $z_\spadesuit$) and so $\mathcal{Z}_{k_{\mathrm{sp}}^\circ}^{\boldsymbol{\mu}_6}$ admits four symplectic leaves:

– There are three τ_6-cuspidal points, namely $z_\diamondsuit$, $z_\heartsuit$ and $z_\spadesuit$.
– There is one 2-dimensional symplectic leaf, which is the smooth locus of $\mathcal{S}_6$ (i.e. $\mathcal{S}_6 \setminus \{z_\heartsuit, z_\spadesuit\}$). Through the parametrization of Theorem 7.1, it corresponds to the pair $(1, p)$, where p is the unique point of the Calogero-Moser space $\mathcal{Z}_{k_{\mathrm{sp}}^\circ}(0, 1)$.

This parametrization fits perfectly with the partition of $\mathrm{Unip}(G_4)$ into 6-Harish-Chandra series, so this proves the spetsial analogues of Conjectures 12.2 and 12.3(a) for $d = 6$.

Concerning Conjecture 12.3(b), the only interesting case is the second one. Recall that $W_{\tau_6} \simeq \boldsymbol{\mu}_6$. The above description proves that $\mathcal{S}_6$ (which is the closure of the symplectic leaf $\mathcal{S}_6 \setminus \{z_\heartsuit, z_\spadesuit\}$) is normal and that

$$\mathcal{S}_6 \simeq \mathcal{Z}_{k_{(1,1)_6}}(\boldsymbol{\mu}_6) \tag{18.12}$$

as Poisson varieties (for the Poisson bracket, see [BoTh]). So Conjecture 12.3(b) holds for $d = 6$.

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Categorical Diagonalization and p-cells

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Dedicated to George Lusztig, who built our playground

Abstract: In the Iwahori-Hecke algebra, the full twist acts on cell modules by a scalar, and the half twist acts by a scalar and an involution. A categorification of this statement, describing the action of the half and full twist Rouquier complexes on the Hecke category, was conjectured by Elias-Hogancamp, and proven in type A. In this paper we make analogous conjectures for the p-canonical basis, and the Hecke category in characteristic p. We prove the categorified conjecture in type C_2, where the situation is already interesting. The decategorified conjecture is confirmed by computer in rank ≤ 6; information is found in the appendix, written by Joel Gibson.

1. New conjectures about p-cells

In [5], the first-named author and Hogancamp made a conjecture (proven in type A) which can be summed up with the motto: the categorical full twist understands cell theory. The goal of this paper is to state a more surprising conjecture: the categorical full twist in finite characteristic understands p-cell theory. We prove our conjecture in type C_2. We also discuss some interesting phenomena which first occur in type C_3.

This first chapter is a working introduction which states the main conjectures of the paper. It is aimed at experts already familiar with the cell theory of the Hecke algebra, and with its categorification, though perhaps not with diagrammatics, categorical diagonalization, or the p-canonical basis. One can skip from there to the chapter on type C_3, aimed at the same audience. The remaining chapters are heavily computational, and aimed at the expert in diagrammatics. For additional background material we recommend [8].

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1.1. Action of the half twist on the Kazhdan-Lusztig basis

We begin by discussing the decategorified version of our conjecture.

Let W be a finite Coxeter group with longest element w_0. Let $\mathbb{H}(W)$ denote the Iwahori-Hecke algebra of W, a $\mathbb{Z}[v, v^{-1}]$-deformation of the group algebra of W. The *half twist* $\mathrm{ht}_W \in \mathbb{H}(W)$ is the standard basis element associated to w_0. The *full twist* $\mathrm{ft}_W \in \mathbb{H}(W)$ is the square of the half twist,

$$\mathrm{ft}_W = \mathrm{ht}_W^2,$$

and it is an important element in the center of $\mathbb{H}(W)$. A common theme in representation theory is to diagonalize central operators, and this is what we explore here for the full twist.

Let $\{b_w\}_{w \in W}$ denote the Kazhdan-Lusztig basis of $\mathbb{H}(W)$, which we call the *KL basis*. If λ is a two-sided cell, we let $I_{<\lambda} \subset \mathbb{H}(W)$ denote the ideal spanned by KL basis elements in strictly lower cells. Here is a classic theorem; we discuss the attribution below.

Theorem 1.1. *There is an integer* $\mathbf{x}(\lambda)$ *for each two-sided cell* λ*, such that for any* $w \in \lambda$ *we have*

$$\mathrm{ft}_W \cdot b_w \equiv v^{2\mathbf{x}(\lambda)} b_w \ \textit{modulo}\ I_{<\lambda}. \tag{1.1}$$

While the KL basis is not an eigenbasis for ft_W, it descends to an eigenbasis in the associated graded of the cell filtration. Morally speaking, cell theory controls the spectral theory of the full twist. Note that there may be multiple cells which share the same[1] value of $\mathbf{x}(\lambda)$.

Theorem 1.1 is an immediate consequence of a more refined property of the half twist.

Theorem 1.2. *There is an integer* $\mathbf{x}(\lambda)$ *and a non-negative integer* $\mathbf{c}(\lambda)$ *for each two-sided cell* λ*, such that for any* $w \in \lambda$ *we have*

$$\mathrm{ht}_W \cdot b_w \equiv (-1)^{\mathbf{c}(\lambda)} v^{\mathbf{x}(\lambda)} b_{\mathrm{Schu}_L(w)} \ \textit{modulo}\ I_{<\lambda}. \tag{1.2}$$

Here Schu_L *is some involution on* W *which preserves each left cell, generalizing the (left) Schützenberger involution in type* A*. Moreover,* $\lambda < \mu$ *implies that* $\mathbf{c}(\lambda) < \mathbf{c}(\mu)$ *and* $\mathbf{x}(\lambda) < \mathbf{x}(\mu)$*.*

[1] In type A, this happens for the first time in type A_5, with the partitions $(3,1,1,1)$ and $(2,2,2)$. However, one can still distinguish between cells in type A by simultaneously diagonalizing full twists of various sizes, see [5].

Theorem 1.2 was proven for all finite types by Mathas [24], and generalized to Hecke algebras with unequal parameters by Lusztig [22]. Previously, in [21, Corollary 5.9], Lusztig proved an analogous result about the action of the half twist braid on the canonical basis of a tensor product representation for the quantum group. In type A, Theorem 1.2 can be deduced from this result using Schur-Weyl duality. Another proof in type A was given in Graham's thesis [15].

Recall that the action of w_0 permutes the two-sided cells. The integers $\mathbf{x}$ and $\mathbf{c}$ can be determined using Lusztig's $\mathbf{a}$-function, via

$$\mathbf{c}(\lambda) = \mathbf{a}(w_0 \cdot \lambda), \qquad \mathbf{x}(\lambda) = \mathbf{a}(w_0 \cdot \lambda) - \mathbf{a}(\lambda). \tag{1.3}$$

From (1.3) it is clear that $\mathbf{c}(\lambda)$ is a non-negative integer, but it only matters up to parity in (1.2), and is invisible in (1.1). Regardless, the precise value of $\mathbf{c}(\lambda)$ plays an important role in the categorification. To emphasize its importance we prefer to rewrite (1.1) as

$$\mathrm{ft}_W \cdot b_w \equiv (-1)^{2\mathbf{c}(\lambda)} v^{2\mathbf{x}(\lambda)} b_w \text{ modulo } I_{<\lambda}. \tag{1.4}$$

In [24, Theorem 3.1] Mathas pins down the involution Schu_L uniquely using cell-theoretic properties. In [24, p9 and following] he explores further properties of Schu_L and in [24, Corollary 3.14] he gives a criterion implying that Schu_L is the identity on a given left cell, which he expects to be necessary as well. As far as we are aware, Schu_L only has an explicit combinatorial interpretation in type A. If desired, (1.2) can be viewed as the definition of $\mathrm{Schu}_L(w)$, which picks out the unique non-vanishing coefficient in $\mathrm{ht}_W \cdot b_w$ modulo lower terms.

Example 1.3. This is the example we follow throughout the paper. Let W have type C_2, with simple reflections $\{s, t\}$. There are three two-sided cells:

$$\lambda_1 = \{\mathrm{id}\}, \quad \lambda_{\mathrm{big}} = \{s, t, st, ts, sts, tst\}, \quad \lambda_0 = \{w_0\}, \tag{1.5a}$$

where w_0 denotes the longest element of W. We have

(1.5b)
$$\mathbf{x}(\lambda_1) = 4, \mathbf{c}(\lambda_1) = 4, \quad \mathbf{x}(\lambda_{\mathrm{big}}) = 0, \mathbf{c}(\lambda_{\mathrm{big}}) = 1, \quad \mathbf{x}(\lambda_0) = -4, \mathbf{c}(\lambda_0) = 0.$$

The Schützenberger involution on λ_{big} satisfies

(1.5c)
$$\mathrm{Schu}_L(s) = sts, \qquad \mathrm{Schu}_L(t) = tst, \qquad \mathrm{Schu}_L(st) = st, \qquad \mathrm{Schu}_L(ts) = ts.$$

Note that b_s is in the same (left) cell as b_{ts} because $b_t b_s = b_{ts}$ and $b_s b_{ts} = b_s + b_{sts}$, so each appears as a summand in the ideal generated by the other.

1.2. Action of the half twist on the p-canonical basis

Cells are a notion intrinsic to an algebra with a chosen basis, and the KL basis is not the only interesting basis of $\mathbb{H}(W)$. A recent player of great importance in modular representation theory [25] is the *p-canonical basis* $\{{}^p b_w\}$ (associated to a prime p), which is defined when W is crystallographic, see [18]. We recall the construction of the p-canonical basis in Definition 2.10. This basis is mysterious: it can be computed in examples, but there is no known algebraic formula (the methods to compute it involve working with the Hecke category rather than the Hecke algebra). The (two-sided) cells associated to the p-canonical basis are called *p-cells*, and were first studied systematically by the second author in [16].

When the prime p is understood, we write $c_w := {}^p b_w$ for ease of reading.

Example 1.4. We continue the previous example. The only prime for which $\{c_w\}$ and $\{b_w\}$ disagree is $p = 2$, so henceforth we set $p = 2$. For most w we have $b_w = c_w$, the one exception being that

$$c_{sts} = b_{sts} + b_s. \tag{1.6a}$$

However, now $b_s c_{ts} = c_{sts}$ which does not have c_s as a summand, so we can not use the same argument as before to deduce that s and ts are in the same cell. Indeed they are not. The big 0-cell λ_{big} now splits into two smaller p-cells:

$$\lambda_s = \{s\}, \qquad \lambda_{\mathrm{pbig}} = \{t, st, ts, sts, tst\}. \tag{1.6b}$$

The partial order on p-cells is

$$\lambda_0 < \lambda_{\mathrm{pbig}} < \lambda_s < \lambda_1. \tag{1.6c}$$

Readers familiar with distinguished involutions can note that sts now behaves like a distinguished involution in λ_{pbig}. Finally, note that w_0 does not act to permute the p-cells.

The first question one can ask is whether there is an analog of (1.2) and (1.4) for the p-canonical basis. If λ is a p-cell, we let $I^p_{<\lambda}$ denote the ideal spanned by strictly lower p-cells.

Conjecture 1.5. *Fix a Weyl group W and a prime p. There are integers $\mathbf{x}^p(\lambda)$ and $\mathbf{c}^p(\lambda)$ for each p-cell λ, together with a p-*Schützenberger involution Schu^p_L *preserving the left p-cells, such that*

$$\mathrm{ht}_W \cdot c_w \equiv (-1)^{\mathbf{c}^p(\lambda)} v^{\mathbf{x}^p(\lambda)} c_{\mathrm{Schu}^p_L(w)} \textit{ modulo } I^p_{<\lambda} \tag{1.7}$$

for any w in λ. Moreover, $\lambda < \mu$ implies that $\mathbf{c}^p(\lambda) < \mathbf{c}^p(\mu)$.

Conjecture 1.5 has been verified for all Weyl groups in rank ≤ 6, though the "Moreover" statement is not currently accessible by computer. We discuss this below.

Remark 1.6. In type $(C_3, p = 2)$ there is an example where $\lambda < \mu$ but $\mathbf{x}^p(\lambda) > \mathbf{x}^p(\mu)$. See (4.5).

Example 1.7. We continue Example 1.4. Since $b_w = c_w$ for many values of w, we see that (1.7) holds for many values of w, where the p-cells λ_0 and λ_{pbig} and λ_1 have the same values for $\mathbf{x}$ and $\mathbf{c}$ as their 0-cell counterparts. The interesting computations occur when $w \in \{s, sts\}$. Below we rewrite the p-canonical basis in the KL basis, apply (1.2), and then reinterpret again in the p-canonical basis. We have

$$\begin{aligned} \mathrm{ht}_W \cdot c_{sts} = \mathrm{ht}_W \cdot (b_s + b_{sts}) &\equiv (-1)(b_{sts} + b_s) \text{ modulo } I_{\lambda_0} \\ &= (-1)c_{sts} \text{ modulo } I^p_{\lambda_0}. \end{aligned} \tag{1.8a}$$

Note that the ideals I_{λ_0} and $I^p_{\lambda_0}$ agree. Thus (1.7) holds for c_{sts}, though $\mathrm{Schu}^p_L(sts) = sts$ in contrast to $\mathrm{Schu}_L(sts) = s$. Meanwhile,

$$\begin{aligned} \mathrm{ht}_W \cdot c_s &\equiv (-1)b_{sts} \text{ modulo } I_{\lambda_0} \\ &= -c_{sts} + c_s \text{ modulo } I^p_{\lambda_0} \\ &\equiv (+1)c_s \text{ modulo } I^p_{<\lambda_s}. \end{aligned} \tag{1.8b}$$

This is consistent with $\mathbf{x}^p(\lambda_s) = 0$ and $\mathbf{c}^p(\lambda_s) = 2$ and $\mathrm{Schu}^p_L(s) = s$. In (1.8b) only the parity of $\mathbf{c}^p$ is evident, but the precise value 2 comes from the categorification.

Let us explain the simple reason why the value of $\mathbf{x}^p$ should be constant on two-sided p-cells.

Proposition 1.8. *Suppose that $\mathrm{ft}_W \cdot c_w = v^{2\mathbf{x}^p(w)} c_w$ modulo lower terms, for some integer $\mathbf{x}^p(w)$ depending on w. Then $\mathbf{x}^p(w)$ is constant on two-sided p-cells.*

Proof. We first argue that the value of $\mathbf{x}^p(w)$ is constant on left p-cells. By definition, the left p-cell is indecomposable as a *based* left module over the Hecke algebra. The full twist ft_W is in the center of the Hecke algebra, so acts by an intertwiner on any module. Under the assumption, the p-canonical basis of the left cell module is an eigenbasis, so the eigenspaces are based submodules. If there were distinct eigenvalues then the left cell module would split as a based module, a contradiction.

Since ft_W is central, right multiplication by ft_W also has the p-canonical basis as an eigenbasis modulo lower terms, with the same eigenvalues. By the same argument, these eigenvalues are constant on right cells. □

1.3. Numerics and evidence

Let us write $\mathbf{x}(w) = \mathbf{x}(\lambda)$ when $w \in \lambda$. For example, in type C_2 as above, we have $\mathbf{x}(s) = \mathbf{x}^p(s) = 0$ and $\mathbf{c}(s) = 1$ and $\mathbf{c}^p(s) = 2$. The behavior of s is typical: it seems that $\mathbf{c}^p(w) > \mathbf{c}(w)$ when w moves to a higher p-cell relative to its 0-cell compatriots. In type C_4, the longest element w of the parabolic subgroup of type A_3 appears to have $\mathbf{c}^p(w) = \mathbf{c}(w) + 2$. There is no reason to expect a global bound on $\mathbf{c}^p(w) - \mathbf{c}(w)$ in general.

Remark 1.9. A small rank heuristic is that $\mathbf{c}^p(w)$ equals the length of the longest chain of p-cells from w to λ_0. However, this will certainly fail in general (it fails in type D_4), as the values of $\mathbf{c}^p$ may skip numbers.

From the example of type C_2 one might get the false impression that each p-cell λ' is contained in a unique 0-cell λ, but this is not true in general. Moreover, it need not be the case[2] that $\mathbf{x}(w) = \mathbf{x}^p(w)$. Counterexamples arise in type C_3, related to the appearance of a non-perverse p-canonical basis element. In the same counterexample is a fascinating surprise: two elements swap eigenvalues! A thorough discussion of this computation, and musings on its categorical significance, can be found in §4.

No general theory currently exists for determining the values of $\mathbf{x}^p(\lambda)$ or $\mathbf{c}^p(\lambda)$. Many other features of ordinary cell theory also do not yet have analogues for the p-canonical basis. The naive analogue of Lusztig's $\mathbf{a}$-function is not constant on p-cells, and no suitable replacement is known[3]. The longest element w_0 does not permute the p-cells. There is a conjectural definition of p-distinguished involutions due to the second author, but the theory is not fully developed. This conjecture does not yet appear in print elsewhere so we place it below as Conjecture 1.11. These are the major tools used to define $\mathbf{x}$ and $\mathbf{c}$ and Schu_L, and they are not yet operational for p-cell theory. The techniques used by Mathas do not seem to adapt well to the p-canonical basis.

[2]Note that $\mathbf{x}^p(w) = \mathbf{x}(x)$ for some x, since the eigenvalues of the full twist do not depend on the choice of basis.

[3]If Conjecture 1.5 holds, then following (1.3) one could define $\mathbf{a}^p(w) := \mathbf{c}^p(w) - \mathbf{x}^p(w)$. This statistic is not monotone over the cell order. Whether this is a satisfactory analogue of Lusztig's $\mathbf{a}$-function for other purposes remains to be seen.

Notation 1.10. For a Laurent polynomial f write $\mathrm{val}(f)$ for the smallest exponent appearing with nonzero coefficient. E.g. $\mathrm{val}(2v^{-1} + 3v) = -1$. Let $\mu^y_{w,x}$ be the coefficient of c_y inside the p-canonical basis expansion of $c_w \cdot c_x$. Let h^y_x be the coefficient of the standard basis element H_y inside c_x. We use the same notation for the ordinary KL basis as well, when the context calls for it. The notation $\mathbf{a}(w)$ or $\mathbf{a}(\lambda)$ will never be used in relation to the p-canonical basis and its coefficient, and always refers to Lusztig's usual $\mathbf{a}$-function.

Conjecture 1.11. *In each left p-cell there is a unique involution d for which $-\mathrm{val}(\mu^d_{d,d}) \geq \mathrm{val}(h^1_d)$. We call these the* p-distinguished involutions*.*

Remark 1.12. For the KL basis, $-\mathrm{val}(\mu^d_{d,d}) \leq \mathrm{val}(h^1_d)$ for all $d \in W$ (involution or not), while only distinguished involutions have an equality, in which case $\mathrm{val}(h^1_d) = -\mathrm{val}(\mu^d_{d,d}) = \mathbf{a}(d)$. For the p-canonical basis, $-\mathrm{val}(\mu^d_{d,d}) > \mathrm{val}(h^1_d)$ can occur.

The involution Schu^p_L is also mysterious. Often it is forced to disagree with the characteristic zero involution Schu_L, as an orbit of size two in a left 0-cell gets split into distinct p-cells. This happens in Example 1.7 with the orbit $\{s, sts\}$. What happens can be unpredictable, as seen in the following example.

Example 1.13. In type G_2 one has $\mathrm{Schu}_L(s) = ststs$ and $\mathrm{Schu}_L(sts) = sts$. When $p = 3$ the element s forms its own two-sided cell, and instead we have $\mathrm{Schu}^p_L(s) = s$ and $\mathrm{Schu}^p_L(sts) = ststs$. See §5.1.

Let us discuss the evidence for Conjecture 1.5.

First we consider type A. In type A_n for $n \leq 6$, the KL basis and the p-canonical basis agree for all primes. The second author proved in [16, Theorem 4.33] that the p-cells agree with the 0-cells in type A_n for all n, even though the p-canonical basis can be quite different from the KL basis. Although the combinatorics of p-cells is unchanged[4], the actual cell ideals in the Hecke algebra do change (e.g. $I_{\leq\lambda} \neq I^p_{\leq\lambda}$), because $c_w - b_w$ often involves KL basis elements in higher cells than w. This makes it hard to compare the associated graded of the cell filtration between the ordinary and p-canonical settings. Worse still, starting in type A_{11}, Williamson [27] found examples where $c_w - b_w$ involves KL basis elements in the same cell as w. This does not invalidate Conjecture 1.5, it only makes it more interesting if true, because (1.2) and

[4]It is only the combinatorics of two-sided cells which is known to be unchanged. There are no known examples where the partial order on left p-cells does change, but there is currently no proof that it doesn't change. Thankfully, it was proven in [17, Corollary 4.8] that left p-cells within a given two-sided p-cell are still incomparable.

(1.7) truly diverge. Sadly, computing the p-canonical basis in type A_{11} is beyond the reach of current computer programs.

Remark 1.14. In recent work, Lanini and McNamara [20] give a general method whereby a nonzero difference $c_w - b_w$ for some $w \in S_n$ will give rise to a nonzero difference $c_y - b_y$ living in the same cell, for a certain $y \in S_N$ with $N > n$.

Remark 1.15. Suppose that the left module for the Hecke algebra coming from a left p-cell is isomorphic to the ordinary left 0-cell module as based modules. Then (1.2) implies (1.7) for this cell. The second author proved in [16, Corollary 4.39] that all left p-cell modules in the same two-sided p-cell are isomorphic in type A. Unpublished computer calculations done by Williamson suggest that there are no unusual W-graphs in type A_n with $n \leq 10$, which should imply that the p-cell and the 0-cell give isomorphic based modules in these ranks.

Outside of type A, the p-canonical basis is only known in ranks ≤ 6, though much of this data is not yet in print. In rank ≤ 4, one can find the p-canonical basis and the additional code which verifies Conjecture 1.5 online [12]. These p-canonical bases were found by a mix of theoretic work and computer calculation due to Geordie Williamson and the second author. The additional code was written by Joel Gibson, who has summarized the results of this verification in the appendix.

1.4. Action of the half twist complex

The formula (1.2) was recently categorified in type A by the first author and Matt Hogancamp in [5, Proposition 4.31, Theorem 6.16], which we now explain.

Let $\mathcal{H}^0(W)$ denote the Hecke category defined over a field of characteristic 0. It is a graded additive monoidal category whose Grothendieck group is isomorphic as a ring to the Hecke algebra. In characteristic zero the Hecke category appears in many guises (e.g. using Soergel bimodules, or equivariant perverse sheaves on flag varieties, or projective functors on category $\mathcal{O}$). In Definition 2.8 we review a presentation of the Hecke category by generators and relations. Further references can be found there.

For now we just recall the essential properties of $\mathcal{H}^0(W)$. It is monoidally generated by objects denoted B_s, one for each simple reflection. Its indecomposable objects are parametrized up to grading shift and isomorphism by $\{B_w\}_{w \in W}$, and the objects $\{B_w\}$ categorify the KL basis $\{b_w\}$. The cell ideals $I_{<\lambda}$ lift to monoidal ideals $\mathcal{I}_{<\lambda}$. Rouquier [26] indicated how one should construct a chain complex for any word in the generators of the braid group of W, which depends only on the braid up to unique homotopy equivalence. Thus to any braid, including the half twist and full twist braids, one has a well-defined object in the bounded homotopy category $K^b(\mathcal{H}^0)$. One typically studies Rouquier complexes (and other objects of the homotopy category) by considering their *minimal complex*, the complex obtained after using homotopy equivalence to remove contractible direct summands. The minimal complex is unique up to isomorphism of complexes (rather than up to homotopy equivalence), and its chain objects in various homological degrees are invariants of the braid.

Conjecture 1.16 (See [5, Conjecture 4.30])**.** *Let W be a finite Coxeter group, and let HT_W denote the half-twist Rouquier complex in the homotopy category $K^b(\mathcal{H}^0)$. We have*

$$\mathrm{HT}_W \otimes B_w \cong \left(\underbrace{\dots}_{\mathcal{I}_{<\lambda}} \to B_{\mathrm{Schu}_L(w)}(\mathbf{x}(\lambda))[\mathbf{c}(\lambda)] \to 0 \right). \tag{1.9}$$

That is, the indecomposable object B_w is sent by HT_W to a complex, whose minimal complex consists of

- *one copy of $B_{\mathrm{Schu}_L(w)}$ in homological degree $\mathbf{c}(\lambda)$ and with a grading shift by $\mathbf{x}(\lambda)$,*
- *various objects in strictly lower cells and strictly lower homological degree.*

This conjecture was proven in type A in [5, Proposition 4.31, Theorem 6.16]. A proof for dihedral types is work in preparation.

That (1.9) categorifies and implies (1.2) is obvious, though the precise positions of objects in particular homological degrees are not visible in the Grothendieck group (only the parity of the homological shift is seen). We must emphasize a part of (1.9) which is invisible in the Grothendieck group: the fact that $B_{\mathrm{Schu}_L(w)}(\mathbf{x}(\lambda))[\mathbf{c}(\lambda)]$ is the unique object in the *maximal* homological degree[5] of $\mathrm{HT}_W \otimes B_w$. Since the inclusion of the final term in a bounded chain complex is a chain map, this means there is a chain map from the one-term complex containing $B_{\mathrm{Schu}_L(w)}(\mathbf{x}(\lambda))[\mathbf{c}(\lambda)]$ into $\mathrm{HT} \otimes B_w$. This chain map becomes a homotopy equivalence modulo $\mathcal{I}_{<\lambda}$. In the next section we explain in what sense this chain map is functorial. First we discuss characteristic p.

[5]In the terminology of [5, §3.4], the conjecture implies that the half twist is *sharp*: it is a fine enough tool to effectively separate cells using homological degree.

Suppose W is crystallographic. Using the same presentation by generators and relations, one can define the Hecke category $\mathcal{H}^p(W)$ over a field of characteristic $p > 0$. Again, see Definition 2.8 for details. It is still generated by objects denoted B_s. The indecomposable objects in $\mathcal{H}^p(W)$ are denoted $\{{}^pB_w\}_{w \in W}$, and are still parametrized by W, but what they are is mysterious. Again, when the prime p is understood we write C_w instead of ${}^pB_w$ for ease of reading. Note that $C_s = B_s$. The Grothendieck group of $\mathcal{H}^p(W)$ is still isomorphic as a ring to the Hecke algebra. The p-canonical basis $\{c_w\}$ is defined to be the images of the symbols of the indecomposable objects $\{C_w\}$ (for a more formal definition, see Definition 2.10). In particular, the p-cells correspond to monoidal ideals $\mathcal{I}^p_{<\lambda}$ which differ from their characteristic zero counterparts.

Rouquier complexes make perfect sense inside $K^b(\mathcal{H}^p)$, so that the half twist and full twist complexes are still well-defined. These complexes are largely unstudied and mysterious, having differently-behaved minimal complexes relative to their characteristic zero counterparts. For example, the half twist in $K^b(\mathcal{H}^0)$ is *perverse* in that each indecomposable summand of a chain object has a grading shift equal to its homological shift. This is false for the half twist in $K^b(\mathcal{H}^p)$. For example, in type C_2, (3.1a) demonstrates the minimal complex of the half twist in characteristic zero, which is perverse. In contrast, (3.1c) demonstrates the minimal complex in characteristic 2, where one summand sticks out like a sore thumb: a copy of $B_s(1)$ in homological degree 2.

Conjecture 1.17. *Fix a Weyl group W and a prime p. Let $\mathbf{x}^p$, $\mathbf{c}^p$, and Schu^p be defined as in Conjecture 1.5. Then for any w inside the p-cell λ we have*

$$\mathrm{HT}_W \otimes C_w \cong \left(\underbrace{\cdots}_{\mathcal{I}^p_{<\lambda}} \to C_{\mathrm{Schu}^p_L(w)}(\mathbf{x}^p(\lambda))[\mathbf{c}^p(\lambda)] \to 0 \right). \tag{1.10}$$

This conjecture is not obvious even in type A, where the p-cells agree with the 0-cells. How this conjecture plays out in examples is rather interesting.

Example 1.18. We continue the example of type C_2 in characteristic 2. Note that $C_s = B_s$ and $C_{sts} = B_sB_tB_s$. Here is the minimal complex for $\mathrm{HT} \otimes C_s$ (formulas for the differential can be found in (3.4b)):

$$\mathrm{HT} \otimes C_s \cong \left(\underline{B}_{w_0}(-1) \longrightarrow B_sB_tB_s(0) \longrightarrow B_s(0) \right). \tag{1.11}$$

This matches Conjecture 1.17 with $\mathbf{c}^p(s) = 2$ and $\mathrm{Schu}_L^p(s) = s$. The inclusion ι of $B_s(0)[2]$ into the last degree of (1.11) is a homotopy equivalence modulo lower cells.

In fact, one can define the complex (1.11) over $\mathbb{Z}$, and then specialize to other base rings. After inverting 2, the differential from $B_sB_tB_s$ to B_s is projection to a summand, and the complementary summand is B_{sts}. Applying Gaussian elimination easily transforms (1.11) into

$$\mathrm{HT} \otimes B_s \cong \left(\underline{B_{w_0}}(-1) \longrightarrow B_{sts}(0) \right). \tag{1.12}$$

This matches Conjecture 1.16 with $\mathbf{c}(s) = 1$, since $\mathrm{Schu}_L(s) = sts$.

Note that (1.11) is an indecomposable complex over $\mathbb{Z}$, and only becomes decomposable when 2 is inverted. The chain map ι can also be defined over $\mathbb{Z}$, and 2ι is nulhomotopic!

Theorem 1.19. *Conjecture 1.17 holds for the dihedral Weyl groups A_2, C_2, and G_2.*

For dihedral groups it is tractable to compute HT and $\mathrm{HT} \otimes C_w$ by brute force, which is how we prove the theorem. The main method is a tedious process of Gaussian elimination of complexes, but we use some tricks to speed the process. In this paper we discuss the computation for C_2, omitting the case of G_2 for reasons of space. We also omit the computation that type A_2 is characteristic independent. The results are far more interesting than the proofs; beyond explaining our methods and giving some illustrative examples, we spare the reader most of the details. Full details of the computation for C_2 can be found in the supplementary document [7].

We present our musings on type C_3 in §4.

1.5. Diagonalizing the full twist: characteristic zero

Earlier we alluded to the fact that the chain map

$$B_{\mathrm{Schu}_L(w)}(\mathbf{x}(\lambda))[\mathbf{c}(\lambda)] \to \mathrm{HT} \otimes B_w$$

might be functorial. If there were a functor F^λ (depending on λ) which sent

$$B_w \mapsto B_{\mathrm{Schu}_L(w)}(\mathbf{x}(\lambda))[\mathbf{c}(\lambda)],$$

then "functoriality" would be the existence of a natural transformation $F^\lambda \to \mathrm{HT} \otimes (-)$ giving rise to the chain map under discussion. However, no such functor F^λ is expected to exist.

Instead, let us consider the full twist complex $\mathrm{FT}_W := \mathrm{HT}_W \otimes \mathrm{HT}_W$. Even though F^λ need not exist, its square would be $\mathbb{1}(2\mathbf{x}(\lambda))[2\mathbf{c}(\lambda)]$, where $\mathbb{1}$ is the identity functor. This functor is a categorification of the eigenvalue $(-1)^{2\mathbf{c}(\lambda)}v^{2\mathbf{x}(\lambda)}$ from (1.4). It is a relatively straightforward consequence of (1.9) (see [5, Lemma 3.26 and Proposition 3.31]) that

$$\mathrm{FT}_W \otimes B_w \cong \left(\underbrace{\ldots}_{\mathcal{I}_{<\lambda}} \to B_w(2\mathbf{x}(\lambda))[2\mathbf{c}(\lambda)] \to 0 \right) \tag{1.13}$$

when w is in cell λ. The inclusion of the final term in this chain complex would be a chain map

$$B_w(2\mathbf{x}(\lambda))[2\mathbf{c}(\lambda)] \to \mathrm{FT}_W \otimes B_w,$$

which is a homotopy equivalence modulo $\mathcal{I}_{<\lambda}$. This homotopy equivalence could be the action of a natural transformation

$$\alpha_\lambda \colon \mathbb{1}(2\mathbf{x}(\lambda))[2\mathbf{c}(\lambda)] \to \mathrm{FT}_W$$

when applied to the object B_w.

Definition 1.20. Assume that (1.13) holds for all w in (two-sided) cell λ. A natural transformation $\alpha_\lambda \colon \mathbb{1}(2\mathbf{x}(\lambda))[2\mathbf{c}(\lambda)] \to \mathrm{FT}_W$ is called a λ-*eigenmap* if $\alpha_\lambda \otimes B_w$ is a homotopy equivalence modulo $\mathcal{I}_{<\lambda}$, for all w in cell λ. Equivalently, one can ask that the map induced by $\alpha_\lambda \otimes B_w$ in homological degree $2\mathbf{c}(\lambda)$ (to the minimal complex of $\mathrm{FT} \otimes B_w$) is an automorphism of $B_w(2\mathbf{x}(\lambda))$.

An eigenmap represents the "functorial relationship" between the operator $\mathrm{FT}_W \otimes (-)$ and its categorified eigenvalue. In [5, Theorem 7.37] it was proven (with difficulty) that eigenmaps do indeed exist for each 0-cell λ in type A. This is conjectured (in characteristic zero) for all finite Coxeter groups in [5, Conjecture 4.32], and proven for dihedral groups in work in preparation [3]. Here are several remarks on this result, before discussing its importance.

Remark 1.21. There are distinct cells $\lambda \neq \lambda'$ which ft_W can not tell apart, satisfying $\mathbf{x}(\lambda) = \mathbf{x}(\lambda')$ and $\mathbf{c}(\lambda) = \mathbf{c}(\lambda')$. Conjecturally, FT_W can distinguish between these cells! We can find a chain map α which is an eigenmap for cell λ but not for cell λ', and vice versa (see [4, §2] for a discussion of this complicated situation).

Remark 1.22. The half twist HT_W is perverse, so the chain objects in its minimal complex are determined in the Grothendieck group. From this one can prove (see [5, Proposition 4.21, Corollary 4.29]) that the chain objects appearing in homological degree $\mathbf{c}(\lambda)$ and cell λ are precisely $\bigoplus B_{\mathrm{Schu}_L(d)}(\mathbf{c}(\lambda))$, where d ranges over the distinguished involutions in λ. Meanwhile, the full twist is not perverse and much less is known about its minimal complex. One consequence of (1.13) and the existence of eigenmaps (see [5, Proposition 4.31]) is that, for the chain objects of the full twist in homological degree $2\mathbf{c}(\lambda)$, the summands in cell λ are precisely $\bigoplus B_d(\mathbf{x}(\lambda) + \mathbf{c}(\lambda))$.

In the Grothendieck group, the fact that ft_W is diagonalizable with known eigenvalues implies that

$$\prod_{\lambda}(\mathrm{ft}_W - v^{2\mathbf{x}(\lambda)}) = 0. \tag{1.14a}$$

A reasonable categorical lift of the operator $(\mathrm{ft}_W - v^{2\mathbf{x}(\lambda)})$ would be the cone of the chain map α_λ, the *eigencone*. Then the categorical analogue of (1.14a) would be

$$\bigotimes_{\lambda} \mathrm{Cone}(\alpha_\lambda) \simeq 0, \tag{1.14b}$$

in which case we say that FT_W is *categorically (pre)diagonalizable*. If each α_λ is an eigenmap and certain homological obstructions[6] vanish, one can prove that FT_W is categorically prediagonalizable, see [5, Proposition 3.42].

Given a diagonalizable operator in linear algebra, Lagrange interpolation gives a method to construct idempotents projecting to eigenspaces, allowing one to deduce that the vector space is the direct sum of its eigenspaces. It is this eigenspace decomposition, rather than the equation (1.14a), which is commonly used in practice. In the categorical setup, given a prediagonalizable endofunctor, one can hope for idempotent functors which project to *eigencategories* (see Remark 1.23 below). A direct sum decomposition into eigencategories is too much to ask for, but one can hope that the category being acted upon has a filtration whose subquotients are eigencategories. A formalization of these hopes is made in [4, Definition 6.16], and a prediagonalizable endofunctor for which nice projection functors exist is called *categorically diagonalizable*.

The main theorem of [4] is a categorification of the Lagrange interpolation construction whereby, given a prediagonalizable functor whose eigenvalues are *homologically distinct* (i.e. different cells have different values of $\mathbf{c}$), one can construct projection functors and prove categorical diagonalizability. In

[6] These obstructions measure to what extent the eigencones tensor-commute.

dihedral type FT_W has homologically distinct eigenvalues. In type A_n for $n \geq 5$ the full twist does not, though [5] is able to prove the categorical diagonalizability of the full twist using additional techniques.

Remark 1.23. Consider a chain map α (such as α_λ above) from a shift of the identity $\mathbb{1}(a)[b]$ to a complex F. An *eigenobject* for α is a complex M such that $\alpha \otimes M : M(a)[b] \to F \otimes M$ is a homotopy equivalence. Eigenobjects are the objects in the *α-eigencategory*, a full triangulated subcategory of the homotopy category. Recall that the KL basis is not actually an eigenbasis for ft_W, it is only an eigenbasis modulo lower cells. Similarly, B_w is not a true eigenobject for α_λ, only being an eigenobject modulo $\mathcal{I}_{<\lambda}$. However, using projection functors, one can construct genuine eigenobjects for each α_λ, justifying the fact that they are called eigenmaps. Note that both projection functors and eigenobjects are typically infinite complexes! Eigenobjects-modulo-lower-terms are easier to work with than true eigenobjects.

1.6. Diagonalizing the full twist: characteristic p

Now we pass to characteristic p. There are potentially more p-cells than 0-cells. Though they have the same set of eigenvalues $v^{2\mathbf{x}(\lambda)}$ for ft_W (powers of v^2), if one keeps track of the invisible factor of $(-1)^{2\mathbf{c}(\lambda)}$ then new "eigenvalues" appear. The main conjecture in this paper is the following, which states that these new p-cells also admit new eigenmaps.

Conjecture 1.24. *Fix a Weyl group W and a prime p. Assume Conjecture 1.17. For any p-cell λ there exists a chain map*

$$\alpha_\lambda \colon \mathbb{1}(2\mathbf{x}^p(\lambda))[2\mathbf{c}^p(\lambda)] \to \mathrm{FT}_W \tag{1.15}$$

which is a λ-eigenmap. *This means that, for any $w \in \lambda$, $\alpha_\lambda \otimes C_w$ is a homotopy equivalence modulo $\mathcal{I}^p_{<\lambda}$.*

Remark 1.25. The properties of the half and full twist discussed in Remark 1.22 should have analogues in characteristic p as well. That is, the p-distinguished involutions from Conjecture 1.11 should govern which chain objects appear in certain homological degrees within the half and full twists.

To give some justification for this conjecture, we prove our main theorem.

Theorem 1.26. *Conjecture 1.24 holds in type C_2 in characteristic* 2.

If Conjecture 1.24 holds then, at least for dihedral groups and other Coxeter groups where the eigenvalues are homologically distinct (i.e. $\mathbf{c}^p(\lambda) \neq$

$\mathbf{c}^p(\mu)$ for distinct p-cells $\lambda \neq \mu$), the machinery from [4, 5] immediately applies to prove that FT_W is categorically diagonalizable. We do not spell this out in type C_2 for reasons of brevity.

To prove Theorem 1.26, we compute the full twist over $\mathbb{Z}$, with the answer given in (3.9). We explain our methods in §3.9, and omit the gory details (found in [7]). Then we can explicitly construct the candidate eigenmaps in §3.4, and they have a very satisfying description: they are built from the unit maps of certain Frobenius algebra objects. We explain the connection between Frobenius algebra objects, distinguished involutions, and eigenmaps in the following sections. The eventual proof is found in §3.7, see Corollary 3.17.

Remark 1.27. We remark on the difficulty of this theorem. As mentioned, the work in preparation [3] proves the characteristic zero version of Theorem 1.26, also by explicitly computing the full twist (for any dihedral group). These characteristic zero full twists have a great deal of interesting structure; while we do not showcase that structure here, we give a hint by displaying an interesting Koszul-like complex in §2.7.

Meanwhile, the computation in finite characteristic is an order of magnitude more difficult, and lacks many useful tools available in the characteristic zero setting. While our computations for the half twist in type C_2 involved only a handful of Gaussian eliminations (6 pages of work), the computation of the full twist was enormous (60 pages of work) and quite involved. At the moment there is no organizing structure, only a mass of computations, though we hope one day this will be rectified.

So, where do the extra eigenmaps come from? The example of type C_2 sheds light on this question.

Example 1.28. The eigenmap for the 0-cell λ_{big} lifts over $\mathbb{Z}$ in a fairly straightforward way, and descends to an eigenmap for the p-cell λ_{pbig}. Meanwhile, the extra p-cell λ_s has eigenvalue $\mathbb{1}(0)[4]$, a homological degree unmatched by any 0-cell, so its eigenmap is not an adaptation of any characteristic zero eigenmap. In characteristic zero there is a one-dimensional space of chain maps $\mathbb{1}(0)[4] \to \mathrm{FT}_W$ modulo homotopy, and the most obvious generator has a lift over $\mathbb{Z}$ which we denote φ.

Let HOM denote the bigraded space of morphisms (of all graded and homological degrees) in the homotopy category. In type C_2 over $\mathbb{Z}$, one can compute that $\mathrm{HOM}(\mathbb{1}, \mathrm{FT}_W)$ is free[7] over $\mathbb{Z}$. The subspace of morphisms in the ideal $\mathcal{I}_{<\lambda_s}$ (which one can define over $\mathbb{Z}$) form a sublattice. In homological

[7]The space of chain maps from a shift of $\mathbb{1}$ to FT_W is free over $\mathbb{Z}$, but it is not obvious that the same should be true of chain maps modulo homotopy. In

degree 4, this sublattice has index 2. In fact, $\varphi \in \mathcal{I}_{<\lambda_s}$. The new eigenmap α_{λ_s} (in characteristic 2) does lift over $\mathbb{Z}$, and $2\alpha_{\lambda_s}$ is homotopic to φ. See §3.8 for details.

In summary, the new eigenmap does not come from a torsion element in $\mathrm{HOM}(\mathbb{1}, \mathrm{FT}_W)$, but from an element which is torsion modulo the sublattice $\mathcal{I}_{<\lambda_s}$. This is in contrast to Example 1.18, where $\alpha_{\lambda_s} \otimes B_s$ produces a 2-torsion element in $\mathrm{HOM}(B_s, \mathrm{HT}_W \otimes B_s)$. Another takeaway is that, when working over $\mathbb{Z}$, $\mathrm{HOM}(\mathbb{1}, \mathrm{FT}_W)$ is bigger than the experts familiar with characteristic zero may expect. While the map φ is the obvious generator in characteristic zero, it only generates an index 2 sublattice over $\mathbb{Z}$.

Remark 1.29. The beautiful conjectures of Gorsky-Negut-Rasmussen [14] (now mostly addressed in works of Oblomkov-Rozansky) have placed the study of the full twist for the symmetric group S_n in a new geometric perspective. They posit that the Drinfeld center of the Hecke category is equivalent to the category of equivariant coherent sheaves on (a particular version of) the flag Hilbert scheme of n points on the plane, with the full twist corresponding to $\mathcal{O}(1)$. Thus maps from $\mathbb{1}$ to FT_W correspond to sections of $\mathcal{O}(1)$. The eigenmaps form a special set of linear sections on this quasi-projective variety. One expects there to be a similarly interesting quasi-projective variety appearing for other finite Coxeter groups. Our observations in this paper suggest that the coordinate rings of these varieties have an interesting integral form whose study could shed light on the geometry of p-cells.

1.7. Outline of the paper

In §1, which perhaps you already read, we stated the main conjectures and theorems in this paper. In §2 we provide background on the diagrammatic Hecke category in type C_2, using the thick calculus established in [1]. In §3 we provide minimal versions of various complexes (the half twist, the full twist, and their action on various indecomposables) in type C_2 in both characteristic 0 and 2. We provide the eigenmaps in §3.4. In §3.5 through §3.7 we discuss Frobenius algebra objects associated to distinguished involutions, and how eigenmaps can optimistically built from their unit maps. Since this does occur in type C_2, we can prove Theorem 1.26 in §3.7. Afterwards, we discuss the extra eigenmap in §3.8, and discuss our methods of computation in §3.9.

type A, Hom spaces in the homotopy category were computed in [6], using spectral sequences which degenerated due to parity. One hopes that similar parity arguments will also apply over $\mathbb{Z}$, allowing one to prove freeness and other properties of full twists.

The bulk of the computations can be found explicitly in the supplemental document [7].

In §3 we assume the reader is familiar with the technique known as Gaussian elimination for complexes. Everything we use about this technique can be found in [2, §5.4].

Because there are few references on dihedral Hecke categories, especially in finite characteristic, we have tried to make §2 a useful resource. We have included everything needed to understand the complexes of §3, but also a few extra topics of interest, such as the Koszul-like complex in §2.7.

In §4 we give a high-level discussion of one interesting numerical situation which arises in type C_3, and hypothesize on the categorical explanation. Though mostly guesswork this chapter contains some interesting food for thought. One can skip from §1 directly to §4.

The appendix §5, written by Joel Gibson, and based on calculations originally done by Geordie Williamson and the second author, describes the differences between the Kazhdan-Lusztig cells and the p-cells for Weyl groups in rank ≤ 6. Graphs detail the places where $(\mathbf{x}, \mathbf{c})$ and $(\mathbf{x}^p, \mathbf{c}^p)$ disagree in rank ≤ 4. For several Weyl groups, this is the first time that the p-cell partial order has appeared in print.

2. Type C_2 diagrammatics

In an effort to make our computations more accessible to anyone attempting a deep study, we provide here a primer on morphisms in the Hecke category in type C_2 (over $\mathbb{Z}$). The reader interested in seeing the results (the half twist and full twist complexes) but without needing to understand the differentials should skip this chapter entirely.

For background on the diagrammatic Hecke category, see [8] or [10].

2.1. Setup and notation

Notation 2.1. Let $\Bbbk$ be a commutative domain, which we call the *base ring*. In this paper, we will typically set $\Bbbk$ to be $\mathbb{Z}$ or a field. For technical reasons, we fix an element $\kappa \in \Bbbk$.

A *realization* of a Coxeter system (W, S) over $\Bbbk$ is, roughly speaking, a free $\Bbbk$-module V equipped with a choice of roots in V and coroots in $V^* := \mathrm{Hom}_{\Bbbk}(V, \Bbbk)$, making V into a "reflection representation" of W. See [8, §5.7] for information on realizations. The non-expert reader need not concern themselves with the definition or properties of realizations in general. In this paper we work with a suitably universal realization V of type C_2.

Definition 2.2. Let W be the Coxeter group of type C_2 with simple reflections $\{s, t\}$. Define V to be the free $\Bbbk$-module with basis $\{\alpha_s, \alpha_t, \varpi\}$, where α_s and α_t are the *simple roots*. Inside V^*, define the *simple coroots* $\alpha_s^\vee$ and $\alpha_t^\vee$ so that

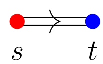
$$\langle \alpha_s^\vee, \alpha_s \rangle = \langle \alpha_t^\vee, \alpha_t \rangle = 2, \qquad \langle \alpha_s^\vee, \alpha_t \rangle = -2, \qquad \langle \alpha_t^\vee, \alpha_s \rangle = -1, \tag{2.1}$$

and

$$\langle \alpha_s^\vee, \varpi \rangle = 1, \qquad \langle \alpha_t^\vee, \varpi \rangle = \kappa. \tag{2.2}$$

Remark 2.3. One technical property one might desire of a realization is *Demazure surjectivity*, which states that the map $\alpha_s^\vee : V \to \Bbbk$ is surjective (and similarly for $\alpha_t^\vee$). This property holds for V since $\alpha_s^\vee$ pairs against ϖ to be 1. This is the reason to introduce ϖ; if V were spanned by α_s and α_t instead, then Demazure surjectivity would fail over $\mathbb{Z}$ or in characteristic 2. We need to introduce the scalar κ to encode the value of $\langle \alpha_t^\vee, \varpi \rangle$, but κ plays absolutely no role in the rest of this paper. Our realization V can be specialized to any other realization of C_2 satisfying Demazure surjectivity (if κ is specialized appropriately). We recall the importance of Demazure surjectivity in Remark 2.14 below.

These pairings of simple coroots with simple roots conform with the standard Cartan matrix for C_2, with short root α_s and long root α_t. Below s is red and t is blue.



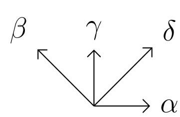
We sometimes refer to the elements of $\{s, t\}$, which often appear as indices in this paper, as *colors*. Below, s will always be red, and t will always be blue.

The group W acts on V by $\Bbbk$-linear transformations, via the formula

$$s(v) = v - \langle \alpha_s^\vee, v \rangle \alpha_s, \qquad t(v) = v - \langle \alpha_t^\vee, v \rangle \alpha_t. \tag{2.3}$$

Notation 2.4. Henceforth, let

$$\alpha = \alpha_s, \quad \beta = \alpha_t, \quad \gamma = \alpha_s + \alpha_t, \quad \delta = \alpha_t + 2\alpha_s \tag{2.4}$$



be the four positive roots in V. It is worth remembering that $\gamma = t(\alpha_s)$ is s-invariant and $\delta = s(\alpha_t)$ is t-invariant. These facts will be used tacitly in all sorts of polynomial forcing relations below. We also use

$$\chi := -s(\varpi) = \alpha - \varpi. \tag{2.5}$$

Notation 2.5. Let R be the polynomial ring of V, with degrees doubled so that $\deg V = 2$. For $\xi \in R$ we write ξ_r for the operator of right-multiplication by ξ, and ξ_l for left multiplication, acting on any given (R, R)-bimodule.

Typical examples of (R, R)-bimodules will be morphism spaces in the Hecke category. For example, the monoidal identity in the Hecke category is denoted $\mathbb{1}$, and its endomorphism ring is R.

Notation 2.6. The operator $\alpha_s^\vee$ extends by a twisted Leibniz rule to a map $\partial_s \colon R \to R$, and similarly for ∂_t, see [8, §4.3, Exercise 5.46] for details. In this paper, ∂_s is primarily applied to elements of $V \subset R$, where it agrees with $\alpha_s^\vee$.

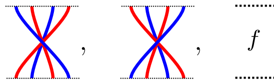
2.2. The Hecke category

We assume the reader is familiar with diagrammatics for monoidal categories, see [8, §7] for an introduction. We give here an ad hoc definition of the diagrammatic Hecke category, tailored for type C_2. In type C_2, the original construction was given in [1]. For a full definition in general, see [8, §10.2]. We do not cite original sources; more detailed background and history of these results can be found in the textbook [8].

Definition 2.7. Let R denote the polynomial ring of V over $\Bbbk$. The *(diagrammatic) Bott-Samelson category* $\mathcal{H}_{\mathrm{BS}} = \mathcal{H}_{\mathrm{BS}}(W, S, \Bbbk, V)$ is the (strict) monoidal category defined as follows. The objects are monoidally generated by objects called B_s and B_t, whose identity maps are drawn as red and blue lines, respectively. Objects are therefore sequences in the set $\{s, t\}$, which we sometimes call *Bott-Samelson objects*. As usual for diagrammatic categories, the morphism space between any two Bott-Samelson objects is the $\Bbbk$-span of diagrams built from the generators in (2.6), modulo certain relations which we discuss below. Diagrams have a degree and relations are homogeneous, making these morphism spaces into graded $\Bbbk$-modules.

The generating morphisms in $\mathcal{H}_{\mathrm{BS}}$ are trivalent vertices and univalent vertices (called *dots*) of each color s and t, 8-valent vertices, and polynomials.

(2.6a) 

(2.6b) 

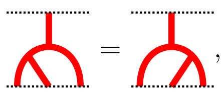
for $f \in R$.

Because the generating diagrams include polynomials in R acting on the monoidal identity $\mathbb{1}$, morphism spaces are naturally graded (R, R)-bimodules, and pre- or post-composition by any morphism is a bimodule map.

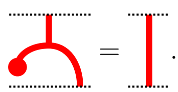
Definition 2.8. The *(diagrammatic) Hecke category* $\mathcal{H} = \mathcal{H}(W, S, \Bbbk, V)$ is the additive graded Karoubi envelope of $\mathcal{H}_{\mathrm{BS}}(W, S, \Bbbk, V)$. The objects are formal direct sums of grading shifts of direct summands (i.e. formal images of idempotents) of Bott-Samelson objects, and the morphisms are induced from those in $\mathcal{H}_{\mathrm{BS}}$.

Notation 2.9. We write $\mathcal{H}^{\mathbb{Z}}$ for the Hecke category when $\Bbbk = \mathbb{Z}$, and $\mathcal{H}^0$ or $\mathcal{H}^p$ for the Hecke category when $\Bbbk$ is a field of the corresponding characteristic.

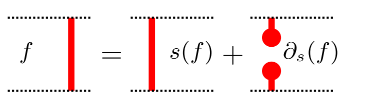
The relations can be found in [8, §10.2.2], and we discuss them here only at a high level. One family of relations are the *isotopy relations*, stating that isotopic diagrams with the same boundary represent the same morphism. A second family of relations are the *Frobenius relations*, stating that B_s is a Frobenius algebra object in $\mathcal{H}_{\mathrm{BS}}$. For example, the trivalent vertex satisfies an associativity relation

(2.7) 

and the univalent and trivalent vertices satisfy a unit relation

(2.8) 

A third family of relations involve the 8-valent vertex. In this paper we will entirely avoid the use of the 8-valent vertex and its relations, by way of the thick calculus developed in [1], which we recall in the next section. A final relation is the *polynomial forcing relation*

(2.9) 

Suppose $f \in R$. Using (2.9), one can show that $f_l - f_r$ acts by zero on the identity map of B_s when f is s-invariant. In particular, it acts by zero on any morphism factoring through the object B_s (since pre- and post-composition are (R, R)-bimodule maps). If f is W-invariant, then $f_l - f_r$ acts by zero on the identity map of any Bott-Samelson object, and therefore on any morphism in $\mathcal{H}$. We record these statements for future reference.

(2.10)
$$\text{For } f \in R^s, f_l - f_r \text{ acts by zero on any morphism factoring through } B_s.$$

$$\text{For } f \in R^W, f_l - f_r \text{ acts by zero on any morphism in } \mathcal{H}. \tag{2.11}$$

Now we recall the key properties of $\mathcal{H}$, all of which are discussed at length in [8, §10 and §11]. Over any base ring $\Bbbk$, morphism spaces are free as left R-modules and as right R-modules. The size of morphism spaces is controlled by the Soergel Hom Formula, see [8, Theorem 5.27], and an explicit *double leaves* basis for morphisms is defined in [8, §10]. When $\Bbbk$ is a field, the category $\mathcal{H}$ is Karoubian and satisfies the Krull-Schmidt property, so its Grothendieck group $[\mathcal{H}]$ has basis given by the symbols of indecomposable objects. One can prove that the $\mathbb{Z}[v, v^{-1}]$-algebra map sending $b_s \mapsto [B_s]$ and $v \mapsto [\mathbb{1}(1)]$ is an isomorphism from the Hecke algebra $\mathbb{H}(W)$ to $[\mathcal{H}]$, for any field $\Bbbk$.

When $\Bbbk$ is a field, the indecomposable objects in $\mathcal{H}$ are parametrized, up to isomorphism and grading shift, by the elements of W. More precisely, the Bott-Samelson object associated to a reduced expression of $w \in W$ will have a unique direct summand which can not be found as a summand of any shorter expression, and this summand is denoted ${}^pB_w$. When the characteristic is understood, this is shortened to C_w, and in characteristic zero we may also use B_w. The object C_w is independent of the choice of reduced expression, up to (non-canonical) isomorphism. However, the behavior of this indecomposable summand is very dependent on the choice of $\Bbbk$! For example, in characteristic 2, $B_sB_tB_s$ is indecomposable, so this Bott-Samelson object is C_{sts}. In any other characteristic, B_s is a direct summand of $B_sB_tB_s$, and C_{sts} is the complementary summand.

Definition 2.10. Let $\Bbbk$ be a field of characteristic p. The *p-canonical basis* $\{c_w\}$ is the basis of $\mathbb{H}(W)$ corresponding to $\{[C_w]\}$ under the aforementioned isomorphism $\mathbb{H}(w) \to [\mathcal{H}]$.

It was proven in [18] that this basis does not depend on the precise choice of field, only on the characteristic of that field. Many other basic properties of the p-canonical basis can be found in [18].

When we work over $\Bbbk = \mathbb{Z}$, the Krull-Schmidt property will not hold. It still makes sense to discuss indecomposable objects in the Hecke category $\mathcal{H}^{\mathbb{Z}}$, but their endomorphism rings will not be local. Statements about the classification of indecomposable objects in $\mathcal{H}^{\mathbb{Z}}$ and the uniqueness of direct sum decompositions are not in the literature to our knowledge. Regardless, type C_2 is sufficiently small that indecomposable objects can be constructed by brute force. For each indecomposable object which is not already a Bott-Samelson object, we will explicitly write down the idempotent (inside the endomorphism ring of some Bott-Samelson object) which that indecomposable is the image of. We will also explicitly write down inclusion and projection maps for all direct sum decompositions we use.

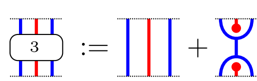
Notation 2.11. Recall that over $\mathbb{Z}$ the Bott-Samelson object $B_sB_tB_s$ is indecomposable, and this stays true in characteristic 2, so that $B_sB_tB_s = C_{sts}$ for $p = 2$. For clarity, we will never use the notation C_{sts} below, always referring to this object as $B_sB_tB_s$. For any base ring where 2 is invertible, $B_sB_tB_s \cong B_{sts} \oplus B_s$ where B_{sts} is the image of the idempotent $\frac{e_{sts}}{2}$ defined in the next section. We only use the notation B_{sts} when 2 is invertible.

Finally, we note the relation between the diagrammatic category $\mathcal{H}$ and the category of Soergel bimodules. There is a monoidal functor $\mathcal{F}$ sending $\mathcal{H}$ to the category of graded (R,R)-bimodules, for which $\mathcal{F}(B_s) = R \otimes_{R^s} R(1)$. The grading shift places $1 \otimes 1$ in degree -1. The indecomposable object C_{w_0} has the same size as B_{w_0} in any characteristic (i.e. $c_{w_0} = b_{w_0}$), and is sent by $\mathcal{F}$ to the bimodule $R \otimes_{R^W} R(4)$. For the realization we are using, $\mathcal{F}$ is fully faithful, and induces an (R,R)-bimodule isomorphism on morphism spaces. The essential image of $\mathcal{F}$ is known as the category of Soergel bimodules.

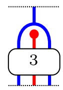
Remark 2.12. In type C_2 the functor $\mathcal{F}$ is fully faithful, so diagrams can be viewed as a computational tool to study Soergel bimodules. In affine type in finite characteristic, it is extremely important to distinguish between the diagrammatic category $\mathcal{H}$ and the category of Soergel bimodules. This is because the reflection representation of an infinite Coxeter group can not be faithful in finite characteristic, leading to additional morphisms between Soergel bimodules which do not exist in $\mathcal{H}$. In other words, the functor $\mathcal{F}$ will no longer be full.

2.3. Basics of thick calculus

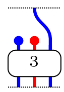
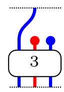
Because $\partial_t(\alpha_s) = -1$ is invertible, B_t is a direct summand of $B_tB_sB_t$. The complementary summand B_{tst} is the image of the idempotent e_{tst}.

(2.12) $e_{tst} =$ 

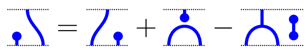
By design e_{tst} satisfies *death by pitchfork*:

(2.13)  $= 0.$

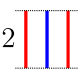
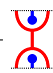
We also have the *leapfrog move*

(2.14)  $=$  .

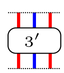
The easiest way to prove (2.14) is to combine death by pitchfork with the *three-way dot force*

(2.15) 

Meanwhile, $B_sB_tB_s$ is equipped with a quasi-idempotent e_{sts}

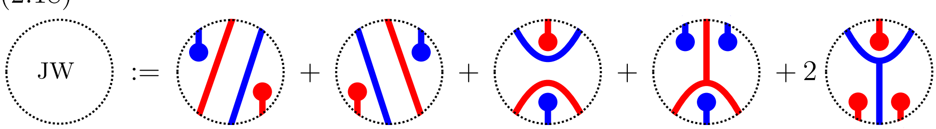
(2.16) $e_{sts} := 2$  $+$  .

This endomorphism satisfies death by pitchfork, and consequently $e_{sts}^2 = 2e_{sts}$. After inverting the number 2 we can define the true idempotent $\frac{e_{sts}}{2}$, which projects to a summand named B_{sts}. We denote this idempotent by

(2.17)  $:= \dfrac{e_{sts}}{2}.$

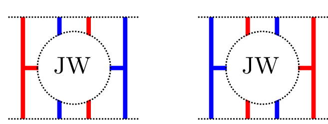
The *Jones-Wenzl morphism* is the following degree $+2$ rotation-invariant sum of diagrams on the planar disk:

(2.18)



It satisfies death by pitchfork from all angles, and is uniquely specified by this property up to scalar. Note the coefficient 2 on the final diagram.

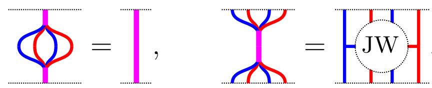
The Jones-Wenzl morphism gives rise to an idempotent inside both the objects $B_sB_tB_sB_t$ and $B_tB_sB_tB_s$,

(2.19) 

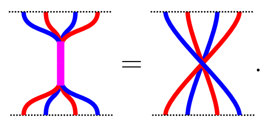
and the images of these two idempotents are isomorphic, see Remark 2.13. We introduce an abstract object B_{w_0} isomorphic to these images, which we denote with a thicker purple strand, following the thick calculus of [1, §6.3]. When interacting with polynomials, B_{w_0} behaves like its Soergel bimodule counterpart $R \otimes_{R^W} R(4)$. There are inclusion and projection maps, denoted as vertices which "split" the purple strand into red and blue strands.

(2.20) 

Then, tautologically, we have the following relations and their color swap:

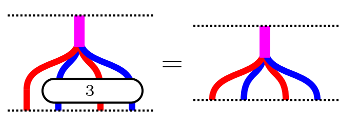
(2.21) 

Remark 2.13. The composition $B_sB_tB_sB_t \to B_{w_0} \to B_tB_sB_tB_s$ is the 8-valent vertex in the ordinary Soergel calculus. The 8-valent vertex induces an isomorphism between the images of the two Jones-Wenzl idempotents.

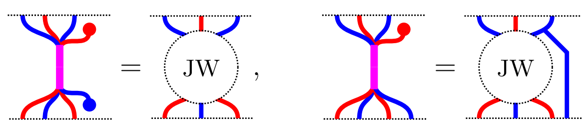
(2.22) 

Because of (2.22), we can entirely avoid using 8-valent vertices, and for the rest of this paper we only use diagrams with splitting vertices instead.

Using (2.21), we deduce that the splitters satisfy death by pitchfork. As a consequence, they absorb e_{tst}:

(2.23) 

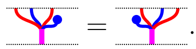
By the uniqueness of the Jones-Wenzl morphism, it is straightforward to deduce

(2.24) 

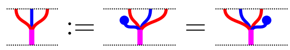
Two other relations allow one to move dots around on splitters. These relations also hold upside-down, or with the colors swapped. The *leapfrog move* can be proven similarly to (2.14) above.

(2.25) 

The *dot migration* move can be proven with (2.21) and (2.24).

(2.26) 

In particular, there is a one-dimensional space of degree $+1$ morphisms from B_{w_0} to $B_sB_tB_s$ (or to $B_tB_sB_t$), spanned by the morphism in (2.26). We often shorten this morphism to a *partial splitter* as follows:

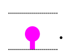
(2.27) 

While the splitter has degree 0, this partial splitter has degree $+1$. Note that these partial splitters satisfy death by pitchfork, and thus the partial splitter to $B_tB_sB_t$ absorbs e_{tst}.

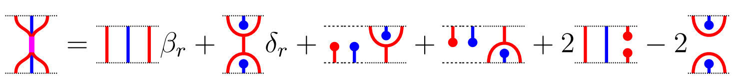
Similarly, there is a one-dimensional space of degree $4 - \ell(u)$ morphisms from B_{w_0} to B_u for any $u \in W$, allowing us to define other partial splitters:

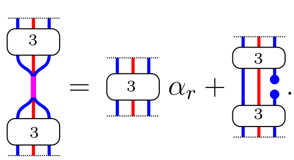
(2.28) 

All partial splitters are just the splitter composed with the appropriate number of dots, whose placement is irrelevant. The partial splitters in (2.28) have degree 2 and 3 respectively. Our conventions for partial splitters are invariant under flipping diagrams horizontally or vertically, and under color swap. The "partial splitter" when $u = 1$ is usually just denoted as a purple dot, c.f. [1, (6.20)]:

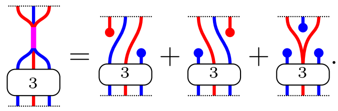
(2.29) 

By placing dots on the various relations above, one can compute a number of useful relations. For clarity we place all polynomials on the right, and use the subscript r to indicate right multiplication.

(2.30a) 

(2.30b) 

(2.30c)

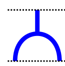
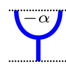


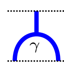
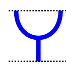
2.4. Direct sum decompositions

Whenever we use the direct sum decomposition

(2.31a) $$B_t B_t \cong B_t(1) \oplus B_t(-1),$$

we use the following inclusion and projection maps. The subscript indicates which summand is the source or target, i.e. p_{-1} is a map to $B_t(-1)$ so it has degree -1, while i_{-1} is a map from $B_t(-1)$ so it has degree $+1$.

(2.31b) $p_{-1} =$ , $i_{-1} =$ .

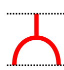
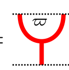
(2.31c) $p_{+1} =$ , $i_{+1} =$ .

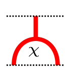
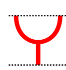
Crucial to this direct sum decomposition is the fact that $\partial_t(-\alpha) = 1$, and $t(\alpha) = \gamma$.

Whenever we use the direct sum decomposition

(2.32a) $$B_s B_s \cong B_s(1) \oplus B_s(-1),$$

we use the following inclusion and projection maps.

(2.32b) $p_{-1} =$ , $i_{-1} =$ .

(2.32c) $p_{+1} =$ , $i_{+1} =$ .

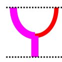
Crucial to this direct sum decomposition is the fact that $\partial_s(\varpi) = 1$ and $-s(\varpi) = \chi$.

Remark 2.14. The reason to include ϖ in the realization V, and more generally the reason for Demazure surjectivity, is to ensure a decomposition as in (2.32).

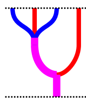
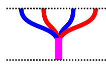
Whenever we use the direct sum decomposition

$$B_{w_0}B_s \cong B_{w_0}(1) \oplus B_{w_0}(-1), \tag{2.33a}$$

we use projection and inclusion maps similar to (2.32), except that every appearance of  is replaced with

(2.33b) ,

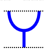
and similarly upside-down. The same polynomials appear in the same places in p_{+1} and i_{-1}. The *thick trivalent vertex* (2.33b) is defined as in [1, (6.18)]. It may help to note that

(2.33c)  = ,

a variant on the unit axiom (it follows easily from (2.21) and one-color relations).

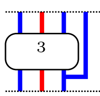
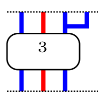
Similarly, for the direct sum decomposition

$$B_{w_0}B_t \cong B_{w_0}(1) \oplus B_{w_0}(-1), \tag{2.34a}$$

we use projection maps similar to (2.31) except with a blue thick trivalent vertex replacing . We do the same for

$$B_{tst}B_t \cong B_{tst}(1) \oplus B_{tst}(-1), \tag{2.34b}$$

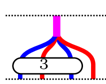
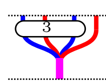
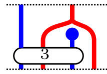
except using a different "trivalent" vertex

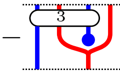
(2.34c)  = .

One has a direct sum decomposition

$$B_{tst}B_s \cong B_{w_0} \oplus B_{ts}. \tag{2.35a}$$

We use inclusion and projection maps

(2.35b) $p_{w_0} =$ , $\quad i_{w_0} =$ , (2.35c) $p_{ts} =$ 

, $\quad i_{ts} = -$ 

.

It is not hard to verify that

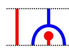
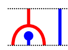
(2.35d) $$p_{ts} \circ i_{ts} = -(\partial_s(\beta) + 1)\,\mathrm{id}_{ts} = \mathrm{id}_{ts},$$

(2.35e) $$p_{w_0} \circ i_{w_0} = \mathrm{id}_{w_0},$$

(2.35f) $$i_{w_0} \circ p_{w_0} + i_{ts} \circ p_{ts} = \mathrm{id}_{tst} \otimes \mathrm{id}_s\,.$$

For (2.35f), when resolving $i_{w_0} p_{w_0}$ with (2.21) and (2.18), death by pitchfork eliminates all but two terms: the identity, and a term cancelling $i_{ts} p_{ts}$. Note that this direct sum decomposition works over $\mathbb{Z}$.

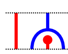
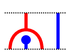
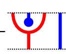
Finally we now consider the decomposition of the Bott-Samelson object $(B_s B_t B_s) B_t$. There is a two-dimensional space of degree zero morphisms to B_{st}, spanned by

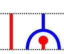
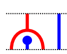
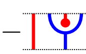
(2.36a)  , 

The two-dimensional space of morphisms from B_{st} is spanned by the vertical flip of these maps. Composing these to get a 2×2 matrix of endomorphisms of B_{st} (the so-called *local intersection form*), the resulting morphisms are all multiples of the identity, and the coefficients are

(2.36b) $$\begin{pmatrix} -1 & 1 \\ 1 & -2 \end{pmatrix}.$$

Since this matrix has determinant 1, B_{st} appears as a summand inside $B_s B_t B_s B_t$ with multiplicity 2 (even over $\mathbb{Z}$). Specifically, we chould choose projection and inclusion maps as follows.

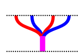
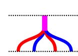
(2.36c) $p_1 =$  $+$  , $\quad i_1 = -$  ,

(2.36d) $p_2 = 2$  $+$  , $\quad i_2 = -$  .

It is easy to verify that $\mathrm{id}_{B_s B_t B_s B_t} - i_1 p_1 - i_2 p_2$ is the Jones-Wenzl idempotent from (2.19). Thus we deduce that

(2.36e) $$B_s B_t B_s B_t \cong B_{w_0} \oplus B_{st} \oplus B_{st}$$

where the final projection and inclusion maps are given by splitters

(2.36f) $p_{w_0} =$  , $\quad i_{w_0} =$  .

2.5. Endomorphisms of B_{w_0}

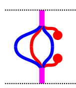
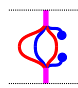
As an R-bimodule, $\mathrm{End}(B_{w_0}) \cong R \otimes_{R^W} R$. This isomorphism equips $\mathrm{End}(B_{w_0})$ with a presentation, where it is generated by left and right multiplication by polynomials, and the relations come from the tensor product. Note that $\mathrm{End}(B_{w_0})$ is a commutative ring.

We will return to left and right multiplication by polynomials in §2.7. First, we explore a different description of $\mathrm{End}(B_{w_0})$.

As a right R-module, $\mathrm{End}(B_{w_0})$ is free of graded rank $1 + 2v^2 + 2v^4 + 2v^6 + v^8$. It has a basis $\{\varphi_u\}$ for $u \in W$, where φ_u is defined as a composition of partial splitters $B_{w_0} \to B_u \to B_{w_0}$, and has degree $8 - 2\ell(u)$. The identity map is φ_{w_0}.

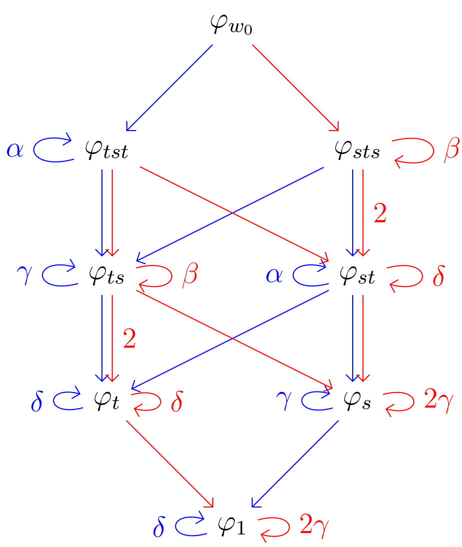
Remark 2.15. All statements about the graded rank of morphism spaces follow from Soergel's Hom formula, and its analogue for the diagrammatic Hecke category (see [8, Theorem 11.1(5)]), which still applies even when working over $\mathbb{Z}$.

Let

(2.37) $x := \varphi_{tst} =$  , $y := \varphi_{sts} =$  .

Then $\mathrm{End}(B_{w_0})$ is generated over the right action of R by x and y. The action of x and y on the right R-basis $\{\varphi_u\}$ can be computed fairly easily using (2.30). We encode it in the following graph, where a blue (resp. red) arrow indicates the action of x (resp. y).

(2.38)



For example, $\varphi_{ts}x = \varphi_t + \gamma_r\varphi_{ts}$ and $\varphi_{ts}y = \varphi_s + 2\varphi_t + \beta_r\varphi_{ts}$. Remember that ξ_r denotes right multiplication by the polynomial $\xi \in R$. As we will see in the next section, this graph is also useful for studying morphisms from B_{w_0} to other objects.

2.6. Morphisms involving B_{w_0}

Now let $z \in W$ be any element except for sts; one can also allow $z = sts$ if 2 is inverted. The indecomposable object B_z categorifies the Kazhdan-Lusztig basis element, and this basis element is smooth, meaning that all of its Kazhdan-Lusztig polynomials are trivial. As a consequence of the Soergel Hom Formula, for any $u \leq z$ there is a one-dimensional space of morphisms from B_z to B_u in degree $\ell(z)-\ell(u)$, which can be obtained by including B_z into a Bott-Samelson, applying the appropriate number of dots (whose placement is irrelevant), and then projecting to B_u. This map is a generalization of the partial splitters above. We might call it a *canonical map*.

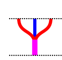
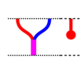
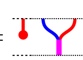
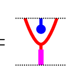
Remark 2.16. When there is a functor from diagrammatics to bimodules, the inclusion and projection maps of top summands are normalized so that they preserve certain vectors known as 1-tensors. The 1-tensors are also preserved by , so they are preserved by canonical maps.

For any $z \in W$ and $u \leq z$ we can define $\varphi_u^z \colon B_{w_0} \to B_z$, defined as the composition $B_{w_0} \to B_u \to B_z$ of canonical maps. It has degree $4+\ell(z)-2\ell(u)$. Then $\{\varphi_u^z\}_{u\leq z}$ is a right R-basis for $\mathrm{Hom}(B_{w_0}, B_z)$. There is an action of $\mathrm{End}(B_{w_0})$ on $\mathrm{Hom}(B_{w_0}, B_z)$ by precomposition, which one might encode with a graph as in (2.38). In fact, this graph embeds inside (2.38), sending $\varphi_u^z \mapsto \varphi_u$ (the proof by direct computation is straightforward). For example,

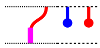
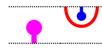
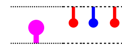
$$\varphi_{ts}^{tst} \circ x = \varphi_t^{tst} + \gamma_r\varphi_{ts}^{tst}.$$

Similarly, we can define $\bar{\varphi}_u^z \colon B_z \to B_{w_0}$, as the composition $B_z \to B_u \to B_{w_0}$. In other words, $\bar{\varphi}_u^z$ is the vertical flip of the diagram for φ_u^z. The action of $\mathrm{End}(B_{w_0})$ by postcomposition on $\mathrm{Hom}(B_z, B_{w_0})$ obeys exactly the same rules as (2.38) again, since the generators x and y of $\mathrm{End}(B_{w_0})$ are invariant under the vertical flip.

It remains to examine $\mathrm{Hom}(B_{w_0}, B_sB_tB_s)$. By flipping everything upside-down, we obtain similar statements for $\mathrm{Hom}(B_sB_tB_s, B_{w_0})$. A basis for morphisms is given by the following eight elements, sorted by degree.

(2.39a) $\varphi_{sts}^{sts} :=$ , $\varphi_{st}^{sts} :=$ , $\varphi_{ts}^{sts} :=$ , $\rho_s^{sts} =$ .

(2.39b)

$\varphi_s^{sts} :=$ , $\quad \varphi_t^{sts} =$ , $\quad \rho_1^{sts} =$ , $\quad \varphi_1^{sts} =$ .

The action of x and y on this basis is given as follows. The action on ρ_s^{sts} and ρ_1^{sts} mimic the action on φ_s and φ_1 from (2.38). In the quotient by the submodule spanned by ρ_s^{sts} and ρ_1^{sts}, the action on φ_u^{sts} embeds inside (2.38). Here is the full graph.

(2.40)

$$\begin{tikzcd}
 & & & & \varphi_{sts}^{sts} \arrow[loop right, "\beta"] \arrow[dllll] \arrow[dllll] \arrow[dll] \arrow[d] \arrow[d, "2"] \\
\rho_s^{sts} \arrow[loop left, "\gamma"] \arrow[loop right, "2\gamma"] \arrow[d] & & \varphi_{ts}^{sts} \arrow[loop left, "\gamma"] \arrow[loop right, "\beta"] \arrow[ll, bend left, "-\alpha"] \arrow[dll] \arrow[d] \arrow[d, "2"] \arrow[drr] & & \varphi_{st}^{sts} \arrow[loop left, "\alpha"] \arrow[loop right, "\delta"] \arrow[dll] \arrow[d] \arrow[d] \\
\rho_1^{sts} \arrow[loop left, "\delta"] \arrow[loop right, "2\gamma"] & & \varphi_t^{sts} \arrow[loop left, "\delta"] \arrow[loop right, "\delta"] \arrow[dr] & & \varphi_s^{sts} \arrow[loop left, "\gamma"] \arrow[loop right, "2\gamma"] \arrow[dl] \\
 & & & \varphi_1^{sts} \arrow[loop left, "\delta"] \arrow[loop right, "2\gamma"] &
\end{tikzcd}$$

2.7. Left multiplication versus right multiplication

We return to the study of $\mathrm{End}(B_{w_0})$. We have the following formulas which compute left multiplication by polynomials in terms of this right R-basis.

$$\alpha_l = 2x - y - \alpha_r. \tag{2.41a}$$

$$\beta_l = 2y - 2x - \beta_r. \tag{2.41b}$$

$$\gamma_l = y - \gamma_r. \tag{2.41c}$$

$$\delta_l = 2x - \delta_r. \tag{2.41d}$$

In particular, there are nice formulas for the sum $f_l + f_r$ for various roots f.

$$\alpha_l + \alpha_r = 2x - y. \tag{2.42a}$$

$$\beta_l + \beta_r = 2y - 2x. \tag{2.42b}$$

$$\gamma_l + \gamma_r = y. \tag{2.42c}$$

$$\delta_l + \delta_r = 2x. \tag{2.42d}$$

Some sums are divisible by 2, so that when we write $\frac{\beta_l+\beta_r}{2}$ we refer to the morphism $y - x$, a genuine morphism over $\mathbb{Z}$. One also obtains formulas for differences $f_l - f_r$, which appear frequently in differentials in Rouquier complexes. For example

$$\frac{\delta_l - \delta_r}{2} = x - \delta_r, \qquad \gamma_l - \gamma_r = y - 2\gamma_r. \tag{2.43}$$

The difference $\gamma_l - \gamma_r$ is an endomorphism of B_{w_0} which will annihilate any morphism $B_{w_0} \to B_s$ under precomposition, or any morphism $B_s \to B_{w_0}$ under postcomposition. This is because γ is s-invariant, so $\gamma_l - \gamma_r$ acts by zero on the identity map of B_s by (2.9). Since pre-composition is an (R, R)-bimodule map, $\gamma_l - \gamma_r$ kills any morphism out of B_s. It should not be surprising then that $\gamma_l - \gamma_r = y - 2\gamma_r$ kills φ_s, as is evident from (2.38). Similarly, $\delta_l - \delta_r$ and $x - \delta_r$ annihilate any morphism between B_{w_0} and B_t. For any polynomial f, $f_l - f_r$ annihilates any morphism between B_{w_0} and R.

Let $q = 2\alpha^2 + 2\alpha\beta + \beta^2$. This is the *quadratic form* induced by the Cartan matrix, and it is W-invariant. Therefore $q_l - q_r = 0$ acting on any morphism space in the Hecke category (since the bimodule action of R factors through $R \otimes_{R^W} R$).

Recall that $\gamma = \beta + \alpha$ and $\delta = \beta + 2\alpha$. This implies that

$$q = \alpha\delta + \beta\gamma. \tag{2.44}$$

Now consider the action of the following operator on an (R, R)-bimodule.

$$\begin{aligned}(\delta_l - \delta_r)(\alpha_l + \alpha_r) &+ (\gamma_l - \gamma_r)(\beta_l + \beta_r) \\ &= (\alpha_l\delta_l + \gamma_l\beta_l) - (\alpha_r\delta_r + \gamma_r\beta_r) + (\delta_l\alpha_r - \delta_r\alpha_l + \gamma_l\beta_r - \gamma_r\beta_l).\end{aligned} \tag{2.45}$$

The first term involves all left actions, and equals q_l by (2.44). The second term equals $-q_r$. The third term mixes the left and right action. However, expanding γ and δ into α and β, the third term equals zero:

$$\begin{aligned}&\delta_l\alpha_r - \delta_r\alpha_l + \gamma_l\beta_r - \gamma_r\beta_l \\ &\quad = 2\alpha_l\alpha_r + \beta_l\alpha_r - 2\alpha_r\alpha_l - \beta_r\alpha_l + \beta_l\beta_r + \alpha_l\beta_r - \beta_r\beta_l - \alpha_r\beta_l = 0.\end{aligned} \tag{2.46}$$

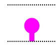
Thus we conclude that

$$(\delta_l - \delta_r)(\alpha_l + \alpha_r) + (\gamma_l - \gamma_r)(\beta_l + \beta_r) = q_l - q_r, \tag{2.47}$$

which acts by zero on any morphism space in the Hecke category.

This permits us to build the 2-periodic "Koszul complex"

$$\ldots \to B_{w_0}^{\oplus 2} \to B_{w_0}^{\oplus 2} \to B_{w_0} \to R \to 0. \tag{2.48}$$

The final differential in this complex is . The penultimate differential is

$$K_0 = \begin{bmatrix} \frac{\delta_l - \delta_r}{2} & \gamma_l - \gamma_r \end{bmatrix},$$

which annihilates any map to R. The differential before that is

$$K_1 = \begin{bmatrix} \gamma_l - \gamma_r & \alpha_l + \alpha_r \\ -\frac{\delta_l - \delta_r}{2} & \frac{\beta_l + \beta_r}{2} \end{bmatrix}$$

and the differential before that is

$$K_2 = \begin{bmatrix} \frac{\beta_l + \beta_r}{2} & -(\alpha_l + \alpha_r) \\ \frac{\delta_l - \delta_r}{2} & \gamma_l - \gamma_r \end{bmatrix}.$$

After that the differentials alternate between K_1 and K_2, as $K_1 \circ K_2 = K_2 \circ K_1 = 0$. We don't use this Koszul complex in this paper, but pieces of it appear in the full twist and its eigenmaps, and the computation that $K^2 = 0$ is a useful stepping stone to the more complicated computations below.

Note also that

$$(\alpha_l - \alpha_r)(\delta_l + \delta_r) + (\beta_l - \beta_r)(\gamma_l + \gamma_r) = q_l - q_r = 0. \tag{2.49}$$

The corresponding Koszul complex is isomorphic to the one above. To see this, think of each instance of $B_{w_0}^{\oplus 2}$ as being $B_{w_0} \boxtimes V$ (or more precisely, since V is three-dimensional, one should tensor with the two-dimensional span of the roots inside V). The multiplicity space V admits a rotation which sends $\alpha \mapsto \gamma$ and $\beta \mapsto -\delta$ and also sends $\gamma \mapsto -\alpha$ and $\delta \mapsto \beta$. Applying this change of basis to each $B_{w_0}^{\oplus 2}$ yields the desired isomorphism up to sign.

3. Type C_2 results

We describe the half twist in §3.1, the action of the half twist on indecomposables in §3.2, and the full twist in §3.3, all in type C_2. We provide complexes defined over $\mathbb{Z}$ which descend to the minimal complex in characteristic 2, and also the minimal complexes obtained after inverting 2. Also, in §3.4 we provide and discuss the eigenmaps for the full twist.

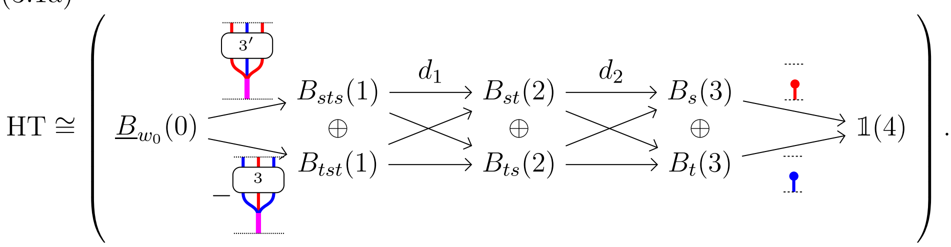
The lengthy computations which justify these complexes are performed in the supplemental document [7], but we explain our methods in §3.9. The supplement [7] also contains minimal complexes for other Rouquier complexes.

An underline indicates the term in homological degree zero.

3.1. The half twist

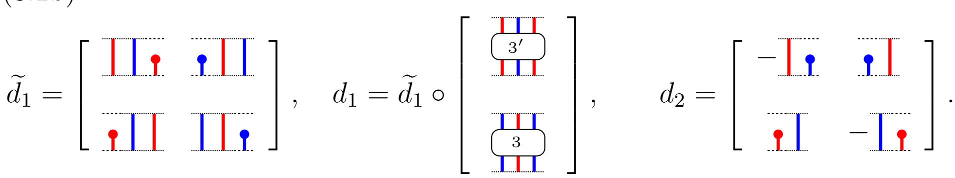
Over any field of characteristic $\neq 2$ one has

(3.1a)



One has

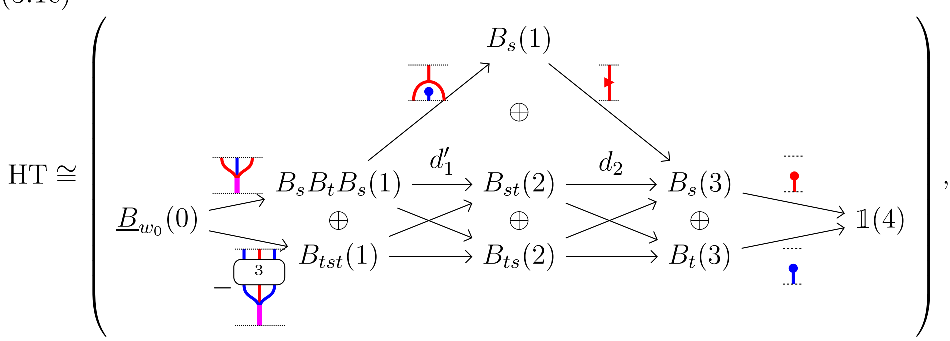
(3.1b)



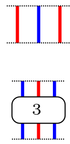
Let us witness the phenomenon discussed in Remark 1.22. We have $\mathbf{c}(\lambda_{\text{big}}) = 1$, and objects in cell λ_{big} only appear in HT in degrees ≥ 1. The distinguished involutions in this cell are s and t, and in degree exactly 1, we find $sts = \mathrm{Schu}(s)$ and $tst = \mathrm{Schu}(t)$. Similar statements hold for λ_0 and λ_1.

In contrast, over $\mathbb{Z}$ or in characteristic 2, one has

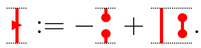
(3.1c)



and no further Gaussian elimination can be applied. Note that the differential has zero component from $B_{tst}(1)$ to $B_s(1)$ or from $B_s(1)$ to $B_t(3)$. Here

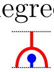
(3.1d)
$$d_1' = \widetilde{d}_1 \circ \left[\quad\right].$$


Also, the degree $+2$ *triangle map* generates the kernel of a dot on top or on bottom:

(3.2)


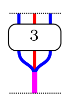
To verify that $d^2 = 0$ from $B_s B_t B_s(1)$ to $B_s(3)$, apply the three-way dot force (2.15).

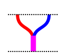
Now consider the same phenomenon (see Remark 1.25) for $p = 2$. We have $\mathbf{c}^p(\lambda_{\mathrm{pbig}}) = 1$, and objects in cell λ_{pbig} first appear in homological degree 1. The distinguished involutions in this p-cell are sts and t, and in homological degree 1 we witness $sts = \mathrm{Schu}^p(sts)$ and $tst = \mathrm{Schu}^p(t)$. Similarly, $\mathbf{c}^p(\lambda_s) = 2$, $\mathrm{Schu}^p(s) = s$, and B_s first appears in homological degree 2.

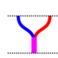
After inverting 2, the component of the differential  in (3.1c) becomes the projection map to a direct summand, and it is easily verified that Gaussian elimination on (3.1c) yields (3.1a).

3.2. Action of the half twist

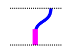
Over $\mathbb{Z}$ or any field, one has

(3.3a)
$$\mathrm{HT} \otimes B_t \cong \left(\underline{B}_{w_0}(-1) \longrightarrow B_{tst}(0) \right).$$


(3.3b)
$$\mathrm{HT} \otimes B_{st} \cong \left(\underline{B}_{w_0}(-2) \longrightarrow B_{st}(0) \right).$$


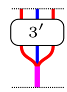
(3.3c)
$$\mathrm{HT} \otimes B_{ts} \cong \left(\underline{B}_{w_0}(-2) \longrightarrow B_{ts}(0) \right).$$


$$\mathrm{HT} \otimes B_{tst} \cong \left(\underline{B}_{w_0}(-3) \longrightarrow B_t(0) \right). \tag{3.3d}$$



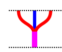
Over any field of characteristic $\neq 2$ one has

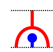
$$\mathrm{HT} \otimes B_s \cong \left(\underline{B}_{w_0}(-1) \longrightarrow B_{sts}(0) \right). \tag{3.4a}$$



In contrast, over $\mathbb{Z}$, one has

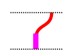
$$\mathrm{HT} \otimes B_s \cong \left(\underline{B}_{w_0}(-1) \longrightarrow B_sB_tB_s(0) \longrightarrow B_s(0) \right). \tag{3.4b}$$





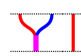
Over any field of characteristic $\neq 2$ one has

$$\mathrm{HT} \otimes B_{sts} \cong \left(\underline{B}_{w_0}(-3) \longrightarrow B_s(0) \right). \tag{3.5a}$$

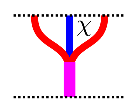


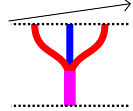
Over $\mathbb{Z}$ one has

$$\mathrm{HT} \otimes B_sB_tB_s \cong \left(\underline{B}_{w_0}\underline{B}_s(-2) \longrightarrow B_sB_tB_s(0) \right) \tag{3.5b}$$



$$\cong \left(\begin{array}{c} \underline{B}_{w_0}(-3) \\ \oplus \\ \underline{B}_{w_0}(-1) \end{array} \longrightarrow B_sB_tB_s(0) \right).$$





Here $\chi = \alpha - \varpi$. The final isomorphism uses the decomposition given in (2.33).

After base change to a field of characteristic $\neq 2$, $B_sB_tB_s \cong B_{sts} \oplus B_s$, and (3.5b) splits as a direct sum of (3.5a) and (3.4a). Over $\mathbb{Z}$ or in characteristic 2, the complex (3.5b) is indecomposable.

Finally, over $\mathbb{Z}$ or any field one has

$$\mathrm{HT} \otimes B_{w_0} \cong \underline{B}_{w_0}(-4). \tag{3.6}$$

3.3. The full twist

After inverting 2, the full twist has the following minimal complex.

(3.7)

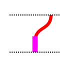
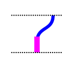
$$\mathrm{FT} \cong \left(\begin{matrix} \underline{B}_{w_0}(-4) \xrightarrow{d_0} \begin{matrix} B_{w_0}(-2) \\ \oplus \\ B_{w_0}(-2) \end{matrix} \xrightarrow{d_1} \begin{matrix} B_s(1) \\ \oplus \\ B_t(1) \\ \oplus \\ B_{w_0}(0) \\ \oplus \\ B_{w_0}(0) \end{matrix} \xrightarrow{d_2} \begin{matrix} B_{st}(2) \\ \oplus \\ B_{ts}(2) \\ \oplus \\ B_{w_0}(2) \\ \oplus \\ B_{w_0}(2) \end{matrix} \xrightarrow{d_3} \begin{matrix} B_{sts}(3) \\ \oplus \\ B_{tst}(3) \\ \oplus \\ B_{w_0}(4) \end{matrix} \\ \xrightarrow{d_4} \begin{matrix} B_{sts}(5) \\ \oplus \\ B_{tst}(5) \end{matrix} \xrightarrow{d_5} \begin{matrix} B_{st}(6) \\ \oplus \\ B_{ts}(6) \end{matrix} \xrightarrow{d_6} \begin{matrix} B_s(7) \\ \oplus \\ B_t(7) \end{matrix} \xrightarrow{d_7} \mathbb{1}(8) \end{matrix} \right).$$

We have

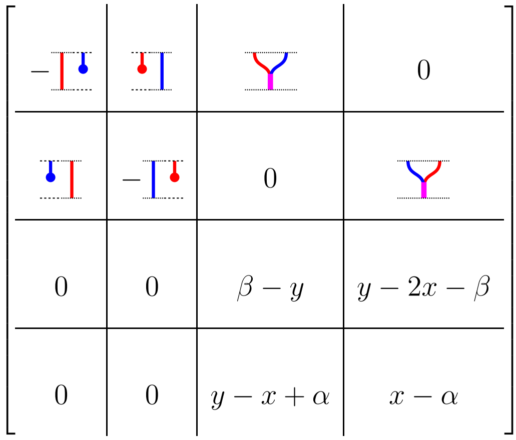
(3.8a)

$$d_0 = \begin{bmatrix} y - 2\gamma \\ \hline 2\delta - 2x \end{bmatrix},$$

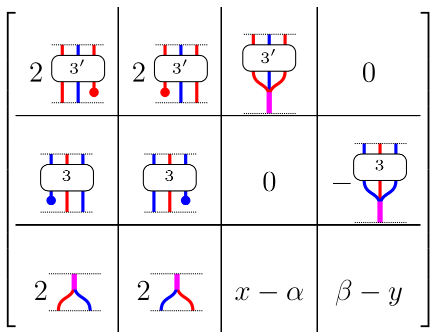
$d_1 =$

	0
0	
$y - \delta$	$y - x - \gamma$
$2x - y - \delta$	$x - \gamma$

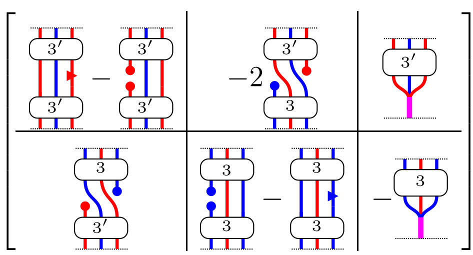
,

(3.8b) $d_2 =$ 

,

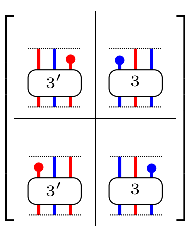
(3.8c) $d_3 =$ 

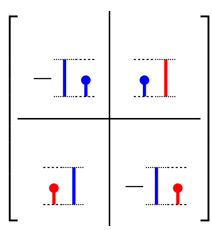
,

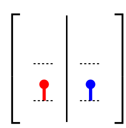
(3.8d) $d_4 =$ 

,

(3.8e)

$d_5 =$ 

, $d_6 =$ 

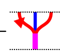
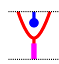
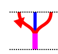
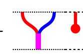
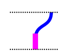
, $d_7 =$ 
,

The full twist in type C_2 over $\mathbb{Z}$ has the following minimal complex.

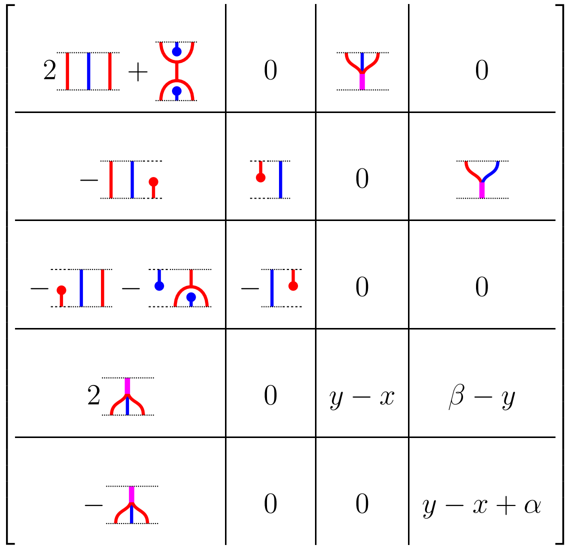
(3.9)
$$\mathrm{FT}\cong\left(\begin{aligned} &\underline{B}_{w_0}(-4) \xrightarrow{d_0} \begin{matrix}B_{w_0}(-2)\\ \oplus \\ B_{w_0}(-2)\end{matrix} \xrightarrow{d_1} \begin{matrix}B_sB_tB_s(1)\\ \oplus\\ B_t(1)\\ \oplus\\ B_{w_0}(0)\\ \oplus\\ B_{w_0}(0)\end{matrix} \xrightarrow{d_2} \begin{matrix}B_sB_tB_s(1)\\ \oplus\\ B_{st}(2)\\ \oplus\\ B_{ts}(2)\\ \oplus\\ B_{w_0}(2)\\ \oplus\\ B_{w_0}(2)\end{matrix} \xrightarrow{d_3} \begin{matrix}B_s(1)\\ \oplus\\ B_sB_tB_s(3)\\ \oplus\\ B_{tst}(3)\\ \oplus\\ B_{w_0}(4)\end{matrix} \\ &\xrightarrow{d_4} \begin{matrix}B_s(3)\\ \oplus\\ B_sB_tB_s(5)\\ \oplus\\ B_{tst}(5)\end{matrix} \xrightarrow{d_5} \begin{matrix}B_s(5)\\ \oplus\\ B_{st}(6)\\ \oplus\\ B_{ts}(6)\end{matrix} \xrightarrow{d_6} \begin{matrix}B_s(7)\\ \oplus\\ B_t(7)\end{matrix} \xrightarrow{d_7} \mathbb{1}(8)\end{aligned}\right).$$

We have

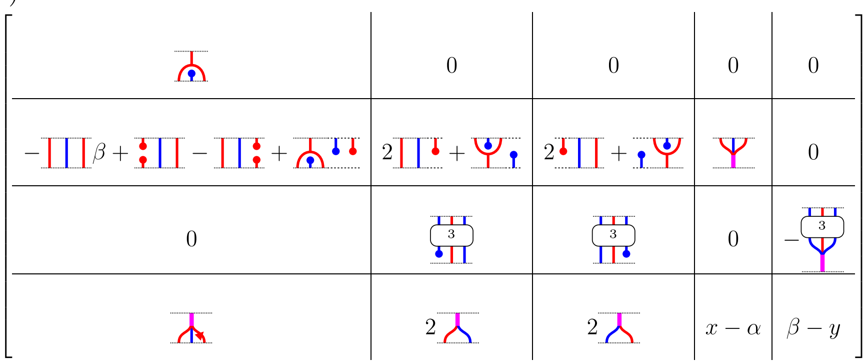
(3.10a) $d_0 = \begin{bmatrix} y-2\gamma \\ \hline x-\delta\end{bmatrix}$, $d_1 =$

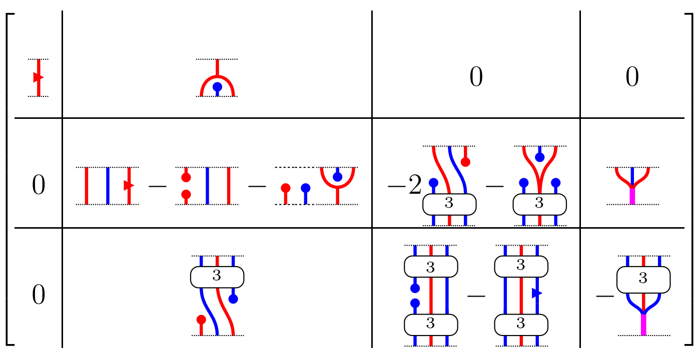
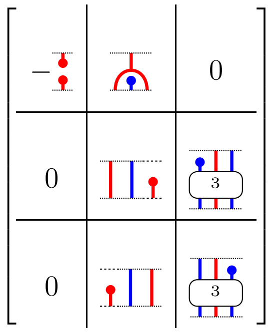
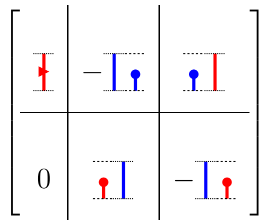
$-$  $-$ 	 $-$ 
0	
$y-2x+\delta$	$2(x-\gamma)$
$x-y$	$y-2x$

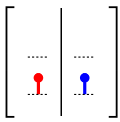
,

(3.10b) $d_2 =$  ,

(3.10c)

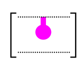
$d_3 =$  ,

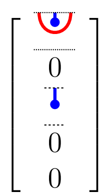
(3.10d) $d_4 =$  , (3.10e) $d_5 =$  , $d_6 =$  ,

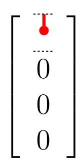
$d_7 =$  ,

3.4. Eigenmaps

The candidate eigenmaps over $\mathbb{Z}$ are described as follows. Checking that they are chain maps is an easy exercise.

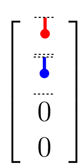
(3.11a) $\alpha_0 \colon \mathbb{1}(-8)[0] \to \mathrm{FT}$ 

(3.11b) $\alpha_{\mathrm{pbig}} \colon \mathbb{1}(0)[2] \to \mathrm{FT}$ 

(3.11c) $\alpha_s \colon \mathbb{1}(0)[4] \to \mathrm{FT}$ 

(3.11d) $\alpha_1 \colon \mathbb{1}(8)[8] \to \mathrm{FT}$ $\begin{bmatrix} 1 \end{bmatrix}$.

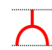
When 2 is inverted, we instead have

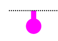
(3.12) $\alpha_{\mathrm{big}} \colon \mathbb{1}(0)[2] \to \mathrm{FT}$  .

To prove that they are eigenmaps, we make a digression over several sections to discuss Frobenius algebra objects. Many of the ideas below owe their conception to conversations between the first author and Matt Hogancamp, and are implicit in [5].

3.5. Frobenius algebra objects and distinguished involutions

See [8, §8] for additional background on Frobenius algebra objects in a monoidal category. The reader may wish to recall Notation 1.10.

The object B_s is a (graded) Frobenius algebra object of degree 1: it has a unit  of degree 1, a multiplication map  of degree -1, and a counit and comultiplication map given by flipping these diagrams upside-down. The relations among diagrams which involve only one color are effectively just the relations stating that B_s is a Frobenius algebra object. The Frobenius algebra structure is related to the decomposition of $B_s \otimes B_s$. For example,  is the inclusion map of a summand $B_s(-1)$ inside $B_s B_s$, with projection map ; they compose to the identity by the unit axiom (2.8).

Similarly, B_{w_0} is also a Frobenius algebra object of degree $\ell(w_0) = 4$, since $R^W \subset R$ is a Frobenius extension. The unit is . The multiplication map is the projection $B_{w_0} \otimes B_{w_0} \to B_{w_0}(-4)$ to the unique negative-most degree summand. See [1, §6.3] for more details on the multiplication map.

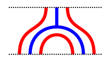
Proposition 3.1. *In $\mathcal{H}^0$, if the indecomposable object B_w is a Frobenius algebra object of degree a, then w is a distinguished involution, and $a = \mathbf{a}(w)$. Similarly, in $\mathcal{H}^p$, if the indecomposable object C_w is a Frobenius algebra object of degree a, then w is a p-distinguished involution in the sense of Conjecture 1.11, and $-\operatorname{val}(\mu^w_{w,w}) \geq a \geq \operatorname{val}(h^1_w)$.*

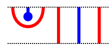
We prove Proposition 3.1 later in this section. These two examples in type C_2 are completely illustrative of the general principles behind this proposition.

Example 3.2. The indecomposable object B_{tst} is not a graded Frobenius algebra object of any degree. This can be seen directly from the graded dimensions of Hom spaces. Any map $\mathbb{1} \to B_{tst}$ has degree at least $+3$, so the unit has such a degree. However, there are no morphisms $B_{tst} \otimes B_{tst} \to B_{tst}$ of degree ≤ -3 to serve as the multiplication map.

Example 3.3. The indecomposable objects B_{st} is not graded Frobenius algebra object, because it is not self-biadjoint.

What is interesting in type C_2 for $p = 2$ is that there is a new p-distinguished involution, namely $C_{sts} = B_sB_tB_s$ in cell λ_{pbig}. The Bott-Samelson objects $B_sB_tB_s$ and $B_tB_sB_t$ are both Frobenius algebra objects of degree 1 (in any characteristic). For $B_sB_tB_s$ the structure maps are

(3.13) unit = , multiplication = .

For the same reasons as above,  is the inclusion map of a direct summand $B_sB_tB_s(-1)$ inside $(B_sB_tB_s) \otimes (B_sB_tB_s)$.

Remark 3.4. We are unaware of any literature on Frobenius algebra objects possessing direct summands which are also Frobenius algebra objects. There should be some relationship between the Frobenius algebra structures on $B_tB_sB_t$ and its direct summand B_t (and similarly for $B_sB_tB_s$ and B_s when 2 is inverted), but we do not know how to make this relationship precise. The unit map for $B_tB_sB_t$ is a composition of the unit map for B_t with the inclusion map $B_t \to B_tB_sB_t$, and is therefore orthogonal to the complementary summand B_{tst}. The multiplication map is not orthogonal to B_{tst} (nor can it be, for the unit axiom to hold).

Here is the postponed proof of Proposition 3.1.

Proof. Let B be a Frobenius algebra object. By composing the counit with the multiplication, one obtains a degree zero map $B \otimes B \to \mathbb{1}$ which is typically called the cap. Similarly, one can define the cup $\mathbb{1} \to B \otimes B$. The cap and cup satisfy isotopy relations, implying that $B \otimes (-)$ is self-biadjoint (the cup and cap are the unit and counit of adjunction). The monoidal adjoint of B_w is $B_{w^{-1}}$, so B_w is self-biadjoint if and only if w is an involution. The same statement can be made for C_w.

Suppose w is an involution and B_w is a Frobenius algebra object in $\mathcal{H}^0$. The minimal degree of any morphism $\mathbb{1} \to B_w$ is $\mathrm{val}(h^1_w)$, so $a \geq \mathrm{val}(h^1_w)$. The minimal degree of any morphism $B_w \otimes B_w \to B_w$ is $\mathrm{val}(\mu^w_{w,w})$, so $-a \geq \mathrm{val}(\mu^w_{w,w}) \geq -\mathbf{a}(w)$. Thus $\mathbf{a}(w) \geq a \geq \mathrm{val}(h^1_w)$. Since $\mathrm{val}(h^1_w) \geq \mathbf{a}(w)$, with equality if and only if w is distinguished, we deduce that $a = \mathbf{a}(w)$ and w is distinguished.

The same initial argument can be made for C_w in $\mathcal{H}^p$, to deduce that $-\mathrm{val}(\mu^w_{w,w}) \geq a \geq \mathrm{val}(h^1_w)$. According to Conjecture 1.11, there is a unique involution in each left or right p-cell for which $-\mathrm{val}(\mu^w_{w,w}) \geq \mathrm{val}(h^1_w)$, and these are precisely the p-distinguished involutions, so w must be p-distinguished. However, this inequality can be strict, so we can not pin down the value of a. $\square$

Remark 3.5. For a p-distinguished involution d in type C_2, $\mathrm{val}(h^1_d) = -\mathrm{val}(\mu^d_{d,d})$, and both agree with $\mathbf{a}(d)$ for the usual $\mathbf{a}$-function. Thus the degree of any Frobenius algebra object C_d must be $\mathbf{a}(d)$.

Let us discuss the converse. In type C_2 in characteristic 2, the p-distinguished involutions are $\{1, s, t, sts, w_0\}$, and C_d is a Frobenius algebra object for each one. For an arbitrary Coxeter group, in characteristic zero, it is conjectured [5, Conjecture 4.40] and [19, §5.2] that B_d is a graded Frobenius algebra object in $\mathcal{H}^0$ whenever d is a distinguished involution. A weaker statement, that the image of B_d is a graded Frobenius algebra object in the associated graded of the cell filtration, was proven in [23, §4.4]. In characteristic p, we do not yet have enough evidence to state a conjecture confidently. Instead we state a dream, which we do not necessarily believe is true. See Conjecture 1.11 for notation.

Dream 3.6. *For Weyl groups, in any characteristic, the indecomposable object C_d associated to a p-distinguished involution d is a graded Frobenius algebra object of degree* $\mathrm{val}(h^1_d)$.

Remark 3.7. A weaker dream, which we expect to be true, is that C_d is a graded Frobenius algebra object when d is a p-distinguished involution and $\mathrm{val}(h^1_d) = -\mathrm{val}(\mu^d_{d,d})$.

3.6. Frobenius modules, distinguished involutions, and the J-ring

Let B be a graded Frobenius algebra object of degree a, with unit map $\eta_B \colon \mathbb{1}(-a) \to B$. Several times in the previous section we emphasized the property that $\eta_B \otimes \mathrm{id}_B \colon B(-a) \to B \otimes B$ is the inclusion of a direct summand, with projection map given by multiplication. The unit axiom implies that the composition of projection and inclusion is id_B, as required. However, the unit map η_B gives rise to many more direct sum decompositions, as we now explain in the example $B = B_s$.

Let w be any element with $sw < w$. Then $B_s B_w \cong B_w(-1) \oplus B_w(1)$, and there is a projection map $p_w \colon B_s B_w \to B_w(-1)$. An interesting example is in type C_2 with $w = w_0$, where p_w is a thick trivalent vertex (the 180 degree rotation of (2.33b)). More generally, one can construct p_w by including B_w into a Bott-Samelson object for a reduced expression of w which starts with s, applying the ordinary trivalent vertex $B_s B_s \to B_s(-1)$, and then projecting again from the Bott-Samelson to B_w. A consequence of this description is that

$$p_w \circ (\eta_s \otimes \mathrm{id}_w) = \mathrm{id}_w \,. \tag{3.14}$$

When $w = s$, one just recovers the unit axiom; this is a generalization.

Definition 3.8. Let B be a graded Frobenius algebra object of degree a in a graded monoidal category, with unit map η_B of degree a. A *Frobenius module* M is just a module object for B (when viewed as an algebra object), where the multiplication map $p_M \colon B \otimes M \to M$ has degree $-a$.

We call M a Frobenius module (rather than just a module) to emphasize some of the other structures it possesses. For example, using the self-adjunction of B one can construct a comultiplication map $M \to B \otimes M$, and one can verify coassociativity and the counit axiom. Note that the unit axiom states that

$$p_M \circ (\eta_B \otimes \mathrm{id}_M) = \mathrm{id}_M . \tag{3.15}$$

Clearly B is a Frobenius module over itself. The example above indicates that B_w is a Frobenius module over B_s whenever $sw < w$. The unit axiom implies in general that $M(-a)$ is a direct summand of $B \otimes M$, with projection map p_M and inclusion map $\eta_B \otimes \mathrm{id}_M$.

Now we explain why one expects many Frobenius modules over a distinguished involution. One key property of distinguished involutions is that they function as the identity element of Lusztig's J-ring. Let us unravel this statement for the non-expert. In characteristic zero, multiplicities in direct sum decompositions are determined by the behavior of the KL basis in the Hecke algebra. Whenever w, x, y are all in cell λ, the minimal integer k for which $B_w(k)$ is a direct summand of $B_x \otimes B_y$ satisfies $k \geq -\mathbf{a}(\lambda)$. Moreover, this minimal shift can be attained using distinguished involutions. If d is the distinguished involution in the same right cell as w, then

$$B_w(-\mathbf{a}(\lambda)) \overset{\oplus}{\subset} B_d \otimes B_w, \tag{3.16}$$

that is, $B_w(-\mathbf{a}(\lambda))$ appears as a direct summand (with multiplicity 1) inside $B_d \otimes B_w$. Similarly, if d' is the distinguished involution in the same left cell as w, then $B_w(-\mathbf{a}(\lambda))$ appears as a direct summand (with multiplicity 1) inside $B_w \otimes B'_d$.

Now thinking with morphisms, the existence of this special direct summand (3.16) implies the existence of a projection map $p_w \colon B_d \otimes B_w \to B_w$ of degree $-\mathbf{a}(\lambda)$. The lack of other summands in this degree implies that the morphism space in which p_w lives is one-dimensional modulo $\mathcal{I}_{<\lambda}$. Assume that B_w is a Frobenius algebra object. The composition $p_w \circ (\eta_d \otimes \mathrm{id}_w)$ is some scalar multiple of id_w, and if this scalar is nonzero we can rescale p_w

to assume that the unit axiom holds. Roughly speaking, this implies B_w is a Frobenius module over B_d (see the next remark). If $p_w \circ (\eta_d \otimes \mathrm{id}_w) = 0$, we would be out of luck; we conjecture below that this never happens.

Remark 3.9. The morphism space $B_d \otimes B_d \otimes B_w \to B_w$ in degree $-2\mathbf{a}(\lambda)$ is also one-dimensional modulo $\mathcal{I}_{<\lambda}$, so that associativity must hold up to scalar, modulo $\mathcal{I}_{<\lambda}$. If the unit axiom holds, then by composing both sides of the associativity relation with the unit $\eta_d \otimes \mathrm{id}_d \otimes \mathrm{id}_w$, we see that the scalar must be 1. So the unit axiom implies the associativity axiom modulo $\mathcal{I}_{<\lambda}$, and B_w is a Frobenius module over B_d in this quotient category. In all examples we have computed, associativity holds on the nose.

The following is only a slight elaboration upon [5, Conjecture 4.40].

Conjecture 3.10. *Let W be a finite Coxeter group, and d a distinguished involution in cell λ. Not only is B_d a graded Frobenius algebra object of degree $\mathbf{a}(\lambda)$ in $\mathcal{H}^0$, but the unit map η_d satisfies the property that $\eta_d \otimes \mathrm{id}_w$ is the inclusion map of a direct summand whenever w is in the same right cell as d. Moreover, B_w is a Frobenius module over B_d.*

Dream 3.11. *Let W be a Weyl group. Let d be a p-distinguished involution (for some characteristic p, possibly zero) for which $\mathrm{val}(h_d^1) = -\mathrm{val}(\mu_{d,d}^d) =: a$. Not only is C_d a graded Frobenius algebra object of degree a, but the unit map η_d satisfies the property that $\eta_d \otimes \mathrm{id}_w$ is the inclusion map of a direct summand whenever w is in the same right p-cell as d. Moreover, C_w is a Frobenius module over C_d.*

By direct inspection, one can verify Dream 3.11 in type C_2. We leave this to the reader, the only interesting remaining case being $d = sts$ and $w = st$ for $p = 2$. A statement closely related to Conjecture 3.10 is proven in [5, Proposition 4.38] in type A.

3.7. Frobenius units and eigenmaps

Finally, let us look again at the eigenmaps constructed in (3.11). In this discussion, p is arbitrary (possibly zero). For each cell λ, the objects appearing in the full twist in homological degree $2\mathbf{c}^p(\lambda)$ and cell λ are precisely the p-distinguished involutions with a particular degree shift, though there are other summands in lower cells. Consequently, any chain map from $\mathbb{1}(2\mathbf{x}^p(\lambda))[2\mathbf{c}^p(\lambda)]$ to FT will consist of some multiple of the Frobenius unit η_d mapping to C_d, as well as unspecified maps to the summands in lower cells. Crucially, we can hope that the scalar on each Frobenius unit is nonzero.

Dream 3.12. *Fix a p-cell λ, and assume for each p-distinguished involution therein that* $\mathrm{val}(h_d^1) = -\mathrm{val}(\mu_{d,d}^d) =: a$*, and that C_d is a Frobenius algebra object of degree a. Consider the homological degree $2\mathbf{c}^p(\lambda)$ inside the minimal complex of the full twist. The objects which appear in this degree in cell λ are precisely the p-distinguished involutions with grading shift $2\mathbf{x}^p(\lambda) + a$. Moreover, there is some chain map from $\mathbb{1}(2\mathbf{x}^p(\lambda))[2\mathbf{c}^p(\lambda)]$ to* FT *which agrees modulo lower cells with a sum of Frobenius units multiplied by nonzero scalars. We denote this chain map α_λ.*

Amazingly, the maps α_λ constructed in (3.11) above do not involve any terms going to lower cells (or scalars on the Frobenius units, though this is a matter of normalization). Instead, it seems that $\sum \eta_d$ is actually a chain map, landing in the kernel of the differential on FT! We know of no reason why the sum of Frobenius units should be a chain map, but this happens in all cases we have computed. Again, we do not yet have enough evidence to state the below as a conjecture, though it is not necessary for any further arguments.

Dream 3.13. *Continuing Dream 3.12, the map from $\mathbb{1}(2\mathbf{x}^p(\lambda))[2\mathbf{c}^p(\lambda)]$ to* FT *which is the sum of the Frobenius units is actually a chain map.*

Now we have a useful technical result for establishing that a candidate is an eigenmap. Though we require Dream 3.11, we do not require the part of that Dream which states that C_w is a Frobenius module.

Proposition 3.14. *Assume that Dream 3.12 and Dream 3.11 and Conjecture 1.17 all hold. Then α_λ is a λ-eigenmap.*

Proof. We need to show that, for any $w \in \lambda$, $\alpha_\lambda \otimes C_w$ is an isomorphism modulo $\mathcal{I}^p_{<\lambda}$. For this purpose we work in the quotient $\mathcal{H}^p/\mathcal{I}^p_{<\lambda}$. In this quotient, the minimal complex of the full twist is supported in degrees $\geq \mathbf{c}^p(\lambda)$, so we are examining the leftmost nonzero degree.

Let ${}^i\,\mathrm{FT}$ denote the i-th chain object in the minimal complex of FT. For the sake of our argument we wish to distinguish between several complexes:

- $X = \mathrm{FT} \otimes C_w$, where the i-th chain object is described as ${}^i\,\mathrm{FT} \otimes C_w$.
- $Y = \mathrm{FT} \otimes C_w$, where the i-th chain object is described as a direct sum of indecomposable objects, and
- Z, the minimal complex of $\mathrm{FT} \otimes C_w$. By Conjecture 1.17, Z is just a single copy of $C_w(2\mathbf{x}^p(\lambda))$ in homological degree $2\mathbf{c}^p(\lambda)$.

Of course X is isomorphic to Y and homotopy equivalent to Z, but it helps to keep track of these isomorphisms and homotopy equivalences explicitly. The map $\rho\colon X \to Y$ is a sum of projection maps for some choice of decomposition. The map $\chi\colon Y \to Z$ is the map induced by Gaussian elimination of contractible summands.

Example 3.15. Consider the cell λ_{big} in characteristic zero. Taking the quotient by lower cells, FT is supported in degrees ≥ 2, see (3.7). Let $w = st$. Then

$$\begin{gathered} {}^2X = B_s B_{st}(1) \oplus B_t B_{st}(1), \\ {}^2Y = B_{st}(0) \oplus B_{st}(2) \oplus B_{tst}(1) \oplus B_t(1), \\ {}^2Z = B_{st}(0). \end{gathered} \tag{3.17}$$

The crucial fact about Gaussian elimination that we use is the following (see [2, Exercise 5.17] for slightly more detail on this well-known tool). A single application of Gaussian elimination might replace

$$Y = (\cdots \to C \oplus D \to C \oplus E \to \dots)$$

(where the differential induces an isomorphism $C \to C$) with

$$Z = (\cdots \to D \to E \to \dots)\,.$$

In the homotopy equivalence $Y \to Z$, the map from $C \oplus E \to E$ is not the naive projection $\begin{pmatrix} 0 & \mathrm{id}_E \end{pmatrix}$, but involves a nontrivial map $C \to E$. However, the map $C \oplus D \to D$ is the naive projection $\begin{pmatrix} 0 & \mathrm{id}_D \end{pmatrix}$! In particular, in the leftmost nonzero degree of any complex, Gaussian elimination will induce the naive projection map.

The chain map $\chi \circ \rho \circ (\alpha_\lambda \otimes \mathrm{id}_w)$ is supported in a single homological degree, where it is the composition

$$\begin{aligned} C_w(2\mathbf{x}^p(\lambda)) &\to \bigoplus_{d'} C_{d'} \otimes C_w(2\mathbf{x}^p(\lambda) + a) \\ &\to C_w(2\mathbf{x}^p(\lambda)) \oplus \bigoplus \text{other terms} \to C_w(2\mathbf{x}^p(\lambda)). \end{aligned} \tag{3.18}$$

The first sum is over all distinguished involutions d' in the two-sided cell λ. The final map is the naive projection to one summand, and that summand only appears inside $C_d \otimes C_w$ when d is the unique distinguished involution in the same right cell as w. So ρ and χ compose to projection to this one summand inside $C_d \otimes C_w$. Hence the entire composition $\chi \circ \rho \circ (\alpha_\lambda \otimes \mathrm{id}_w)$ is equal in this homological degree to

$$C_w(2\mathbf{x}^p(\lambda)) \to C_d \otimes C_w(2\mathbf{x}^p(\lambda) + a) \to C_w(2\mathbf{x}^p(\lambda)). \tag{3.19}$$

The first map is a nonzero scalar multiple of $\eta_d \otimes \mathrm{id}_w$, the component of α_λ mapping to $C_d \otimes C_w$. The second map is the projection p_w, which is uniquely-defined up to a scalar modulo $\mathcal{I}^p_{<\lambda}$, regardless of the choice of decomposition (as noted previously, this morphism space is one-dimensional in the quotient). By Dream 3.11, the composition (3.19) is a nonzero multiple of id_w. Thus $\alpha_\lambda \otimes C_w$ is an isomorphism modulo $\mathcal{I}^p_{<\lambda}$, as desired. □

Remark 3.16. The assumption that Dream 3.11 holds is motivational, but could probably be removed by the following method. One would argue that the property that $\alpha_\lambda \otimes C_w$ is an isomorphism (modulo $\mathcal{I}^p_{<\lambda}$) is constant over right p-cells. Suppose that $\alpha_\lambda \otimes \mathrm{id}_{B_W} : C_w \to \mathrm{FT} \otimes C_w$ is an isomorphism (modulo lower terms, ignoring shifts). Now tensor on the right with C_x, to obtain an isomorphism $C_w C_x \to \mathrm{FT} \otimes C_w C_x$. Taking direct summands of both sides (using an idempotent which commutes with $\alpha_\lambda \otimes \mathrm{id}_{C_w C_x}$), $C_y \to \mathrm{FT} \otimes C_y$ is an isomorphism too. In this fashion, one could reduce the proof that α_λ is an eigenmap to the case when w is a p-distinguished involution, where it follows from the unit axiom as in the proof above.

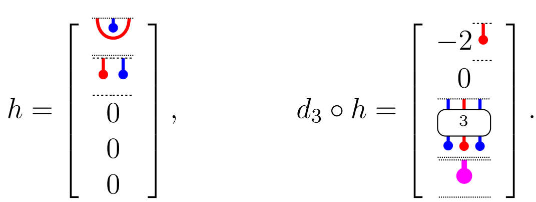
This argument can also apply to left p-cells, provided one could prove that the full twist is in the Drinfeld center, and the eigenmaps are morphisms in the Drinfeld center. Then one can left multiply $\mathrm{FT} \otimes C_w$ by C_x, and obtain something canonically isomorphic to $\mathrm{FT} \otimes C_x C_w$, compatibly with the map α_λ.

Corollary 3.17. *Theorem 1.26, stating the existence of eigenmaps in type* C_2 *for* $p = 2$, *holds true.*

Proof. By construction, the chain maps of (3.11) satisfy Dream 3.13. The other dreams and conjectures have been verified in type C_2 earlier. Thus Proposition 3.14 finishes the proof. □

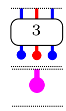
3.8. The s-eigenmap and 2-torsion

Let us consider the new eigenmap α_s in more detail. One can compute that the space of all chain maps $\mathbb{1}(0)[4] \to \mathrm{FT}$ modulo homotopy is the $\mathbb{Z}$-span of α_s. There is a homotopy, a non-chain map $\mathbb{1}(0)[3] \to \mathrm{FT}$ of the form

(3.20) 

In particular, $2\alpha_s$ is homotopic to

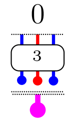
(3.21)
$$\varphi := \begin{bmatrix} 0 \\ 0 \\ \ast \end{bmatrix}.$$



Since φ is built from morphisms factoring through objects in $< \lambda_s$, φ is inside $\mathcal{I}^p_{<\lambda_s}$. However, α_s is nonzero in the quotient by $\mathcal{I}^p_{<\lambda_s}$, since it is a λ_s-eigenmap. Thus the subspace of morphisms in $\mathcal{I}^p_{<\lambda_s}$ has index 2.

Now, when 2 is inverted, we can work instead with the version of FT found in (3.7). One can confirm that the space of chain maps modulo homotopy $\mathbb{1}(0)[4] \to \mathrm{FT}$ is generated by the image of φ composed with the Gaussian elimination homotopy equivalence, which we will also abusively call φ:

$$\varphi = \begin{bmatrix} 0 \\ \ast \end{bmatrix}.$$



From the perspective of (3.7), it is surprising that $\frac{\varphi}{2}$ has intrinsic meaning.

3.9. Methods of computation

We typically combined two methods to perform computations most effectively.

The first method is direct Gaussian elimination. For the computation of HT, suppose we have computed already that (c.f. (3.1) for the differentials)

(3.22a)
$$F_{tst} = F_t F_s F_t \cong \left(\begin{tikzcd} & B_{st}(1) \arrow[r, "d_2"] \arrow[dr] & B_s(2) \arrow[dr] & \\ B_{tst}(0) \arrow[ur] \arrow[dr] & \oplus & \oplus & \mathbb{1}(3) \\ & B_{ts}(1) \arrow[r] \arrow[ur] & B_t(2) \arrow[ur] & \end{tikzcd} \right).$$

Note that, like all Rouquier complexes for reduced expressions, this appears up to a shift as a subcomplex of HT itself.

Now we compute $F_{tst} \otimes B_s$ by Gaussian elimination, postponing the question of why until after the computation.

(3.22b)

$$F_{tst} \otimes B_s \cong \left(\begin{tikzcd} & B_sB_tB_s(1) \arrow[r, "d_2"] \arrow[dr] & B_sB_s(2) \arrow[dr] & \\ B_{tst}B_s(0) \arrow[ur] \arrow[dr] & \oplus & \oplus & B_s(3) \\ & B_{ts}B_s(1) \arrow[r] \arrow[ur] & B_{ts}(2) \arrow[ur] & \end{tikzcd} \right).$$

The map gives a splitting of the map in the last differential, allowing us to eliminate $B_s(3)$ with one summand of $B_sB_s(2)$. Having done this, the remaining differential from $B_sB_tB_s(1)$ to the other summand $B_s(1)$ is the original differential to $B_sB_s(2)$, composed with the projection map from $B_sB_s(2)$ to $B_s(1)$, see (2.32).

(3.22c)

$$\left(\begin{tikzcd} & B_sB_tB_s(1) \arrow[r, "-"] \arrow[dr] & B_s(1) \\ B_{tst}B_s(0) \arrow[ur] \arrow[dr] & \oplus & \oplus \\ & B_{ts}B_s(1) \arrow[r] \arrow[ur] & B_{ts}(2) \end{tikzcd} \right).$$

The sign on the pitchfork can be removed later (up to isomorphism of complexes) by multiplying $B_s(1)$ by -1.

Similarly, using we can eliminate $B_{ts}(2)$ with one summand of $B_{ts}B_s(1)$. In theory this could modify the surviving differential from $B_sB_tB_s(1)$ to $B_s(1)$ by subtracting a *zigzag term*, the (differential-splitting-differential) composition $B_sB_tB_s(1) \to B_{ts}(2) \to B_{ts}(2) \to B_s(1)$. One can compute that this zigzag term is

$$= 0.$$

To cancel the summand $B_{ts}(0)$ inside $B_{ts}B_s(1)$ with the summand $B_{ts}(0)$ inside $B_{tst}B_s(0)$, we can use the splitting given by the map i_{ts} from (2.35). Again, one can compute that no zigzag term will affect the differential $B_{w_0}(0) \to B_sB_tB_s(1)$ that remains. So we compute the minimal complex

(3.22d)

$$F_{tst} \otimes B_s \cong \left(\underline{B}_{w_0}(0) \longrightarrow B_sB_tB_s(1) \longrightarrow B_s(1) \right).$$

On the other hand, suppose we do the computation more lazily, only keeping track of enough information to ensure that the terms we expect to

Gaussian eliminate will be connected by isomorphisms (rather than zero, see the warning in [8, Exercise 19.14]). We will end up with a complex of the form

$$\left(\underline{B}_{w_0}(0) \to B_s B_t B_s(1) \to B_s(1)\right),$$

but where we do not know the differentials. However, we can easily rederive the differentials from first principles! Both $\mathrm{Hom}(B_{w_0}, B_s B_t B_s(1))$ and $\mathrm{Hom}(B_s B_t B_s(1), B_s(1))$ are free $\mathbb{Z}$-modules of rank 1. Thus our differentials agree with those of (3.22d) up to scalars $a_i \in \mathbb{Z}$.

$$F_{tst} \otimes B_s \cong \left(\underline{B}_{w_0}(0) \xrightarrow{a_1} B_s B_t B_s(1) \xrightarrow{a_2} B_s(1) \right). \tag{3.23}$$

Suppose e.g. that $a_2 = 0$. Then the complex $F_{tst} B_s$ would be decomposable in the homotopy category. However, B_s is indecomposable and $F_{tst} \otimes (-)$ is an invertible functor on the homotopy category, so $F_{tst} B_s$ must be indecomposable, a contradiction. Similarly, suppose that $a_2 \neq \pm 1$. Then after specialization to some finite characteristic, $F_{tst} B_s$ would be decomposable. This is again a contradiction, since F_{tst} is invertible over any field. Thus $a_1 = \pm 1$ and $a_2 = \pm 1$. Using isomorphisms which scale the chain objects, we can assume $a_1 = a_2 = 1$. This method also has the added bonus of proving a uniqueness statement: (3.22d) is the unique absolutely indecomposable complex with the same underlying chain objects, up to isomorphism.

What we wished to compute was $\mathrm{HT} = F_{tst} F_s$. Note that F_s is the cone of a map $B_s \to \mathbb{1}(1)$, so HT is the cone of a chain map $F_{tst} B_s \to F_{tst}(1)$. We have already computed $F_{tst} B_s$ and $F_{tst}(1)$ above. Instead of carefully computing what this chain map is, we take the "lazy" approach of computing all possible chain maps. One can compute directly that there is a three-dimensional space (rather, a free rank 3 $\mathbb{Z}$-module) of chain maps $F_{tst} B_s \to F_{tst}(1)$, of which a two-dimensional subspace is nulhomotopic. Homotopic chain maps give rise to homotopy-equivalent cones, which have isomorphic minimal complexes. A quick way to compute the space of possible chain maps modulo homotopy is to first use the homotopies to assert that certain coefficients of the chain map are zero. Then one computes the remaining constraints on the chain map.

For example, let us compute how chain maps from $F_{tst} B_s$ to $F_{tst}(1)$ behave in homological degree 2, where one has a map from $B_s(1)$ to $B_s(3) \oplus B_t(3)$. The space of maps $B_s(1) \to B_t(3)$ is one-dimensional, spanned by . The homotopy produces a nulhomotopic chain map, whose coefficient of in homological degree 2 is 1. By adding a multiple of this nulhomotopic chain map to any chain map, we can assert that the map $B_s(1) \to B_t(3)$ is zero. What remains in degree 2 is some morphism $B_s(1) \to B_s(3)$, which must be killed by the differential  on the target to be a chain map. There is only a one-dimensional space of such morphisms, spanned by the triangle map .

Using these methods one computes that any chain map $F_{tst}B_s \to F_{tst}(1)$ is, up to homotopy, a scalar $a \in \mathbb{Z}$ times the chain map whose cone yields the complex in (3.1c). Once again, $a = \pm 1$ is the only choice for which the cone of this map is absolutely indecomposable, which HT must be. Up to isomorphism of complexes, we can assume $a = 1$.

Let us call the method above the *uniqueness method*. Alternatively, we could have used the *(explicit) Gaussian elimination method* to compute $F_{tst} \otimes F_s$ directly. There would be exactly the same three cancellation moves as in $F_{tst} \otimes B_s$: one term $B_s(3)$ would cancel a summand of $B_sB_s(2)$, etcetera. However, each cancellation would produce many more zigzag terms, involving differentials going from the $F_{tst}B_s$ half of the complex to the $F_{tst}(1)$ half of the complex. For example, the cancellation of the summands $B_s(3)$ produced no zigzag terms in $F_{tst}B_s$ for trivial reasons, because there were no surviving terms in homological degree 3. The analogous cancellation in $F_{tst}F_s$ does produce a zigzag term, and keeping track of this might be inconvenient.

To emphasize the difference between these two methods, the uniqueness method seems like it has to do more work because it computes the space of all possible chain maps, but it does so from a small complex to a large complex. The Gaussian elimination method will simplify a large complex to a small complex, while keeping track of extra zigzags to another large complex, and this becomes quite painful. The only real advantage of the Gaussian elimination method is that it does not require first computing $F_{tst} \otimes B_s$.

However, computing $F_{tst}B_s$ was not a waste of time. For example, we can now immediately compute $\mathrm{HT} \otimes B_s$, since $\mathrm{HT} \cong F_tF_sF_tF_s$ and $F_sB_s \cong B_s(-1)$. Thus $\mathrm{HT}\, B_s \cong F_{tst}B_s(-1)$ is a three-term complex. Tensoring this with B_t, we can now apply Gaussian elimination to cancel a copy of $B_{st}(0)$ in homological degrees 1 and 2, and a copy of $B_{w_0}(0)$ in degrees 0 and 1. This can be done explicitly using the decompositions in (2.34) and (2.36), see [7] for details. The result is a two-term complex

$$\mathrm{HT} \otimes B_{st} \cong \left(\underline{B}_{w_0}(-2) \to B_{st}(0) \right).$$

From here it is easy to compute the complex $\mathrm{HT} \otimes (B_sB_tB_s)$ as well.

Our preferred method of computing $\mathrm{HT} \otimes B_t$ is to first compute F_{sts} and then $F_{sts}B_t$, another big calculation with Gaussian elimination like the computation of $F_{tst}B_s$ was. All that remains at the end is a simple two-term complex $(\underline{B}_{w_0}(-1) \to B_{tst}(0))$. Because almost everything eventually is eliminated, one need not keep careful track of most differentials (one must just ensure that one does not make an erroneous cancellation), and can use an easy uniqueness argument at the end to determine the differential. From here it is straightforward to compute $\mathrm{HT} \otimes B_{ts}$ and $\mathrm{HT} \otimes B_{tst}$ as well.

Finally, we need to compute the full twist, which we accomplish by computing $\mathrm{HT} \otimes F_w$ starting with $w = 1$ (already done), and steadily increasing the length of w. To begin, $\mathrm{HT} \otimes F_t$ is the cone of a map $\mathrm{HT}\, B_t \to \mathrm{HT}(1)$, from a simple two-term complex to a complicated complex. Computing all the chain maps (modulo homotopy, but there are no homotopies in this example) is relatively easy, using the morphisms and relations given in §2, such as (2.30). For the computation of $\mathrm{HT} \otimes F_s$, finding all the chain maps from $\mathrm{HT} \otimes B_s$ to $\mathrm{HT}(1)$ is somewhat more of an ordeal, but the alternative (careful Gaussian elimination with an extra space of differentials to $\mathrm{HT}(1)$) is worse.

Now suppose we wish to compute $\mathrm{HT} \otimes F_{ts}$. We know this is the cone of a chain map from $\mathrm{HT} \otimes B_{ts}$ to $\mathrm{HT} \otimes (B_s(1) \oplus B_t(1) \to R(2))$. This latter tensor product we already know by our computations above, because it has both $\mathrm{HT} \otimes F_s$ and $\mathrm{HT} \otimes F_t$ as subcomplexes. Now we compute the space of all chain maps from the two-term complex $\mathrm{HT} \otimes B_{ts}$ to the large complex $\mathrm{HT} \otimes (B_s(1) \oplus B_t(1) \to R(2))$, which is not as terrible as it seems. The complex $\mathrm{HT} \otimes (B_s(1) \oplus B_t(1) \to R(2))$ may be quite large, but only homological degrees between 0 and 2 are relevant for computing chain maps from the two-term complex $\mathrm{HT} \otimes B_{ts}$.

Continuing, we compute $\mathrm{HT} \otimes F_w$ for all $w \in W$, until we finally find our formula for the full twist. At each step, the result was unique up to homotopy equivalence, assuming the indecomposability of the result over any field. The computation of $\mathrm{HT} \otimes B_s B_t B_s$ is by far the nastiest part of this process, as the nontrivial automorphisms of $B_s B_t B_s$ lead to an extra degree of freedom (this nastiness is avoided in characteristic zero). Finally, one additional trick was used: that $\mathrm{HT} \otimes F_w \otimes B_{w_0}$ is indecomposable, which requires the differential from the lowest degree to have enough nonzero components to be able to cancel all extra copies of B_{w_0}.

Details on these computations can be found in [7].

4. Type C_3

Let us discuss the case where W has type C_3 and $p = 2$. The simple reflections are denoted $\{s, t, u\}$, with $m_{st} = 3$ and $m_{tu} = 4$. The longest element is denoted w_0 as usual.

4.1. p-cells and their eigenvalues

The p-cells in this example can be found[8] in [16, p16], and also are close at hand in the appendix, under labelling $(s, t, u) = (1, 2, 3)$. Let μ denote the (two-sided[9]) 0-cell containing sts, and ν the 0-cell containing $stsuts$. Although there are eleven elements $w \in W$ for which $b_w \neq c_w$, every 0-cell aside from μ and ν is also a p-cell, with $\mathbf{x}$ and $\mathbf{c}$ and Schu_L unchanged. Something quite fascinating occurs with μ and ν.

For the computation below, the following elements of the p-canonical basis play a role.

$$c_{sts} = b_{sts}. \tag{4.1a}$$

$$c_{stsuts} = b_{stsuts} + (v + v^{-1}) b_{sts}. \tag{4.1b}$$

$$c_{w_0 u} = b_{w_0 u} + b_{stsuts}. \tag{4.1c}$$

$$c_{w_0} = b_{w_0}. \tag{4.1d}$$

Note that c_{stsuts} is the "first" example of a *non-perverse* p-canonical basis element, meaning that its coefficients in the ordinary KL basis are Laurent polynomials rather than integers.

The two 0-cells μ and ν divide into three p-cells as follows.

$$\lambda_{sts} = \{sts\}, \qquad \mu' = \mu \setminus \{sts\} \cup \{stsuts\}, \qquad \nu' = \nu \setminus \{stsuts\}. \tag{4.2}$$

Thus the p-cells do not form a partition of the 0-cells.

---

[8] Technically, [16, p16] describes the right p-cells. Using symmetry one can determine the left p-cells and then the two-sided p-cells. There are a couple typos to be found on [16, p16]. The elements 21 and 23 should be swapped in the partition of C_2 into p-cells. The last element in C_7 should read 21213212.

[9] The 0-cell μ is the union of the right 0-cells denoted C_6, C_9 and C_{11} on [16, p16], and ν is the union of C_7, C_{10}, and C_{12}.

We compute[10] the action of the half twist. On the ordinary KL basis we have

$$\mathrm{ht}_W \cdot b_{sts} = v^{-3}b_{w_0} - v^{-2}b_{w_0u} + v^{-1}b_{sts}, \tag{4.3a}$$

$$\mathrm{ht}_W \cdot b_{stsuts} = v^{-6}b_{w_0} - v^{-3}b_{stsuts}, \tag{4.3b}$$

from which one easily computes that

$$\mathrm{ht}_W \cdot c_{sts} = v^{-3}c_{w_0} - v^{-2}c_{w_0u} + v^{-2}c_{stsuts} - v^{-3}c_{sts}, \tag{4.3c}$$

$$\mathrm{ht}_W \cdot c_{stsuts} = (v^{-2} + v^{-4} + v^{-6})c_{w_0} - (v^{-1} + v^{-3})c_{w_0u} + v^{-1}c_{stsuts}. \tag{4.3d}$$

As a consequence we have

$$\mathrm{ht}_W \cdot b_{sts} \equiv v^{-1}b_{sts} + I_{<\mu}, \tag{4.4a}$$

$$\mathrm{ht}_W \cdot c_{sts} \equiv -v^{-3}c_{sts} + I_{<\lambda_{sts}}, \tag{4.4b}$$

and

$$\mathrm{ht}_W \cdot b_{stsuts} \equiv -v^{-3}b_{stsuts} + I_{<\nu}, \tag{4.4c}$$

$$\mathrm{ht}_W \cdot c_{stsuts} \equiv v^{-1}c_{stsuts} + I_{<\mu'}. \tag{4.4d}$$

As you can see, the eigenvalues of sts and $stsuts$ were swapped!

Conjecture 1.5 holds, consistent with the values

$$\begin{aligned} \mathbf{x}^p(\lambda_{sts}) &= -3, \mathbf{c}^p(\lambda_{sts}) = 3, \\ \mathbf{x}^p(\mu') &= -1, \mathbf{c}^p(\mu') = 2, \\ \mathbf{x}^p(\nu') &= -3, \mathbf{c}^p(\nu') = 1. \end{aligned} \tag{4.5}$$

Note that Schu_L^p and Schu_L agree, and both are the identity map outside of ν'.

4.2. Musings on the categorification: guessing the minimal complexes

Now we discuss how we expect things to play out in the categorification, with an attempt to understand why sts and $stsuts$ swapped eigenvalues.

[10] The following computations were done by computer, with thanks to Joel Gibson and Geordie Williamson.

Everything below should be considered as an educated guess, or as the ravings of a soothsayer. In characteristic zero, Conjecture 1.16 has still not been proven in type C_3, but we assume it for sanity later in this discussion.

Our arguments will compare characteristic zero and characteristic p, by working with the Hecke category over $\mathbb{Z}$. By [10, 11], morphism spaces in the Hecke category are free $\mathbb{Z}$-modules, whose graded rank is determined by the Soergel Hom formula. All assertions about the size of Hom spaces stem from this fact. The only prime for which the p-canonical basis differs from the KL basis in type C_3 is $p = 2$. Thus we expect the decomposition of bimodules over $\mathbb{Z}$ to mimic that in characteristic 2, with no additional subtleties. We write C_x for an indecomposable bimodule over $\mathbb{Z}$, the top summand of its Bott-Samelson bimodule, which categorifies c_x.

The fact that $c_{w_0u} = b_{w_0u} + b_{stsuts}$ should be interpreted as follows. From the Soergel Hom formula there are maps

$$q\colon C_{w_0u} \to C_{stsuts}, \qquad j\colon C_{stsuts} \to C_{w_0u}$$

which span their respective Hom spaces in degree zero. If $e := j \circ q$ then $e^2 = 2e$. After inverting 2, $\frac{e}{2}$ is the idempotent projecting to the common summand B_{stsuts}. The complementary summand in C_{w_0u} is B_{w_0u}. Meanwhile, in characteristic 2, e is nilpotent and C_{w_0u} is indecomposable, with "the extra copy of B_{stsuts} still stuck on." The maps q and j are now in the Jacobson radical of the category.

In similar fashion, $c_{stsuts} = b_{stsuts} + (v + v^{-1})b_{sts}$ indicates the existence of maps $p_{\pm 1}\colon C_{stsuts} \to C_{sts}$ of degree ± 1, and maps $i_{\pm 1}$ in the opposite direction, which project to the common summands $B_{sts}(\pm 1)$ up to multiplication by 2.

In characteristic not equal to 2 we expect the following minimal complex:

$$\mathrm{HT} \otimes B_{sts} \cong \left(\underline{B_{w_0}}(-3) \to B_{w_0u}(-2) \to B_{sts}(-1)\right). \tag{4.6a}$$

In characteristic 2 or over $\mathbb{Z}$ we expect

$$\mathrm{HT} \otimes C_{sts} \cong \left(\underline{C_{w_0}}(-3) \to C_{w_0u}(-2) \to C_{stsuts}(-2) \to C_{sts}(-3)\right). \tag{4.6b}$$

We explain these expectations below, but first we discuss the differentials. All differentials live in a Hom space which has rank 1 over $\mathbb{Z}$. Choosing a generator of each Hom space as a $\mathbb{Z}$-module, we obtain the complex above. Note that $d^2 = 0$ simply because it lives in a zero Hom space (e.g. there are no morphisms of degree 2 from B_{w_0} to B_{sts}). The second differential in

(4.6b) is the map q discussed above, and the third differential is p_{-1}. After specialization to characteristic 2, the differentials live in the Jacobson radical.

That $\mathrm{HT} \otimes B_{sts}$ should be built from $B_{w_0}(-3)$ and $B_{sts}(-1)$ in even homological degree, and $B_{w_0u}(-2)$ in odd homological degree, is expected from (4.3a). One might worry that the minimal complex of $\mathrm{HT} \otimes B_{sts}$ might have additional chain objects which are invisible in the Grothendieck group. For example, some $B_w(k)$ might appear in two different homological degrees, one even and one odd. We call this a *cancellation phenomenon.* The cancellation phenomenon *does* occur in finite characteristic, see (3.9), and is a legitimate worry. In characteristic zero it was proven [9] that there is no cancellation in Rouquier complexes which are positive lifts (like HT), because they are perverse. No analogous result has been proven for complexes like $\mathrm{HT} \otimes B_w$, but it is still expected that the chain objects in the minimal complex have no cancellation in characteristic zero.

Lemma 4.1. *In the absence of any cancellation phenomenon,* $\mathrm{HT} \otimes B_{sts}$ *must have the form given in* (4.6a)*, with the differentials stated (up to isomorphism).*

Proof. First we claim that the choice of differentials in a complex of the form (4.6a) is unique up to isomorphism, given that $\mathrm{HT} \otimes B_{sts}$ is absolutely indecomposable. In previous chapters we explained the reasoning, which we briefly recall. The complex HT is invertible so it preserves indecomposable objects in the homotopy category. This remains true after any specialization. If a differential did not generate its Hom space, then it becomes zero in some specialization, and the complex splits nontrivially. There is only one generator of a free rank 1 module up to sign, and we can rescale the chain objects by signs to modify the generators at will.

The only non-zero morphism spaces between the objects $B_{w_0}(-3)$ and $B_{w_0u}(-2)$ and $B_{sts}(-1)$ are those indicated by the arrows in (4.6a). Had one tried to build a complex with these objects in a different order, then some differential is zero. Thus (4.6a) is the unique way to glue these three objects together into an indecomposable complex, up to an overall homological shift. The overall homological shift is determined by considering the homology of this complex in the category of (R, R)-bimodules. The homology of HT is known to be concentrated in degree zero, and thus the same holds for $\mathrm{HT} \otimes B_{sts}$. □

With (4.6a) as a given, let us try to find a complex well-defined over $\mathbb{Z}$ which is homotopy equivalent to (4.6a) after inverting 2. In homological degree 1 we must have $C_{w_0u}(-2)$ instead of $B_{w_0u}(-2)$, because the latter is not defined over $\mathbb{Z}$. This contributes an extra summand $B_{stsuts}(-2)$ after inverting 2, which must be cancelled by some copy of $B_{stsuts}(-2)$ in homological degree 2 or 0. Over $\mathbb{Z}$, this cancelling copy can come from either $C_{stsuts}(-2)$ or $C_{w_0u}(-2)$. Having another copy of $C_{w_0u}(-2)$ is unlikely: in characteristic zero it would produce another copy of $B_{w_0u}(-2)$, needing to be cancelled by yet another copy of $C_{w_0u}(-2)$, which just repeats the same problem again. So we expect some copy of $C_{stsuts}(-2)$ in homological degree 0 or 2. We do not expect it to appear in homological degree zero, as this would contribute additional copies of B_{sts} in degree zero which would need to be cancelled out. The cases we found unlikely above all lead to a cancellation phenomenon. These arguments are slightly optimistic, and we discuss this optimism in several remarks below.

A copy of $C_{stsuts}(-2)$ in homological degree 2 would account already for the term $B_{sts}(-1)$ in homological degree 2, but would also contribute a copy of $B_{sts}(-3)$. Optimism suggests that it is cancelled in degree 3 rather than degree 1, yielding our guess (4.6b). Note that the differentials are unique up to isomorphism, given the absolute indecomposability of $\mathrm{HT} \otimes C_{sts}$.

Here are some additional comments on this line of reasoning.

Remark 4.2. In the examples we know where the cancellation phenomenon occurs, it was not possible to construct an indecomposable complex without these extra terms; one can not arrange for $d^2 = 0$. The complex in (4.6b) is well-defined without the need for extra terms.

Remark 4.3. In Remarks 1.22 and 1.25 we discuss properties of the half and full twist which follow from our conjectures. Suppose that the minimal complex of HT does not contain C_w in any homological degree less than $\mathbf{c}^p(w)$. For example, only C_{w_0} appears in homological degree zero, and only terms in the bottom two cells λ_0 and ν' can appear in homological degree 1. Since p-cells correspond to monoidal ideals, the same must be true in $\mathrm{HT} \otimes C_{sts}$. This rules out the appearance of C_{stsuts} in degree zero, or C_{sts} in degree 1, etcetera.

Now we consider the other element $stsuts$. In characteristic zero we expect

$$\mathrm{HT} \otimes B_{stsuts} \cong \left(\underline{B}_{w_0}(-6) \to B_{stsuts}(-3)\right). \tag{4.7a}$$

Remembering that C_{stsuts} splits in characteristic zero as $B_{stsuts} \oplus B_{sts}(-1) \oplus B_{sts}(+1)$, we see that $\mathrm{HT} \otimes C_{stsuts}$ should have the same size as (4.7a) plus two copies of (4.6a). Using quantum numbers for grading shifts to save space, we expect

$$\mathrm{HT} \otimes C_{stsuts} \cong \left([3]\underline{C}_{w_0}(-4) \to [2]C_{w_0u}(-2) \to C_{stsuts}(-1)\right), \tag{4.7b}$$

matching (4.3d). Note that $C_{stsuts}(-1)$ contains the [2] copies of $B_{sts}(-1)$ expected to appear in homological degree 2. No copies of C_{sts} may appear, because the monoidal ideal generated by C_{stsuts} does not contain C_{sts} (which is in a higher p-cell). The justifications for guessing (4.7a) and (4.7b) are similar to the arguments above. One difference is that there is no a priori reason why $d^2 = 0$ in (4.7b), and it would not be surprising if there were additional cancelling copies of $C_{w_0}(-4)$ in degrees 0 and 1, needed to force $d^2 = 0$. This would not affect the discussion below.

4.3. Musings on the categorification: eigenvalue swap

So why did c_{stsuts} inherit the eigenvalue of b_{sts}? Let us pose the question more generally. Suppose that $w, x \in W$ (think $x = sts$ and $w = stsuts$) satisfy the property that w is in a strictly lower 0-cell and a lower p-cell than x, and that

$$c_w = b_w + f b_x \tag{4.8}$$

for some nonzero $f \in \mathbb{N}[v, v^{-1}]$. We claim that C_w should inherit the eigenvalue of B_x. For sake of simplicity we assume in the argument below that Schu_L is the identity operator.

Assuming our conjectures hold, $\mathrm{HT} \otimes B_w$ is concentrated in homological degrees $\leq \mathbf{c}(w)$. However, C_w has an extra f copies of B_x, and $\mathrm{HT} \otimes B_x$ has terms appearing in homological degree $\mathbf{c}(x)$ as well. The statistic $\mathbf{c}$ is monotone in the cell order, so $\mathbf{c}(w) < \mathbf{c}(x)$. Thus we expect that, after passage to characteristic zero, $\mathrm{HT} \otimes C_w$ should have $v^{\mathbf{x}(x)} f$ copies of B_x in degree $\mathbf{c}(x)$. However, $\mathrm{HT} \otimes C_w$ can only have terms in cells less than that of w. Thus in the lift over $\mathbb{Z}$, these copies of B_x can not come from copies of C_x. Though other options are theoretically possible, the simplest solution to obtaining $v^{\mathbf{x}(x)} f$ copies of B_x is from one copy of $v^{\mathbf{x}(x)} C_w$. Thus, we expect that $\mathrm{HT} \otimes C_w$ has one copy of C_w in homological degree $\mathbf{c}(x)$ and with grading shift $\mathbf{x}(x)$, giving it the eigenvalue of B_x.

Remark 4.4. This heuristic argument did not depend on the value of f. In particular, it did not matter that c_{stsuts} was not perverse.

Remark 4.5. There are no such pairs (w, x) in any dihedral type.

Now we ask: should C_x inherit the eigenvalue of B_w? The answer is no: in our example, $\mathbf{x}^p(sts) = \mathbf{x}(stsuts)$ but $\mathbf{c}^p(sts) \neq \mathbf{c}(stsuts)$, they only agree in parity. The better question is: should we expect $\mathbf{x}^p(x) = \mathbf{x}(w)$? We do not have a convincing reason this equality should hold. However, it is easier to explain why, in our example, the shift $\mathbf{x}(sts) - \mathbf{x}^p(sts) = 2$ occurred. This seems to be related to the difference between the two summands $(v+v^{-1})b_{sts}$ in the non-perverse basis element c_{stsuts}.

Suppose one has a pair (w,x) as in (4.8) and that the valuation of f is k, i.e. $f = v^k + \ldots + v^{-k}$ for some $k > 0$. We do not require the coefficient of v^k is one, but we will assume this for simplicity. Let q_{-k} be the map[11] of degree $-k$ which projects from C_w to B_x up to scalar (it splits after inverting p).

Assuming our conjectures hold, $\mathrm{HT} \otimes C_x$ should have, in its final homological degree $\mathbf{c}^p(x)$, a copy of $C_x(\mathbf{x}^p(x))$. This is also the unique copy of C_x in the minimal complex of $\mathrm{HT} \otimes C_x$. Suppose one posits that q_{-k} appears in the differential which maps to this final degree (discussion below). The source of q_{-k} would be a copy of $C_w(\mathbf{x}^p(x)+k)$ in degree $\mathbf{c}^p(x)-1$. After inverting p, q_{-k} splits, but $C_w(\mathbf{x}^p(x)+k)$ has another summand $B_x(\mathbf{x}^p(x)+2k)$. Were this term to survive to the minimal complex after p is inverted (discussion below), then since $\mathrm{HT} \otimes B_x$ has a unique copy of B_x in its minimal complex, it must be this one. This situation leads to $\mathbf{c}(x) = \mathbf{c}^p(x) - 1$ and $\mathbf{x}(x) = \mathbf{x}^p(x) + 2k$.

The same heuristic argument can be used to predict in type C_2 that $\mathbf{c}(s) = \mathbf{c}^p(s) - 1$ and $\mathbf{x}(s) = \mathbf{x}^p(s)$. This time, we expect a differential from $C_{sts}(\mathbf{x}(sts))$ in degree $\mathbf{c}^p(s) - 1$ to $C_s(\mathbf{x}^p(sts))$ in degree $\mathbf{c}^p(s)$. The other summand $B_{sts}(\mathbf{x}^p(sts))$ inside $C_{sts}(\mathbf{x}^p(sts))$ survives, and this is the Schützenberger dual of s.

Why should q_{-k}, or other maps which fail to split over $\mathbb{Z}$, appear as differentials in $\mathrm{HT} \otimes C_x$? Before applying any Gaussian elimination, the tensor product of complexes $\mathrm{HT} \otimes C_x$ has many extra terms. In order for Gaussian elimination to remove these terms in characteristic zero, many projection maps have to appear as differentials, such as q_{-k}. These happen not to split over $\mathbb{Z}$, so they survive as differentials.

Why should $B_x(\mathbf{x}^p(x) + 2k)$ in degree $\mathbf{c}^p(x) - 1$ survive to the minimal complex in $\mathrm{HT} \otimes C_x$ after p is inverted? It need not.

Example 4.6. In type C_4, number the simple reflections so that $\{s_1, s_2, s_3\}$ generate a parabolic subgroup of type A_3. Let $x = s_1 s_2 s_1 s_3 s_2 s_1$ be the longest element of this parabolic subgroup, and let $w = x s_4 s_3 s_2 s_1$. We have $c_w = b_w + (v^2 + 1 + v^{-2}) b_x$ and $c_x = b_x$. We have $\mathbf{x}^p(x) = \mathbf{x}(w) = -8$ and $\mathbf{x}^p(w) = \mathbf{x}(x) = -4$, another eigenvalue swap. Again, $\mathbf{x}(x) - \mathbf{x}^p(x) = 4$ is the difference between the largest and smallest shifts of b_x appearing in c_w. We expect $\mathbf{c}^p(x) = \mathbf{c}(x) + 2$, so the heuristic explanation above will not suffice.

In C_4, what might have happened to the copy of $B_x(\mathbf{x}^p(x) + 2k)$ in degree $\mathbf{c}^p(x) - 1$? No more copies of C_x appear to Gaussian eliminate against it.

[11] This matches p^{-1} in the previous section, but p was changed to q because we also invoke the name of the prime.

One option is that C_x has summands of the form B_y, and thus $\mathrm{HT} \otimes C_x$ has summands of the form $\mathrm{HT} \otimes B_y$; when x is in a lower cell than y, this can easily explain copies of B_x in degree $\mathbf{c}^p(x) - 1$. Another option is that additional copies of C_w or some other $C_{w'}$ contain summands B_x. In the C_3 example it is hard to produce such a situation without a massive and unlikely cancellation effect.

In truth, the situation is still a mystery, and additional exploration is needed.

5. Appendix: *p*-cells and their eigenvalues

In this appendix we present explicitly the differences visible in the cell structure between the canonical and p-canonical bases for the Cartan types of rank at most 4. The computations have been performed up to rank 6, but we only include a summary of ranks 5 and 6. These calculations are possible due to work of Geordie Williamson and the second author, who have written an algorithm to calculate p-canonical bases based on the diagrammatic Hecke category. Tables of p-canonical bases for all the types described below, as well as the Magma package `IHecke` used to work with the Hecke algebra and calculate the cell decompositions, are available on GitHub [12]. The algorithm which calculates the bases themselves is described in [13].

Let us call a prime p *interesting* for a Cartan type if the p-canonical basis differs from the canonical basis. Up to rank four, the interesting types are C_2, C_3, C_4 with $p = 2$, B_2, B_3, B_4 with $p = 2$, D_4 with $p = 2$, F_4 with $p = 2, 3$, and G_2 with $p = 2, 3$. In all of these interesting types, we have verified the following statement. Let λ denote a left p-cell, and $w \in \lambda$ an element of that cell. Then

$$h_{w_0} \cdot {}^p b_w \equiv (-1)^{\mathbf{c(w)}} v^{\mathbf{x(w)}} \cdot {}^p b_{\mathrm{Schu}_L^p(w)} \pmod{I_{<\lambda}} \tag{5.1}$$

for some functions $\mathbf{c}\colon W \to \{0, 1\}$ and $\mathbf{x}\colon W \to \mathbb{Z}$ which are constant on left cells, and an involution $\mathrm{Schu}_L^p\colon W \to W$. Magma code which verifies this conjecture is available as one of the examples in the `IHecke` package mentioned above. Note that since the Hecke algebra only sees the parity of the actual function $\mathbf{c}\colon W \to \mathbb{Z}$, throughout this appendix we refer to the eigenvalue pairs $(\mathbf{x}, \pm)$ rather than $(\mathbf{x}, \mathbf{c})$, and really all we can verify is that the function $(-1)^{\mathbf{c}}$ is constant on left cells.

The following table collects some high-level statistics about the p-cells in the types listed above. The rows of this table are (in order): the number of left cells, two-sided cells, unique $(\mathbf{x}, \pm)$-eigenvalues, Schu_L^p-fixed points, and Schu_L^p-moving points.

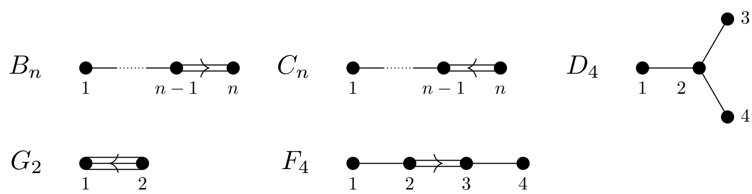
Table 1: Statistics on 0-cells vs p-cells for the Cartan types of rank at most 4

Type	C_2	C_2	G_2	G_2	G_2	C_3	C_3	B_3	C_4	C_4	B_4	D_4	D_4	F_4	F_4	F_4
p	0	2	0	2	3	0	2	2	0	2	2	0	2	0	2	3
# Left	4	5	4	6	5	14	17	17	50	63	63	36	38	72	106	78
# Two-sided	3	4	3	4	4	6	8	8	10	15	15	11	12	11	17	12
# $(\mathbf{x}, \pm)$	3	4	3	4	3	6	7	7	9	12	12	7	8	9	12	9
# Fix Schu_L^p	4	6	4	12	4	32	40	40	280	332	332	112	120	544	1136	544
# Unfix Schu_L^p	4	2	8	0	8	16	8	8	104	52	52	80	72	608	16	608

Similar tables for ranks 5 and 6 appear in §5.4. Notice that the statistics listed in the tables for types C_n and B_n are identical for $n \leq 6$, but the cells themselves are certainly not. It is unclear whether this is a general phenomenon, or just a low-rank coincidence.

Our labeling of the simple reflections follows the Dynkin diagram numbering which is built-in to MAGMA, and reproduced below. We use the stan-

Table 2: Dynkin diagram numbering convention used throughout the appendix



dard Cartan matrix[12] for this Dynkin diagram when defining the realization from which the diagrammatic Hecke category is constructed (see [10, §3.1]). Namely, if a double edge points from i to j, then $\partial_i(\alpha_j) = -1$ and $\partial_j(\alpha_i) = -2$.

Acknowledgments The author of this appendix learned everything he knows about calculating p-canonical bases from Thorge Jensen, and would like to thank him.

5.1. Rank 2

The only interesting primes in rank 2 are $p = 2$ for C_2, and $p = 2, 3$ for G_2. These groups are small enough that we may explicitly draw the elements of

[12] For another programmer wanting to double-check conventions: look at the coefficients involved in the change-of-basis from the 2-canonical bases of C_3 and B_3 to the canonical basis. All coefficients should be integers, except for one $(v^{-1} + v)$ in the table for C_3.

the cells directly on to a Hasse diagram. We group elements by parentheses when they are related by the Schützenberger involution. We also use the y-coordinates to label the $(\mathbf{x}, \pm)$-eigenvalues on each cell.

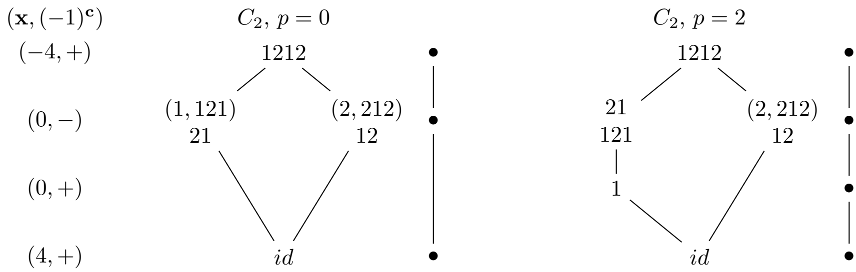


Figure 1: Left and two-sided 0-cells and 2-cells in type C_2.

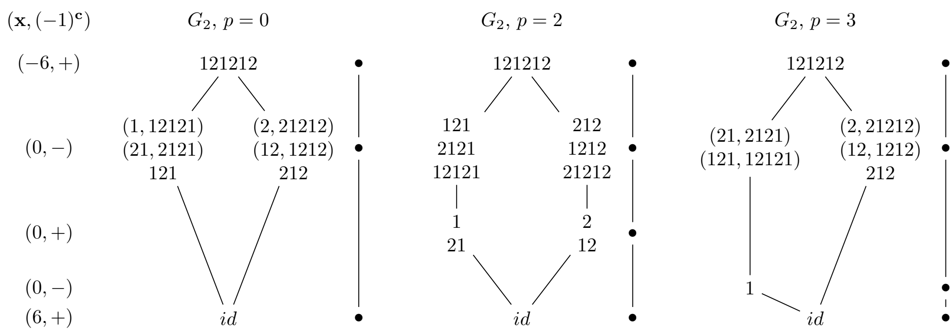


Figure 2: Left and two-sided 0, 2, and 3-cells in type C_2.

5.2. Rank 3

The only interesting types in rank 3 are B_3 and C_3, each with $p = 2$ an interesting prime. Since their underlying Coxeter systems are isomorphic, their canonical bases and hence 0-cells are identical. Below we show the Hasse diagrams for the left 0-cells and 2-cells for each type.

Each left 0-cell is written as $[w]$, where w is the shortest element of the cell which is least lexicographically. Left 2-cells are written as $[w]_{2,C}$ or $[w]_{2,B}$ when they differ (as a set) from any of the 0-cells, otherwise the labelling for the 0-cell is re-used. Cells on which Schu_L^p act nontrivially are marked with an asterisk.

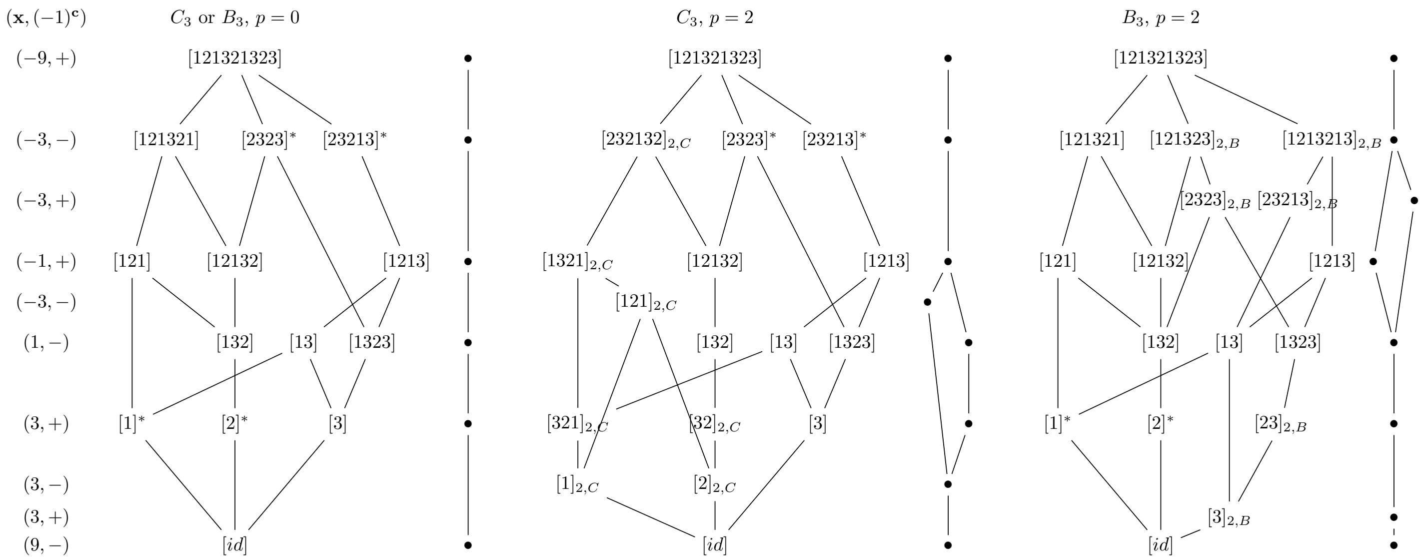


Figure 3: Left and two-sided 0-cells and 2-cells for types C_3 and B_3.

Table 3: Contents of the left 0-cells and 2-cells of C_3 and B_3. Rows have been grouped together whenever a 2-cell intersects a 0-cell. When a 2-cell and 0-cell have the same elements and Schutzenberger involution, the content of the 2-cell is omitted. For example, 1 is still mapped to 12321 under Schu_L^2 in type B_3.

	$\mathbf{x}$	$\mathbf{c}$	$[w]$		$\mathbf{x}^2$	$\mathbf{c}^2$	$[w]_2$
$[id]$	9	$-$	id				
$[1]^*$	3	$+$	(1, 12321), (21, 2321), 321	$[1]_{2,C}$	3	$-$	1, 21
				$[321]_{2,C}$	3	$+$	321, 2321, 12321
$[2]^*$	3	$+$	(2, 232), (12, 1232), 32	$[2]_{2,C}$	3	$-$	2, 12
				$[32]_{2,C}$	3	$+$	32, 232, 1232
$[3]$	3	$+$	3, 23, 123, 323	$[3]_{2,B}$	3	$+$	3
				$[23]_{2,B}$	3	$+$	23, 123, 323
$[13]$	1	$-$	13, 213, 3213				
$[121]$	-1	$+$	121, 1321, 21321	$[121]_{2,C}$	-3	$-$	121
$[121321]$	-3	$-$	121321, 232132, 1232132, 12132132	$[1321]_{2,C}$	-1	$+$	1321, 21321, 121321
				$[232132]_{2,C}$	-3	$-$	232132, 1232132, 12132132
$[132]$	1	$-$	132, 2132, 32132				
$[1213]$	-1	$+$	1213, 13213, 213213				
$[1323]$	1	$-$	1323, 21323, 321323				
$[2323]^*$	-3	$-$	(2323, 21321323), (12323, 1321323), 121323	$[2323]_{2,B}$	-3	$+$	2323, 12323
				$[121323]_{2,B}$	-3	$-$	121323, 1321323, 21321323
$[12132]$	-1	$+$	12132, 132132, 2132132				
$[23213]^*$	-3	$-$	(23213, 2321323), (123213, 12321323), 1213213	$[23213]_{2,B}$	-3	$+$	23213, 123213
				$[1213213]_{2,B}$	-3	$-$	1213213, 2321323, 12321323
$[121321323]$	-9	$+$	121321323				

In the above, there are some 0-cells which are a union of 2-cells, for example in type C_3 we have $[1] = [1]_{2,C} \sqcup [321]_{2,C}$ and $[2] = [2]_{2,C} \sqcup [32]_{2,C}$, while in type B_3 all 0-cells are unions of 2-cells. However, in type C_3 we also have $[121] \sqcup [121321] = [121]_{C,2} \sqcup [1321]_{C,2} \sqcup [232132]_{C,2}$, and no union of cells on the right is a cell on the left. We also have that $132 <_L 121$ in the 0-left-cell ordering on C_3, but $132 \not<_L 121$ in the 2-left-cell ordering. Finally, we can see that on the chain $[121]_{2,C} < [1321]_{2,C} < [232132]_{2,C}$ the eigenvalue $\mathbf{x}^2(-)$ takes values $-3, -1, -3$, meaning that we cannot expect that the implication $\lambda < \mu \implies \mathbf{x}^p(\lambda) < \mathbf{x}^p(\mu)$ will hold for p-cells, as it does for 0-cells.

5.3. Rank 4

For the rank 4 types B_4, C_4, D_4, and F_4 we only show the two-sided cells. Since words in the generators get quite long, we write $\overline{w} = w_0 w$ for a Coxeter group element, and when choosing representatives for a two-sided cell λ, we will either have $\lambda = [\![w]\!]$ where $w \in \lambda$ is the least (in short-lex order) element

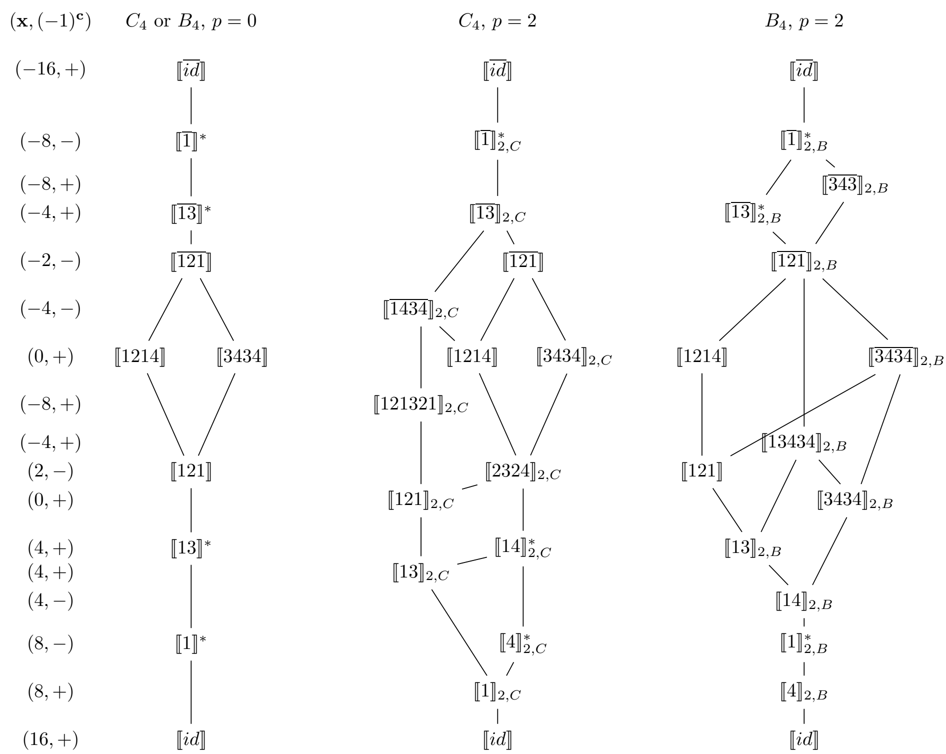


Figure 4: Two-sided 0-cells and 2-cells in types C_4 and B_4.

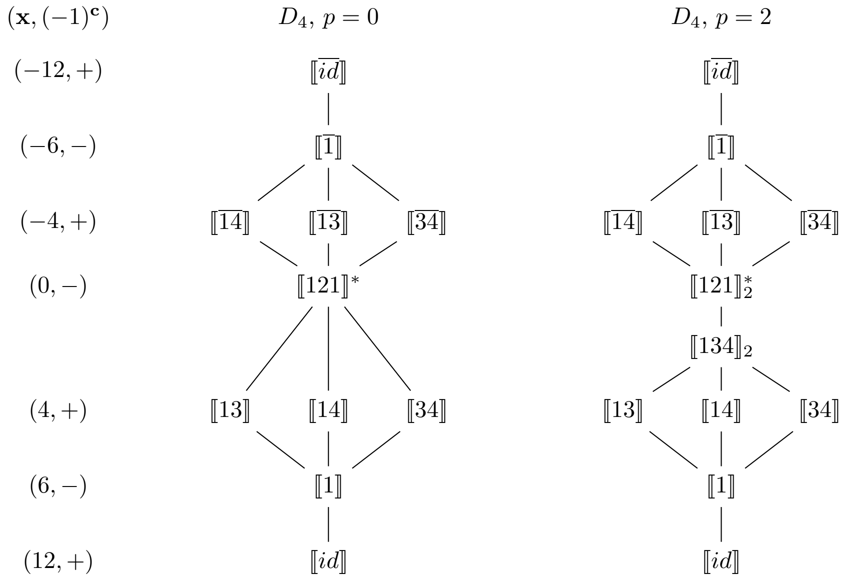


Figure 5: Two-sided 0-cells and 2-cells in type D_4.

belonging to λ, or $\lambda = [\![\overline{w}]\!]$, where $\overline{w}$ is the greatest (in short-lex order) element belonging to λ. When writing down a cell, we choose whichever representative will give us the shortest word. Again we mark cells on which Schu_L^p acts nontrivially with an asterisk, and only label p-cells with a subscript p when those cells do not appear as 0-cells.

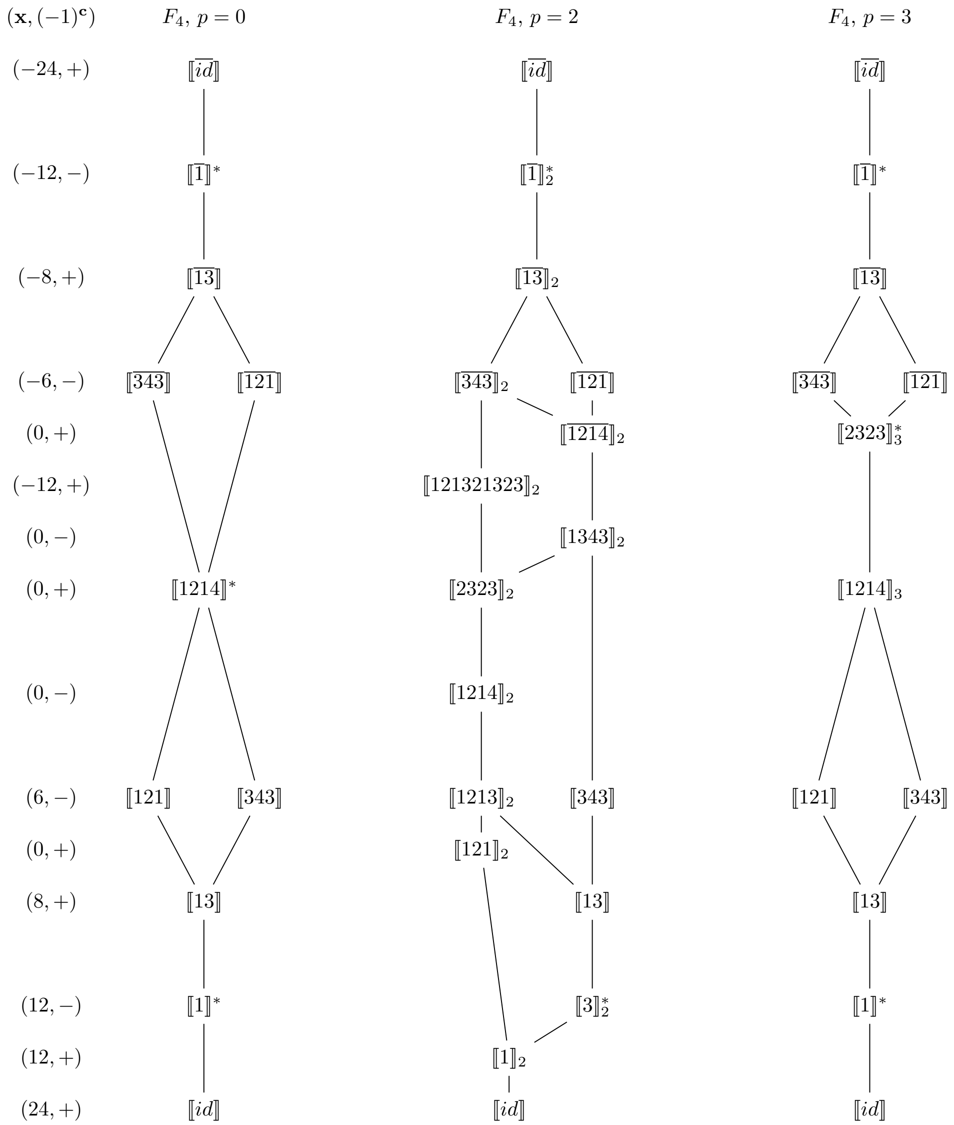


Figure 6: Two-sided 0, 2, and 3-cells in type F_4.

5.4. Ranks 5 and 6

The conjecture has also been verified and eigenvalues computed in ranks 5 and 6, and we display the high-level statistics here. All interesting primes appear below.

Table 4: Statistics on 0-cells vs p-cells for the Cartan types of rank 5.

Type	C_5	C_5	B_5	D_5	D_5
p	0	2	2	0	2
# Left cells	182	247	247	126	136
# Two-sided cells	16	26	26	14	16
# Unique $(\mathbf{x}, \pm)$ pairs	14	20	20	14	16
# Schu_L^p fixed points	1912	2876	2876	540	640
# Schu_L^p moving points	1928	964	964	1380	1280

Table 5: Statistics on 0-cells vs p-cells for the Cartan types of rank 6.

Type	C_6	C_6	B_6	C_6	B_6	D_6	D_6	D_6	E_6	E_6	E_6	E_6	E_6
p	0	2	2	3	3	0	2	3	0	2	3	5	7
# Left cells	752	1058	1058	752	752	578	622	578	652	742	662	652	652
# Two-sided cells	26	45	45	26	26	27	32	27	17	20	18	17	17
# Unique $(\mathbf{x}, \pm)$ pairs	19	26	26	19	19	19	22	19	17	20	17	17	17
# Schu_L^p fixed points	22824	33702	33702	22824	22824	14904	15740	14904	4008	5868	4008	4008	4008
# Schu_L^p moving points	23256	12378	12378	23256	23256	8136	7300	8136	47832	45972	47832	47832	47832

Remark 5.1. We remark on the computation time needed to compute the p-canonical basis and verify the conjecture for a given type and characteristic. Unless otherwise specified, the computation took seconds. Types D_6, B_5, C_5, and $(F_4, p = 2)$ took minutes. Types E_6, $(B_6, p = 3)$ and $(C_6, p = 3)$ took hours. Type $(B_6, p = 2)$ took two days, and type $(C_6, p = 2)$ took seven days.

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On the computation of character values for finite Chevalley groups of exceptional type

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Abstract: We discuss various computational issues around the problem of determining the character values of finite Chevalley groups, in the framework provided by Lusztig's theory of character sheaves. Some of the remaining open questions (concerning certain roots of unity) for the cuspidal unipotent character sheaves of groups of exceptional type are resolved.

Keywords: Groups of Lie type, Deligne–Lusztig characters, character sheaves.

1. Introduction

Let p be a prime and $k = \overline{\mathbb{F}}_p$ be an algebraic closure of the field with p elements. Let $\boldsymbol{G}$ be a connected reductive algebraic group over k and assume

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that $\boldsymbol{G}$ is defined over the finite subfield $\mathbb{F}_q \subseteq k$, where q is a power of p. Let $F: \boldsymbol{G} \to \boldsymbol{G}$ be the corresponding Frobenius map. Then the group of rational points $\boldsymbol{G}^F = \boldsymbol{G}(\mathbb{F}_q)$ is called a "finite group of Lie type". (For the basic theory of these groups, see [4], [10], [21].) We are concerned with the problem of computing the values of the irreducible characters of $\boldsymbol{G}^F$. The work of Lusztig [33], [41], [45] has led to a general program for solving this problem. In this framework, one seeks to establish certain identities of class functions on $\boldsymbol{G}^F$ of the form $R_x = \zeta\chi_A$, where R_x denotes an "almost character" (that is, an explicitly defined linear combination of the irreducible characters of $\boldsymbol{G}^F$) and χ_A denotes the characteristic function of a suitable F-invariant "character sheaf" A on $\boldsymbol{G}$; here, ζ is an algebraic number of absolute value 1. This program has been successfully carried out in many cases (see [21, §2.7] for a survey), but not in complete generality.

This paper is part of an ongoing project (involving various authors) to complete the program of establishing identities $R_x = \zeta\chi_A$ as above including the explicit determination of the scalars ζ. We shall solve this problem here in a number of previously open cases for $\boldsymbol{G}$ simple of exceptional type.

The above identities $R_x = \zeta\chi_A$ take a particularly striking form when A is a cuspidal character sheaf and $\boldsymbol{G}$ is a simple algebraic group, and this is our main focus here. In that case, the set $\{g \in \boldsymbol{G}^F \mid \chi_A(g) \neq 0\}$ is contained in a single F-stable conjugacy class Σ of $\boldsymbol{G}$; furthermore, the values of χ_A are determined by the choice of an element $g_1 \in \Sigma^F$ and a certain irreducible character ψ of the finite group of components $A_{\boldsymbol{G}}(g_1) = C_{\boldsymbol{G}}(g_1)/C^\circ_{\boldsymbol{G}}(g_1)$. By [41, 0.4], the general case can be reduced to the "cuspidal" case assuming that the cohomological induction functor $R_{\boldsymbol{L}}^{\boldsymbol{G}}$ (see [32], [10, §9.1]) is explicitly known, where $\boldsymbol{L} \subseteq \boldsymbol{G}$ is any F-stable Levi subgroup of a not necessarily F-stable parabolic subgroup of $\boldsymbol{G}$.

A crucial ingredient in this whole program is the problem of identifying "good" choices for $g_1 \in \Sigma^F$ as above. If Σ is a unipotent class, then one can use the concept of "split" elements; see Beynon–Spaltenstein [1] and Shoji [55]. In general, there are a few rare cases where one can single out a representative $g_1 \in \Sigma^F$ simply by looking at the order or the structure of the centralisers. At the other extreme, all $g_1 \in \Sigma^F$ may have the same centraliser order. In such cases, we use the following techniques: 1) Steinberg's cross-section [59] for regular elements or, more generally, Lusztig's "C-small" classes [43]; 2) rationality properties of characters; 3) powermaps and congruence conditions for character values.

When dealing with concrete examples, we will usually assume that $\boldsymbol{G}$ is simple, simply connected in order to facilitate the use of the above techniques and computations with semisimple elements. Our aim is to achieve something close to the famous Cambridge ATLAS [6], where no explicit representatives of conjugacy classes are given, but the classes can be almost uniquely identified by some formal properties, like 2) or 3). For many applications of character theory (for examples, see [21, App. A.10]) this is entirely sufficient. In Section 4, we develop some techniques that will help us identifying such "good" choices for $g_1 \in \Sigma^F$. (Note, however, that there does not yet seem to be a universally applicable definition of what "good" should mean.)

But first we need to address another essential point: the explicit evaluation of the Deligne–Lusztig characters $R_{\boldsymbol{T}}^{\boldsymbol{G}}(\theta)$, where $\boldsymbol{T}$ is any F-stable maximal torus of $\boldsymbol{G}$ and $\theta \in \mathrm{Irr}(\boldsymbol{T}^F)$. There is a character formula in [7] which reduces that problem to the computation of Green functions. The formula involves some technical issues of a purely group-theoretical nature. It will be known to the experts how to deal with this, but the details are not readily available so we include them here in Sections 2 and 3 (following, and slightly revising Lübeck [30]). We hope that this will be useful as a reference in other contexts as well.

Finally, in Sections 5–7, we explicitly deal with cuspidal character sheaves in groups of types F_4, E_6, E_7. Much of this is inspired by Lusztig [38] (values of characters on unipotent elements) and Shoji [52] (values of unipotent characters for classical groups); an additional complication here is that unipotent characters of exceptional groups may have non-rational values. We heavily rely on computer calculations, where we use Michel's extremely powerful version of CHEVIE [47]. In addition to the general functions concerning Weyl groups, reflection subgroups and their character tables, there are programs in [47] for producing information about the unipotent characters of $\boldsymbol{G}^F$ (degrees, Fourier matrices etc.), and for computing (generalised) Green functions, which turn out to be particularly helpful for our purposes here. Combined with previous work by a number of authors (for precise references see Sections 5–7), we can now state:

Let $\boldsymbol{G}$ be simple of type G_2, F_4, E_6 or E_7. Then the scalars ζ in the identities $R_x = \zeta\chi_A$ for cuspidal unipotent character sheaves A are explicitly known. In all cases considered, there is a "good" choice of $g_1 \in \Sigma^F$ such that $\zeta = 1$.

As far as simple groups of exceptional type are concerned, what remains to be done is to deal with a number of cuspidal character sheaves for $\boldsymbol{G}$ of type E_8 (which are all unipotent) and with the non-unipotent cuspidal character sheaves for $\boldsymbol{G}$ of type E_6 and E_7. For type E_8, see [27] (plus work in progress); the remaining cases for E_6, E_7 (which only occur when $\boldsymbol{G}$ is simply connected with a non-trivial center), will be considered in [22].

1.1. Notation and conventions. The set of (complex) irreducible characters of a finite group Γ is denoted by $\mathrm{Irr}(\Gamma)$. We work over a fixed subfield $\mathbb{K} \subseteq \mathbb{C}$, which is algebraic over $\mathbb{Q}$, invariant under complex conjugation and "large enough", that is, $\mathbb{K}$ contains sufficiently many roots of unity and $\mathbb{K}$ is a splitting field for Γ and all its subgroups. Thus, $\chi(g) \in \mathbb{K}$ for all $\chi \in \mathrm{Irr}(\Gamma)$ and $g \in \Gamma$. When required, we will assume chosen an embedding of $\mathbb{K}$ into $\overline{\mathbb{Q}}_l$, where $\mathbb{Q}_l$ is the field of l-adic numbers for some prime l. If $\alpha\colon \Gamma \to \Gamma$ is a group automorphism, we say that $g_1, g_2 \in \Gamma$ are α-conjugate if there exists some $g \in \Gamma$ such that $g_2 = g^{-1} g_1 \alpha(g)$.

2. Fusion of F-stable maximal tori

We keep the general notation from the introduction. Let $\boldsymbol{T}_0$ be a maximally split torus of $\boldsymbol{G}$, that is, $\boldsymbol{T}_0$ is an F-stable maximal torus contained in an F-stable Borel subgroup $\boldsymbol{B} \subseteq \boldsymbol{G}$. Let Φ be the root system of $\boldsymbol{G}$ with respect to $\boldsymbol{T}_0$, and let $\Phi^+ \subseteq \Phi$ be the set of positive roots determined by $\boldsymbol{B}$. Let $\boldsymbol{W} := N_{\boldsymbol{G}}(\boldsymbol{T}_0)/\boldsymbol{T}_0$ be the Weyl group of $\boldsymbol{T}_0$ and $\ell\colon \boldsymbol{W} \to \mathbb{Z}_{\geqslant 0}$ be the length function. We have $\boldsymbol{W} = \langle w_\alpha \mid \alpha \in \Phi\rangle$ where $w_\alpha \in \boldsymbol{W}$ denotes the reflection with root α. We have $\boldsymbol{G} = \langle \boldsymbol{T}_0, \boldsymbol{U}_\alpha \ (\alpha \in \Phi)\rangle$ where $\boldsymbol{U}_\alpha \subseteq \boldsymbol{G}$ denotes the root subgroup corresponding to α. The Frobenius map F induces a permutation $\alpha \mapsto \alpha^\dagger$ of Φ such that $F(\boldsymbol{U}_\alpha) = \boldsymbol{U}_{\alpha^\dagger}$ for all $\alpha \in \Phi$. We denote by $\sigma\colon \boldsymbol{W} \to \boldsymbol{W}$ the automorphism induced by F. For each $w \in \boldsymbol{W}$, let $\dot{w} \in N_{\boldsymbol{G}}(\boldsymbol{T}_0)$ be a representative; if $\sigma(w) = w$, then we tacitly assume that $F(\dot{w}) = \dot{w}$.

It is well known that the $\boldsymbol{G}^F$-conjugacy classes of F-stable maximal tori of $\boldsymbol{G}$ are parametrised by the σ-conjugacy classes of $\boldsymbol{W}$. Given $w \in \boldsymbol{W}$, let $g \in \boldsymbol{G}$ be such that $\dot{w} = g^{-1}F(g)$. (The existence of g relies on Lang's Theorem, which will be used many times in what follows, without further explicit reference.) Then $\boldsymbol{T} := g\boldsymbol{T}_0 g^{-1}$ is a corresponding F-stable maximal torus, unique up to conjugation by elements of $\boldsymbol{G}^F$; in this situation, we also say that $\boldsymbol{T}$ is "of type w". (See [4, §2.3] for further details.) For the evaluation of Deligne–Lusztig characters, we shall need to relate $\boldsymbol{G}^F$-conjugacy classes of F-stable maximal tori of $\boldsymbol{G}$ to those in certain connected reductive subgroups of maximal rank. Since this is crucial for the explicit computations that we need to carry out, we will explain the details here; see also Lübeck [30], [31].

2.1. Subsystem subgroups. As in Carter [3, §2], we consider subsets $\Phi' \subseteq \Phi$ that are themselves root systems and are closed in the sense that, whenever $\alpha, \beta \in \Phi'$ and $\alpha + \beta \in \Phi$, then $\alpha + \beta \in \Phi'$. Given such a Φ', there is a corresponding closed connected reductive subgroup $\boldsymbol{H}' \subseteq \boldsymbol{G}$ generated by $\boldsymbol{T}_0$

and the subgroups $\boldsymbol{U}_\alpha$ for $\alpha \in \Phi'$. Here, Φ' is the root system of $\boldsymbol{H}'$ with respect to $\boldsymbol{T}_0$ and

$$\boldsymbol{W}(\Phi') := \langle w_\alpha \mid \alpha \in \Phi' \rangle = N_{\boldsymbol{H}'}(\boldsymbol{T}_0)/\boldsymbol{T}_0$$

is the Weyl group of $\boldsymbol{H}'$. Let Ξ be the set of all pairs (Φ', w) where $\Phi' \subseteq \Phi$ is a subset as above and $w \in \boldsymbol{W}$ is such that $w(\alpha^\dagger) \in \Phi'$ for all $\alpha \in \Phi'$. Given $(\Phi', w) \in \Xi$, we form the corresponding subgroup $\boldsymbol{H}' \subseteq \boldsymbol{G}$ as above; then $F(\boldsymbol{H}') = \dot{w}^{-1}\boldsymbol{H}'\dot{w}$; note that $\dot{w}\boldsymbol{U}_\alpha\dot{w}^{-1} = \boldsymbol{U}_{w(\alpha)}$ for all $\alpha \in \Phi$. Hence, writing $\dot{w} = g^{-1}F(g)$ for some $g \in \boldsymbol{G}$, the subgroup $g\boldsymbol{H}'g^{-1}$ is F-stable and uniquely determined by (Φ', w), up to conjugation by an element of $\boldsymbol{G}^F$. It is known (see Carter [3], Deriziotis [8], Mizuno [48]) that the $\boldsymbol{G}^F$-conjugacy classes of F-stable subgroups $g\boldsymbol{H}'g^{-1}$ as above are parametrised by the pairs in Ξ modulo the equivalence relation defined by: $(\Phi'_1, w_1) \sim (\Phi'_2, w_2)$ if there exists some $x \in \boldsymbol{W}$ such that $x(\Phi'_1) = \Phi'_2$ and $x^{-1}w_2\sigma(x)w_1^{-1} \in \boldsymbol{W}(\Phi'_1)$. (The above statement concerning $\boldsymbol{G}^F$-conjugacy classes of F-stable maximal tori is a very special case of this correspondence.) Note that $(\Phi', w) \sim (\Phi', uw)$ for all $(\Phi', w) \in \Xi$ and $u \in \boldsymbol{W}(\Phi')$; thus, one could also consider pairs $(\Phi', \boldsymbol{W}(\Phi')w)$ but we find it more convenient in our setting here to work with the above definition of Ξ.

2.2. The relation $\sim$ on Ξ. For future reference, we briefly indicate how the relation $\sim$ comes about. (Note that the discussion in [3, §2] assumes that the subgroup $\boldsymbol{H}'$ corresponding to Φ' is itself F-stable, which will not always be the case if $\sigma \neq \mathrm{id}_{\boldsymbol{W}}$; furthermore, [3, §2] only considers $\boldsymbol{G}^F$-conjugacy for the subgroups corresponding to a fixed Φ'.) So let (Φ'_1, w_1) and (Φ'_2, w_2) be pairs in Ξ; let $g_1, g_2 \in \boldsymbol{G}$ be such that $g_1^{-1}F(g_1) = \dot{w}_1$ and $g_2^{-1}F(g_2) = \dot{w}_2$. We have the corresponding F-stable subgroups $g_i\boldsymbol{H}'_ig_i^{-1}$, where $\boldsymbol{H}'_i := \langle \boldsymbol{T}_0, \boldsymbol{U}_\alpha\ (\alpha \in \Phi'_i)\rangle$ for $i = 1, 2$. Suppose now that $g_1\boldsymbol{H}'_1g_1^{-1}$ and $g_2\boldsymbol{H}'_2g^{-1}$ are conjugate in $\boldsymbol{G}^F$; so $\tilde{g}g_1\boldsymbol{H}'_1g_1^{-1}\tilde{g}^{-1} = g_2\boldsymbol{H}'_2g_2^{-1}$ for some $\tilde{g} \in \boldsymbol{G}^F$. Setting $\hat{g} := g_2^{-1}\tilde{g}g_1$, we have $\hat{g}\boldsymbol{H}'_1\hat{g}^{-1} = \boldsymbol{H}'_2$ and there exists some $h_2 \in \boldsymbol{H}'_2$ such that $\hat{g}\boldsymbol{T}_0\hat{g}^{-1} = h_2\boldsymbol{T}_0h_2^{-1}$. Then $n := h_2^{-1}\hat{g} \in N_{\boldsymbol{G}}(\boldsymbol{T}_0)$ and $n\boldsymbol{H}'_1n^{-1} = \boldsymbol{H}'_2$. Hence, $x(\Phi'_1) = \Phi'_2$, where x is the image of n in $\boldsymbol{W}$. We now have $g_1 = \tilde{g}^{-1}g_2\hat{g} = \tilde{g}^{-1}g_2h_2n$ and a straightforward computation yields that

$$\dot{w}_1 = g_1^{-1}F(g_1) = \big(n^{-1}\dot{w}_2F(n)\big)\big(F(n)^{-1}\dot{w}_2^{-1}h_2^{-1}\dot{w}_2F(h_2)F(n)\big).$$

We have $F(\boldsymbol{H}'_2) = \dot{w}_2^{-1}\boldsymbol{H}'_2\dot{w}_2$ and so $\dot{w}_2^{-1}h_2^{-1}\dot{w}_2F(h_2) \in F(\boldsymbol{H}'_2)$. Furthermore, $F(n)^{-1}F(\boldsymbol{H}'_2)F(n) = F(n^{-1}\boldsymbol{H}'_2n) = F(\boldsymbol{H}'_1) = \dot{w}_1^{-1}\boldsymbol{H}'_1\dot{w}_1$. Hence, we obtain $\dot{w}_1 = n^{-1}\dot{w}_2F(n)\dot{w}_1^{-1}h_1\dot{w}_1$ for some $h_1 \in \boldsymbol{H}'_1$. Thus, $n^{-1}\dot{w}_2F(n)\dot{w}_1^{-1} \in N_G(\boldsymbol{T}_0) \cap \boldsymbol{H}'_1$ and so $x^{-1}w_2\sigma(x)w_1^{-1} \in \boldsymbol{W}(\Phi'_1)$, as desired. Conversely, if $(\Phi'_1, w_1) \sim (\Phi'_2, w_2)$, then one needs to run the above argument backwards.

2.3. Let us fix a pair $(\Phi', w) \in \Xi$. As already noted, we have $(\Phi', w) \sim (\Phi', uw)$ for all $u \in \boldsymbol{W}(\Phi')$. By [33, Lemma 1.9], the coset $\boldsymbol{W}(\Phi')w$ contains a unique element of minimal length; let us denote this element by d. Thus, when considering equivalence classes of pairs $(\Phi', d) \in \Xi$, we may assume without loss of generality that d has minimal length in the coset $\boldsymbol{W}(\Phi')d$. (Note that Lübeck [30] does not make this assumption on d.) We define a new Frobenius map $F' \colon \boldsymbol{G} \to \boldsymbol{G}$ by $F'(g) := \dot{d}F(g)\dot{d}^{-1}$ for $g \in \boldsymbol{G}$. Then we have $F'(\boldsymbol{H}') = \boldsymbol{H}'$, where $\boldsymbol{H}' = \langle \boldsymbol{T}_0, \boldsymbol{U}_\alpha \ (\alpha \in \Phi')\rangle$. The map induced by F' on $\boldsymbol{W}$ is given by

(a) $$\sigma' \colon \boldsymbol{W} \to \boldsymbol{W}, \qquad w \mapsto d\sigma(w)d^{-1}.$$

Clearly, $\boldsymbol{T}_0$ is also an F'-stable maximal torus of $\boldsymbol{H}'$. We claim that

(b) $\boldsymbol{T}_0$ is a maximally split torus of $\boldsymbol{H}'$ with respect to F'.

This is seen as follows. The group $\boldsymbol{B}' = \langle \boldsymbol{T}_0, \boldsymbol{U}_\alpha \ (\alpha \in \Phi^+ \cap \Phi')\rangle$ is a Borel subgroup of $\boldsymbol{H}'$ (see [4, §3.5]). Since $\boldsymbol{T}_0 \subseteq \boldsymbol{B}'$, it is sufficient to show that $\boldsymbol{B}'$ is F'-stable. For this purpose, let $\alpha \in \Phi^+ \cap \Phi'$. By [33, Lemma 1.9], we have $d^{-1}(\alpha) \in \Phi^+$ and so $\dot{d}^{-1}\boldsymbol{U}_\alpha\dot{d} = \boldsymbol{U}_{d^{-1}(\alpha)} \subseteq \boldsymbol{B} = F(\boldsymbol{B})$. Consequently, we have $\boldsymbol{U}_\alpha \subseteq \dot{d}F(\boldsymbol{B})\dot{d}^{-1} = F'(\boldsymbol{B})$. Since also $\boldsymbol{T}_0 = F'(\boldsymbol{T}_0) \subseteq F'(\boldsymbol{B})$, we conclude that $\boldsymbol{B}' \subseteq F'(\boldsymbol{B})$. Furthermore, $\boldsymbol{B}' \subseteq \boldsymbol{H}' = F'(\boldsymbol{H}')$ and so $\boldsymbol{B}' \subseteq F'(\boldsymbol{B}) \cap F'(\boldsymbol{H}') = F'(\boldsymbol{B} \cap \boldsymbol{H}') = F'(\boldsymbol{B}')$. Hence, we must have $\boldsymbol{B}' = F'(\boldsymbol{B}')$, as claimed. Thus, if Δ' is the unique set of simple roots in $\Phi^+ \cap \Phi'$, then we have $\boldsymbol{W}(\Phi') = \langle S'\rangle$ where

(c) $$S' := \{w_\alpha \mid \alpha \in \Delta'\} \qquad \text{and} \qquad \sigma'(S') = S'.$$

In particular, $(\boldsymbol{W}(\Phi'), S')$ is a Coxeter system and $\sigma'\big(\boldsymbol{W}(\Phi')\big) = \boldsymbol{W}(\Phi')$.

2.4. In the setting of §2.3, where $(\Phi, d) \in \Xi$, let us also fix an element $g \in \boldsymbol{G}$ such that $g^{-1}F(g) = \dot{d}$. Then $\boldsymbol{T}_d := g\boldsymbol{T}_0g^{-1} \subseteq \boldsymbol{G}$ is an F-stable maximal torus of type d. Furthermore, if $\boldsymbol{H}' = \langle \boldsymbol{T}_0, \boldsymbol{U}_\alpha \ (\alpha \in \Phi')\rangle$ and $\boldsymbol{H}_d := g\boldsymbol{H}'g^{-1}$, then $\boldsymbol{T}_d \subseteq \boldsymbol{H}_d$ and $F(\boldsymbol{H}_d) = \boldsymbol{H}_d$. Now Remark 2.3(b) immediately implies that $\boldsymbol{B}_d := g\boldsymbol{B}'g^{-1}$ is an F-stable Borel subgroup of $\boldsymbol{H}_d$ and so $\boldsymbol{T}_d$ is a maximally split torus of $\boldsymbol{H}_d$. Let $\boldsymbol{W}_d := N_{\boldsymbol{H}_d}(\boldsymbol{T}_d)/\boldsymbol{T}_d$ be the Weyl group of $\boldsymbol{H}_d$. We denote by $\sigma_d \colon \boldsymbol{W}_d \to \boldsymbol{W}_d$ the automorphism induced by F. Then the conjugation map $\gamma_g \colon \boldsymbol{G} \to \boldsymbol{G}$, $x \mapsto g^{-1}xg$, induces an embedding $\bar{\gamma}_g \colon \boldsymbol{W}_d \hookrightarrow \boldsymbol{W}$, where

$$\boldsymbol{W}(\Phi') = \bar{\gamma}_g(\boldsymbol{W}_d) \subseteq \boldsymbol{W} \qquad \text{and} \qquad \bar{\gamma}_g \circ \sigma_d = \sigma' \circ \bar{\gamma}_g.$$

Via the isomorphism $\bar{\gamma}_g\colon \boldsymbol{W}_d \to \boldsymbol{W}(\Phi')$, the σ_d-conjugacy classes of $\boldsymbol{W}_d$ correspond to the σ'-conjugacy classes of $\boldsymbol{W}(\Phi')$. Thus, the $\boldsymbol{H}_d^F$-conjugacy classes of F-stable maximal tori of $\boldsymbol{H}_d$ are parametrised by the σ'-conjugacy classes of $\boldsymbol{W}(\Phi')$. More precisely, if $w' \in \boldsymbol{W}(\Phi')$, then an F-stable maximal torus $\boldsymbol{T}' \subseteq \boldsymbol{H}_d$ of type w' (inside $\boldsymbol{H}_d$) is given by $\boldsymbol{T}' := h\boldsymbol{T}_d h^{-1}$ where $h \in \boldsymbol{H}_d$ is such that $h^{-1}F(h) = \gamma_g^{-1}(\dot{w}') = g\dot{w}'g^{-1}$.

The following result describes the fusion from F-stable maximal tori in $\boldsymbol{H}_d$ to F-stable maximal tori in $\boldsymbol{G}$; see also Lübeck [30, §4.1(2)] (but note that slightly different conventions and assumptions are used in [30]).

Lemma 2.5. *In the above setting, let $\boldsymbol{T}' \subseteq \boldsymbol{H}_d$ be an F-stable maximal torus of type $w' \in \boldsymbol{W}(\Phi')$. Then $\boldsymbol{T}' \subseteq \boldsymbol{G}$ is an F-stable maximal torus of type $w'd \in \boldsymbol{W}$. In particular, a maximally split torus of $\boldsymbol{H}_d$ is of type d (relative to $\boldsymbol{G}$).*

Proof. Recall that $g^{-1}F(g) = \dot{d}$ and that $\boldsymbol{T}_d = g\boldsymbol{T}_0 g^{-1}$ is a maximally split torus of $\boldsymbol{H}_d$. As above, let $h \in \boldsymbol{H}_d$ be such that $\boldsymbol{T}' = h\boldsymbol{T}_d h^{-1}$ and $h^{-1}F(h) = \gamma_g^{-1}(\dot{w}') = g\dot{w}'g^{-1}$. Then $\boldsymbol{T}' = hg\boldsymbol{T}_0 g^{-1}h^{-1}$ and $(hg)^{-1}F(hg) \in N_{\boldsymbol{G}}(\boldsymbol{T}_0)$. Now

$$(hg)^{-1}F(hg) = g^{-1}h^{-1}F(h)F(g) = g^{-1}(g\dot{w}'g^{-1})F(g) = \dot{w}'\dot{d}.$$

Hence, $\boldsymbol{T}'$ is an F-stable maximal torus of type $\dot{w}d$ in $\boldsymbol{G}$. ☐

Example 2.6. Let $\boldsymbol{G}$ be simple of type G_2; then $\sigma = \mathrm{id}_{\boldsymbol{W}}$ and the permutation $\alpha \mapsto \alpha^\dagger$ is the identity. Let $\Delta = \{\alpha_1, \alpha_2\}$ be the set of simple roots in Φ^+, where α_1 is long and α_2 is short. There are two particular subsystems $\Phi' \subseteq \Phi$ that occur in the classification of cuspidal character sheaves on $\boldsymbol{G}$ (see the proof of [36, Prop. 20.6]). Up to $\boldsymbol{W}$-conjugacy, these are Φ_0' of type $A_1 \times A_1$, spanned by $\{\alpha_2, \alpha_0\}$, and $\Phi_0'' \subseteq \Phi$ of type A_2, spanned by $\{\alpha_1, \alpha_0\}$. (Here, $\alpha_0 \in \Phi$ denotes the unique root of maximal height in Φ.) There is only one equivalence class of pairs $(\Phi', w) \in \Xi$ under $\sim$ where $\Phi' = \Phi_0'$; a representative is given by (Φ_0', d_1) with $d_1 = 1_{\boldsymbol{W}}$. There are two equivalence classes of pairs $(\Phi', w) \in \Xi$ where $\Phi' = \Phi_0''$; representatives are given by (Φ_0'', d_1) with $d_1 = 1_{\boldsymbol{W}}$, and by (Φ_0'', d_2) with $d_2 = w_{\alpha_2}$ (and d_2 has minimal length in $\boldsymbol{W}(\Phi_0'')d_2$). The information is summarised in Table 1 (which is a model for the tables in later sections).

In that table, Δ' is the set of simple roots in $\Phi^+ \cap \Phi'$. Furthermore, we define $\sigma_i' \in \mathrm{Aut}(\boldsymbol{W}(\Phi'))$ by $\sigma_i'(w) = d_i w d_i^{-1}$ for $w \in \boldsymbol{W}$. Then σ_i' induces a permutation of the simple reflections in $\boldsymbol{W}(\Phi')$; see Remark 2.3. This permutation, in cycle notation, is indicated in the fourth column of the table; note

Table 1: Subsystems for type G_2

Φ'	Δ'	d_i	permutation	σ_i'-classes
$A_1{\times}A_1$	$\alpha_2, 2\alpha_1{+}3\alpha_2$	$d_1 = 1_{\boldsymbol{W}}$	$()$	4
A_2	$\alpha_1, \alpha_1{+}3\alpha_2$	$d_1 = 1_{\boldsymbol{W}}$	$()$	3
		$d_2 = w_{\alpha_2}$	$(1,2)$	3

that this permutation refers to the simple roots in Δ', not to those in Δ. The last column contains the number of σ_i'-conjugacy classes of $\boldsymbol{W}(\Phi')$.

Now consider the fusion of F-stable maximal tori described by Lemma 2.5. In each case, we need to work out representatives of the σ_i'-conjugacy classses of $\boldsymbol{W}(\Phi')$, multiply these by d_i and identify the conjugacy class of $\boldsymbol{W}$ to which the new element belongs. Here, of course, this can be done by hand, but for larger $\boldsymbol{W}$, such computations are conveniently done using the computer algebra system CHEVIE [20], [47], for example.

2.7. Centralisers of semisimple elements. Let C be an F-stable conjugacy class of semisimple elements of $\boldsymbol{G}$. It is well-known that $C \cap \boldsymbol{T}_0$ is non-empty and a single orbit under the action of $\boldsymbol{W}$; furthermore, $C_{\boldsymbol{G}}^{\circ}(t)$, for $t \in C \cap \boldsymbol{T}_0$, is a connected reductive subgroup of the type considered above (see, e.g., [4, §3.5, §3.7]). Thus, we are led to consider the subset $\Xi^{\circ} \subseteq \Xi$ consisting of all pairs $(\Phi', w) \in \Xi$, for which there exists some $t \in \boldsymbol{T}_0$ such that

$$\Phi' = \{\alpha \in \Phi \mid \alpha(t) = 1\} \qquad \text{and} \qquad F(t) = \dot{w}^{-1} t \dot{w}.$$

Then the $\boldsymbol{G}^F$-conjugacy classes of subgroups of the form $C_{\boldsymbol{G}}^{\circ}(s)$, where $s \in \boldsymbol{G}^F$ is semisimple, are parametrised by the pairs in Ξ° modulo the equivalence relation $\sim$ on Ξ°. (See again [3], [8], [48].) Given a pair $(\Phi', w) \in \Xi^{\circ}$, a corresponding semisimple element $s \in \boldsymbol{G}^F$ is obtained as follows. Let $t \in \boldsymbol{T}_0$ be such that $\Phi' = \{\alpha \in \Phi \mid \alpha(t) = 1\}$ and $F(t) = \dot{w}^{-1} t \dot{w}$; then $C_{\boldsymbol{G}}^{\circ}(t) = \langle \boldsymbol{T}_0, \boldsymbol{U}_\alpha \ (\alpha \in \Phi') \rangle$. Let $g \in \boldsymbol{G}$ be such that $g^{-1}F(g) = \dot{w}$ and set $s := gtg^{-1}$. Then $F(s) = s$ and $C_{\boldsymbol{G}}^{\circ}(s) = g C_{\boldsymbol{G}}^{\circ}(t) g^{-1}$.

Note that, if $d \in \boldsymbol{W}$ has minimal length in the coset $\boldsymbol{W}(\Phi')w$, then $w = w'd$ for some $w' \in \boldsymbol{W}(\Phi')$ and we still have $F(t) = \dot{d}^{-1} t \dot{d}$ (since $\dot{w}' \in C_{\boldsymbol{G}}(t)$). Hence, again, we may assume without loss of generality that $w = d$ and so the discussions in Remarks 2.3, 2.4, and Lemma 2.5 apply. The subsystems $\Phi' \subseteq \Phi$ which can arise at all as the root system of $C_{\boldsymbol{G}}^{\circ}(t)$, for some $t \in \boldsymbol{T}_0$, are characterised in [8, §2.3]; given such a subset $\Phi' \subseteq \Phi$, the condition of whether there is some $w \in \boldsymbol{W}$ such that $(\Phi', w) \in \Xi^{\circ}$ may also depend on the isogeny type of $\boldsymbol{G}$ and the $\mathbb{F}_q$-rational structure on $\boldsymbol{G}$; see [3, §5] and [8, Chap. 2] for further details.

3. On the evaluation of Deligne–Lusztig characters

Let $\boldsymbol{T} \subseteq \boldsymbol{G}$ be an F-stable maximal torus and $\theta \in \mathrm{Irr}(\boldsymbol{T}^F)$ be an irreducible character. Then we have a corresponding virtual character $R_{\boldsymbol{T}}^{\boldsymbol{G}}(\theta)$ of $\boldsymbol{G}^F$, as defined by Deligne–Lusztig [7] (see also [4, Chap. 7]). These virtual characters span a significant subspace of the space of all class functions on $\boldsymbol{G}^F$ (see the introduction of [33] and [21, Cor. 2.7.13] for a more precise measure of what "significant" means). It is known that, if $u \in \boldsymbol{G}^F$ is unipotent, then $Q_{\boldsymbol{T}}^{\boldsymbol{G}}(u) := R_{\boldsymbol{T}}^{\boldsymbol{G}}(\theta)(u) \in \mathbb{Z}$ does not depend on θ; the function $u \mapsto Q_{\boldsymbol{T}}^{\boldsymbol{G}}(u)$ is called a *Green function*. We now have the following important character formula.

Let $\tilde{g} \in \boldsymbol{G}^F$ and write $\tilde{g} = su = us$, where $s \in \boldsymbol{G}^F$ is semisimple and $u \in \boldsymbol{G}^F$ is unipotent. Then, setting $\boldsymbol{H}_s := C_{\boldsymbol{G}}^\circ(s)$, we have

$$R_{\boldsymbol{T}}^{\boldsymbol{G}}(\theta)(\tilde{g}) = \frac{1}{|\boldsymbol{H}_s^F|} \sum_{x \in \boldsymbol{G}^F \,:\, x^{-1}sx \in \boldsymbol{T}} Q_{x\boldsymbol{T}x^{-1}}^{\boldsymbol{H}_s}(u)\, \theta(x^{-1}sx).$$

Note that, firstly, $\boldsymbol{H}_s$ is connected and reductive; secondly, if $x^{-1}sx \in \boldsymbol{T}^F$, then $x\boldsymbol{T}x^{-1}$ is an F-stable maximal torus contained in $\boldsymbol{H}_s$; furthermore, u is known to belong to $\boldsymbol{H}_s$. (See [7, 4.2] or [4, 7.2.8] for further details.) In particular, the formula shows that all values of $R_{\boldsymbol{T}}^{\boldsymbol{G}}(\theta)$ belong to the field $\mathbb{Q}(\theta(t) \mid t \in \boldsymbol{T}^F)$. We also see that, if s is not conjugate in $\boldsymbol{G}^F$ to an element of $\boldsymbol{T}^F$, then $R_{\boldsymbol{T}}^{\boldsymbol{G}}(\theta)(g) = 0$.

We now explain how the above formula can be evaluated explicitly; for this purpose, we need to

(1) know the values of the Green functions of $\boldsymbol{H}_s$,

(2) deal with the sum over all $x \in \boldsymbol{G}^F$ such that $x^{-1}sx \in \boldsymbol{T}$.

As far as (1) is concerned, see the surveys in [10, Chap. 13] and [21, §2.8]. In any case, for $\boldsymbol{G}$ simple of exceptional type, explicit tables are known by Beynon–Spaltenstein [1] and Shoji [51]. (The fact that these tables remain valid whenever p is a "good" prime for $\boldsymbol{G}$ follows from [54, Theorem 5.5]; see also [19] for "bad" p.) The tables can be obtained via the function `ICCTable` of Michel's version of CHEVIE [47]. As far as (2) is concerned, the following result provides a first simplification.

Lemma 3.1 (See [21, 2.2.23])**.** *In the above setting, assume that s is conjugate in $\boldsymbol{G}^F$ to an element in $\boldsymbol{T}^F$. Let $\boldsymbol{T}_1, \ldots, \boldsymbol{T}_m$ be representatives of the $\boldsymbol{H}_s^F$-conjugacy classes of F-stable maximal tori of $\boldsymbol{H}_s$ that are conjugate in $\boldsymbol{G}^F$ to $\boldsymbol{T}$. For each i, let $\tilde{g}_i \in \boldsymbol{G}^F$ be such that $\boldsymbol{T}_i = \tilde{g}_i \boldsymbol{T} \tilde{g}_i^{-1}$. Then*

$$R_{\boldsymbol{T}}^{\boldsymbol{G}}(\theta)(\tilde{g}) = \sum_{1 \leqslant i \leqslant m} Q_{\boldsymbol{T}_i}^{\boldsymbol{H}_s}(u) \frac{1}{|\boldsymbol{W}(\boldsymbol{H}_s, \boldsymbol{T}_i)^F|} \sum_{y \in \boldsymbol{W}(\boldsymbol{G}, \boldsymbol{T}_i)^F} \theta(\tilde{g}_i^{-1} \dot{y}^{-1} s \dot{y} \tilde{g}_i)$$

where $\boldsymbol{W}(\boldsymbol{G},\boldsymbol{T}_i) = N_{\boldsymbol{G}}(\boldsymbol{T}_i)/\boldsymbol{T}_i$ *and* $\boldsymbol{W}(\boldsymbol{H}_s,\boldsymbol{T}_i) = N_{\boldsymbol{H}_s}(\boldsymbol{T}_i)/\boldsymbol{T}_i$.

Now note that the subgroup $\boldsymbol{T}^F \subseteq \boldsymbol{G}^F$ is not really computationally accessible, and the same is true for $\boldsymbol{T}_1^F, \ldots, \boldsymbol{T}_m^F$. All we can do explicitly are computations within $\boldsymbol{T}_0$. Hence, in order to proceed, we use a different model for $R_{\boldsymbol{T}}^{\boldsymbol{G}}(\theta)$, as already constructed in [7]. Assume that $\boldsymbol{T}$ is of type $w \in \boldsymbol{W}$ and let $g_1 \in \boldsymbol{G}$ be such that $g_1^{-1}F(g_1) = \dot{w}$. Then define

$$\boldsymbol{T}_0[w] := \{t \in \boldsymbol{T} \mid F(t) = \dot{w}^{-1}t\dot{w}\} = g_1^{-1}\boldsymbol{T}^F g_1 \subseteq \boldsymbol{G}.$$

Let $\theta' \in \mathrm{Irr}(\boldsymbol{T}_0[w])$ be the irreducible character defined by $\theta'(t) := \theta(g_1 t g_1^{-1})$ for all $t \in \boldsymbol{T}_0[w]$. Then we have $R_{\boldsymbol{T}}^{\boldsymbol{G}}(\theta) = R_w^{\theta'}$, where the right hand side is constructed directly from (w,θ') (see [21, 2.3.18] for further details). Thus, the virtual characters $R_{\boldsymbol{T}}^{\boldsymbol{G}}(\theta)$ may equally well be defined in terms of pairs (w,θ'), where $w \in \boldsymbol{W}$ and $\theta' \in \mathrm{Irr}(\boldsymbol{T}_0[w])$ — and the latter set of pairs (w,θ') is, indeed, computationally accessible.

We can now apply the results in the previous section, especially the discussions in Remark 2.4 and Lemma 2.5. Let $\tilde{g} = su = us \in \boldsymbol{G}^F$ as above and $\boldsymbol{H}_s := C_{\boldsymbol{G}}^\circ(s)$. Let $\boldsymbol{T}' \subseteq \boldsymbol{H}_s$ be a maximally split torus and $g \in \boldsymbol{G}$ be such that $\boldsymbol{T}' = g\boldsymbol{T}_0 g^{-1}$. Then $g^{-1}F(g) \in N_{\boldsymbol{G}}(\boldsymbol{T}_0)$ and we denote by $d \in \boldsymbol{W}$ the image of $g^{-1}F(g)$ in $\boldsymbol{W}$. Let $t := g^{-1}sg \in \boldsymbol{T}_0$ and $\Phi' := \{\alpha \in \Phi \mid \alpha(t) = 1\}$. Then $F(t) = \dot{w}^{-1}t\dot{w}$ and $(\Phi', d) \in \Xi^\circ$ parametrises the $\boldsymbol{G}^F$-conjugacy class of $\boldsymbol{H}_s$; furthermore, we have that d has minimal length in $\boldsymbol{W}(\Phi')d$. As in Remark 2.3, we define $\sigma' \in \mathrm{Aut}(\boldsymbol{W})$ by $\sigma'(w) = d\sigma(w)d^{-1}$ for $w \in \boldsymbol{W}$; then $\sigma'(\boldsymbol{W}(\Phi')) = \boldsymbol{W}(\Phi')$.

(∗) Let $w_1', \ldots, w_m' \in \boldsymbol{W}(\Phi')$ be representatives of the σ'-conjugacy classes of $\boldsymbol{W}(\Phi')$ such that $w_i'd$ is σ-conjugate in $\boldsymbol{W}$ to w. For each i let us fix an element $x_i \in \boldsymbol{W}$ such that $w = x_i^{-1}w_i'd\sigma(x_i)$.

For $1 \leqslant i \leqslant m$ let $h_i \in \boldsymbol{H}_s$ be such that $h_i^{-1}F(h_i) = g\dot{w}_i'g^{-1}$, and set $\boldsymbol{T}_i := h_i\boldsymbol{T}'h_i^{-1} \subseteq \boldsymbol{H}_d$. Then, by Lemma 2.5, $\boldsymbol{T}_1, \ldots, \boldsymbol{T}_m$ are maximal tori as required in Lemma 3.1. For $1 \leqslant i \leqslant m$ define $\boldsymbol{T}_0[w_i'd] \subseteq \boldsymbol{T}_0$ analogously to $\boldsymbol{T}_0[w]$ above; then $\boldsymbol{T}_0[w_i'd] = \dot{x}_i\boldsymbol{T}_0[w]\dot{x}_i^{-1}$ (see [21, 2.3.20]). So, given $\theta' \in \mathrm{Irr}(\boldsymbol{T}_0[w])$ as above, we can define a character $\theta_i' \in \mathrm{Irr}(\boldsymbol{T}_0[w_i'd])$ by

$$\theta_i'(t) := \theta'(\dot{x}_i^{-1}t\dot{x}_i) \qquad \text{for } t \in \boldsymbol{T}_0[w_i'd].$$

Now let $C_{\boldsymbol{W},\sigma}(w) = \{x \in \boldsymbol{W} \mid xw = w\sigma(x)\}$ be the σ-centraliser of w in $\boldsymbol{W}$; then $\dot{x}\boldsymbol{T}_0[w]\dot{x}^{-1} = \boldsymbol{T}_0[w]$ for all $x \in C_{\boldsymbol{W},\sigma}(w)$. Defining $C_{\boldsymbol{W},\sigma}(w_i'd)$ analogously, we have $\dot{x}\boldsymbol{T}_0[w_i'd]\dot{x}^{-1} = \boldsymbol{T}_0[w_i'd]$ for all $x \in C_{\boldsymbol{W},\sigma}(w_i'd)$. Let also

$$C_{\boldsymbol{W}(\Phi'),\sigma'}(w_i') = \{x \in \boldsymbol{W}(\Phi') \mid xw_i' = w_i'\sigma'(x)\} = \boldsymbol{W}(\Phi') \cap C_{\boldsymbol{W},\sigma}(w_i'd).$$

With this notation, we can now state the following result; see also Lübeck [30, Satz 2.1] for a slightly different formulation.

Lemma 3.2. *In the above setting, we have* $\boldsymbol{W}(\boldsymbol{H}_s, \boldsymbol{T}_i)^F \cong C_{\boldsymbol{W}(\Phi'),\sigma'}(w_i')$ *and*

$$\sum_{y \in \boldsymbol{W}(\boldsymbol{G},\boldsymbol{T}_i)^F} \theta(\tilde{g}_i^{-1}\dot{y}^{-1}s\dot{y}\tilde{g}_i) = \sum_c \theta_i'(\dot{c}^{-1}t\dot{c}) \qquad \text{for } 1 \leqslant i \leqslant m,$$

where c runs over all elements of $C_{\boldsymbol{W},\sigma}(w_i'd)$. In particular, if $\theta = 1_{\boldsymbol{T}}$ is the trivial character of $\boldsymbol{T}^F$, then the above sum equals $|C_{\boldsymbol{W},\sigma}(w_i'd)|$ for $1 \leqslant i \leqslant m$.

Proof. Recall that $\boldsymbol{T} = g_1\boldsymbol{T}_0g_1^{-1}$, $\boldsymbol{T}' = g\boldsymbol{T}_0g^{-1}$ and $\boldsymbol{T}_i = h_i\boldsymbol{T}'h_i^{-1}$ for all i. Hence, setting $\tilde{g}_i := h_ig\dot{x}_ig_1^{-1} \in \boldsymbol{G}$, we have $\tilde{g}_i\boldsymbol{T}\tilde{g}_i^{-1} = \boldsymbol{T}_i$ for all i. Since $\boldsymbol{T}$ and $\boldsymbol{T}_i$ are $\boldsymbol{G}^F$-conjugate, we can replace h_i by h_it_i' for a suitable $t_i' \in \boldsymbol{T}'$ such that $F(\tilde{g}_i) = \tilde{g}_i$ (see the argument in the proof of [4, Prop. 3.3.3]). Thus, the elements $\tilde{g}_i$ are as required in Lemma 3.1. Next, by [4, Prop. 3.3.6], we have a group isomorphism

$$C_{\boldsymbol{W},\sigma}(w_i'd) \xrightarrow{\sim} \boldsymbol{W}(\boldsymbol{G},\boldsymbol{T}_i)^F, \qquad c \mapsto h_ig\dot{c}g^{-1}h_i^{-1}.$$

(Recall that $h_ig\boldsymbol{T}_0g^{-1}h_i^{-1} = \boldsymbol{T}_i$ and $(h_ig)^{-1}F(h_ig) = \dot{w}_i'\dot{d}$.) Hence, we have

$$\sum_{y \in \boldsymbol{W}(\boldsymbol{G},\boldsymbol{T}_i)^F} \theta(\tilde{g}_i^{-1}\dot{y}^{-1}s\dot{y}\tilde{g}_i) = \sum_c \theta(\tilde{g}_i^{-1}h_ig\dot{c}^{-1}g^{-1}h_i^{-1}sh_ig\dot{c}g^{-1}h_i^{-1}\tilde{g}_i)$$

where c runs over all elements of $C_{\boldsymbol{W},\sigma}(w_i'd)$. Now $g^{-1}h_i^{-1}sh_ig = t$; hence, the terms in the above sum on the right hand side are given by

$$\theta(\tilde{g}_i^{-1}h_ig\dot{c}^{-1}t\dot{c}g^{-1}h_i^{-1}\tilde{g}_i) = \theta(g_1\dot{x}_i^{-1}\dot{c}^{-1}t\dot{c}\dot{x}_ig_1^{-1}) = \theta_i'(\dot{c}^{-1}t\dot{c})$$

for all $c \in C_{\boldsymbol{W},\sigma}(w_i'd)$, as required. Finally, consider the assertion concerning $\boldsymbol{W}(\boldsymbol{H}_s,\boldsymbol{T}_i)^F$. Let $\boldsymbol{W}_s = N_{\boldsymbol{H}_s}(\boldsymbol{T}')/\boldsymbol{T}'$ and $\sigma_s \colon \boldsymbol{W}_s \to \boldsymbol{W}_s$ be induced by F. Let $\boldsymbol{H}' = C_{\boldsymbol{G}}^\circ(t) = g^{-1}\boldsymbol{H}_sg$ and $F' \colon \boldsymbol{H}' \to \boldsymbol{H}'$ be as in Remark 2.3; recall that F' induces $\sigma' \in \mathrm{Aut}(\boldsymbol{W}(\Phi'))$. As discussed in Remark 2.4, conjugation by g induces a bijection between the σ'-conjugacy classes in $\boldsymbol{W}(\Phi')$ and the σ_s-conjugacy classes in $\boldsymbol{W}_s$. Thus, we have $\boldsymbol{W}(\boldsymbol{H}_s,\boldsymbol{T}_i)^F \cong \boldsymbol{W}(\boldsymbol{H}',\boldsymbol{T}_i')^{F'}$ where $\boldsymbol{T}_i' \subseteq \boldsymbol{H}'$ is an F'-stable maximal torus of type $w_i' \in \boldsymbol{W}(\Phi')$ (relative to F'). Again by [4, Prop. 3.3.6], the group $\boldsymbol{W}(\boldsymbol{H}',\boldsymbol{T}_i')^{F'}$ is isomorphic to $C_{\boldsymbol{W}(\Phi'),\sigma'}(w_i')$. $\square$

The point about the above result is that the formula on the right hand side of the identity can be explicitly and effectively computed, once a character $\theta' \in \mathrm{Irr}(\boldsymbol{T}_0[w])$ has been specified: all we need to know is the action of $\boldsymbol{W}$ on $\boldsymbol{T}_0$, plus information concerning various σ-conjugacy classes in $\boldsymbol{W}$.

Example 3.3. Assume that $\tilde{g} = su = us$ where $u \in \boldsymbol{H}_s$ is regular unipotent. Then $Q_{\boldsymbol{T}_i}^{\boldsymbol{H}_s}(u) = 1$ for $1 \leqslant i \leqslant m$ (see [7, Theorem 9.16]). Let $\theta = 1_{\boldsymbol{T}}$ be the trivial character of $\boldsymbol{T}^F$. Then

$$R_{\boldsymbol{T}}^{\boldsymbol{G}}(1_{\boldsymbol{T}})(g) = |C_{\boldsymbol{W},\sigma}(w)| \sum_{1 \leqslant i \leqslant m} |C_{\boldsymbol{W}(\Phi'),\sigma'}(w_i')|^{-1}.$$

Indeed, this is now clear by Lemmas 3.1 and 3.2. Note that, since the maximal tori $\boldsymbol{T}$ and $\boldsymbol{T}_i$ are conjugate in $\boldsymbol{G}^F$, we have $\boldsymbol{W}(\boldsymbol{G},\boldsymbol{T})^F \cong \boldsymbol{W}(\boldsymbol{G},\boldsymbol{T}_i)^F$ and, hence, $\boldsymbol{W}(\boldsymbol{G},\boldsymbol{T}_i)^F \cong C_{\boldsymbol{W},\sigma}(w)$ for $1 \leqslant i \leqslant m$.

Example 3.4. Assume that $\boldsymbol{G}$ is of split type; then $\sigma = \mathrm{id}_{\boldsymbol{W}}$. Then we define

$$R_\phi^{\boldsymbol{G}} := \frac{1}{|\boldsymbol{W}|} \sum_{w \in \boldsymbol{W}} \phi(w) R_{\boldsymbol{T}_w}^{\boldsymbol{G}}(1) \qquad \text{for } \phi \in \mathrm{Irr}(\boldsymbol{W}),$$

where $\boldsymbol{T}_w \subseteq \boldsymbol{G}$ is an F-stable maximal torus of type w and 1 stands for the trivial character of $\boldsymbol{T}_w^F$. (This is a very special case of [33, (3.7.1)].) We also have

$$R_{\boldsymbol{T}_w}^{\boldsymbol{G}}(1) = \sum_{\phi \in \mathrm{Irr}(\boldsymbol{W})} \phi(w) R_\phi^{\boldsymbol{G}} \qquad \text{for all } w \in \boldsymbol{W};$$

so knowing the $R_{\boldsymbol{T}_w}^{\boldsymbol{G}}(1)$'s is equivalent to knowing the $R_\phi^{\boldsymbol{G}}$'s. Also assume now that $\tilde{g} = su = us$ is such that $s \in \boldsymbol{T}_0^F$. Then one easily sees that

$$R_\phi^{\boldsymbol{G}}(\tilde{g}) = \sum_{\psi \in \mathrm{Irr}(\boldsymbol{W}_s)} m(\psi,\phi) R_\psi^{\boldsymbol{H}_s}(u) \qquad \text{for any } \phi \in \mathrm{Irr}(\boldsymbol{W}),$$

where $\boldsymbol{W}_s = N_{\boldsymbol{H}_s}(\boldsymbol{T}_0)/\boldsymbol{T}_0 \subseteq \boldsymbol{W}$ is the Weyl group of $\boldsymbol{H}_s$ and $m(\psi,\phi)$ denotes the multiplicity of $\psi \in \mathrm{Irr}(\boldsymbol{W}_s)$ in the restriction of ϕ. Thus, here the question of finding the fusion of F-stable maximal tori from $\boldsymbol{H}_s$ to $\boldsymbol{G}$ has been absorbed into the question of decomposing the restriction of any $\phi \in \mathrm{Irr}(\boldsymbol{W})$ to $\boldsymbol{W}_s$.

Example 3.5. Let $\boldsymbol{G}$ be simple of type F_4, where $p \neq 2$. There exists an involution $s \in \boldsymbol{T}_0^F$ such that $\boldsymbol{H}' := C_{\boldsymbol{G}}(s)$ has a root system of type B_4 (see §7.3 below for further details). Furthermore, there is a unipotent element $u \in \boldsymbol{H}^F$ such that, if we let Σ be the $\boldsymbol{G}^F$-conjugacy class of su, then Σ^F splits into five conjugacy classes in $\boldsymbol{G}^F$, with centraliser orders $8q^8, 8q^8, 4q^4, 4q^4, 4q^4$ (see §7.10 for more details). The condition on the centraliser orders uniquely determines the conjugacy class of u in $\boldsymbol{H}'$. Now let $u' \in \boldsymbol{H}'^F$ be one of the unipotent elements such that $su' \in \Sigma$ and $|C_{\boldsymbol{G}}(su')^F| = 8q^8$.

Let $\boldsymbol{W}' \subseteq \boldsymbol{W}$ be the Weyl group of $\boldsymbol{H}'$. Then $\mathrm{Irr}(\boldsymbol{W}')$ is parametrised by the bi-partitions of 4. By the output of the function `ICCTable` in Michel's version of CHEVIE [47], the only $\psi \in \mathrm{Irr}(\boldsymbol{W}')$ such that $R_\psi^{\boldsymbol{H}'}(u') \neq 0$ are $\psi_{(4,-)}$, $\psi_{(3,1)}$, $\psi_{(-,4)}$, $\psi_{(22,-)}$, $\psi_{(2,2)}$. Furthermore, we have

$$R^{\boldsymbol{H}'}_{\psi_{(4,-)}}(u') = 1,\; R^{\boldsymbol{H}'}_{\psi_{(3,1)}}(u') = q,\; R^{\boldsymbol{H}'}_{\psi_{(-,4)}}(u') = R^{\boldsymbol{H}'}_{\psi_{(22,-)}}(u') = R^{\boldsymbol{H}'}_{\psi_{(2,2)}}(u') = q^2.$$

In order to evaluate the formula for $R_\phi^{\boldsymbol{G}}$ in Example 3.4, we need to know the multiplicities $m(\psi,\phi)$ for $\psi \in \mathrm{Irr}(\boldsymbol{W}')$ and $\phi \in \mathrm{Irr}(\boldsymbol{W})$; these are readily available via the function `InductionTable` of CHEVIE [47]. This yields the following values:

$$\begin{aligned} R^{\boldsymbol{G}}_{\phi''_{9,6}}(su') &= R^{\boldsymbol{G}}_{\phi_{4,8}}(su') = R^{\boldsymbol{G}}_{\phi''_{1,12}}(su') = q^2, \\ R^{\boldsymbol{G}}_{\phi_{12,4}}(su') &= R^{\boldsymbol{G}}_{\phi'_{9,6}}(su') = R^{\boldsymbol{G}}_{\phi'_{1,12}}(su') = 0, \end{aligned}$$

where we use the notation of Carter [4, p. 413] for the irreducible characters of $\boldsymbol{W}$. (CHEVIE uses the same notation; the conversion to the notation defined and used by Lusztig [33] is displayed in Table 2.) The knowledge of the above values will turn out to be useful in the further discussion in §7.10.

Table 2: Conventions for the labelling of $\mathrm{Irr}(\boldsymbol{W})$ for type F_4

$\phi_{1,0}$	$\phi''_{1,12}$	$\phi'_{1,12}$	$\phi_{1,24}$	$\phi''_{2,4}$	$\phi'_{2,16}$	$\phi'_{2,4}$	$\phi''_{2,16}$	$\phi_{4,8}$	$\phi_{9,2}$	$\phi''_{9,6}$	$\phi'_{9,6}$	$\phi_{9,10}$
1_1	1_3	1_2	1_4	2_3	2_4	2_1	2_2	4_1	9_1	9_3	9_2	9_4

$\phi'_{6,6}$	$\phi''_{6,6}$	$\phi_{12,4}$	$\phi_{4,1}$	$\phi''_{4,7}$	$\phi'_{4,7}$	$\phi_{4,13}$	$\phi''_{8,3}$	$\phi'_{8,9}$	$\phi'_{8,3}$	$\phi''_{8,9}$	$\phi_{16,5}$
6_1	6_2	12_1	4_2	4_4	4_3	4_5	8_3	8_4	8_1	8_2	16_1

The labels $\phi_{1,0}$ etc. are those in [4, p. 413];
the labels 1_1 etc. those of Lusztig [33, 4.10].

4. Characteristic functions and conjugacy classes

Let $\hat{\boldsymbol{G}}$ denote the set of character sheaves on $\boldsymbol{G}$ (up to isomorphism), as defined by Lusztig [35]. If $A \in \hat{\boldsymbol{G}}$ is F-invariant, that is, we have $F^*A \cong A$, then the choice of an isomorphism $\phi\colon F^*A \xrightarrow{\sim} A$ gives rise to a characteristic function $\chi_A\colon \boldsymbol{G}^F \to \overline{\mathbb{Q}}_l$ (where $l \neq p$ is a prime); see [39, §5]. The isomorphism ϕ can be chosen such that the values of χ_A are cyclotomic integers and the standard inner product of χ_A with itself is 1. Hence, we may assume that χ_A is a function with values in $\mathbb{K}$. The various functions arising in this way form an orthonormal basis of the space of class functions on $\boldsymbol{G}^F$. (See [37, §25]; these results hold unconditionally because of the "cleanness" established in [44].) Similarly to the situation for $\mathrm{Irr}(\boldsymbol{G}^F)$, we have a partition $\hat{\boldsymbol{G}} = \coprod_s \hat{\boldsymbol{G}}_s$ where s runs over the semisimple elements (up to conjugation) in a group $\boldsymbol{G}^*$ dual to $\boldsymbol{G}$ (see [41, 1.2].) The character sheaves in $\hat{\boldsymbol{G}}_s$, where $s = 1$, are called unipotent character sheaves.

In the case where A is a cuspidal character sheaf (and $\boldsymbol{G}$ is simple), the characteristic functions χ_A can be evaluated in a simple way. We begin with some general remarks concerning conjugacy classes.

4.1. Parametrisation of $\boldsymbol{G}^F$-conjugacy classes. Let Σ be an F-stable conjugacy class of $\boldsymbol{G}$. Let us fix a representative $g_1 \in \Sigma^F$ and set $A_{\boldsymbol{G}}(g_1) := C_{\boldsymbol{G}}(g_1)/C^\circ_{\boldsymbol{G}}(g_1)$. Then F induces an automorphism of $A_{\boldsymbol{G}}(g_1)$ that we denote by the same symbol. Given $a \in A_{\boldsymbol{G}}(g_1)$, let $\dot{a} \in C_{\boldsymbol{G}}(g_1)$ be a representative of a and write $\dot{a} = x^{-1}F(x)$ for some $x \in \boldsymbol{G}$. Then $g_a := xg_1x^{-1} \in \boldsymbol{G}^F$; let C_a be the $\boldsymbol{G}^F$-conjugacy class of g_a. A standard argument (using Lang's Theorem, see [58, I, 2.7]) shows that C_a only depends on a; furthermore $\Sigma^F = \bigcup_{a\in A_{\boldsymbol{G}}(g_1)} C_a$, where $C_a = C_{a'}$ if and only if a, a' are F-conjugate in $A_{\boldsymbol{G}}(g_1)$. Now there are two natural operations on the $\boldsymbol{G}^F$-conjugacy classes contained in Σ^F.

(a) The first one is denoted by $\mathrm{Sh}_{\boldsymbol{G}}$ and called the Shintani map. Let C be a $\boldsymbol{G}^F$-conjugacy class contained in Σ^F; thus, $C = C_a$ for some $a \in A_{\boldsymbol{G}}(g_1)$. Let $g \in C$ and write $g = x^{-1}F(x)$ for some $x \in \boldsymbol{G}$. Then $g' := xgx^{-1} \in \boldsymbol{G}^F$ and the $\boldsymbol{G}^F$-conjugacy class of g' does not depend on the choice of g or x; we denote that class by $\mathrm{Sh}_{\boldsymbol{G}}(C)$. By Digne–Michel [9, Chap. IV, Prop. 1.1], we have

$$\mathrm{Sh}_{\boldsymbol{G}}(C_a) = C_{\bar{g}_1 a} \qquad (a \in A_{\boldsymbol{G}}(g_1))$$

where $\bar{g}_1$ denotes the image of $g_1 \in C_{\boldsymbol{G}}(g_1)$ in $A_{\boldsymbol{G}}(g_1)$.

(b) The second one, defined in [41, 3.1], only plays a role when $Z(\boldsymbol{G}) \neq \{1\}$. Let $z \in Z(\boldsymbol{G})$ and write $z = t^{-1}F(t)$ for some $t \in \boldsymbol{G}$. (We could even take $t \in \boldsymbol{T}_0$.) As above, let $C = C_a$ be a $\boldsymbol{G}^F$-conjugacy class contained in Σ^F. One easily sees that $\gamma_z(C) := tCt^{-1}$ is a conjugacy class in $\boldsymbol{G}^F$ and does not depend on the choice of t; furthermore, we have

$$\gamma_z(C_a) = C_{\bar{z}a} \qquad (a \in A_{\boldsymbol{G}}(g_1))$$

where $\bar{z} \in A_{\boldsymbol{G}}(g_1)$ denotes the image of z under the natural map $Z(\boldsymbol{G}) \to A_{\boldsymbol{G}}(g_1)$.

4.2. Characteristic functions of cuspidal character sheaves. Assume that $\boldsymbol{G}$ is a simple algebraic group. Let A be a cuspidal character sheaf on $\boldsymbol{G}$ such that $F^*A \cong A$. (See [35, Def. 3.10]; such an A may be unipotent or not.) Then there exists an F-stable conjugacy class Σ of $\boldsymbol{G}$ and an irreducible, $\boldsymbol{G}$-equivariant $\overline{\mathbb{Q}}_l$-local system $\mathcal{E}$ on Σ such that $F^*\mathcal{E} \cong \mathcal{E}$ and $A = \mathrm{IC}(\overline{\Sigma}, \mathcal{E})[\dim \Sigma]$; see [35, 3.12]. Let us fix $g_1 \in \Sigma^F$ and set $A_{\boldsymbol{G}}(g_1) := C_{\boldsymbol{G}}(g_1)/C^\circ_{\boldsymbol{G}}(g_1)$, as above. We further assume that:

($*$) the local system $\mathcal{E}$ is one-dimensional and, hence, corresponds to an F-invariant linear character $\psi\colon A_{\boldsymbol{G}}(g_1) \to \mathbb{K}^\times$ (via [42, 19.7]).

(This assumption will be satisfied in all examples that we consider.) Now ($*$) implies that the function $\psi\colon A_{\boldsymbol{G}}(g_1) \to \mathbb{K}^\times$ is constant on the F-conjugacy classes of $A_{\boldsymbol{G}}(g_1)$. Hence, we obtain a class function $\chi_{g_1,\psi}\colon \boldsymbol{G}^F \to \mathbb{K}$ by setting

$$\chi_{g_1,\psi}(g) := \begin{cases} q^{(\dim \boldsymbol{G} - \dim \Sigma)/2}\psi(a) & \text{if } g \in C_a \text{ for some } a \in A_{\boldsymbol{G}}(g_1), \\ 0 & \text{if } g \notin \Sigma^F. \end{cases}$$

(Note that there are cases where $\dim \boldsymbol{G} - \dim \Sigma$ is not even; in such a case, we also need to fix a square root of q in $\mathbb{K}$. This is typically done by fixing once and for all a square root $\sqrt{p} \in \mathbb{K}$ and then setting $\sqrt{q} := \sqrt{p}^f$ if $q = p^f$ with $f \geqslant 1$.) Since $\mathcal{E}$ is one-dimensional, we can choose an isomorphism $F^*\mathcal{E} \xrightarrow{\sim} \mathcal{E}$ such that the induced map on the stalk $\mathcal{E}_{g_1}$ is given by scalar multiplication by $q^{(\dim \boldsymbol{G} - \dim \Sigma)/2}$. Then this isomorphism canonically induces an isomorphism $\phi\colon F^*A \xrightarrow{\sim} A$ and $\chi_{g_1,\psi}$ is the corresponding characteristic function χ_A, of norm 1 with respect to the standard inner product. (This follows from the fact that A is "clean" [44], using the construction in [42, 19.7].) We shall also set

$$\lambda_A := \psi(\bar{g}_1) \qquad \text{where } \bar{g}_1 \text{ denotes the image of } g_1 \text{ in } A_{\boldsymbol{G}}(g_1).$$

Then λ_A is a root of unity that only depends on A (see Shoji [53, Theorem 3.3], [53, Prop. 3.8]); it is a useful invariant of A. In this context, we have the following basic problem, formulated by Lusztig [41, 0.4(a)]:

(♣) Express the functions $\chi_{g_1,\psi}$ as explicit linear combinations of $\mathrm{Irr}(\boldsymbol{G}^F)$.

This problem is solved in many cases, but not in complete generality. Some examples in small rank cases (types A_1, C_2, ${}^3D_4$, $\ldots$) are mentioned in [17, Example 7.8]. We will produce further examples below.

For $\boldsymbol{G}$ simple of exceptional type, many cuspidal character sheaves turn out to be unipotent. (Exceptions only occur in types E_6 and E_7.) So it is of particular importance to address theses cases.

4.3. Cuspidal unipotent character sheaves. Assume that $\boldsymbol{G}$ is simple, of split type (so $\sigma = \mathrm{id}_{\boldsymbol{W}}$). Let $\mathrm{Unip}(\boldsymbol{G}^F)$ denote the set of unipotent characters of $\boldsymbol{G}^F$. By [33, Main Theorem 4.23], $\mathrm{Unip}(\boldsymbol{G}^F)$ is parametrised by a certain set $X(\boldsymbol{W})$ which only depends on $\boldsymbol{W}$; we shall write $\mathrm{Unip}(\boldsymbol{G}^F) = \{\rho_x \mid x \in X(\boldsymbol{W})\}$. For each $x \in X(\boldsymbol{W})$, we also have a corresponding almost character R_x, defined as an explicit linear combination of $\mathrm{Unip}(\boldsymbol{G}^F)$; see [33, 4.24.1]. The matrix of multiplicities $\langle \rho_x, R_{x'} \rangle$ $(x, x' \in X(\boldsymbol{W}))$ is Lusztig's non-abelian Fourier matrix; see also [4, §13.6]. There is an embedding $\mathrm{Irr}(\boldsymbol{W}) \hookrightarrow X(\boldsymbol{W})$, $\phi \mapsto x_\phi$, such that

$$R_{x_\phi} = R_\phi^{\boldsymbol{G}} = \frac{1}{|\boldsymbol{W}|} \sum_{w \in \boldsymbol{W}} \phi(w) R_{\boldsymbol{T}_w}^{\boldsymbol{G}}(1) \qquad \text{for } \phi \in \mathrm{Irr}(\boldsymbol{W}).$$

Thus, the values of R_{x_ϕ} can be computed using the character table of $\boldsymbol{W}$ and the results discussed in Section 3. Finally, the unipotent character sheaves on $\boldsymbol{G}$ are also parametrised by $X(\boldsymbol{W})$; see [37, Theorem 23.1] (plus the "cleanness" in [44]). If $x \in X(\boldsymbol{W})$ is such that A_x is cuspidal and F-invariant, then we have a corresponding characteristic function $\chi_{g_1,\psi}$ as in §4.2. In this situation, there is the following solution of (♣) in §4.2. For all $x \in X(\boldsymbol{W})$ such that A_x is cuspidal, we have

(♣′) $R_x = \zeta \chi_{g_1,\psi}$ for some scalar $\zeta \in \mathbb{K}$ of absolute value 1.

If p is sufficiently large, then this is part of Lusztig [41, Theorem 0.8]. For arbitrary p, this is part of the main results of Shoji [53], [54], (The latter results hold without condition on p, thanks to the "cleanness" in [44].) The scalars ζ are determined by Shoji [52], [56] for $\boldsymbol{G}$ of classical type. For exceptional types, there are a number of cases where the scalars ζ remain to be determined, and it is one purpose of this paper to reduce the number of open cases.

The following technical result will be needed in Section 6.

Lemma 4.4. *In the setting of §4.1, let $a \in A_{\boldsymbol{G}}(g_1)$ and $z \in Z(\boldsymbol{G})$. Then every $\rho \in \mathrm{Unip}(\boldsymbol{G}^F)$ takes the same value on C_a and on $C_{\bar{z}a}$.*

Proof. Let $g \in C_a$. By §4.1(b) we have $C_{\bar{z}a} = \gamma_z(C_a)$. Hence, writing $z = t^{-1}F(t)$ for some $t \in \boldsymbol{G}$, we have $g' := tgt^{-1} \in C_{\bar{z}a}$. Let $\rho \in \mathrm{Unip}(\boldsymbol{G}^F)$. In order to show that $\rho(g) = \rho(g')$, we use a regular embedding $\boldsymbol{G} \subseteq \tilde{\boldsymbol{G}}$ (see, e.g., [21, §1.7]). Thus, $\tilde{\boldsymbol{G}}$ is a connected reductive group with a connected center and $\boldsymbol{G}$, $\tilde{\boldsymbol{G}}$ have the same derived subgroup; furthermore, $\tilde{\boldsymbol{G}}$ is also defined over $\mathbb{F}_q$ and we denote the corresponding Frobenius map again by F. Now $Z(\boldsymbol{G}) \subseteq Z(\tilde{\boldsymbol{G}})$ and so, since $Z(\tilde{\boldsymbol{G}})$ is connected, we can write $z = \tilde{t}^{-1}F(\tilde{t})$ where $\tilde{t} \in Z(\tilde{\boldsymbol{G}})$. Then $h := t\tilde{t}^{-1} \in \tilde{\boldsymbol{G}}^F$ and so $hgh^{-1} = t\tilde{t}^{-1}g\tilde{t}t^{-1} = tgt^{-1} =$

g', that is, g and g' are conjugate in $\tilde{\boldsymbol{G}}^F$. Since ρ is unipotent, there exists some F-stable maximal torus $\boldsymbol{T} \subseteq \boldsymbol{G}$ such that ρ occurs in $R_{\boldsymbol{T}}^{\boldsymbol{G}}(1)$ (where 1 stands for the trivial character of $\boldsymbol{T}^F$). There is an F-stable maximal torus $\tilde{\boldsymbol{T}} \subseteq \tilde{\boldsymbol{G}}$ such that $\boldsymbol{T} \subseteq \tilde{\boldsymbol{T}}$. Since $R_{\boldsymbol{T}}^{\boldsymbol{G}}(1)$ is the restriction of $R_{\tilde{\boldsymbol{T}}}^{\tilde{\boldsymbol{G}}}(1)$ to $\boldsymbol{G}^F$ (see [21, Remark 2.3.16]), there exists some $\tilde{\rho} \in \mathrm{Unip}(\tilde{\boldsymbol{G}})$ such that ρ occurs in the restriction of $\tilde{\rho}$ to $\boldsymbol{G}^F$. But it is known that $\tilde{\rho}|_{\boldsymbol{G}^F}$ is irreducible (see [21, Lemma 2.3.14]) and so ρ is equal to the restriction of $\tilde{\rho}$. Thus, we certainly have $\rho(g) = \tilde{\rho}(g) = \tilde{\rho}(g') = \rho(g')$. $\square$

Example 4.5. Let $\boldsymbol{G}$ be simple of type G_2. In this case, the complete character table of $\boldsymbol{G}^F$ is known; see Chang–Ree [5] ($p \neq 2, 3$), Enomoto [11] ($p = 3$) and Enomoto–Yamada [12] ($p = 2$). Now, there are four cuspidal character sheaves, and they are all unipotent; see [36, §20], [53, §6, §7]. From the known character tables, the above identities ($\clubsuit'$) and the required scalars ζ can be easily extracted. For example, if $p \neq 2, 3$, then the four functions Y_1, Y_2, Y_3, Y_4 printed on [5, p. 411] are characteristic functions of the four cuspidal character sheaves on $\boldsymbol{G}$.

Let us go back to the general case. Implicit in ($\clubsuit$) and ($\clubsuit'$) is the problem of choosing a convenient representative $g_1 \in \Sigma^F$. In a number of cases, Σ consists of regular elements in $\boldsymbol{G}$. In such a case, there are additional techniques to single out a canonical choice for $g_1 \in \Sigma^F$; see Corollary 4.8 below.

4.6. Regular elements. An element $g \in \boldsymbol{G}$ is called regular, if $\dim C_{\boldsymbol{G}}(g)$ is as small as possible; it is known that this is equivalent to the condition that $\dim C_{\boldsymbol{G}}(g) = \dim \boldsymbol{T}_0$. Furthermore, let $g = su = us$ be the Jordan decomposition of g (where s is semisimple and u is unipotent). Then g is regular if and only if u is regular in $C_{\boldsymbol{G}}^{\circ}(s)$. (For all this see, for example, [10, §12.1].) By Steinberg [59, Theorem 1.2], every semisimple element of $\boldsymbol{G}$ is the semisimple part of some regular element; finally, two regular elements of $\boldsymbol{G}$ are conjugate if and only if their semisimple parts are conjugate. In particular, all regular unipotent elements are conjugate.

Assume now that $\boldsymbol{G}$ is simple and simply connected. Then a cross-section for the conjugacy classes of regular elements has been found by Steinberg [59]. Let us write $\boldsymbol{B} = \boldsymbol{U}\boldsymbol{T}_0$ where $\boldsymbol{U}$ is the unipotent radical of $\boldsymbol{B}$. Let $\boldsymbol{B}^- \subseteq \boldsymbol{G}$ be the opposite Borel subgroup; then $\boldsymbol{B}^- = \boldsymbol{U}^-\boldsymbol{T}_0$, where $\boldsymbol{U}^-$ is the unipotent radical of $\boldsymbol{B}^-$, and we have $\boldsymbol{U} \cap \boldsymbol{U}^- = \{1\}$. Let $w_c := w_{\alpha_1} \cdots w_{\alpha_r} \in \boldsymbol{W}$ be a Coxeter element, where $r = \dim \boldsymbol{T}_0$ and $\alpha_1, \ldots, \alpha_r$ is a fixed enumeration of the simple roots in Φ^+. Then the required cross-section is given by

$$\mathcal{N}_{\dot{w}_c} := \boldsymbol{U}\dot{w}_c \cap \dot{w}_c\boldsymbol{U}^- \;\subseteq\; \boldsymbol{U}\dot{w}_c\boldsymbol{U} \;\subseteq\; \boldsymbol{B}\dot{w}_c\boldsymbol{B}.$$

Indeed, by [59, Theorem 1.4, Lemma 7.3], all elements of $\mathcal{N}_{\dot{w}_c}$ are regular and every regular element of $\boldsymbol{G}$ is conjugate to exactly one element in $\mathcal{N}_{\dot{w}_c}$. And everything takes place inside the single double coset $\boldsymbol{U}\dot{w}_c\boldsymbol{U}$; note that this depends on the choice of the representative $\dot{w}_c$ of $w_c \in \boldsymbol{W}$. The following result is a very special case of the results on "C-small" classes in [43, §5], so we include the proof here. (It is already mentioned in the proof of [38, Lemma 8.10].)

Proposition 4.7 (Steinberg, He–Lusztig)**.** *Assume that $\boldsymbol{G}$ is simple and simply connected. Let Σ be a $\boldsymbol{G}$-conjugacy class of regular elements.*

(a) The set $\Sigma \cap \boldsymbol{U}\dot{w}_c\boldsymbol{U} \neq \varnothing$ is a single $\boldsymbol{U}$-orbit (under conjugation).
(b) The set $\Sigma \cap \boldsymbol{B}\dot{w}_c\boldsymbol{B} \neq \varnothing$ is a single $\boldsymbol{B}$-orbit (under conjugation).
(c) In (a)*, the stabilisers are trivial; in* (b) *they are equal to $Z(\boldsymbol{G})$.*

Proof. By He–Lusztig [24, Theorem 3.6(ii)], the map

$$\boldsymbol{U} \times \mathcal{N}_{\dot{w}_c} \to \boldsymbol{U}\dot{w}_c\boldsymbol{U}, \qquad (u, z) \mapsto uzu^{-1},$$

is bijective. (A closely related result is stated in [59, Prop. 8.9], but since the proof is omitted there, we cite [24]; see also [2, §10].) Let us denote by g the unique element in $\Sigma \cap \mathcal{N}_{\dot{w}_c}$; in particular, $g \in \Sigma \cap \boldsymbol{U}\dot{w}_c\boldsymbol{U}$.

(a) Given any $g' \in \Sigma \cap \boldsymbol{U}\dot{w}_c\boldsymbol{U}$, we can write $g' = uzu^{-1}$ where $u \in \boldsymbol{U}$ and $z \in \mathcal{N}_{\dot{w}_c}$. Thus, the two elements z and g in $\mathcal{N}_{\dot{w}_c}$ are conjugate in $\boldsymbol{G}$. But then we must have $z = g$ and so g' is conjugate to g under $\boldsymbol{U}$.

(b) Take any $g' \in \Sigma \cap \boldsymbol{B}\dot{w}_c\boldsymbol{B}$. Since $\boldsymbol{B}\dot{w}_c\boldsymbol{B} = \boldsymbol{U}\boldsymbol{T}_0\dot{w}_c\boldsymbol{U}$, we can write $g' = u_1 t\dot{w}_c u_2$ where $u_1, u_2 \in \boldsymbol{U}$ and $t \in \boldsymbol{T}_0$. By [59, Lemma 7.6], we can further write $t\dot{w}_c = \tilde{t}\dot{w}_c\tilde{t}^{-1}$ for some $\tilde{t} \in \boldsymbol{T}_0$. Then

$$\tilde{t}^{-1}g'\tilde{t} = \tilde{t}^{-1}u_1 t\dot{w}_c u_2\tilde{t} = (\tilde{t}^{-1}u_1\tilde{t})\dot{w}_c(\tilde{t}^{-1}u_2\tilde{t}) \in \boldsymbol{U}\dot{w}_c\boldsymbol{U}.$$

So we have $\tilde{t}^{-1}g'\tilde{t} = uzu^{-1}$ where $u \in \boldsymbol{U}$ and $z \in \mathcal{N}_{\dot{w}_c}$. Thus, $z \in \mathcal{N}_{\dot{w}_c}$ and $g \in \mathcal{N}_{\dot{w}_c}$ are conjugate in $\boldsymbol{G}$ and so $z = g$. Hence, g' is conjugate to g under $\boldsymbol{B}$.

(c) The bijectivity of the above map $\boldsymbol{U} \times \mathcal{N}_{\dot{w}_c} \to \boldsymbol{U}\dot{w}_c\boldsymbol{U}$ immediately implies that $C_{\boldsymbol{U}}(g) = \{1\}$; thus, the stabilisers are trivial in (a). For (b), we must show that $\mathrm{Stab}_{\boldsymbol{B}}(g) = Z(\boldsymbol{G})$. So let $b \in \boldsymbol{B}$ be such that $bgb^{-1} = g$. Writing $g = v\dot{w}_c$ (where $v \in \boldsymbol{U}$) and $b = ut$ (where $u \in \boldsymbol{U}$ and $t \in \boldsymbol{T}_0$), we obtain

$$v\dot{w}_c = g = bgb^{-1} = utv\dot{w}_c t^{-1}u^{-1} = (utvt^{-1})\dot{w}_c(\dot{w}_c^{-1}t\dot{w}_c t^{-1})u^{-1}.$$

Setting $\tilde{t} := \dot{w}_c^{-1}t\dot{w}_c t^{-1} \in \boldsymbol{T}_0$, we see that the left hand side lies in the double coset $\boldsymbol{U}\dot{w}_c\boldsymbol{U}$, and the right hand side lies in the double coset $\boldsymbol{U}\dot{w}_c\tilde{t}\boldsymbol{U}$. But then the sharp form of the Bruhat decomposition implies that $\tilde{t} = 1$ and so $t = \dot{w}_c^{-1} t \dot{w}_c$. By [59, Remarks 7.7(b)], this forces $t \in Z(\boldsymbol{G})$. But then $u \in C_{\boldsymbol{U}}(g)$ and so $u = 1$. Hence, $\mathrm{Stab}_{\boldsymbol{B}}(g) \subseteq Z(\boldsymbol{G})$; the reverse inclusion is clear. □

Corollary 4.8. *In the setting of Proposition 4.7, assume that Σ is F-stable and $F(\dot{w}_c) = \dot{w}_c$. Then there exists a unique $\boldsymbol{G}^F$-conjugacy class $C \subseteq \Sigma^F$ such that $C \cap \boldsymbol{U}^F \dot{w}_c \boldsymbol{U}^F \neq \varnothing$. Furthermore, we have $C \cap \mathcal{N}_{\dot{w}_c} \neq \varnothing$.*

Proof. By Proposition 4.7, the group $\boldsymbol{U}$ acts transitively on $X := \Sigma \cap \boldsymbol{U} \dot{w}_c \boldsymbol{U}$ by conjugation, and we have $\mathrm{Stab}_{\boldsymbol{U}}(x) = \{1\}$ for all $x \in X$; in particular, $\mathrm{Stab}_{\boldsymbol{U}}(x)$ is connected. A standard application of Lang's Theorem (see, e.g., [21, Prop. 1.4.9]) shows that $X^F \neq \varnothing$ and that X^F is a single $\boldsymbol{U}^F$-orbit. Thus, there exists a unique $\boldsymbol{G}^F$-conjugacy class $C \subseteq \Sigma^F$ such that $C \cap (\boldsymbol{U} \dot{w}_c \boldsymbol{U})^F \neq \varnothing$. Note that $(\boldsymbol{U} \dot{w}_c \boldsymbol{U})^F = \boldsymbol{U}^F \dot{w}_c \boldsymbol{U}^F$, by the sharp form of the Bruhat decomposition.

Finally note that, since $F(\dot{w}_c) = \dot{w}_c$, we have $F(\mathcal{N}_{\dot{w}_c}) = \mathcal{N}_{\dot{w}_c}$. Let g be the unique element in $\Sigma \cap \mathcal{N}_{\dot{w}_c}$. Then we also have $F(g) \in \Sigma \cap \mathcal{N}_{\dot{w}_c}$ and so $F(g) = g$. Hence, $g \in \Sigma^F$ and $g \in \boldsymbol{U}^F \dot{w}_c \boldsymbol{U}^F$. So C must be the $\boldsymbol{G}^F$-conjugacy class of g. □

Example 4.9. Let $\boldsymbol{G}$, Σ, C be as in Corollary 4.8. Then, clearly, we also have $C \cap \boldsymbol{B}^F \dot{w}_c \boldsymbol{B}^F \neq \varnothing$. Since the stabilisers for the action of $\boldsymbol{B}$ on $\Sigma \cap \boldsymbol{B} \dot{w}_c \boldsymbol{B}$ are equal to $Z(\boldsymbol{G})$, the set $(\Sigma \cap \boldsymbol{B} \dot{w}_c \boldsymbol{B})^F$ will split into finitely many $\boldsymbol{B}^F$-orbits, indexed by the F-conjugacy classes of $Z(\boldsymbol{G})$. More precisely, let g be the unique element in $C \cap \mathcal{N}_{\dot{w}_c}$. Let $z_1, \ldots, z_r \in Z(\boldsymbol{G})$ be representatives of the F-conjugacy classes of $Z(\boldsymbol{G})$, where $z_1 = 1$. For $1 \leqslant i \leqslant r$, we set $g_i := t_i g t_i^{-1}$, where $t_i \in \boldsymbol{T}_0$ is such that $z_i = t_i^{-1} F(t_i)$; here, we also assume $t_1 = 1$. Then $g_1, \ldots, g_r$ are representatives of the $\boldsymbol{B}^F$-orbits on $(\Sigma \cap \boldsymbol{B} \dot{w}_c \boldsymbol{B})^F$ (see [58, I, 2.7]). Hence,

$$\text{(a)} \qquad (\Sigma \cap \boldsymbol{B} \dot{w}_c \boldsymbol{B})^F = (C_1 \cap \boldsymbol{B}^F \dot{w}_c \boldsymbol{B}^F) \cup \ldots \cup (C_r \cap \boldsymbol{B}^F \dot{w}_c \boldsymbol{B}^F),$$

where C_i is the $\boldsymbol{G}^F$-conjugacy class of g_i, for all i. Since $z_i = t_i^{-1} F(t_i) \in Z(\boldsymbol{G})$, the map $C_{\boldsymbol{G}}(g)^F \to C_{\boldsymbol{G}}(g_i)^F$, $x \mapsto t_i x t_i^{-1}$, is bijective. In particular, we have

$$\text{(b)} \qquad |C| = |C_i| \qquad \text{for } 1 \leqslant i \leqslant r.$$

The union in (a) may not be disjoint; but $C_i = C_j$ can only happen if the images of z_i and z_j in $A_{\boldsymbol{G}}(g)$ are F-conjugate. Now, it is known that $A_{\boldsymbol{G}}(g)$ is abelian, see [58, III, 1.16 and 1.17]; so, for example, if the natural map $Z(\boldsymbol{G}) \to A_{\boldsymbol{G}}(g)$ is injective and F acts trivially on $A_{\boldsymbol{G}}(g)$, then the C_i are all distinct. (This will cover most examples that we consider.) Finally, we note the implication:

$$\text{(c)} \qquad Z(\boldsymbol{G})^F = \{1\} \qquad \Rightarrow \qquad (\Sigma \cap \boldsymbol{B}\dot{w}_c\boldsymbol{B})^F = C \cap \boldsymbol{B}^F\dot{w}_c\boldsymbol{B}^F.$$

Indeed, if $Z(\boldsymbol{G})^F = \{1\}$, then all elements of $Z(\boldsymbol{G})$ are F-conjugate and so $r = 1$.

Lemma 4.10. *Let $\boldsymbol{G}$, Σ, C be as in Corollary 4.8. Assume, furthermore, that $\boldsymbol{G}$ is of split type, that $\Sigma = \Sigma^{-1}$, and that the union in Example 4.9(a) is disjoint.*

(a) There is a permutation $i \mapsto i'$ (of order 2) of the set $\{1, \ldots, r\}$ such that $C_i^{-1} = C_{i'}$ for $1 \leqslant i \leqslant r$.
(b) If r is odd (e.g., if $Z(\boldsymbol{G})^F = \{1\}$), then we have $C_i = C_i^{-1}$ for $1 \leqslant i \leqslant r$.

Proof. First note that $C_i^{-1} \subseteq \Sigma^{-1} = \Sigma$ for $1 \leqslant i \leqslant r$. We claim that

$$C_i^{-1} \cap \boldsymbol{B}^F\dot{w}_c\boldsymbol{B}^F \neq \varnothing \qquad \text{for all } i.$$

To see this, it is enough to show that $C_i \cap \boldsymbol{B}^F\dot{w}_c^{-1}\boldsymbol{B}^F \neq \varnothing$. Now note that w_c^{-1} also is a Coxeter element (of minimal length); it is well-known that w_c and w_c^{-1} are conjugate in $\boldsymbol{W}$. By [43, 0.2], [16, Cor. 3.7(a)], we have

$$|C_i \cap \boldsymbol{B}^F\dot{w}\boldsymbol{B}^F| = |C_i \cap \boldsymbol{B}^F\dot{w}'\boldsymbol{B}^F|$$

for any two elements $w, w' \in \boldsymbol{W}$ that are conjugate in $\boldsymbol{W}$ and of minimal length in their conjugacy class. In particular, we can conclude that

$$|C_i^{-1} \cap \boldsymbol{B}^F\dot{w}_c\boldsymbol{B}^F| = |C_i \cap \boldsymbol{B}^F\dot{w}_c^{-1}\boldsymbol{B}^F| = |C_i \cap \boldsymbol{B}^F\dot{w}_c\boldsymbol{B}^F| \neq 0$$

for all i, as required. Thus, the above claim is proved.

(a) Let $i \in \{1, \ldots, r\}$. By the above claim, C_i^{-1} is a $\boldsymbol{G}^F$-conjugacy class that is contained in $(\Sigma \cap \boldsymbol{B}\dot{w}_c\boldsymbol{B})^F$. So, by Example 4.9(a), we must have $C_i^{-1} = C_{i'}$ for some $i' \in \{1, \ldots, r\}$. Since the C_i are all distinct, i' is uniquely determined by i; furthermore, the map $i \mapsto i'$ is a permutation (of order 2) of the set $\{1, \ldots, r\}$.

(b) Assume that r is odd. Then there must be some $i_0 \in \{1, \ldots, r\}$ such that $i_0 = i_0'$, that is, $C_{i_0} = C_{i_0}^{-1}$. Now recall that C_{i_0} is the $\boldsymbol{G}^F$-conjugacy class of g_{i_0}, where $g_{i_0} = t_{i_0} g t_{i_0}^{-1}$ and $t_{i_0} \in \boldsymbol{T}_0$ is such that $z_{i_0} = t_{i_0}^{-1}F(t_{i_0})$. There exists some $x \in \boldsymbol{G}^F$ such that $g_{i_0}^{-1} = x g_{i_0} x^{-1}$. It follows that $g^{-1} = ygy^{-1}$, where $y := t_{i_0}^{-1} x t_{i_0}$. Now $F(y) = F(t_{i_0})^{-1} x F(t_{i_0}) = z_{i_0}^{-1} t_{i_0}^{-1} x t_{i_0} z_{i_0} = y$, since $z_{i_0} \in Z(\boldsymbol{G})$. Thus, we have shown that $C = C^{-1}$. Then the same argument, applied to any $i \in \{1, \ldots, r\}$, also yields that $C_i = C_i^{-1}$. □

The following example (pointed out to the author by G. Malle) shows that the situation is really different when r is even.

Example 4.11. Let $\boldsymbol{G} = \mathrm{SL}_2(k)$ with $p \neq 2$. Then $Z(\boldsymbol{G})$ has order 2 and so $r = 2$. Let Σ be the class of regular unipotent elements; we certainly have $\Sigma = \Sigma^{-1}$. Let $\boldsymbol{B}$ be the Borel subgroup consisting of the upper triangular matrices in $\boldsymbol{G}$; also let $\boldsymbol{W} = \langle s_1 \rangle$. Then one checks that

$$g := \left(\begin{array}{cc} 1 & 0 \\ 1 & 1 \end{array} \right) \in \Sigma^F \cap \boldsymbol{U}\dot{s}_1\boldsymbol{U}^F \qquad \text{where} \qquad \dot{s}_1 := \left(\begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right).$$

So the unique class C in Corollary 4.8 is the $\boldsymbol{G}^F$-conjugacy class of g. Now Σ^F splits into two classes in $\boldsymbol{G}^F$, with representatives g and

$$g' := \left(\begin{array}{cc} 1 & 0 \\ \xi & 1 \end{array} \right) \in \Sigma^F, \qquad \text{where } \xi \in \mathbb{F}_q^\times \text{ is not a square in } \mathbb{F}_q^\times.$$

(One checks that, indeed, $g' \in \boldsymbol{B}^F\dot{s}_1\boldsymbol{B}^F$ but $g' \notin \boldsymbol{U}^F\dot{s}_1\boldsymbol{U}^F$.) Furthermore, one checks that g and g^{-1} are conjugate in $\boldsymbol{G}^F$ if and only if -1 is a square in $\mathbb{F}_q^\times$, that is, if and only if $q \equiv 1 \bmod 4$. Hence, if $q \equiv 3 \bmod 4$, then $C \neq C^{-1}$.

Remark 4.12. Assume that $\boldsymbol{G}$ is of split type (then $\sigma = \mathrm{id}_{\boldsymbol{W}}$). Let Σ be an arbitrary F-stable conjugacy class of $\boldsymbol{G}$ and $w \in \boldsymbol{W}$. For any $g \in \Sigma^F$ denote by C_g the $\boldsymbol{G}^F$-conjugacy class of g. Then the cardinalities $|C_g \cap \boldsymbol{B}^F\dot{w}\boldsymbol{B}^F|$ can be computed using the representation theory of $\boldsymbol{G}^F$; see, e.g., [43, 1.2(a)]. For this purpose, we consider the induced character $\mathrm{Ind}_{\boldsymbol{B}^F}^{\boldsymbol{G}^F}(1)$ (where 1 stands for the trivial character of $\boldsymbol{B}^F$) and let $\mathcal{H}_q$ be the corresponding Hecke algebra, that is, the endomorphism algebra of a $\mathbb{K}\boldsymbol{G}^F$-module affording $\mathrm{Ind}_{\boldsymbol{B}^F}^{\boldsymbol{G}^F}(1)$. This algebra has a standard basis usually denoted by $\{T_w \mid w \in \boldsymbol{W}\}$, where

$$T_{w_\alpha}^2 = qT_1 + (q-1)T_{w_\alpha} \qquad \text{for every simple root } \alpha \in \Phi.$$

There is a bijection, $\phi \leftrightarrow \phi^{(q)}$, between $\mathrm{Irr}(\boldsymbol{W})$ and the irreducible characters of $\mathcal{H}_q$ (which is canonical once a square root $\sqrt{q} \in \mathbb{K}$ has been fixed; see, e.g., [23, §9.3].) Via this correspondence, the irreducible characters of $\boldsymbol{G}^F$ that occur in $\mathrm{Ind}_{\boldsymbol{B}^F}^{\boldsymbol{G}^F}(1)$ are parametrised by $\mathrm{Irr}(\boldsymbol{W})$ (see, e.g., [23, §8.4]); we denote by $[\phi] \in \mathrm{Irr}(\boldsymbol{G}^F)$ the character corresponding to $\phi \in \mathrm{Irr}(\boldsymbol{W})$. Then we have:

$$\text{(a)} \qquad |C_g \cap \boldsymbol{B}^F\dot{w}\boldsymbol{B}^F| = \frac{|\boldsymbol{B}^F|}{|C_{\boldsymbol{G}}(g)^F|} \sum_{\phi \in \mathrm{Irr}(\boldsymbol{W})} \phi^{(q)}(T_w)\,[\phi](g)$$

for any $w \in \boldsymbol{W}$ and any $g \in \boldsymbol{G}^F$. (See [43, 1.2(a)] and [16, Remark 3.6] for further details and references.) The values $\phi^{(q)}(T_w)$ are explicitly known (or there are explicit combinatorial algorithms); see [23]. Hence, if we have sufficient information on the values of $[\phi] \in \mathrm{Irr}(\boldsymbol{G}^F)$, then we can work out the cardinality $|C_g \cap \boldsymbol{B}^F \dot{w} \boldsymbol{B}^F|$, and this will be useful to identify $C_g \subseteq \Sigma^F$.

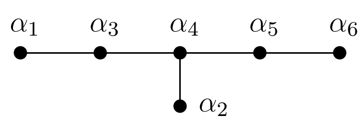
Assume now that we are in the setting of §4.6, where $\boldsymbol{G}$ is simple and simply connected, Σ consists of regular elements and $w = w_c$ is a Coxeter element. Also assume that the natural map $Z(\boldsymbol{G}) \to A_{\boldsymbol{G}}(g_1)$ is injective and F acts trivially on $A_{\boldsymbol{G}}(g_1)$. Let $C_1, \ldots, C_r$ be the $\boldsymbol{G}^F$-conjugacy classes that are contained in Σ^F and have a non-empty intersection with $\boldsymbol{B}^F \dot{w}_c \boldsymbol{B}^F$. Then, by Example 4.9, we have $r = |Z(\boldsymbol{G})|$ and each set $C_i \cap \boldsymbol{B}^F \dot{w}_c \boldsymbol{B}^F$ is a single $\boldsymbol{B}^F$-orbit, of size $|\boldsymbol{B}^F|/r$. Hence, the above identity (a) can be expressed as follows.

$$\text{(b)} \qquad \sum_{\phi \in \mathrm{Irr}(\boldsymbol{W})} \phi^{(q)}(T_w)\,[\phi](g) = \begin{cases} \frac{1}{r}|C_{\boldsymbol{G}}(g)^F| & \text{if } g \in C_1 \cup \ldots \cup C_r, \\ 0 & \text{if } g \in \Sigma^F \setminus (C_1 \cup \ldots \cup C_r). \end{cases}$$

This identity can be exploited to obtain information on the character values $[\phi](g)$ and, hence, potentially, on the unknown scalars ζ in §4.3(♣′); see the proof of Proposition 6.5 below for an example.

5. Cuspidal unipotent character sheaves in type E_6

Throughout this section, let $\boldsymbol{G}$ be simple, simply connected of type E_6. Let $q = p^f$ (where $f \geqslant 1$) be such that $F\colon \boldsymbol{G} \to \boldsymbol{G}$ defines an $\mathbb{F}_q$-rational structure. Except for §5.6 (at the very end), we assume that $\boldsymbol{G}$ is of split type; thus, $\sigma = \mathrm{id}_{\boldsymbol{W}}$ and the permutation $\alpha \mapsto \alpha^\dagger$ of Φ is the identity. Let $\Delta = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6\}$ be the set of simple roots in Φ^+, where the labelling is chosen as follows.



If $p = 3$, then the cuspidal character sheaves and almost characters have been considered by Hetz [25]. So assume from now on that $p \neq 3$. Let $\alpha_0 \in \Phi$ be the unique root of maximal height and consider the subsystem $\Phi_0 \subseteq \Phi$ of type $A_2 \times A_2 \times A_2$ spanned by $\{\alpha_1, \alpha_2, \alpha_3, \alpha_5, \alpha_6, \alpha_0\}$. The relevance of this particular example is that Φ_0 occurs in the classification of cuspidal unipotent character sheaves on $\boldsymbol{G}$; see [36, Prop. 20.3] (and also the remarks in [54, 5.2]).

Using CHEVIE, we find that the unique set Δ_0 of simple roots in $\Phi_0 \cap \Phi^+$ is given by

$$\Delta_0 = \{\ \alpha_1,\ \alpha_2,\ \alpha_3,\ \alpha_5,\ \alpha_6,\quad \alpha_0' := \alpha_1+\alpha_2+2\alpha_3+3\alpha_4+2\alpha_5+\alpha_6\ \}.$$

(In the Dynkin diagram of Δ_0, the node α_0' is joined to α_2.) There are three equivalence classes of pairs $(\Phi', w) \in \Xi$ under $\sim$, where $\Phi' = \Phi_0$; representatives (Φ_0, d_i), where $d_i \in \boldsymbol{W}$ has minimal length in $\boldsymbol{W}(\Phi_0)d_i$ for $i = 1, 2, 3$, are given in Table 3. In that table, an expression like $4315\cdots$ means the product $w_{\alpha_4}w_{\alpha_3}w_{\alpha_1}w_{\alpha_5}\cdots$ in $\boldsymbol{W}$. Otherwise, the conventions are the same as in Example 2.6. In particular, recall that the permutation in the second column refers to the simple roots in Δ_0, as listed above, not to those in Δ. (So, e.g., the cycle $(3, 4, 6)$ means that $\alpha_3 \mapsto \alpha_5 \mapsto \alpha_0' \mapsto \alpha_3$.)

Table 3: The subsystem Φ_0 for type E_6

d_i	permutation	σ_i'-classes
$d_1 = 1_{\boldsymbol{W}}$	$()$	27
$d_2 = 431543654$	$(1,4)(2,6)(3,5)$	9
$d_3 = 425431654234$	$(1,5,2)(3,4,6)$	3

5.1. The subgroup $\boldsymbol{H}' = \langle \boldsymbol{T}_0, \boldsymbol{U}_\alpha\ (\alpha \in \Phi_0)\rangle$. For each root $\alpha \in \Phi$, denote by $\alpha^\vee \colon k^\times \to \boldsymbol{T}_0$ the corresponding coroot. Since $\boldsymbol{G}$ is simply connected, every $t \in \boldsymbol{T}_0$ has a unique expression $t = h(\xi_1, \ldots, \xi_6) := \prod_{1\leqslant i\leqslant 6} \alpha_i^\vee(\xi_i)$ where $\xi_i \in k^\times$ for $1 \leqslant i \leqslant 6$. By [21, Example 1.5.6], we have

$$Z(\boldsymbol{G}) = \{h(\xi, 1, \xi^{-1}, 1, \xi, \xi^{-1}) \mid \xi \in k^\times, \xi^3 = 1\}.$$

A similar computation shows that

$$Z(\boldsymbol{H}') = \{h(\xi, 1, \xi^{-1}, 1, \zeta, \zeta^{-1}) \mid \xi, \zeta \in k^\times, \xi^3 = \zeta^3 = 1\}.$$

Thus, $Z(\boldsymbol{H}')$ is generated by $Z(\boldsymbol{G})$ and any fixed $t \in Z(\boldsymbol{H}') \setminus Z(\boldsymbol{G})$. Since q is not a power of 3, we have $|Z(\boldsymbol{G})| = 3$ and $|Z(\boldsymbol{H}')| = 9$. Given $t = h(\xi, 1, \xi^{-1}, 1, \zeta, \zeta^{-1}) \in Z(\boldsymbol{H}')$ (where $\xi^3 = \zeta^3 = 1$), we have $C_{\boldsymbol{G}}(t) = \boldsymbol{H}'$ if and only if $\xi \neq \zeta$. Furthermore, one easily checks that

$$\begin{aligned} \dot{d}_2^{-1} t \dot{d}_2 &= z_2(t)t^{-1} \quad \text{where} \quad z_2(t) := h(\xi\zeta, 1, (\xi\zeta)^{-1}, 1, \xi\zeta, (\xi\zeta)^{-1}) \in Z(\boldsymbol{G}), \\ \dot{d}_3^{-1} t \dot{d}_3 &= z_3(t)t \qquad \text{where} \quad z_3(t) := h(\xi\zeta^{-1}, 1, \xi^{-1}\zeta, 1, \xi\zeta^{-1}, \xi^{-1}\zeta) \in Z(\boldsymbol{G}). \end{aligned}$$

These two relations show that all elements in $Z(\boldsymbol{H}')\setminus Z(\boldsymbol{G})$ are conjugate in $\boldsymbol{G}$. In particular, if we choose $\zeta=\xi^{-1}\neq 1$, then $\dot{d}_2^{-1}t\dot{d}_2=t^{-1}$. Thus, if $\mathcal{C}$ denotes the $\boldsymbol{G}$-conjugacy class of the elements in $Z(\boldsymbol{H}')\setminus Z(\boldsymbol{G})$, then

$$F(\mathcal{C})=\mathcal{C}, \qquad \mathcal{C}=\mathcal{C}^{-1} \qquad \text{and} \qquad \mathcal{C}=Z(\boldsymbol{G})\mathcal{C}.$$

In order to see that $\mathcal{C}$ is F-stable, we argue as follows. If $q\equiv 1 \bmod 3$, then $F(z)=z$ for all $z\in Z(\boldsymbol{H}')$. On the other hand, if $q\equiv 2 \bmod 3$, then $F(z)=z^{-1}$ for all $z\in Z(\boldsymbol{H}')$ and, hence, $Z(\boldsymbol{H}')^F=\{1\}$. But, if $t=h(\xi,1,\xi^{-1},1,\zeta,\zeta^{-1})\in\mathcal{C}$ with $\zeta=\xi^{-1}\neq 1$ as above, then $F(t)=t^{-1}=\dot{d}_2^{-1}t\dot{d}_2\in\mathcal{C}$, as required.

5.2. The conjugacy class $\Sigma\subseteq\boldsymbol{G}$. Let us fix an element $s_1\in\mathcal{C}$; since $\mathcal{C}$ is F-stable, we may assume that $F(s_1)=s_1$. Now $\boldsymbol{H}_1':=C_{\boldsymbol{G}}(s_1)$ is conjugate to $\boldsymbol{H}'$ in $\boldsymbol{G}$. Let $u_1\in\boldsymbol{H}_1'$ be regular unipotent; since all regular unipotent elements in $\boldsymbol{H}_1'$ are conjugate in $\boldsymbol{H}_1'$, we may assume that $F(u_1)=u_1$. Let Σ be the $\boldsymbol{G}$-conjugacy class of $g_1:=s_1u_1$; then Σ is F-stable since $F(g_1)=g_1$. Since $Z(\boldsymbol{G})\mathcal{C}=\mathcal{C}=\mathcal{C}^{-1}$, one also deduces that $Z(\boldsymbol{G})\Sigma=\Sigma=\Sigma^{-1}$. We claim that

$$A_{\boldsymbol{G}}(g_1) \text{ is generated by } \bar{s}_1 \text{ and all } \bar{z} \text{ for } z\in Z(\boldsymbol{G});$$

here, for any $c\in C_{\boldsymbol{G}}(g_1)$, we denote by $\bar{c}$ the image of c in $A_{\boldsymbol{G}}(g_1)$. Indeed, we have $C_{\boldsymbol{G}}(g_1)=C_{\boldsymbol{H}_1'}(u_1)$ and so $C^\circ_{\boldsymbol{G}}(g_1)=C^\circ_{\boldsymbol{H}_1'}(u_1)$. Since we are in good characteristic (inside $\boldsymbol{H}_1'\cong\boldsymbol{H}'$, which is of type $A_2{\times}A_2{\times}A_2$) and $Z(\boldsymbol{H}_1')$ is finite, it follows that $A_{\boldsymbol{G}}(g_1)=A_{\boldsymbol{H}_1'}(u_1)\cong Z(\boldsymbol{H}_1')$, where the isomorphism is induced by the natural map $Z(\boldsymbol{H}_1')\subseteq C_{\boldsymbol{H}_1'}(u_1)\to A_{\boldsymbol{H}_1'}(u_1)$; see [10, §12.3]. Note also that $\bar{g}_1=\bar{s}_1$. By §5.1, we have that $Z(\boldsymbol{H}')$ is generated by $Z(\boldsymbol{G})$ and any fixed element in $Z(\boldsymbol{H}')\setminus Z(\boldsymbol{G})$. Consequently, $Z(\boldsymbol{H}_1')$ is generated by $Z(\boldsymbol{G})$ and s_1, which implies the above claim.

Note that F acts trivially on $A_{\boldsymbol{G}}(g_1)$ if $q\equiv 1 \bmod 3$. On the other hand, if $q\equiv 2 \bmod 3$, then $F(z)=z^{-1}$ for all $z\in Z(\boldsymbol{G})$ and, hence, F acts non-trivially on $A_{\boldsymbol{G}}(g_1)$; in this case, we have $A_{\boldsymbol{G}}(g_1)^F=\langle\bar{s}_1\rangle\cong\mathbb{Z}/3\mathbb{Z}$. Hence, the set Σ^F splits into an odd number (either 9 or 3) of conjugacy classes in $\boldsymbol{G}^F$. So, among these classes, there must be at least one that is equal to its inverse; we now choose $g_1\in\Sigma^F$ to be in such a class; thus, g_1 is conjugate to g_1^{-1} in $\boldsymbol{G}^F$ (and not just in $\boldsymbol{G}$). Note also that, using Lemma 4.10(b), we could fix the $\boldsymbol{G}^F$-conjugacy class of g_1 completely, by requiring that $g_1\in C$, where C is the $\boldsymbol{G}^F$-conjugacy class determined by Σ (and the choice of $\dot{w}_c$) as in Corollary 4.8.

5.3. Cuspidal unipotent character sheaves. First we consider the group $\tilde{\boldsymbol{G}}:=\boldsymbol{G}/Z(\boldsymbol{G})$. Let $\pi\colon\boldsymbol{G}\to\tilde{\boldsymbol{G}}$ be the canonical map; let $\tilde{\Sigma}:=\pi(\Sigma)$ and

$\tilde{g}_1 := \pi(g_1)$. By the proof of [36, Prop. 20.3(a)] (see also [54, 4.6, 5.2]), there are two cuspidal unipotent character sheaves $\tilde{A}_1$ and $\tilde{A}_2$ of $\tilde{\boldsymbol{G}}$; they have support $\tilde{\Sigma}$ and they are F-invariant. As explained in the proof of [36, Cor. 20.4] (see also [54, p. 347]), the local systems (see §4.2) associated with $\tilde{A}_1$ and $\tilde{A}_2$ are one-dimensional; they correspond to linear characters $\tilde{\psi}_1$ and $\tilde{\psi}_2$ of $A_{\tilde{\boldsymbol{G}}}(\tilde{g}_1)$, such that $\tilde{\psi}_1(\bar{\tilde{g}}_1) = \theta$ and $\tilde{\psi}_2(\bar{\tilde{g}}_1) = \theta^2$, where $1 \neq \theta \in \mathbb{K}^\times$ is a fixed third root of unity. (Here, $\bar{\tilde{g}}_1$ denotes the image of $\tilde{g}_1$ in $A_{\tilde{\boldsymbol{G}}}(\tilde{g}_1)$.) Now use π to go back to $\boldsymbol{G}$. First note that π canonically induces a group homomorphism $\bar{\pi}\colon A_{\boldsymbol{G}}(g_1) \to A_{\tilde{\boldsymbol{G}}}(\tilde{g}_1)$. Hence, we obtain irreducible characters of $A_{\boldsymbol{G}}(g_1)$ by setting

$$\psi_1 := \tilde{\psi}_1 \circ \bar{\pi} \in \mathrm{Irr}(A_{\boldsymbol{G}}(g_1)) \qquad \text{and} \qquad \psi_2 := \tilde{\psi}_2 \circ \bar{\pi} \in \mathrm{Irr}(A_{\boldsymbol{G}}(g_1)).$$

By the proof of [36, Prop. 20.3(b)], $A_1 := \pi^*\tilde{A}_1$ and $A_2 := \pi^*\tilde{A}_2$ are cuspidal unipotent character sheaves on $\boldsymbol{G}$ (and these are the only ones); they have support Σ and they correspond to the linear characters ψ_1 and ψ_2 of $A_{\boldsymbol{G}}(g_1)$. (See the general reduction techniques described in [34, 2.10].) Finally note that, clearly, the image of $Z(\boldsymbol{G})$ in $A_{\boldsymbol{G}}(g_1)$ is contained in $\ker(\bar{\pi})$. Hence, we have

$$\psi_1(\bar{s}_1) = \theta, \qquad \psi_2(\bar{s}_1) = \theta^2, \qquad \psi_1(\bar{z}) = \psi_2(\bar{z}) = 1 \quad \text{for } z \in Z(\boldsymbol{H}_1')^F.$$

(Recall that $\bar{g}_1 = \bar{s}_1$.) Thus, ψ_1 and ψ_2 are completely determined, where ψ_2 is the complex conjugate of ψ_1. Furthermore, the roots of unity attached to A_1 and A_2 as in §4.2 are $\lambda_{A_1} = \theta$ and $\lambda_{A_2} = \theta^2$. Using ψ_1 and ψ_2, we can now write down characteristic functions of A_1 and A_2, as in §4.2; we have $\chi_{g_1,\psi_i} = \chi_{\tilde{g}_1,\tilde{\psi}_i} \circ \pi$ for $i = 1, 2$. (Recall that $\Sigma = Z(\boldsymbol{G})\Sigma$ and so $\Sigma = \pi^{-1}(\tilde{\Sigma})$.)

5.4. Unipotent characters and almost characters. Let $\tilde{\boldsymbol{G}} = \boldsymbol{G}/Z(\boldsymbol{G})$ and $\pi\colon \boldsymbol{G} \to \tilde{\boldsymbol{G}}$ as above. First note that the unipotent characters of $\boldsymbol{G}^F$ and $\tilde{\boldsymbol{G}}^F$ can be canonically identified via π (see, e.g., [21, Prop. 2.3.15]). As discussed in §4.3, the unipotent characters are parametrised by a certain set $X(\boldsymbol{W})$ (which only depends on $\boldsymbol{W}$). This set is further partitioned into "families"; the interesting family for us is the one which contains $x \in X(\boldsymbol{W})$ such that $\rho_x \in \mathrm{Unip}(\boldsymbol{G}^F)$ is cuspidal; it is given as follows (see the tables in [4, p. 480] and [33, p. 363]).

$x \in X(\boldsymbol{W})$:	$(1,1)$	$(1,\varepsilon)$	$(g_2,1)$	$(g_3,1)$	$(1,r)$	(g_2,ε)	(g_3,θ)	(g_3,θ^2)
$\rho_x \in \mathrm{Unip}(\boldsymbol{G}^F)$:	$[80_s]$	$[20_s]$	$[60_s]$	$[10_s]$	$[90_s]$	$D_4[r]$	$E_6[\theta]$	$E_6[\theta^2]$

Here, labels such as 80_s, 20_s etc. denote irreducible characters of $\boldsymbol{W}$ (as in [33, Chap. 4]); then $[80_s]$, $[20_s]$ etc. are the corresponding irreducible constituents of $\mathrm{Ind}_{\boldsymbol{B}^F}^{\boldsymbol{G}^F}(1)$ where 1 stands for the trivial character of $\boldsymbol{B}^F$; the characters $E_6[\theta]$ and $E_6[\theta^2]$ are cuspidal unipotent; $D_4[r]$ is a further non-cuspidal character. (Note also that, for example, the "g_3" in (g_3,θ) and (g_3,θ^2) has nothing to do with elements in $\boldsymbol{G}^F$; these are just notations for parameters in $X(\boldsymbol{W})$.) For each $x \in X(\boldsymbol{W})$ we also have a unipotent almost character R_x, formed using the entries of the corresponding Fourier matrix (the relevant part of which is printed in [4, p. 457]). In particular, we have:

$$\begin{aligned} R_{(g_3,\theta)} &:= \tfrac{1}{3}\big([80_s]+[20_s]-[10_s]-[90_s]+2E_6[\theta]-E_6[\theta^2]\big),\\ R_{(g_3,\theta^2)} &:= \tfrac{1}{3}\big([80_s]+[20_s]-[10_s]-[90_s]-E_6[\theta]+2E_6[\theta^2]\big). \end{aligned}$$

Now consider the two cuspidal unipotent character sheaves A_1 and A_2 described above, with characteristic functions χ_{g_1,ψ_1} and χ_{g_1,ψ_2}. By the main result of [54, §4] (see also [54, 5.2]), there are scalars $\zeta,\zeta' \in \mathbb{K}$ of absolute value 1 such that

$$R_{(g_3,\theta)} = \zeta\chi_{g_1,\psi_1} \qquad \text{and} \qquad R_{(g_3,\theta^2)} = \zeta'\chi_{g_1,\psi_2}.$$

(More precisely, in [54, §4], this is proved for $\tilde{\boldsymbol{G}}$ but the discussion in §5.3 shows that this also holds for $\boldsymbol{G}$, with ψ_1 and ψ_2 as above.) By [15, Table 1], the characters $E_6[\theta]$ and $E_6[\theta^2]$ are complex conjugate to each other, and their values lie in the field $\mathbb{Q}(\theta)$; furthermore, all characters $[\phi]$ (where $\phi \in \mathrm{Irr}(\boldsymbol{W})$) are rational-valued; see [15, Prop. 5.6]. We conclude that the class functions $R_{(g_3,\theta)}$ and $R_{(g_3,\theta^2)}$ are complex conjugate to each other, and their values lie in $\mathbb{Q}(\theta)$. Now, since $\dim\boldsymbol{G}-\dim\Sigma=\dim\boldsymbol{T}_0=6$, we have

$$\begin{aligned} R_{(g_3,\theta)}(g_1) &= \zeta\chi_{g_1,\psi_1}(g_1) = \zeta q^3 \in \mathbb{Q}(\theta),\\ R_{(g_3,\theta^2)}(g_1) &= \zeta'\chi_{g_1,\psi_2}(g_1) = \zeta' q^3 \in \mathbb{Q}(\theta). \end{aligned}$$

Thus, we can already conclude that $\zeta'=\overline{\zeta}$. Since g_1 is conjugate to g_1^{-1} in $\boldsymbol{G}^F$, we have $E_6[\theta](g_1)=E_6[\theta](g_1^{-1})=\overline{E_6[\theta]}(g_1)=E_6[\theta^2](g_1)$. Consequently, we also have $R_{(g_3,\theta)}(g_1)=R_{(g_3,\theta^2)}(g_1)$ and so $\zeta=\zeta'=\overline{\zeta}$. Since $\zeta\in\mathbb{Q}(\theta)$, this implies that $\zeta\in\mathbb{Q}$. And since ζ has absolute value 1, we must have $\zeta=\zeta'=\pm 1$.

Proposition 5.5. *In the above setting, recall that $g_1\in\Sigma^F$ is conjugate to g_1^{-1} in $\boldsymbol{G}^F$. Then we have $\zeta=\zeta'=1$, that is, $R_{(g_3,\theta)}=\chi_{g_1,\psi_1}$ and $R_{(g_3,\theta^2)}=\chi_{g_1,\psi_2}$.*

Proof. (See also [18, Remark 3.3].) Inverting the matrix relating unipotent characters and unipotent almost characters, we obtain:

$$E_6[\theta] = \tfrac{1}{3}\big(R_{80_s}+R_{20_s}-R_{10_s}-R_{90_s}+2R_{(g_3,\theta)}-R_{(g_3,\theta^2)}\big).$$

Using the formula for R_ϕ in §4.3, the (known) character table of $\boldsymbol{W}$ and Example 3.3, we find that

$$R_{80_s}(g_1) = R_{20_s}(g_1) = R_{90_s}(g_1) = 0 \qquad \text{and} \qquad R_{10_s}(g_1) = \epsilon,$$

where $\epsilon = \pm 1$ is such that $q \equiv \epsilon \bmod 3$. This yields $E_6[\theta](g_1) = \frac{1}{3}(-\epsilon + \zeta q^3) \in \mathbb{Q}$, where the left hand side is an algebraic integer. Hence, 3 must divide $\zeta q^3 - \epsilon \in \mathbb{Z}$. Since $\zeta = \pm 1$, the only possibility is that $\zeta = 1$. □

Table 4: Values of $E_6[\theta]$ on Σ^F

$q \equiv 1 \bmod 3$	$\{\bar{1},\ \bar{z},\ \bar{z}^2\}$	$\{\bar{s}_1,\ \bar{s}_1\bar{z},\ \bar{s}_1\bar{z}^2\}$	$\{\bar{s}_1^2,\ \bar{s}_1^2\bar{z},\ \bar{s}_1^2\bar{z}^2\}$
$E_6[\theta]$	$\frac{1}{3}(q^3-1)$	$\frac{1}{3}(q^3-1)+q^3\theta$	$\frac{1}{3}(q^3-1)+q^3\theta^2$

$q \equiv 2 \bmod 3$	$\bar{1}$	$\bar{s}_1$	$\bar{s}_1^2$
$E_6[\theta]$	$\frac{1}{3}(q^3+1)$	$\frac{1}{3}(q^3+1)+q^3\theta$	$\frac{1}{3}(q^3+1)+q^3\theta^2$

(Here, $g_1 = s_1u_1 \in \Sigma^F$ is such that g_1 and g_1^{-1} are conjugate in $\boldsymbol{G}^F$.)

The resulting values of $E_6[\theta]$ on the conjugacy classes of $\boldsymbol{G}^F$ that are contained in Σ^F are displayed in Table 4, where z denotes a non-trivial element in $Z(\boldsymbol{G})$ when $q \equiv 1 \bmod 3$. (Recall that Σ^F splits into 9 classes if $q \equiv 1 \bmod 3$, and into 3 classes if $q \equiv 2 \bmod 3$; these classes are parametrised by representatives of the F-conjugacy classes of $A_{\boldsymbol{G}}(g_1) \cong Z(\boldsymbol{H}_1')$.)

5.6. Twisted type. Let $\boldsymbol{G}$ be as above (of type E_6) but let now $\tilde{F}\colon \boldsymbol{G} \to \boldsymbol{G}$ be a Frobenius map (corresponding to an $\mathbb{F}_q$-rational structure on $\boldsymbol{G}$) such that $(\boldsymbol{G}, \tilde{F})$ is non-split. Then the induced automorphism $\sigma\colon \boldsymbol{W} \to \boldsymbol{W}$ is given by conjugation with the longest element $w_0 \in \boldsymbol{W}$. The permutation $\alpha \mapsto \alpha^\dagger$ of Φ is of order 2, such that $\alpha_1^\dagger = \alpha_6$, $\alpha_3^\dagger = \alpha_5$, $\alpha_2^\dagger = \alpha_2$ and $\alpha_4^\dagger = \alpha_4$. The two cuspidal unipotent character sheaves A_1 and A_2 considered above are also $\tilde{F}$-stable; see [36, Cor. 20.4] and its proof. In all essential points, we can further argue as above, so we just state the main results. To begin with, the subsystem $\Phi_0 \subseteq \Phi$ is invariant under †. Again, there are three equivalence classes of pairs $(\Phi', w) \in \Xi$ under $\sim$, where $\Phi' = \Phi_0$; representatives (Φ_0, d_i), where $d_i \in \boldsymbol{W}$ has minimal length in $\boldsymbol{W}(\Phi_0)d_i$ for $i = 1, 2, 3$, are given as follows.

d_i	permutation	σ_i'-classes
$d_1 = 1_{\boldsymbol{W}}$	$(1,5)(3,4)$	9
$d_2 = 431543654$	$(1,3)(2,6)(4,5)$	27
$d_3' = 423143542314354$	$(1,6,5,3,2,4)$	3

Given $t = h(\xi_1, \ldots, \xi_6) \in \boldsymbol{T}_0$, with $\xi \in k^\times$ for all i, we now have

$$\tilde{F}(t) = h(\xi_6^q, \xi_2^q, \xi_5^q, \xi_4^q, \xi_3^q, \xi_1^q) \qquad \text{for all } \xi \in k^\times.$$

Recall that $Z(\boldsymbol{H}') = \{h(\xi, 1, \xi^{-1}, 1, \zeta, \zeta^{-1}) \mid \xi, \zeta \in k^\times, \xi^3 = \zeta^3 = 1\}$; also recall the definition of the $\boldsymbol{G}$-conjugacy class $\mathcal{C}$ from §5.1.

Let $t = h(\xi, 1, \xi^{-1}, 1, \zeta, \zeta^{-1}) \in Z(\boldsymbol{H}')$. Assume first that $q \equiv 1 \bmod 3$. Then $\tilde{F}(t) = t$ if and only if $\xi = \zeta^{-1}$. Thus, there exists some $t \in Z(\boldsymbol{H}')^{\tilde{F}}$ such that $C_{\boldsymbol{G}}(t) = \boldsymbol{H}'$. On the other hand, if $q \equiv 2 \bmod 3$, then one checks that $\tilde{F}(t) = \dot{d}_2^{-1} t \dot{d}_2$ for all $t \in Z(\boldsymbol{H}')$. In particular, there exists some $t \in Z(\boldsymbol{H}')$ such that $C_{\boldsymbol{G}}(t) = \boldsymbol{H}'$ and $\tilde{F}(t) = \dot{d}_2^{-1} t \dot{d}_2$. Hence, in both cases, the class $\mathcal{C}$ is $\tilde{F}$-stable. We now define Σ as in §5.2; let $g_1 \in \Sigma^{\tilde{F}}$. It follows again that

$$A_{\boldsymbol{G}}(g_1) \text{ is generated by } \bar{s}_1 \text{ and all } \bar{z} \text{ for } z \in Z(\boldsymbol{G}).$$

We obtain characteristic functions χ_{g_1,ψ_1} and χ_{g_1,ψ_2} for A_1 and A_2, respectively, by exactly the same formulae as in §5.3.

Now let us turn to the unipotent characters and almost characters of $\boldsymbol{G}^{\tilde{F}}$. The unipotent characters are parametrised by the same set $X(\boldsymbol{W})$ as before in §5.4; the notation for the characters in the 8-element family is now as follows.

$x \in X(\boldsymbol{W})$:	$(1,1)$	$(1,\varepsilon)$	$(g_2,1)$	$(g_3,1)$	$(1,r)$	(g_2,ε)	(g_3,θ)	(g_3,θ^2)
$\tilde{\rho}_x \in \mathrm{Unip}(\boldsymbol{G}^{\tilde{F}})$:	${}^2E_6[1]$	$[\phi_{12,4}]$	$[\phi_{4,8}]$	$[\phi_{6,6}']$	$[\phi_{6,6}'']$	$[\phi_{16,5}]$	${}^2E_6[\theta]$	${}^2E_6[\theta^2]$

Here, we use the notation in [4, p. 481]. Thus, $\phi_{12,4}$, $\phi_{6,6}'$ etc. are irreducible characters of $\boldsymbol{W}^\sigma := \{w \in \boldsymbol{W} \mid \sigma(w) = w\}$ (a Weyl group of type F_4); then $[\phi_{12,4}]$, $[\phi_{6,6}']$ etc. are the corresponding irreducible constituents of $\mathrm{Ind}_{\boldsymbol{B}^{\tilde{F}}}^{\boldsymbol{G}^{\tilde{F}}}(1)$; characters denoted like ${}^2E_6[1]$ are cuspidal unipotent. (We refer to [4] instead of [33], because the table of unipotent characters for twisted E_6 is not explicitly printed in [33].)

For each $x \in X(\boldsymbol{W})$ we also have a corresponding unipotent almost character $\tilde{R}_x$ of $\boldsymbol{G}^{\tilde{F}}$. Since $\boldsymbol{G}, \tilde{F}$ is not of split type, there is no canonical choice for these almost characters; they are only well-defined up to multiplication by roots of unity. But, by [33, 4.19], we can choose $\tilde{R}_x$ such that, for each $x' \in X(\boldsymbol{W})$, the multiplicity of $\tilde{\rho}_{x'} \in \mathrm{Unip}(\boldsymbol{G}^{\tilde{F}})$ in $\tilde{R}_x$ is equal to that of $\rho_{x'} \in \mathrm{Unip}(\boldsymbol{G}^F)$ in the almost character R_x of $\boldsymbol{G}^F$. In particular, this yields the two formulae:

$$\tilde{R}_{(g_3,\theta)} := \tfrac{1}{3}\big({}^2E_6[1] + [\phi_{12,4}] - [\phi_{6,6}'] - [\phi_{6,6}''] + 2\cdot{}^2E_6[\theta] - {}^2E_6[\theta^2]\big),$$

$$\tilde{R}_{(g_3,\theta^2)} := \tfrac{1}{3}\big({}^2E_6[1] + [\phi_{12,4}] - [\phi'_{6,6}] - [\phi''_{6,6}] - {}^2E_6[\theta] + 2\cdot{}^2E_6[\theta^2]\big).$$

Furthermore, if $\phi \in \mathrm{Irr}(\boldsymbol{W})$ is such that x_ϕ belongs to the above 8-element family of $X(\boldsymbol{W})$, then the above choice leads to the following formula for $\tilde{R}_{x_\phi}$:

$$\tilde{R}_{x_\phi} = \tilde{R}_\phi := -\frac{1}{|\boldsymbol{W}|}\sum_{w\in\boldsymbol{W}} \phi(ww_0)R_{\boldsymbol{T}_w}^{\boldsymbol{G}}(1).$$

(Indeed, according to [33, 4.19], one needs to take the "preferred extension" of ϕ as in [36, 17.2], in order for the formula $\langle\tilde{\rho}_{x'}, \tilde{R}_\phi\rangle = \langle\rho_{x'}, R_\phi\rangle$ to hold; since the a-invariants of all ϕ as above are 3, this leads to the minus sign in the definition of $\tilde{R}_\phi$. See also [33, Prop. 7.11].) Now, by the main results of Shoji [54, §4] (see also [54, 4.8, 5.2]), there are scalars $\zeta, \zeta' \in \mathbb{K}$ of absolute value 1 such that

$$\tilde{R}_{(g_3,\theta)} = \zeta\chi_{g_1,\psi_1} \qquad \text{and} \qquad \tilde{R}_{(g_3,\theta^2)} = \zeta'\chi_{g_1,\psi_2}.$$

By the same argument as in §5.4, we can choose $g_1 \in \Sigma^{\tilde{F}}$ such that g_1 is conjugate to g_1^{-1} in $\boldsymbol{G}^{\tilde{F}}$. Hence, as before, we already know that $\zeta' = \overline{\zeta} = \zeta = \pm 1$. So it only remains to decide whether ζ equals 1 or -1.

Proposition 5.7. *Recall that $g_1 \in \Sigma^{\tilde{F}}$ is conjugate to g_1^{-1} in $\boldsymbol{G}^{\tilde{F}}$. With $\tilde{R}_{(g_3,\theta)}$ and $\tilde{R}_{(g_3,\theta^2)}$ as specified above, we have $\zeta = \zeta' = 1$.*

Proof. As in the proof of Proposition 5.5, we invert the matrix relating unipotent characters and unipotent almost characters of $\boldsymbol{G}^{\tilde{F}}$; this yields the identity:

$${}^2E_6[\theta] = \tfrac{1}{3}\big(\tilde{R}_{80_s} + \tilde{R}_{20_s} - \tilde{R}_{10_s} - \tilde{R}_{90_s} + 2\tilde{R}_{(g_3,\theta)} - \tilde{R}_{(g_3,\theta^2)}\big).$$

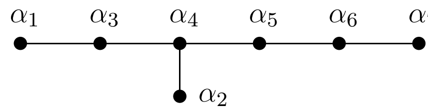
By the above definition of $\tilde{R}_\phi$ and the formula in Example 3.3, we obtain

$$\tilde{R}_{80_s}(g_1) = \tilde{R}_{20_s}(g_1) = \tilde{R}_{90_s}(g_1) = 0 \qquad \text{and} \qquad \tilde{R}_{10_s}(g_1) = \varepsilon,$$

where $\varepsilon = \pm 1$ is such that $q \equiv \varepsilon \bmod 3$. This yields the relation ${}^2E_6[\theta](g_1) = \frac{1}{3}(\zeta q^3 - \varepsilon)$, which implies that $\zeta = 1$ regardless of whether ε is $+1$ or -1. $\square$

6. Cuspidal unipotent character sheaves in type E_7

Throughout this section, let $\boldsymbol{G}$ be simple, simply connected of type E_7. Let $q = p^f$ (where $f \geqslant 1$) be such that $F\colon \boldsymbol{G} \to \boldsymbol{G}$ defines an $\mathbb{F}_q$-rational structure. Here, $\boldsymbol{G}$ is of split type; thus, $\sigma = \mathrm{id}_{\boldsymbol{W}}$ and the permutation $\alpha \mapsto \alpha^\dagger$ of Φ is the identity. Let $\Delta = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7\}$ be the set of simple roots in Φ^+, where the labelling is chosen as follows.



If $p = 2$, then the cuspidal character sheaves and almost characters have been considered by Hetz [26]. So assume from now on that $p \neq 2$. Let $\alpha_0 \in \Phi$ be the unique root of maximal height and consider the subsystem $\Phi_0 \subseteq \Phi$ of type $A_3 \times A_3 \times A_1$ spanned by $\{\alpha_1, \alpha_2, \alpha_3, \alpha_5, \alpha_6, \alpha_7, \alpha_0\}$. Again, the relevance of this particular example is that Φ_0 occurs in the classification of cuspidal unipotent character sheaves on $\boldsymbol{G}$; see [36, Prop. 20.3] (and also [54, 5.2]). Using CHEVIE, we find that the unique set Δ_0 of simple roots in $\Phi_0 \cap \Phi^+$ is given by

$$\Delta_0 = \{\ \alpha_1,\ \alpha_2,\ \alpha_3,\ \alpha_5,\ \alpha_6,\ \alpha_7,\ \alpha_0' := \alpha_1+2\alpha_2+2\alpha_3+4\alpha_4+3\alpha_5+2\alpha_6+\alpha_7\ \}.$$

Furthermore, there are four equivalence classes of pairs $(\Phi', w) \in \Xi$ under $\sim$, where $\Phi' = \Phi_0$; representatives (Φ_0, d_i), where $d_i \in \boldsymbol{W}$ has minimal length in $\boldsymbol{W}(\Phi_0)d_i$ for $i = 1, 2, 3, 4$, are given in Table 5, where we use similar notational conventions as in the previous section. (Note that, in the Dynkin diagram of Δ_0, the node α_0' is joined to α_3, not to α_1.)

Table 5: The subsystem Φ_0 for type E_7

d_i	permutation	σ_i'-classes
$d_1 = 1_{\boldsymbol{W}}$	$()$	50
$d_2 = 423143542654317654234$	$(1,6)(3,5)(4,7)$	10
$d_3 = 4234542346542347654234$	$(1,7)(4,6)$	50
$d_4 = 42314354231435465423143542654$	$(1,4)(3,5)(6,7)$	10

6.1. The subgroup $\boldsymbol{H}' = \langle \boldsymbol{T}_0, \boldsymbol{U}_\alpha\ (\alpha \in \Phi_0)\rangle$. As in §5.1, every $t \in \boldsymbol{T}_0$ has a unique expression $t = h(\xi_1, \ldots, \xi_7) := \prod_{1 \leqslant i \leqslant 7} \alpha_i^\vee(\xi_i)$ where $\xi_i \in k^\times$ for $1 \leqslant i \leqslant 7$. By [21, Example 1.5.6], we have

$$Z(\boldsymbol{G}) = \{h(1, \xi, 1, 1, \xi, 1, \xi) \mid \xi = \pm 1 \in k\} \cong \mathbb{Z}/2\mathbb{Z}.$$

A similar computation shows that

$$Z(\boldsymbol{H}') = \{h(1, \pm 1, 1, 1, \xi, \xi^2, \xi^{-1}) \mid \xi \in k^\times, \xi^4 = 1\} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}.$$

Since q is not a power of 2, we have $|Z(\boldsymbol{G})| = 2$ and $|Z(\boldsymbol{H}')| = 8$. Given $t = h(1, \pm 1, 1, 1, \xi, \xi^2, \xi^{-1}) \in Z(\boldsymbol{H}')$ (where $\xi^4 = 1$), we have $C_{\boldsymbol{G}}(t) = \boldsymbol{H}'$ if and only if $\xi^2 \neq 1$. The elements $h(1,-1,1,1,1,1,1)$ and $h(1,1,1,1,-1,1,-1)$ have a centraliser of type $D_6 \times A_1$. Furthermore, one easily checks that

$$\dot{d}_2^{-1} t \dot{d}_2 = zt, \qquad \dot{d}_3^{-1} t \dot{d}_3 = t^{-1}, \qquad \dot{d}_4^{-1} t \dot{d}_4 = zt^{-1},$$

where $z := h(1,\xi^2,1,1,\xi^2,1,\xi^2) \in Z(\boldsymbol{G})$. These three relations show that all elements $t \in Z(\boldsymbol{H}')$ such that $C_{\boldsymbol{G}}(t) = \boldsymbol{H}'$ are conjugate in $\boldsymbol{G}$. Thus, if $\mathcal{C}$ denotes the $\boldsymbol{G}$-conjugacy class of these elements, then

$$F(\mathcal{C}) = \mathcal{C}, \qquad \mathcal{C} = \mathcal{C}^{-1} \qquad \text{and} \qquad \mathcal{C} = Z(\boldsymbol{G})\mathcal{C}.$$

In order to see that $\mathcal{C}$ is F-stable, we argue as follows. If $q \equiv 1 \bmod 4$, then $F(z) = z$ for all $z \in Z(\boldsymbol{H}')$. On the other hand, if $q \equiv 3 \bmod 4$, then $F(z) = z^{-1}$ for all $z \in Z(\boldsymbol{H}')$ and, hence, $Z(\boldsymbol{H}')^F$ is a Klein four group. If $t \in Z(\boldsymbol{H}') \cap \mathcal{C}$ as above, then $F(t) = t^{-1} = \dot{d}_3^{-1} t \dot{d}_3 \in \mathcal{C}$, as required.

As in §5.2, we fix an element $s_1 \in \mathcal{C}^F$ and set $\boldsymbol{H}_1' := C_{\boldsymbol{G}}(s_1)$. We pick a regular unipotent element $u_1 \in \boldsymbol{H}_1'^F$ and let Σ be the $\boldsymbol{G}$-conjugacy class of $g_1 := s_1 u_1$. Again, we see that Σ is F-stable and $Z(\boldsymbol{G})\Sigma = \Sigma = \Sigma^{-1}$. Furthermore, $A_{\boldsymbol{G}}(g_1) \cong Z(\boldsymbol{H}_1')$ has order 8 and

$$A_{\boldsymbol{G}}(g_1) = \langle \bar{s}_1, \bar{z} \rangle \quad \text{where } z \text{ is the non-trivial element of } Z(\boldsymbol{G}).$$

Now F acts trivially on $A_{\boldsymbol{G}}(g_1)$, regardless of the congruence class of q modulo 4. Hence, the set Σ^F always splits into 8 conjugacy classes in $\boldsymbol{G}^F$ (each with centraliser order $8q^7$), which are parametrised by the 8 elements of $A_{\boldsymbol{G}}(g_1)$. However, now it is less obvious whether we can choose $g_1 \in \Sigma^F$ such that g_1 is conjugate to g_1^{-1} in $\boldsymbol{G}^F$. We will come back to this issue in §6.4. (Since $Z(\boldsymbol{G})$ has order 2, we can not use the argument in Lemma 4.10(b).)

6.2. Cuspidal unipotent character sheaves. By an argument entirely analogous to that in §5.3 (but now using [36, Prop. 20.5] and its proof), we see that there are two F-invariant cuspidal unipotent character sheaves A_1 and A_2 on $\boldsymbol{G}$. (Again, they are pulled back from $\tilde{\boldsymbol{G}} = \boldsymbol{G}/Z(\boldsymbol{G})$ via the canonical map $\pi\colon \boldsymbol{G} \to \tilde{\boldsymbol{G}}$.) The local systems associated with A_1 and A_2 are one-dimensional; they correspond to linear characters ψ_1 and ψ_2 of $A_{\boldsymbol{G}}(g_1)$ such that

$$\psi_1(\bar{s}_1) = \mathrm{i}, \quad \psi_2(\bar{s}_1) = -\mathrm{i}, \quad \psi_1(\bar{z}) = \psi_2(\bar{z}) = 1 \quad \text{for } z \in Z(\boldsymbol{H}_1')^F,$$

where $\mathrm{i} \in \mathbb{K}$ is fixed such that $\mathrm{i}^2 = -1$. (Recall that $\bar{g}_1 = \bar{s}_1$.) Thus, ψ_1 and ψ_2 are completely determined, where ψ_2 is the complex conjugate of ψ_1.

Furthermore, the roots of unity attached to A_1 and A_2 as in §4.2 are $\lambda_{A_1} = \mathrm{i}$ and $\lambda_{A_2} = -\mathrm{i}$. Using ψ_1 and ψ_2, we can now write down characteristic functions of A_1 and A_2, as in §4.2. The values are given as follows, where $1 \neq z \in Z(\boldsymbol{G})$.

	$\{\bar{1},\ \bar{z}\}$	$\{\bar{s}_1^2,\ \bar{s}_1^2\bar{z}\}$	$\{\bar{s}_1,\ \bar{s}_1\bar{z}\}$	$\{\bar{s}_1^{-1},\ \bar{s}_1^{-1}\bar{z}\}$
χ_{g_1,ψ_1}	$q^{7/2}$	$-q^{7/2}$	$\mathrm{i}q^{7/2}$	$-\mathrm{i}q^{7/2}$
χ_{g_1,ψ_2}	$q^{7/2}$	$-q^{7/2}$	$-\mathrm{i}q^{7/2}$	$\mathrm{i}q^{7/2}$

Note that, here, we have $\dim \boldsymbol{G} - \dim \Sigma = \dim \boldsymbol{T}_0 = 7$. We set $q^{n/2} := \sqrt{q}^n$ for any $n \geqslant 1$, where $\sqrt{q} \in \mathbb{K}$ has been fixed as described in §4.2.

Table 6: Unipotent almost characters for type E_7

$$\begin{aligned} R_{512'_a} = R_{(1,1)} &:= \tfrac{1}{2}\big([512'_a] + [512_a] - E_7[\xi] - E_7[-\xi]\big), \\ R_{512_a} = R_{(1,\varepsilon)} &:= \tfrac{1}{2}\big([512'_a] + [512_a] + E_7[\xi] + E_7[-\xi]\big), \\ R_{(g_2,1)} &:= \tfrac{1}{2}\big([512'_a] - [512_a] - E_7[\xi] + E_7[-\xi]\big), \\ R_{(g_2,\varepsilon)} &:= \tfrac{1}{2}\big([512'_a] - [512_a] + E_7[\xi] - E_7[-\xi]\big). \end{aligned}$$

6.3. Unipotent characters and almost characters. Exactly as in §5.4, the unipotent characters of $\boldsymbol{G}^F$ can be canonically identified with those of $\tilde{\boldsymbol{G}}^F$, where $\tilde{\boldsymbol{G}} = \boldsymbol{G}/Z(\boldsymbol{G})$. Again, they are parametrised by a certain set $X(\boldsymbol{W})$ (which only depends on $\boldsymbol{W}$). We use the notation in the table on [33, pp. 364–365]; also note the special remarks concerning type E_7 (and E_8) on [33, p. 362]. The unipotent almost characters are also parametrised by $X(\boldsymbol{W})$. The interesting cases for us are displayed in Table 6, where $\xi = \mathrm{i}\sqrt{q} \in \mathbb{K}$. In that table, $512'_a$ and 512_a are irreducible characters of $\boldsymbol{W}$; then $[512'_a]$ and $[512_a]$ are the corresponding constituents of $\mathrm{Ind}_{\boldsymbol{B}^F}^{\boldsymbol{G}^F}(1)$; the characters $E_7[\pm\xi]$ are cuspidal unipotent. (Note also that the "g_2" in $(g_2, 1)$ and (g_2, ε) has nothing to do with elements in $\boldsymbol{G}^F$.)

Now consider the two character sheaves A_1 and A_2 described above, with characteristic functions χ_{g_1,ψ_1} and χ_{g_1,ψ_2}. By the main result of [54, §4] (see also [54, 5.2]), there are scalars $\zeta, \zeta' \in \mathbb{K}$ of absolute value 1 such that

$$R_{(g_2,1)} = \zeta\chi_{g_1,\psi_1} \qquad \text{and} \qquad R_{(g_2,\varepsilon)} = \zeta'\chi_{g_1,\psi_2}.$$

(Again, in [54, §4], this is proved for $\tilde{\boldsymbol{G}}$ but the discussion in §6.2 shows that this also holds for $\boldsymbol{G}$, with ψ_1 and ψ_2 as above.) By [15, Table 1], the characters $E_7[\xi]$ and $E_7[-\xi]$ are complex conjugate to each other, and their values lie in the field $\mathbb{Q}(\xi)$. Furthermore, all characters $[\phi]$ (where $\phi \in \mathrm{Irr}(\boldsymbol{W})$) have their values in $\mathbb{Q}(\sqrt{q})$; see [15, Prop. 5.6]. We conclude that $R_{(g_2,1)}$ and $R_{(g_2,\varepsilon)}$ are complex conjugate to each other, and their values lie in $\mathbb{Q}(\mathrm{i},\sqrt{q})$. Since

$$R_{(g_2,1)}(g_1) = \zeta q^{7/2} \qquad \text{and} \qquad R_{(g_2,\varepsilon)}(g_1) = \zeta' q^{7/2},$$

we can already conclude that $\zeta' = \overline{\zeta}$.

6.4. On the choice of $g_1 \in \Sigma^F$. We now come back to the issue of finding a "good" representative $g_1 \in \Sigma^F$. Recall that $Z(\boldsymbol{G})$ has order 2. By Example 4.9, there are precisely two $\boldsymbol{G}^F$-conjugacy classes $C, C' \subseteq \Sigma^F$ which have a non-empty intersection with $\boldsymbol{B}^F \dot{w}_c \boldsymbol{B}^F$. We let $g_1 \in C \cup C'$. To fix the notation, we let C be the $\boldsymbol{G}^F$-conjugacy class associated with Σ (and the choice of $\dot{w}_c$) as in Corollary 4.8; by the construction in Example 4.9, C' is the $\boldsymbol{G}^F$-conjugacy class parametrised by $\bar{z} \in A_{\boldsymbol{G}}(g_1)$. The table in §6.2 now shows that χ_{g_1,ψ_1} and χ_{g_1,ψ_2} have the same value on all elements in $C \cup C'$. Using the formula in Example 3.3, the (known) character table of $\boldsymbol{W}$ and the required computations concerning σ'-conjugacy classes in $\boldsymbol{W}$, we find that the restrictions of the almost characters $R_{512'_a}$ and R_{512_a} to Σ^F are identically zero. Finally, the relations in Table 6 can be inverted and yield the following relations:

$$\begin{aligned}
[512'_a] &= \tfrac{1}{2}\big(R_{512'_a} + R_{512_a} + \zeta\chi_{g_1,\psi_1} + \zeta'\chi_{g_1,\psi_2}\big),\\
[512_a] &= \tfrac{1}{2}\big(R_{512'_a} + R_{512_a} - \zeta\chi_{g_1,\psi_1} - \zeta'\chi_{g_1,\psi_2}\big),\\
E_7[\xi] &= \tfrac{1}{2}\big(-R_{512'_a} + R_{512_a} - \zeta\chi_{g_1,\psi_1} + \zeta'\chi_{g_1,\psi_2}\big),\\
E_7[-\xi] &= \tfrac{1}{2}\big(-R_{512'_a} + R_{512_a} + \zeta\chi_{g_1,\psi_1} - \zeta'\chi_{g_1,\psi_2}\big).
\end{aligned}$$

So the above discussion implies that $E_7[\xi](g) = (\overline{\zeta} - \zeta)q^{7/2}$ for all $g \in C \cup C'$. Next recall that $\Sigma = \Sigma^{-1}$. So, by Lemma 4.10(a), we have $\{C^{-1}, C'^{-1}\} = \{C, C'\}$ which implies that $E_7[\xi](g^{-1}) = (\overline{\zeta} - \zeta)q^{7/2}$ for all $g \in C \cup C'$. But the left hand side also equals $\overline{E_7[\xi]}(g) = (\zeta - \overline{\zeta})q^{7/2}$. This implies that $\overline{\zeta} = \zeta$ and, consequently, $\zeta = \zeta' = \pm 1$ (since ζ has absolute value 1). Then all the values of $[512_a]$, $[512'_a]$ and $E_7[\pm\xi]$ on Σ^F are determined (up to $\zeta = \pm 1$); see Table 7.

Now, how important is it to determine the scalar $\zeta = \pm 1$ exactly? The above expressions for $E_7[\pm\xi]$ show that $E_7[\xi](g) = E_7[-\xi](g) \in \mathbb{Z}$ for all $g \in \boldsymbol{G}^F \setminus \Sigma^F$. In other words, the two characters $E_7[\xi]$ and $E_7[-\xi]$ can only (!) be distinguished by their values on elements in Σ^F, where they are given by Table 7. The same is also true for $[512'_a]$ and $[512_a]$. Thus, up to simultaneously exchanging the names of $[512'_a]$, $[512_a]$ and of $E_7[\xi]$, $E_7[-\xi]$, we could

Table 7: Values of $[512_a']$, $[512_a]$ and $E_7[\pm\xi]$ on Σ^F

$(\zeta=\zeta'=\pm 1)$	$\{\bar{1},\ \bar{z}\}$	$\{\bar{s}_1^2,\ \bar{s}_1^2\bar{z}\}$	$\{\bar{s}_1,\ \bar{s}_1\bar{z}\}$	$\{\bar{s}_1^{-1},\ \bar{s}_1^{-1}\bar{z}\}$
$[512_a']$	$\zeta q^{7/2}$	$-\zeta q^{7/2}$	0	0
$[512_a]$	$-\zeta q^{7/2}$	$\zeta q^{7/2}$	0	0
$E_7[\xi]$	0	0	$-\mathrm{i}\zeta q^{7/2}$	$\mathrm{i}\zeta q^{7/2}$
$E_7[-\xi]$	0	0	$\mathrm{i}\zeta q^{7/2}$	$-\mathrm{i}\zeta q^{7/2}$

(The classes labelled by $\bar{1}$ and $\bar{z}$ correspond to C and C'.)

assume "without loss of generality" that $\zeta=1$. For most applications of character theory, this is entirely sufficient. — However, it is actually possible to determine ζ exactly.

Proposition 6.5. *Recall that $g_1\in C\cup C'$ where C and C' are the two $\boldsymbol{G}^F$-conjugacy classes that are contained in Σ^F and have a non-empty intersection with $\boldsymbol{B}^F\dot{w}_c\boldsymbol{B}^F$. Then $\zeta=\zeta'=1$, that is, $R_{(g_2,1)}=\chi_{g_1,\psi_1}$ and $R_{(g_2,\varepsilon)}=\chi_{g_1,\psi_2}$.*

Proof. Let us denote the $\boldsymbol{G}^F$-conjugacy classes in Table 7 by $C_1,\ldots,C_8$ (from left to right); in particular, $C=C_1$ and $C'=C_2$. We will now try to evaluate the formula in Remark 4.12(b) for elements $g\in\Sigma^F$. For this purpose, we write $X(\boldsymbol{W})=X^\circ\cup\{x_1,x_2\}$ where $x_1=(g_2,1)$ and $x_2=(g_2,\varepsilon)$. Since every unipotent character of $\boldsymbol{G}^F$ is a linear combination of unipotent almost characters, we have

$$[\phi]=[\phi]^\circ+\alpha_1(\phi)R_{x_1}+\alpha_2(\phi)R_{x_2}\qquad\text{for each } \phi\in\mathrm{Irr}(\boldsymbol{W}),$$

where $\alpha_1(\phi),\alpha_2(\phi)\in\mathbb{K}$ and $[\phi]^\circ$ is a linear combination of $\{R_x\mid x\in X^\circ\}$. Setting

$$B:=\sum_{\phi\in\mathrm{Irr}(\boldsymbol{W})}\phi^{(q)}(T_{w_c})\alpha_1(\phi)\qquad\text{and}\qquad D(g):=\sum_{\phi\in\mathrm{Irr}(\boldsymbol{W})}\phi^{(q)}(T_{w_c})[\phi]^\circ(g)$$

for $g\in\Sigma^F$, we obtain

$$\sum_{\phi\in\mathrm{Irr}(\boldsymbol{W})}\phi^{(q)}(T_w)[\phi](g)=D(g)+B\cdot\big(R_{x_1}(g)+R_{x_2}(g)\big).$$

Now we note the following. Let $x\in X^\circ$ and consider the possible values of R_x on $C_1,\ldots,C_8$. Since R_x is a linear combination of unipotent characters, R_x takes the same value on C_{2i-1}, C_{2i} for $i=1,\ldots,4$ (see Lemma 4.4).

Using Table 7, we see that the values of R_{x_1} on $C_1, \ldots, C_8$ are given by $u, u, -u, -u, \mathrm{i}u, \mathrm{i}u, -\mathrm{i}u, -\mathrm{i}u$ and those of R_{x_2} are $u, u, -u, -u, -\mathrm{i}u, -\mathrm{i}u, \mathrm{i}u, \mathrm{i}u$ (where $u = \zeta q^{7/2}$). Since R_x is orthogonal to both R_{x_1} and R_{x_2}, we conclude that the values of R_x must be x, x, x, x, y, y, y, y for some $x, y \in \mathbb{K}$. What is important here is that $D(g)$ takes the same value on all $g \in C_1 \cup C_2 \cup C_3 \cup C_4$; let us denote by D_0 this common value of $D(g)$ on $C_1 \cup C_2 \cup C_3 \cup C_4$.

Next, the explicitly known matrix relating unipotent characters and unipotent almost characters shows that

$$\alpha_1(512'_a) = \alpha_2(512'_a) = \tfrac{1}{2}, \qquad \alpha_1(512_a) = \alpha_2(512_a) = -\tfrac{1}{2},$$

and $\alpha_1(\phi) = \alpha_2(\phi) = 0$ for all $\phi \neq 512'_a, 512_a$. Furthermore, we have

$$(512'_a)^{(q)}(T_{w_c}) = q^{7/2} \qquad \text{and} \qquad (512_a)^{(q)}(T_{w_c}) = -q^{7/2},$$

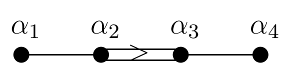
by known results on character values of Hecke algebras (see [23, Example 9.2.9(b)]; these values are readily available within CHEVIE [47]). This yields $B = q^{7/2}$. Furthermore, R_{x_1}, R_{x_2} take value $\zeta q^{7/2}$ on elements in $C_1 \cup C_2$, and value $-\zeta q^{7/2}$ on elements in $C_3 \cup C_4$. Hence, we obtain:

$$\sum_{\phi \in \mathrm{Irr}(\boldsymbol{W})} \phi^{(q)}(T_w)[\phi](g) = \begin{cases} D_0 + 2\zeta q^7 & \text{if } g \in C_1 \cup C_2, \\ D_0 - 2\zeta q^7 & \text{if } g \in C_3 \cup C_4. \end{cases}$$

By Remark 4.12(b), the left hand side equals $4q^7$ or 0, according to whether $g \in C_1 \cup C_2$ or $g \in \Sigma^F \setminus (C_1 \cup C_2)$. Thus, $0 = D_0 - 2\zeta q^7$ and, consequently, $4q^7 = D_0 + 2\zeta q^7 = 4\zeta q^7$. In particular, $\zeta = 1$. $\square$

7. Cuspidal character sheaves in type F_4

Throughout this section, let $\boldsymbol{G}$ be simple of type F_4; here we have $Z(\boldsymbol{G}) = \{1\}$. Let $q = p^f$ (where $f \geqslant 1$) be such that $F\colon \boldsymbol{G} \to \boldsymbol{G}$ defines an $\mathbb{F}_q$-rational structure. Here, $\boldsymbol{G}$ is of split type; thus, $\sigma = \mathrm{id}_{\boldsymbol{W}}$ and the permutation $\alpha \mapsto \alpha^\dagger$ of Φ is the identity. Let $\Delta = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ be the set of simple roots in Φ^+, where the labelling is chosen as follows.



Except for §7.12 (at the very end), we will assume that $p \neq 2, 3$. The subsystems of Φ occurring in the classification of cuspidal character sheaves are given by Table 8; see [36, Prop. 21.3] and its proof. (We use the same notational conventions as in the previous sections.)

Table 8: Subsystems for type F_4

Φ'	Δ'	d_i	perm.	σ_i'-classes
$A_2{\times}A_2$	$\alpha_1,\alpha_3,\alpha_4,\alpha_1{+}3\alpha_2{+}4\alpha_3{+}2\alpha_4$	$d_1=1_{\boldsymbol{W}}$	$()$	9
		$d_2=232432$	$(1,4)(2,3)$	9
$A_3{\times}A_1$	$\alpha_1,\alpha_2,\alpha_4,\alpha_1{+}2\alpha_2{+}4\alpha_3{+}2\alpha_4$	$d_1=1_{\boldsymbol{W}}$	$()$	10
		$d_2=3234323$	$(1,4)$	10
B_4	$\alpha_1,\alpha_2,\alpha_3,\alpha_2{+}2\alpha_3{+}2\alpha_4$	$d_1=1_{\boldsymbol{W}}$	$()$	20
$C_3{\times}A_1$	$\alpha_2,\alpha_3,\alpha_4,2\alpha_1{+}3\alpha_2{+}4\alpha_3{+}2\alpha_4$	$d_1=1_{\boldsymbol{W}}$	$()$	20

We begin by working out the center of $\boldsymbol{H}'=\langle \boldsymbol{T}_0,\boldsymbol{U}_\alpha\ (\alpha\in\Phi')\rangle$ in each case.

7.1. The subsystem Φ' of type $A_2\times A_2$. In this case, we have

$$Z(\boldsymbol{H}')=\{h(1,1,\xi,\xi^{-1})\mid \xi\in k^\times,\xi^3=1\}\cong \mathbb{Z}/3\mathbb{Z}.$$

Given $t\in Z(\boldsymbol{H}')$, we have $C_{\boldsymbol{G}}(t)=\boldsymbol{H}'$ if and only if $t\neq 1$. Furthermore, one checks that $\dot{d}_2^{-1}t\dot{d}_2=t^{-1}$. Hence, the two elements $t\in Z(\boldsymbol{H}')$ such that $C_{\boldsymbol{G}}(t)=\boldsymbol{H}'$ are conjugate in $\boldsymbol{G}$. If $\mathcal{C}$ denotes the $\boldsymbol{G}$-conjugacy class of these elements, then $\mathcal{C}=\mathcal{C}^{-1}$; furthermore, $F(\mathcal{C})=\mathcal{C}$. Indeed, if $q\equiv 1 \bmod 3$, then $F(t)=t$ for all $t\in Z(\boldsymbol{H}')$. On the other hand, if $q\equiv 2 \bmod 3$, then $F(t)=t^{-1}$ for all $t\in Z(\boldsymbol{H}')$. If $t\in Z(\boldsymbol{H}')\cap\mathcal{C}$, then $F(t)=t^{-1}=\dot{d}_2^{-1}t\dot{d}_2\in\mathcal{C}$, as required.

7.2. The subsystem Φ' of type $A_3\times A_1$. In this case, we have

$$Z(\boldsymbol{H}')=\{h(1,1,\xi^2,\xi)\mid \xi\in k^\times,\xi^4=1\}\cong \mathbb{Z}/4\mathbb{Z}.$$

Given $t=h(1,1,\xi^2,\xi)\in Z(\boldsymbol{H}')$ (where $\xi^4=1$), we have $C_{\boldsymbol{G}}(t)=\boldsymbol{H}'$ if and only if $\xi^2\neq 1$. (Note that the element $h(1,1,1,-1)$ has a centraliser of type B_4.) Furthermore, one checks that $\dot{d}_2^{-1}t\dot{d}_2=h(1,1,\xi^2,\xi^{-1})=t^{-1}$. Hence, the two elements $t\in Z(\boldsymbol{H}')$ such that $C_{\boldsymbol{G}}(t)=\boldsymbol{H}'$ are conjugate in $\boldsymbol{G}$. If $\mathcal{C}$ denotes the $\boldsymbol{G}$-conjugacy class of these elements, then $\mathcal{C}=\mathcal{C}^{-1}$; furthermore, $F(\mathcal{C})=\mathcal{C}$. Indeed, if $q\equiv 1 \bmod 4$, then $F(t)=t$ for all $t\in Z(\boldsymbol{H}')$. On the other hand, if $q\equiv 3 \bmod 4$, then $F(t)=t^{-1}$ for all $t\in Z(\boldsymbol{H}')$. If $t\in Z(\boldsymbol{H}')\cap\mathcal{C}$, then $F(t)=t^{-1}=\dot{d}_2^{-1}t\dot{d}_2\in\mathcal{C}$, as required.

7.3. The subsystem Φ' of type B_4. In this case, we have

$$Z(\boldsymbol{H}')=\{h(1,1,\xi,1)\mid \xi\in k^\times,\xi^2=1\}\cong \mathbb{Z}/2\mathbb{Z}.$$

Hence, if $s_1 := h(1,1,-1,1) \in Z(\boldsymbol{H}')$, then $C_{\boldsymbol{G}}(s_1) = \boldsymbol{H}'$. (This element s_1 is conjugate to the element $h(1,1,1,-1)$ mentioned above.)

7.4. The subsystem Φ' of type $C_3 \times A_1$. In this case, we have

$$Z(\boldsymbol{H}') = \{h(1,\xi,1,\xi) \mid \xi \in k^\times, \xi^2 = 1\} \cong \mathbb{Z}/2\mathbb{Z}.$$

Hence, if $s_1 := h(1,-1,1,-1) \in Z(\boldsymbol{H}')$, then $C_{\boldsymbol{G}}(s_1) = \boldsymbol{H}'$.

7.5. The unipotent class $F_4(a_3)$. In total, there are seven cuspidal character sheaves on $\boldsymbol{G}$; they are all unipotent and F-invariant; see [41, 1.7] and [53, §6, §7]. Let A be any of these seven cuspidal character sheaves and write $A = \mathrm{IC}(\Sigma, \mathcal{E})[\dim \mathcal{E}]$ where Σ is an F-stable conjugacy class of $\boldsymbol{G}$ and $\mathcal{E}$ is an F-invariant irreducible local system on Σ. In all cases, $\mathcal{E}$ is one-dimensional, so condition $(*)$ in §4.2 is satisfied. Furthermore, let $\mathcal{O}_0$ be the conjugacy class of the unipotent part of an element in Σ. Then $\mathcal{O}_0$ is the unipotent class denoted by $F_4(a_3)$ in [57, §5]. (The identification of $\mathcal{O}_0$ follows from [41, Prop. 1.16] if p is sufficiently large; by Taylor [60], it is enough to assume that $p > 3$. For small values of p, one can also use explicit computations in a matrix realisation of $\boldsymbol{G}$ and the results of Lawther [29].) We have $A_{\boldsymbol{G}}(u) \cong \mathfrak{S}_4$ for $u \in \mathcal{O}_0$, and there exists some $u \in \mathcal{O}_0^F$ such that F acts trivially on $A_{\boldsymbol{G}}(u)$; see Shoji [50]. Thus, $\mathcal{O}_0^F$ splits into five conjugacy classes in $\boldsymbol{G}^F$, corresponding to the five conjugacy classes of $\mathfrak{S}_4$. As in [50], we denote representatives of those five $\boldsymbol{G}^F$-conjugacy classes by $x_{14}, \ldots, x_{18}$. We have $|C^\circ_{\boldsymbol{G}}(x_i)^F| = q^{12}$ in each case; the groups $A_{\boldsymbol{G}}(x_i)$ are as follows.

$$\begin{array}{ll}
A_{\boldsymbol{G}}(x_{14})^F \cong \mathfrak{S}_4, & \text{(cycle type (1111))},\\
A_{\boldsymbol{G}}(x_{15})^F \cong D_8, & \text{(cycle type (22))},\\
A_{\boldsymbol{G}}(x_{16})^F \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}, & \text{(cycle type (211))},\\
A_{\boldsymbol{G}}(x_{17})^F \cong \mathbb{Z}/4\mathbb{Z}, & \text{(cycle type (4))},\\
A_{\boldsymbol{G}}(x_{18})^F \cong \mathbb{Z}/3\mathbb{Z}, & \text{(cycle type (31))}.
\end{array}$$

Thus, the five representatives x_i $(i = 14, \ldots, 18)$ can be distinguished from each other by the structure of the group $A_{\boldsymbol{G}}(x_i)^F$. Now let $n \in \mathbb{Z}$ be such that $p \nmid n$. Since $\mathcal{O}_0$ is uniquely determined by its dimension, each $u \in \mathcal{O}_0$ is conjugate to u^n in $\boldsymbol{G}$; it then also follows that each x_i is conjugate to x_i^n in $\boldsymbol{G}^F$, for $i = 14, \ldots, 18$. We shall make use of this remark for $n = 2$ in the discussion below.

Now we turn to the detailed description of the seven cuspidal character sheaves of $\boldsymbol{G}$, where we follow Lusztig [36, §20, §21] and Shoji [53, §6]. In each case, we will determine the scalar ζ in the identity $(\clubsuit')$; see §4.3. We deal with the various cases in order of increasing difficulty.

7.6. The cuspidal character sheaves A_3, A_4. Let $s_1 \in \boldsymbol{G}^F$ be semisimple such that $\boldsymbol{H}_1' = C_{\boldsymbol{G}}(s_1)$ has a root system Φ' of type $A_2 \times A_2$; recall from §7.1 that $Z(\boldsymbol{H}_1') \cong \mathbb{Z}/3\mathbb{Z}$ and this is generated by s_1. Let $u_1 \in \boldsymbol{H}_1'^F$ be a regular unipotent element and Σ be the conjugacy class of $g_1 := s_1 u_1$. As in §5.2, one sees that $\Sigma = \Sigma^{-1}$ and that $A_{\boldsymbol{G}}(g_1) \cong \mathbb{Z}/3\mathbb{Z}$ is generated by the image $\bar{g}_1$ of g_1 in $A_{\boldsymbol{G}}(g_1)$. By [53, (6.2.4)(c)], there are two cuspidal character sheaves $A_i = \mathrm{IC}(\Sigma, \mathcal{E}_i)[\dim \Sigma]$ where $i = 3, 4$. Let $1 \neq \theta \in \mathbb{K}^\times$ be a fixed third root of unity. Then $\mathcal{E}_3$ corresponds to the linear character $\psi_3 \colon A_{\boldsymbol{G}}(g_1) \to \mathbb{K}^\times$ such that $\psi_3(\bar{g}_1) = \theta$ and $\mathcal{E}_4$ corresponds to the linear character $\psi_4 \colon A_{\boldsymbol{G}}(g_1) \to \mathbb{K}^\times$ such that $\psi_4(\bar{g}_1) = \theta^2$. By [53, (6.4.1)], A_3 is parametrised by $(g_3, \theta) \in X(\boldsymbol{W})$ and A_4 is parametrised by $(g_3, \theta^2) \in X(\boldsymbol{W})$. By the main result of [53, §6], there are scalars $\zeta, \zeta' \in \mathbb{K}$ of absolute value 1 such that $R_{(g_3,\theta)} = \zeta \chi_{g_1,\psi_3}$ and $R_{(g_3,\theta^2)} = \zeta' \chi_{g_1,\psi_4}$, where the almost characters $R_{(g_3,\theta)}$ and $R_{(g_3,\theta^2)}$ are defined as the following linear combinations of unipotent characters:

$$
\begin{aligned}
R_{(g_3,\theta)} &:= \tfrac{1}{3}\big([\phi_{12,4}] + F_4^{\mathrm{II}}[1] - [\phi_{6,6}'] - [\phi_{6,6}''] + 2F_4[\theta] - F_4[\theta^2]\big),\\
R_{(g_3,\theta^2)} &:= \tfrac{1}{3}\big([\phi_{12,4}] + F_4^{\mathrm{II}}[1] - [\phi_{6,6}'] - [\phi_{6,6}''] - F_4[\theta] + 2F_4[\theta^2]\big).
\end{aligned}
$$

Here, we use the notation in [4, p. 479], with analogous conventions as in the previous sections. Thus, $\phi_{12,4}$ etc. are irreducible characters of $\boldsymbol{W}$; then $[\phi_{12,4}]$ etc. are the corresponding irreducible constituents of $\mathrm{Ind}_{\boldsymbol{B}^F}^{\boldsymbol{G}^F}(1)$; characters denoted like $F_4^{\mathrm{II}}[1]$ are cuspidal unipotent. (In this section we refer to [4] instead of [33], because the full 21×21 Fourier matrix related to type F_4 is printed on [4, p. 456], and that matrix will be needed for several arguments below.) By Lemma 4.10(b), we can choose $g_1 \in \Sigma^F$ to be conjugate to g_1^{-1} in $\boldsymbol{G}^F$. By an argument analogous to that in §5.4, one sees that $R_{(g_3,\theta)}$ and $R_{(g_3,\theta^2)}$ are complex conjugate to each other. So we conclude that

$$
\zeta = \zeta' = \pm 1 \qquad \text{and} \qquad R_{(g_3,\theta)}(g_1) = R_{(g_3,\theta^2)}(g_1) = \zeta q^2.
$$

Inverting the matrix relating unipotent characters and unipotent almost characters, we obtain the following relation:

$$
F_4[\theta] = \tfrac{1}{3}\big(R_{(12,4)} + R_{(1,\lambda^3)} - R_{(6,6)'} - R_{(6,6)''} + 2R_{(g_3,\theta)} - R_{(g_3,\theta^2)}\big),
$$

where we just write, for example, $R_{(12,4)}$ instead of $R_{\phi_{(12,4)}}$. Using CHEVIE and the formula in Example 3.3, we find that

$$
R_{(12,4)}(g_1) = R_{(6,6)''}(g_1) = 0, \qquad R_{(6,6)'}(g_1) = 1.
$$

By [53, (6.2.2)], the pair $(1,\lambda^3)$ also parametrises a cuspidal character sheaf, which will be supported on a conjugacy class distinct from Σ. By the main result of [53, §6], a characteristic function of that character sheaf equals $R_{(1,\lambda^3)}$, up to multiplication by a scalar. Hence, we have $R_{(1,\lambda^3)}(g_1)=0$ and we obtain $F_4[\theta](g_1)=\frac{1}{3}(-1+\zeta q^2)$. Since the left hand side is an algebraic integer, this forces that $\zeta=1$. Thus, we have shown that

(a) $$R_{(g_3,\theta)}=\chi_{g_1,\psi_3} \qquad \text{and} \qquad R_{(g_3,\theta^2)}=\chi_{g_1,\psi_4};$$

recall that, here, we fixed $g_1\in\Sigma^F$ such that g_1 is conjugate to g_1^{-1} in $\boldsymbol{G}^F$. The values of $F_4[\theta]$ and $F_4[\theta^2]$ on Σ^F are given by the following table.

	$\bar{1}$	$\bar{s}_1$	$\bar{s}_1^2$
$F_4[\theta]$	$\frac{1}{3}(q^2-1)$	$\frac{1}{3}(q^2-1)+q^2\theta$	$\frac{1}{3}(q^2-1)+q^2\theta^2$
$F_4[\theta^2]$	$\frac{1}{3}(q^2-1)$	$\frac{1}{3}(q^2-1)+q^2\theta^2$	$\frac{1}{3}(q^2-1)+q^2\theta$

This table also shows that $g_1\in\Sigma^F$ is uniquely determined (up to $\boldsymbol{G}^F$-conjugation) by the property that g_1 is conjugate to g_1^{-1} in $\boldsymbol{G}^F$.

7.7. The cuspidal character sheaves A_5, A_6. Let $s_1\in\boldsymbol{G}^F$ be semisimple such that $\boldsymbol{H}_1'=C_{\boldsymbol{G}}(s_1)$ has a root system Φ' of type $A_3\times A_1$; recall from §7.2 that $Z(\boldsymbol{H}_1')\cong\mathbb{Z}/4\mathbb{Z}$ and this is generated by s_1. Let $u_1\in\boldsymbol{H}_1'^F$ be a regular unipotent element and Σ be the conjugacy class of $g_1:=s_1u_1$. As above, one sees that $\Sigma=\Sigma^{-1}$ and that $A_{\boldsymbol{G}}(g_1)\cong\mathbb{Z}/4\mathbb{Z}$ is generated by $\bar{g}_1\in A_{\boldsymbol{G}}(g_1)$. By [53, (6.2.4)(d)], there are two cuspidal character sheaves $A_i=\mathrm{IC}(\Sigma,\mathcal{E}_i)[\dim\Sigma]$ where $i=5,6$; here, $\mathcal{E}_5$ corresponds to the linear character $\psi_5\colon A_{\boldsymbol{G}}(g_1)\to\mathbb{K}^\times$ such that $\psi_5(\bar{g}_1)=\mathrm{i}$ (where $\mathrm{i}^2=-1$ in $\mathbb{K}$) and $\mathcal{E}_6$ corresponds to the linear character $\psi_6\colon A_{\boldsymbol{G}}(g_1)\to\mathbb{K}^\times$ such that $\psi_6(\bar{g}_1)=-\mathrm{i}$. By [53, (6.4.1)], A_5 is parametrised by $(g_4,\mathrm{i})\in X(\boldsymbol{W})$ and A_6 is parametrised by $(g_4,-\mathrm{i})\in X(\boldsymbol{W})$. By the main result of [53, §6], there are scalars $\zeta,\zeta'\in\mathbb{K}$ of absolute value 1 such that $R_{(g_4,\mathrm{i})}=\zeta\chi_{g_1,\psi_5}$ and $R_{(g_4,-\mathrm{i})}=\zeta'\chi_{g_1,\psi_6}$, where

$$\begin{aligned}R_{(g_4,\mathrm{i})}:=\tfrac{1}{4}\big([\phi_{12,4}]-[\phi_{9,6}']+[\phi_{1,12}']-F_4^{\mathrm{II}}[1]-[\phi_{9,6}'']-F_4^{\mathrm{I}}[1]\\ +[\phi_{1,12}'']+[\phi_{4,8}]+2F_4[\mathrm{i}]-2F_4[-\mathrm{i}]\big);\end{aligned}$$

there is a similar expression for $R_{(g_4,-\mathrm{i})}$ where the roles of $F_4[\mathrm{i}]$ and $F_4[-\mathrm{i}]$ are interchanged. By Lemma 4.10(b), we can choose $g_1\in\Sigma^F$ to be conjugate in $\boldsymbol{G}^F$ to g_1^{-1}. As in §7.6, we conclude that $\zeta=\zeta'=\pm1$. Inverting the matrix relating unipotent characters and unipotent almost characters, we obtain:

$$F_4[\mathrm{i}]=\tfrac{1}{4}\big(R_{(12,4)}-R_{(9,6)'}+R_{(1,12)'}-R_{(1,\lambda^3)}-R_{(9,6)''}$$

$$- R_{(g_2',\varepsilon)} + R_{(1,12)''} + R_{(4,8)} + 2R_{(g_4,\mathrm{i})} - 2R_{(g_4,-\mathrm{i})}\big).$$

Using Example 3.3, we find that $R_\phi(g) = 0$ for all $g \in \Sigma^F$ and all $\phi \in \mathrm{Irr}(\boldsymbol{W})$ occurring in the sum on the right hand side. Again, by [53, (6.2.2)], the pair (g_2', ε) also parametrises a cuspidal character sheaf, which will be supported on a conjugacy class distinct from Σ. By the main result of [53, §6], a characteristic function of that character sheaf equals $R_{(g_2',\varepsilon)}$, up to multiplication by a scalar. Hence, we also have $R_{(g_2',\varepsilon)}(g) = 0$ for all $g \in \Sigma^F$. A similar argument shows that $R_{(1,\lambda^3)}(g_1) = 0$. This yields the following table for the values of $F_4[\mathrm{i}]$ on Σ^F:

	$\bar{1}$	$\bar{g}_1^2$	$\bar{g}_1$	$\bar{g}_1^{-1}$
$F_4[\mathrm{i}]$	0	0	$\zeta \mathrm{i} q^2$	$-\zeta \mathrm{i} q^2$
$F_4[-\mathrm{i}]$	0	0	$-\zeta \mathrm{i} q^2$	$\zeta \mathrm{i} q^2$

We can now draw the following conclusions. Let C_1, C_2, C_3, C_4 be the four conjugacy classes of $\boldsymbol{G}^F$ into which Σ^F splits (not necessarily ordered as in the above table). We have $\Sigma = \Sigma^{-1}$, so taking inverses permutes the four classes. The table shows that we can arrange the notation such that $C_4 = C_3^{-1}$, where $C_3 = \mathrm{Sh}_{\boldsymbol{G}}(C_1)$ and $C_4 = \mathrm{Sh}_{\boldsymbol{G}}(C_2)$; see §4.1(a). Since $g_1 \in \Sigma^F$ is conjugate in $\boldsymbol{G}^F$ to g_1^{-1}, this forces that $C_1 = C_1^{-1}$, $C_2 = C_2^{-1}$ and $g_1 \in C_1 \cup C_2$. By Lemma 4.10(b), we can further fix the notation such that $C_1 \cap \boldsymbol{B}^F \dot{w}_c \boldsymbol{B}^F \neq \varnothing$ and $C_2 \cap \boldsymbol{B}^F \dot{w}_c \boldsymbol{B}^F = \varnothing$. Then we claim:

(a) $$\zeta = \begin{cases} 1 & \text{if } g_1 \in C_1, \\ -1 & \text{if } g_1 \in C_2. \end{cases}$$

This is seen by an argument entirely analogous to the proof of Proposition 6.5, based on the formula in Remark 4.12(b). The data required for that argument (that is, the constants $\alpha_1(\phi)$, $\alpha_2(\phi)$ and the values $\phi^{(q)}(T_{w_c})$) are now given as follows. We have $\alpha_1(\phi) = \alpha_2(\phi) = \frac{1}{4}$ for $\phi \in \{\phi_{1,12}', \phi_{1,12}'', \phi_{4,8}, \phi_{12,4}\}$, and $\alpha_1(\phi) = \alpha_2(\phi) = 0$ otherwise; furthermore, if $\alpha_1(\phi) \neq 0$, then $\phi^{(q)}(T_{w_c}) = q^2$; we omit further details.

7.8. The cuspidal character sheaf A_1. Let Σ be the unipotent class of $\boldsymbol{G}$ denoted by $F_4(a_3)$, as already introduced in §7.5. We take $g_1 := x_{14} \in \Sigma^F$; hence, F acts trivially on $A_{\boldsymbol{G}}(g_1) \cong \mathfrak{S}_4$. We also remarked in §7.5 that g_1 is conjugate in $\boldsymbol{G}^F$ to g_1^{-1}. As in [53, (6.2.4)(a)], there is a cuspidal character sheaf $A_1 = \mathrm{IC}(\Sigma, \mathcal{E})[\dim \Sigma]$ where $\mathcal{E}$ corresponds to the sign character $\mathrm{sgn} \in \mathrm{Irr}(A_{\boldsymbol{G}}(g_1))$. In [53, §6], it is not stated explicitly to which parameter in $X(\boldsymbol{W})$ the character sheaf A_1 corresponds, but this is easily found as follows, using the information already available from §7.6. (See also the argument in [38, Lemma 8.8].) We claim that A_1 is parametrised by $(1,\lambda^3) \in X(\boldsymbol{W})$. Assume, if possible, that this is not the case. By Shoji's results [51] on the Green functions of $\boldsymbol{G}^F$, we can compute $R_\phi(g_1)$ for any $\phi \in \mathrm{Irr}(\boldsymbol{W})$. (These results are known to hold whenever $p \neq 2,3$; see [54, Theorem 5.5] and [14, §3].) In particular, we obtain $R_{(6,6)''}(g_1) = 0$, $R_{(6,6)'}(g_1) = 2q^4$ and $R_{(12,4)}(g_1) = q^4$. Since $R_{(g_3,\theta)}$ and $R_{(g_3,\theta^2)}$ are zero on unipotent elements (see §7.6), we conclude that $F_4[\theta](g_1) = \frac{1}{3}(q^4 - 2q^4) = -q^4/3$, contradiction since $p \neq 3$. Hence, A_1 is parametrised by $(1,\lambda^3)$. By the main result of [53, §6], there is a scalar $\zeta \in \mathbb{K}$ of absolute value 1 such that $R_{(1,\lambda^3)} = \zeta\chi_{g_1,\mathrm{sgn}}$. The exact expression of $R_{(1,\lambda^3)}$ as a linear combination of 21 unipotent characters is obtained from the Fourier matrix on [4, p. 456] and the list of labels for unipotent characters on [4, p. 479]; we will not print it here.

It was first shown by Kawanaka [28, §4] that $\zeta = 1$, assuming that p, q are sufficiently large; Lusztig [38, 8.6, 8.12] shows this assuming that q satisfies a certain congruence condition. Since Kawanaka's results on generalised Gelfand–Graev representations are now known to hold whenever q is a power of a good prime p (see Taylor [60]), we can conclude that $\zeta = 1$ holds unconditionally (but recall our standing assumption that $p > 3$).

We can also argue as follows. Consider again the formula for $F_4[\mathrm{i}]$ in §7.7. Using Shoji's results on Green functions, we can compute the values of R_ϕ on unipotent elements, for all ϕ occurring in that formula. Furthermore, we have $R_{(g_2',\varepsilon)}(g_1) = R_{(g_4,\pm\mathrm{i})}(g_1) = 0$. This yields that $F_4[\mathrm{i}](g_1) = -\frac{1}{4}q^4(\zeta q^2 - 1)$. Since g_1 is $\boldsymbol{G}^F$-conjugate to g_1^{-1}, we have $\zeta = \overline{\zeta}$. Since $F_4[\mathrm{i}](g_1)$ is an algebraic integer, we must have $\zeta = 1$.

Table 9: Some character values on the unipotent class $F_4(a_3)$

	x_{14} (1111)	x_{15} (22)	x_{16} (211)	x_{17} (4)	x_{18} (31)
$[\phi'_{1,12}]$	$\frac{1}{8}q^4(q^2-1)+3q^4$	$\frac{1}{8}q^4(q^2-1)$	$\frac{1}{8}q^4(1-q^2)$	$\frac{1}{8}q^4(1-q^2)$	$\frac{1}{8}q^4(q^2-1)$
$[\phi''_{1,12}]$	$\frac{1}{8}q^4(q^2-1)$	$\frac{1}{8}q^4(q^2-1)+q^4$	$\frac{1}{8}q^4(1-q^2)$	$\frac{1}{8}q^4(1-q^2)$	$\frac{1}{8}q^4(q^2-1)$
$F_4[-1]$	$\frac{1}{4}q^4(1-q^2)$	$\frac{1}{4}q^4(1-q^2)$	$\frac{1}{4}q^4(q^2-1)+q^4$	$\frac{1}{4}q^4(q^2-1)$	$\frac{1}{4}q^4(1-q^2)$
$F_4[\mathrm{i}]$	$\frac{1}{4}q^4(1-q^2)$	$\frac{1}{4}q^4(1-q^2)$	$\frac{1}{4}q^4(q^2-1)$	$\frac{1}{4}q^4(q^2-1)+q^4$	$\frac{1}{4}q^4(1-q^2)$

7.9. Character values on $F_4(a_3)$. Once $R_{(1,\lambda^3)}$ has been determined, we can determine all character values on $\boldsymbol{G}_{\mathrm{uni}}^F$, where $\boldsymbol{G}_{\mathrm{uni}}$ denotes the unipotent variety of $\boldsymbol{G}$. Indeed, the 25 unipotent almost characters R_ϕ (for $\phi \in \mathrm{Irr}(\boldsymbol{W})$)

remain linearly independent upon restriction to $\boldsymbol{G}_{\mathrm{uni}}^F$; they are explicitly computed by Shoji [51]. (As mentioned above, Shoji's results remain valid whenever $p \neq 2, 3$.) Hence, together with the "cuspidal" almost character $R_{(1,\lambda^3)}$, we obtain 26 linearly independent functions on $\boldsymbol{G}_{\mathrm{uni}}^F$. Since there are also 26 unipotent conjugacy classes of $\boldsymbol{G}^F$ (see [50, Theorem 2.1]), we obtain a basis for the space of class functions on $\boldsymbol{G}_{\mathrm{uni}}^F$. Note that all the remaining unipotent almost characters are orthogonal to the functions in that basis, which implies that they are identically zero on $\boldsymbol{G}_{\mathrm{uni}}^F$. In §7.11 below, we shall need the values of some unipotent characters on $\mathcal{O}_0^F$, with $\mathcal{O}_0$ as in §7.5. The values displayed in Table 9 will allow us to distinguish the various $\boldsymbol{G}^F$-conjugacy classes contained in $\mathcal{O}_0^F$. These values are easily obtained from the functions `UnipotentCharacters` and `ICCTable` in Michel's version of CHEVIE [47].

7.10. The cuspidal character sheaf A_7. Let $s_1 := h(1, 1, -1, 1) \in \boldsymbol{T}_0^F$ and $\boldsymbol{H}_1' := C_{\boldsymbol{G}}(s_1)$; then $\boldsymbol{H}_1'$ has a root system Φ' of type B_4 (see §7.3); recall that $Z(\boldsymbol{H}_1') \cong \mathbb{Z}/2\mathbb{Z}$ and this is generated by s_1. Consider the natural isogeny $\beta \colon \boldsymbol{H}_1' \to \overline{\boldsymbol{H}}_1' := \mathrm{SO}_9(k)$ (defined over $\mathbb{F}_q$). Let $\mathcal{O}$ be the unipotent class of $\boldsymbol{H}_1'$ such that the elements $\beta(u) \in \overline{\boldsymbol{H}}_1'$, for $u \in \mathcal{O}$, have Jordan type $(5, 3, 1)$. Let Σ be the conjugacy class of $s_1 u$, where $u \in \mathcal{O}$. Now $\mathcal{O}$ is F-stable and so Σ is also F-stable. By Shoji [50, Table 4], and the correction discussed by Fleischmann–Janiszczak [13, p. 233], we have:

(a) There exists an element $g_1 \in \Sigma^F$ such that $A_{\boldsymbol{G}}(g_1)$ is dihedral of order 8 and F acts trivially on $A_{\boldsymbol{G}}(g_1)$; we have $|C_{\boldsymbol{G}}(g_1)^F| = 8q^8$.

Thus, the set Σ^F splits into five conjugacy classes in $\boldsymbol{G}^F$, with centraliser orders $8q^8, 8q^8, 4q^4, 4q^4, 4q^4$. So there are two possibilities for the $\boldsymbol{G}^F$-conjugacy class of g_1 as in (a). (We just choose one of them; this choice does not affect the result at the end. By §4.1(a) and [9, Chap. I, Prop. 7.2], we also see that the two classes are interchanged by the Shintani map $\mathrm{Sh}_{\boldsymbol{G}}$.) Now, by [53, (6.2.4)(e)], there is a cuspidal character sheaf $A_7 = \mathrm{IC}(\Sigma, \mathcal{E})[\dim \Sigma]$ where $\mathcal{E}$ corresponds to the sign character $\mathrm{sgn} \in \mathrm{Irr}(A_{\boldsymbol{G}}(g_1))$. By [53, (6.2.2)], A_7 is parametrised either by the pair (g_2, ε) or by the pair (g_2', ε) in $X(\boldsymbol{W})$. We can easily fix this as follows. We note that the eigenvalue $\lambda_{A_7} = \mathrm{sgn}(\bar{g}_1)$ in §4.2 must be 1 since $\bar{g}_1$ is in the center of $A_{\boldsymbol{G}}(g_1)$. But there are also certain eigenvalues for the almost characters, where $\lambda_{R_x} = -1$ for $x = (g_2, \varepsilon)$, and $\lambda_{R_x} = 1$ for $x = (g_2', \varepsilon)$; see [53, (6.2.2)]. By the main result of [53, §6], there is a scalar $\zeta \in \mathbb{K}$ of absolute value 1 such that $R_x = \zeta \chi_{(g_1,\mathrm{sgn})}$ where $x \in \{(g_2, \varepsilon), (g_2', \varepsilon)\}$ and where the eigenvalues of the character sheaves do match those of the almost characters (see [54, 4.6]). Hence,

(b) A_7 is parametrised by $(g_2', \varepsilon) \in X(\boldsymbol{W})$ and $R_{(g_2',\varepsilon)} = \zeta \chi_{g_1,\mathrm{sgn}}$.

The exact expression of $R_{(g_2',\varepsilon)}$ as a linear combination of 18 unipotent characters is obtained from the Fourier matrix on [4, p. 456] and the list of labels for unipotent characters on [4, p. 479]; we will not print it here. We claim:

(c) With g_1 as in (a), we have $\zeta = 1$.

This is seen as follows. We use again the following identity from §7.7:

$$F_4[\mathrm{i}] = \tfrac{1}{4}\big(R_{(12,4)} - R_{(9,6)'} + R_{(1,12)'} - R_{(1,\lambda^3)} - R_{(9,6)''} \\ - R_{(g_2',\varepsilon)} + R_{(1,12)''} + R_{(4,8)} + 2R_{(g_4,\mathrm{i})} - 2R_{(g_4,-\mathrm{i})}\big).$$

Now all R_ϕ ($\phi \in \mathrm{Irr}(\boldsymbol{W})$) are rational-valued. Since $\overline{F_4[\mathrm{i}]} = F_4[-\mathrm{i}]$ and $\overline{R_{(g_4,\mathrm{i})}} = R_{(g_4,-\mathrm{i})}$, we conclude that $R_{(g_2',\varepsilon)}$ is invariant under complex conjugation and, hence, $\zeta = \pm 1$. Now evaluate $F_4[\mathrm{i}]$ on $g_1 \in \Sigma^F$. Note that $R_{(1,\lambda^3)}$ and $R_{(g_4,\pm\mathrm{i})}$ have support on conjugacy classes that are distinct from Σ^F and, hence, their value is zero on g_1; see §7.7 and §7.8. By Example 3.5, we obtain

$$R_{(9,6)''}(g_1) = R_{(4,8)}(g_1) = R_{(1,12)''}(g_1) = q^2, \\ R_{(12,4)}(g_1) = R_{(9,6)'}(g_1) = R_{(1,12)'}(g_1) = 0.$$

(Recall that g_1 is chosen such that $|C_{\boldsymbol{G}}(g_1)^F| = 8q^8$.) Since $R_{(g_2',\varepsilon)}(g_1) = \zeta q^4$, we obtain $F_4[\mathrm{i}](g_1) = \frac{1}{4}q^2(1-\zeta q^2)$. Since the left hand side is an algebraic integer, we deduce that $\zeta = 1$. Thus, (c) is proved. Finally, we note:

(d) If $g_1 = s_1u_1$ is as in (a), then u_1 is $\boldsymbol{G}^F$-conjugate to x_{14} or x_{15}.

Indeed, since $C_{\boldsymbol{G}}(g_1) \subseteq C_{\boldsymbol{G}}(g_1^2) = C_{\boldsymbol{G}}(u_1^2)$ and u_1^2 is $\boldsymbol{G}^F$-conjugate to $u_1 \in \mathcal{O}_0$, we conclude that 8 divides $|C_{\boldsymbol{G}}(u_1)^F|$. Hence, the only possibilities are that u_1 is $\boldsymbol{G}^F$-conjugate to x_{14} or x_{15}. I conjecture that for one of the two possibilities of $g_1 = s_1u_1$ as in (a), we do have that u_1 is $\boldsymbol{G}^F$-conjugate to x_{14} (but the choice of that g_1 may depend on q mod 4).

7.11. The cuspidal character sheaf A_2. Let $s_1 \in \boldsymbol{G}^F$ be semisimple such that $\boldsymbol{H}_1' = C_{\boldsymbol{G}}(s_1)$ has a root system Φ' of type $C_3 \times A_1$; recall from §7.4 that $Z(\boldsymbol{H}_1') \cong \mathbb{Z}/2\mathbb{Z}$ and this is generated by s_1. Now we have a natural isogeny $\beta\colon \mathrm{Sp}_4(k) \times \mathrm{SL}_2(k) \to \boldsymbol{H}_1'$ (defined over $\mathbb{F}_q$). Let $\mathcal{O}$ be the unipotent class of $\boldsymbol{H}_1'$ that corresponds to unipotent elements of Jordan type $(4,2)\times(2)$ under β. We start by picking any element $u_1 \in \mathcal{O}^F$ and let Σ be the conjugacy class of $g_1 := s_1u_1$. We have $\dim \boldsymbol{G} - \dim\Sigma = 6$ and $|C_{\boldsymbol{G}}(g_1)^F| = 4q^6$; one easily sees that $\Sigma = \Sigma^{-1}$. Now there is some $1 \neq a \in A_{\boldsymbol{G}}(g_1)$ such that

$$A_{\boldsymbol{G}}(g_1) = \langle \bar{g}_1\rangle \times \langle a\rangle \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \qquad \text{(with trivial } F\text{-action).}$$

By [53, (6.2.4)(b)], there is a cuspidal character sheaf $A_2 = \mathrm{IC}(\Sigma, \mathcal{E})[\dim \Sigma]$ where $\mathcal{E}$ corresponds to a non-trivial $\psi \in \mathrm{Irr}(A_{\boldsymbol{G}}(g_1))$ (further specified below). By [53, (6.2.2)], A_2 is parametrised either by the pair (g_2, ε) or by the pair (g_2', ε) in $X(\boldsymbol{W})$. By §7.10(b), we conclude that A_2 must be parametrised by (g_2, ε); in particular, $\psi(\bar{g}_1) = \lambda_{A_2} = -1$. We can now also fix the element $a \in A_{\boldsymbol{G}}(g_1)$ such that $\psi(a) = 1$. By the main result of [53, §6], there is a scalar $\zeta \in \mathbb{K}$ of absolute value 1 such that $R_{(g_2,\varepsilon)} = \zeta\chi_{g_1,\psi}$, where

$$\begin{aligned} R_{(g_2,\varepsilon)} := \tfrac{1}{4}\big([\phi_{12,4}] + [\phi_{9,6}'] - [\phi_{1,12}'] - F_4^{\mathrm{II}}[1] - 2[\phi_{16,5}] \\ + 2F_4[-1] + [\phi_{9,6}''] - F_4^{\mathrm{I}}[1] - [\phi_{1,12}''] + [\phi_{4,8}]\big). \end{aligned}$$

Let C_1, C_2, C_3, C_4 be the four $\boldsymbol{G}^F$-conjugacy classes into which Σ^F splits (initially ordered in no particular way). For each i, we denote $C_i^{[2]} := \{g^2 \mid g \in C_i\}$. Writing $g_1 = s_1u_1$ as above, we have $u_1 \in \mathcal{O}_0$; furthermore, u_1, u_1^2 are $\boldsymbol{G}^F$-conjugate and so $g_1^2 = u_1^2 \in \mathcal{O}_0$ (see §7.5). We claim that the notation can be arranged such that

(a) $\quad x_{14} \in C_1^{[2]}, \qquad x_{15} \in C_2^{[2]} \qquad \text{and} \qquad x_{16} \in C_3^{[2]} = C_4^{[2]}.$

Depending on how we choose $g_1 \in \Sigma^F$, the scalar ζ is then determined as follows.

(b) $$\zeta = \begin{cases} 1 & \text{if } g_1 \in C_1 \cup C_2, \\ -1 & \text{if } g_1 \in C_3 \cup C_4. \end{cases}$$

This is proved as follows. Inverting the matrix relating unipotent characters and unipotent almost characters, we obtain:

$$\begin{aligned} F_4[-1] = \tfrac{1}{4}\big(R_{(12,4)} + R_{(9,6)'} - R_{(1,12)'} - R_{(1,\lambda^3)} - 2R_{(16,5)} \\ + 2R_{(g_2,\varepsilon)} + R_{(9,6)''} - R_{(g_2',\varepsilon)} - R_{(1,12)''} + R_{(4,8)}\big). \end{aligned}$$

Now we evaluate this on $g_1 \in \Sigma^F$. By §7.8 and §7.10, we have $R_{(1,\lambda^3)}(g_1) = R_{(g_2',\varepsilon)}(g_1) = 0$. By a computation entirely analogous to that in Example 3.5, we obtain $R_{(16,5)}(g_1) = q$ and

$$R_{(12,4)}(g_1) = R_{(9,6)'}(g_1) = R_{(9,6)''}(g_1) = R_{(1,12)'}(g_1) = R_{(1,12)''}(g_1) = 0;$$

this does not depend on how we choose $g_1 \in \Sigma^F$. Since $R_{(g_2,\varepsilon)}$ takes the values $\zeta q^3, \zeta q^3, -\zeta q^3, -\zeta q^3$ on the representatives in Σ^F parametrised by $\bar{1}, a, \bar{g}_1, a\bar{g}_1$, this yields the following values for $F_4[-1]$ on Σ^F.

	$\bar{1}$	a	$\bar{g}_1$	$a\bar{g}_1$
$F_4[-1]$	$\frac{1}{2}q(\zeta q^2-1)$	$\frac{1}{2}q(\zeta q^2-1)$	$-\frac{1}{2}q(\zeta q^2+1)$	$-\frac{1}{2}q(\zeta q^2+1)$

By [15, Table 1], the character $F_4[-1]$ is rational-valued, so we must have $\zeta=\pm 1$. Regardless of whether ζ equals 1 or -1, two of the above values are $\frac{1}{2}q(q^2-1)$, and two of them are $-\frac{1}{2}q(q^2+1)$. Thus, two of the above values are even integers, and two of them are odd integers. Now compare with Table 9:

$$F_4[-1](x_{16}) \equiv 1 \bmod 2 \qquad \text{and} \qquad F_4[-1](x_i) \equiv 0 \bmod 2 \quad \text{for } i \neq 16.$$

By a well-known fact from the general character theory of finite groups, we have $F_4[-1](g_1^2) \equiv F_4[-1](g_1) \bmod 2$. Hence, if $g_1 \in \Sigma^F$ is such that $F_4[-1](g_1)$ is odd, then g_1^2 must be $\boldsymbol{G}^F$-conjugate to x_{16}. Since there are two $\boldsymbol{G}^F$-conjugacy classes in Σ^F on which the value of $F_4[-1]$ is odd, we conclude that $x_{16} \in C_i^{[2]}$ for two values of $i \in \{1,2,3,4\}$; we arrange the notation such that these two values are $i=3$ and $i=4$. Now choose $g_1 \in \Sigma^F$ such that $g_1 \in C_3 \cup C_4$. Since $F_4[-1](g_1)$ is given by the entry corresponding to $\bar{1} \in A_{\boldsymbol{G}}(g_1)$ in the above table, we conclude that $\frac{1}{2}q(\zeta q^2-1)$ must be odd and so $\zeta=-1$. Thus, (a) and (b) are proved as far as C_3 and C_4 are concerned. On the other hand, let us choose $g_1 \in \Sigma^F \setminus (C_3 \cup C_4)$. Then $F_4[-1](g_1) = \frac{1}{2}q(\zeta q^2-1)$ must be even and so $\zeta=1$. So all that remains to be done is to identity $i,j \in \{14,\ldots,18\}$ such that $x_i \in C_1^{[2]}$ and $x_j \in C_2^{[2]}$. For this purpose, we consider the characters $[\phi'_{1,12}]$ and $[\phi''_{1,12}]$.

Using the ingredients of the CHEVIE function `LusztigMapb` explained in [47, §7] (which relies on the theoretical fact that the indicator function of a $\boldsymbol{G}^F$-conjugacy class is "uniform", see [17, §8]), we can compute $\sum_{g\in\Sigma^F}\rho(g)$ for any $\rho \in \mathrm{Unip}(\boldsymbol{G}^F)$. Since all elements in Σ^F have the same centraliser order, we can actually compute the sum of the four values of ρ on C_1, C_2, C_3, C_4. Applying this to $\rho=[\phi'_{1,12}]$, we find that the result is $-q$. Consequently, the four values of $[\phi'_{1,12}]$ on C_1, C_2, C_3, C_4 cannot all have the same parity. Hence, there exists some $g \in \Sigma^F$ such that $[\phi'_{1,12}](g) \equiv [\phi'_{1,12}](x_{14}) \bmod 2$. But then we also have

$$[\phi'_{1,12}](g^2) \equiv [\phi'_{1,12}](g) \equiv [\phi'_{1,12}](x_{14}) \bmod 2.$$

Since $[\phi'_{1,12}](x_i) \not\equiv [\phi'_{1,12}](x_{14}) \bmod 2$ for $i \neq 14$ (see Table 9), we conclude that g^2 is $\boldsymbol{G}^F$-conjugate to x_{14}. Thus, we can arrange the notation such that $x_{14} \in C_1^{[2]}$. Then a completely analogous argument using the character $[\phi''_{1,12}]$ shows that $x_{15} \in C_2^{[2]}$. Thus, (a) and (b) are proved. The above table of values also shows that the values of $F_4[-1]$ on the classes parametrised by $\bar{1}$ and $\bar{g}_1$

have a different parity; similarly for a and $a\bar{g}_1$. Hence, we can fix the notation for C_3 and C_4 such that $C_3 = \mathrm{Sh}_{\boldsymbol{G}}(C_1)$ and $C_4 = \mathrm{Sh}_{\boldsymbol{G}}(C_2)$ (see §4.1(a)).

Finally, we remark that we can also obtain an explicit representative in Σ^F. Indeed, using CHEVIE, we can easily compute the full $\boldsymbol{W}$-orbit of s_1; by inspection, $s_1' := h(-1,-1,1,1) \in \boldsymbol{T}_0^F$ belongs to that orbit, that is, s_1' is conjugate to s_1 in $\boldsymbol{G}^F$. Using the explicit expression (in terms of Chevalley generators of $\boldsymbol{G}^F$) for x_{16} in [50, Table 5], we can check that s_1' commutes with x_{16}. Hence, we have $g_1 := s_1' x_{16} \in \Sigma^F$; since $g_1 = x_{16}^2$, we must have $\zeta = -1$ for this choice of g_1.

7.12. The cases where $p = 2, 3$. In the above discussion, we assumed that $p \neq 2, 3$. For $p = 2$, the scalars ζ in the identities $R_x = \zeta\chi_A$ have been determined by Marcelo–Shinoda [46, §4] and [18, §5]. Now assume that $p = 3$. For those cuspidal character sheaves A where the corresponding conjugacy class Σ is unipotent (there are three of them), the scalars ζ are also determined by [46, §4]. By [53, §7.2], the remaining four cuspidal character sheaves are analogous to those denoted above by A_2, A_5, A_6 and A_7. One checks that the discussions in §7.7, 7.10, 7.11 can be applied almost verbatim to the case $p = 3$, and yield the same results. The Green functions for $p = 3$ are known by [46] (see also [19, §5])).

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Affine Deligne-Lusztig varieties associated with generic Newton points

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To George Lusztig with admiration

Abstract: This paper gives an explicit formula of the dimension of affine Deligne-Lusztig varieties associated with generic Newton point in terms of Demazure product of Iwahori-Weyl groups.

Keywords: Affine Deligne-Lusztig varieties, Iwahori-Weyl groups, Demazure product.

Introduction

Let F be a non-archimedean local field, and $\breve{F}$ be the completion of its maximal unramified extension. Let σ be the Frobenius automorphism of $\breve{F}$ over F. Let $\mathbf{G}$ be a connected reductive group over F. Let $\breve{\mathcal{I}}$ be the standard Iwahori subgroup of $\mathbf{G}(\breve{F})$. Let w be an element in the Iwahori-Weyl group $\tilde{W}$ and $b \in \mathbf{G}(\breve{F})$. The affine Deligne-Lusztig variety associated to (w, b) is defined to be

$$X_w(b) = \{g\breve{\mathcal{I}} \in \mathbf{G}(\breve{F})/\breve{\mathcal{I}}; g^{-1} b\sigma(g) \in \breve{\mathcal{I}} w \breve{\mathcal{I}}\}.$$

This is a subscheme locally of finite type, of the affine flag variety. It is introduced by Rapoport in [Ra05] and plays an important role when studying the special fiber of both Shimura varieties and moduli spaces of Shtukas.

Let $B(\mathbf{G})$ be the set of σ-conjugacy classes of $\mathbf{G}(\breve{F})$. The set $B(\mathbf{G})$ is classified by the Kottwitz map and the Newton map. The set $B(\mathbf{G})$ is equipped with a natural partial order by requiring the equality under the Kottwitz map and the dominance order on the associated Newton points. It is easy to see that if b and b' are in the same σ-conjugacy class, then $X_w(b)$ and $X_w(b')$ are isomorphic. The set $B(G)_w = \{[b] \in B(G); X_w(b) \neq \emptyset\}$ contains a unique maximal element, which we denote by $[b_w]$. The variety $X_w(b_w)$ is called the affine Deligne-Lusztig varieties associated with generic Newton points. It has been studied in [Mi16+] and [MV20].

The main purpose of this note is as follows.

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Theorem 0.1. *Let $w \in \tilde{W}$ and $[b_w]$ be the maximal element in $B(G)_w$. Then*

$$\dim X_w(b_w) = \ell(w) - \lim_{n \longrightarrow \infty} \frac{\ell(w^{*_\sigma, n})}{n}.$$

Here $*$ is the Demazure product on $\tilde{W}$ and

$$w^{*_\sigma, n} = w * \sigma(w) * \cdots * \sigma^{n-1}(w)$$

is the n-th σ-twisted Demazure power of w.

Now we discuss the motivation for this formula and the outline of the proof.

For simplicity, we only discuss the split group case here. In this case, σ acts trivially on $\tilde{W}$ and we may simply write $w^{*,n}$ for $w^{*_\sigma,n}$.

There is a natural bijection between the poset $B(G)$ and the poset of straight conjugacy classes in $\tilde{W}$, established in [He14] & [He16a]. Here by definition, an element $x \in \tilde{W}$ is straight if $\ell(x^n) = n\ell(x)$ for all $n \in \mathbb{N}$ and a conjugacy class of $\tilde{W}$ is straight if it contains a straight element.

Let $\mathcal{O}_w$ be the straight conjugacy class associated to $[b_w]$ and w' be a minimal length element in C_w. Then we have $\dim X_w(b_w) = \ell(w) - \ell(w')$. In particular, if w is a straight element, then we may take $w' = w$. In this case, $\dim X_w(b_w) = 0$. By definition, for straight element w we have $\ell(w^{*,n}) = \ell(w^n) = n\ell(w)$. So the statement is obvious in this case. Theorem 0.1 gives an estimate on the "non-straightness" of the element w.

By our assumption, any generic element in the double coset $\breve{\mathcal{I}} w \breve{\mathcal{I}}$ is σ-conjugate to an element in $\breve{\mathcal{I}} w' \breve{\mathcal{I}}$. Let g be a generic element in $\breve{\mathcal{I}} w \breve{\mathcal{I}}$ and $g' \in \breve{\mathcal{I}} w' \breve{\mathcal{I}}$ such that g and g' are σ-conjugate by an element $h \in \mathbf{G}(\breve{F})$. We consider the σ-twisted power defined by $g^{\sigma,n} = g\sigma(g) \cdots \sigma^{n-1}(g)$. Then for any $n \in \mathbb{N}$, $g^{\sigma,n}$ and $(g')^{\sigma,n}$ are σ^n-conjugate by the same element h. By the straightness assumption on $\mathcal{O}_w$, we have $(g')^{\sigma,n} \in \breve{\mathcal{I}}(w')^n \breve{\mathcal{I}}$. On the other hand, we have

$$g^{\sigma,n} \in (\breve{\mathcal{I}} w \breve{\mathcal{I}})(\breve{\mathcal{I}} w \breve{\mathcal{I}}) \cdots (\breve{\mathcal{I}} w \breve{\mathcal{I}}) \subset \overline{\breve{\mathcal{I}} w^{*,n} \breve{\mathcal{I}}}.$$

However, it is not clear if $g^{\sigma,n} \in \breve{\mathcal{I}} w^{*,n} \breve{\mathcal{I}}$.

The trick we will use to bypass this difficulty is to apply the technique in [HN20] and to translate the question on the σ-conjugation action on $\mathbf{G}(\breve{F})$ to the question on the ordinary conjugation action on a reductive $\mathbf{G}'$ over $\mathbb{C}((\epsilon))$. Let $\breve{\mathcal{I}}'$ be an Iwahori subgroup of $\mathbf{G}'(\mathbb{C}((\epsilon)))$. Using Lusztig's theory of total positivity [Lu94] and [Lu19], we show that for generic element $g \in \breve{\mathcal{I}}' w \breve{\mathcal{I}}'$, $g^n \in \breve{\mathcal{I}}'(w^{*,n})\breve{\mathcal{I}}'$. The condition that there exists $h \in \mathbf{G}'(\mathbb{C}((\epsilon))$ such that $\breve{\mathcal{I}}'(w^{*,n})\breve{\mathcal{I}}' \cap h(\breve{\mathcal{I}}'(w')^n\breve{\mathcal{I}}')h^{-1} \neq \emptyset$ for all $n \in \mathbb{N}$ implies that $\ell(w') = \lim_{n\to\infty} \frac{\ell(w^{*,n})}{n}$. This finishes the proof.

1. Preliminary

1.1. Notations

Let $\mathbf{G}$ be a connected reductive group over a non-archimedean local field F. We write Γ for $\mathrm{Gal}(\overline{F}/F)$, where $\overline{F}$ is an algebraic closure of F. We write Γ_0 for the inertia subgroup of Γ. Let $\breve{F}$ be the completion of the maximal unramified extension of F and σ be the Frobenius morphism of $\breve{F}/F$. The residue field of F is a finite field $\mathbb{F}_q$ and the residue field of $\breve{F}$ is the algebraically closed field $\bar{\mathbb{F}}_q$. We write $\breve{G}$ for $\mathbf{G}(\breve{F})$. We use the same symbol σ for the induced Frobenius morphism on $\breve{G}$.

Let S be a maximal $\breve{F}$-split torus of $\mathbf{G}$ defined over F, which contains a maximal F-split torus. Let $\mathcal{A}$ be the apartment of $\mathbf{G}_{\breve{F}}$ corresponding to $S_{\breve{F}}$. We fix a σ-stable alcove $\mathfrak{a}$ in $\mathcal{A}$, and let $\breve{\mathcal{I}} \subset \breve{G}$ be the Iwahori subgroup corresponding to $\mathfrak{a}$. Then $\breve{\mathcal{I}}$ is σ-stable.

Let T be the centralizer of S in $\mathbf{G}$. Then T is a maximal torus. We denote by N the normalizer of T in $\mathbf{G}$. The *Iwahori–Weyl group* (associated to S) is defined as

$$\tilde{W} = N(\breve{F})/T(\breve{F}) \cap \breve{\mathcal{I}}.$$

For any $w \in \tilde{W}$, we choose a representative $\dot{w}$ in $N(\breve{F})$. The action σ on $\breve{G}$ induces a natural action of σ on $\tilde{W}$, which we still denote by σ.

We denote by ℓ the length function on $\tilde{W}$ determined by the base alcove $\mathfrak{a}$ and denote by $\tilde{\mathbb{S}}$ the set of simple reflections in $\tilde{W}$. Let W_{aff} be the subgroup of $\tilde{W}$ generated by $\tilde{\mathbb{S}}$. Then W_{aff} is an affine Weyl group. Let $\Omega \subset \tilde{W}$ be the subgroup of length-zero elements in $\tilde{W}$. Then

$$\tilde{W} = W_{\text{aff}} \rtimes \Omega.$$

Since the length function is compatible with the σ-action, the semi-direct product decomposition $\tilde{W} = W_{\text{aff}} \rtimes \Omega$ is also stable under the action of σ.

1.2. The σ-conjugacy classes of $\breve{G}$

The σ-conjugation action on $\breve{G}$ is defined by $g \cdot_\sigma g' = gg'\sigma(g)^{-1}$ for $g, g' \in \breve{G}$. Let $B(\mathbf{G})$ be the set of σ-conjugacy classes on $\breve{G}$. The classification of the σ-conjugacy classes is due to Kottwitz [Ko85] and [Ko97]. Any σ-conjugacy class $[b]$ is determined by two invariants:

- The element $\kappa([b]) \in \pi_1(\mathbf{G})_\sigma$;
- The Newton point $\nu_b \in \big((X_*(T)_{\Gamma_0,\mathbb{Q}})^+\big)^{\langle\sigma\rangle}$.

Here $-_\sigma$ denotes the σ-coinvariants, $(X_*(T)_{\Gamma_0,\mathbb{Q}})^+$ denotes the intersection of $X_*(T)_{\Gamma_0} \otimes \mathbb{Q} = X_*(T)^{\Gamma_0} \otimes \mathbb{Q}$ with the set $X_*(T)_{\mathbb{Q}}^+$ of dominant elements in $X_*(T)_{\mathbb{Q}}$.

We denote by $\leqslant$ the dominance order on $X_*(T)_{\mathbb{Q}}^+$, i.e., for $\nu, \nu' \in X_*(T)_{\mathbb{Q}}^+$, $\nu \leqslant \nu'$ if and only if $\nu' - \nu$ is a non-negative (rational) linear combination of positive roots over $\breve{F}$. The dominance order on $X_*(T)_{\mathbb{Q}}^+$ extends to a partial order on $B(\mathbf{G})$. Namely, for $[b], [b'] \in B(\mathbf{G})$, $[b] \leqslant [b']$ if and only if $\kappa([b]) = \kappa([b'])$ and $\nu_b \leqslant \nu_{b'}$.

1.3. The straight σ-conjugacy classes of $\tilde{W}$

Let $w \in \tilde{W}$ and $n \in \mathbb{N}$. The *n-th σ-twisted power* of w is defined by

$$w^{\sigma,n} = w\sigma(w) \cdots \sigma^{n-1}(w).$$

In the case where σ acts trivially on $\tilde{W}$, we have $w^{\sigma,n} = w^n$ is the ordinary n-th power of w.

By definition, an element $w \in \tilde{W}$ is called *σ-straight* if for any $n \in \mathbb{N}$, $\ell(w^{\sigma,n}) = n\ell(w)$. A σ-conjugacy class of $\tilde{W}$ is *straight* if it contains a σ-straight element. Let $B(\tilde{W}, \sigma)_{\mathrm{str}}$ be the set of straight σ-conjugacy classes of $\tilde{W}$. The following result is proved in [He14, Theorem 3.7].

Theorem 1.1. *The map $w \mapsto [w]$ induces a natural bijection*

$$\Psi : B(\tilde{W}, \sigma)_{str} \longrightarrow B(\mathbf{G}).$$

1.4. Affine Deligne-Lusztig varieties

Following [Ra05], we define the affine Deligne-Lusztig variety $X_w(b)$. Let $\mathrm{Fl} = \breve{G}/\breve{\mathcal{I}}$ be the affine flag variety. For any $w \in \tilde{W}$ and $b \in \mathbf{G}$, we set

$$X_w(b) = \{g\breve{\mathcal{I}} \in \mathrm{Fl}; g^{-1}b\sigma(g) \in \breve{\mathcal{I}}w\breve{\mathcal{I}}\}.$$

In the equal characteristic, $X_w(b)$ is the set of $\bar{\mathbb{F}}_q$-points of a scheme. In the equal characteristic, $X_w(b)$ is the set of $\bar{\mathbb{F}}_q$-points of a perfect scheme (see [BS17] and [Zh17]).

It is easy to see that $X_w(b)$ only depends on w and the σ-conjugacy class $[b]$ of b. For any $w \in \tilde{W}$, we set

$$B(\mathbf{G})_w = \{[b] \in B(G); X_w(b) \neq \emptyset\}.$$

Let $[b_w]$ be the σ-conjugacy class in the (unique) generic point of $\breve{\mathcal{I}}w\breve{\mathcal{I}}$. Then $[b_w]$ is the unique maximal element in $B(\mathbf{G})_w$ with respect to the partial ordering $\leqslant$ on $B(\mathbf{G})$ (see [MV20, Definition 3.1]). We choose a representative $b_w \in [b_w]$ and call $X_w(b_w)$ the *affine Deligne-Lusztig variety associated with the generic Newton point* of w.

1.5. Demazure product

Now we recall the Demazure product $*$ on $\tilde{W}$. By definition, $(W, *)$ is a monoid such that $w * w' = ww'$ for any $w, w' \in \tilde{W}$ if $\ell(ww') = \ell(w) + \ell(w')$ and $s * w = w$ for $s \in \tilde{\mathbb{S}}$ and $w \in \tilde{W}$ if $sw < w$. In other words, $\tau * w = \tau w$ and $s * w = \max\{w, sw\}$ for $\tau \in \Omega$, $s \in \tilde{\mathbb{S}}$ and $w \in \tilde{W}$.

The geometric interpretation of the Demazure product is as follows. For any $w \in \tilde{W}$, $\overline{\breve{\mathcal{I}}w\breve{\mathcal{I}}} = \cup_{w' \leqslant w} \breve{\mathcal{I}}w'\breve{\mathcal{I}}$ is a closed admissible subset of $\mathbf{G}$ in the sense of [He16a, A.2]. Then for any $w, w' \in \tilde{W}$, we have

$$\overline{\breve{\mathcal{I}}w\breve{\mathcal{I}}}\,\overline{\breve{\mathcal{I}}w'\breve{\mathcal{I}}} = \overline{\breve{\mathcal{I}}(w * w')\breve{\mathcal{I}}}.$$

Let $w \in \tilde{W}$ and $n \in \mathbb{N}$. The *n-th σ-twisted Demazure power* of w is defined by

$$w^{*_\sigma, n} = w * \sigma(w) * \cdots * \sigma^{n-1}(w).$$

1.6. Minimal length elements

For $w, w' \in \tilde{W}$ and $s \in \tilde{\mathbb{S}}$, we write $w \xrightarrow{s}_\sigma w'$ if $w' = sw\sigma(s)$ and $\ell(w') \leqslant \ell(w)$. We write $w \rightarrow_\sigma w'$ if there is a sequence $w = w_1, w_2, \dots, w_n = w'$ in $\tilde{W}$ such that for each $2 \leq k \leq n$ we have $w_{k-1} \xrightarrow{s_k}_\sigma w_k$ for some $s_k \in \tilde{\mathbb{S}}$. We write $w \approx_\sigma w'$ if $w \rightarrow_\sigma w'$ and $w' \rightarrow_\sigma w$. We write $w \tilde{\approx}_\sigma w'$ if $w \approx_\sigma \tau w' \sigma(\tau)^{-1}$ for some $\tau \in \Omega$. For any σ-conjugacy class $\mathcal{O}$, we write $\ell(\mathcal{O}) = \ell(x)$ for any minimal length element x of $\mathcal{O}$.

The following result is proved in [HN14, Theorem A].

Theorem 1.2. *Let $w \in \tilde{W}$. Then there exists a minimal length element w' in the same σ-conjugacy class of w such that $w \rightarrow_\sigma w'$.*

2. The generic σ-conjugacy class

In this section, we study the σ-conjugacy class $[b_w]$ in more detail.

2.1. Via the Bruhat order

Let $w \in \tilde{W}$. The generic σ-conjugacy class $[b_w]$ in $\breve{\mathcal{I}} w \breve{\mathcal{I}}$ is first studied by Viehmann in [Vi14]. The following result is proved in [Vi14, Corollary 5.6].

Proposition 2.1. *Let $w \in \tilde{W}$. Then the set*

$$\{[w']; w' \leqslant w\} \subset B(\mathbf{G})$$

contains a unique maximal element and this maximal element equals $[b_w]$.

A more explicit description of the generic Newton point ν_{b_w} is obtained by Milićević [Mi16+, Theorem 3.2] for split group $\mathbf{G}$ and sufficiently large w.

2.2. Via the partial order on $B(\tilde{W}, \sigma)_{\mathrm{str}}$

Let $w \in \tilde{W}$ and $\mathcal{O} \in B(\tilde{W}, \sigma)_{\mathrm{str}}$. We write $\mathcal{O} \preceq_\sigma w$ if there exists a minimal length element $w' \in \mathcal{O}$ such that $w' \leqslant w$ with respect to the Bruhat order $\leqslant$ of $\tilde{W}$. Let $\mathcal{O}, \mathcal{O}' \in B(\tilde{W}, \sigma)_{\mathrm{str}}$. We write $\mathcal{O}' \preceq_\sigma \mathcal{O}$ if $\mathcal{O}' \preceq_\sigma w$ for some minimal length element w of $\mathcal{O}$. By [He16a, §3.2], if $\mathcal{O}' \preceq_\sigma \mathcal{O}$, then $\mathcal{O}' \preceq_\sigma w'$ for any minimal length element w' of $\mathcal{O}$. Hence $\preceq_\sigma$ is a partial order on $B(\tilde{W}, \sigma)_{\mathrm{str}}$.

It is proved in [He16a, Theorem B] that

Theorem 2.2. *The partial order $\preceq_\sigma$ on $B(\tilde{W}, \sigma)_{str}$ coincides with the partial order $\leqslant$ on $B(\mathbf{G})$ via the bijection map $\Psi : B(\tilde{W}, \sigma)_{str} \to B(\mathbf{G})$.*

Now we show that

Proposition 2.3. *Let $w \in \tilde{W}$. Then the set $\{\mathcal{O} \in B(\tilde{W}, \sigma)_{str}; \mathcal{O} \preceq_\sigma w\}$ contains a unique maximal element $\mathcal{O}_w$ with respect to the partial order $\preceq_\sigma$. Moreover, $\Psi(\mathcal{O}_w) = [b_w]$.*

Proof. By [He16a, §2.7], $\cup_{w' \leqslant w} [w'] = \cup_{\mathcal{O} \preceq_\sigma w} \Psi(\mathcal{O})$. By Proposition 2.1, the set

$$\{[w']; w' \leqslant w\} = \{\Psi(\mathcal{O}) \in B(\tilde{W}, \sigma)_{\mathrm{str}}; \mathcal{O} \preceq_\sigma w\}$$

contains a unique maximal element, which is $[b_w]$. Let $\mathcal{O}_w = \Psi^{-1}([b_w])$. By Theorem 2.2, $\mathcal{O}_w$ is the unique maximal element of $\{\mathcal{O} \in B(\tilde{W}, \sigma)_{\mathrm{str}}; \mathcal{O} \preceq_\sigma w\}$. $\square$

2.3. An algorithm

We provide an algorithm to compute $\mathcal{O}_w$. We argue by induction on $\ell(w)$. By [He16a, §2.7], if w is a minimal length element in its σ-conjugacy class, then $\mathcal{O}_w$ is the straight σ-conjugacy class of $\tilde{W}$ that corresponds to $[w] \in B(\mathbf{G})$.

If w is not a minimal length element in its σ-conjugacy class, then by Theorem 1.2, there exists $w' \in \tilde{W}$ and a simple reflection s such that $w' \approx_\sigma w$ and $sw'\sigma(s) < w'$. By [He16a, Proposition 2.4], $\mathcal{O} \preceq_\sigma w$ if and only if $\mathcal{O} \preceq_\sigma w'$. Let $\mathcal{O}$ be a straight σ-conjugacy class with $\mathcal{O} \preceq_\sigma w'$. Then there exists a minimal length element w_1 of $\mathcal{O}$ with $w_1 \leqslant w'$. If $sw_1 > w_1$, then $w_1 \leqslant \min\{w', sw'\} = sw'$. If $sw_1 < w_1$, then $\ell(sw_1\sigma(s)) \leqslant \ell(sw_1)+1 = \ell(w_1)$ and $sw_1\sigma(s)$ is also a minimal length element in $\mathcal{O}$. Moreover, we have $sw_1 \leqslant sw'$ and $sw_1\sigma(s) \leqslant \max\{sw', sw'\sigma(s)\} = sw'$. In either case, $\mathcal{O} \preceq_\sigma sw'$. By inductive hypothesis on sw', $\mathcal{O}_w = \mathcal{O}_{w'} = \mathcal{O}_{sw'}$ is the unique maximal element in $\{\mathcal{O}; \mathcal{O} \preceq_\sigma w\} = \{\mathcal{O}; \mathcal{O} \preceq_\sigma sw'\}$.

2.4. Via the 0-Hecke algebras

Let H_0 be the 0-*Hecke algebra* of $\tilde{W}$. It is a $\mathbb{C}$-algebra generated by $\{t_w; w \in \tilde{W}\}$ subject to the relations

- $t_w t_{w'} = t_{ww'}$ for any $w, w' \in \tilde{W}$ with $\ell(ww') = \ell(w) + \ell(w')$.
- $t_s^2 = -t_s$ for any $s \in \tilde{\mathbb{S}}$.

The automorphism σ on $\tilde{W}$ induces a natural algebra homomorphism on H_0, which we still denote by σ. For any $h, h' \in H_0$, the σ-commutator of h and h' is defined by $[h, h']_\sigma = hh' - h'\sigma(h)$. The σ-*commutator* $[H_0, H_0]_\sigma$ of H_0 is by definition the subspace of H_0 spanned by $[h, h']_\sigma$ for all $h, h' \in H_0$. The σ-cocenter of H_0 is defined to be $\bar{H}_{0,\sigma} = H_0/[H_0, H_0]_\sigma$.

Let $\tilde{W}_{\sigma,\min}$ be the set of elements in $\tilde{W}$ which are of minimal length in their σ-conjugacy classes. It is easy to see that if $w \in \tilde{W}_{\sigma,\min}$ and $w' \tilde{\approx}_\sigma w$, then $w' \in \tilde{W}_{\sigma,\min}$. Let $\tilde{W}_{\sigma,\min}/\tilde{\approx}_\sigma$ be the set of $\tilde{\approx}_\sigma$-equivalence classes in $\tilde{W}_{\sigma,\min}$.

By [He15, Proposition 2.1], if $w \tilde{\approx}_\sigma w'$, then t_w and $t_{w'}$ have the same image in $\bar{H}_{0,\sigma}$. For any $\Sigma \in \tilde{W}_{\sigma,\min}/\tilde{\approx}_\sigma$, we write t_Σ for the image of t_w in $\bar{H}_{0,\sigma}$ for any $w \in \Sigma$.

We have the following result.

Proposition 2.4. *(1) The set* $\{t_\Sigma\}_{\Sigma \in \tilde{W}_{\sigma,\min}/\tilde{\approx}_\sigma}$ *is a* $\mathbb{C}$*-basis of* $\bar{H}_{0,\sigma}$*.*

(2) For any $w \in \tilde{W}$*, there exists a unique* $\Sigma_w \in \tilde{W}_{\sigma,\min}/\tilde{\approx}_\sigma$ *such that the image of* t_w *in* $\bar{H}_{0,\sigma}$ *equals* $\pm t_{\Sigma_w}$*.*

This result is proved for 0-Hecke algebras of finite Weyl groups in [He15, Proposition 6.1 & Proposition 7.3]. The proof for the 0-Hecke algebras of Iwahori-Weyl groups is the same.

Now we give another description of $[b_w]$.

Proposition 2.5. *Let $w \in \tilde{W}$. Then $[b_w] = \Psi(\Sigma_w)$.*

Remark 2.6. It is worth pointing out that in general, Σ_w is different from $\mathcal{O}_w$ in Proposition 2.3.

Proof. The argument is similar to §2.3. We argue by induction on $\ell(w)$. By [He16a, §2.7], if w is a minimal length element in its σ-conjugacy class and t_{Σ_w} is the $\approx_\sigma$-equivalence class of w, then $\Psi(\Sigma_w) = [w] = [b_w]$.

If w is not a minimal length element in its σ-conjugacy class, then by Theorem 1.2, there exists $w' \in \tilde{W}$ and a simple reflection s such that $w' \approx_\sigma w$ and $sw'\sigma(s) < w'$. We have

$$t_w \equiv t_{w'} = t_s t_{sw'} \equiv t_{sw'} t_{\sigma(s)} = -t_{sw'} \mod [H_0, H_0]_\sigma.$$

Therefore $\Sigma_w = \Sigma_{w'} = \Sigma_{sw'}$. By §2.3, $\mathcal{O}_w = \mathcal{O}_{w'} = \mathcal{O}_{sw'}$. Now the statement follows from induction hypothesis on sw'. □

3. Passing from non-archimedean local fields to $\mathbb{C}((\epsilon))$

3.1. Dimension formula

The following result follows from [He16b, Theorem 2.23].

Lemma 3.1. *Let $w \in \tilde{W}$. Then*

$$\dim X_w(b_w) = \ell(w) - \ell(\mathcal{O}_w).$$

Note that the definition of $\mathcal{O}_w$ only depends on the triple $(\tilde{W}, \sigma, w)$ and is independent of the reductive $\mathbf{G}$ over F. This allows us to reduce the calculation of the dimension of affine Deligne-Lusztig varieties to the calculation of the length of certain elements in the Iwahori-Weyl group. And the latter problem will be translated to a problem on reductive groups over $\mathbb{C}((\epsilon))$. This is what we will do in this section.

3.2. Reduction to W_{aff}

Let $w = x\tau$ for $x \in W_{\text{aff}}$ and $\tau \in \Omega$. We write $\theta = \mathrm{Ad}(\tau) \circ \sigma \in \mathrm{Aut}(W_{\text{aff}})$. Define the map

$$\iota : W_{\text{aff}} \longrightarrow \tilde{W}, \quad x' \longmapsto x'\tau.$$

For any $x' \in W_{\text{aff}}$ and $n \in \mathbb{N}$, we have $\ell(\iota(x')^{\sigma,n}) = \ell((x')^{\theta,n})$. Thus x' is θ-straight if and only if $\iota(x')$ is σ-straight. It is also easy to see that if x_1, x_2 are in the same θ-conjugacy class of W_{aff}, then $\iota(x_1), \iota(x_2)$ are in the same σ-conjugacy class of $\tilde{W}$. The map ι induces a map $B(W_{\text{aff}}, \theta)_{\text{str}} \to B(\tilde{W}, \sigma)_{\text{str}}$, which we still denote by ι.

By Proposition 2.3, there exists a unique maximal element $\mathcal{O}_x$ in

$$\{\mathcal{O}' \in B(W_{\text{aff}}, \theta)_{\text{str}}; \mathcal{O}' \preceq_\theta x\}.$$

By definition, $\iota(\mathcal{O}_x) \preceq_\sigma \iota(x) = w$ and it is a maximal element in

$$\{\mathcal{O} \in B(\tilde{W}, \sigma)_{\text{str}}; \mathcal{O} \preceq_\sigma w\}.$$

Thus $\iota(\mathcal{O}_x) = \mathcal{O}_w$. We have $\ell(w) - \ell(\mathcal{O}_w) = \ell(x) - \ell(\mathcal{O}_x)$.

Thus to prove Theorem 0.1, it remains to show that for any diagram automorphism θ of W_{aff}, we have

$$(*) \qquad \ell(\mathcal{O}_x) = \lim_{n \longrightarrow \infty} \frac{\ell(x^{\theta,n})}{n}.$$

3.3. The group $\mathbf{G}'$ over $\mathbb{C}((\epsilon))$

Let $\mathbf{G}'$ be a connected semisimple group split over $\mathbb{C}((\epsilon))$ whose Iwahori-Weyl group is isomorphic to W_{aff}. We write $\breve{G}'$ for $\mathbf{G}'(\mathbb{C}((\epsilon)))$. Let T' be a split maximal torus of $\mathbf{G}'$ and $B' \supset T'$ be a Borel subgroup. Let

$$\breve{\mathcal{I}}' = \{g \in G'(\mathbb{C}[[\epsilon]]); g \mid_{\epsilon \longmapsto 0} \in B'(\mathbb{C})\}$$

be an Iwahori subgroup of $\breve{G}'$. We have the decomposition $\breve{G}' = \sqcup_{x \in W_{\text{aff}}} \breve{\mathcal{I}}' x \breve{\mathcal{I}}'$.

The diagram automorphism θ on W_{aff} can be lifted to a diagram automorphism on $\breve{G}'$, which we still denote by θ. We consider the θ-conjugation action $\cdot_\theta$ on $\breve{G}'$ here.

Let $\mathcal{O} \in B(W_{\text{aff}}, \theta)_{\text{str}}$. By [HN20, §3.2], $\breve{G}' \cdot_\theta \breve{\mathcal{I}}' x \breve{\mathcal{I}}' = \breve{G}' \cdot_\theta \breve{\mathcal{I}}' x' \breve{\mathcal{I}}'$ for any minimal length elements $x, x' \in \mathcal{O}$. We write $[\mathcal{O}] = \breve{G}' \cdot_\theta \breve{\mathcal{I}}' x \breve{\mathcal{I}}'$ for any minimal length element $x \in \mathcal{O}$. We have

Theorem 3.2. $\breve{G}' = \bigsqcup_{\mathcal{O} \in B(W_{\text{aff}}, \theta)_{str}} [\mathcal{O}]$.

It is proved in [HN20, Theorem 3.2] for reductive groups over $\mathbf{k}((\epsilon))$, where $\mathbf{k}$ is an algebraically closed field of positive characteristic. The same proof works over $\mathbb{C}((\epsilon))$.

As a variation of Proposition 2.3, $\breve{G} \cdot_\sigma \overline{\breve{\mathcal{I}} w \breve{\mathcal{I}}} = \bigsqcup_{\mathcal{O} \preceq_\sigma \mathcal{O}_w} \Psi(\mathcal{O})$. Similarly we have the following result.

Proposition 3.3. *Let $x \in W_{\text{aff}}$. Then*

$$\breve{G}' \cdot_\theta \overline{\breve{\mathcal{I}}' x \breve{\mathcal{I}}'} = \bigsqcup_{\mathcal{O} \preceq_\theta \mathcal{O}_x} [\mathcal{O}].$$

4. Generic elements coming from total positivity

Let $x \in W_{\text{aff}}$. Then for any $g \in \breve{\mathcal{I}}' x \breve{\mathcal{I}}'$, we have $g \in \sqcup_{\mathcal{O} \preceq_\theta \mathcal{O}_x} [\mathcal{O}]$ and $g^{\theta, n} \in \sqcup_{x' \leqslant x^{*_\theta, n}} \breve{\mathcal{I}}' x' \breve{\mathcal{I}}'$ for all n. The advantage of working with $\mathbb{C}((\epsilon))$ instead of $\breve{F}$ is that we may prove the following result on the generic elements of $\breve{\mathcal{I}}' x \breve{\mathcal{I}}'$.

Proposition 4.1. *Let $x \in W_{\text{aff}}$. Then there exists $g \in \breve{\mathcal{I}}' x \breve{\mathcal{I}}'$ such that $g \in [\mathcal{O}_x]$ and for any $n \in \mathbb{N}$, $g^{\theta, n} \in \breve{\mathcal{I}}' x^{*_\theta, n} \breve{\mathcal{I}}'$.*

Proof. The idea is to use Lusztig's theory of total positivity. Recall that $\tilde{\mathbb{S}}$ is the set of positive affine simple roots of $\breve{G}'$. For any $i \in \tilde{\mathbb{S}}$, let α_i be the corresponding affine simple root and $U_{\alpha_i} \subset \breve{G}'$ be the corresponding affine root subgroup. Let $\{x_i : \mathbb{G}_m \to U_{\alpha_i}; i \in \tilde{\mathbb{S}}\}$ be an affine pinning of $\mathbf{G}'$ (see [GH21+, §5.3]). Since θ is a diagram automorphism of $\mathbf{G}'$, we may choose $\{x_i\}$ to be θ-stable. Let $U_{-\alpha_i}$ be the affine root subgroup correspond to $-\alpha_i$. Let $y_i : \mathbb{G}_m \to U_{-\alpha_i}$ be the isomorphism such that $x_i(1) y_i(-1) x_i(1) \in N(T')$[1].

Let $y \in W_{\text{aff}}$ and $y = s_{i_1} \cdots s_{i_k}$ be a reduced expression of y. Set

$$U_y^- = \{y_{i_1}(a_1) \cdots y_{i_k}(a_k); a_1, \dots, a_k \in \mathbb{R}_{>0}\}.$$

By [Lu19, §2.5], U_y^- is independent of the choices of reduced expressions of y. Moreover, since $y_i(a) \in \breve{\mathcal{I}}' s_i \breve{\mathcal{I}}'$ for any $i \in \tilde{\mathbb{S}}$ and $a \neq 0$, we have $U_y^- \subset \breve{\mathcal{I}}' y \breve{\mathcal{I}}'$.

By [Lu19, §2.11], for any $y_1, y_2 \in W_{\text{aff}}$, we have $U_{y_1}^- U_{y_2}^- = U_{y_1 * y_2}^-$.

In particular, for any $g \in U_x^-$ and $n \in \mathbb{N}$, we have

$$g^{\theta, n} \in U_x^- \cdot U_{\theta(x)}^- \cdots U_{\theta^{n-1}(x)}^- \subset U_{x^{*_\theta, n}}^- \subset \breve{\mathcal{I}}' x^{*_\theta, n} \breve{\mathcal{I}}'.$$

We show that

[1] This normalization of y_i differs from [GH21+] but is consistent with the normalization used in Lusztig's theory of total positivity.

(a) If $sy < y$ for some simple reflection s, then any element in U_y^- is θ-conjugate to an element in $U_{(sy)*\theta(s)}^-$.

By definition, there exists a reduced expression of y with $y = s_{i_1} \cdots s_{i_k}$ and $s_{i_1} = s$. Thus $y_{i_1}(a_1) \cdots y_{i_k}(a_k)$ is θ-conjugate to

$$y_{i_2}(a_2) \cdots y_{i_k}(a_k)\theta(y_{i_1}(a_1)) = y_{i_2}(a_2) \cdots y_{i_k}(a_k) y_{\theta(i_1)}(a_1).$$

If $a_1, \dots, a_k > 0$, then $y_{i_2}(a_2) \cdots y_{i_k}(a_k) y_{\theta(i_1)}(a_1) \in U_{sy}^- U_{\theta(s)}^- = U_{(sy)*\theta(s)}^-$. (a) is proved.

Now we show that $g \in [\mathcal{O}_x]$. We argue by induction on $\ell(x)$.

If x is a minimal length element in its θ-conjugacy class of W_{aff}, then by the reduction argument in [He14, Lemma 3.1], $g \in \breve{\mathcal{I}}' x \breve{\mathcal{I}}' \subset [\mathcal{O}_x]$. If x is not a minimal length element in its θ-conjugacy class of W_{aff}, then by Theorem 1.2, there exists $x' \in W_{\text{aff}}$ and a simple reflection s such that $x' \approx_\theta x$ and $sx'\theta(s) < x'$. We have $sx' < x'$ and $(sx') * \theta(s) = sx'$. By (a), any element in U_x^- is θ-conjugate to an element in $U_{x'}^-$ and any element in $U_{x'}^-$ is θ-conjugate to an element in $U_{sx'}^-$. By inductive hypothesis on sx', we have $U_{sx'}^- \subset [\mathcal{O}_{sx'}]$. By §2.3, $\mathcal{O}_{sx'} = \mathcal{O}_{x'} = \mathcal{O}_x$.

This finishes the proof. □

4.1. Proof of Theorem 0.1

Now we prove Theorem 0.1. As explained in §3.2, it suffices to prove §3.2(*). By Proposition 4.1, there exists $g \in \breve{\mathcal{I}}' x \breve{\mathcal{I}}'$ such that $g \in [\mathcal{O}_x]$ and for any $n \in \mathbb{N}$, $g^{\theta,n} \in \breve{\mathcal{I}}' x^{*_\theta, n} \breve{\mathcal{I}}'$. Let x' be a minimal length element in $\mathcal{O}_x$. Then x' is a θ-straight element. Since $g \in [\mathcal{O}_x]$, there exists $h \in \breve{G}'$ and $g' \in \breve{\mathcal{I}}' x' \breve{\mathcal{I}}'$ such that $g' = hg\theta(h)^{-1}$.

Since x' is θ-straight, we have

$$(g')^{\theta,n} \in (\breve{\mathcal{I}}' x' \breve{\mathcal{I}}')(\breve{\mathcal{I}}' \theta(x') \breve{\mathcal{I}}') \cdots (\breve{\mathcal{I}}' \theta^{n-1}(x') \breve{\mathcal{I}}') = \breve{\mathcal{I}}' (x')^{\theta,n} \breve{\mathcal{I}}'.$$

On the other hand,

$$(g')^{\theta,n} = hg^{\theta,n}\theta^n(h)^{-1} \in h(\breve{\mathcal{I}}' x^{*_\theta, n} \breve{\mathcal{I}}')\theta^n(h)^{-1}.$$

We have $h \in \breve{\mathcal{I}}' y \breve{\mathcal{I}}'$ for some $y \in W_{\text{aff}}$. Let $N_0 = \ell(y)$. Then $\theta^n(h) \in \breve{\mathcal{I}}' \theta^n(y) \breve{\mathcal{I}}'$ and $\ell(\theta^n(y)) = N_0$.

Note that

$$\begin{aligned} h(\breve{\mathcal{I}}' x^{*_\theta, n} \breve{\mathcal{I}}')\theta^n(h)^{-1} &\subset (\breve{\mathcal{I}}' y \breve{\mathcal{I}}')(\breve{\mathcal{I}}' x^{*_\theta, n} \breve{\mathcal{I}}')(\breve{\mathcal{I}}' \theta^n(y) \breve{\mathcal{I}}') \\ &\subset \bigsqcup_{z \in W_{\text{aff}}; \ell(x^{*_\theta, n}) - 2N_0 \leqslant \ell(z) \leqslant \ell(x^{*_\theta, n}) + 2N_0} \breve{\mathcal{I}}' z \breve{\mathcal{I}}'. \end{aligned}$$

Therefore

$$\ell(x^{*_\theta, n}) - 2N_0 \leqslant \ell((x')^{\theta, n}) \leqslant \ell(x^{*_\theta, n}) + 2N_0$$

for all $n \in \mathbb{N}$. Since x' is θ-straight, $\ell((x')^{\theta, n}) = n\ell(x')$. Thus

$$\frac{\ell(x^{*_\theta, n})}{n} - \frac{2N_0}{n} \leqslant \ell(x') \leqslant \frac{\ell(x^{*_\theta, n})}{n} + \frac{2N_0}{n}.$$

Therefore $\ell(\mathcal{O}_x) = \ell(x') = \lim_{n\to\infty} \frac{\ell(x^{*_\theta, n})}{n}$. The proof is finished.

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Remarks on the ABG induction theorem*

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Abstract: A key result in a 2004 paper by S. Arkhipov, R. Bezrukavnikov, and V. Ginzburg [ABG] compares the bounded derived category $D^b block(\mathbb{U})$ of modules for the principal block of a Lusztig quantum enveloping algebra $\mathbb{U}$ at an ℓth root of unity with a subcategory $D_{triv}(\mathbb{B})$ of the derived category of integrable Type 1 modules for a Borel part $\mathbb{B} \subset \mathbb{U}$. Specifically, according to this "Induction Theorem" [ABG, Theorem 3.5.5] the right derived functor of induction $\mathrm{Ind}_{\mathbb{B}}^{\mathbb{U}}$ yields an equivalence of categories $\mathrm{RInd}_{\mathbb{B}}^{\mathbb{U}} : D_{triv}(\mathbb{B}) \xrightarrow{\sim} D^b block(\mathbb{U})$ (under appropriate hypotheses on ℓ). The authors of [ABG] suggest a similar result holds for algebraic groups in positive characteristic p, and this paper provides a statement with proof for such a modular induction theorem. Our argument uses the philosophy of [ABG] as well as new ingredients. A secondary goal of this paper has been to put the original characteristic zero quantum result on firmer ground, and we provide arguments as needed to give a complete proof of that result also. Finally, using the modular result, we have been able in [HKS] to introduce truncation functors, associated to finite weight posets, which effectively commute with the modular induction equivalence, assuming $p > 2h - 2$, with h the Coxeter number. This enables interpreting the equivalence at the level of derived categories of modules for suitable finite dimensional quasi-hereditary algebras. We expect similar results to hold in the original quantum setting, assuming $\ell > 2h - 2$.

Keywords: Induction theorem, quantum enveloping algebra, principal block, semi-simple algebraic groups, positive characteristic, bounded derived category, triangulated category.

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1. Introduction

If G is a semisimple algebraic group and B a Borel subgroup, it is well known that the category of rational G-modules fully embeds via the restriction functor into the category of rational B-modules. Explicitly describing the objects in the image of restriction is a difficult problem, unsolved in general. However, as we will see here, it is possible to make progress at the derived category level. Our starting point is a result [ABG, Theorem 3.5.5] by S. Arkhipov, R. Bezrukavnikov, and V. Ginzburg in the world of quantum groups. The result establishes a natural equivalence between the bounded derived category of modules for the principal block of a Lusztig quantum enveloping algebra at a root of unity with an *explicit* subcategory of the bounded derived category of integrable modules for a Borel part of this quantum algebra. We will refer to this result as "the induction theorem." We begin this paper with its explicit statement.

Suppose $\mathbb{U}$ is a Lusztig quantum algebra, associated to a root datum $\mathfrak{R} = (\Pi, \mathbb{X}, \Pi^\vee, \mathbb{X}^\vee)$, and specialized to a characteristic 0 field K with ℓ^{th} root of unity $q \in K$, as defined in Section 2. In particular, we assume q is defined by $q^\ell = 1$ with ℓ odd, and not divisible by 3 if the root system corresponding to $\mathfrak{R}$ has a component of type G_2. We denote the root system in general by R. Moreover, we assume $\ell > h$, where h is the Coxeter number of R, unless otherwise noted. Suppose $\mathbb{B} = \mathbb{U}^- \otimes_K \mathbb{U}^0 \subset \mathbb{U}$ is a 'Borel part' of $\mathbb{U}$ arising from a triangular decomposition of $\mathbb{U}$ as in Section 2. Denote by $D^b block(\mathbb{U})$ the bounded derived category of the abelian category of type 1 integrable modules in the "principal block" of $\mathbb{U}$.[1] For $D^b(\mathbb{B})$ the usual bounded derived category of the module category for $\mathbb{B}$, let $D_{triv}(\mathbb{B})$ be the full triangulated subcategory of $D^b(\mathbb{B})$ whose objects are complexes representable by

$$M = \cdots \to M_{i-1} \to M_i \to M_{i+1} \to \cdots \quad i \in \mathbb{Z}$$

so that for all $i \in \mathbb{Z}$,

(i) M_i is an integrable $\mathbb{B}$-module;
(ii) M_i has a grading $M_i = \oplus_{\nu \in \mathbb{Y}} M_i(\nu)$ by the root lattice $\mathbb{Y}$ of the root datum $\mathfrak{R}$;

[1]More precisely, the principal block of $\mathbb{U}$ is the full subcategory of finite-dimensional integrable Type 1 $\mathbb{U}$-modules whose composition factors are all "linked" to the trivial module. Equivalently, the highest weights of these composition factors all belong to the "dot" orbit of 0 under the affine Weyl group. A precise definition of the "dot" action is given in Section 2.

(iii) for any $m \in M_i(\nu)$ and $u \in \mathbb{U}^0$, $um = \nu(u) \cdot m$;
(iv) the total cohomology module $H^\bullet(M) = \oplus_{i \in \mathbb{Z}} H^i(M)$ has a finite composition series, all of whose successive quotients are of the form $K_{\mathbb{B}}(\ell\lambda)$, $\lambda \in \mathbb{Y}$.

Here $K_{\mathbb{B}}(\ell\lambda)$ denotes a 1-dimensional $\mathbb{B}$-module associated to $\ell\lambda$. Further details on notation may be found in Section 2.

Theorem 1 (*Induction Theorem*, Theorem 3.5.5 [ABG])**.** *For an appropriately defined induction functor $Ind_{\mathbb{B}}^{\mathbb{U}}$, its right derived functor $RInd_{\mathbb{B}}^{\mathbb{U}}$ yields an* ***equivalence*** *of triangulated categories*

$$D_{\mathrm{triv}}(\mathbb{B}) \xrightarrow{\sim} D^b block(\mathbb{U}).$$

A precise definition for $\mathrm{Ind}_{\mathbb{B}}^{\mathbb{U}}$ appears in Section 2. It is an analog of induction (right adjoint to restriction) in the theory of representations of algebraic groups, and has similar properties. To "induce" a module, one applies induction in the sense of associative rings and algebras, then passes to the largest Type 1 integrable submodule.

The present paper contains, as a secondary feature, a complete proof of the above result, along the lines of [ABG], though with some variations and a number of corrections. We are grateful to Pramod Achar for alerting us to possible issues (first observed by his collaborator, Simon Riche) in the proof of [ABG, Lemma 4.1.1(ii)]. The argument we eventually found (the proof of our Lemma 3.2(ii)) is quite substantial, spanning two appendices and improving a theorem of Rickard [R94]. Other corrections we make are more minor, often rooted in inadequacies in the quantum literature. The "variations" mentioned often occur from our desire to give a proof that "carries over" to the characteristic p algebraic groups case. Indeed, the latter has been the central aim of our work here.

The existence of a modular analog of the induction theorem was suggested by the following assertion "An analogue of Theorem 3.5.5 holds also for the principal block of complex representations of the algebraic group $G(F)$ over an algebraically closed field of characteristic $p > 0$. Our proof of the theorem applies to the latter case as well" [ABG, p. 616]. Replacing the term "complex representations" in the above quote with "rational representations" (likely intended) yields the Theorem 2, which this paper confirms is indeed a theorem. However, while the philosophy and some ingredients of the proof we present may be found in the [ABG] treatment of the quantum case, additional critical ingredients are also required. See, for example, Corollary 2.15(1) and Lemma 3.6.

To set the notation, $block(G)$ is the principal block of finite-dimensional rational G-modules, and $D_{triv}(B)$ is defined analogously to $D_{triv}(\mathbb{B})$ in the quantum case. That is, rational B-modules replace (Type 1) integrable $\mathbb{B}$ modules, the distribution algebra of a maximal split torus $T \subseteq B$ is used for $\mathbb{U}^0$, and $k_B(p\lambda)$, with k as below, replaces $K_{\mathbb{B}}(\ell\lambda)$ above (in the definition of $D_{\mathrm{triv}}(\mathbb{B})$, which becomes $D_{\mathrm{triv}}(B)$). As before h denotes the Coxeter number of the underlying root system, now regarded as associated to G.

Theorem 2. *Let G be a semisimple algebraic group over an algebraically closed field k of positive characteristic $p > h$. Let B be a Borel subgroup of G. Then the functor $RInd_B^G$ yields an* ***equivalence*** *of triangulated categories*

$$D_{\mathrm{triv}}(B) \to D^b block(G).$$

Generally, we use the "quantum case" to refer to the context of Theorem 1, and the "algebraic groups case" (or "positive characteristic case," or "modular case") when referring to the context of Theorem 2. Of course, some discussions in a given "case" do not require the full hypotheses of these theorems. (We sometimes keep track of such situations.)

Theorem 2 is a starting point for yet another result, proved in [HKS]. It shows, for $p > 2h-2$, that RInd_B^G in Theorem 2 induces an equivalence between certain natural full triangulated subcategories

$$D_{triv}(\mathrm{Dist}(B)_{\Lambda_m}) \to D^b(block(G)_{\Gamma_m}),$$

depending on p and indexed by an integer $m > 0$. Here Λ_m is a finite subposet in a variation of van der Kallen's "excellent order" on weights [vdK1], and Γ_m is a finite subposet of dominant weights in the usual dominance order. The arguments $\mathrm{Dist}(B)_{\Lambda_m}$ and $block(G)_{\Gamma_m}$ of the constructions in the display refer to finite-dimensional quasi-hereditary algebra quotients of the distribution algebra $\mathrm{Dist}(B)$ and $\mathrm{Dist}(G)$, respectively, the latter associated to G–a mild abuse of notation. For further details, see [HKS]. Collectively, these more "finite" equivalences can be used to reconstruct the full equivalence given by RInd_B^G in Theorem 2, thereby deepening our understanding of it.

This paper is organized as follows. Section 2 collects notation and some needed background material. Section 3 proves Theorems 1 and 2. The statements above these theorems contain the start of a dictionary for going back and forth between the characteristic 0 quantum root of unity case and the positive characteristic algebraic group case. Indeed, there is nothing to stop us from using the same names as in Theorem 1 for parallel objects in Theorem 2, putting $\mathbb{U} = \mathrm{Dist}(G)$, $\mathbb{B} = \mathrm{Dist}(B)$, $block(\mathbb{U}) = block(G)$, $D_{triv}(\mathbb{B}) = D_{triv}(B)$, and even writing $\mathrm{RInd}_{\mathbb{B}}^{\mathbb{U}}$ for RInd_B^G. We can then restate

Theorem 2.1 (Cosmetic variation on Theorem 2)**.** *Let $\mathbb{U}$ be the distribution algebra of a semisimple algebraic group over an algebraically closed field of prime characteristic $\ell = p > h$. Let $\mathbb{B}$ be the distribution algebra of a Borel subgroup. Then the induction functor* $\mathrm{RInd}_{\mathbb{B}}^{\mathbb{U}}$ *induces an equivalence*

$$D_{triv}(\mathbb{B}) \to D^b \ block(\mathbb{U}).$$

In Section 3 we give a simultaneous proof of both Theorem 1 and Theorem 2.1.

Some of the rationale for the overall approach is discussed in Subsection 3.5. Sections 4, 5, and 6 present three appendices, labeled A, B, C, respectively. The first two are used to prove Lemma 3.2(ii), which restates [ABG, Lem. 4.1.1(ii)], asserting that it holds in both the quantum and modular algebraic groups cases. The modular case, at least, is of independent interest of categorifying a theorem [R94, Thm. 2.1] of Rickard in the regular weight case, and the quantum case of the lemma may be viewed as giving an analogous quantum result. Also, Appendix C, independent of the rest of this paper, corrects the statement and proof of [ABG, Lem. 9.10.5] as a service to the reader. Finally, a few acknowledgements and thanks are collected in the final section.

Theorem 2 was first announced in [HKS], though the proof underwent several corrections later, specifically when a proof of Lemma 3.2(ii) was written down. This was done in the modular case, and completed our proof of the modular induction theorem. The proof actually also works in the quantum case, thus proving [ABG, Lem. 4.1.1(ii)], though we found it necessary to work through some foundational issues regarding quantum induction (Remarks 2.11(d), (e)). We also found it necessary to fill in other details in the quantum literature to complete our simultaneous treatment of the quantum and modular induction theorems. Another proof of the modular induction theorem, as part of a larger geometric program, was subsequently posted by Achar and Riche [AR]. With this evidence of the value in having the modular induction theorem from multiple perspectives, we note that the differing approach and results in this current paper predate and or differ from [AR], and remain necessary for the support and results in [HKS], as well as include elements of independent interest.

2. Background

Generally, we follow Lusztig [L5], [L3] for basic material on quantum enveloping algebras, and Andersen's paper [A] for many additional results on their representation theory, especially results on induced representations that parallel those found in Jantzen [J] in the case of semisimple algebraic groups. These results on induced representations have their origin in an earlier paper of Andersen-Polo-Wen [APW], as supplemented by [AW]. For the study of (characteristic zero) quantum groups at ℓ^{th} roots of unity with ℓ a prime power, the [APW] paper is generally sufficient, while the context of [AW] allows all values ℓ (orders of roots of unity) used in (the main results of) this paper. (It does restrict ℓ to be odd, and not divisible by 3 in case the root system has a component of type G2.) The context of [A] is even more general, though it references an argument from [AW], and there are a number of references of convenience (which could be avoided) to arguments in [APW].

We are interested in the semisimple algebraic groups case as much or more so than in the quantum case, but focus now on giving notation below as befits the quantum case, where there is much less uniformity in the literature than in the algebraic groups case. All the notation and results have analogs in [J], however. In later parts of this paper, excluding the appendices, we will try to treat both the algebraic groups and quantum cases simultaneously and with the same notation. Some of our quantum group notation has been chosen to maintain consistency with these later discussions.[2] Some important background on induction and cohomology is given in Subsection 2.5, modifying and completing a number of references given by [ABG] to the literature on quantum group representations. Many of the results we discuss in that subsection are well-known in the algebraic groups case, and we generally do not track their analogs there in detail. Starting with Subsection 2.6 and continuing in the rest of Section 2 and all of Section 3, we use the "uniform" notation for both the quantum and positive characteristic cases, though some differentiation of the two cases is sometimes required for proofs. Appendices A and B, used to prove Lemma 3.2, are given in algebraic groups notation, with the quantum case treated in remarks. A concise quantum group reference written in the spirit of comparing general results in the quantum and algebraic groups cases may be found [J, Appendix H] in summary form.

[2]In some cases, our notation differs from [ABG]. In particular we use a "simply connected" set-up, which allows modules with weights in $\mathbb{X}$ (defined below), rather than the "adjoint" set-up of [ABG], which restricts attention to modules with weights in $\mathbb{Y}$ below (the root lattice). This does not affect the triangulated categories (up to natural equivalences) entering into the statements of Theorems 1, 2 and 2.1. Also, we always induce from Borel subgroups associated to negative roots, and correspondingly use a different (but more standard) affine Weyl group "dot" action, defined later in Subsection 2.4.2.

2.1. Quantum enveloping algebras and algebraic groups

2.1.1. Root datum Assume $\mathfrak{g}_{\mathbb{C}}$ is a complex semisimple Lie algebra of rank n, with Cartan matrix $C = (c_{ij})_{1\leq i,j\leq n}$, and Killing form $\kappa : \mathfrak{g}_{\mathbb{C}} \times \mathfrak{g}_{\mathbb{C}} \to \mathbb{C}$. Then from a choice of Cartan subalgebra $\mathfrak{h}_{\mathbb{C}} \subset \mathfrak{g}_{\mathbb{C}}$ one obtains a root-datum realization $\mathfrak{R} = (\Pi, \mathbb{X}, \Pi^\vee, \mathbb{X}^\vee)$ of C from the following data.

- $R \subset \mathfrak{h}_{\mathbb{C}}^*$ denotes the set of roots arising from the Cartan decomposition $\mathfrak{g}_{\mathbb{C}} = \mathfrak{h}_{\mathbb{C}} \oplus \bigoplus_{\alpha\in R} \mathfrak{g}_{\mathbb{C},\alpha}$ into $\mathfrak{h}_{\mathbb{C}}^*$-weight spaces under the restriction of the adjoint action of $\mathfrak{g}_{\mathbb{C}}$ to $\mathfrak{h}_{\mathbb{C}}$, with corresponding elements $t_\alpha \in \mathfrak{h}_{\mathbb{C}}$, $t_\alpha \leftrightarrow \alpha \in R$ arising from the identification of $\mathfrak{h}_{\mathbb{C}}$ with $\mathfrak{h}_{\mathbb{C}}^*$ obtained from the nondegeneracy of the Killing form by setting, for any $\phi \in \mathfrak{h}_{\mathbb{C}}^*$, $t_\phi \in \mathfrak{h}_{\mathbb{C}}$ to be the unique element such that $\phi(h) = \kappa(t_\phi, h)$ for all $h \in \mathfrak{h}_{\mathbb{C}}$.
- Take as the set of coroots $R^\vee := \{\alpha^\vee := \frac{2\alpha}{(\alpha,\alpha)} \,|\, \alpha \in R\}$. For $h_\alpha := \frac{2t_\alpha}{\kappa(t_\alpha,t_\alpha)}$, there is a correspondence $h_\alpha \leftrightarrow \alpha^\vee$ under the identification of $\mathfrak{h}_{\mathbb{C}}$ with $\mathfrak{h}_{\mathbb{C}}^*$.
- Take $E = E_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{R}$, for $E_{\mathbb{Q}}$ the $\mathbb{Q}$-span of the roots R in $\mathfrak{h}_{\mathbb{C}}^*$. The rational space $E_{\mathbb{Q}}$ has a nondegenerate bilinear form obtained from restriction of $(\lambda, \mu) = (t_\lambda, t_\mu)$ on $\mathfrak{h}_{\mathbb{C}}^*$; this extends uniquely to a positive definite form $(-,-)$ on E, making E into an n-dimensional Euclidean space. Both R and $R^\vee$ are root systems in E; in particular they both span E. Observe that, for $< \zeta, \eta >:= \frac{2(\zeta,\eta)}{(\eta,\eta)}\ \forall \zeta, \eta \in E$, one has $< \beta, \alpha >= (\beta, \alpha^\vee)$ for $\alpha, \beta \in R$.
- The Weyl group W is the subgroup of $GL(E)$ generated by reflections $s_\alpha(v) = v - (v, \alpha^\vee)\alpha, \alpha \in R$.
- $\Pi = \{\alpha_1, \ldots, \alpha_n\}$ to be a set of simple roots (i.e., basis for E such that any $\alpha \in R$ satisfies $\alpha = \sum m_i\alpha_i \in \oplus_{1\leq i\leq n}\mathbb{Z}\alpha_i$ with all $m_i \geq 0$ or all $m_i \leq 0$). Then W is generated by the $s_i := s_{\alpha_i}, 1 \leq i \leq n$.
- $\Pi^\vee := \{\alpha_i^\vee \,|\, 1 \leq i \leq n\}$ to be the corresponding set of simple coroots
- $\mathbb{X}^\vee = \oplus_{i=1}^n \mathbb{Z}\alpha_i^\vee$ coweight lattice
- $\mathbb{X} = \{\lambda \in E \,|\, < \lambda, \alpha >= (\lambda, \alpha^\vee) \in \mathbb{Z} \quad \forall \alpha \in R\} = \{\lambda \in E \,|\, (\lambda, \alpha_i^\vee) \in \mathbb{Z} \quad \forall \alpha_i \in \Pi\} \cong \mathrm{Hom}_{\mathbb{Z}}(\mathbb{X}^\vee, \mathbb{Z})$ weight lattice
- $\mathbb{Y}$ is the subgroup of $\mathbb{X}$ generated by R; $\mathbb{Y} = \bigoplus_{i=1}^n \mathbb{Z}\alpha_i$ root lattice
- $\mathbb{X}^+ := \{\lambda \in \mathbb{X} \,|\, (\lambda, \alpha_i^\vee) \geq 0 \quad \forall 1 \leq i \leq n\}$ dominant weights; for $\varpi_1, \ldots, \varpi_n$ the dual basis defined by $(\varpi_i, \alpha_j^\vee) = \delta_{i,j}$, we get $\mathbb{X} = \bigoplus_{i=1}^n \mathbb{Z}\varpi_i$.
- In our notation the Cartan matrix $C = (c_{i,j})$ is given by $c_{i,j} = (\alpha_j, \alpha_i^\vee)$, $1 \leq i, j \leq n$.

Furthermore, for any $\alpha \in R$, set $d_\alpha = \frac{(\alpha,\alpha)}{2}$, so then $d_\alpha \in \{1,2,3\}$ and $(\lambda,\alpha) = d_\alpha(\lambda,\alpha^\vee) = d_\alpha < \lambda,\alpha > \in \mathbb{Z}$. Writing d_i for d_{α_i}, and $D = diag(d_1,\dots,d_n)$, one has that $DC = (d_i c_{i,j})$ is symmetric.

2.1.2. Quantum enveloping algebra Our description of quantum enveloping algebras here follows Lusztig [L3], with similar notation, especially for generators. There are some differences in the notational names of algebras and subalgebras, and the labeling of relations. We make no distinction in the terms "quantum enveloping algebra," "quantum algebra," and "quantum group."

Take v to be an indeterminate, and consider the following expressions in the ring $\mathbb{Q}(v)$:

(2.0.1)

$$[n]_d := \frac{v^{nd}-v^{-nd}}{v^d-v^{-d}},\ \text{for } d,n \in \mathbb{N};\ \text{when } d=1,\ \text{have } [n] := [n]_1 = \frac{v^n-v^{-n}}{v-v^{-1}}$$

$$[n]_d! := \Pi_{s=1}^n \frac{v^{d\cdot s}-v^{-d\cdot s}}{v^d-v^{-d}} = \Pi_{s=1}^n [s]_d,\ \text{for } n,d \in \mathbb{N}$$

(2.0.2)

$$\left[\begin{smallmatrix} n \\ t \end{smallmatrix}\right]_d := \Pi_{s=1}^t \frac{v^{d(n-s+1)}-v^{-d(n-s+1)}}{v^{ds}-v^{-ds}}\ \text{for } n \in \mathbb{Z}, d,t \in \mathbb{N};$$

$$\text{when } d=1,\ \text{we set } \left[\begin{smallmatrix} n \\ t \end{smallmatrix}\right] := \left[\begin{smallmatrix} n \\ t \end{smallmatrix}\right]_1.$$

The simply connected quantum enveloping algebra[3] $\mathbb{U}'_v = \mathbb{U}'_v(\mathfrak{R})$ is the $\mathbb{Q}(v)$-algebra generated by the symbols $E_i, F_i, K_i^{\pm 1}, 1 \le i \le n$, subject to the five sets of relations below, as found in [L3, p.90]. We take this opportunity to warn the reader that we will sometimes also need to refer to Lusztig's book [L], where the symbols K_i here (and in [L3]) correspond to symbols $\tilde{K}_i$ there.[4]

(a1) $K_iK_j = K_jK_i,\ K_iK_i^{-1} = 1 = K_i^{-1}K_i,$ for $1 \le i,j \le n, i \ne j$;
(a2) $K_iE_j = v^{d_ic_{i,j}}E_jK_i$, and $K_iF_j = v^{-d_ic_{i,j}}F_jK_i$, for $1 \le i,j \le n$;
(a3) $E_iF_j - F_jE_i = \delta_{i,j}\frac{K_i-K_i^{-1}}{v^{d_i}-v^{-d_i}}$, for $1 \le i \le n$.
(a4) $\sum_{s+t=1-c_{i,j}} (-1)^s \left[\begin{smallmatrix} 1-c_{i,j} \\ s \end{smallmatrix}\right] E_i^sE_jE_i^t = 0$, for $1 \le i,j \le n$, $i \ne j$;
(a5) $\sum_{s+t=1-c_{i,j}} (-1)^s \left[\begin{smallmatrix} 1-c_{i,j} \\ s \end{smallmatrix}\right] F_i^sF_jF_i^t = 0$, for $1 \le i,j \le n$, $i \ne j$;

[3]Over $\mathbb{Q}(v)$ Lustzig [L3, p.90] and in [L5] credits this form to Drinfeld and Jimbo. Lusztig himself considers in these papers more general rings as coefficients, especially $\mathbb{Z}[v,v^{-1}]$. The $'$ notation, convenient for us here (freeing the unprimed $\mathbb{U}$ for other uses) is not used in [L3]. Our usage of it is similar to that of [L5], suggesting the use of a quotient field, such as $\mathbb{Q}(v)$, in the coefficient system.

[4]In [L] larger quantum algebras are built, which we do not need. There are (new) elements K_i in these larger algebras, which serve as d_i^{th} roots for the elements $\tilde{K}_i$.

Note that it is also common to let $v_i = v^{d_i}$, so e.g., the first part of (a2) can be rewritten as $K_iE_jK_i^{-1} = v^{(\alpha_i,\alpha_j)}E_j$ for $1 \le i, j \le n$, and similarly for the second part of (a2).

The algebra $\mathbb{U}_v^{'}$ is also a Hopf algebra, with comultiplication Δ, antipode S, and counit ϵ given as below [*ibid*]. These formulas hold for all indices i with $1 \le i \le n$.

(b) $\Delta E_i = E_i \otimes 1 \oplus K_i \otimes E_i, \;\; \Delta F_i = F_i \otimes K_i^{-1} \oplus 1 \otimes F_i,$
$\Delta K_i = K_i \otimes K_i,$

(c1) $SE_i = -K_i^{-1}E_i, \;\; SF_i = -F_iK_i, \;\; SK_i = K_i^{-1},$

(c2) $\epsilon E_i = \epsilon F_i = 0, \;\; \epsilon K_i = 1.$

We next give a brief discussion of the Lusztig integral form of this Hopf algebra [L3].

Starting with $E_i, F_i, K_i \in \mathbb{U}_v$, $1 \le i \le n$, and $s, t \in \mathbb{N}$, $c \in \mathbb{Z}$, define the divided powers $E_i^{(s)}, F_i^{(s)}, \left[{K_i;c \atop t} \right]$ by

$$\begin{aligned} E_i^{(s)} &:= \frac{E_i^s}{[s]_{d_i}!}, \\ F_i^{(s)} &:= \frac{F_i^s}{[s]_{d_i}!}, \\ \left[{K_i;c \atop t} \right] &:= \Pi_{j=1}^t \frac{K_i v^{d_i(c-j+1)} - K_i^{-1} v^{d_i(-c+j-1)}}{v^{d_ij} - v^{-d_ij}}. \end{aligned} \tag{2.0.3}$$

Each term on the left above with $s = 0$ or $t = 0$ is defined to be 1.

Set $\mathcal{Z} = \mathbb{Z}[v, v^{-1}]$. Keeping the hypotheses that $\mathfrak{R}$ is the root datum for a semisimple complex Lie algebra, with $\Pi^\vee = \{\alpha_1^\vee, \dots, \alpha_n^\vee\}$, then the Lusztig integral form[5] $\mathbb{U}_\mathcal{Z} = \mathbb{U}_\mathcal{Z}(\mathfrak{R})$ of $\mathbb{U}_v^{'}$ is the $\mathcal{Z}$-subalgebra of $\mathbb{U}_v^{'}$ generated by $E_i^{(s)}, F_i^{(s)}, K_i^{\pm 1}$and $\left[{K_i;c \atop t} \right]$, for all$i$ with $1 \le i \le n, s, t \in \mathbb{N}, c \in \mathbb{Z}$; equivalently (it turns out) $\mathbb{U}_\mathcal{Z}$ is generated by all $E_i^{(s)}, F_i^{(s)}, K_i^{\pm 1}$ and[6] $\left[{K_i;0 \atop t} \right]$, $1 \le i \le n$, $s, t \in \mathbb{N}$. Corresponding to either set of generators, both $\mathbb{U}_v'$ and $\mathbb{U}_\mathcal{Z}$ have (compatibly generated) triangular decompositions $\mathbb{U}_v' = \mathbb{U}_v'^{-} \otimes \mathbb{U}_v'^{0} \otimes \mathbb{U}_v'^{+}$, resp., $\mathbb{U}_\mathcal{Z} = \mathbb{U}_\mathcal{Z}^- \otimes \mathbb{U}_\mathcal{Z}^0 \otimes \mathbb{U}_\mathcal{Z}^+$. Either set of generators, with

[5] This is the form employed in [A] (and also [J, Appendix H]). In [ABG, 2.4], a version of Lusztig's integral form is given very loosely, but apparently intended to be defined by an "adjoint type" version of the relations we use here. The latter relations, however, appear to be consistent with the alternate "simply connected" development suggested [ABG, Remark 2.6], a point of view we have used throughout this paper.

[6] These additional expressions in the K_i are redundant–see the brief discussion [A, p.3, bottom]–but are needed for the integral triangular decomposition.

the relations (a1),..., (a5), define $\mathbb{U}'_v$ over $\mathbb{Q}(v)$. These relations are often sufficient to work with $\mathbb{U}_{\mathcal{Z}}$. However, there are many additional useful relations [L3, §6] on the elements $\left[\begin{smallmatrix} K_i;0 \\ t \end{smallmatrix}\right]$ and their interactions with the "divided powers" $E_i^{(s)}, F_i^{(s)}$. Also, there are analogs of the latter elements for all positive roots. All of these elements belong to $\mathbb{U}_{\mathcal{Z}}$, and may be used to define the latter by generators and relations in its own right, and to construct for it a monomial basis [L3].

Finally, the $\mathcal{Z}$-algebra $\mathbb{U}_{\mathcal{Z}}$ is a Hopf algebra, inheriting its Hopf algebra structure from $\mathbb{U}'_v$ [L3, 8.11]. (All the Hopf algebras mentioned in this paragraph have bijective antipodes, with clearly invertible squares.) Similar statements apply for $\mathbb{U}^0_{\mathcal{Z}}$, $\mathbb{U}^-{}_{\mathcal{Z}}\otimes\mathbb{U}^0_{\mathcal{Z}}$, and $\mathbb{U}^0_{\mathcal{Z}}\otimes\mathbb{U}^+_{\mathcal{Z}}$ [*ibid*]. Also, the root of unity specializations discussed in the next section inherit Hopf algebra structures from $\mathbb{U}_{\mathcal{Z}}$, as do the "small" quantum groups $\mathbb{u}$, $\mathbb{u}^-, \mathbb{u}^0, \mathbb{u}^+$[*ibid*]. The ("Frobenius") homomorphism discussed later in Section 2.7 is a homomorphism of Hopf algebras.

2.2. Quantum specializations at roots of unity–notation

For any commutative ring K and invertible element $q \in K$, with unique accompanying 'evaluation morphism' $\epsilon_q : \mathcal{Z} \to K$ satisfying $v \mapsto q$, define the **specialization** of $\mathbb{U}_{\mathcal{Z}} = \mathbb{U}_{\mathcal{Z}}(\mathfrak{R})$ by

$$\mathbb{U}_{q,K} = \mathbb{U}_{\mathcal{Z}} \otimes_{\mathcal{Z}} K, \tag{2.0.4}$$

where the tensor product is formed by using the $\mathcal{Z}$-module structure on K given by ϵ_q. For this paper we will be interested in specializations where

- K is a field of characteristic zero
- q is a primitive ℓ^{th}-root of unity in K with ℓ odd, and $\ell \neq 3$ if the root system of $\mathfrak{g}$ has a component of type G_2.

We henceforth fix this meaning for K, ℓ, q, unless otherwise noted. *We also take* $\ell > h$, the Coxeter number[7], from Corollary 2.10 forward. We also now introduce further notation that will be used in this quantum root of unity setting, and also used in a parallel setting from algebraic groups, discussed below. Relatively abbreviated notations are chosen to facilitate later parallel

[7]All these restrictions agree in substance with those in [ABG], though h there denotes the dual Coxeter number. Also, the literature differs as to whether q is chosen to be the image of v or the image of v^2, the latter fitting somewhat better with Hecke algebra notation. This makes little difference when the order of the image of v is odd, as is the case here.

discussions. In the present *quantum context*, we let $\mathbb{U}$ denote the specialization $\mathbb{U}_{q,K}$ as in (2.0.4). Similar conventions are adapted for $\mathbb{U}^+, \mathbb{U}^0, \mathbb{U}^-$ and $\mathbb{B} = \mathbb{U}^0 \cdot \mathbb{U}^-$. Lusztig's finite dimensional Hopf algebra [L3, §8.2] (the "small" quantum group) is denoted $\mathbb{u}$, with components of its triangular decomposition denoted $\mathbb{u}^-, \mathbb{u}^0, \mathbb{u}^+$. For example, $\mathbb{u}^0$ is generated by all $K_i^{\pm}$, and $\mathbb{u}^-$ is generated by all the F_i [L3, pp. 107-108]. Imitating the notation in [ABG] we set $\mathbb{b} := \mathbb{u}^- \cdot \mathbb{u}^0$ and $\mathbb{p} := \mathbb{b} \cdot \mathbb{U}^0$.

Overall, our notation here is quite similar to that used for quantum groups at a root of unity in [ABG], with the exception that our characteristic 0 field K (which may be compared with $\mathbb{k}$ in [ABG]) is not assumed to be algebraically closed. Also, our $\mathbb{B}$ is $\mathbb{U}^0 \cdot \mathbb{U}^-$, whereas in [ABG] the same symbol $\mathbb{B}$ is used to denote $\mathbb{U}^0 \cdot \mathbb{U}^+$.

2.3. Some parallel algebraic groups notation

Let G be a simply connected semisimple algebraic group, with root datum $\mathfrak{R}$, over an algebraically closed field k of characteristic $p > h$. We assume G is defined and split over the prime field $\mathbb{F}_p$. In particular there is a Borel subgroup $B = TU$, with U the unipotent radical of B, and T a maximal torus, all defined over $\mathbb{F}_p$, with T isomorphic (over the same field) to a direct product of copies of $k^\times$. The root groups in B are viewed as negative. We refer to this set-up as the *algebraic groups context* or, even more loosely, as the *algebraic groups case*. The distribution algebras $\mathrm{Dist}(G)$, $\mathrm{Dist}(B)$, $\mathrm{Dist}(T)$, $\mathrm{Dist}(U)$, and $\mathrm{Dist}(G_1)$ (the restricted enveloping algebra) parallel $\mathbb{U}$, $\mathbb{B}$, $\mathbb{U}^0$, $\mathbb{U}^-$, and $\mathbb{u}$, respectively. We will use the latter symbol set in place of the former, when the context is clear, or if both the algebraic groups and quantum contexts have been explicitly allowed. In either of these circumstances, additional notational substitutions in the same spirit may also be made, such as $\mathbb{p}$ for $\mathrm{Dist}(B_1T)$.

2.4. Affine Weyl groups

Affine Weyl groups W_ℓ are used to index modules in both the quantum and algebraic groups context, with p used for ℓ in the latter. Our main references for affine Weyl groups are [J] and [A]. To clarify discussions and differences in these references, we temporarily allow ℓ to be any positive integer.

2.4.1. Affine Weyl groups, as in [J] Following e.g., the conventions and notation in [J, §6.1], for $\beta \in R$ and $m \in \mathbb{Z}$, define the affine reflection on $\mathbb{X}$ by

$$s_{\beta,m}(\lambda) = \lambda - (\lambda, \beta^\vee) - m)\beta, \quad \forall \lambda \in \mathbb{X};$$

one could take $\mathbb{X} \otimes_{\mathbb{Z}} \mathbb{R}$ in place of $\mathbb{X}$. Thus, for the reflections s_β given by $s_\beta(\lambda) = \lambda - (\lambda, \beta^\vee)\beta$ one has

$$s_{\beta,m}(\lambda) = s_\beta(\lambda) + m\beta \quad \forall\lambda.$$

For any positive integer ℓ, the affine Weyl group W_ℓ is the group

$$W_\ell := < s_{\beta,n\ell} \,|\, \beta \in R, n \in \mathbb{Z} > .$$

In the notation used in [J, §6.1], there is a (largely formal) isomorphism $W_\ell \cong W_a(R^\vee)$, for $W_a(R^\vee) := < s_{\beta,m} \,|\, \beta \in R, m \in \mathbb{Z} >$, as defined by Bourbaki [B, ch. VI, §2]. When $\ell = 1$ this isomorphism is an equality. The Bourbaki reference makes a good case for the labelling with $R^\vee$, though it is common in algebraic group theory to associate both W_ℓ and $W_a(R^\vee)$ with the root system R. A familiar semidirect product description is obtained by regarding $\ell\mathbb{Z}R$ as a group of translations on $\mathbb{X} \otimes_{\mathbb{Z}} \mathbb{R}$, namely, $W_\ell \cong \ell\mathbb{Z}R \rtimes W = \ell\mathbb{Y} \rtimes W$ ([J] references [B, ch. VI§2 prop. 1] for a proof). Here W is the usual Weyl group associated to R.

2.4.2. Dot action Set $\rho = \frac{1}{2}\sum_{\alpha \in R^+} \alpha$. For $w \in W$, $\lambda \in \mathbb{X}$, one sets,

$$w \cdot \lambda = w(\lambda + \rho) - \rho.$$

More generally, the affine Weyl group $W_\ell \cong \ell\mathbb{Y} \rtimes W$ acts, for $w_a := \ell\tau \rtimes w \in \ell\mathbb{Y} \rtimes W$ by

$$w_a \cdot \lambda := w(\lambda + \rho) - \rho + \ell\tau, \quad \text{for all } \lambda \in \mathbb{X}.$$

Our "dot" action $\cdot$, which is often used, is not given by the same formula as the action $\bullet$ defined in [ABG], possibly intending some variation on [ABG, Lem. 3.5.1] (which is incorrect as stated for "positive" Borel subalgebras). A simpler approach, it seems to us, is to use "negative" Borel subalgebras and the usual "dot" action.

2.4.3. Affine Weyl groups as in [A] Take $q \in K$ to be a root of unity. (In [A] the field K can have any characteristic, though that is not relevant to our discussions here, and we may keep our assumption that K has characteristic 0.) Set ℓ to be the order of q^2, so that q is a primitive ℓ^{th} or $2\ell^{th}$ root of unity. For the Cartan matrix C of $\mathfrak{R}$ and symmetrization DC (as at the end of our Section 2.1.1), set $\ell_i = \frac{\ell}{gcd(\ell,d_i)}$. For each $\beta \in R$, there is some $i, 1 \leq i \leq n$ so that β is conjugate under the Weyl group W of R to α_i. Set $\ell_\beta = \ell_i$ (well-defined). For each $\beta \in R$ and $m \in \mathbb{Z}$, as in Section 2.4.1, we have the affine reflection $s_{\beta,m\ell_\beta}$ with

$$s_{\beta,m\ell_\beta} \cdot \lambda = s_\beta \cdot \lambda + m\ell_\beta\beta \quad \forall\lambda \in \mathbb{X}.$$

Here, s_β and $s_\beta \cdot \lambda$ are defined just as in Sections 2.4.1 and 2.4.2. Now define a new group of affine reflections

$$W_{D,\ell} :=< s_{\beta,m\ell_\beta} \,|\, \beta \in R, m \in \mathbb{Z} > .$$

Take $W_\ell^\vee$ to be the group

$$W_\ell^\vee :=< s_{\beta^\vee,n\ell} \,|\, \beta \in R^+, n \in \mathbb{Z} >,$$

generated by reflections as in Section 2.4.1, but utilizing coroots in place of roots. The following proposition relates the three groups $W_\ell, W_{D,\ell}$, and $W_\ell^\vee$.

Proposition 2.1. *Assume R is indecomposable. There are identifications giving inclusions*

$$W_\ell \subseteq W_{D,\ell} \subseteq W_\ell^\vee$$

so that

(i) If $gcd(d_i, \ell) = 1$ for all $1 \le i \le n$, then $W_\ell = W_{D,\ell}$.
(ii) On the other hand, if $gcd(d_i, \ell) \neq 1$ for some $1 \le i \le n$, then $W_{D,\ell} = W_\ell^\vee$.

Proof. Without loss, some $d_i \neq 1$. Since R is assumed to be indecomposable, all $d_i \neq 1$ take the same value $d \in \{1, 2, 3\}$. If d does not divide ℓ, $\ell_i =$ for all indices i, and it follows that $\ell = \ell_\beta$ for all $\beta \in R$. Consequently, $W_{D,\ell} = W_\ell$. On the other hand, if d does divide ℓ, then $d\beta^\vee = \beta$ and $d\ell_\beta = \ell$ for all long $\beta \in R$, and $\beta^\vee = \beta$ for all short roots β. It follows that $W_{D,\ell} = W_\ell^\vee$ in this case. This proves the proposition. ☐

Remark 2.2. We have included Proposition 2.1 in part to address possible confusion that a reader casually comparing [J] and [A] may encounter. [A, p.6] says "Note that if ℓ is prime to all entries of the Cartan matrix, then the group $W_{D,\ell}$ (denoted W_ℓ in [A]!) is the 'usual' affine Weyl group of R. However, in general $W_{D,\ell}$ is the affine Weyl group of the dual root system". As we have pointed out above, the " 'usual' affine Weyl group" in algebraic groups discussions is W_ℓ as defined in [J] and Section 2.4.1 above, and that"the affine Weyl group on the dual root system" referred to by [A] is $W_\ell^\vee \cong W_a(R)$, rather than $W_a(R^\vee)$. The proposition and our previous discussion perhaps make precise what Andersen intended. In any case, in this paper, under the assumption below (2.0.4) that ℓ be odd, and not divisible by 3 in case the root system has a component of type G_2, it is clear from the proposition that $W_\ell = W_{D,\ell}$.

2.5. Induction and cohomology

We continue the notation of Section 2.4.3 above, appropriate for Andersen's paper [A]. This is somewhat more general than our standard assumptions stated below (2.0.4). Those more special assumptions are all that we need for this paper, and are explicitly used as hypotheses in [AW]. The latter paper, along with some arguments of [APW] could possibly be used as an alternate source for some of the results of this subsection, with the standard assumptions below (2.0.4) as hypotheses. Unfortunately, it is not possible to quote [APW] directly, since its standing assumptions effectively require ℓ to be a prime power. (See [APW, Lem. 6.6], [AW, p.35].) On the other hand, [A] contains explicit statements (with weaker hypotheses) of most of the results we need, with the exceptions tractable with modest effort.

Accordingly, we follow [A] using the notation for K, q, ℓ in the previous subsection. In addition, we use the notation $\mathbb{U}_{q,K}$ as in (2.0.4), though with more general assumptions than those below (2.0.4). We will define induction functors $\mathrm{Ind}\,_{\mathbb{B}_{q,K}}^{\mathbb{U}_{q,K}}$ from the category of integrable $\mathbb{B}_{q,K}$-modules of Type 1 to the category of integrable $\mathbb{U}_{q,K}$-modules of Type 1 (both categories defined below). For the moment, we will not use our preferred $\mathbb{U}, \mathbb{B}, \ldots$ notation, to help remind the reader of our slightly different context, with weaker hypotheses. Of course, we will obtain from this construction the induction functors $\mathrm{Ind}\,_{\mathbb{B}}^{\mathbb{U}}$ whose right derived functors are the focus of this paper.

We begin as in [A, §1]. First, we coordinate the notation $\mathbb{X}$ in our Section 2.1.1 with the "weights" $\mathbb{Z}^n$, given in [A, p.3]. The correspondence is simply to let $\lambda \in \mathbb{X}$ correspond to the n-tuple with i^{th} coordinate $\lambda_i =< \lambda, \alpha_i^\vee >$. Then, as in *loc. cit.*, λ defines a 1-dimensional representation of (what we call here) $\mathbb{U}_{q,K}^0$ via the homomorphism $\chi_\lambda : \mathbb{U}_{q,K}^0 \to K$ sending $K_i^\pm$ to $q^{\pm d_i \lambda_i}$ and $\left[{K_i;c \atop t} \right]$ to $\left[{\lambda_i + c \atop t} \right]_{d_i}$. Here $1 \le i \le n,\quad c \in \mathbb{Z}$, and $t \in \mathbb{N}$. For any $\mathbb{U}_{q,K}^0$-module M, let M_λ denote the sum of all 1-dimensional submodules on which $\mathbb{U}_{q,K}^0$ acts via the homomorphism χ_λ. We will call M_λ the "weight space" for M associated to λ. If M is the sum (necessarily direct) of its weight spaces M_λ, $\lambda \in \mathbb{X}$, we say that M is integrable of Type 1 as a $\mathbb{U}_{q,K}^0$-module. If we start with M a $\mathbb{B}$-module (resp., a $\mathbb{U}$-module), we say that M is integrable of Type

1 as a $\mathbb{B}$-module (resp., as a $\mathbb{U}$-module) if it is integrable of Type 1 as a $\mathbb{U}^0_{q,K}$-module, and each vector $v \in M$ is, for each index i, killed by all $F_i^{(s)}$ for s sufficiently large (resp., killed by $F_i^{(s)}$ and $E_i^{(s)}$ for s sufficiently large).

Next, suppose that V is any $\mathbb{U}_{q,K}$-module. Define

(2.2.1)

$$\mathcal{F}(V) := \{v \in \oplus_{\lambda\in\mathbb{X}} M_\lambda \mid E_i^{(r)}v = F_i^{(r)}v = 0\ \forall i = 1,\dots,n \text{ and } \forall r >> 0\}.$$

According to [A, p.5], the submodule $\mathcal{F}(V)$ is a Type 1 integrable $\mathbb{U}_{q,K}$-module.[8] We can now define

$$H^0_q(M) := \mathcal{F}(\mathrm{Hom}_{\mathbb{B}_{q,K}}(\mathbb{U}_{q,K}, M)), \tag{2.2.2}$$

for any Type 1 integrable $\mathbb{B}_{q,K}$-module M. This yields a Type 1 integrable $\mathbb{U}_{q,K}$-module which we call the induced module $\mathrm{Ind}_{\mathbb{B}_{q,K}}^{\mathbb{U}_{q,K}}(M)$, later to be written in this paper as $\mathrm{Ind}_{\mathbb{B}}^{\mathbb{U}}(M)$. (Andersen uses the word "induction," but does not use our notation for the induced module, preferring instead $H^0_q(M)$.) In the definition of $H^0_q(M)$ above, left multiplication of $\mathbb{B}_{q,K}$ on $\mathbb{U}_{q,K}$ provides the $\mathbb{B}_{q,K}$-module structure on $\mathbb{U}_{q,K}$, and a $\mathbb{U}_{q,K}$-module structure on $\mathrm{Hom}_{\mathbb{B}_{q,K}}(\mathbb{U}_{q,K}, M)$ is given by $uf(x) = f(xu)$ for all $u, x \in \mathbb{U}_{q,K}$ and $f \in \mathrm{Hom}_{\mathbb{B}_{q,K}}(\mathbb{U}_{q,K}, M)$. The categories of Type I integrable $\mathbb{B}_{q,K}$-modules and $\mathbb{U}_{q,K}$-modules have enough injectives (as may be seen from the ring cases, applying the "largest Type 1 integrable submodule functors," such as $\mathcal{F}$ above)) and, hence, the left exact functor $\mathrm{Ind}_{\mathbb{B}_{q,K}}^{\mathbb{U}_{q,K}} = H^0_q$ has right derived functors $R^n\mathrm{Ind}_{\mathbb{B}_{q,K}}^{\mathbb{U}_{q,K}} = H^n_q$.

Definition 2.3. For $\leq$ the usual order on $\mathbb{X}$ determined by the positive roots R^+, set $\mu, \lambda \in \mathbb{X}$ to be **linked** if $\mu = w \cdot \lambda$ for some $w \in W_{D,\ell}$. If there is a chain $\lambda = \lambda_1, \dots, \lambda_s = \mu$ and a sequence $s_{\beta_1,m_1\ell_{\beta_1}}, \dots, s_{\beta_{s-1},m_{s-1}\ell_{\beta_{s-1}}}$ for which $\lambda_i \geq \lambda_{i+1} := s_{\beta_i,m_i\ell_{\beta_i}} \cdot \lambda_i$, $i = 1, \dots, s-1$, then μ is **strongly linked** to λ, denoted $\mu \uparrow_{D,\ell} \lambda$.

Remarks 2.4. (1) The relationship of strong linkage for weights $\mathbb{X}$ refines that of the usual ordering $\leq$. That is, $\mu \uparrow_{D,\ell} \lambda$ implies $\mu \leq \lambda$.

[8] No argument is given in [A], noting the property is "not hard to check." Perhaps this is true, once one knows how to do it. An argument for the case of (positive or negative Borel) subalgebras may be obtained with the method of [L, proof of Lem. 3.5.3], but using the generalized quantum Serre relations (through their corollary [L, Cor. 7.1.7]) in place of the quantum Serre relations. The case of the full quantum enveloping algebra then reduces to the rank 1 case, which can be handled with the formulas [L, 3.14(b), (c)].

(2) In the analogous $\mathrm{char}(k) = p > 0$ representation theory of algebraic groups, one defines $\mu \uparrow \lambda$ for weights $\mu, \lambda \in \mathbb{X}$ by using the affine Weyl group $W_\ell, \ell = p$, in place of $W_{D,\ell}$. In this circumstance, under mild restrictions on the prime p relative to the root system R, one has $W_p = W_{D,\ell}$, by Proposition 2.1(1).

Let $\mathcal{C}_q$ denote the category of Type 1 integrable $\mathbb{U}_{q,K}$-modules. The following fundamental result ultimately yields a splitting of $\mathcal{C}_q$ into a direct sum of blocks associated to orbits of an appropriate affine Weyl group. For application to the Induction Theorem 1, we will just need the version $\mathbb{U}$ of $\mathbb{U}_{q,K}$ described below (2.0.4) in which case $W_{D,\ell} = W_\ell$. We will then focus on *block*($\mathbb{U}$), the **principal block** of $\mathcal{C}_q$, corresponding to the orbit $W_{D,\ell} \cdot 0$. (Composition factors $L_q(\mu)$ of modules in the block are indexed by dominant weights μ in the orbit.) However, the results below hold more generally. They are claimed in [A] under the standing hypotheses of this subsection on q, K, ℓ, even with K allowed to have positive characteristic. However, it should be pointed out that the only reference given in support of one key auxiliary result [A, Thm. 2.1], a Grothendieck vanishing theorem needed in the proofs, is to the paper [AW]. The latter has as one of its standing assumptions that ℓ be odd, and not divisible by 3 in case the root system has a component of type G_2. This assumption on ℓ is, of course, satisfied by our $\mathbb{U}$, so we have not pursued the issue further. Possibly, it was the intent of Andersen to claim that the argument in [AW] worked in the more general set-up of [A], though there is no explicit comment to that effect.

Theorem 3. (1) (Strong Linkage Principle [A, Theorem 3.1, Theorem 3.13]). *Let $\lambda \in \mathbb{X}^+ - \rho$. (Thus, $\lambda + \rho \in \mathbb{X}^+$.) Let $\mu \in \mathbb{X}^+$. If $L_q(\mu)$ is a composition factor of some $H^i_q(w \cdot \lambda)$ with $w \in W$ and $i \in \mathbb{N}$, then $\mu \uparrow_{D,\ell} \lambda$.*

(2) (Linkage Principle [A, Thm. 4.3, Cor. 4.4]). *Let $\lambda, \mu \in \mathbb{X}^+$. If $\mathrm{Ext}^1_{\mathcal{C}_q}(L_q(\lambda), L_q(\mu)) \neq 0$, then λ is linked, but not equal to, μ. Consequently, if $M \in \mathcal{C}_{\mathbb{U}}$ is indecomposable, then the highest weights of all composition factors of M are linked, and the category $\mathcal{C}_q$ splits into blocks corresponding to the orbits for the dot action of $W_{D,\ell}$ on $\mathbb{X}^+$.*

Proof. We refer the reader to [A] for the proofs, on which we we make several remarks which may be helpful. First, note that there appears to be a serious misprint, an expression apparently carried over unintentionally to one result from a previous one, in the statement of the auxiliary result [A, Prop. 3.6]: In the expression "$< \lambda, \alpha_i^\vee >= -1$", the subexpression "$= -1$" should be replaced with "≥ 0".

Next, note that the exact sequences labeled (3) and (4) of [A, p.8] exist (and are later used) in the case $s = 1$ of the discussion there, with all terms in both exact sequences set equal to zero. (The reader might have been led by the wording to think these sequences were defined only for $s > 1$.)

Next, there is an organizational issue on [A, p.10]. The first three lines of the proof of [A, Cor. 3.8] do not use the "minimality" hypothesis of that corollary, and are implicitly quoted later on the same page, in the proof of [A, Thm. 3.8], where it is claimed "we have already checked the result for $w = 1$."

There are further minor points which occur on the same page [A, p.10]. In one repeated case, the vanishing of H^0_q on 1-dimensional nondominant $\mathbb{B}$-modules is given without proof, or hint. One approach that works is to use the version proved in the rank 1 case, then use induction from a corresponding parabolic subalgebra (and a Grothendieck spectral sequence).

At another place on the same page, [APW] is quoted to help determine, using a Weyl group action, the highest weight of a module $H^0_q(k_\lambda)$. However, an alternate argument may be given directly from the induced module definition. Quoting [APW] in this context is undesirable, because of the (implicit) restrictive set-up of that paper regarding ℓ. A similar issue, which we already noted above, before the statement of theorem, regards the reference to [AW] for a proof of [A, Thm. 2.1]. As noted above, the generality of the [AW] set-up is sufficient for applications in this paper.

Finally, the "splitting into blocks" is justified in [A] by corollary [A, Cor. 4.4]. Both the corollary and the splitting are made a straightforward consequence of [A, Thm. 4.3] by the local finiteness of Type 1 integrable modules, for which we refer ahead to Proposition 2.8 below. □

The paper [A] gives some history of the Theorem 3, most of which had been proved piecemeal previously by Andersen and his students and collaborators. There is, of course, a completely corresponding theorem–first proved in full generality by Andersen–for semsimple algebraic groups, as discussed in Jantzen's book [J]. With a certain amount of hindsight, some conceptual similarities can be imposed on the proofs and statements of supporting results. In particular, the presentation in [J] of Strong Linkage for the algebraic groups case ([J, II 6.13]), working with $\mathbb{X}^+$ rather than $\mathbb{X}^+ - \rho$, breaks the proof down into a lemma and two propositions [J, II 6.15, 6.16]. We have combined these propositions into an $\mathbb{X}^+ - \rho$ quantum analogue stated below. We need this extra detail (for $\mathbb{X}^+$) in order to provide more precise information about the appearance of irreducible modules $L_q(\mu)$ as composition factors in appropriate cohomology modules. $H^i_q(\nu)$. The Theorem [A, Thm. 2.1], discussed above, is needed in the proof (beyond the use of Strong Linkage).

Proposition 2.5. *1. Let $i \in \mathbb{N}$ and $w \in W$. If $L_q(\mu)$, $\mu \in \mathbb{X}^+$ is a composition factor of $H^i_q(w \cdot \lambda)$ with $\lambda \in \mathbb{X}^+ - \rho$, then $\mu \uparrow_{D,\ell} \lambda$. If $\ell(w) \neq i$, then $\mu < \lambda$.*

2. *Suppose* $\lambda \in \mathbb{X}^+$. *Then* $L_q(\lambda)$ *is a composition factor with multiplicity one of each* $H_q^{\ell(w)}(w \cdot \lambda)$ *with* $w \in W$.

Proof. The first part of item (1) just repeats Strong Linkage. The second part of item (1), and item (2), can be deduced from the approach in the proof of [A, Thm. 3.9]. Note that any weight μ in that proof which arises from the application of [A, Lem. 3.7] is strictly less than λ. As a consequence, the argument shows, in the presence of Strong Linkage, that the lemma holds for i and w if and only if it holds for $i+1$ and sw (assuming $sw > w$). This property can be applied repeatedly, moving i up or down. Using it, as in the first three lines of the proof of [A, Cor. 3.8], we obtain item (1) of the proposition. (This uses the discussed [A, Thm. 2.9].) Similarly, item (2) is reduced to the case $i = 0$ and $w = 1$. Here it follows by showing, directly from the definition, that λ has a 1-dimensional weight space in $H_q^0(\lambda)$. This completes the proof of the proposition. □

We remark that the discussions above of results in [A] corrects the proof of [ABG, Lem. 3.5.1]. This is with our choice of "negative" Borel subalgebras and our (standard) "dot" action. Both [A] and [APW] use "negative" Borel subalgebras as we do. The statement of the lemma given by [ABG] is apparently an attempt to use [APW] in a "positive" Borel subalgebra context, but the lemma is still incorrectly stated for that context. Also, they quote [APW] for the proof, though the latter paper does not contain as strong a result [ABG, Lem. 3.5.1] Instead, the main result [APW, Thm. 6.7] of its Borel-Weil-Bott section is a "lowest ℓ alcove" version, and explicitly requires that ℓ be a prime power.

The next corollary restates the main conclusions (those dealing with $\mathbb{X}^+$) of the proposition above in a form handy for later use. As already indicated, the (completely analogous) algebraic groups version is the combination of the two propositions [J, II 6.15, 6.16].

Corollary 2.6. *If* $\mu \in \mathbb{X}$ *and* $\lambda = w \cdot \mu \in \mathbb{X}(T)^+$ *(i.e.,* λ *is the dominant weight in the* W*-orbit of* μ*), then* $L_q(\lambda)$ *occurs just once as a composition factor of any of the modules* $H_q^i(\mu)$*,* i *running over all nonnegative integers. Precisely, one has* $[H_q^i(\mu) : L_q(\lambda)] \neq 0$ *only for* $i = \ell(w)$*, and then* $[H_q^{\ell(w)}(\mu) : L_q(\lambda)] = 1$. *If* $\eta \in \mathbb{X}^+$ *with* $\eta \neq \lambda$*, then for all* $i \in \mathbb{N}$*,* $[H_q^i(\mu) : L_q(\eta)] \neq 0$ *implies* $\eta < \lambda$ *and that* η *is strongly linked to* λ

The proposition below is quite important for applications, especially in the next subsection. There is a completely analogous result for induction from Borel subgroups in reductive algebraic groups, a special case of [J, II,4.2].

Proposition 2.7 ([A, Thm. 3.9, Thm. 2.1])**.** *Let $\mu \in \mathbb{X}$. Then $H^i_q(\mu)$ has finite dimension over K, and vanishes for $i > N$, the number of positive roots.*

Proof. We ask the reader again to read [A] for proofs, after first reviewing our comments on the proof of Theorem 3 above. $\square$

The final proposition in this section, also useful in the next section, is an analogue in the generality of [A] of [APW, Cor. 1.28] and of [L, Prop. 32.1.2]. It does not appear to be implied by either of these latter results, however.

Proposition 2.8. *Let M be any Type 1 integrable $\mathbb{U}_{q,K}$-module. Then M is locally finite, in the sense that each vector $v \in M$ generates a finite dimensional $\mathbb{U}_{q,K}$-module. Similarly, Type 1 integrable $\mathbb{B}_{q,K}$-modules are locally finite.*

Proof. For this proof, let $\mathbb{U}^+_j$ denote, for each index j, the $\mathcal{Z}$ span (a subalgebra) in $\mathbb{U}_{\mathcal{Z}}$ of all the elements $E_j^{(s)}$, $s \in \mathbb{N}$, with a similar notation for $\mathbb{U}^-_j$. Lusztig constructs his PBW-type basis [L3, Thm. 6.7] for the quantum enveloping algebra $\mathbb{U}_{\mathcal{Z}}$ using (finitely many) compositions of his explicit braid group automorphisms T_i, applied to the various $\mathbb{U}^\pm_j$. This process yields, for each positive root α, $\mathcal{Z}$-subalgebras $\mathbb{U}^\pm_\alpha$, and the whole quantum algebra $\mathbb{U}_{\mathcal{Z}}$ is a (ring-theoretic) product of finitely many of these, together with $\mathbb{U}^0_{\mathcal{Z}}$.

In several formulas listed in [L, 37.1.3] Lusztig gives explicit formulas for several similar automorphisms, including their action on basis elements of each $\mathbb{U}^\pm_j$. The setting for the action of these automorphisms is a $\mathbb{Q}(v)$-algebra $\mathbf{U}$ containing the algebra we have called $\mathbb{U}'_v$; moreover, the action of these automorphisms on the various elements K_i (in the notation of [L]) shows that all these automorphisms act bijectively on $\mathbb{U}'_v$. It is easy to pick out the braid group automorphism T_i defined in [L3] in this context, as (the restriction to $\mathbb{U}'_v$ of) $T''_{i,1}$ in [L, 37.1.2]). Accordingly, we learn, for each index j, that $T_i(\mathbb{U}^\pm_j)$ is contained in a product of $\mathbb{U}^0_{\mathcal{Z}}$ and at most three $\mathcal{Z}$ subalgebras, each of the latter having the form $\mathbb{U}^\pm_{j''}$, for some index j'' (in $1, \ldots, n$). To this information we add the fact that T_i stabilizes $\mathbb{U}_0$, which may be deduced from [L3, Thm. 3.3, Thm. 6.6(ii), Thm. 6.7(c)].

It follows now that $\mathbb{U}_{\mathcal{Z}}$ is a product of finitely many of the various subalgebras $\mathbb{U}^\pm_j$, together with $\mathbb{U}^0_{\mathcal{Z}}$. However, it is obvious that, if V is any finite-dimensional subspace of M, then any (ring-theoretic) product $\mathbb{U}^\pm_j V$ is finite-dimensional. Repeated application of this fact completes the proof of the proposition. $\square$

2.6. Some derived category considerations

We finally begin to use the assumptions and notation first given below (2.0.4, which the reader should review at this point. The notations include a common notation $\mathbb{U}$ for a quantum enveloping algebra, specialized at an ℓ^{th} root of unity, and the distribution algebra of a simply connected semisimple algebraic group G. There are similar common notations associated to various subalgebras of $\mathbb{U}$, and distribution algebras associated to subgroups of G, such as the (negative) Borel subgroup B. Both $p > h$ and $\ell > h$ are required, and there are further conditions on ℓ. (It must be odd, and not divisible by 3 when the root system of $\mathbb{U}$ has a component of type G2).

In addition, we introduce here the notations $\mathcal{C}_{\mathbb{U}}$, $\mathcal{C}_{\mathbb{B}}, \ldots$ for the categories of Type 1 integrable $\mathbb{U}, \mathbb{B}, \ldots$-modules, respectively, in the quantum case. In the algebraic groups case, the same notations reference the categories of rational $G, B, \ldots$-modules, respectively. These latter categories may be rewritten, according to our conventions for naming distribution algebras, as the categories of rational $\mathbb{U}, \mathbb{B}, \ldots$-modules. Here, "locally finite" would be a more accurate term than "rational," but we will use either term in unambiguous contexts.

This section provides a starting point for the proof of the Induction Theorems 1 and 2, the latter as reformulated in Theorem 2.1. The result below, a corollary of the those in the previous subsection, is the starting point. The statement and proof work in both the quantum and algebraic groups context, in the notation discussed above.

By $block(\mathbb{U})$ we mean the category of finite dimensional modules in the principal block of $\mathcal{C}_{\mathbb{U}}$; equivalently, it is the full subcategory of all finite-dimensional modules whose composition factors have highest weights in $W_\ell \cdot 0$ (taking $\ell = p$ in the algebraic groups case).

By $D^b block(\mathbb{U})$ we mean the bounded derived category of the abelian category $block(\mathbb{U})$, as defined by Verdier–see, for example, [Ha, Chapter I]. Let $D^b_{block(\mathbb{U})}(\mathcal{C}_{\mathbb{U}})$ denote the full subcategory of $D^b(\mathcal{C}_{\mathbb{U}})$ consisting of objects which have each of their (finitely many) cohomology groups in $block(\mathbb{U})$. Using the local finiteness of rational modules (see Proposition 2.8 in the quantum case), we observe the following lemma.

Lemma 2.9. *The natural map $D^b block(\mathbb{U}) \to D^b(\mathcal{C}_{\mathbb{U}})$, arising from the inclusion functor at the abelian category level, induces an equivalence*

$$D^b block(\mathbb{U}) \cong D^b_{block(\mathbb{U})}(\mathcal{C}_{\mathbb{U}}).$$

Proof. Let $K^\bullet$ be a bounded complex of objects in $\mathcal{C}_{\mathbb{U}}$ with each cohomology group belonging to $block(\mathbb{U})$. We claim there is a bounded subcomplex $F^\bullet$ of $K^\bullet$, with finite dimensional objects in $\mathcal{C}_{\mathbb{U}}$ in each degree, such that the inclusion map $F^\bullet \subseteq K^\bullet$ is a quasi-isomorphism. To construct the subcomplex $F^\bullet$, we may assume, inductively, its terms in all degrees $\geq i$ are constructed, so that they form a subcomplex $F^{\geq i}$. In addition, we require, inductively, that inclusion of this complex into $K^\bullet$ induces an isomorphism on cohomology in grades $> i$ and an epimorphism on the i^{th} cohomology groups. Then, we wish to construct $F^{i-1} \subseteq K^{i-1}$ so that the resulting complex $F^{\geq i-1}$ has the analogous properties for $i-1$ in place of i. Let δ denote the differential $K^{i-1} \to K^i$. Then $\delta(K^{i-1}) \cap F^i$ is, of course, both finite and contained in the image of δ. Choose a finite dimensional subspace E of K^{i-1} such that $\delta(E) = \delta(K^{i-1}) \cap F^i$, Also choose a finite dimensional subspace E' of the kernel of δ such that the image of E' in the natural surjection $Ker\ \delta \to H^{i-1}(K^\bullet)$ is all of $H^{i-1}(K^\bullet)$. Take F^{i-1} to be the $\mathbb{U}$-module generated by $E+E'$. Then $\delta(F^{i-1}) = \delta(K^{i-1}) \cap F^i$. Consequently, inclusion induces a monomorphism $H^i(F^{\geq(i-1)}) \to H^i(K^\bullet)$. The (downward) induction hypothesis implies the same map is a surjection, so it must be an isomorphism. Our construction of F^{i-1} gives a surjection of H^{i-1} for the same inclusion of complexes. The inductive step may be repeated, eventually reaching cohomological degrees j where K^j and all lower degree terms are 0. At that point we may take F^j also zero, and zero in lower degrees. This gives that $F^\bullet \subseteq K^\bullet$ induces an isomorphism on all cohomology groups. That is, it induces a quasi-isomorphism, as required in the claim.

We remark further, that, by taking block projections, the complex $F^\bullet$ may be assumed to consist in each degree of objects in $block(\mathbb{U})$.

The lemma proposes that the natural map induces an equivalence. It follows from the claim and remark that every object on the right-hand side of the proposed equivalence is, indeed, in the strict image of the left hand side. It remains to show the natural map induces a full embedding at the derived category morphism level. For this, observe the claim above can be strengthened so that the constructed complex $F^\bullet$ contains any given finite dimensional subcomplex $N^\bullet$ of $K^\bullet$. (Strengthen the induction hypothesis in the proof by adding the assertion $N^{\geq i} \subseteq F^{\geq i}$. Then, at the inductive step, replace E by $E + N^{i-1}$; this does not effect $\delta(E)$, since $\delta(N^{i-1}) \subseteq \delta(K^{i-1}) \cap F_i$.) As before, if $N^\bullet$ is a complex of objects in the principal block, we may assume the complex $F^\bullet$ constructed is also a complex of objects in the principal block.

Taking the same idea yet another step further, we can even assume $F^\bullet$ contains any given finite number of subcomplexes like $N^\bullet$, since the sum of any number of finite subcomplexes of $K^\bullet$ is again a finite subcomplex.

Now use the standard direct limit constructions (in the second variable) of derived category morphisms. Here we mean the Verdier localization construction of derived categories (and bounded derived categories), which proceeds (first) by localization of homotopy categories of complexes. (See [Ha, pp. 32,37].) In particular any morphism on the right hand side from an object $M_1^\bullet$ to $M_2^\bullet$ is represented by an object $K^\bullet$ and two morphisms $M_1^\bullet \to K^\bullet$ and $M_2^\bullet \to K^\bullet$, the latter a quasi-isomorphism. Now assume $M_1^\bullet$ and $M_2^\bullet$ are finite dimensional complexes of objects in $block(\mathbb{U})$, and let $N_1^\bullet$ and $N_2^\bullet$ be their respective images in $K^\bullet$. Applying the strengthened versions of the claim, above, we may construct a finite dimensional complex $F^\bullet$ containing both $N_1^\bullet$, $N_2^\bullet$, and contained in contained in $K^\bullet$. Moreover, the latter inclusion is constructed to be a quasi-isomorphism. It follows that the pair of morphisms $M_1^\bullet \to F^\bullet$ and $M_2^\bullet \to F^\bullet$ represent a derived category morphism on the left hand side of the display in the lemma. This proves the surjectivity required in the full embedding property at the morphism level.

It remains to prove injectivity. Suppose we are given a morphism on the left hand side of the display which becomes zero on the right hand side. The morphism on the left may be represented by the following configuration: We are given $M_1^\bullet$, $M_2^\bullet$ and $J^\bullet$, all finite dimensional complexes of objects in $block(\mathbb{U})$, and a pair of morphisms $M_1^\bullet \to J^\bullet$ and $M_2^\bullet \to J^\bullet$, the latter a quasi-isomorphism. To say that the derived category morphism represented by this configuration becomes zero, when considered on the right hand side, means the following: There is a complex $K^\bullet$ of objects in $\mathcal{C}_\mathbb{U}$ and a quasi-isomorphism $J^\bullet \to K^\bullet$ such that the composite map of complexes $M_1^\bullet \to J^\bullet \to K^\bullet$ is homotopy equivalent to zero. Let $h = \{h_i\}_{i\in\mathbb{Z}}$ be a family of maps defining the homotopy in question. That is, each $h_i : M_1^i \to K^{i-1}$ is a morphism in $\mathcal{C}_\mathbb{U}$, and $\delta_{K^\bullet} \circ h + h \circ \delta_{M_1^\bullet}$ is the given map $M_1^\bullet \to J^\bullet \to K^\bullet$. Here the subscripted symbols δ denote the evident families of differentials. Observe that the sum $L^\bullet$ over i of all $\delta_{K^{i-1}} \circ h_i(M_1^i) + h_i(M_1^i)$ is a finite dimensional subcomplex of $K^\bullet$, with all of its objects and differentials in $block(\mathbb{U})$ Using the extended claims above, we can construct a finite dimensional $block(\mathbb{U})$-complex $F^\bullet$, contained in $K^\bullet$ as a $\mathcal{C}_\mathbb{U}$-subcomplex, and itself containing each of $L^\bullet$, the image of $J^\bullet$ in $K^\bullet$, and the image of $M_1^\bullet$ in $K^\bullet$ (already in $L^\bullet$, actually). In addition, the above constructions allow us to assume that that the inclusion $F^\bullet \subseteq K^\bullet$ is a quasi-isomorphism. It follows that $J^\bullet \to F^\bullet$ is a quasi-isomorphism. Consequently, the derived category morphism (viewed as a direct limit) represented by the original configuration is also represented by the pair of maps $M_1^\bullet \to J^\bullet \to F^\bullet$ and $M_2^\bullet \to J^\bullet \to F^\bullet$. However, the map $M_1^\bullet \to J^\bullet \to F^\bullet$ is visibly homotopic to zero, using the same function h to define the required homotopy. (By construction $h_i(M_1^i) \subseteq F^{i-1}$ for each i.)

Thus, the morphism represented by the original configuration is zero in its associated direct limit. This proves the required injectivity and completes the proof of the lemma. □

It is suggested below [A, Defn. 3.5.6] that, in the quantum case, *block*($\mathbb{U}$) is "known" to have enough injectives. There is such a result about injectives in [APW]. But the context, while possibly too restrictive, ostensibly applies only to the cases where ℓ is a prime power, as do the discussions in [APW2]. Our argument above does not depend on such a property, and, indeed, applies to the algebraic groups case, where there are no finite-dimensional injectives.

Though it is somewhat informal, we henceforth identify $D^b block(\mathbb{U})$ and $D^b_{block(\mathbb{U})}(\mathcal{C}_{\mathbb{U}})$ through the Lemma 2.9. This language is used in the result below.

Corollary 2.10. *(i) For any* $\lambda \in \mathbb{Y}$, $RInd_{\mathbb{B}}^{\mathbb{U}}(l\lambda) \in D^b block(\mathbb{U})$.

(ii) The category $D^b block(\mathbb{U})$ *is generated, as a triangulated category, by the family of objects* $\{RInd_{\mathbb{B}}^{\mathbb{U}}(l\lambda)\}_{\lambda\in\mathbb{Y}}$.

Proof. The linkage principle, Corollary 2.6 and Proposition 2.7 imply item (i) above. Corollary 2.6, used with induction on weights and standard cohomological degree truncation operators [BBD, p.29], implies item (ii). □

Remarks 2.11. (a) The result above is stated as Cor. 3.5.2 in [ABG] in their quantum enveloping algebra set-up. Their proof, overall, relies on similar considerations, though some of the references supplied to [APW] for their preparatory lemma [ABG, Lem. 3.5.1] are inaccurate. It is not clear if their proof applies to the case where ℓ is not a prime power.

(b) This is perhaps a good point to mention that Kempf's vanishing theorem, well-known in the algebraic groups case [J, II,§4], also holds [AW, Thm. 5.3] in the quantum case under the hypotheses of this section. Thus, the higher derived functors $R^n\mathrm{Ind}\,{}_{\mathbb{B}}^{\mathbb{U}}(\mu)$ are zero for μ dominant and $n > 0$. The generalized tensor identities [J, I, Prop. 4.8] also work here [AW, Prop. 4.7]. These results are stated using individual higher derived functors R^n in each degree $n \geq 0$, but their proofs show that there are isomorphisms $\mathrm{RInd}_{\mathbb{B}}^{\mathbb{U}}(M \otimes N|_{\mathbb{B}}) \cong \mathrm{RInd}_{\mathbb{B}}^{\mathbb{U}}(M) \otimes N$ whenever $M \in \mathcal{C}_{\mathbb{B}}$ and $N \in \mathcal{C}_{\mathbb{U}}$. These isomorphisms may also be deduced from the natural maps in (d) below, applied with M replaced by an injective resolution.

(c) The roles of left and right in the tensor identity may be reversed. (See (d) below. The first argument for such a reversal is probably that for [APW, Prop. 2.7].) Also, although the quantum algebras we deal with are not generally cocommutative ($\mathbb{U}^0$ being an exception), the orders of tensor products of integrable modules we deal with can often be interchanged (up to isomorphism). This holds in particular for tensor products of finite dimensional modules in $\mathcal{C}_{\mathbb{U}}$. See [L, 32.16]. We have, however, not investigated the naturality properties this reversal may or may not have. The reversal is natural in the tensor identity case, as can be seen from (d) below. If M there is then replaced by a complex of injective modules, a natural reversal is obtained in the generalized tensor identity case.

(d) It is sometimes useful to have explicit natural isomorphisms

$$\alpha_{M,N} : \mathrm{Ind}\,_{\mathbb{B}}^{\mathbb{U}}(M \otimes N|_{\mathbb{B}}) \longrightarrow \mathrm{Ind}\,_{\mathbb{B}}^{\mathbb{U}}(M) \otimes N; \text{ and}$$
$$\gamma_{N,M} : \mathrm{Ind}\,_{\mathbb{B}}^{\mathbb{U}}(N|_{\mathbb{B}} \otimes M) \longrightarrow N \otimes \mathrm{Ind}\,_{\mathbb{B}}^{\mathbb{U}}(M)$$

where $M \in \mathcal{C}_{\mathbb{B}}, N \in \mathcal{C}_{\mathbb{U}}$. We give such natural isomorphisms for the convenience of the reader:

Drop the subscripts M, N and regard both modules in the top row as a contained in $\mathrm{Hom}\,_k(\mathbb{U}, M \otimes N)$.

We have, for $f \in \mathrm{Ind}\,_{\mathbb{B}}^{\mathbb{U}}(M \otimes N|_{\mathbb{B}}),\ x \in \mathbb{U}$,

$$\alpha(f)(x) = (1 \otimes S(x_2))f(x_1)$$

in (implicit sum) Sweedler notation. (Thus $\Delta(x) = x_1 \otimes x_2$, a sum over an invisible implicit index shared by $x_1,\ x_2$.) To check $\mathbb{B}$-equivariance of $\alpha(f)$, let $h \in \mathbb{B}$. Then

$$\begin{aligned}\alpha(f)(hx) &= (1 \otimes S((hx)_2))f((hx)_1)\\ &= (1 \otimes S(x_2)S(h_2))f(h_1x_1),\end{aligned}$$

where the last line is a sum over two implicit and independent indices, one for the $x's$ and one for the $h's$. Continuing, we obtain further similar expressions

$$\begin{aligned}&= (1 \otimes S(x_2)S(h_3))(h_1 \otimes h_2)f(x_1)\\ &= (1 \otimes S(x_2))(h_1 \otimes S(h_3h_2)f(x_1)\\ &= (1 \otimes S(x_2))(h_1 \otimes S\epsilon(h_2))f(x_1)\\ &= (1 \otimes S(x_2))(h_1 \otimes 1)f(x_1)\\ &= (h \otimes 1)(1 \otimes S(x_2))f(x_1)\\ &= (h \otimes 1)\alpha(x)\end{aligned}$$

which is the desired equivariance. Notice the top line is, for any fixed $x_1,\ x_2$, the image of $h \in \mathbb{B}$ under a linear map $\mathbb{B} \xrightarrow{\triangle} \mathbb{B} \otimes \mathbb{B} \otimes \mathbb{B} \longrightarrow M$. Here $\triangle$ denotes, with some abuse of notation, the map usually denoted $(1 \otimes \triangle) \circ$

$\triangle$ or $(\triangle \otimes 1) \circ \triangle$, with $\triangle : \mathbb{B} \longrightarrow \mathbb{B} \otimes \mathbb{B}$ the comultiplication. We write both $\Delta(h) = h_1 \otimes h_2$ and $\Delta(h) = h_1 \otimes h_2 \otimes h_3$, depending on context. The inverse β of α is given, for $g \in \mathrm{Ind}\,_{\mathbb{B}}^{\mathbb{U}}(M) \otimes N \subseteq \mathrm{Hom}\,_{\Bbbk}(\mathbb{U}, M \otimes N),\ x \in \mathbb{U}$, by

$$\beta(g)(x) = (1 \otimes x_2)g(x_1).$$

We leave it to the reader to check that $\beta(g)$ satisfies the appropriate $\mathbb{B}$-equivariance (by an argument similar in spirit to that for α), and that β is inverse to α. The formula for γ, is for $f \in \mathrm{Ind}\,_{\mathbb{B}}^{\mathbb{U}}(N|_{\mathbb{B}} \otimes M) \subseteq \mathrm{Hom}\,_{mathbbk}(\mathbb{U}, N \otimes M),\ x \in \mathbb{U}$,

$$\gamma(f)(x) = (S(x_1) \otimes 1)f(x_2)$$

For the inverse δ of γ it is, for $g \in N \otimes \mathrm{Ind}\,_{\mathbb{B}}^{\mathbb{U}}(M) \subseteq \mathrm{Hom}\,_{\Bbbk}(\mathbb{U}, N \otimes M)$, $x \in \mathbb{U}$,

$$\delta(g) = (x_1 \otimes 1)g(x_2)$$

Again, the reader may check, with arguments similar in spirit to those illustrated, that γ and δ satisfy the appropriate equivariance properties and are inverse to each other.

(e) We remark that $Ind_{\mathbb{B}}^{\mathbb{U}}(M)$ is equipped with a natural "counit" $\epsilon_M : Ind_{\mathbb{B}}^{\mathbb{U}}(M)|_{\mathbb{B}} \longrightarrow M$ which has a well-known property of being right adjoint to restriction. In the full module categories for $\mathbb{U}$ and $\mathbb{B}$ ϵ_M may be given as evaluation at 1 on the right adjoint $\mathrm{Hom}\,_{\mathbb{B}}(\mathbb{U}, M)$, and it follows that ϵ_M may be similarly interpreted for $\mathrm{Ind}\,_{\mathbb{B}}^{\mathbb{U}}(M)$ when dealing with integrable modules. We only want to observe here that there is a similar "evaluation at 1" counit for each of the modules $Ind_{\mathbb{B}}^{\mathbb{U}}(M \otimes N|_{\mathbb{B}}), \mathrm{Ind}\,_{\mathbb{B}}^{\mathbb{U}}(M) \otimes N), \mathrm{Ind}\,_{\mathbb{B}}^{\mathbb{U}}(N|_{\mathbb{B}} \otimes M), N \otimes \mathrm{Ind}\,_{\mathbb{B}}^{\mathbb{U}}(M)$ above, providing each of these constructions with the structure of a right adjoint to restriction. The proof is easy, noting the isomorphisms in (d) commute with evaluation at 1 on the ambient $\mathrm{Hom}\,_{\mathbb{B}}(\mathbb{U}, -)$ module. Rewriting this fact in the ϵ notation, we have that $\epsilon_M \otimes N|_{\mathbb{B}}$ and $N|_{\mathbb{B}} \otimes \epsilon_M$ are counits (that is, provide a right adjoint structure) for $\mathrm{Ind}\,_{\mathbb{B}}^{\mathbb{U}}(M) \otimes N),\ N \otimes \mathrm{Ind}\,_{\mathbb{B}}^{\mathbb{U}}(M)$, respectively. We will use this fact in Appendix B.

(f) Finally, we explain briefly how the induction functors we have used above, based on the formalism in [A] and compatible with [APW], fit with the algebraic groups formalism in [J, I,3.3]. Actually, the original definition [CPS77, §1] of induction $\mathrm{Ind}\,_B^G(M)$ in the algebraic groups case, for a finite-dimensional rational G-module M, was the set $\mathrm{Morph}_B(G, M)$ of B-equivariant morphisms from G to M, with an evident direct union used for a general rational module M. This definition is formally quite close to the [A] definition in the quantum case. Using Sullivan's theorem [CPS80, Thm. 6.8], that all locally finite Dist(G) modules are rational, it is easy to see that this definition coincides with the definition of [A] used above, with Dist(G) in the role of $\mathbb{U}$ and Dist(G) in the role of $\mathbb{U}$. Finally, to connect the [CPS77] definition with that of [J], simply replace the $B \times G^{op}$ action $(b, g)x = bxg$ on G with the isomorphic action $(b, g)x = g^{-1}xb^{-1}$, where $b \in B$ and $x, g \in G$.

2.7. Some special twisted induced modules

In this subsection and the next, we will adapt the "uniform" notation for the quantum and positive characteristic cases introduced in the discussion of Theorem 2.1 and elaborated in Subsection 2.2. We will presume and utilize the definitions for the quantum Frobenius morphism as in, e.g., [L3, Thm. 8.10], [L]. We use a similar notation in positive characteristic, where the Frobenius morphism originated and is well-known.

Remarks 2.12. In the quantum case, the Frobenius morphism is a homomorphism $\varphi : \mathbb{U} \to Dist(G')$, where $Dist(G')$ is the distribution algebra (over K) of an algebraic group G' (semisimple, simply connected, and defined and split over K, with the same root datum as $\mathbb{U}$). If M is a rational G'-module over K, we may twist it through φ and obtain an integral $\mathbb{U}$ module ${}^{\varphi}M$, trivial on $\mathbb{u}$. We will use the notation $M^{[1]} := {}^{\varphi}M$, and the same notation for twisting a module through the Frobenius in the corresponding characteristic p algebraic groups situation (where $\mathbb{U} = Dist(G)$ is both the domain and the target of the Frobenius homomorphism).

Returning to the quantum case, we remind the reader of our notation $\mathbb{p} = \mathbb{b}{\cdot}\mathbb{U}^0$ (and that this notation is similar to that in [ABG], except that our $\mathbb{b}$ is associated to negative roots). The Frobenius homomorphism is compatible with triangular decompositions of its domain and target; see [L3, Thm. 8.10]. So, the above $M^{[1]}$ notation also makes sense, if M is (in the obvious analogous notation) a rational B' or T'-module. This results, respectively, in an integral $\mathbb{B}$ or $\mathbb{p}$ module $M^{[1]}$, trivial as a $\mathbb{b}$-module. (There is some ambiguity of notation here: if M is not obviously a G'-module, we deliberately do not include all of $\mathbb{u}$ as part of the domain of definition of $M^{[1]}$ without explicit mention otherwise.) Conversely, we claim any integrable module N for $\mathbb{B}$ or $\mathbb{p}$, which is trivial for $\mathbb{b}$, has, respectively, this form, and in a unique way. The corresponding assertion for $\mathbb{U}^-$ for modules trivial for $\mathbb{u}^-$ follows from the (negative root analogs of) [L3, Lem.s 8.8, 8.9]. (Note also from these results that the kernel ideal of the Frobenius homomorphism on $\mathbb{U}^-$ is the left $\mathbb{U}^-$ ideal generated by the augmentation ideal of $\mathbb{u}^-$. Similarly, the kernel must be the right ideal generated by this augmentation ideal.) At the level of $\mathbb{U}^0$ for modules trivial on $\mathbb{u}^0$ it follows from the explicit form of the Frobenius homomorphism on $\mathbb{U}^0$ in [*op cit*, p.110, bottom] and the monomial bases [*op cit*, Thm. 6.7(c), Thm. 8.3(ii)] for $\mathbb{U}^0$ and $\mathbb{u}^0$, respectively. The claim follows.

We remark that the parenthetic note above shows an interesting additional property: If N is any integrable $\mathbb{B}$ module, with unspecified action of $\mathbb{b}$, the largest $\mathbb{b}$-module quotient of N with trivial $\mathbb{b}$-module action is naturally a (twisted) $\mathbb{B}$-module, call it $M^{[1]}$. Consequently, if E is any T'-module, then, using rational induction for algebraic groups, $\operatorname{Hom}_{\mathbb{B}}(N, \operatorname{Ind}_{T'}^{B'}(E)^{[1]}) \cong \operatorname{Hom}_{\mathbb{B}}(M^{[1]}, \operatorname{Ind}_{T'}^{B'}(E)^{[1]}) \cong \operatorname{Hom}_{B'}(M, \operatorname{Ind}_{T'}^{B'}(E)) \cong \operatorname{Hom}_{T'}(M|_{p'}, E) \cong \operatorname{Hom}_{\mathbb{p}}(N|_{\mathbb{p}}, E^{[1]})$. This shows the functor sending $E^{[1]}$ to $\operatorname{Ind}_{T'}^{B'}(E)^{[1]})$ serves, on appropriate categories of twisted integrable modules, as a right adjoint to restriction on corresponding integrable (but not necessarily twisted) categories. In fact, a right adjoint $\operatorname{Ind}_{\mathbb{p}}^{\mathbb{B}}$ on the full integrable categories is constructed in [ABG, §2.7]. (There is a similar construction in [APW2], but the set-up there ostensibly requires ℓ to be a prime power.) The adjointness properties observed above in this paragraph imply

$$\operatorname{Ind}_{\mathbb{p}}^{\mathbb{B}}(E^{[1]}) \cong \operatorname{Ind}_{T'}^{B'}(E)^{[1]},$$

for E any rational T'-module.

A completely adequate version of the isomorphism is noted without proof in [ABG, (2.8.2)], presumably based on [ABG, Lem. 2.6.5], which discusses also details of the Frobenius homomorphism at the $\mathbb{B}$ module level. The main application in both [ABG] and this paper occurs with $E^{[1]}$ a 1-dimensional $\mathbb{p}$ module $k(\ell\lambda) := k_{\mathbb{p}}(\ell\lambda)$, with $\lambda \in \mathbb{Y}$.

We note further that, once an induction $\operatorname{Ind}_{\mathbb{p}}^{\mathbb{B}}$ is available (as a right adjoint to restriction from $\mathcal{C}_{\mathbb{B}}$ to $\mathcal{C}_{\mathbb{p}}$), an induction functor $\operatorname{Ind}_{\mathbb{p}}^{\mathbb{U}}$ can be constructed as the composition $\operatorname{Ind}_{\mathbb{B}}^{\mathbb{U}} \circ \operatorname{Ind}_{\mathbb{p}}^{\mathbb{B}}$.

We conclude these remarks by noting that all features of the above paragraphs have obvious parallels that hold in the (characteristic p) algebraic groups case, with $G = G'$, etc. We continue this dual use of the notations $G', \ldots$ below.

Adapting [ABG, §4.3] to our negative Borel framework, set

$$I_\mu := \operatorname{Ind}_{T'}^{B'}(\mu) \cong \varinjlim_{\nu \in \mathbb{Y}^{++}} (V_\nu|_{B'} \otimes \Bbbk_{B'}(\mu + \nu)) \tag{2.12.1}$$

where V_ν denotes the "costandard" G'-module $\operatorname{Ind}_{B'}^{G'}(-w_0\nu) = H^0(-w_0\nu)$ with highest weight $-w_0\nu$. We take μ to be any weight in $\mathbb{X}$, though we will only use the case $\mu \in \mathbb{Y}$. In the quantum case, G' is a semsimple algebraic group in characteristic 0, so V_ν is irreducible, though we will not need that to explain the isomorphism in 2.12.1, which we do now:

In general, the lowest weight of V_ν is $-\nu$, appearing with multiplicity 1, and $\Bbbk_{B'}(-\nu)$ is the B'-socle of V_ν. Let ω be in $\mathbb{Y}^{++}$, so that $\nu + \omega$ is also in $\mathbb{Y}^{++}$, and is "larger" than ν if $\omega \neq 0$. This defines an evident directed system of weights. There is a natural homomorphism of G'-modules $V_\nu \otimes V_\omega \to V_{\nu+\omega}$ which is an isomorphism on highest (and, applying w_0, on lowest) weight spaces. In particular, the induced homomorphism of B'-modules $V_\nu \otimes \Bbbk_{B'}(-\omega) \to V_{\nu+\omega}$ is an isomorphism on B'-socles, hence injective. Tensoring on the right with 1-dimensional modules $\Bbbk_{B'}(\nu + \omega + \mu)$ gives injections

$$V_\nu \otimes \Bbbk_{B'}(\nu + \mu) \to V_{\nu+\omega} \otimes \Bbbk_{B'}(\nu + \omega + \mu).$$

For fixed μ these describe the directed system underlying the direct limit in 2.12.1, and shows it is a directed system of injections, all with a common socle $\Bbbk_{B'}(\mu)$ and with the weight μ appearing with multiplicity 1. In particular, the direct limit exists as a rational B' module I and has the same socle $\Bbbk_{B'}(\mu)$. Consequently, there is a map $I \to \operatorname{Ind}_{T'}^{B'}(k(\mu)) = I_\mu$ which is an isomorphism on socles. (The induced module definition of I_μ shows its only 1-dimensional submodule is $\Bbbk_{B'}(\mu)$.) Thus, $I \subseteq I_\mu$. To get equality, it is enough to show that, for any weight τ of I_μ, there is a $\nu \in \mathbb{Y}^{++}$ such that the τ weight space of $V_\nu \otimes \Bbbk_{B'}(\nu + \mu)$ has dimension equal to that of the τ weight space of I_μ. The (weight space by weight space) linear dual I_μ^* of I_μ is (after conversion to a left B'-module) generated by its (1-dimensional) $-\mu$ weight space, call it $\Bbbk v$. Thus, $I_\mu^* = \operatorname{Dist}(U'^-)v$ may be viewed as the homomorphic image of $M(\nu)|_{B'} \otimes \Bbbk_{B'}(-\nu - \mu)$, where $M(\nu)$ denotes the Verma module for $\operatorname{Dist}(G')$ with highest weight ν. The linear dual of the injection $V_\nu \otimes \Bbbk_{B'}(\nu + \mu) \to I_\mu$ gives a surjection $I_\mu^* \to \Delta(\nu)|_{B'} \otimes \Bbbk_{B'}(-\nu - \mu)$, where $\Delta(\nu)$ is the Weyl module of highest weight ν. Thus, we have a composition of surjections

$$M(\nu)|_{B'} \otimes \Bbbk_{B'}(-\nu - \mu) \to I_\mu^* \to \Delta(\nu)|_{B'} \otimes \Bbbk_{B'}(-\nu - \mu).$$

We want to show that the $-\tau$ weight space dimensions on the left and right (and, thus, also in the middle) are the same for some choice of ν. This is equivalent to showing that the $\nu + \mu - \tau$ weight spaces of the Verma and Weyl modules with highest weight ν are the same for some ν, given μ and τ. This dimension is obviously independent of the base field $\Bbbk$ in the Verma module case, and the same independence is true in the Weyl module case by

[J, II,8.3(3)]. Over the complex numbers, the kernel of the map $M(\nu) \to \Delta(\nu)$ is generated as a B'-module, by the elements $F_i^{N_i+1}v^+$, where v^+ is a highest weight vector, N_i is the coefficient of μ at the i^{th} fundamental weight, and F_i is a Chevalley basis root vector associated with the negative of the i^{th} fundamental root α_i. (All observed in [ABG, §4.3].) It is easy to choose all ν so that each coefficient of $\tau - \mu$ at α_i is smaller than $N_i + 1$. In this case the kernel has a zero weight space for weight $\nu + \mu - \tau$. Thus, the dimensions of the weight spaces for this weight are the same in both the Verma and Weyl module, as desired. This proves $I = I_\mu$.

By applying Frobenius twists, one has

$$\mathrm{Ind}_{\mathbb{P}}^{\mathbb{B}}(\ell\mu) \cong I_\mu^{[1]} = \mathrm{Ind}\,{}_{T'}^{B'}(\mu)^{[1]} \cong \varinjlim_{\nu \in \mathbb{Y}^{++}} (V_\nu^{[1]}|_{\mathbb{B}} \otimes \Bbbk_{\mathbb{B}}(\ell\mu + \ell\nu)). \tag{2.12.2}$$

Definition 2.13. For any dominant weight σ define $J_{\sigma,\mu} = V_\sigma^{[1]}|_{\mathbb{B}} \otimes \Bbbk_{\mathbb{B}}(\ell\mu + \ell\sigma)$.

In the lemma below, and elsewhere in this paper, we freely use "Ext^n" for a derived category "Hom^n."

Lemma 2.14. *Let Y be any finite dimensional $\mathbb{B}$-module, and μ any weight in $\mathbb{Y}$ (or $\mathbb{X}$). Then, for sufficiently large σ, we have*

(a) $\mathrm{Ext}_{\mathbb{B}}^n(Y, J_{\sigma,\mu}) \cong \mathrm{Ext}_{\mathbb{B}}^n(Y, I_\mu^{[1]})$*, and*
(b) $\mathrm{Ext}_{\mathbb{B}}^n(RInd_{\mathbb{B}}^{\mathbb{U}}(Y), J_{\sigma,\mu}) \cong \mathrm{Ext}_{\mathbb{B}}^n(RInd_{\mathbb{B}}^{\mathbb{U}}(Y), I_\mu^{[1]})$

for all nonnegative integers n. In addition we have (independently of σ)

(c) $\mathrm{Ext}_{\mathbb{B}}^n(Y, I_\mu^{[1]}) \cong \mathrm{Ext}_{\mathbb{P}}^n(Y, \Bbbk_{\mathbb{B}}(\ell\mu))$*, and*
(d) $\mathrm{Ext}_{\mathbb{B}}^n(RInd_{\mathbb{B}}^{\mathbb{U}}(Y), I_\mu^{[1]}) \cong \mathrm{Ext}_{\mathbb{U}}^n(RInd_{\mathbb{B}}^{\mathbb{U}}(Y), RInd_{\mathbb{B}}^{\mathbb{U}}(I_\mu^{[1]}))$ *for all nonnegative integers n.*

Proof. For part a) observe that $\mathrm{Ext}_{\mathbb{B}}^n(Y, \Bbbk_{\mathbb{B}}(\omega)) = 0$ unless ω is dominated by some weight ν with the weight space $Y_\nu \neq 0$. There is no such ν, if the height of ω is sufficiently large, depending only on the finite dimensional module Y. For σ sufficiently large, all the nonzero weight spaces in $J'_{\sigma,\mu} := I_\mu^{[1]}/J_{\sigma,\mu}$ occur for weights with a large height, determined by μ and the choice of σ. For such a σ and μ, we have $\mathrm{Ext}_{\mathbb{B}}^n(Y, J'_{\sigma,\mu}) = 0$ for all nonnegative integers n. (This can be easily seen with direct limit arguments.) Part a) follows, and part b) may be obtained by using (modules in a finite complex representing) $\mathrm{RInd}_{\mathbb{B}}^{\mathbb{U}}(Y)$ for Y in part a) to give part b). Parts c) and d) are standard reciprocity results. (For part (c), note that $I_\mu^{[1]} \cong \mathrm{Ind}_{\mathbb{P}}^{\mathbb{B}}(\Bbbk_{\mathbb{B}}(\ell\mu)) \cong \mathrm{RInd}_{\mathbb{P}}^{\mathbb{B}}(\Bbbk_{\mathbb{B}}(\ell\mu))$.) This completes the proof of the lemma. ∎

Corollary 2.15. *Let Y be a finite-dimensional $\mathbb{B}-$module. Assume all composition factors of Y have weight $\ell\omega$ for some weight $\omega \in \mathbb{Y}$, and fix $\mu \in \mathbb{Y}$.*

1. *If n is an odd nonnegative integer, then both $\mathrm{Ext}\,^n_{\mathbb{B}}(Y, I_\mu^{[1]})$ and $\mathrm{Ext}\,^n_{\mathbb{U}}(RInd_{\mathbb{B}}^{\mathbb{U}}(Y),\ RInd_{\mathbb{B}}^{\mathbb{U}}(I_\mu^{[1]}))$ are zero.*
2. *For any nonnegative integer n, if the Y-composition factor weights $\ell\omega$ all have ω of sufficiently large height, depending only on n and μ, then both $\mathrm{Ext}\,^n_{\mathbb{B}}(Y, I_\mu^{[1]})$ and $\mathrm{Ext}\,^n_{\mathbb{U}}(RInd_{\mathbb{B}}^{\mathbb{U}}(Y), RInd_{\mathbb{B}}^{\mathbb{U}}(I_\mu^{[1]}))$ are zero. [We refer ahead to (3.4.2) in the proof of this part.]*

Proof. The $\mathbb{b}$-cohomology of the trivial module is, as a $\mathbb{B}$-module, the symmetric algebra $S^*(\mathfrak{n}^{*[1]})$. Here $\mathfrak{n}$ is the Lie algebra of the "unipotent radical" of $\mathbb{B}'$, with $\mathfrak{n}^{*[1]}$ the linear dual of that Lie algebra twisted by the Frobenius, and given cohomological degree 2. See [AJ, Prop. 2.3] in characteristic $p > h$ and [GK, Thm. 3][9] for the analogous characteristic 0 quantum group result. This gives part (1), using Lemma 2.14(c).

Part (2) also follows (in both the quantum and algebraic group cases), using Lemma 2.14, and (3.4.2) below. In more detail, observe first that it is sufficient to take Y of dimension 1. Then (2.14(c) clearly gives the required vanishing of $\mathrm{Ext}\,^n_{\mathbb{B}}(Y, I_\mu^{[1]})$, provided the weight of Y, call it $\ell\lambda$, is of sufficient large height. Note this also gives vanishing of $\mathrm{Ext}\,^n_{\mathbb{B}}(Y, J_{\sigma,\mu})$ for all sufficiently large σ. Next, regarding $I_\mu^{[1]}$ as the direct union of the various modules $J_{\sigma,\mu}$, we obtain $\mathrm{Ext}\,^n_{\mathbb{U}}(\mathrm{RInd}_{\mathbb{B}}^{\mathbb{U}}(Y),\ \mathrm{RInd}_{\mathbb{B}}^{\mathbb{U}}(I_\mu^{[1]}))$ as a direct limit of the various $\mathrm{Ext}\,^n_{\mathbb{U}}(\mathrm{RInd}_{\mathbb{B}}^{\mathbb{U}}(Y), \mathrm{RInd}_{\mathbb{B}}^{\mathbb{U}}(J_{\sigma,\mu})) = \mathrm{Ext}\,^n_{\mathbb{U}}(\mathrm{RInd}_{\mathbb{B}}^{\mathbb{U}}(Y),\ \mathrm{RInd}_{\mathbb{B}}^{\mathbb{U}}(V_\sigma^{[1]} \otimes \Bbbk_{\mathbb{B}}(\ell\mu + \ell\sigma))) \cong \mathrm{Ext}\,^n_{\mathbb{U}}(\mathrm{RInd}_{\mathbb{B}}^{\mathbb{U}}(Y), V_\sigma^{[1]} \otimes \mathrm{RInd}_{\mathbb{B}}^{\mathbb{U}}(\Bbbk_{\mathbb{B}}(\ell\mu + \ell\sigma))$. Applying 3.4.2 with $\Bbbk_{\mathbb{B}}(\ell\lambda) = Y$ and N in 3.4.2 equal to V_σ, we obtain that the dimension of $\mathrm{Ext}\,^n_{\mathbb{U}}(\mathrm{RInd}_{\mathbb{B}}^{\mathbb{U}}(Y), V_\sigma^{[1]} \otimes \mathrm{RInd}_{\mathbb{B}}^{\mathbb{U}}(\Bbbk_{\mathbb{B}}(\ell\mu + \ell\sigma))$ is $\dim \mathrm{Ext}\,^n_{\mathbb{B}}(Y, V_\sigma^{[1]} \otimes \Bbbk_{\mathbb{B}}(\ell\mu + \ell\sigma)) = \dim \mathrm{Ext}\,^n_{\mathbb{B}}(Y, J_{\sigma,\mu}) = 0$ for large σ. This proves part (2), and the proof of the corollary is complete. ☐

3. A Proof of the induction theorems

After providing an initial framework and brief outline of the steps to be used, this section proceeds step-by-step to complete the proof of Theorem 1

[9]While several references are made in the proof of this theorem to [APW] and [APW2], they are of a general formal nature, similar to those of [A] we discuss above of 2.3, and do not requiring that ℓ be a prime power. It is, however, necessary in [GK] to quote a case of Kempf's theorem, but there [GK] gives a (correct) reference to [AW, Thm. 5.3].

and Theorem 2, the latter treated in its equivalent formulation, Theorem 2.1. Indeed, at this stage the proofs can be given simultaneously, largely thanks to results in the previous section, such as Corollary 2.15. In those results, the statements make sense and are correct in both the quantum and positive characteristic cases, though some attention to differences–at least to different sources–are sometimes required in their proofs. Such differentiation is no longer necessary in the wording of proofs in this section. Once the proof has been completed, Subsection 3.5 summarizes some of the similarities and differences between our approach and that taken in [ABG].

3.1. Sketch of the proof of the induction theorems

The proof of the induction theorems begins, as is implicit in [ABG], with the idea of utilizing the following 'general nonsense' result on categorical equivalences. The proof is left to the reader.

Lemma 3.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be two triangulated categories, and let $\mathcal{F} : \mathcal{A} \to \mathcal{B}$ be a morphism of triangulated categories, that is, $\mathcal{F}$ sends triangles in $\mathcal{A}$ to triangles in $\mathcal{B}$, and commutes with the respective translation functors on $\mathcal{A}$ and $\mathcal{B}$. Then $\mathcal{F}$ is an equivalence of triangulated categories if there is a set of objects S in $\mathcal{A}$, such that the following two conditions hold:*

(A) the minimal full triangulated subcategory of $\mathcal{A}$ containing S is, up to isomorphic objects, the whole category $\mathcal{A}$, and likewise for $\mathcal{B}$ in place of $\mathcal{A}$ and $\mathcal{F}(S) := \{\mathcal{F}(a) \,|\, a \in S\}$ in place of S;

(B) for any objects $a, a' \in S$, the functor $\mathcal{F}$ induces isomorphisms

$$Hom_{\mathcal{A}}(a, a'[k]) \mapsto Hom_{\mathcal{B}}(\mathcal{F}(a), \mathcal{F}(a')[k]) \qquad \textit{for all } k \in \mathbb{Z}.$$

This lemma will be applied with $\mathcal{A} = D_{triv}(\mathbb{B})$, $\mathcal{B} = D^b block(\mathbb{U})$[10] $S = k_{\mathbb{B}}(\ell\lambda), \lambda \in \mathbb{Y}$. Standard arguments with (distinguished triangles arising from) homological degree truncations (see [BBD, Examples 1.3.2]) show that S does, indeed, generate $\mathcal{A}$ as a triangulated category.

[10] Throughout this paper we use quantum and algebraic groups in a "simply connected" setting. In particular this means that all of our $\mathbb{B}$-modules, always assumed to be a direct sum of their weight spaces, can have associated weights which are in $\mathbb{X}$, not just $\mathbb{Y}$, as in the [ABG] "adjoint group" setting. However, the more general $\mathbb{B}$-modules are obviously the natural direct sum of submodules whose associated weights belong to a fixed coset (one for each summand) of $\mathbb{Y}$ in $\mathbb{X}$. This has the consequence that the two versions of $D_{triv}(\mathbb{B})$ in these respective $\mathbb{B}$-module contexts are naturally equivalent. Similar considerations apply to $\mathbb{U}$-modules and $block(\mathbb{U})$.

The functor $\mathrm{Ind}_{\mathbb{B}}^{\mathbb{U}}$ has been discussed in Section 2. It is an additive left exact functor, so, from general principles, its right derived functor $\mathrm{RInd}_{\mathbb{B}}^{\mathbb{U}}$, which exists, is a morphism of triangulated categories. Corollary 2.10 shows that $\mathrm{RInd}_{\mathbb{B}}^{\mathbb{U}}(S)$ generates $D^b block(\mathbb{U})$. Consequently, the starting hypotheses of Lemma 3.1 and its condition (A) are met, with $\mathcal{F}$ the restriction of $\mathrm{RInd}_{\mathbb{B}}^{\mathbb{U}}$ to $D_{triv}(\mathbb{B})$.

So, to establish the induction theorems, it suffices to prove condition (B) of Lemma 3.1 holds, that is, $\mathrm{RInd}_{\mathbb{B}}^{\mathbb{U}}$ gives isomorphisms

$$\begin{aligned}&\mathrm{Hom}_{D_{triv}(\mathbb{B})}(k_{\mathbb{B}}(\ell\lambda), k_{\mathbb{B}}(\ell\mu)[n])\\ &\quad\cong \mathrm{Hom}_{D^b block(\mathbb{U})}(\mathrm{RInd}_{\mathbb{B}}^{\mathbb{U}} k_{\mathbb{B}}(\ell\lambda), \mathrm{RInd}_{\mathbb{B}}^{\mathbb{U}} k_{\mathbb{B}}(\ell\mu)[n])\end{aligned}$$

for any $\lambda, \mu \in \mathbb{Y}$ and $n \in \mathbb{Z}$. That is, it suffices prove for all $\lambda, \mu \in \mathbb{Y}$ and $n \geq 0$ that applying $\mathrm{RInd}_{\mathbb{B}}^{\mathbb{U}}$ produces isomorphisms

$$\mathrm{Ext}^n_{\mathbb{B}}(k_{\mathbb{B}}(\ell\lambda), k_{\mathbb{B}}(\ell\mu)) \cong \mathrm{Ext}^n_{block(\mathbb{U})}(\mathrm{RInd}_{\mathbb{B}}^{\mathbb{U}} k_{\mathbb{B}}(\ell\lambda), \mathrm{RInd}_{\mathbb{B}}^{\mathbb{U}} k_{\mathbb{B}}(\ell\mu)), \tag{3.1.1}$$

where the right hand side is an 'Ext' computed in the sense of hypercohomology. To establish (3.1.1) we will proceed as follows, with the first step the same as in [ABG]:

[STEP 1:] Show

(3.1.2)

$$\dim(\mathrm{Ext}^n_{\mathbb{B}}(k_{\mathbb{B}}(\ell\lambda), k_{\mathbb{B}}(\ell\mu))) = \dim(\mathrm{Ext}^n_{block(\mathbb{U})}(\mathrm{RInd}_{\mathbb{B}}^{\mathbb{U}} k_{\mathbb{B}}(\ell\lambda), \mathrm{RInd}_{\mathbb{B}}^{\mathbb{U}} k_{\mathbb{B}}(\ell\mu)))$$

for any $\lambda, \mu \in \mathbb{Y}$ and $n \geq 0$.

[STEP 2]: Employing the equalities (3.1.2), show, for the twisted $\mathbb{B}$-modules $I_\mu^{[1]}$ defined in (2.12.2), that there are isomorphisms

$$\mathrm{Ext}^n_{\mathbb{B}}(k_{\mathbb{B}}(\ell\lambda), I_\mu^{[1]}) \cong \mathrm{Ext}^n_{block(\mathbb{U})}(\mathrm{RInd}_{\mathbb{B}}^{\mathbb{U}}(k_{\mathbb{B}}(\ell\lambda)), \mathrm{Ind}_{\mathbb{B}}^{\mathbb{U}}(I_\mu^{[1]})), \tag{3.1.3}$$

arising from the functoriality of $\mathrm{RInd}_{\mathbb{B}}^{\mathbb{U}}$.

[STEP 3]: Making use of the structure of the modules $I_\eta^{[1]}$, recover the desired isomorphisms (3.1.1) from the isomorphisms (3.1.3), and (once again) the dimension equalities (3.1.2).

3.2. Step 1 of the proof of the induction theorems

We largely follow [ABG] for this step, with the exception of the proof of part (ii) of Lemma 3.2 below, which is [ABG, Lem.4.1.1(ii)] in the quantum case. We give a proof in the algebraic groups case in Appendices A and B of this paper. However, the proof [ABG, Lem.4.1.1(ii] given in [ABG] is not nearly adequate, in our view, even in the quantum case, and we point out in appendices A and B how our proofs there apply to complete it. As mentioned in the introduction, we are grateful to P. Achar and S. Riche for a suggestion to the effect that we look more closely at the [ABG] proof of this result. (In a very preliminary version of this paper, we had assumed the proof in [ABG] was sufficient in the quantum case, and even that it applied to the modular case.) We also thank S. Riche for pointing out an error in our first naive attempt at a correction.

The result Lemma 3.2(ii) below is actually quite strong, in either the algebraic groups or quantum case, and gives, in the regular weight case, a categorification of Rickard's theorem [R94, Thm. 2,1] on derived equivalences arising from translations. (Essentially, the latter theorem gives Lemma 3.2(i), when the derived equivalences involved in the statement of the theorem are identified in its proof. But the theorem only claims a version of Lemma 3.2(ii) at a character-theoretic level.)

We introduce the lemma by observing, as in the discussion above [ABG, Lem. 4.1.1] that translation functors may constructed as in the algebraic groups case. This is carried out in [APW, §9], though with the explicit assumption that ℓ be a prime power. This may be removed by appealing to Theorem 3 above. It is easy to check that the resulting constructions have the familiar adjointness and exactness properties of the algebraic groups case [J, II, Lem. 7.6]. Continuing the discussion in [ABG], let $\Xi_\alpha : \mathit{block}(\mathbb{U}) \longrightarrow \mathit{block}(\mathbb{U})$ denote a composition of translation functors first 'to the wall' labelled associated to a simple reflection s_α and, then back 'out of the wall'. There are canonical adjunction morphisms $f : id \longrightarrow \Xi_\alpha$, and $g : \Xi_\alpha \longrightarrow id$. It is noted in [ABG] that the mapping cone, $C(f)$, of f gives rise to a triangulated functor from $D^b\mathit{block}(\mathbb{U})$ to itself, denoted θ^+_α. A similar construction (using $C(g)[-1]$) gives a triangulated functor θ^-_α. These constructions carry over easily to the algebraic groups case. The statement below is [ABG, Lem. 4.1.1] in both the quantum and algebraic groups cases.

Lemma 3.2. *With the notation discussed above, we have in both the quantum and algebric groups cases:*

(i) In $D^b block(\mathbb{U})$ we have the following canonical isomorphisms $\theta^+_\alpha \circ \theta^-_\alpha \cong id$ and $\theta^-_\alpha \circ \theta^+_\alpha \cong id$. In particular θ^+_α and θ^-_α are autoequivalences.

(ii) If $\lambda \in W_{aff} \cdot 0$ and $\lambda^{s_\alpha} > \lambda$ then $\theta^+_\alpha(RInd_{\mathbb{B}}^{\mathbb{U}} \lambda) \cong RInd_{\mathbb{B}}^{\mathbb{U}}(\lambda^{s_\alpha})$.

Remark 3.3. As discussed above, the proof of part (ii) is given in appendices A and B. The proof of part (i) may be obtained from the argument for [R94, Thm. 2.1], or, alternately, from the argument for [ABG, Lem. 4.1.1(i)]. (Both arguments involve similar ingredients.) We mention that [ABG] defines both θ^+ and θ^- as mapping cones. The reader should be aware that the natural definition of θ^- is as a shifted mapping cone, as in the description given here, to obtain property (i).

Lemma 3.4. *For any $\lambda, \mu \in R$ and $n \geq 0$,*

(3.4.1)
$$\dim(\mathrm{Ext}^n_{\mathbb{B}}(k_{\mathbb{B}}(\ell\lambda), k_{\mathbb{B}}(\ell\mu))) = \dim(\mathrm{Ext}^n_{block(\mathbb{U})}(RInd_{\mathbb{B}}^{\mathbb{U}} k_{\mathbb{B}}(\ell\lambda), RInd_{\mathbb{B}}^{\mathbb{U}} k_{\mathbb{B}}(\ell\mu)))$$

Proof. The proof follows [ABG, proof of Lem. 4.2.2]. We include some details for completeness. First, the identity

$$\begin{aligned}\mathrm{Hom}_{\mathbb{B}}(k_{\mathbb{B}}(0), M) &\overset{adjunction}{=} \mathrm{Hom}_{\mathbb{U}}(k_{\mathbb{U}}(0), \mathrm{RInd}_{\mathbb{B}}^{\mathbb{U}} M) \\ &\overset{BWB}{=} \mathrm{Hom}_{\mathbb{U}}(\mathrm{RInd}_{\mathbb{B}}^{\mathbb{U}} k_{\mathbb{B}}(0), \mathrm{RInd}_{\mathbb{B}}^{\mathbb{U}} M)\end{aligned}$$

is established, using Borel Weil Bott (BWB) type theory. In fact, Corollary 2.6 is sufficient in the quantum case, and the better known algebraic groups case of that corollary is discussed just above it. To summarize, the identity above holds in both cases, for (at least) any object M in $D_{triv}(\mathbb{B})$. This proves the lemma in the special case $\lambda = 0$ and arbitrary $\mu \in \mathbb{Y}$.

The general case will be reduced to the special case by means of translation functors. For any $\lambda, \mu \in \mathbb{Y}$ and $\nu \in \mathbb{Y}^+$, we claim

$$(*) \qquad \begin{aligned}&\mathrm{RHom}_{block(\mathbb{U})}(\mathrm{RInd}_{\mathbb{B}}^{\mathbb{U}} k_{\mathbb{B}}(\ell\lambda), \mathrm{RInd}_{\mathbb{B}}^{\mathbb{U}} k_{\mathbb{B}}(\ell\mu)) \\ &\quad \cong \mathrm{RHom}_{block(\mathbb{U})}(\mathrm{RInd}_{\mathbb{B}}^{\mathbb{U}} k_{\mathbb{B}}(\ell\lambda + \ell\nu), \mathrm{RInd}_{\mathbb{B}}^{\mathbb{U}} k_{\mathbb{B}}(\ell\mu + \ell\nu)).\end{aligned}$$

Let $\nu = s_{\alpha_1} s_{\alpha_2} \cdots s_{\alpha_r} \in \mathbb{Y} \subset W_{aff}$ be a reduced expression. Then $\ell\nu = 0^{s_{\alpha_1} s_{\alpha_2} \cdots s_{\alpha_r}}$ and hence

$$\ell\lambda + \ell\nu = (\ell\lambda)^{s_{\alpha_1} s_{\alpha_2} \cdots s_{\alpha_r}} > (\ell\lambda)^{s_{\alpha_1} s_{\alpha_2} \cdots s_{\alpha_{r-1}}} > \cdots > \ell\lambda.$$

Now the repeated use of the second part of Lemma 3.2 give us $\mathrm{RInd}_{\mathbb{B}}^{\mathbb{U}} k_{\mathbb{B}}(\ell\lambda + \ell\nu) \cong \theta^+_{\alpha_r} \circ \theta^+_{\alpha_{r-1}} \circ \cdots \circ \theta^+_{\alpha_1}(\mathrm{RInd}_{\mathbb{B}}^{\mathbb{U}} k_{\mathbb{B}}(\ell\lambda))$. This together with the first property of Lemma 3.2 gives (*).

For any λ, $\mu \in \mathbb{Y}$, choose a large $\nu \in \mathbb{Y}^+$ such that $\nu - \lambda \in \mathbb{Y}^+$. Using $\nu - \lambda$ in place of ν in (*) (i.e. a shift by $\nu - \lambda$), we get

$$(**)\quad \begin{aligned} &\mathrm{RHom}_{\mathit{block}(\mathbb{U})}(\mathrm{RInd}_{\mathbb{B}}^{\mathbb{U}}\, k_{\mathbb{B}}(\ell\lambda), \mathrm{RInd}_{\mathbb{B}}^{\mathbb{U}}\, k_{\mathbb{B}}(\ell\mu)) \\ &\quad \cong \mathrm{RHom}_{\mathit{block}(\mathbb{U})}(\mathrm{RInd}_{\mathbb{B}}^{\mathbb{U}}\, k_{\mathbb{B}}(\ell\nu), \mathrm{RInd}_{\mathbb{B}}^{\mathbb{U}}\, k_{\mathbb{B}}(\ell\mu + \ell\nu - \ell\lambda)). \end{aligned}$$

Again, a shift by ν gives

$$(**)\quad \begin{aligned} &\mathrm{RHom}_{\mathit{block}(U)}(\mathrm{RInd}_{\mathbb{B}}^{\mathbb{U}}\, k_{\mathbb{B}}(0), \mathrm{RInd}_{\mathbb{B}}^{\mathbb{U}}\, k_{\mathbb{B}}(\ell\mu - \ell\lambda)) \\ &\quad \cong \mathrm{RHom}_{\mathit{block}(\mathbb{U})}(\mathrm{RInd}_{\mathbb{B}}^{\mathbb{U}}\, k_{\mathbb{B}}(\ell\nu), \mathrm{RInd}_{\mathbb{B}}^{\mathbb{U}}\, k_{\mathbb{B}}(\ell\mu + \ell\nu - \ell\lambda)). \end{aligned}$$

Notice the right hand terms of the two isomorphisms labeled (**) are the same, so that we can view (**) as providing isomorphisms of three expressions, all obtained by applying $R\mathrm{Hom}_{\mathit{block}(\mathbb{U})}$. Passing, for any fixed i, to $R^i\mathrm{Hom}_{\mathit{block}(\mathbb{U})}$ expressions, we obtain three vector spaces of the same dimension. One of these vectors spaces $R^i\mathrm{Hom}_{\mathit{block}(U)}(\mathrm{RInd}_{\mathbb{B}}^{\mathbb{U}}\, k_{\mathbb{B}}(0), \mathrm{RInd}_{\mathbb{B}}^{\mathbb{U}}\, k_{\mathbb{B}}(\ell\mu - \ell\lambda))$ is isomorphic to $R^i\mathrm{Hom}_{\mathbb{B}}(k_{\mathbb{B}}(0), k_{\mathbb{B}}(\ell\mu - \ell\lambda))$, by the special case above with $(\ell\mu - \ell\lambda))$ used in the role of $\ell\mu$. Finally, using the isomorphism

$$\mathrm{RHom}_{\mathbb{B}}(k_{\mathbb{B}}(0), k_{\mathbb{B}}(\ell\mu - \ell\lambda)) \cong \mathrm{RHom}_{\mathbb{B}}(k_{\mathbb{B}}(\ell\lambda), k_{\mathbb{B}}(\ell\mu))$$

we complete the proof of the lemma. □

Observe that for any finite-dimensional G-module V, with weights in $\mathbb{Y}$,

$$\theta_\alpha^+(M \otimes V^{[1]}) = \theta_\alpha^+(M) \otimes V^{[1]}.$$

Thus, using the proof of Step 1 and recalling the generalized tensor identity (see Remark 2.11(ii), tensoring on the right the first (resp., second) component of each term appearing in the equality given by Lemma 3.4, by any finite-dimensional twisted G-module $M^{[1]}$ (resp., $N^{[1]}$) with weights in $\mathbb{Y}$, preserves the equality of dimensions:

$$(3.4.2)\quad \begin{aligned} &\dim(\mathrm{Ext}^n_{\mathbb{B}}(M^{[1]}|_{\mathbb{B}} \otimes k_{\mathbb{B}}(\ell\lambda), k_{\mathbb{B}}(\ell\mu))) \\ &\quad = \dim(\mathrm{Ext}^n_{\mathit{block}(\mathbb{U})}(M^{[1]} \otimes \mathrm{RInd}_{\mathbb{B}}^{\mathbb{U}}\, k_{\mathbb{B}}(\ell\lambda), \mathrm{RInd}_{\mathbb{B}}^{\mathbb{U}}\, k_{\mathbb{B}}(\ell\mu))), \end{aligned}$$

and

$$\begin{aligned} &\dim(\mathrm{Ext}^n_{\mathbb{B}}(k_{\mathbb{B}}(\ell\lambda), N^{[1]}|_{\mathbb{B}} \otimes k_{\mathbb{B}}(\ell\mu))) \\ &\quad = \dim(\mathrm{Ext}^n_{\mathit{block}(\mathbb{U})}(\mathrm{RInd}_{\mathbb{B}}^{\mathbb{U}}\, k_{\mathbb{B}}(\ell\lambda), N^{[1]} \otimes \mathrm{RInd}_{\mathbb{B}}^{\mathbb{U}}\, k_{\mathbb{B}}(\ell\mu))). \end{aligned}$$

3.3. Step 2 of the proof of the induction theorems

Lemma 3.5. *Let λ be any element of $\mathbb{Y}$ (or $\mathbb{X}$). Choose $\nu = N\rho$, with ρ as in Section 2.4.2, and with $N \in \mathbb{N}$ large enough so that $\ell\tau := \ell(\nu + \lambda)$ is dominant. Let V_ν be as in (2.12.1). Then the $\mathbb{B}$-module $M = V_\nu^{[1]} \otimes k_{\mathbb{B}}(\ell\tau)$ satisfies the following three properties:*

1. *$k_{\mathbb{B}}(\ell\lambda) \subset M$.*
2. *All composition factors of $M/k_{\mathbb{B}}(\ell\lambda)$ have the form $k_{\mathbb{B}}(\ell\eta))$ with $\eta > \lambda$ in the dominance order.*
3. *The map $RInd_{\mathbb{B}}^{\mathbb{U}}(M) \to M$ is $\mathbb{p}$-split.*

Proof. Once again, the argument uses ideas from [ABG], especially in the analysis of the map in part (3).

Note $M \cong M_0^{[1]}$, where M_0 is the B'-module $V_\nu \otimes k_{B'}(\tau)$. The B'-socle of V_ν is $k_{B'}(-\nu)$, so $k_{B'}(\lambda) \cong k_{B'}(-\nu) \otimes k_{B'}(\tau)$ is the B'-socle of M_0. Parts 1) and 2) of the lemma for M follow immediately from corresponding properties of M_0.

Next, put $F_0 = \mathrm{Ind}_{\mathbb{B}}^{\mathbb{U}}(M_0) \cong V_\nu \otimes \mathrm{Ind}_{\mathbb{B}}^{\mathbb{U}}(k_{\mathbb{B}}(\tau))$ and $F = F_0^{[1]}$. The natural $\mathbb{B}$-map $\varphi_0 : F_0 \to M_0$ is surjective, as follows from the surjectivity of $\mathrm{Ind}_{\mathbb{B}}^{\mathbb{U}}(k_{\mathbb{B}}(\tau) \to k_{\mathbb{B}}(\tau)$.

Consequently, the Frobenius twisted map $\varphi : F \to M$ is surjective. It is also $\mathbb{p}$-split, since both domain and range are completely reducible as $\mathbb{p}$-modules. The map φ gives rise (by adjointness) to a map $F \to \mathrm{RInd}_{\mathbb{B}}^{\mathbb{U}}(M)$ which, when composed with the natural map $\mathrm{RInd}_{\mathbb{B}}^{\mathbb{U}}(M) \to M$, is the $\mathbb{p}$-split surjection φ. Consequently, $\mathrm{RInd}_{\mathbb{B}}^{\mathbb{U}}(M) \to M$ is also $\mathbb{p}$-split. This proves Property (3) and completes the proof of Lemma 3.5. □

Our arguments now begin to diverge from [ABG].

Lemma 3.6. *Suppose $\mu \in \mathbb{Y}$ and Y is a finite-dimensional $\mathbb{B}$-module all of whose composition factors are of the form factors $k_{\mathbb{B}}(\ell\lambda)$ with $\lambda \in \mathbb{Y}$). Then for all nonnegative integers n, and any μ,*

$$\mathrm{Ext}^n_{\mathbb{B}}(Y, I_\mu^{[1]}) \cong \mathrm{Ext}^n_{\mathbb{U}}(RInd_{\mathbb{B}}^{\mathbb{U}}(Y), RInd_{\mathbb{B}}^{\mathbb{U}}(I_\mu^{[1]})). \tag{3.6.1}$$

Proof. First, observe that (3.6.1) is true for n odd, with both sides zero, by Corollary 2.15(1). This greatly simplifies long exact sequence arguments in the remaining n even cases. Now fix n even. Then (3.6.1) is equivalent to the case where Y is one-dimensional. (In fact, for any given Y, (3.6.1) is implied by the corresponding results for each of its composition factors.)

Next, observe in the one-dimensional case, that it is sufficient to check injectivity of the left-to-right map implicit in (3.6.1). This is a consequence of Corollary 2.15(1) and the dimensional equalities (3.4.2). In fact, (3.6.1) will be an isomorphism for any one-dimensional Y and μ for which it is an injection or for any Y and μ (satisfying the hypotheses of the lemma) for which (3.6.1) is an injection on each composition factor of Y.

We are now in a position to treat the one-dimensional case $Y = k_{\mathbb{B}}(\ell\lambda)$, for our fixed even n, by downward induction on the height of λ. Note that (3.6.1) is true (with both sides zero) for λ sufficiently large, by Corollary 2.15(2). We may, hence, assume inductively that (3.6.1) holds for $Y = k_{\mathbb{B}}(\ell\eta)$ with η of larger height than λ, or, more generally for all finite-dimensional Y with composition factors satisfying this height condition.

Let M be the $\mathbb{B}$-module guaranteed by Lemma 3.5. Let N be the cokernel of the $\mathbb{B}$-module inclusion $\mathrm{k}_{\mathbb{B}}(\ell\lambda) \hookrightarrow M$. By our height induction, there is an isomorphism (3.6.1) for $Y = N$ and our fixed even n.

Also, the $\mathbb{p}$-split map $\mathrm{RInd}_{\mathbb{B}}^{\mathbb{U}}(M) \to M$ from Lemma (3.5) gives an injection

$$\mathrm{Ext}^{\,n}_{\mathbb{p}}(M, k_{\mathbb{p}}(\ell\mu)) \hookrightarrow \mathrm{Ext}^{\,n}_{\mathbb{p}}(\mathrm{RInd}_{\mathbb{B}}^{\mathbb{U}}(M), k_{\mathbb{p}}(\ell\mu)),$$

or, equivalently,

$$\begin{aligned} \mathrm{Ext}^n_{\mathbb{B}}(M, I_\mu^{[1]}) &\hookrightarrow \mathrm{Ext}^n_{\mathbb{B}}(\mathrm{RInd}_{\mathbb{B}}^{\mathbb{U}}(M), I_\mu^{[1]}) \\ &\cong \mathrm{Ext}^n_{\mathbb{U}}(\mathrm{RInd}_{\mathbb{B}}^{\mathbb{U}}(M), \mathrm{RInd}_{\mathbb{B}}^{\mathbb{U}}(I_\mu^{[1]})). \end{aligned} \tag{3.6.2}$$

Likewise, the adjunction map $\mathrm{RInd}_{\mathbb{B}}^{\mathbb{U}}(\mathrm{k}_{\mathbb{B}}(\ell\lambda)) \to \mathrm{k}_{\mathbb{B}}(\ell\lambda)$ gives a morphism

$$\gamma : \mathrm{Ext}^n_{\mathbb{B}}(\mathrm{k}_{\mathbb{B}}(\ell\lambda), I_\mu^{[1]}) \to \mathrm{Ext}^n_{\mathbb{U}}(\mathrm{RInd}_{\mathbb{B}}^{\mathbb{U}}(\mathrm{k}_{\mathbb{B}}(\ell\lambda)), \mathrm{RInd}_{\mathbb{B}}^{\mathbb{U}}(I_\mu^{[1]}). \tag{3.6.3}$$

Let

$$\begin{aligned} A &= \mathrm{Ext}^n_{\mathbb{B}}(N, I_\mu^{[1]}), & A' &= \mathrm{Ext}^{n+1}_{\mathbb{B}}(N, I_\mu^{[1]}) = 0, \\ \widetilde{A} &= \mathrm{Ext}^n_{\mathbb{U}}(\mathrm{RInd}_{\mathbb{B}}^{\mathbb{U}}(N), \mathrm{RInd}_{\mathbb{B}}^{\mathbb{U}}(I_\mu^{[1]})), & \widetilde{A}' &= \mathrm{Ext}^{n+1}_{\mathbb{U}}(\mathrm{RInd}_{\mathbb{B}}^{\mathbb{U}}(N), \mathrm{RInd}_{\mathbb{B}}^{\mathbb{U}}(I_\mu^{[1]})) = 0, \\ B &= \mathrm{Ext}^n_{\mathbb{B}}(M, I_\mu^{[1]}), & \widetilde{B} &= \mathrm{Ext}^n_{\mathbb{U}}(\mathrm{RInd}_{\mathbb{B}}^{\mathbb{U}}(M), \mathrm{RInd}_{\mathbb{B}}^{\mathbb{U}}(I_\mu^{[1]})), \\ C &= \mathrm{Ext}^n_{\mathbb{B}}(\mathrm{k}_{\mathbb{B}}(\ell\lambda), I_\mu^{[1]}), & \widetilde{C} &= \mathrm{Ext}^n_{\mathbb{U}}(\mathrm{Ind}_{\mathbb{B}}^{\mathbb{U}}(\mathrm{k}_{\mathbb{B}}(\ell\lambda)), \mathrm{RInd}_{\mathbb{B}}^{\mathbb{U}}(I_\mu^{[1]})) \end{aligned}$$

Then the $\mathbb{B}$-module exact sequence $0 \to \mathrm{k}_{\mathbb{B}}(\ell\lambda) \to M \to N \to 0$ gives rise to a commutative diagram

$$\begin{array}{ccccccc} A & \xrightarrow{u} & B & \xrightarrow{v} & C & \xrightarrow{w} & A' = 0 \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \\ \widetilde{A} & \xrightarrow{\widetilde{u}} & \widetilde{B} & \xrightarrow{\widetilde{v}} & \widetilde{C} & \xrightarrow{\widetilde{w}} & \widetilde{A}' = 0 \end{array}$$

with γ the morphism in (3.6.3), α the isomorphism given by the induction argument so far, and β an injection given by (3.6.2). Both rows are exact. By a standard diagram chase, these conditions force γ to be an injection. As discussed above, this implies γ is an isomorphism, and completes the induction for our fixed n. Since n was an arbitrary even nonnegative integer, and since the odd case has already been handled, the proof of the lemma is complete. □

To be clear: As a consequence of Lemma 3.6, we immediately obtain the isomorphisms (3.1.3). using $Y = k_{\mathbb{B}}(\ell\lambda)$. This completes Step 2.

3.4. Step 3 of the proof of the induction theorems

Recall that by Lemma 3.1, it suffices to establish that there are isomorphisms (3.1.1):

$$\mathrm{Ext}\,^n_{\mathbb{B}}(k_{\mathbb{B}}(\ell\lambda), k_{\mathbb{B}}(\ell\mu)) \cong \mathrm{Ext}\,^n_{block(\mathbb{U})}(\mathrm{RInd}_{\mathbb{B}}^{\mathbb{U}}\, k_{\mathbb{B}}(\ell\lambda), \mathrm{RInd}_{\mathbb{B}}^{\mathbb{U}}\, k_{\mathbb{B}}(\ell\mu))$$

arising from the application of $\mathrm{RInd}_{\mathbb{B}}^{\mathbb{U}}$.

It suffices to fix throughout an otherwise arbitrary weight $\lambda \in \mathbb{Y}$. For a given $\mu \in \mathbb{Y}$, all nonnegative integers n will be treated simultaneously. As a notational convenience, all $\mathrm{Ext}\,^m_{\mathbb{B}}$-groups with a negative index m are equal to zero by definition.

Starting from §3.3, we have isomorphisms (3.1.3) (arising, as noted, from the functoriality of $\mathrm{RInd}_{\mathbb{B}}^{U}$), and wish to pass to analogous isomorphisms with $k_{\mathbb{B}}(\ell\mu)$ in place of the terms $I_\mu^{[1]}$ appearing in (3.1.3).

To begin, note that "twisting" the canonical $\mathbb{B}$-module injection $\mu \hookrightarrow I_\mu$ of $k_{\mathbb{B}}(\mu)$ into its injective hull I_μ leads to a s.e.s.

$$0 \longrightarrow k_{\mathbb{B}}(\ell\mu) \longrightarrow I_\mu^{[1]} \longrightarrow \frac{I_\mu^{[1]}}{k_{\mathbb{B}}(\ell\mu)} =: \Sigma_\mu \longrightarrow 0, \tag{3.6.4}$$

wherein the module $I_\mu^{[1]}$ has $\mathrm{k}_{\mathbb{B}}\,(\ell\mu)$ as its socle, and the $\mathbb{B}$-composition factors of Σ_μ have the form $\mathrm{k}_{\mathbb{B}}\,(\ell\tau)$, with $ht(\tau) > ht(\mu)$, using the usual height function. Set $ht_\lambda(\Sigma_\mu)$ to be sum of all submodules M of Σ_μ for which $ht(\eta) \leq$

$ht(\lambda)$ whenever $\ell\eta$ is a composition factor of M. Then $ht_\lambda(\Sigma_\mu)$ is the largest submodule of Σ_μ with this property. In the corresponding s.e.s.

$$0 \longrightarrow ht_\lambda(\Sigma_\mu) \longrightarrow \Sigma_\mu \longrightarrow \frac{\Sigma_\mu}{ht_\lambda(\Sigma_\mu)} \longrightarrow 0, \tag{3.6.5}$$

one has $ht_\lambda(\frac{\Sigma_\mu}{ht_\lambda(\Sigma_\mu)}) = 0$.

Since $\operatorname{Ext}^n_{\mathbb{B}}(k_{\mathbb{B}}(\ell\zeta), k_{\mathbb{B}}(\ell\eta)) \neq 0$ implies $\zeta \geq \eta$, one has, for any weights $\zeta, \eta \in \mathbb{X}$,

$$\operatorname{Ext}^n_{\mathbb{B}}(k_{\mathbb{B}}(\ell\zeta), k_{\mathbb{B}}(\ell\eta)) = 0 \text{ if } ht(\eta) > ht(\zeta). \tag{3.6.6}$$

Thus, from (3.6.6), it follows that for any finite-dimensional submodule $N \subset \frac{\Sigma_\mu}{ht_\lambda(\Sigma_\mu)}$,

$$\operatorname{Ext}^n_{\mathbb{B}}(k_{\mathbb{B}}(\ell\lambda), N) = 0 \quad \forall n \geq 0 \tag{3.6.7}$$

Moreover, since $k_{\mathbb{B}}(\ell\lambda)$ is finite dimensional, $\operatorname{Ext}^n_{\mathbb{B}}(\mathrm{k}_{\mathbb{B}}(\ell\lambda), -)$ commutes with direct limits. Since I_μ is a direct limit of its finite dimensional submodules, so is Σ_μ, and also $\frac{\Sigma_\mu}{ht_\lambda(\Sigma_\mu)}$; consequently, the vanishing property (3.6.7) yields the vanishing results

$$\operatorname{Ext}^n_{\mathbb{B}}\left(k_{\mathbb{B}}(\ell\lambda), \frac{\Sigma_\mu}{ht_\lambda(\Sigma_\mu)}\right) = 0 \quad \forall n \geq 0. \tag{3.6.8}$$

The s.e.s. (3.6.5) and the vanishing results (3.6.8) yield the l.e.s.

$$\begin{gathered} \cdots \longrightarrow \operatorname{Ext}^{n-1}_{\mathbb{B}}(k_{\mathbb{B}}(\ell\lambda), ht_\lambda(\Sigma_\mu)) \longrightarrow \operatorname{Ext}^{n-1}_{\mathbb{B}}(k_{\mathbb{B}}(\ell\lambda), \Sigma_\mu) \longrightarrow \\ \operatorname{Ext}^{n-1}_{\mathbb{B}}(k_{\mathbb{B}}(\ell\lambda), \tfrac{\Sigma_\mu}{ht_\lambda(\Sigma_\mu)}) = 0 \\ \longrightarrow \operatorname{Ext}^{n}_{\mathbb{B}}(k_{\mathbb{B}}(\ell\lambda), ht_\lambda(\Sigma_\mu)) \longrightarrow \operatorname{Ext}^{n}_{\mathbb{B}}(k_{\mathbb{B}}(\ell\lambda), \Sigma_\mu) \longrightarrow \\ \operatorname{Ext}^{n}_{\mathbb{B}}(k_{\mathbb{B}}(\ell\lambda), \tfrac{\Sigma_\mu}{ht_\lambda(\Sigma_\mu)}) = 0 \longrightarrow \cdots \end{gathered}$$

whence isomorphisms

$$\operatorname{Ext}^n_{\mathbb{B}}(k_{\mathbb{B}}(\ell\lambda), ht_\lambda(\Sigma_\mu)) \xrightarrow{\cong} \operatorname{Ext}^n_{\mathbb{B}}(k_{\mathbb{B}}(\ell\lambda), \Sigma_\mu), \quad \forall n \geq 0. \tag{3.6.9}$$

Consider now the distinguished triangle obtained from (3.6.5) under $\mathrm{RInd}_{\mathbb{B}}^{\mathbb{U}}$:

(3.6.10)

$$\cdots \longrightarrow \mathrm{RInd}_{\mathbb{B}}^{\mathbb{U}}(ht_\lambda(\Sigma_\mu)) \longrightarrow \mathrm{RInd}_{\mathbb{B}}^{\mathbb{U}}(\Sigma_\mu) \longrightarrow \mathrm{RInd}_{\mathbb{B}}^{\mathbb{U}}\left(\frac{\Sigma_\mu}{ht_\lambda(\Sigma_\mu)}\right) \longrightarrow \cdots .$$

Since the functor $\mathrm{RInd}_{\mathbb{B}}^{\mathbb{U}}$ commutes with taking direct limits, $\mathrm{RInd}_{\mathbb{B}}^{\mathbb{U}}(\Sigma_\mu/ht_\lambda(\Sigma_\mu) = \underrightarrow{\lim}\mathrm{RInd}_{\mathbb{B}}^{\mathbb{U}}(N)$ over finite dimensional submodules $N \subset \Sigma_\mu/ht_\lambda(\Sigma_\mu)$. Thus, if we can replicate the following vanishing result, analogous to (3.6.7):

$$\text{(3.6.11)} \qquad \mathrm{Ext}^{\,n}_{block(\mathbb{U})}(\mathrm{RInd}_{\mathbb{B}}^{\mathbb{U}}(k_{\mathbb{B}}(\ell\lambda)), \mathrm{RInd}_{\mathbb{B}}^{\mathbb{U}}(N)) = 0 \quad \forall n \geq 0,$$

then, from the preceding arguments, mutatis mutandis, we will obtain the following isomorphisms:

$$\text{(3.6.12)} \qquad \begin{aligned} &\mathrm{Ext}^{\,n}_{\mathbb{B}}(\mathrm{RInd}_{\mathbb{B}}^{\mathbb{U}}(k_{\mathbb{B}}(\ell\lambda)), \mathrm{RInd}_{\mathbb{B}}^{\mathbb{U}}(ht_\lambda(\Sigma_\mu))) \\ &\quad \xrightarrow{\cong} \mathrm{Ext}^{\,n}_{\mathbb{B}}(\mathrm{RInd}_{\mathbb{B}}^{\mathbb{U}}(k_{\mathbb{B}}(\ell\lambda)), \mathrm{RInd}_{\mathbb{B}}^{\mathbb{U}}(\Sigma_\mu)), \quad \forall n \geq 0. \end{aligned}$$

Recall that the key step in producing the isomorphisms (3.6.7) was the earlier vanishing result (3.6.6), but now from (3.6.6) and the dimension equality (3.1.2),

$$\text{(3.6.13)} \quad \mathrm{Ext}^{\,n}_{block(\mathbb{U})}(\mathrm{RInd}_{\mathbb{B}}^{\mathbb{U}}(k_{\mathbb{B}}(\ell\zeta)), \mathrm{RInd}_{\mathbb{B}}^{\mathbb{U}}(k_{\mathbb{B}}(\ell\eta))) = 0 \text{ if } ht(\eta) > ht(\zeta).$$

By applying this vanishing result of (3.6.13) to composition factors of N, (3.6.11), and hence (3.6.12), do indeed follow as claimed.

We now carry out a descending induction on $ht(\mu)$. Assume for all weights η for which $ht(\eta) > ht(\mu)$, $\mathrm{RInd}_{\mathbb{B}}^{\mathbb{U}}$ induces, for all n, isomorphisms

$$\text{(3.6.14)} \quad \mathrm{Ext}^{\,n}_{\mathbb{B}}(k_{\mathbb{B}}(\ell\lambda), k_{\mathbb{B}}(\ell\eta)) \cong \mathrm{Ext}^{\,n}_{block(\mathbb{U})}(\mathrm{RInd}_{\mathbb{B}}^{\mathbb{U}}k_{\mathbb{B}}(\ell\lambda), \mathrm{RInd}_{\mathbb{B}}^{\mathbb{U}}k_{\mathbb{B}}(\ell\eta)).$$

Then by the definition of $ht_\lambda(\Sigma_\mu)$ and its finite dimensionality, it follows from (3.6.14) that

(3.6.15)

$$\mathrm{Ext}^{\,n}_{\mathbb{B}}(k_{\mathbb{B}}(\ell\lambda), ht_\lambda(\Sigma_\mu)) \cong \mathrm{Ext}^{\,n}_{block(\mathbb{U})}(\mathrm{RInd}_{\mathbb{B}}^{\mathbb{U}}k_{\mathbb{B}}(\ell\lambda), \mathrm{RInd}_{\mathbb{B}}^{\mathbb{U}}(ht_\lambda(\Sigma_\mu)).$$

From the s.e.s. (3.6.4), upon letting $\widehat{V}$ denote $\mathrm{RInd}_{\mathbb{B}}^{\mathbb{U}}(V)$, and $b(\mathbb{U})$ denote $block(\mathbb{U})$, we obtain two l.e.s.s tied together:

$$\begin{array}{ccccccccccc} \cdots\to & \mathrm{Ext}^{\,n-1}_{\mathbb{B}}(k_{\mathbb{B}}(\ell\lambda), I^{[1]}_\mu) & \to & \mathrm{Ext}^{\,n-1}_{\mathbb{B}}(k_{\mathbb{B}}(\ell\lambda), \Sigma_\mu) & \to & \mathrm{Ext}^{\,n}_{\mathbb{B}}(k_{\mathbb{B}}(\ell\lambda), k_{\mathbb{B}}(\ell\mu)) & \to & \mathrm{Ext}^{\,n}_{\mathbb{B}}(k_{\mathbb{B}}(\ell\lambda), I^{[1]}_\mu) & \to & \mathrm{Ext}^{\,n}_{\mathbb{B}}(k_{\mathbb{B}}(\ell\lambda), \Sigma_\mu) & \to\cdots \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ \cdots\to & \mathrm{Ext}^{\,n-1}_{b(\mathbb{U})}(\widehat{k_{\mathbb{B}}(\ell\lambda)}, \widehat{I^{[1]}_\mu}) & \to & \mathrm{Ext}^{\,n-1}_{b(\mathbb{U})}(\widehat{k_{\mathbb{B}}(\ell\lambda)}, \widehat{\Sigma_\mu}) & \to & \mathrm{Ext}^{\,n}_{b(\mathbb{U})}(\widehat{k_{\mathbb{B}}(\ell\lambda)}, \widehat{k_{\mathbb{B}}(\ell\mu)}) & \to & \mathrm{Ext}^{\,n}_{b(\mathbb{U})}(\widehat{k_{\mathbb{B}}(\ell\lambda)}, \widehat{I^{[1]}_\mu}) & \to & \mathrm{Ext}^{\,n}_{b(\mathbb{U})}(\widehat{k_{\mathbb{B}}(\ell\lambda)}, \widehat{\Sigma_\mu}) & \to\cdots \end{array}$$

In the above diagram, all vertical morphisms arise from the functoriality of $\mathrm{RInd}_{\mathbb{B}}^{\mathbb{U}}$. The first and fourth vertical maps shown are isomorphisms, as given by Lemma 3.6. By (3.6.9) and (3.6.12), Σ_μ in the second and fifth vertical morphisms shown can be replaced with $ht_\lambda(\Sigma_\mu)$, and the resulting morphisms in the second and fifth vertical spots are isomorphisms. By the Five Lemma the third vertical morphism is an isomorphism, i.e., $\mathrm{RInd}_{\mathbb{B}}^{\mathbb{U}}$ determines

$$\mathrm{Ext}\,^n_{\mathbb{B}}(k_{\mathbb{B}}(\ell\lambda), k_{\mathbb{B}}(\ell\mu)) \cong \mathrm{Ext}\,^n_{block(\mathbb{U})}(\mathrm{RInd}_{\mathbb{B}}^{\mathbb{U}}\, k_{\mathbb{B}}(\ell\lambda), \mathrm{RInd}_{\mathbb{B}}^{\mathbb{U}}\, k_{\mathbb{B}}(\ell\mu)),$$

for each $n \geq 0$, as desired. This completes the proof of Step 3, and, consequently of both induction Theorems 1 and 2.

3.5. Summary, and comparison with the approach in [ABG]

Although a natural approach, [ABG] were unable to use $S := \{\mathrm{k}_{\mathbb{B}}\,(\ell\lambda) \,|\, \lambda \in R\}$ directly as as set of generators for the triangle category equivalence tool given by Theorem 3.1. This roadblock apparently motivated their attempt to use the set $S' := \{I_\lambda^{[1]} \,|\, \lambda \in R\}$ in place of S. (See [ABG, Remark 4.2.7].) However, the corresponding claim in [ABG, Lem. 4.3.6] that S' (equivalently the set $\{\mathrm{Ind}_{\mathbb{P}}^{\mathbb{B}}\,(\ell\lambda) \,|\, \lambda \in R\}$) is inaccurate, since these modules do not actually lie in $D_{triv}(\mathbb{B})$. Nevertheless, in the characteristic 0 setting of [ABG], it is true that $D_{triv}(\mathbb{B})$ is contained in the triangulated category generated by S', so that a line of argument establishing isomorphisms

$$\mathrm{Ext}\,^n_{\mathbb{B}}(I_\lambda^{[1]}, I_\mu^{[1]}) \cong \mathrm{Ext}\,^n_{block(\mathbb{U})}(\mathrm{RInd}_{\mathbb{B}}^{\mathbb{U}}\,(I_\lambda^{[1]}), \mathrm{RInd}_{\mathbb{B}}^{\mathbb{U}}\,(I_\mu^{[1]})), \tag{3.6.16}$$

as pursued in [ABG, Lem. 4.3.6] *would* imply the existence of isomorphisms (3.1.1). Unfortunately, there is no such inclusion of $D_{triv}(\mathbb{B})$ in characteristic $p > 0$. In particular, the first line of the proof of [ABG, Lem. 4.3.6], asserting that the universal enveloping algebra $\mathcal{U}\mathfrak{n}$ (for $\mathfrak{n}$ a nilpotent Lie algebra in a triangular decomposition) has finite global dimension is not true for the correctly analogous characteristic p situation. It is a question of what modules are to be pulled back under the Frobenius morphism. In the characteristic p situation, it is necessary to use modules for the distribution algebra of a positive characteristic unipotent algebraic group, not its unrestricted enveloping algebra, and the finite global dimension property is lost. Overcoming this obstacle, while using much of the apparatus of [ABG], is not trivial, and our proof eventually involves parity properties for $\mathbb{b}$-cohomology [AJ, Prop. 2.3]. See above Corollary 2.15 and the proof of Lemma 3.6, which also present our argument in the quantum case.

4. Appendix A

The discussion below, in the algebraic groups case, is based on Jantzen's book [J, pp. 258–259]. We follow the notations there. Comments on the quantum case are given in Remark 4.3, to which the reader might look ahead, now. In this appendix and the next we will provide a proof of Lemma 3.2(ii). A closely linked goal is to understand the adjunction map $id \circ \mathrm{RInd}_B^G \longrightarrow T_\mu^\lambda \circ T_\lambda^\mu \circ \mathrm{RInd}_B^G$ in the spirit of the long exact sequences on [J, p. 259]. Put $L = L(\nu_1)$ as on the cited page. The functor $T_\lambda^\mu,\ T_\mu^\lambda$ are constructed from tensor product functors $L \otimes (-),\ L^* \otimes (-)$, respectively, using block projections $pr_\lambda,\ pr_\mu$:

$$T_\mu^\lambda = pr_\lambda(L^* \otimes -) \circ pr_\mu,\ T_\lambda^\mu = pr_\mu(L \otimes -) \circ pr_\lambda.$$

These are the definitions given in Jantzen, familiar to many readers. All functors are regarded as functors from the category of rational G-modules to itself. The functor $L \otimes - := L \otimes (-)$ is (left and right) adjoint to $L^* \otimes - := L^* \otimes (-)$, while pr_λ and pr_μ are both self-adjoint (left and right). Consequently, T_λ^μ is (left and right) adjoint to T_μ^λ.

Construction of the adjunction map So far, this is all standard, but we can go a little further.

(1) Let $X, Y \in G - Mod$, the category of rational G-modules. Then the identifications $\mathrm{Hom}_G(pr_\lambda X, Y) \cong \mathrm{Hom}_G(X, pr_\lambda Y)$ are quite canonical: Write X, Y, respectively, as direct sums of submodules

$$X = pr_\lambda X \oplus pr'_\lambda X, Y = pr_\lambda Y \oplus pr'_\lambda Y,$$

where $pr'_\lambda X$ has no composition factors in the block associated with λ and is maximal, as a submodule of X, with that property. The submodule $pr'_\lambda Y$ of Y is defined similarly. Obviously, any G-homomorphism $pr_\lambda X \to Y$ has image in $pr_\lambda Y$. Consequently, it identifies with a map $pr_\lambda X \to pr_\lambda Y$. Also, any G-homomorphism $X \to pr_\lambda Y$ sends pr'_λ to 0 and sends $pr_\lambda X$ to $pr_\lambda Y$. Thus,

$$\mathrm{Hom}_G(pr_\lambda X, Y) \cong \mathrm{Hom}_G(pr_\lambda X, pr_\lambda Y) \cong \mathrm{Hom}_G(X, pr_\lambda Y)$$

with each identification very obvious and canonical. We also record

$$\mathrm{Hom}_G(X, Y) \cong \mathrm{Hom}_G(pr_\lambda X, pr_\lambda Y) \oplus \mathrm{Hom}_G(pr'_\lambda X, pr'_\lambda Y)$$

All the observations in the above paragraph hold if λ is replaced by μ.

(2) In particular, suppose we are given a natural transformation $\eta = \{\eta_{X,Y}\}_{X,Y \in G-Mod}$ from $\mathrm{Hom}_G(L \otimes Y, Y)$ to $\mathrm{Hom}_G(X, L^* \otimes Y)$. Then η gives maps

$$\eta_{pr_\lambda X, pr_\mu Y} : \mathrm{Hom}_G(L \otimes pr_\lambda X, pr_\mu Y) \longrightarrow \mathrm{Hom}_G(pr_\lambda X, L^* \otimes pr_\mu Y).$$

This induces, using **(1)**, a natural transformation we will call $\tilde{\eta}$, again defined on $G - Mod \times G - Mod$, with $\tilde{\eta}_{X,Y}$ a map from $\mathrm{Hom}_G(pr_\lambda L \otimes pr_\lambda X, pr_\mu Y)$ to $\mathrm{Hom}_G(X, pr_\lambda(L^* \otimes pr_\mu Y))$. Explicitly,

$$\begin{array}{ccc}
\mathrm{Hom}_G(L \otimes pr_\lambda X, pr_\mu Y) & \xrightarrow{\eta_{pr_\lambda X, pr_\mu Y}} & \mathrm{Hom}_G(pr_\lambda X, L^* pr_\mu Y) \\
\downarrow\cong & & \downarrow\cong \\
\mathrm{Hom}_G(pr_\lambda L \otimes pr_\lambda X, pr_\mu Y) & \xrightarrow{\tilde{\eta}_{X,Y}} & \mathrm{Hom}_G(X, pr_\lambda(L^* \otimes pr_\mu Y)) \\
\downarrow= & & \downarrow= \\
\mathrm{Hom}_G(T_\lambda^\mu X, Y) & & \mathrm{Hom}_G(X, T_\mu^\lambda Y)
\end{array}$$

with the vertical isomorphisms between the top two rows given by **(1)**. Note there is a similar diagram with $pr_\mu Y$ replacing Y in the bottom two rows (using $\tilde{\eta}_{X, pr_\mu Y}$).

(3) Using the naturality of η, we can put another row and commutative diagram(s) on top of the top row above:

$$\begin{array}{ccc}
\mathrm{Hom}_G(L \otimes pr_\lambda X, Y) & \xrightarrow{\eta_{pr_\lambda X, Y}} & \mathrm{Hom}_G(pr_\lambda X, L^* \otimes Y) \\
\downarrow\uparrow & & \downarrow\uparrow \\
\mathrm{Hom}_G(L \otimes pr_\lambda X, pr_\mu Y) & \xrightarrow{\eta_{pr_\lambda X, pr_\mu Y}} & \mathrm{Hom}_G(pr_\lambda X, L^* \otimes pr_\mu Y)
\end{array}$$

Here the pair of vertical maps pointing upward are indexed by the inclusion $pr_\mu \to Y$ and yield a commutative diagram. Similarly the pair of downward arrows are indexed by the projection $Y \to pr_\mu Y$ and give a commutative diagram. The composite of the homomorphisms represented by the upward pointing arrows with the homomomorphism represented by the corresponding downward pointing arrows are identities.

We can now prove

Proposition 4.1. *Assume the natural transformation* $\eta = \{\eta_{X,Y}\}_{X,Y \in G-Mod}$ *gives natural isomorphisms*

$$\eta_{X,Y} : \ Hom_G(L \otimes X, Y) \longrightarrow Hom_G(X, L^* \otimes Y),$$

and let $\tilde{\eta} = \{\tilde{\eta}_{X,Y}\}_{X,Y \in G-Mod}$ *be the corresponding natural transformation constructed above. Then* $\tilde{\eta}$ *gives natural isomorphisms*

$$\tilde{\eta}_{X,Y}:\ Hom_G(T_\lambda^\mu X, Y) \longrightarrow Hom_G(X, T_\mu^\lambda Y).$$

Moreover, the corresponding adjunction transformation $\widetilde{adj}$ *from the identity functor on* $G - Mod$ *to the functor* $T_\mu^\lambda T_\lambda^\mu$ *may be constructed from the adjunction map adj similarly associated with* η*. In fact, for each* $X \in G - Mod$*, we have a commutative diagram*

$$\begin{array}{ccc} X & \xrightarrow{adj_X} & L^* \otimes L \otimes X \\ \big\downarrow = & & \big\downarrow \\ X & \xrightarrow{\widetilde{adj}_X} & T_\mu^\lambda T_\lambda^\mu X \end{array}$$

where the down arrow on the right is the composite projection

$$\begin{aligned} L^* \otimes L \otimes X \to L^* \otimes L \otimes pr_\lambda X \to L^* \otimes pr_\mu(L \otimes pr_\lambda X) \\ \to pr_\lambda(L^* \otimes pr_\mu(L \otimes pr_\lambda X)) = T_\mu^\lambda T_\lambda^\mu X. \end{aligned}$$

Proof. We use the (noted) alternate version of the diagram in **(2)** in which $pr_\mu Y$ replaces Y, and use the diagram in **(3)** as given. The combination gives a commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}_G(L \otimes pr_\lambda X, Y) & \xrightarrow{\eta_{pr_\lambda X, Y}} & \mathrm{Hom}_G(pr_\lambda X, L^* \otimes Y) \\ \big\downarrow \cong & & \big\downarrow \cong \\ \mathrm{Hom}_G(L \otimes pr_\lambda X, pr_\mu Y) & \xrightarrow{\eta_{pr_\lambda X, pr_\mu Y}} & \mathrm{Hom}_G(pr_\lambda X, L^* \otimes pr_\mu Y) \\ \big\downarrow \cong & & \big\downarrow \cong \\ \mathrm{Hom}_G(pr_\mu(L \otimes pr_\lambda X), pr_\mu Y) & \xrightarrow{\tilde{\eta}_{X, pr_\mu Y}} & \mathrm{Hom}_G(X, pr_\lambda(L^* \otimes pr_\mu Y)) \\ \big\downarrow = & & \big\downarrow = \\ \mathrm{Hom}_G(T_\lambda^\mu X, Y) & & \mathrm{Hom}_G(X, T_\mu^\lambda Y) \end{array}$$

Now take $Y = L \otimes pr_\lambda X$. Thus $pr_\mu Y = T_\lambda^\mu X$, and $\tilde{\eta}_{X, pr_\mu Y}(1_{T_\lambda^\mu} X) = \tilde{\eta}_{X, T_\lambda^\mu X}(1_{T_\lambda^\mu X}) = \widetilde{adj}_X$. Chasing the element $1_{T_\lambda^\mu X}$ up to the second row gives an element which is the projection $Y \to pr_\mu Y$ in $\mathrm{Hom}_G(L \otimes pr_\lambda X, pr_\mu Y) = \mathrm{Hom}_B(Y, pr_\mu Y)$. This element is also the image of $1_{L \otimes pr_\lambda X} = 1_Y$ under the downward vertical map on the left. Observe $\eta_{pr_\lambda X, Y}(1_Y) = adj_{pr_\lambda X}$. Following the right hand vertical maps in the case $X = pr_\lambda X$ gives a commutative diagram

$$\begin{tikzcd} pr_\lambda X \arrow[r, "adj_{pr_\lambda X}"] \arrow[d, "="] & L^* \otimes L \otimes pr_\lambda X \arrow[d] \\ pr_\lambda X \arrow[r, "\widetilde{adj}_{pr_\lambda X}"] & T_\mu^\lambda T_\lambda^\mu pr_\lambda X \end{tikzcd}$$

with the right hand map the composite of projections

$$L^* \otimes L \otimes pr_\lambda X \to L^* \otimes pr_\mu(L \otimes pr_\lambda X) \to pr_\lambda(L^* \otimes pr_\mu(L \otimes pr_\lambda X)) = T_\mu^\lambda T_\lambda^\mu X.$$

Now return to the case of a general X and apply functoriality[11] of the adjunction maps adj, $\widetilde{adj}$ to obtain a commutative diagram

$$\begin{tikzcd} X \arrow[r, "adj_X"] \arrow[d] & L^* \otimes L \otimes X \arrow[d] \\ pr_\lambda X \arrow[r, "adj_{pr_\lambda X}"] \arrow[d, "="] & L^* \otimes L \otimes pr_\lambda X \arrow[d] \\ pr_\lambda X \arrow[r, "\widetilde{adj}_{pr_\lambda X}"] & T_\mu^\lambda T_\lambda^\mu pr_\lambda X \\ X \arrow[u] \arrow[r, "\widetilde{adj}_X"] & T_\mu^\lambda T_\lambda^\mu X \arrow[u, "="] \end{tikzcd}$$

[11]It is a general property of adjunction maps $Id \to EF$ where E is a right adjoint to a functor F, that any map $\phi : X \to X'$ in the underlying category gives a commutative diagram

$$\begin{tikzcd} X' \arrow[r] & EF(X') \\ X \arrow[u, "\phi"] \arrow[r] & EF(X) \arrow[u, "EF(\phi),"] \end{tikzcd}$$

where both horizontal maps are adjunctions. We include a brief proof: $F(\phi)$ is the value at $1_{F(X)}$ of the evident map $\mathrm{Hom}\,(F(X), F(X)) \to \mathrm{Hom}\,(F(X), F(X'))$ and also the value at $1_{F(X')}$ of the evident map $\mathrm{Hom}\,(F(X'), F(X')) \to \mathrm{Hom}\,(F(X), F(X'))$. Applying the (natural) adjointness isomorphism $\mathrm{Hom}\,(F(-), F(-)) \cong \mathrm{Hom}\,(-, EF(-))$ to $F(\phi)$ yields a map $X \to EF(X')$ which, correspondingly, factors in two different ways, giving the desired commutative diagram.

We remark that there is a dual commutative diagram for the "counital adjunction" $FE \to Id$ The formulation and proof may be given using dual categories and the adjunction case.

In this diagram, the middle rectangle is identical to the diagram just discussed. All the unlabeled vertical maps are evident projections. In particular, the whole commutative diagram could be extended on the left, preserving commutativity, by a long downward equality map from the upper left X to the lower left X. Also, the lower right equality arrow can be reversed, still preserving commutativity. With these changes, the perimeter rectangle becomes the commutative diagram required in the proposition. This completes its proof. $\square$

We remark that the adjunction obtained from the usual natural isomorphism

$$\mathrm{Hom}_G(L \otimes X, Y) \cong \mathrm{Hom}_G(X, L^* \otimes X)$$

is quite explicit: For $x \in X$, $adj_X(x) = 1_L \otimes x$, if $L^* \otimes L$ is identified with $\mathrm{Hom}_k(L, L)$. Even if we do not use that identification, we can just write

$$adj_X(x) = \sum_{\epsilon \in I} \epsilon^* \otimes \epsilon \otimes x,$$

where ϵ ranges over any basis I of L, and ϵ^* denotes the corresponding dual basis element. The sum on the right is independent of the basis I chosen.

As a corollary to the proposition, we have

Corollary 4.2. *Let $X, Y \in G-mod$, and identify $L^* \otimes L \otimes (X \otimes Y) \cong (L^* \otimes L \otimes X) \otimes Y$. Then $adj_{X \otimes Y}(-) = (adj_X(-)) \otimes Y$. If all weights of Y lie in the root lattice, then $\widetilde{adj}_{X \otimes Y^{[1]}}(-) = \widetilde{adj}_X(-) \otimes Y^{[1]}$, identifying $T_\mu^\lambda T_\lambda^\mu(X \otimes Y^{[1]})$ with $(T_\mu^\lambda T_\lambda^\mu X) \otimes Y^{[1]}$.*

Proof. The first equality is immediate from the formula for $adj_X(x), x \in X$ above, applied to to $X \otimes Y$ and $adj_{X \otimes Y}$.

We can argue with adj to handle $\widetilde{adj}$: First, observe the rearrangements

$$pr_\lambda(X \otimes Y^{[1]}) = pr_\lambda X \otimes Y^{[1]},$$

and

$$pr_\mu(L \otimes pr_\lambda(X \otimes Y^{[1]})) = pr_\mu(L \otimes pr_\lambda X \otimes Y^{[1]}) = pr_\mu(L \otimes pr_\lambda X) \otimes Y^{[1]}.$$

Also,

$$pr_\lambda(L^* \otimes pr_\mu(L \otimes pr_\lambda(X \otimes Y^{[1]}))) = pr_\lambda(L^* \otimes pr_\mu(L \otimes pr_\lambda X)) \otimes Y^{[1]}.$$

Here we have heavily used the fact that the operator $-\otimes Y^{[1]}$ commutes with our "block" projections. (Recall the latter are formulated in terms of the affine Weyl group, which contains translations by p-multiples of the root lattice.) We have regarded $pr_\lambda X$ as a submodule of X, and have taken a similar viewpoint with all the projections in these equalities. (Similar equalities hold for complementary projections, Thus, $pr'_\lambda(X\otimes Y^{[1]}) = pr'_\lambda X\otimes Y^{[1]}$, etc.)

Recall that we have described $\widetilde{adj}_X$ in Proposition 4.1 as the composition of adj_X followed by a sequence of projections

$$L^*\otimes L\otimes X \to L^*\otimes L\otimes pr_\lambda X \to L^*\otimes pr_\mu(L\otimes pr_\lambda X) \to pr_\lambda(L^*\otimes pr_\mu(L\otimes pr_\lambda X)).$$

Tensoring $adj_X(-)$ on the right with $Y^{[1]}$ gives $\widetilde{adj}_{X\otimes Y^{[1]}}(-)$ as shown above (even with $Y^{[1]}$ any G-module). Next, tensor the sequence of projections displayed above with $Y^{[1]}$, obtaining

$$\begin{aligned} L^*\otimes L\otimes X\otimes Y^{[1]} &\to L^*\otimes L\otimes pr_\lambda(X\otimes Y^{[1]}) \\ &\to L^*\otimes pr_\mu(L\otimes pr_\lambda(X\otimes Y^{[1]})) \\ &\to pr_\lambda(L^*\otimes pr_\mu(L\otimes pr_\lambda X))\otimes Y^{[1]}. \end{aligned}$$

Using the rearrangements discussed above, we get

$$\begin{aligned} L^*\otimes L\otimes X\otimes Y^{[1]} \to L^*\otimes L\otimes pr_\lambda X\otimes Y^{[1]} \to L^*\otimes pr_\mu(L\otimes pr_\lambda X)\otimes Y^{[1]} \\ \to pr_\lambda(L^*\otimes pr_\mu(L\otimes pr_\lambda(X\otimes Y^{[1]}))). \end{aligned}$$

Here we have identified tensor products isomorphic through the associative law. The lower display above is easily recognized as the sequence in Proposition 4.1 whose composition with $adj_{X\otimes Y^{[1]}}$ gives $\widetilde{adj}_{X\otimes Y^{[1]}}$. Combining this with the equality $adj_{X\otimes Y^{[1]}}(-) = adj_X(-)\otimes Y^{[1]}$ noted above gives the identification $\widetilde{adj}_{X\otimes Y^{[1]}}(-) = \widetilde{adj}_X(-)\otimes Y^{[1]}$, completing the proof of the corollary. □

Remark 4.3. The quantum case

The lack of cocommutativity requires some care in treating the quantum case, and it becomes important to distinguish right from left. For example, consider the "usual" natural isomorphism

$$\mathrm{Hom}_G(L\otimes X, Y) \cong \mathrm{Hom}_G(X, L^*\otimes X)$$

in the algebraic groups case. It may be given in more detail as a composite

$$\mathrm{Hom}_G(L\otimes X, Y) \cong \mathrm{Hom}_G(X, \mathrm{Hom}_k(L, Y)) \cong \mathrm{Hom}_G(X, L^*\otimes Y). \tag{4.3.1}$$

The right hand isomorphism depends on the isomorphism of G-modules

$$\mathrm{Hom}_k(L, Y) \cong L^* \otimes Y.$$

The usual way to identify a simple tensor element $f(-) \otimes y$ on the right ($f \in L^*, y \in Y$) with a function on the left is to let it send $v \in L$ to $f(v)y$. Let $g \in G$, and suppose for a moment that G is not a group, but a Hopf algebra with antipode S, and that L, Y are left G-modules. Using Sweedler (implicit sum) notation, with g mapping to $g_1 \otimes g_2$ under comultiplication, the action of g on the right hand element gives $f(Sg_1(-) \otimes g_2 y$, but, on the left, it gives a function sending v to $f(Sg_2(v))g_1 y$. The latter is not formally the function corresponding to the right hand element without cocommutativity.

This can be fixed by either using the right action of the Hopf algebra G or by keeping the left action and changing the tensor product $L^* \otimes Y$ to $Y \otimes L^*$. We prefer the latter approach, since left actions are often implicitly used–e.g., in [J]. In keeping with the spirit of previous sections of this paper, define an "opposite" tensor product $\otimes^{op}$ by

$$X \otimes^{op} Y := Y \otimes X,$$

A similar analysis can be carried out on the left hand isomorphism of the display (4.3.1). We find that the standard correspondence gives an isomorphism of left G-modules

$$\mathrm{Hom}_k(L \otimes^{op} X, Y) \cong \mathrm{Hom}_k(X, \mathrm{Hom}_k(L, Y)).$$

Here the left action of the Hopf algebra G on the various modules $\mathrm{Hom}_k(-,-)$ is given by "conjugation." That is, if $g \in G$ and f is a linear function from one left G-module to another, the action of g on f gives a linear function $g_1 f(Sg_2(-))$. When the antipode is surjective (as it is for all the Hopf algebras we consider), the space of "fixed points" of this action of G (all f for which each $g \in G$ acts through the counit) results precisely in the space of G-homomorphisms. (A general statement and proof of this fact may be found in [APW, 2.9].) In particular, we have a general version of (4.3.1) which holds for any such Hopf algebra:

(4.3.2)
$$\mathrm{Hom}_G(L \otimes^{op} X, Y) \cong \mathrm{Hom}_G(X, \mathrm{Hom}_k(L, Y)) \cong \mathrm{Hom}_G(X, L^* \otimes^{op} Y).$$

Finally, notice that $\otimes^{op}$ is just as associative an operation as $\otimes$, which is strictly associative, if standard identifications are made in iterated tensor products of k-spaces.

Thus, **the results and arguments of this section hold in the quantum case.** The reader can even read or reread the statements and arguments in both the algebraic groups case and quantum case simultaneously, after replacing $\otimes$ with $\otimes^{op}$, and using the same simultaneous notations $\mathbb{U}, \mathbb{B}, \Bbbk, \dots,$ as in previous sections, in place of $G, B, k, \dots.$

5. Appendix B

We now return to Jantzen [J, pp. 258–259]. Proposition 7.11 there implies, if T_λ^μ is "to a wall," then

$$T_\lambda^\mu \mathrm{RInd}_B^G(w \cdot \lambda) \cong \mathrm{RInd}_B^G(w \cdot \mu).$$

Recall Jantzen denotes one dimensional weight modules by the weights alone.

The argument for the above isomorphism is helpful: $T_\lambda^\mu \mathrm{RInd}_B^G(w \cdot \lambda) = pr_\mu(L \otimes pr_\lambda \mathrm{RInd}_B^G(w \cdot \lambda)) = pr_\mu(L \otimes \mathrm{RInd}_B^G(w \cdot \lambda)) = pr_\mu \mathrm{RInd}_B^G(L \otimes w \cdot \lambda)$. At this point a B composition series of L is examined and it is found that there is only one composition factor, call it λ_l appearing with multiplicity one, such that $pr_\mu \mathrm{RInd}_B^G(\lambda_l \otimes w \cdot \lambda) \neq 0$. It is determined that $\lambda_l \otimes w \cdot \lambda$ is $w \cdot \mu$, completing the proof.

Next, let us come "out of the wall" with T_μ^λ. We assume μ is on a true "wall" with stabilizer $\{1, s\}$ for a simple reflection s. We want to know what happens to $T_\mu^\lambda \mathrm{RInd}_B^G(w \cdot \mu)$. Again, write

$$\begin{aligned} T_\mu^\lambda \mathrm{RInd}_B^G(w \cdot \mu) &= pr_\lambda(L^* \otimes pr_\mu \mathrm{RInd}_B^G(w \cdot \mu)) \\ &= pr_\lambda(L^* \otimes \mathrm{RInd}_B^G(w \cdot \mu)) \\ &= pr_\lambda \mathrm{RInd}_B^G(L^* \otimes w \cdot \mu). \end{aligned}$$

This time L^* has two composition factors exactly, $\gamma = \nu$ and $\gamma = \nu'$, each appearing with multiplicity 1, such that $pr_\lambda \mathrm{RInd}_B^G(\gamma \otimes w \cdot \mu) \leq 0$. The two weights $\nu,\ \nu'$(not Jantzen's notation) satisfy $\{\nu + w \cdot \mu, \nu' + w \cdot \mu\} = \{w \cdot \lambda, ws \cdot \lambda\}$.

Jantzen treats the case $ws \cdot \lambda < w \cdot \lambda$ (with the roles of λ, μ reversed) in [J, Prop. 7.12]. For our purposes, to be compatible with Lemma 3.2(ii), we will consider the case $ws \cdot \lambda > w \cdot \lambda$, which requires different arguments (in the same setting).

5.1. The main issue. In this case there is an exact sequence of B-modules

$$0 \longrightarrow M \longrightarrow L^* \otimes sw \cdot \mu \longrightarrow M' \longrightarrow 0$$

in which the weight $w\cdot\lambda$ appears in M and $ws\cdot\lambda$ appears in M'. These appearances are each with multiplicity 1, and no other weight τ with $pr_\lambda \mathrm{RInd}_B^G(\tau) \neq 0$ appears in either M or M'. Apply $pr_\lambda \mathrm{RInd}_B^G$ to the above short exact sequence.

The result is a distinguished triangle

$$(*) \qquad \cdots \longrightarrow \mathrm{RInd}_B^G(w\cdot\lambda) \longrightarrow T_\mu^\lambda \mathrm{RInd}_B^G(w\cdot\mu) \longrightarrow \mathrm{RInd}_B^G(ws\cdot\lambda) \longrightarrow \cdots$$

As previously noted, the middle term is isomorphic to

$$T_\mu^\lambda T_\lambda^\mu \mathrm{RInd}_B^G(w\cdot\lambda).$$

This leads to the question as to whether or not the resulting map

$$\mathrm{RInd}_B^G(w\cdot\lambda) \longrightarrow T_\mu^\lambda T_\lambda^\mu \mathrm{RInd}(w\cdot\lambda)$$

is the adjunction map. **We claim** that it is, indeed, the adjunction map, at least up to a nonzero scalar multiple. (Thus, there is a distinguished triangle (*) in which the left hand map is adjunction.)

The proof of this claim will essentially occupy the rest of this appendix! We will use the context and notation of the algebraic groups case, and treat the quantum case at the end in Remark 5.15. Part (ii) of Lemma 3.2 then follows, since $\theta_\alpha^+(\mathrm{RInd}_{\mathbb{B}}^{\mathbb{U}}\lambda)$, from its mapping cone definition, fits into a triangle with the same two objects and left hand map as (*), but replacing the object $\mathrm{RInd}_{\mathbb{B}}^{\mathbb{U}}(\lambda^{s_\alpha})$ as the third object. Standard triangulated category axioms then give a map $\theta_\alpha^+(\mathrm{RInd}_{\mathbb{B}}^{\mathbb{U}}\lambda) \longrightarrow \mathrm{RInd}_{\mathbb{B}}^{\mathbb{U}}(\lambda^{s_\alpha})$, part of a commutative diagram with identify maps on the two left hand objects in (*) and their translations under [1]. The "five lemma" then gives the desired isomorphism $\theta_\alpha^+(\mathrm{RInd}_{\mathbb{B}}^{\mathbb{U}}\lambda) \cong \mathrm{RInd}_{\mathbb{B}}^{\mathbb{U}}(\lambda^{s_\alpha})$, completing the proof of part (ii) of Lemma 3.2. Since a proof of part (i) has already been given, this will complete the proof of the lemma.

For the moment, we prove the claim in the case where both $w\cdot\lambda$ and $ws\cdot\lambda$ are dominant: Note that $\mathrm{RInd}_B^G(w\cdot\lambda) \cong \mathrm{Ind}_B^G(w\cdot\lambda)$ and $\mathrm{RInd}_B^G(ws\cdot\lambda) \cong \mathrm{Ind}_B^G(ws\cdot\lambda)$ by Kempf's theorem. Also, of course, the functor RInd_B^G is right adjoint to restriction. In particular

$$\mathrm{Hom}_{D^b(G)}(\mathrm{RInd}_B^G(w\cdot\lambda), \mathrm{RInd}_B^G(ws\cdot\lambda)) = \mathrm{Hom}_B(\mathrm{Ind}_B^G(w\cdot\lambda), ws\cdot\lambda)$$

so that any map from $\mathrm{RInd}(w\cdot\lambda)$ to the middle term of (*) factors through the left hand map. However, $\mathrm{Hom}_B(\mathrm{RInd}_B^G(w\cdot\lambda), w\cdot\lambda) \cong k$. The claim follows. In our argument we have used the fact that RInd_B^G is right adjoint to restriction.

5.2. We will now try to exploit the validity of the dominant case, by using it to build well-behaved resolutions in the general $w \cdot \lambda < ws \cdot \lambda$ case, to which we now return.

The B-modules we will use to resolve $w \cdot \lambda$ will be sums of those of the form $w \cdot \lambda \otimes p\tau \otimes \mathrm{V}^{[1]}$, where τ is in the root lattice and $\mathrm{V}^{[1]}$ is a Frobenius twisted G-module (restricted to B) with V having all weights in the root lattice.

Lemma 5.3. *The trivial module $k = k(0)$ has a positive resolution $k \xrightarrow{\sim} K^\bullet$, where each K^n is a direct sum of B-modules $p\tau \otimes V^{[1]}$, with τ and $V^{[1]}$ as above. Moreover, we may assume all τ are dominant and that $q \cdot \lambda + p\tau$, $w \cdot \mu + p\tau$, are $ws \cdot \lambda + p\tau$ are also dominant.*

In fact, we can assume $\nu + p\tau$ is dominant for all ν in any fixed finite list of weights.

Proof. Notice that k and all $p\tau \otimes \mathrm{V}^{[1]}$ are Frobenius-twisted B-modules. Each (Frobenius-)twisted injective B-module hull $I_\mu^{[1]}$ for a weight μ in the root lattice is a direct union of modules $p(\mu+\sigma) \otimes \mathrm{V}^{[1]}$; see (2.12.2). Thus, any finite dimensional B-module $N^{[1]}$, with N having all weights in the root lattice, can be embedded in a direct sum of these, with the weights $\tau = \mu + \sigma$, as large as we like. The cokernel of the embedding will also be a finite dimensional twisted B-module of the same form as $N^{[1]}$ above. Hence the process can continue. Starting with $k = k(0)$ in the initial role of $N^{[1]}$, we obtained the desired resolution. ☐

We now describe some of the main issues we face at this point. Let τ be any weight in the root lattice such that $w \cdot \lambda + p\tau$, $w \cdot \mu + p\tau$, and $ws \cdot \lambda + p\tau$ are dominant, as well as $p\tau$. Form the composite of the adjunction map

$$\mathrm{Ind}_B^G(w \cdot \lambda + p\tau) \longrightarrow T_\mu^\lambda T_\lambda^\mu \mathrm{Ind}_B^G(w \cdot \mu + p\tau)$$

and the usual isomorphism

$$\mathrm{T}_\mu^\lambda\, \mathrm{T}_\lambda^\mu\, \mathrm{Ind}_B^G(w \cdot \lambda + p\tau) \xrightarrow{\sim} \mathrm{T}_\mu^\lambda\, \mathrm{Ind}_B^G(w \cdot \mu + p\tau)$$

Note $w \cdot \lambda + p\tau = w' \cdot \lambda$ and $w \cdot \mu + p\tau = w' \cdot \mu$ for w', the composite of w followed by translation by $p\tau$. We will discuss the "usual" isomorphism later in some details, but its exact nature may be regarded as unknown at the moment, together with any details regarding the adjunction map. We do, however, note that the latter map is nonzero. The composite then gives a nonzero map

$$\mathrm{Ind}_B^G(w \cdot \lambda + p\tau) \longrightarrow \mathrm{T}_\lambda^\mu\, \mathrm{Ind}_B^G(w \cdot \mu + p\tau) \tag{5.3.1}$$

"Another" map with the same domain and target objects is obtained, as in 5.1, by applying $pr_\lambda \mathrm{Ind}_B^G(-)$ to the sequence $0 \longrightarrow M \longrightarrow L^* \otimes w \cdot \mu$ there, but with $w' \cdot \mu = w \cdot \mu + p\tau$ playing the role of $w \cdot \mu$. We will call the resulting map the "Jantzen map" $Jan_Y^{w\cdot\lambda}$ for Y the B-module $p\tau = k(p\tau)$. (We will shortly generalize this notation.) To discuss $Jan_Y^{w\cdot\lambda}$ and (5.3.1) in a parallel way, denote the latter as $Adj_Y^{w\cdot\lambda}$ (for the same Y). Then the discussion at the end of 5.1 gives, using w' in place of w there:

Proposition 5.4. *The maps $Adj_Y^{w\cdot\lambda}$ and $Jan_Y^{w\cdot\lambda}$ differ by at most a nonzero scalar multiple, for $Y = p\tau$, when $p\tau$, $w \cdot \lambda + p\tau$, $w \cdot \mu + p\tau$, and $ws \cdot p\tau$ are dominant, and τ with the root lattice.*

Now let $\mathcal{Y}^{w\cdot\lambda}$ denote the full subcategory of B-modules $p\tau \otimes \mathrm{V}^{[1]}$ with $p\tau$ as in the proposition and $\mathrm{V}^{[1]}$ a finite dimensional Frobenius twisted G-module with tall weights of V in the root lattice. Also write $\mathrm{V}^{[1]}$ for its restriction $Y^{[1]}|_B$ depending on context.

We will usually abbreviate $\mathcal{Y} := \mathcal{Y}^{w\cdot\lambda}$.

Our next goal is to extend the maps $Adj_Y^{w\cdot\lambda}$, $Jan_Y^{w\cdot\lambda}$ to all $y \in \mathcal{Y}$ and regard them as natural transformations $Adj^{w\cdot\lambda} = \{Adj_Y^{w\cdot\lambda}\}_{Y\in\mathcal{Y}}$, $Jan^{w\cdot\lambda} = \{Jan_Y^{w\cdot\lambda}\}_{Y\in\mathcal{Y}}$

$$Adj^{w\cdot\lambda} : \mathrm{Ind}_B^G(w\cdot\lambda\otimes -) \longrightarrow \mathrm{T}_\mu^\lambda\,\mathrm{Ind}_B^G(w\cdot\mu\otimes -) \quad \text{and}$$
$$Jan^{w\cdot\lambda} : \mathrm{Ind}_B^G(w\cdot\lambda\otimes -) \longrightarrow \mathrm{T}_\mu^\lambda\,\mathrm{Ind}_B^G(w\cdot\mu\otimes -)$$

These functors and natural transformations will then automatically extend to $add\,\mathcal{Y}$, the additive full subcategory of B-mod consisting of all finite direct sums of objects in $\mathcal{Y}$. Notice that all the K^n from the previous lemma belong to $add\,\mathcal{Y}$, so that the (to be demonstrated) naturality will result in two maps of complexes

$$\mathrm{Ind}_B^G(w\cdot\lambda\otimes K^\bullet) \longrightarrow \mathrm{T}_\mu^\lambda\,\mathrm{Ind}_B^G(w\cdot\mu\otimes K^\bullet)$$

resulting in maps

$$\mathrm{RInd}_B^G(w\cdot\lambda) \longrightarrow \mathrm{T}_\mu^\lambda\,\mathrm{RInd}_B^G(w\cdot\mu)$$

to which we want to compare. We will return to this point after achieving the goal above.

We treat first the Jantzen maps.

The Jantzen maps $Jan_Y^{w\cdot\lambda}(y \in \mathcal{Y})$ and their naturality Recall the short exact sequence

$$0 \longrightarrow M \longrightarrow L^* \otimes w \cdot \mu \longrightarrow M' \longrightarrow 0$$

in 5.1. Let $Y = p\tau \otimes \mathrm{V}^{[1]} \in \mathcal{Y}$. Tensor on the right with Y and apply $\mathrm{RInd}_B^G(-)$ to get a distinguished triangle

$$\cdots \longrightarrow \mathrm{RInd}_B^G(M \otimes Y) \longrightarrow \mathrm{RInd}_B^G(L^* \otimes w \cdot \mu \otimes Y) \longrightarrow \mathrm{RInd}_B^G(M' \otimes Y) \longrightarrow \cdots$$

The middle term naturally identifies with $L^* \otimes \mathrm{RInd}_B^G(w \cdot \mu \otimes \gamma)$ through the "generalized tensor identity" (discussed in this paper in Remark 2.11(ii)). Note $L^* \otimes \mathrm{RInd}_B^G(w \cdot \mu \otimes \gamma) \cong L^* \otimes \mathrm{Ind}_B^G(w \cdot \mu \otimes \gamma)$, by the construction of $\mathcal{Y}$. As discussed earlier in this appendix, M has one weight $\nu \in W_{aff} \cdot \lambda$, namely $\nu = w \cdot \lambda$, appearing with multiplicity 1. Also note that $\nu + \eta$ is in the same (dot action) affine Weyl group orbit as ν for any weight ν and weight η of $\mathcal{Y}$. Consequently, $pr_\lambda \mathrm{RInd}_B^G(M \otimes \gamma) \cong \mathrm{RInd}_B^G(w \cdot \lambda \otimes Y) \cong \mathrm{Ind}_B^G(w \cdot \lambda \otimes Y)$. A specific construction of an isomorphism may be given from any full flag of B-submodules of M with one dimensional sections. If such a flag is fixed, we obtain an isomorphism natural in Y of $Y \in \mathcal{Y}$. Similar remarks apply for M' and isomorphism $pr_\lambda \mathrm{RInd}_B^G(M' \otimes Y) \cong \mathrm{Ind}_B^G(ws \cdot \lambda \otimes Y)$.

As a consequence of the discussion above, we have exact sequences, natural in $Y \in \mathcal{Y}$

$$0 \longrightarrow \mathrm{Ind}_B^G(w \cdot \lambda \otimes Y) \longrightarrow \mathrm{T}_\mu^\lambda \mathrm{Ind}_B^G(w \cdot \mu \otimes Y) \longrightarrow \mathrm{Ind}_B^G(ws \cdot \lambda \otimes Y) \longrightarrow 0$$

We define the map on the left (ignoring the obvious zero map) to be $Jan_Y^{w\cdot\lambda}$.

We summarize some of its main properties (in addition to the above exact sequence).

Proposition 5.5. *(i) The maps $Jan_Y^{w\cdot\lambda}$, $y \in \mathcal{Y}$, collectively define a natural transformation of functors.*

$$Ind_B^G(w \cdot \lambda \otimes -) \longrightarrow T_\mu^\lambda \, Ind_B^G(w \cdot \mu \otimes -)$$

on the category $\mathcal{Y}$ (whose morphisms are B-maps).

(ii) For any fixed $Y = p\tau \otimes V^{[1]}$, there is a commutative diagram with "obvious" vertical isomorphisms, natural in V

$$\begin{array}{ccc}
Ind_B^G(w \cdot \lambda \otimes \tau \otimes V^{[1]}) & \xrightarrow{Jan_Y^{w\cdot\lambda}} & T_\mu^\lambda \, Ind_B^G(w \cdot \mu \otimes p\tau \otimes V^{[1]}) \\
\downarrow \cong & & \downarrow \cong \\
Ind_B^G(w \cdot \lambda \otimes p\tau) \otimes V^{[1]} & \xrightarrow{Jan_{p\tau}^{w\cdot\lambda} \otimes V^{[1]}} & (T_\mu^\lambda(w \cdot \mu \otimes p\tau)) \otimes V^{[1]}
\end{array}$$

Proof. Part(i) has been proved already. For part (ii), it is enough to check the commutativity after identifying the right hand terms with $pr_\lambda \mathrm{Ind}_B^G(L^* \otimes w \cdot \mu \otimes p\tau \otimes \mathrm{V}^{[1]})$ and $pr_\lambda \mathrm{Ind}_B^G(L^* \otimes w \cdot \mu \otimes p\tau) \otimes \mathrm{V}^{[1]}$, respectively. The top row in this revised diagram may be obtained by applying $pr_\lambda \mathrm{RInd}_B^G(-)$ to the inclusion $M \otimes \gamma \longrightarrow L^* \otimes w \cdot \mu \otimes \gamma$, by construction. Similarly the bottom row may be obtained by applying $pr_\lambda \mathrm{RInd}_B^G(-)$ to the inclusion $M \otimes p\tau \leq L^* \otimes w \cdot \mu$, then tensoring on the right with $\mathrm{V}^{[1]}$. Now, naturality of the generalized tensor identity gives commutativity of the closed rectangle in the diagram below.

$$\begin{tikzcd}
\mathrm{Ind}_B^G(w \cdot \lambda \otimes p\tau \otimes \mathrm{V}^{[1]}) \arrow[r, "\cong"] & pr_\lambda \mathrm{RInd}_B^G(M \otimes Y) \arrow[r] \arrow[d, "\cong"] & pr_\lambda \mathrm{RInd}_B^G(L^* \otimes w \cdot \mu \otimes p\tau \otimes \mathrm{V}^{[1]}) \arrow[d, "\cong"] \\
 & pr_\lambda(\mathrm{RInd}_B^G(M \otimes p\tau) \otimes \mathrm{V}^{[1]}) \arrow[r] \arrow[d, "\cong"] & pr_\lambda(\mathrm{RInd}_B^G(L^* \otimes w \cdot \mu \otimes p\tau) \otimes \mathrm{V}^{[1]}) \arrow[d, "\cong"] \\
\mathrm{Ind}_B^G(w \cdot \lambda \otimes p\tau) \otimes \mathrm{V}^{[1]} \arrow[r, "\cong"] & pr_\lambda \mathrm{RInd}_B^G(M \otimes p\tau) \otimes \mathrm{V}^{[1]} & pr_\lambda \mathrm{RInd}_B^G(L^* \otimes w \cdot \mu \otimes p\tau) \otimes \mathrm{V}^{[1]}
\end{tikzcd}$$

The identity and its naturality may also be used to complete the open rectangle on the left to a commutative rectangle[12], using the "obvious" tensor identity isomorphism for a vertical map. Finally, all the "RInd_B^G" symbols in the diagram may be replaced with "Ind_B^G," and the bottom row completed to make a lower-right commutative rectangle.

The bottom row then agrees with that of the revised diagram. That is, its composition gives the composition of $Jan_{p\tau}^{w \cdot \lambda} \otimes \mathrm{V}^{[1]}$ and the identification $(\mathrm{T}_\mu^\lambda \mathrm{Ind}_B^G(w \cdot \mu \otimes p\tau) \otimes \mathrm{V}^{[1]} \cong pr_\lambda \mathrm{Ind}_B^G(L^* \otimes w \cdot \mu \otimes p\tau) \otimes \mathrm{V}^{[1]}$. We have now shown that both top and bottom rows of the now completed and commutative outer rectangle agree with those of the revised version of the diagram in (ii). The left hand columns also agree, and the "obvious" isomorphism on the right in the outer rectangle define, through composition, an obvious isomorphism in the "revised" diagram, making the latter commutative. In the original diagram in ii), the composition is

$$\begin{aligned}
\mathrm{T}_\mu^\lambda \mathrm{Ind}_B^G(w \cdot \mu \otimes p\tau \otimes \mathrm{V}^{[1]}) &\cong \mathrm{T}_\mu^\lambda (\mathrm{Ind}_B^G(w \cdot \mu \otimes p\tau) \otimes \mathrm{V}^{[1]}) \\
&\cong (\mathrm{T}_\mu^\lambda \mathrm{Ind}_B^G(w \cdot \mu \otimes p\tau)) \otimes \mathrm{V}^{[1]}
\end{aligned}$$

which may be taken as the definition of the right hand column "obvious" isomorphism in the original diagram in ii). The latter diagram then becomes commutative, and the proposition is proved. □

[12]It is is carried out by using a B-module flag of $M \otimes p\tau$ and applying naturality to the various inclusion and factor maps involved.

This completes our treatment of $Jan^{w\cdot\lambda}$. Before turning to $Adj^{w\cdot\lambda}$, we discuss some isomorphisms $\mathrm{T}_\lambda^\mu \operatorname{Ind}_B^G(w\cdot\lambda\otimes Y) \xrightarrow{\sim} \operatorname{Ind}_B^G(w\cdot\mu\otimes Y)$ and $\mathrm{T}_\mu^\lambda \mathrm{T}_\lambda^\mu \operatorname{Ind}_B^G(w\cdot\lambda\otimes Y) \xrightarrow{\sim} \mathrm{T}_\mu^\lambda \operatorname{Ind}_B^G(w\cdot\mu\otimes Y)$, $Y\in\mathcal{Y}$, which enter into the definition and discussion of $Adj^{w\cdot\lambda}$. We will call the first isomorphism above $Iso_\gamma^{w\cdot\lambda}$, and the second, $TIso_Y^{w\cdot\lambda}(=\mathrm{T}_\mu^\lambda\circ Iso_Y^{w\cdot\lambda})$.

The isomorphism $Iso_Y^{w\cdot\lambda}$ is obtained in a similar spirit to our construction above of the isomorphism $pr_\lambda\mathrm{RInd}_B^G(M\otimes\gamma)\cong\operatorname{Ind}_B^G(w\cdot\lambda\otimes Y)$, except we apply $pr_\mu\mathrm{RInd}_B^G$ to $L\otimes -$. The module L has only one weight, which we called λ_ℓ at the beginning of this appendix (in the discussion of the "to a wall" isomorphism), with the property that $\lambda_\ell+w\cdot\lambda$ belongs to $W_{aff}\cdot\mu$. The same is true if any weight of Y is added to $w\cdot\lambda$. We have $w\cdot\mu=\lambda_\ell+w\cdot\lambda$, and so $w\cdot\mu\otimes Y\cong\lambda_\ell\otimes w\cdot\lambda\otimes Y$. The weight λ_ℓ appears with multiplicity one in L, so

$$\begin{aligned} pr_\mu\mathrm{RInd}_B^G(L\otimes w\cdot\lambda\otimes Y) &\cong \mathrm{RInd}_B^G(\lambda_\ell\otimes w\cdot\lambda\otimes Y)\cong\mathrm{RInd}_B^G(w\cdot\mu\otimes Y)\\ &\cong\operatorname{Ind}_B^G(w\cdot\mu\otimes Y)\end{aligned}$$

The first isomorphism can be constructed by using any B-flag of L with λ_ℓ as a section and applying $pr_\mu\mathrm{RInd}_B^G$ to the various sub and factor modules associated to the flag terms. If we fix the flag and procedure, the first isomorphism becomes natural in $Y\in\mathcal{Y}$. The other isomorphisms obviously are natural in Y, as are the isomorphisms

$$\begin{aligned} \mathrm{T}_\lambda^\mu\operatorname{Ind}_B^G(w\cdot\lambda\otimes Y) &= pr_\mu(L\otimes\operatorname{Ind}_B^G(w\cdot\lambda\otimes Y))\\ &\cong pr_m u(\operatorname{Ind}_B^G(L\otimes w\cdot\lambda\otimes Y))\end{aligned}$$

and

$$pr_m u(\operatorname{Ind}_B^G(L\otimes w\cdot\lambda\otimes Y))\cong pr_\mu(\mathrm{RInd}_B^G(L\otimes w\cdot\lambda\otimes Y)).$$

This latter isomorphism arises from the vanishing of $(R^n\operatorname{Ind}_B^G)(L\otimes w\cdot\lambda\otimes Y)=0$ for $n>0$ (a consequence of our construction of $\mathcal{Y}$ and the generalized tensor identity).

The composition of all these isomorphisms (in an evident order) is defined to be

$$Iso_Y^{w\cdot\lambda}:\mathrm{T}_\lambda^\mu\operatorname{Ind}_B^G(w\cdot\lambda\otimes Y)\xrightarrow{\sim}\operatorname{Ind}_B^G(w\cdot\mu\otimes Y)$$

The construction shows it is natural in $Y\in\mathcal{Y}$ as is $TIso_Y^{w\cdot\lambda}:=\mathrm{T}_\mu^\lambda\circ Iso_Y^{w\cdot\lambda}$. This gives part (i) of the following proposition.

Proposition 5.6. *(i) The maps $Iso_Y^{w\cdot\lambda}$ and $TIso_Y^{w\cdot\lambda}$ $(Y \in \mathcal{Y})$ collectively define natural isomorphisms of functors on the category $\mathcal{Y}$*

$$T_\lambda^\mu \, Ind\,(w\cdot\lambda\otimes -) \longrightarrow Ind_B^G\,(w\cdot\mu\otimes -)$$

and

$$T_\mu^\lambda \, T_\lambda^\mu \, Ind\,(w\cdot\lambda\otimes -) \longrightarrow T_\mu^\lambda \, Ind_B^G\,(w\cdot\mu\otimes -)$$

(ii) For any fixed $Y = p\tau\otimes V^{[1]} \in \mathcal{Y}$, these are commutative diagrams, with "obvious" vertical isomorphisms, natural in V.

$$\begin{tikzcd}
T_\lambda^\mu \, Ind\,(w\cdot\lambda\otimes p\tau\otimes V^{[1]}) \arrow[r, "Iso_Y^{w\cdot\lambda}"] \arrow[d, "\cong"] & Ind_B^G\,(w\cdot\mu\otimes p\tau\otimes V^{[1]}) \arrow[d, "\cong"] \\
T_\lambda^\mu \, Ind\,(w\cdot\lambda\otimes p\tau)\otimes V^{[1]} \arrow[r, "Iso_Y^{w\cdot\lambda}\otimes V^{[1]}"] & Ind_B^G\,(w\cdot\mu\otimes p\tau)\otimes V^{[1]}
\end{tikzcd}$$

and

$$\begin{tikzcd}
T_\mu^\lambda \, T_\lambda^\mu \, Ind\,(w\cdot\lambda\otimes p\tau\otimes V^{[1]}) \arrow[r, "TIso_Y^{w\cdot\lambda}"] \arrow[d, "\cong"] & T_\mu^\lambda \, Ind\,(w\cdot\mu\otimes p\tau\otimes V^{[1]}) \arrow[d, "\cong"] \\
T_\mu^\lambda \, T_\lambda^\mu \, Ind\,(w\cdot\lambda\otimes p\tau)\otimes V^{[1]} \arrow[r, "TIso_Y^{w\cdot\lambda}\otimes V^{[1]}"] & T_\mu^\lambda \, Ind\,(w\cdot\mu\otimes p\tau)\otimes V^{[1]}
\end{tikzcd}$$

Proof. Part (i) already has been proved. Next, note that a commutative lower diagram in (ii) can be obtained by first applying T_μ^λ to a commutative upper diagram, then using the natural isomorphism $\mathrm{T}_\mu^\lambda\,(-\otimes \mathrm{V}^{[1]}) \cong \mathrm{T}_\mu^\lambda\,(-\otimes \mathrm{V}^{[1]}$ on the lower row of the upper diagram. the reader may convince him/her self that the entire procedure preserves the "obvious" property of the vertical maps!

Thus, it is suffice to treat the upper diagram in (ii). The first thing to do here is to note the "obvious" isomorphism $\mathrm{T}_\lambda^\mu\,\mathrm{Ind}\,(w\cdot\lambda\otimes p\tau\otimes \mathrm{V}^{[1]}) \cong \mathrm{T}_\lambda^\mu\,(\mathrm{Ind}_B^G\,(w\cdot\lambda\otimes p\tau)\otimes \mathrm{V}^{[1]}) \cong \mathrm{T}_\lambda^\mu\,\mathrm{Ind}\,(w\cdot\lambda\otimes p\tau)\otimes \mathrm{V}^{[1]}$. This gives the first column in the upper diagram. The isomorphism may be regarded as the (by now "obvious") process of "pulling out" $\mathrm{V}^{[1]}$", from inductions of tensor products, block projection or translation functors, or some combination of these operators. The row of isomorphism above requires two steps to fully "pullout" $\mathrm{V}^{[1]}$. If we continue with the several steps required to define $Iso_Y^{w\cdot\lambda}$, we see at every step along the way there is an opportunity to "pull out" $\mathrm{V}^{[1]}$.

This gives a series of possibly commutative diagrams, written below in top to bottom order.

$$
\begin{tikzcd}
{} \arrow[d, phantom, "\mathbf{(1)}"] & \mathrm{T}_\lambda^\mu \,\mathrm{Ind}_{B\ \mu}^G(w\cdot\lambda\otimes p\tau\otimes \mathrm{V}^{[1]}) \arrow[r, "\cong"] \arrow[d, equal] & \mathrm{T}_\lambda^\mu \,\mathrm{Ind}_B^G\,(w\cdot\lambda\otimes p\tau)\otimes \mathrm{V}^{[1]} \arrow[d, equal] \\
{} \arrow[d, phantom, "\mathbf{(2)}"] & pr_\mu(L\otimes \mathrm{Ind}_B^G\,(w\cdot\lambda\otimes p\tau\otimes \mathrm{V}^{[1]}) \arrow[r, "\cong"] \arrow[d, "\cong"] & pr_\mu(L\otimes \mathrm{Ind}_B^G\,(w\cdot\lambda\otimes p\tau))\otimes \mathrm{V}^{[1]} \arrow[d, "\cong"] \\
{} \arrow[d, phantom, "\mathbf{(3)}"] & pr_\mu(\mathrm{Ind}_B^G\,(L\otimes w\cdot\lambda\otimes p\tau\otimes \mathrm{V}^{[1]}) \arrow[r, "\cong"] \arrow[d, "\cong"] & pr_\mu(\mathrm{Ind}_B^G\,(L\otimes w\cdot\lambda\otimes p\tau))\otimes \mathrm{V}^{[1]} \arrow[d, "\cong"] \\
{} \arrow[d, phantom, "\mathbf{(4)}"] & pr_\mu(\mathrm{RInd}_B^G\,(L\otimes w\cdot\lambda\otimes p\tau\otimes \mathrm{V}^{[1]})) \arrow[r, "\cong"] \arrow[d, "\cong"] & pr_\mu(\mathrm{RInd}_B^G\,(L\otimes w\cdot\lambda\otimes p\tau))\otimes \mathrm{V}^{[1]} \arrow[d, "\cong"] \\
{} \arrow[d, phantom, "\mathbf{(5)}"] & \mathrm{RInd}_B^G\,(\lambda_\ell\otimes w\cdot\lambda\otimes p\tau\otimes \mathrm{V}^{[1]}) \arrow[r, "\cong"] \arrow[d, "\cong"] & \mathrm{RInd}_B^G\,(\lambda_\ell\otimes w\cdot\lambda\otimes p\tau)\otimes \mathrm{V}^{[1]} \arrow[d, "\cong"] \\
{} \arrow[d, phantom, "\mathbf{(6)}"] & \mathrm{RInd}_B^G\,(w\cdot\mu\otimes p\tau\otimes \mathrm{V}^{[1]}) \arrow[r, "\cong"] \arrow[d, "\cong"] & \mathrm{RInd}_B^G\,(w\cdot\mu\otimes p\tau)\otimes \mathrm{V}^{[1]} \arrow[d, "\cong"] \\
{} & \mathrm{RInd}_B^G\,(w\cdot\mu\otimes p\tau\otimes \mathrm{V}^{[1]}) \arrow[r, "\cong"] & \mathrm{Ind}_B^G\,(w\cdot\mu\otimes p\tau)\otimes \mathrm{V}^{[1]}
\end{tikzcd}
$$

Diagram (1) commutes as a matter of notation, identifying the functor $\mathrm{T}_\lambda^\mu\,(-)$ with $pr_\lambda L(-)$, when applied to the "block" associated to $W_{aff}\cdot\lambda$ (the top row isomorphism has already been given in T_λ^μ notation.) For diagram (2), note that the isomorphism in its top row may formally be applied to the same row with pr_μ removed. Next, remove pr_μ from the bottom row of (2) also. If we can get commutativity in the resulting rectangle

$$
\begin{tikzcd}
L\otimes \mathrm{Ind}_B^G\,(w\cdot\lambda\otimes p\tau\otimes \mathrm{V}^{[1]}) \arrow[r, "\cong"] \arrow[d, "\cong"] & L\otimes \mathrm{Ind}_B^G\,(w\cdot\lambda\otimes p\tau)\otimes \mathrm{V}^{[1]} \arrow[d, "\cong"] \\
\mathrm{Ind}_B^G\,(L\otimes w\cdot\lambda\otimes p\tau\otimes \mathrm{V}^{[1]}) \arrow[r, "\cong"] & \mathrm{Ind}_B^G\,(L\otimes w\cdot\lambda\otimes p\tau)\otimes \mathrm{V}^{[1]}
\end{tikzcd}
$$

We get it for (2) by applying pr_μ to the whole diagram, then pulling out $\mathrm{V}^{[1]}$ on the right.

To get commutativity of the rectangle itself note that all four of its corners are induced modules, by the tensor identity, isomorphism to the lower left hand corner. Using the formalism in [J, I.3.4], every induced module $\mathrm{Ind}_B^G M$ (where M here just denotes some $B-$module) is equipped with a $B-$module map $\epsilon_M : \mathrm{Ind}_B^G M \longrightarrow M$. If N is $G-$module the tensor identity isomorphism $\mathrm{Ind}_B^G(M \otimes N) \cong (\mathrm{Ind}_B^G M) \otimes N$ composed with $\epsilon_M \otimes N$ gives $\epsilon_{M\otimes N}$. (This can be extracted from the discussion in [J, I.3.6].) This implies that the usual universal property of induction (see [J, I. Prop. 3.46]) applies directly to $(\mathrm{Ind}_B^G M) \otimes N$ using $\epsilon_M \otimes N$ in the role of a "counit" adjunction (terminology of Wikipedia). Note that the target of $\epsilon_M \otimes N \otimes M \times N$. We will just call $\epsilon_M \otimes N$ the evaluation map associated with $(\mathrm{Ind}_B^G M) \otimes N$ and $M \otimes N$ the associated evaluation target. Returning to the rectangle above, all four of its corners, all obtained from the induced module in the lower left corner by various applications of the tensor identity, have the same target (up to associativity isomorphisms). Consequently, all maps in the rectangle may be viewed as "induced" from the identity map on their (common) target. (This certainly true in the case of an individual application of the tensor identity, from which it to follows in the case of the tensor identity applied within a tensor product of several factors. All individual maps in the rectangle arise this way, and the property of being "induced" from the identity map on their (common) target. (This is certainly true in the case of an individual application of the tensor identity, from which it follows in the case of the tensor identity applied within a tensor product of several factors. All individual maps in the rectangle arise this way, and the property of being "induced" from the identity map on a common target carries over to composition.) It follows now that the rectangle above is (thoroughly) commutative, as in (2).

Commutativity of (3) is easily seen to hold, since the derived functor RInd_B^G on both sides is applied to objects acyclic for Ind_B^G (i.e., their "higher derived functors vanish"). The meaning of the vertical maps in (4) was discussed in the construction of $Iso_Y^{w\cdot\lambda}$. The horizontal maps in the bottom row as obtained from the generalized tensor identity. The top row map is obtained similarly, after pulling $V^[1]$ out of the block projection. Both column constructions may be viewed, before applying pr_μ, as arising from maps $L \longrightarrow L' \longleftarrow \lambda_\ell$ where L' is a quotient of L, tensoring with $w \cdot \lambda \otimes p\tau$ or $w \cdot \lambda \otimes p$ and applying $\mathrm{RInd}_B^G(-)$ or $\mathrm{RInd}_B^G(-) \otimes \mathrm{V}^{[1]}$. Since the generalized tensor identity may be regarded as a natural transformation of functors. We obtain a commutative diagram rising from maps $L \longrightarrow L' \longleftarrow \lambda_\ell$ where L' is a quotient of L, tensoring with $w \cdot \lambda \otimes p\tau$ or $w \cdot \lambda \otimes p$ and applying $\mathrm{RInd}_B^G(-)$ or $\mathrm{RInd}_B^G(-) \otimes \mathrm{V}^{[1]}$. Since the generalized tensor identity may be regarded as a natural transformation of functors. We obtain a commutative diagram

$$\begin{array}{ccc}
\mathrm{RInd}_B^G\,(L \otimes w \cdot \lambda \otimes p\tau \otimes \mathrm{V}^{[1]}) & \cong & \mathrm{RInd}_B^G\,(L \otimes w \cdot \lambda \otimes p\tau) \otimes \mathrm{V}^{[1]} \\
\downarrow & & \downarrow \\
\mathrm{RInd}_B^G\,(L' \otimes w \cdot \lambda \otimes p\tau \otimes \mathrm{V}^{[1]}) & \cong & \mathrm{RInd}_B^G\,(L' \otimes w \cdot \lambda \otimes p\tau) \otimes \mathrm{V}^{[1]} \\
\uparrow & & \uparrow \\
\mathrm{RInd}_B^G\,(\lambda_\ell \otimes w \cdot \lambda \otimes p\tau \otimes \mathrm{V}^{[1]}) & \cong & \mathrm{RInd}_B^G\,(\lambda_\ell \otimes w \cdot \lambda \otimes p\tau) \otimes \mathrm{V}^{[1]}
\end{array}$$

Now apply pr_μ and pullout $\mathrm{V}^{[1]}$ on the right. All column isomorphism become the column isomorphisms in (4), equating the objects in the bottom row of the latter with the same objects with pr_μ applied. The top row of (4) has been discussed and agrees with the top row of the diagram above, after the modification.

The commutativity of diagram (5) is easy, since λ_ℓ is equal to $w \cdot \mu$ as a weight. It is interesting to note that the construction of $Iso_Y^{w\cdot\lambda}$ must fix an isomorphism between the 1-dimensional section λ_ℓ of L, and the abstract 1-dimensional weight space $w \cdot \mu$.

Observation In this sense $Iso_Y^{w\cdot\lambda}$ can be modified by a nonzero scalar multiplication, and remain a version obtained by the "same" construction (still a natural transformation defined on the category $\mathcal{Y}$). Such a modification carries over to $TIso_Y^{w\cdot\lambda}$.

The pull-out operation in (6) is the generalized tensor identity in both rows, except that $\mathrm{RInd}_B^G\,(-)$ may be identified with $\mathrm{Ind}\,_B^G(-)$ on the bottom row. The columns just reflect this identification and the diagram is clearly commutative.

Note that the bottom row in (6) is precisely the right hand column in the upper diagram in (*ii*). The right hand column of the iterated rectangles $(1), (2), \ldots, (6)$ is by construction, $Iso_{p\tau}^{w\cdot\lambda} \otimes \mathrm{V}^{[1]}$. Thus, the outer perimeter of $(1), (2), \ldots, (6)$ gives a commutative version of the upper diagram in (*ii*), after turning the perimeter diagram on its side (left hand side put on top). This completes the proof of the proposition. $\square$

The maps $Adj_Y^{w\cdot\lambda}(Y \in \mathcal{Y})$ and their naturality The map $Adj_Y^{w\cdot\lambda} : \mathrm{Ind}_B^G\,(w \cdot \lambda \otimes Y) \longrightarrow \mathrm{T}_\mu^\lambda\,(w \cdot \mu \otimes Y)$ is defined as the composition of adjunction adj_X : $X \longrightarrow \mathrm{T}_\mu^\lambda\,\mathrm{T}_\lambda^\mu X$, with $X = \mathrm{Ind}_B^G\,(w \cdot \lambda \otimes Y)$, $Y \in \mathcal{Y}$, and the previously discussed isomorphism

$$TIso_Y^{w\cdot\lambda} : \mathrm{T}_\mu^\lambda\,\mathrm{T}_\lambda^\mu\,\mathrm{Ind}_B^G\,(w \cdot \lambda \otimes Y) \xrightarrow{\sim} \mathrm{T}_\mu^\lambda\,\mathrm{Ind}_B^G\,(w\mu \cdot \lambda \otimes Y)$$

The adjunction map $\widetilde{adj}_X$, $X \in G$-mod, is defined as the image of $1 \in \mathrm{Hom}_G(\mathrm{T}_\lambda^\mu X, \mathrm{T}_\lambda^\mu X)$ under a natural isomorphism $\mathrm{Hom}_G(\mathrm{T}_\lambda^\mu -, -) \xrightarrow{\sim} \mathrm{Hom}_G(-, \mathrm{T}_\mu^\lambda -)$, with $X, \mathrm{T}_\lambda^\mu X$ as the variables. (Thus $\mathrm{Hom}_G(\mathrm{T}_\lambda^\mu X, \mathrm{T}_\lambda^\mu X) \cong \mathrm{Hom}_G(X, \mathrm{T}_\mu^\lambda \mathrm{T}_\lambda^\mu X)$). In Appendix A, we have given a thorough discussion of $\widetilde{adj}_X$, constructing it from a similar adjunction map adj_X associated to the adjoint functors $L \otimes -$ and $L^* \otimes -$. We will quote from Appendix A to prove the proposition below.

Proposition 5.7. *(i) The maps $\widetilde{adj}_X$ ($X \in G - mod$) collectively give the adjunction natural transformation from the identity functor to $T_\mu^\lambda\, T_\mu^\lambda$.*

(ii) For any V in G-mod with all weights in the root lattice, there is a commutative diagram.

$$\begin{array}{ccc}
X \otimes V^{[1]} & \xrightarrow{\widetilde{adj}_{X \otimes V^{[1]}}} & T_\mu^\lambda\, T_\lambda^\mu\, (X \otimes V^{[1]}) \\
\downarrow{=} & & \downarrow{\cong} \\
X \otimes V^{[1]} & \xrightarrow{\widetilde{adj}_{X \otimes V^{[1]}}} & (T_\mu^\lambda\, T_\lambda^\mu\, X) \otimes V^{[1]}
\end{array}$$

The right hand column is morphism becomes equality, if both right hand objects are viewed as a submodules of $L^* \otimes L \otimes X \otimes V^{[1]}$.

Proof. Part (i) has already been discussed. Note the obvious fact that adjunctions are natural transformations. (A proof is written down in footnote 11 of this paper, noted in the proof of Proposition 4.1.)

Part (ii) follows from Corollary 4.2. □

We can now give parallel properties of $Adj^{w\cdot\lambda}$, meant especially to mirror Proposition 5.5 for $Jan^{w\cdot\lambda}$.

Proposition 5.8. *(i) The maps $Adj_Y^{w\cdot\lambda}, Y \in \mathcal{Y}$, collectively define a natural transformation of functors*

$$Ind_B^G(w \cdot \lambda \otimes -) \longrightarrow T_\mu^\lambda\, Ind_B^G(w \cdot \mu \otimes -)$$

on the category $\mathcal{Y}$

(ii) For any fixed $Y = p\tau \otimes V^{[1]}$ in $\mathcal{Y}$, there is a commutative diagram with "obvious" vertical isomorphisms, natural in V:

$$\begin{tikzcd} Ind_B^G(w\cdot\lambda\otimes p\tau\otimes V^{[1]}) \arrow[r, "Adj_Y^{w\cdot\lambda}"] \arrow[d, "\cong"] & T_\mu^\lambda(w\cdot\mu\otimes p\tau\otimes V^{[1]}) \arrow[d, "\cong"] \\ Ind_B^G(w\cdot\lambda\otimes p\tau)\otimes V^{[1]} \arrow[r, "Adj_{p\tau}^{w\cdot\lambda}\otimes V^{[1]}"] & T_\mu^\lambda(w\cdot\mu\otimes p\tau)\otimes V^{[1]} \end{tikzcd}$$

Proof. Part (i) follows from the definition $Adj_Y^{w\cdot\lambda} = TIso_Y^{w\cdot\lambda}\circ\widetilde{adj}_X$ with $X = \mathrm{Ind}_B^G(w\cdot\lambda\otimes Y)$.

For part (ii), note that the left hand column of the lower diagram in Proposition 3(ii) may be written as a composition

$$\begin{aligned}\mathrm{T}_\mu^\lambda\,\mathrm{T}_\lambda^\mu\,\mathrm{Ind}_B^G(w\cdot\lambda\otimes p\tau\otimes \mathrm{V}^{[1]}) &\cong \mathrm{T}_\mu^\lambda\,\mathrm{T}_\lambda^\mu(\mathrm{Ind}_B^G(w\cdot\lambda\otimes p\tau)\otimes\mathrm{V}^{[1]})\\ &\cong \mathrm{T}_\mu^\lambda\,\mathrm{T}_\lambda^\mu\,\mathrm{Ind}(w\cdot\lambda\otimes p\tau)\otimes\mathrm{V}^{[1]}.\end{aligned}$$

The second isomorphism is the right hand column of Proposition 5.7(ii). To deal with the first isomorphism, we need the following Lemma.

Lemma 5.9. *For $Y = p\tau\otimes V^{[1]}\in\mathcal{Y}$, there is a commutative diagram*

$$\begin{tikzcd} Ind_B^G(w\cdot\lambda\otimes p\tau\otimes V^{[1]}) \arrow[r, "\widetilde{adj}_{Ind_B^G(w\cdot\lambda\otimes p\tau\otimes V^{[1]})}"] \arrow[d, "\cong"] & T_\mu^\lambda\, T_\lambda^\mu\, Ind_B^G(w\cdot\lambda\otimes p\tau\otimes V^{[1]}) \arrow[d, "\cong"] \\ Ind_B^G(w\cdot\lambda\otimes p\tau)\otimes V^{[1]} \arrow[r, "\widetilde{adj}_{Ind_B^G(w\cdot\lambda\otimes p\tau)\otimes V^{[1]}}"] & T_\mu^\lambda\, T_\lambda^\mu(Ind_B^G(w\cdot\lambda)\otimes p\tau\otimes V^{[1]}) \end{tikzcd}$$

Proof. This is just naturality of $\widetilde{adj}_X$ with respect to $X\in G$-mod, applied to the G-module isomorphism comprising the left column. □

We mow return to the proof of Proposition 5.8. Put the diagram of Lemma 5.9 on top of that of Proposition 5.7 (ii), taking $X = \mathrm{Ind}_B^G(w\cdot\lambda\otimes Y)$. In this case, the top horizontal edge of the diagram in Proposition 5.7(ii) agrees with the bottom edge of the lemma, so concatenation makes sense. Moreover the right hand column of the concatenated diagram agrees with the left hand column of the lower diagram in Proposition 5.6(ii). (The latter column was discussed above as composition of isomorphisms.) This allows a further concatenation, a giving commutative diagram

$$\begin{tikzcd}
\mathrm{Ind}_B^G(w\cdot\lambda\otimes p\tau\otimes \mathrm{V}^{[1]}) \arrow[r] \arrow[d, "\cong"] & \mathrm{T}_\mu^\lambda \mathrm{T}_\lambda^\mu \,\mathrm{Ind}\,(w\cdot\lambda\otimes p\tau\otimes \mathrm{V}^{[1]}) \arrow[r] \arrow[d, "\cong"] & \mathrm{T}_\mu^\lambda \,\mathrm{Ind}\,(w\cdot\mu\otimes p\tau\otimes \mathrm{V}^{[1]}) \arrow[dd, "\cong"] \\
\mathrm{Ind}_B^G(w\cdot\lambda\otimes p\tau)\otimes \mathrm{V}^{[1]} \arrow[r] \arrow[d, "="] & \mathrm{T}_\mu^\lambda \mathrm{T}_\lambda^\mu\,(\mathrm{Ind}_B^G(w\cdot\lambda\otimes p\tau)\otimes \mathrm{V}^{[1]}) \arrow[d, "\cong"] & \\
\mathrm{Ind}_B^G(w\cdot\lambda\otimes p\tau)\otimes \mathrm{V}^{[1]} \arrow[r] & \mathrm{T}_\mu^\lambda \mathrm{T}_\lambda^\mu \,\mathrm{Ind}\,(w\cdot\lambda\otimes p\tau)\otimes \mathrm{V}^{[1]} \arrow[r] & \mathrm{T}_\mu^\lambda \,\mathrm{Ind}\,(w\cdot\mu\otimes p\tau\otimes \mathrm{V}^{[1]}).
\end{tikzcd}$$

The top row maps are $\widetilde{adj}_{\mathrm{Ind}_B^G(w\cdot\lambda\otimes p\tau\otimes \mathrm{V}^{[1]}}$, on the left and $TIso_Y^{w\cdot\lambda}$, with $Y = p\tau\otimes \mathrm{V}^{[1]}$, on the right. In the same order, the bottom row maps are $\widetilde{adj}_{\mathrm{Ind}_B^G(w\cdot\lambda\otimes p\tau)}\otimes \mathrm{V}^{[1]}$ and $TIso_{p\tau}^{w\cdot\lambda}\otimes \mathrm{V}^{[1]}$. The composition of the two top row maps is $Adj_Y^{w\cdot\lambda}$, and the composition of the bottom row maps is $Adj_{p\tau}^{w\cdot\lambda}\otimes \mathrm{V}^{[1]}$. The commutativity of the outer rectangle now gives the desired commutativity of the diagram in (ii) $\square$

We are just about ready for a vast improvement to Proposition 5.4. First we need an easy but key observation.

Lemma 5.10. *For any $Y = p\tau\otimes V^{[1]}$ in $\mathcal{Y}$, the left column "obvious" isomorphisms in the diagrams of Proposition 5.5(ii) and Proposition 5.8(ii) are equal as are the right column.*

Proof. On the left, both isomorphism s just pull out $\mathrm{V}^{[1]}$ using the tensor identity. A similar isomorphism is used on the right (in both cases) except it is also necessary to commute $\mathrm{T}_\mu^\lambda(-)$ and $(-)\otimes \mathrm{V}^{[1]}$. $\square$

We can now prove a main theorem.

Theorem 5.11. *There is a nonzero scalar $c\in k$ such that $Adj_Y^{w\cdot\lambda} = cJan_Y^{w\cdot\lambda}$ for all $y\in\mathcal{Y}$.*

Proof. Proposition 5.4 gives a nonzero scalar that works in the especial case $Y = p\tau$. The constant it gives is possibly dependent on Y and call it $c(p\tau)$. Propositions 5.5(ii) and 5.8(iii), together with Lemma 5.10, show we claim, an equality

$$adj_Y^{w\cdot\lambda} = c(p\tau)Jan_{p\tau}^{w\cdot\lambda}.$$

whenever $Y = p\tau\otimes \mathrm{V}^{[1]}\in\mathcal{Y}$. To prove this equality, note $Adj^{w\cdot\lambda}{=}c(p\tau)Jan_{p\tau}^{w\cdot\lambda}$. Tensor on the right with $\mathrm{V}^{[1]}$ to get

$$Adj_{p\tau}^{w\cdot\lambda}\otimes \mathrm{V}^{[1]} = c(p\tau)Jan_{p\tau}^{w\cdot\lambda}\otimes \mathrm{V}^{[1]}$$

Precompose each side with the downward left column isomorphism common to the diagrams in Propositions 5.5(ii) and 5.8(ii), and postcompose with the upward right column isomorphism. This gives the claimed equality (reading it off from the two commutative diagrams and the previous equality.)

It remains to prove $c(p\tau) = c(p\tau')$ whenever $p\tau,\ p\tau'$ are 1-dimensional objects in $\mathcal{Y}$. Note that $p(\tau+\tau')$ with necessarily, also belong to $\mathcal{Y}$. We will show $c(p\tau) = c(p(\tau+\tau'))$. This is enough, since the equality $c(\tau') = c(p(\tau'+\tau))$ will follow by re-choosing notations.

Let $V = \mathrm{Ind}_B^G(\tau')$. The $p\tau \otimes \mathrm{V}^{[1]}$ and $p\tau \otimes p\tau' = p(\tau+\tau')$ both belong to Y. There a $B-$ module map ("evaluation") from $V = \mathrm{Ind}_B^G(\tau')$ onto τ', and we twist it by the Frobenius to get a surjective map $\mathrm{V}^{[1]} \longrightarrow p\tau'$. Tensor on the left with $p\tau$ to get a surjective map $p\tau \otimes \mathrm{V}^{[1]} \longrightarrow p(\tau+\tau')$. We will call this map ϕ. It is a map in the category $\mathcal{Y}$.

We have commutative diagram by naturality of $Jan^{w\cdot\lambda}$ with respect to $\mathcal{Y}$

$$\begin{tikzcd}
\mathrm{Ind}_B^G(w\cdot\lambda\otimes p(\tau+\tau')) \arrow[r, "Jan^{w\cdot\lambda}_{p(\tau+\tau')}"] & \mathrm{T}_\mu^\lambda\,\mathrm{Ind}\,(w\cdot\mu\otimes p(\tau+\tau')) \\
\mathrm{Ind}_B^G(w\cdot\lambda\otimes p\tau\otimes V^{[1]}) \arrow[u, "\mathrm{Ind}_B^G(w\cdot\lambda\otimes\phi)"] \arrow[r, "Jan^{w\cdot\lambda}_{p\tau\otimes\mathrm{V}^{[1]}}"] & \mathrm{T}_\mu^\lambda\,\mathrm{Ind}\,(w\cdot\mu\otimes p\tau\otimes\mathrm{V}^{[1]}) \arrow[u, "\mathrm{T}_\mu^\lambda\mathrm{Ind}\,(w\cdot\mu\otimes\phi)"']
\end{tikzcd}$$

The left column map is nonzero, since its composition with the evaluation map $\mathrm{Ind}_B^G(w\cdot\lambda\otimes p(\tau+\tau')) \longrightarrow w\cdot\lambda\otimes p(\tau+\tau')$ is nonzero. The top row map is injective, an instance of the left part of the short exact sequence displayed above Proposition 5.5. Hence the composition

$$Jan^{w\cdot\lambda}_{p(\tau+\tau')} \circ \mathrm{Ind}_B^G(w\cdot\lambda\otimes\phi)$$

is not zero.

However, there is a similar diagram, identical to the above, but with "Jan" replaced by "Adj". We have

$$Adj^{w\cdot\lambda}_{p(\tau+\tau')} \circ \mathrm{Ind}_B^G(w\cdot\lambda\otimes\phi) = c(p(\tau+\tau'))Jan^{w\cdot\lambda}_{p(\tau+\tau')} \circ \mathrm{Ind}_B^G(w\cdot\lambda\otimes\phi).$$

On the other hand, commutativity of the "Adj" diagram equates the left expression with $\mathrm{T}_\mu^\lambda\,\mathrm{Ind}_B^G(w\cdot\mu\otimes\phi)\circ Adj^{w\cdot\lambda}_{p\tau\otimes\mathrm{V}^{[1]}} = \mathrm{T}_\mu^\lambda\,\mathrm{Ind}_B^G(w\cdot\mu\otimes\phi)\circ c(p\tau)Jan^{w\cdot\lambda}_{p\tau\otimes\mathrm{V}^{[1]}}$. Now bring out the scalar $c(\phi)$ and apply commutativity in the "Jan" diagram. The right expression becomes $c(p\tau)Jan^{w\cdot\lambda}_{p(\tau+\tau')}\mathrm{Ind}_B^G(w\cdot\lambda\otimes\phi)$. We have shown

$$c(p(\tau+\tau'))Jan^{w\cdot\lambda}_{p(\tau+\tau')} \circ \mathrm{Ind}_B^G(w\cdot\lambda\otimes\phi) = c(p\tau)Jan^{w\cdot\lambda}_{p(\tau+\tau')} \circ \mathrm{Ind}_B^G(w\cdot\lambda\otimes\phi)$$

But we have shown that the map appearing to the right of both $c(p(\tau+\tau'))$ and $c(p\tau)$ above is not zero. So, the only way the equality can occur is to have $c(p(\tau+\tau') = c(p\tau)$.

This completes the proof of the theorem. $\square$

Though not entirely necessary, it simplifies notation if we modify $TIso_Y^{w\cdot\lambda}$ by a scalar by changing its construction, as per the observation in the proof of Proposition 5.6. We do this so that the newly constructed $TIso_Y^{w\cdot\lambda}$ is the old one multiplied by c above (uniformly in $Y \in \mathcal{Y}$). This allows us to have actual equalities.

$Adj_Y^{w\cdot\lambda} = Jan_Y^{w\cdot\lambda}$ for all $Y \in \mathcal{Y}$. This is also a good time to enlarge the domain of the natural transformation $Adj^{w\cdot\lambda}$ and $Jan_Y^{w\cdot\lambda}$ from $\mathcal{Y}$ to $add\,\mathcal{Y}$ (which has objects direct sums of objects of $\mathcal{Y}$). Similar domain enlargements can be made for $Iso^{w\cdot\lambda}$ and $TIso^{w\cdot\lambda}$. The domain $add\,\mathcal{Y}$ can also be extended to complexes of objects from $add\,\mathcal{Y}$, such as the complex $K^\bullet$ discussed as Lemma 5.3.

In the next proposition, we use this formalism to illuminate the triangle (*) above the claim in 5.1, rewritten below

$$(*) \qquad \cdots \longrightarrow \mathrm{RInd}_B^G(w\cdot\lambda) \longrightarrow \mathrm{T}_\mu^\lambda\,\mathrm{RInd}_B^G(w\cdot\lambda) \longrightarrow \mathrm{RInd}_B^G(ws\cdot\lambda) \longrightarrow \cdots$$

Proposition 5.12. *With a suitable choice of the complex $K^\bullet$ in Lemma 5.3, there is an exact sequence of complexes*

$$0 \longrightarrow Ind(w\cdot\lambda\otimes K^\bullet) \longrightarrow T_\mu^\lambda\, Ind_B^G(w\cdot\mu\otimes K^\bullet) \longrightarrow Ind_B^G(ws\cdot\lambda\otimes K^\bullet) \longrightarrow 0$$

which represent (*) at the level of complexes (in the sense that its sequence of these objects and two maps – ignoring the $0'$s – identifies, after passing to the bounded derived category, with the displayed portion above of (*)).

The left hand complex map is $Jan_{K^\bullet}^{w\cdot\lambda}$, the extension of $Jan^{w\cdot\lambda}$ to complexes of $add\,\mathcal{Y}$ objects, in the particular case of the complex $K^\bullet$.

Proof. Choose $K^\bullet$ so that each K^n is a direct sum of terms $Y = p\tau\otimes \mathrm{V}^{[1]}$ in $\mathcal{Y}$ with also $\nu + p\tau$ dominant for each weight ν of $L^*\otimes w\cdot\mu$. The construction of (*) is from the exact sequence at the top of 5.1

$$0 \longrightarrow M \longrightarrow L^*\otimes w\cdot\mu \longrightarrow M' \longrightarrow 0$$

by applying $pr_\lambda \mathrm{RInd}_B^G(-)$. With our choice of $K^\bullet$, $\mathrm{RInd}_B^G(-)$ applied to each term is the same as $\mathrm{Ind}_B^G(-\otimes K^\bullet)$. Also, pr_λ applied to $\mathrm{Ind}_B^G(\nu\otimes K^\bullet)$ for $\nu\neq w\cdot\lambda$ in M or $\nu\neq ws\cdot\lambda$ in M', is the zero complex.

Thus, $pr_\lambda \mathrm{Ind}_B^G(-\otimes K^\bullet)$, applied to the displayed sequence, gives an exact sequence

$$0 \longrightarrow \mathrm{Ind}_B^G(w{\cdot}\lambda{\otimes}K^\bullet) \longrightarrow pr_\lambda \mathrm{Ind}(L^*{\otimes}w{\cdot}\mu{\otimes}K^\bullet) \longrightarrow \mathrm{Ind}_B^G(ws{\cdot}\lambda{\otimes}K^\bullet) \longrightarrow 0$$

The middle term identifies with $pr_\lambda(L^* \otimes \mathrm{Ind}_B^G(w \cdot \mu \otimes K^\bullet))$ via the tensor identity. Such an identification must also be made in the construction of (*), though with $\mathrm{RInd}_B^G(w \cdot \mu)$ replacing $\mathrm{Ind}_B^G(w \cdot \mu \otimes K^\bullet)$. Note also $pr_\lambda(L^* \otimes \mathrm{Ind}_B^G(w \cdot \mu \otimes K^\bullet)) = pr_\lambda(L^* \otimes pr_\mu \mathrm{Ind}_B^G(w \cdot \mu \otimes K^\bullet)) = \mathrm{T}_\mu^\lambda \mathrm{Ind}_B^G(w \cdot \mu \otimes K^\bullet)$, and a similar equality holds for $\mathrm{RInd}_B^G(w \cdot \mu)$.

Following each step above gives the identifications claimed in the proposition. An alternate argument could be made by replacing $K^\bullet$ in the short exact sequence displayed above with a complex of injective B-modules (a resolution of $k = k(0)$ also). This gives a semisplit short exact sequence of complexes. Its three term sequence then automatically becomes a three term sequence in a distinguished triangle, upon passing to the derived category. In more detail, let $K^\bullet \longrightarrow I^\bullet$ be an isomorphism of complexes, with $I^\bullet$ a B-module injective resolution of $k = k(0)$. There are resulting commutative diagram of map of G-module complexes

$$\begin{tikzcd}
0 \arrow[r] & p\tau\mathrm{Ind}(M \otimes I^\bullet) \arrow[r] & pr_\lambda \mathrm{Ind}_B^G(L^* \otimes w \cdot \mu \otimes I^\bullet) \arrow[rr] \arrow[dr] & & p\tau\mathrm{Ind}(M \otimes I^\bullet) \arrow[r] & 0 \\
& & & pr_\lambda(L^* \otimes \mathrm{Ind}_B^G(w \cdot \mu \otimes I^\bullet & & \\
0 \arrow[r] & p\tau\mathrm{Ind}(M \otimes K^\bullet) \arrow[r] \arrow[uu] & pr_\lambda \mathrm{Ind}_B^G(L^* \otimes w \cdot \mu \otimes K^\bullet) \arrow[rr] \arrow[dr] \arrow[uu] & & p\tau\mathrm{Ind}(M \otimes K^\bullet) \arrow[r] \arrow[uu] & 0 \\
& & & pr_\lambda(L^* \otimes \mathrm{Ind}_B^G(w \cdot \mu \otimes I^\bullet \arrow[uu] & &
\end{tikzcd}$$

The skew maps are isomorphism of complexes, and the vertical maps are all quasi-isomorphisms. The top row is an exact sequence of complexes of injective objects, is therefore semi-split, and therefore becomes part of a distinguished triangle (ignoring the zeros and zero maps) at the derived category label (D^+ or D^b here). There are a few other commutative squares of quasi isomorphism need to give a complete picture of the identification claimed in the proposition, but we leave then to the reader (who should have the idea by now). On the left, for example, diagrams must be added handling the identifications $\mathrm{Ind}_B^G(w \cdot \lambda \otimes K^\bullet) \cong pr_\lambda \mathrm{Ind}_B^G(M \otimes K^\bullet)$. (This will require two rectangles, associated with the location of $w \cdot \lambda$ as a section of M.)

The analogous and simpler identification $\mathrm{Ind}_B^G(w\cdot\lambda\otimes Y)\cong pr_\lambda \mathrm{Ind}_B^G(M\otimes Y)$ is part of the Jantzen map $Jan_Y^{w\cdot\lambda}$, discussed above Proposition 5.5, partly using "$\mathrm{RInd}_B^G{}''$ notation. Sticking to the "Ind_B^G" notation, the map $Jan_Y^{w\cdot\lambda}$ is the composite of $\mathrm{Ind}_B^G(w\cdot\lambda\otimes Y)\cong pr_\lambda(L^*\otimes \mathrm{Ind}_B^G(w\cdot\mu\otimes Y)$, with $pr_\lambda \mathrm{Ind}_B^G(M\otimes Y)\to pr_\lambda I\mathrm{Ind}_B^G(L^*\otimes w\cdot\mu\otimes Y)\cong pr_\lambda(L^*\otimes Ind_B^G(w\cdot\mu\otimes Y))$, the latter equal to $pr_\lambda(L^*\otimes pr_\mu \mathrm{Ind}_B^G(w\cdot\mu\otimes Y)\cong \mathrm{T}_\mu^\lambda\,\mathrm{Ind}\,(w\cdot\mu\otimes Y)$.

Passing from $\mathcal{Y}$ to $add\,\mathcal{Y}$ and then to complexes of $add\,\mathcal{Y}$ objects, and, following the pathway above, we find that $Jan_{K^\bullet}^{w\cdot\lambda}$ is the composition of the identification $\mathrm{Ind}_B^G(w\cdot\lambda\otimes K^\bullet)\cong pr_\lambda \mathrm{Ind}_B^G(M\otimes K^\bullet)$, the bottom left map of the above diagram followed by the adjacent skew map, and finally the identification

$$pr_\lambda(L^*\otimes \mathrm{Ind}_B^G(w\cdot\mu\otimes K^\bullet)=pr_\lambda(L^*\otimes pr_\mu(\mathrm{Ind}_B^G(w\cdot\mu\otimes K^\bullet)=\mathrm{T}_\mu^\lambda\,\mathrm{Ind}_B^G(w\cdot\mu\otimes K^\bullet).$$

This is, altogether, precisely the map

$$\mathrm{Ind}_B^G(w\cdot\lambda\otimes K^\bullet)\longrightarrow \mathrm{T}_\mu^\lambda\,\mathrm{Ind}_B^G(w\cdot\mu\otimes K^\bullet)$$

which our construction, in completed form, gives for the left hand amp in the exact sequences displayed in the proposition. So that map is $Jan_{K^\bullet}^{w\cdot\lambda}$, and our proof of the proposition is complete. $\square$

The general case of the claim of 5.1 Recall that we noted in 5.1 that the middle term of the distinguished triangle (*) was isomorphic to $\mathrm{T}_\mu^\lambda\,\mathrm{T}_\lambda^\mu\,\mathrm{RInd}_B^G(w\cdot\lambda)$. Noting further that this resulted in a map $\mathrm{RInd}_B^G(w\cdot\lambda)\longrightarrow \mathrm{T}_\mu^\lambda\,\mathrm{T}_\lambda^\mu\,\mathrm{RInd}_B^G(w\cdot\mu)$, we claimed that this map was adjunction, at least up to scalar multiple.

To some extent, this requires first interpreting what isomorphism of $\mathrm{T}_\mu^\lambda\,\mathrm{T}_\lambda^\mu\,\mathrm{RInd}_B^G(w\cdot\lambda)$ with the middle term $\mathrm{T}_\mu^\lambda\,\mathrm{RInd}_B^G(w\cdot\mu)$ of (*) was intended. The top of Subsection 5.1 indicates that an isomorphism $\mathrm{T}_\lambda^\mu\,\mathrm{RInd}_B^G(w\cdot\lambda)\cong \mathrm{RInd}_B^G(w\cdot\mu)$ be used. We will follow that framework. Represent $\mathrm{RInd}_B^G(w\cdot\lambda)$ by $\mathrm{Ind}_B^G(w\cdot\lambda\otimes K^\bullet)$ and $\mathrm{RInd}_B^G(w\cdot\mu)$ by $\mathrm{Ind}_B^G(w\cdot\mu\otimes K^\bullet)$, as in Proposition 5.12 and its proof. Then an isomorphism $\mathrm{T}_\lambda^\mu\,\mathrm{Ind}_B^G(w\cdot\lambda\otimes K^\bullet)\cong \mathrm{Ind}_B^G(w\cdot\mu\otimes K^\bullet)$ is s given by $Iso_{K^\bullet}^{w\cdot\lambda}$. Composing with T_μ^λ on both sides gives the isomorphism of complexes

$$TIso_{K^\bullet}^{w\cdot\lambda}:\mathrm{T}_\mu^\lambda\,\mathrm{T}_\lambda^\mu\,\mathrm{Ind}\,(w\cdot\lambda\otimes K^\bullet)\cong \mathrm{T}_\mu^\lambda\,\mathrm{Ind}\,(w\cdot\mu\otimes K^\bullet).$$

Note that the two sides represent $\mathrm{T}_\mu^\lambda\,\mathrm{T}_\lambda^\mu\,\mathrm{RInd}_B^G(w\cdot\lambda)$ and $\mathrm{T}_\mu^\lambda\,\mathrm{RInd}_B^G(w\cdot\mu)$, respectively.

Finally, we need a way to represent the adjunction map from $\mathrm{RInd}_B^G(w\cdot\lambda)$, represented as $\mathrm{Ind}_B^G(w\cdot\lambda\otimes K^\bullet)$, to $\mathrm{T}_\mu^\lambda\,\mathrm{T}_\lambda^\mu\,\mathrm{RInd}_B^G(w\cdot\lambda)$, which we have represented as $\mathrm{T}_\mu^\lambda\,\mathrm{T}_\lambda^\mu\,\mathrm{Ind}_B^G(w\cdot\lambda\otimes K^\bullet)$. For this, we use the map of complexes

$$\widetilde{adj}_{\mathrm{Ind}_B^G(w\cdot\lambda\otimes K^\bullet)}:\mathrm{Ind}_B^G(w\cdot\lambda\otimes K^\bullet)\longrightarrow \mathrm{T}_\mu^\lambda\,\mathrm{T}_\lambda^\mu\,\mathrm{Ind}\,(w\cdot\lambda\otimes K^\bullet).$$

This map just applies adjunction to the G-modules in each degree of the complex $\mathrm{Ind}_B^G(w\cdot\lambda\otimes K^\bullet)$. This results in a map of complexes, by naturality of adjunction. In particular, passing to the derived category level (D^+ or D^b), the map of complexes $\tilde{adj}_{\mathrm{Ind}_B^G(w\cdot\lambda\otimes K^\bullet}$ induces "adjunction" as a map

$$\mathrm{RInd}_B^G(w\cdot\lambda)\longrightarrow \mathrm{T}_\mu^\lambda\,\mathrm{T}_\lambda^\mu\,\mathrm{RInd}_B^G(w\cdot\lambda).$$

The result below is a corollary to Theorem 5.11 and Proposition 5.12.

Corollary 5.13. *With the isomorphism $T_\mu^\lambda\,RInd_B^G(w\cdot\mu)\xrightarrow{\sim}T_\mu^\lambda\,T_\lambda^\mu\,RInd_B^G(w\cdot\lambda)$ taken as inverse to the isomorphism induced by $TIso_{K^\bullet}^{w\cdot\lambda}$ discussed above, the claim of Subsection 5.1 is correct. More precisely, we have a commutative diagram*

$$\begin{tikzcd}
RInd_B^G(w\cdot\lambda) \arrow[r] \arrow[d, "="] & T_\mu^\lambda\,T_\lambda^\mu\,RInd_B^G(w\cdot\lambda) \arrow[d, "\cong"] \\
RInd_B^G(w\cdot\lambda) \arrow[r] & T_\mu^\lambda\,RInd_B^G(w\cdot\mu)
\end{tikzcd}$$

with the top row adjunction and the bottom row from $(*)$. The right column isomorphism is as above.

Proof. We have $Adj_Y^{w\cdot\lambda}=TIso_Y^{w\cdot\lambda}\circ\widetilde{adj}_{\mathrm{Ind}_B^G(w\cdot\lambda\otimes Y)}$ for all $Y\in\mathcal{Y}$, by definition. Passing to *adj* $\mathcal{Y}$ and complexes such as $K^\bullet$, we have the similar identity

$$Adj_{K^\bullet}^{w\cdot\lambda}=TIso_{K^\bullet}^{w\cdot\lambda}\circ\widetilde{adj}_{\mathrm{Ind}\,(w\cdot\cdot\lambda\otimes K^\bullet)}$$

By Theorem 5.11 we also have

$$Adj_{K^\bullet}^{w\cdot\lambda}=cJan_{K^\bullet}^{w\cdot\lambda}$$

where c is a non zero scalar. If we adjust $TIso^{w\cdot\lambda}$ as per the (bold-faced), observation in the proof of Proposition 5.6, we may assume $c=1$. Assume that this adjustment is in force. Then we have a commutative diagram at complexes

$$\begin{tikzcd}
\mathrm{Ind}_B^G(w\cdot\lambda\otimes K^\bullet) \arrow[r, "\widetilde{adj}_{\mathrm{Ind}_B^G(w\cdot\lambda\otimes K^\bullet)}"] \arrow[d, "="] & \mathrm{T}_\mu^\lambda\,\mathrm{T}_\lambda^\mu\,\mathrm{Ind}\,(w\cdot\lambda\otimes K^\bullet) \arrow[d, "TIso_{K^\bullet}^{w\cdot\lambda}"] \\
\mathrm{Ind}_B^G(w\cdot\lambda\otimes K^\bullet) \arrow[r, "Jan_{K^\bullet}^{w\cdot\lambda}"] & \mathrm{T}_\mu^\lambda\,\mathrm{Ind}\,(w\cdot\mu\otimes K^\bullet)
\end{tikzcd}$$

By Proposition 5.6 the bottom arrow represents the map $\mathrm{RInd}_B^G(w\cdot\lambda) \longrightarrow \mathrm{T}_\mu^\lambda\,\mathrm{RInd}_B^G(w\cdot\mu)$ in (*). We discussed, above the statement of the corollary, the fact that the top row becomes the adjunction map $\mathrm{RInd}_B^G(w\cdot\lambda) \longrightarrow \mathrm{T}_\mu^\lambda\,\mathrm{T}_\lambda^\mu\,\mathrm{RInd}_B^G(w\cdot\lambda)$ upon passing to the derived category. The right column map is, as discussed in the statement of the corollary the right column derived derived category, isomorphism in the corollary's diagram. Altogether, the commutativity of the diagram of complexes above gives the commutativity of the diagram in the corollary. This completes its proof. □

Remark 5.14. Without the adjustment observed in the proof of Proposition 5.6, we only get commutativity up to a scalar, as allowed in the claim.

Remark 5.15. The quantum case

The same changes of $\otimes$ to $\otimes^{op}$ observed in Appendix A need to be made in this appendix, in the quantum case. In addition it is necessary to replace the references to [J] in the proof of Proposition 5.6 with references to Remarks 2.11(d), (e). Remark 2.11(f) helps explain the differences in the formalism of these remarks (which also could be used in the algebraic groups case) with that of [J]. Recall also that Remark 2.11(d) provides both right and left generalized tensor identities in the quantum case, heavily used in the arguments above (for example, in the proofs of Propositions 5.5 and 5.6. With these changes and observations, all of the proofs and results in this appendix carry over to the quantum case.

In particular, the claim of Subsection 5.1 holds in both the algebraic groups and quantum cases. As argued below the claim, this completes the proof of Lemma 3.2.

6. Appendix C

The purpose of this appendix is to supplement, and, indeed, to "fix," the statement and proof of [ABG, Lem. 9.10.5], as a service to the reader. This is all in characteristic 0, and not part of the induction theorem (except in the way of application), but it is important to [ABG] as a whole and to the discussion in [PS2, ftn.13] concerning Koszulity in the quantum case. The proof given in [ABG] of the lemma, corrected for misprints and issues with the induction theorem proof, still seemed inaccurate to us, but we found it could be fixed using an algebraic result from [PS2]. The latter result is nontrivial, but relatively elementary, not using the Lusztig quantum conjecture. This seems desirable, so that [ABG] could have the latter conjecture, in the $\ell > h$ case, as a corollary.

Our notation in this appendix largely follows [ABG], with two major changes: The formula for the "dot" notation $\bullet$ is replaced by that for the standard "dot" action $\cdot$ in [J] and Subsection 2.4.2 above. Thus, the new formula reads, for w in the Weyl group or affine Weyl group, and $\lambda \in \mathbb{X}$,

$$w \bullet \lambda = w(\lambda + \rho) - \rho.$$

Also, we will use Borel subalgebras B whose associated roots are negative, rather than positive. With these two changes, [ABG, Lem. 3.5.1], which we will use below, is correct as stated. (It actually was not, before, even for $w = 1$.) The statement of the quantum induction theorem [ABG, Thm. 3.5.5], which we will also use, is unchanged. Finally, the change from positive to negative Borels on the quantum side is deliberately not repeated on the Langlands dual side, when choosing Borel objects there (associated to Grassmannian varieties).

At the point the result [ABG, Lem. 9.10.5] in question is introduced in [ABG] the authors have established an equivalence of derived categories [ABG, (9.10.1)]

$$\gamma : D^b\,\mathsf{block}(\mathsf{U}) \to D^b\ \mathcal{P}erv(\mathrm{Gr}),$$

and it is desired to show the functor γ induces an equivalence from $\mathsf{block}(\mathsf{U})$ to $\mathcal{P}erv(\mathrm{Gr})$. To this end, categories $D^b_{\leq\lambda}\ \mathsf{block}(\mathsf{U})$ and $D^p_{\leq\lambda}\mathcal{P}erv$ are introduced "for each $\lambda \in \mathbb{Y}^{++}$." This appears to be a misprint, repeated several times on [ABG, p. 668], and the definition of $D^b_{\leq\lambda}\ \mathsf{block}(\mathsf{U})$ is incorrect with any choice of λ. Instead, these categories should be introduced for each $\lambda \in \mathbb{Y}$, with $\mu \leq \lambda$ interpreted to mean $\tilde{\mu} \uparrow \tilde{\lambda}$, where $\tilde{\mu} \in W \bullet \ell\mu$ is, in our notation here, the (unique) dominant weight, and $\tilde{\lambda}$ is defined similarly.

The order $\uparrow$ above is that discussed in [J, II,6.1-6.11]; note that p there is allowed to be any positive integer. The order $\uparrow$ should replace the order $\preceq$ in [ABG, (3.4.5)]. The following equivalence (in the case $\ell \geq h$)) follows from the more general theorem [PS2, Thm. 9.6], which also has a formulation for $\ell < h$

$$y \bullet 0 \uparrow w \bullet 0 \text{ iff } y \leq' w,$$

whenever $y \bullet 0$, $w \bullet 0$ are dominant and $y, w \in W_{\text{aff}}$. The order $\leq'$ is the Bruhat–Chevalley order with respect to the dominant standard chamber fundamental reflections. This equivalence seems essential to correct the lemma.

We shall use $\leq$ for the Bruhat–Chevalley order with respect to the antidominant standard chamber. Thus, for $y, w \in W_{\text{aff}}$, $y \leq w$ iff $w_0 y w_0 \leq' w_0 w w_0$, with w_0 the long word in W. When y, w are in W, $y \leq w$ means the same as $y \leq' w$. When $\nu, \mu \in \mathbb{Y}$, $\nu y \leq \mu w$ iff $(-\nu)y \leq' (-\mu)w$. (The classical root system has an automorphism $y \mapsto -w_0(y)$, preserving positive roots.) Notice there is an implicit change from $\leq'$ to $\leq$ in the proof of [ABG, Cor. 8.3.2]. (This occurs in the assertion that "λw^{-1} is minimal in the right coset $\lambda W \leq W^n_{\text{aff}}$." The hypothesis of Cor. 8.3.2(ii) gives minimality of $w\lambda$ with respect to $\leq'$. Passing to inverses gives minimality of $(-\lambda)w^{-1}$ with respect to $\leq'$. Now it is necessary, it seems, to use $\leq$ to get minimality of λw^{-1}.)

The definition of $D^b_{\leq\lambda}\mathcal{P}erv$ is correct as given in [ABG, p. 668] provided it is allowed that $\lambda \in \mathbb{Y}$. Similarly, λ should be taken in $\mathbb{Y}$ in the statement of the lemma, which we provide below, with this change. Note the direction of Υ is reverse to the equivalence in part (i).

Lemma 6.1 ([ABG, 9.10.5])**.** *For any $\lambda \in \mathbb{Y}$, we have*

(i) the functor Υ induces an equivalence

$$D^b_{\leq\lambda}\mathcal{P}erv \xrightarrow{\sim} D^b_{\leq\lambda}\ \mathsf{block}(\mathsf{U}).$$

Moreover,

(ii) the induced functor

$$D^b_{\leq\lambda}\mathcal{P}erv/D^b_{<\lambda}\mathcal{P}erv \longrightarrow D^b_{\leq\lambda}\ \mathsf{block}(\mathsf{U})/D^b_{<\lambda}\ \mathsf{block}(\mathsf{U})$$

sends the class of IC_λ to the class of L_λ.

Proof. We follow [ABG], taking into account the changes above, and also the misprints noted in [ABG, p. 675]. There are also some inaccuracies in [ABG, Cor. 8.2.4, Cor. 8.3.2] which we address as they arise.

We know for any $\lambda \in \mathbb{Y}$, the functor Υ sends, by construction, the object $R\ \mathrm{Ind}_{\mathsf{B}}^{\mathsf{U}}(\ell\lambda)$ to $\overline{\mathcal{W}}_\lambda$. Fix $\lambda \in \mathbb{Y}$, and let $w \in W$ be the element with $w\lambda \bullet 0 = w \bullet \ell\lambda$ dominant. Then, by [ABG, Lem. 3.5.1] – see our Remark 2.11(a) and Subsections 2.5, 2.6 for additional details–we have that $R^{\ell(w)}\mathrm{Ind}_{\mathsf{B}}^{\mathsf{U}}(\ell\lambda)$ has L_λ as a composition factor with multiplicity one, and all other composition factors L_μ of $R^{\ell(w)}\mathrm{Ind}_{\mathsf{B}}^{\mathsf{U}}(\ell\lambda)$, or any composition factor L_μ of $R^j\mathrm{Ind}_{\mathsf{B}}^{\mathsf{U}}(\ell\lambda)$ with $j \neq \ell(w)$, satisfy $y\mu \bullet 0 \uparrow w\lambda \bullet 0$, with $y \in W_{\text{aff}}$, $y\mu \bullet 0$ dominant, and

$y\mu\bullet 0 \neq w\lambda\bullet 0$. For any such μ, we have $y\mu <' w\lambda$, as noted above, and $\mu y^{-1} < \lambda w^{-1}$. As observed in the proof of [ABG, Cor. 8.3.2], this implies $\operatorname{supp} \overline{\mathcal{W}}_\mu = \overline{\mathrm{Gr}}_\mu \subsetneq \overline{\mathrm{Gr}}_\lambda = \operatorname{supp} \overline{\mathcal{W}}_\lambda$, and the inclusion is proper. Thus, Υ takes $D^b_{\leq\lambda}$ $\mathsf{block}(\mathsf{U})$ into $D^b_{\leq\lambda}\mathcal{P}erv$. By induction (on, say, the height of the dominant weight $w\bullet\ell\lambda$), we may assume Υ induces an equivalence of triangulated categories, when $\leq\lambda$ is replaced by $<\lambda$. (Here $\mu<\lambda$ is taken to mean $y\mu\bullet 0 \uparrow w\lambda\bullet 0$ as above, with $y\in W$ and $y\mu\bullet 0\uparrow w\lambda\bullet 0$, $y\mu\bullet 0 \neq w\lambda\bullet 0$. Equivalently, $\overline{\mathrm{Gr}}_\mu$ is properly contained in $\overline{\mathrm{Gr}}_\lambda$, as we have seen.) By [ABG, §9.1, p. 655] $\mathcal{P}erv(\mathrm{Gr})$ is generated by simple objects IC_ν, $\nu\in\mathbb{Y}$, each with support contained in $\overline{\mathrm{Gr}}_\nu$.

We take this opportunity to mention there are errors of sign in [ABG, Cor. 8.2.4, Cor. 8.3.2], where $\mathbb{C}_{yw}[-\dim\ \mathcal{B}_{yw}]$ should be replaced by $\mathbb{C}_y[\dim\ \mathcal{B}_{yw}]$ and $\mathbb{C}_\lambda[-\dim \mathrm{Gr}_\lambda - \ell(w)]$ should be replaced by $\mathbb{C}_\lambda[\dim \mathrm{Gr} - \ell(w)]$.[13]

With these changes, the conclusion of [ABG, Cor. 8.3.2(ii)] shows $\overline{\mathcal{W}}_\lambda$ and $IC_\lambda[-\ell(w)]$ have the same restriction to Gr_λ (from $\overline{\mathrm{Gr}}_\lambda$). This shows, together with the fact that Υ induces an equivalence $D^b_{<\lambda}\,\mathsf{block}(\mathsf{U}) \to D^b_{<\lambda}\mathcal{P}erv$, which we obtained above by induction, that the strict image under Υ of $D^b_{\leq\lambda}$ $\mathsf{block}(\mathsf{U})$ is $D^b_{\leq\lambda}\mathcal{P}erv$. Since we already know Υ provides an equivalence $D^b\,\mathsf{block}(\mathsf{U}) \to D^b\mathcal{P}erv$, it follows now that it induces one between $D^b_{\leq\lambda}$ $\mathsf{block}(\mathsf{U})$ and $D^b_{\leq\lambda}\mathcal{P}erv$. This proves (i).

Moreover, we also get (ii), since we have shown that $\Upsilon(R\,\mathrm{Ind}_{\mathsf{B}}^{\mathsf{U}}(\ell\lambda)) = \overline{\mathcal{W}}_\lambda$ is $IC_\lambda[-\ell(w)]$ in the quotient category $D^b_{\leq\lambda}\mathcal{P}erv/D^b_{<\lambda}\,\mathcal{P}erv$. Our remarks on the composition factors of the cohomology groups of the preimage $R\,\mathrm{Ind}_{\mathsf{B}}^{\mathsf{U}}(\ell\lambda)$ of $\overline{\mathcal{W}}_\lambda$ show that the image in the quotient category $D^b_{\leq\lambda}\,\mathsf{block}(\mathsf{U})/D^b_{<\lambda}$ $\mathsf{block}(\mathsf{U})$ is $L_\lambda[-\ell(w)]$. This completes the proof of (ii) and the lemma. □

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[13]This latter change needs to be made both in the statement and proof of [ABG, Cor. 8.3.2]. The error is in ignoring the shift in degree that can accompany direct images with proper maps.

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On modular categories $\mathcal{O}$ for quantized symplectic resolutions

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Dedicated to George Lusztig, on his 75th birthday, with admiration

Abstract: In this paper we study highest weight and standardly stratified structures on modular analogs of categories $\mathcal{O}$ over quantizations of symplectic resolutions. We also show how to recover the usual categories $\mathcal{O}$ (reduced mod $p \gg 0$) from our modular categories. More precisely, we consider a conical symplectic resolution that is defined over a finite localization of $\mathbb{Z}$ and is equipped with a Hamiltonian action of a torus T that has finitely many fixed points. We consider algebras $\mathcal{A}_\lambda$ of global sections of a quantization in characteristic $p \gg 0$, where λ is a parameter. Then we consider a category $\tilde{\mathcal{O}}_\lambda$ consisting of all finite dimensional T-equivariant $\mathcal{A}_\lambda$-modules. Under reasonable assumptions that hold in most examples of interest, we show that for λ lying in a *p-alcove* ${}^pA$, the category $\tilde{\mathcal{O}}_\lambda$ is highest weight (in some generalized sense). Moreover, we show that every face of ${}^pA$ that survives in ${}^pA/p$ when $p \to \infty$ defines a standardly stratified structure on $\tilde{\mathcal{O}}_\lambda$. We identify the associated graded categories for these standardly stratified structures with reductions mod p of the usual categories $\mathcal{O}$ in characteristic 0. Applications of our construction include computations of wall-crossing bijections in characteristic p and the existence of gradings on categories $\mathcal{O}$ in characteristic 0.

1. Introduction

In this paper we study some aspects of the representation theory of quantizations of symplectic resolutions in large positive characteristic.

1.1. Representations in characteristic 0

We start by recalling some features of the representation theory in characteristic 0, which is more classical and more extensively studied. Let X be a conical

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symplectic resolution over $\mathbb{C}$ that is defined over a finite localization of $\mathbb{Z}$. We assume that it comes with an action of a torus T that is Hamiltonian, has finitely many fixed points, and is defined over a finite localization of $\mathbb{Z}$. Examples are provided by $X = T^*(G/B)$ (where G is a semisimple algebraic group, B is a Borel subgroup, and T is a maximal torus in B) or $X = \mathrm{Hilb}_n(\mathbb{A}^2)$ (and T is a one-dimensional torus acting on $\mathbb{A}^2$ in a Hamiltonian way, the action lifts to $\mathrm{Hilb}_n(\mathbb{A}^2)$).

Consider the space $\mathfrak{P} := H^2(X, \mathbb{C})$. To $\lambda \in \mathfrak{P}$, we can assign a quantization $\mathcal{A}_\lambda^\theta$ of X that is a microlocal sheaf of filtered algebras on X. For example, in the case of $X = T^*(G/B)$ we get the microlocalization of the sheaf of $\lambda - \rho$-twisted differential operators on G/B. Taking the global sections of the sheaf $\mathcal{A}_\lambda^\theta$, we get a filtered quantization of $\mathbb{C}[X]$ to be denoted by $\mathcal{A}_\lambda$. In the case when $X = T^*(G/B)$ we get the central reductions of the universal enveloping algebra $U(\mathfrak{g})$, while for the case of $X = \mathrm{Hilb}_n(\mathbb{A}^2)$ we recover the spherical subalgebras of the rational Cherednik algebras for S_n (introduced in [EG]). We note that T acts on $\mathcal{A}_\lambda$ and the action is Hamiltonian.

Since T has finitely many fixed points in X, it makes sense to speak about categories $\mathcal{O}$ for $\mathcal{A}_\lambda$, see [BLPW, Sections 3.2,3.3]. These categories depend on a choice of a generic one-parameter subgroup $\nu : \mathbb{C}^\times \to T$ (where "generic" means that $X^{\nu(\mathbb{C}^\times)} = X^T$). The category corresponding to ν is denoted by $\mathcal{O}_\nu(\mathcal{A}_\lambda)$. It can be defined as the category of finitely generated $\mathcal{A}_\lambda$-modules that admit a $\nu(\mathbb{C}^\times)$-equivariant structure so that the weights of ν are bounded from above. For $X = T^*(G/B)$ and regular λ, we recover an infinitesimal block of the classical BGG category $\mathcal{O}$ (and the simples are labelled by the elements of the Weyl group), while for $X = \mathrm{Hilb}_n(\mathbb{A}^2)$ we get the category $\mathcal{O}$ for the rational Cherednik algebra for S_n (and the simples are labelled by the partitions of n). The categories $\mathcal{O}$ over quantized symplectic resolutions have been studied very extensively recently from various perspectives, see, e.g., [BLPW, L6, L9]– not to mention numerous papers dealing with special cases such as BGG categories $\mathcal{O}$, categories $\mathcal{O}$ over rational Cherednik algebras, Nakajima quiver varieties and quantized Coulomb branches.

Let us list some known results about the structure of $\mathcal{O}_\nu(\mathcal{A}_\lambda)$. We can talk about *regular* (i.e., non-degenerate) parameters λ. Under mild assumptions, we will show below that there is a finite union of affine hyperplanes (to be called *singular*) in $\mathfrak{P}$ such that a parameter outside this finite union is regular. For a regular parameter, the simples in $\mathcal{O}_\nu(\mathcal{A}_\lambda)$ are labelled by X^T, [BLPW, Section 5.1]. Also it is known that $\mathcal{O}_\nu(\mathcal{A}_\lambda)$ is a highest weight category provided λ is regular, [BLPW, Section 5.2].

In the examples we consider, these finite collections of hyperplanes are as follows. For $X = T^*(G/B)$ we just take the root hyperplanes, $\ker \alpha^\vee$, where

$\alpha^\vee$ runs over the set of coroots. For $X = \mathrm{Hilb}_n(\mathbb{A}^2)$ we usually consider a shifted parameter, $c = \lambda - 1/2$. Here for the hyperplanes we can take the following collection of points: $\{-\frac{a}{b} + i | 1 \leqslant a < b \leqslant n, i \in \mathbb{Z}, |i| \leqslant s\}$ for some non-negative integer s (one can actually prove that $s = 0$ but this is not going to be relevant for our purposes).

Note that we can also consider the category of T-equivariant objects $\mathcal{O}^T_\nu(\mathcal{A}_\lambda)$ in $\mathcal{O}_\nu(\mathcal{A}_\lambda)$. This category is the direct sum of several copies of $\mathcal{O}_\nu(\mathcal{A}_\lambda)$ labelled by the characters of T, so we do not get anything new. The simples are labelled by $X^T \times \mathfrak{X}(T)$, where $\mathfrak{X}(T)$ stands for the character lattice of T. In this case, we can define a highest weight order on $X^T \times \mathfrak{X}(T)$ as follows: $(\mathsf{x}, \kappa) <_\nu (\mathsf{x}', \kappa')$ if $\langle \nu, \kappa \rangle < \langle \nu, \kappa' \rangle$.

1.2. Representations in characteristic $p \gg 0$

From now on and until the end of the introduction suppose λ is rational, i.e., $\lambda \in \mathfrak{P}_{\mathbb{Q}} := H^2(X, \mathbb{Q})$. We assume that all the objects from the previous section (X, the T-action, $\mathcal{A}^\theta_\lambda$) are defined over a finite localization R of $\mathbb{Z}$ depending only on the denominator of λ. In particular, we have an R-algebra $\mathcal{A}_{\lambda,\mathsf{R}}$ and the base change $\mathcal{A}_{\lambda,\mathbb{F}} := \mathbb{F} \otimes_{\mathsf{R}} \mathcal{A}_{\lambda,\mathsf{R}}$ for $\mathbb{F} := \overline{\mathbb{F}}_p$ with $p \gg 0$ (we will also impose some congruence conditions on p, for example, when $X = \mathrm{Hilb}_n(\mathbb{A}^2)$ we will assume that $p+1$ is divisible by $n!$). So we get a quantization $\mathcal{A}_{\lambda,\mathbb{F}}$ of $\mathbb{F}[X]$ for all $\lambda \in \mathfrak{P}_{\mathbb{F}}$. On $\mathcal{A}_{\lambda,\mathbb{F}}$ we still have a Hamiltonian action of $T_{\mathbb{F}}$. In what follows we assume that the center of $\mathcal{A}_{\lambda,\mathbb{F}}$ is identified with $\mathbb{F}[X^{(1)}_{\mathbb{F}}]$, where "(1)" denotes the Frobenius twist. This holds in most of the examples of interest. In particular, the algebra $\mathcal{A}_{\lambda,\mathbb{F}}$ is a finite module over its center, hence all irreducible $\mathcal{A}_{\lambda,\mathbb{F}}$-modules are finite dimensional.

For $\lambda \in \mathfrak{P}_{\mathbb{F}_p} (= H^2(X, \mathbb{F}_p))$ (this is the most interesting case), we consider the category $\tilde{\mathcal{O}}(\mathcal{A}_{\lambda,\mathbb{F}})$ consisting of all finite dimensional $T_{\mathbb{F}}$-equivariant $\mathcal{A}_{\lambda,\mathbb{F}}$-modules. Consider the *p-singular* hyperplanes in $\mathfrak{P}_{\mathbb{F}_p}$ that are obtained as the reductions mod p of the singular hyperplanes mentioned in the previous section. We can still talk about the *regular* parameters in $\mathfrak{P}_{\mathbb{F}_p}$: the parameters lying outside these hyperplanes. The p-singular hyperplanes split the integral lattice $\mathfrak{P}_{\mathbb{Z}}$ into the union of regions to be called *p-alcoves*. For example, for $X = T^*(G/B)$ we get the usual p-alcoves (the *p-singular* hyperplanes are of the form $\langle \alpha^\vee, \bullet \rangle = pm$, where $\alpha^\vee$ is a coroot and $m \in \mathbb{Z}$). In particular, we have the so called fundamental p-alcove, it is consists from the points λ in the weight lattice such that $\langle \alpha_i^\vee, \lambda \rangle > 0$ (where $\alpha_i^\vee$ are simple coroots) and $\langle \alpha_0^\vee, \lambda \rangle > -p$, where $\alpha_0^\vee$ is the minimal coroot. In the case of $X = \mathrm{Hilb}_n(\mathbb{A}^2)$, each p-alcove consists of integers z satisfying $\dfrac{(p+1)a}{b} + s < z < \dfrac{(p+1)a'}{b'} - s$, where $\frac{a}{b} < \frac{a'}{b'}$ are rational numbers with denominators between 2 and n such that the interval $(\frac{a}{b}, \frac{a'}{b'})$ contains no such rational numbers (and the nonnegative integer s is such as in the previous section).

It turns out that inside of each p-alcove categories $\tilde{\mathcal{O}}(\mathcal{A}_{\lambda,\mathbb{F}})$ are naturally equivalent, Proposition 8.4. The categories for two p-alcove that are opposite with respect to a common face are perverse equivalent via so called *wall-crossing functors*.

1.3. Highest weight structures

It turns out that the categories $\tilde{\mathcal{O}}(\mathcal{A}_{\lambda,\mathbb{F}})$ carry highest weight structures (in some generalized sense to be explained below). We have one highest weight structure for each generic one-parameter subgroup of T. Recall that a one-parameter subgroup $\nu : \mathbb{G}_m \to T$ is called *generic* if $X^{\operatorname{im}\nu} = X^T$. We fix a one-parameter subgroup ν with this property.

Recall that a highest weight structure on an abelian category is a partial order on the set of irreducible objects subject to certain upper triangularity conditions. Usually, the definition is given in the case when the category of interest is equivalent to that of modules over a finite dimensional algebra. This is not the case with the category $\tilde{\mathcal{O}}(\mathcal{A}_{\lambda,\mathbb{F}})$: the number of simples is infinite (and, moreover, the category does not have projectives, they only exist in a suitable completion). However, we still have a partial order on the set of simples that is defined as follows. The simples in $\tilde{\mathcal{O}}(\mathcal{A}_{\lambda,\mathbb{F}})$ are labelled by pairs (x, κ), where $\mathsf{x} \in X^T$ and $\kappa \in \mathfrak{X}(T)$. To x we can assign $\mathsf{c}_{\nu,\lambda}(\mathsf{x}) \in \mathfrak{t}^*_{\mathbb{Q}}$, the highest weight of the corresponding Verma module (over $\mathbb{C}$). We can now define a partial order $\leqslant_{\lambda,\nu}$ on $X^T \times \mathfrak{X}(T)$ as follows: $(\mathsf{x}, \kappa) \leqslant_{\lambda,\nu} (\mathsf{x}', \kappa')$ if $(\mathsf{x}, \kappa) = (\mathsf{x}', \kappa')$ or

$$\kappa - \mathsf{c}_{\nu,\lambda}(\mathsf{x}) = \kappa' - \mathsf{c}_{\nu,\lambda}(\mathsf{x}') \bmod p, \langle \nu, \kappa \rangle < \langle \nu, \kappa' \rangle.$$

Now pick two integers $z_1 \leqslant z_2$. We can consider the subquotient $\tilde{\mathcal{O}}(\tilde{\mathcal{A}}_{\lambda,\mathbb{F}})_{[z_1,z_2]}$ corresponding to all labels (x, κ) with $z_1 \leqslant \langle \nu, \kappa \rangle \leqslant z_2$.

Our first principal result, Theorem 8.7, is that this subquotient category is highest weight with respect to the partial order $\leqslant_{\nu,\lambda}$. Note that when $\dim T = 1$, then the number of simples in $\tilde{\mathcal{O}}(\mathcal{A}_{\lambda,\mathbb{F}})_{[z_1,z_2]}$ is finite, while when $\dim T > 1$, we need to slightly extend the definition of a highest weight category.

It turns out that the equivalences of the categories $\tilde{\mathcal{O}}(\mathcal{A}_{\lambda,\mathbb{F}})$ for two parameters in the same alcove respect the highest weight structures in a suitable sense. So we have a collection of highest weight categories labelled by alcoves.

1.4. Standardly stratified structures

We define the real alcoves (in $\mathfrak{P}_{\mathbb{R}}$) as connected components of the complement in $\mathfrak{P}_{\mathbb{R}}$ to the hyperplanes that are obtained from the singular ones by translating them by the lattice $\mathfrak{P}_{\mathbb{Z}}$. Note that there is a natural bijection between the real alcoves and the p-alcoves for $p \gg 0$ (we rescale a p-alcove by $1/p$, then there is a unique real alcove whose intersection with a real p-alcove is large in a suitable sense).

Now let A be a real alcove, ${}^pA$ be the corresponding p-alcove. We impose a congruence condition on p requiring that $p+1$ is divisible by a certain number, see Section 5.1. Then pick a face Θ of A (of any codimension). To Θ and a generic one-parameter subgroup $\nu : \mathbb{C}^\times \to T$ (and, in fact, to some additional data) we will assign a *standardly stratified structure* on $\tilde{\mathcal{O}}(\mathcal{A}_{\lambda,\mathbb{F}})$ (that comes from a pre-order on the set of simples and generalizes the notion of a highest weight structure). We will give a rigorous definition of this standardly stratified structure in Section 9.

Such a structure, in particular, gives rise to a filtration on $\tilde{\mathcal{O}}(\mathcal{A}_{\lambda,\mathbb{F}})$ by Serre subcategories. The main result of this paper, Theorem 9.4,

(i) checks the axioms of a standardly stratified structure,
(ii) relates the associated graded to the category $\mathcal{O}_\nu^T(\mathcal{A}_{\bar{\lambda}})$ for a suitable parameter $\bar{\lambda}$ (informally, the relation is that the associated graded is the reduction to characteristic p of $\mathcal{O}_\nu^T(\mathcal{A}_{\bar{\lambda}})$),
(iii) and shows that the reductions to characteristic p of projective and simple objects of $\mathcal{O}_\nu^T(\mathcal{A}_{\bar{\lambda}})$ become standard and proper standard objects for the standardly stratified structure.

(ii) basically means that we can recover characteristic 0 categories $\mathcal{O}$ from their characteristic p analogs, $\tilde{\mathcal{O}}$.

We also relate the wall-crossing functor corresponding to Θ to the standardly stratified structures on the categories corresponding to A (and the alcove opposite to A with respect to Θ). Namely, we show that the wall-crossing functor is a partial Ringel duality functor for the standardly stratified structures, Theorem 9.11.

1.5. Applications

There are two applications of the results outlined in the previous section that we explore in this paper.

First, we use the characterization of the wall-crossing functors as partial Ringel dualities to prove that the wall-crossing bijections (i.e., the bijections between the sets of simples induced by the perverse wall-crossing functors) in characteristic $p \gg 0$ are the same as in characteristic 0. Computing the wall-crossing bijections is an important ingredient in the Bezrukavnikov-Okounkov program of studying the representations of quantizations of symplectic resolutions in characteristic p.

Second, under some additional assumptions on X that hold in all examples we know, the category $\tilde{\mathcal{O}}(\mathcal{A}_{\lambda,\mathbb{F}})$ comes with an additional grading (induced by the contracting torus action on X). We use (ii) of the previous section to show that the grading carries over to the categories $\mathcal{O}_\nu(\mathcal{A}_\lambda)$. Conjecturally, the grading on $\tilde{\mathcal{O}}(\mathcal{A}_{\lambda,\mathbb{F}})$ is Koszul and we deduce from here that the resulting grading on $\mathcal{O}_\nu(\mathcal{A}_\lambda)$ is Koszul.

2. Preliminaries on quantizations

2.1. Symplectic resolutions

Let Y be a normal Poisson affine variety over $\mathbb{C}$ with an action of $\mathbb{C}^\times$. The action gives rise to a natural grading on the algebra $\mathbb{C}[Y]$ of regular functions on Y: $\mathbb{C}[Y] = \bigoplus_{i\in\mathbb{Z}} \mathbb{C}[Y]_i$. We assume that the grading is positive: $\mathbb{C}[Y]_i = 0$ when $i < 0$, and $\mathbb{C}[Y]_0 = \mathbb{C}$. Further, we assume there is a positive integer d such that $\{\mathbb{C}[Y]_i, \mathbb{C}[Y]_j\} \subset \mathbb{C}[Y]_{i+j-d}$ for all i, j. By a *symplectic resolution of singularities* of Y one means a pair (X, ρ) of

- a smooth symplectic algebraic variety X (with form ω)
- a morphism $\rho : X \to Y$ of Poisson varieties that is a projective resolution of singularities.

Below we assume that (X, ρ) is a symplectic resolution of singularities. The $\mathbb{C}^\times$-action lifts from Y to X making ρ equivariant, see Step 1 of the proof of [Nam1, Proposition A.7]. Clearly, such a lift is unique.

2.1.1. Structural results Note that $\rho^* : \mathbb{C}[Y] \to \mathbb{C}[X]$ is an isomorphism because Y is normal and ρ is proper and birational. By the Grauert-Riemenschneider theorem, we have $H^i(X, O_X) = 0$ for $i > 0$.

The following is [BPW, Proposition 2.5].

Lemma 2.1. *We have $H^i(X, \mathbb{C}) = 0$ for i odd.*

Corollary 2.2. *The Chern character map $c_1 : \mathrm{Pic}(X) \to H^2(X, \mathbb{Z})$ induces an isomorphism $\mathbb{Q} \otimes_{\mathbb{Z}} \mathrm{Pic}(X) \xrightarrow{\sim} H^2(X, \mathbb{Q})$.*

2.1.2. Deformations We will be interested in deformations $\hat{X}/\mathfrak{p}'$ of X, where $\mathfrak{p}'$ is a finite dimensional vector space, and $\hat{X}$ is a symplectic scheme over $\mathfrak{p}'$ with a symplectic form $\hat{\omega} \in \Omega^2(\hat{X}/\mathfrak{p}')$ and also with a $\mathbb{C}^\times$-action subject to the following conditions.

- We have $X = \{0\} \times_{\mathfrak{p}'} \hat{X}$ and $\hat{\omega}$ restricts to ω;
- the morphism $\hat{X} \to \mathfrak{p}'$ is $\mathbb{C}^\times$-equivariant for the action on $\mathfrak{p}'$ given by $t.p = t^{-d}p$;
- the restriction of the action to X coincides with the contracting action;
- $t.\hat{\omega} := t^d\hat{\omega}$.

It turns out that there is a universal such deformation $X_\mathfrak{p}$ over $\mathfrak{p} := H^2(X, \mathbb{C})$ (any other deformation is obtained via the pull-back with respect to a unique linear map $\mathfrak{p}' \to \mathfrak{p}$). This result for formal deformations is due to Kaledin-Verbitsky, [KV], but then it carries over to the algebraic setting thanks to the contracting $\mathbb{C}^\times$-action on X, see [Nam1].

Let $Y_\mathfrak{p}$ stand for $\mathrm{Spec}(\mathbb{C}[X_\mathfrak{p}])$. The natural morphism $\tilde{\rho} : X_\mathfrak{p} \to Y_\mathfrak{p}$ is projective and birational. Since the variety X has no higher cohomology, $Y_\mathfrak{p}$ is a deformation of Y over $\mathfrak{p}$ meaning, in particular, that $\mathbb{C}[Y_\mathfrak{p}]/(\mathfrak{p}) = \mathbb{C}[Y]$. For $\lambda \in \mathfrak{p}$, let X_λ, Y_λ denote the fibers of $X_\mathfrak{p}, Y_\mathfrak{p}$ over λ. Let $\mathfrak{p}^{sing}$ denote the locus of $\lambda \in \mathfrak{p}$ such that $\rho_\lambda : X_\lambda \to Y_\lambda$ is not an isomorphism and set $\mathfrak{p}^{reg} := \mathfrak{p} \setminus \mathfrak{p}^{sing}$. Then, according to Namikawa, [Nam3, Main theorem], $\mathfrak{p}^{sing}$ is the union of codimension 1 subspaces in $\mathfrak{p}$ to be called *walls* (or *classical walls*). The elements of $\mathfrak{p}^{reg}$ will be called *generic*.

2.1.3. Classification of symplectic resolutions and the Weyl group Let us describe the possible symplectic resolutions of Y following Namikawa.

If X^1, X^2 are two symplectic resolutions of Y, then there are open subvarieties $\breve{X}^i \subset X^i, i = 1, 2$, with $\mathrm{codim}_{X^i} X^i \setminus \breve{X}^i \geqslant 2$ and $\breve{X}^1 \xrightarrow{\sim} \breve{X}^2$, see, e.g., [BPW, Proposition 2.19]. This allows one to identify the Picard groups, $\mathrm{Pic}(X^1) = \mathrm{Pic}(X^2)$. Let $\mathfrak{p}_\mathbb{Z}$ be the image of $\mathrm{Pic}(X)$ in $H^2(X, \mathbb{C})$.

Set $\mathfrak{p}_\mathbb{R} := \mathbb{R} \otimes_\mathbb{Z} \mathfrak{p}_\mathbb{Z} = H^2(X, \mathbb{R})$. According to [Nam3], there is a finite group W acting on $\mathfrak{p}_\mathbb{R}$ as a reflection group, such that the movable cone C_{mov} of X (that does not depend on the choice of a resolution by the previous paragraph) is a fundamental chamber for W. Further, the set of classical walls introduced in Section 2.1.2 is W-stable and the walls for C are among the classical walls. So the classical walls further split C_{mov} into chambers (to be called *classical chambers*) and the set of (isomorphism classes of) conical symplectic resolutions of Y is in one-to-one correspondence with the set of chambers inside C_{mov}.

For $\theta \in C_{mov} \setminus \mathfrak{p}^{sing}$ we define X^θ to be the symplectic resolutions corresponding to the classical chamber containing θ (to be called below simply the chamber of θ). For $w \in W$, we set $X^{w\theta} := X^\theta$. But we twist the identification of $\mathfrak{p}$ with $H^2(X, \mathbb{C})$ by w so that the ample cone for X in $\mathfrak{p}$ now contains $w\theta$.

We note that W acts on $Y_\mathfrak{p}$ by $\mathbb{C}^\times$-equivariant Poisson automorphisms making $Y_\mathfrak{p} \to \mathfrak{p}$ W-equivariant. The quotient $Y_\mathfrak{p}/W$ is a universal $\mathbb{C}^\times$- equivariant deformation of Y, [Nam2]. In particular, it is independent of the choice of a resolution X. Note that the locus in $X_\mathfrak{p}$, where $X_\mathfrak{p} \to Y_\mathfrak{p}$ is not an isomorphism, has codimension $\geqslant 2$. Let $Y_\mathfrak{p}^0$ denote the complement to this locus. This allows one to identify $\mathrm{Pic}(X_\mathfrak{p})$ with $\mathrm{Pic}(Y_\mathfrak{p}^0)$. On the other hand, every line bundle on X uniquely deforms to $X_\mathfrak{p}$ giving an identification $\mathrm{Pic}(X) \cong \mathrm{Pic}(X_\mathfrak{p})$. So for two different resolutions X, X' we get identifications

$$\mathrm{Pic}(X) \xrightarrow{\sim} \mathrm{Pic}(X_\mathfrak{p}) \xrightarrow{\sim} \mathrm{Pic}(Y_\mathfrak{p}^0) \xrightarrow{\sim} \mathrm{Pic}(X'_\mathfrak{p}) \xrightarrow{\sim} \mathrm{Pic}(X').$$

It is easy to see that the resulting identification $\mathrm{Pic}(X) \xrightarrow{\sim} \mathrm{Pic}(X')$ is the same as what we had in the beginning of the section.

Note that W acts on $Y_\mathfrak{p}^0$, which gives a W-action on $\mathrm{Pic}(X^\theta)$. We identify $\mathrm{Pic}(X^\theta)$ with $\mathrm{Pic}(X^{w\theta})$ by w. In particular, for $\chi \in \mathrm{Pic}(X)$ it makes sense to speak about the line bundle $O(\chi)$ on X^θ for every generic θ.

2.1.4. Example: cotangent bundles to flag varieties Let us proceed to examples.

Take a semisimple Lie algebra $\mathfrak{g}$ over $\mathbb{C}$. Let G be the corresponding connected semisimple group of adjoint type and $B \subset G$ be a Borel subgroup. Consider the flag variety $\mathcal{B} = G/B$. For X we can take the cotangent bundle $T^*\mathcal{B}$, then Y is the nilpotent cone $\mathcal{N} \subset \mathfrak{g} \cong \mathfrak{g}^*$ and $\rho : X \to Y$ is the Springer resolution.

We can identify $\mathfrak{p}$ with $\mathfrak{h}^*$, where $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{b}$. Then W is the Weyl group of $\mathfrak{g}$ and C_{mov} is the positive Weyl chamber. The locus $\mathfrak{p}^{sing}$ coincides with the locus of non-regular elements in $\mathfrak{h}^*$ so the classical walls are $\ker \alpha^\vee$, where $\alpha^\vee$ runs over the set of (positive) coroots. The universal deformation $X_\mathfrak{p}$ is the homogeneous vector bundle $G \times_B \mathfrak{b}$ and $Y_\mathfrak{p} = \mathfrak{g}^* \times_{\mathfrak{h}^*/W} \mathfrak{h}^*$, the morphism $\tilde{\rho} : X_\mathfrak{p} \to Y_\mathfrak{p}$ is Grothendieck's simultaneous resolution.

This example can be generalized in several ways. For X, we can take cotangent bundles to partial flag varieties. More generally, we can also take (parabolic) Slodowy varieties: preimages of transverse slices to nilpotent orbits in the cotangent bundles of (partial) flag varieties.

2.1.5. Example: $\mathbf{Hilb}_{\boldsymbol{n}}(\mathbb{C}^2)$ Set $Y = (\mathbb{C}^2)^n/S_n$. This variety admits a symplectic resolution, $X = \mathrm{Hilb}_n(\mathbb{C}^2)$. The space $\mathfrak{p} = H^2(X,\mathbb{C})$ is one-dimensional and $W = \mathbb{Z}/2\mathbb{Z}$, see, e.g., [L10, Section 3.6].

There is a classical description of X, Y as Nakajima quiver varieties. Consider the vector space $V = \mathrm{End}(\mathbb{C}^n) \oplus \mathbb{C}^{n*}$. It comes equipped with a natural $G := \mathrm{GL}_n(\mathbb{C})$-action. The action extends to a Hamiltonian action on $T^*V = V \oplus V^*$ with moment map φ, the comoment map $\mathfrak{g} \to T^*V$ is given by $\varphi^*(x) := x_V$, where x_V is the vector field on V defined by $x \in \mathfrak{g}$.

Now pick a nontrivial character θ of G so that $\theta = \det^k$, where $k \neq 0$. Consider the semistable locus $(T^*V)^{\theta-ss} \subset T^*V$. Then we can define the Hamiltonian reduction $\varphi^{-1}(0)^{\theta-ss}/G$. It is naturally identified with $\mathrm{Hilb}_n(\mathbb{C}^2)$. Furthermore, $Y = \varphi^{-1}(0)/\!/G$.

The parameter space $\mathfrak{p}$ is identified with $\mathfrak{g}^{*G}$, so is 1-dimensional. We have $Y_{\mathfrak{p}} = \varphi^{-1}(\mathfrak{g}^{*G})/\!/G, X_{\mathfrak{p}} = \varphi^{-1}(\mathfrak{g}^{*G})/\!/G$. The Weyl group W_X is $\{\pm 1\}$.

This construction of a symplectic resolution can be generalized for V being a framed representation space of any quiver. In this way we get general smooth Nakajima quiver varieties, all of them are symplectic resolutions.

2.1.6. Conical slices Recall that, by a result of Kaledin, [K], Y_λ has finitely many symplectic leaves for all $\lambda \in \mathfrak{p}$.

Take $y \in Y_{\mathfrak{p}}$. Let ζ denote the image of y in $\mathfrak{p}$. Consider the completion $\mathbb{C}[Y_{\mathfrak{p}}]^{\wedge_\zeta}$. This is a Poisson algebra over the completion $\mathbb{C}[\mathfrak{p}]^{\wedge_\zeta}$. Further, set $Y_{\mathfrak{p}}^{\wedge_y} := \mathrm{Spec}(\mathbb{C}[Y_{\mathfrak{p}}^{\wedge_y}])$ and $X_{\mathfrak{p}}^{\wedge_y} := Y_{\mathfrak{p}}^{\wedge_y} \times_{Y_{\mathfrak{p}}} X_{\mathfrak{p}}$. Note that $\mathbb{C}[Y_{\mathfrak{p}}^{\wedge_y}]$ is naturally identified with $\mathbb{C}[X_{\mathfrak{p}}^{\wedge_y}]$.

The space $\mathfrak{p}^*$ naturally embeds into $\mathbb{C}[\mathfrak{p}]^{\wedge_\zeta}$. We have a unique $\mathbb{C}^\times$-action on $\mathbb{C}[\mathfrak{p}]^{\wedge_\zeta}$ by topological algebra automorphisms characterized by the property that $\mathfrak{p}^*$ has degree d.

Below we always impose the following assumption that holds in all examples we know.

($\diamondsuit$) There is a $\mathbb{C}^\times$-action on $X_{\mathfrak{p}}^{\wedge_y}$ that

- makes the morphism $X_{\mathfrak{p}}^{\wedge_y} \to \mathfrak{p}^{\wedge_\zeta}$ $\mathbb{C}^\times$-equivariant,
- rescales the fiberwise symplectic form on $X_{\mathfrak{p}}^{\wedge_y}$ by $t \mapsto t^d$,
- and the induced action on $Y_{\mathfrak{p}}^{\wedge_y}$ is contracting.

We would like to deduce some corollaries. We can decompose $\mathbb{C}[Y_{\mathfrak{p}}]^{\wedge_y}$ into the completed tensor product of complete local Poisson algebras $\mathbb{C}[[T_y\mathcal{L}]]\widehat{\otimes}\underline{A}'_{\mathfrak{p}}$, where $\mathcal{L}$ is the symplectic leaf through y in Y_λ and $\underline{A}'_{\mathfrak{p}}$ is the Poisson centralizer of $\mathbb{C}[[T_y\mathcal{L}]]$ under a suitable embedding $\mathbb{C}[[T_y\mathcal{L}]] \hookrightarrow \mathbb{C}[Y_{\mathfrak{p}}]^{\wedge_y}$. Such embeddings are conjugate under Hamiltonian automorphisms of $\mathbb{C}[Y_{\mathfrak{p}}]^{\wedge_y}$ hence $\underline{A}'_{\mathfrak{p}}$ is well-defined.

Under assumption $(\diamondsuit)$, we can choose the decomposition

$$\mathbb{C}[Y_{\mathfrak{p}}]^{\wedge_y} \cong \mathbb{C}[[T_y\mathcal{L}]]\widehat{\otimes}\underline{A}'_{\mathfrak{p}}$$

to be $\mathbb{C}^\times$-stable. Define $\underline{A}_{\mathfrak{p}}$ to be the $\mathbb{C}^\times$-finite part of $\underline{A}'_{\mathfrak{p}}$. Then $\underline{A}_{\mathfrak{p}}$ is a finitely generated graded Poisson algebra. We set $\underline{Y}_{\mathfrak{p}} := \operatorname{Spec}(\underline{A}_{\mathfrak{p}})$ so that $Y_{\mathfrak{p}}^{\wedge_y}$ is identified with $(\underline{Y}_{\mathfrak{p}} \times T_y\mathcal{L})^{\wedge_0}$.

The variety $\underline{Y}$ admits a conical symplectic resolution $\underline{X}$ with a deformation $\underline{X}_{\mathfrak{p}}$ over $\mathfrak{p}$. For example, the deformation $\underline{X}$ is constructed in the following way: we take a homogeneous coordinate ring for the projective (over $\underline{Y}^{\wedge_0}$) scheme $\underline{Y}^{\wedge_0} \times_{Y_\zeta^{\wedge_y}} X_\zeta$. It carries a natural $\mathbb{C}^\times$-action. Then $\underline{X}$ is the Proj of the $\mathbb{C}^\times$-finite part of this homogeneous coordinate ring. By the construction, we have $X_{\mathfrak{p}}^{\wedge_y} \cong (\underline{X}_{\mathfrak{p}} \times T_y\mathcal{L})^{\wedge_0}$.

Now we return to the general situation. Set $\underline{\mathfrak{p}} := H^2(\underline{X}, \mathbb{C})$. Let $\underline{\pi}$ denote the resolution of singularities morphism $\underline{X} \to \underline{Y}$. Note that $\underline{\mathfrak{p}} = H^2(\underline{\pi}^{-1}(0), \mathbb{C})$ and $\underline{\pi}^{-1}(0) \hookrightarrow X_\lambda$. The spaces $H^2(X_{\mathfrak{p}}, \mathbb{C})$ and $H^2(X, \mathbb{C}) = \mathfrak{p}$ are naturally identified. This gives rise to the pullback map

$$\mathfrak{p} = H^2(X_{\mathfrak{p}}, \mathbb{C}) \to H^2(\underline{\pi}^{-1}(0), \mathbb{C}) = \underline{\mathfrak{p}}$$

to be denoted by η. We have a similar map between the Picard groups: for this one needs to notice that $\operatorname{Pic}(\underline{X}) = \operatorname{Pic}(\underline{X}^{\wedge_0})$ thanks to the contracting $\mathbb{C}^\times$-action. The maps $\eta : \operatorname{Pic}(X) \to \operatorname{Pic}(\underline{X})$ and $\eta : H^2(X, \mathbb{C}) \to H^2(\underline{X}, \mathbb{C})$ are intertwined by the 1st Chern character map.

By the construction, we have $\underline{X}_{\mathfrak{p}} = \mathfrak{p} \times_{\underline{\mathfrak{p}}} \underline{X}_{\underline{\mathfrak{p}}}$.

2.1.7. Rank 1 case Here we consider the case when ζ is generic (in the sense explained below) in a classical wall.

Proposition 2.3. *Suppose that $\zeta \in \mathfrak{p}$ is such that the only rational hyperplane in $\mathfrak{p}$ that contains ζ is a classical wall, say Γ. Pick $y \in Y_\zeta$ and form the symplectic resolution $\underline{X}$ as in Section 2.1.6. Then the kernel of $\eta : \mathfrak{p} \to \underline{\mathfrak{p}}$ coincides with Γ.*

Proof. We can identify $H^2_{DR}(X_\zeta)$ with $H^2_{DR}(X)$ by means of the Gauss-Manin connection. Under this identification the class of the symplectic form on X_ζ is ζ by the construction of the period map in [KV]. The pullback of the form from X_ζ to $\underline{X}^{\wedge_0}$ is rescaled by a torus action and hence it is zero. We conclude that $\eta(\zeta) = 0$. On the other hand, we have seen in the end of Section 2.1.6

that η is defined over $\mathbb{Q}$. It follows that Γ lies in the kernel. Since η takes ample line bundles to ample line bundles it cannot be zero. Therefore the kernel coincides with Γ. $\square$

Example 2.4. Let $X = T^*\mathcal{B}$. For ζ as above, we have $\underline{X} = T^*\mathbb{P}^1$ hence $\underline{\mathfrak{p}}$ is 1-dimensional. If α is the unique simple root vanishing at ζ, then the map $\eta : \mathfrak{p} \to \underline{\mathfrak{p}}$ sends ζ to $\langle \zeta, \alpha^\vee \rangle$.

2.2. Quantizations

We will study quantizations of $Y, Y_\mathfrak{p}, X, X_\mathfrak{p}$. By a quantization of Y, we mean

- a filtered algebra $\mathcal{A} = \bigcup_i \mathcal{A}_{\leqslant i}$ such that $[\mathcal{A}_{\leqslant i}, \mathcal{A}_{\leqslant j}] \subset \mathcal{A}_{\leqslant i+j-d}$ for all i, j
- together with an isomorphism $\operatorname{gr} \mathcal{A} \cong \mathbb{C}[Y]$ of graded Poisson algebras.

Similarly, a quantization $\tilde{\mathcal{A}}$ of $Y_\mathfrak{p}$ is a filtered $\mathbb{C}[\mathfrak{p}]$-algebra (with $\mathfrak{p}^*$ in degree d) together with an isomorphism $\operatorname{gr} \tilde{\mathcal{A}} \cong \mathbb{C}[Y_\mathfrak{p}]$ of graded Poisson $\mathbb{C}[\mathfrak{p}]$-algebras. For $\lambda \in \mathfrak{p}$, we set $\mathcal{A}_\lambda := \mathbb{C}_\lambda \otimes_{\mathbb{C}[\mathfrak{p}]} \tilde{\mathcal{A}}$, this is a filtered quantization of $\mathbb{C}[Y]$. So a quantization of $\mathbb{C}[Y_\mathfrak{p}]$ can be viewed as a family of quantizations of $\mathbb{C}[Y]$ parameterized by $\mathfrak{p}$.

By a quantization of $X = X^\theta$, we mean

- a sheaf $\mathcal{A}^\theta$ of filtered algebras in the conical topology on X (in this topology, "open" means Zariski open and $\mathbb{C}^\times$-stable) that is complete and separated with respect to the filtration
- together with an isomorphism $\operatorname{gr} \mathcal{A}^\theta \cong O_{X^\theta}$ (of sheaves of graded Poisson algebras).

Similarly, we can talk about quantizations of $X^\theta_\mathfrak{p}$.

2.2.1. Classification of quantizations of X [BK1, Theorem 1.8] (with ramifications given in [L2, Section 2.3]) shows that the quantizations $\mathcal{A}^\theta$ of X are parameterized (up to an isomorphism) by the points in $\mathfrak{p} = H^2(X, \mathbb{C})$. Below we will use the notation $\mathfrak{P}$ for $\mathfrak{p}$ viewed as a parameter space for quantizations. We view $\mathfrak{P}$ as an affine space, the associated vector space is $\mathfrak{p}$.

More precisely, there is a *canonical* quantization $\mathcal{A}^\theta_\mathfrak{P}$ of $X^\theta_\mathfrak{p}$ such that the quantization of X^θ corresponding to $\lambda \in \mathfrak{P}$ is the specialization of $\mathcal{A}^\theta_\mathfrak{P}$ to λ.

It follows from [L2, Section 2.3] that $\mathcal{A}^\theta_{-\lambda}$ is isomorphic to $(\mathcal{A}^\theta_\lambda)^{opp}$.

2.2.2. Algebras of global sections We set $\mathcal{A}_{\mathfrak{P}} := \Gamma(\mathcal{A}^\theta_{\mathfrak{P}}), \mathcal{A}_\lambda = \Gamma(\mathcal{A}^\theta_\lambda)$. It follows from [BPW, Section 3.3] that the algebras $\mathcal{A}_{\mathfrak{P}}, \mathcal{A}_\lambda$ are independent of the choice of θ. From $H^i(X^\theta, O_{X^\theta}) = 0$, we deduce that the higher cohomology of both $\mathcal{A}^\theta_{\mathfrak{P}}$ and $\mathcal{A}^\theta_\lambda$ vanish. In particular, $\mathcal{A}_\lambda$ is the specialization of $\mathcal{A}_{\mathfrak{P}}$ at λ. Also we see that $\mathcal{A}_{\mathfrak{P}}$ is a quantization of $\mathbb{C}[X_{\mathfrak{p}}]$ and $\mathcal{A}_\lambda$ is a quantization of $\mathbb{C}[X] = \mathbb{C}[Y]$.

From the isomorphism $\mathcal{A}^\theta_\lambda \cong \mathcal{A}^\theta_{-\lambda}$ we deduce $\mathcal{A}_{-\lambda} \cong \mathcal{A}^{opp}_\lambda$. Also we have $\mathcal{A}_\lambda \cong \mathcal{A}_{w\lambda}$ for all $\lambda \in \mathfrak{p}, w \in W$, see [BPW, Section 3.3].

2.2.3. Example: central reductions of universal enveloping algebras Let us start with the example of $X = T^*\mathcal{B}$. Identify the center of the universal enveloping algebra $U(\mathfrak{g})$ with $\mathbb{C}[\mathfrak{h}^*]^W$ by means of the Harish-Chandra isomorphism. Then the quantization $\mathcal{A}_\lambda$ of $Y = \mathcal{N}$ corresponding to $\lambda \in \mathfrak{h}^*$ is the central reduction $U(\mathfrak{g}) \otimes_{\mathbb{C}[\mathfrak{h}^*]^W} \mathbb{C}_\lambda$. The quantization $\mathcal{A}^\theta_\lambda$ is (the microlocalization to $T^*\mathcal{B}$ of) the sheaf $\mathcal{D}^{\lambda-\rho}_{\mathcal{B}}$ of $\lambda-\rho$-twisted differential operators on $\mathcal{B}$. Here, as usual, ρ is half the sum of positive roots. Similarly, the universal quantization $\mathcal{A}_{\mathfrak{P}}$ is $U(\mathfrak{g}) \otimes_{\mathbb{C}[\mathfrak{h}^*]^W} \mathbb{C}[\mathfrak{h}^*]$.

Let us elaborate on the construction of $\mathcal{A}^\theta_{\mathfrak{P}}$. Consider the universal sheaf $\tilde{\mathcal{D}}_{\mathcal{B}}$ of twisted differential operators. It is constructed as $\varpi_*(\mathcal{D}_{G/U})^T$, where U stands for the unipotent radical of B, $T = B/U$ is the maximal torus and $\varpi : G/U \to G/B$ is the projection. The microlocalization of $\tilde{\mathcal{D}}_{\mathcal{B}}$ is the universal quantization of $T^*\mathcal{B}$. It is a sheaf of $\mathbb{C}[\mathfrak{t}^*]$-algebras with the map $\mathbb{C}[\mathfrak{t}^*] \to \Gamma(\tilde{\mathcal{D}}_{\mathcal{B}})$ coming from the the quantum comoment map for the action of T on $\mathcal{D}_{G/U}$ shifted so that the fiber of $\varpi_*(\mathcal{D}_{G/U})^T$ over 0 is $\mathcal{D}^{-\rho}_{\mathcal{B}}$.

2.2.4. Example: quantizations of $\mathrm{Hilb}_n(\mathbb{C}^2), (\mathbb{C}^2)^n/S_n$ Now let us describe the quantizations of the Hilbert scheme $X := \mathrm{Hilb}_n(\mathbb{C}^2)$. Consider the algebra $D(V)$ of linear differential operators on the space V from Section 2.1.5. The algebra $D(V)$ carries a Hamiltonian action of G. We choose the *symmetrized* quantum comoment map $\Phi(x)$ given by the formula $\Phi(x) = \frac{1}{2}(x_V + x_{V^*})$.

We have $\mathfrak{P} = \mathbb{C}$. The quantization $\mathcal{A}^\theta_{\mathfrak{P}}$ is given by

$$\pi_{G*}[D_V/D_V\Phi([\mathfrak{g},\mathfrak{g}])|_{(T^*V)^{\theta-ss}}]^G,$$

where we write D_V for the microlocal sheaf of algebras on T^*V that is obtained from the algebra $D(V)$ by microlocalization and π_G is the quotient morphism $\mu^{-1}(0)^{\theta-ss} \to \mu^{-1}(0)^{\theta-ss}/G$. In our case, the global sections of $\mathcal{A}^\theta_\lambda$ is the quantum Hamiltonian reduction on the level of algebras: $\mathcal{A}_\lambda = [D(V)/D(V)\{\Phi(x) - \langle\lambda, x\rangle | x \in \mathfrak{g}\}]^G$, it is a quantization of $Y = (\mathbb{C}^2)^n/S_n$, see, e.g., [L2, Lemma 4.2.4].

There is an alternative way to construct $\mathcal{A}_\lambda$ due to Etingof and Ginzburg, [EG], as the so called *spherical rational Cherednik algebras*. The full rational Cherednik algebra H_c, where $c \in \mathbb{C}$ is a parameter, is the quotient of the smash-product algebra $\mathbb{C}\langle x_1, \ldots, x_n, y_1, \ldots, y_n\rangle \# S_n$ (where the triangular brackets indicate the free algebra) by the relations

$$\begin{aligned}
&[x_i, x_j] = [y_i, y_j] = 0,\\
&[y_i, x_j] = c(ij), i \neq j,\\
&[y_i, x_i] = 1 - c\sum_{j\neq i}(ij).
\end{aligned}$$

Let e be the averaging idempotent in $\mathbb{C}S_n$. We can view e as an element of H_c. Then we can consider the unital algebra eH_ce, this is the so called spherical rational Cherednik algebra. It is a quantization of $\mathbb{C}[Y]$.

In fact, by [GG, Theorem 1.3.1], $\mathcal{A}_\lambda \cong eH_ce$, where $c = \lambda - 1/2$.

Both constructions can be generalized. The quantum Hamiltonian reduction construction produces quantizations of arbitrary Nakajima quiver varieties, while rational Cherednik algebras can be generalized to so called symplectic reflection algebras that give quantizations of the varieties $\mathbb{C}^{2n}/\Gamma$, where Γ is a finite subgroup of $\mathrm{Sp}_{2n}(\mathbb{C})$.

2.3. Localization theorems

We are interested in the categories $\mathcal{A}_\lambda$-mod of all finitely generated $\mathcal{A}_\lambda$-modules and $\mathrm{Coh}(\mathcal{A}^\theta_\lambda)$ of all coherent sheaves of $\mathcal{A}^\theta_\lambda$-modules, see [BL, Section 2.3] for a definition of the latter. These categories are related via the global section functor $\Gamma^\theta_\lambda : \mathrm{Coh}(\mathcal{A}^\theta_\lambda) \to \mathcal{A}_\lambda$-mod (mapping a coherent sheaf into its global sections) and its left adjoint, the localization functor $\mathrm{Loc}^\theta_\lambda := \mathcal{A}^\theta_\lambda \otimes_{\mathcal{A}_\lambda} \bullet$. These functors have derived versions: $R\Gamma^\theta_\lambda : D^b(\mathrm{Coh}(\mathcal{A}^\theta_\lambda)) \to D^b(\mathcal{A}_\lambda\text{-mod})$ and $L\,\mathrm{Loc}^\theta_\lambda : D^-(\mathcal{A}_\lambda\text{-mod}) \to D^-(\mathrm{Coh}(\mathcal{A}^\theta_\lambda))$, the latter restricts to the bounded derived categories whenever $\mathcal{A}_\lambda$ has finite homological dimension.

We say that *abelian localization holds* for (λ, θ) if the functors $\Gamma^\theta_\lambda, \mathrm{Loc}^\theta_\lambda$ are mutually inverse equivalences between $\mathrm{Coh}(\mathcal{A}^\theta_\lambda), \mathcal{A}_\lambda$-mod. Similarly, we say that *derived localization holds* for (λ, θ) if $R\Gamma^\theta_\lambda, L\,\mathrm{Loc}^\theta_\lambda$ are mutually inverse equivalences.

2.3.1. Examples In the case when $X = T^*\mathcal{B}$ one can explicitly describe the parameters where abelian and derived localization hold.

Proposition 2.5. *Let $X = T^*\mathcal{B}$. The following claims are true:*

- *Derived localization holds for* (λ, θ) *if and only if* λ *is regular meaning that* $\langle \lambda, \alpha^\vee \rangle \neq 0$ *for every coroot* $\alpha^\vee$.
- *Let* θ *be in the positive Weyl chamber. Then abelian localization holds for* (λ, θ) *if and only if* $\langle \lambda, \alpha^\vee \rangle \not\in \mathbb{Z}_{\leqslant 0}$ *for all positive coroots* $\alpha^\vee$.

These are the classical derived and abelian Beilinson-Bernstein localization theorems, [BB1, BB2].

2.3.2. Example: quantizations of $\mathrm{Hilb}_n(\mathbb{C}^2)$ Here we have the following result, [GS1, GS2, KR, BE].

Proposition 2.6. *Let* $X = \mathrm{Hilb}_n(\mathbb{C}^2)$. *The following claims are true:*

1. *The homological dimension of* $\mathcal{A}_\lambda$ *is infinite, equivalently, derived localization theorem fails for* $(\lambda, \theta \neq 0)$, *if and only if* $c = \lambda - 1/2$ *lies in* $(-1, 0)$ *and is a rational number with denominator* $\leqslant n$.
2. *For* $\theta > 0$, *abelian localization holds for* (λ, θ) *if and only if*

$$(c + \mathbb{Z}_{\geqslant 0}) \cap \{-\frac{a}{b} | 1 \leqslant a < b \leqslant n\} = \varnothing.$$

2.4. Harish-Chandra bimodules

2.4.1. Definition Let R be a commutative Noetherian ring. Let $\mathcal{A}, \mathcal{A}'$ be two $\mathbb{Z}_{\geqslant 0}$-filtered R-algebras such that $\mathrm{gr}\,\mathcal{A}, \mathrm{gr}\,\mathcal{A}'$ are finitely generated commutative R-algebras. We assume $\mathrm{gr}\,\mathcal{A}, \mathrm{gr}\,\mathcal{A}'$ are isomorphic, we fix an isomorphism and denote the resulting algebra by A.

By a Harish-Chandra (shortly, HC) bimodule we mean an $\mathcal{A}$-$\mathcal{A}'$-bimodule $\mathcal{B}$ that can be equipped with a *good filtration*, i.e., an $\mathcal{A}$-$\mathcal{A}'$-bimodule filtration bounded from below subject to the following two properties:

- the induced left and right A-actions on $\mathrm{gr}\,\mathcal{B}$ coincide,
- $\mathrm{gr}\,\mathcal{B}$ is a finitely generated A-module.

By a homomorphism of Harish-Chandra bimodules we mean a bimodule homomorphism. The category of HC $\mathcal{A}$-$\mathcal{A}'$-bimodules is denoted by $\mathrm{HC}(\mathcal{A}\text{-}\mathcal{A}')$. We also consider the full subcategory $D^b_{HC}(\mathcal{A}\text{-}\mathcal{A}')$ of the derived category of $\mathcal{A}$-$\mathcal{A}'$-bimodules with Harish-Chandra homology.

By the associated variety $\mathrm{V}(\mathcal{B})$ of $\mathcal{B}$ we mean the support of the finitely generated A-module $\mathrm{gr}\,\mathcal{B}$ in $\mathrm{Spec}(A)$.

We will concentrate on the algebras $\mathcal{A}, \mathcal{A}'$, etc., of the form $\mathcal{A}_\lambda$ (or some other specialization of $\mathcal{A}_{\mathfrak{P}}$). For the time being, $R = \mathbb{C}$, while starting in

Section 6 we will also consider the situations when R is a localization of $\mathbb{Z}$ or a positive characteristic field.

For $\chi \in \mathfrak{p}$, let $\mathrm{HC}(\mathcal{A}_{\mathfrak{P}}, \chi)$ denote the category of all HC $\mathcal{A}_{\mathfrak{P}}$-bimodules $\mathcal{B}$ such that $[\alpha, b] = \langle \alpha, \chi \rangle b$ for all $\alpha \in \mathfrak{p}^* \subset \mathbb{C}[\mathfrak{p}] \subset \mathcal{A}_{\mathfrak{P}}$ and $b \in \mathcal{B}$.

2.4.2. Properties Now let $\mathcal{A}, \mathcal{A}', \mathcal{A}''$ be three R-algebras whose associated graded are identified with the same R-algebra A as before. Let $\mathcal{B}_1 \in \mathrm{HC}(\mathcal{A}\text{-}\mathcal{A}'), \mathcal{B}_2 \in \mathrm{HC}(\mathcal{A}'\text{-}\mathcal{A}'')$. Then $\mathrm{Tor}_i^{\mathcal{A}'}(\mathcal{B}_1, \mathcal{B}_2) \in \mathrm{HC}(\mathcal{A}\text{-}\mathcal{A}'')$. Indeed, $\mathrm{Tor}_i^{\mathcal{A}'}(\mathcal{B}_1, \mathcal{B}_2)$ comes with a natural bounded from below filtration such that $\operatorname{gr} \mathrm{Tor}_i^{\mathcal{A}'}(\mathcal{B}_1, \mathcal{B}_2)$ is a subquotient of $\mathrm{Tor}_i^{A}(\operatorname{gr} \mathcal{B}_1, \operatorname{gr} \mathcal{B}_2)$.

Similarly, if $\mathcal{B}_1 \in \mathrm{HC}(\mathcal{A}\text{-}\mathcal{A}'), \mathcal{B}_2 \in \mathrm{HC}(\mathcal{A}\text{-}\mathcal{A}'')$, then $\mathrm{Ext}^i_{\mathcal{A}}(\mathcal{B}_1, \mathcal{B}_2) \in \mathrm{HC}(\mathcal{A}'\text{-}\mathcal{A}'')$ (and the similar claim holds for the Ext's in the category of right $\mathcal{A}$-modules).

2.4.3. Translation bimodules In order to approach abelian localization in the next section, we will need translation bimodules introduced in the present generality in [BPW, Section 6.3].

Set $X := X^\theta$. Pick $\chi \in \mathrm{Pic}(X)$ (recall that the Picard groups of different symplectic resolutions of Y are naturally identified, see Section 2.1.3). Let $O(\chi)$ denote the corresponding line bundle on X. Since $H^i(X, O_X) = 0$ for $i > 0$, the line bundle $O(\chi)$ uniquely deforms to a right $\mathcal{A}^\theta_{\mathfrak{P}}$-module. It was shown in [BPW, Section 5.1] that the deformation carries an $\mathcal{A}^\theta_{\mathfrak{P}}$-bimodule structure, where the adjoint action of $\mu \in \mathfrak{p}^*$ is by $\langle c_1(\chi), \mu \rangle$. We will denote the resulting bimodule by $\mathcal{A}^\theta_{\mathfrak{P},\chi}$.

Set $\mathcal{A}_{\mathfrak{P},\chi} := \Gamma(\mathcal{A}^\theta_{\mathfrak{P},\chi})$, this is an $\mathcal{A}_{\mathfrak{P}}$-bimodule that is independent of the choice of θ by [BPW, Proposition 6.24]. Note that $\mathcal{A}_{\mathfrak{P},\chi} \in \mathrm{HC}(\mathcal{A}_{\mathfrak{P}}, \chi)$, [BPW, Proposition 6.23]. Set $\mathcal{A}_{\lambda,\chi} := \mathcal{A}_{\mathfrak{P},\chi} \otimes_{\mathbb{C}[\mathfrak{P}]} \mathbb{C}_\lambda$, this is an $\mathcal{A}_{\lambda+\chi}$-$\mathcal{A}_\lambda$-bimodule (here and below we abuse the notation and write $\mathcal{A}_{\lambda+\chi}$ instead of $\mathcal{A}_{\lambda+c_1(\chi)}$). We call $\mathcal{A}_{\lambda,\chi}$ a *translation bimodule*.

The following result was obtained in [BPW, Proposition 6.26].

Lemma 2.7. *Suppose that $H^i(X^\theta, O(\chi)) = 0$ for all $i > 0$. Then $\mathcal{A}_{\lambda,\chi} \xrightarrow{\sim} R\Gamma(\mathcal{A}^\theta_{\lambda,\chi})$.*

Let us recall some properties of the translation bimodules obtained in [BPW].

Lemma 2.8 (Proposition 6.31 in [BPW])**.** *Suppose abelian localization holds for $(\lambda + \chi, \theta)$. Then we have a functor isomorphism*

$$\Gamma^\theta_{\lambda+\chi}(\mathcal{A}^\theta_{\lambda,\chi} \otimes_{\mathcal{A}^\theta_\lambda} L\,\mathrm{Loc}^\theta_\lambda(\bullet)) \cong \mathcal{A}_{\lambda,\chi} \otimes^L_{\mathcal{A}_\lambda} \bullet.$$

Corollary 2.9. *Suppose that abelian localization holds for* $(\lambda,\theta),(\lambda+\chi,\theta)$. *Then the bimodules* $\mathcal{A}_{\lambda,\chi},\mathcal{A}_{\lambda+\chi,-\chi}$ *are mutually inverse Morita equivalences.*

2.4.4. Restriction functors In [L9, Section 3.3] we have constructed the restriction functors for HC bimodules over $\mathcal{A}_\lambda$ (or $\mathcal{A}_{\mathfrak{P}}$) under the assumption that Y has conical slices. We can generalize this construction using the assumption ($\diamondsuit$) from Section 2.1.6.

We use the notation and conventions of Section 2.1.6. Pick $y\in Y_{\mathfrak{p}}$ and let λ denote its image in $\mathfrak{p}$ and $\mathcal{L}$ denote the symplectic leaf of y. Consider the algebra $\underline{\mathcal{A}}_{\underline{\mathfrak{P}}}$, the analog of $\mathcal{A}_{\mathfrak{P}}$ for $\underline{X}$, and set $\underline{\mathcal{A}}_{\mathfrak{P}}:=\mathbb{C}[\mathfrak{P}]\otimes_{\mathbb{C}[\underline{\mathfrak{P}}]}\underline{\mathcal{A}}_{\underline{\mathfrak{P}}}$. Further let $\mathbb{A}$ denote the Weyl algebra of the symplectic vector space $T_y\mathcal{L}$. Form the filtered algebra $\mathbb{A}\otimes\underline{\mathcal{A}}_{\mathfrak{P}}$, take its Rees algebra $R_\hbar(\mathbb{A}\otimes\underline{\mathcal{A}}_{\mathfrak{P}})$ and complete it at zero. Denote this completion by $R_\hbar(\mathbb{A}\otimes\underline{\mathcal{A}}_{\mathfrak{P}})^{\wedge_0}$.

On the other hand, we can take the Rees algebra $R_\hbar(\mathcal{A}_{\mathfrak{P}})$ and complete it at y getting the algebra $R_\hbar(\mathcal{A}_{\mathfrak{P}})^{\wedge_y}$. It was shown in [L9, Section 3.3] that we have a $\mathbb{C}[[\mathfrak{P},\hbar]]$-linear isomorphism

$$R_\hbar(\mathcal{A}_{\mathfrak{P}})^{\wedge_y}\cong R_\hbar(\mathbb{A}\otimes\underline{\mathcal{A}}_{\mathfrak{P}})^{\wedge_0} \tag{2.1}$$

that lifts the isomorphism $\mathbb{C}[Y_{\mathfrak{p}}]^{\wedge_y}\cong\mathbb{C}[T_y\mathcal{L}\times\underline{Y}_{\mathfrak{p}}]^{\wedge_0}$ from Section 2.1.6.

Similarly to [L9, Section 3.3], (2.1) give rise to the restriction functor $\bullet_{\dagger,y}:\mathrm{HC}(\mathcal{A}_{\mathfrak{P}},\chi)\to\mathrm{HC}(\underline{\mathcal{A}}_{\mathfrak{P}},\chi)$. The functor has the following properties:

(i) It is exact and $\mathbb{C}[\mathfrak{P}]$-linear.
(ii) We have $\mathcal{B}_{\dagger,y}=0$ if and only if $y\not\in\mathrm{V}(\mathcal{B})$.
(iii) For $\mathcal{B}\in\mathrm{HC}(\mathcal{A}_{\lambda+\chi}\text{-}\mathcal{A}_\lambda)$ and $y\in Y$, we have $\dim\mathcal{B}_{\dagger,y}<\infty$ if and only if $\mathcal{L}$ is open in $\mathrm{V}(\mathcal{B})$.
(iv) The functor $\bullet_{\dagger,y}$ is monoidal.
(v) Assume that $H^i(X^\theta,O(\chi))=0$ for all $i>0$. Then $(\mathcal{A}_{\mathfrak{P},\chi})_{\dagger,y}=\underline{\mathcal{A}}_{\mathfrak{P},\chi}$.

We note that the bimodule $\underline{\mathcal{A}}_{\mathfrak{P},\chi}$ is obtained from $\mathbb{C}[\mathfrak{P}]\otimes_{\mathbb{C}[\underline{\mathfrak{P}}]}\underline{\mathcal{A}}_{\underline{\mathfrak{P}},\eta(\chi)}$ by shifting the left $\mathbb{C}[\mathfrak{P}]$-action.

2.5. Translation bimodules and abelian localization

2.5.1. Tensor products of translation bimodules We want to compare the bimodules $\mathcal{A}_{\lambda,\chi+\chi'}$ and $\mathcal{A}_{\lambda+\chi,\chi'}\otimes^L_{\mathcal{A}_{\lambda+\chi}}\mathcal{A}_{\lambda,\chi}$. Note that we have a natural homomorphism

$$\mathcal{A}_{\mathfrak{P},\chi'}\otimes^L_{\mathcal{A}_{\mathfrak{P}}}\mathcal{A}_{\mathfrak{P},\chi}\to\mathcal{A}_{\mathfrak{P},\chi+\chi'}. \tag{2.2}$$

It is induced by the isomorphism

$$\mathcal{A}^\theta_{\mathfrak{P},\chi'} \otimes_{\mathcal{A}^\theta_{\mathfrak{P}}} \mathcal{A}^\theta_{\mathfrak{P},\chi} \to \mathcal{A}^\theta_{\mathfrak{P},\chi+\chi'}. \tag{2.3}$$

(2.2) specializes to

$$\mathcal{A}_{\lambda+\chi,\chi'} \otimes^L_{\mathcal{A}_{\lambda+\chi}} \mathcal{A}_{\lambda,\chi} \to \mathcal{A}_{\lambda,\chi+\chi'}. \tag{2.4}$$

Below we will need a characterization of (2.4).

Lemma 2.10. *Suppose that there is a resolution $X = X^\theta$ such that*

$$H^i(X, O(\chi+\chi')) = 0, \forall i > 0.$$

Then (2.4) is the unique homomorphism whose microlocalization to Y^{reg} coincides with the microlocalization of the specialization of (2.3).

Proof. Clearly, (2.4) has the required property. We need to show that it characterizes it uniquely. This will follow if we check that

$$\mathcal{A}_{\lambda,\chi+\chi'} \hookrightarrow \Gamma(\mathcal{A}_{\lambda,\chi+\chi'}|_{Y^{reg}}).$$

By Lemma 2.7, we have that $\mathcal{A}_{\lambda,\chi+\chi'} \xrightarrow{\sim} R\Gamma(\mathcal{A}^\theta_{\lambda,\chi+\chi'})$. By the assumption of the lemma, $R\Gamma(\mathcal{A}^\theta_{\lambda,\chi+\chi'}) = \Gamma(\mathcal{A}^\theta_{\lambda,\chi+\chi'})$. And it is clear that $\Gamma(\mathcal{A}^\theta_{\lambda,\chi+\chi'}) \hookrightarrow \Gamma(\mathcal{A}^\theta_{\lambda,\chi+\chi'}|_{Y^{reg}})$. □

2.5.2. Main result The following is the main result relating translation bimodules to abelian localization.

Proposition 2.11. *Let χ in the chamber of θ be such that $H^i(X^\theta, O(m\chi)) = 0$ for all $m > 0$ and all $i > 0$. Suppose, further, that the bimodules $\mathcal{A}_{\lambda+m\chi,\chi}$, $\mathcal{A}_{\lambda+(m+1)\chi,-\chi}$ are mutually inverse Morita equivalences for each $m \geqslant 0$. Then abelian localization holds for (λ, θ).*

Proof. The proof is similar to that of [BL, Lemma 4.4].

Thanks to [BPW, Proposition 5.13], what we need to show is that, for all $m > 1$, the homomorphism

$$\mathcal{A}_{\lambda+(m-1)\chi,\chi} \otimes_{\mathcal{A}_{\lambda+(m-1)\chi}} \mathcal{A}_{\lambda,(m-1)\chi} \to \mathcal{A}_{\lambda,m\chi} \tag{2.5}$$

is an isomorphism. We will construct its inverse. Consider the homomorphism

$$\mathcal{A}_{\lambda+m\chi,-\chi} \otimes_{\mathcal{A}_{\lambda+m\chi}} \mathcal{A}_{\lambda,m\chi} \to \mathcal{A}_{\lambda,(m-1)\chi}.$$

Tensoring both sides with $\mathcal{A}_{\lambda+(m-1)\chi,\chi}$ (that is inverse to the first factor in the left hand side), we get

$$\mathcal{A}_{\lambda,m\chi} \to \mathcal{A}_{\lambda+(m-1)\chi,\chi} \otimes_{\mathcal{A}_{\lambda+(m-1)\chi}} \mathcal{A}_{\lambda,(m-1)\chi}. \tag{2.6}$$

We claim that (2.5) and (2.6) are mutually inverse. Note that, by the construction, these homomorphisms are mutually inverse after microlocalizing to Y^{reg}. It follows from Lemma 2.10 that they are mutually inverse. □

2.5.3. Further results First, we have the following lemma.

Lemma 2.12. *Let $\chi \in \operatorname{Pic}(X)$. Then the locus of $\lambda \in \mathfrak{P}$, where the homomorphisms*

$$\mathcal{A}_{\lambda,\chi} \otimes_{\mathcal{A}_\lambda} \mathcal{A}_{\lambda+\chi,-\chi} \to \mathcal{A}_{\lambda+\chi}, \mathcal{A}_{\lambda+\chi,-\chi} \otimes_{\mathcal{A}_{\lambda+\chi}} \mathcal{A}_{\lambda,\chi} \to \mathcal{A}_\lambda$$

are isomorphisms is non-empty Zariski open in $\mathfrak{P}$.

Proof. This follows as in the proof of (2) of [BL, Proposition 4.5] using [L6, Proposition 2.6]. □

From (iv), (v) from Section 2.4.4 and Proposition 2.11 we deduce the following.

Corollary 2.13. *If abelian localization holds for $\mathcal{A}^\theta_\lambda$, then it also holds for $\underline{\mathcal{A}}^\theta_\lambda$.*

Let us mention one more important property of translation bimodules.

Lemma 2.14. *Let $\lambda \in \mathfrak{P}$ and χ_1, χ_2, χ_3 be such that abelian localization holds for $(\lambda, \theta), (\lambda + \chi_1, \theta)$ and for $(\lambda' - \chi_3, \theta')$ and (λ', θ'), where $\lambda' = \lambda + \chi$ with $\chi = \chi_1 + \chi_2 + \chi_3$. Then*

$$\mathcal{A}_{\lambda,\chi} = \mathcal{A}_{\lambda+\chi_1+\chi_2,\chi_3} \otimes_{\mathcal{A}_{\lambda+\chi_1+\chi_2}} \mathcal{A}_{\lambda+\chi_1,\chi_2} \otimes_{\mathcal{A}_{\lambda+\chi_1}} \mathcal{A}_{\lambda,\chi_1}.$$

Proof. We will consider the case when $\chi_3 = 0$ (the case when $\chi_1 = 0$ is similar and together they imply the general case). We have a natural bimodule homomorphism

$$\mathcal{A}_{\lambda'-\chi_2,\chi_2} \otimes_{\mathcal{A}_{\lambda+\chi_1}} \mathcal{A}_{\lambda,\chi_1} \to \mathcal{A}_{\lambda,\chi}. \tag{2.7}$$

But $\mathcal{A}_{\lambda'-\chi_2,\chi_2}$ is a Morita equivalence bimodule by Corollary 2.9 and its inverse is $\mathcal{A}_{\lambda',-\chi_2}$. From here, using Lemma 2.10, we see that we get an inverse of (2.7) from the natural homomorphism

$$\mathcal{A}_{\lambda',-\chi_2} \otimes_{\mathcal{A}_{\lambda'}} \mathcal{A}_{\lambda,\chi} \to \mathcal{A}_{\lambda,\chi_1}$$

by tensoring with $\mathcal{A}_{\lambda'-\chi_2,\chi_2}$. □

2.6. Wall-crossing functors

Here we will recall wall-crossing functors. These functors are a classical tool in the representation theory of semisimple Lie algebras. In the generality we need they were constructed in [BPW, Section 6.4] and further studied in [L9], where it was shown that some of these functors are perverse equivalences.

2.6.1. Definition Let $\lambda \in \mathfrak{p}, \chi \in \operatorname{Pic}(X)$, where $X = X^\theta$. Suppose that abelian localization holds for $(\lambda+\chi,\theta)$, while derived localization holds for (λ,θ). Consider the functor $\mathfrak{WC}_{\lambda+\chi\leftarrow\lambda} := \mathcal{A}_{\lambda,\chi} \otimes^L_{\mathcal{A}_\lambda} \bullet$, as we have seen in Section 2.4.3, this is a derived equivalence. Note that if abelian equivalence holds for (λ,θ), then $\mathfrak{WC}_{\lambda+\chi\leftarrow\lambda}$ is an abelian equivalence.

2.6.2. Wall-crossing between simple algebras Now let us examine the behavior of $\mathfrak{WC}_{\lambda+\chi\leftarrow\lambda}$, when the algebras $\mathcal{A}_\lambda, \mathcal{A}_{\lambda+\chi}$ are simple.

Lemma 2.15. *If the algebras $\mathcal{A}_\lambda, \mathcal{A}_{\lambda+\chi}$ are simple, then $\mathcal{A}_{\lambda,\chi} \otimes^L_{\mathcal{A}_\lambda} \bullet$ is an abelian equivalence.*

Proof. Note that we have natural homomorphisms

$$\mathcal{A}_{\lambda,\chi} \otimes_{\mathcal{A}_\lambda} \mathcal{A}_{\lambda+\chi,-\chi} \to \mathcal{A}_{\lambda+\chi} \text{ and } \mathcal{A}_{\lambda+\chi,-\chi} \otimes_{\mathcal{A}_{\lambda+\chi}} \mathcal{A}_{\lambda,\chi} \to \mathcal{A}_\lambda.$$

Pick a generic point $y \in Y$ and consider the corresponding restriction functor between the categories of Harish-Chandra bimodules, see Section 2.4.4. The target category for this functor is Vect because of the choice of y. Moreover, the images of $\mathcal{A}_{\lambda+\chi,-\chi}, \mathcal{A}_{\lambda,\chi}$ are one-dimensional. The functor sends the bimodule homomorphisms above to the identity maps. So their kernels and cokernels vanish, equivalently they have proper associated varieties. This already shows that the homomorphisms are surjective. But the kernels are also Harish-Chandra bimodules. Since the algebras $\mathcal{A}_\lambda, \mathcal{A}_{\lambda+\chi}$ are simple they have no Harish-Chandra bimodules with proper associated varieties, this follows, for example, from [L5, Lemma 4.4]. □

2.6.3. Perversity Let us recall the general definition of a perverse equivalence due to Chuang and Rouquier. Let $\mathcal{T}^1, \mathcal{T}^2$ be triangulated categories equipped with t-structures that are homologically finite (each object in $\mathcal{T}^i$ has only finitely many nonzero homology groups). Let $\mathcal{C}^1, \mathcal{C}^2$ denote the hearts of $\mathcal{T}^1, \mathcal{T}^2$, respectively.

Fix filtrations $\mathcal{C}^i = \mathcal{C}^i_0 \supset \mathcal{C}^i_1 \supset \ldots \supset \mathcal{C}^i_k = \{0\}$ by Serre subcategories. We are going to define a perverse equivalence with respect to these filtrations. By definition, this is a triangulated equivalence $\mathcal{F} : \mathcal{T}^1 \xrightarrow{\sim} \mathcal{T}^2$ subject to the following conditions:

(P1) For any j, the equivalence $\mathcal{F}$ restricts to an equivalence $\mathcal{T}^1_{\mathcal{C}^1_j} \to \mathcal{T}^2_{\mathcal{C}^2_j}$, where we write $\mathcal{T}^i_{\mathcal{C}^i_j}, i = 1, 2$, for the category of all objects in $\mathcal{T}^i$ with homology (computed with respect to the t-structures of interest) in $\mathcal{C}^i_j$.

(P2) For $M \in \mathcal{C}^1_j$, we have $H_\ell(\mathcal{F}M) = 0$ for $\ell < j$ and $H_\ell(\mathcal{F}M) \in \mathcal{C}^2_{j+1}$ for $\ell > j$.

(P3) The functor $M \mapsto H_j(\mathcal{F}M)$ induces an equivalence $\mathcal{C}^1_j/\mathcal{C}^1_{j+1} \xrightarrow{\sim} \mathcal{C}^2_j/\mathcal{C}^2_{j+1}$ of abelian categories.

We note that thanks to (P3), $\mathcal{F}$ induces a bijection $\varphi : \mathrm{Irr}(\mathcal{C}^1) \xrightarrow{\sim} \mathrm{Irr}(\mathcal{C}^2)$.

It turns out that the wall-crossing functor $\mathfrak{WC}_{\lambda+\chi\leftarrow\lambda}$ is perverse under certain additional assumptions, [L9, Section 3.1]. Namely, suppose that abelian localization holds for $(\lambda, \theta), (\lambda+\chi, \theta')$ and derived localization holds for (λ, θ'), where θ, θ' lie in chambers that are opposite with respect to a common face, denote it by Γ. Then $\mathfrak{WC}_{\lambda+\chi\leftarrow\lambda}$ is perverse, [L9, Theorem 3.1]. Under some more assumptions one can describe the filtrations in terms of annihilators by certain ideals. Namely, let $\mathfrak{p}_0$ be the subspace of $\mathfrak{p}$ spanned by Γ. Let us assume that abelian localization holds for $(\hat\lambda, \theta)$ and $(\hat\lambda + \chi, \theta')$ and derived localization holds for $(\hat\lambda, \theta')$ for a Weil generic $\hat\lambda \in \mathfrak{P}_0 := \lambda + \mathfrak{p}_0$. Then it was shown in [L9, Theorem 3.1] that there are chains of ideals $\mathcal{A}_{\mathfrak{P}_0} = \mathcal{I}^{k+1}_{\mathfrak{P}_0} \supset \mathcal{I}^k_{\mathfrak{P}_0} \supset \ldots \supset \mathcal{I}^0_{\mathfrak{P}_0} = \{0\}$ and $\mathcal{A}_{\mathfrak{P}_0+\chi} = \mathcal{I}^{k+1}_{\mathfrak{P}_0+\chi} \supset \mathcal{I}^k_{\mathfrak{P}_0+\chi} \supset \ldots \supset \mathcal{I}^0_{\mathfrak{P}_0+\chi} = \{0\}$, where $k = \frac{1}{2}\dim X$ with the following properties:

(a) For a Weil generic $\lambda' \in \mathfrak{P}_0$, the specialization $\mathcal{I}^j_{\lambda'}$ is the minimal ideal $\mathcal{I} \subset \mathcal{A}_{\lambda'}$ with $\operatorname{GK-dim}(\mathcal{A}_{\lambda'}/\mathcal{I}) \leqslant 2(k-j)$, where we write $\operatorname{GK-dim}(\mathcal{A}_{\lambda'}/\mathcal{I})$ for the GK dimension of $\mathcal{A}_{\lambda'}/\mathcal{I}$. The similar characterization is true for $\mathcal{I}^j_{\lambda'+\chi} \subset \mathcal{A}_{\lambda'+\chi}$.

(b) For a Zariski generic $\lambda' \in \mathfrak{P}_0$ and $\mathcal{B} := \mathcal{A}_{\lambda',\chi}$, we have

(b1) For all i, j, we have $\mathcal{I}^j_{\lambda'+\chi} \operatorname{Tor}^{\mathcal{A}_{\lambda'}}_i(\mathcal{B}, \mathcal{A}_{\lambda'}/\mathcal{I}^j_{\lambda'}) = 0$.

(b2) For all i, j, we have $\operatorname{Tor}^{\mathcal{A}_{\lambda'+\chi}}_i(\mathcal{A}_{\lambda'+\chi}/\mathcal{I}^j_{\lambda'+\chi}, \mathcal{B})\mathcal{I}^j_{\lambda'} = 0$.

(b3) We have $\operatorname{Tor}^{\mathcal{A}_{\lambda'}}_i(\mathcal{B}, \mathcal{A}_{\lambda'}/\mathcal{I}^j_{\lambda'}) = 0$ for $i < j$.

(b4) We have

$$\mathcal{I}^{j-1}_{\lambda'+\chi} \operatorname{Tor}^{\mathcal{A}_{\lambda'}}_i(\mathcal{B}, \mathcal{A}_{\lambda'}/\mathcal{I}^j_{\lambda'}) = \operatorname{Tor}^{\mathcal{A}_{\lambda'+\chi}}_i(\mathcal{A}_{\lambda'+\chi}/\mathcal{I}^j_{\lambda'+\chi}, \mathcal{B})\mathcal{I}^{j-1}_{\lambda'} = 0$$

for $i > j$

(b5) Set $\mathcal{B}_j := \operatorname{Tor}_j^{\mathcal{A}_{\lambda'}}(\mathcal{B}, \mathcal{A}_{\lambda'}/\mathcal{I}^j_{\lambda'})$. The kernel and the cokernel of the natural homomorphism

$$\mathcal{B}_j \otimes_{\mathcal{A}_{\lambda'}} \operatorname{Hom}_{\mathcal{A}_{\lambda'+\chi}}(\mathcal{B}_j, \mathcal{A}_{\lambda'+\chi}/\mathcal{I}^j_{\lambda'+\chi}) \to \mathcal{A}_{\lambda'+\chi}/\mathcal{I}^j_{\lambda'+\chi}$$

are annihilated by $\mathcal{I}^{j-1}_{\lambda'+\chi}$ on the left and on the right.

(b6) The kernel and the cokernel of the natural homomorphism

$$\operatorname{Hom}_{\mathcal{A}_{\lambda'}}(\mathcal{B}_j, \mathcal{A}_{\lambda'}/\mathcal{I}^j_{\lambda'}) \otimes_{\mathcal{A}_{\lambda'+\chi}} \mathcal{B}_j \to \mathcal{A}_{\lambda'}/\mathcal{I}^j_{\lambda'}.$$

are annihilated on the left and on the right by $\mathcal{I}^{j-1}_{\lambda'}$.

It was shown in [L9, Section 3.4] that (a) implies that, for a Zariski generic λ', the subcategories $\mathcal{A}_{\lambda'}/\mathcal{I}^j_{\lambda'}\text{-mod} \subset \mathcal{A}_{\lambda'}\text{-mod}$ and $\mathcal{A}_{\lambda'+\chi}/\mathcal{I}^j_{\lambda'+\chi}\text{-mod} \subset \mathcal{A}_{\lambda'+\chi}\text{-mod}$ are Serre. Further, it was shown there that the functor $\mathcal{A}_{\lambda',\chi} \otimes^L_{\mathcal{A}_{\lambda'}} \bullet$ is perverse with respect to the filtrations by these subcategories once the conditions of (b) hold. Namely, (b1) and (b2) imply (P1), then (b3) implies the first condition in (P2), and (b4)-(b6) imply the second condition in (P2) and (P3). Conversely, it is straightforward to see that if $\mathcal{A}_{\lambda',\chi} \otimes^L_{\mathcal{A}_{\lambda'}} \bullet$ is perverse with respect to the filtrations above, then (b1)-(b6) hold.

Remark 2.16. Using techniques of Section 4.5 below we can show that the locus of λ' in (b) is given by removing finitely many hyperplanes from $\mathfrak{P}_0$.

Remark 2.17. Note that the filtrations making a derived equivalence perverse are uniquely recovered from that equivalence (provided they exist). Recall that λ and $\lambda+\chi$ are such that abelian localization holds for $(\lambda, \theta), (\lambda + \chi, \theta')$. In particular, $\mathfrak{WC}_{\lambda+\chi\leftarrow\lambda}$ is perverse. In particular, thanks to Lemma 2.14 and (b), the filtrations making $\mathfrak{WC}_{\lambda+\chi\leftarrow\lambda}$ perverse are always given by annihilation by a suitable chain of two-sided ideals (that are obtained from $\mathcal{I}^j_{\lambda'}, \mathcal{I}^j_{\lambda'+\chi}$ via Morita equivalences

$$\mathcal{A}_{\lambda'}\text{-bimod} \xrightarrow{\sim} \mathcal{A}_{\lambda}\text{-bimod}, \mathcal{A}_{\lambda'}\text{-bimod} \xrightarrow{\sim} \mathcal{A}_{\lambda}\text{-bimod}$$

for a suitable element λ'). In particular, these chains of ideals in $\mathcal{A}_\lambda, \mathcal{A}_{\lambda+\chi}$ are determined uniquely.

3. Preliminaries on categories $\mathcal{O}$

3.1. Highest weight and standardly stratified structures

Let $\mathbb{K}$ be a field. Let $\mathcal{C}$ be a $\mathbb{K}$-linear abelian category equivalent to the category of finite dimensional modules over a split unital associative finite dimensional $\mathbb{K}$-algebra. We will write $\mathcal{T}$ for an indexing set for the simple objects of $\mathcal{C}$. Let us write $L(\tau)$ for the simple object indexed by $\tau \in \mathcal{T}$ and $P(\tau)$ for the projective cover of $L(\tau)$.

3.1.1. Highest weight categories The additional structure of a highest weight category on $\mathcal{C}$ is a partial order $\leqslant$ on $\mathcal{T}$ that should satisfy axioms (HW1), (HW2) to be explained below. To state the axioms we need some notation.

The partial order $\leqslant$ defines a filtration $\mathcal{C}_{\leqslant \tau}$ on $\mathcal{C}$ by Serre subcategories indexed by $\mathcal{T}$: the subcategory $\mathcal{C}_{\leqslant \tau}$ is, by definition, the Serre span of the simples $L(\tau')$ with $\tau' \leqslant \tau$. Define $\mathcal{C}_{<\tau}$ analogously and let $\mathcal{C}_\tau$ denote the quotient $\mathcal{C}_{\leqslant \tau}/\mathcal{C}_{<\tau}$. The first axiom of a highest weight category is as follows:

(HW1) $\mathcal{C}_\tau$ is equivalent to the category of finite dimensional vector spaces for all τ.

To formulate the second axiom we need some more notation. Let π_τ denote the quotient functor $\mathcal{C}_{\leqslant \tau} \twoheadrightarrow \mathcal{C}_\tau$. Let us write $\Delta_\tau : \mathcal{C}_\tau \to \mathcal{C}_{\leqslant \tau}$ for the left adjoint functor of π_τ. Let P_τ denote the indecomposable object in $\mathcal{C}_\tau$ and set $\Delta(\tau) = \Delta_\tau(P_\tau)$. The object $\Delta(\tau)$ is called *standard*. The second axiom of a highest weight category is as follows:

(HW2) $P(\tau)$ surjects onto $\Delta(\tau)$ in such a way that the kernel is filtered by $\Delta(\tau')$'s with $\tau' > \tau$.

We note that $\mathcal{C}^{opp}$ is also a highest weight category with respect to the same order. The standard objects in $\mathcal{C}^{opp}$ are called costandard objects and are denoted by $\nabla(\tau)$. They have the following property: $\dim \operatorname{Ext}^i_{\mathcal{C}}(\Delta(\tau), \nabla(\tau')) = \delta_{i,0}\delta_{\tau,\tau'}$.

Below we will need the following lemma.

Lemma 3.1. *For $M \in \mathcal{C}, \tau \in \mathcal{T}$, the following two conditions are equivalent:*

1. $\dim \operatorname{Hom}(\Delta(\tau'), M) = \delta_{\tau,\tau'}$,
2. $M \hookrightarrow \nabla(\tau)$.

Proof. It is clear that (2) $\Rightarrow$ (1). Let us prove (1) $\Rightarrow$ (2). Note that $M \in \mathcal{C}_{\leqslant \tau}$, so we can replace $\mathcal{C}$ with $\mathcal{C}_{\leqslant \tau}$ and assume that τ is maximal. In this case, $\nabla(\tau)$ is injective, $\Delta(\tau)$ is projective. So $L(\tau)$ occurs in M with multiplicity 1 that gives a nonzero homomorphism $M \to \nabla(\tau)$. We need to show that it is injective. Let K be the kernel. Then $L(\tau)$ is not a composition factor of K. It follows that $\operatorname{Hom}(\Delta(\tau'), K) = 0$ for all τ'. Hence $K = 0$. $\square$

Remark 3.2. There are various generalizations of the notion of a highest weight category. We will need the situation when $\mathcal{C}$ has finite length property and also enough projectives but not necessarily finitely many simples. Instead of requiring that there are finitely many simples we can impose more general conditions:

- the length of all increasing chains of elements in the poset $(\mathrm{Irr}(\mathcal{C}), \leqslant)$ is bounded from above by some constant, say d,
- and the quotient functors $\mathcal{C}_{\leqslant\tau} \twoheadrightarrow \mathcal{C}_{\tau}$ have left adjoints.

Then we say that $\mathcal{C}$ is a highest weight category if (HW1), (HW2) hold.

3.1.2. Standardly stratified categories Let us define standardly stratified categories following [LW, Section 2].

Let $\mathcal{C}$ be the same as in the first paragraph of Section 3.1. The additional structure of a standardly stratified category on $\mathcal{C}$ is a partial *pre-order* $\preceq$ on $\mathcal{T}$ that should satisfy axioms (SS1), (SS2) to be explained below. Let Ξ denote the set of equivalence classes of $\preceq$. Then, for $\xi \in \Xi$, we can define the Serre subcategories $\mathcal{C}_{\prec\xi} \subset \mathcal{C}_{\preceq\xi}$ similarly to Section 3.1.1, their quotient $\mathcal{C}_{\xi}$ together with the quotient functor $\pi_{\xi} : \mathcal{C}_{\leqslant\xi} \to \mathcal{C}_{\xi}$, and its left adjoint Δ_{ξ}. Further, for $\tau \in \xi$, we write $L_{\xi}(\tau)$ for $\pi_{\xi}(L(\tau))$ and $P_{\xi}(\tau)$ for the projective cover of $L_{\xi}(\tau)$ in $\mathcal{C}_{\xi}$. We define the standard (resp. proper standard) objects by $\Delta_{\preceq}(\tau) := \Delta_{\xi}(P_{\xi}(\tau))$, resp., $\bar{\Delta}_{\preceq}(\tau) := \Delta_{\xi}(L_{\xi}(\tau))$.

The axioms of a standardly stratified category as defined in [LW] are as follows.

(SS1) The functor Δ_{ξ} is exact.
(SS2) The projective object $P(\tau)$ admits an epimorphism onto $\Delta_{\preceq}(\tau)$ with kernel filtered by $\Delta_{\preceq}(\tau')$'s with $\tau' \succ \tau$.

We will be mostly interested in standardly stratified structures on highest weight categories subject to suitable compatibility conditions. Namely, let $\leqslant$ be a partial order on $\mathcal{T}$ defining a highest weight structure on $\mathcal{C}$. We say that a pre-order $\preceq$ on $\mathcal{T}$ is compatible with $\leqslant$ if $\tau \prec \tau' \Rightarrow \tau \leqslant \tau' \Rightarrow \tau \preceq \tau'$.

Lemma 3.3. *Let $\preceq$ be a pre-order compatible with $\leqslant$. Then it defines a standardly stratified structure on $\mathcal{C}$ if and only if both $\pi_{\xi}^{!}$ and π_{ξ}^{*} (the right adjoint of π_{ξ}) are exact.*

Proof. $\mathcal{C}$ satisfies conditions of [L6, Lemma 3.3] because it is highest weight. Our claim follows from [L6, Lemma 3.3]. □

Remark 3.4. Remark 3.2 can be generalized to the setting of standardly stratified categories in a straightforward way.

3.1.3. Partial Ringel dualities Let $\mathcal{C}$ be a highest weight category with respect to a partial order $\leqslant$ and let $\preceq$ be a compatible partial pre-order giving a standardly stratified structure.

Now let $\mathcal{C}'$ be another category with $\mathrm{Irr}(\mathcal{C}') \xrightarrow{\sim} \mathcal{T}$. Define a new partial order on $\mathcal{T}$ by $\tau \leqslant' \tau'$ if

- either $\tau \prec \tau'$
- or $\tau \sim \tau'$ and $\tau \geqslant \tau'$.

The axioms of a partial order are straightforward to verify. Suppose that $\mathcal{C}'$ is highest weight with respect to $\leqslant'$ and standardly stratified with respect to $\preceq$.

By a partial Ringel duality functor we mean a derived equivalence $\psi : D^b(\mathcal{C}) \to D^b(\mathcal{C}')$ that maps $\Delta(\tau)$ to $\Delta_{\preceq}(\underline{\nabla}(\tau))$, where $\underline{\nabla}(\tau)$ is the costandard object in the subquotient category $\mathcal{C}'_{\xi}$. Note that such a functor is defined uniquely up to pre- or post-composing with an abelian equivalence that is the identity on K_0. Also $\mathcal{C}'$ is defined uniquely up to an abelian equivalence which is the identity on the set of simples.

Note that if $\preceq$ is the trivial pre-order, then we recover the usual notion of Ringel duality. The usual Ringel dual always exists. In general, the existence is unclear, see [L9, Section 4.5].

Remark 3.5. We can generalize partial Ringel dualities to categories of the kind considered in Remarks 3.2, 3.4.

3.2. Categories $\mathcal{O}$: general setting

Let $\mathcal{A}$ be a Noetherian associative algebra equipped with a rational Hamiltonian $\mathbb{C}^\times$-action ν. Let $h \in \mathcal{A}$ be the image of 1 under the comoment map and let $\mathcal{A}^i$ denote the ith graded component so that $\mathcal{A}^i := \{a | [h,a] = ia\}$. We set $\mathcal{A}^{\geqslant 0} := \bigoplus_{i\geqslant 0} \mathcal{A}^i, \mathcal{A}^{>0} := \bigoplus_{i>0} \mathcal{A}^i, \mathsf{C}_\nu(\mathcal{A}) := \mathcal{A}^{\geqslant 0}/(\mathcal{A}^{\geqslant 0} \cap \mathcal{A}\mathcal{A}^{>0})$. Note that $\mathcal{A}^{\geqslant 0} \cap \mathcal{A}\mathcal{A}^{>0}$ is a two-sided ideal in $\mathcal{A}^{\geqslant 0}$ so $\mathsf{C}_\nu(\mathcal{A})$ is an algebra. We have a natural isomorphism

$$\mathsf{C}_\nu(\mathcal{A}) \cong \mathcal{A}^0 / \bigoplus_{i>0} \mathcal{A}^{-i}\mathcal{A}^i. \tag{3.1}$$

Let us note that the definitions of $\mathcal{A}^{>0}, \mathcal{A}^{\geqslant 0}, \mathsf{C}_\nu(\mathcal{A})$ make sense even if the action ν is not Hamiltonian.

By the category $\mathcal{O}_\nu(\mathcal{A})$ we mean the full subcategory of the category $\mathcal{A}$-mod of the finitely generated $\mathcal{A}$-modules consisting of all modules such that $\mathcal{A}^{>0}$ acts locally nilpotently. We have the *Verma module* functor Δ_ν : $\mathsf{C}_\nu(\mathcal{A})$-mod $\to \mathcal{O}_\nu(\mathcal{A})$ given by

$$\Delta_\nu(N) := \mathcal{A} \otimes_{\mathcal{A}^{\geqslant 0}} N = (\mathcal{A}/\mathcal{A}\mathcal{A}^{>0}) \otimes_{\mathsf{C}_\nu(\mathcal{A})} N.$$

Note that if h acts on N with eigenvalue α (so that N is a single generalized h-eigenspace), then h acts locally finitely on $\Delta_\nu(N)$ with eigenvalues in $\alpha+\mathbb{Z}_{\leqslant 0}$. The space N is embedded into $\Delta_\nu(N)$ as the α-eigenspace for h. The module $\Delta_\nu(N)$ has the maximal submodule that does not intersect N. The quotient is denoted by $L_\nu(N)$. The map $N \mapsto L_\nu(N)$ is easily seen to be a bijection $\operatorname{Irr}(\mathsf{C}_\nu(\mathcal{A})) \xrightarrow{\sim} \operatorname{Irr}(\mathcal{O}_\nu(\mathcal{A}))$.

Assume now that $\dim \mathsf{C}_\nu(\mathcal{A}) < \infty$. Let $N_1, \ldots, N_r$ be the full list of the simple $\mathsf{C}_\nu(\mathcal{A})$-modules. Let $\alpha_1, \ldots, \alpha_r$ be the eigenvalues of h on these modules (note that h maps into the center of $\mathsf{C}_\nu(\mathcal{A})$ and so acts by a scalar on every irreducible module). We define a partial order $\leqslant$ on the set $N_1, \ldots, N_r$ by setting $N_i \leqslant N_j$ if $\alpha_j - \alpha_i \in \mathbb{Z}_{>0}$ or $i = j$. Using this partial order it is easy to show that all the generalized eigenspaces for h are finite dimensional and all modules in the category $\mathcal{O}$ have finite length. Moreover, we say that N_i, N_j (or the corresponding simples $L_\nu(N_i), L_\nu(N_j)$) lie in the same *h-block* if $\alpha_i - \alpha_j \in \mathbb{Z}$. Note that the simples of $\mathcal{O}$ from different h-blocks lie in different blocks so we can decompose $\mathcal{O}$ according to h-blocks: $\mathcal{O}_\nu(\mathcal{A}) = \sum_{\beta \in \mathbb{C}/\mathbb{Z}} \mathcal{O}_\nu^{\beta+\mathbb{Z}}(\mathcal{A})$, where $\mathcal{O}_\nu^{\beta+\mathbb{Z}}(\mathcal{A})$ is the Serre span of the simples in the h-block corresponding to β.

We will also need a graded version $\mathcal{O}_\nu^{gr}(\mathcal{A})$ of the category $\mathcal{O}$. By definition, it consists of the modules in $\mathcal{O}_\nu(\mathcal{A})$ together with a grading (compatible with the grading on $\mathcal{A}$ coming from ν). The homomorphisms are grading preserving. Note that the irreducibles in $\mathcal{O}_\nu^{gr}(\mathcal{A})$ are labelled by the graded irreducible $\mathsf{C}_\nu(\mathcal{A})$-modules. In the situation when $\dim \mathsf{C}_\nu(\mathcal{A}) < \infty$, the category $\mathcal{O}_\nu^{gr}(\mathcal{A})$ splits into the direct sum $\bigoplus_{\alpha \in \mathbb{C}} \mathcal{O}_\nu^\alpha(\mathcal{A})$, where $\mathcal{O}_\nu^\alpha(\mathcal{A})$ consists of all modules M such that h acts on the graded component $M(i)$ with single eigenvalue $\alpha + i$. Note that $\mathcal{O}_\nu^\alpha(\mathcal{A}) \cong \mathcal{O}_\nu^{\alpha+\mathbb{Z}}(\mathcal{A})$.

3.3. Categories $\mathcal{O}$: setting of quantized symplectic resolutions

Now we are going to concentrate on the case when $\mathcal{A} = \mathcal{A}_\lambda$ is the algebra of global sections of a quantization $\mathcal{A}_\lambda^\theta$. Suppose that X is equipped with a Hamiltonian action of a torus T such that X^T is finite. Choose a one-parameter subgroup $\nu : \mathbb{C}^\times \to T$ and assume that $X^{\nu(\mathbb{C}^\times)}$ is a finite set. In this case, $X^{\nu(\mathbb{C}^\times)} = X^T$.

Lemma 3.6. *The algebra $\mathsf{C}_\nu(\mathcal{A}_{\mathfrak{P}})$ is finitely generated over $\mathbb{C}[\mathfrak{P}]$. In particular, $\mathsf{C}_\nu(\mathcal{A}_\lambda)=\mathsf{C}_\nu(\mathcal{A}_{\mathfrak{P}})\otimes_{\mathbb{C}[\mathfrak{P}]}\mathbb{C}_\lambda$ is finite dimensional.*

Proof. The algebra $\mathsf{C}_\nu(\mathcal{A}_{\mathfrak{P}})$ carries a natural filtration and $\mathsf{C}_\nu(\mathbb{C}[Y_{\mathfrak{p}}])\twoheadrightarrow \operatorname{gr}\mathsf{C}_\nu(\mathcal{A}_{\mathfrak{P}})$. So it is enough to show that $\mathsf{C}_\nu(\mathbb{C}[Y_{\mathfrak{p}}])$ is finitely generated over $\mathbb{C}[\mathfrak{p}]$. Note that $\mathsf{C}_\nu(\mathbb{C}[Y_{\mathfrak{p}}])/(\mathfrak{p})=\mathsf{C}_\nu(\mathbb{C}[Y])$ is finitely generated as an algebra because it is a quotient of $\mathbb{C}[Y]^T$. Since T has finitely many fixed points in Y, the algebra $\mathsf{C}_\nu(\mathbb{C}[Y])$ is finite dimensional. It follows that $\mathsf{C}_\nu(\mathbb{C}[Y_{\mathfrak{p}}])$ is finitely generated over $\mathbb{C}[\mathfrak{p}]$. □

For a Zariski generic $\lambda\in\mathfrak{P}$ one can give a more precise description of $\mathsf{C}_\nu(\mathcal{A}_\lambda)$, see [L6, Proposition 5.3] or [BLPW, Section 5.1] for a somewhat weaker result. Namely, we can define the Cartan subquotient $\mathsf{C}_\nu(\mathcal{A}^\theta_\lambda)$ that will be a sheaf on $X^{\nu(\mathbb{C}^\times)}$, see [L6, Section 5.2]. Since ν is generic and T has finitely many fixed points in X, $\mathsf{C}_\nu(\mathcal{A}^\theta_\lambda)$ is an algebra naturally identified with $\mathbb{C}[X^{\nu(\mathbb{C}^\times)}]$. By the construction, there is a natural homomorphism $\mathsf{C}_\nu(\mathcal{A}_\lambda)\to\mathsf{C}_\nu(\mathcal{A}^\theta_\lambda)=\mathbb{C}[X^{\nu(\mathbb{C}^\times)}]$, see [L6, Section 5.3]. Similarly, we have the algebra $\mathsf{C}_\nu(\mathcal{A}^\theta_{\mathfrak{P}})$ that is naturally isomorphic to $\mathbb{C}[\mathfrak{P}][X^T]$. We have a $\mathbb{C}[\mathfrak{P}]$-linear isomorphism $\mathsf{C}_\nu(\mathcal{A}_{\mathfrak{P}})\to\mathsf{C}_\nu(\mathcal{A}^\theta_{\mathfrak{P}})$.

Lemma 3.7 (Proposition 5.3 in [L6])**.** *For a Zariski generic $\lambda\in\mathfrak{P}$, the homomorphism $\mathsf{C}_\nu(\mathcal{A}_\lambda)\to\mathbb{C}[X^T]$ is an isomorphism.*

Below we will give a more precise description of this Zariski open subset.

Following [BLPW, Section 3.3] (see also [L6, Section 4.3]), one can define the subcategory $\mathcal{O}_\nu(\mathcal{A}^\theta_\lambda)\subset\operatorname{Coh}(\mathcal{A}^\theta_\lambda)$. By definition, it consists of the coherent sheaves of $\mathcal{A}^\theta_\lambda$-modules supported on the contracting locus for ν that admit a $\nu(\mathbb{C}^\times)$-equivariant structure (in a weak sense, i.e., without a quantum comoment map). It was shown in [BLPW, Corollary 3.19] that the functors $L_i\operatorname{Loc}^\theta_\lambda, R^i\Gamma^\theta_\lambda$ map between categories $\mathcal{O}$. In particular, if abelian localization holds for (λ,θ), then the categories $\mathcal{O}_\nu(\mathcal{A}_\lambda),\mathcal{O}_\nu(\mathcal{A}^\theta_\lambda)$ are equivalent (via Loc and Γ).

Note that similarly to Section 3.2 we can consider the graded versions $\mathcal{O}^{gr}_\nu(\mathcal{A}_\lambda),\mathcal{O}^{gr}_\nu(\mathcal{A}^\theta_\lambda)$.

Also we can consider the T-equivariant versions $\mathcal{O}^T_\nu(\mathcal{A}_\lambda),\mathcal{O}^T_\nu(\mathcal{A}^\theta_\lambda)$ of all T-equivariant objects in $\mathcal{O}_\nu(\mathcal{A}_\lambda),\mathcal{O}_\nu(\mathcal{A}_\lambda)$ (when T is one-dimensional, we recover the categories $\mathcal{O}^{gr}_\nu(\mathcal{A}_\lambda)$, $\mathcal{O}^{gr}_\nu(\mathcal{A}^\theta_\lambda)$).

3.4. Verma modules and their duals

3.4.1. Objects $\boldsymbol{\Delta_{\nu,\lambda}(\mathsf{x})}$ Let $\mathsf{x}\in X^T$. We can view $\mathbb{C}_{\mathsf{x}}$, the simple $\mathbb{C}[X^T]$-module corresponding to x, as a module over $\mathsf{C}_\nu(\mathcal{A}_\lambda)$ via the homomorphism

$\mathsf{C}_\nu(\mathcal{A}_\lambda) \to \mathbb{C}[X^T]$. We denote the module $\Delta_\nu(\mathbb{C}_{\mathsf{x}})$ by $\Delta_{\nu,\lambda}(\mathsf{x})$. Note that we have

$$\operatorname{Hom}_{\mathcal{A}_\lambda}(\Delta_{\nu,\lambda}(\mathsf{x}), N) = \operatorname{Hom}_{\mathsf{C}_\nu(\mathcal{A}_\lambda)}(\mathbb{C}_{\mathsf{x}}, N^{\mathcal{A}_\lambda^{>0}}). \tag{3.2}$$

The module $\Delta_{\nu,\lambda}$ has unique simple quotient module denoted by $L_{\nu,\lambda}(\mathsf{x})$.

The modules $\Delta_{\nu,\lambda}(\mathsf{x})$ come in a family over $\mathfrak{P}$: we can use the homomorphism $\mathsf{C}_\nu(\mathcal{A}_{\mathfrak{P}}) \to \mathbb{C}[\mathfrak{P}][X^T]$ to form the module $\Delta_{\nu,\mathfrak{P}}(\mathsf{x})$ whose specialization to λ is $\Delta_{\nu,\lambda}(\mathsf{x})$. Similarly, we can consider a homogenized version. Namely, form the Rees algebra $\mathcal{A}_{\mathfrak{p},\hbar}$ of $\mathcal{A}_{\mathfrak{P}}$. We still have a homomorphism $\mathsf{C}_\nu(\mathcal{A}_{\mathfrak{p},\hbar}) \to \mathbb{C}[\mathfrak{p},\hbar][X^T]$ and so can form the $\mathcal{A}_{\mathfrak{p},\hbar}$-module $\Delta_{\nu,\mathfrak{p},\hbar}(\mathsf{x})$ that specializes to $\Delta_{\nu,\mathfrak{P}}(\mathsf{x})$ when $\hbar = 1$.

Recall that we write $\mathfrak{p}^{sing}$ for the locus in $\mathfrak{p}$, where X_λ is not affine, this is a union of hyperplanes, see Section 2.1.2. We write $\mathfrak{p}^{reg}$ for $\mathfrak{p} \setminus \mathfrak{p}^{sing}$. Note that, for $\lambda \in \mathfrak{p}^{reg}$, we get $\Delta_{\nu,\lambda,0}(\mathsf{x}) = \mathbb{C}[(X_\lambda)^+_{\mathsf{x}}]$, where $(X_\lambda)^+_{\mathsf{x}} := \{x \in X_\lambda | \lim_{t\to 0} \nu(t)x = \mathsf{x}\}$. The variety $(X_\lambda)^+_{\mathsf{x}}$ is a smooth lagrangian subvariety in X_λ that is isomorphic to an affine space.

Now let $\kappa \in \mathfrak{X}(T)$. We can form the graded versions $\Delta_{\nu,\lambda}(\mathsf{x},\kappa), \Delta_{\nu,\mathfrak{P}}(\mathsf{x},\kappa)$ etc. by putting $\mathbb{C}_{\mathsf{x}}$ in degree κ.

3.4.2. Flatness Here we are going to give a description of the locus in $\mathfrak{p} \times \mathbb{C}$ where $\Delta_{\nu,\mathfrak{p},\hbar}(\mathsf{x})$ is flat.

The proof the following lemma repeats that of [BL, Lemma 3.5].

Lemma 3.8. *Let Z be a closed subvariety in $\mathfrak{p} \times \mathbb{C}$. Let M be a finitely generated $\mathbb{C}[Z] \otimes_{\mathbb{C}[\mathfrak{p},\hbar]} \mathcal{A}_{\mathfrak{p},\hbar}$-module. Then there is an open dense affine subvariety $Z^0 \subset Z$ such that $\mathbb{C}[Z^0] \otimes_{\mathbb{C}[Z]} M$ is a free $\mathbb{C}[Z^0]$-module.*

Corollary 3.9. *There is a partition $\mathfrak{p} \times \mathbb{C} = \bigsqcup_{i=1}^k Z^i$ into a union of locally closed affine subvarieties such that*

1. *$\mathbb{C}[Z^i] \otimes_{\mathbb{C}[\mathfrak{p},\hbar]} \Delta_{\nu,\mathfrak{p},\hbar}(\mathsf{x})$ is free over $\mathbb{C}[Z^i]$,*
2. *and, for all j, the union $\bigsqcup_{i=1}^j Z^i$ is open.*

Now note that $\Delta_{\nu,\mathfrak{p},\hbar}(\mathsf{x})$ is $\mathbb{C}^\times$-equivariant with respect to the Hamiltonian torus action and all eigenspaces are finitely generated $\mathbb{C}[\mathfrak{p},\hbar]$-modules. Let $\Delta_{\nu,\mathfrak{p},\hbar}(\mathsf{x})^j$ denote the eigenspace of weight j. Note that, for $z_1, z_2 \in Z^i$ we have $\Delta_{\nu,\mathfrak{p},\hbar}(\mathsf{x})^j_{z_1} \cong \Delta_{\nu,\mathfrak{p},\hbar}(\mathsf{x})^j_{z_2}$, an isomorphism of vector spaces. We get the following corollary.

Corollary 3.10. *The locus in $\mathfrak{p} \times \mathbb{C}$, where the fibers of all $\mathbb{C}[\mathfrak{p},\hbar]$-modules $\Delta_{\nu,\mathfrak{p},\hbar}(\mathsf{x})^j$ have the minimal possible dimension is Zariski open in $\mathfrak{p} \times \mathbb{C}$.*

Denote this locus by $\tilde{Z}_{\mathsf{x}}$.

Proposition 3.11. *We have* $\tilde{Z}_{\mathsf{x}} \cap (\mathfrak{p} \times \{0\}) \supset \mathfrak{p}^{reg}$.

Proof. Note that the T-fixed points in X_ζ are in a natural bijection with the T-fixed points in X. Let x' be the point in the T-fixed locus in X_ζ corresponding to x for $\zeta \in \mathfrak{p}^{reg}$. Form the completion $\mathcal{A}_{\mathfrak{p},\hbar}^{\wedge_{\mathsf{x}'}}$. This completion is $T \times \mathbb{C}^\times$-equivariantly identified with $\mathbb{C}[\mathfrak{p}]\widehat{\otimes}_{\mathbb{C}}\mathbb{A}_\hbar(T_{\mathsf{x}}X)^{\wedge_0}$, where we write $\mathbb{A}_\hbar(T_{\mathsf{x}}X)$ for the homogenized Weyl algebra of the symplectic vector space $T_{\mathsf{x}}X$. From here it is easy to see that

$$\mathcal{A}_{\mathfrak{p},\hbar}^{\wedge_{\mathsf{x}'}}/\mathcal{A}_{\mathfrak{p},\hbar}^{\wedge_{\mathsf{x}'}}\left(\mathcal{A}_{\mathfrak{p},\hbar}^{\wedge_{\mathsf{x}'}}\right)^{>0}$$

is free over $\mathbb{C}[\mathfrak{p},\hbar]$. Moreover, the T-eigenspaces are free over $\mathbb{C}[\mathfrak{p},\hbar]$.

On the other hand, we claim that the T-finite part, M, of this module is identified with

$$\mathbb{C}[\mathfrak{p},\hbar]^{\wedge_{(\zeta,0)}} \otimes_{\mathbb{C}[\mathfrak{p},\hbar]} \Delta_{\nu,\mathfrak{p},\hbar}(\mathsf{x}). \tag{3.3}$$

Indeed, thanks to the direct analog of (3.2), we have a homomorphism

$$\mathbb{C}[\mathfrak{p},\hbar]^{\wedge_{(\zeta,0)}} \otimes_{\mathbb{C}[\mathfrak{p},\hbar]} \Delta_{\nu,\mathfrak{p},\hbar}(\mathsf{x}) \to M.$$

Its specialization to the closed point is an isomorphism and the target is free over $\mathbb{C}[\mathfrak{p},\hbar]^{\wedge_{(\zeta,0)}}$. It follows that this homomorphism is an isomorphism.

So (3.3) is free over $\mathbb{C}[\mathfrak{p},\hbar]^{\wedge_{(\zeta,0)}}$. The claim of the proposition follows. $\square$

Definition 3.12. Let $(\mathfrak{p} \times \mathbb{C})^{fl}$ denote the intersection of the open subsets $\tilde{Z}_{\mathsf{x}}$ for all $\mathsf{x} \in X^T$. Set $\mathfrak{P}^{fl} := (\mathfrak{p} \times \mathbb{C})^{fl} \cap (\mathfrak{p} \times \{1\})$.

We write $(T_{\mathsf{x}}X)^{>0}$ for the ν-contacting locus in $T_{\mathsf{x}}X$.

Corollary 3.13. *The following two conditions are equivalent:*

- $\lambda \in \mathfrak{P}^{fl}$,
- *for all* $\mathsf{x} \in X^T$, *we have a* T*-equivariant isomorphism*

$$\Delta_{\nu,\lambda}(\mathsf{x}) \cong \mathbb{C}[(T_{\mathsf{x}}X)^{>0}].$$

3.4.3. Contravariant duality Let us discuss contravariant duality for categories $\mathcal{O}$. Take $M \in \mathcal{O}_\nu(\mathcal{A}_\lambda)$. This module decomposes as $\bigoplus_{\alpha\in\mathbb{C}} M_\alpha$, where M_α is the generalized eigenspace for $h = d_1\nu$ with eigenvalue α. Recall that all M_α are finite dimensional, see, e.g., [BLPW, Lemma 3.13]. Then we can consider the restricted dual $M^\vee$, this is a right $\mathcal{A}_\lambda$-module. One can show that it lies in a category $\mathcal{O}$ for $\mathcal{A}_\lambda^{opp}$, more precisely, in $\mathcal{O}_{-\nu}(\mathcal{A}_\lambda^{opp})$. Using the isomorphism $\mathcal{A}_\lambda^{opp} \cong \mathcal{A}_{-\lambda}$, we get $M^\vee \in \mathcal{O}_{-\nu}(\mathcal{A}_{-\lambda})$. The quasi-inverse functor of $M \mapsto M^\vee$ is constructed completely analogously and we again denote it by $\bullet^\vee$.

The following result was established in [L3].

Lemma 3.14. *For $M_1 \in \mathcal{O}_\nu(\mathcal{A}_\lambda), M_2 \in \mathcal{O}_{-\nu}(\mathcal{A}_\lambda^{opp})$, we get*

$$\operatorname{Tor}^i_{\mathcal{A}_\lambda}(M_1, M_2)^* \cong \operatorname{Ext}^i_{\mathcal{A}_\lambda}(M_1, M_2^\vee).$$

Note that $\mathsf{C}_{-\nu}(\mathcal{A}_\lambda^{opp})$ is naturally identified with $\mathsf{C}_\nu(\mathcal{A}_\lambda)^{opp}$. So for $N \in \mathsf{C}_\nu(\mathcal{A}_\lambda)$ let us set $\nabla_\nu(N) = \Delta^r_{-\nu}(N^*)^\vee$, where we write $\Delta^r_{-\nu}$ for the Verma module functor for the category $\mathcal{O}_{-\nu}(\mathcal{A}_\lambda^{opp})$. Alternatively, one can define $\nabla_\nu(N)$ as $\operatorname{Hom}_{\mathsf{C}_\nu(\mathcal{A}_\lambda)}(\mathcal{A}_\lambda/\mathcal{A}_\lambda^{<0}\mathcal{A}_\lambda, N)$, see [L6, Section 4.2] (here we take the restricted Hom, i.e., the direct sum of Hom's from the graded components).

We write $\nabla_{\nu,\lambda}(\mathsf{x})$ for $\Delta_{\nu,-\lambda}(\mathsf{x})^\vee$.

3.5. Highest weight structures

Now let us discuss highest weight structures on $\mathcal{O}_\nu(\mathcal{A}_\lambda)$, $\mathcal{O}_\nu(\mathcal{A}_\lambda^\theta)$.

3.5.1. Ext vanishing A proof of the following result can be found in [L9, Lemma 4.8].

Lemma 3.15. *For a Zariski generic parameter $\lambda \in \mathfrak{P}$ and $\mathsf{x}, \mathsf{x}' \in X^{\nu(\mathbb{C}^\times)}$, we have $\mathsf{C}_\nu(\mathcal{A}_\lambda) \xrightarrow{\sim} \mathbb{C}[X^{\nu(\mathbb{C}^\times)}]$ and*

$$\dim \operatorname{Ext}^i_{\mathcal{A}_\lambda}(\Delta_\nu(\mathsf{x}), \nabla_\nu(\mathsf{x}')) = \delta_{i,0}\delta_{\mathsf{x},\mathsf{x}'}.$$

A number of interesting corollaries of this result was deduced in [BLPW, Section 5.2].

Corollary 3.16. *For λ as in Lemma 3.15, the natural functor $D^b(\mathcal{O}_\nu(\mathcal{A}_\lambda)) \to D^b(\mathcal{A}_\lambda\text{-mod})$ is a fully faithful embedding.*

Also Lemma 3.15 implies that the category $\mathcal{O}_\nu(\mathcal{A}_\lambda)$ is highest weight, where the order is as in Section 3.2. In more detail, it is as follows. To $\lambda \in \mathfrak{P}$ and $\mathsf{x} \in X^{\nu(\mathbb{C}^\times)}$ we can assign the scalar $\mathsf{c}_{\nu,\lambda}(\mathsf{x})$ equal to the image of $h \in \mathsf{C}_\nu(\mathcal{A}_\lambda)$ under the projection $\mathsf{C}_\nu(\mathcal{A}_\lambda) \to \mathbb{C}$ of evaluation at x. Then the order $\leqslant_\lambda$ on $X^{\nu(\mathbb{C}^\times)}$ is introduced as follows: $\mathsf{x} \leqslant_\lambda \mathsf{x}'$ if $\mathsf{x} = \mathsf{x}'$ or $\mathsf{c}_{\nu,\lambda}(\mathsf{x}') - \mathsf{c}_{\nu,\lambda}(\mathsf{x}) \in \mathbb{Z}_{>0}$.

Corollary 3.17. *The category $\mathcal{O}_\nu(\mathcal{A}^\theta_\lambda)$ has a highest weight structure for all λ via an equivalence $\mathcal{O}_\nu(\mathcal{A}^\theta_\lambda) \cong \mathcal{O}_\nu(\mathcal{A}_{\lambda+n\chi})$ for χ in the chamber of θ and sufficiently large n.*

Below we will see that this highest weight structure is independent of the choice of n (as long as n is large enough), see [BLPW, Section 5.3].

3.5.2. Identification of K_0 with $\mathbb{C}X^T \times \mathfrak{X}(T)$ Now we describe the K_0 groups of the categories $\mathcal{O}_\nu(\mathcal{A}^\theta)$ and their T-equivariant versions (under some assumptions on λ).

Suppose that $\lambda \in \mathfrak{P}^{fl}$ (see Definition 3.12) satisfies the conditions of Lemma 3.15. Since $\mathcal{O}_\nu(\mathcal{A}_\lambda)$ is a highest weight category with standards $\Delta_{\nu,\lambda}(\mathsf{x})$, see Section 3.5.1, the classes $[\Delta_{\nu,\lambda}(\mathsf{x})]$ form a basis in $K_0(\mathcal{O}_\nu(\mathcal{A}_\lambda))$. This gives an identification of $K_0(\mathcal{O}_\nu(\mathcal{A}_\lambda))$ with $\mathbb{C}X^T$. We can also identify $K_0(\mathcal{O}^T_\nu(\mathcal{A}_\lambda))$ with $\mathbb{C}X^T \times \mathfrak{X}(T)$ by sending the class of $\hat{\Delta}_{\nu,\lambda}(\mathsf{x},\kappa)$ to (x,κ).

3.6. Examples

3.6.1. Cotangent bundles to flag varieties Let us consider the special case when $X = T^*(G/B)$. Let T denote a maximal torus in B, it acts on X in a Hamiltonian fashion. Let $\nu : \mathbb{C}^\times \to T$ be a one-parameter subgroup. The set $X^{\nu(\mathbb{C}^\times)}$ is finite if and only if ν is regular, i.e., its centralizer in G is T. In this case, the fixed points are labelled by W, where $W = N_G(T)/T$ is the Weyl group of $\mathfrak{g}$.

We recover the usual BGG category $\mathcal{O}$ (in the version, where it consists of finitely generated $\mathcal{U}_\lambda$-modules with locally finite action of $\mathfrak{b}$). Conditions of Lemma 3.15 hold under the assumption that λ is regular. Here $\mathsf{C}_\nu(\mathcal{A}_\lambda)$ is naturally identified with $\mathbb{C}[\mathfrak{h}]/\{f \in \mathbb{C}[\mathfrak{h}] | f(w\lambda) = 0, \forall w\}$. Further, we have $\mathsf{c}_{\nu,\lambda}(\mathsf{x}) = \langle w\lambda, \nu\rangle$ for $\mathsf{x} \in X^{\nu(\mathbb{C}^\times)}$ corresponding to $w \in W$.

3.6.2. Example: rational Cherednik algebras of type A Consider the case when $X = \mathrm{Hilb}_n(\mathbb{C}^2)$. Let $\mathfrak{h}$ denote the span of $y_1, \ldots, y_n$ and $\mathfrak{h}^*$ denote the span of $x_1, \ldots, x_n$. Then $S(\mathfrak{h}^*), S(\mathfrak{h})$ are included into H_c as subalgebras. Moreover, $\mathbb{C}[Y] = S(\mathfrak{h} \oplus \mathfrak{h}^*)^{S_n}$.

The category $\mathcal{O}_c$ was introduced in [GGOR] as the category of all H_c-modules that are finitely generated over $S(\mathfrak{h}^*)$ and have locally nilpotent action of $\mathfrak{h}$. We can introduce the Verma modules $\Delta_c(\mu) = H_c \otimes_{S(\mathfrak{h})\#S_n} \mu$, where μ runs over the irreducible S_n-modules (that are naturally labelled by partitions of n). For $c \notin \{-\frac{a}{b} | 1 \leqslant a < b \leqslant n\}$, the functor $M \mapsto eM : H_c\text{-mod} \to eH_ce\text{-mod}$ is an equivalence of categories, see, e.g., [BE, Corollary 4.2].

On X we have a one-parameter Hamiltonian torus acting. We choose ν so that $\nu(t).x_i = t^{-1}x_i, \nu(t).y_i = ty_i$. The fixed points in X are also naturally parameterized by the partitions of n.

The simples in $\mathcal{O}_\nu(\mathcal{A}_\lambda)$ (where $\lambda = c + \frac{1}{2}$) are the modules of the form $eL_c(\mu)$. Let x_μ denote the fixed point in X^T such that $eL_c(\mu)$ corresponds to x_μ. In fact, the labeling $\mu \mapsto x_\mu$ is the standard one, but we will not need this.

Set $\mathsf{x} := \mathsf{x}_\mu$. Let us compute the number $\mathsf{c}_{\nu,\lambda}(\mathsf{x})$. We introduce two important statistics of a Young diagram μ, namely, $\operatorname{cont}(\mu)$ and $n(\mu)$. Recall that by the content of a box in a Young diagram we mean the difference of its horizontal and vertical coordinates. The content of μ denoted by $\operatorname{cont}(\mu)$ is the sum of contents of all boxes, for example, $\operatorname{cont}((n)) = 0+1+2+\ldots+(n-1)$ (here and below by (n) me mean a Young diagram with a single row of n boxes). If $\mu = (\mu_1, \ldots, \mu_k)$, then we write $n(\mu) := \sum_{i=1}^k (i-1)\mu_i$. This is the minimal degree in which μ occurs in $S(\mathfrak{h})$.

Lemma 3.18. *Suppose that c is Zariski generic. Then, for a suitable choice of h (it is defined up to adding a constant), we have $\mathsf{c}_{\nu,\lambda}(\mathsf{x}) = c\operatorname{cont}(\mu) - n(\mu)$ (where μ, x correspond to the same Young diagram).*

Proof. Note that $\mathsf{c}_{\nu,\lambda}(\mathsf{x})$ is an affine function in λ, see [L6, Section 6.2]. So it is enough to prove the equality assuming λ is Weil generic. Here the simple object corresponding to x is $e\Delta_c(\mu)$. It follows that $\mathsf{c}_{\nu,\lambda}(\mathsf{x})$ is the highest weight space in $e\Delta_c(\mu)$. The highest weight in $\Delta_c(\mu)$ is $c\operatorname{cont}(\mu)$ and the required equality follows from the fact that $n(\mu)$ is the minimal degree of μ^* in $S(\mathfrak{h}^*)$. $\square$

4. Very generic and regular parameters

4.1. Generic simplicity

4.1.1. Essential hyperplanes Here we introduce the notion of an *essential hyperplane*. Pick a classical wall Γ. Fix ζ as in Proposition 2.3 so that, for all $y \in Y_\lambda$, the rank of the map $\eta : \mathfrak{p} \to \underline{\mathfrak{p}}$ is equal to 1. Let $\underline{\mathfrak{p}}'$ denote the image of η, these spaces are identified for all y. We consider the lattice $\underline{\mathfrak{p}}'_{\mathbb{Z}}$ that is the image of $\operatorname{Pic}(X)$, it is isomorphic to $\mathbb{Z}$.

Consider the algebra $\underline{\mathcal{A}}_{\underline{\mathfrak{P}}'}$. Pick an ample line bundle $O(\chi)$ on X. Let $O(\underline{\chi})$ denote the induced line bundle on $\underline{X}$. Note that the map $\chi \mapsto \underline{\chi}$ gives η after tensoring with $\mathbb{Q}$.

Consider the bimodules $\underline{\mathcal{A}}_{\underline{\mathfrak{P}}',\underline{\chi}}, \underline{\mathcal{A}}_{\underline{\mathfrak{P}}',-\underline{\chi}}$. Thanks to Lemma 2.12, the locus in $\underline{\mathfrak{P}}'$, where these bimodules fail to be mutually inverse Morita equivalence bimodules is a finite subset to be denoted by Σ_y.

We define Σ_Γ as follows. We pick points $y_1, \ldots, y_k$, one in each minimal symplectic leaf in Y_ζ. Consider the union $\bigcup_i \Sigma_{y_i}$. By definition, Σ_Γ consists of all $\underline{\lambda}$ in $\underline{\mathfrak{P}}'$ that have integral (i.e., belonging to $\underline{\mathfrak{p}}'_{\mathbb{Z}}$) difference with a point in Σ_y.

For a wall Γ, let α_Γ denote an indecomposable element in $\mathfrak{p}^*_{\mathbb{Z}}$ such that $\Gamma = \ker \alpha_\Gamma$ (this element is defined up to multiplication by ± 1). We can identify $\underline{\mathfrak{p}}'$ with $\mathbb{C}$ by taking the image of α_Γ for 1.

Definition 4.1. By an *essential hyperplane* in $\mathfrak{P}$ we mean a hyperplane of the form $\tilde{\Gamma}$ given by $\langle \alpha_\Gamma, \bullet \rangle = \sigma$ for $\sigma \in \Sigma_\Gamma$ for some classical wall Γ. We say that this essential hyperplane is associated with Γ.

Definition 4.2. Let $\lambda \in \mathfrak{P}$. We say that an essential hyperplane $\tilde{\Gamma}$ is *relevant* for λ if $(\lambda + \mathfrak{p}_{\mathbb{Z}}) \cap \tilde{\Gamma} \neq \varnothing$.

Example 4.3. Let $X = T^*\mathcal{B}$. Then $\Sigma_\Gamma = \mathbb{Z}$ for all Γ.

Example 4.4. Let X be the Hilbert scheme $\operatorname{Hilb}_n(\mathbb{C}^2)$. Then there is only one hyperplane Γ, which is $\Gamma = \{0\}$. Then for Σ_Γ we can take $\mathbb{Z} + \{-\frac{a}{b} + \frac{1}{2} | 1 \leqslant a < b \leqslant n\}$. This follows, for example, from [BE, Section 4.1] combined with the construction of translation bimodules in [GS1].

The sets Σ_Γ can also be computed for quantizations of Nakajima quiver varieties, see [BL, Section 9.4], at least, under some additional assumptions, e.g., that the underlying quiver is of finite or affine type.

4.1.2. Very generic parameters

Definition 4.5. We say that $\lambda \in \mathfrak{P}$ is *very generic* if it doesn't lie in any essential hyperplane.

Lemma 4.6. *Let λ be very generic. Then the following claims hold.*

1. *The algebra $\mathcal{A}_\lambda$ is simple.*
2. *All functors $\mathcal{A}_{\lambda,\chi} \otimes^L_{\mathcal{A}_\lambda} \bullet$ are t-exact equivalences.*
3. *Abelian localization holds for (λ, θ) with any θ.*

Proof. We just need to prove (1): (2) will follow from Lemma 2.15 because once λ is very generic, then so is $\lambda + \chi$. And (3) follows from (2) and Proposition 2.11. The proof of (1) follows that of [BL, Proposition 9.6] with some modifications. The proof is in several steps.

Step 1. Suppose that abelian localization holds for (λ, θ). Our goal for the time being is to show that the long wall-crossing functor $\mathfrak{WC}_{\lambda^- \leftarrow \lambda}$ is a t-exact equivalence. By [BPW, Theorem 6.35], $\mathfrak{WC}_{\lambda^- \leftarrow \lambda}$ decomposes into the composition of short wall-crossing functors, each crossing a single wall. So we need to prove that each short wall-crossing functor is t-exact.

Step 2. Let Γ be a wall for the classical chamber C containing θ and let θ' be in the chamber opposite to C with respect to Γ. Suppose that χ is such that abelian localization holds for $(\lambda + \chi, \theta')$. As was discussed in Section 2.6.3, the functor $\mathfrak{WC}_{\lambda+\chi \leftarrow \lambda}$ is a perverse equivalence, moreover, if $\mathfrak{WC}_{\lambda+\chi \leftarrow \lambda}$ is not t-exact, then the support of $\mathcal{A}_{\mathfrak{P}_0}/I^1_{\mathfrak{P}_0}$ in $\mathfrak{P}_0 := \lambda + \Gamma$ is Zariski dense. On the other hand, pick $\zeta \in \Gamma$ as in Proposition 2.3 and apply the restriction functor $\bullet_\dagger$ associated to a point $y_i \in Y_\zeta$ be as in Section 4.1.1. Since $\lambda + \Gamma$ is not an essential hyperplane, thanks to (v) of Section 2.4.4, the bimodules $(\mathcal{A}_{\lambda,\chi})_\dagger, (\mathcal{A}_{\lambda+\chi,-\chi})_\dagger$ are mutually inverse Morita equivalences. It follows that $(I^1_{\mathfrak{P}_0})_\dagger = 0$. This contradicts the claim that the support of $\mathcal{A}_{\mathfrak{P}_0}/I^1_{\mathfrak{P}_0}$ in $\mathfrak{P}_0 := \lambda + \Gamma$ is Zariski dense. This completes the proof of the claim that $\mathfrak{WC}_{\lambda+\chi \leftarrow \lambda}$ is a t-exact equivalence. Hence $\mathfrak{WC}_{\lambda^- \leftarrow \lambda}$ is a t-exact equivalence.

Step 3. Thanks to the results recalled in Section 2.6.3, $\mathfrak{WC}_{\lambda^- \leftarrow \lambda}$ is a perverse equivalence with respect to the filtration by the dimensions of the associated varieties of the annihilators. Since $\mathcal{A}_\lambda$ is a prime Noetherian algebra, the claim that this perverse equivalence is t-exact, means that there are no proper two-sided ideals in $\mathcal{A}$, i.e., $\mathcal{A}$ is simple. $\square$

4.2. Abelian localization and finite homological dimension

In what follows we will need to understand when the specialization $\mathcal{A}_{\lambda,z}$ for $(\lambda, z) \in \mathfrak{p} \oplus \mathbb{C}$ has finite homological dimension and has some related properties. We start with abelian localization.

4.2.1. Abelian localization

Lemma 4.7. *There is a finite collection of essential hyperplanes such that for any λ away from the union of these hyperplanes abelian localization holds for (λ, θ) for θ that depends on λ.*

Proof. For each classical chamber C choose $\chi \in \operatorname{Pic}(X)$ such that $c_1(\chi)$ is inside C, equivalently, $O(\chi)$ is ample on the resolution X corresponding to C. Recall, Lemma 2.12, that the locus of all λ such that one of the natural homomorphisms

$$\mathcal{A}_{\lambda,\chi} \otimes_{\mathcal{A}_{\lambda+\chi}} \mathcal{A}_{\lambda+\chi,-\chi} \to \mathcal{A}_\lambda, \mathcal{A}_{\lambda+\chi,-\chi} \otimes_{\mathcal{A}_\lambda} \mathcal{A}_{\lambda,\chi} \to \mathcal{A}_{\lambda+\chi}$$

is not an isomorphism is Zariski closed in $\mathfrak{p}$. Let Z_χ denote this locus. Note that Z_χ must be contained in the union of finitely many essential hyperplanes by Lemma 4.6. We can choose a finite collection Ψ of essential hyperplanes containing all subsets Z_χ and subject to the following condition:

(*) For any of the chosen elements χ, any of the walls Γ and any λ such that neither $\lambda, \lambda+\chi$ lie in a hyperplane from Ψ, we have that λ and $\lambda+\chi$ lie in the same half space with respect to any of the hyperplanes in Ψ that is relevant for λ.

Now take λ that does not lie in any of the hyperplanes in Ψ. We can find C such that, for the corresponding element χ, we have that $\lambda+n\chi$ does not lie in a hyperplane from Ψ for any $n \geqslant 0$. In particular, $\lambda+n\chi \not\in Z_\chi$ for any $n \geqslant 0$. Now we can use Proposition 2.11 to conclude that abelian localization holds for (λ,χ) finishing the proof. □

In what follows we enlarge Ψ so that for $\lambda \not\in \Psi$ abelian localization holds for (λ,θ) and $(-\lambda,\theta')$ for some θ,θ'. It follows that abelian localization holds for the algebra $\mathcal{A}_\lambda \otimes \mathcal{A}_\lambda^{opp}$ and a suitable resolution of $Y\times Y$.

4.2.2. Finite projective dimension As before, we consider $\mathfrak{P}$ as the affine subspace $\mathfrak{p}\times\{1\} \subset \mathfrak{p}\times\mathbb{C}$. So any hyperplane in Ψ spans a codimension 1 subspace in $\mathfrak{p}\times\mathbb{C}$. Let $(\mathfrak{p}\times\mathbb{C})^0$ denote the complement to the union of all codimension 1 subspaces that arise in this way. In particular, $(\mathfrak{p}\times\mathbb{C})^0\cap(\mathfrak{p}\times\{0\}) = \mathfrak{p}\setminus\mathfrak{p}^{reg}$. This discussion and Lemma 4.7 have the following corollary.

Corollary 4.8. *For every $(\lambda,z)\in(\mathfrak{p}\times\mathbb{C})^0$ the specialization $\mathcal{A}_{\lambda,z}$ has finite homological dimension.*

The main result of this section is as follows.

Proposition 4.9. *The $\mathbb{C}[(\mathfrak{p}\times\mathbb{C})^0]\otimes_{\mathbb{C}[\mathfrak{p},\hbar]}(\mathcal{A}_{\mathfrak{p},\hbar}\otimes_{\mathbb{C}[\mathfrak{p},\hbar]}\mathcal{A}_{\mathfrak{p},\hbar}^{opp})$-module $\mathbb{C}[(\mathfrak{p}\times\mathbb{C})^0]\otimes_{\mathbb{C}[\mathfrak{p},\hbar]}\mathcal{A}_{\mathfrak{p},\hbar}$ has finite projective dimension not exceeding $2\dim X$.*

Proof. First, we claim that for each $(\lambda,z)\in(\mathfrak{p}\times\mathbb{C})^0$ the regular $\mathcal{A}_{\lambda,z}$-bimodule has projective dimension not exceeding $2\dim X$. If $z=0$, then $\mathcal{A}_{\lambda,0}=\mathbb{C}[X_\lambda]$ is the algebra of functions on a smooth affine variety and our claim follows. If $z\neq 0$, we can assume it is equal to 1. Then abelian localization holds for the quantization $\mathcal{A}_\lambda\otimes_{\mathbb{C}}\mathcal{A}_\lambda^{opp}$ of $Y\times Y$ and a suitable choice of a resolution, so the projective dimension of the regular bimodule does not exceed that of the category of coherent sheaves on the resolution of $Y\times Y$, which equals $2\dim X$ (compare with the proof of [BL, Lemma 9.2]).

We claim that the fiberwise bounds on the projective dimensions in the previous paragraph imply the claim of the proposition. Namely, set $k = 2\dim X$, $\mathsf{A} := \mathcal{A}_{\mathfrak{p},\hbar}\otimes_{\mathbb{C}[\mathfrak{p},\hbar]}\mathcal{A}_{\mathfrak{p},\hbar}$, and consider an A-module resolution

$$0\to M\to P_{k-1}\to\ldots\to P_0$$

of $\mathcal{A}_{\mathfrak{p},\hbar}$, where all P_i are projective. Since $\mathcal{A}_{\mathfrak{p},\hbar}$ is a free $\mathbb{C}[\mathfrak{p},\hbar]$-module, we see that M is free over $\mathbb{C}[\mathfrak{p},\hbar]$. Also if we specialize this resolution to any point (λ,z) we still get a resolution. In particular, for any $(\lambda,z)\in(\mathfrak{p}\times\mathbb{C})^0$, the specialization $M_{\lambda,z}$ is a projective $\mathcal{A}_{\lambda,z}$-bimodule. We claim that this implies that $M^0:=\mathbb{C}[(\mathfrak{p}\times\mathbb{C})^0]\otimes_{\mathbb{C}[\mathfrak{p},\hbar]}M$ is a projective $\mathsf{A}^0:=\mathbb{C}[\mathfrak{p},\hbar]^0\otimes_{\mathbb{C}[\mathfrak{p},\hbar]}\mathsf{A}$-module.

Note that A^0 is a Noetherian ring. So to show that M^0 is projective, it is enough to show that M^0 is flat, equivalently, for any right ideal $J\subset\mathsf{A}^0$, the natural map $J\otimes_{\mathsf{A}^0}M^0\to M^0$ is injective. Assume the contrary, let K denote the kernel.

Now consider the completion $\mathsf{A}^{\wedge_{\lambda,z}}$. This algebra is flat over A^0. Moreover, on the category of finitely generated A^0-modules the functor $\mathsf{A}^{\wedge_{\lambda,z}}\otimes_{\mathsf{A}^0}\bullet$ coincides with the completion of $\mathbb{C}[\mathfrak{p},\hbar]$-modules with respect to the maximal ideal of (λ,z). Applying this functor to the homomorphism $J\otimes_{\mathsf{A}^0}M^0\to M^0$ we get

$$J^{\wedge_{\lambda,z}}\otimes_{\mathsf{A}^{\wedge_{\lambda,z}}}M^{\wedge_{\lambda,z}}\to M^{\wedge_{\lambda,z}}. \tag{4.1}$$

The kernel of this homomorphism is $K^{\wedge_{\lambda,z}}$. Note that $M^{\wedge_{\lambda,z}}$ is a formal deformation of $M_{\lambda,z}$. Since $M_{\lambda,z}$ is flat over $\mathsf{A}_{\lambda,z}$, we conclude that $M^{\wedge_{\lambda,z}}$ is flat over $\mathsf{A}^{\wedge_{\lambda,z}}$. It follows that (4.1) is injective. Hence $K^{\wedge_{\lambda,z}}=0$ for all $(\lambda,z)\in(\mathfrak{p}\times\mathbb{C})^0$. Using Lemma 3.8, we conclude that $K=0$. $\square$

4.3. Translation and wall-crossing functors

In this section we will study the behavior of the translation and wall-crossing functors on the category $\mathcal{O}$. Recall that it makes sense to speak about the wall-crossing functor $\mathfrak{WC}_{\lambda+\chi\leftarrow\lambda}$ when abelian localization holds for $(\lambda+\chi,\theta)$ and (λ,θ') for generic θ,θ'. Note that $\mathfrak{WC}_{\lambda+\chi\leftarrow\lambda}$ restricts to an equivalence $D^b_{\mathcal{O}_\nu}(\mathcal{A}_\lambda\text{-mod})\to D^b_{\mathcal{O}_\nu}(\mathcal{A}_{\lambda+\chi}\text{-mod})$.

4.3.1. Behavior on $K_0(\mathcal{O}^T_\nu)$ Pick a T-equivariant structure on $O(\chi)$ (defined up to a twist with a character). This gives a T-equivariant structure on $\mathcal{A}^\theta_{\mathfrak{P},\chi}$ and on $\mathcal{A}_{\lambda,\chi}$. So $\mathfrak{WC}_{\lambda+\chi\leftarrow\lambda}$ upgrades to an equivalence $D^b_{\mathcal{O}_\nu}(\mathcal{A}_\lambda\text{-mod}^T)\xrightarrow{\sim}D^b_{\mathcal{O}_\nu}(\mathcal{A}_{\lambda+\chi}\text{-mod}^T)$.

Let $\mathsf{wt}_\chi(\mathsf{x})$ stand for the character of T at the fiber of $O(\chi)$ at the fixed point x.

Proposition 4.10. *Suppose that*

(i) $\lambda, \lambda+\chi \in \mathfrak{P}^{fl}$,
(ii) *and abelian localization holds for* $(\lambda+\chi, \theta)$ *and* (λ, θ') *for some generic* θ, θ'.

Then we have

1. *For* $? = \lambda, \lambda+\chi$*, the classes* $[\Delta_{\nu,?}(\mathsf{x})]$ *form a basis in* $K_0(\mathcal{O}_\nu(\mathcal{A}_?))$.
2. $[\mathfrak{WC}_{\lambda+\chi\leftarrow\lambda}\Delta_{\nu,\lambda}(\mathsf{x},\kappa)] = [\Delta_{\nu,\lambda+\chi}(\mathsf{x},\kappa+\mathsf{wt}_\chi(\mathsf{x}))]$.
3. *Moreover, the characteristic cycle of* $\operatorname{Loc}^{\theta'}_\lambda \Delta_{\lambda,\nu}(\mathsf{x})$ *has the contracting component of* x *with multiplicity* 1 *and the other contracting components appearing with nonzero multiplicities correspond to the fixed points that are less than* x *in the order induced by* ν.

Proof. For $(\lambda, z) \in \mathfrak{p}\times\mathbb{C}$ we define the category $\mathcal{O}^T_\nu(\mathcal{A}^\theta_{\lambda,z})$ of coherent T-weakly equivariant $\mathcal{A}^\theta_{\lambda,z}$-modules that are set theoretically supported on X^+. Similarly, we can define the category $\mathcal{O}^T_\nu(\mathcal{A}^\theta_{\mathfrak{p},\hbar})$. We have the degeneration map $K_0(\mathcal{O}^{gr}_\nu(\mathcal{A}^\theta_{\lambda,z})) \to K_0(\operatorname{Coh}^T_{X^+}(X))$. Thanks to this map we can talk about the characteristic cycle of an object in $\mathcal{O}^T_\nu(\mathcal{A}^\theta_{\lambda,z})$.

Note that we have a functor

$$\operatorname{Loc}^\theta := \mathcal{A}^\theta_{\mathfrak{p},\hbar}\otimes^L_{\mathcal{A}_{\mathfrak{p},\hbar}} \bullet : D^b(\mathcal{A}_{\mathfrak{p},\hbar}\text{-mod}) \to D^-(\operatorname{Coh}(\mathcal{A}^\theta_{\mathfrak{p},\hbar})).$$

This functor upgrades to the weakly T-equivariant categories. Hence we also get a functor

$$\operatorname{Loc}^\theta := \mathcal{A}^\theta_{\mathfrak{p},\hbar}\otimes^L_{\mathcal{A}_{\mathfrak{p},\hbar}} \bullet : D^b_{\mathcal{O}_\nu}(\mathcal{A}_{\mathfrak{p},\hbar}\text{-mod}^T) \to D^-_{\mathcal{O}_\nu}(\operatorname{Coh}^T(\mathcal{A}^\theta_{\mathfrak{p},\hbar})).$$

Set $\mathcal{A}_{(\mathfrak{p}\times\mathbb{C})^0} := \mathbb{C}[(\mathfrak{p}\times\mathbb{C})^0]\otimes_{\mathbb{C}[\mathfrak{p},\hbar]}\mathcal{A}_{\mathfrak{p},\hbar}$. Recall, Proposition 4.9, that the projective dimension of the $\mathcal{A}_{(\mathfrak{p}\times\mathbb{C})^0}\otimes_{\mathbb{C}[(\mathfrak{p}\times\mathbb{C})^0]}\mathcal{A}^{opp}_{(\mathfrak{p}\times\mathbb{C})^0}$-module $\mathcal{A}_{(\mathfrak{p}\times\mathbb{C})^0}$ does not exceed $2\dim X$. It follows that the projective dimension of the regular $\mathcal{A}_{(\mathfrak{p}\times\mathbb{C})^0}$-bimodule does not exceed $k := 2\dim X + \dim\mathfrak{p} + 1$. Consider the functor

$$\operatorname{Loc}^\theta_\geqslant : D^b_{\mathcal{O}_\nu}(\mathcal{A}_{\mathfrak{p},\hbar}\text{-mod}^T) \to D^b_{\mathcal{O}_\nu}(\operatorname{Coh}^T(\mathcal{A}^\theta_{\mathfrak{p},\hbar})),$$

the composition of $\operatorname{Loc}^\theta$ and the truncation functor $\tau_{\geqslant -k}$. Note that the localization of $\operatorname{Loc}^\theta_\geq$ to $(\mathfrak{p}\times\mathbb{C})^0$ coincides with the localization of $\operatorname{Loc}^\theta$.

Now consider the objects

$$\mathcal{M}^1_{\mathfrak{p},\hbar} := \mathcal{A}^\theta_{\mathfrak{p},\hbar,\chi}\otimes_{\mathcal{A}^\theta_{\mathfrak{p},\hbar}}\operatorname{Loc}^\theta_\geqslant(\Delta_{\nu,\mathfrak{p},\hbar}(\mathsf{x},\kappa)),$$
$$\mathcal{M}^2_{\mathfrak{p},\hbar} := \operatorname{Loc}^\theta_\geqslant(\mathsf{x},\kappa+\mathsf{wt}_\chi(\mathsf{x})) \in D^b_{\mathcal{O}_\nu}(\operatorname{Coh}^T(\mathcal{A}^\theta_{\mathfrak{p},\hbar})).$$

For $(\lambda, z) \in \mathfrak{p} \times \mathbb{C}$, we set $\mathcal{M}^1_{\lambda,z} = \mathbb{C}_{\lambda,z} \otimes^L_{\mathbb{C}[\mathfrak{p},\hbar]} \mathcal{M}^1_{\mathfrak{p},\hbar}$ and define $\mathcal{M}^2_{\lambda,z}$ similarly. Since $\lambda+\chi, \lambda \in \mathfrak{P}^{fl}$ we have the following isomorphisms
(4.2)
$\mathcal{M}^1_{\lambda+\chi} \cong \mathcal{A}^\theta_{\lambda,\chi} \otimes_{\mathcal{A}^\theta_\lambda} \operatorname{Loc}^\theta_\lambda(\Delta_{\nu,\lambda}(\mathsf{x},\kappa)), \mathcal{M}^2_{\lambda+\chi} \cong \operatorname{Loc}^\theta_{\lambda+\chi}(\Delta_{\nu,\lambda+\chi}(\mathsf{x},\kappa+\mathsf{wt}_\chi(\mathsf{x})))$.

Now let $\zeta \in \mathfrak{p}^{reg}$. We write $X^+_{\zeta,\mathsf{x}}$ for the contracting locus of the T-fixed point corresponding to x in X_ζ. Note that the restriction of $O_{X_\zeta}(\chi)$ to $X^+_{\zeta,\mathsf{x}}$ is the trivial line bundle with T acting via $\mathsf{wt}_\chi(\mathsf{x})$. It follows that

$$\mathcal{M}^1_{\zeta,0} \cong \mathcal{M}^2_{\zeta,0} \cong \mathcal{O}_{X^+_{\zeta,\mathsf{x}}}(\kappa+\mathsf{wt}_\chi(\mathsf{x}))). \tag{4.3}$$

For an object $\mathcal{M} \in K_0(\operatorname{Coh}^T_{\mathcal{O}_\nu}(\mathcal{A}^\theta_{\mathfrak{p},\hbar}))$ the image of its specialization to (λ, z) in $K_0(\operatorname{Coh}^T_{X^+}(X))$ is independent of the choice of (λ, z). This is because this image coincides with the class of $\mathbb{C}_{0,0} \otimes^L_{\mathbb{C}[\mathfrak{p},\hbar]} \mathcal{M}$. In particular, the images of $[\mathcal{M}^1_{\lambda+\chi}], [\mathcal{M}^2_{\lambda+\chi}]$ coincide with that of $\mathcal{O}_{X^+_{\zeta,\mathsf{x}}}(\kappa+\mathsf{wt}_\chi(\mathsf{x})))$. The characteristic cycle of the latter has the property listed in (3). This implies (3).

This argument also implies that the classes of $[\Delta_{\nu,\lambda}(\mathsf{x},\kappa)]$ for $\mathsf{x} \in X^T, \kappa \in \mathfrak{X}(T)$, in $K_0(\mathcal{O}^T_\nu(\mathcal{A}_{\lambda+\chi}))$ are linearly independent because their images in $K_0(\operatorname{Coh}^T_{X^+}(X))$ are. Therefore the classes $[\Delta_{\nu,\lambda}(\mathsf{x})] \in K_0(\mathcal{O}_\nu(\mathcal{A}_{\lambda+\chi}))$ are linearly independent. Note that

$$\dim K_0(\mathcal{O}_\nu(\mathcal{A}_{\lambda+\chi})) = \dim K_0(\mathcal{O}_\nu(\mathcal{A}^\theta_{\lambda+\chi})).$$

The right hand side equals $|X^T|$ by Corollary 3.17. We get (1). Since the classes $[\Delta_{\nu,\lambda}(\mathsf{x},\kappa)]$ are linearly independent, we see that (4.2) and (4.3) imply $[\mathcal{M}^1_{\lambda+\chi}] = [\mathcal{M}^2_{\lambda+\chi}]$. This, in turn, implies $[R\Gamma\mathcal{M}^1_{\lambda+\chi}] = [R\Gamma\mathcal{M}^2_{\lambda+\chi}]$, which is (2). □

4.3.2. Translations of Verma modules Our goal here is to prove the following result.

Proposition 4.11. *Suppose that $\lambda \in \mathfrak{P}$ and $\chi \in \operatorname{Pic}(X)$. Assume that the following hold:*

(i) Conditions of Lemma 3.15 hold for $\lambda, \lambda+\chi$,
(ii) $\mathfrak{WC}_{\lambda+\chi\leftarrow\lambda}$ is an abelian equivalence,
(iii) $\lambda, \lambda+\chi \in \mathfrak{P}^{fl}$.

Then the following claims hold.

1. $\mathcal{A}_{\lambda,\chi} \otimes_{\mathcal{A}_\lambda} \Delta_{\nu,\lambda}(\mathsf{x}) \cong \Delta_{\nu,\lambda+\chi}(\mathsf{x})$.

2. *Let $\mathfrak{P}' \subset \mathfrak{P}$ be a principal open subset such that conditions (i)-(iii) hold for all λ'. Then we have $\mathcal{A}_{\mathfrak{P}',\chi} \otimes_{\mathcal{A}_{\mathfrak{P}}} \Delta_{\nu,\mathfrak{P}}(\mathsf{x}) \cong \Delta_{\nu,\mathfrak{P}'+\chi}(\mathsf{x})$.*

Proof. We prove part (1). Note that by Proposition 4.10 (more precisely, its weaker version that does not take the equivariant structures into account), we have $[\mathcal{A}_{\lambda,\chi} \otimes_{\mathcal{A}_\lambda} \Delta_{\nu,\lambda}(\mathsf{x})] \cong [\Delta_{\nu,\lambda+\chi}(\mathsf{x})]$. But both $\mathcal{A}_{\lambda,\chi} \otimes_{\mathcal{A}_\lambda} \Delta_{\nu,\lambda}(\mathsf{x}), \Delta_{\nu,\lambda+\chi}(\mathsf{x})$ are standard objects for highest weight structures on $\mathcal{O}_\nu(\mathcal{A}_{\lambda+\chi})$. As was checked in [GL, Lemma 4.3.2], this implies that the modules are isomorphic.

Now we proceed to proving (2). The specializations of the both sides to any point $\lambda \in \mathfrak{P}'$ are isomorphic. We can equip $\mathcal{A}_{\mathfrak{P}',\chi} \otimes_{\mathcal{A}_{\mathfrak{P}}} \Delta_{\nu,\mathfrak{P}}(\mathsf{x})$ and $\Delta_{\nu,\mathfrak{P}'+\chi}(\mathsf{x})$ with gradings compatible with the Hamiltonian $\mathbb{C}^\times$-actions: we grade $\Delta_{\nu,\mathfrak{P}'+\chi}(\mathsf{x}), \Delta_{\nu,\mathfrak{P}'}(\mathsf{x})$ by putting the highest weight space $\mathbb{C}[\mathfrak{P}']$ in degree 0 and $\mathcal{A}_{\mathfrak{P}',\chi} \otimes_{\mathcal{A}_{\mathfrak{P}}} \Delta_{\nu,\mathfrak{P}}(\mathsf{x})$ gets the tensor product grading. Note that for any compatible grading on a Verma modules that highest weight component coincides with the highest degree components so the grading is uniquely recovered up to a shift. The specializations of $\mathcal{A}_{\mathfrak{P}',\chi} \otimes_{\mathcal{A}_{\mathfrak{P}}} \Delta_{\nu,\mathfrak{P}}(\mathsf{x})$ and $\Delta_{\nu,\mathfrak{P}'+\chi}(\mathsf{x})$ to any λ' inherit the gradings. It follows that we can shift the grading on $\mathcal{A}_{\mathfrak{P}',\chi} \otimes_{\mathcal{A}_{\mathfrak{P}}} \Delta_{\nu,\mathfrak{P}}(\mathsf{x})$ so that the highest weight component in all specializations is in degree 0. This highest weight component is a projective rank one $\mathbb{C}[\mathfrak{P}']$-module. It is therefore free. It is annihilated by $\mathcal{A}_{\mathfrak{P}'}^{>0}$. Also it is annihilated by the kernel of $\mathsf{C}_\nu(\mathcal{A}_{\mathfrak{P}'}) \to \mathbb{C}[\mathfrak{P}']_{\mathsf{x}}$ because this is so in every specialization by part (1). This gives rise to a homomorphism $\Delta_{\nu,\mathfrak{P}'+\chi}(\mathsf{x}) \to \mathcal{A}_{\mathfrak{P}',\chi} \otimes_{\mathcal{A}'_{\mathfrak{P}}} \Delta_{\nu,\mathfrak{P}}(\mathsf{x})$. This homomorphism is an isomorphism after every specialization, hence an isomorphism. This finishes the proof. ☐

4.3.3. Long wall-crossing as Ringel duality functor Let us recall a result from [L6, Section 7.3]. Let abelian localization hold for $(\lambda, \theta), (\lambda + \chi, -\theta)$.

Lemma 4.12. *The wall-crossing functor*

$$\mathfrak{WC}_{\lambda+\chi\leftarrow\lambda} : D^b(\mathcal{O}_\nu(\mathcal{A}_\lambda)) \xrightarrow{\sim} D^b(\mathcal{O}_\nu(\mathcal{A}_{\lambda+\chi}))$$

is a Ringel duality functor.

4.4. Category $\mathcal{O}$ for very generic parameters

We are going to prove a number of results on the category $\mathcal{O}$ at very generic (=lying outside of all essential hyperplanes) parameters.

Proposition 4.13. *Let λ be a very generic parameter. Then we have the following:*

(i) The category $\mathcal{O}_\nu(\mathcal{A}_\lambda)$ *is semisimple.*
(ii) $\lambda \in \mathfrak{P}^{fl}$.
(iii) The natural homomorphism $\mathsf{C}_\nu(\mathcal{A}_\lambda) \to \mathbb{C}[X^T]$ *is an isomorphism.*
(iv) The conclusion of Lemma 3.15 holds, equivalently, we have

$$\dim \operatorname{Tor}_i^{\mathcal{A}_\lambda}(\Delta_\nu(\mathsf{x}), \Delta^r_{-\nu}(\mathsf{x}')) = \delta_{i0}\delta_{\mathsf{x},\mathsf{x}'}.$$

Proof. Let us prove (i). By Lemma 4.6, the categories $\mathcal{O}_\nu(\mathcal{A}_\lambda)$ and $\mathcal{O}_\nu(\mathcal{A}^\theta_\lambda)$ are equivalent for any generic θ and all wall-crossing functors are t-exact equivalences. This equips $\mathcal{O}_\nu(\mathcal{A}_\lambda)$ with a highest weight structure. It follows from Lemma 4.12 that the Ringel duality equivalences between $\mathcal{O}_\nu(\mathcal{A}_\lambda), \mathcal{O}_\nu(\mathcal{A}_{\lambda+\chi})$, where χ has the same meaning as in Lemma 4.12, are t-exact. Consider the self-equivalence $\mathcal{F}$ of $\mathcal{O}_\nu(\mathcal{A}_\lambda)$ obtained by composing the Ringel duality functors $\mathcal{O}_\nu(\mathcal{A}_\lambda) \to \mathcal{O}_\nu(\mathcal{A}_{\lambda+\chi}) \to \mathcal{O}_\nu(\mathcal{A}_\lambda)$. This equivalence sends projectives to injectives and is t-exact. So every projective in $\mathcal{O}_\nu(\mathcal{A}_\lambda)$ is also injective. But since $\mathcal{O}_\nu(\mathcal{A}_\lambda)$ is highest weight, an easy induction on the highest weight order shows that this is only possible when $\mathcal{O}_\nu(\mathcal{A}_\lambda)$ is semisimple.

Let us prove (ii). We need to show that the T-character of $\Delta_{\nu,\lambda}(\mathsf{x})$ coincides with that of $\mathbb{C}[(T_\mathsf{x}X)^{>0}]$. Note that there is an object $\Theta_\mathsf{x} \in \mathcal{O}_\nu(\mathcal{A}_\lambda)$ (constructed in [BLPW, Section 5.3]) with that T-character and a nonzero homomorphism $\Delta_{\nu,\lambda}(\mathsf{x}) \to \Theta_\mathsf{x}$. The module $\Delta_{\nu,\lambda}(\mathsf{x})$ is indecomposable, hence simple. Therefore the homomorphism is injective. Also the dimensions of graded components of $\Delta_{\nu,\lambda}(\mathsf{x})$ are minimized for $\lambda \in \mathfrak{P}^{fl}$. We conclude that $\lambda \in \mathfrak{P}^{fl}$.

Let us prove (iii). First, let us show that the algebra $\mathsf{C}_\nu(\mathcal{A}_\lambda)$ is semisimple. Indeed, otherwise there is a non-split exact sequence $0 \to N_1 \to N \to N_2 \to 0$, where N_1, N_2 are simple $\mathsf{C}_\nu(\mathcal{A}_\lambda)$-modules necessarily with the same action of h, say, by scalar α. The object $\Delta_\nu(N)$ is completely reducible because the category $\mathcal{O}_\nu(\mathcal{A}_\lambda)$ is semisimple. The α-eigenspace for h in $\Delta_\nu(N)$ is N. Since $\Delta_\nu(N)$ has no homomorphisms to modules with zero α-eigenspace, we see that any nonzero direct summand of $\Delta_\nu(N)$ must have a nonzero intersection with N. This intersection is $\mathsf{C}_\nu(\mathcal{A}_\lambda)$-stable. Since N is indecomposable, it follows that $\Delta_\nu(N)$ is indecomposable, hence simple. But $\Delta_\nu(N) \twoheadrightarrow \Delta_\nu(N_2)$, therefore, this epimorphism is an iso. Since the α-eigenspace in $\Delta_\nu(N_2)$ is N_2 we arrive at a contradiction. We conclude that the algebra $\mathsf{C}_\nu(\mathcal{A}_\lambda)$ is semisimple.

Proposition 4.10 applies to λ thanks to (ii) and Lemma 4.6. We conclude that the classes $\Delta_{\nu,\lambda}(\mathsf{x})$ form a basis in $K_0(\mathcal{O}_\nu(\mathcal{A}_\lambda))$. On the other hand the classes of $[\Delta_{\nu,\lambda}(N)]$ for $N \in \operatorname{Irr}(\mathsf{C}_\nu(\mathcal{A}_\lambda))$ form a basis in $K_0(\mathcal{O}_\nu(\mathcal{A}_\lambda))$. This implies that the pullback under the homomorphism $\mathsf{C}_\nu(\mathcal{A}_\lambda) \to \mathsf{C}_\nu(\mathcal{A}^\theta_\lambda)$ is an identification of the sets of irreducibles for these algebras. Since $\mathsf{C}_\nu(\mathcal{A}_\lambda)$ is semisimple, we conclude that the homomorphism is an isomorphism. This proves (iii).

Let us prove (iv). Recall, Lemma 3.14, that

$$\operatorname{Tor}_i^{\mathcal{A}_\lambda}(\Delta_\nu(\mathsf{x}), \Delta^r_{-\nu}(\mathsf{x}))^* = \operatorname{Ext}^i_{\mathcal{A}_\lambda}(\Delta_\nu(\mathsf{x}), \nabla_\nu(\mathsf{x}')).$$

Since abelian localization holds for (λ, θ), we see that this Ext is the same as

$$\operatorname{Ext}^i_{\operatorname{Coh}(\mathcal{A}^\theta_\lambda)}(\operatorname{Loc}^\theta_\lambda \Delta_\nu(\mathsf{x}), \operatorname{Loc}^\theta_\lambda \nabla_\nu(\mathsf{x}')).$$

But the latter Ext is the same as in the category $\mathcal{O}_\nu(\mathcal{A}^\theta_\lambda)$, see Section 3.5.1. This category is semisimple and the objects we consider are simple. So the latter Ext is $\delta_{i,0}\delta_{\mathsf{x},\mathsf{x}'}$ we get the claim of (iii). $\square$

4.5. Regular parameters and quantum chambers

4.5.1. Terminology Let us start by introducing some terminology.

Pick $\lambda \in \mathfrak{P}$. Consider the set of all classical walls Γ such that there is an essential hyperplane parallel to Γ containing λ. These walls Γ will be called *integral walls for* λ. They split $\mathfrak{p}_\mathbb{R}$ into the set of polyhedral chambers to be called *integral chambers* for λ.

A subset Σ of $\mathbb{C}$ will be called *saturated* if for any two elements $z, z+n \in \Sigma$ with $n \in \mathbb{Z}_{>0}$, we have $z+1, z+2, \ldots, z+n-1 \in \Sigma$.

Now suppose that we have fixed saturated subsets $\tilde{\Sigma}_\Gamma \subset \Sigma_\Gamma$ for each wall Γ. Suppose $\lambda \in \mathfrak{P}$ satisfies $\langle \lambda, \alpha_\Gamma\rangle \not\in \tilde{\Sigma}_\Gamma$ (note that this is automatic if Γ is not an integral wall for λ). Then there is a unique integral chamber C^{int} for λ with the following property: for $\lambda' \in \lambda + (C^{int} \cap \mathfrak{p}_\mathbb{Z})$ we have $\langle \lambda', \alpha_\Gamma\rangle \not\in \tilde{\Sigma}_\Gamma$ for all Γ. We will say that C^{int} is a *positive chamber for* λ.

Now let us define *quantum chambers*. Let $\lambda \in \mathfrak{P}$ and pick an integral chamber C^{int}. Quantum chambers will be subsets in $\lambda + \mathfrak{P}_\mathbb{Z}$ defined by linear inequalities. Namely, let $\Gamma_1, \ldots, \Gamma_k$ be all walls such that $\langle \lambda, \alpha_{\Gamma_i}\rangle \in \Sigma_{\Gamma_i}$. Let ϵ_{Γ_i} be the sign of α_{Γ_i} on C^{int}. Pick minimal integers $m_i, i = 1, \ldots, k$ such that $\tilde{m}_i := \epsilon_{\Gamma_i}\langle \lambda, \alpha_{\Gamma_i}\rangle + m_i \not\in \tilde{\Sigma}_{\Gamma_i}$. Then we define the quantum chamber C^q as the subset of the set of regular parameters in $\lambda + \mathfrak{P}_\mathbb{Z}$ given by the inequalities $\epsilon\langle \alpha_\Gamma, \bullet\rangle \geqslant \tilde{m}_i$. We say that it is *shifted* from C^{int}.

By the definition, different quantum chambers are disjoint. Also C^{int} is a positive integral chamber for $\lambda' \in \lambda + \mathfrak{P}_\mathbb{Z}$ if and only if λ' lies in some shifted (from C^{int}) quantum chamber. If λ' lies in a quantum chamber, then it is unique.

Example 4.14. Let $X = T^*\mathcal{B}$ for $G = \mathrm{SL}_3(\mathbb{C})$. Let α_1, α_2 denote the simple roots and let $\alpha_{12} = \alpha_1 + \alpha_2$. Then we have three walls: $\Gamma_1 := \ker \alpha_1^\vee, \Gamma_2 := \ker \alpha_2^\vee, \Gamma_{12} := \ker \alpha_{12}^\vee$. Set $\tilde{\Sigma}_{\Gamma_1} = \{0\}, \tilde{\Sigma}_{\Gamma_2} = \{0\}, \tilde{\Sigma}_{\Gamma_{12}} = \{-2,-1,0,1,2\}$. Let us take an integral weight λ. One of the integral chambers is the positive (in the usual Lie-theoretic sense) chamber C^{int}, it is given by $\langle \alpha_1^\vee, \bullet\rangle \geqslant 0, \langle \alpha_2^\vee, \bullet\rangle \geqslant 0$. The corresponding quantum chamber is given by $\{(x_1,x_2)|x_1 \geqslant 1, x_2 \geqslant 1, x_1+x_2 \geqslant 3\}$, where $x_1 := \langle \alpha_1^\vee, \lambda\rangle, x_2 := \langle \alpha_2^\vee, \lambda\rangle$. In particular, C^q is not a cone.

4.5.2. Result

Proposition 4.15. *For each wall Γ, there is a finite saturated subset $\widetilde{\Sigma}_\Gamma \subset \Sigma_\Gamma$ such that for the union $\tilde{\Psi}$ of the hyperplanes $\langle \alpha_\Gamma, \bullet\rangle \in \tilde{\Sigma}_\Gamma$, where Γ_i runs over the set of all classical walls, the following hold:*

1. *If $\lambda \not\in \tilde{\Psi}$, then abelian localization holds for (λ', θ), where θ is in the integral chamber C^{int} of λ, and any $\lambda' \in \lambda + (C^{int} \cap \mathfrak{p}_\mathbb{Z})$.*
2. *If $\lambda \not\in \tilde{\Psi}$, then $\lambda \in \mathfrak{P}^{fl}$,*
3. *If $\lambda \not\in \tilde{\Psi}$, then $\mathsf{C}_\nu(\mathcal{A}_\lambda) \xrightarrow{\sim} \mathsf{C}_\nu(\mathcal{A}_\lambda^\theta)$,*
4. *if $\lambda \not\in \tilde{\Psi}$, then $\dim \operatorname{Ext}^i_{\mathcal{A}_\lambda}(\Delta_{\nu,\lambda}(\mathsf{x}), \nabla_{\nu,\lambda}(\mathsf{x}')) = \delta_{i,0}\delta_{\mathsf{x},\mathsf{x}'}$.*

Proof. Taking Ψ from Lemma 4.7 and including it into a saturated collection $\tilde{\Psi}$, we get $\tilde{\Psi}$ satisfying (1). The conditions (2)-(4) hold for very generic λ thanks to Proposition 4.13. On the other hand, the loci where conditions (2) and (3) fail are Zariski closed. For (2), this follows from Corollary 3.10. For (3), the locus of λ, where $\mathsf{C}_\nu(\mathcal{A}_\lambda) \to \mathsf{C}_\nu(\mathcal{A}_\lambda^\theta)$ is precisely the locus $\mathfrak{P}'$ in $\mathfrak{P}$, where $\mathsf{C}_\nu(\mathcal{A}_\mathfrak{P}) \to \mathsf{C}_\nu(\mathcal{A}_\mathfrak{P}^\theta)$ is an isomorphism. This follows because $\mathsf{C}_\nu(\mathcal{A}_\mathfrak{P}^\theta)$ is projective over $\mathbb{C}[\mathfrak{P}]$. And since both $\mathsf{C}_\nu(\mathcal{A}_\mathfrak{P}), \mathsf{C}_\nu(\mathcal{A}_\mathfrak{P}^\theta)$ are finitely generated $\mathbb{C}[\mathfrak{P}]$-modules, the locus $\mathfrak{P}'$ is Zariski open.

Now we can find $\tilde{\Psi}$ satisfying (2) and (3) as well. It remains to show that we can find $\tilde{\Psi}$ so that (4) is satisfied as well.

Set $\mathfrak{P}^0 := \mathfrak{P} \cap (\mathfrak{p} \times \mathbb{C})^0$. We claim that the locus in $\mathfrak{P}^0 \cap \mathfrak{P}^{fl}$ where (4) holds is Zariski open. Consider $\Delta^{opp}_{\nu,\mathfrak{P}}(\mathsf{x}') \otimes^L_{\mathcal{A}_\mathfrak{P}} \Delta_{\nu,\mathfrak{P}}(\mathsf{x})$, this is an object in $D^-(\mathbb{C}[\mathfrak{P}]\text{-mod})$. The localization to $\mathfrak{P}^0$ is bounded. Moreover, for $\lambda \in \mathfrak{P}^0 \cap \mathfrak{P}^{fl}$, we have

$$\mathbb{C}_\lambda \otimes^L_{\mathbb{C}[\mathfrak{P}]} \Delta^{opp}_{\nu,\mathfrak{P}}(\mathsf{x}') \otimes^L_{\mathcal{A}_\mathfrak{P}} \Delta_{\nu,\lambda}(\mathsf{x}) \cong \Delta^{opp}_{\nu,\lambda}(\mathsf{x}') \otimes^L_{\mathbb{C}} \Delta_{\nu,\lambda}(\mathsf{x}).$$

So the locus in $\mathfrak{P}^0 \cap \mathfrak{P}^{fl}$, where the right hand side is in homological degree 0 and vanishes for $\mathsf{x} \neq \mathsf{x}'$ is Zariski open. Now we are done by Lemma 3.14. $\square$

Definition 4.16. Let $\mathfrak{P}^{reg}$ denote the complement to $\tilde{\Psi}$ in $\mathfrak{P}$. The elements of $\mathfrak{P}^{reg}$ will be called regular.

Remark 4.17. Note that we can make (1) hold even in the case when there is no torus action. Also it follows from Proposition 4.9 that the homological dimension of $\mathcal{A}_{\mathfrak{P}^{reg}}$ does not exceed $2\dim X + \dim \mathfrak{p}$.

4.5.3. Examples Let us explain what being regular means in the examples we consider: the cotangent bundles $T^*\mathcal{B}$ and the Hilbert schemes $\mathrm{Hilb}_n(\mathbb{C}^2)$.

Let us start with the case of $X = T^*\mathcal{B}$.

For $X = T^*\mathcal{B}$, it is easy to see that we can take $\tilde{\Sigma}_\Gamma = \{0\}$ for all Γ so that regular in our sense is the same as regular in the usual sense ($\langle \alpha^\vee, \lambda\rangle \neq 0$ for all roots α). Integral chambers for λ are given by $\langle \alpha_i^\vee, \bullet\rangle \geqslant 0, i = 1,\ldots,k$, where $\alpha_1,\ldots,\alpha_k$ is a system of simple roots for the integral Weyl group $W_{[\lambda]}$. Quantum chambers are given by $\langle \alpha_i^\vee, \bullet\rangle \geqslant 1$.

For $X = \mathrm{Hilb}_n(\mathbb{C}^2)$, we have a single wall $\Gamma = \{0\}$ and $\Sigma_\Gamma = \{-\frac{a}{b} + 1/2 | 1 \leqslant a < b \leqslant n\}$. We will take $\tilde{\Sigma}_\Gamma$ of the form $\bigcup_{k=-\ell}^{\ell}(\Sigma_\Gamma + k)$, where ℓ is a suitable (sufficiently big) positive integer. Clearly, $\tilde{\Sigma}_\Gamma$ is saturated. When $c = \lambda - 1/2$ is irrational, integer, or rational with denominator bigger than n, then there is only one integral chamber, while otherwise there are two: $\mathbb{R}_{\geqslant 0}, \mathbb{R}_{\leqslant 0}$. The quantum chambers (for $c \in -\frac{a}{b} + \mathbb{Z}$) are of the form $\{-\frac{a}{b} + \ell + m | m \in \mathbb{Z}_{>0}\}$ and $\{-\frac{a}{b} - \ell - m | m \in \mathbb{Z}_{>0}\}$.

5. Alcoves

This is a technical section, our goal here is to discuss various things related to alcoves defined by the walls Γ and the subsets $\tilde{\Sigma}_\Gamma \subset \Sigma_\Gamma$. Here we only deal with questions from elementary combinatorial geometry.

We start with a finitely generated lattice $\mathfrak{p}_\mathbb{Z}$, a finite collection of codimension one sublattices to be called *walls*. For each wall Γ, we have a primitive element $\alpha_\Gamma \in \mathfrak{p}_\mathbb{Z}^*$ defined up to a sign with $\Gamma = \ker \alpha_\Gamma$. Further, for each Γ we fix a finite saturated subset $\tilde{\Sigma}_\Gamma \subset \mathbb{Q}$. Let Σ_Γ denote the set of classes of elements from $\tilde{\Sigma}_\Gamma$ in $\mathbb{Q}/\mathbb{Z}$.

5.1. Real alcoves and *p*-alcoves

We define two closely related sets of *alcoves*. One of them, *real alcoves*, will be in $\mathfrak{p}_\mathbb{R}$, while the other, *p-alcoves*, will be in $\mathfrak{p}_\mathbb{Z}$.

Let us start with real alcoves. Consider the hyperplanes of the form $\langle \alpha_\Gamma, \bullet\rangle = \sigma$ for $\sigma \in \Sigma_\Gamma$ for all walls Γ. By real alcoves we mean the closures of the connected components of $\mathfrak{p}_\mathbb{R}$ with these hyperplanes removed. Note that the set of real alcoves is stable under translations by $\mathfrak{p}_\mathbb{Z}$.

Example 5.1. Let us consider the very classical case of $X = T^*\mathcal{B}$ and the corresponding data of $\mathfrak{p}_{\mathbb{Z}}$, walls, and sets $\tilde{\Sigma}_\Gamma = \{0\}$. The hyperplanes we consider are of the form $\langle \alpha^\vee, \bullet \rangle = m$, where $\alpha^\vee$ is a coroot. Let $\alpha_1^\vee, \ldots, \alpha_r^\vee$ denote the simple coroots and $\alpha_0^\vee$ be the minimal coroot. Then we have the so called fundamental alcove defined by $\langle \alpha_i^\vee, \bullet \rangle \geqslant 0$ for $i = 1, \ldots, r$, and $\langle \alpha_0^\vee, \bullet \rangle \geqslant -1$. All other alcoves are obtained from this one by the standard action of the affine Weyl group of W on $\mathfrak{p}_{\mathbb{R}}$.

Example 5.2. Now let us consider the case of $X = \mathrm{Hilb}_n(\mathbb{C}^2)$ (and the parameter c). Here, recall, $\mathfrak{p}_{\mathbb{Z}} = \mathbb{Z}, \Gamma = \{0\}$ and $\tilde{\Sigma}_\Gamma = \bigsqcup_{i=-\ell}^{\ell} \{-\frac{a}{b} + i | 1 \leqslant a < b \leqslant n\}$. It follows that the real alcoves are intervals between consecutive rational numbers with denominators from 2 to n.

Alternatively, the real alcoves can be described as follows. For $\sigma \in \Sigma_\Gamma$, we can consider the half spaces

$$\Upsilon^+_\sigma = \{\lambda \in \mathfrak{P} | \langle \alpha_\Gamma, \lambda \rangle \geqslant \sigma\}, \Upsilon^-_\sigma = \{\lambda \in \mathfrak{P} | \langle \alpha_\Gamma, \lambda \rangle \leqslant \sigma\} \tag{5.1}$$

Every real alcove is an intersection of finitely many half spaces of this form. Moreover, for a real alcove A there is a unique minimal collection of half-spaces with this property. We call them the half-spaces *associated* to A.

Now we proceed to defining the p-alcoves. Fix an integer $p \gg 0$ such that $p+1$ is divisible by the denominator of any element in $\tilde{\Sigma}_\Gamma$ for all Γ. Later on, p will assumed to be prime but at this point we do not require that. Consider the hyperplanes of the form $\langle \alpha_\Gamma, \bullet \rangle = (p+1)\sigma + pm$, where $\sigma \in \tilde{\Sigma}_\Gamma$ and $m \in \mathbb{Z}$.

Let $\tilde{A}_{\mathbb{R}}$ denote a connected component of the complement of the union of these hyperplanes in $\mathfrak{P}_{\mathbb{R}}$. By a p-alcove we mean the intersection of an open subset of the form $\tilde{A}_{\mathbb{R}}$ with $\mathfrak{P}_{\mathbb{Z}}$. So every p-alcove is the set of integral points λ subject to the inequalities of one of the following forms

- $\langle \alpha_\Gamma, \lambda \rangle \geqslant (p+1)\sigma + pm + 1$ for σ maximal in $\tilde{\Sigma}_{\mathbb{Z}} \cap (\sigma + \mathbb{Z})$ (and all possible m),
- $\langle \alpha_\Gamma, \lambda \rangle \leqslant (p+1)\sigma + pm - 1$ for σ minimal in $\tilde{\Sigma}_{\mathbb{Z}} \cap (\sigma + \mathbb{Z})$ (and all possible m).

Alternatively, we can describe the p-alcoves as follows. Let Γ be a classical wall. Choose $\sigma \in \Sigma_\Gamma$ (defined up to an integral summand) and let σ_+, σ_- denote the maximal and minimal elements in $\sigma + \mathbb{Z}$. For $m \in \mathbb{Z}$, we can consider the affine half spaces $\Upsilon^+_{\sigma,m}, \Upsilon^-_{\sigma,m} \subset \mathfrak{P}_{\mathbb{R}}$ defined as follows:

$$\begin{aligned} \tilde{\Upsilon}^+_{\sigma,m} &= \{\lambda \in \mathfrak{P} | \langle \alpha_\Gamma, \lambda \rangle > (p+1)\sigma_+ + pm\}, \\ \tilde{\Upsilon}^-_{\sigma,m} &= \{\lambda \in \mathfrak{P} | \langle \alpha_\Gamma, \lambda \rangle < (p+1)\sigma_- + pm\}. \end{aligned} \tag{5.2}$$

Any p-alcove is the intersection of $\mathfrak{p}_{\mathbb{Z}}$ with a finite collection of the half spaces of this kind. Moreover, for any p-alcove $\tilde{A}$ there is a unique minimal collection of half spaces as above whose intersection coincides with A. Similarly to the real case, we call them the half-spaces *associated* to $\tilde{A}$.

Example 5.3. Let $X = T^*\mathcal{B}$. Then the hyperplanes are of the form $\langle \alpha, \bullet \rangle = pm$ for $m \in \mathbb{Z}$. We have the fundamental p-alcove, whose points are all integral λ such that $\langle \alpha_i^\vee, \lambda \rangle \geqslant 1, \langle \alpha_0^\vee, \lambda \rangle \geqslant 1-p$. All other p-alcoves are obtained from the fundamental one by the affine Weyl group action, where an element μ of the root lattice acts by the shift by $p\mu$.

Example 5.4. Let $X = \mathrm{Hilb}_n(\mathbb{C}^2)$. Let us describe the p-alcoves for the parameter $c = \lambda - 1/2$. By our assumption we need to take p with $p+1$ divisible by $n!$. The p-alcoves have form $[\frac{(p+1)a'}{b'} + s + 1, \frac{(p+1)a}{b} - s - 1]$, where $\frac{a'}{b'} < \frac{a}{b}$ are rational numbers with denominators between 2 and n such that $(\frac{a'}{b'}, \frac{a}{b})$ has no rational numbers with these denominators.

Now we discuss a connection between the p-alcoves and the real alcoves. Recall that p is very large. Take (the real form of) a p-alcove $\tilde{A}_{\mathbb{R}}$ and divide it by p. There is a unique real alcove A such that the volumes of both $A \setminus (\tilde{A}_{\mathbb{R}}/p)$ and $(\tilde{A}_{\mathbb{R}}/p) \setminus A$ are small compared to p. This defines a bijection. We write ${}^pA$ for the unique p-alcove corresponding to A. Under this correspondence, to each half space Υ associated to A we can assign a unique half space associated to ${}^pA$ that is a shift of Υ. We denote it by ${}^p\Upsilon$. The assignment $\Upsilon \mapsto {}^p\Upsilon$ is injective but may fail to be surjective. One can describe the image as follows. The image of $\Upsilon \mapsto {}^p\Upsilon$ consists precisely of the associated half spaces $\tilde{\Upsilon}$ satisfying the following condition: the volume of $(\dim \mathfrak{p} - 1)$-dimensional polytope $\partial\tilde{\Upsilon} \cap {}^p\overline{A}$ is of order $p^{\dim \mathfrak{p} - 1}$. Here $\partial\tilde{\Upsilon}$ is the boundary hyperplane and ${}^p\overline{A}$ stands for the closure.

On the other hand, to each half-space $\tilde{\Upsilon}$ associated to ${}^pA$ we can assign a unique half-space ${}^{\mathbb{R}}\tilde{\Upsilon}$ with the following properties:

- $A \subset {}^{\mathbb{R}}\tilde{\Upsilon}$,
- ${}^{\mathbb{R}}\tilde{\Upsilon}$ is obtained from $\tilde{\Upsilon}$ by a shift,
- The boundary of A intersects the boundary hyperplane of ${}^{\mathbb{R}}\tilde{\Upsilon}$.

Example 5.5. We use the setting of Example 4.14. Consider the real alcove A given by $\alpha_1^\vee > 0, \alpha_2^\vee > 0, \alpha_1^\vee + \alpha_2^\vee < 1$. The corresponding p-alcove ${}^pA$ is given by $\alpha_1^\vee > 0, \alpha_2^\vee > 0, \alpha_1^\vee + \alpha_2^\vee > 2$ and $\alpha_1^\vee + \alpha_2^\vee < p-2$. There are four associated half spaces for ${}^pA$. All but one, $\alpha_1^\vee + \alpha_2^\vee > 2$, correspond to the associated half spaces of A. For this half-space $\tilde{\Upsilon}$ we have $\mathbb{R}^{\tilde{\Upsilon}} = \{\lambda | \langle \alpha_1^\vee + \alpha_2^\vee, \lambda \rangle > 0\}$.

Remark 5.6. The condition that $p+1$ is divisible by the denominators of all elements in $\tilde{\Sigma}_\Gamma$ is motivated by the following: we want to have in the same order as the real alcoves, compare Example 5.4 and Example 5.2.

5.2. Compatible elements

In this section we will choose a finite collection of points in $\mathfrak{P}_{\mathbb{Q}}$ with a prescribed behavior mod p.

Namely, pick a real alcove A and its face Θ. Let $\Upsilon_1, \ldots, \Upsilon_k$ be all half spaces of the form ${}^{\mathbb{R}}\tilde{\Upsilon}$, where $\tilde{\Upsilon}$ is a half space associated to ${}^pA$ that contains Θ. Let $\tilde{\Upsilon}_1, \ldots, \tilde{\Upsilon}_k$ be the corresponding associated half spaces of ${}^pA$. Let $\tilde{\Upsilon}'_1, \ldots, \tilde{\Upsilon}'_\ell$ be the remaining associated subspaces of ${}^pA$. For instance, in Example 5.5, we can take $\Theta = \{0\}$ and we get $k=3, \ell=1$ with $\tilde{\Upsilon}_1, \tilde{\Upsilon}_2, \tilde{\Upsilon}_3$ being $\alpha_1^\vee > 0, \alpha_2^\vee > 0, \alpha_1^\vee + \alpha_2^\vee > 2$, and $\tilde{\Upsilon}'_1$ is $\alpha_1^\vee + \alpha_2^\vee < p-2$.

For $\tilde{\lambda} \in {}^pA$ we can define numbers $d_i(\tilde{\lambda}), d'_j(\tilde{\lambda})$ as follows. Let Γ_i be the classical wall parallel to the boundary hyperplane $\tilde{\Gamma}_i$ of $\tilde{\Upsilon}_i$. Set $\alpha_i = \pm\alpha_{\Gamma_i}$, where the sign is chosen so that α_i is bounded from below on $\tilde{\Upsilon}_i$. We set

$$d_i(\tilde{\lambda}) := \alpha_i(\tilde{\lambda}) - \alpha_i|_{\tilde{\Gamma}_i},$$

this is a positive number. Define $d'_j(\tilde{\lambda})$ in a similar way.

Now let $\lambda \in \mathfrak{p}_{\mathbb{Q}}$. Below in this section we consider p that is large enough and is such that $(p+1)\lambda \in \mathfrak{p}_{\mathbb{Z}}$ and $p+1$ is divisible by the denominators of all elements in $\tilde{\Sigma}_\Gamma$ for all Γ.

Definition 5.7. We say that λ is compatible with (A, Θ) if there is ${}^p\lambda \in {}^pA$ depending on p in an affine way such that, for all $p \gg 0$ satisfying the congruence conditions in the previous paragraph, we have

(i) ${}^p\lambda = \lambda + p\mu$, where $\mu \in \lambda + \mathfrak{p}_{\mathbb{Z}}$,
(ii) the numbers $d_i({}^p\lambda)$ are independent of p for all $i = 1, \ldots, k$,
(iii) the numbers $d'_j({}^p\lambda)$ are affine functions in p with nonzero linear coefficient (that is automatically a positive number) for all $j = 1, \ldots, \ell$.

The conditions (ii) and (iii) say, in particular, that ${}^p\lambda$ is close to the hyperplanes $\tilde{\Gamma}_i$ for $i = 1, \ldots, k$ but is far from the hyperplanes $\tilde{\Gamma}'_j$ for all $j = 1, \ldots, \ell$.

Lemma 5.8. *The following claims are true:*

1. *For every (A, Θ), there is λ compatible with (A, Θ).*
2. *There is a finite subset $\Lambda \in \mathfrak{p}_{\mathbb{Q}}$ such that for every (A, Θ), there is an element $\lambda \in \Lambda$ compatible with (A, Θ).*

Proof. We prove (1). We will be looking for ${}^p\lambda$ in the form $\lambda + p\mu$ for μ such that

$$\lambda - \mu \in \mathfrak{p}_{\mathbb{Z}} \tag{5.3}$$

so that (i) holds.

Let us see what the condition ${}^p\lambda \in \mathfrak{p}_{\mathbb{Z}}$ means in terms of λ, μ. Recall that we have chosen p so that $(p+1)\lambda \in \mathfrak{p}_{\mathbb{Z}}$. So $\lambda + p\mu = (p+1)\lambda + p(\mu - \lambda)$. This expression is integral for p satisfying the congruence conditions from Section 5.1 provided (5.3) holds.

Let $m_i \in \mathbb{Q}$ (resp. m'_j) be such that Υ_i (resp. Υ'_j) is given by $\alpha_i \geqslant m_i$ (resp., $\alpha'_j \geqslant m'_j$). Then the inequalities for $\tilde{\Upsilon}_i$ (resp., $\tilde{\Upsilon}'_j$) are of the form $\alpha_i \geqslant pm_i + \tilde{m}_i$ for a suitable element $\tilde{m}_i \in m_i + \mathbb{Z}$ (resp., $\alpha'_j \geqslant pm'_j + \tilde{m}'_j$).

In (ii) we have

$$d_i({}^p\lambda) = p(\langle \alpha_i, \mu\rangle - m_i) + (\langle \alpha_i, \lambda\rangle - \tilde{m}_i).$$

So (ii) means

$$\langle \alpha_i, \mu\rangle = m_i,\ \langle \alpha_i, \lambda\rangle > \tilde{m}_i. \tag{5.4}$$

Similarly (iii) means that

$$\langle \alpha'_j, \mu\rangle > m'_j. \tag{5.5}$$

So for μ we can take any rational point lying inside Θ, while for λ we can take any element in $\mu + \mathfrak{p}_{\mathbb{Z}}$ such that $\langle \alpha_i, \lambda\rangle > \tilde{m}_i$. It is clear by the choice of $\alpha_1, \ldots, \alpha_k$ that such λ exists. This finishes the proof of (1).

Let us prove (2). For this we just need to notice that an element compatible with (A, Θ) is also compatible with $(A + \chi, \Theta)$ for every $\chi \in \mathfrak{p}_{\mathbb{Z}}$. The set of orbits for the corresponding action of $\mathfrak{p}_{\mathbb{Z}}$ on the set of pairs (A, Θ) is finite, which finishes the proof. $\square$

Remark 5.9. Let λ be compatible with (A, Θ) and let $\chi \in \mathfrak{p}_{\mathbb{Z}}$ satisfy $\langle \alpha_i, \chi\rangle \geqslant 0$. Then $\lambda + \chi$ is also compatible with (A, Θ).

Also let A^- denote the alcove opposite to A with respect to Θ. Let λ, λ^- be elements compatible with $(A, \Theta), (A^-, \Theta)$. We can assume that $\lambda^- - \lambda \in \mathfrak{p}_{\mathbb{Z}}$.

Example 5.10. Suppose that we are in the setting of Example 5.4. Pick a real alcove $A = (a/b, a'/b')$. Then, for any $m > s$, the element $a/b + m$ is compatible with $(A, \{a/b\})$ and the element $a'/b' - m$ is compatible with $(A, \{a'/b'\})$.

6. R-forms

Below in this section R denotes a finite localization of $\mathbb{Z}$.

6.1. Assumptions

6.1.1. Assumptions on X, Y We still assume that the formal slice to any symplectic leaf in Y is conical, see Section 2.1.6, and that X comes with a Hamiltonian action of a torus T with finitely many fixed points.

We further assume that $X, Y, \rho : X \to Y$ and the $\mathbb{C}^\times \times T$-action are defined over some finite localization R of $\mathbb{Z}$. We have a natural homomorphism $\operatorname{Pic}(X_{\mathsf{R}}) \to \operatorname{Pic}(X)$ by changing the base. We assume that it becomes an isomorphism after tensoring with $\mathbb{Q}$. We define the sublattice $\mathfrak{p}'_{\mathbb{Z}}$ to be the image of $\operatorname{Pic}(X_{\mathsf{R}})$ in $\mathfrak{p}_{\mathbb{Z}}$.

These two assumptions clearly hold in our examples of the cotangent bundles to flag varieties and Hilbert schemes (and, more generally, for Nakajima quiver varieties of affine type A and Slodowy varieties associated to principal Levi nilpotent elements).

Localizing R further, we may assume that $H^i(X_{\mathsf{R}}, O) = 0$ for $i > 0$ and $H^0(X_{\mathsf{R}}, O) = \mathsf{R}[Y]$, this algebra is automatically flat over R.

Now we proceed to the deformations. Since $H^i(X_{\mathbb{Q}}, O) = 0$, we see that the deformations $X_{\mathfrak{p}}, Y_{\mathfrak{p}}$ are defined over $\mathbb{Q}$. Replacing R with a finite localization, we achieve that $\mathfrak{p}_{\mathsf{R}} := H^2(X, \mathsf{R})$ is an R-lattice in $\mathfrak{p}_{\mathbb{Q}}$. We can pick R-forms $X_{\mathfrak{p},\mathsf{R}}, Y_{\mathfrak{p},\mathsf{R}}$ flat over $\mathfrak{p}_{\mathsf{R}}$. The schemes $X_{\mathfrak{p},\mathsf{R}}, Y_{\mathfrak{p},\mathsf{R}}$ come with contracting $\mathbb{G}_m$-actions and also with Hamiltonian T_{R}-actions (perhaps after a finite localization of R). We assume that the fixed point subvariety in $X_{\mathfrak{p}}$ is defined over $\mathbb{Q}$ (and so are the isomorphisms of the connected components of $X_{\mathfrak{p}}^T$ with $\mathfrak{p}$). It is easy to see that these assumptions hold for $X = T^*\mathcal{B}$ and for $X = \operatorname{Hilb}_n(\mathbb{A}^2)$ (and, more generally, for all Slodowy varieties associated to principal Levi nilpotent elements and for all quiver varieties of affine type A).

In particular, for a generic one-parameter subgroup ν, we see that $\mathsf{C}_\nu(O_{X_{\mathbb{Q}}})$ is the structure sheaf of $X_{\mathbb{Q}}^{T_{\mathbb{Q}}}$ and, similarly, $\mathsf{C}_\nu(O_{X_{\mathfrak{p},\mathbb{Q}}})$ is the structure sheaf of $X_{\mathfrak{p},\mathbb{Q}}^{T_{\mathbb{Q}}}$. After a finite localization of R, we achieve that these properties hold over R. We can also achieve that $\mathsf{C}_\nu(\mathsf{R}[Y_{\mathfrak{P}}])$ is a finitely generated module over $\mathsf{R}[\mathfrak{P}]$.

6.1.2. Assumption (S) and set $\boldsymbol{\Lambda}$ One important assumption we are going to make is as follows.

(S) For each classical wall Γ, the set Σ_Γ defined in Section 4.1.1 consists of rational numbers.

Thanks to Examples 4.3 and 4.4, (S) holds in the case when $X = T^*(G/B)$ and $X = \mathrm{Hilb}_n(\mathbb{C}^2)$. More generally, one can show that (S) holds for Slodowy varieties and Nakajima quiver varieties.

Now we pick a finite set of elements $\Lambda \subset \mathfrak{P}_{\mathbb{Q}}$. We choose Λ so that for every pair (A, Θ) of a real alcove and its face and every coset in $\mathfrak{p}_{\mathbb{Z}}/\mathfrak{p}'_{\mathbb{Z}}$, there is an element $\lambda \in \Lambda$ in that coset that is compatible with (A, Θ). Let $\tilde{\Upsilon}_1, \ldots, \tilde{\Upsilon}_k$ have the same meaning as in Section 5.2, $\tilde{\Gamma}_1, \ldots, \tilde{\Gamma}_k$ be their boundary hyperplanes, and let $\Gamma_1, \ldots, \Gamma_k$ be the classical walls. Thanks to Remark 5.9 and results recalled in Section 2.6.3 we can, in addition, assume that the following hold:

(a) Each $\lambda \in \Lambda$ lies in a quantum chamber C^q shifted from an integral chamber C^{int} such that $\bigcap_{i=1}^k \Gamma_i$ (where the meaning of Γ_i's is as above) cuts a face of C^{int}.

(b) For each $\lambda \in \Lambda$ compatible with (A, Θ), there is $\lambda^- \in \Lambda$ compatible with (A^-, Θ) such that

  (b1) $\lambda^- = \lambda + \chi$ for $\chi \in \mathfrak{p}'_{\mathbb{Z}}$,

  (b2) λ^- lies in the quantum chamber $C^{q,-}$ shifted from $C^{int,-}$ (the integral chamber that is opposite to C^{int} with respect to the face defined by $\langle \alpha_{\Gamma_i}, \bullet \rangle = 0, i = 1, \ldots, k$),

  (b3) and $\mathfrak{WC}_{\lambda^- \leftarrow \lambda}$ is a perverse derived equivalence $D^b(\mathcal{A}_\lambda\text{-mod}) \xrightarrow{\sim} D^b(\mathcal{A}_{\lambda^-}\text{-mod})$.

We can assume that $\mathbb{Z}[1/N!] \subset \mathsf{R}$, where N is the maximum of the denominators of elements of all sets Σ_Γ and of all elements in Λ.

Now we define two finite subsets $\mathcal{P}_1, \mathcal{P}_2 \subset \mathrm{Pic}(X_{\mathsf{R}})$. For $\mathcal{P}_1$ we take a finite collection of line bundle whose images include all χ that appear in (b1) of Section 6.1.2. We can assume that $-\mathcal{P}_1 = \mathcal{P}_1$. For $\mathcal{P}_2$ a finite collection of line bundles that map to the union of generating sets for the monoids $C_{\mathbb{Z}} = C \cap \mathfrak{p}'_{\mathbb{Z}}$, where C runs over all possible classical chambers. Note that $\mathcal{P}_2$ contains a generating set of $C^{int}_{\mathbb{Z}}$ for every integral chamber C^{int}.

6.1.3. Assumptions on quantizations We assume that (S) is satisfied for quantizations of X (over $\mathbb{C}$).

We suppose that the canonical quantization $\mathcal{A}^\theta_{\mathfrak{P},\mathbb{Q}}$ of $X^\theta_{\mathfrak{p},\mathbb{Q}}$ has an R-form $\mathcal{A}^\theta_{\mathfrak{P},\mathsf{R}}$ that is a microlocal quantization of $X_{\mathfrak{p},\mathsf{R}}$. Let us write $\mathcal{A}_{\mathfrak{P},\mathsf{R}}$ for the algebra of global sections. Note that $\mathcal{A}_{\mathfrak{P}} = \mathbb{C} \otimes_{\mathsf{R}} \mathcal{A}_{\mathfrak{P},\mathsf{R}}$ and $\mathcal{A}^\theta_{\mathfrak{P}} = \mathbb{C} \hat{\otimes}_{\mathsf{R}} \mathcal{A}^\theta_{\mathfrak{P},\mathsf{R}}$ (where $\hat{\otimes}$ stands for the completed tensor product with respect to the induced filtration).

This clearly holds in the example of $T^*\mathcal{B}$, where the quantization $\mathcal{A}^\theta_{\mathfrak{P},\mathsf{R}}$ is obtained as (the microlocalization of) $\mathcal{D}^{T_\mathsf{R}}_{G/U,\mathsf{R}}$. This also holds for the Hilbert schemes, where the quantization $\mathcal{A}^\theta_{\mathfrak{P},\mathsf{R}}$ is obtained as

$$\left([D(V_\mathsf{R})/D(V_\mathsf{R})\Phi([\mathfrak{g}_\mathsf{R},\mathfrak{g}_\mathsf{R}])]|_{T^*V_\mathsf{R}^{\theta-ss}}\right)^{G_\mathsf{R}}.$$

More generally, the conditions hold for the quantizations of Slodowy varieties and of Nakajima quiver varieties.

Now let us discuss homological dimension for the algebras $\mathcal{A}_{\lambda,\mathsf{R}}$.

Lemma 6.1. *After a finite localization of* R*, for all* $\lambda\in\Lambda$*, the algebra* $\mathcal{A}_{\lambda,\mathsf{R}}$ *has finite homological dimension.*

Proof. The algebra $\mathcal{A}_\lambda$ has finite homological dimension. This is because, by the construction of Λ, see (a) in Section 6.1.2, $\lambda\in\mathfrak{P}^{reg}$. To deduce that $\mathcal{A}_{\lambda,\mathsf{R}}$ has finite homological dimension (after a finite localization of R) we can argue as in the proof of [BL, Lemma 7.8]. □

6.2. Forms of translation bimodules

We will further replace R with its finite localization so that the translation bimodules $\mathcal{A}_{\mathsf{R},\chi}$ have good properties as explained below in this section.

Recall the finite sets $\mathcal{P}_1,\mathcal{P}_2\subset \operatorname{Pic}(X_\mathsf{R})$ from Section 6.1.2. Set $\mathcal{P}=\mathcal{P}_1\cup\mathcal{P}_2$.

As above, we can quantize the line bundles $O(\chi)$ on $X_{\mathfrak{p},\mathsf{R}}$ getting bimodules $\mathcal{A}^\theta_{\mathfrak{P},\chi,\mathsf{R}}$. We note that we have natural homomorphisms

$$\begin{aligned}&\mathcal{A}^\theta_{\mathfrak{P},\chi_2,\mathsf{R}}\otimes_{\mathcal{A}^\theta_{\mathfrak{P},\mathsf{R}}}\mathcal{A}^\theta_{\mathfrak{P},\chi_1,\mathsf{R}}\to\mathcal{A}^\theta_{\mathfrak{P},\chi_1+\chi_2,\mathsf{R}},\\&\Gamma(\mathcal{A}^\theta_{\mathfrak{P},\chi_2,\mathsf{R}})\otimes_{\mathcal{A}_{\mathfrak{P},\mathsf{R}}}\Gamma(\mathcal{A}^\theta_{\mathfrak{P},\chi_1,\mathsf{R}})\to\Gamma(\mathcal{A}^\theta_{\mathfrak{P},\chi_1+\chi_2,\mathsf{R}}).\end{aligned}$$

Note that the base changes of $\Gamma(\mathcal{A}^\theta_{\mathfrak{P},\chi,\mathsf{R}})$ to $\mathbb{Q}$ are independent of θ for the same reason as the $\mathbb{C}$-forms are independent of θ. Each of the bimodules $\Gamma(\mathcal{A}^\theta_{\mathfrak{P},\chi,\mathsf{R}})$ is finitely generated over $\mathcal{A}_{\mathfrak{P},\mathsf{R}}$. It follows that after a finite localization of R, the bimodule $\Gamma(\mathcal{A}^\theta_{\mathfrak{P},\chi,\mathsf{R}})$ is independent of θ for any $\chi\in\mathcal{P}$. We denote this bimodule by $\mathcal{A}_{\mathfrak{P},\chi,\mathsf{R}}$. Note that $\mathbb{C}\otimes_\mathsf{R}\mathcal{A}_{\mathfrak{P},\chi,\mathsf{R}}=\mathcal{A}_{\mathfrak{P},\chi}$.

Note that each bimodule $\mathcal{A}_{\mathfrak{P},\chi,\mathsf{R}},\chi\in\mathcal{P}$, is Harish-Chandra. From here one deduces that after a finite localization of R we can achieve that $\mathcal{A}_{\mathfrak{P},\chi,\mathsf{R}}$ is flat over R. This is a consequence of the following more general result that is proved completely analogously to [BL, Lemma 3.5].

Lemma 6.2. *Let M be a finitely generated $\mathcal{A}_{\mathfrak{P},\mathsf{R}}$-module. Then after a finite localization of R, the R-module M becomes flat.*

Corollary 6.3. *After a finite localization of R, we achieve that $\mathcal{A}_{\lambda,\chi,\mathsf{R}}$ is flat over R for all $\lambda \in \Lambda, \chi \in \mathcal{P}_1$.*

We will also need some properties of bimodules $\mathcal{A}_{\mathfrak{P},\chi,\mathsf{R}}$ for $\chi \in \mathcal{P}_2$ and $\mathcal{A}_{\lambda,\chi,\mathsf{R}}$, where $\lambda \in \Lambda$ and $\lambda^- = \lambda + \chi$.

For $\chi \in \mathcal{P}_2$, let $\mathfrak{P}^{reg[\chi]}$ denote the subset of $\lambda \in \mathfrak{p}$ such that $\lambda, \lambda+\chi$ lie in the same quantum chamber. By the definition of quantum chambers, $\mathfrak{P}^{reg[\chi]}$ is the complement to finitely many essential hyperplanes in $\mathfrak{P}$. Also note that $\mathfrak{P}^{reg[-\chi]} = \mathfrak{P}^{reg[\chi]} + \chi$. Consider the bimodules $\mathcal{A}_{\mathfrak{P}^{reg[\chi]},\chi,\mathsf{R}} := \mathcal{A}_{\mathfrak{P},\chi,\mathsf{R}} \otimes_{\mathsf{R}[\mathfrak{P}]} \mathsf{R}[\mathfrak{P}^{reg[\chi]}]$ (this is a $\mathcal{A}_{\mathfrak{P}^{reg[-\chi]},\mathsf{R}}$-$\mathcal{A}_{\mathfrak{P}^{reg[\chi]},\mathsf{R}}$-bimodule) and $\mathcal{A}_{\mathfrak{P}^{reg[-\chi]},-\chi,\mathsf{R}}$.

Lemma 6.4. *After a finite localization of R, the bimodules $\mathcal{A}_{\mathfrak{P}^{reg[\chi]},\chi,\mathsf{R}}$ and $\mathcal{A}_{\mathfrak{P}^{reg[-\chi]},\chi,\mathsf{R}}$ are mutually inverse Morita equivalence bimodules for all $\chi \in \mathcal{P}_2$.*

Proof. By Corollary 2.9, $\mathcal{A}_{\lambda,\chi}, \mathcal{A}_{\lambda+\chi,-\chi}$ are mutually dual Morita equivalence bimodules for $\lambda \in \mathfrak{P}^{reg[\chi]}$. The natural homomorphisms

$$\mathcal{A}_{\lambda+\chi,-\chi} \otimes_{\mathcal{A}_{\lambda+\chi}} \mathcal{A}_{\lambda,\chi} \to \mathcal{A}_{\lambda}, \mathcal{A}_{\lambda,\chi} \otimes_{\mathcal{A}_{\lambda}} \mathcal{A}_{\lambda+\chi,-\chi} \to \mathcal{A}_{\lambda+\chi}$$

are obtained by specialization to λ from

$$\begin{aligned} &\mathcal{A}_{\mathfrak{P}^{reg[-\chi]},-\chi} \otimes_{\mathcal{A}_{\mathfrak{P}^{reg[-\chi]}}} \mathcal{A}_{\mathfrak{P}^{reg[\chi]},\chi} \to \mathcal{A}_{\mathfrak{P}^{reg[\chi]}},\\ &\mathcal{A}_{\mathfrak{P}^{reg[\chi]},\chi} \otimes_{\mathcal{A}_{\mathfrak{P}^{reg[\chi]}}} \mathcal{A}_{\mathfrak{P}^{reg[-\chi]},-\chi} \to \mathcal{A}_{\mathfrak{P}^{reg[-\chi]}}. \end{aligned} \tag{6.1}$$

It follows that homomorphisms (6.1) are isomorphisms. These homomorphisms are defined over R. It follows that the kernels and the cokernels of

$$\begin{aligned} &\mathcal{A}_{\mathfrak{P}^{reg[-\chi]},-\chi,\mathsf{R}} \otimes_{\mathcal{A}_{\mathfrak{P}^{reg[-\chi]},\mathsf{R}}} \mathcal{A}_{\mathfrak{P}^{reg[\chi]},\chi,\mathsf{R}} \to \mathcal{A}_{\mathfrak{P}^{reg[\chi]},\mathsf{R}},\\ &\mathcal{A}_{\mathfrak{P}^{reg[\chi]},\chi,\mathsf{R}} \otimes_{\mathcal{A}_{\mathfrak{P}^{reg[\chi]},\mathsf{R}}} \mathcal{A}_{\mathfrak{P}^{reg[-\chi]},-\chi,\mathsf{R}} \to \mathcal{A}_{\mathfrak{P}^{reg[-\chi]},\mathsf{R}} \end{aligned}$$

are R-torsion. A direct analog of Lemma 6.2 holds for the algebras of the form

$$\mathcal{A}_{\mathfrak{P}^{reg[\chi]},\mathsf{R}} \otimes_{\mathsf{R}} \mathcal{A}^{opp}_{\mathfrak{P}^{reg[-\chi]},\mathsf{R}}$$

and so the kernels and the cokernels vanish after a finite localization of R. $\square$

Now pick $\lambda \in \Lambda$ and let $\lambda^- \in \Lambda, \chi \in \mathcal{P}_1$ be as before. The bimodule $\mathcal{A}_{\lambda,\chi}$ is defined over $\mathbb{Q}$. It follows that the filtrations by Serre subcategories on $\mathcal{A}_{\lambda}$-mod, $\mathcal{A}_{\lambda+\chi}$-mod that make $\mathfrak{WC}_{\lambda^- \leftarrow \lambda}$ perverse are defined over $\mathbb{Q}$.

Recall, Section 2.6.3, that these filtrations come from chains of ideals, say $\{0\} = \mathcal{I}^0_? \subset \mathcal{I}^1_? \subset \ldots \subset \mathcal{I}^k_? \subset \mathcal{A}_?$, where $? = \lambda$ or λ^-. So the ideals are defined over $\mathbb{Q}$ as well. We set $\mathcal{I}^j_{?,\mathsf{R}} = \mathcal{I}^j_? \cap \mathcal{A}_{?,\mathsf{R}}$ so that $\mathcal{I}^j_? = \mathbb{C} \otimes_{\mathsf{R}} \mathcal{A}_{?,\mathsf{R}}$.

Lemma 6.5. *After a finite localization of* R *we achieve that for all* $\lambda \in \Lambda$ *the following hold:*

1. $\mathcal{I}^j_{\lambda,\mathsf{R}} = (\mathcal{I}^j_{\lambda,\mathsf{R}})^2$ *for all* j*.*
2. *The functor* $\mathcal{A}_{\lambda,\chi,\mathsf{R}} \otimes^L_{\mathcal{A}_{\lambda,\mathsf{R}}} \bullet : D^b(\mathcal{A}_{\lambda,\mathsf{R}}\text{-mod}) \to D^b(\mathcal{A}_{\lambda+\chi,\mathsf{R}}\text{-mod})$ *is a perverse derived equivalence with respect to the filtrations given by the ideals* $\mathcal{I}^j_{\lambda,\mathsf{R}}, \mathcal{I}^j_{\lambda+\chi,\mathsf{R}}$*.*

Proof. We have $\mathcal{I}^j_\lambda = (\mathcal{I}^j_\lambda)^2$. From here we get $\mathcal{I}^j_{\lambda,\mathsf{R}} = (\mathcal{I}^j_{\lambda,\mathsf{R}})^2$ similarly to the proof of Lemma 6.4. Let us proceed to (2).

The functor $\mathcal{A}_{\lambda,\chi,\mathsf{R}} \otimes^L_{\mathcal{A}_{\lambda,\mathsf{R}}} \bullet$ is an equivalence if and only if

$$\mathcal{A}_{\lambda+\chi,\mathsf{R}} \xrightarrow{\sim} R\operatorname{End}_{\mathcal{A}_{\lambda,\mathsf{R}}}(\mathcal{A}_{\lambda,\chi,\mathsf{R}}), \quad \mathcal{A}_{\lambda,\chi,\mathsf{R}} \otimes^L_{\mathcal{A}_{\lambda+\chi,\mathsf{R}}} R\operatorname{Hom}(\mathcal{A}_{\lambda,\chi,\mathsf{R}}, \mathcal{A}_{\lambda,\mathsf{R}}) \xrightarrow{\sim} \mathcal{A}_{\lambda,\mathsf{R}}$$

We know that these homomorphisms become iso after base change to $\mathbb{C}$. So they are iso after a finite localization of R.

We also know that (b1)-(b6) from Section 2.6.3 hold over $\mathbb{C}$. From here we deduce that (b1)-(b6) hold over R, perhaps after a finite localization of R. Now that implies that $\mathcal{A}_{\lambda,\chi,\mathsf{R}} \otimes^L_{\mathcal{A}_{\lambda,\mathsf{R}}} \bullet$ is perverse as required. $\square$

6.3. Forms of Verma modules

The goal of this and the next sections is to show that most results and constructions that we have for the categories $\mathcal{O}$ over $\mathbb{C}$ still carry over to R, perhaps after some finite localization of R.

Recall the principal open subset $\mathfrak{P}^{reg} \subset \mathfrak{P}$ from Definition 4.16. By our choice of R, $\mathfrak{P}^{reg}$ is defined over R. Let $\mathfrak{P}^{reg}_{\mathsf{R}}$ denote the natural R-form of $\mathfrak{P}^{reg}$.

Recall, Section 6.1.1, that $\mathsf{C}_\nu(O_{X_{\mathfrak{P},\mathsf{R}}}) \cong \mathsf{R}[\mathfrak{P}][X^T]$. Replacing R with its finite localization we achieve that the sheaf $\mathsf{C}_\nu(\mathcal{A}^\theta_{\mathfrak{P},\mathsf{R}})$ is a filtered deformation of $\mathsf{C}_\nu(O_{X_{\mathfrak{P},\mathsf{R}}})$. This follows because the filtration on $\mathsf{C}_\nu(\mathcal{A}^\theta_{\mathfrak{P}})$ is defined over R, $\mathsf{C}_\nu(O_{X_{\mathfrak{P},\mathsf{R}}}) \cong \mathsf{R}[\mathfrak{P}][X^T] \to \operatorname{gr}\mathsf{C}_\nu(\mathcal{A}^\theta_{\mathfrak{P},\mathsf{R}})$ and this homomorphism becomes an isomorphism after tensoring with $\mathbb{C}$.

We get a homomorphism $\mathsf{C}_\nu(\mathcal{A}_{\mathfrak{P},\mathsf{R}}) \to \mathsf{R}[\mathfrak{P}][X^T]$ similarly to the case of $\mathbb{C}$. The $\mathsf{R}[\mathfrak{p}]$-module $\mathsf{C}_\nu(\mathsf{R}[X_{\mathfrak{p}}])$ is finitely generated for the same reason as in the case of $\mathbb{C}$. From $\mathsf{C}_\nu(\mathsf{R}[X_{\mathfrak{p}}]) \twoheadrightarrow \operatorname{gr}\mathsf{C}_\nu(\mathcal{A}_{\mathfrak{P},\mathsf{R}})$ we deduce that $\mathsf{C}_\nu(\mathcal{A}_{\mathfrak{P},\mathsf{R}})$ is a finitely generated R-module.

Since the homomorphism $\mathsf{C}_\nu(\mathcal{A}_{\mathfrak{P}^{reg},\mathsf{R}}) \to \mathsf{R}[\mathfrak{P}^{reg}][X^T]$ is an isomorphism after a base change to $\mathbb{C}$, it is still an isomorphism after a finite localization of R. We assume from now on that this is an isomorphism.

Thanks to the homomorphism $\mathsf{C}_\nu(\mathcal{A}_{\mathfrak{P},\mathsf{R}}) \to \mathsf{R}[\mathfrak{P}][X^T]$ we can define the Verma module $\Delta_{\nu,\mathfrak{P},\mathsf{R}}(\mathsf{x})$ for $\mathsf{x} \in X^T$ similarly to the case of $\mathbb{C}$. Clearly, $\mathbb{C} \otimes_{\mathsf{R}} \Delta_{\nu,\mathfrak{P},\mathsf{R}}(\mathsf{x}) = \Delta_{\nu,\mathfrak{P}}(\mathsf{x})$. By Lemma 6.2, after a finite localization of R we can assume that $\Delta_{\nu,\mathfrak{P},\mathsf{R}}$ is flat over R. It follows that all graded components of $\Delta_{\nu,\mathfrak{P},\mathsf{R}}(\mathsf{x})$ are finitely generated $\mathsf{R}[\mathfrak{P}]$-modules.

Lemma 6.6. *After a finite localization of* R*, each* $\Delta_{\nu,\mathfrak{P}^{reg},\mathsf{R}}(\mathsf{x})$ *is projective over* $\mathsf{R}[\mathfrak{P}^{reg}]$.

Proof. We can filter the algebra $\mathcal{A}_{\mathfrak{P},\mathsf{R}}$ (with an ascending T_{R}-stable $\mathbb{Z}_{\geqslant 0}$-filtration) in such a way that $\mathsf{R}[\mathfrak{P}]$ is in degree 0 and $\operatorname{gr}\mathcal{A}_{\mathfrak{P},\mathsf{R}} = \mathsf{R}[\mathfrak{P}][X]$, compare to [BL, Lemma 3.5]. Equip $\Delta_{\nu,\mathfrak{P}^{reg},\mathsf{R}}(\mathsf{x})$ with a compatible T_{R}-stable good filtration. Then $\operatorname{gr}\Delta_{\nu,\mathfrak{P}^{reg},\mathsf{R}}(\mathsf{x})$ is a finitely generated $R[\mathfrak{P}^{reg}][X]$-module. Hence, by generic flatness, there is a finite localization $\mathsf{R}[\mathfrak{P}^{reg}]^0$ of $\mathsf{R}[\mathfrak{P}^{reg}]$ such that $\mathsf{R}[\mathfrak{P}^{reg}]^0 \otimes_{\mathsf{R}[\mathfrak{P}^{reg}]} \operatorname{gr}\Delta_{\nu,\mathfrak{P}^{reg},\mathsf{R}}(\mathsf{x})$ is flat over $\mathsf{R}[\mathfrak{P}^{reg}]^0$. It follows that $\mathsf{R}[\mathfrak{P}^{reg}]^0 \otimes_{\mathsf{R}[\mathfrak{P}]} \Delta_{\nu,\mathfrak{P}^{reg},\mathsf{R}}(\mathsf{x})$ is flat over $\mathsf{R}[\mathfrak{P}^{reg}]^0$. So every graded component of $\Delta_{\nu,\mathfrak{P}^{reg},\mathsf{R}}(\mathsf{x})$ becomes flat (hence projective) after localization to $\mathsf{R}[\mathfrak{P}^{reg}]^0$.

Now let S be a quotient of $\mathsf{R}[\mathfrak{P}^{reg}]$. An argument similar to the previous paragraph show that the specialization

$$\mathsf{S} \otimes_{\mathsf{R}[\mathfrak{P}^{reg}]} \Delta_{\nu,\mathfrak{P}^{reg},\mathsf{R}}(\mathsf{x})$$

becomes flat after a finite localization of the quotient S.

We deduce that one can split $\mathfrak{P}^{reg}_{\mathsf{R}}$ into a disjoint union of locally closed irreducible subschemes in such a way that the restriction of any graded component to any of the subschemes is projective. On the other hand, we know that $\mathbb{Q} \otimes_{\mathsf{R}} \Delta_{\nu,\mathfrak{P}^{reg},\mathsf{R}}(\mathsf{x})$ is flat over $\mathbb{Q}[\mathfrak{P}^{reg}]$ (because the similar statement holds over $\mathbb{C}$). Combining these two observations, we see that after a finite localization of R each graded component of $\Delta_{\nu,\mathfrak{P}^{reg},\mathsf{R}}(\mathsf{x})$ is flat over $\mathsf{R}[\mathfrak{P}^{reg}]$. $\square$

Now let us study translations of Verma modules. We can consider equivariant Verma modules $\Delta_{\nu,\mathfrak{P},\mathsf{R}}(\mathsf{x},\kappa)$ and equivariant lifts $\mathcal{A}_{\mathfrak{P},\chi,\mathsf{R}}$ with $\chi \in \mathcal{P}_1$, as before, see Section 4.3.1. Recall that for $\chi \in \operatorname{Pic}^T(X)$, we write $\mathsf{wt}_\chi(\mathsf{x})$ for the weight of T in the fiber $O(\chi)$ at x.

Lemma 6.7. *After a finite localization of* R*, we have a* T_{R}*-equivariant isomorphism*

$$\Delta_{\nu,\mathfrak{P}^{reg[-\mathsf{x}]},\mathsf{R}}(\mathsf{x},\kappa+\mathsf{wt}_\chi(\mathsf{x})) \cong \mathcal{A}_{\mathfrak{P}^{reg[\mathsf{x}]},\chi,\mathsf{R}} \otimes_{\mathcal{A}_{\mathfrak{P}^{reg[\mathsf{x}]},\mathsf{R}}} \Delta_{\nu,\mathfrak{P}^{reg[\mathsf{x}]},\mathsf{R}}(\mathsf{x},\kappa). \tag{6.2}$$

Proof. By (2) of Proposition 4.11, we know that

$$\Delta_{\nu,\mathfrak{P}^{reg[-\chi]}}(\mathsf{x},\kappa+\mathsf{wt}_\chi(\mathsf{x}))\cong\mathcal{A}_{\mathfrak{P}^{reg[\chi]},\chi}\otimes_{\mathcal{A}_{\mathfrak{P}^{reg[\chi]}}}\Delta_{\nu,\mathfrak{P}^{reg[\chi]}}(\mathsf{x},\kappa). \tag{6.3}$$

Now it is enough to show that a version of (6.2) holds over $\mathbb{Q}$ (then, since our modules are finitely generated, we can take a finite localization of R to achieve (6.2)). But an isomorphism in (6.2) is rational if and only if its highest degree component is rational. Again, $\mathbb{Q}[\mathfrak{P}^{reg[\chi]}]$ is a factorial algebra, so the highest degree component of the right hand side is a free rank one module. The analog of (6.2) over $\mathbb{Q}$ follows. □

6.3.1. Tor vanishing We can also consider the right-handed versions of Verma modules, $\Delta^r_{-\nu,\mathfrak{P},\mathsf{R}}(\mathsf{x})$. Straightforward analogs of Lemmas 6.6 and 6.7 hold for these modules.

Lemma 6.8. *After a finite localization of* R*, we have*

$$\operatorname{Tor}^i_{\mathcal{A}_{\mathfrak{P}^{reg},\mathsf{R}}}(\Delta^r_{-\nu,\mathfrak{P}^{reg},\mathsf{R}}(\mathsf{x}),\Delta_{\nu,\mathfrak{P}^{reg},\mathsf{R}}(\mathsf{x}'))\cong\mathsf{R}[\mathfrak{P}^{reg}]^{\oplus\delta_{i0}\delta_{\mathsf{x},\mathsf{x}'}}. \tag{6.4}$$

Proof. Note that, for $\lambda\in\mathfrak{P}^{reg}_{\mathbb{C}}$, we have

$$\mathbb{C}_\lambda\otimes^L_{\mathsf{R}[\mathfrak{P}^{reg}]}\left(\Delta^r_{-\nu,\mathfrak{P}^{reg},\mathsf{R}}(\mathsf{x})\otimes^L_{\mathcal{A}_{\mathfrak{P}^{reg},\mathsf{R}}}\Delta_{\nu,\mathfrak{P}^{reg},\mathsf{R}}(\mathsf{x}')\right)=\Delta^r_{-\nu,\lambda}(\mathsf{x})\otimes^L_{\mathcal{A}_\lambda}\Delta_{\nu,\lambda}(\mathsf{x}').$$

The right hand side vanishes if $\mathsf{x}\neq\mathsf{x}'$ and is the one-dimensional space in homological degree 0 if $\mathsf{x}=\mathsf{x}'$.

Let us show that the left hand side of (6.4) is a finitely generated $\mathsf{R}[\mathfrak{P}^{reg}]$-module. This will follow once we check that

$$\operatorname{Tor}^i_{\mathcal{A}_{\mathfrak{P},\mathsf{R}}}(\mathcal{A}_{\mathfrak{P},\mathsf{R}}/\mathcal{A}^{<0}_{\mathfrak{P},\mathsf{R}}\mathcal{A}_{\mathfrak{P},\mathsf{R}},\mathcal{A}_{\mathfrak{P},\mathsf{R}}/\mathcal{A}_{\mathfrak{P},\mathsf{R}}\mathcal{A}^{>0}_{\mathfrak{P},\mathsf{R}})$$

is a finitely generated $\mathsf{R}[\mathfrak{P}]$-module for all i. The modules $\mathcal{A}_{\mathfrak{P},\mathsf{R}}/\mathcal{A}^{<0}_{\mathfrak{P},\mathsf{R}}\mathcal{A}_{\mathfrak{P},\mathsf{R}}$ and $\mathcal{A}_{\mathfrak{P},\mathsf{R}}/\mathcal{A}_{\mathfrak{P},\mathsf{R}}\mathcal{A}^{>0}_{\mathfrak{P},\mathsf{R}}$ are filtered. The Tor inherits a filtration whose associated graded is a $\mathsf{R}[\mathfrak{P}]$-module subquotient of

$$\operatorname{Tor}^i_{\mathsf{R}[Y_{\mathfrak{p}}]}(\operatorname{gr}(\mathcal{A}_{\mathfrak{P},\mathsf{R}}/\mathcal{A}^{<0}_{\mathfrak{P},\mathsf{R}}\mathcal{A}_{\mathfrak{P},\mathsf{R}}),\operatorname{gr}(\mathcal{A}_{\mathfrak{P},\mathsf{R}}/\mathcal{A}_{\mathfrak{P},\mathsf{R}}\mathcal{A}^{>0}_{\mathfrak{P},\mathsf{R}})).$$

The intersection of the supports of the arguments in $\mathsf{R}[Y_{\mathfrak{p}}]$ is finite over $\operatorname{Spec}(\mathsf{R}[\mathfrak{p}])$, so the latter Tor is a finitely generated $\mathsf{R}[\mathfrak{p}]$-module.

Now let us recall that

$$\mathbb{Q}\otimes_{\mathsf{R}}\operatorname{Tor}^i_{\mathcal{A}_{\mathfrak{P}^{reg},\mathsf{R}}}(\Delta^r_{-\nu,\mathfrak{P}^{reg},\mathsf{R}}(\mathsf{x}),\Delta_{\nu,\mathfrak{P}^{reg},\mathsf{R}}(\mathsf{x}'))$$

is zero if $i \neq 0, \mathsf{x} \neq \mathsf{x}'$ and is $\mathbb{Q}[\mathfrak{P}^{reg}]$ else. Similarly to the proof of Lemma 6.6 (see the last paragraph there, in particular), it follows that

$$\operatorname{Tor}^i_{\mathcal{A}_{\mathfrak{P}^{reg},\mathsf{R}}}(\Delta^r_{-\nu,\mathfrak{P},\mathsf{R}}(\mathsf{x}), \Delta_{\nu,\mathfrak{P}^{reg},\mathsf{R}}(\mathsf{x}'))$$

becomes flat over $\mathsf{R}[\mathfrak{P}^{reg}]$ after a finite localization of R (depending on i).

It remains to check that the algebra $\mathcal{A}_{\mathfrak{P}^{reg},\mathsf{R}}$ has finite homological dimension after a finite localization of R (so that we only have finitely many non-vanishing Tor's). Similarly to Lemma 6.1, this will follow if we check that $\mathcal{A}_{\mathfrak{P}^{reg}}$ has finite homological dimension. By Remark 4.17, the homological dimension of $\mathcal{A}_{\mathfrak{P}^{reg}}$ does not exceed $2\dim X + \dim \mathfrak{p}$.

So after a finite localization of R, (6.4) holds (for all $\mathsf{x}, \mathsf{x}', i$). □

6.4. Forms of simple, projective, etc. objects in $\mathcal{O}$

Let us discuss forms of various objects in $\mathcal{O}_\nu(\mathcal{A}_\lambda), \lambda \in \Lambda$, over R. First of all, let us note that, by our rationality assumptions on $\mathcal{A}_{\mathfrak{P}}$ and $\mathsf{C}_\nu(\mathcal{A}_{\mathfrak{P}})$, for $\lambda \in \mathfrak{P}^{reg}_{\mathbb{Q}}$, the category $\mathcal{O}_\nu(\mathcal{A}_\lambda)$ is defined over $\mathbb{Q}$. For $\mathsf{x} \in X^T$, let $L_{\nu,\lambda,\mathbb{Q}}(\mathsf{x}), P_{\nu,\lambda,\mathbb{Q}}(\mathsf{x}), I_{\nu,\lambda,\mathbb{Q}}(\mathsf{x}), \nabla_{\nu,\lambda,\mathbb{Q}}(\mathsf{x})$ denote the simple, the projective, the injective and the costandard objects in $\mathcal{O}_\nu(\mathcal{A}_{\lambda,\mathbb{Q}})$ labelled by x.

Let us pick some T_{R}-stable R-forms $L_{\nu,\lambda,\mathsf{R}}(\mathsf{x}) \subset L_{\nu,\lambda,\mathbb{Q}}(\mathsf{x}), \nabla_{\nu,\lambda,\mathsf{R}}(\mathsf{x}) \subset \nabla_{\nu,\lambda,\mathbb{Q}}(\mathsf{x}), P_{\nu,\lambda,\mathsf{R}}(\mathsf{x}) \subset P_{\nu,\lambda,\mathbb{Q}}(\mathsf{x}), I_{\nu,\lambda,\mathsf{R}}(\mathsf{x}) \subset I_{\nu,\lambda,\mathbb{Q}}(\mathsf{x})$.

Lemma 6.9. *After a finite localization of* R, *for every* $\lambda \in \Lambda$, *we achieve the following:*

1. *The objects* $\Delta_{\nu,\lambda,\mathsf{R}}(\mathsf{x}), \nabla_{\nu,\lambda,\mathsf{R}}(\mathsf{x})$ *are filtered by* $L_{\nu,\lambda,\mathsf{R}}(\mathsf{x}')$*'s. Moreover,* $\Delta_{\nu,\lambda,\mathsf{R}}(\mathsf{x}) \twoheadrightarrow L_{\nu,\lambda,\mathsf{R}}(\mathsf{x}) \hookrightarrow \nabla_{\nu,\lambda,\mathsf{R}}(\mathsf{x})$.
2. *The objects* $P_{\nu,\lambda,\mathsf{R}}(\mathsf{x})$ *are filtered by* $\Delta_{\nu,\lambda,\mathsf{R}}(\mathsf{x}')$*'s and the objects* $I_{\nu,\lambda,\mathsf{R}}(\mathsf{x})$ *are filtered by* $\nabla_{\nu,\lambda,\mathsf{R}}(\mathsf{x}')$*'s (in the increasing order with respect to* $\leqslant_{\nu,\lambda}$*).*
3. *Every object* $L_{\nu,\lambda,\mathsf{R}}(\mathsf{x})$ *has a finite resolution*

$$\ldots \to P_{1,\mathsf{R}} \to P_{0,\mathsf{R}} \to L_{\nu,\lambda,\mathsf{R}}(\mathsf{x}) \to 0,$$

where each $P_{i,\mathsf{R}}$ *is the direct sum of* $P_{\nu,\lambda,\mathsf{R}}(\mathsf{x}')$*'s. Similarly,* $L_{\nu,\lambda,\mathsf{R}}(\mathsf{x})$ *has a resolution*

$$0 \to L_{\nu,\lambda,\mathsf{R}}(\mathsf{x}) \to I_{0,\mathsf{R}} \to I_{1,\mathsf{R}} \to \ldots$$

where each $I_{i,\mathsf{R}}$ *is the direct sum of* $I_{\nu,\lambda,\mathsf{R}}(\mathsf{x}')$*'s.*

Proof. The proofs of (1)-(3) are similar, let us prove (2). This claim clearly holds over $\mathbb{C}$ and the filtrations are defined over $\mathbb{Q}$. So it holds over $\mathbb{Q}$ as well. After a finite localization, it holds over R too. □

Lemma 6.10. *Let $M_{\mathsf{R}}, N_{\mathsf{R}}$ be graded R-lattices in $M_{\mathbb{Q}}, N_{\mathbb{Q}} \in \mathcal{O}_\nu^T(\mathcal{A}_{\lambda,\mathbb{Q}})$. Then, for all i, $\operatorname{Ext}^i_{\mathcal{A}_{\lambda,\mathsf{R}},T}(M_{\mathsf{R}}, N_{\mathsf{R}})$ is a finitely generated R-module.*

Proof. Let us pick a free graded resolution

$$\ldots \to \mathcal{A}_{\lambda,\mathsf{R}} \otimes_{\mathsf{R}} V_1 \to \mathcal{A}_{\lambda,\mathsf{R}} \otimes_{\mathsf{R}} V_0 \to M_{\mathsf{R}} \to 0,$$

where the V_i's a finite rank graded free R-modules. The complex

$$\ldots \to V_1^* \otimes_{\mathsf{R}} N_{\mathsf{R}} \to V_0^* \otimes_{\mathsf{R}} N_{\mathsf{R}} \to 0$$

is graded. The Ext's we want to compute are the degree 0 components homology of this complex. All graded components of N_{R} are finitely generated over R and our claim follows. □

Pick $m \in \mathbb{Z}$ such that $|c_{\nu,\lambda}(\mathsf{x}_1) - c_{\nu,\lambda}(\mathsf{x}_2)| \leqslant m$ for all $\mathsf{x}_1, \mathsf{x}_2$ in the same h-block.

Corollary 6.11. *After a finite localization of R, we have the following. Let $M_{\mathsf{R}}, N_{\mathsf{R}}$ be modules of the form $L_{\nu,\lambda,\mathsf{R}}(\mathsf{x}_i, \kappa_i), \Delta_{\nu,\lambda,\mathsf{R}}(\mathsf{x}_i, \kappa_i), \nabla_{\nu,\lambda,\mathsf{R}}(\mathsf{x}_i, \kappa_i), P_{\nu,\lambda,\mathsf{R}}(\mathsf{x}_i, \kappa_i)$ or $I_{\nu,\lambda,\mathsf{R}}(\mathsf{x}_i, \kappa_i), i = 1, 2$, where $\mathsf{x}_1, \mathsf{x}_2$ are in the same h-block and $|\kappa_1 - \kappa_2| \leqslant m$. Then for all i, the R-modules $\operatorname{Ext}^i_{\mathcal{A}_{\lambda,\mathsf{R}},T}(M_{\mathsf{R}}, N_{\mathsf{R}})$ are free and finitely generated over R (and, automatically, $\mathbb{C} \otimes_{\mathsf{R}} \operatorname{Ext}^i_{\mathcal{A}_{\lambda,\mathsf{R}},T}(M_{\mathsf{R}}, N_{\mathsf{R}}) \xrightarrow{\sim} \operatorname{Ext}^i_{\mathcal{A}_{\lambda,\mathsf{R}},T}(M_{\mathbb{C}}, N_{\mathbb{C}})$).*

To finish this section let us describe the images of $\Delta_{\nu,\lambda,\mathsf{R}}(\mathsf{x}, \kappa)$ under the wall-crossing functors $\mathcal{A}_{\lambda,\chi,\mathsf{R}} \otimes^L_{\mathcal{A}_{\lambda,\mathsf{R}}} \bullet$. Here we assume that $\lambda \in \Lambda$ and $\chi \in \mathcal{P}_1$ is the corresponding element. We choose a T_{R}-equivariant structure on $O_{\mathsf{R}}(\chi)$ that gives rise to an equivariant structure on $\mathcal{A}_{\lambda,\chi,\mathsf{R}}$.

Lemma 6.12. *After a finite localization of R, for all $\lambda \in \Lambda$ and $\chi \in \mathcal{P}_1$, we have*

$$\mathcal{A}_{\lambda,\chi,\mathsf{R}} \otimes^L_{\mathcal{A}_{\lambda,\mathsf{R}}} \Delta_{\nu,\lambda,\mathsf{R}}(\mathsf{x}, \kappa) \cong \nabla_{\nu,\lambda,\mathsf{R}}(\mathsf{x}, \kappa + \mathsf{wt}_\chi(\mathsf{x})).$$

Proof. We have such an isomorphism over $\mathbb{C}$, this follows from Lemma 4.12. It follows that the degree $\kappa + \mathsf{wt}_\chi(\mathsf{x})$ component in $\mathcal{A}_{\lambda,\chi,\mathbb{Q}} \otimes^L_{\mathcal{A}_{\lambda,\mathbb{Q}}} \Delta_{\nu,\lambda,\mathbb{Q}}(\mathsf{x}, \kappa)$ is 1-dimensional. Also it is the irreducible $\mathsf{C}_\nu(\mathcal{A}_{\lambda,\mathbb{Q}})$-module corresponding to x. So we get a homomorphism

$$\mathcal{A}_{\lambda,\chi,\mathbb{Q}} \otimes^L_{\mathcal{A}_{\lambda,\mathbb{Q}}} \Delta_{\nu,\lambda,\mathbb{Q}}(\mathsf{x}, \kappa) \to \nabla_{\nu,\lambda,\mathbb{Q}}(\mathsf{x}, \kappa + \mathsf{wt}_\chi(\mathsf{x}))$$

unique up to proportionality. So it is an isomorphism over $\mathbb{C}$, hence an isomorphism. The claim of the lemma easily follows from here. □

Below in the paper we assume that R satisfies all results in Section 6.

7. Reduction to characteristic p

7.1. Assumptions on p

Let R be as directly above. Also recall a finite subset $\Lambda \subset \mathfrak{P}_{\mathbb{Q}}$ from Section 6.1.2 and finite subsets $\mathcal{P}_1, \mathcal{P}_2 \subset \mathrm{Pic}(X_{\mathsf{R}})$ from Section 6.1.1.

Let p be a prime number that is non-invertible in R. Let us write $\mathbb{F}$ for $\bar{\mathbb{F}}_p$. We will be interested in categories of representations of the algebras $\mathcal{A}_{\mathbb{F},\lambda} := \mathbb{F}_\lambda \otimes_{R[\mathfrak{P}]} \mathcal{A}_{\mathfrak{P},R}$ for $\lambda \in \mathfrak{P}_{\mathbb{F}}$.

We will impose several additional assumptions on p. First of all, we assume that $p+1$ is divisible by the denominators of all elements in the sets Σ_Γ (defined in Section 4.1.1) for all classical walls Γ. Second, we assume that $(p+1)\lambda \in \mathfrak{P}_{\mathbb{Z}}$ for all $\lambda \in \Lambda$ for all $(p+1)\mathsf{c}_{\nu,\lambda}(\mathsf{x}) \in \mathbb{Z}$ for all $\lambda \in \mathfrak{P}_{\mathsf{R}}$, generic ν and all $\mathsf{x} \in X^T$. There are infinitely many such primes.

Also we need to assume that p is large enough. More precisely, this means the following

- The map $A \mapsto {}^pA$ from Section 5.1 is a bijection between the set of real alcoves and the set of p-alcoves.
- p is large enough to satisfy the conditions of the next lemma.

Lemma 7.1. *For p sufficiently large, for every two points λ_1, λ_2 in the same p-alcove $\tilde{A}$ such that $\lambda_2 - \lambda_1 \in \mathfrak{p}'_{\mathbb{Z}}$, there is a sequence $\chi_1, \ldots, \chi_k \in \mathcal{P}_2$ such that*

(*) The elements $\lambda_1 + \sum_{i=1}^{j} \chi_i$ are in $\tilde{A}$ for all $j = 0, \ldots, k$.

Proof. Let $\tilde{\Upsilon}_1, \ldots, \tilde{\Upsilon}_m$ be the associated half spaces of $\tilde{A}$ (Section 5.1). Let $\langle \alpha_i, \bullet \rangle > \hat{m}_i, i = 1, \ldots, m$ be the inequality defining $\tilde{\Upsilon}_i$. Pick a very small (but independent of p) number $\epsilon > 0$. Clearly, if p is large enough and $\langle \alpha_i, \lambda_\ell \rangle > \hat{m}_i + \epsilon p, \ell = 1, 2$, then there are $\chi_1, \ldots, \chi_k$ such that (*) holds.

Now consider the case when one of λ_ℓ, say λ_1, satisfies

$$\hat{m}_i < \langle \alpha_i, \lambda_1 \rangle \leqslant \hat{m}_i + \epsilon p \tag{7.1}$$

for some i. This condition says that λ_1 is relatively close to a wall.

Let $i = 1, \ldots, r$ be precisely the indexes i, for (7.1) holds. If p is sufficiently large, we can find a classical chamber C whose interior points are positive on all $\alpha_1, \ldots, \alpha_r$. Pick the generators $\chi'_1, \ldots, \chi'_s$ of $C_{\mathbb{Z}}$ that lie in $\mathcal{P}_2$. A point of the form $\lambda + (m+1)(\chi'_1 + \ldots + \chi'_r) + m(\chi'_{r+1} + \ldots + \chi'_s)$ lies in the alcove $\tilde{A}$ for all $r = 1, \ldots, s$, and all m bounded by a function linear in ϵ^{-1}. We can increase N and decrease ϵ (so that $N\epsilon$ is fixed) in such a way that

$\lambda' := \lambda + m(\chi'_1 + \ldots + \chi'_s)$ now satisfies $\langle \alpha_{\Gamma_i}, \lambda' \rangle > \hat{m}_i + \epsilon p$ for all i. Thanks to the previous paragraph, this finishes the proof. $\square$

Now we proceed to our next condition on p. Fix $\chi \in \mathcal{P}_2$. Recall the locus $\mathfrak{P}^{reg[\chi]}$ of $\lambda \in \mathfrak{P}$ such that $\lambda, \lambda + \chi$ are in the same quantum chamber. This locus is given by the conditions of the form $\langle \lambda, \alpha_{\Gamma_i} \rangle \notin \tilde{\Sigma}_\Gamma(\chi)$, where Γ runs over all classical walls and $\tilde{\Sigma}_\Gamma(\chi)$ is a suitable finite subset such that $\tilde{\Sigma}_\Gamma(\chi) \subset \Sigma_\Gamma$ containing $\tilde{\Sigma}_\Gamma$.

We also assume that p is chosen in such a way that, for $\lambda \in \mathfrak{P}_\mathbb{Z}$, the following are equivalent:

(i) $\lambda, \lambda + \chi$ are in the same p-alcove,
(ii) $\langle \lambda, \alpha_\Gamma \rangle \neq (p+1)\sigma + pm$ for all walls Γ, $\sigma \in \tilde{\Sigma}_\Gamma(\chi), m \in \mathbb{Z}$.

This holds for $p \gg 0$.

Finally, we need an assumption on the residues of $\mathsf{c}_{\nu,\lambda}(\mathsf{x})$ mod p. Let $\bar{\mathsf{c}}_{\nu,\lambda}(\mathsf{x})$ denote this residue viewed as an element in $\{0, \ldots, p-1\}$. By our previous assumptions on p, $\bar{\mathsf{c}}_{\nu,\lambda}(\mathsf{x})$ equals $(p+1)\mathsf{c}_{\nu,\lambda}(\mathsf{x})$ mod p. We suppose that the h-blocks for $\lambda \in \Lambda$ are *ordered* in the following sense: if $\mathsf{x}_1, \mathsf{x}_2$ lie in the same h-block for λ (which, recall, means that $\mathsf{c}_{\nu,\lambda}(\mathsf{x}_1) - \mathsf{c}_{\nu,\lambda}(\mathsf{x}_2) \in \mathbb{Z}$) and $\mathsf{x}'_1, \mathsf{x}'_2$ lie in a different h-block, then $\bar{\mathsf{c}}_{\nu,\lambda}(\mathsf{x}_1) > \bar{\mathsf{c}}_{\nu,\lambda}(\mathsf{x}'_1)$ implies $\bar{\mathsf{c}}_{\nu,\lambda}(\mathsf{x}_2) > \bar{\mathsf{c}}_{\nu,\lambda}(\mathsf{x}'_2)$. This is clearly the case for p large enough.

7.2. p-center

We will put one more assumption: that the algebras $\mathcal{A}_{\lambda,\mathbb{F}}$ have large center for $\lambda \in \mathfrak{P}_{\mathbb{F}_p}$. Let the superscript (1) denote the Frobenius twist. We assume that the inclusion $\mathbb{F}[Y_\mathbb{F}^{(1)}] \hookrightarrow \mathbb{F}[Y]$ given by $f \mapsto f^p$ lifts to a central embedding $\mathbb{F}[Y_\mathbb{F}^{(1)}] \hookrightarrow \mathcal{A}_{\lambda,\mathbb{F}}$ (so that the former is the associated graded homomorphism of the latter). The image of $\mathbb{F}[Y_\mathbb{F}^{(1)}] \hookrightarrow \mathcal{A}_{\lambda,\mathbb{F}}$ is easily seen to coincide with the entire center of the algebra $\mathcal{A}_{\lambda,\mathbb{F}}$.

The existence of the p-center is the classical fact in the case when Y is the nilpotent cone (and $\mathcal{A}_\lambda$ is a central reduction of the universal enveloping algebra). For Nakajima quiver varieties, the existence follows from [BFG].

In particular, for an irreducible $\mathcal{A}_{\lambda,\mathbb{F}}$-module it makes sense to consider its *p-character*, a point in $Y_\mathbb{F}^{(1)}$ that gives the action of the center on the module.

7.3. Equivalences

7.3.1. Abelian equivalences For $\lambda \in \mathfrak{P}_\mathbb{F}, \chi \in \mathcal{P}_2$ we define an $\mathcal{A}_{\lambda+\chi,\mathbb{F}}$-$\mathcal{A}_{\lambda,\mathbb{F}}$-bimodule $\mathcal{A}_{\lambda,\chi,\mathbb{F}}$ as the specialization of $\mathbb{F} \otimes_\mathsf{R} \mathcal{A}_{\mathfrak{P},\chi,\mathsf{R}}$ to λ.

Proposition 7.2. *Let $\lambda, \lambda' = \lambda + \chi$ lie in the same p-alcove of $\mathfrak{p}_{\mathbb{Z}}$. Then the categories $\mathcal{A}_{\lambda,\mathbb{F}}$-mod, $\mathcal{A}_{\lambda',\mathbb{F}}$-mod are equivalent via tensoring with a suitable sequence of bimodules $\mathcal{A}_{\lambda_i,\chi_i,\mathbb{F}}$, where the elements λ_i lie in the same p-alcove and $\chi_i \in \mathcal{P}_2$.*

Proof. Thanks to Lemma 7.1, it is enough to prove that $\mathcal{A}_{\lambda_i,\chi_i,\mathbb{F}}$ is a Morita equivalence $\mathcal{A}_{\lambda_i+\chi_i,\mathbb{F}}$-$\mathcal{A}_{\lambda_i,\mathbb{F}}$-bimodule as long as $\lambda_i, \lambda_i + \chi_i$ are in the same p-alcove. Lemma 6.4 implies that $\mathcal{A}_{\mathfrak{P}^{reg}[\chi],\chi,\mathsf{R}}, \mathcal{A}_{\mathfrak{P}^{reg}[-\chi],\chi,\mathsf{R}}$ are mutually inverse Morita equivalence bimodules. Thanks to the equivalence of (i) and (ii) in the Section 7.1, we see that $\lambda_i \pmod p$ belongs to $\mathfrak{P}^{reg}_{\mathsf{R}}$ mod p. It follows that $\mathcal{A}_{\lambda_i,\chi_i,\mathbb{F}}$ and $\mathcal{A}_{\lambda_i+\chi_i,-\chi_i,\mathbb{F}}$ are mutually inverse Morita equivalence bimodules. $\square$

7.3.2. Perverse equivalences Let A be a real chamber and Θ be its face. Now pick $\lambda \in \Lambda$ compatible with (A, Θ). Let $\chi \in \mathcal{P}_1$ such that $\lambda + \chi$ is compatible with (A^-, Θ), where A^- is the real alcove opposite to A with respect to Θ. Consider the chains of ideals $\mathcal{I}^j_{\lambda,\mathbb{F}} := \mathbb{F} \otimes_{\mathsf{R}} \mathcal{I}^j_{\lambda,\mathsf{R}}$ in $\mathcal{A}_{\lambda,\mathbb{F}}$ and the similarly defined chain $\mathcal{I}^j_{\lambda+\chi,\mathbb{F}}$.

Proposition 7.3. *The functor*

$$\mathcal{A}_{\lambda,\chi,\mathbb{F}} \otimes^L_{\mathcal{A}_{\lambda,\mathbb{F}}} \bullet : D^b(\mathcal{A}_{\lambda,\mathbb{F}}\text{-mod}) \xrightarrow{\sim} D^b(\mathcal{A}_{\lambda+\chi,\mathbb{F}}\text{-mod})$$

is a perverse derived equivalence with respect to the filtrations defined by chains of ideals $\mathcal{I}^j_{\lambda,\mathbb{F}}, \mathcal{I}^j_{\lambda+\chi,\mathbb{F}}$.

Proof. We have $(\mathcal{I}^j_{?,\mathbb{F}})^2 = \mathcal{I}^j_{?,\mathbb{F}}$ $(? = \lambda, \lambda+\chi)$ because the similar equalities hold over R. Also note that the vanishing conditions (b1)-(b6) from Section 2.6.3 hold over $\mathbb{F}$ because they hold over R, see Lemma 6.5. Details are left to the reader. $\square$

Corollary 7.4. *Let (A, Θ) be as above and let A^- be the real alcove that is opposite to A with respect to Θ. Let λ_1, λ_2 be the elements of Λ associated to $(A, \Theta), (A^-, \Theta)$, respectively. Then the categories $\mathcal{A}_{\lambda_1,\mathbb{F}}$-mod and $\mathcal{A}_{\lambda_2,\mathbb{F}}$-mod are perverse derived equivalent.*

8. Modular categories $\mathcal{O}$

8.1. Definition and basic properties

8.1.1. Definition Recall that we have an action of $T_{\mathbb{F}}$ on $X_{\mathbb{F}}, Y_{\mathbb{F}}, \mathcal{A}_{\lambda,\mathbb{F}}$ etc.

For $\lambda \in \mathfrak{P}_{\mathbb{Z}}$, we consider the category $\tilde{\mathcal{O}}(\mathcal{A}_{\lambda,\mathbb{F}})$ consisting of all (weakly) $T_{\mathbb{F}}$-equivariant finite dimensional $\mathcal{A}_{\lambda,\mathbb{F}}$-modules.

Note that the p-character of a simple module in $\tilde{\mathcal{O}}(\mathcal{A}_{\lambda,\mathbb{F}})$ (a point in $Y_{\mathbb{F}}^{(1)}$) is $T_{\mathbb{F}}$-stable. Since $T_{\mathbb{F}}$ has finitely many fixed points in $X_{\mathbb{F}}$, we conclude that the only fixed point in $Y_{\mathbb{F}}^{(1)}$ is 0 and so the p-character of any module in $\tilde{\mathcal{O}}(\mathcal{A}_{\lambda,\mathbb{F}})$ is zero.

8.1.2. Baby Verma modules Now suppose that $\lambda \in \mathfrak{P}_{\mathbb{Z}}$ lies in a p-alcove. Let ν be a generic one-parameter subgroup.

Lemma 8.1. *We have* $\mathsf{C}_{\nu}(\mathcal{A}_{\lambda,\mathbb{F}}) \cong \mathbb{F}[X^T]$.

Proof. Recall, Section 6.3, that we have

$$\mathsf{C}_{\nu}(\mathcal{A}_{\mathfrak{P}^{reg},\mathsf{R}}) \cong \mathsf{R}[\mathfrak{P}^{reg}][X^T]. \tag{8.1}$$

Since λ is in a p-alcove, its reduction modulo p is in $\mathfrak{P}_{\mathbb{F}}^{reg}$. Specializing (8.1), we get the required isomorphism. $\square$

Let us give an example of an object in $\tilde{\mathcal{O}}(\mathcal{A}_{\lambda,\mathbb{F}})$, a baby Verma module. For $\mathsf{x} \in X^T$ and $\kappa \in \mathfrak{X}(T)$, we can form the Verma module $\Delta_{\nu,\mathbb{F}}(\mathsf{x},\kappa)$ as in characteristic 0:

$$\Delta_{\nu,\mathbb{F}}(\mathsf{x},\kappa) := \mathcal{A}_{\lambda,\mathbb{F}}/\mathcal{A}_{\lambda,\mathbb{F}}\mathcal{A}_{\lambda,\mathbb{F}}^{>0} \otimes_{\mathsf{C}_{\nu}(\mathcal{A}_{\lambda,\mathbb{F}})} \mathbb{F}_{\mathsf{x}},$$

where we put $\mathbb{F}_{\mathsf{x}}$ in degree κ. This is not an object in $\tilde{\mathcal{O}}(\mathcal{A}_{\lambda,\mathbb{F}})$, in fact, it is a pro-object.

We define the baby Verma module $\underline{\Delta}_{\nu,\mathbb{F}}(\mathsf{x},\kappa)$ as the fiber of $\Delta_{\nu,\mathbb{F}}(\mathsf{x},\kappa)$ over $0 \in Y_{\mathbb{F}}^{(1)}$.

Lemma 8.2. *A choice of a generic one-parameter subgroup gives rise to an identification of the set of simples in* $\tilde{\mathcal{O}}(\mathcal{A}_{\lambda,\mathbb{F}})$ *with* $X^T \times \mathfrak{X}(T)$.

Proof. A standard argument shows that every simple is the quotient of a unique baby Verma module. This gives the required classification. $\square$

The simple corresponding to (x,κ) will be denoted by $\underline{L}_{\nu,\lambda,\mathbb{F}}(\mathsf{x},\kappa)$.

8.1.3. Equivariant block decomposition We will also need a direct sum decomposition of the category $\tilde{\mathcal{O}}(\mathcal{A}_{\lambda,\mathbb{F}})$. Consider the quantum comoment map $\Phi : \mathfrak{t}_{\mathbb{F}} \to \mathcal{A}_{\lambda,\mathbb{F}}$. Note that h has degree 0 hence preserves every graded component of $M \in \tilde{\mathcal{O}}(\mathcal{A}_{\lambda,\mathbb{F}})$. Then $M := \bigoplus_{\alpha \in \mathbb{F}_p \otimes \mathfrak{X}(T)} M^{\alpha}$, where M^{α} denotes the $T_{\mathbb{F}}$-graded subspace of M, where all $h \in \mathfrak{t}_{\mathbb{F}_p}$ acts on the graded component

M_κ^α with the single eigenvalue $\alpha+\kappa$. Then M_κ^α is a graded $\mathcal{A}_{\lambda,\mathbb{F}}$-submodule and hence an object in $\tilde{\mathcal{O}}(\mathcal{A}_{\lambda,\mathbb{F}})$.

We consider the category $\tilde{\mathcal{O}}^\beta(\mathcal{A}_{\lambda,\mathbb{F}}), \beta \in \mathfrak{t}^*_{\mathbb{F}_p}$, consisting of all M with $M = M^\beta$. These categories will be called *equivariant blocks*. Then $\tilde{\mathcal{O}}(\mathcal{A}_{\lambda,\mathbb{F}}) = \bigoplus_{\beta\in\mathfrak{t}^*_{\mathbb{F}_p}} \tilde{\mathcal{O}}^\beta(\mathcal{A}_{\lambda,\mathbb{F}})$. Note that all categories $\tilde{\mathcal{O}}^\beta(\mathcal{A}_{\lambda,\mathbb{F}})$ are obtained from $\tilde{\mathcal{O}}^0(\mathcal{A}_{\lambda,\mathbb{F}})$ by shifting the $T_{\mathbb{F}}$-equivariant structure.

For $\mathsf{x} \in X^T$, let $\Phi_{\mathsf{x}} \in \mathfrak{t}^*_{\mathbb{F}}$ denote the composition of $\Phi : \mathfrak{t}_{\mathbb{F}} \to \mathcal{A}^T_{\lambda,\mathbb{F}}$ and the projection $\mathcal{A}^T_{\lambda,\mathbb{F}} \twoheadrightarrow \mathbb{F}$ corresponding to x. This is an element of $\mathfrak{t}^*_{\mathbb{F}_p}$. We note that $\underline{L}_{\nu,\lambda,\mathbb{F}}(\mathsf{x},\kappa)$ lies in $\tilde{\mathcal{O}}^\beta(\mathcal{A}_{\lambda,\mathbb{F}})$ if and only if $\Phi_{\mathsf{x}} - \kappa = \beta$ modulo p.

8.1.4. Equivalences Recall that a choice of a $T_{\mathbb{F}}$-equivariant structure on $O_{\mathsf{R}}(\chi)$ gives a $T_{\mathbb{F}}$-equivariant structure on $\mathcal{A}_{\mathfrak{P},\chi,\mathsf{R}}$ and hence on $\mathcal{A}_{\mathfrak{P},\chi,\mathbb{F}}$.

Lemma 8.3. *For $\lambda \in \mathfrak{P}_{\mathbb{F}}$ and $\chi \in \mathcal{P}_2$, the functor $\mathcal{A}_{\lambda,\chi,\mathbb{F}} \otimes^L_{\mathcal{A}_{\lambda,\mathbb{F}}} \bullet$ restricts to*

$$D^-_{fin}(\mathcal{A}_{\lambda,\mathbb{F}}\text{-mod}^T) \to D^-_{fin}(\mathcal{A}_{\lambda+\chi,\mathbb{F}}\text{-mod}^T).$$

Here the subscript "fin" stands for the full subcategory of all complexes with finite dimensional homology.

Proof. The claim that $\mathcal{A}_{\lambda,\chi,\mathbb{F}} \otimes^L_{\mathcal{A}_{\lambda,\mathbb{F}}} \bullet$ sends finite dimensional modules to complexes with finite dimensional homology follows from the fact that $\mathcal{A}_{\lambda,\chi,\mathbb{F}}$ is a HC bimodule. Since $\mathcal{A}_{\lambda,\chi,\mathbb{F}}$ is also $T_{\mathbb{F}}$-equivariant, it follows that $\mathcal{A}_{\lambda,\chi,\mathbb{F}} \otimes^L_{\mathcal{A}_{\lambda,\mathbb{F}}} \bullet$ upgrades to a functor between equivariant categories. □

Proposition 8.4. *Let $\lambda, \lambda+\chi$ lie in the same alcove. Then $\mathcal{A}_{\lambda,\chi,\mathbb{F}} \otimes_{\mathcal{A}_{\lambda,\mathbb{F}}} \bullet$ is an equivalence $\tilde{\mathcal{O}}(\mathcal{A}_{\lambda,\mathbb{F}})$ and $\tilde{\mathcal{O}}(\mathcal{A}_{\lambda+\chi,\mathbb{F}})$ that sends $\underline{L}_{\nu,\lambda,\mathbb{F}}(\mathsf{x},\kappa)$ to $\underline{L}_{\nu,\lambda+\chi,\mathbb{F}}(\mathsf{x},\kappa + \mathsf{wt}_\chi(\mathsf{x}))$.*

Proof. By Proposition 7.2, $\mathcal{A}_{\lambda,\chi,\mathbb{F}}$ is a Morita equivalence bimodule. It follows from Lemma 6.7, that $\mathcal{A}_{\lambda,\chi,\mathbb{F}}$ (with our choice of a $T_{\mathbb{F}}$-equivariant structure) sends $\Delta_{\nu,\lambda,\mathbb{F}}(\mathsf{x};\kappa)$ to $\Delta_{\nu,\lambda+\chi,\mathbb{F}}(\mathsf{x};\kappa + \mathsf{wt}_\chi(\mathsf{x}))$. Since $\underline{L}_{\nu,\lambda,\mathbb{F}}(\mathsf{x};\kappa)$ is the unique graded quotient of $\Delta_{\nu,\lambda,\mathbb{F}}(\mathsf{x};\kappa)$, our claim follows. □

8.1.5. Duality Now we want to discuss the contravariant duality between categories $\tilde{\mathcal{O}}(\mathcal{A}_{\lambda,\mathbb{F}})$ and $\tilde{\mathcal{O}}(\mathcal{A}_{-\lambda,\mathbb{F}})$. Recall that we have an isomorphism $(\mathcal{A}_{\lambda,\mathbb{F}})^{opp} \cong \mathcal{A}_{-\lambda,\mathbb{F}}$. Then $M \mapsto M^*$ defines a contravariant equivalence between $\tilde{\mathcal{O}}(\mathcal{A}_{\lambda,\mathbb{F}})$ and $\tilde{\mathcal{O}}(\mathcal{A}^{opp}_{\lambda,\mathbb{F}})$. But $\mathcal{A}^{opp}_{\lambda,\mathbb{F}} \cong \mathcal{A}_{-\lambda,\mathbb{F}}$.

Lemma 8.5. *We have $\underline{L}_{\nu,\lambda,\mathbb{F}}(\mathsf{x},\kappa) = \underline{L}_{-\nu,-\lambda,\mathbb{F}}(\mathsf{x},-\kappa)^*$.*

Proof. Since $\bullet^*$ is an equivalence, the object $\underline{L}_{-\nu,-\lambda,\mathbb{F}}(\mathsf{x},-\kappa)^*$ is simple. First of all, let us show that it corresponds to the point $\mathsf{x}\in X^T$. Recall, see, e.g., [L6, Section 4.2], that we have an isomorphism $\mathsf{C}_\nu(\mathcal{A}_{\lambda,\mathbb{F}})^{opp}\cong \mathsf{C}_{-\nu}(\mathcal{A}_{-\lambda,\mathbb{F}})$. This isomorphism intertwines the identifications of these algebras with $\mathbb{F}[X^T]$. The condition that $\underline{L}_{-\nu,-\lambda,\mathbb{F}}(\mathsf{x},-\kappa)^{\mathcal{A}_{\lambda,\mathbb{F}}^{\nu,<0}}=\mathbb{F}_{\mathsf{x}}$ translates to

$$\underline{L}_{-\nu,-\lambda,\mathbb{F}}(\mathsf{x},-\kappa)^*/\mathcal{A}_{\lambda,\mathbb{F}}^{\nu,<0}\underline{L}_{-\nu,-\lambda,\mathbb{F}}(\mathsf{x},-\kappa)^*\cong\mathbb{F}_{\mathsf{x}}.$$

Since $\underline{L}_{-\nu,-\lambda,\mathbb{F}}(\mathsf{x},-\kappa)^*$ is an irreducible module, it follows that

$$\left(\underline{L}_{-\nu,-\lambda,\mathbb{F}}(\mathsf{x},-\kappa)^*\right)^{\mathcal{A}_{\lambda,\mathbb{F}}^{>0,\nu}}\cong\mathbb{F}_{\mathsf{x}}.$$

Hence $\underline{L}_{-\nu,-\lambda,\mathbb{F}}(\mathsf{x},-\kappa)^*\cong\underline{L}_{\nu,\lambda,\mathbb{F}}(\mathsf{x},\kappa')$ for some κ'. Note also that the highest degree (with respect to ν) component of $\underline{L}_{-\nu,-\lambda,\mathbb{F}}(\mathsf{x},-\kappa)^*$ equals $\langle\kappa,\nu\rangle$, while the character of the action of T on this component is κ. So $\underline{L}_{-\nu,-\lambda,\mathbb{F}}(\mathsf{x},-\kappa)^*\cong\underline{L}_{\nu,\lambda,\mathbb{F}}(\mathsf{x},\kappa)$. $\square$

Using the duality we can define analogs of costandard objects of $\tilde{\mathcal{O}}(\mathcal{A}_{\lambda,\mathbb{F}})$ that will be ind-objects to be denoted by $\nabla^{\mathbb{F}}_{\nu,\lambda}(\mathsf{x},\kappa)$. Namely, we define $\nabla^{\mathbb{F}}_{\nu,\lambda}(\mathsf{x},\kappa)$ as the restricted dual of $\Delta_{-\nu,-\lambda,\mathbb{F}}(\mathsf{x},-\kappa)$. We note that $\nabla^{\mathbb{F}}_{\nu,\lambda}(\mathsf{x},\kappa)$ is not the same as $\mathbb{F}\otimes_{\mathsf{R}}\nabla_{\nu,\lambda,\mathsf{R}}(\mathsf{x},\kappa)$ (the latter is still a pro-object).

8.2. Highest weight structure

Let $\mathcal{C}$ be an abelian Artinian and Noetherian category, equivalently, an abelian category where all objects have finite length. Suppose that we have an auto-equivalence $\mathcal{S}$ of $\mathcal{C}$ such that the action of $\mathbb{Z}$ on $\mathrm{Irr}(\mathcal{C})$ induced by $\mathcal{S}$ is free. Further, suppose that that $\mathrm{Irr}(\mathcal{C})/\mathbb{Z}$ is finite.

Further, let $\leqslant$ be a partial order on $\mathrm{Irr}(\mathcal{C})$ subject to the following properties:

- The $\mathbb{Z}$-action preserves the order.
- We have $L<\mathcal{S}L$ for every $L\in\mathrm{Irr}(\mathcal{C})$.
- For any two L,L' with $L<L'$, there is $n\in\mathbb{Z}_{\geqslant 0}$ such that $L'<\mathcal{S}^nL$.

For $L\in\mathrm{Irr}(\mathcal{C})$, let $\mathcal{C}_{\leqslant L}$ denote the Serre span of $L'\in\mathrm{Irr}(\mathcal{C})$ with $L'\leqslant L$. We say that the pair $(\mathcal{C},\leqslant)$ is *periodic highest weight* (shortly, PHW) if the following holds:

(PHW) Each quotient $\mathcal{C}_{\leqslant L}/\mathcal{C}_{\leqslant L'}$ with $L'<L$ is a highest weight category with respect to the order $\leqslant$ in the usual sense (see Section 3.1.1).

Recall that by a poset interval we mean a subset $\mathfrak{I}$ such that if $L, L' \in \mathfrak{I}$ satisfy $L \leqslant L'$ and L'' some other element of the poset with $L \leqslant L'' \leqslant L'$, then $L'' \in \mathfrak{I}$. For an interval $\mathfrak{I} \subset \operatorname{Irr}(\mathcal{C})$ we can define a subquotient category $\mathcal{C}_{\mathfrak{I}}$. Namely, consider the set $\overline{\mathfrak{I}}$ of all simples L with $L \leqslant L'$ for some $L' \in \mathfrak{I}$. Then we can consider the Serre span $\mathcal{C}_{\overline{\mathfrak{I}}}$ of all simples in $\overline{\mathfrak{I}}$. By definition, $\mathcal{C}_{\mathfrak{I}} := \mathcal{C}_{\overline{\mathfrak{I}}}/\mathcal{C}_{\overline{\mathfrak{I}}\setminus\mathfrak{I}}$. We note that (PHW) is equivalent to $\mathcal{C}_{\mathfrak{I}}$ being highest weight (with respect to the restricted order) for any finite interval $\mathfrak{I}$.

8.3. Main result

For simplicity, we will assume that $\dim T = 1$. We will remark on the general case later.

Let us define a partial order $\leqslant_\nu$ on $\operatorname{Irr}(\tilde{\mathcal{O}}(\mathcal{A}_{\lambda,\mathbb{F}})) = X^T \times \mathbb{Z}$ as follows: $(\mathsf{x}, \kappa) \leqslant_\nu (\mathsf{x}', \kappa')$ if the following hold:

- (x, κ) and (x', κ') belong to the same equivariant block meaning $\Phi_{\mathsf{x}} - \kappa = \Phi_{\mathsf{x}'} - \kappa'$ in $\mathbb{F}_p$.
- $(\mathsf{x}, \kappa) = (\mathsf{x}', \kappa')$ or $\kappa < \kappa'$.

We can define a $\mathbb{Z}$-action on this poset as follows: $z.(\mathsf{x}, \kappa) := (\mathsf{x}, \kappa + zp)$. Clearly, it satisfies the conditions of the previous section.

Now pick a finite interval $\mathfrak{I} \subset X^T \times \mathfrak{X}(T)$.

Lemma 8.6. *Let $(\mathsf{x}, \kappa), (\mathsf{x}', \kappa') \in \mathfrak{I}$. Then there is a graded submodule $M \subset \Delta_{\nu,\lambda,\mathbb{F}}(\mathsf{x}', \kappa')$ of finite codimension such that $\underline{L}_{\nu,\lambda,\mathbb{F}}(\mathsf{x}, \kappa)$ does not occur in M (i.e., does not occur in the quotient M/M' for any graded $M' \subset M$ of finite codimension). The dual statement holds for $\nabla^{\mathbb{F}}_{\nu,\lambda}(\mathsf{x}', \kappa')$.*

Note that the intersection of all graded finite codimension submodules in $\Delta_{\nu,\lambda,\mathbb{F}}(\mathsf{x}, \kappa)$ is zero (because of the large center).

Proof. Note that the set of $T_{\mathbb{F}}$-weights in $\underline{L}_{\nu,\lambda,\mathbb{F}}(\mathsf{x}, \kappa)$ is finite. All weight spaces in $\Delta_{\nu,\lambda,\mathbb{F}}(\mathsf{x}', \kappa')$ are finite dimensional. So we can find a $T_{\mathbb{F}}$-stable ideal of finite codimension $\mathfrak{m} \subset \mathbb{F}[Y_{\mathbb{F}}^{(1)}]$ such that this set of weights doesn't appear in $M := \mathfrak{m}\Delta_{\nu,\lambda,\mathbb{F}}(\mathsf{x}', \kappa')$. The module $\Delta_{\nu,\lambda,\mathbb{F}}(\mathsf{x}', \kappa')/M$ is finite dimensional, and M does not contain $\underline{L}_{\nu,\lambda,\mathbb{F}}(\mathsf{x}, \kappa)$ as a subquotient.

The claim for $\nabla^{\mathbb{F}}_{\nu,\lambda}(\mathsf{x}', \kappa')$ follows by duality. $\square$

Thanks to the lemma, we can define the object $\Delta^{\mathfrak{I}}_{\nu,\lambda,\mathbb{F}}(\mathsf{x}', \kappa') \in \tilde{\mathcal{O}}(\mathcal{A}_{\lambda,\mathbb{F}})_{\mathfrak{I}}$ for $(\mathsf{x}', \kappa') \in \mathfrak{I}$ as the image of $\Delta_{\nu,\lambda,\mathbb{F}}(\mathsf{x}', \kappa')/M$ from the previous lemma. Analogously, we can define $\nabla^{\mathbb{F},\mathfrak{I}}_{\nu,\lambda}(\mathsf{x}', \kappa')$.

The following theorem is the main result of this section.

Theorem 8.7. *The category $\tilde{\mathcal{O}}(\mathcal{A}_{\lambda,\mathbb{F}})$ is a PHW category with respect to the order $\leqslant_\nu$. Moreover, for any finite interval $\mathfrak{I} \subset X^T \times \mathfrak{X}(T)$, the standard and costandard objects in the subquotient $\tilde{\mathcal{O}}(\mathcal{A}_{\lambda,\mathbb{F}})_{\mathfrak{I}}$ corresponding to $(\mathsf{x},\kappa) \in \mathfrak{I}$ are $\Delta^{\mathfrak{I}}_{\nu,\lambda,\mathbb{F}}(\mathsf{x},\kappa)$ and $\nabla^{\mathfrak{I},\mathbb{F}}_{\nu,\lambda}(\mathsf{x},\kappa)$.*

8.3.1. Examples Let us consider two examples of the order $\leqslant_\nu$. First, let $X = T^*(G/B)$. Recall that X^T is identified with W and $\mathsf{c}_{\nu,\lambda}(w) = \langle \nu, w\lambda\rangle$. Pick $\nu = \rho^\vee$. So the order on $W \times \mathbb{Z}$ is as follows: $(w,\kappa) \leqslant (w',\kappa')$ if

- $\langle w\lambda, \rho^\vee\rangle - \kappa = \langle w'\lambda, \rho^\vee\rangle - \kappa'$ in $\mathbb{F}_p$,
- and either $\kappa < \kappa'$ or $(w,\kappa) = (w',\kappa')$.

Now let $X = \mathrm{Hilb}_n(\mathbb{A}^2)$. Here X^T is in bijection with the set $\mathcal{P}(n)$ of partitions of n. So the order on $\mathcal{P}(n) \times \mathbb{Z}$ is as follows (recall that we set $c = \lambda - 1/2$): $(\psi,\kappa) \leqslant (\psi',\kappa')$ if

- $c\,\mathrm{cont}(\psi) - n(\psi) - \kappa = c\,\mathrm{cont}(\psi') - n(\psi') - \kappa'$ in $\mathbb{F}_p$,
- and either $\kappa < \kappa'$ or $(\psi,\kappa) = (\psi',\kappa')$.

8.4. Proof of Theorem 8.7

The proof is in several steps.

Step 1. Pick $(\mathsf{x},\kappa) \in \mathfrak{I}$. Set $\mathfrak{I}' := \{(\mathsf{x}',\kappa') \in \mathfrak{I} | (\mathsf{x}',\kappa') \leqslant (\mathsf{x},\kappa)\}$. By the construction, the object $\Delta^{\mathfrak{I}}_{\nu,\lambda,\mathbb{F}}(\mathsf{x},\kappa)$ coincides with $\Delta^{\mathfrak{I}'}_{\nu,\lambda,\mathbb{F}}(\mathsf{x},\kappa)$. By the construction, $\Delta^{\mathfrak{I}'}_{\nu,\lambda,\mathbb{F}}(\mathsf{x},\kappa)$ is the projective cover of $\underline{L}_{\nu,\lambda,\mathbb{F}}(\mathsf{x},\kappa)$ in the Serre subcategory $\tilde{\mathcal{O}}(\mathcal{A}_{\lambda,\mathbb{F}})_{\mathfrak{I}'} \subset \tilde{\mathcal{O}}(\mathcal{A}_{\lambda,\mathbb{F}})_{\mathfrak{I}}$. Dually, $\nabla^{\mathfrak{I},\mathbb{F}}_{\nu,\lambda}(\mathsf{x},\kappa)$ is the injective hull of $\underline{L}_{\nu,\lambda,\mathbb{F}}(\mathsf{x},\kappa)$ in that subcategory.

Step 2. To finish the proof of the theorem we need to show that the projectives in $\tilde{\mathcal{O}}(\mathcal{A}_{\lambda,\mathbb{F}})_{\mathfrak{I}}$ are Δ-filtered (Steps 2-4) and that Δ's appear in a correct order (Step 5).

Note that, similarly to Lemma 3.14, we have

(8.2)
$$\mathrm{Ext}^i_{\mathcal{A}_{\lambda,\mathbb{F}},T}(\Delta_{\nu,\lambda,\mathbb{F}}(\mathsf{x},\kappa), \nabla^{\mathbb{F}}_{\nu,\lambda}(\mathsf{x}',\kappa')) = \mathrm{Tor}_i^{\mathcal{A}_{\lambda,\mathbb{F}},T}(\Delta_{\nu,\lambda,\mathbb{F}}(\mathsf{x},\kappa), \Delta^r_{-\nu,\lambda,\mathbb{F}}(\mathsf{x}',\kappa'))^*.$$

Using Lemmas 6.8 and 6.6 (as well as the direct analog of the latter for the right-handed Verma modules) we see that the right hand side of (8.2) vanishes. We deduce that

$$\dim \mathrm{Ext}^i_{\mathcal{A}_{\lambda,\mathbb{F}},T}(\Delta_{\nu,\lambda,\mathbb{F}}(\mathsf{x},\kappa), \nabla^{\mathbb{F}}_{\nu,\lambda}(\mathsf{x}',\kappa')) = 0, \text{ for } i > 0. \tag{8.3}$$

In steps 3 and 4 we will deduce that the projectives are Δ-filtered from (8.3).

Step 3. Take an integer z and consider the category $\mathcal{A}_{\lambda,\mathbb{F}}\text{-mod}^T_{\leqslant z}$ of all $T_{\mathbb{F}}$-equivariant $\mathcal{A}_{\lambda,\mathbb{F}}$-modules M such that

- any $T_{\mathbb{F}}$-weight κ of M satisfies $\kappa \leqslant z$,
- all weight spaces are finite dimensional.

This is a Serre subcategory of $\mathcal{A}_{\lambda,\mathbb{F}}\text{-mod}^T$. Note that $\Delta_{\nu,\lambda,\mathbb{F}}(\mathsf{x},\kappa), \nabla^{\mathbb{F}}_{\nu,\lambda}(\mathsf{x},\kappa) \in \mathcal{A}_{\lambda,\mathbb{F}}\text{-mod}^T_{\leqslant z}$ if and only if $\kappa \leqslant z$.

Repeating the proof of Lemma 8.6 we see that every object in $\mathcal{A}_{\lambda,\mathbb{F}}\text{-mod}^T_{\leqslant z_2}$ contains a graded submodule M of finite codimension such that $\underline{L}_{\nu,\lambda,\mathbb{F}}(\mathsf{x},\kappa)$ with $\kappa \in [z_1, z_2]$ does not appear in M. It follows that

$$\tilde{\mathcal{O}}(\mathcal{A}_{\lambda,\mathbb{F}})_{[z_1,z_2]} \xrightarrow{\sim} \mathcal{A}_{\lambda,\mathbb{F}}\text{-mod}^T_{\leqslant z_2} / \mathcal{A}_{\lambda,\mathbb{F}}\text{-mod}^T_{< z_1} \tag{8.4}$$

Thanks to (8.3) and the standard results that the Ext vanishing in an ambient category implies that in any Serre subcategory, we get

$$\dim \operatorname{Ext}^i_{\mathcal{A}_{\lambda,\mathbb{F}}\text{-mod}^T_{\leqslant z}}(\Delta_{\nu,\lambda,\mathbb{F}}(\mathsf{x},\kappa), \nabla^{\mathbb{F}}_{\nu,\lambda}(\mathsf{x}',\kappa')) = 0, \text{ for } i > 0. \tag{8.5}$$

Step 4. Note that the category $\mathcal{A}_{\lambda,\mathbb{F}}\text{-mod}^T_{\leqslant z}$ has enough projectives. Indeed, for $z' \leqslant z$, the object $P_{z,z'} := \mathcal{A}_{\lambda,\mathbb{F}}/\mathcal{A}_{\lambda,\mathbb{F}}\mathcal{A}^{>z-z'}_{\lambda,\mathbb{F}}$, where the image of 1 has degree z', is projective. This is because $\operatorname{Hom}(P_{z,z'}, M) = M_{z'}$ for $M \in \mathcal{A}_{\lambda,\mathbb{F}}\text{-mod}^T_{\leqslant z}$.

Every module in $\mathcal{A}_{\lambda,\mathbb{F}}\text{-mod}^T_{\leqslant z}$ is covered by a direct sum of the modules $P_{z,z'}$ (for finitely generated modules we can take a finite sum). A standard argument shows that $P_{z,z'}$ is filtered by Δ's. For reader's convenience, let us provide this argument. We prove the existence of filtration by induction on $z - z'$. The case of $z - z' = 0$ is obvious: the object $P_{z,z}$ is the direct sum of Verma modules. Now note that we have a natural surjection $P_{z,z'} \twoheadrightarrow P_{z-1,z'}$, let K stand for the kernel.

Note that $\operatorname{Ext}^i_{\mathcal{A}_{\lambda,\mathbb{F}}\text{-mod}^T_{\leqslant z}}(K, \nabla^{\mathbb{F}}_{\nu,\lambda}(\mathsf{x}',\kappa')) = 0$ for $i > 0$. Indeed, for $i > 1$ this follows from $P_{z,z'}$ being projective and $P_{z-1,z'}$ being filtered by Δ's. We also have an exact sequence

$$\begin{aligned} 0 \to \operatorname{Hom}(K, \nabla^{\mathbb{F}}_{\nu,\lambda}(\mathsf{x}',\kappa')) &\to \operatorname{Hom}(P_{z,z'}, \nabla^{\mathbb{F}}_{\nu,\lambda}(\mathsf{x}',\kappa')) \\ \to \operatorname{Hom}(P_{z-1,z'}, \nabla^{\mathbb{F}}_{\nu,\lambda}(\mathsf{x}',\kappa')) &\to \operatorname{Ext}^1(K, \nabla^{\mathbb{F}}_{\nu,\lambda}(\mathsf{x}',\kappa')) \to 0. \end{aligned}$$

Now consider two cases. If $\kappa' < z$, then the middle homomorphism is the identity isomorphism $\nabla^{\mathbb{F}}_{\nu,\lambda}(\mathsf{x}',\kappa')_{z'} \xrightarrow{\sim} \nabla^{\mathbb{F}}_{\nu,\lambda}(\mathsf{x}',\kappa')_{z'}$. So

$$\operatorname{Hom}(K, \nabla^{\mathbb{F}}_{\nu,\lambda}(\mathsf{x}',\kappa')) = \operatorname{Ext}^1(K, \nabla^{\mathbb{F}}_{\nu,\lambda}(\mathsf{x}',\kappa')) = 0. \tag{8.6}$$

If $\kappa' = z$, then $\operatorname{Hom}(P_{z-1,z'}, \nabla^{\mathbb{F}}_{\nu,\lambda}(\mathsf{x}', \kappa')) = 0$ and hence

$$\operatorname{Hom}(K, \nabla^{\mathbb{F}}_{\nu,\lambda}(\mathsf{x}', \kappa')) = \nabla^{\mathbb{F}}_{\nu,\lambda}(\mathsf{x}', \kappa')_{z'}, \quad \operatorname{Ext}^1(K, \nabla^{\mathbb{F}}_{\nu,\lambda}(\mathsf{x}', \kappa')) = 0. \tag{8.7}$$

in both cases.

Set $d(\mathsf{x}') := \dim \nabla^{\mathbb{F}}_{\nu,\lambda}(\mathsf{x}', \kappa')_{z'}$ From (8.6) and the dual version of Lemma 3.1 we conclude that there is an epimorphism

$$\bigoplus_{\mathsf{x} \in X^T} \Delta_{\nu,\lambda,\mathbb{F}}(\mathsf{x}, z)^{\oplus d(\mathsf{x})} \twoheadrightarrow K.$$

The kernel of this epimorphism has no nonzero homomorphisms to the objects $\nabla^{\mathbb{F}}_{\nu,\lambda}(\mathsf{x}', \kappa')$ with $\kappa' \leqslant z$ by (8.6) and (8.7). On the other hand, if the kernel is nonzero, then it has a finite dimensional graded quotient (compare to the proof of Lemma 8.6) that must have a nonzero Hom to some $\nabla^{\mathbb{F}}_{\nu,\lambda}(\mathsf{x}', \kappa')$.

So we see that $K = 0$. This finishes the proof of the claim that all objects $P_{z,z'}$ admit Δ-filtrations.

Step 5. Note that $\operatorname{Ext}^1_{\mathcal{A}_{\lambda,\mathbb{F}}\text{-mod}^T}(\Delta_{\nu,\lambda,\mathbb{F}}(\mathsf{x}, \kappa), \Delta_{\nu,\lambda,\mathbb{F}}(\mathsf{x}', \kappa')) \neq 0$, then $\kappa' > \kappa$. So the Verma modules in a Δ-filtration of $P_{z,z'}$ appear in the correct order. Now suppose that $\mathfrak{I} = \{(\mathsf{x}, \kappa) | z_1 \leqslant \kappa \leqslant z_2\}$ for some integers $z_1 < z_2$. The images of $P_{z_2,z'}$ for $z' \in [z_1, z_2]$ under the quotient functor $\mathcal{A}_{\lambda,\mathbb{F}}\text{-mod}^T_{\leqslant z_2} \twoheadrightarrow \tilde{\mathcal{O}}(\mathcal{A}_{\lambda,\mathbb{F}})_{\mathfrak{I}}$ are projectives in $\tilde{\mathcal{O}}(\mathcal{A}_{\lambda,\mathbb{F}})_{\mathfrak{I}}$ and every simple is covered by one of these projectives. It follows that the indecomposable projectives in $\tilde{\mathcal{O}}(\mathcal{A}_{\lambda,\mathbb{F}})_{\mathfrak{I}}$ are filtered by $\Delta^{\mathfrak{I}}_{\nu,\lambda,\mathbb{F}}(\mathsf{x}, \kappa)$'s in a correct order. So for this choice of the interval $\mathfrak{I}$, the category $\tilde{\mathcal{O}}(\mathcal{A}_{\lambda,\mathbb{F}})_{\mathfrak{I}}$ is highest weight. For a general interval $\mathfrak{I}'$, the category $\tilde{\mathcal{O}}(\mathcal{A}_{\lambda,\mathbb{F}})_{\mathfrak{I}'}$ is a highest weight subquotient in $\tilde{\mathcal{O}}(\mathcal{A}_{\lambda,\mathbb{F}})_{\mathfrak{I}}$, where $\mathfrak{I} = [z_1, z_2]$ for suitable z_1, z_2, and hence a highest weight category itself.

8.4.1. Case of higher dimensional torus Let us explain what modifications are needed to deal with the case when $\dim T > 1$. We can still define a partial order $\leqslant_\nu$ on $X^T \times \mathfrak{X}(T)$ (instead of characters themselves we need now to compare their pairings with ν). A technical problem here is that most of intervals for this order are infinite. But we can use a more general definition of a highest weight category in Remark 3.2 and we still get an analog of Theorem 8.7.

9. Standardly stratified structures

In this section we prove the main result of the present paper, Theorem 9.4. Namely, let λ lie inside the p-alcove ${}^pA$ (corresponding to a real alcove A). For each pair $(\Theta, \bar{\lambda})$, where Θ is a face of A and $\bar{\lambda}$ is a parameter compatible with (A, Θ), see Section 5.2, we will define a standardly stratified structure (in a suitable sense to be explained below) on $\tilde{\mathcal{O}}(\mathcal{A}_{\lambda,\mathbb{F}})$. Again, for convenience we assume that $\dim T = 1$, the general case can be treated as in Section 8.4.1.

We will get two results about the standardly stratified structure. First, we will show that the associated graded of $\tilde{\mathcal{O}}(\mathcal{A}_{\lambda,\mathbb{F}})$ is essentially a reduction to characteristic p of $\mathcal{O}_\nu(\mathcal{A}_{\bar{\lambda},\mathbb{Q}})$. We will further show that, roughly speaking, standard and proper standard objects in $\tilde{\mathcal{O}}(\mathcal{A}_{\lambda,\mathbb{F}})$ are reductions to characteristic p of projective and simple objects in $\mathcal{O}_\nu(\mathcal{A}_{\bar{\lambda},\mathbb{Q}})$.

9.1. Main result

In this section, after some preparation we state the main theorem regarding standardly stratified structures on $\tilde{\mathcal{O}}(\mathcal{A}_{\lambda,\mathbb{F}})$ determined by faces of the real alcove A.

9.1.1. Standardly stratified structures on PHW categories We use the ramification of the definition of a standardly stratified structure that appeared in [LW] and recalled in Section 3.1.2. Let $\mathcal{C}$ be a PHW category with the shift functor $\mathcal{S}$.

Definition 9.1. The additional data of a standardly stratified category is a $\mathbb{Z}$-invariant pre-order $\preceq$ on $\operatorname{Irr}(\mathcal{C})$ that is compatible with $\leqslant$, meaning that

$$L \prec L' \Rightarrow L < L' \Rightarrow L \preceq L'.$$

We also require that $L \prec \mathcal{S}L$ for all L. Since $|\operatorname{Irr}(\mathcal{C})/\mathbb{Z}| < \infty$, we see that the equivalence classes for $\prec$ are finite.

The axiom of a standardly stratified structure in this case is that, for each finite interval $\mathfrak{I} \subset \operatorname{Irr}(\mathcal{C})$ that is the union of some equivalence classes for $\prec$, the subquotient category $\mathcal{C}_{\mathfrak{I}}$ is standardly stratified with respect to $\prec$ (in the sense explained in Section 3.1.2).

9.1.2. Pre-order determined by Θ We will now define a pre-order on $\operatorname{Irr}(\tilde{\mathcal{O}}(\mathcal{A}_{\lambda,\mathbb{F}}))$. Namely, recall that for λ' in ${}^pA$ we have an equivalence $\tilde{\mathcal{O}}(\mathcal{A}_{\lambda,\mathbb{F}}) \xrightarrow{\sim} \tilde{\mathcal{O}}(\mathcal{A}_{\lambda',\mathbb{F}})$ of highest weight categories. We will take $\lambda' := {}^p\bar{\lambda}$, where the right hand side is recovered from $\bar{\lambda}$ as explained in Section 5.2. We will define a pre-order on $\operatorname{Irr}(\tilde{\mathcal{O}}(\mathcal{A}_{\lambda',\mathbb{F}}))^0$ (then we can transfer this pre-order to $\operatorname{Irr}(\tilde{\mathcal{O}}(\mathcal{A}_{\lambda',\mathbb{F}}))^\beta$ using the natural bijection between these sets, the simples from different equivariant blocks, by definition, are not comparable).

Now note that, for a fixed point $\mathsf{x} \in X^T$ (and $\bar{\lambda}$), the map $p \mapsto c_{\nu, p\bar{\lambda}}(\mathsf{x})$ is an affine map in p. It follows that the possible values of κ such that $(\mathsf{x}, \kappa) \in \operatorname{Irr}(\tilde{\mathcal{O}}(\mathcal{A}_{\lambda',\mathbb{F}}))^0$ are affine functions in p, they are of the form $\kappa(p) + mp$, where κ is one of these functions, and m is an arbitrary integer. Define a pre-order $\preceq_{\nu,\bar{\lambda}}$ on $\operatorname{Irr}(\tilde{\mathcal{O}}(\mathcal{A}_{\lambda',\mathbb{F}}))^0$ by $(\mathsf{x}, \kappa) \preceq_{\nu,\bar{\lambda}} (\mathsf{x}, \kappa')$ if the coefficient of p in the affine function $\kappa' - \kappa$ is nonnegative. We will discuss examples below.

Let us describe the equivalence classes of the resulting pre-order on $\operatorname{Irr}(\tilde{\mathcal{O}}(\mathcal{A}_{\lambda',\mathbb{F}})^0)$ (the zero equivariant block). Recall that the labels of simples in that equivariant block are (x, κ) such that $\kappa = \mathsf{c}_{\nu,\lambda'}(\mathsf{x})$ in $\mathbb{F}_p$. We note that $\sim_{\nu,\bar{\lambda}}$ descends to an equivalence relation between the fixed points that will also be denoted by $\sim_{\nu,\bar{\lambda}}$. So $\mathsf{x} \sim_{\nu,\bar{\lambda}} \mathsf{x}'$ if $\mathsf{c}_{\nu,\bar{\lambda}}(\mathsf{x}) - \mathsf{c}_{\nu,\bar{\lambda}}(\mathsf{x}') \in \mathbb{Z}$. By the definition of $\sim_{\nu,\bar{\lambda}}$, the points x and x' are equivalent if and only if the corresponding simples lie in the same h-block of $\mathcal{O}_\nu(\mathcal{A}_\lambda)$.

Then we have the following easy elementary lemma that follows from the construction of λ' (this was a starting observation for the present paper).

Lemma 9.2. *The following two conditions are equivalent:*

- $(\mathsf{x}, \kappa) \sim (\mathsf{x}', \kappa')$
- $\mathsf{x} \sim_{\nu,\bar{\lambda}} \mathsf{x}'$ *and* $\mathsf{c}_{\nu,\bar{\lambda}}(\mathsf{x}) - \kappa = \mathsf{c}_{\nu,\bar{\lambda}}(\mathsf{x}') - \kappa'$.

In particular, the order on the equivalence class for $\preceq_{\nu,\bar{\lambda}}$ *coincides with* $\leqslant_{\nu,\bar{\lambda}}$.

So any equivalence class for $\preceq$ gives rise to a rational number β (defined up to adding an integer) as follows: if (x, κ) is in the equivalence class, then $\beta - c_{\nu,\bar{\lambda}}(\mathsf{x}) \in \mathbb{Z}$.

Remark 9.3. A priori, the pre-order $\preceq_{\nu,\bar{\lambda}}$ depends on the choice of $\bar{\lambda}$ and not only on Θ. However, in the most interesting case when Θ is a point, it is easy to see, compare with Lemma 9.10 below, that $\preceq_{\nu,\bar{\lambda}}$ is independent of the choice of $\bar{\lambda}$.

9.1.3. Reductions to characteristic *p* Let $M \in \mathcal{O}_\nu(\mathcal{A}_{\bar{\lambda},\mathbb{Q}})$ and let $M_{\mathsf{R}} \subset M$ be a graded R-lattice. Pick $z_2 \in \mathbb{Z}$ such that $M_z = 0$ if $z > z_2$. Then $M_{\mathbb{F}} := \mathbb{F} \otimes_{\mathsf{R}} M_{\mathsf{R}}$ lies in $\mathcal{A}_{\lambda',\mathbb{F}}\text{-mod}^T_{\leqslant z_2}$.

Now choose an integer $z_1 < z_2$ and set $\mathfrak{I} = [z_1, z_2]$. We can view $\mathfrak{I}$ as an interval in $\operatorname{Irr}(\tilde{\mathcal{O}}(\mathcal{A}_{\lambda',\mathbb{F}})^0)$.

Let $\pi_{\mathfrak{I}}$ denote the quotient functor $\mathcal{A}_{\lambda',\mathbb{F}}\text{-mod}^T_{\leqslant z_2} \twoheadrightarrow \tilde{\mathcal{O}}(\mathcal{A}_{\lambda,\mathbb{F}})_{\mathfrak{I}}$ (that mods out $\mathcal{A}_{\lambda',\mathbb{F}}\text{-mod}^T_{<z_1}$, see (8.4)). Set $M_{\mathbb{F},\mathfrak{I}} := \pi_{\mathfrak{I}}(M_{\mathbb{F}})$.

Recall that we have distinguished lattices $L_{\nu,\bar{\lambda},\mathsf{R}}(\mathsf{x}, \kappa) \subset L_{\nu,\bar{\lambda},\mathbb{Q}}(\mathsf{x}, \kappa)$ and $P_{\nu,\bar{\lambda},\mathsf{R}}(\mathsf{x}, \kappa) \subset P_{\nu,\bar{\lambda},\mathbb{Q}}(\mathsf{x}, \kappa)$, see Section 6.4. So for $\kappa \in [z_1, z_2]$ we get objects $L_{\nu,\bar{\lambda},\mathbb{Q}}(\mathsf{x}, \kappa)_{\mathfrak{I}} := \pi_{\mathfrak{I}}(L_{\nu,\bar{\lambda},\mathbb{F}}(\mathsf{x}, \kappa))$. Also, similarly to Lemma 6.9, we have a unique subobject $M_{\mathsf{R}} \subset P_{\nu,\bar{\lambda},\mathsf{R}}(\mathsf{x},\kappa)$ that is filtered with $\Delta_{\nu,\bar{\lambda},\mathsf{R}}(\mathsf{x}',\kappa')$, where $\kappa' > \kappa$, while the quotient $P_{\nu,\bar{\lambda},\mathsf{R}}(\mathsf{x},\kappa)/M_{\mathsf{R}}$ is filtered with $\Delta_{\nu,\bar{\lambda},\mathsf{R}}(\mathsf{x}',\kappa')$ with $\kappa' \leqslant \kappa$. We set $P_{\nu,\bar{\lambda},\mathbb{Q}}(\mathsf{x},\kappa)_{\mathfrak{I}} := \pi_{\mathfrak{I}}(P_{\nu,\bar{\lambda},\mathbb{F}}(\mathsf{x},\kappa)/M_{\mathbb{F}})$.

Now for an h-block $\mathcal{O}_\nu(\mathcal{A}_{\bar{\lambda},\mathbb{Q}})^\beta$ we define its reduction $\mathcal{O}_\nu(\mathcal{A}_{\bar{\lambda},\mathbb{F}})^\beta$. Namely, let P be the direct sum of all indecomposable projectives in $\mathcal{O}_\nu(\mathcal{A}_{\bar{\lambda},\mathbb{Q}})^\beta$ and let P_{R} be the direct sum of their distinguished R-forms. By definition, $\mathcal{O}_\nu(\mathcal{A}_{\bar{\lambda},\mathbb{F}})^\beta$ is the category of right modules over $\mathbb{F} \otimes_{\mathsf{R}} \operatorname{End}_{\mathcal{A}_{\lambda,\mathsf{R}}}(P_{\mathsf{R}})$.

The following is the main result of this section, and of the entire paper.

Theorem 9.4. *Pick an interval* $\mathfrak{I} \subset \operatorname{Irr}(\tilde{\mathcal{O}}(\mathcal{A}_{\lambda',\mathbb{F}}))$ *of the form* $\{(\mathsf{x},\kappa)|z - pm \leqslant \kappa < z\}$, *where* z *is the minimal value of* κ *in some equivalence class for the pre-order* $\preceq$. *Then the following claims are true:*

1. *The category* $\tilde{\mathcal{O}}(\mathcal{A}_{\lambda',\mathbb{F}})_{\mathfrak{I}}$ *is standardly stratified with respect to the pre-order* $\preceq$. *The objects* $P_{\nu,\bar{\lambda},\mathbb{F}}(\mathsf{x},\kappa)_{\mathfrak{I}}$ *are standard and the objects* $L_{\nu,\bar{\lambda},\mathbb{F}}(\mathsf{x},\kappa)_{\mathfrak{I}}$ *are proper standard for* $(\mathsf{x},\kappa) \in \mathfrak{I}$.
2. *The component of the associated graded category* $\operatorname{gr}\tilde{\mathcal{O}}(\mathcal{A}_{\lambda',\mathbb{F}})^0_{\mathfrak{I}}$ *corresponding to any equivalence class with respect to* $\preceq$ *is* $\mathcal{O}_\nu(\mathcal{A}_{\bar{\lambda},\mathbb{F}})^\beta$ *(as a highest weight category), where* β *is the rational number corresponding to the equivalence class, see the discussion after Lemma 9.2. In particular,* $\operatorname{gr}\tilde{\mathcal{O}}(\mathcal{A}_{\lambda',\mathbb{F}})_{\mathfrak{I}} \cong \mathcal{O}_\nu(\mathcal{A}_{\bar{\lambda},\mathbb{F}})^{\oplus m}$.

9.1.4. Example of $T^*(G/B)$ The most interesting case here is when Θ is a point.

Consider $X = T^*(G/B)$ and take the fundamental p-alcove: $\langle\alpha_i^\vee,\lambda\rangle \geqslant 1, \langle\alpha_0^\vee,\lambda\rangle \geqslant 1-p$, where we write $\alpha_0^\vee$ for the minimal coroot and i runs over $\{1,2,\ldots,r\}$. Pick $i \in \{0,\ldots,r\}$ and consider the element $\bar{\lambda}_i$ defined by $\langle\alpha_j^\vee,\bar{\lambda}_i\rangle = 1$ for $j \neq i$ (in particular, $\bar{\lambda}_0 = \rho$). The corresponding point $\lambda_i' = {}^p\bar{\lambda}_i$ is given

- by $\lambda_0' = \rho$ if $i = 0$,
- and by $\langle\alpha_j^\vee,\lambda_i'\rangle = 1$ for $j \neq 0, i, \langle\alpha_0^\vee,\lambda'\rangle = 1-p$.

We have $\mathsf{c}_{\nu,{}^p\bar{\lambda}}(w) = \langle w\,{}^p\bar{\lambda},\rho^\vee\rangle$ is an affine function in p.

Two elements $w_1, w_2 \in X^T \cong W$ lie in the same h-block if and only if $\langle w_1\bar{\lambda} - w_2\bar{\lambda},\rho^\vee\rangle \in \mathbb{Z}$. So each of these equivalence classes splits into the union of right cosets for the integral Weyl group $W_{[\lambda]}$ of λ (generated by the reflections s_α, where α is such that $\langle\alpha^\vee,\lambda\rangle \in \mathbb{Z}$). The categories $\mathcal{O}_\nu(\mathcal{A}_{\bar{\lambda}})$ are the direct sums of the regular blocks of the BGG categories $\mathcal{O}$ for $W_{[\lambda]}$.

9.1.5. Example of $\mathrm{Hilb}_n(\mathbb{A}^2)$ Now consider the case of $X = \mathrm{Hilb}_n(\mathbb{A}^2)$ so that the algebra $\mathcal{A}_\lambda$ is a spherical rational Cherednik algebra for S_n. As usual we set $c = \lambda - 1/2$.

Recall that the p-alcoves (for the parameter c) are the intervals of the form $[\frac{(p+1)a}{b} + s, \frac{(p+1)a'}{b'} - s]$, where $a/b < a'/b'$ are two rational numbers with denominators between 2 and n such that there are no rational numbers with such denominators between $a/b, a'/b'$.

Pick $\bar{\lambda} = \frac{a}{b} + s$. It is again clear that $\mathsf{c}_{\nu,{}^p\bar{\lambda}}(\mathsf{x})$ is an affine function in p, compare to Lemma 3.18. By that lemma, we have $\mu \sim_{\nu,\bar{\lambda}} \mu'$ if and only if $\mathrm{cont}(\mu) - \mathrm{cont}(\mu')$ is divisible by b. The order within the equivalence class is as follows: $\mu < \mu'$ if $\mathrm{cont}(\mu) > \mathrm{cont}(\mu')$.

We note that, thanks to results of [R, L4], the category $\mathcal{O}_\nu(\mathcal{A}_\lambda)$ is equivalent to the category of modules over the q-Schur algebra $\mathcal{S}_q(n)$ for the quantum $\mathfrak{gl}_n$ (a labelling preserving equivalence of highest weight categories). It follows that $\mathcal{O}_\nu(\mathcal{A}_{\lambda,\mathbb{F}}) \cong \mathcal{S}_{q,\mathbb{F}}(n)$-mod.

9.2. Proof of Theorem 9.4

9.2.1. Three technical lemmas In the proof we will use the following three technical lemmas. The first one is a straightforward consequence of Corollary 6.11.

Lemma 9.5. *Let $M_{\mathsf{R}}, N_{\mathsf{R}}$ be modules of the form $L_{\nu,\lambda,\mathsf{R}}(\mathsf{x},\kappa), P_{\nu,\lambda,\mathsf{R}}(\mathsf{x}',\kappa')$, where x,x' are in the same h-block and $|\kappa - \kappa'| \leqslant m$, where m has the same meaning as in Corollary 6.11. Then for all i, the R-modules $\mathrm{Ext}^i_{\mathcal{A}_{\lambda,\mathsf{R}},T}(M_{\mathsf{R}}, N_{\mathsf{R}})$ are free and finitely generated over R. Then we automatically have*

$$\mathbb{C} \otimes_{\mathsf{R}} \mathrm{Ext}^i_{\mathcal{A}_{\lambda,\mathsf{R}},T}(M_{\mathsf{R}}, N_{\mathsf{R}}) \xrightarrow{\sim} \mathrm{Ext}^i_{\mathcal{A}_{\lambda,\mathsf{R}},T}(M_{\mathbb{C}}, N_{\mathbb{C}}),$$
$$\mathbb{F} \otimes_{\mathsf{R}} \mathrm{Ext}^i_{\mathcal{A}_{\bar{\lambda},\mathsf{R}},T}(M_{\mathsf{R}}, N_{\mathsf{R}}) \xrightarrow{\sim} \mathrm{Ext}^i_{\mathcal{A}_{\lambda',\mathbb{F}},T}(M_{\mathbb{F}}, N_{\mathbb{F}}).$$

Lemma 9.6. *Let β be an equivalence class for $\sim_\nu$ and let M_{R} be an object in $\mathcal{A}_{\lambda,\mathsf{R}}\text{-mod}^T$ filtered by $L_{\nu,\lambda,\mathsf{R}}(\mathsf{x},\kappa)$, where $(\mathsf{x},\kappa) \in \beta$. Then for any finite interval $\mathfrak{I}$ containing β and any $N_{\mathbb{F}} \in \tilde{\mathcal{O}}(\mathcal{A}_{\lambda',\mathbb{F}})_{\mathfrak{I},\prec\beta}$ we have*

$$\mathrm{Ext}^i_{\tilde{\mathcal{O}}(\mathcal{A}_{\lambda',\mathbb{F}})_{\mathfrak{I}}}(\mathcal{M}_{\mathbb{F},\mathfrak{I}}, N) = 0, \quad \forall i \geqslant 0.$$

Proof. Of course, it is enough to assume that $M_{\mathsf{R}} = L_{\nu,\lambda,\mathsf{R}}(\mathsf{x},\kappa)$. By Lemma 6.9, M_{R} admits a resolution by objects filtered by $\Delta_{\nu,\lambda,\mathsf{R}}(\mathsf{x}',\kappa')$ for $(\mathsf{x}',\kappa') \in \beta$. So it is enough to prove that

$$\mathrm{Ext}^i_{\tilde{\mathcal{O}}(\mathcal{A}_{\lambda',\mathbb{F}})_{\mathfrak{I}}}(\Delta_{\nu,\lambda,\mathbb{F}}(\mathsf{x}',\kappa')_{\mathfrak{I}}, N) = 0, \quad \forall i \geqslant 0.$$

These equalities hold because $\Delta_{\nu,\lambda,\mathbb{F}}(\mathsf{x}',\kappa')_{\mathfrak{J}}$ is the standard corresponding to (x',κ') and N is filtered by simples whose labels are less than the labels in β. $\square$

Corollary 9.7. *Let M_{R} be as in Lemma 9.6 and $M_{\mathbb{F}} := \mathbb{F}\otimes_{\mathsf{R}} M_{\mathsf{R}}$. Then we have $M_{\mathbb{F}} = L\pi^!_\beta\circ\pi_\beta(M_{\mathbb{F}})$, where we write π_β for the quotient functor $\mathcal{A}_{\lambda,\mathbb{F}}\text{-mod}^T_{\preceq\beta}\twoheadrightarrow\tilde{\mathcal{O}}(\mathcal{A}_{\lambda,\mathbb{F}})_\beta$ and $L\pi^!_\beta$ for the left adjoint functor.*

Lemma 9.8. *Pick a label $(\mathsf{x},\kappa)\in\operatorname{Irr}(\tilde{\mathcal{O}}(\mathcal{A}_{\lambda,\mathbb{F}})^0)$ and let β be the $\sim_\nu$ equivalence class of (x,κ). Then $P_{\nu,\bar\lambda,\mathbb{F}}(\mathsf{x},\kappa)_\beta$ is the projective in $\tilde{\mathcal{O}}(\mathcal{A}_{\lambda',\mathbb{F}})_\beta$ labelled by (x,κ), while $L_{\nu,\bar\lambda,\mathbb{F}}(\mathsf{x},\kappa)_\beta$ is the simple labelled by (x,κ).*

Proof. Pick $(\mathsf{x}',\kappa')\in\beta$. By Corollary 6.11, we have

$$\operatorname{Hom}_{\mathcal{A}_{\lambda,\mathsf{R}},T}(\Delta_{\nu,\bar\lambda,\mathsf{R}}(\mathsf{x}',\kappa'),L_{\nu,\bar\lambda,\mathsf{R}}(\mathsf{x},\kappa))=\mathsf{R}^{\oplus\delta_{\mathsf{x},\mathsf{x}'}\delta_{\kappa,\kappa'}}.$$

Applying Lemma 9.5, we see that

$$\dim\operatorname{Hom}_{\mathcal{A}_{\lambda',\mathbb{F}},T}(\Delta_{\nu,\bar\lambda,\mathbb{F}}(\mathsf{x}',\kappa'),L_{\nu,\bar\lambda,\mathbb{F}}(\mathsf{x},\kappa)))=\delta_{\mathsf{x},\mathsf{x}'}\delta_{\kappa,\kappa'}.$$

By Corollary 9.7, $\Delta_{\nu,\bar\lambda,\mathbb{F}}(\mathsf{x}',\kappa')=\pi^!_\beta(\Delta_{\nu,\bar\lambda,\mathbb{F}}(\mathsf{x}',\kappa')_\beta)$. From here we deduce that

$$\dim\operatorname{Hom}_{\tilde{\mathcal{O}}(\mathcal{A}_{\lambda',\mathbb{F}})_\beta}(\Delta_{\nu,\bar\lambda,\mathbb{F}}(\mathsf{x}',\kappa')_\beta,L_{\nu,\bar\lambda,\mathbb{F}}(\mathsf{x},\kappa)_\beta))=\delta_{\mathsf{x},\mathsf{x}'}\delta_{\kappa,\kappa'}.$$

Recall that the objects $\Delta_{\nu,\bar\lambda,\mathbb{F}}(\mathsf{x}',\kappa')_\beta$ are standard. By Lemma 3.1,

$$L_{\nu,\bar\lambda,\mathbb{F}}(\mathsf{x},\kappa)_\beta\hookrightarrow\nabla^{\mathbb{F}}_{\nu,\lambda'}(\mathsf{x},\kappa)_\beta.$$

It remains to show that the head of $L_{\nu,\bar\lambda,\mathbb{F}}(\mathsf{x},\kappa)_\beta$ coincides with $\pi_\beta(\underline{L}_{\nu,\lambda',\mathbb{F}}(\mathsf{x},\kappa))$.

By Corollary 9.7, $\pi^!_\beta(L_{\nu,\bar\lambda,\mathbb{F}}(\mathsf{x},\kappa)_{\mathfrak{J}})=L_{\nu,\bar\lambda,\mathbb{F}}(\mathsf{x},\kappa)_\beta$. So using Lemma 9.5 again and arguing as in the first paragraph of the proof of the present lemma, we conclude

$$\dim\operatorname{Hom}_{\tilde{\mathcal{O}}(\mathcal{A}_{\lambda',\mathbb{F}})_\beta}(L_{\nu,\bar\lambda,\mathbb{F}}(\mathsf{x},\kappa)_\beta,L_{\nu,\bar\lambda,\mathbb{F}}(\mathsf{x}',\kappa')_\beta))=\delta_{\mathsf{x},\mathsf{x}'}\delta_{\kappa,\kappa'}.$$

Since $L_{\nu,\bar\lambda,\mathbb{F}}(\mathsf{x}',\kappa')_\beta\hookrightarrow\nabla^{\mathbb{F}}_{\nu,\lambda'}(\mathsf{x}',\kappa')_\beta$ we conclude that the head of $L_{\nu,\bar\lambda,\mathbb{F}}(\mathsf{x},\kappa)_\beta$ is $\pi_\beta(\underline{L}_{\nu,\lambda',\mathbb{F}}(\mathsf{x},\kappa))$. This finishes the proof that $L_{\nu,\bar\lambda,\mathbb{F}}(\mathsf{x},\kappa)_\beta=\pi_\beta(\underline{L}_{\nu,\lambda',\mathbb{F}}(\mathsf{x},\kappa))$.

Let us show that $P_{\nu,\bar\lambda,\mathbb{F}}(\mathsf{x},\kappa)_\beta$ is the projective cover of $\pi_\beta(\underline{L}_{\nu,\lambda',\mathbb{F}}(\mathsf{x},\kappa))$. From Lemma 9.5 it follows that

$$\dim\operatorname{Ext}^i_{\mathcal{A}_{\lambda',\mathbb{F}},T}(P_{\nu,\bar\lambda,\mathbb{F}}(\mathsf{x},\kappa),L_{\nu,\bar\lambda,\mathbb{F}}(\mathsf{x}',\kappa'))=\delta_{i,0}\delta_{\mathsf{x},\mathsf{x}'}\delta_{\kappa,\kappa'}.$$

By Corollary 9.7, $L\pi_\beta^! P_{\nu,\bar\lambda,\mathbb{F}}(\mathsf{x},\kappa)_\beta = P_{\nu,\bar\lambda,\mathbb{F}}(\mathsf{x},\kappa)$. We conclude that, for $i = 0,1$, we have

$$\dim \operatorname{Ext}^i_{\tilde{\mathcal{O}}(\mathcal{A}_{\lambda',\mathbb{F}})_\beta}(P_{\nu,\bar\lambda,\mathbb{F}}(\mathsf{x},\kappa)_\beta, L_{\nu,\bar\lambda,\mathbb{F}}(\mathsf{x}',\kappa')_\beta) = \delta_{i,0}\delta_{\mathsf{x},\mathsf{x}'}\delta_{\kappa,\kappa'}.$$

We already know that $L_{\nu,\bar\lambda,\mathbb{F}}(\mathsf{x}',\kappa')_\beta$ is the simple object corresponding to (x',κ'). Hence $P_{\nu,\bar\lambda,\mathbb{F}}(\mathsf{x},\kappa)_\beta$ is the projective object corresponding to (x,κ). ☐

9.2.2. Completion of the proof

Proof of Theorem 9.4. Let β be as in Lemma 9.8 and let $\mathfrak{I}$ be an interval containing β. By Lemma 9.8, for $(\mathsf{x},\kappa)\in\beta$, the object $L_{\nu,\bar\lambda,\mathbb{F}}(\mathsf{x},\kappa)_\beta$ (that coincides with $\pi_\beta(L_{\nu,\bar\lambda,\mathbb{F}}(\mathsf{x},\kappa)_\mathfrak{I})$ by the definitions) is simple. By Corollary 9.6, $L\pi_\beta^!(L_{\nu,\bar\lambda,\mathbb{F}}(\mathsf{x},\kappa)_\beta) = L_{\nu,\bar\lambda,\mathbb{F}}(\mathsf{x},\kappa)_\mathfrak{I}$. It follows, in particular, that $\pi_\beta^!$ is exact.

Thanks to Lemma 3.3, to finish the proof of the claim that $\preceq$ defines a standardly stratified structure, it remains to show that $\pi_\beta^* : \tilde{\mathcal{O}}(\mathcal{A}_{\lambda,\mathbb{F}})_\beta \to \tilde{\mathcal{O}}(\mathcal{A}_{\lambda,\mathbb{F}})_{\mathfrak{I},\preceq\beta}$ is exact. For this we will use the duality functor $\bullet^*$ from Section 8.1.5. Recall that $\underline{L}_{\nu,\lambda,\mathbb{F}}(\mathsf{x},\kappa) = \underline{L}_{-\nu,-\lambda,\mathbb{F}}(\mathsf{x},-\kappa)^*$. It follows that $\bullet^*$ induces an equivalence $\bullet^* : \tilde{\mathcal{O}}(\mathcal{A}_{\lambda,\mathbb{F}})_\mathfrak{I} \xrightarrow{\sim} \tilde{\mathcal{O}}(\mathcal{A}_{-\lambda,\mathbb{F}})^{opp}_{-\mathfrak{I}}$ for any interval $\mathfrak{I}$. Clearly, the functors $\bullet^\dagger$ intertwine $(\pi_\beta^{opp})^!$ with π_β^*. Since we know that $(\pi_\beta^{opp})^!$ is exact, we see that π_β^* is exact.

It follows that the pre-order $\preceq$ indeed defines a standardly stratified structure. Corollary 9.7 and Lemma 9.8 imply that $L_{\nu,\lambda,\mathbb{F}}(\mathsf{x},\kappa)_\mathfrak{I}, P_{\nu,\lambda,\mathbb{F}}(\mathsf{x},\kappa)_\mathfrak{I}$ are the proper standard and standard objects. This finishes the proof of (1).

Part (2) easily follows from Lemma 9.5 (applied to $M_\mathsf{R} = N_\mathsf{R} := P_\mathsf{R}$) and the projective part of Lemma 9.8. ☐

9.2.3. The case of $\dim T > 1$ The case when $\dim T > 1$ can be handled similarly to the situation of highest weight structures. In particular, we can use a more general notion of a standardly stratified category, see Remark 3.4.

9.3. Wall-crossing for categories $\tilde{\mathcal{O}}$

Now let A_1, A_2 be two real alcoves that are opposite with respect a common face Θ. Let $\bar\lambda_1, \bar\lambda_2 = \bar\lambda_1 + \chi$ be elements in Λ compatible with $(A_1,\Theta),(A_2,\Theta)$ with $\chi\in\mathcal{P}_1$. Set $\lambda_i' := {}^p\bar\lambda_i, i = 1,2$. Then we have the wall-crossing functor $\mathcal{A}_{\lambda_1',\chi,\mathbb{F}} \otimes^L_{\mathcal{A}_{\lambda_1',\mathbb{F}}} \bullet : D^b(\mathcal{A}_{\lambda_1',\mathbb{F}}\text{-mod}^T) \xrightarrow{\sim} D^b(\mathcal{A}_{\lambda_2',\mathbb{F}}\text{-mod}^T)$.

Our goal in this section is to show, that, in a suitable sense, the wall-crossing functor is a partial Ringel duality functor between the categories $\tilde{\mathcal{O}}(\mathcal{A}_{\lambda_1',\mathbb{F}})$ and $\tilde{\mathcal{O}}(\mathcal{A}_{\lambda_2',\mathbb{F}})$.

9.3.1. Preparation Let us start with the following elementary lemma.

Lemma 9.9. *Let* (x,κ) *and* (x',κ') *lie in the same equivariant block for* $\lambda \in \mathfrak{P}_{\mathbb{F}}$. *Then, for any* $\chi \in \operatorname{Pic}(X)$, $(\mathsf{x},\kappa+\mathsf{wt}_\chi(\mathsf{x})), (\mathsf{x}',\kappa'+\mathsf{wt}_\chi(\mathsf{x}'))$ *lie in the same equivariant block.*

Proof. Recall, [L6, Section 6.1], $\mathsf{c}_\lambda(\mathsf{x})-\mathsf{c}_\lambda(\mathsf{x}')$ is an affine function in λ and $(\mathsf{c}_{\lambda+\chi}(\mathsf{x})-\mathsf{c}_{\lambda+\chi}(\mathsf{x}'))-(\mathsf{c}_{\nu,\lambda}(\mathsf{x})-\mathsf{c}_{\nu,\lambda}(\mathsf{x}'))=\mathsf{wt}_\chi(\mathsf{x})-\mathsf{wt}_\chi(\mathsf{x}')$. Our claim easily follows from here. $\square$

So by choosing a suitable equivariant structure on $O(\chi)$, we can assume that $\underline{L}_\nu(\mathsf{x},\kappa+\mathsf{wt}_\chi(\mathsf{x}))\in\tilde{\mathcal{O}}(\mathcal{A}_{\lambda+\chi,\mathbb{F}})^0$ if and only if $\underline{L}_\nu(\mathsf{x},\kappa)\in\tilde{\mathcal{O}}(\mathcal{A}_{\lambda,\mathbb{F}})^0$.

Arguing as in the proof of Lemma 9.9 and using the condition that χ is independent of p, we get the following.

Lemma 9.10. *We have* $(\mathsf{x},\kappa)\sim_{\nu,\lambda_1'}(\mathsf{x}',\kappa')$ *if and only if* $(\mathsf{x},\kappa+\mathsf{wt}_\chi(\mathsf{x}))\sim_{\nu,\lambda_2'}(\mathsf{x}',\kappa'+\mathsf{wt}_\chi(\mathsf{x}'))$.

So we have a bijection between equivalence classes of simples in $\tilde{\mathcal{O}}(\mathcal{A}_{\lambda_1',\mathbb{F}})^0$ and in $\tilde{\mathcal{O}}(\mathcal{A}_{\lambda_2',\mathbb{F}})^0$. This bijection is compatible with $\preceq$. So to an interval (with respect to $\preceq$) $\mathfrak{I}\subset\operatorname{Irr}(\tilde{\mathcal{O}}(\mathcal{A}_{\lambda_1',\mathbb{F}})^0)$ we can assign an interval $\mathfrak{I}^\chi\subset\operatorname{Irr}(\tilde{\mathcal{O}}(\mathcal{A}_{\lambda_2',\mathbb{F}})^0)$.

9.3.2. Main result The following is the main result of the section.

Theorem 9.11. *The functor*

$$\mathfrak{WC}_{\lambda_2'\leftarrow\lambda_1'}:\mathcal{A}_{\bar{\lambda},\chi,\mathbb{F}}\otimes^L_{\mathcal{A}_{\bar{\lambda},\mathbb{F}}}\bullet : D^b(\mathcal{A}_{\lambda_1',\mathbb{F}}\text{-mod}^T)\xrightarrow{\sim}D^b(\mathcal{A}_{\lambda_2',\mathbb{F}}\text{-mod}^T)$$

induces a partial Ringel duality functor $D^b(\tilde{\mathcal{O}}(\mathcal{A}_{\lambda_1',\mathbb{F}})_{\mathfrak{I}})\xrightarrow{\sim}D^b(\tilde{\mathcal{O}}(\mathcal{A}_{\lambda_2',\mathbb{F}})_{\mathfrak{I}^\chi})$ *(with respect to the standardly stratified structures defined by* $\bar{\lambda}$*).*

Proof. Recall that, in Step 3 of the proof of Theorem 8.7, for $z\in\mathbb{Z}$ we have defined the category $\mathcal{A}_{\lambda,\mathbb{F}}\text{-mod}^T_{\leqslant z}$. Inside we can consider the subcategory $\mathcal{A}_{\lambda,\mathbb{F}}\text{-mod}^{T,0}_{\leqslant z}\subset\mathcal{A}_{\lambda,\mathbb{F}}\text{-mod}^T_{\leqslant z}$ defined similarly to $\tilde{\mathcal{O}}(\mathcal{A}_{\lambda,\mathbb{F}})^0\subset\tilde{\mathcal{O}}(\mathcal{A}_{\lambda,\mathbb{F}})$. This subcategory is a direct summand. We have sufficiently many projectives in $\mathcal{A}_{\lambda,\mathbb{F}}\text{-mod}^T_{\leqslant z}$ and they have no higher self-extensions in $\mathcal{A}_{\lambda,\mathbb{F}}\text{-mod}^T$, this was established in the proof of Theorem 8.7. So the natural functor $D^b(\mathcal{A}_{\lambda,\mathbb{F}}\text{-mod}^T_{\leqslant z})\hookrightarrow D^b(\mathcal{A}_{\lambda,\mathbb{F}}\text{-mod}^T)$ is a full embedding. So, for an interval $[z_1,z_2]$ corresponding to $\mathfrak{I}$, we have

$$D^b(\tilde{\mathcal{O}}(\mathcal{A}_{\lambda,\mathbb{F}})_{\mathfrak{I}})=D^b(\mathcal{A}_{\lambda,\mathbb{F}}\text{-mod}^T_{\leqslant z_2})/D^b(\mathcal{A}_{\lambda,\mathbb{F}}\text{-mod}^T_{<z_1}).$$

It follows that $D^b(\tilde{\mathcal{O}}(\mathcal{A}_{\lambda,\mathbb{F}})^0_{\mathfrak{J}}) = D^b(\mathcal{A}_{\lambda,\mathbb{F}}\text{-mod}^{T,0}_{\leqslant z_2})/D^b(\mathcal{A}_{\lambda,\mathbb{F}}\text{-mod}^{T,0}_{<z_1})$. From here and the previous section we deduce that $\mathfrak{WC}_{\lambda_2'\leftarrow\lambda_1'}$ indeed induces a functor $D^b(\tilde{\mathcal{O}}^0(\mathcal{A}_{\lambda_1',\mathbb{F}})_{\mathfrak{J}}) \xrightarrow{\sim} D^b(\tilde{\mathcal{O}}^0(\mathcal{A}_{\lambda_2',\mathbb{F}})_{\mathfrak{J}^\chi})$.

It follows from Lemma 6.12 and the R-flatness of $\mathcal{A}_{\bar{\lambda},\chi,\mathsf{R}}, \Delta_{\nu,\bar{\lambda},\mathsf{R}}(\mathsf{x},\kappa)$ (Corollary 6.3 and Lemma 6.6) that

$$\mathfrak{WC}_{\lambda_2'\leftarrow\lambda_1'}(\Delta_{\nu,\bar{\lambda},\mathbb{F}}(\mathsf{x},\kappa)) = \nabla_{\nu,\bar{\lambda}+\chi,\mathbb{F}}(\mathsf{x},\kappa+\mathsf{wt}_\chi(\mathsf{x})).$$

From here it follows that $\mathfrak{WC}_{\lambda_2'\leftarrow\lambda_1'}$ is compatible with the filtrations on the categories $D^b(\tilde{\mathcal{O}}(\mathcal{A}_{\lambda_1',\mathbb{F}})_{\mathfrak{J}})$, $D^b(\tilde{\mathcal{O}}(\mathcal{A}_{\lambda_2',\mathbb{F}})_{\mathfrak{J}^\chi})$ coming from the standardly stratified structures. In particular, the functor induces an equivalence of the associated graded categories. Assume $\mathfrak{J}$ is an equivalence class. By Theorem 9.4, for $(\mathsf{x},\kappa)\in\mathfrak{J}$, the costandard object in $\tilde{\mathcal{O}}(\mathcal{A}_{\lambda_2',\mathbb{F}})_{\mathfrak{J}^\chi}$ is $\pi_{\mathfrak{J}}(\nabla_{\nu,\bar{\lambda}+\chi,\mathbb{F}}(\mathsf{x},\kappa+\mathsf{wt}_\chi(\mathsf{x})))$. It follows that the equivalence of the associated graded categories induced by $\mathfrak{WC}_{\lambda_2'\leftarrow\lambda_1'}$ is a Ringel duality functor. So the functor $D^b(\tilde{\mathcal{O}}(\mathcal{A}_{\lambda_1',\mathbb{F}})_{\mathfrak{J}}) \xrightarrow{\sim} D^b(\tilde{\mathcal{O}}(\mathcal{A}_{\lambda_2',\mathbb{F}})_{\mathfrak{J}^\chi})$ induced by $\mathfrak{WC}_{\lambda_2'\leftarrow\lambda_1'}$ is a partial Ringel duality. $\square$

10. Applications

10.1. Wall-crossing bijections

Let Θ,λ',χ be as above. We consider the category $\mathcal{A}_{\lambda',\mathbb{F}}\text{-mod}_0$ of all finite dimensional $\mathcal{A}_{\lambda',\mathbb{F}}$-modules with generalized zero p-character. We also consider the category $D^b_0(\mathcal{A}_{\lambda',\mathbb{F}}\text{-mod})$ of all objects in $D^b_0(\mathcal{A}_{\lambda',\mathbb{F}}\text{-mod})$ with homology in $\mathcal{A}_{\lambda',\mathbb{F}}\text{-mod}_0$. Note that $\mathfrak{WC}_{\lambda'+\chi\leftarrow\lambda'} = \mathcal{A}_{\lambda',\chi,\mathbb{F}}\otimes^L_{\mathcal{A}_{\lambda',\mathbb{F}}}\bullet$ restricts to a perverse equivalence

$$D^b_0(\mathcal{A}_{\lambda',\mathbb{F}}\text{-mod}) \xrightarrow{\sim} D^b_0(\mathcal{A}_{\lambda'+\chi,\mathbb{F}}\text{-mod})$$

Let us classify the irreducible objects in $\mathcal{A}_{\lambda',\mathbb{F}}\text{-mod}_0$.

Lemma 10.1. *Fix a generic one-parameter subgroup ν. Then we get a bijection between* $\operatorname{Irr}(\mathcal{A}_{\lambda',\mathbb{F}}\text{-mod}_0)$ *and X^T.*

Proof. Every irreducible from $\mathcal{A}_{\lambda',\mathbb{F}}\text{-mod}_0$ has a $T_{\mathbb{F}}$-equivariant structure, unique up to a twist with a character. It is still irreducible as a $T_{\mathbb{F}}$-equivariant module and hence is $\underline{L}(\mathsf{x},\kappa)$ for some character κ. This implies the claim of the lemma. $\square$

As any perverse equivalence, $\mathfrak{WC}_{\lambda'+\chi\leftarrow\lambda'}$ gives rise to a bijection

$$\operatorname{Irr}(\mathcal{A}_{\lambda',\mathbb{F}}\text{-mod}_0) \xrightarrow{\sim} \operatorname{Irr}(\mathcal{A}_{\lambda'+\chi,\mathbb{F}}\text{-mod}_0),$$

i.e., a self-bijection of X^T. We want to compare this bijection with a similarly defined bijection for categories $\mathcal{O}$ in characteristic 0.

Proposition 10.2. *For $p \gg 0$, the self-bijection of X^T induced by $\mathfrak{WC}_{\lambda'+\chi\leftarrow\lambda'}$ coincides with the bijection $\operatorname{Irr}(\mathcal{O}_\nu(\mathcal{A}_{\bar\lambda})) \xrightarrow{\sim} \operatorname{Irr}(\mathcal{O}_\nu(\mathcal{A}_{\bar\lambda+\chi}))$ coming from the wall-crossing functor $\mathfrak{WC}_{\bar\lambda+\chi\leftarrow\bar\lambda}$.*

Proof. Let φ denote the bijection coming from $\mathfrak{WC}_{\bar\lambda+\chi\leftarrow\bar\lambda}$. What we need to show is that the wall-crossing bijection $\tilde\varphi$ for the categories $\tilde{\mathcal{O}}(\mathcal{A}_{?,\mathbb{F}})$ has the form $(\mathsf{x},\kappa) \mapsto (\varphi(\mathsf{x}),\kappa')$. Note that $\mathfrak{WC}_{\lambda'+\chi\leftarrow\lambda'}$ respects the filtrations on the categories $\tilde{\mathcal{O}}(\mathcal{A}_{\lambda',\mathbb{F}}), \tilde{\mathcal{O}}(\mathcal{A}_{\lambda'+\chi,\mathbb{F}})$ coming from the face Θ. So the induced functor between the associated graded categories is perverse and the induced bijection is the same as $\tilde\varphi$. On the other hand, the associated graded categories are $\mathcal{O}^T_\nu(\mathcal{A}_{\bar\lambda,\mathbb{F}})$ and $\mathcal{O}^T_\nu(\mathcal{A}_{\bar\lambda+\chi,\mathbb{F}})$ and the functor induced by $\mathfrak{WC}_{\lambda'+\chi\leftarrow\lambda'}$ is the Ringel duality. So forgetting the T-character component, we see that the bijection $X^T = \operatorname{Irr}(\mathcal{A}_{\lambda',\mathbb{F}}\text{-mod}_0) \xrightarrow{\sim} \operatorname{Irr}(\mathcal{A}_{\lambda'+\chi,\mathbb{F}}\text{-mod}_0) = X^T$ coincides with the bijection between the sets of simples in $\mathcal{O}_\nu(\mathcal{A}_{\bar\lambda,\mathbb{F}})$ and $\mathcal{O}_\nu(\mathcal{A}_{\bar\lambda+\chi,\mathbb{F}})$ induced by the Ringel duality. But $\mathfrak{WC}_{\bar\lambda+\chi\leftarrow\bar\lambda}$ is the Ringel duality between $\mathcal{O}_\nu(\mathcal{A}_{\bar\lambda})$ and $\mathcal{O}_\nu(\mathcal{A}_{\bar\lambda+\chi})$ and since p is very large, the bijections induced by the Ringel duality over $\mathbb{F}$ and over $\mathbb{C}$ coincide. This completes the proof. $\square$

Wall-crossing bijections were studied (and sometimes computed combinatorially) in a number of cases. Paper [L7] studied the case of $X = T^*(G/B)$. There we have seen that the wall-crossing bijections through the faces containing 0 define an action of the so called cactus group Cact_W on W. Using Proposition 10.2 one can show that this action extends to an action of the affine cactus group. We note that combinatorial recipes to compute the action are not known in general.

The case of rational Cherednik algebras of type A (and of more general cyclotomic rational Cherednik algebras) was considered in [L8]. It was shown that the wall-crossing bijections for rational Cherednik algebras of type A are extended Mullineux involutions, see [L8, Corollary 5.7] for a precise statement. Therefore the wall-crossing bijection induced by $\mathfrak{WC}_{\lambda'+\chi\leftarrow\lambda'}$ is given by the same rule.

10.2. Gradings

In this section, following an idea of Bezrukavnikov, we produce graded lifts of the category $\tilde{\mathcal{O}}(\mathcal{A}_{\lambda',\mathbb{F}})$ that come from the contacting torus action on X. Then we show that our grading lift induces a grading lift of $\mathcal{O}_\nu(\mathcal{A}_{\bar\lambda})$. Finally, we compare Koszulity properties of the grading lifts on $\tilde{\mathcal{O}}(\mathcal{A}_{\lambda',\mathbb{F}})$ and on $\mathcal{O}_\nu(\mathcal{A}_{\bar\lambda})$.

10.2.1. Grading on $\tilde{\mathcal{O}}(\mathcal{A}_{\lambda',\mathbb{F}})$ Let $Y_{\mathbb{F}}^{(1),\wedge_0}$ denote the spectrum of the completion $\mathbb{F}[Y^{(1)}]^{\wedge_0}$, an $\mathbb{F}$-scheme, and let $X_{\mathbb{F}}^{(1),\wedge_0}$ be its preimage in $X_{\mathbb{F}}^{(1)}$.

We assume that the microlocal quantization $\mathcal{A}_{\lambda',\mathbb{F}}^{\theta}$ of $X_{\mathbb{F}}$ is obtained (by completing with respect to the filtration) from a Frobenius constant quantization (in the sense of [BK2]). By abusing the notation, we denote the corresponding Azumaya algebra on $X_{\mathbb{F}}^{(1)}$ also by $\mathcal{A}_{\lambda',\mathbb{F}}^{\theta}$. This holds in the examples we consider (and in more general examples of Slodowy varieties, [BMR2], and of Nakajima quiver varieties, [BFG]).

Consider the restriction $\mathcal{A}_{\lambda',\mathbb{F}}^{\theta,\wedge_0}$ of $\mathcal{A}_{\lambda',\mathbb{F}}^{\theta}$ to $X_{\mathbb{F}}^{(1)\wedge_0}$. We assume that it splits, let $\mathcal{V}_{\mathbb{F}}^{\wedge_0}$ denote the splitting bundle. Recall that $\mathcal{V}_{\mathbb{F}}^{\wedge_0}$ is defined up to a twist with a line bundle. The splitting bundle again exists in all examples we consider, see [BMR2] for the case of Slodowy varieties and [BL] for the case of Nakajima quiver varieties.

The splitting bundle $\mathcal{V}_{\mathbb{F}}^{\wedge_0}$ has no higher self-extensions. It follows that it admits a $T_{\mathbb{F}} \times \mathbb{F}^{\times}$-equivariant structure, where we write $\mathbb{F}^{\times}$ for the contracting torus. We can choose a $T_{\mathbb{F}}$-equivariant structure on $\mathcal{V}_{\mathbb{F}}^{\wedge_0}$ so that the isomorphism $\mathcal{E}nd(\mathcal{V}_{\mathbb{F}}^{\wedge_0}) \cong \mathcal{A}_{\lambda',\mathbb{F}}^{\theta,\wedge_0}$ is $T_{\mathbb{F}}$-equivariant.

Since the action of $\mathbb{F}^{\times}$ is contracting we can uniquely extend $\mathcal{V}_{\mathbb{F}}^{\wedge_0}$ to a $T_{\mathbb{F}} \times \mathbb{F}^{\times}$-equivariant vector bundle $\mathcal{V}_{\mathbb{F}}$. Set $\tilde{A}_{\mathbb{F}} := \operatorname{End}(\mathcal{V}_{\mathbb{F}})$. We can consider the categories $\tilde{A}_{\mathbb{F}}\text{-mod}_0^T$ of all $T_{\mathbb{F}}$-equivariant finite dimensional $\tilde{A}_{\mathbb{F}}$-modules (automatically supported at $0 \in Y_{\mathbb{F}}^{(1)}$) and $\tilde{A}_{\mathbb{F}}\text{-mod}_0^{T\times\mathbb{F}^{\times}}$ of all $T_{\mathbb{F}} \times \mathbb{F}^{\times}$-equivariant finite dimensional $\tilde{A}_{\mathbb{F}}$-modules. The construction of $\tilde{A}_{\mathbb{F}}$ yields a category equivalence $\tilde{\mathcal{O}}(\mathcal{A}_{\lambda',\mathbb{F}}) \cong \tilde{A}_{\mathbb{F}}\text{-mod}_0^T$. So $\tilde{A}_{\mathbb{F}}\text{-mod}_0^{T\times\mathbb{F}^{\times}}$ is a graded lift of $\tilde{\mathcal{O}}(\mathcal{A}_{\lambda',\mathbb{F}})$. Note that this graded lift is independent of the choice of λ' in its p-alcove.

10.2.2. From graded lift $\tilde{\mathcal{O}}(\mathcal{A}_{\lambda',\mathbb{F}})$ to graded lift of $\mathcal{O}_{\nu}(\mathcal{A}_{\bar{\lambda}})$ Now we are going to produce a graded lift of $\mathcal{O}_{\nu}(\mathcal{A}_{\bar{\lambda}})$ from a graded lift of $\tilde{\mathcal{O}}(\mathcal{A}_{\lambda',\mathbb{F}})$. Note that a graded lift of $\tilde{\mathcal{O}}(\mathcal{A}_{\lambda',\mathbb{F}})$ induces that of any quotient category $\tilde{\mathcal{O}}(\mathcal{A}_{\lambda',\mathbb{F}})_{\leqslant z_2}/\tilde{\mathcal{O}}(\mathcal{A}_{\lambda',\mathbb{F}})_{\leqslant z_1}$. In particular, $\mathcal{O}_{\nu}(\mathcal{A}_{\bar{\lambda},\mathbb{F}})$ is the direct sum of such subquotients so we get graded lifts of $\mathcal{O}_{\nu}(\mathcal{A}_{\bar{\lambda},\mathbb{F}})$.

Now we are in the following situation. Let R be a finite localization of $\mathbb{Z}$ and let B_{R} be an R-algebra that is a free finite rank R-module (in our case $B_{\mathsf{R}} = \operatorname{End}(P_{\mathsf{R}})^{opp}$, where P_{R} is an R-form of a pro-generator of $\mathcal{O}_{\nu}(\mathcal{A}_{\lambda})$). We need to check that for $p \gg 0$ there is a natural bijection between graded lifts of $B_{\overline{\mathbb{F}}_p}\text{-mod}$ and of $B_{\mathbb{C}}\text{-mod}$.

For this we need to note that, for a finite dimensional algebra B over an algebraically closed field $\mathbb{K}$, graded lifts of $B\text{-mod}$ are parameterized by conjugacy classes of one-parameter subgroups of $H := \operatorname{Aut}(B)/B^{\times}$.

In our case $H_{\mathbb{K}}$ is the base change of an algebraic group scheme H_{R} from R to $\mathbb{K}$ assuming that $\operatorname{char}\mathbb{K}$ is 0 or is large enough. After taking a finite extension of R, we can find a subgroup subscheme $T_{\mathsf{R}} \subset H_{\mathsf{R}}$ that becomes a maximal torus in $H_{\mathbb{K}}$ after a base change to $\mathbb{K}$. The conjugacy classes of one-parameter subgroups in $H_{\mathbb{K}}$ are the orbits of the action of $N_{H_{\mathbb{K}}}(T_{\mathbb{K}})$ on the lattice of one-parameter subgroups in $T_{\mathbb{K}}$. This is clearly independent of $\mathbb{K}$.

This establishes a required bijection between the graded lifts.

10.2.3. Koszulity Let us now show that if $\tilde{A}_{\mathbb{F}}\text{-mod}_0^{T\times\mathbb{F}^\times}$ is Koszul for infinitely many p, then the category $\mathcal{O}_\nu(\mathcal{A}_\lambda)$ is Koszul as well.

Recall that being Koszul for the category $\tilde{A}_{\mathbb{F}}\text{-mod}_0^{T\times\mathbb{F}^\times}$ means that for every simple $L \in \tilde{A}_{\mathbb{F}}\text{-mod}_0^{T}$ one can find a lift $\tilde{L} \in \tilde{A}_{\mathbb{F}}\text{-mod}_0^{T\times\mathbb{F}^\times}$ such that $\operatorname{Ext}^i(\tilde{L}, \tilde{L}')$ is concentrated in degree i for all simples L, L' (here the Ext's are taken in $\tilde{A}_{\mathbb{F}}\text{-mod}_0^{T}$). Let $\tilde{\mathcal{O}}(\mathcal{A}_{\lambda,\mathbb{F}})^{\mathbb{F}^\times}$ denote the corresponding graded lift of $\tilde{\mathcal{O}}(\mathcal{A}_{\lambda,\mathbb{F}})$.

Lemma 10.3. *Suppose $\tilde{A}_{\mathbb{F}}\text{-mod}_0^{T\times\mathbb{F}^\times}$ is Koszul. Then for every interval $\mathfrak{I} \subset \mathbb{Z}$ (with respect to $\leqslant_\nu$) the subquotient category the induced graded lift $\tilde{\mathcal{O}}(\mathcal{A}_{\lambda,\mathbb{F}})_{\mathfrak{I}}$ is Koszul.*

Proof. Recall that the natural functor $D^b(\tilde{\mathcal{O}}(\mathcal{A}_{\lambda,\mathbb{F}})_{\leqslant z}) \hookrightarrow D^b(\tilde{\mathcal{O}}(\mathcal{A}_{\lambda,\mathbb{F}}))$ is a full embedding. So the induced graded lift of $\tilde{\mathcal{O}}(\mathcal{A}_{\lambda,\mathbb{F}})_{\leqslant z}$ is Koszul. Note that $\tilde{\mathcal{O}}(\mathcal{A}_{\lambda,\mathbb{F}})_{\leqslant z}$ has enough projectives. Let $P(\tilde{L})$ denote the projective cover of $\tilde{L}$ in $\tilde{\mathcal{O}}(\mathcal{A}_{\lambda,\mathbb{F}})_{\leqslant z}$. Since $\tilde{\mathcal{O}}(\mathcal{A}_{\lambda,\mathbb{F}})_{\leqslant z}$ is Koszul, we can find a projective resolution of any $\tilde{L}$ of the form $\ldots \to P_1 \to P_0 \to 0$ such that P_i is the direct sum of $P(\tilde{L}')\langle i\rangle$'s with some multiplicities, where $\langle\bullet\rangle$ means a grading shift. Then $\pi_{\mathfrak{I}}(P_\bullet)$ is the projective resolution of $\pi_{\mathfrak{I}}(\tilde{L})$. Hence $\tilde{\mathcal{O}}(\mathcal{A}_{\lambda,\mathbb{F}})_{\mathfrak{I}}$ is Koszul. $\square$

In particular, we see that the categories $\mathcal{O}_\nu(\mathcal{A}_{\lambda,\mathbb{F}})$ acquire Koszul graded lifts for infinitely many p. Let us deduce from here that $\mathcal{O}_\nu(\mathcal{A}_\lambda)$ does. This follows from the next lemma.

Lemma 10.4. *Let R be a ring that is a finite algebraic extension of $\mathbb{Z}$. Let B_{R} be an R-algebra that is a free finite rank R-module. Suppose that B_{R} has finite homological dimension. Finally, suppose that the fibers of B_{R} at infinitely many maximal ideals of R carry Koszul gradings. Then $B_{\operatorname{Frac}(\mathsf{R})}$ carries a Koszul grading.*

Proof. By replacing R with its finite localization, we can achieve that there are objects $L^i_{\mathsf{R}}, i = 1, \ldots, k$, such that the fibers of these objects at any point of R are the simples over the corresponding algebra. Moreover, localizing R

further, we can assume that $\mathrm{Ext}^{\ell}_{B_{\mathsf{R}}}(L^i_{\mathsf{R}}, L^j_{\mathsf{R}})$ are projective R-modules for all i, j, ℓ (here we use that B_{R} has finite homological dimension).

As we have seen above, after replacing R with its finite algebraic extension, we can assume that there is a bijection between gradings on all the fibers of B_{R}, moreover, we can assume that every grading of every fiber comes from a grading on B_{R}. So pick a grading that gives rise to a Koszul grading in some fiber. We claim that it gives a Koszul grading on some fiber. This follows from the observation that all R-modules $\mathrm{Ext}^{\ell}_{B_{\mathsf{R}}}(L^i_{\mathsf{R}}, L^j_{\mathsf{R}})$ are graded and all graded components are projective R-modules. $\square$

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Perverse sheaves, nilpotent Hessenberg varieties, and the modular law

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Dedicated to George Lusztig

Abstract: We consider generalizations of the Springer resolution of the nilpotent cone of a simple Lie algebra by replacing the cotangent bundle with certain other vector bundles over the flag variety. We show that the analogue of the Springer sheaf has as direct summands only intersection cohomology sheaves that arise in the Springer correspondence. The fibers of these general maps are nilpotent Hessenberg varieties, and we build on techniques established by De Concini, Lusztig, and Procesi to study their geometry. For example, we show that these fibers have vanishing cohomology in odd degrees. This leads to several implications for the dual picture, where we consider maps that generalize the Grothendieck–Springer resolution of the whole Lie algebra. In particular we are able to prove a conjecture of Brosnan.

As we vary the maps, the cohomology of the corresponding nilpotent Hessenberg varieties often satisfy a relation we call the geometric modular law, which also has origins in the work of De Concini, Lusztig, and Procesi. We connect this relation in type A with a combinatorial modular law defined by Guay-Paquet that is satisfied by certain symmetric functions and deduce some consequences of that connection.

1. Introduction

Let G be a simple algebraic group over $\mathbb{C}$ with Lie algebra $\mathfrak{g}$. Let B be a Borel subgroup with Lie algebra $\mathfrak{b}$ containing a maximal torus T with Lie algebra $\mathfrak{t}$. Denote by Φ the root system associated to the pair (T, B), with simple roots Δ. Let U be the unipotent radical of B with $\mathfrak{u}$ its Lie algebra. Let $W = N_G(T)/T$ be the Weyl group of T and $\mathcal{B} := G/B$ the flag variety of G.

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Let $\mathcal{N}$ denote the variety of nilpotent elements in $\mathfrak{g}$. For complex varieties, $\dim(X)$ refers to the complex dimension. Let $N := \dim(\mathcal{N})$, which equals $2\dim(\mathfrak{u})$. The Springer resolution of the nilpotent cone $\mathcal{N}$ in $\mathfrak{g}$ is the proper, G-equivariant map

$$\mu : G \times^B \mathfrak{u} \to \mathcal{N}$$

sending $(g, x) \in G \times \mathfrak{u}$ to $g.x$, where $g.x := \mathrm{Ad}(g)(x)$ denotes the adjoint action of G on $\mathfrak{g}$.

Let $\underline{\mathbb{C}}[N]$ be the shifted constant sheaf on $G \times^B \mathfrak{u}$ with coefficients in $\mathbb{C}$. The shift makes it a G-equivariant perverse sheaf on $G \times^B \mathfrak{u}$. A central object in Springer theory is the Springer sheaf $R\mu_*(\underline{\mathbb{C}}[N])$, the derived pushforward of $\underline{\mathbb{C}}$ under μ. The Springer sheaf is a G-equivariant perverse sheaf on $\mathcal{N}$. The nilpotent cone $\mathcal{N}$ is stratified by nilpotent G-orbits. Let $\mathcal{O}$ be a nilpotent orbit and $\mathcal{L}$ an irreducible G-equivariant local system on $\mathcal{O}$. Denote by Θ the set of all such pairs $(\mathcal{O}, \mathcal{L})$. Let $IC(\overline{\mathcal{O}}, \mathcal{L})$ denote the intersection cohomology sheaf on $\mathcal{N}$ defined by a pair $(\mathcal{O}, \mathcal{L}) \in \Theta$. We use the convention that if $\mathcal{O}' \subsetneq \mathcal{O}$, then $\mathcal{H}^j IC(\overline{\mathcal{O}}, \mathcal{L})|_{\mathcal{O}'} = 0$ unless

$$-\dim \mathcal{O} \leq j < -\dim \mathcal{O}'.$$

The decomposition theorem implies that $R\mu_*(\underline{\mathbb{C}}[N])$ is a direct sum of shifted IC-complexes. That is,

$$R\mu_*(\underline{\mathbb{C}}[N]) \simeq \bigoplus_{(\mathcal{O},\mathcal{L})\in\Theta} IC(\overline{\mathcal{O}}, \mathcal{L}) \otimes V_{\mathcal{O},\mathcal{L}}, \tag{1.1}$$

where each $V_{\mathcal{O},\mathcal{L}}$ is a graded complex vector space. Since μ is semismall, $V_{\mathcal{O},\mathcal{L}}$ is concentrated in degree 0. Now, both sides of (1.1) carry an action of W that makes the nonzero vector space $V_{\mathcal{O},\mathcal{L}}$ into an irreducible representation of W. Let Θ_{sp} denote the pairs $(\mathcal{O}, \mathcal{L})$ for which $V_{\mathcal{O},\mathcal{L}} \neq 0$. The Springer correspondence says that the map

$$(\mathcal{O}, \mathcal{L}) \in \Theta_{sp} \to V_{\mathcal{O},\mathcal{L}} \in \mathrm{Irr}(W) \tag{1.2}$$

is a bijection. Here, $\mathrm{Irr}(K)$ denotes the irreducible complex representations of a group K. Our convention for the Springer correspondence sends the zero orbit with trivial local system to the sign representation of W and the regular nilpotent orbit with trivial local system to the trivial representation of W. See [Ach21, Chapter 8] for a more detailed discussion of the Springer correspondence.

This paper is concerned with the generalization of (1.1) when $\mathfrak{u}$ is replaced by a subspace $I \subset \mathfrak{u}$ that is B-stable, as well as the connection of this map to related objects in Lie theory and combinatorics. The B-stable subspaces are also called ad-nilpotent ideals of $\mathfrak{b}$ and are well-studied Lie-theoretic objects (see, for example, [Kos98, CP00]). Denote by $\mathfrak{I}d$ the set of all B-stable subspaces of $\mathfrak{u}$. The cardinality of $\mathfrak{I}d$ is the W-Catalan number $\prod_{i=1}^{n} \frac{d_i+h}{d_i}$ where $d_1, \ldots, d_n$ are the fundamental degrees of W and h is the Coxeter number. In type A these ideals are in bijection with Dyck paths (see §7).

If $I \in \mathfrak{I}d$, then $G.I$ is the closure of a nilpotent orbit, denoted by $\mathcal{O}_I$. The restriction of μ gives a map

$$\mu^I : G \times^B I \to \overline{\mathcal{O}_I} \tag{1.3}$$

that is still proper, but it is no longer a resolution or semismall in general. Set $N_I := \dim(G \times^B I)$, which equals $\dim I + \dim G/B$. The decomposition theorem still applies to the analogue of the Springer sheaf. Namely,

$$R\mu^I_*(\underline{\mathbb{C}}[N_I]) \simeq \bigoplus_{(\mathcal{O},\mathcal{L})\in\Theta} IC(\overline{\mathcal{O}}, \mathcal{L}) \otimes V^I_{\mathcal{O},\mathcal{L}}, \tag{1.4}$$

where $V^I_{\mathcal{O},\mathcal{L}}$ is a graded vector space, but no longer concentrated in degree 0 in general.

Our first main result is that if a pair $(\mathcal{O}, \mathcal{L}) \in \Theta$ contributes a nonzero term in (1.4), then it must appear in the Springer correspondence, i.e., it contributes a nonzero term in (1.1). In other words,

Theorem 1.1. *Let $I \in \mathfrak{I}d$. If $V^I_{\mathcal{O},\mathcal{L}} \neq 0$ in (1.4), then $(\mathcal{O}, \mathcal{L}) \in \Theta_{sp}$.*

The case when $I = \mathfrak{u}_P$, the nilradical of the Lie algebra of a parabolic subgroup P of G, was established by Borho and MacPherson [BM83] and they gave a formula for the dimensions of the vector spaces $V^I_{\mathcal{O},\mathcal{L}}$ (see (4.8)).

Theorem 1.1 is proved by analyzing the fibers of the map μ^I. Let $x \in \mathcal{N}$, and let

$$\mathcal{B}^I_x := (\mu^I)^{-1}(x).$$

For the $I = \mathfrak{u}$ case, the fibers $\mu^{-1}(x)$ are the Springer fibers and denoted more simply as $\mathcal{B}_x$. The fiber $\mathcal{B}^I_x$ is a subvariety of $\mathcal{B}_x$, and any variety defined in this way is called a nilpotent Hessenberg variety.

For $x \in \mathcal{N}$, denote by $\mathcal{O}_x$ the G-orbit of x under the adjoint action. The component group $A(x) := Z_G(x)/Z^\circ_G(x)$ is a finite group, which identifies with the fundamental group of $\mathcal{O}_x$ when G is simply-connected. The cohomology of $\mathcal{B}^I_x$ carries an action of $A(x)$ and Theorem 1.1 is equivalent, using proper base change, to showing that if $\chi \in \mathrm{Irr}(A(x))$ has nonzero multiplicity in $H^*(\mathcal{B}^I_x)$, then the pair $(\mathcal{O}_x, \mathcal{L}_\chi)$ belongs to Θ_{sp}. Here, $\mathcal{L}_\chi$ denotes the irreducible G-equivariant local system on $\mathcal{O}_x$ defined by χ.

The analysis of the $\mathcal{B}_x^I$ occurs in §3, where we establish a decomposition of $\mathcal{B}_x^I$ into vector bundles over a small set of smooth varieties, from which we also deduce that $\mathcal{B}_x^I$ has no odd cohomology. These results are generalizations of those for the Springer fibers $\mathcal{B}_x$, handled by De Concini, Lusztig and Procesi in [DCLP88], and we rely on the techniques developed in that paper. Theorem 1.1 is then proved in §4.

Theorem 1.1 has an important implication for certain generalizations of the Grothendieck–Springer resolution. For $I \in \mathfrak{I}d$, we can consider $I^\perp \subset \mathfrak{g}$, the orthogonal complement to I under the Killing form. Then $H = I^\perp$ is also B-stable and it contains $\mathfrak{b}$, the Lie algebra of B. The map μ^H given by

$$\mu^H : G \times^B H \to \mathfrak{g} \tag{1.5}$$

is surjective and generalizes the Grothendieck–Springer resolution for $H = \mathfrak{b}$. Using Theorem 1.1 and the Fourier transform, we deduce in Theorem 5.1, that

$$R\mu_*^H(\underline{\mathbb{C}}_H[\dim G \times^B H])$$

has full support, proving a conjecture of Brosnan [Xue20, VX21]. This generalizes results of Bălibanu–Crooks [BC20] who proved the theorem in type A and Xue [Xue20] who has given a proof in type G_2. The remainder of §5 studies applications of Theorem 5.1. In Proposition 5.3 we establish a generalization to all types of an unpublished result of Tymoczko and MacPherson in type A. We conclude §5 by introducing two graded W-representations, the dot action representation of Tymoczko and LLT representations of Procesi and Guay-Paquet.

In §6 the main result relates the cohomology of $\mathcal{B}_x^I$ for certain triples of subspaces $I \in \mathfrak{I}d$. The setup is a triple $I_2 \subset I_1 \subset I_0$ of ideals in $\mathfrak{I}d$, each of codimension one in the next. For a simple root $\alpha \in \Delta$, let P_α denote the minimal parabolic subgroup containing B associated to α. The triples $I_2 \subset I_1 \subset I_0$ of interest are those satisfying the following conditions:

1. I_2 and I_0 are P_α-stable for some $\alpha \in \Delta$, and
2. the representation of the Levi subgroup of P_α (which is of type A_1) on the two-dimensional space I_0/I_2 is irreducible.

Such triples were introduced in [DCLP88, §2.8]. See Definition 6.1 for a purely root-theoretic definition. The first example of such a triple occurs when $\mathfrak{g} = \mathfrak{sl}_3(\mathbb{C})$. Let α, β denote the simple roots and let $I_0 = \mathfrak{u}_{P_\alpha}$ be the nilradical of parabolic subalgebra $\mathrm{Lie}(P_\alpha)$. Set $I_2 = \{0\}$. There is a unique $I_1 \in \mathfrak{I}d$ with $I_2 \subsetneq I_1 \subsetneq I_0$ and these three spaces form such a triple.

Our result below is a generalization of [DCLP88, Lemma 2.11] in that it applies to all nilpotent elements, not just to those $x \in \mathcal{O}_{I_0}$.

Proposition 1.2 (The geometric modular law). *Given a triple $I_2 \subset I_1 \subset I_0$ as above and $x \in \mathcal{N}$, there is an $A(x)$-equivariant isomorphism*

$$H^j(\mathcal{B}_x^{I_1}) \oplus H^{j-2}(\mathcal{B}_x^{I_1}) \simeq H^j(\mathcal{B}_x^{I_0}) \oplus H^{j-2}(\mathcal{B}_x^{I_2}) \tag{1.6}$$

for all $j \in \mathbb{Z}$.

The proposition has a formulation as a statement about perverse sheaves. If Q is any parabolic subgroup stabilizing I, then we can also consider the map $\mu^{I,Q} : G \times^Q I \to \overline{\mathcal{O}_I}$ and its derived pushforward

$$\mathcal{S}_{I,Q} := R\mu_*^{I,Q}(\underline{\mathbb{C}}[\dim(G \times^Q I)]).$$

Proposition 1.2 has the following consequence (see Proposition 6.7): for any triple $I_2 \subset I_1 \subset I_0$ as above, there is an isomorphism

$$\mathcal{S}_{I_1,B} \simeq \mathcal{S}_{I_2,P_\alpha} \oplus \mathcal{S}_{I_0,P_\alpha}$$

in the derived category of G-equivariant perverse sheaves on $\mathcal{N}$.

Proposition 1.2 implies that various polynomials $F^I \in \mathbb{N}[q]$, depending on $I \in \mathfrak{I}d$, that arise in our study satisfy the following law for a triple of ideals:

$$(1+q)F^{I_1} = F^{I_2} + qF^{I_0}.$$

These include the Poincare polynomials of nilpotent Hessenberg varieties and coefficients in the decomposition of the dot action and LLT representations, see Proposition 6.5.

In type A this law is closely related to a linear relation, called the modular law, satisfied by certain graded symmetric functions. It is due to Guay-Paquet [GP13] and studied more recently by Abreu–Nigro [AN21a]. It was seeing this law in the combinatorial setting that led us to connect it with the work of [DCLP88]. Indeed, we show in §7 that the geometric modular law of Proposition 1.2 implies the combinatorial modular law. This allows us to give another proof of the Shareshian and Wachs Conjecture [SW16, Conjecture 1.4] and to show that the Frobenius characteristic of the LLT representation in type A is a unicellular LLT polynomial, see Corollary 7.9. The key idea, due to Abreu–Nigro [AN21a], is that in type A any set of polynomials F^I for $I \in \mathfrak{I}d$ satisfying the modular law are completely determined by the $F^{\mathfrak{u}_P}$ where P is a parabolic subgroup.

2. Preliminaries

Let Φ^+, Φ^- and Δ denote the positive, negative and simple roots associated to the pair (T, B). For a simple root $\alpha \in \Delta$, let $s_\alpha \in W$ denote the corresponding simple reflection. Let $\ell(w)$ denote the minimal length of $w \in W$ when written as a product of simple reflections. We fix a representative $w \in N_G(T)$ for each $w \in W$, and denote both by the same letter. Let $\mathfrak{g}_\gamma \subset \mathfrak{g}$ denote the root space corresponding to $\gamma \in \Phi$.

If P is a parabolic subgroup of G, then $\mathfrak{p}$ denotes its Lie algebra and $\mathfrak{u}_P$ the nilradical of $\mathfrak{p}$. For $P = B$, we instead use $\mathfrak{b}$ and $\mathfrak{u}$. Generally, P will denote a standard parabolic subgroup, i.e., $B \subset P$.

For a rational representation M of P, the smooth variety $G \times^P M$ consists of equivalence classes of pairs $(g, m) \in G \times M$ with $(gp, p^{-1}.m) \sim (g, m)$. If $M \subset \mathfrak{g}$, there is a proper map from $G \times^P M$ to $\mathfrak{g}$ given by $(g, m) \to g.m$. See [Jan04].

We use $H_*(-)$ for Borel-Moore homology with complex coefficients and $H^*(-)$ for singular cohomology with complex coefficients.

2.1. Grading induced by a nilpotent orbit

Let $x \in \mathfrak{g}$ be a nonzero nilpotent element and recall $\mathcal{O}_x$ is the G-orbit of x. By the Jacobson–Morozov theorem, x can be completed to a $\mathfrak{sl}_2$-triple $\{x, h, y\} \subseteq \mathfrak{g}$. Namely, there exists $h, y \in \mathfrak{g}$ such that

$$[h, x] = 2x, \quad [h, y] = -2y, \quad [x, y] = h,$$

which implies $\mathrm{span}_{\mathbb{C}}\{x, h, y\} \simeq \mathfrak{sl}_2(\mathbb{C})$. For $j \in \mathbb{Z}$, let

$$\mathfrak{g}_j := \{z \in \mathfrak{g} \mid [h, z] = jz\}.$$

Without loss of generality, we may conjugate the triple so that $h \in \mathfrak{t}$ and $\alpha(h) \geq 0$ for all $\alpha \in \Delta$. Then h and the resulting grading $\mathfrak{g} = \oplus_{i\in\mathbb{Z}}\mathfrak{g}_i$ are uniquely determined by $\mathcal{O}_x$. We then have $x \in \mathfrak{g}_2$ and $\mathfrak{b} \subseteq \mathfrak{p}$ where $\mathfrak{p} = \bigoplus_{i\geq 0} \mathfrak{g}_i$ is a parabolic subalgebra of $\mathfrak{g}$.

Let P and G_0 be the connected subgroups of G whose Lie algebras are $\mathfrak{p}$ and $\mathfrak{g}_0$, respectively. Let U_P be the unipotent radical of P. The Lie algebra of U_P is $\mathfrak{u}_P = \bigoplus_{i\geq 1} \mathfrak{g}_i$. Then $P = G_0 U_P$ is a Levi decomposition of P corresponding to the decomposition $\mathfrak{p} = \mathfrak{g}_0 \oplus \mathfrak{u}_P$.

Set $B_0 = B \cap G_0$, which is a Borel subgroup in G_0, with Lie algebra $\mathfrak{b}_0 = \mathfrak{b} \cap \mathfrak{g}_0$. A key fact is that $\mathfrak{g}_{\geq 2}$ is a P-prehomogeneous space, meaning there is a unique dense P-orbit. Indeed, $\mathcal{O}_x \cap \mathfrak{g}_{\geq 2}$ is a P-orbit in $\mathfrak{g}_{\geq 2}$ and it is dense. Moreover, $\mathfrak{g}_2$ is a G_0-prehomogeneous space, with dense orbit $\mathcal{O}_x \cap \mathfrak{g}_2$. See [Car93] for these results.

Since $\mathfrak{t} \subset \mathfrak{g}_0$, we can define $\Phi_0 \subseteq \Phi$ to be the roots of $\mathfrak{g}_0$ relative to $\mathfrak{t}$ with simple roots $\Delta_0 := \Delta \cap \Phi_0$ and $\Phi_0^{\pm} := \Phi^{\pm} \cap \Phi_0$. Given a nonzero integer m, define $\Phi_m := \{\gamma \in \Phi \mid \mathfrak{g}_\gamma \subseteq \mathfrak{g}_m\}$ and $\Phi_{\geq m} := \cup_{i \geq m} \Phi_i$.

2.2. P-orbits on $\mathcal{B}$

Let $\mathcal{B}_0$ denote the flag variety G_0/B_0. The Weyl group of $\mathfrak{g}_0$ is $W_0 := \langle s_\alpha \mid \alpha \in \Delta_0 \rangle$. Let W^0 be the set of right coset representatives for W_0 in W of shortest length in their respective cosets. Then

$$W^0 = \{w \in W \mid w^{-1}(\Phi_0^+) \subset \Phi^+\}. \tag{2.1}$$

The set W^0 parametrizes the P-orbits on the $\mathcal{B} = G/B$. For $w \in W^0$, the corresponding P-orbit is $\mathcal{P}_w := PwB/B$.

Let $\lambda : \mathbb{C}^* \to T$ be a co-character satisfying $\alpha(\lambda(z)) = z^{\alpha(h)}$ for $z \in \mathbb{C}^*$ and $\alpha \in \Phi$. This gives a $\mathbb{C}^*$-action on $\mathcal{B}$ preserving each P-orbit. The fixed points of this $\mathbb{C}^*$-action on $\mathcal{P}_w$ is G_0wB/B, which is isomorphic to $\mathcal{B}_0$. Moreover, the smooth variety $\mathcal{P}_w$ is a vector bundle over its $\mathbb{C}^*$-fixed points $\mathcal{P}_w^{\mathbb{C}^*} \simeq \mathcal{B}_0$ with map $\pi_w : \mathcal{P}_w \to \mathcal{B}_0$ given by

$$\pi_w(pwB) = \lim_{z \to 0} \lambda(z)pwB. \tag{2.2}$$

The fibers of this vector bundle identify with the affine space

$$\mathbf{A}^{\ell(w)} \simeq \bigoplus_{\beta \in \Phi^+ \cap w(\Phi^-)} \mathfrak{g}_\beta.$$

By (2.1) all roots β such that $\beta \in \Phi^+ \cap w(\Phi^-)$ belong to $\Phi_{\geq 1}$. Hence $\mathbb{C}^*$ acts linearly with positive eigenvalues on the fibers of this vector bundle, a key fact used in [DCLP88] to decompose the the Springer fiber $\mathcal{B}_x$.

3. A decomposition of Hessenberg varieties

3.1. Definition of Hessenberg varieties

The fiber of the map μ_I in (1.3) over $x \in \mathcal{N}$ is given by

$$\mathcal{B}_x^I = \{gB \in \mathcal{B} \mid g^{-1}.x \in I\},$$

called a **nilpotent Hessenberg variety**. These varieties generalize Springer fibers by replacing $\mathfrak{u}$ in the definition of $\mathcal{B}_x$ with $I \in \mathfrak{I}d$.

More generally, let M be a subspace of $\mathfrak{g}$ that is B-stable. Since M is also T-stable, it is a sum of weight spaces of T. Let Φ_M denote the nonzero weights (i.e., roots) of T that appear in the sum. For any such M and $x \in \mathfrak{g}$, the Hessenberg variety associated to x and M is the closed subvariety $\mathcal{B}_x^M$ of the flag variety defined by

$$\mathcal{B}_x^M = \{gB \in \mathcal{B} \mid g^{-1}.x \in M\}. \tag{3.1}$$

When x is nilpotent, the $\mathcal{B}_x^M$ are called nilpotent Hessenberg varieties. We will mainly deal with the case where x is nilpotent, but in §5.2 below, the case where x is regular semisimple also arises.

We are interested in two kinds of subspaces of $\mathfrak{g}$ that are stable under the action of B. Those of the first kind are contained in $\mathfrak{u}$ and are called ad-nilpotent ideals (since they are Lie algebra ideals in $\mathfrak{b}$) or just ideals. Those of the second kind contain $\mathfrak{b}$, and are called Hessenberg spaces. Let $\mathfrak{I}d$ denote the set of subspaces of the first kind and $\mathcal{H}$, those of the second kind. That is,

$$\mathfrak{I}d = \{I \mid I \subset \mathfrak{u} \text{ and } B.I = I\} \text{ and } \mathcal{H} = \{H \mid \mathfrak{b} \subset H \text{ and } B.H = H\}.$$

The two sets are in bijection. If $I \in \mathfrak{I}d$, then the orthogonal subspace $I^\perp$ to I under the Killing form is B-stable since I is B-stable and $I^\perp$ contains $\mathfrak{b}$. Hence $I^\perp \in \mathcal{H}$. Since the Killing form is non-degenerate, we have $(I^\perp)^\perp = I$, proving that taking the orthogonal complement defines a bijection between ideals in $\mathfrak{I}d$ and Hessenberg spaces $\mathcal{H}$.

Let P be a parabolic subgroup of G with Lie algebra $\mathfrak{p}$. The varieties $\mathcal{B}_x^I$ for $I \in \mathfrak{I}d$ are also a kind of generalization of Spaltenstein varieties: if $I = \mathfrak{u}_P$ is the nilradical of $\mathfrak{p}$, then the image of $\mathcal{B}_x^I$ in G/P is the Spaltenstein variety $\mathcal{P}_x^0$ from [BM83]. The varieties $\mathcal{B}_x^H$ for $H \in \mathcal{H}$ are a kind of generalization of Steinberg varieties: if $H = \mathfrak{p}$, the image of $\mathcal{B}_x^H$ in G/P is the Steinberg variety $\mathcal{P}_x$ from [BM83].

We now describe a decomposition of $\mathcal{B}_x^M$ when x is nilpotent that generalizes the decomposition (for the Springer fiber $\mathcal{B}_x$) defined and studied by De Concini, Lusztig, and Procesi in [DCLP88]. The story from *loc. cit.* goes through: $\mathcal{B}_x^M$ decomposes as a union of smooth varieties, each of which is a vector bundle over one of a small set of smooth varieties.

For the rest of the section, we fix x nilpotent and its induced grading on $\mathfrak{g}$ as in §2.

3.2. Building block varieties

Recall that $\mathfrak{g}_2$ is a prehomogeneous space for G_0, with dense G_0-orbit $\mathcal{O}_x \cap \mathfrak{g}_2$. Let $\mathfrak{I}d_2$ denote the set of B_0-stable linear subspaces of $\mathfrak{g}_2$, and $\mathfrak{I}d_2^{gen} \subset \mathfrak{I}d_2$ denote those $U \in \mathfrak{I}d_2$ with $U \cap \mathcal{O}_x \neq \emptyset$.

Following [DCLP88, §2.1], for $U \in \mathfrak{I}d_2^{gen}$ define subvarieties of $\mathcal{B}_0$ as follows

$$X_U := \{gB_0 \in G_0/B_0 \mid g^{-1}.x \in U\}.$$

These are smooth, projective varieties and

$$\dim X_U = \dim(\mathcal{B}_0) - \dim(\mathfrak{g}_2/U). \tag{3.2}$$

For example, if $U = \mathfrak{g}_2$ then $X_U = \mathcal{B}_0$. The variety X_U is empty for subspaces $U \in \mathfrak{I}d_2 \setminus \mathfrak{I}d_2^{gen}$.

Remark 3.1. Let $U \in \mathfrak{I}d_2^{gen}$. If we set $I := U \oplus \mathfrak{g}_{\geq 3}$, then $I \in \mathfrak{I}d$ and $X_U \simeq \mathcal{B}_x^I$ by [Fen08, Proposition 4.2], so the X_U are special cases of the varieties being considered.

3.3. The decomposition

Let M be a B-stable subspace of $\mathfrak{g}$. The main result of this section is that the Hessenberg variety $\mathcal{B}_x^M$ decomposes as a union of vector bundles over disjoint copies of various X_U for $U \in \mathfrak{I}d_2^{gen}$.

Lemma 3.2. *For each $w \in W^0$, the subspace $w.M \cap \mathfrak{g}_2$ of $\mathfrak{g}_2$ is B_0-stable.*

Proof. Since M is B-stable and hence T-stable, $w.M$ is also T-stable since $T = wTw^{-1}$. Also being T-stable, $w.M$ is a sum of weight spaces for T and thus $w.M \cap \mathfrak{g}_2$ is a sum of root spaces. Let $\mathfrak{g}_\beta \subset w.M$ and $\mathfrak{g}_\gamma \subset \mathfrak{b}_0$. Then $\gamma \in \Phi_0^+$ and $w^{-1}(\gamma) \in \Phi^+$ by (2.1). Now $w^{-1}(\beta) \in \Phi_M$ and M is B-stable, so $w^{-1}(\beta) + w^{-1}(\gamma) \in \Phi_M$ if the sum is a root. If so, $w^{-1}(\beta + \gamma) \in \Phi_M$, which means $\beta + \gamma \in \Phi_{w.M}$. This shows $w.M$ is B_0-stable. The result follows since $\mathfrak{g}_2$ is B_0-stable, being G_0-stable. □

Recall that for $w \in W^0$, $\mathcal{P}_w$ denotes the P-orbit on $\mathcal{B}$ containing wB. The next proposition shows that the intersection $\mathcal{P}_w \cap \mathcal{B}_x^M$ is smooth and describes its structure. The proof is a generalization of the methods in [DCLP88]. Some cases of these generalizations have previously appeared in [Pre13, Fre16, Xue20].

Proposition 3.3. *Let $w \in W^0$ and $U = w.M \cap \mathfrak{g}_2 \in \mathfrak{I}d_2$. Set $\mathcal{B}_{x,w}^M = \mathcal{P}_w \cap \mathcal{B}_x^M$.*

1. *$\mathcal{B}^M_{x,w} \neq \emptyset$ if and only if $U \in \mathfrak{I}d_2^{gen}$.*
2. *If $\mathcal{B}^M_{x,w}$ is nonempty, then*
   (a) *$\mathcal{B}^M_{x,w}$ is smooth and*

$$\dim(\mathcal{B}^M_{x,w}) = \ell(w) + |\Phi_0^+| - |\{\gamma \in \Phi_{\geq 2} \mid w^{-1}(\gamma) \notin \Phi_M\}|.$$

   (b) *$\mathcal{B}^M_{x,w}$ is a vector bundle over X_U with dimension of its fiber equal to*

$$\ell(w) - |\{\gamma \in \Phi_{\geq 3} \mid w^{-1}(\gamma) \notin \Phi_M\}|.$$

Proof. First, we prove 2(a). By definition (3.1),

$$\mathcal{B}^M_{x,w} = \left\{pwB, p \in P \mid w^{-1}p^{-1}.x \in M\right\}.$$

Since $p^{-1}.x \in \mathfrak{g}_{\geq 2}$ for $p \in P$, this can be rewritten as

$$\mathcal{B}^M_{x,w} = \left\{pwB \mid p^{-1}.x \in w.M \cap \mathfrak{g}_{\geq 2}\right\}.$$

The setup in [DCLP88, §2.1] now applies to the P-prehomogeneous space $\mathfrak{g}_{\geq 2}$, the linear subspace of $\mathfrak{g}_{\geq 2}$ equal to $w.M \cap \mathfrak{g}_{\geq 2}$, and the closed subgroup of P equal to $B_w := P \cap wBw^{-1}$. The subspace $w.M \cap \mathfrak{g}_{\geq 2}$ is B_w-stable since M is B-stable and $\mathfrak{g}_{\geq 2}$ is P-stable. Then $\mathcal{B}^M_{x,w}$ is isomorphic to the subvariety of P/B_w given by

$$\{pB_w \mid p^{-1}.x \in w.M \cap \mathfrak{g}_{\geq 2}\},$$

which is smooth by Lemma 2.2 in *loc. cit.*, and its dimension equals

$$\dim(P/B_w) - \dim(\mathfrak{g}_{\geq 2}/w.M \cap \mathfrak{g}_{\geq 2}).$$

Simplifying the dimension formula above using the fact that $\dim(P/B_w) = |\Phi_0^+| + \ell(w)$, we get

$$\dim(\mathcal{B}^M_{x,w}) = |\Phi_0^+| + \ell(w) - |\{\gamma \in \Phi_{\geq 2} \mid w^{-1}(\gamma) \notin \Phi_M\}|. \tag{3.3}$$

This completes the proof of 2(a).

Recall the cocharacter λ from §2.2. Since $\lambda(z)\cdot x = z^2 x$, the smooth closed subvariety $\mathcal{B}^M_{x,w}$ of $\mathcal{P}_w$ is $\mathbb{C}^*$-stable. Now we can use the result in [DCLP88, §1.5] (see also [BH85, Theorem 1.9]): let $\pi_w : \mathcal{P}_w \to G_0wB$ be the vector bundle map from §2.2. Since the $\mathbb{C}^*$-action on $\mathcal{P}_w$ preserves the fibers of π_w and acts with strictly positive weights, it follows that

$$\mathcal{B}^M_{x,w} \to G_0wB \cap \mathcal{B}^M_{x,w}$$

is a vector sub-bundle of π_w. Finally,

$$G_0 w B \cap \mathcal{B}^M_{x,w} \simeq \{gB_0 \in G_0/B_0 \mid g^{-1}.x \in w.M \cap \mathfrak{g}_2\} = X_U$$

as in §3.7 of *loc. cit.*

By (3.2),

$$\begin{aligned}\dim X_U &= \dim(G_0/B_0) - \dim(\mathfrak{g}_2/U)\\ &= |\Phi_0^+| - |\{\gamma \in \Phi_2 \mid w^{-1}(\gamma) \notin \Phi_M\}|.\end{aligned}$$

Subtracting this value from the one in (3.3) completes the proof of 2(b).

Finally, the proof of (1). If $\mathcal{B}^M_{x,w}$ is nonempty, then by 2(b) it follows that X_U is nonempty, so it contains some gB_0 for $g \in G_0$, i.e., $g^{-1}.x \in U$. Hence the G-orbit of x meets U and $U \in \mathfrak{I}d_2^{gen}$. Conversely, if $U \in \mathfrak{I}d_2^{gen}$ then $g^{-1}.x \in U$ for some $g \in G_0$, which means $gB \in \mathcal{B}^M_{x,w}$. □

The centralizer $Z_G(x)$ acts on $\mathcal{B}^M_x$ and this gives an action of the component group $A(x) = Z_G(x)/Z_G^\circ(x)$ on the Borel-Moore homology $H_*(\mathcal{B}^M_x)$ of $\mathcal{B}^M_x$ since the induced action of a connected group is trivial.

There is also an action of $A(x)$ on $H_*(\mathcal{B}^M_{x,w})$ and $H_*(X_U)$. Namely, $Z_G(x) = Z_P(x)$ [Jan04, Proposition 5.9] and $Z_P(x)$ acts on $\mathcal{B}^M_{x,w}$, so $A(x)$ acts on $H_*(\mathcal{B}^M_{x,w})$. Set $L = G_0$, then the centralizer $Z_L(x)$ acts on X_U. Since $Z_L(x)$ is a Levi factor of $Z_P(x)$, then $A(x) \simeq Z_L(x)/Z_L^\circ(x)$ acts on $H_*(X_U)$.

For each $w \in W^0$, let $t_M(w) = \ell(w) - |\{\gamma \in \Phi_{\geq 3} \mid w^{-1}(\gamma) \notin \Phi_M\}|$, the dimension of the fiber of the vector bundle from Proposition 3.3. The following corollary is implicit in [DCLP88].

Corollary 3.4. *Let $w \in W^0$ and $U = w.M \cap \mathfrak{g}_2$. Let $U \in \mathfrak{I}d_2^{gen}$. Then as $A(x)$-modules, we have the isomorphism*

$$H_{j+2t_M(w)}(\mathcal{B}^M_{x,w}) \simeq H_j(X_U) \tag{3.4}$$

for all $j \in \mathbb{N}$.

Proof. The isomorphism as vector spaces follows from the vector bundle result in Proposition 2(a). Now the action of $Z_P(x)$ and $Z_L(x)$ on $\mathcal{B}^M_{x,w}$ induce the same action of $A(x)$. Since the $Z_L(x)$-action commutes with the $\mathbb{C}^*$-action coming from λ, the action of $\ell \in Z_L(x)$ commutes with the map π_w defined in (2.2) and the result follows. □

We also need the following crucial result from [DCLP88]. This is proved by a reduction to distinguished nilpotent orbits, where the classical cases are handled by explicit computation and the exceptional groups are handled by a method that we review in §6.

Theorem 3.5 (Theorem 3.9 in [DCLP88]). *Let $U \in \mathfrak{I}d_2^{gen}$. Then $H_i(X_U) = 0$ for i odd.*

For $U \in \mathfrak{I}d_2^{gen}$, define a subset $W_{M,U}$ of W^0 and polynomial $g_{M,U}(q)$ by

$$\begin{aligned} W_{M,U} &= \{w \in W^0 \mid U = w.M \cap \mathfrak{g}_2\} \text{ and} \\ g_{M,U}(q) &= \sum_{w \in W_{M,U}} q^{t_M(w)}. \end{aligned} \tag{3.5}$$

Let $\mathcal{V}_{M,U}$ be a graded vector space whose Poincaré polynomial is $g_{M,U}(q^2)$. We consider $\mathcal{V}_{M,U}$ as an $A(x)$-module with trivial action.

Corollary 3.6. *We have*

$$H^*(\mathcal{B}_x^M) \simeq \bigoplus_{U \in \mathfrak{I}d_2^{gen}} \mathcal{V}_{M,U} \otimes H^*(X_U), \tag{3.6}$$

as $A(x)$-modules. In particular, $H^i(\mathcal{B}_x^M) = 0$ for i odd.

Proof. As in [DCLP88], $\mathcal{B}_x^M$ admits an α-partition of the nonempty $\mathcal{B}_{x,w}^M$. Since each $\mathcal{B}_{x,w}^M$ has no odd Borel-Moore homology by Theorem 3.5 and Corollary 3.4, the long exact sequence in Borel-Moore homology gives the isomorphism as vector spaces. Then the naturality of the long exact sequence and Corollary 3.4 yield the isomorphism as $A(x)$-modules. Since $\mathcal{B}_x^M$ and the X_U are projective varieties, the Borel-Moore homology and singular cohomology coincide, and the result follows. □

Remark 3.7. In fact, the varieties $\mathcal{B}_x^M$ satisfy Property (S) from §1.7 in [DCLP88]. Namely, switching to Borel-Moore integral homology, we have $H_i(\mathcal{B}_x^M) = 0$ for i odd, $H_i(\mathcal{B}_x^M)$ has no torsion for i even, and the cycle map from the i-th Chow group of $\mathcal{B}_x^M$ to $H_{2i}(\mathcal{B}_x^M)$ is an isomorphism. These results follow from the fact that X_U has Property (S) for all $U \in \mathfrak{I}d_2^{gen}$ as shown in [DCLP88] and Lemmas 1.8 and 1.9 in *loc. cit.*

Let

$$\mathsf{P}(X) = \sum_j \dim\left(H^{2j}(X)\right) q^j \tag{3.7}$$

denote the modified Poincaré polynomial of a variety X with no odd cohomology. If a group K acts on X and $\chi \in \mathrm{Irr}(K)$, let

$$\mathsf{P}(X;\chi) = \sum_j \left(\chi \colon H^{2j}(X)\right) q^j,$$

where $(\chi : \chi') = \dim \mathrm{Hom}_K(\chi, \chi')$ denotes the multiplicity of χ in the K-representation χ'. Then Corollary 3.6 immediately implies the following.

Corollary 3.8. *We have*

$$\mathsf{P}(\mathcal{B}_x^M) = \sum_{U \in \mathfrak{I}d_2^{gen}} g_{M,U}(q) \mathsf{P}(X_U).$$

and

$$\mathsf{P}(\mathcal{B}_x^M; \chi) = \sum_{U \in \mathfrak{I}d_2^{gen}} g_{M,U}(q) \mathsf{P}(X_U; \chi). \tag{3.8}$$

for all $\chi \in \mathrm{Irr}(A(x))$.

Remark 3.9. Corollary 3.8 gives a way to compute $\mathsf{P}(\mathcal{B}_x^M)$ and $\mathsf{P}(\mathcal{B}_x^M; \chi)$, which is feasible in small ranks. There are three steps: (1) determine the set $\mathfrak{I}d_2^{gen}$, (2) compute $\mathsf{P}(X_U; \chi)$ for each $U \in \mathfrak{I}d_2^{gen}$, and (3) compute $g_{M,U}$.

For (1), methods from Fenn's thesis [Fen08] apply and Fenn and the second author wrote computer code to do this, which works up to rank 10. For (3) the second author wrote code that is efficient up to rank 7 and can handle rank 8 for the cases where $M \in \mathfrak{I}d$. For (2), however, we only know how to do the computation in type A, small rank cases, and for certain nilpotent orbits in all cases. The method for doing this comes from [DCLP88], which we explain in §6. We have carried out this computation in several small rank cases and give the example of B_3 for $M \in \mathfrak{I}d$ in §8.

4. Proof of Theorem 1.1

We now restrict to the case where $M = I$ for $I \in \mathfrak{I}d$ and return to the map $\mu^I : G \times^B I \to \overline{\mathcal{O}_I}$ from the introduction. Pushing forward the shifted constant sheaf yields

$$R\mu_*^I(\underline{\mathbb{C}}[N_I]) \simeq \bigoplus_{(\mathcal{O},\mathcal{L})} IC(\overline{\mathcal{O}}, \mathcal{L}) \otimes V_{\mathcal{O},\mathcal{L}}^I. \tag{4.1}$$

where the sum is over pairs $(\mathcal{O}, \mathcal{L}) \in \Theta$ consisting of a nilpotent orbit $\mathcal{O} \subset \overline{\mathcal{O}}_I$ and an irreducible local system $\mathcal{L}$ on $\mathcal{O}$. The $V_{\mathcal{O},\mathcal{L}}^I$ are graded complex vectors spaces. In this section we will prove Theorem 1.1: if $V_{\mathcal{O},\mathcal{L}}^I \neq 0$ in (4.1), then $(\mathcal{O}, \mathcal{L})$ is an element of Θ_{sp}, the pairs that arise in the Springer correspondence (1.2).

Taking stalks in (4.1) and using proper base change, the cohomology of the fibers of μ^I and local intersection cohomology are related by

$$H^{j+N_I}(\mathcal{B}^I_x, \mathbb{C}) = \bigoplus_{\substack{(\mathcal{O},\mathcal{L}) \\ \mathcal{O}\subset\overline{\mathcal{O}}_I}} \mathcal{H}^j_x \left(IC(\overline{\mathcal{O}}, \mathcal{L}) \otimes V^I_{\mathcal{O},\mathcal{L}} \right) \tag{4.2}$$

for $x \in \overline{\mathcal{O}_I}$. We would like to write down the $A(x)$-equivariant version of this statement as discussed in [BM83] (see also [Ach21]).

For $x \in \mathcal{N}$ and $\chi \in \operatorname{Irr}(A(x))$, we sometimes write (x, χ) in place of $(\mathcal{O}_x, \mathcal{L}_\chi)$. Given $(x, \chi) \in \Theta$, we define the Laurent polynomial $m^I_{x,\chi}(q) \in \mathbb{N}[q, q^{-1}]$ so that the coefficient of q^j is the dimension of the j-th graded component of $V^I_{x,\chi}$. By the properties of perverse sheaves, we have $m^I_{x,\chi}(q^{-1}) = m^I_{x,\chi}(q)$. Let $(\mathcal{L}_\chi : -)$ denote the multiplicity of $\mathcal{L}_\chi$ in a local system on $\mathcal{O}_x$. For (x, χ) and (u, ϕ) in Θ, define

$$c^{u,\phi}_{x,\chi}(q) = \sum_{j\in\mathbb{Z}} \left(\mathcal{L}_\chi : \mathcal{H}^j IC(\overline{\mathcal{O}}_u, \mathcal{L}_\phi)|_{\mathcal{O}_x} \right) q^j.$$

Then taking the dimension on both sides of the $A(x)$-equivariant version of (4.2) gives

$$\mathsf{P}(\mathcal{B}^I_x; \chi)(q^2) = q^{N_I} \sum_{(u,\phi)\in\Theta} c^{u,\phi}_{x,\chi}(q) m^I_{u,\phi}(q). \tag{4.3}$$

Now suppose we know that $P(\mathcal{B}^I_x; \chi) = 0$ for some $\chi \in \operatorname{Irr}(A(x))$. Then since $c^{x,\chi}_{x,\chi} \neq 0$ and all the coefficients on the right side of (4.3) are nonnegative, we conclude that $m^I_{x,\chi} = 0$ and thus $V^I_{x,\chi} = 0$. Hence, the proof of Theorem 1.1 is reduced to showing that $\mathsf{P}(\mathcal{B}^I_x; \chi) = 0$ for $(x, \chi) \notin \Theta_{sp}$.

Next, by Corollary 3.8, if we can show that $\mathsf{P}(X_U; \chi) = 0$ for all $(x, \chi) \notin \Theta_{sp}$ and all $U \in \mathfrak{Id}^{gen}_2$, then it follows that $\mathsf{P}(\mathcal{B}^I_x; \chi) = 0$ for all $(x, \chi) \notin \Theta_{sp}$.

For the case of $I = \mathfrak{u}$, which is the Springer resolution case, there is a stronger equivariant statement. The Weyl group W acts on $H^*(\mathcal{B}_x)$ and this action commutes with the $A(x)$-action, so $W \times A(x)$ acts. For (x, χ) and (u, ϕ) in Θ_{sp},

$$\mathsf{P}(\mathcal{B}_x; V_{u,\phi} \otimes \chi)(q^2) = q^N c^{u,\phi}_{x,\chi}(q) \tag{4.4}$$

where $V_{u,\phi}$ is the irreducible W-representation corresponding to $(u, \phi) \in \Theta_{sp}$ in (1.2). Moreover, the left side (and the right side) will always be zero for those $(x, \chi) \notin \Theta_{sp}$. This was first established by Beynon–Spaltenstein by computer calculation in [BS84] for the exceptional groups and by Shoji in [Sho83] for the classical groups. Later, Lusztig gave a uniform framework that also included handling his Generalized Springer correspondence [Lus86]. The resulting algorithm from [Lus86, §24] is now known as the Lusztig–Shoji algorithm and it computes either side of (4.4) knowing only the partial order on the orbits in $\mathcal{N}$, the component groups $A(x)$, the Springer correspondence, and the character table of W.

Returning to the proof of Theorem 1.1, since the left side of (4.4) is zero for $(x,\chi) \notin \Theta_{sp}$, it follows that $P(\mathcal{B}_x;\chi) = 0$ for $(x,\chi) \notin \Theta_{sp}$. Therefore, if $(x,\chi) \notin \Theta_{sp}$ and $U \in \mathfrak{I}d_2^{gen}$, then $\mathsf{P}(X_U;\chi) = 0$ whenever X_U appears in the decomposition of $H^*(\mathcal{B}_x)$ in Corollary 3.6. We can therefore finish the proof of the theorem if we can show that X_U appears in the decomposition of $H^*(\mathcal{B}_x)$ for all $U \in \mathfrak{I}d_2^{gen}$. In other words, we will show $g_{\mathfrak{u},U} \neq 0$ for all $U \in \mathfrak{I}d_2^{gen}$, or equivalently, $W_{\mathfrak{u},U} \neq 0$. This amounts to showing that, for all $U \in \mathfrak{I}d_2^{gen}$, there exists $w \in W^0$ such that $U = w.\mathfrak{u} \cap \mathfrak{g}_2$.

By Lemma 3.2 there is a map $\Psi : W^0 \to \mathfrak{I}d_2$ defined by

$$\Psi(w) = w.\mathfrak{u} \cap \mathfrak{g}_2 \text{ for } w \in W^0. \tag{4.5}$$

In [DCLP88, §3.7(b)] it is shown that Ψ is surjective for the case when $\mathcal{O}_x$ is an even orbit, i.e., when $\mathfrak{g}_i = 0$ for i odd. We now extend that result to show Ψ is surjective for all nilpotent orbits.

Lemma 4.1. *Given $U \in \mathfrak{I}d_2$, there exists $w \in W^0$ such that $U = w.\mathfrak{u} \cap \mathfrak{g}_2$.*

Proof. Let $U \in \mathfrak{I}d_2$. It suffices to show that there exists $w \in W^0$ such that $\Phi_U = w(\Phi^+) \cap \Phi_2$. When $\mathcal{O}_x$ is even, [DCLP88, §3.7(b)] gives a construction of such a $w \in W^0$. Moreover, the w constructed is the unique element achieving the largest possible value for $\ell(w)$ among those $w \in W^0$ satisfying $\Phi_U = w(\Phi^+) \cap \Phi_2$.

Suppose now that $\mathcal{O}_x$ is not even. Consider the Lie subalgebra

$$\mathfrak{s} := \bigoplus_{i \in 2\mathbb{Z}} \mathfrak{g}_i.$$

Then $\mathfrak{s}$ is the centralizer in $\mathfrak{g}$ of an element of order 2 in G, namely, the image of $\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ under the map $SL_2(\mathbb{C}) \to G$ coming from the $\mathfrak{sl}_2$-triple defined in §2. Hence $\mathfrak{s}$ is a reductive subalgebra and its simple roots are a subset of the extended simple roots of $\mathfrak{g}$ (see [Car93]). Let $\Phi_{\mathfrak{s}}$ be the root system of $\mathfrak{s}$ with positive roots $\Phi_{\mathfrak{s}}^+ = \Phi^+ \cap \Phi_{\mathfrak{s}}$. Let $W_{\mathfrak{s}}$ denote the Weyl group of $\mathfrak{s}$ relative to $\mathfrak{t} \subset \mathfrak{s}$.

The $\mathfrak{sl}_2$-triple for x in $\mathfrak{g}$ lies in $\mathfrak{s}$ and so the W_0 defined relative to $\mathfrak{g}$ and $\mathfrak{s}$ coincide. Since $U \subset \mathfrak{g}_2 \subset \mathfrak{s}$ and $\mathcal{O}_x \cap \mathfrak{s}$ is an even orbit in $\mathfrak{s}$, where the lemma holds, there exists $w \in W^0 \cap W_{\mathfrak{s}}$ such that $w(\Phi^+_{\mathfrak{s}}) \cap \Phi_2 = \Phi_U$.

Now let $\sigma \in W$ be any element satisfying $\sigma^{-1}(\Phi^+_{\mathfrak{s}}) \subset \Phi^+$. Then $\Phi^+_{\mathfrak{s}} \subset \sigma(\Phi^+) \cap \Phi_{\mathfrak{s}}$ and so $\sigma(\Phi^+) \cap \Phi_{\mathfrak{s}} = \Phi^+_{\mathfrak{s}}$. We claim that

$$w\sigma(\Phi^+) \cap \Phi_2 = \Phi_U.$$

Indeed, let $\beta \in \Phi_2$. Then $\beta \in w\sigma(\Phi^+)$ if and only if $w^{-1}(\beta) \in \sigma(\Phi^+)$. Since $w \in W_{\mathfrak{s}}$ and $\Phi_2 \subset \Phi_{\mathfrak{s}}$, we have $w^{-1}(\Phi_2) \subset \Phi_{\mathfrak{s}}$. Hence, $w^{-1}(\beta) \in \sigma(\Phi^+)$ if and only if

$$w^{-1}(\beta) \in \sigma(\Phi^+) \cap \Phi_{\mathfrak{s}} = \Phi^+_{\mathfrak{s}}.$$

This shows that $\beta \in w\sigma(\Phi^+)$ if and only if $\beta \in w(\Phi^+_{\mathfrak{s}})$, proving that $w\sigma(\Phi^+) \cap \Phi_2 = w(\Phi^+_{\mathfrak{s}}) \cap \Phi_2$, as desired. Finally, we may take σ to be the identity of W to see that w itself satisfies $\Phi_U = w(\Phi^+) \cap \Phi_2$. $\square$

The Lemma applies in particular to $U \in \mathfrak{I}d_2^{gen}$, completing the proof of Theorem 1.1. We can now write (4.1) as

$$R\mu^I_*(\underline{\mathbb{C}}[N_I]) \simeq \bigoplus_{(\mathcal{O},\mathcal{L}) \in \Theta_{sp}} IC(\overline{\mathcal{O}}, \mathcal{L}) \otimes V^I_{\mathcal{O},\mathcal{L}}. \tag{4.6}$$

Remark 4.2. The proof of Lemma 4.1 shows that for each $w \in W^0 \cap W_{\mathfrak{s}}$ such that $U = \Psi(w)$, we have $\Psi(w\sigma) = \Psi(w)$ where σ runs over the right coset representatives of $W_{\mathfrak{s}}$ in W that lie in W^0. It follows from Corollary 3.8 for the Springer fiber case where $M = \mathfrak{u}$ that the index $[W : W_{\mathfrak{s}}]$ divides the Euler characteristic of $\mathcal{B}_x$ when $\mathcal{O}_x$ is not an even orbit.

For example, in type E_7 there are 3 involutions in G, up to conjugacy. Curiously, among all non-even orbits, only the involution for which $\mathfrak{s}$ is of type $D_6 \times A_1$ occurs for the $\mathfrak{s}$ in the proof of Lemma 4.1. The Weyl group of the standard $D_6 \times A_1$ is exactly the stabilizer of the line through the highest root, showing that $[W : W_{\mathfrak{s}}] = 63$, the number of positive roots in E_7. We deduce that the Euler characteristic of any Springer fiber of a non-even nilpotent element in E_7 is divisible by 63, which was observed by Fenn.

It will be convenient sometimes to use a parametrization indexed by $\mathrm{Irr}(W)$ instead of Θ_{sp}. For each $\varphi \in \mathrm{Irr}(W)$, we can write $\varphi = \varphi_{\mathcal{O},\mathcal{L}}$ for a unique $(\mathcal{O},\mathcal{L}) \in \Theta_{sp}$ by the Springer correspondence (1.2). We sometimes write m^I_φ or V^I_φ in place of $m^I_{x,\chi}$ or $V^I_{\mathcal{O}_x,\mathcal{L}_\chi}$, respectively.

Consider the partial order on $\mathrm{Irr}(W)$ by $\varphi_{\mathcal{O}',\mathcal{L}'} \preceq \varphi_{\mathcal{O},\mathcal{L}}$ if $\mathcal{O}' \subseteq \overline{\mathcal{O}}$. We choose a linear ordering on $\mathrm{Irr}(W)$ that respects this partial order. Define the matrix $\mathbf{K}$ to have entries coming from the left side of (4.4):

$$K_{\varphi,\varphi'} = \sum_j \left(\varphi \otimes \chi : H^{2j}(\mathcal{B}_x)\right) q^j$$

where $\varphi' = \varphi_{\mathcal{O}_x,\mathcal{L}_\chi}$. The entries of $\mathbf{K}$ lie in $\mathbb{N}[q]$ and $\mathbf{K}$ is lower triangular, with powers of q along the diagonal. The matrix $\mathbf{K}$ is computable by the Lusztig–Shoji algorithm [Lus86, §24]. The entries of $\mathbf{K}$ are sometimes referred to as the Green functions of G. In type A, they coincide with modified Kostka–Foulkes polynomials [GP92].

Set $c_I = \dim \mathfrak{u} - \dim I$, the codimension of I in $\mathfrak{u}$. Then since $c_I = N - N_I$, we can rewrite Equation (4.3) as

$$q^{2c_I}\, \mathsf{P}(\mathcal{B}_x^I;\chi)(q^2) = \sum_{\varphi \in \mathrm{Irr}(W)} q^{c_I} m_\varphi^I(q) \mathbf{K}_{\varphi,\varphi'}(q^2). \tag{4.7}$$

where $\varphi' = \varphi_{\mathcal{O}_x,\mathcal{L}_\chi}$. Since the other expressions in (4.7) involve even exponents, it follows that the exponents of $q^{c_I} m_\varphi^I(q)$ are also even, allowing us to make the following definition.

Definition 4.3. For $\varphi \in \mathrm{Irr}(W)$ and $I \in \mathfrak{I}d$, define $f_\varphi^I(q)$ by $f_\varphi^I(q^2) = q^{c_I} m_\varphi^I(q)$.

Proposition 4.4. *We have $f_\varphi^I \in \mathbb{N}[q]$ and its coefficients are symmetric about $q^{c_I/2}$.*

Proof. The highest power of q on the left in (4.7) is $2c_I + 2\dim \mathcal{B}_x^I$, each term in the sum is bounded by this value. In particular this holds for $\varphi' = \varphi$. In that case, $\mathbf{K}_{\varphi',\varphi'}(q^2) = q^{2\dim(\mathcal{B}_x)}$ and so

$$2\deg(f_{\varphi'}^I) + 2\dim(\mathcal{B}_x) \leq 2c_I + 2\dim \mathcal{B}_x^I$$

Since $\mathcal{B}_x^I \subset \mathcal{B}_x$ and so $\dim \mathcal{B}_x^I \leq \dim \mathcal{B}_x$, we get $\deg(f_{\varphi'}^I) \leq c_I$. Hence, the exponents of $m_{\varphi'}^I$ are bounded by c_I and therefore below by $-c_I$ by the $q \to q^{-1}$ symmetry of $m_{\varphi'}^I$. Hence, $f_{\varphi'}^I \in \mathbb{N}[q]$ and its coefficients are symmetric about $q^{c_I/2}$. ∎

The polynomials f_φ^I, or their transformations by a fixed matrix independent of I, make several appearances in the rest of this paper.

The results in this section where obtained in [BM83] in the parabolic setting, i.e., for $I = \mathfrak{u}_P$ where P is any parabolic subgroup of G. Borho and MacPherson also did more, giving a formula for $f_\varphi^{\mathfrak{u}_P}$. Namely,

$$f_\varphi^{\mathfrak{u}_P} = \mathsf{P}(P/B)(\varphi : \mathrm{Ind}_{W_P}^W(\mathrm{sgn})). \tag{4.8}$$

where $\mathrm{sgn} \in \mathrm{Irr}(W)$ denotes the sign representation of W and W_P denotes the Weyl group of P.

Remark 4.5. Although less efficient than the Lusztig–Shoji algorithm, our methods give an inductive way to compute $\mathbf{K}$ while also computing m_φ^I, or equivalently f_φ^I. The induction starts at the zero orbit and moves up in the partial order on Θ_{sp}. The procedure relies on the dimension constraints for the IC-sheaves and the symmetry of the coefficients of m_φ^I. We need to know $H^*(\mathcal{B}_x^I)$ for all x and enough I and then we can use (4.7). This inductive process is analogous to the usual method for computing the Kazhdan–Lusztig polynomials, i.e., the IC stalks of Schubert varieties, using the Bott–Samelson resolutions.

5. Generalized Grothendieck–Springer setting

In this section we prove a conjecture of Brosnan and show that the polynomials f_φ^I from Definition 4.3 control the decomposition of the pushforward of the shifted constant sheaf when we move from the setting of $I \in \mathfrak{Id}$ to that of $H \in \mathcal{H}$.

5.1. Brosnan's conjecture

Let $H \in \mathcal{H}$ and consider the map

$$\mu^H : G \times^B H \to \mathfrak{g}, (g, x) \mapsto g.x. \tag{5.1}$$

This map is proper and since $\mathfrak{b} \subseteq H$, the image of μ_H is $\mathfrak{g}$. Let $N_H = \dim G/B + \dim H$, the dimension of the smooth variety $G \times^B H$.

When $H = \mathfrak{b}$, this map is the Grothendieck–Springer resolution, which Lusztig [Lus81] showed was a small map. In particular, $N_b = \dim \mathfrak{g}$. Since the map is small, $R\mu_*^{\mathfrak{b}}(\underline{\mathbb{C}}[N_{\mathfrak{b}}])$ decomposes into a sum of irreducible perverse sheaves on $\mathfrak{g}$ with maximal support. More precisely

$$R\mu_*^{\mathfrak{b}}(\underline{\mathbb{C}}[N_{\mathfrak{b}}]) = \bigoplus_{\varphi \in \mathrm{Irr}(W)} IC(\mathfrak{g}, \mathcal{M}_\varphi) \otimes \varphi \tag{5.2}$$

where $\mathcal{M}_\varphi$ is the irreducible local system supported on the regular semisimple elements $\mathfrak{g}_{rs}$ of $\mathfrak{g}$ corresponding to $\varphi \in \mathrm{Irr}(W)$ [Ach21, Lemma 8.2.5].

We wish to generalize Equation (5.2) to any $H \in \mathcal{H}$ and to compute the sheaf $R\mu_*^H(\underline{\mathbb{C}}_H[N_H])$, as we did for the case of $I \in \mathfrak{I}d$. Recall in that situation, as a consequence of Theorem 1.1 that (4.1) becomes

$$R\mu_*^I(\underline{\mathbb{C}}[N_I]) \simeq \bigoplus_{(\mathcal{O},\mathcal{L})\in\Theta_{sp}} IC(\overline{\mathcal{O}},\mathcal{L}) \otimes V^I_{\mathcal{O},\mathcal{L}}, \tag{5.3}$$

where $V^I_{\mathcal{O},\mathcal{L}}$ is a $\mathbb{Z}$-graded vector space. For $\varphi \in \mathrm{Irr}(W)$, we will write V^I_φ for $V^I_{\mathcal{O},\mathcal{L}}$ where $\varphi = \varphi_{\mathcal{O},\mathcal{L}}$ under the Springer correspondence.

Our result is the following theorem, originally conjectured by Brosnan (see [Xue20, Conjecture 5.2.2] and [VX21]).

Theorem 5.1. *Let $H \in \mathcal{H}$. Let $I = H^\perp$, the annihilator of H under the Killing form. There is an isomorphism*

$$R\mu_*^H(\underline{\mathbb{C}}_H[N_H]) \simeq \bigoplus_{\varphi\in Irr(W)} IC(\mathfrak{g},\mathcal{M}_\varphi) \otimes V^I_{\varphi\otimes sgn} \tag{5.4}$$

in the derived category of G-equivariant perverse sheaves on $\mathfrak{g}$. In particular, every simple summand of $R\mu_*^H(\underline{\mathbb{C}}_H[N_H])$ is a simple perverse sheaf on $\mathfrak{g}$ with full support.

Proof. To prove the result, we apply the Fourier transform (see [Ach21, Cor. 6.9.14]) to obtain

$$\mathfrak{F}(R\mu_*^I\underline{\mathbb{C}}_I[N_I]) = R\mu_*^H(\underline{\mathbb{C}}_H[N_H]), \tag{5.5}$$

where $\mathfrak{F}$ maps each simple summand of $R\mu_*^I(\underline{\mathbb{C}}_I[N_I])$ in (5.3) to a simple summand of $R\mu_*^H(\underline{\mathbb{C}}_H[N_H])$. Next, for $(\mathcal{O},\mathcal{L}) \in \Theta_{sp}$, we have

$$\mathfrak{F}(IC(\overline{\mathcal{O}},\mathcal{L})) = IC(\mathfrak{g},\mathcal{M}_{\varphi\otimes\mathrm{sgn}}) \text{ where } \varphi = \varphi_{\mathcal{O},\mathcal{L}}$$

(see Sections 8.2, 8.3, and specifically equation (8.3.2) of [Ach21]). The result follows. $\square$

Theorem 5.1 generalizes [BC20, Theorem 3.6] to all Lie types.

5.2. Monodromy action

Let $s \in \mathfrak{t}$ be a regular semisimple element. The variety $\mathcal{B}_s^H$ defined as in (3.1) is called a regular semisimple Hessenberg variety. There is an action of W on

$H^*(\mathcal{B}_s^H)$ arising from monodromy [BC18], [BC20]. Then taking stalks in (5.4) at s and using W-equivariant proper base change gives

$$H^{*+N_H-\dim \mathfrak{g}}(\mathcal{B}_s^H) \simeq \bigoplus_{\varphi \in \mathrm{Irr}(W)} \varphi \otimes V_{\varphi\otimes \mathrm{sgn}}^{H^\perp} \tag{5.6}$$

as W-modules. Here, φ is in degree 0 on the right and carries the W-action. We have used that $\mathcal{H}_s^j IC(\mathfrak{g}, \mathcal{M}_\varphi)$ is zero, except for $j = -\dim \mathfrak{g}$, where it equals φ.

Now $N_H - \dim \mathfrak{g} = \dim H - \dim \mathfrak{b}$, which is the dimension of $\mathcal{B}_s^H$. We deduce that for $\varphi \in \mathrm{Irr}(W)$ that

$$\mathsf{P}(\mathcal{B}_s^H; \varphi)(q^2) = q^{\dim H - \dim \mathfrak{b}} m_{\varphi\otimes \mathrm{sgn}}^{H^\perp}(q). \tag{5.7}$$

Since $\dim H - \dim \mathfrak{b} = \dim \mathfrak{u} - \dim H^\perp = c_{H^\perp}$, we immediately have

Proposition 5.2. *For $H \in \mathcal{H}$ and $\varphi \in \mathrm{Irr}(W)$, we have $\mathsf{P}(\mathcal{B}_s^H; \varphi) = f_{\varphi\otimes sgn}^{H^\perp}$.*

Note that Proposition 5.2 gives another proof that f_φ^I is a polynomial, while Proposition 4.4 gives a (new) proof that $\mathsf{P}(\mathcal{B}_s^H; \varphi)$ is palindromic.

5.3. Nilpotent Hessenberg varieties

We can also take the stalks in (5.4) at elements of $\mathcal{N}$ and get an interesting result. First, we need the fact that

$$IC(\mathfrak{g}, \mathcal{M}_\varphi)|_{\mathcal{N}}[-\dim \mathfrak{t}] \simeq IC(\overline{\mathcal{O}}, \mathcal{L})$$

where $\varphi = \varphi_{\mathcal{O},\mathcal{L}}$ (for example, see [Ach21, Lemma 8.3.5]). Next, $N_H - N = \dim H - \dim \mathfrak{u}$. But with the extra shift by $\dim \mathfrak{t}$ from above, we get the analogue of (4.7).

Proposition 5.3. *For $x \in \mathcal{N}$ and $\chi \in \mathrm{Irr}(A(x))$, we have*

$$\mathsf{P}(\mathcal{B}_x^H; \chi) = \sum_{\varphi \in \mathrm{Irr}(W)} f_{\varphi\otimes sgn}^{H^\perp} \mathbf{K}_{\varphi,\varphi'} \tag{5.8}$$

where $\varphi' = \varphi_{\mathcal{O}_x, \mathcal{L}_\chi}$.

Proposition 5.3 was known in the parabolic case. In [BM83], Borho and MacPherson consider the restriction $\mu_{\mathcal{N}}^H$ of μ^H to $X_{\mathcal{N}} = G \times^B (H \cap \mathcal{N})$. Since the intersection $\mathfrak{p} \cap \mathcal{N}$ is rationally smooth for $H = \mathfrak{p}$, they obtained the stronger statement

$$R(\mu_{\mathcal{N}}^{\mathfrak{p}})_*(\underline{\mathbb{C}}[\dim X_{\mathcal{N}}]) \simeq \bigoplus_{(\mathcal{O},\mathcal{L}) \in \Theta_{sp}} IC(\overline{\mathcal{O}}, \mathcal{L}) \otimes V_{\varphi_{\mathcal{O},\mathcal{L}} \otimes \mathrm{sgn}}^{\mathfrak{u}_P}. \tag{5.9}$$

We suspect that (5.9) holds for all $H \in \mathcal{H}$, presumably because $X_{\mathcal{N}}$ is rationally smooth for all $H \in \mathcal{H}$. In type A, Proposition 5.3, or perhaps the stronger version in (5.9), is an unpublished theorem of Tymoczko and MacPherson [Obe19, pg. 2882].

Finally, we note that when x is regular nilpotent, the only term in (5.8) for which $\mathbf{K}_{\varphi,\varphi'}$ is nonzero occurs when φ is the trivial representation, in which case the value is 1. Hence $f_{\mathrm{sgn}}^{H^\perp} = \mathsf{P}(\mathcal{B}_x^H)$. When x is regular nilpotent, $\mathsf{P}(\mathcal{B}_x^H)$ has a formula as a product of q-numbers depending only on the roots $\Phi_H \cap \Phi^-$ [AHM+20], [ST06].

5.4. The dot action and LLT representations

In [Tym08], Tymoczko defined a Weyl group representation on the ordinary cohomology of regular semisimple Hessenberg varieties, called the dot action. The representation of W on $H^*(\mathcal{B}_s^H)$ from §5.2 was shown to coincide with Tymoczko's dot action representation by Brosnan and Chow in [BC18] in type A, and their proof was adapted to all Lie types by Bălibanu and Crooks in [BC20].

The dot action is closely related to another W-representation via a tensor product formula, which first arose in Procesi's study of the toric variety associated to the Weyl chambers from [Pro90] and was later treated by Guay–Paquet in type A [GP16]. We summarize the key properties we need here and include the details in Appendix A.

Let $\mathcal{C}$ denote the coinvariant algebra of W, with the reflection representation in degree 1. Then $\mathcal{C} \simeq H^*(G/B)$ as graded W-representations, up to the doubling of degrees. If $g(q) = \sum a_i q^i$ is a polynomial, we regard it as a graded representation consisting of a_i copies of the trivial representation in degree a_i. The following result is a direct generalization of [Pro90, Theorem 2] and [GP16, Lemma 168].

Proposition 5.4. *For each Hessenberg space $H \in \mathcal{H}$ there exists a unique graded W-representation LLT_H satisfying*

$$\mathsf{P}(G/B) \otimes \mathrm{LLT}_H \simeq \mathcal{C} \otimes H^*(\mathcal{B}_s^H). \tag{5.10}$$

Furthermore, LLT_H is nonzero only in nonnegative degrees.

We call LLT_H the LLT representation; the reason for this terminology is that, in Type A, the Frobenius characteristic of LLT_H is a unicellular LLT polynomial (see Corollary 7.9 below).

The coinvariant algebra $\mathcal{C}$ carries the regular representation and the tensor product of any representation of dimension d with the regular representation is the direct sum of d copies of the regular representation. Therefore when $q = 1$ both sides of (5.10) are isomorphic to $|W|$ copies of the regular representation. Thus LLT_H is a graded version of the regular representation. At the same time, forgetting the W-actions, $H^*(\mathcal{B}_s^H)$ and LLT_H coincide as graded vector spaces since $\mathsf{P}(G/B)$ measures the dimension of the components of $\mathcal{C}$.

To compute LLT_H from $H^*(\mathcal{B}_s^H)$ requires knowing the matrix $\widetilde{\boldsymbol{\Omega}}$ with entries

$$\widetilde{\boldsymbol{\Omega}}_{\varphi,\varphi'} = \sum_j \left(\varphi \otimes \varphi' : \mathcal{C}^j\right) q^j.$$

The matrix $\widetilde{\boldsymbol{\Omega}}$ is closely related to the matrix needed as input to the Lusztig–Shoji algorithm. Define polynomials $g_\varphi^H(q)$ by

$$g_\varphi^H(q) = \sum_j (\varphi : LLT_H) q^{2j}.$$

Then

$$\mathsf{P}(G/B) g_\varphi^H = \sum_{\varphi' \in \mathrm{Irr}(W)} P(\mathcal{B}_s^H; \varphi') \widetilde{\boldsymbol{\Omega}}_{\varphi,\varphi'} = \sum_{\varphi' \in \mathrm{Irr}(W)} f_{\varphi'}^{H^\perp} \widetilde{\boldsymbol{\Omega}}_{\varphi,\varphi'}. \tag{5.11}$$

We have used Proposition 5.2 and the fact that

$$\mathrm{Hom}_W(\varphi \otimes \varphi', \mathcal{C}^j) \simeq \mathrm{Hom}_W(\varphi, \varphi' \otimes \mathcal{C}^j)$$

since representations $\varphi \in \mathrm{Irr}(W)$ satisfy $\varphi^* \simeq \varphi$.

For §7 we need to know the $H^*(\mathcal{B}_s^H)$ and LLT_H in the parabolic case, i.e., when $H = \mathfrak{p}$ the Lie algebra of the parabolic subgroup P. Let $\mathcal{C}_P$ be the coinvariant algebra of W_P as a Coxeter group, a graded representation of W_P.

Proposition 5.5. *In the parabolic setting, we have*

1. $H^*(\mathcal{B}_s^{\mathfrak{p}}) \simeq \mathsf{P}(P/B) \otimes \mathrm{Ind}_{W_P}^W(\mathbf{1})$.
2. $LLT_{\mathfrak{p}} \simeq \mathrm{Ind}_{W_P}^W(\mathcal{C}_P)$.

where the modules on the left are zero for odd degrees and the component in degree $2i$ on the left matches the one in degree i on the right.

Proof. Part (1) follows directly from Equation (4.8) and Proposition 5.2.

For (2), we have the isomorphism $\mathcal{C}_P \simeq H^*(P/B)$, as W_P-representations, up to doubling of degrees. Then the fiber bundle of G/B over G/P with fiber P/B gives rise to the isomorphism

$$H^*(G/B) \simeq H^*(G/P) \otimes H^*(P/B)$$

as W_P-representations where the action on $H^*(G/P)$ is trivial. Hence, $\mathcal{C} \simeq \mathsf{P}(G/P)\mathcal{C}_P$ as W_P-representations. Inducing up to W, we have

$$\mathrm{Ind}_{W_P}^W(\mathcal{C}) \simeq \mathsf{P}(G/P) \otimes \mathrm{Ind}_{W_P}^W(\mathcal{C}_P)$$

as W-modules, or equivalently,

$$\mathcal{C} \otimes \mathrm{Ind}_{W_P}^W(\mathbf{1}) \simeq \mathsf{P}(G/P) \otimes \mathrm{Ind}_{W_P}^W(\mathcal{C}_P)$$

since $\mathcal{C}$ is a W-representation. Now using Part (1) and the fact that $\mathsf{P}(G/B) = \mathsf{P}(G/P)\mathsf{P}(P/B)$, the result follows. □

6. The modular law

Let $M \subseteq \mathfrak{g}$ be a B-invariant subspace. As mentioned in Remark 3.9, the results of §3 give a method to compute the isotypic component $H^*(\mathcal{B}_x^M)^\chi$ if we can compute the isotypic component $H^*(X_U)^\chi$ for all $U \in \mathfrak{Id}_2^{gen}$, where $\mathfrak{Id}_2^{gen}$ is defined as in §3.2 using the grading induced by x.

In [DCLP88], a method is given to compute $H^*(X_U)^\chi$ that works for all distinguished nilpotent elements in the exceptional groups. Although it does not work in general, it is a powerful technique. In this section we prove a generalization of this method: for certain triples of $I_2 \subset I_1 \subset I_0$ of ideals in $\mathfrak{Id}$, knowing any two of the $H^*(\mathcal{B}_x^{I_i})^\chi$ determines the third. We call this relation the geometric modular law.

We were led to this generalization after seeing the type A combinatorial version in [AN21a] and [GP13], where the relation is known as the modular law. In Proposition 7.7 below, we show that our geometric modular law implies the combinatorial one.

6.1. The basic move

We first define a relation on ideals $I_1, I_0 \in \mathfrak{Id}$ as in [FS20].

Definition 6.1. Two ideals $I_1, I_0 \in \mathfrak{Id}$ are related by the **basic move** if $I_0 = I_1 \oplus \mathfrak{g}_\beta$ for $\beta \in \Phi^+$ and there exists $\alpha \in \Delta$ such that

1. $\langle \beta, \alpha^\vee \rangle = -1$, and
2. The set Φ_{I_0} is invariant under the simple reflection $s_\alpha \in W$.

The second condition is equivalent to I_0 being P_α-stable, where P_α is the parabolic subgroup containing B corresponding to α.

If ideals $I_1 \subset I_0$ satisfy $I_0 = I_1 \oplus \mathfrak{g}_\beta$, then β is a minimal root in Φ_{I_0} under the partial order on positive roots. If I_1 and I_0 are related by the basic move, then condition (1) implies that $s_\alpha(\beta) = \alpha + \beta$, which is a positive root bigger than β in the partial order; hence, $\alpha+\beta \in \Phi_{I_1}$. In fact, $\alpha+\beta$ is a minimal root of Φ_{I_1}. Suppose otherwise; then Φ_{I_1} contains a positive root $\gamma = \alpha + \beta - \alpha'$ for a simple root α'. Now, $\alpha \neq \alpha'$ since $\beta \notin \Phi_{I_1}$. Since $\langle \alpha', \alpha^\vee \rangle \leq 0$ for any distinct simple roots and $\langle \alpha + \beta, \alpha^\vee \rangle = 1$ by condition (1), then $\langle \gamma, \alpha^\vee \rangle \geq 1$. But then $s_\alpha(\gamma) < \beta$. Since $\gamma \in \Phi_{I_1} \subset \Phi_{I_0}$ and

$$s_\alpha(\Phi_{I_0}) = \Phi_{I_0}$$

by condition (2), this means $s_\alpha(\gamma) \in \Phi_{I_0}$ and we obtain a contradiction to β being a minimal root in Φ_{I_0}.

Thus we can define $I_2 \in \mathfrak{Id}$ to be the subspace satisfying $I_1 = I_2 \oplus \mathfrak{g}_{\alpha+\beta}$. It is clear that I_2 is P_α-stable. We call $I_2 \subset I_1 \subset I_0$ a **modular triple**, or just a **triple**. These triples were first constructed in [DCLP88, §2.7].

Since $P_\alpha.I_1 = I_0$, we have $\mathcal{O}_{I_1} = \mathcal{O}_{I_0}$, while the orbit $\mathcal{O}_{I_2}$ need only satisfy $\mathcal{O}_{I_2} \subset \overline{\mathcal{O}}_{I_0}$.

Example 6.2. Consider the A_3 root system with $\Delta = \{\alpha_1, \alpha_2, \alpha_3\}$, where α_1 and α_3 orthogonal. Let $I_0 \in \mathfrak{Id}$ be such that Φ_{I_0} has one minimal root $\beta = \alpha_2$. Then β satisfies condition (1) with respect to either α_1 or α_3. Hence I_1 and I_0 are related by the basic move where $\Phi_{I_1} = \{\alpha_1+\alpha_2, \alpha_2+\alpha_3, \alpha_1+\alpha_2+\alpha_3\}$. But there are two different triples that arise: I_2 will satisfy $I_1 = I_2 \oplus \mathfrak{g}_{\alpha_1+\alpha_2}$ or $I_1 = I_2 \oplus \mathfrak{g}_{\alpha_2+\alpha_3}$ depending on whether $\alpha = \alpha_1$ or $\alpha = \alpha_3$, respectively.

6.2. Geometric modular law for the cohomology of fibers

Whenever three subspaces form a modular triple, they satisfy the three conditions in §2.7 of [DCLP88] with $U'' \subset U \subset U'$ in place of $I_2 \subset I_1 \subset I_0$ and $M = G$, $H = B$, and $P = P_\alpha$.

Fix $x \in \mathcal{N}$ and let $X_i = \mathcal{B}_x^{I_i}$. When $x \in \mathcal{O}_{I_0}$, it is shown in [DCLP88, Lemmas 2.2] that the X_i are smooth (X_2 can be empty), as was noted in §3.2. In [DCLP88, Lemma 2.11] it is proved that there is a geometric relationship among the three varieties, which we now show holds for all $x \in \mathcal{N}$, not just for $x \in \mathcal{O}_{I_0}$, and from this we deduce Proposition 1.2. Since the X_i are no longer smooth in general, our argument relies on the fact that $\mathcal{B}_x^I$ has no odd homology for any $x \in \mathcal{N}$ by Corollary 3.6. We are now ready to prove the geometric modular law, which we restate here for the reader's convenience.

Proposition 1.2 (The geometric modular law)**.** Let $I_2 \subset I_1 \subset I_0$ be a modular triple and let $x \in \mathcal{N}$. Write X_i for $\mathcal{B}_x^{I_i}$. Then

$$H^j(X_1) \oplus H^{j-2}(X_1) \simeq H^j(X_0) \oplus H^{j-2}(X_2) \text{ for all } j \in \mathbb{Z}. \tag{6.1}$$

Proof. We again use Borel-Moore homology until the final step. By Corollary 3.6, the odd homology of all X_i vanish, so we need only consider j even. Let $\alpha \in \Delta$ from the definition of the modular triple. Consider

$$Z = \{(gB, g'B) \in G/B \times G/B \mid g^{-1}.x \in I_1 \text{ and } g^{-1}g' \in P_\alpha\},$$

as in Lemma 2.11 in [DCLP88]. Then Z is a $\mathbb{P}^1$-bundle over X_1 by forgetting the second factor. Since X_1 has no odd homology, and neither does $\mathbb{P}^1$, the Leray spectral sequence degenerates and yields

$$H_j(Z) \simeq H_j(X_1) \oplus H_{j-2}(X_1) \text{ for all } j \in \mathbb{N}. \tag{6.2}$$

Next, as in *loc cit*, the variety Z maps to a variety Z', which is $\mathbb{P}^1$-bundle over X_0, by sending $(gB, g'B) \in Z$ to $(gB, g'B) \in Z'$, where

$$Z' := \{(gB, g'B) \in G/B \times G/B \mid g'^{-1}.x \in I_0 \text{ and } g^{-1}g' \in P_\alpha\}.$$

This works since $g' = gp$ for some $p \in P_\alpha$, so $g'^{-1}.x = p^{-1}g^{-1} \in I_0$ if $g^{-1}.x \in I_1$ because $P_\alpha.I_1 = I_0$. Inside Z' consider the subvariety Y consisting of those $(gB, g'B) \in Z'$ where the line $gI_1/g'I_2 \subset g'I_0/g'I_2$ contains x. Then Z is isomorphic to Y.

At the same time, Y maps surjectively to X_0 and the pre-image of X_2 is a $\mathbb{P}^1$-bundle E over X_2. The complement of E in Y is isomorphic to $X_0 \backslash X_2$. In [DCLP88], the proof ends since X_2 and X_0 are smooth and one knows the singular cohomology of Y, which is a blow-up of X_0 over X_2. To get around the lack of smoothness in the general case, we again use that the X_i have no odd Borel-Moore homology.

First, $H_j(E) \simeq H_j(X_2) \oplus H_{j-2}(X_2)$ for all j as in (6.2) since X_2 has no odd homology. Next, when j is even, there are two long exact sequences in Borel-Moore homology. For $X_2 \subset X_0$, we have

$$0 \to H_{j+1}(X_0 \backslash X_2) \to H_j(X_2) \to H_j(X_0) \to H_j(X_0 \backslash X_2) \to 0$$

and for $E \subset Y$, we have

$$0 \to H_{j+1}(Y \backslash E) \to H_j(E) \to H_j(Y) \to H_j(Y \backslash E) \to 0$$

since E has no odd homology. Now, $Y \backslash E \simeq X_0 \backslash X_2$ and so it follows that

$$H_j(Y) \oplus H_j(X_2) \simeq H_j(E) \oplus H_j(X_0)$$

as $\mathbb{Q}$-vector spaces. Thus $H_j(Y) \simeq H_j(X_0) \oplus H_{j-2}(X_2)$. The result follows from (6.2) and the isomorphism $Z \simeq Y$. We can switch back to singular cohomology since the X_i are projective varieties. $\square$

6.3. Implications of the modular law

It follows from Proposition 1.2 that if $I_2 \subset I_1 \subset I_0$ is a triple of ideals, then

$$(1+q)\mathsf{P}(\mathcal{B}_x^{I_1};\chi) = \mathsf{P}(\mathcal{B}_x^{I_0};\chi) + q\mathsf{P}(\mathcal{B}_x^{I_2};\chi). \tag{6.3}$$

This leads us to define

Definition 6.3. A collection of objects $\{F^I\}_{I \in \mathfrak{I}d}$ each carrying an action of $\mathbb{Q}(q)$ is said to **satisfy the modular law** if

$$(1+q)F^{I_1} = F^{I_2} + qF^{I_0} \tag{6.4}$$

whenever $I_2 \subset I_1 \subset I_0$ is a triple of ideals.

The F^I could be polynomials or Laurent polynomials in $\mathbb{Q}(q)$ or a vector of such polynomials indexed by $\mathrm{Irr}(W)$ or, equivalently, Θ_{sp}. We could also take F^I to be a graded representation of $\mathrm{Irr}(W)$, which in type A amounts to a symmetric function with coefficients in $\mathbb{Q}(q)$.

Notice that we have reversed the role of I_0 and I_2 in (6.4) as compared to (6.3).

Remark 6.4. If $\{F_1^I\}$ and $\{F_2^I\}$ satisfy the modular law, so does $\{aF_1^I + bF_2^I\}$ for any $a, b \in \mathbb{Q}(q)$.

Proposition 6.5. *The following polynomials satisfy the modular law in* (6.4).

1. $\mathsf{P}(\mathcal{B}_x^I;\chi)(q^{-1})$ *for* $(x,\chi) \in \Theta_{sp}$.
2. $f_\varphi^I(q)$ *for* $\varphi \in \mathrm{Irr}(W)$.
3. $g_\varphi^{I^\perp}(q)$ *for* $\varphi \in \mathrm{Irr}(W)$.
4. $\mathsf{P}(\mathcal{B}_x^{I^\perp};\chi)(q)$ *for* $(x,\chi) \in \Theta_{sp}$.

Proof. Statement (1) follows (6.3) by substituting q^{-1} for q and then multiplying by q.

For (2), the determinant of $\mathbf{K}$ equals q^m for some $m \in \mathbb{N}$. So $\mathbf{K}$ is invertible (and in fact the entries of $q^m\mathbf{K}^{-1}$ are polynomials). We want to convert (4.7) into a matrix equation. To that end, we construct vectors $(\mathsf{P}(\mathcal{B}_x^I;\chi))$ and $\left(q^{-c_I} f_\varphi^I\right)$ using the linear order on Θ_{sp} and $\mathrm{Irr}(W)$, respectively, from §4. Then (4.7) becomes the matrix-vector equation

$$(\mathsf{P}(\mathcal{B}_x^I;\chi)) = (q^{-c_I} f_\varphi^I)\mathbf{K}$$

and therefore

$$\left(q^{-c_I} f_\varphi^I\right) = (\mathsf{P}(\mathcal{B}_x^I;\chi))\mathbf{K^{-1}}.$$

Now part (1) and the Remark 6.4 imply that $q^{-c_I} f_\varphi^I(q^{-1})$ satisfies the modular law for all $\varphi \in \mathrm{Irr}(W)$. Multiplying by $q^{c_{I_2}}$ we have

$$(1+q)\cdot q^{-1} f_\varphi^{I_1}(q^{-1})) = f_\varphi^{I_2}(q^{-1}) + q\cdot q^{-2} f_\varphi^{I_0}(q^{-1})$$

and the result follows by replacing q^{-1} with q.

Statement (3) now follows from (2) and (5.11) and Remark 6.4. Similarly (4) follows from Proposition 5.3. □

Remark 6.6. It is also possible to prove that $\mathsf{P}(\mathcal{B}_x^H;\chi)$ satisfies the modular law by adapting the geometric proof in Proposition 1.2 to the varieties $\mathcal{B}_x^H$.

6.4. Alternative formulation of Proposition 1.2

As discussed in the introduction, if Q is any parabolic subgroup stabilizing $I \in \mathfrak{Id}$, then we can consider the map

$$\mu^{I,Q} : G \times^Q I \to \overline{\mathcal{O}_I}$$

and its derived pushforward

$$\mathcal{S}_{I,Q} := R\mu_*^{I,Q}(\underline{\mathbb{C}}[\dim G/Q + \dim I]).$$

Suppose Q' is another parabolic subgroup stabilizing I with $Q' \subset Q$. Then

$$\mathcal{S}_{I,Q'} = H^*(Q/Q')[\dim Q/Q'] \otimes \mathcal{S}_{I,Q} \tag{6.5}$$

since $G \times^{Q'} I$ is a fiber bundle over $G \times^Q I$ with fiber Q/Q'.

Now let $I_2 \subset I_1 \subset I_0$ be a modular triple of ideals. Since the f^I_φ satisfy $(1+q)f^{I_1}_\varphi = f^{I_2}_\varphi + qf^{I_0}_\varphi$ by Proposition 6.5, the $m^I_{u,\phi}$ from §4 satisfy

$$(q+q^{-1})m^{I_1}_{u,\phi} = m^{I_2}_{u,\phi} + m^{I_0}_{u,\phi}$$

for all $(u,\phi) \in \Theta_{sp}$. Since the $m^I_{u,\phi}$ determine the $V^I_{u,\phi}$ in Equation (5.3), it follows that

$$(q+q^{-1})\mathcal{S}_{I_1,B} = \mathcal{S}_{I_0,B} \oplus \mathcal{S}_{I_2,B}.$$

Since I_1 and I_0 are P_α-stable and $\mathsf{P}(P_\alpha/B) = 1+q$, we have $\mathcal{S}_{I_0,B} = (q+q^{-1})\mathcal{S}_{I_0,P_\alpha}$ and $\mathcal{S}_{I_2,B} = (q+q^{-1})\mathcal{S}_{I_2,P_\alpha}$. Hence, clearing away the $(q+q^{-1})$ from all three terms, we have the following.

Proposition 6.7. *Let $I_2 \subset I_1 \subset I_0$ be a modular triple. There is an isomorphism*

$$\mathcal{S}_{I_1,B} \simeq \mathcal{S}_{I_2,P_\alpha} \oplus \mathcal{S}_{I_0,P_\alpha}$$

in the derived category of G-equivariant perverse sheaves on $\mathcal{N}$.

7. Type *A* results

In Proposition 6.5 we proved that the polynomials f^I_φ and $g^{I^\perp}_\varphi$ satisfy the modular law of (6.4). We now connect those results to a combinatorial modular law for symmetric functions, proving that the two notions are equivalent in the type A case.

Let $G = SL_n(\mathbb{C})$ throughout this section, B be the set of upper triangular matrices in G, and T the set of diagonal matrices. Let E_{ij} denote the elementary matrix with 1 in entry (i,j) and all other entries equal to 0. Then the E_{ij} for $i<j$ are basis vectors of the positive root spaces of $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$ relative to $\mathfrak{t}$ and $\mathfrak{b}$. The positive roots are $\epsilon_i - \epsilon_j$ for $i<j$, where ϵ_k denotes the linear dual of E_{kk}, and the simple roots Δ are $\alpha_k := \epsilon_k - \epsilon_{k+1}$ for $1 \le k \le n-1$. The simple reflection $s_k := s_{\alpha_k}$ corresponds to the simple transposition in $W \simeq S_n$ exchanging k and $k+1$.

The modular law was first introduced for chromatic symmetric functions by Guay–Paquet in [GP13]. More recently, Abreu and Nigro showed that any collection of multiplicative symmetric functions satisfying the modular law are uniquely determined up to some initial values [AN21a]. As an application of our results, we apply their theorem to compute the Frobenius characteristic of the dot action and LLT representations, recovering results of Brosnan–Chow and Guay–Paquet [BC18, GP16].

7.1. The combinatorial modular law

We introduce the combinatorial modular law in the context of Hessenberg functions. In type A_{n-1}, each ideal $I \in \mathfrak{I}d$ uniquely determines, and is determined by, a weakly increasing function

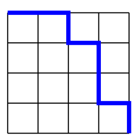
$$h : \{1, 2, \dots, n\} \to \{1, 2, \dots, n\}$$

such that $i \leq h(i)$ for all i. Such a function is called a Hessenberg function and its Hessenberg vector is $(h(1), \dots, h(n))$. Given a Hessenberg function h, the ideal $I_h \in \mathfrak{I}d$ corresponding to h is given by

$$I_h := \operatorname{span}_{\mathbb{C}}\{E_{ij} \mid h(i) < j\}. \tag{7.1}$$

The function h also determines a lattice path from the upper left corner of an $n \times n$ grid to the lower right corner, by requiring that the vertical step in row i occurs $h(i)$ columns from the left. The requirement that $h(i) \geq i$ guarantees that this lattice path never crosses the diagonal. Thus, Hessenberg functions (and ideals $\mathfrak{I}d$ and Hessenberg spaces $\mathcal{H}$) are in bijection with the set of Dyck paths of length $2n$. By a slight abuse of notation, we write $\mathcal{H}$ for the set of Hessenberg functions, or equivalently Dyck paths, throughout this section.

Example 7.1. For $n = 4$, the Hessenberg function h with vector $(2, 3, 3, 4)$ corresponds to the ideal $I_h = \operatorname{span}_{\mathbb{C}}\{E_{13}, E_{14}, E_{24}, E_{34}\}$ and defines the following lattice path. The matrices in I_h are those with all zeros below the path.



Lemma 7.2. *Let $h \in \mathcal{H}$ and $\beta = \epsilon_i - \epsilon_j \in \Phi_{I_h}$. Then β is a minimal root of Φ_{I_h} if and only if $j = h(i) + 1$ and $h(i) < h(i + 1)$.*

Proof. Suppose $\beta = \epsilon_i - \epsilon_j \in \Phi_{I_h}$, i.e. that $h(i) < j$. Assume first that $j = i + 1$. In this case, the lemma is trivial since β is a simple root (and thus a minimal root of Φ^+) and $h(i) < i + 1$ if and only if $h(i) = i$. We may therefore assume $j > i + 1$ for the remainder of the proof. In this case, we have $\alpha \in \Delta$ such that $\beta - \alpha \in \Phi$ if and only if $\alpha = \alpha_i$ or $\alpha = \alpha_{j-1}$ since Φ is a type A root system. Now $\beta \in \Phi_{I_h}$ is minimal if and only if

$$\beta - \alpha_{j-1} = \epsilon_i - \epsilon_{j-1} \notin \Phi_{I_h} \quad \text{and} \quad \beta - \alpha_i = \epsilon_{i+1} - \epsilon_j \notin \Phi_{I_h},$$

or equivalently, $h(i) \geq j-1$ and $h(i+1) \geq j$. As $h(i) < j$ these conditions are equivalent to $h(i) = j-1$ and $h(i) < h(i+1)$, as desired. $\square$

Lemma 7.3. *Let $h \in \mathcal{H}$ and $k \in \{1, 2, \ldots, n-1\}$. Then $s_k(\Phi_{I_h}) = \Phi_{I_h}$ if and only if $h(k) = h(k+1)$ and $h^{-1}(k) = \emptyset$.*

Proof. Suppose first that $h(k) = h(k+1)$ and $h^{-1}(k) = \emptyset$. Note that $h(k) = h(k+1) \Rightarrow h(k) \geq k+1$ so $\alpha_k \notin \Phi_{I_h}$. Let $\beta = \epsilon_i - \epsilon_j \in \Phi_{I_h}$, so $h(i) < j$. Consider $s_k(\beta) = \beta - \langle \beta, \alpha_k^\vee \rangle \alpha_k$. If $\langle \beta, \alpha_k^\vee \rangle \leq 0$ then $s_k(\beta) \in \Phi_{I_h}$ since $I_h \in \mathfrak{I}d$. We may therefore assume $\langle \beta, \alpha_k^\vee \rangle > 0$, in which case we have either $i = k$ or $j = k+1$ since Φ is a type A root system and $\beta \neq \alpha_k$. If $i = k$ then $s_k(\beta) = \epsilon_{k+1} - \epsilon_j \in \Phi_{I_h}$ since $h(k+1) = h(i) < j$. If $j = k+1$, then $s_k(\beta) = \epsilon_i - \epsilon_k \in \Phi_{I_h}$ since $h(i) < k+1$ and $h^{-1}(k) = \emptyset$ implies $h(i) < k$.

Now suppose either $h(k) < h(k+1)$ or $h^{-1}(k) \neq \emptyset$. If $h(k) < h(k+1)$, then $\epsilon_k - \epsilon_{h(k)+1}$ is a minimal root of Φ_{I_h} by Lemma 7.2 so

$$s_k(\epsilon_k - \epsilon_{h(k)+1}) = \epsilon_{k+1} - \epsilon_{h(k)+1} = (\epsilon_k - \epsilon_{h(k)+1}) - \alpha_k \notin \Phi_{I_h},$$

and Φ_{I_h} is not s_k-invariant. If $h^{-1}(k) \neq \emptyset$ then there exists $i \in \{1, 2, \ldots, n-1\}$ such that $h(i) = k$ and $h(i) < h(i+1)$. We have that $\epsilon_i - \epsilon_{k+1}$ is a minimal root of Φ_{I_h} so

$$s_k(\epsilon_i - \epsilon_{k+1}) = \epsilon_i - \epsilon_k = (\epsilon_i - \epsilon_{k+1}) - \alpha_k \notin \Phi_{I_h},$$

and, as before, Φ_{I_h} is not s_k-invariant. $\square$

We now introduce the triples used to define the combinatorial modular law as in [AN21a, Def. 2.1].

Definition 7.4. We say $h_0, h_1, h_2 \in \mathcal{H}$ is a **combinatorial triple** whenever one of the following two conditions holds:

1. There exists $i \in \{1, 2, \ldots, n-1\}$ such that $h_1(i-1) < h_1(i) < h_1(i+1)$ and $h_1(h_1(i)) = h_1(h_1(i)+1)$. Moreover, h_0 and h_2 are defined to be:

$$h_0(j) = \begin{cases} h_1(j) & j \neq i \\ h_1(i) - 1 & j = i \end{cases} \quad \text{and} \tag{7.2}$$

$$h_2(j) = \begin{cases} h_1(j) & j \neq i \\ h_1(i) + 1 & j = i. \end{cases}$$

2. There exists $i \in \{1, 2, \ldots, n-1\}$ such that $h_1(i+1) = h_1(i) + 1$ and $h_1^{-1}(i) = \emptyset$. Moreover h_0 and h_2 are defined to be:

$$h_0(j) = \begin{cases} h_1(j) & j \neq i+1 \\ h_1(i) & j = i+1 \end{cases} \quad \text{and} \tag{7.3}$$

$$h_2(j) = \begin{cases} h_1(j) & j \neq i \\ h_1(i+1) & j = i. \end{cases}$$

Example 7.5. Let $n = 6$ and set $h_1 = (2, 3, 4, 6, 6, 6)$. Then h_1 satisfies condition (1) of Definition 7.4 for $i = 3$ since $h_1(2) < h_1(3) < h_1(4)$ and

$$h_1(h_1(3)) = h_1(4) = 6 = h_1(4+1) = h_1(h_1(3) + 1).$$

Using (7.2) we obtain $h_0 = (2, 3, 3, 6, 6, 6)$ and $h_2 = (2, 3, 5, 6, 6, 6)$.

Given a triple h_0, h_1, h_2 of Hessenberg functions we let $I_0, I_1, I_2 \in \mathfrak{I}d$ be the corresponding ideals defined as in (7.1).

Lemma 7.6. *The Hessenberg functions h_0, h_1, h_2 form a combinatorial triple if and only if $I_2 \subset I_1 \subset I_0$ is a modular triple of ideals.*

Proof. We first prove that any combinatorial triple corresponds to a modular triple of ideals in $\mathfrak{I}d$. Let h_0, h_1, h_2 be a triple of Hessenberg functions satisfying condition (1) of Definition 7.4. We must have $h_1(i) > i$ in this case. Indeed, $h_1(i) \geq i$ and if $h_1(i) = i$ then $h_1(h_1(i)) = i$ and $h_1(h_1(i) + 1) = h_1(i+1) \geq i+1$, violating the fact that $h_1(h_1(i)) = h_1(h_1(i) + 1)$. We may therefore assume $i < h_1(i) < n$. The formula for h_0 in (7.2) now yields

$$h_0(h_1(i)) = h_1(h_1(i)) = h_1(h_1(i) + 1) = h_0(h_1(i) + 1). \tag{7.4}$$

Similarly, if $j < i$ then $h_0(j) = h_1(j) \leq h_1(i-1) < h_1(i)$ and if $j > i$ then $h_0(j) = h_1(j) \geq h_1(i+1) > h_1(i)$. As $h_0(i) = h_1(i) - 1 \neq h_1(i)$, this proves $h_0^{-1}(h_1(i)) = \emptyset$. Together with (7.4), this gives us $s_{h_1(i)}(I_{h_0}) = I_{h_0}$ by Lemma 7.3. Set $\beta = \epsilon_i - \epsilon_{h_1(i)}$. Then $\langle \beta, \alpha^\vee_{h_1(i)} \rangle = -1$. As $h_1(i) = h_0(i) + 1$ and

$$h_0(i) = h_1(i) - 1 < h_1(i+1) = h_0(i+1),$$

we have that β is a minimal root of I_0 by Lemma 7.2. The formulas for h_0 and h_2 given in (7.2) imply $I_0 = I_1 \oplus \mathfrak{g}_\beta$ and $I_1 = I_2 \oplus \mathfrak{g}_{\alpha_{h_1(i)} + \beta}$. This proves $I_2 \subset I_1 \subset I_0$ is a modular triple.

Now suppose h_0, h_1, h_2 satisfy condition (2) of Definition 7.4. The formula for h_0 in (7.3) gives us $h_1^{-1}(i) = \emptyset \Rightarrow h_0^{-1}(i) = \emptyset$ and $h_0(i) = h_1(i) = h_0(i+1)$.

Therefore $s_i(\Phi_{I_0}) = \Phi_{I_0}$ by Lemma 7.3. The assumption $h_1(i+1) = h_1(i)+1$ yields

$$h_0(i+1) + 1 = h_1(i) + 1 = h_1(i+1)$$

and

$$h_0(i+1) = h_1(i) < h_1(i+1) \leq h_1(i+2) = h_0(i+1)$$

so $\epsilon_{i+1} - \epsilon_{h_1(i+1)}$ is a minimal root of Φ_{I_0} by Lemma 7.2. Setting $\beta = \epsilon_{i+1} - \epsilon_{h_1(i+1)}$ we get $\langle \beta, \alpha_i^\vee \rangle = -1$, $I_0 = I_1 \oplus \mathfrak{g}_\beta$, and $I_1 = I_2 \oplus \mathfrak{g}_{\alpha_i+\beta}$. This proves $I_2 \subset I_1 \subset I_0$ is a modular triple.

Next we argue that any modular triple of ideals $I_2 \subset I_1 \subset I_0$ corresponds to a combinatorial triple h_0, h_1, h_2. There exists $\alpha_k \in \Delta$ and $\beta \in \Phi^+$ such that $\langle \beta, \alpha_k^\vee \rangle = -1$, $I_0 = I_1 \oplus \mathfrak{g}_\beta$, and $s_k(\Phi_{I_0}) = \Phi_{I_0}$. Furthermore, by definition I_2 is the ideal defined by the condition $I_1 = I_2 \oplus \mathfrak{g}_{\alpha+\beta}$. Since Φ is a type A root system, $\beta = \epsilon_i - \epsilon_k$ for some $i < k$ or $\beta = \epsilon_{k+1} - \epsilon_p$ for some $p > k+1$. We consider each case.

Suppose first that $\beta = \epsilon_i - \epsilon_k$ for some $i < k$. We argue that the triple of Hessenberg functions h_0, h_1, h_2 satisfies condition (1) of Definition 7.4. As $\alpha+\beta = \epsilon_i - \epsilon_{k+1}$ is a minimal root of I_1, we have $h_1(i) = k$ and $h_1(i) < h_1(i+1)$ by Lemma 7.2. Since $I_0 = I_1 \oplus \mathfrak{g}_\beta$,

$$h_0(j) = h_1(j) \text{ for all } j \neq i \text{ and } h_0(i) = h_1(i) - 1. \tag{7.5}$$

If $h_1(i-1) = h_1(i) = k$ then $h_0(i-1) = k$ by (7.5), contradicting, by Lemma 7.3, the assumption that $s_k(\Phi_{I_0}) = \Phi_{I_0}$. Thus $h_1(i-1) < h_1(i)$. Lemma 7.3 also implies $h_0(k) = h_0(k+1)$ and since $i < k$, (7.5) now yields $h_1(k) = h_1(k+1) \Rightarrow h_1(h_1(i)) = h_1(h_1(i)+1)$. Finally, as I_2 satisfies $I_1 = I_2 \oplus \mathfrak{g}_{\alpha+\beta}$, the Hessenberg function h_2 is defined as in (7.2), concluding this case.

Suppose $\beta = \epsilon_{k+1} - \epsilon_p$ for some $p > k+1$. We argue that the triple of Hessenberg functions h_0, h_1, h_2 satisfies condition (2) of Definition 7.4. Lemma 7.2 implies that $p = h_0(k+1) + 1$ since β is a minimal root of Φ_{h_0}. Now that fact that $I_0 = I_1 \oplus \mathfrak{g}_\beta$ implies

$$h_1(k+1) = p = h_0(k+1) + 1 \text{ and } h_1(j) = h_0(j) \text{ for all } j \neq k+1. \tag{7.6}$$

Lemma 7.3 and (7.6) together imply that $h_1(k+1) = h_0(k) + 1 = h_1(k) + 1$ and $h_1^{-1}(k) = \emptyset$. This proves h_1 and h_0 are as in Definition 7.4 with $i = k$. The fact that I_2 satisfies $I_1 = I_2 \oplus \mathfrak{g}_{\alpha+\beta}$ where $\alpha + \beta = \epsilon_k - \epsilon_p$ implies that h_2 is defined as in (7.3) with $i = k$. The proof is now complete. $\square$

Let Λ^n denote the $\mathbb{Z}$-module of homogeneous symmetric functions of degree n. The Schur functions $\{s_\lambda \mid \lambda \vdash n\}$ are a basis of Λ^n. Here, $\lambda \vdash n$ means $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell > 0)$ is a partition of n.

We say the function $F : \mathcal{H} \to \mathbb{Q}(q) \otimes \Lambda^n$ satisfies the **combinatorial modular law** if

$$(1+q)\, F(h_1) = F(h_2) + qF(h_0) \tag{7.7}$$

whenever $h_0, h_1, h_2 \in \mathcal{H}$ is a combinatorial triple of Hessenberg functions. With Lemma 7.6 in hand, we recover the combinatorial modular law as a special case of the modular law from Definition 6.3.

Let $F : \mathcal{H} \to \mathbb{Q}(q) \otimes \Lambda^n$. Given $I \in \mathfrak{I}d$ and $\lambda \vdash n$, define $F^I_\lambda \in \mathbb{Q}(q)$ by

$$F(h) = \sum_{\lambda \vdash n} F^I_\lambda(q) s_\lambda$$

where $h \in \mathcal{H}$ is the unique Hessenberg function with $I = I_h$.

Proposition 7.7. *The function $F : \mathcal{H} \to \mathbb{Q}(q) \otimes \Lambda^n$ satisfies the combinatorial modular law (7.7) if and only if $\{F^I_\lambda\}_{I \in \mathfrak{I}d}$ satisfies (6.4) for all $\lambda \vdash n$.*

Proof. By Lemma 7.6, $h_0, h_1, h_2 \in \mathcal{H}$ is a combinatorial triple if and only if $I_2 \subset I_1 \subset I_0$ is a modular triple in $\mathfrak{I}d$. Since the s_λ form a basis of Λ^n, it follows that

$$(1+q)\, F(h_1) = q\, F(h_0) + F(h_2)$$

if and only if

$$(1+q)\, F^{I_1}_\lambda = qF^{I_0}_\lambda + F^{I_2}_\lambda$$

for all $\lambda \vdash n$. □

7.2. Chromatic quasisymmetric functions and LLT polynomials

When working in the context of chromatic and LLT polynomials, a Hessenberg function (or Dyck path) is frequently identified with an indifference graph (see [HP19, §2.3]). For a composition $\mu = (\mu_1, \ldots, \mu_r)$ of n, let $h^{(\mu)}$ denote the Hessenberg function with $h(i) = \mu_1 + \cdots + \mu_k$ where k is the smallest index satisfying $i \leq \mu_1 + \cdots + \mu_k$. Let K_m denote the complete graph on m vertices. Then the graph corresponding to $h^{(\mu)}$ is a disjoint union $K_{\mu_1} \sqcup K_{\mu_2} \sqcup \cdots \sqcup K_{\mu_r}$ of complete graphs.

Abreu-Nigro showed the following key result in [AN21a, Theorem 1.2].

Theorem 7.8 (Abreu–Nigro)**.** *Let $F : \mathcal{H} \to \mathbb{Q}(q) \otimes \Lambda^n$ be a function satisfying the combinatorial modular law of Equation (7.7). Then F is determined by its values $F(h^{(\mu)})$ where μ is a composition of n.*

Let $\mathcal{R}_n$ denote the representation ring of S_n and recall that the Frobenius characteristic map defines an isomorphism $\mathcal{R} = \oplus_n \mathcal{R}_n \to \Lambda = \oplus_n \Lambda^n$ (see [Ful97, Section 7.3]). Given a graded complex vector space $U = \oplus_i U_{2i}$ concentrated in even degree such that each U_{2i} is a finite dimensional S_n-representation, we let $\mathsf{Ch}(U) = \sum_i [U_{2i}] q^i$ where $[U_{2i}]$ denotes the class of U_{2i} in $\mathcal{R}_n$.

If μ is a composition of n and $I \in \mathfrak{I}d$ is the ideal corresponding $h^{(\mu)}$, then $H = I^\perp \in \mathcal{H}$ is a parabolic subalgebra $\mathfrak{p}_\mu$ of a parabolic subgroup P_μ in G with $W_{P_\mu} \simeq S_{\mu_1} \times S_{\mu_2} \times \cdots \times S_{\mu_r}$. Since we know the values of the dot action and LLT representations at all $\mathfrak{p}_\mu$ by Proposition 5.5, we can use Theorem 7.8 to identify the graded characters of the dot action and LLT representations as symmetric functions under the Frobenius characteristic map. In the former case, we obtain another proof of the Shareshian–Wachs conjecture, originally proved by Brosnan and Chow in [BC18] and again using independent methods by Guay-Paquet in [GP16]). We note that the proof below is not wholly independent of that of Brosnan and Chow, since our computations in the previous section rely on their result [BC18] identifying the dot action as a monodromy action.

Rather than define the chromatic quasisymmetric and unicellular LLT functions corresponding to a given Hessenberg function h in careful detail here, we refer the interested reader to [SW16, AN21a, AN21b, AP18].

Corollary 7.9. *Let $h \in \mathcal{H}$ and let $H = I_h^\perp$.*

1. *The image of $\mathsf{Ch}(H^*(\mathcal{B}_s^H) \otimes sgn)$ under the Frobenius characteristic map is equal to the chromatic quasisymmetric function of h.*
2. *The image of $\mathsf{Ch}(\mathrm{LLT}_H)$ under the Frobenius characteristic map is equal to the unicellular LLT polynomial of h.*

Proof. Proposition 7.7 and Proposition 6.5 imply that the Frobenius characteristic map of $\mathsf{Ch}(H^*(\mathcal{B}_s^H) \otimes \mathrm{sgn})$ and $\mathsf{Ch}(\mathrm{LLT}_H)$ in $\mathbb{Q}(q) \otimes \Lambda^n$ both satisfy the combinatorial modular law. It therefore suffices by Theorem 7.8 to show that $\mathsf{Ch}(H^*(\mathcal{B}_s^{\mathfrak{p}_\mu}) \otimes \mathrm{sgn})$ is equal to the chromatic quasisymmetric function of $h^{(\mu)}$ and $\mathsf{Ch}(\mathrm{LLT}_{\mathfrak{p}_\mu})$ is equal to the unicellular LLT polynomial of $h^{(\mu)}$ under the Frobenius characteristic map, for each composition μ of n.

By Proposition 5.5(1)

$$\mathsf{Ch}(H^*(\mathcal{B}_s^{\mathfrak{p}_\mu}) \otimes \mathrm{sgn}) = \mathsf{P}(P_\mu/B)\mathsf{Ch}(\mathrm{Ind}_{S_{\mu_1} \times \cdots \times S_{\mu_r}}^{S_n}(\mathrm{sgn})).$$

The Frobenius characteristic of $\mathrm{Ind}_{S_{\mu_1}\times\cdots\times S_{\mu_r}}^{S_n}(\mathrm{sgn})$ is the elementary symmetric polynomial e_μ. Also $P(P_\mu/B) = \prod_{i=1}^r [\mu_i]!$ where $[m]! = [m][m-1]\dots[2][1]$ and $[m]$ is the q-number $1+q+\ldots+q^{m-1}$. The chromatic quasisymmetric function of $h^{(\mu)}$ is $\prod_{i=1}^r [\mu_i]! e_\mu$, so both sides agree in the parabolic setting and the result follows.

By Proposition 5.5(2), $\mathsf{Ch}(\mathrm{LLT}_{\mathfrak{p}_\mu})$ equals the graded character of

$$\mathrm{Ind}_{S_{\mu_1}\times\cdots\times S_{\mu_r}}^{S_n}(\mathcal{C}_{\mu_1}\otimes\mathcal{C}_{\mu_2}\otimes\cdots\otimes\mathcal{C}_{\mu_r}),$$

where $\mathcal{C}_m$ denotes the covariant algebra of S_m. Let $\mathrm{Frob}(U)$ denote the Frobenius characteristic of a graded representation U. Then

$$\mathrm{Frob}\left(\mathrm{Ind}_{S_{\mu_1}\times\cdots\times S_{\mu_r}}^{S_n}(\mathcal{C}_{\mu_1}\otimes\mathcal{C}_{\mu_2}\cdots\otimes\mathcal{C}_{\mu_r})\right) = \prod_{i=1}^r \mathrm{Frob}(\mathcal{C}_{\mu_i}).$$

Since unicellular LLT polynomials corresponding to Hessenberg functions are multiplicative (see [AN21b, Theorem 2.4]), it suffices to know that $\mathrm{Frob}(\mathcal{C}_m)$ equals the unicellular LLT polynomials for S_m for the case of the complete graph K_m, which is true by, for example, [AP18, Equation (30)]. $\square$

8. Example in B_3

In type B_3, there are 20 ideals in $\mathfrak{I}d$ and 10 irreducible representations in $\mathrm{Irr}(W)$. The ideals are grouped according to the nilpotent orbit $\mathcal{O}_I$. We specify I by listing the minimal roots in Φ_I, using the coefficients of the simple roots. The elements of Θ_{sp} are listed in the top row of Table 1. The elements of $\mathrm{Irr}(W)$, as pairs of partitions, are listed in the top row of Table 2.

The ideals I for which $\mathcal{O}_I = \mathcal{O}_x$ and

$$I \subset \bigoplus_{i\geq 2}\mathfrak{g}_i$$

for the grading induced by x are listed with a $*$ symbol. For such I, we have $I\cap\mathfrak{g}_2 \in \mathfrak{I}d_2^{gen}$ and all such elements of $\mathfrak{I}d_2^{gen}$ arise in this way.

There are 10 examples of the modular law in type B_3, which we list using the numbering of the ideals in the tables:

$$(8.1)\qquad \begin{gathered}(3,5,11),(4,5,11),(4,6,9),(7,8,12),(9,10,13),(11,12,13),\\ (12,13,17),(14,15,16),(17,18,19),(18,19,20).\end{gathered}$$

Table 1: The polynomials $\mathsf{P}(\mathcal{B}_x^I;\chi)$ for B_3.

Ideal	Min roots	$[1^7]$	$[2^2,1^3]$	$[3,1^4],\epsilon$	$[3,1^4],1$	$[3,2^2]$	$[3^2,1],\epsilon$	$[3^2,1],1$	$[5,1^2].\epsilon$	$[5,1^2],1$	$[7]$
20*	$\emptyset$	$[2][4][6]$									
19*	122	$[2][4][6]$	$[2][2]$								
18	112	$[2][4][6]$	$[2][2][2]$								
17	012	$[2][4][6]$	$[2][2][3]$								
16*	111	$[2][4][6]$	$[2][2][2]$	$[2][2]$	$[2][2]$						
15*	110	$[2][4][6]$	$[2][2][3]$	$q[2]$	$[2][3]$						
14*	100	$[2][4][6]$	$[2][2][4]$	0	$[2][4]$						
13*	111, 012	$[2][4][6]$	$[2][2][3]$	$[2][2]$	$[2][2]$	$[2]$					
12	011	$[2][4][6]$	$[2][2][3]$	$[2][2][2]$	$[2][2][2]$	$[2][2]$					
11	001	$[2][4][6]$	$[2][2][3]$	$[2][2][3]$	$[2][2][3]$	$[2][3]$					
10*	110, 012	$[2][4][6]$	$[2][2](1+q+2q^2)$	$q[2]$	$[2][3]$	$[2]$	1	1			
9	100, 012	$[2][4][6]$	$[2][2](1+q+2q^2+q^3)$	0	$[2][4]$	$[2]$	$[2]$	$[2]$			
8*	110, 011	$[2][4][6]$	$[2][2](1+q+2q^2)$	$2q+3q^2+q^3$	$[2](1+2q+2q^2)$	$[2](1+2q)$	0	$[2]$			
7*	010	$[2][4][6]$	$[2][2](1+q+2q^2+q^3)$	$q[2][2]$	$[2][2][3]$	$[2][2][2]$	0	$[2][2]$			
6*	100, 011	$[2][4][6]$	$[2][2](1+q+2q^2+q^3)$	$q[2][2]$	$[2][2][3]$	$[2](1+2q)$	q	$1+2q$	1	1	
5	110, 001	$[2][4][6]$	$[2][2](1+q+2q^2)$	$q[2](2+2q+q^2)$	$[2](1+2q+3q^2+q^3)$	$[2][2][2]$	0	$[2]$	1	1	
4	100, 001	$[2][4][6]$	$[2][2](1+q+2q^2+q^3)$	$q[2][2][2]$	$[2](1+2q+3q^2+2q^3)$	$[2](1+2q+2q^2)$	0	$[2][2]$	$[2]$	$[2]$	
3	010, 001	$[2][4][6]$	$[2][2](1+q+2q^2+q^3)$	$q[2][2][2]$	$[2](1+2q+3q^2+2q^3)$	$[2](1+2q+2q^2)$	0	$[2][2]$	$[2]$	$[2]$	
2*	100, 010	$[2][4][6]$	$[2][2](1+q+2q^2+2q^3)$	$q^2[2]$	$[2](1+2q+2q^2+2q^3)$	$[2][2][2]$	$q[2]$	$1+3q+2q^2$	0	$[2]$	
1*	Δ	$[2][4][6]$	$[2][2](1+q+2q^2+2q^3)$	$2q^2+3q^3+q^4$	$1+3q+5q^2+6q^3+3q^4$	$1+3q+5q^2+3q^3$	q^2	$1+3q+3q^2$	$2q$	$1+3q$	1

Table 2: The polynomials f^I_φ for B_3.

Ideal	Min roots	$\emptyset, [1^3]$	$\emptyset, [2,1]$	$[1^3], \emptyset$	$[1], [1^2]$	$[1^2], [1]$	$\emptyset, [3]$	$[1], [2]$	$[2,1], \emptyset$	$[2], [1]$	$[3], \emptyset$
20	$\emptyset$	$[2][4][6]$									
19	122	$[2][4][5]$	$q^3[2][2]$								
18	112	$[2][4][4]$	$q^2[2][2][2]$								
17	012	$[2][3][4]$	$q[2][2][3]$								
16	111	$[2][3][4]$	$q^2[2][2]$	$q^2[2][2]$	$q^2[2][2]$						
15	110	$[2][2][4]$	$q[2][3]$	$q^2[2]$	$q[2][3]$						
14	100	$[2][4]$	$[2][4]$	0	$[2][4]$						
13	$111, 012$	$[2][3][3]$	$q[2][2][2]$	$q^2[2]$	$q^2[2]$	$q^2[2]$					
12	011	$[2][2][3]$	$q[2][2]$	$q[2][2]$	$q[2][2]$	$q[2][2]$					
11	001	$[2][3]$	0	$[2][3]$	$[2][3]$	$[2][3]$					
10	$110, 012$	$[2][2][3]$	$2q[2][2]$	q^2	$q[2][2]$	q^2	q^2	q^2			
9	$100, 012$	$[2][3]$	$[2][2][2]$	0	$[2][3]$	0	$q[2]$	$q[2]$			
8	$110, 011$	$[2][2][2]$	$2q[2]$	$q[2]$	$2q[2]$	$2q[2]$	0	$q[2]$			
7	010	$[2][2]$	$[2][2]$	0	$[2][2]$	$[2][2]$	0	$[2][2]$			
6	$100, 011$	$[2][2]$	$1+3q+q^2$	q	$1+3q+q^2$	$2q$	q	$2q$	q	q	
5	$110, 001$	$[2][2]$	q	$[2][2]$	$1+3q+q^2$	$1+3q+q^2$	0	q	q	q	
4	$100, 001$	$[2]$	$[2]$	$[2]$	$2[2]$	$2[2]$	0	$[2]$	$[2]$	$[2]$	
3	$010, 001$	$[2]$	$[2]$	$[2]$	$2[2]$	$2[2]$	0	$[2]$	$[2]$	$[2]$	
2	$100, 010$	$[2]$	$2[2]$	0	$2[2]$	$[2]$	$[2]$	$2[2]$	0	$[2]$	
1	Δ	1	2	1	3	3	1	2	2	3	1

The modular triples (8.1) provide a check on our calculation in both tables, as do the 8 cases in the parabolic setting in Table 2 using (4.8). We made use of Binegar's tables to check our calculation of the Green polynomial matrix **K** [Bin].

Appendix A. Definition of the dot action and LLT representations

This appendix introduces the dot action on the equivariant cohomology of a regular semisimple Hessenberg variety and uses the construction to define the dot action and LLT representations. Our main goal is to obtain a proof of Proposition 5.4 above, which was used to define and study the LLT representations.

Let $H \in \mathcal{H}$ be a Hessenberg space and $s \in \mathfrak{t}$ a regular semisimple element. Recall that the torus T acts on regular semisimple Hessenberg variety $\mathcal{B}_s^H$ by left multiplication. The variety is in fact equivariantly formal with respect to this action [Tym05]. Applying the theory developed by Goresky, Kottwitz, and MacPherson [GKM98], the equivariant cohomology $H_T^*(\mathcal{B}_s^H)$ has the following description. The inclusion map of the T-fixed point $\mathcal{B}_s^{H,T} = \{wB \mid w \in W\}$ into $\mathcal{B}_s^H$ induces an injection

$$\iota : H_T^*(\mathcal{B}_s^H) \hookrightarrow H_T^*(\mathcal{B}_s^{H,T}) \simeq \bigoplus_{w\in W} \mathbb{C}[\mathfrak{t}^*].$$

Here $\mathbb{C}[\mathfrak{t}^*]$ denotes the polynomial ring in the simple roots $\mathbb{C}[\alpha_1, \dots, \alpha_n]$. We identify $H_T^*(\mathcal{B}_s^H)$ with its image under this map, which is given by the following concrete description,

$$H_T^*(\mathcal{B}_s^H) \simeq \left\{ (f_w)_{w\in W} \mid \begin{array}{c} f_w - f_{s_\gamma w} \in \langle\gamma\rangle \text{ for } \gamma \in \Phi^+, \\ w^{-1}(\gamma) \in \Phi_H \cap \Phi^- \end{array} \right\}. \tag{A.1}$$

A more leisurely exposition of the above can be found in [AHM+20, Tym05].

Let $\mathcal{T} := \bigoplus_{w\in W} \mathbb{C}[\mathfrak{t}^*]$. The ring $\mathcal{T}$ is a $\mathbb{C}[\mathfrak{t}^*]$-module via the action,

$$(p, f) \mapsto pf \quad \text{where} \quad (pf)_w = pf_w \tag{A.2}$$

for all $p \in \mathbb{C}[\mathfrak{t}^*]$ and $f = (f_w)_{w\in W} \in \mathcal{T}$. Equation (A.1) identifies $H_T^*(\mathcal{B}_s^H)$ as a $\mathbb{C}[\mathfrak{t}^*]$-submodule of $\mathcal{T}$, and we make this identification from now on. We can also view $\mathcal{T}$ as a $\mathbb{C}[\mathfrak{t}^*]$-module in another way. To distinguish this structure from that defined in (A.2) we call this the right $\mathbb{C}[\mathfrak{t}^*]$-module action on $\mathcal{T}$, which is defined by

$$(q, f) \mapsto fq \quad \text{where} \quad (fq)_w = w(q) f_w \tag{A.3}$$

for all $q \in \mathbb{C}[\mathfrak{t}^*]$ and $f = (f_w)_{w\in W} \in \mathcal{T}$. The equivariant cohomology $H_T^*(\mathcal{B}_s^H)$ is also a $\mathbb{C}[\mathfrak{t}^*]$-submodule of $\mathcal{T}$ with respect to the right action.

Let $\mathbf{1} = (1)_{w\in W} \in \mathcal{T}$. We can identify $\mathbb{C}[\mathfrak{t}^*]$ with the $\mathbb{C}[\mathfrak{t}^*]$-submodule generated by $\mathbf{1}$, via the left action from (A.2) or the right action from (A.3). To distinguish between the left and right submodules generated by $\mathbf{1}$ we write:

- $\mathbb{C}[L]$ for the left $\mathbb{C}[\mathfrak{t}^*]$-module generated by $\mathbf{1}$ via the action from (A.2).
- $\mathbb{C}[R]$ for the right $\mathbb{C}[\mathfrak{t}^*]$-module generated by $\mathbf{1}$ via the action from (A.3).

This notation is inspired by the exposition in [GP16]. Both $\mathbb{C}[L]$ and $\mathbb{C}[R]$ are submodules of $H_T^*(\mathcal{B}_s^H)$ in the appropriate sense. Recall that $H^*(\mathcal{B}_s^H) \simeq H_T^*(\mathcal{B}_s^H)/\left\langle \alpha_1, \dots, \alpha_n \right\rangle_L$ where $\left\langle \alpha_1, \dots, \alpha_n \right\rangle_L$ is the ideal of $H_T^*(\mathcal{B}_s^H)$ generated by the positive degree elements in $\mathbb{C}[L]$ (see [Tym05, Prop. 2.3]).

The W-action on $\Phi \subseteq \mathfrak{t}^*$ extends to a W-action on $\mathbb{C}[\mathfrak{t}^*]$ in a natural way. We denote the action of $w \in W$ on $f \in \mathbb{C}[\mathfrak{t}^*]$ by $w(f)$. The dot action of W on $\mathcal{T}$ is defined by

$$(v \cdot f)_w := v(f_{v^{-1}w}) \quad \text{for all} \quad v \in W,\ f \in \mathcal{T}. \tag{A.4}$$

The dot action preserves the equivariant cohomology $H_T^*(\mathcal{B}_s^H)$ in $\mathcal{T}$ [AHM+20, Lemma 8.7].

The submodules $\mathbb{C}[L]$ and $\mathbb{C}[R]$ introduced above are also invariant under the dot action. Let $\mathbb{C}[\mathfrak{t}^*]^W$ denote the ring of W-invariants in $\mathbb{C}[\mathfrak{t}^*]$. By Chevalley's Theorem $\mathbb{C}[\mathfrak{t}^*] \simeq \mathbb{C}[\mathfrak{t}^*]^W \otimes \mathcal{C}$ where $\mathcal{C}$ is the coinvariant algebra of W. The following lemma computes the graded character of the dot action on $\mathbb{C}[L]$ and $\mathbb{C}[R]$.

Lemma A.1. *We have $\mathbb{C}[L] \simeq \mathbb{C}[\mathfrak{t}^*]^W \otimes \mathcal{C}$ and $\mathbb{C}[R] \simeq \mathbb{C}[\mathfrak{t}^*]^W \otimes \mathcal{C}'$, where $\mathcal{C}' \simeq \mathcal{C}$ as vector spaces, but W acts trivially on $\mathcal{C}'$. In particular,*

$$\mathsf{Ch}(\mathbb{C}[L]) = \prod_{i=1}^{n} \frac{1}{(1-q^{d_i})} \mathsf{Ch}(\mathcal{C})$$

and

$$\mathsf{Ch}(\mathbb{C}[R]) = \prod_{i=1}^{n} \frac{1}{(1-q^{d_i})} \sum_{w\in W} [1_W] q^{\ell(w)}$$

where $d_1, \dots, d_n$ denote the degrees of W (cf. [Hum90, Sec. 3.7-3.8]).

Proof. We first compute the dot action on $\mathbb{C}[L]$ and $\mathbb{C}[R]$, respectively. If $p \in \mathbb{C}[L]$ then

$$(v \cdot p)_w = v(p_{v^{-1}w}) = v(p) = (v(p))_w.$$

Thus the dot action on $\mathbb{C}[L]$ is the usual graded representation of W on the polynomial ring $\mathbb{C}[\mathfrak{t}^*]$. If $q \in \mathbb{C}[R]$ then

$$(v \cdot q)_w = v(q_{v^{-1}w}) = v(v^{-1}w(q)) = w(q) = q_w$$

so the dot action on $\mathbb{C}[R]$ is trivial. Since $\mathbb{C}[L] \simeq \mathbb{C}[R] \simeq \mathbb{C}[\mathfrak{t}^*]$ as vector spaces, the first assertion of the lemma follows from the above computations. Finally, the second assertion follows from the fact that the Poincaré polynomial of the ring $\mathbb{C}[\mathfrak{t}^*]^W$ is precisely $\prod_{i=1}^n \frac{1}{(1-q^{d_i})}$ and $\mathsf{Ch}(\mathcal{C}') = \sum_{w \in W} [1_W] q^{\ell(w)}$. □

Consider the ideals $\langle \alpha_1, \dots, \alpha_n \rangle_L$, and respectively $\langle \alpha_1, \dots, \alpha_n \rangle_R$, in the equivariant cohomology $H_T^*(\mathcal{B}_s^H)$ generated by the positive degree elements of $\mathbb{C}[L]$, and respectively $\mathbb{C}[R]$. Both are W-invariant, and the dot action on $H_T^*(\mathcal{B}_s^H)$ induces an action on the quotients

$$H^*(\mathcal{B}_s^H) \simeq H_T^*(\mathcal{B}_s^H) / \langle \alpha_1, \dots, \alpha_n \rangle_L \tag{A.5}$$

and

$$\mathrm{LLT}_H := H_T^*(\mathcal{B}_s^H) / \langle \alpha_1, \dots, \alpha_n \rangle_R\,. \tag{A.6}$$

We refer to the W-module $H^*(\mathcal{B}_s^H)$ as the dot action representation and the W-module LLT_H as the LLT representation.

Remark A.2. In the type A case $\mathrm{LLT}_H \simeq H^*(\mathcal{X}_H)$ where $\mathcal{X}_H$ is the smooth manifold of Hermitian matrices having a particular staircase form (determined by H) and a given fixed simple spectrum (determined by s) [AB20].

We can now prove Proposition 5.4; our argument closely follows that of Guay-Paquet in [GP16] for the type A case. In the proof below, $\star$ denotes the product (which corresponds to taking the tensor product of representations) in the character ring of W.

Proof of Proposition 5.4. The equivariant cohomology $H_T^*(\mathcal{B}_s^H)$ is a free module of rank $n!$ over $\mathbb{C}[\mathfrak{t}^*]$ with respect to either (A.2) or (A.3), see the discussion in [GP16, Section 8.5] or [AHM$^+$20, Section 2.3]. In particular, the usual extension of scalars construction for free modules yields isomorphisms of W-modules:

$$H_T^*(\mathcal{B}_s^H) \simeq \mathbb{C}[L] \otimes_{\mathbb{C}} H^*(\mathcal{B}_s^H) \tag{A.7}$$

and

$$H_T^*(\mathcal{B}_s^H) \simeq \mathbb{C}[R] \otimes_{\mathbb{C}} \mathrm{LLT}_H. \tag{A.8}$$

It now follows that

$$\mathsf{Ch}(\mathbb{C}[L]) \star \mathsf{Ch}(H^*(\mathcal{B}_s^H)) = \mathsf{Ch}(\mathbb{C}[R]) \star \mathsf{Ch}(\mathrm{LLT}_H).$$

Applying the formulas from Lemma A.1 and dividing by $\prod_{i=1}^n \frac{1}{(1-q^{d_i})}$ now yields the desired result. □

To conclude, we sketch a proof of the fact that if $\mathcal{B}_s^H$ is disconnected then both the dot action and LLT representations are induced by corresponding representations of a parabolic subgroup of W. It is well known to experts but, to the best of our knowledge, has not appeared in the literature so we include an outline of the argument here. This fact can be used together with Proposition 5.2 to give another proof of Borho and MacPherson's result in equation (4.8) above.

Let $J = \{\alpha \in \Delta \mid -\alpha \in \Phi_H\} \subseteq \Delta$. Then $\mathcal{B}_s^H$ is connected if and only if $J = \Delta$ [AT10, Appendix A]. Let L denote the standard Levi subgroup associated to J with Lie algebra $\mathfrak{l}$. There is a natural embedding of the flag variety $\mathcal{B}_L := L/(B \cap L)$ of L into $\mathcal{B}$ given by $\ell(B \cap L) \mapsto \ell B$. Let $W_J := \langle s_\alpha \mid \alpha \in J\rangle$ be the corresponding parabolic subgroup of W, which is the Weyl group of L. Denote by W^J the set of shortest coset representatives for the left cosets W/W_J. For each $v \in W^J$ we have that $s_v := v^{-1}.s$ is a regular semisimple element in $\mathfrak{l}$. Furthermore, by definition $\overline{H} := H \cap \mathfrak{l}$ is a Hessenberg space in $\mathfrak{l}$ and the regular semisimple Hessenberg variety $\mathcal{B}_{L,s}^{\overline{H}}$ in the flag variety of L is connected. Now the decomposition of $\mathcal{B}_s^H$ into connected components is given by

$$\mathcal{B}_s^H = \bigsqcup_{v \in W^J} v(\mathcal{B}_{L,s_v}^{\overline{H}}) \tag{A.9}$$

where $\mathcal{B}_{L,s_v}^{\overline{H}} = \{\ell(B \cap L) \in \mathcal{B}_L \mid \ell^{-1}.s_v \in \overline{H}\}$. Each $v(\mathcal{B}_{L,s_v}^{\overline{H}})$ is isomorphic to $\mathcal{B}_{L,s}^{\overline{H}}$, and the cohomology decomposes accordingly. We now obtain the following directly from the definition of the dot action together with the decomposition cohomology induced by (A.9).

Corollary A.3. *There is an isomorphism of W-modules:*

$$H_T^*(\mathcal{B}_s^H) \simeq \mathrm{Ind}_{W_J}^W(H_T^*(\mathcal{B}_{L,s}^{\overline{H}})). \tag{A.10}$$

In particular, both the dot action and LLT representations are obtained by induction from the corresponding representations for $\mathcal{B}^{\overline{H}}_{L,s}$, namely,

$$H^*(\mathcal{B}^H_s) \simeq \mathrm{Ind}^W_{W_J}(H^*(\mathcal{B}^{\overline{H}}_{L,s})) \text{ and } \mathrm{LLT}_H \simeq \mathrm{Ind}^W_{W_J}(\mathrm{LLT}_{\overline{H}}).$$

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The based rings of two-sided cells in an Affine Weyl group of type $\tilde{B}_3$, II

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Abstract: We compute the based rings of two-sided cells corresponding to the unipotent classes in $Sp_6(\mathbb{C})$ with Jordan blocks (33), (411), (222), respectively. The results for the first two two-sided cells also verify Lusztig's conjecture on the structure of the based rings of two-sided cells of an affine Weyl group. The result for the last two-sided cell partially suggests a modification of Lusztig's conjecture on the structure of the based rings of two-sided cells of an affine Weyl group.

We are concerned with the based rings of two-sided cells in an affine Weyl group of type $\tilde{B}_3$. In a previous paper we discussed the based ring of the two-sided cell corresponding to the nilpotent element in $Sp_6(\mathbb{C})$ with 3 equal Jordan blocks and showed that Lusztig's conjecture on the structure of the based rings of the two-sided cells of an affine Weyl group needs modification (see section 4 in [11]). In this paper we compute the based rings of two-sided cells corresponding to the unipotent classes in $Sp_6(\mathbb{C})$ with Jordan blocks (411), (33), (222), respectively. The results for the first two two-sided cells also verify Lusztig's conjecture on the structure of the based rings of two-sided cells of an affine Weyl group. The result for the last two-sided cell partially suggests a modification of Lusztig's conjecture on the structure of the based rings of two-sided cells of an affine Weyl group. For the first two two-sided cells, the validity of Lusztig's conjecture on the based rings is already included in the main theorem in [3]. Here we construct the bijection in Lusztig's conjecture explicitly so that the results in this paper can be used for computing certain irreducible representations of affine Hecke algebras of type $\tilde{B}_3$. In Section 5 we

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give a description (see Theorem 5.5) for the based ring of the two-sided cell corresponding to the nilpotent element in $Sp_6(\mathbb{C})$ with Jordan blocks (222), which can also be used to compute certain irreducible representations of affine Hecke algebras of type $\tilde{B}_3$. Theorem 5.5 indicates that Lusztig's conjecture on this based ring is not true as stated, a fact already noted by R. Bezrukavnikov in 2020 and showed in detail in [2] and [11].

The contents of the paper are as follows. Section 1 is devoted to preliminaries, which include some basic facts on (extended) affine Weyl groups and their Hecke algebras and formulation of Lusztig's conjecture on the structure of the based ring of a two-sided cell in an affine Weyl group. In Section 2 we recall some results on cells of the (extended) affine Weyl group of type $\tilde{B}_3$, which are due to J. Du. Sections 3, 4, 5 are devoted to discussing based rings of two-sided cells corresponding to the unipotent classes in $Sp_6(\mathbb{C})$ with Jordan blocks (411), (33), (222), respectively.

1. Affine Weyl groups and their Hecke algebras

In this section we fix some notations and refer to [4], L1, L2, L4, L5, QX for more details.

1.1. Extended affine Weyl groups and their Hecke algebras

Let W be an extended affine Weyl group. Then W contains a Weyl group W_0 and a free abelian subgroup X of finite rank such that $W = W_0 \ltimes X$. The set of simple reflections of W is denoted by S. We shall denote the length function of W by l and use $\leq$ for the Bruhat order on W. We also often write $y < w$ or $w > y$ if $y \leq w$ and $y \neq w$.

Let H be the Hecke algebra of (W, S) over $\mathcal{A} = \mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$ with parameter q. Let $\{T_w\}_{w\in W}$ be its standard basis. Then we have $(T_s - q)(T_s + 1) = 0$ and $T_wT_u = T_{wu}$ if $l(wu) = l(w) + l(u)$. Let $C_w = q^{-\frac{l(w)}{2}} \sum_{y\leq w} P_{y,w}T_y,\ w \in W$ be the Kazhdan-Lusztig basis of H, where $P_{y,w}$'s are the Kazhdan-Lusztig polynomials. The degree of $P_{y,w}$ is less than or equal to $\frac{1}{2}(l(w) - l(y) - 1)$ if $y < w$ and $P_{w,w} = 1$. Convention: set $P_{y,w} = 0$ if $y \not\leq w$.

We should remark here that our notation C_w stands for C'_w in [4]. The reason we use the elements C'_w in [4] since the positivity in multiplications of the elements is simpler in writing and in practical computation. Again for simplicity we write them as C_w in this paper instead of their original form C'_w in [4].

If $y < w$, we write $P_{y,w} = \mu(y, w)q^{\frac{1}{2}(l(w)-l(y)-1)} +$ lower degree terms. We shall write $y \prec w$ if $\mu(y, w) \neq 0$. We have

(a) Let $y \le w$. Assume that $sw \le w$ for some $s \in S$. Then

$$P_{y,w} = P_{sy,w}, \text{ if } sy > y;$$
$$P_{y,w} = q^{1-c}P_{sy,sw} + q^c P_{y,sw} - \sum_{\substack{z \in W \\ y \le z \prec sw \\ sz < z}} \mu(z, sw) q^{\frac{l(w)-l(z)}{2}} P_{y,z},$$

where $c = 1$ if $sy < y$ and $c = 0$ if $sy > y$.

(b) Let $y \le w$. Assume that $ws \le w$ for some $s \in S$. Then

$$P_{y,w} = P_{ys,w}, \text{ if } ys > y;$$
$$P_{y,w} = q^{1-c}P_{ys,ws} + q^c P_{y,ws} - \sum_{\substack{z \in W \\ y \le z \prec ws \\ zs < z}} \mu(z, ws) q^{\frac{l(w)-l(z)}{2}} P_{y,z},$$

where $c = 1$ if $ys < y$ and $c = 0$ if $ys > y$.

From the two formulas above one gets (see [4])

(c) Let $y, w \in W$ and $s \in S$ be such that $y < w$, $sw < w$, and $sy > y$. Then $y \prec w$ if and only if $w = sy$. Moreover this implies that $\mu(y, w) = 1$.

(d) Let $y, w \in W$ and $s \in S$ be such that $y < w$, $ws < w$, and $ys > y$. Then $y \prec w$ if and only if $w = ys$. Moreover this implies that $\mu(y, w) = 1$.

The following formulas for computing C_w (see [5]) will be used in Sections 3, 4 and 5. (Note the notation C_w here stands for C'_w in [4].)

(e) Let $w \in W$ and $s \in S$. Then

$$C_s C_w = \begin{cases} (q^{\frac{1}{2}} + q^{-\frac{1}{2}})C_w, & \text{if } sw < w, \\ C_{sw} + \sum\limits_{\substack{z \prec w \\ sz < z}} \mu(z, w) C_z, & \text{if } sw \ge w. \end{cases} \tag{1}$$

$$C_w C_s = \begin{cases} (q^{\frac{1}{2}} + q^{-\frac{1}{2}})C_w, & \text{if } ws < w, \\ C_{ws} + \sum\limits_{\substack{z \prec w \\ zs < z}} \mu(z, w) C_z, & \text{if } ws \ge w. \end{cases} \tag{2}$$

1.2. Cells of affine Weyl groups

We refer to [4, 9] for definition of left cells, right cells and two-sided cells of W.

For $h,\, h' \in H$ and $x \in W$, write

$$hC_x = \sum_{y\in W} a_y C_y, \quad C_x h = \sum_{y\in W} b_y C_y, \quad hC_x h' = \sum_{y\in W} c_y C_y, \quad a_y, b_y, c_y \in \mathcal{A}.$$

Define $y \underset{L}{\leq} x$ if $a_y \neq 0$ for some $h \in H$, $y \underset{R}{\leq} x$ if $b_y \neq 0$ for some $h \in H$, and $y \underset{LR}{\leq} x$ if $c_y \neq 0$ for some $h, h' \in H$.

We write $x \underset{L}{\sim} y$ if $x \underset{L}{\leq} y \underset{L}{\leq} x$, $x \underset{R}{\sim} y$ if $x \underset{R}{\leq} y \underset{R}{\leq} x$, and $x \underset{LR}{\sim} y$ if $x \underset{LR}{\leq} y \underset{LR}{\leq} x$. Then $\underset{L}{\sim}$, $\underset{R}{\sim}$, $\underset{LR}{\sim}$ are equivalence relations on W. The equivalence classes are called left cells, right cells, and two-sided cells of W respectively. Note that if Γ is a left cell of W, then $\Gamma^{-1} = \{w^{-1} \,|\, w \in \Gamma\}$ is a right cell.

For $w \in W$, set $R(w) = \{s \in S \,|\, ws \leq w\}$ and $L(w) = \{s \in S \,|\, sw \leq w\}$. Then we have (see [4])

(a) $R(w) \subset R(u)$ if $u \underset{L}{\leq} w$ and $L(w) \subset L(u)$ if $u \underset{R}{\leq} w$. In particular, $R(w) = R(u)$ if $u \underset{L}{\sim} w$ and $L(w) = L(u)$ if $u \underset{R}{\sim} w$.

1.3. $*$-operations

The $*$-operation introduced in [4] and generalized in [5] is a useful tool in the theory of cells of Coxeter groups.

Let s, t be simple reflections in S and assume that st has order $m \geq 3$. Let $w \in W$ be such that $sw \geq w$, $tw \geq w$. Then the $m-1$ elements $sw,\ tsw,\ stsw, \ldots,$ is called a left string (with respect to $\{s,t\}$), and the $m-1$ elements $tw,\ stw,\ tstw, \ldots,$ is also called a left string (with respect to $\{s,t\}$). Similarly we define right strings (with respect to $\{s,t\}$). Then (see [5])

(a) A left string in W is contained in a left cell of W and a right string in W is contained in a right cell of W.

Assume that x is in a left (resp. right) string (with respect to $\{s,t\}$) of length $m-1$ and is the ith element of the left (resp. right) string, define ${}^*x$ (resp. x^*) to be the $(m-i)$th element of the string, where $* = \{s,t\}$. The following result is proved in [12].

(b) Let x be in W such that x is in a left string with respect to $* = \{s,t\}$ and is also in a right string with respect to $\star = \{s',t'\}$. Then ${}^*x$ is in a right string with respect to $\{s',t'\}$ and $x^\star$ is in a left string with respect to $\{s,t\}$. Moreover ${}^*(x^\star) = ({}^*x)^\star$. We shall write ${}^*x^\star$ for ${}^*(x^\star) = ({}^*x)^\star$.

The following result is due to Lusztig [5].

(c) Let Γ be a left cell of W and an element $x \in \Gamma$ be in a right string σ_x with respect to $* = \{s,t\}$. Then any element $w \in \Gamma$ is in a right string σ_w with respect to $* = \{s,t\}$. Moreover $\Gamma^* = \{w^* \mid w \in \Gamma\}$ is a left cell of W and $\Omega = (\cup_{w\in\Gamma}\sigma_w) - \Gamma$ is a union of at most $m-2$ left cells.

Following Lusztig [5] we set $\tilde{\mu}(y,w) = \mu(y,w)$ if $y < w$ and $\tilde{\mu}(y,w) = \mu(w,y)$ if $w < y$. For convenience we also set $\tilde{\mu}(y,w) = 0$ if $y \not< w$ and $w \not< y$. Assume that $x_1, x_2, \ldots, x_{m-1}$ and $y_1, y_2, \ldots, y_{m-1}$ are two left strings with respect to $* = \{s,t\}$. Define

$$a_{ij} = \begin{cases} \tilde{\mu}(x_i, y_j), & \text{if } \{s,t\} \cap L(x_i) = \{s,t\} \cap L(y_j), \\ 0, & \text{otherwise.} \end{cases}$$

Lusztig proved the following identities (see Subsection 10.4 in [5]).

(d) If $m = 3$, then $a_{11} = a_{22}$ and $a_{12} = a_{21}$.
(e) If $m = 4$, then

$$a_{11} = a_{33},\ a_{13} = a_{31},\ a_{22} = a_{11} + a_{13},\ a_{12} = a_{21} = a_{23} = a_{32}. \tag{3}$$

1.4. Lusztig's a-function

For $x, y \in W$, write

$$C_x C_y = \sum_{z\in W} h_{x,y,z} C_z, \qquad h_{x,y,z} \in \mathcal{A} = \mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}].$$

Following Lusztig [5], we define

$$a(z) = \min\{i \in \mathbf{N} \mid q^{-\frac{i}{2}} h_{x,y,z} \in \mathbb{Z}[q^{-\frac{1}{2}}] \text{ for all } x, y \in W\}.$$

If for any i, $q^{-\frac{i}{2}} h_{x,y,z} \notin \mathbb{Z}[q^{-\frac{1}{2}}]$ for some $x, y \in W$, we set $a(z) = \infty$.

Springer showed that $l(z) \geq a(z)$ (see [6]). Let $\delta(z)$ be the degree of $P_{e,z}$, where e is the neutral element of W. Then actually one has $l(z) - a(z) - 2\delta(z) \geq 0$ (see [6]). Set

$$\mathcal{D} = \{z \in W \mid l(z) - a(z) - 2\delta(z) = 0\}.$$

The elements of $\mathcal{D}$ are involutions, called distinguished involutions of (W,S) (see [6]).

The following properties are proved in [5].

(a) We have $a(w) \leq l(w_0)$ for any $w \in W$, where w_0 is the longest element in the Weyl group W_0.
(b) $a(x) \geq a(y)$ if $x \underset{LR}{\leq} y$. In particular, $a(x) = a(y)$ if $x \underset{LR}{\sim} y$.
(c) $x \underset{L}{\sim} y$ (resp. $x \underset{R}{\sim} y$, $x \underset{LR}{\sim} y$) if $a(x) = a(y)$ and $x \underset{L}{\leq} y$ (resp. $x \underset{R}{\leq} y$, $x \underset{LR}{\leq} y$).
(d) If $h_{x,y,z} \neq 0$, then $z \underset{R}{\leq} x$ and $z \underset{L}{\leq} y$. In particular, $a(z) > a(x)$ if $z \underset{R}{\not\sim} x$, and $a(z) > a(y)$ if $z \underset{L}{\not\sim} y$.

Let i be a nonnegative integer. Using (b), (d) and the definitions in Subsection 1.2, we see that the $\mathcal{A}$-submodule I of H spanned by all C_w's with $a(w) \geq i$ is a two-sided ideal of H. This fact will be used in Sections 3, 4 and 5.

Following Lusztig, we define $\gamma_{x,y,z}$ by the following formula,

$$h_{x,y,z} = \gamma_{x,y,z} q^{\frac{a(z)}{2}} + \text{ lower degree terms.}$$

We remark that the notation $\gamma_{x,y,z}$ here stands for Lusztig's original notation $\gamma_{x,y,z^{-1}}$ in [6]. The following properties are due to Lusztig [6] except the (j) (which is trivial) and (k) (proved in [12]).

(e) $\gamma_{x,y,z} \neq 0 \Longrightarrow x \underset{L}{\sim} y^{-1}$, $y \underset{L}{\sim} z$, $x \underset{R}{\sim} z$.
(f) $x \underset{L}{\sim} y^{-1}$ if and only if $\gamma_{x,y,z} \neq 0$ for some $z \in W$.
(g) $\gamma_{x,y,z} = \gamma_{y,z^{-1},x^{-1}} = \gamma_{z^{-1},x,y^{-1}}$.
(h) $\gamma_{x,d,x} = \gamma_{d,x^{-1},x^{-1}} = \gamma_{x^{-1},x,d} = 1$ if $x \underset{L}{\sim} d$ and d is a distinguished involution.
(i) $\gamma_{x,y,z} = \gamma_{y^{-1},x^{-1},z^{-1}}$.
(j) If $\omega, \tau \in W$ has length 0, then

$$\gamma_{\omega x, y\tau, \omega z\tau} = \gamma_{x,y,z}, \quad \gamma_{x\omega, \tau y, z} = \gamma_{x,\omega\tau y, z}.$$

(k) Let $x, y, z \in W$ be such that (1) x is in a left string with respect to $* = \{s, t\}$ and also in a right string with respect to $\# = \{s', t'\}$, (2) y is in a left string with respect to $\# = \{s', t'\}$ and also in a right string with respect to $\star = \{s'', t''\}$, (3) z is in a left string with respect to $* = \{s, t\}$ and also in a right string with respect to $\star = \{s'', t''\}$. Then

$$\gamma_{x,y,z} = \gamma_{{}^*x^\#, {}^\#y^\star, {}^*z^\star}.$$

For $w \in W$, set $\tilde{T}_w = q^{-l(w)/2}T_w$. For $x, y \in W$, write

$$\tilde{T}_x\tilde{T}_y = \sum_{z\in W} f_{x,y,z}\tilde{T}_z, \qquad f_{x,y,z} \in \mathcal{A} = \mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}].$$

(l) If x, y, w are in a two-sided cell of W, $f_{x,y,w} = \lambda q^{\frac{a(w)}{2}} +$ lower degree terms and as Laurent polynomials in $q^{\frac{1}{2}}$, deg $f_{x,y,z} \leq a(w)$ for all $z \in W$, then

$$\gamma_{x,y,w} = \lambda.$$

(m) Each left cell (resp. each right cell) of W contains a unique distinguished involution.

(n) Each two-sided cell of W contains only finitely many left cells.

(o) Let I be a subset of S such that the subgroup W_I of W generated by I is finite. Then the longest element w_I is a distinguished involution.

Let d be a distinguished involution in W.

(p) For any $\omega \in \Omega$, the element $\omega d\omega^{-1}$ is a distinguished involution.

(q) Suppose $s, t \in S$ and st has order 3. Then $d \in D_L(s,t)$ if and only if $d \in D_R(s,t)$. If $d \in D_L(s,t)$, then ${}^*d^*$ is a distinguished involution.

1.5. Some identities

Assume $s, t \in S$ and st has order 4. Let w, u, v be in W such that $l(ststw) = l(w) + 4$ and $l(ststv) = l(v) + 4$. We have (see 1.6.3 in [12])

(a) $\gamma_{tsw,u,tv} = \gamma_{sw,u,stv}$,

(b) $\gamma_{tsw,u,tsv} = \gamma_{sw,u,sv} + \gamma_{sw,u,stsv}$,

(c) $\gamma_{tsw,u,tstv} = \gamma_{sw,u,stv}$,

(d) $\gamma_{tstw,u,tv} + \gamma_{tw,u,tv} = \gamma_{stw,u,stv}$,

(e) $\gamma_{tstw,u,tsv} = \gamma_{stw,u,stsv}$,

(f) $\gamma_{tstw,u,tstv} + \gamma_{tw,u,tstv} = \gamma_{stw,u,stv}$.

Assume $s, t \in S$ and st has order 4. Let w, u, v be in W such that $l(ustst) = l(u) + 4$ and $l(vstst) = l(v) + 4$. We have (loc.cit.)

(a') $\gamma_{w,ut,vst} = \gamma_{w,uts,vs}$,

(b') $\gamma_{w,ust,vst} = \gamma_{w,us,vs} + \gamma_{w,usts,vs}$,

(c') $\gamma_{w,utst,vst} = \gamma_{w,uts,vs}$,

(d') $\gamma_{w,ut,vtst} + \gamma_{w,ut,vt} = \gamma_{w,uts,vts}$,

(e') $\gamma_{w,ust,vtst} = \gamma_{w,usts,vts}$,

(f') $\gamma_{w,utst,vtst} + \gamma_{w,utst,vt} = \gamma_{w,uts,vts}$.

1.6. The based ring of a two-sided cell

For each two-sided cell c of W, let J_c be the free $\mathbb{Z}$-module with a basis $t_w,\ w \in c$. Define

$$t_x t_y = \sum_{z \in c} \gamma_{x,y,z} t_z.$$

Then J_c is an associative ring with unit $\sum_{d \in \mathcal{D} \cap c} t_d$.

The ring $J = \bigoplus_c J_c$ is a ring with unit $\sum_{d \in \mathcal{D}} t_d$. Sometimes J is called the asymptotic Hecke algebra since Lusztig established an injective $\mathcal{A}$-algebra homomorphism (see [6])

$$\phi : H \to J \otimes \mathcal{A}, \quad C_x \mapsto \sum_{\substack{d \in \mathcal{D} \\ w \in W \\ w \underset{L}{\sim} d}} h_{x,d,w} t_w.$$

1.7. Lusztig's conjecture on the structure of J_c

Let G be a simply connected simple algebraic group over $\mathbb{C}$ and W be the extended affine Weyl group attached to G. In [8] Lusztig establishes a bijection between the set of the two-sided cells of W and the set of the unipotent classes of G. (In the case $SP_6(\mathbb{C})$ concerned in subsequent sections, the bijection was already established in [3].) For each two-sided cell c of W, let u be a unipotent element in the unipotent class corresponding to c and let F_c be a maximal reductive subgroup of the centralizer of u in G.

Conjecture (Lusztig [8])**.** *Keep notations above. Then there exists a finite set Y with an algebraic action of F_c and a bijection*

$\pi : c \to$ *the set of isomorphism classes of irreducible F_c-vector bundles on $Y \times Y$.*

such that

(i) The bijection π induces a based ring isomorphism (see [7] for definition)

$$\pi : J_c \to K_{F_c}(Y \times Y), \quad t_x \mapsto \pi(x).$$

(ii) $\pi(x^{-1})_{(a,b)} = \pi(x)^*_{(b,a)}$ is the dual representation of $\pi(x)_{(b,a)}$.

1.8. A modification of Lusztig's conjecture on the structure of J_c

Motivated by Theorem 5.5 in Section 5 and the discussion of the cocenter of J in Section 5 of [1] and the evidence from the asymptotic Hecke algebra of an affine Weyl group of type $\tilde{A}_2$, we suggest a modification of Lusztig's conjecture on the structure of J_c which is stated for any connected reductive group over $\mathbb{C}$.

Let W be the extended affine Weyl group attached to a connected reductive group over $\mathbb{C}$ and let c be a two-sided cell of W. Note that Lusztig's bijection in [8] is valid for any connected reductive group over $\mathbb{C}$. Let F_c be a maximal reductive subgroup of the centralizer of an element in the corresponding unipotent class of G. Then there should exist a reductive group $\tilde{F}_c$ with the following properties[1]

(i) The reductive group $\tilde{F}_c$ is a simply connected covering of F_c. That is, the identity component $\tilde{F}_c^\circ$ has simply connected derived group, and there is a natural surjective homomorphism $\tilde{F}_c \to F_c$ with finite kernel. In particular, if F_c° has simply connected derived group, then $\tilde{F}_c = F_c$.

(ii) There exists a finite set Y with an algebraic action of $\tilde{F}_c$ and an injection

$$\pi : c \hookrightarrow \text{the set of iso. classes of irreducible } \tilde{F}_c\text{-vector bundles on } Y \times Y.$$

such that

(iii) The injection π induces a based ring injection

$$\Pi : J_c \to K_{\tilde{F}_c}(Y \times Y), \quad t_x \mapsto \pi(x).$$

(iv) $\pi(x^{-1})_{(a,b)} = \pi(x)^*_{(b,a)}$ is the dual representation of $\pi(x)_{(b,a)}$.

(v) $K_{\tilde{F}_c}(Y \times Y)$ is a finitely generated left (and right as well) $\Pi(J_c)$-module.

It seems natural that the F_c-set $\mathbf{B}_e$ defined in a recent paper (see [10]) would have an $\tilde{F}_c$-action compatible with the F_c-action and then $\mathbf{B}_e$ should be a good candidate for the set Y above.

[1] Added in proof: this modification of Lusztig's conjecture on the structure of J_c was partially resolved by Oron Yehonatan Propp in his PhD Thesis "A Coherent Categorification of the Asymptotic Affine Hecke Algebras", MIT, 2023. In the thesis it is showed that J_c embeds in a K-group of equivariant vector bundles on the square of a finite set.

2. Cells in an extended affine Weyl group of type $\tilde{B}_3$

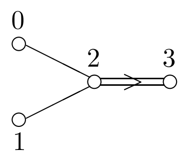
In this section $G = Sp_6(\mathbb{C})$, so that the extended affine Weyl group W attached to G is of type $\tilde{B}_3$. The left cells and two-sided cells are described by J. Du (see [3]). We recall his results.

2.1. The Coxeter graph of W

As usual, we number the 4 simple reflections $s_0,\ s_1,\ s_2,\ s_3$ in W so that

$$\begin{aligned}
&s_0s_1 = s_1s_0,\\
&s_0s_3 = s_3s_0,\\
&s_1s_3 = s_3s_1,\\
&(s_0s_2)^3 = (s_1s_2)^3 = e,\\
&(s_2s_3)^4 = e,
\end{aligned}$$

where e is the neutral element in W. The relations among the simple reflections can be read through the following Coxeter graph:

$\tilde{B}_3$: 

There is a unique nontrivial element τ in W with length 0. We have $\tau^2 = e,\ \tau s_0\tau = s_1,\ \tau s_i\tau = s_i$ for $i = 2,3$. Note that s_1, s_2, s_3 generate the Weyl group W_0 of type B_3 and s_0, s_1, s_2, s_3 generate an affine Weyl group W' of type $\tilde{B}_3$. And W is generated by $\tau,\ s_0, s_1, s_2, s_3$.

2.2. Cells in W

According to [3], the extended affine Weyl group W attached to $Sp_6(\mathbb{C})$ has 8 two-sided cells:

$$A,\quad B,\quad C,\quad D,\quad E,\quad F,\quad G,\quad H.$$

The following table displays some useful information on these two-sided cells.

X	$a(X)$	Number of left cells in X	Size of Jordan blocks of the corresp. unip. class in $Sp_6(\mathbb{C})$	Max. red. subgroup of the centralizer of any element in the corresp. unip. class
A	9	48	(111111)	$Sp_6(\mathbb{C})$
B	6	24	(21111)	$Sp_4(\mathbb{C}) \times \mathbb{Z}/2\mathbb{Z}$
C	4	18	(2211)	$SL_2(\mathbb{C}) \times O_2(\mathbb{C})$
D	3	12	(222)	$O_3(\mathbb{C})$
E	2	8	(411)	$SL_2(\mathbb{C}) \times \mathbb{Z}/2\mathbb{Z}$
F	2	6	(33)	$SL_2(\mathbb{C})$
G	1	4	(42)	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$
H	0	1	(6)	$\mathbb{Z}/2\mathbb{Z}$

The notations for two-sided cells in the table are the same as those in [3], which will be replaced by other notations in subsequent sections, otherwise confusion would happen since notations C, F, G are already used for other objects.

In subsequent sections, for a reduced expression $s_{i_1} s_{i_2} \cdots s_{ik}$ of an element in W, we often write $i_1 i_2 \cdots i_k$ instead of the reduced expression.

In the rest of the paper, W always stands for the affine Weyl group attached to $Sp_6(\mathbb{C})$, τ, s_i are as in Subsection 2.1, and all representations are rational representations of algebraic groups.

3. The based ring of the two-sided cell containing $s_0 s_1$

3.1. The two-sided cell containing $s_0 s_1$

In this section c stands for the two-sided cell of W containing $s_0 s_1$. According to Figure I and Theorem 6.4 of [3], c has six left cells. We list the six left cells and representative elements in the left cells given in figure I of [3]:

$$\Gamma_1,\ 01; \quad \Gamma_2,\ 012; \quad \Gamma_3,\ 0123; \quad \Gamma_4,\ 01232; \quad \Gamma_5,\ 012321; \quad \Gamma_6,\ 012320.$$

The value of a-function on c is 2.

The corresponding unipotent class in $Sp_6(\mathbb{C})$ has Jordan block sizes (411). The maximal reductive subgroup of the centralizer of an element in the unipotent class is $F_c = \mathbb{Z}/2\mathbb{Z} \times SL_2(\mathbb{C})$. Let ϵ be the nontrivial one dimensional representation of $\mathbb{Z}/2\mathbb{Z}$ and $V(k)$ be an irreducible representation of $SL_2(\mathbb{C})$ with highest weight k. They can be regarded as irreducible representations of F_c naturally. Up to isomorphism, the irreducible representations of F_c are $V(k),\ \epsilon \otimes V(k),\ k = 0, 1, 2, 3, \ldots$. We will denote $\epsilon \otimes V(k)$ by $\epsilon V(k)$.

Let $x_k=(s_0s_1s_2s_3s_2)^ks_1s_0,\ u_1=e,\ u_2=s_2,\ u_3=s_3s_2,\ u_4=s_2s_3s_2,\ u_5=s_1s_2s_3s_2,\ u_6=s_0s_2s_3s_3$.

According to Theorem 6.4 of [3], we have

(a) $c=\{u_ix_ku_j^{-1},\ u_i\tau x_ku_j^{-1}\,|\,1\le i,j\le 6,\ k=0,1,2,3,\dots\}$.
(b) $\Gamma_j=\{u_ix_ku_j^{-1},\ u_i\tau x_ku_j^{-1}\,|\,1\le i\le 6,\ k=0,1,2,3,\dots\}$, j=$1,2,\dots,6$.

Let Y=$\{1,\ 2,\dots,\ 6\}$ and let F_c act on Y trivially. Then $K_{F_c}(Y\times Y)$ is isomorphic to the 6×6 matrix ring $M_6(\mathrm{Rep}\,F_c)$, where $\mathrm{Rep}\,F_c$ is the representation ring of F_c. Recall that $F_c=\mathbb{Z}/2\mathbb{Z}\times SL_2(\mathbb{C})$ in this section.

The main result in this section is the following theorem.

Theorem 3.2. *Let c be the two-sided cell of W (the affine Weyl group W attached to $Sp_6(\mathbb{C})$) containing s_0s_1. Then the map*

$$\pi: c\to M_6(Rep\,F_c),\quad u_ix_ku_j^{-1}\mapsto V(k)_{ij},\ u_i\tau x_ku_j^{-1}\mapsto \epsilon V(k)_{ij}$$

induces a based ring isomorphism

$$\pi: J_c\to M_6(Rep\,F_c),\quad t_{u_ix_ku_j^{-1}}\mapsto V(k)_{ij},\quad t_{u_i\tau x_ku_j^{-1}}\mapsto \epsilon V(k)_{ij},$$

where $V(k)_{ij}$ (resp. $\epsilon V(k)_{ij}$) is the matrix in $M_6(Rep\,F_c)$ whose entry at (p,q) is $V(k)$ (resp. $\epsilon V(k)$) if $(p,q)=(i,j)$ and is 0 otherwise.

Remark. Theorem 4 in [3] implies that Lusztig's conjecture on the structure of J_c is true. Since under the isomorphism $K_{F_c}(Y\times Y)\simeq M_6(\mathrm{Rep}\,F)$, irreducible F_c-vector bundles on $Y\times Y$ correspond to those $V(k)_{ij},\ \epsilon V(k)_{ij}$, hence Theorem 3.2 provides a computable verification for Lusztig's conjecture on the structure of J_c.

We prove Theorem 3.2 by establishing three lemmas.

Lemma 3.3. *Let $1\le i,j,m,n\le 6$ and k,l be nonnegative integers. For $z_k=x_k$ or τx_k, $z_l=x_l$ or τx_l, and $z_p=x_p$ or τx_p, we have*

(a) $\gamma_{u_iz_ku_j^{-1},u_mz_lu_n^{-1},z}=0$ *if* $j\ne m$ *or* $z\ne u_i\tau^az_pu_n^{-1}$, $a=0,1$, *for some p;*
(b) $\gamma_{u_iz_ku_j^{-1},u_jz_lu_n^{-1},u_iz_pu_n^{-1}}=\gamma_{z_k,z_l,z_p}$, *for any nonnegative integer p.*

Proof. Note that $z_l^{-1}=z_l$. If $\gamma_{u_iz_ku_j^{-1},u_mz_lu_n^{-1},z}\ne 0$, then by 1.4(e) we get $u_iz_ku_j^{-1}\underset{L}{\sim}(u_mz_lu_n^{-1})^{-1}=u_nz_lu_m^{-1}$, $u_iz_ku_j^{-1}\underset{R}{\sim}z$, $u_mz_lu_n^{-1}\underset{L}{\sim}z$. By (b) in Subsection 3.1 we see that the first assertion is true.

Now we prove the second assertion. Let $*=\{s_1,s_2\}$, $\#=\{s_2,s_3\}$ and $\star=\{s_0,s_2\}$. Then

(c) $\Gamma_2 = \Gamma_1^*, \quad \Gamma_4 = \Gamma_2^\#, \quad \Gamma_5 = \Gamma_4^*, \quad \Gamma_6 = \Gamma_4^\star.$

Applying 1.4(k) we see that (b) is true if none of i, j, n is 3.
Now assume that $i = 3$. By 1.5 (b) we get

$$\gamma_{u_3 z_k u_j^{-1}, u_j z_l u_n^{-1}, u_3 z_p u_n^{-1}} = \gamma_{u_2 z_k u_j^{-1}, u_j z_l u_n^{-1}, u_2 z_p u_n^{-1}} + \gamma_{u_2 z_k u_j^{-1}, u_j z_l u_n^{-1}, u_4 z_p u_n^{-1}}.$$

By Part (a) of the lemma, we have $\gamma_{u_2 z_k u_j^{-1}, u_j z_l u_n^{-1}, u_4 z_p u_n^{-1}} = 0$. Then using (c) above and 1.4(k) we get

$$\gamma_{u_3 z_k u_j^{-1}, u_j z_l u_n^{-1}, u_3 z_p u_n^{-1}} = \gamma_{u_2 z_k u_j^{-1}, u_j z_l u_n^{-1}, u_2 z_p u_n^{-1}} = \gamma_{z_k u_j^{-1}, u_j z_l u_n^{-1}, z_p u_n^{-1}}.$$

Similarly, if $n = 3$, we have

$$\gamma_{u_i z_k u_j^{-1}, u_j z_l u_3^{-1}, u_i z_p u_3^{-1}} = \gamma_{u_i z_k u_j^{-1}, u_j z_l u_2^{-1}, u_i z_p u_2^{-1}} = \gamma_{u_i z_k u_j^{-1}, u_j z_l, u_i z_p}.$$

We have showed for any $1 \le i, n \le 6$ the following identity holds:

$$\gamma_{u_i z_k u_j^{-1}, u_j z_l u_n^{-1}, u_i z_p u_n^{-1}} = \gamma_{z_k u_j^{-1}, u_j z_l, z_p}.$$

Note that $z_k^{-1} = z_k$. By above identity and 1.4(g), we get

$$\gamma_{z_k u_j^{-1}, u_j z_l, z_p} = \gamma_{u_j z_l, z_p^{-1}, u_j z_k^{-1}} = \gamma_{z_l, z_p, z_k} = \gamma_{z_k, z_l, z_p}.$$

Assertion (b) is proved and the lemma is proved. □

Lemma 3.4. *For nonnegative integers k, l, p, and $a, b = 0, 1$, we have*

$$\gamma_{\tau^a x_k, \tau^b x_l, \tau^{a+b} x_p} = \gamma_{x_k, x_l, x_p}, \quad \gamma_{\tau^a x_k, \tau^b x_l, \tau^c x_p} = 0 \text{ if } \tau^c \ne \tau^{a+b}.$$

Proof. The assertion follows from 1.4(j). □

Lemma 3.5. *For nonnegative integers k, l we have*

$$t_{x_k} t_{x_l} = \sum_{0 \le p \le \min\{k,l\}} t_{x_{k+l-2p}}.$$

Proof. If $k = 0$ or $l = 0$, the identity above is trivial since x_0 is a distinguished involution.

Now assume that $k = 1$ and $l \ge 1$. Let $\zeta = q^{\frac{1}{2}} - q^{-\frac{1}{2}}$. By a simple computation we see

$$\begin{aligned}\tilde{T}_{x_1} \tilde{T}_{x_l} =& \zeta^2 (\tilde{T}_{x_{l+1}} + \tilde{T}_{x_{l-1}} + \tilde{T}_{s_0 s_1 s_3 (s_2 s_3 s_2 s_0 s_1)^l} + \tilde{T}_{s_0 s_2 s_3 s_2 s_1 (s_2 s_3 s_2 s_0 s_1)^l}) \\ &+ \text{lower degree terms,}\end{aligned}$$

Since $a(s_0s_1s_3(s_2s_3s_2s_0s_1)^l) \geq a(s_0s_1s_3) = 3$, $a(s_0s_2s_3s_2s_1(s_2s_3s_2s_0s_1)^l) \geq a(s_2s_1s_2) = 3$, we see that $s_0s_1s_3(s_2s_3s_2s_0s_1)^l$ and $s_0s_2s_3s_2s_1(s_2s_3s_2s_0s_1)^l$ are not in the two-sided cell c. By 1.4(l), we have

$$t_{x_1}t_{x_l} = t_{x_{l+1}} + t_{x_{l-1}}.$$

For $k \geq 2$, since $t_{x_k} = t_{x_1}t_{x_{k-1}} - t_{x_{k-2}}$, we can use induction on k to prove the lemma. The argument is completed. □

Proof of Theorem 3.2. Combining Lemmas 3.3, 3.4 and 3.5 we see that Theorem 3.2 is true. □

4. The based ring of the two-sided cell containing s_1s_3

4.1. The two-sided cell containing s_1s_3

In this section c stands for the two-sided cell of W containing s_1s_3. According to Figure I and Theorem 6.4 of [3], c has eight left cells. We list the eight left cells and representative elements in the left cells given in [Figure I of [3]:

Γ_1,	13;	Γ_2,	132;	Γ_3,	1323;	Γ_4,	1320;
Γ_5,	03;	Γ_6,	032;	Γ_7,	0323;	Γ_8,	0321.

The value of a-function on c is 2.

According to [3], the corresponding unipotent class in $Sp_6(\mathbb{C})$ has Jordan block sizes (33). The maximal reductive subgroup of the centralizer of an element in the unipotent class is $F_c = SL_2(\mathbb{C})$. Let $V(k)$ be an irreducible representation of $F_c = SL_2(\mathbb{C})$ with highest weight k. Up to isomorphism, the irreducible representations of F_c are $V(k)$, $k = 0, 1, 2, 3, \ldots$.

Let $x_k = (\tau s_0s_3s_2)^k s_1s_3$, $u_1 = e$, $u_2 = s_2$, $u_3 = s_3s_2$, $u_4 = s_0s_2$, $u_5 = \tau$, $u_6 = \tau s_2$, $u_7 = \tau s_3s_2$, $u_8 = \tau s_0s_2$.

According to Theorem 6.4 of [3], we have

(a) $c = \{u_ix_ku_j^{-1} \,|\, 1 \leq i, j \leq 8,\ k = 0, 1, 2, 3, \ldots\}$.

(b) $\Gamma_j = \{u_ix_ku_j^{-1} \,|\, 1 \leq i \leq 8,\ k = 0, 1, 2, 3, \ldots\}$, $j = 1, 2, 3, 4, 5, 6, 7, 8$.

Let $Y = \{1,\ 2, \ldots,\ 7,\ 8\}$ and let F_c act on Y trivially. Then $K_{F_c}(Y \times Y)$ is isomorphic to the 8×8 matrix ring $M_8(\operatorname{Rep} F_c)$, where $\operatorname{Rep} F_c$ is the representation ring of $F_c = SL_2(\mathbb{C})$.

The main result in this section is the following.

Theorem 4.2. *Let c be the two-sided cell of W (the extended affine Weyl group attached to $Sp_6(\mathbb{C})$) containing s_1s_3. Then the map*

$$\pi : c \to M_8(Rep\, F_c), \quad u_i x_k u_j^{-1} \mapsto V(k)_{ij}$$

induces a based ring isomorphism

$$\pi : J_c \to M_8(Rep\, F_c), \quad t_{u_i x_k u_j^{-1}} \mapsto V(k)_{ij},$$

where $V(k)_{ij}$ is the matrix in $M_8(Rep\, F_c)$ whose entry at (p, q) is $V(k)$ if $(p, q) = (i, j)$ and is 0 otherwise.

Remark. Theorem 4 in [3] implies that Lusztig's conjecture on the structure of J_c is true. Since under the isomorphism $K_{F_c}(Y \times Y) \simeq M_8(\mathrm{Rep}\, F_c)$, irreducible F_c-vector bundles on $Y \times Y$ correspond to $V(k)_{ij}$'s, Theorem 4.2 provides a computable verification for Lusztig's conjecture on the structure of J_c.

We prove Theorem 4.2 by establishing two lemmas.

Lemma 4.3. *Let $1 \le i, j, m, n \le 8$ and k, l be nonnegative integers. Then*

(a) $\gamma_{u_i x_k u_j^{-1}, u_m x_l u_n^{-1}, z} = 0$ *if $j \neq m$ or $z \neq u_i x_p u_n^{-1}$ for some p;*
(b) $\gamma_{u_i x_k u_j^{-1}, u_j x_l u_n^{-1}, u_i x_p u_n^{-1}} = \gamma_{x_k, x_l, x_p}$, *for any nonnegative integer p.*

Proof. Note that $x_l^{-1} = x_l$. If $\gamma_{u_i x_k u_j^{-1}, u_m x_l u_n^{-1}, z} \neq 0$, then by 1.4(e) we get $u_i x_k u_j^{-1} \underset{L}{\sim} (u_m x_l u_n^{-1})^{-1} = u_n x_l u_m^{-1}$, $u_i x_k u_j^{-1} \underset{R}{\sim} z$, and $u_m x_l u_n^{-1} \underset{L}{\sim} z$. By (b) in Subsection 4.1 we see that the first assertion is true.

Now we prove the second assertion. Let $* = \{s_1, s_2\}$, $\# = \{s_2, s_3\}$ and $\star = \{s_0, s_2\}$. Then

(c) $\Gamma_2 = \Gamma_1^*$, $\Gamma_3 = \Gamma_1^\#$, $\Gamma_4 = \Gamma_2^\star$, $\Gamma_5 = \tau\Gamma_1\tau$, $\Gamma_6 = \tau\Gamma_2\tau$, $\Gamma_7 = \tau\Gamma_3\tau$, $\Gamma_8 = \tau\Gamma_4\tau$.

Applying 1.4(j) and 1.4(k) (repeatedly if necessary) we get the following identity.

$$\gamma_{u_i x_k u_j^{-1}, u_j x_l u_n^{-1}, u_i x_p u_n^{-1}} = \gamma_{x_k u_j^{-1}, u_j x_l, x_p}.$$

Note that $\tau^2 = e$. Again using 1.4(j) and 1.4(k) (repeatedly if necessary) we get the following identity.

$$\gamma_{x_k u_j^{-1}, u_j x_l, x_p} = \gamma_{x_k, x_l, x_p}.$$

Part (b) is proved and the lemma is proved. $\square$

Lemma 4.4. *For nonnegative integers k, l, we have*

$$t_{x_k}t_{x_l} = \sum_{0\le p\le \min\{k,l\}} t_{x_{k+l-2p}}.$$

Proof. If $k=0$ or $l=0$, the identity above is trivial since x_0 is a distinguished involution.

Now assume that $k=1$ and $l\ge 1$. Put $\xi = q^{\frac{1}{2}}+q^{-\frac{1}{2}}$. Let $H^{<13}$ be the two-sided ideal of H spanned by all C_w's with $a(w)\ge 3$ (cf. Subsection 1.4). Before continuing, we make a convention: *we shall use the symbol $\Box$ for any element in the two-sided ideal $H^{<13}$ of H. Then $\Box+\Box=\Box$ and $h\Box=\Box$ for any $h\in H$.*

First we have

$$C_{x_1} = C_{\tau s_3 s_0 s_2 s_1 s_3} = C_{\tau s_3}C_{s_0}C_{s_2}C_{s_1}C_{s_3} - C_{\tau s_0 s_1 s_3}, \quad \text{and } C_{\tau s_0 s_1 s_3}\in H^{<13}.$$

Note that $L(x_k)=\{s_1,s_3\}$. Hence

$$C_{x_1}C_{x_l} = C_{\tau s_3}C_{s_0}C_{s_2}C_{s_1}C_{s_3}C_{x_l}+\Box = \xi^2 C_{\tau s_3}C_{s_0}C_{s_2}C_{x_l}+\Box. \tag{4}$$

We compute $C_{\tau s_3}C_{s_0}C_{s_2}C_{x_l}$ step by step.

Step 1. Compute $C_{s_2}C_{x_l}$. We have

$$C_{s_2}C_{x_l} = C_{s_2x_l} + \sum_{\substack{z\prec x_l\\ s_2z<z}} \mu(z,x_l)C_z.$$

Note that $L(x_l)=\{s_1,s_3\}$. Assume that $z\prec x_l$ and $s_2z\le z$. If $s_1z\le z$, then $\{s_1,s_2\}\subset L(z)$ and $a(z)\ge a(s_1s_2s_1)=3$. In this case, we have $C_z\in H^{<13}$. If $s_1z\ge z$, by 1.1(c) we must have $s_1z=x_l$. Then $z=\tau s_3s_2x_{l-1}$. This contradicts $s_2z\le z$. Therefore we have

$$C_{s_2}C_{x_l} = C_{s_2x_l}+\Box\in C_{s_2x_l}+H^{<13}. \tag{5}$$

Step 2. Compute $C_{s_0}C_{s_2x_l}$. We have

$$C_{s_0}C_{s_2x_l} = C_{s_0s_2x_l} + \sum_{\substack{z\prec s_2x_l\\ s_0z<z}} \mu(z,s_2x_l)C_z.$$

Note that $L(s_2x_l)=\{s_2\}$. Assume that $z\prec s_2x_l$ and $s_0z\le z$. If $s_2z\le z$, then $\{s_0,s_2\}\subset L(z)$ and $a(z)\ge a(s_0s_2s_0)=3$. In this case, we have $C_z\in$

$H^{<13}$. If $s_2z \geq z$, by 1.1(c) we must have $z = x_l$. This contradicts $s_0z \leq z$. Therefore we have

$$C_{s_0}C_{s_2x_l} = C_{s_0s_2x_l} + \square \in C_{s_0s_2x_l} + H^{<13}. \tag{6}$$

Step 3. Compute $C_{\tau s_3}C_{s_0s_2x_l}$. We have

$$C_{\tau s_3}C_{s_0s_2x_l} = C_{x_{l+1}} + \sum_{\substack{z \prec s_0s_2x_l \\ s_3z<z}} \mu(z, s_0s_2x_l)C_{\tau z}.$$

Assume that $z \prec s_0s_2x_l$ and $s_3z \leq z$. Using 1.4(b), 1.4(c) and 1.4(d), we see that $a(\tau z) \geq 2$, and if $a(\tau z) = 2$ then $x_l \underset{L}{\sim} \tau z \underset{R}{\sim} x_1$. We are only concerned with those $C_{\tau z}$'s in above summation with $a(\tau z) = 2$. Then $\tau z = x_m$ for some $m < l$ and $L(z) = \{s_0, s_3\}$. Note that $L(s_0s_2x_l) = \{s_0\}$. We then have $\mu(z, s_0s_2x_l) = \tilde{\mu}({}^{*\star}z, {}^{*\star}(s_0s_2x_l)) = \tilde{\mu}(s_1s_2z, x_l)$, where $\star = \{s_0, s_2\}$, $* = \{s_1, s_2\}$. Since $m < l$, we have $\tilde{\mu}(s_1s_2z, x_l) = \mu(s_1s_2z, x_l)$. Noting that $s_3s_1s_2z = s_3s_1s_2\tau x_m \geq s_1s_2\tau x_m$ and $s_3x_l \leq x_l$, by 1.1(c) we see $s_3s_1s_2z = x_l$, which implies that $\tau z = x_{l-1}$.

In conclusion, if $z \prec s_0s_2x_l$ and $s_3z \leq z$, then either $C_z \in H^{<13}$ or $z = \tau x_{l-1}$. Hence we have

$$C_{\tau s_3}C_{s_0s_2x_l} = C_{x_{l+1}} + C_{x_{l-1}} + \square. \tag{7}$$

Combining formulas (4)–(7) we get

$$C_{x_1}C_{x_l} = \xi^2(C_{x_{l+1}} + C_{x_{l-1}}) + \square \in \xi^2(C_{x_{l+1}} + C_{x_{l-1}}) + H^{<13}.$$

Therefore we have $t_{x_1}t_{x_l} = t_{x_{l+1}} + t_{x_{l-1}}$.

For $k \geq 2$, since $t_{x_k} = t_{x_1}t_{x_{k-1}} - t_{x_{k-2}}$, we can use induction on k to prove the lemma. The argument is completed. $\square$

Proof of Theorem 4.2. Combining Lemmas 4.3 and 4.4 we see that Theorem 4.2 is true. $\square$

5. The based ring of the two-sided cell containing $s_1s_2s_1$

5.1. Representatives of left cells in the two-sided cell containing $s_1s_2s_1$

Now we consider the two-sided cell in W containing $s_1s_2s_1$. In [1] and [11] it is shown that Lusztig's conjecture on the structure of the based ring of the two-sided cell needs modification. In this section we give a description of this based ring. For consistence, we keep notations from [11] for the two-sided cell of W containing $s_1s_2s_1$. In particular, we denote D for the two-sided cell of W containing $s_1s_2s_1$.

According to Figure I and Theorem 6.4 of [3], there are 12 left cells in the two-sided cell D. A representative of each left cell is also given in [3]. We number the left cells and state Du's result as follows.

(a) The 12 left cells in the two-sided cell D and a representative of each left cell in D are:

$$\begin{array}{llllll}
\Gamma_1 = D_{013}, & 013; & \Gamma_2 = D_2, & 0132; & \Gamma_3 = D_3, & 01323; \\
\Gamma_4 = D_{12}, & 01321; & \Gamma_5 = D_{02}, & 01320; & \Gamma_6 = D_{01}, & 013201; \\
\Gamma_7 = D_{13}, & 013213; & \Gamma_8 = D_{03}, & 013203; & \Gamma_9 = \widehat{D'_2}, & 0132132; \\
\Gamma_{10} = D'_2, & 0132032; & \Gamma_{11} = D_0, & 01321320; & \Gamma_{12} = D_1, & 01320321.
\end{array}$$

The value of a-function on D is 3.

Let Γ and Γ' be two left cells of W. If $\Gamma' = \Gamma^*$ for some $* = \{s,t\}$ (see Subsection 1.3 for definition of $*$-operation), then we write $\Gamma \xrightarrow{\{s,t\}} \Gamma'$. The following result is easy to verify.

Lemma 5.2. *Keep notations as above. Then we have two graphs:*

$$\begin{tikzcd}
\begin{matrix}\Gamma_3\\ 01323\end{matrix} \arrow[r, dash, "\{s_2,s_3\}"] & \begin{matrix}\Gamma_1\\ 013\end{matrix} \arrow[r, dash, "\{s_1,s_2\}"] & \begin{matrix}\Gamma_2\\ 0132\end{matrix}
\end{tikzcd}$$

$$\begin{tikzcd}
\begin{matrix}\Gamma_{11}\\ 01321320\end{matrix} \arrow[d, dash, "\{s_0,s_2\}"] & & & & \begin{matrix}\Gamma_{12}\\ 01320321\end{matrix} \arrow[d, dash, "\{s_1,s_2\}"] \\
\begin{matrix}\Gamma_9\\ 0132132\end{matrix} \arrow[r, dash, "\{s_2,s_3\}"] \arrow[d, dash, "\{s_1,s_2\}"] & \begin{matrix}\Gamma_4\\ 01321\end{matrix} \arrow[r, dash, "\{s_0,s_2\}"] & \begin{matrix}\Gamma_6\\ 013201\end{matrix} \arrow[r, dash, "\{s_1,s_2\}"] & \begin{matrix}\Gamma_5\\ 01320\end{matrix} \arrow[r, dash, "\{s_2,s_3\}"] & \begin{matrix}\Gamma_{10}\\ 0132032\end{matrix} \arrow[d, dash, "\{s_0,s_2\}"] \\
\begin{matrix}\Gamma_7\\ 013213\end{matrix} & & & & \begin{matrix}\Gamma_8\\ 013203\end{matrix}
\end{tikzcd},$$

where each vertex contains a left cell and a representative element of the left cell.

5.3. The two-sided cell containing $s_1s_2s_1$

Let

$$
\begin{array}{lll}
u_k = (s_0s_1s_3s_2)^k s_0s_1s_3, & & \\
x_k = (s_1s_2s_3s_0)^k s_1s_2s_1, & x'_0 = \tau s_2s_0s_1s_2s_1, & x'_{k+1} = \tau s_0s_2s_3s_0x_k, \\
p_1 = e, & p_2 = s_2, & p_3 = s_3s_2, \\
p_4 = s_1s_2, & p_5 = s_0s_2, & p_6 = s_0s_1s_2, \\
p_7 = s_3s_1s_2, & p_8 = s_3s_0s_2, & p_9 = s_2s_3s_1s_2, \\
p_{10} = s_2s_3s_0s_2, & p_{11} = s_0s_2s_3s_1s_2, & p_{12} = s_1s_2s_3s_0s_2; \\
q_4 = p_1 = e, & q_5 = \tau, & q_6 = s_0, \\
q_7 = s_3, & q_8 = s_3\tau, & q_9 = s_2s_3, \\
q_{10} = s_2s_3\tau, & q_{11} = s_0s_2s_3, & q_{12} = s_1s_2s_3\tau.
\end{array}
$$

According to Theorem 6.4 of [3], we have

(a) The two-sided cell D consists of the following elements:

$$p_iu_kp_j^{-1},\ p_i\tau u_kp_j^{-1},\ q_lx_0q_m^{-1},\ q_lx_0q_6^{-1},\ q_lx_0q_6^{-1}\tau,\ q_lx'_0q_m^{-1},$$

where $1 \le i, j \le 12,\ 4 \le l \le 12,\ 4 \le m \ne 6 \le 12,\ k \ge 0$.

(b1) For $j = 1,\ 2,\ 3$, the left cell Γ_j consists of the following elements:

$$p_iu_kp_j^{-1}, \quad p_i\tau u_kp_j^{-1}, \qquad 1 \le i \le 12,\ k \ge 0.$$

(b2) For $j = 4,\ 5,\ 7,\ 8, \dots,\ 12$, the left cell Γ_j consists of the following elements:

$$p_iu_kp_j^{-1}, \quad p_i\tau u_kp_j^{-1}, \quad q_lx_0q_j^{-1}, \quad q_lx'_0q_j^{-1},$$
$$1 \le i \le 12,\ 4 \le l \le 12,\ k \ge 0.$$

Note that $p_4u_kp_4^{-1} = x_{k+1},\ p_4\tau u_kp_4^{-1} = x'_{k+1}$.

(b3) The left cell Γ_6 consists of the following elements:

$$p_iu_kp_6^{-1}, \quad p_i\tau u_kp_6^{-1}, \quad q_lx_0q_6^{-1}, \quad q_lx_0q_6^{-1}\tau,$$
$$1 \le i \le 12,\ 4 \le l \le 12,\ k \ge 0.$$

5.4. The reductive group F_c and its rational irreducible representations

According to [3], for the two-sided cell D, the corresponding unipotent class in $Sp_6(\mathbb{C})$ has Jordan block sizes (222). The maximal reductive subgroup of the centralizer of an element in the unipotent class is $F_c = O_3(\mathbb{C}) = \mathbb{Z}/2\mathbb{Z} \times SO_3(\mathbb{C})$. Let $\tilde{F}_c = \mathbb{Z}/2\mathbb{Z} \times SL_2(\mathbb{C})$ be the simply connected covering of F_c.

Let Y be a set of 12 elements and let $\tilde{F}_c$ act on Y trivially. Then $K_{\tilde{F}_c}(Y \times Y)$ is isomorphic to the 12×12 matrix ring $M_{12}(\mathrm{Rep}\,\tilde{F}_c)$, where $\mathrm{Rep}\,\tilde{F}_c$ is the representation ring of $\tilde{F}_c$.

For a nonnegative integer k, let $V(k)$ be an irreducible representation of $SL_2(\mathbb{C})$ with highest weight k. Let ϵ be the sign representation of $\mathbb{Z}/2\mathbb{Z}$. Regarding $V(k)$ and ϵ as representations of $\tilde{F}_c$ naturally, then, up to isomorphism, the irreducible representations of $\tilde{F}_c$ are $V(k),\ \epsilon V(k),\ k = 0,\ 1,\ 2, \ldots$. When k is even, $V(k)$ and $\epsilon V(k)$ are also irreducible representation of F_c.

Let $V(k)_{ij} \in M_{12}(\mathrm{Rep}\,\tilde{F}_c)$ be the matrix whose entry at (i,j) is $V(k)$ and is 0 elsewhere. Similarly we define $\epsilon V(k)_{ij}$. The main result in this section is the following theorem.

Theorem 5.5. *There is a natural injection*

$$\begin{aligned}
&\pi : c \hookrightarrow M_{12}(Rep\ \tilde{F}_c),\\
&p_i u_k p_j^{-1} \longmapsto \begin{cases} V(2k)_{ij}, & 1 \le i,j \le 3,\\ V(2k+2)_{ij}, & 4 \le i,j \le 12,\\ V(2k+1)_{ij}, & \textit{otherwise};\end{cases}\\
&p_i \tau u_k p_j^{-1} \longmapsto \begin{cases} \epsilon V(2k)_{ij}, & 1 \le i,j \le 3,\\ \epsilon V(2k+2)_{ij}, & 4 \le i,j \le 12,\\ \epsilon V(2k+1)_{ij}, & \textit{otherwise};\end{cases}\\
&y \longmapsto V(0)_{lm}, \quad \textit{if } y \textit{ can be obtained from } x_0 \textit{ by a}\\
&\qquad\qquad \textit{sequence of left and/or right star operations},\\
&y \longmapsto \epsilon V(0)_{lm}, \quad \textit{if } y \textit{ can be obtained from } x_0' \textit{ by a}\\
&\qquad\qquad \textit{sequence of left and/or right star operations},
\end{aligned}$$

where $y = q_l x_0 q_m^{-1}$ *or* $q_l x_0' q_m^{-1}$ $(m \ne 6)$ *or* $q_l x_0 q_6^{-1}$ *or* $q_l x_0 q_6^{-1}\tau$, $4 \le l, m \le 12$.

The injection π *induces an injective ring homomorphism*

$$\Pi : J_c \to M_{12}(Rep\,\tilde{F}_c) \simeq K_{\tilde{F}_c}(Y \times Y), \quad t_w \mapsto \pi(w),$$

where Y *is a set of 12 elements with trivial* $\tilde{F}_c$ *action.*

Remark. It is immediate from [3] that the diagonal points $i = j$ would not have nontrivial central extensions. Here we offer a computable version for this fact.

Proof. We need to prove that

(8) $$\Pi(t_wt_u)=\pi(w)\cdot\pi(u),\quad \text{for all } w,u\in D.$$

Since D is the union of all $\Gamma_i,\ 1\le i\le 12$ and $D=D^{-1}$, we know that D is the union of all $\Gamma_i^{-1}\cap\Gamma_j,\ 1\le i,j\le 12$.

Assume that $w\in\Gamma_i^{-1}\cap\Gamma_j$ and $u\in\Gamma_k^{-1}\cap\Gamma_l$. Using 1.4(j), 1.4(k) and Lemma 5.2, we know that it suffices to prove formula (8) for $w\in\Gamma_i^{-1}\cap\Gamma_j$, $u\in\Gamma_k^{-1}\cap\Gamma_l$, $i,j,k,l\in\{1,4\}$. When $j\neq k$, by 1.4(e) we see that $t_wt_u=0$, hence formula (8) holds in this case. When $i=j=k=l$, according to Theorem 3.1 in [11], we know that formula (8) holds in this case. To complete the proof of the theorem we need to prove formula (8) for the following cases:

(i) $i=1,\ j=k=1,\ l=4$;
(ii) $i=1,\ j=k=4,\ l=4$;
(iii) $i=1,\ j=k=4,\ l=1$;
(iv) $i=4,\ j=k=1,\ l=1$;
(v) $i=4,\ j=k=1,\ l=4$;
(vi) $i=4,\ j=k=4,\ l=1$.

Keep notations from the above paragraph. Applying 1.4(g) and 1.4(i) we see that to prove formula (8) we only need to prove it for the following two cases: (♣) $w\in\Gamma_4^{-1}\cap\Gamma_1$ and $u\in\Gamma_1^{-1}\cap\Gamma_1$; (♠) $w\in\Gamma_4^{-1}\cap\Gamma_1$ and $u\in\Gamma_1^{-1}\cap\Gamma_4$.

Lemma ♣. *We have*

(a) $\Gamma_4^{-1}\cap\Gamma_1=\{s_1s_2u_k,\ s_1s_2\tau u_k\mid k\ge 0\}$ *and* $\Gamma_1^{-1}\cap\Gamma_1=\{u_k,\ \tau u_k\mid k\ge 0\}$.

(b) $t_{s_1s_2u_k}t_{u_l}=t_{s_1s_2\tau u_k}t_{\tau u_l}=\displaystyle\sum_{0\le i\le\min\{2k+1,2l\}}t_{s_1s_2u_{k+l-i}}$.

(c) $t_{s_1s_2\tau u_k}t_{u_l}=t_{s_1s_2u_k}t_{\tau u_l}=\displaystyle\sum_{0\le i\le\min\{2k+1,2l\}}t_{s_1s_2\tau u_{k+l-i}}$.

Proof. Part (a) is obtained from 5.3(b1) and 5.3(b2).

Now we prove (b). First we prove the identity:

(9) $$t_{s_1s_2u_0}t_{u_l}=t_{s_1s_2u_l}+t_{s_1s_2u_{l-1}},\quad \text{for any } l\ge 0.$$

Here we set $t_{s_1s_2u_{-1}}=0$.

When $l=0$, formula (9) is trivial as u_0 is a distinguished involution and we have set $t_{s_1s_2u_{-1}}=0$.

Now assume that $l>0$. Let $\xi=q^{\frac12}+q^{-\frac12}$. By a simple computation we get

(10) $$C_{s_1s_2u_0}=(C_{s_1}C_{s_2}-1)C_{s_0s_1s_3}$$

$$C_{s_0s_1s_3}C_{u_l}=\xi^3C_{u_l}. \tag{11}$$

Hence

$$C_{s_1s_2u_0}C_{u_l}=\xi^3(C_{s_1}C_{s_2}-1)C_{u_l}. \tag{12}$$

Before continuing, we make a convention: *we shall use the symbol $\square$ for any element in the two-sided ideal $H^{<013}$ of H spanned by all C_w with $a(w)>3$. Then $\square+\square=\square$ and $h\square=\square$ for any $h\in H$.*

In subsection 3.3, Step 1 of [11], we have shown the following identity:

$$C_{s_2}C_{u_l}=C_{s_2u_l}+\square\in C_{s_2u_l}+H^{<013}. \tag{13}$$

Now we compute $C_{s_1}C_{s_2u_l}$. By formula (1) in 1.1(e), we have

$$C_{s_1}C_{s_2u_l}=C_{s_1s_2u_l}+\sum_{\substack{y\prec s_2u_l\\ s_1y<y}}\mu(y,s_2u_l)C_y.$$

Note that $L(s_2u_l)=\{s_2\}$. Using 1.1(c) we get $\mu(u_l,s_2u_l)=1$. By definition of u_l, we see $s_1u_l<u_l$. According to 1.4(c) and 1.4(d), if C_y appears in the above summation with nonzero coefficient, $y\neq u_l$ and $C_y\notin H^{<013}$, then $y\in\Gamma_4^{-1}\cap\Gamma_1$.

Assume $y\prec s_2u_l$, $s_1y<y$ and $y\in\Gamma_4^{-1}\cap\Gamma_1$. By (a) we must have $y=s_1s_2u_k=s_1s_2(s_0s_1s_3s_2)^ks_0s_1s_3$ for some nonnegative integer $k\leq l-1$. Since $s_2s_0y\geq s_0y\geq y$, by 1.3(d) we get $\mu(y,s_2u_l)=\mu(s_0y,u_l)$. Now $s_3u_l\leq u_l$ and $s_3s_0y\geq s_0y$, by 1.1(c) we get $s_3s_0y=u_l$. Hence $y=s_1s_2u_{l-1}$.

We have shown

$$C_{s_1}C_{s_2u_l}=C_{s_1s_2u_l}+C_{u_l}+C_{s_1s_2u_{l-1}}+\square. \tag{14}$$

Combining formulas (12), (13) and (14), we get formula (9).

Recall the following formula in subsection 3.3 of [11]:

$$t_{u_k}t_{u_l}=\sum_{0\leq i\leq\min\{2k,2l\}}t_{u_{k+l-i}}. \tag{15}$$

Now we employ formulas (9) and (15) to prove the identity in (b). We use induction on k. When $k=0$, it is just formula (9). Assume that the formula in (b) is true for nonnegative integer less than k. Noting that $t_{s_1s_2u_{-1}}=0$, we get

$$\begin{aligned}
t_{s_1s_2u_k}t_{u_l} =&(t_{s_1s_2u_0}t_{u_k}-t_{s_1s_2u_{k-1}})t_{u_l}\\
=&t_{s_1s_2u_0}\cdot \sum_{0\le i\le \min\{2k,2l\}} t_{u_{k+l-i}}-t_{s_1s_2u_{k-1}}t_{u_l}\\
=&\sum_{0\le i\le \min\{2k,2l\}} t_{s_1s_2u_{k+l-i}}+\sum_{0\le i\le \min\{2k,2l\}} t_{s_1s_2u_{k+l-i-1}}\\
&-\sum_{0\le j\le \min\{2k-1,2l\}} t_{s_1s_2u_{k+l-1-j}}\\
=&\sum_{0\le i\le \min\{2k,2l\}} t_{s_1s_2u_{k+l-i}}+\sum_{1\le i\le \min\{2k+1,2l+1\}} t_{s_1s_2u_{k+l-i}}\\
&-\sum_{1\le j\le \min\{2k,2l+1\}} t_{s_1s_2u_{k+l-j}}\\
=&\sum_{0\le i\le \min\{2k+1,2l\}} t_{s_1s_2u_{k+l-i}}
\end{aligned}$$

Since $\tau u_k=u_k\tau$, by 1.4(j) we have $t_{s_1s_2\tau u_k}t_{\tau u_l}=t_{s_1s_2u_k}t_{u_l}$. Part (b) is proved.

Since $\tau u_k=u_k\tau$ and $\tau u_{k+l-i}=u_{k+l-i}\tau$, using 1.4(j) we see that Part (c) follows from Part (b).

The proof is completed. □

Lemma ♠. *(a) For $k\ge 0$, we have $s_1s_2u_ks_2s_1=x_{k+1}$ and $s_1s_2\tau u_ks_2s_1=x'_{k+1}$. Moreover, $\Gamma_4^{-1}\cap\Gamma_4=\{x_k,\ x'_k\mid k\ge 0\}$.*

For nonnegative integers k,l we have

(b) $t_{s_1s_2u_k}t_{u_ls_2s_1}=t_{s_1s_2\tau u_k}t_{u_l\tau s_2s_1}=\displaystyle\sum_{0\le i\le \min\{2k+1,2l+1\}} t_{x_{k+l+1-i}}.$

(c) $t_{s_1s_2\tau u_k}t_{u_ls_2s_1}=t_{s_1s_2u_k}t_{u_l\tau s_2s_1}=\displaystyle\sum_{0\le i\le \min\{2k+1,2l+1\}} t_{x'_{k+l+1-i}}.$

Proof. Part (a) follows from the discussion in Subsection 5.3.

Now we prove Part (b). First we prove

$$t_{s_1s_2u_0}t_{u_ls_2s_1}=t_{x_{l+1}}+t_{x_l}. \tag{16}$$

In subsection 4.2 of [11], it is showed $t_{s_1s_2u_0}t_{u_0s_2s_1}=t_{x_1}+t_{x_0}$. Now assume that $l\ge 1$. As before, we write $\xi=q^{\frac12}+q^{-\frac12}$. Since $C_{s_0s_1s_3}C_{u_ls_2s_1}=\xi^3C_{u_ls_2s_1}$, using formula (10) we get

$$C_{s_1s_2u_0}C_{u_ls_2s_1}=\xi^3(C_{s_1}C_{s_2}-1)C_{u_ls_2s_1}. \tag{17}$$

We compute the right hand side of equality (17) step by step. As in the proof of Lemma ♣, *we shall use the symbol* $\Box$ *for any element in the two-sided ideal* $H^{<013}$ *of* H *spanned by all* C_w *with* $a(w) > 3$.

Step 1: Compute $C_{s_2}C_{u_l s_2 s_1}$.

By formula (1) in 1.1(e), we have

$$C_{s_2}C_{u_l s_2 s_1} = C_{s_2 u_l s_2 s_1} + \sum_{\substack{y \prec u_l s_2 s_1 \\ s_2 y < y}} \mu(y, u_l s_2 s_1) C_y.$$

Note that $L(u_l s_2 s_1) = \{s_0, s_1, s_3\}$.

Assume $y \prec u_l s_2 s_1$ and $s_2 y < y$.

- If $s_0 y > y$, then by 1.1(c) we get $s_0 y = u_l s_2 s_1$. This contradicts the assumption. So $s_0 y > y$ would not occur.
- If $s_1 y > y$, then by 1.1(c) we get $s_1 y = u_l s_2 s_1$. This contradicts the assumption. So $s_1 y > y$ would not occur.
- If $s_0 y < y, s_1 y < y$ and $s_2 y < y$, then $a(y) \geq a(w_{012}) = 6$. So $C_y \in H^{<013}$.

Therefore,

$$C_{s_2}C_{u_l s_2 s_1} = C_{s_2 u_l s_2 s_1} + \Box. \tag{18}$$

Step 2: Similar to the proof of formula (14), we have

$$C_{s_1}C_{s_2 u_l s_2 s_1} = C_{s_1 s_2 u_l s_2 s_1} + C_{u_l s_2 s_1} + C_{s_1 s_2 u_{l-1} s_2 s_1} + \Box. \tag{19}$$

Note $s_1 s_2 u_l s_2 s_1 = x_{l+1}$. Combining formulas (17)–(19), we get (16).

Now we can prove part (b) using induction on k. By 1.4(j), we know $t_{s_1 s_2 u_k} t_{u_l s_2 s_1} = t_{s_1 s_2 \tau u_k} t_{u_l \tau s_2 s_1}$. Thus for $k = 0$, Part (b) is equivalent to formula (16), which is true. Now assume that $k \geq 1$ and Part (b) is true for $k-1$. Using Lemma ♣ and 1.4(i), induction hypothesis and formula (16), we get

$$\begin{aligned}
t_{s_1 s_2 u_k} t_{u_l s_2 s_1} =& (t_{s_1 s_2 u_0} t_{u_k} - t_{s_1 s_2 u_{k-1}}) t_{u_l s_2 s_1} \\
=& t_{s_1 s_2 u_0} \cdot \sum_{0 \leq i \leq \min\{2l+1, 2k\}} t_{u_{k+l-i} s_2 s_1} - t_{s_1 s_2 u_{k-1}} t_{u_l s_2 s_1} \\
=& \sum_{0 \leq i \leq \min\{2l+1, 2k\}} (t_{x_{k+l+1-i}} + t_{x_{k+l-i}}) - \sum_{0 \leq i \leq \min\{2l+1, 2k-1\}} t_{x_{k+l-i}} \\
=& \sum_{0 \leq i \leq \min\{2l+1, 2k+1\}} t_{x_{k+l+1-i}}.
\end{aligned}$$

This completes the proof of Part (b).

Proof of Part (c) is similar. First, it is easy to check that

$$C_{s_0 s_2 u_0} C_{u_0 s_2 s_1} = \xi^3 (C_{\tau x'_1} + C_{\tau x'_0}),$$

which implies

$$t_{s_1 s_2 \tau u_0} t_{u_0 s_2 s_1} = t_{x'_1} + t_{x'_0}.$$

Further, we prove that

$$t_{s_1 s_2 \tau u_0} t_{u_l s_2 s_1} = t_{x'_{l+1}} + t_{x'_l}.$$

Then using induction on k, as the proof of Part (b), we prove Part (c). The proof of Lemma ♠ is completed. □

We have completed the proof of Theorem 5.5. □

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PBW bases for modified quantum groups

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Dedicated to George Lusztig with admiration and appreciation

Abstract: We construct a basis for a modified quantum group of finite type, extending the PBW bases of positive and negative halves of a quantum group. Generalizing Lusztig's classic results on PBW bases, we show that this basis is orthogonal with respect to its natural bilinear form (and hence called a PBW basis), and moreover, the matrix for the PBW-expansion of the canonical basis is unital triangular. All these follow by a new construction of the modified quantum group of arbitrary type, which is built on limits of sequences of elements in tensor products of lowest and highest weight modules. Explicitly formulas are worked out in the rank one case.

Keywords: Quantum groups, canonical basis, PBW basis.

1. Introduction

1.1.

Let $\mathbf{U} = \mathbf{U}^-\mathbf{U}^0\mathbf{U}^+$ be a Drinfeld-Jimbo quantum group (of finite type). There are several remarkable bases for $\mathbf{U}^-$, known as PBW basis and canonical basis. The PBW basis is orthogonal with respect to a natural bilinear form on $\mathbf{U}^-$, providing an approach to the construction of the canonical basis [Lu90, Lu91] (also cf. [K91] for another construction of the canonical basis). Lusztig [Lu92, Lu93] further constructed a canonical basis for the modified quantum group $\dot{\mathbf{U}}$, which is compatible with canonical bases on the tensor products of lowest and highest weight modules ${}^\omega L(\lambda)\otimes L(\mu)$, for various dominant weights $\lambda, \mu \in X^+$. The canonical bases admit remarkable positivity properties in ADE and symmetric types; they have major impacts in several (geometric, combinatorial, and categorical) directions of representation theory.

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1.2.

The goal of this paper is to formulate a PBW basis for the modified quantum group $\dot{\mathbf{U}}$ of finite type with an orthogonality property and to establish its relations to canonical basis. The formulation follows by a new construction for the modified quantum group of arbitrary type, which is built on limits of sequences of elements in tensor products of lowest and highest weight modules.

1.3.

The inspiration came from the computation by the author [Wa21] of an orthogonal basis for an $\imath$quantum group of rank 1 in terms of $\imath$divided powers (aka $\imath$canonical basis) [BW18, BeW18]; in this setting, the $\imath$quantum group is a polynomial algebra in one variable. The $\imath$quantum groups arise from quantum symmetric pairs, and as we view quantum groups as $\imath$quantum groups of diagonal type, it is natural to explore the counterpart in the Drinfeld-Jimbo quantum group setting, starting again in rank 1.

Indeed, by imposing the orthogonality condition, we are able to construct naturally and compute explicitly an (apparently new) PBW basis for $\dot{\mathbf{U}}$ of rank 1 in terms of the canonical basis. Moreover, relations of these two bases when acting on the tensor product modules of the form ${}^{\omega}L(p) \otimes L(p+m)$ can be further observed; compare [Lu93, §25.3]. A (PBW) basis of $\dot{\mathbf{U}}$ with an orthogonality property emerges as a limit in a suitable sense of the tensor products of PBW basis elements acting on the tensor product modules. The rank 1 case is carried out in § 3.3–3.5, and some reader might prefer to go over the rank 1 case first. A bilinear pairing formula between canonical basis elements which appeared in Lauda [La10, Proposition 2.8] (who gave a long combinatorial proof) follows most naturally from the orthogonality of PBW basis for $\dot{\mathbf{U}}$ and the PBW expansion of the canonical basis of $\dot{\mathbf{U}}$.

We develop in Section 2 a framework for studying the limits of the so-called standard sequences of elements in tensor products of lowest and highest weight modules, ${}^{\omega}L(\lambda) \otimes L(\lambda+\zeta)$ with ζ fixed, as λ tends to ∞. This leads to a $\mathbb{Q}(q)$-linear isomorphism (valid for quantum groups of arbitrary type)

$$\mathcal{F}_{\zeta} \colon \mathbf{U}^{+} \otimes \mathbf{U}^{-} \xrightarrow{\cong} \dot{\mathbf{U}}\mathbf{1}_{\zeta},$$

which allows us to transfer any pair of bases for $\mathbf{U}^{\pm}$ to a new basis for $\dot{\mathbf{U}}$.

A *fused canonical basis* for $\dot{\mathbf{U}}$ is obtained via $\mathcal{F}_{\zeta}$ (for various ζ) from the pure tensors of canonical bases in $\mathbf{U}^{+} \otimes \mathbf{U}^{-}$.

1.4.

Now assume $\mathbf{U}$ is of finite type. The observations made in the rank 1 example suggest us to define the PBW basis for $\dot{\mathbf{U}}$ as the transfer under $\mathcal{F}_\zeta$ (for various ζ) of a tensor product of PBW bases in $\mathbf{U}^+ \otimes \mathbf{U}^-$; here the PBW bases of $\mathbf{U}^\pm$ can be associated to any reduced expression of the longest Weyl group element w_0. We show that the PBW basis for $\dot{\mathbf{U}}$ is orthogonal with respect to the standard bilinear form on $\dot{\mathbf{U}}$; it contains as a subset the PBW bases for $\mathbf{U}^+$ and $\mathbf{U}^-$. We further show that the transition matrix from the canonical basis to the PBW basis on $\dot{\mathbf{U}}$ is unital triangular.

Let us make clear that the PBW basis and the fused canonical basis for $\dot{\mathbf{U}}$ are bases over $\mathbb{Q}(q)$, but they do not lie in the integral form of $\dot{\mathbf{U}}$. Already in the rank 1 case, the coefficients of the PBW-expansion of the canonical basis are typically (up to factors of q-powers) rational functions of the following form

$$\prod_{a=1}^{m} \frac{1}{1-q^{-2a}}.$$

These rational functions expand as power series in q^{-1} with *positive integral* coefficients.

For a general ADE type, we show that the canonical basis is PBW-positive with *positive* coefficients in $\mathbb{N}[[q^{-1}]] \cap \mathbb{Q}(q)$. The proof relies on two positivity results. To that end, we show that the canonical basis has an expansion with coefficients in $\mathbb{N}[[q^{-1}]] \cap \mathbb{Q}(q)$ in terms of the fused canonical basis by using a positivity result in finite type of Webster [We15, Corollary 8.9] on the expansion of any canonical basis element in terms of pure tensors of canonical basis elements in ${}^\omega L(\lambda) \otimes L(\lambda+\zeta)$. On the other hand, the canonical basis of $\mathbf{U}^+$ is PBW-positive; this was first established in [Lu90, Corollary 10.7] when the reduced expressions of w_0 are "adapted", conjectured by Lusztig and proved by Syu Kato [Ka14] using categorification for arbitrary reduced expressions of w_0 (see [BKM14] for a second categorification proof and [Oya18] for another proof based on the positivity of canonical bases under comultiplication [Lu91]). Consequently, it follows that the fused canonical basis of $\dot{\mathbf{U}}$ is PBW-positive.

It will be very interesting to explore if the fused canonical basis and the PBW basis of $\dot{\mathbf{U}}$ admit a categorification in a generalized KLR categorical setting, generalizing the geometrical and categorical interpretation for the canonical basis and PBW basis of $\mathbf{U}^+$ [KL10, R12, VV11, Ka14, BKM14]. We shall return in [Wa21] to construct PBW bases for modified $\imath$quantum groups arising from quantum symmetric pairs.

2. A limit construction of the modified quantum group

In this section, we develop a framework for studying limits of sequences of elements in tensor product $\mathbf{U}$-modules ${}^{\omega}L(\lambda)\otimes L(\lambda+\zeta)$, as λ tends to ∞. This leads to a linear isomorphism $\mathbf{U}^{+}\otimes\mathbf{U}^{-}\rightarrow\dot{\mathbf{U}}\mathbf{1}_{\zeta}$, which allows to construct new bases for $\dot{\mathbf{U}}$, including the fused canonical basis of $\dot{\mathbf{U}}$ arising from the pure tensors of canonical bases in $\mathbf{U}^{+}\otimes\mathbf{U}^{-}$.

2.1. Quantum groups and bilinear forms

We denote by $\mathbf{U}$ the quantum group [Lu93] associated to the Cartan/root datum $(X,Y,\mathbb{I},\cdot)$ with a perfect bilinear pairing $\langle\cdot,\cdot\rangle : Y\times X\rightarrow\mathbb{Z}$; it is a $\mathbb{Q}(q)$-algebra generated by E_i, F_i, K_μ, for $i\in\mathbb{I},\mu\in Y$. Denote by $\dot{\mathbf{U}}$ the modified quantum group [Lu93, Chapter 23]. Denote by $F_i^{(m)}=F_i^m/[m]_{q_i}!$, for $i\in\mathbb{I},m\in\mathbb{N}$, the divided powers of F_i. Denote by X^+ the set of dominant weights in X. The comultiplication Δ satisfies

$$\Delta(F_i)=F_i\otimes\tilde{K}_i^{-1}+1\otimes F_i,\qquad \Delta(E_i)=E_i\otimes 1+\tilde{K}_i\otimes E_i. \tag{2.1}$$

By identifying $\mathbf{f}\cong\mathbf{U}^{-}$ $(z\mapsto z^{-})$, we have $\mathbb{Q}(q)$-linear maps ${}_ir,r_i:\mathbf{U}^{-}\rightarrow\mathbf{U}^{-}$, for $i\in\mathbb{I}$; cf. [Lu93, 1.2.13]. We have [Lu93, 3.1.6], for $y\in\mathbf{U}^{-}$,

$$E_iy-yE_i=\frac{\tilde{K}_i\,{}_ir(y)-r_i(y)\tilde{K}_i^{-1}}{q_i-q_i^{-1}}. \tag{2.2}$$

Similarly, by identifying $\mathbf{f}\cong\mathbf{U}^{+}$ $(z\mapsto z^{+})$, we have $\mathbb{Q}(q)$-linear maps ${}_ir,r_i$: $\mathbf{U}^{+}\rightarrow\mathbf{U}^{+}$, and for $x\in\mathbf{U}^{+}$,

$$F_ix-xF_i=\frac{\tilde{K}_i^{-1}\,{}_ir(x)-r_i(x)\tilde{K}_i}{q_i-q_i^{-1}}. \tag{2.3}$$

Note that $\mathbf{U}^{\pm}$ are $\mathbb{N}\mathbb{I}$-graded: $\mathbf{U}^{+}=\sum_{\nu\in\mathbb{N}\mathbb{I}}\mathbf{U}^{+}_{\nu}$, and $\mathbf{U}^{+}=\sum_{\nu\in\mathbb{N}\mathbb{I}}\mathbf{U}^{-}_{\nu}$. We say an element $x\in\mathbf{U}^{+}$ (respectively, $y\in\mathbf{U}^{-}$) is homogeneous if $x\in\mathbf{U}^{+}_{\nu}$ (respectively, $y\in\mathbf{U}^{-}_{\nu}$) for some ν. In this case, we denote $|x|=\nu$ and $|y|=-\nu$.

Denote by $\mathbf{B}$ the canonical basis for $\mathbf{f}$; the isomorphism $\mathbf{f}\cong\mathbf{U}^{\pm}$ induces the canonical bases $\mathbf{B}^{\pm}$ $(b\mapsto b^{\pm})$ for $\mathbf{U}^{\pm}$. Let $\mathbf{U}^{+}_{\mathbb{Z}[q^{-1}]}$ (respectively, $\mathbf{U}^{-}_{\mathbb{Z}[q^{-1}]}$) denote the $\mathbb{Z}[q^{-1}]$-span of the canonical basis in $\mathbf{U}^{+}$ (respectively, $\mathbf{U}^{-}$).

Let $L(\lambda)$ be the highest weight $\mathbf{U}$-module with highest weight vector η_λ of highest weight $\lambda\in X^+$, and let ${}^{\omega}L(\lambda)$ be the lowest weight $\mathbf{U}$-module with lowest weight vector $\xi_{-\lambda}$ of lowest weight $-\lambda$. For $x \in \mathbf{U}^+$, $y \in \mathbf{U}^-$, $\lambda \in X^+$ and $\zeta \in X$ such that $\lambda + \zeta \in X^+$, it follows by (2.1) that

$$y(\xi_{-\lambda} \otimes \eta_{\lambda+\zeta}) = \xi_{-\lambda} \otimes y\eta_{\lambda+\zeta}, \qquad x(\xi_{-\lambda} \otimes \eta_{\lambda+\zeta}) = x\xi_{-\lambda} \otimes \eta_{\lambda+\zeta}. \tag{2.4}$$

There is an anti-involution ρ on $\mathbf{U}$ such that, for $i \in \mathbb{I}, \nu \in Y$,

$$\rho(E_i) = q_i \tilde{K}_i F_i, \quad \rho(F_i) = q_i^{-1} E_i \tilde{K}_i^{-1}, \quad \rho(\tilde{K}_\nu) = \tilde{K}_{-\nu}.$$

According to [K91, Lu93], there is a bilinear form $(\cdot,\cdot)$ on $L(\mu)$, for $\mu \in X^+$, such that $(\eta_\mu, \eta_\mu) = 1$ and $(ux, y) = (x, \rho(u)y)$, for all $x, y \in L(\mu), u \in \mathbf{U}$; a bilinear form $(\cdot,\cdot)$ on ${}^\omega L(\mu)$ is defined similarly. A bilinear form $(\cdot,\cdot)$ on ${}^\omega L(\lambda) \otimes L(\mu)$ is defined by $(x \otimes y, x' \otimes y') = (x, x')(y, y')$.

There exists a unique $\mathbb{Q}(q)$-bilinear form $(\cdot,\cdot)$ on $\dot{\mathbf{U}}$ [Lu93, 26.1.2], which extends the one on $\mathbf{U}^-(\cong \mathbf{f})$ [Lu93, 1.2.3, 1.2.5] such that

$$(\mathbf{1}_\lambda x \mathbf{1}_\mu, \mathbf{1}_{\lambda'} x' \mathbf{1}_{\mu'}) = 0, \text{ for all } x, x' \in \dot{\mathbf{U}}, \text{ unless } \lambda = \lambda' \text{ and } \mu = \mu', \tag{2.5}$$
$$(ux, y) = (x, \rho(u)y), \text{ for } x, y \in \dot{\mathbf{U}}, u \in \mathbf{U}, \tag{2.6}$$
$$(f\mathbf{1}_\lambda, f'\mathbf{1}_\lambda) = (f, f'), \text{ for } f, f' \in \mathbf{U}^-, \lambda \in X. \tag{2.7}$$

This bilinear form is symmetric. Note

$$(F_i^{(m)}, F_i^{(n)}) = (E_i^{(m)}, E_i^{(n)}) = \delta_{m,n}(q_i^{-2}; q_i^{-2})_m^{-1}, \tag{2.8}$$

where we have denoted

$$(a; q_i^{-2})_m = \prod_{s=0}^{m-1}(1 - a q_i^{-2s}), \qquad (q_i^{-2}; q_i^{-2})_m = \prod_{s=1}^{m}(1 - q_i^{-2s}). \tag{2.9}$$

2.2. Standard sequences

We shall often deal with sequences of elements $\{z_\lambda\}_{\lambda \in X^+}$, where $z_\lambda \in {}^\omega L(\lambda) \otimes L(\lambda + \zeta)$ is a linear combination of elements of the form

(2.10) $\quad u\mathbf{1}_\zeta(\xi_{-\lambda} \otimes \eta_{\lambda+\zeta}), \quad$ for $u \in \dot{\mathbf{U}}$,

(2.11) $\quad x\xi_{-\lambda} \otimes y\eta_{\lambda+\zeta}, \quad$ for $(x, y) \in \mathbf{U}^+ \times \mathbf{U}^-$,

(2.12) $\quad$ where u and (x, y) run over some finite sets independent of λ.

We shall refer to such a sequence *standard*. The coefficients of z_λ usually take a certain form, which we now specify.

Definition 2.1. (1) A standard sequence $\{z_\lambda\}_{\lambda\in X^+}$ is said to be *bounded* if z_λ can be written as a linear combination of elements (2.10)–(2.11) subject to (2.12), with coefficients being a finite sum of the form

(2.13)
$$\sum_{s\geq 0}\sum_{\vec{i}=(i_1,\ldots,i_s)\in\mathbb{I}^s} f_{\vec{i}}(q)\prod_{a=1}^{s} q_{i_a}^{-2\langle i_a,\lambda\rangle},\ \text{ where } f_{\vec{i}}(q)\in\mathbb{Q}(q) \text{ is independent of } \lambda.$$

(2) A standard sequence $\{z_\lambda\}_{\lambda\in X^+}$ is said to be *asymptotically zero*, if z_λ can be written as a linear combination of elements (2.10)–(2.11) subject to (2.12), with coefficients being a finite sum of the form

(2.14)
$$\sum_{s\geq 1}\sum_{\vec{i}=(i_1,\ldots,i_s)\in\mathbb{I}^s} f_{\vec{i}}(q)\prod_{a=1}^{s} q_{i_a}^{-2\langle i_a,\lambda\rangle},\ \text{ where } f_{\vec{i}}(q)\in\mathbb{Q}(q) \text{ is independent of } \lambda.$$

We say λ tends to ∞ if $\langle i,\lambda\rangle$ tends to $+\infty$, for each $i\in\mathbb{I}$; in this case we shall denote $\lambda\mapsto\infty$. Note that the coefficients in (2.13) (respectively, (2.14)) converges in $\mathbb{Q}((q^{-1}))$ to some scalar (respectively, to 0) as λ tends to ∞. Given bounded standard sequences $\{z_\lambda\}_{\lambda\in X^+}, \{z'_\lambda\}_{\lambda\in X^+}$ and $\{z''_\lambda\}_{\lambda\in X^+}$, we shall denote

$$z_\lambda = o(1), \qquad \text{and } \ z'_\lambda = z''_\lambda + o(1),$$

if $\{z_\lambda\}_{\lambda\in X^+}$ is asymptotically zero and $\{z'_\lambda - z''_\lambda\}_{\lambda\in X^+}$ is asymptotically zero, respectively.

2.3. Approximations

There are 2 types of elements in (2.10)–(2.11), and we shall understand how to approximate one another as $\lambda\mapsto\infty$ in a precise fashion.

Lemma 2.2. *Let $x'\in\mathbf{U}^+, y\in\mathbf{U}^-$ be homogeneous, and $i\in\mathbb{I}$. Then we have*

(2.15)
$$E_i(x'\xi_{-\lambda}\otimes y\eta_{\lambda+\zeta}) = E_i x'\xi_{-\lambda}\otimes y\eta_{\lambda+\zeta} + \frac{q_i^{\langle i,\zeta+|x'|+|y|\rangle+2}}{q_i-q_i^{-1}} x'\xi_{-\lambda}\otimes {}_ir(y)\eta_{\lambda+\zeta}$$
$$- \frac{q_i^{-2\langle i,\lambda\rangle} q_i^{\langle i,-\zeta+|x'|\rangle}}{q_i-q_i^{-1}} x'\xi_{-\lambda}\otimes r_i(y)\eta_{\lambda+\zeta}.$$

Equivalently, we have

(2.16)
$$E_i x'\xi_{-\lambda} \otimes y\eta_{\lambda+\zeta} = E_i(x'\xi_{-\lambda} \otimes y\eta_{\lambda+\zeta}) - \frac{q_i^{\langle i,\zeta+|x'|+|y|\rangle+2}}{q_i - q_i^{-1}} x'\xi_{-\lambda} \otimes {}_ir(y)\eta_{\lambda+\zeta}$$
$$+ \frac{q_i^{-2\langle i,\lambda\rangle} q_i^{\langle i,-\zeta+|x'|\rangle}}{q_i - q_i^{-1}} x'\xi_{-\lambda} \otimes r_i(y)\eta_{\lambda+\zeta}.$$

Proof. Using the comultiplication formula (2.1) and the identity (2.2), we compute that

$$\begin{aligned}
&E_i(x'\xi_{-\lambda} \otimes y\eta_{\lambda+\zeta}) \\
&= E_i x'\xi_{-\lambda} \otimes y\eta_{\lambda+\zeta} + \tilde{K}_i x'\xi_{-\lambda} \otimes E_i y\eta_{\lambda+\zeta} \\
&= E_i x'\xi_{-\lambda} \otimes y\eta_{\lambda+\zeta} + \tilde{K}_i x'\xi_{-\lambda} \otimes \frac{\tilde{K}_i\, {}_ir(y) - r_i(y)\tilde{K}_i^{-1}}{q_i - q_i^{-1}} \eta_{\lambda+\zeta} \\
&= E_i x'\xi_{-\lambda} \otimes y\eta_{\lambda+\zeta} + \frac{q_i^{\langle i,\zeta+|x'|+|y|\rangle+2}}{q_i - q_i^{-1}} x'\xi_{-\lambda} \otimes {}_ir(y)\eta_{\lambda+\zeta} \\
&\quad - \frac{q_i^{\langle i,-2\lambda-\zeta+|x'|\rangle}}{q_i - q_i^{-1}} x'\xi_{-\lambda} \otimes r_i(y)\eta_{\lambda+\zeta}.
\end{aligned}$$

This proves (2.15). The identity (2.16) follows from (2.15). □

Lemma 2.3. *Let $x \in \mathbf{U}^+, y \in \mathbf{U}^-$ be homogeneous and $i \in \mathbb{I}$. Then we have*

(2.17)
$$F_i(x\xi_{-\lambda} \otimes y\eta_{\lambda+\zeta}) = x\xi_{-\lambda} \otimes F_i y\eta_{\lambda+\zeta} + \frac{q_i^{-\langle i,\zeta+|x|+|y|\rangle+2}}{q_i - q_i^{-1}} {}_ir(x)\xi_{-\lambda} \otimes y\eta_{\lambda+\zeta}$$
$$- \frac{q_i^{-2\langle i,\lambda\rangle} q_i^{\langle i,-\zeta-|y|\rangle}}{q_i - q_i^{-1}} r_i(x)\xi_{-\lambda} \otimes y\eta_{\lambda+\zeta}.$$

Proof. Using the comultiplication formula (2.1) and the identity (2.3), we compute that

$$\begin{aligned}
&F_i(x\xi_{-\lambda} \otimes y\eta_{\lambda+\zeta}) \\
&= x\xi_{-\lambda} \otimes F_i y\eta_{\lambda+\zeta} + F_i x\xi_{-\lambda} \otimes \tilde{K}_i^{-1} y\eta_{\lambda+\zeta} \\
&= x\xi_{-\lambda} \otimes F_i y\eta_{\lambda+\zeta} + \frac{\tilde{K}_i^{-1}\, {}_ir(x) - r_i(x)\tilde{K}_i}{q_i - q_i^{-1}} \xi_{-\lambda} \otimes \tilde{K}_i^{-1} y\eta_{\lambda+\zeta}
\end{aligned}$$

$$= x\xi_{-\lambda} \otimes F_i y\eta_{\lambda+\zeta} + \frac{q_i^{-\langle i,\zeta+|x|+|y|\rangle+2}}{q_i - q_i^{-1}} {}_i r(x)\xi_{-\lambda} \otimes y\eta_{\lambda+\zeta}$$
$$- \frac{q_i^{\langle i,-2\lambda-\zeta-|y|\rangle}}{q_i - q_i^{-1}} r_i(x)\xi_{-\lambda} \otimes y\eta_{\lambda+\zeta}.$$

The lemma is proved. □

Lemma 2.4. *Let $g \in \dot{\mathbf{U}}$.*

(1) If $\{z_\lambda\}_{\lambda\in X^+}$ is a bounded standard sequence, then so is $\{gz_\lambda\}_{\lambda\in X^+}$.
(2) If $\{z_\lambda\}_{\lambda\in X^+}$ is an asymptotically-zero standard sequence, then so is $\{gz_\lambda\}_{\lambda\in X^+}$. (We shall write $g\cdot o(1) = o(1)$.)

Proof. We shall prove (1) only, as the proof of (2) is entirely similar.

If $g = g_1 + g_2$ and if $g_i z_\lambda$ is a bounded standard sequences (for $i = 1, 2$), then so is gz_λ. So we can assume $g = E_{i_a} \cdots E_{i_1} F_{j_b} \cdots F_{j_1} \mathbf{1}_\zeta$, and we only need to concern about g acting on elements of the form (2.11). A simple induction on $a+b$ reduces the proof to two basic cases for $g = E_i\mathbf{1}_\zeta$ and $g = F_i\mathbf{1}_\zeta$, which in turn follow by applying the identities (2.15) and (2.17), respectively. □

Given $\mu = \sum_{i\in\mathbb{I}} n_i i \in \mathbb{N}\mathbb{I}$, we define its height $\operatorname{ht}\mu = \sum_i n_i$. Recall $\mathbf{B}$ is the canonical basis of $\mathbf{f}$, and any $b \in \mathbf{B}$ has weight $|b| \in \mathbb{N}\mathbb{I}$. Given $N, N' \in \mathbb{N}$, we shall denote by $P^+(N)$ (respectively, $P^-(N)$) the $\mathbb{Q}(q)$-submodule of $\mathbf{U}^+$ (respectively, $\mathbf{U}^-$) spanned by the elements $b^+ \in \mathbf{B}^+$ (respectively, $b^- \in \mathbf{B}^-$), for $b \in \mathbf{B}$ such that $\operatorname{ht}|b| \le N$. For $\zeta \in X$, we denote by $P(N, N')$ the $\mathbb{Q}(q)$-submodule of $\dot{\mathbf{U}}$ spanned by the elements $b_1^+ b_2^- \mathbf{1}_\zeta$, where $b_1, b_2 \in \mathbf{B}$ are such that $\operatorname{ht}|b_1| \le N, \operatorname{ht}|b_2| \le N'$ and $|b_1| - |b_2| = \zeta$. The following is the most crucial technical construction in this paper.

Proposition 2.5. *Let $\zeta \in X$.*

(1) Given $x \in \mathbf{U}^+, y \in \mathbf{U}^-$, there exists a unique element $x\natural_\zeta y \in \dot{\mathbf{U}}\mathbf{1}_\zeta$ such that

$$(x\natural_\zeta y)(\xi_{-\lambda} \otimes \eta_{\lambda+\zeta}) - x\xi_{-\lambda} \otimes y\eta_{\lambda+\zeta} = o(1). \tag{2.18}$$

(2) Given $u \in \dot{\mathbf{U}}\mathbf{1}_\zeta$, there exists a unique element $u'' = \sum_k x_k \otimes y_k \in \mathbf{U}^+ \otimes \mathbf{U}^-$ such that

$$u(\xi_{-\lambda} \otimes \eta_{\lambda+\zeta}) - \sum_k x_k\xi_{-\lambda} \otimes y_k\eta_{\lambda+\zeta} = o(1). \tag{2.19}$$

Proof. (1) For the existence, we shall prove the following more precise statement.

Claim 1. For $x \in \mathbf{U}^+$ and $y \in \mathbf{U}^-$ homogeneous, there exists $x\natural_\zeta y \in P(\mathrm{ht}\,|x|, \mathrm{ht}\,|y|)$ of the form

$$x\natural_\zeta y \in xy\mathbf{1}_\zeta + P(\mathrm{ht}\,|x|-1, \mathrm{ht}\,|y|-1) \text{ such that (2.18) holds.} \tag{2.20}$$

We prove Claim 1 by induction on $\mathrm{ht}\,|x|$. The case when $\mathrm{ht}\,|x| = 0$ clearly follows by (2.4). If $\mathrm{ht}\,|x| > 0$, we can assume x is of the form $x = E_i x'$, for some $x' \in \mathbf{U}^+$. We observe by (2.16) that

(2.21)

$$E_i x'\xi_{-\lambda} \otimes y\eta_{\lambda+\zeta} = E_i(x'\xi_{-\lambda} \otimes y\eta_{\lambda+\zeta}) - \frac{q_i^{\langle i, \zeta+|x'|+|y|\rangle+2}}{q_i - q_i^{-1}} x'\xi_{-\lambda} \otimes {}_ir(y)\eta_{\lambda+\zeta} + o(1).$$

Since $\mathrm{ht}\,|x'| < \mathrm{ht}\,|x|$, the inductive assumption can be applied to $x'\xi_{-\lambda} \otimes y\eta_{\lambda+\zeta}$ and $x'\xi_{-\lambda} \otimes {}_ir(y)\eta_{\lambda+\zeta}$ on the RHS (2.21), and we have

$$\begin{aligned} x'\xi_{-\lambda} \otimes y\eta_{\lambda+\zeta} &= (x'\natural_\zeta y)(\xi_{-\lambda} \otimes y\eta_{\lambda+\zeta}) + o(1), \\ x'\xi_{-\lambda} \otimes {}_ir(y)\eta_{\lambda+\zeta} &= (x'\natural_\zeta {}_ir(y))(\xi_{-\lambda} \otimes y\eta_{\lambda+\zeta}) + o(1). \end{aligned}$$

By Lemma 2.4, $E_i \cdot o(1)$ is of the form $o(1)$. Hence Equation (2.21) can be written as

$$E_i x'\xi_{-\lambda} \otimes y\eta_{\lambda+\zeta} = \Big(E_i(x'\natural_\zeta y) - \frac{q_i^{\langle i, \zeta+|x'|+|y|\rangle+2}}{q_i - q_i^{-1}} x'\natural_\zeta {}_ir(y)\Big)(\xi_{-\lambda} \otimes \eta_{\lambda+\zeta}) + o(1).$$

By the inductive assumption, we have

$$E_i(x'\natural_\zeta y) \in E_i(x'y\mathbf{1}_\zeta + P(\mathrm{ht}\,|x'|-1, \mathrm{ht}\,|y|-1)) \subseteq xy\mathbf{1}_\zeta + P(\mathrm{ht}\,|x|-1, \mathrm{ht}\,|y|-1)$$

and $x'\natural_\zeta {}_ir(y) \in P(\mathrm{ht}\,|x|-1, \mathrm{ht}\,|y|-1)$. Therefore, setting

$$x\natural_\zeta y := E_i(x'\natural_\zeta y) - \frac{q_i^{\langle i, \zeta+|x'|+|y|\rangle+2}}{q_i - q_i^{-1}} x'\natural_\zeta {}_ir(y)$$

will satisfy (2.18) and (2.20).

Assume that another element $w \in \dot{\mathbf{U}}\mathbf{1}_\zeta$ also satisfies the same property (2.18) as $x\natural_\zeta y$. Set $z := w - x\natural_\zeta y$, and $z_\lambda := z(\xi_{-\lambda} \otimes \eta_{\lambda+\zeta})$. Then $z_\lambda = o(1)$ by using (2.18) twice. Thus (z_λ, z_λ) converges to 0, as λ tends to ∞. On the other hand, by [Lu93, 26.2.3], we have (z_λ, z_λ) converges to (z, z), as λ tends to ∞. Hence we must have $(z, z) = 0$, and whence $z = 0$, i.e., $w = x\natural_\zeta y$.

(2) For the existence, we shall prove a more precise statement.

Claim 2. For $u = xy\mathbf{1}_\zeta \in \dot{\mathbf{U}}\mathbf{1}_\zeta$ with $x \in \mathbf{U}^+$ and $y \in \mathbf{U}^-$ homogeneous, there exists $u'' \in P^+(\mathrm{ht}\,|x|) \otimes P^-(\mathrm{ht}\,|y|)$ of the form

$$u'' \in x \otimes y + P^+(\mathrm{ht}\,|x| - 1) \otimes P^-(\mathrm{ht}\,|y| - 1) \text{ such that (2.19) holds.} \tag{2.22}$$

We prove the existence by induction on $\mathrm{ht}\,|x|$. If $|x| = 0$, then the statement follows by (2.4). Assume $\mathrm{ht}\,|x| > 0$, we can assume $x = E_i x'$, for some $x' \in \mathbf{U}^+$, and so $u = E_i x' y \mathbf{1}_\zeta$. Hence, by the inductive assumption on $x' y \mathbf{1}_\zeta$ (and recalling $E_i \cdot o(1) = o(1)$ by Lemma 2.4), we have

$$\begin{aligned} u(\xi_{-\lambda} \otimes \eta_{\lambda+\zeta}) &= E_i \cdot x' y \mathbf{1}_\zeta (\xi_{-\lambda} \otimes \eta_{\lambda+\zeta}) \\ &= E_i (x' \xi_{-\lambda} \otimes y \eta_{\lambda+\zeta} + \sum_\ell x'_\ell \xi_{-\lambda} \otimes y'_\ell \eta_{\lambda+\zeta}) + o(1), \end{aligned} \tag{2.23}$$

for some $\sum_\ell x'_\ell \otimes y'_\ell \in P^+(\mathrm{ht}\,|x| - 2) \otimes P^-(\mathrm{ht}\,|y| - 1)$. By applying (2.15), we see that RHS (2.23) $= u''(\xi_{-\lambda} \otimes \eta_{\lambda+\zeta}) + o(1)$, for u'' of the form (2.22). The uniqueness can be established by the same arguments as in (1). $\square$

2.4. To infinity

Recall Definition 2.1 for (asymptotically-zero) bounded standard sequences.

Lemma 2.6. *Let $\zeta \in X$, and let $\{z_\lambda\}_{\lambda \in X^+} \in {}^\omega L(\lambda) \otimes L(\lambda + \zeta)$ be a bounded standard sequence. Then, $(z_\lambda, u(\xi_{-\lambda} \otimes \eta_{\lambda+\zeta})) \in \mathbb{Q}(q)$ (and respectively, $(z_\lambda, x\xi_{-\lambda} \otimes y\eta_{\lambda+\zeta}) \in \mathbb{Q}(q)$) converges in $\mathbb{Q}((q^{-1}))$ as λ tends to ∞, for any $u \in \dot{\mathbf{U}}$, $x \in \mathbf{U}^+$, and $y \in \mathbf{U}^-$.*

Moreover, the following statements (a)–(d) for $\{z_\lambda\}_{\lambda \in X^+}$ are equivalent:

- *(a) $\{z_\lambda\}_{\lambda \in X^+}$ is asymptotically-zero;*
- *(b) $(z_\lambda, u(\xi_{-\lambda} \otimes \eta_{\lambda+\zeta}))$ converges in $\mathbb{Q}((q^{-1}))$ to 0 as λ tends to ∞, for any $u \in \dot{\mathbf{U}}$;*
- *(c) $(z_\lambda, x\xi_{-\lambda} \otimes y\eta_{\lambda+\zeta})$ converges in $\mathbb{Q}((q^{-1}))$ to 0 as λ tends to ∞, for any $x \in \mathbf{U}^+$ and $y \in \mathbf{U}^-$;*
- *(d) (z_λ, z'_λ) converges in $\mathbb{Q}((q^{-1}))$ to 0 as λ tends to ∞, for any bounded standard sequence $\{z'_\lambda\}_{\lambda \in X^+}$.*

If one of the conditions (a)–(d) above is satisfied, we say $\lim\limits_{\lambda \mapsto \infty} z_\lambda = 0$.

Proof. Let us prove that $(z_\lambda, u(\xi_{-\lambda}\otimes\eta_{\lambda+\zeta}))\in\mathbb{Q}(q)$ converges in $\mathbb{Q}((q^{-1}))$ as λ tends to ∞. Note $(z_\lambda, u(\xi_{-\lambda}\otimes\eta_{\lambda+\zeta}))=(\rho(u)z_\lambda,\xi_{-\lambda}\otimes\eta_{\lambda+\zeta})$, and $\{\rho(u)z_\lambda\}$ is a bounded standard sequence spanned by elements of the form (2.10)–(2.11) with coefficients $f_\lambda(q)$ as in (2.13). If $f_\lambda(q)u'(\xi_{-\lambda}\otimes\eta_{\lambda+\zeta})$ is a summand of $\rho(u)z_\lambda$, then $(f_\lambda(q)u'(\xi_{-\lambda}\otimes\eta_{\lambda+\zeta}),\xi_{-\lambda}\otimes\eta_{\lambda+\zeta})$ converges in $\mathbb{Q}((q^{-1}))$ to $\lim\limits_{\lambda\mapsto\infty} f_\lambda(q)\cdot(u',\mathbf{1}_\zeta)$, by [Lu93, 26.2.3]. If $f_\lambda(q)x'\xi_{-\lambda}\otimes y'\eta_{\lambda+\zeta}$ is a summand of $\rho(u)z_\lambda$, then $(f_\lambda(q)x'\xi_{-\lambda}\otimes y'\eta_{\lambda+\zeta},\xi_{-\lambda}\otimes\eta_{\lambda+\zeta})$ converges in $\mathbb{Q}((q^{-1}))$ to $\lim\limits_{\lambda\mapsto\infty} f_\lambda(q)\cdot(x',1)(y',1)$. Summarizing, $(z_\lambda, u(\xi_{-\lambda}\otimes\eta_{\lambda+\zeta}))=(\rho(u)z_\lambda,\xi_{-\lambda}\otimes\eta_{\lambda+\zeta})$ converges in $\mathbb{Q}((q^{-1}))$.

Assume (a) holds. Then (b) and (c) follow by the same arguments above together with $\lim\limits_{\lambda\mapsto\infty} f_\lambda(q)=0$, and subsequently (d) also follows. Therefore, for any bounded standard sequences $\{z_\lambda\}_{\lambda\in X^+},\{z'_\lambda\}_{\lambda\in X^+}$, we have

$$\lim_{\lambda\mapsto\infty}(z_\lambda,z'_\lambda)=0,\qquad\text{if either sequence is } o(1). \tag{2.24}$$

Now by Proposition 2.5(1), $x\xi_{-\lambda}\otimes y\eta_{\lambda+\zeta}=(x\natural_\zeta y)(\xi_{-\lambda}\otimes\eta_{\lambda+\zeta})+o(1)$, and we have already shown that $\lim\limits_{\lambda\mapsto\infty}(z_\lambda,(x\natural_\zeta y)(\xi_{-\lambda}\otimes\eta_{\lambda+\zeta}))$ exists. Thus, by applying (2.24), $\lim\limits_{\lambda\mapsto\infty}(z_\lambda,x\xi_{-\lambda}\otimes y\eta_{\lambda+\zeta})=\lim\limits_{\lambda\mapsto\infty}(z_\lambda,(x\natural_\zeta y)(\xi_{-\lambda}\otimes\eta_{\lambda+\zeta}))$ exists in $\mathbb{Q}((q^{-1}))$.

The equivalence of (b) and (c) follows by (2.24) and Proposition 2.5. Clearly Parts (b) and (c) are special cases of (d), and on the other hand, Part (d) follows from (b)–(c) combined easily.

It remains to show that (b) $\Rightarrow$ (a). By Proposition 2.5 and (2.24), we can assume that z_λ is spanned by elements of the form (2.11), i.e., $z_\lambda=\sum_u f_{\lambda,u}(q)u(\xi_{-\lambda}\otimes\eta_{\lambda+\zeta})$, for various nonzero u which are orthogonal to each other. Then, for each such u, $(z_\lambda,u(\xi_{-\lambda}\otimes\eta_{\lambda+\zeta}))$ converges in $\mathbb{Q}((q^{-1}))$ to $\lim\limits_{\lambda\mapsto\infty} f_{\lambda,u}(q)\cdot(u,u)$, by [Lu93, 26.2.3]. By the assumption (b), $\lim\limits_{\lambda\mapsto\infty} f_{\lambda,u}(q)=0$, so $f_{\lambda,u}(q)$ is of the form (2.14) and z_λ is asymptotically zero.

The lemma is proved. ☐

We have the following reformulation of Proposition 2.5. Given $x,y\in\mathbf{f}$, we have $x^+\in\mathbf{U}^+, y^-\in\mathbf{U}^-$, and we shall simply write $x\natural_\zeta y$ for $x^+\natural_\zeta y^-$.

Proposition 2.7. *Let $\zeta\in X$.*

(1) Given $x,y\in\mathbf{f}$, there exists a unique element $x\natural_\zeta y\in\dot{\mathbf{U}}\mathbf{1}_\zeta$ such that

$$\lim_{\lambda\mapsto\infty}\Big((x\natural_\zeta y)(\xi_{-\lambda}\otimes\eta_{\lambda+\zeta})-x^+\xi_{-\lambda}\otimes y^-\eta_{\lambda+\zeta}\Big)=0.$$

(2) *Given* $u \in \dot{\mathbf{U}}\mathbf{1}_\zeta$, *there exists a unique element* $u'' = \sum_k x_k \otimes y_k \in \mathbf{U}^+ \otimes \mathbf{U}^-$ *such that*

$$\lim_{\lambda \mapsto \infty} \Big(u(\xi_{-\lambda} \otimes \eta_{\lambda+\zeta}) - \sum_k x_k \xi_{-\lambda} \otimes y_k \eta_{\lambda+\zeta} \Big) = 0.$$

2.5. A linear isomorphism for $\dot{\mathbf{U}}\mathbf{1}_\zeta$

We now present the first main result of this paper, which is built on the constructions in Proposition 2.5 or its reformulation in Proposition 2.7; we retain the notation therein in Part (1) of the theorem below.

Theorem 2.8.

(1) *For any* $\zeta \in X$, *there exists a* $\mathbb{Q}(q)$*-linear isomorphism*

$$\mathcal{F}_\zeta \colon \mathbf{U}^+ \otimes \mathbf{U}^- \longrightarrow \dot{\mathbf{U}}\mathbf{1}_\zeta$$

such that

$$\mathcal{F}_\zeta(x \otimes y) = x \natural_\zeta y, \qquad \mathcal{F}_\zeta^{-1}(u) = u''.$$

(2) *Given bases* B_1, B_2 *for* $\mathbf{f}$, *the set*

$$B_1 \natural B_2 := \big\{ b_1 \natural_\zeta b_2 \mid b_1 \in B_1, b_2 \in B_2, \zeta \in X \big\} \tag{2.25}$$

forms a $\mathbb{Q}(q)$*-basis for* $\dot{\mathbf{U}}$. *Moreover, for* $b_1, b_1' \in B_1, b_2, b_2' \in B_2$ *and* $\zeta, \zeta' \in X$, *we have*

$$(b_1 \natural_\zeta b_2,\, b_1' \natural_{\zeta'} b_2') = \delta_{\zeta,\zeta'} (b_1, b_1')\, (b_2, b_2'). \tag{2.26}$$

Proof. (1) Write $\mathcal{F} = \mathcal{F}_\zeta$, and denote $\mathcal{G} : \dot{\mathbf{U}}\mathbf{1}_\zeta \longrightarrow \mathbf{U}^+ \otimes \mathbf{U}^-$ the linear map such that $\mathcal{G}(u) = u''$ as given in Proposition 2.5(2) or Proposition 2.7(2). It follows by the uniqueness in Proposition 2.7(2) that $\mathcal{G}\mathcal{F}(x \otimes y) = \mathcal{G}(x \natural_\zeta y) = x \otimes y$. Let $u \in \dot{\mathbf{U}}\mathbf{1}_\zeta$ and $\mathcal{G}(u) = u'' := \sum_k x_k \otimes y_k$ as in Proposition 2.7(2). Then by Proposition 2.7(1), we have $(\sum_k x_k \natural_\zeta y_k)(\xi_{-\lambda} \otimes \eta_{\lambda+\zeta}) - \sum_k x_k \xi_{-\lambda} \otimes y_k \eta_{\lambda+\zeta} = o(1)$. By the uniqueness in Proposition 2.7(1), we must have $u = \sum_k x_k \natural_\zeta y_k$, i.e., $u = \mathcal{F}\mathcal{G}(u)$. So $\mathcal{F}$ is a linear isomorphism with inverse $\mathcal{G}$.

(2) The first statement on bases follows by (1). The formula (2.26) is trivial if $\zeta' \neq \zeta$.

Assume $\zeta' = \zeta$. Set

$$z_\lambda := (b_1 \natural_\zeta b_2)(\xi_{-\lambda} \otimes \eta_{\lambda+\zeta}) - b_1 \xi_{-\lambda} \otimes b_2 \eta_{\lambda+\zeta},$$

$$z'_\lambda := (b'_1 \natural_\zeta b'_2)(\xi_{-\lambda} \otimes \eta_{\lambda+\zeta}) - b'_1\xi_{-\lambda} \otimes b'_2\eta_{\lambda+\zeta}.$$

It follows by Proposition 2.7 that $\lim_{\lambda \mapsto \infty} z_\lambda = 0$ and $\lim_{\lambda \mapsto \infty} z'_\lambda = 0$. Then we have

$$\begin{aligned}
&\big((b_1 \natural_\zeta b_2)(\xi_{-\lambda} \otimes \eta_{\lambda+\zeta}), (b'_1 \natural_\zeta b'_2)(\xi_{-\lambda} \otimes \eta_{\lambda+\zeta})\big) \\
&\quad - \big(b_1\xi_{-\lambda} \otimes b_2\eta_{\lambda+\zeta}, b'_1\xi_{-\lambda} \otimes b'_2\eta_{\lambda+\zeta}\big) \\
&= \big(z_\lambda, (b'_1 \natural_\zeta b'_2)(\xi_{-\lambda} \otimes \eta_{\lambda+\zeta})\big) + \big(z'_\lambda, b_1\xi_{-\lambda} \otimes b_2\eta_{\lambda+\zeta}\big),
\end{aligned}$$

whose RHS converges in $\mathbb{Q}((q^{-1}))$ to 0 as λ tends to ∞ by Lemma 2.6.

By [Lu93, 26.2.3], the pairing $\big((b_1 \natural_\zeta b_2)(\xi_{-\lambda} \otimes \eta_{\lambda+\zeta}), (b'_1 \natural_\zeta b'_2)(\xi_{-\lambda} \otimes \eta_{\lambda+\zeta})\big)$ converges in $\mathbb{Q}((q^{-1}))$ to $\big(b_1 \natural_\zeta b_2, b'_1 \natural_\zeta b'_2\big)$ as λ tends to ∞. On the other hand, we have

$$\big(b_1\xi_{-\lambda} \otimes b_2\eta_{\lambda+\zeta}, b'_1\xi_{-\lambda} \otimes b'_2\eta_{\lambda+\zeta}\big) = \big(b_1\xi_{-\lambda}, b'_1\xi_{-\lambda}\big)\big(b_2\eta_{\lambda+\zeta}, b'_2\eta_{\lambda+\zeta}\big),$$

which converges to $(b_1, b'_1)(b_2, b'_2)$ as λ tends to ∞. Hence (2.26) for $\zeta' = \zeta$ follows.

The theorem is proved. $\square$

2.6. Fused canonical basis for $\dot{\mathbf{U}}$

We have several distinguished choices of bases for $\mathbf{f}$. If we choose the canonical basis $\mathbf{B}$ for $\mathbf{f}$, the $\mathbb{Q}(q)$-basis

$$\mathbf{B} \natural \mathbf{B} = \big\{ b_1 \natural_\zeta b_2 \mid b_1, b_2 \in \mathbf{B}, \zeta \in X \big\}$$

is called a *fused canonical basis* for $\dot{\mathbf{U}}$. Note that

$$b^+ \mathbf{1}_\zeta \in \mathbf{B} \natural \mathbf{B},\ b^- \mathbf{1}_\zeta \in \mathbf{B} \natural \mathbf{B}, \qquad \text{for } b \in \mathbf{B}.$$

Recall Lusztig's canonical basis $\{b_1 \diamondsuit_\zeta b_2 | b_1, b_2 \in \mathbf{B}, \zeta \in X\}$ on $\dot{\mathbf{U}}$. Define a partial order $\leq$ on $\mathbf{B} \times \mathbf{B}$ as follows [Lu93, 24.3.1]. We say $(b'_1, b'_2) \leq (b_1, b_2)$ if $|b'_1| - |b'_2| = |b_1| - |b_2|$ and if either ($b_1 = b'_1$ and $b_2 = b'_2$) or

$$\mathrm{ht}\,|b'_1| < \mathrm{ht}\,|b_1| \text{ and } \mathrm{ht}\,|b'_2| < \mathrm{ht}\,|b_2|.$$

Theorem 2.9. *Let $\zeta \in X$, and $b_1, b_2 \in \mathbf{B}$. We have*

$$b_1 \diamondsuit_\zeta b_2 = b_1 \natural_\zeta b_2 + \sum_{(b'_1, b'_2) < (b_1, b_2)} P_{(b'_1, b'_2),(b_1, b_2)}(q)\, b'_1 \natural_\zeta b'_2$$

where $P_{(b'_1,b'_2),(b_1,b_2)}(q) \in \mathbb{Z}[[q^{-1}]] \cap \mathbb{Q}(q)$*. Moreover, for the ADE type,* $P^{\lambda}_{(b'_1,b'_2),(b_1,b_2)}(q)$ *lies in* $\mathbb{N}[[q^{-1}]] \cap \mathbb{Q}(q)$*.*

Proof. As $\{b'_1 \natural_\zeta b'_2 \mid b'_1, b'_2 \in \mathbf{B}\}$ is a basis for $\dot{\mathbf{U}}\mathbf{1}_\zeta$ by Theorem 2.8, we have

$$b_1 \diamondsuit_\zeta b_2 = \sum_{(b'_1,b'_2)} P_{(b'_1,b'_2),(b_1,b_2)}(q)\, b'_1 \natural_\zeta b'_2 \tag{2.27}$$

where $P_{(b'_1,b'_2),(b_1,b_2)}(q) \in \mathbb{Q}(q)$.

There is a canonical basis $\{(b_1 \diamondsuit b_2)_{\lambda,\lambda+\zeta} | b_1 \in \mathbf{B}(\lambda), b_2 \in \mathbf{B}(\lambda+\zeta)\}$ on ${}^{\omega}L(\lambda) \otimes L(\lambda+\zeta)$, for $\lambda \in X^+$ and $\lambda+\zeta \in X^+$, by [Lu93, 24.3.3]. By [Lu93, 25.2.1], we have

$$(b_1 \diamondsuit_\zeta b_2)(\xi_{-\lambda} \otimes \eta_{\lambda+\zeta}) = (b_1 \diamondsuit b_2)_{\lambda,\lambda+\zeta}, \text{ for } b_1 \in \mathbf{B}(\lambda), b_2 \in \mathbf{B}(\lambda+\zeta). \tag{2.28}$$

We shall assume λ is chosen to be large enough below such that $b_1 \in \mathbf{B}(\lambda), b_2 \in \mathbf{B}(\lambda+\zeta)$. When combining (2.28) with [Lu93, 24.3.3] on expansion of $(b_1 \diamondsuit b_2)_{\lambda,\lambda+\zeta}$ in terms of pure tensors of canonical bases on ${}^{\omega}L(\lambda)$ and $L(\lambda+\zeta)$, we have

$$\begin{aligned}(b_1 \diamondsuit_\zeta b_2)(\xi_{-\lambda} \otimes \eta_{\lambda+\zeta}) &= b_1^+ \xi_{-\lambda} \otimes b_2^- \eta_{\lambda+\zeta} \\ &\quad + \sum_{\substack{(b'_1,b'_2) \in \mathbf{B}(\lambda)\times\mathbf{B}(\lambda+\zeta) \\ (b'_1,b'_2) < (b_1,b_2)}} \widetilde{P}^{\lambda}_{(b'_1,b'_2),(b_1,b_2)}(q)\, b'^+_1 \xi_{-\lambda} \otimes b'^-_2 \eta_{\lambda+\zeta},\end{aligned} \tag{2.29}$$

where $\widetilde{P}^{\lambda}_{(b'_1,b'_2),(b_1,b_2)}(q) \in \mathbb{Z}[q^{-1}]$, with the superscript indicating its dependence on λ. We set $\widetilde{P}^{\lambda}_{(b'_1,b'_2),(b_1,b_2)}(q) = 0$ if $(b'_1, b'_2) \not\leq (b_1, b_2)$. By (2.27) and Proposition 2.5, we have

$$(b_1 \diamondsuit_\zeta b_2)(\xi_{-\lambda} \otimes \eta_{\lambda+\zeta}) = \sum_{(b'_1,b'_2)} P_{(b'_1,b'_2),(b_1,b_2)}(q)\, (b'_1 \xi_{-\lambda} \otimes b'_2 \eta_{\lambda+\zeta}) + o(1). \tag{2.30}$$

Comparing (2.29) and (2.30), we conclude that

$$\lim_{\lambda \mapsto \infty} \widetilde{P}^{\lambda}_{(b'_1,b'_2),(b_1,b_2)}(q) = P_{(b'_1,b'_2),(b_1,b_2)}(q). \tag{2.31}$$

Hence, we must have $P_{(b'_1,b'_2),(b_1,b_2)}(q) \in \mathbb{Z}[[q^{-1}]]$; moreover, $P_{(b'_1,b'_2),(b_1,b_2)}(q) = 0$ unless $(b'_1, b'_2) \leq (b_1, b_2)$, and $P_{(b_1,b_2),(b_1,b_2)}(q) = 1$.

Now assume $\mathbf{U}$ is of ADE type. By [We15, Corollary 8.9], $\widetilde{P}^{\lambda}_{(b_1',b_2'),(b_1,b_2)}(q)$ in (2.29) lies in $\mathbb{N}[q^{-1}]$, and then by (2.31), we have $P^{\lambda}_{(b_1',b_2'),(b_1,b_2)}(q) \in \mathbb{N}[[q^{-1}]] \cap \mathbb{Q}(q)$. □

3. PBW basis for $\dot{\mathbf{U}}$

In this section, we restrict ourselves to $\mathbf{U}$ of finite type; see however Remark 3.5. We construct and study the PBW basis for $\dot{\mathbf{U}}$ and its relation to the canonical basis. We present explicit formulas in rank 1.

3.1. PBW basis in finite type

Let Φ^+ be the set of positive roots and W be the associated Weyl group for $\mathbf{U}$. Let $\preceq$ be a convex order on Φ^+ associated to an (arbitrarily) fixed reduced expression of the longest element w_0 in the Weyl group W, and we denote the (decreasingly) ordered roots as $\Phi^+ = \{\beta_1, \ldots, \beta_N\}$, where $N = |\Phi^+|$. For each $\beta \in \Phi^+$, Lusztig [Lu90, Lu93] defined a root vector E_β using braid group action on $\mathbf{U}$, and its corresponding divided power $E_\beta^{(m)} = E_\beta^m/[m]_{q_\beta}!$, where $q_\beta = q_i$ if β lies in the W-orbit of a simple root α_i. We shall identify the set KP of Kostant partitions with $\mathbb{N}^{\Phi^+}$, thanks to the bijection given by $\mathbf{m} = (m_\beta)_{\beta\in\Phi^+} \in \mathbb{N}^{\Phi^+} \mapsto (\beta^{m_\beta})_{\beta\in\Phi^+} \in \mathrm{KP}$. For each $\mathbf{m} \in \mathrm{KP}$, we define

$$E_{\mathbf{m}} := \prod_{a=1}^{N} E_{\beta_a}^{(m_{\beta_a})}.$$

Then $\mathbf{W}^+ := \{E_{\mathbf{m}} | \mathbf{m} \in \mathrm{KP}\}$ forms a PBW basis for $\mathbf{U}^+$, which is orthogonal with respect to the bilinear form $(\cdot,\cdot)$ on $\mathbf{U}^+$ (see [Lu93, 38.2.3]).

The canonical basis for $\mathbf{f}$ can be parametrized by KP as well [Lu90, Lu93], and we shall write $\{b_{\mathbf{m}} | \mathbf{m} \in \mathrm{KP}\}$; via the isomorphism $\mathbf{f} \cong \mathbf{U}^+$, we obtain the canonical basis $\{b^+_{\mathbf{m}} | \mathbf{m} \in \mathrm{KP}\}$ for $\mathbf{U}^+$ and $\{b^-_{\mathbf{m}} | \mathbf{m} \in \mathrm{KP}\}$ for $\mathbf{U}^-$. Each canonical basis element in $\mathbf{U}^+$ can be characterized by the following 2 properties:

(i) $\overline{b^+_{\mathbf{m}}} = b^+_{\mathbf{m}}$;
(ii) $b^+_{\mathbf{m}} \in E_{\mathbf{m}} + \sum_{\mathbf{m}'\in\mathrm{KP}} q^{-1}\mathbb{Z}[q^{-1}]E_{\mathbf{m}'}$.

There is a suitable partial order $\preceq$ on KP (compatible with the convex order on Φ^+), so that $b^+_{\mathbf{m}} \in E_{\mathbf{m}} + \sum_{\mathbf{m}'\prec\mathbf{m}} q^{-1}\mathbb{Z}[q^{-1}]E_{\mathbf{m}'}$ [Lu93] (also cf. [Ka14, BKM14]). Note $\mathbf{m}' \preceq \mathbf{m}$ implies that $\sum_{\beta\in\Phi^+} m'_\beta \beta = \sum_{\beta\in\Phi^+} m_\beta \beta$. Denote

$$b^+_{\mathbf{m}} = \sum_{\mathbf{m}'\preceq\mathbf{m}} R_{\mathbf{m}',\mathbf{m}}(q) E_{\mathbf{m}'} \tag{3.1}$$

where $R_{\mathbf{m}',\mathbf{m}}(q) \in q^{-1}\mathbb{Z}[q^{-1}]$ for $\mathbf{m}' \prec \mathbf{m}$, and $R_{\mathbf{m},\mathbf{m}}(q) = 1$.

Similarly, by applying suitable braid group actions to construct root vectors and their divided powers for $\mathbf{U}^-$ (which lie in $\mathbf{U}^-_{\mathbb{Z}[q^{-1}]}$), we obtain a PBW basis $\mathbf{W}^- := \{F_\mathbf{m} | \mathbf{m} \in \mathrm{KP}\}$ for $\mathbf{U}^-$, which is orthogonal with respect to the bilinear form $(\cdot,\cdot)$ on $\mathbf{U}^-$. The canonical basis $\{b^-_\mathbf{n} | \mathbf{n} \in \mathrm{KP}\}$ for $\mathbf{U}^-$ satisfies $b^-_\mathbf{n} \in F_\mathbf{n} + \sum_{\mathbf{n}' \prec \mathbf{n}} q^{-1}\mathbb{Z}[q^{-1}] F_{\mathbf{n}'}$. It follows that

$$b^-_\mathbf{n} = \sum_{\mathbf{n}' \preceq \mathbf{n}} R_{\mathbf{n}',\mathbf{n}}(q) F_{\mathbf{n}'}. \tag{3.2}$$

We formulate the following distinguished case of Theorem 2.8 separately and a little more precisely, which is the original goal of this paper.

Theorem 3.1 (PBW basis for $\dot{\mathbf{U}}$)**.** *Let* $\mathbf{U}$ *be finite type. Then the set*

$$\mathbf{W}\natural\mathbf{W} = \{E_\mathbf{m} \natural_\zeta F_\mathbf{n} \mid \mathbf{m}, \mathbf{n} \in KP, \zeta \in X\}$$

forms an orthogonal basis (called a PBW basis*) for* $\dot{\mathbf{U}}$*. The* dual PBW basis *for* $\dot{\mathbf{U}}$ *is given by*

$$\left\{ \prod_{\beta \in \Phi^+} (q_\beta^{-2}; q_\beta^{-2})_{m_\beta} (q_\beta^{-2}; q_\beta^{-2})_{n_\beta} E_\mathbf{m} \natural_\zeta F_\mathbf{n} \mid \mathbf{m}, \mathbf{n} \in KP, \zeta \in X \right\}.$$

Proof. Clearly $\mathbf{W}\natural\mathbf{W}$ is a basis for $\dot{\mathbf{U}}$, by Theorem 2.8. Its orthogonality follows by (2.26) and the orthogonality of the PBW bases for $\mathbf{U}^\pm$ [Lu93, 38.2.3]. Note that $(E_\mathbf{m}, E_\mathbf{m}) = \prod_{\beta \in \Phi^+} (q_\beta^{-2}; q_\beta^{-2})^{-1}_{m_\beta}$, cf. [Lu93]. The dual PBW basis follows from this and (2.26). ☐

Thanks to $E_\mathbf{m} \natural_\zeta 1 = E_\mathbf{m} \mathbf{1}_\zeta$ and $1 \natural_\zeta F_\mathbf{n} = F_\mathbf{n} \mathbf{1}_\zeta$, the PBW bases for $\mathbf{U}^\pm$ are parts of the PBW basis for $\dot{\mathbf{U}}$.

One can also formulate hybrid bases $\mathbf{W}\natural\mathbf{B}$ and $\mathbf{B}\natural\mathbf{W}$ for $\dot{\mathbf{U}}$.

3.2. PBW basis vs (fused) canonical basis

The fused canonical basis for $\dot{\mathbf{U}}$ in finite type can now be written as $\mathbf{B}\natural\mathbf{B} = \{b_\mathbf{m} \natural_\zeta b_\mathbf{n} | \mathbf{m}, \mathbf{n} \in \mathrm{KP}, \zeta \in X\}$. We formulate its relation to the PBW basis $\mathbf{W}\natural\mathbf{W}$.

Proposition 3.2. *Let* $\mathbf{m}, \mathbf{n} \in KP$ *and* $\zeta \in X$*. We have*

$$b_\mathbf{m} \natural_\zeta b_\mathbf{n} = E_\mathbf{m} \natural_\zeta F_\mathbf{n} + \sum_{\substack{(\mathbf{m}',\mathbf{n}') \neq (\mathbf{m},\mathbf{n}) \\ \mathbf{m}' \preceq \mathbf{m}, \mathbf{n}' \preceq \mathbf{n}}} R_{\mathbf{m}',\mathbf{m}}(q) R_{\mathbf{n}',\mathbf{n}}(q)\, E_{\mathbf{m}'} \natural_\zeta F_{\mathbf{n}'}.$$

Moreover, for the ADE type, we have $R_{\mathbf{m}',\mathbf{m}}(q), R_{\mathbf{n}',\mathbf{n}}(q) \in \mathbb{N}[[q^{-1}]] \cap \mathbb{Q}(q)$.

Proof. By construction (see Proposition 2.5), $x\natural_\zeta y$ is bilinear on x and y. Thus the expansion formula for $b_{\mathbf{m}}\natural_\zeta b_{\mathbf{n}}$ follows by (3.1)–(3.2).

Assume $\mathbf{U}$ is of ADE type. Then we have $R_{\mathbf{m}',\mathbf{m}}(q), R_{\mathbf{n}',\mathbf{n}}(q) \in \mathbb{N}[q^{-1}]$. This was first proved in [Lu90, Corollary 10.7] when the reduced expressions of the longest element $w_0 \in W$ are "adapted", and then proved by Syu Kato [Ka14] in general (also see [BKM14] and [Oya18] for different approaches). ☐

Now we formulate the relation between the canonical basis and PBW basis for $\dot{\mathbf{U}}$.

Corollary 3.3. *Let* $\mathbf{m}, \mathbf{n} \in KP$ *and* $\zeta \in X$. *We have*

$$b_{\mathbf{m}}\diamondsuit_\zeta b_{\mathbf{n}} = \sum_{\substack{(b_{\mathbf{m_1}},b_{\mathbf{n_1}})\leq(b_{\mathbf{m}},b_{\mathbf{n}})\\ \mathbf{m}'\preceq\mathbf{m_1},\mathbf{n}'\preceq\mathbf{n_1}}} P_{(b_{\mathbf{m_1}},b_{\mathbf{n_1}}),(b_{\mathbf{m}},b_{\mathbf{n}})}(q)\, R_{\mathbf{m}',\mathbf{m_1}}(q)R_{\mathbf{n}',\mathbf{n_1}}(q)\, E_{\mathbf{m}'}\natural_\zeta F_{\mathbf{n}'}.$$

Moreover, for the ADE type, $P_{(b_{\mathbf{m_1}},b_{\mathbf{n_1}}),(b_{\mathbf{m}},b_{\mathbf{n}})}(q)\, R_{\mathbf{m}',\mathbf{m_1}}(q)R_{\mathbf{n}',\mathbf{n_1}}(q) \in \mathbb{N}[[q^{-1}]] \cap \mathbb{Q}(q)$.

Proof. Follows by combining Theorem 2.9 and Proposition 3.2. ☐

Remark 3.4. Assume $\mathbf{U}$ is of ADE type. The (dual) PBW bases for $\mathbf{U}^+$ have been categorified in terms of (proper) standard modules over the KLR or quiver Hecke algebras [Ka14, BKM14]. In light of the positivity properties of the transition matrices in Theorem 2.9, Proposition 3.2 and Corollary 3.3, it is reasonable to ask for a categorification of the fused canonical basis and the PBW basis of $\dot{\mathbf{U}}$.

Remark 3.5. Let $\mathbf{U}$ be of affine type. By a theorem of [BCP99], there exists a PBW basis $\mathbf{W}$ for $\mathbf{f}$, which is actually a $\mathbb{Z}[q^{-1}]$-basis for the $\mathbf{f}_{\mathbb{Z}[q^{-1}]}$; it gives rise to PBW bases for $\mathbf{U}^+$ and for $\mathbf{U}^-$. It still holds that the set $\mathbf{W}\natural\mathbf{W}$ forms a basis for $\dot{\mathbf{U}}$ by Theorem 2.8 and may be called its PBW basis. However, this basis is not orthogonal, as the PBW basis for $\mathbf{U}^+$ (or $\mathbf{U}^-$) in [BCP99] is not either.

3.3. PBW basis in rank 1

In the remainder of this section, we set $\mathbf{U} = \mathbf{U}_q(\mathfrak{sl}_2)$, the $\mathbb{Q}(q)$-algebra generated by E, F, K, K^{-1}. We shall write $q_i = q$, identify the weight lattice $X = \mathbb{Z}$ and $X^+ = \mathbb{N}$. We work out explicit formulas of PBW basis via canonical basis for $\dot{\mathbf{U}}$, and vice versa.

For $n \in \mathbb{N}$, let $L(n)$ be the highest weight $\mathbf{U}$-module with highest weight vector η_n, and let ${}^\omega L(n)$ be the lowest weight $\mathbf{U}$-module with lowest weight vector ξ_{-n}. For $m \in \mathbb{Z}$, we shall consider the tensor product $\mathbf{U}$-modules ${}^\omega L(p) \otimes L(m+p)$, for various $p \in \mathbb{N}$ such that $m+p \in \mathbb{N}$.

Recall $\{E^{(a)} | a \in \mathbb{N}\}$ is the PBW basis (and also the canonical basis) for $\mathbf{U}^+$, and $\{F^{(b)} | b \in \mathbb{N}\}$ is the PBW basis (and the canonical basis) for $\mathbf{U}^-$. By Proposition 2.7 and Theorem 3.1, there exists a unique PBW basis element

$$\mathfrak{w}_m(a,b) := E^{(a)} \natural_m F^{(b)} \in \dot{\mathbf{U}}\mathbf{1}_m$$

such that

$$\lim_{p \mapsto \infty} \Big(\mathfrak{w}_m(a,b)(\xi_{-p} \otimes \eta_{p+m}) - E^{(a)}\xi_{-p} \otimes F^{(b)}\eta_{p+m} \Big) = 0. \tag{3.3}$$

In particular, $\mathfrak{w}_m(a,0) = E^{(a)}\mathbf{1}_m$ and $\mathfrak{w}_m(0,b) = F^{(b)}\mathbf{1}_m$, for $a, b \in \mathbb{N}$.

Proposition 3.6. *The set $\{\mathfrak{w}_m(a,b) \mid a,b \in \mathbb{N}, m \in \mathbb{Z}\}$ forms an orthogonal (PBW) basis for $\dot{\mathbf{U}}$ with respect to $(\cdot,\cdot)$. Moreover, we have*

$$\Big(\mathfrak{w}_m(a,b), \mathfrak{w}_{m'}(a',b') \Big) = \frac{\delta_{a,a'}\delta_{b,b'}\delta_{m,m'}}{(q^{-2};q^{-2})_a (q^{-2};q^{-2})_b}. \tag{3.4}$$

The dual PBW basis for $\dot{\mathbf{U}}$ is $\{(q^{-2};q^{-2})_a(q^{-2};q^{-2})_b \mathfrak{w}_m(a,b) \mid a,b \in \mathbb{N}, m \in \mathbb{Z}\}$.

Proof. The first statement is a rephrasing of Theorem 3.1. The explicit bilinear pairing formula (3.4) follows from (2.26) and the formula (2.8). ☐

3.4. PBW-expansion of canonical basis in rank 1

Recall the canonical basis for $\dot{\mathbf{U}}$ consists of the following elements (cf. [K93], [Lu93, 25.3.2]):

$$E^{(a)}F^{(b)}\mathbf{1}_m \ \ (m \le b-a), \qquad F^{(b)}E^{(a)}\mathbf{1}_m \ \ (m \ge b-a), \qquad \text{for } a,b \in \mathbb{N},$$

with the (only) identification

$$E^{(a)}F^{(b)}\mathbf{1}_m = F^{(b)}E^{(a)}\mathbf{1}_m, \qquad \text{for } m = b-a. \tag{3.5}$$

Theorem 3.7. *Let $a,b \in \mathbb{N}$ and $m \in \mathbb{Z}$. Then we have*

$$E^{(a)}F^{(b)}\mathbf{1}_m = \sum_{s=0}^{\min(a,b)} \frac{q^{-s^2+s(a-b+m)}}{(q^{-2};q^{-2})_s} \mathfrak{w}_m(a-s,b-s), \quad \textit{for } m \le b-a, \tag{3.6}$$

(3.7) $$F^{(b)}E^{(a)}\mathbf{1}_m = \sum_{s=0}^{\min(a,b)} \frac{q^{-s^2-s(a-b+m)}}{(q^{-2};q^{-2})_s}\mathfrak{w}_m(a-s,b-s), \quad \textit{for } m \geq b-a.$$

Note that the leading coefficients are 1 and all the (non-leading) coefficients in (3.6)–(3.7) lies in $q^{-1}\mathbb{N}[[q^{-1}]] \cap \mathbb{Q}(q)$.

Proof. Assume $m \leq b-a$. We consider the action of $\mathbf{U}$ on the module ${}^{\omega}L(p)\otimes L(m+p)$, for various $p \geq |m|$. Recall the following formula from [Lu93, §25.3], where we have replaced Lusztig's η_q by η_{p+m} here and his $(-n+2b)$ by m here:

(3.8) $$\begin{aligned} &E^{(a)}F^{(b)}(\xi_{-p}\otimes\eta_{p+m}) \\ &\quad= \sum_{s\geq 0; s\leq a, s\leq b} q^{s(a-s-p)} \begin{bmatrix} s-b+m+p \\ s \end{bmatrix} E^{(a-s)}\xi_{-p}\otimes F^{(b-s)}\eta_{p+m}. \end{aligned}$$

To make even more clear the dependence on p of these coefficients, we rewrite the formula (3.8) as

(3.9)

$$\begin{aligned} &E^{(a)}F^{(b)}(\xi_{-p}\otimes\eta_{p+m}) \\ &= \sum_{s\geq 0; s\leq a, s\leq b} q^{-s^2+s(a-b+m)} \prod_{d=1}^{s} \frac{1-q^{2b-2m-2d-2p}}{1-q^{-2d}} E^{(a-s)}\xi_{-p}\otimes F^{(b-s)}\eta_{p+m} \\ &= \sum_{s\geq 0; s\leq a, s\leq b} \frac{q^{-s^2+s(a-b+m)}}{(q^{-2};q^{-2})_s} E^{(a-s)}\xi_{-p}\otimes F^{(b-s)}\eta_{p+m} \\ &- \sum_{s\geq 0; s\leq a, s\leq b} \frac{q^{-s^2+s(a-b+m)}}{(q^{-2};q^{-2})_s} \big(1-(q^{2b-2m-2-2p};q^{-2})_s\big) E^{(a-s)}\xi_{-p}\otimes F^{(b-s)}\eta_{p+m}. \end{aligned}$$

Note that the first summand on the RHS (3.9) has coefficients independent of p; the second summand is an asymptotically-zero standard sequence as p varies, and so it has limit 0 as p tends to ∞ by Lemma 2.6. Therefore, (3.6) follows from the identities (3.3) and (3.9).

Now assume $m \geq b-a$. Recall the following formula from [Lu93, §25.3]:

(3.10) $$\begin{aligned} &F^{(b)}E^{(a)}(\xi_{-p}\otimes\eta_{p+m}) \\ &\quad= \sum_{s\geq 0; s\leq a, s\leq b} q^{s(b-s-m-p)} \begin{bmatrix} s-a+p \\ s \end{bmatrix} E^{(a-s)}\xi_{-p}\otimes F^{(b-s)}\eta_{p+m}. \end{aligned}$$

This can be rewritten as

(3.11)

$$
\begin{aligned}
&F^{(b)}E^{(a)}(\xi_{-p}\otimes\eta_{p+m})\\
&=\sum_{s\geq 0; s\leq a, s\leq b} q^{-s^2-s(a-b+m)}\prod_{d=1}^{s}\frac{1-q^{2a-2d-2p}}{1-q^{-2d}}E^{(a-s)}\xi_{-p}\otimes F^{(b-s)}\eta_{p+m}\\
&=\sum_{s\geq 0; s\leq a, s\leq b}\frac{q^{-s^2-s(a-b+m)}}{(q^{-2};q^{-2})_s}E^{(a-s)}\xi_{-p}\otimes F^{(b-s)}\eta_{p+m}\\
&\quad-\sum_{s\geq 0; s\leq a, s\leq b}\frac{q^{-s^2-s(a-b+m)}}{(q^{-2};q^{-2})_s}(1-(q^{2a-2-2p};q^{-2})_s)E^{(a-s)}\xi_{-p}\otimes F^{(b-s)}\eta_{p+m}.
\end{aligned}
$$

The second summand above has limit 0 as p tends to ∞, and (3.7) follows. □

Corollary 3.8. *The bilinear pairings between canonical basis elements are given by:*

$$
\begin{aligned}
&\big(E^{(a)}F^{(b)}\mathbf{1}_m, E^{(a')}F^{(b')}\mathbf{1}_m\big)\\
&=\sum_{s=\min(0,a-a')}^{\min(a,b)}\frac{q^{-s^2-(a'-a+s)^2+(a'-a+2s)(a-b+m)}}{(q^{-2};q^{-2})_{a-s}(q^{-2};q^{-2})_{b-s}(q^{-2};q^{-2})_s(q^{-2};q^{-2})_{a'-a+s}}
\end{aligned}
$$

if $m\leq b-a=b'-a'$*;*

$$
\begin{aligned}
&\big(F^{(b)}E^{(a)}\mathbf{1}_m, F^{(b')}E^{(a')}\mathbf{1}_m\big)\\
&=\sum_{s=\min(0,a-a')}^{\min(a,b)}\frac{q^{-s^2-(a'-a+s)^2-(a'-a+2s)(a-b+m)}}{(q^{-2};q^{-2})_{a-s}(q^{-2};q^{-2})_{b-s}(q^{-2};q^{-2})_s(q^{-2};q^{-2})_{a'-a+s}}
\end{aligned}
$$

if $m\geq b-a=b'-a'$*; The bilinear pairings are* 0 *whenever* $b-a\neq b'-a'$.

Proof. This follows immediately from Theorem 3.7 and Proposition 3.6. □

Remark 3.9. The formulas in Corollary 3.8 appeared first in [La10, Proposition 2.8] (whose convention differs from us by $q\leftrightarrow q^{-1}$), and their direct combinatorial proof is long and occupied [La10, §10]. There is a very different formula for the bilinear pairing obtained by adjunctions, cf. [La10].

Remark 3.10. An alternative short proof of Corollary 3.8 is as follows (we focus on the case when $m\leq b-a$ below). Use the formula (3.8) to compute the bilinear pairing

$$\Big(E^{(a)}F^{(b)}(\xi_{-p}\otimes\eta_{p+m}),E^{(a')}F^{(b')}(\xi_{-p}\otimes\eta_{p+m})\Big)$$

on the module ${}^{\omega}L(p)\otimes L(m+p)$ (which is easy as the summands in (3.8) are orthogonal). Then its limit in $\mathbb{Q}((q^{-1}))$ as p tends to ∞ can be directly read off from the reformulation (3.9) of (3.8). This limit gives us $(E^{(a)}F^{(b)}\mathbf{1}_m,E^{(a')}F^{(b')}\mathbf{1}_m)$ according to [Lu93, 26.2.3].

Remark 3.11. The bilinear pairing on the modified $\imath$quantum group of rank one was computed [Wa21] in the same approach as in Remark 3.10. For $\imath$quantum groups, this is the only way of computing for now as there is no characterization of the bilinear form via adjunction like (2.5)–(2.7).

3.5. PBW basis via canonical basis in rank 1

Now we give a formula for the PBW basis in terms of canonical basis.

Theorem 3.12. *For $a,b\in\mathbb{N}$ and $m\in\mathbb{Z}$, we have*

(3.12)

$$\mathfrak{w}_m(a,b)=\begin{cases}\displaystyle\sum_{s=0}^{\min(a,b)}(-1)^s\frac{q^{-s+s(a-b+m)}}{(q^{-2};q^{-2})_s}E^{(a-s)}F^{(b-s)}\mathbf{1}_m, & \text{if } m\le b-a,\\ \displaystyle\sum_{s=0}^{\min(a,b)}(-1)^s\frac{q^{-s-s(a-b+m)}}{(q^{-2};q^{-2})_s}F^{(b-s)}E^{(a-s)}\mathbf{1}_m, & \text{if } m\ge b-a.\end{cases}$$

Proof. We shall prove the first formula for $\mathfrak{w}_m(a,b)$ with $m\le b-a$; the proof of the other formula is entirely similar and will be skipped.

Let us fix $a,b\in\mathbb{N}$ and $m\in\mathbb{Z}$ such that $m\le b-a$. One could rephrase the formula (3.6) as giving the unital triangular transition matrix from a basis $C:=\{E^{(a-s)}F^{(b-s)}\mathbf{1}_m\mid 0\le s\le\min(a,b)\}$ to another basis $P:=\{\mathfrak{w}_m(a-s,b-s)\mid 0\le s\le\min(a,b)\}$ (these 2 sets have the same span). Accordingly, the first formula in (3.12) gives the unital triangular transition matrix from P to C, and we shall prove these two transition matrices are inverses to each other. This boils down to verifying the following identity, for $k\ge 1$:

$$\sum_{s=0}^{k}(-1)^{k-s}\frac{q^{-s^2+s(a-b+k)}q^{(k-s)(a-b+k-1)}}{(q^{-2};q^{-2})_s(q^{-2};q^{-2})_{k-s}}=0.$$

Upon multiplying with $(-1)^k q^{k(b-a-k+1)}(q^{-2};q^{-2})_k$, we reduce the above identity to the following standard q-binomial identity, for $k \geq 1$ (cf. [Lu93, 1.3.4]):

$$\sum_{s=0}^{k}(-1)^s q^{s(1-m)} \begin{bmatrix} m \\ s \end{bmatrix} = 0.$$

This proves the first formula in (3.12) (actually, we have shown that (3.6) and the first formula in (3.12) are equivalent). $\square$

The two different formulas of $\mathfrak{w}_m(a,b)$ for $m = b-a$ in (3.12) coincide thanks to (3.5). The simplest new PBW basis elements are given by

$$\mathfrak{w}_m(1,1) = \begin{cases} EF\mathbf{1}_m - \frac{q^{m-1}}{1-q^{-2}}\mathbf{1}_m & \text{if } m \leq 0, \\ FE\mathbf{1}_m - \frac{q^{-m-1}}{1-q^{-2}}\mathbf{1}_m, & \text{if } m \geq 0. \end{cases} \tag{3.13}$$

For $m \leq 0$, we have $\mathfrak{w}_m(1,1)(\xi_{-p} \otimes \eta_{p+m}) = E\xi_{-p} \otimes F\eta_{p+m} - \frac{q^{-1-2p}}{1-q^{-2}}(\xi_{-p} \otimes \eta_{p+m})$.

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Character sheaves for symmetric pairs: spin groups*

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Dedicated to George Lusztig with gratitude and admiration

Abstract: We determine character sheaves for symmetric pairs associated to spin groups. In particular, we determine the cuspidal character sheaves and show that they can be obtained via the nearby cycle construction of [6] and its generalisation in [14].

1. Introduction

In [13] we give a classification of character sheaves for classical symmetric pairs, which can be viewed as an analogue of Lusztig's generalized Springer correspondence [9]. We show that all character sheaves can be obtained using the nearby cycle construction of [6] and parabolic induction. As in Lusztig's generalised Springer correspondence, for symmetric pairs associated to groups in the other isogeny classes, such as special linear groups and spin groups, we need extra work to determine the character sheaves at various central characters. The inner involutions for special linear groups are treated in [14]. In this note we classify character sheaves for symmetric pairs associated to spin groups. In [15] we explain how the general reductive case can be reduced to the case of almost simple simply connected groups G. Thus this completes the classification of character sheaves for symmetric pairs associated to groups of classical types.

We recall the set-up in [13]. Let G be a connected complex reductive algebraic group and $\theta : G \to G$ an involution. Let $K = G^\theta$ (or $(G^\theta)^0$), and let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be the decomposition of the Lie algebra $\mathfrak{g} = \operatorname{Lie} G$ induced by θ such that $d\theta|_{\mathfrak{g}_i} = (-1)^i$. Let $\mathcal{A}_K(\mathfrak{g}_1)$ denote the set of nilpotent orbital complexes, that is, the simple K-equivariant perverse sheaves on $\mathcal{N}_1 = \mathcal{N} \cap \mathfrak{g}_1$, where $\mathcal{N}$ denotes the nilpotent cone of $\mathfrak{g}$. By definition, the set of character sheaves $\operatorname{Char}_K(\mathfrak{g}_1)$ for the symmetric pair (G, K) consists of Fourier transforms of sheaves in $\mathcal{A}_K(\mathfrak{g}_1)$. We write $\mathfrak{F} : \operatorname{Perv}_K(\mathcal{N}_1) \subset \operatorname{Perv}_K(\mathfrak{g}_1) \to$

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$\mathrm{Perv}_K(\mathfrak{g}_1)$ for the Fourier transform functor (where we identify $\mathfrak{g}_1$ with $\mathfrak{g}_1^*$). The set $\mathrm{Char}_K(\mathfrak{g}_1)$ is determined explicitly for all classical symmetric pairs in [13] (where $K = G^\theta$). The symmetric pairs (SL_n, SO_n) are treated in [3] and the pairs (SL_{2n}, Sp_{2n}) have been studied previously in [5, 8, 10].

In this note we will focus on involutions of spin groups $Spin_N$, $N \geq 5$. Note that in this case $K = G^\theta$ is connected by a theorem of Steinberg as G is simply connected. As in [13, 14], we determine the character sheaves by writing down explicitly the supports of the IC sheaves and the corresponding K-equivariant local systems. The supports are dual strata $\check{\mathcal{O}}$ associated to nilpotent K-orbits $\mathcal{O}$ in $\mathcal{N}_1$. The local systems are given by irreducible representations of the equivariant fundamental groups $\pi_1^K(\check{\mathcal{O}})$, which are (extended) braid groups. We write down these irreducible representations explicitly. They are given by irreducible representations of Hecke algebras associated to finite Coxeter groups with parameters ± 1. In particular, we determine the cuspidal character sheaves (see Theorem 1.1 below and Corollary 4.7), that is, the character sheaves which do not arise as a direct summand (up to shift) of parabolic induction of character sheaves of a θ-stable Levi subgroup contained in a proper θ-stable parabolic subgroup. We show that they all arise from the nearby cycle construction of [6] and its generalisation given in [14].

Let $\pi : G = Spin_N \to \bar{G} = SO_N$ denote the double covering homomorphism. We have a natural partition of the character sheaves by their central characters, in particular, by the action of $\ker \pi \subset K$. That is, we have $\mathrm{Char}_K(\mathfrak{g}_1) = \mathrm{Char}_K(\mathfrak{g}_1)_{\kappa_0} \sqcup \mathrm{Char}_K(\mathfrak{g}_1)_{\kappa_1}$, where κ_0 (resp. κ_1) denote the trivial (resp. nontrivial) character of $\ker \pi \cong \mathbb{Z}/2\mathbb{Z}$. The set $\mathrm{Char}_K(\mathfrak{g}_1)_{\kappa_0}$ can be identified with the set $\mathrm{Char}_{\bar{K}}(\mathfrak{g}_1)$, where $\bar{K} = \pi(K)$. We determine this set following the strategy of [13]. Note that in [13] we work with a disconnected $\widetilde{K}$ and for the purpose of this paper we need to work with $\widetilde{K}^0 = \bar{K}$. To determine the set $\mathrm{Char}_K(\mathfrak{g}_1)_{\kappa_1}$ we make use of a generalisation of the nearby cycle construction as in [14].

Let $\mathrm{Char}_K^{\mathrm{cusp}}(\mathfrak{g}_1)$ (resp. $\mathrm{Char}_K^{\mathrm{f}}(\mathfrak{g}_1)$, $\mathrm{Char}_K^{\mathrm{n}}(\mathfrak{g}_1)$) denote the subset of $\mathrm{Char}_K(\mathfrak{g}_1)$ consisting of cuspidal (resp. full support, nilpotent support) character sheaves. Let $\mathrm{Char}_K^{\mathrm{cusp}}(\mathfrak{g}_1)_{\kappa_i} = \mathrm{Char}_K^{\mathrm{cusp}}(\mathfrak{g}_1) \cap \mathrm{Char}_K(\mathfrak{g}_1)_{\kappa_i}$, $i = 0, 1$. Similarly for $\mathrm{Char}_K^{\mathrm{f}}(\mathfrak{g}_1)_{\kappa_i}$, $\mathrm{Char}_K^{\mathrm{n}}(\mathfrak{g}_1)_{\kappa_i}$. We show that $\mathrm{Char}_K^{\mathrm{cusp}}(\mathfrak{g}_1)_{\kappa_0}$ is nonempty if and only if (G, K) is a split symmetric pair, and $\mathrm{Char}_K^{\mathrm{cusp}}(\mathfrak{g}_1)_{\kappa_1}$ is nonempty when (G, K) is of type BDI, that is, $\bar{K} \cong SO_p \times SO_q$, and $N \geq (p-q)^2$. More precisely, we have

Theorem 1.1. *The cuspidal character sheaves are*

$$\mathrm{Char}^{\mathrm{cusp}}_{K^{n+t,n}}(\mathfrak{g}_1)_{\kappa_0} = \mathrm{Char}^{\mathrm{f}}_{K^{n+t,n}}(\mathfrak{g}_1)_{\kappa_0} = \left\{ \mathrm{IC}(\mathfrak{g}_1^{rs}, \mathcal{T}_\psi) \mid \psi \in \Theta_{n,t}^{\kappa_0} \right\}, \; |t| \leq 1$$

$$\mathrm{Char}^{\mathrm{cusp}}_{K^{m+\frac{t^2+t}{2}, m+\frac{t^2-t}{2}}}(\mathfrak{g}_1)_{\kappa_1} = \left\{ \mathrm{IC}(\check{\mathcal{O}}_{1_+^m 1_-^m \sqcup \mu_t}, \mathcal{F}_\rho) \mid \rho \in \Theta_{m,t}^{\kappa_1} \right\}, \; \textit{any } t.$$

Here $K^{p,q}$ indicates that $\pi(K) \cong SO_p \times SO_q$. We refer the readers to the main text for notations in the above theorem. The sheaves in (i) are obtained using the nearby cycle construction and those in (ii) using the generalised nearby cycle construction.

The paper is organised as follows. In Section 2 we discuss the symmetric pairs, the nilpotent orbits and the component groups of their centralisers. We also fix some notations that will be used throughout the paper. In Section 3 we construct a set of character sheaves using the nearby cycle construction of [6, 7] and its generalisation in [14]. In Section 4 we describe the character sheaves explicitly (Theorems 4.5 and 4.6). In particular, we write down the set of nilpotent orbits $\mathcal{O}$ such that the corresponding dual strata $\check{\mathcal{O}}$ are the supports of character sheaves. We determine the cuspidal character sheaves (Corollary 4.7), the nilpotent support character sheaves (Corollary 4.8), and their numbers. In Section 5 we prove Theorem 4.5 and its corollaries. In particular, we determine the number of character sheaves (Proposition 5.1).

2. Preliminaries and notations

Throughout the paper let $G = Spin_N$ and $\bar{G} = SO_N$, $N \geq 5$. We first recall the definition of spin groups using Clifford algebras (see for example [9, §14.3]). Let V be a $\mathbb{C}$-vector space of dimension N equipped with a non-degenerate symmetric bilinear form $(,)$. Let C_V be the corresponding Clifford algebra; it can be defined as $T_V/\langle vv' + v'v = 2(v, v'),\ v, v' \in V\rangle$, where T_V is the tensor algebra of V. Let $C_V^+ \subset C_V$ denote the sub-algebra spanned by elements of the form $v_1v_2\cdots v_{2a}$, $v_i \in V$, that is, products of an even number of vectors in V. The spin group $G = Spin_V = Spin_N$ is the subgroup of the group of units in C_V^+ consisting of elements of the form $v_1v_2\cdots v_{2a}$, $a \in \mathbb{N}$, $v_i \in V$, and $(v_i, v_i) = 1$. The homomorphism $\pi : Spin_V \to SO_V$, $x \mapsto (v \mapsto xvx^{-1})$ realises G as a simply connected double cover of $\bar{G}$. We have $\ker\pi = \{1, \epsilon\}$, where ϵ denotes the element (-1) times the unit element of C_V^+.

The symmetric pairs associated to G are as follows. Recall that $K = G^\theta$ is connected.

(Type BDI) Let $V = V^+ \oplus V^-$ be an orthogonal decomposition such that $\dim V^+ = p$ and $\dim V^- = q = N - p$. Let $\theta : G \to G$ be an involution such that $\pi(G^\theta) = SO_{V^+} \times SO_{V^-}$. We have

$$K \cong (Spin_{V^+} \times Spin_{V^-})/\langle(\epsilon_p, \epsilon_q)\rangle := K^{p,q}$$

where ϵ_p (resp. ϵ_q) denotes the element (-1) times the unit element in $C^+_{V^+}$ (resp. $C^+_{V^-}$). Note that $\epsilon = \overline{(\epsilon_p, 1)} = \overline{(1, \epsilon_q)} \in K$. We write

$$\bar{K}^{p,q} = SO_p \times SO_q,\ \widetilde{K}^{p,q} = S(O_p \times O_q),\ n = [N/2],\quad t = p - q,$$

where $[N/2]$ denotes the integer part of $N/2$.

(Type DIII) Let $N = 2n$ and let $V = V^+ \oplus V^-$ be a decomposition such that $\dim V^+ = \dim V^- = n$ and $(,)|_{V^+} = (,)|_{V^-} = 0$. Let $\theta : G \to G$ be an involution such that $\bar{K} := \pi(G^\theta) = \bar{G} \cap (GL_{V^+} \times GL_{V^-}) \cong GL_n$. We have $\ker \pi \subset K$ as θ is inner.

The involutions θ in the above can be realized explicitly as follows. In type BDI, we let $\{e_1 \cdots, e_p\}$ be an orthonormal basis of V^+, that is $(e_i, e_j) = \delta_{ij}$. Let $g_0 := e_1 \cdots e_p \in C^\times_V$. Then we can define $\theta(g) = g_0 g g_0^{-1}$. In type DIII, let $\{e_1, \ldots, e_n\}$ be a basis of V^+ and $\{f_1, \ldots, f_n\}$ be a basis of V^- such that $(e_i, f_j) = \delta_{ij}$. We let

$$v_i = (e_i + f_i)/\sqrt{2},\ w_i = (\mathbf{e}^{\mathbf{i}\pi/4} e_i + \mathbf{e}^{\mathbf{i}\pi/4} f_i)/\sqrt{2},\ i = 1, \ldots, n,$$

and $g_0 = v_1 w_1 \cdots v_n w_n \in Spin_V$. Then we can define $\theta(g) = g_0 g g_0^{-1}$.

Note that θ is an *outer* involution if and only if it is of type BDI with both p and q odd, and (G, K) is a split symmetric pair if and only if it is of type BDI and $|p - q| \leq 1$.

Recall $\mathfrak{g}_i$, $i = 0, 1$, denote the $(-1)^i$-eigenspace of $d\theta$ and $\mathcal{N}_1 = \mathcal{N} \cap \mathfrak{g}_1$, where $\mathcal{N}$ is the nilpotent cone of $\mathfrak{g} = \operatorname{Lie} G$. To ease the notation, we will identify $\operatorname{Lie} \bar{G}$ with $\mathfrak{g}$, $d\pi(\mathfrak{g}_1)$, $d\pi(\mathcal{N}_1)$ with $\mathfrak{g}_1$, $\mathcal{N}_1$. Recall also κ_0 (resp.κ_1) : $\ker \pi \to \mathbb{G}_m,\ \epsilon \mapsto 1$ (resp. -1) denote the trivial (resp. nontrivial) character of $\ker \pi$.

For a finite group H, we write $\widehat{H}$ for the set of irreducible representations of H over $\mathbb{C}$ (up to isomorphism). If there is a natural map $\ker \pi \to Z(H)$, we write $\widehat{H}_{\kappa_0}$ (resp. $\widehat{H}_{\kappa_1}$) for the subset of $\widehat{H}$ consisting of the representations such that $\epsilon \in \ker \pi$ acts by 1 (resp. -1), where $Z(H)$ denotes the center of H.

We denote by $\mathcal{P}(n)$ (resp. $\mathcal{P}_2(n)$) the set of partitions of n (resp. bi-partitions of n) and let $\mathbf{p}(n) = |\mathcal{P}(n)|$. By definition $\mathcal{P}(x) = \mathcal{P}_2(x) = \emptyset$ for $x \notin \mathbb{N}$.

2.1. Nilpotent orbits and component groups

In this subsection we describe the K-orbits in $\mathcal{N}_1$, the component groups $A_K(x) = Z_K(x)/Z_K(x)^0$ for $x \in \mathcal{N}_1$, and $\widehat{A_K(x)}$ following [9, §14]. Let

$A_{\bar{K}}(x) = Z_{\bar{K}}(x)/Z_{\bar{K}}(x)^0$. We can identify $\widehat{A_K(x)}_{\kappa_0}$ with $\widehat{A_{\bar{K}}(x)}$ via the natural projection map $A_K(x) \to A_{\bar{K}}(x)$ induced by π. We will write $A_K(\mathcal{O}) := A_K(x)$, $x \in \mathcal{O}$. Similarly for $A_{\bar{K}}(\mathcal{O})$.

The nilpotent K-orbits in $\mathcal{N}_1$ are parametrized in the same way as the nilpotent $\bar{K}$-orbits in $\mathcal{N}_1$, see, for example, [2] and [11]. We use the same notations as in [13, §2.5]. We write a signed Young diagram as follows

$$\lambda = (\lambda_1)_+^{p_1}(\lambda_1)_-^{q_1}(\lambda_2)_+^{p_2}(\lambda_2)_-^{q_2}\cdots(\lambda_s)_+^{p_s}(\lambda_s)_-^{q_s} \tag{2.1a}$$

where $\lambda = (\lambda_1)^{p_1+q_1}(\lambda_2)^{p_2+q_2}\cdots(\lambda_s)^{p_s+q_s}$ is the corresponding partition of N, $\lambda_1 > \lambda_2 > \cdots > \lambda_s > 0$, for $i = 1, \ldots, s$, $p_i + q_i > 0$ is the multiplicity of λ_i in λ, and $p_i \geq 0$ (resp. $q_i \geq 0$) is the number of rows of length λ_i that begins with sign $+$ (resp. $-$). We will sometimes replace the subscript $+$ by 0 and $-$ by 1 and write the signed Young diagram in (2.1a) as

$$\lambda = (\lambda_1)_0^{p_1}(\lambda_1)_1^{q_1}(\lambda_2)_0^{p_2}(\lambda_2)_1^{q_2}\cdots(\lambda_s)_0^{p_s}(\lambda_s)_1^{q_s}. \tag{2.1b}$$

2.1.1. Type BDI Let Σ denote the set of signed Young diagrams

$$\Sigma = \{\lambda = (\lambda_1)_+^{p_1}(\lambda_1)_-^{q_1}\cdots(\lambda_s)_+^{p_s}(\lambda_s)_-^{q_s} \mid p_i = q_i \text{ if } \lambda_i \text{ is even}\}. \tag{2.2}$$

Let $\Sigma^{p,q} \subset \Sigma$ denote the subset of signed Young diagrams with signature (p, q).

For $\lambda \in \Sigma$ of the form (2.2), we define

$$a_\lambda = |\{i \in [1, s] \mid \lambda_i \equiv 1 \text{ mod } 4,\ p_i{>}0\}| + |\{i \in [1, s] \mid \lambda_i \equiv 3 \text{ mod } 4,\ q_i{>}0\}|,$$
$$b_\lambda = |\{i \in [1, s] \mid \lambda_i \equiv 1 \text{ mod } 4,\ q_i{>}0\}| + |\{i \in [1, s] \mid \lambda_i \equiv 3 \text{ mod } 4,\ p_i{>}0\}|.$$

Note that $a_\lambda \equiv b_\lambda + 1 \bmod 2$ if N is odd, and $a_\lambda \equiv b_\lambda \equiv 1 \bmod 2$ (resp. $a_\lambda \equiv b_\lambda \equiv 0 \bmod 2$) if N is even and θ is outer (resp. inner). Let

$$\Sigma_1 = \{\lambda \in \Sigma \mid a_\lambda > 0, b_\lambda > 0\},\ \Sigma_2 = \{\lambda \in \Sigma \mid a_\lambda + b_\lambda > 0,\ a_\lambda b_\lambda = 0\},$$
$$\Sigma_3 = \{\lambda \in \Sigma \mid a_\lambda = b_\lambda = 0\},\ \text{and } \Sigma_i^{p,q} = \Sigma^{p,q} \cap \Sigma_i,\ i = 1, 2, 3.$$

We define

$$r_\lambda = a_\lambda + b_\lambda - 2 \text{ (resp. } a_\lambda + b_\lambda - 1,\ 0) \text{ if } \lambda \in \Sigma_1 \text{ (resp. } \Sigma_2, \Sigma_3). \tag{2.3}$$

The set of $K^{p,q}$-orbits in $\mathcal{N}_1$ is (see, for example, [2, Theorem 9.3.4])

$$\{\mathcal{O}_\lambda \mid \lambda \in \Sigma_1^{p,q}\} \sqcup \{\mathcal{O}_\lambda^\delta \mid \lambda \in \Sigma_2^{p,q}, \delta = \mathrm{I}, \mathrm{II}\} \sqcup \{\mathcal{O}_\lambda^\delta \mid \lambda \in \Sigma_3^{p,q}, \delta = \mathrm{I}, \mathrm{II}, \mathrm{III}, \mathrm{IV}\}.$$

Here and henceforth the superscript δ in $\mathcal{O}_\lambda^\delta$ denotes the distinct orbits corresponding to the same Young diagram. This reflects the fact that each $O_{p,q}$-orbits split into 2 or 4 $SO_{p,q}^0$-orbits.

Let $\lambda \in \Sigma$ and let $\mathcal{O} = \mathcal{O}_\lambda$ or $\mathcal{O}_\lambda^\delta$ be a K-orbit in $\mathcal{N}_1$ corresponding to λ. Let $x_\lambda \in \mathcal{O}$.

Lemma 2.1. (i) *We have* $A_{\bar{K}}(x_\lambda) \cong (\mathbb{Z}/2\mathbb{Z})^{r_\lambda}$, *where* r_λ *is defined in* (2.3).
(ii) *We have* $A_K(x_\lambda) \cong A_{\bar{K}}(x_\lambda)$ *if there exists* λ_i *odd such that* $p_i \geq 2$ *or* $q_i \geq 2$.
(iii) *Suppose that* $p_i \leq 1,\ q_i \leq 1$ *for each odd* λ_i. *Then* $A_K(x_\lambda)$ *is isomorphic to a central extension of* $A_{\bar{K}}(x_\lambda)$ *by* $\mathbb{Z}/2\mathbb{Z}$. *We have the following cases:*

(a) *Suppose that* N *is odd. The set* $\widehat{A_K(x_\lambda)}_{\kappa_1}$ *consists of 2 (resp. 1) representations of dimension* $2^{\frac{r_\lambda-1}{2}}$ *(resp.* $2^{\frac{r_\lambda}{2}}$*) if* $\lambda \in \Sigma_1$ *(resp.* Σ_2*).*

(b) *Suppose that* N *is even and* θ *is outer. The set* $\widehat{A_K(x_\lambda)}_{\kappa_1}$ *consists of 1 representation of dimension* $2^{\frac{r_\lambda}{2}}$.

(c) *Suppose that* N *is even and* θ *is inner. The set* $\widehat{A_K(x_\lambda)}_{\kappa_1}$ *consists of* 4 *(resp.* 2, 1*) representations of dimension* $2^{\frac{r_\lambda-2}{2}}$ *(resp.* $2^{\frac{r_\lambda-1}{2}}$, $2^{\frac{r_\lambda}{2}}$*) if* $\lambda \in \Sigma_1$ *(resp.* Σ_2, Σ_3*).*

Proof. (i) Consider the symmetric pair $(\bar{G}, \bar{K}^{p,q}) = (SO_N, SO_p \times SO_q)$. We have (see [11, Chap IV, §2])

$$A_{\bar{K}}(x_\lambda) \cong$$
$$\left(S\left(\prod_{\substack{\lambda_i \equiv 1 \\ \text{mod } 4 \\ p_i > 0}} O_{p_i} \prod_{\substack{\lambda_i \equiv 3 \\ \text{mod } 4 \\ q_i > 0}} O_{q_i} \right) \times S\left(\prod_{\substack{\lambda_i \equiv 1 \\ \text{mod } 4 \\ q_i > 0}} O_{q_i} \prod_{\substack{\lambda_i \equiv 3 \\ \text{mod } 4 \\ p_i > 0}} O_{p_i} \right) \right) \Big/ \prod_{\lambda_i \text{ odd}} (SO_{p_i} \times SO_{q_i}).$$

Thus (i) follows.

(ii) We follow the proof in [9, §14.3]. We have $Z_K(x_\lambda) = \pi^{-1}(Z_{\bar{K}}(x_\lambda))$ and $A_K(x_\lambda) \cong A_{\bar{K}}(x_\lambda)$ if $\epsilon \in Z_K(x_\lambda)^0$, $A_K(x_\lambda)$ is a central extension of $A_{\bar{K}}(x_\lambda)$ by $\mathbb{Z}/2\mathbb{Z}$ if $\epsilon \notin Z_K(x_\lambda)^0$.

Without loss of generality, we assume that $p_i \geq 2$ for some $\lambda_i = 4k+1$. Then there exist two x_λ-stable subspaces V_{4k+1}^a, $a = 1, 2$, $\dim V_{4k+1}^a = 4k+1$ such that $V = V_{4k+1}^1 \oplus V_{4k+1}^2 \oplus W$ is an x_λ-stable orthogonal decomposition. Moreover, there exists an isometry $\gamma : V_{4k+1}^1 \to V_{4k+1}^2$ such that $\gamma(x_\lambda v) = x_\lambda \gamma(v)$, $v \in V_{4k+1}^1$. We can choose an orthogonal basis $e_i^a \in V^+$, $i \in [1, 2k+1]$, $f_i^a \in V^-$, $i \in [1, 2k]$, of V_{4k+1}^1 such that $(e_i^1, e_i^1) = 1 = (f_j^1, f_j^1)$, $e_i^2 = \gamma(e_i^1)$

and $f_i^2=\gamma(f_i^1)$. Let $b,c\in\mathbb{C}$ be such that $b^2+c^2=1$. Let

$$V_{b,c}=\operatorname{span}\{v_{b,c,i}:=be_i^1+ce_i^2, i=\in[1,2k+1],\ w_{b,c,j}:=bf_j^1+cf_j^2, j\in[1,2k]\}.$$

Then $V_{b,c}$ is x_λ-stable and $\{v_{b,c,i}, i\in[1,2k+1],\ w_{b,c,j}, j\in[1,2k]\}$ is an orthonormal basis of $V_{b,c}$. Let $\sigma_+^1=e_1^1\cdots e_{2k+1}^1$, $\sigma_-^1=f_1^1\cdots f_{2k}^1$, $\sigma_{b,c,+}=(be_1^1+ce_1^2)\cdots(be_{2k+1}^1+ce_{2k+1}^2)$, $\sigma_{b,c,-}=(bf_1^1+cf_1^2)\cdots(bf_{2k}^1+cf_{2k}^2)$, $\sigma^1=\sigma_+^1\sigma_-^1$, $\sigma_{b,c}=\sigma_{b,c,+}\sigma_{b,c,-}$ and $k_{b,c}=\sigma^1\sigma_{b,c}\sigma^1\sigma_{b,c}\in K$. Since $v\mapsto\sigma^1v(\sigma^1)^{-1}$ restricts to $V_0=V_{4k+1}^1\oplus V_{4k+1}^2$ (resp. $V_0^\perp=W$) is 1 (resp. -1), $v\mapsto\sigma_{b,c}v\sigma_{b,c}^{-1}$ restricts to $V_{b,c}$ (resp. $V_{b,c}^\perp$) is 1 (resp. -1), and V_0, $V_{b,c}$ are x_λ-stable, we conclude that $\pi(k_{b,c})$ commutes with x_λ. Thus $k_{b,c}\in Z_K(x_\lambda)$. Since $k_{1,0}=1\in Z_K(x_\lambda)^0$, and $\{(b,c)\in\mathbb{C}\mid b^2+c^2=1\}$ is irreducible, we conclude that $k_{0,1}=\epsilon\in Z_K(x_\lambda)^0$. It follows that $A_K(x_\lambda)\cong A_{\bar K}(x_\lambda)$.

(iii) Suppose that $p_i\le 1, q_i\le 1$ for each odd λ_i. Let

$$\begin{aligned} J_1 &= \{i\in[1,s]\mid \lambda_i\equiv 1 \bmod 4,\ p_i>0\}\\ &\cup\ \{i\in[1,s]\mid \lambda_i\equiv 3 \bmod 4,\ q_i>0\}, \end{aligned}\tag{2.4}$$

$$\begin{aligned} J_2 &= \{i\in[1,s]\mid \lambda_i\equiv 1 \bmod 4,\ q_i>0\}\\ &\cup\ \{i\in[1,s]\mid \lambda_i\equiv 3 \bmod 4,\ p_i>0\}. \end{aligned}\tag{2.5}$$

We have an orthogonal decomposition $V=\oplus_{i\in J_1}V_{\lambda_i}\oplus_{i\in J_2}W_{\lambda_i}\oplus W$ into x_λ-stable subspaces. Suppose that $i\in J_1$. We choose an orthonormal basis $e_j^{i,1}\in V^+, j=1,\ldots,2[\frac{\lambda_i}{4}]+1, f_j^{i,1}\in V^-, j=1,\ldots,2[\frac{\lambda_i+1}{4}]$ of V_{λ_i} and define

$$x_i=e_1^{i,1}\cdots e_{2[\frac{\lambda_i}{4}]+1}^{i,1}f_1^{i,1}\cdots f_{2[\frac{\lambda_i+1}{4}]}^{i,1}\in C_V,\ i\in J_1.$$

Suppose that $i\in J_2$. We choose an orthonormal basis $e_j^{i,2}\in V^+$, $j=1,\ldots,2[\frac{\lambda_i+1}{4}]$, $f_j^{i,2}\in V^-, j=1,\ldots,2[\frac{\lambda_i}{4}]+1$ of W_{λ_i} and define

$$y_i=e_1^{i,2}\cdots e_{2[\frac{\lambda_i+1}{4}]}^{i,2}f_1^{i,2}\cdots f_{2[\frac{\lambda_i}{4}]+1}^{i,2}\in C_V,\ i\in J_2.$$

We have

$$\begin{aligned} &x_i^2=\epsilon^{\frac{\lambda_i(\lambda_i-1)}{2}},\ x_ix_{i'}=\epsilon x_{i'}x_i,\ i\neq i',\\ &y_i^2=\epsilon^{\frac{\lambda_i(\lambda_i-1)}{2}},\ y_iy_{i'}=\epsilon y_{i'}y_i,\ i\neq i',\ x_iy_j=\epsilon y_jx_i. \end{aligned}$$

Let $\hat\Gamma_1$ (resp. $\hat\Gamma_2$) be the subgroup of the group of units of C_V generated by x_i (resp. y_i), $i\in J_1$, (resp. $i\in J_2$) and $\Gamma_1\subset\hat\Gamma_1$ (resp. $\Gamma_2\subset\hat\Gamma_2$) the subgroup consisting of elements that are products of an even number of x_i's (resp. y_i's). Then $\Gamma_1 \subset K$ and $\Gamma_2 \subset K$. Moreover,

$$A_K(x_\lambda) \cong \Gamma_1 \times \Gamma_2 / \langle (\epsilon_1, \epsilon_2) \rangle$$

where ϵ_1 (resp. ϵ_2) is the element (-1) times unit in $C_{\oplus_{i\in J_1} V_{\lambda_i}}$ (resp. $C_{\oplus_{i\in J_2} V_{\lambda_i}}$). By [9, §14.3], Γ_1 (resp. Γ_2) is an central extension of $(\mathbb{Z}/2\mathbb{Z})^{a_\lambda - 1}$ (resp. $(\mathbb{Z}/2\mathbb{Z})^{b_\lambda - 1}$) by $\mathbb{Z}/2\mathbb{Z}$ if $a_\lambda > 0$ (resp. if $b_\lambda > 0$). Moreover, Γ_1 (resp. Γ_2) has 2 irreducible representations with non-trivial ϵ_1-action (resp. ϵ_2-action), each of dimension $2^{\frac{a_\lambda - 2}{2}}$ (resp. $2^{\frac{b_\lambda - 2}{2}}$), if a_λ (resp. b_λ) is even; it has 1 irreducible representation with non-trivial ϵ_1-action (resp. ϵ_2-action) of dimension $2^{\frac{a_\lambda - 1}{2}}$ (resp. $2^{\frac{b_\lambda - 1}{2}}$), if a_λ (resp. b_λ) is odd. Thus (iii) follows. ☐

2.1.2. Type DIII Let Λ denote the following set of signed Young diagrams

$$\Lambda = \{\lambda = (\lambda_1)_+^{p_1} (\lambda_1)_-^{q_1} \cdots (\lambda_s)_+^{p_s} (\lambda_s)_-^{q_s} \mid p_i = q_i \text{ for odd } \lambda_i, \quad p_i \equiv q_i \equiv 0 \bmod 2 \text{ for even } \lambda_i\}. \tag{2.6}$$

We write $\Lambda = \Lambda^{n,n}$ to indicate the signature of the Young diagrams. The set of K-orbits in $\mathcal{N}_1$ is (see, for example, [2, Theorem 9.3.4])

$$\{\mathcal{O}_\lambda \mid \lambda \in \Lambda^{n,n}\}.$$

Let $\lambda \in \Lambda$ and let $\mathcal{O} = \mathcal{O}_\lambda$ be the K-orbit in $\mathcal{N}_1$ corresponding to λ. Let $x_\lambda \in \mathcal{O}$.

Lemma 2.2. *We have $A_K(x_\lambda) = A_{\bar{K}}(x_\lambda) = 1$ unless all λ_i are even. In the latter case, $A_K(x_\lambda) \cong \mathbb{Z}/2\mathbb{Z}$, $A_{\bar{K}}(x_\lambda) = 1$ and $|\widehat{A_K(x_\lambda)}_{\kappa_1}| = 1$.*

Proof. The proof is similar to that of Lemma 2.1. Suppose that λ has an odd part of size $2k+1$. We show that $\epsilon \in Z_K(x_\lambda)^0$.

There exists an x_λ-stable orthogonal decomposition $V = W \oplus W^\perp$ such that $W = \text{span}\{e_i, f_i, i \in [1, 2k+1]\}$, $e_i \in V^+$, $f_i \in V^-$, $(e_i, f_j) = \delta_{2k+2,i+j}$ and $x_\lambda e_i = f_i, x_\lambda e_{2k+2-i} = -f_{2k+2-i}$, $x_\lambda f_i = e_{i+1}, x_\lambda f_{2k+1-i} = -e_{2k+2-i}$, $i \in [1, k]$, $x_\lambda e_{k+1} = x_\lambda f_{2k+1} = 0$.

Let $b, c \in \mathbb{C}^*$ be such that $bc = 1/2$. Let $v_{b,c,i} = be_i + cf_{2k+2-i}$, $i \in [1, k+1]$, $v_{b,c,i} = bf_{i-k-1} + ce_{3k+3-i}$, $i \in [k+2, 2k+1]$. We have $(v_{b,c,i}, v_{b,c,j}) = \delta_{i,j}$. Let $\sigma_{b,c} = v_{b,c,1} v_{b,c,2} \cdots v_{b,c,2k+1} \in C_V$. One checks that $\sigma_{b,c} e_i \sigma_{b,c}^{-1} = 2b^2 f_{2k+2-i}$ (resp. $2c^2 f_{2k+2-i}$) if $i \in [k+2, 2k+1]$ (resp. $i \in [1, k+1]$), and $\sigma_{b,c} f_i \sigma_{b,c}^{-1} = 2b^2 e_{2k+2-i}$ (resp. $2c^2 e_{2k+2-i}$) if $i \in [k+1, 2k+1]$ (resp. $i \in [1, k]$). Let

$$g_{b,c} = \sigma_{b_0,c_0} \sigma_{b,c} \sigma_{b_0,c_0} \sigma_{b,c} \in Spin_V$$

where $b_0 = c_0 = 1/\sqrt{2}$. It follows that $g_{b,c}e_ig_{b,c}^{-1} = 4b^4e_i$ (resp. $4c^4e_i$) if $i \in [k+2, 2k+1]$ (resp. $i \in [1, k+1]$), and $g_{b,c}f_ig_{b,c}^{-1} = 4b^4f_i$ (resp. $4c^4f_i$) if $i \in [k+1, 2k+1]$ (resp. $i \in [1,k]$). Thus $\pi(g_{b,c}) \in Z_{\bar{K}}(x_\lambda)$. Since $\{(b,c) \in (\mathbb{C}^*)^2 \mid bc = 1/2\}$ is irreducible, $g_{b_0,c_0} = 1$ and $g_{\sqrt{-1/2},-\sqrt{-1/2}} = \epsilon$, we conclude that $\epsilon \in Z_K(x_\lambda)^0$. Thus $A_K(x_\lambda) = A_{\bar{K}}(x_\lambda) = 1$.

Suppose that all parts λ_i are even. In this case $\bar{K}^\phi \cong \prod_i Sp_{p_i} \times \prod_i Sp_{q_i}$ is simply connected. It follows that $A_K(x_\lambda) \cong \mathbb{Z}/2\mathbb{Z}$ and $|\widehat{A_K(x_\lambda)}_{\kappa_1}| = 1$. $\square$

Corollary 2.3. *We have*

$$|\operatorname{Char}_K(\mathfrak{g}_1)| = |\operatorname{Char}_K(\mathfrak{g}_1)_{\kappa_0}| = |\operatorname{Char}_{\bar{K}}(\mathfrak{g}_1)| \text{ when } n \text{ is odd}$$
$$|\operatorname{Char}_K(\mathfrak{g}_1)_{\kappa_0}| = |\operatorname{Char}_{\bar{K}}(\mathfrak{g}_1)|,\ |\operatorname{Char}_K(\mathfrak{g}_1)_{\kappa_1}| = |\mathcal{P}_2(n/2)| \text{ when } n \text{ is even.}$$

Proof. Note that $|\operatorname{Char}_K(\mathfrak{g}_1)| = |\mathcal{A}_K(\mathfrak{g}_1)| = \sum_{\lambda\in\Lambda} |\widehat{A_K(\mathcal{O}_\lambda)}|$ and $|\operatorname{Char}_K(\mathfrak{g}_1)_{\kappa_i}| = |\mathcal{A}_K(\mathfrak{g}_1)_{\kappa_i}| = \sum_{\lambda\in\Lambda} |\widehat{A_K(\mathcal{O}_\lambda)}_{\kappa_i}|$. The claim for n odd and the first claim for n even follows from Lemma 2.2. Suppose n is even. Then $|\operatorname{Char}_K(\mathfrak{g}_1)_{\kappa_1}| = |\{\lambda \in \Lambda \mid \lambda_i \text{ even for all } i\}|$. The set of such λ's are in bijection with $\mathcal{P}_2(n/2)$ via the following map

$$(2\mu_1)_+^{2p_1}(2\mu_1)_-^{2q_1}\cdots(2\mu_s)_+^{2p_s}(2\mu_s)^{2q_s} \mapsto ((\mu_1)^{p_1}\cdots(\mu_s)^{p_s}, (\mu_1)^{q_1}\cdots(\mu_s)^{q_s})\,.$$

$\square$

2.2. Representations of extended braid groups

For a finite Coxeter group W, let B_W denote its associated braid group with associated projection $p : B_W \to W$ and let $\mathcal{H}_W$ denote a Hecke algebra associated to W. We write $\operatorname{Irr}\mathcal{H}_W$ for the set of irreducible representations of $\mathcal{H}_W$ over $\mathbb{C}$ (up to isomorphism). For $\rho \in \operatorname{Irr}\mathcal{H}_W$, we continue to write ρ for the irreducible representation of B_W obtained by pulling back ρ via the surjective map $\mathbb{C}[B_W] \to \mathcal{H}_W$.

Let $\widetilde{B}_W = B_W \ltimes H$ be an extended braid group, where H is a finite group, and let $E \in \widehat{H}$. We assume that the action of B_W on $\widehat{H}$ factors through W (which is the case in the situations we consider). Suppose that $W_{E,0}$ is a Coxeter subgroup of $W_E = \operatorname{Stab}_W(E)$. We write $B_W^{E,0} = p^{-1}(W_{E,0})$ and $\widetilde{B}_W^{E,0} = B_W^{E,0} \ltimes H$. Suppose that we have a surjective map $B_W^{E,0} \to B_{W_{E,0}}$. For $\rho \in \operatorname{Irr}\mathcal{H}_{W_{E,0}}$, we write

$$V_{\rho,E} = \mathbb{C}[\widetilde{B}_W] \otimes_{\mathbb{C}[\widetilde{B}_W^{E,0}]} (\rho \otimes E)$$

for the induced representation of $\widetilde{B}_W$, where $\widetilde{B}_W^{E,0} = B_W^{E,0} \ltimes H$ acts on $\rho \otimes E$ via the H action on E and the $B_W^{E,0}$-action on ρ via $B_W^{E,0} \to B_{W_{E,0}}$. Then $V_{\rho,E}$ is an irreducible representation of $\widetilde{B}_W$ if $W_{E,0} = W_E$. When $W_{E,0} \neq W_E$, we write $V_{\rho,E}^{\delta}$ for the non-isomorphic irreducible summands of $V_{\rho,E}$ as representations of $\widetilde{B}_W$.

2.3. Notations

We write W_r for the Weyl group of type B_r and W_r' for the Weyl group of type D_r.

We write $(A;x)_\infty = \prod_{s=0}^{\infty}(1 - Ax^s)$ and $(A;x)_n = \prod_{s=0}^{n-1}(1 - Ax^s)$.

We let μ_t denote the signed Young diagram

$$\mu_t = (2|t| - 1)_{sgn_t}(2|t| - 3)_{sgn_t} \cdots 3_{sgn_t} 1_{sgn_t} \text{ if } t \neq 0, \tag{2.7}$$

where $sgn_t = +$ (resp. $-$) if $t > 0$ (resp. $t < 0$) and we also write $\mu_t = \emptyset$ if $t = 0$.

We will often indicate the parameters of the Hecke algebra $\mathcal{H}_W$ by writing $\mathcal{H}_{W,c}$. For example, $\mathcal{H}_{W,-1}$ denote the Hecke algebra associated to W with parameter -1, that is, the Hecke algebra generated by T_i (i runs through a set of simple reflections in W) subject to braid relations plus the Hecke relations $(T_i - 1)^2 = 0$. Recall also the Hecke algebras $\mathcal{H}_{W_k,1,-1}, \mathcal{H}_{W_k,-1,1}, \mathcal{H}_{W_k,1,1} = \mathbb{C}[W_k]$ and their simple modules (see [13, §2.7])

$$\operatorname{Irr} \mathcal{H}_{W_k,1,-1} = \{L_\tau \mid \tau \in \mathcal{P}(k)\}, \ \operatorname{Irr} \mathcal{H}_{W_k,1,1} = \{L_\sigma \mid \sigma \in \mathcal{P}_2(k)\}. \tag{2.8}$$

Moreover, we have (see [1, 4])

$$\sum_{n \geq 0} |\operatorname{Irr} \mathcal{H}_{W_n,-1}|\, x^n = \prod_{s \geq 1}(1 + x^{2s})(1 + x^s) \tag{2.9a}$$

$$\sum_{n \geq 0} |\operatorname{Irr} \mathcal{H}_{W_n',-1}|\, x^n = \frac{1}{2}\prod_{s \geq 1}(1 + x^{2s-1})(1 + x^s) \tag{2.9b}$$

$$= \frac{1}{2}\sum_{n \geq 0} |\operatorname{Irr} \mathcal{H}_{W_n,-1,1}|\, x^n.$$

3. Nearby cycle sheaves

In this section we will produce character sheaves supported on $\check{\mathcal{O}}_{m,t}$ for type BDI and full support character sheaves for type DIII, making use of the nearby cycle construction of [6, 7] and its generalisation in [14]. We use the notations from [13] and refer the readers to [loc.cit] for details. The $\check{\mathcal{O}}_{m,t}$ are the dual stratum associated to the following nilpotent orbits in $\mathcal{N}_1$ (see [13, §3.1])

$$\mathcal{O}_{m,t} := \mathcal{O}_{1_+^m 1_-^m \sqcup \mu_t}, \ m = (N - |t|^2)/2$$

where μ_t is defined in (2.7). For convenience, let us define

$$\eta_{m,t} = \begin{cases} 2 & \text{if } t \text{ is odd} \\ 4 & \text{if } m \equiv \frac{t}{2} \bmod 2 \\ 1 & \text{if } m \equiv \frac{t}{2} + 1 \bmod 2. \end{cases} \tag{3.1}$$

We have the following short exact sequence (see [13, §3.1, equation (3.4)])

$$1 \to I_{\check{\mathcal{O}}} \to \pi_1^K(\check{\mathcal{O}}) \to B_{W_{\check{\mathcal{O}}}} \to 1$$

where $I_{\check{\mathcal{O}}} \cong A_K(x) = Z_K(x)/Z_K(x)^0$, $x \in \check{\mathcal{O}}$, and $B_{W_{\check{\mathcal{O}}}} \cong B_{W_{\mathfrak{a}^\phi}}$ is the braid group associated to $W_{\mathfrak{a}^\phi}$, $\phi = (e, f, h)$ is a normal $\mathfrak{sl}_2$-triple with $e \in \mathcal{O}, h \in \mathfrak{g}_0$.

Let $\mathfrak{a} \subset \mathfrak{g}_1$ be a Cartan subspace and let $W_\mathfrak{a} = N_K(\mathfrak{a})/Z_K(\mathfrak{a})$ be the little Weyl group. Then $W_{\mathfrak{a}^\phi} = W_\mathfrak{a}$ if $e = 0$. Note also that if $|t| \leq 1$, that is, if θ is split, then $\check{\mathcal{O}}_{m,t} = \mathfrak{g}_1^{rs}$, the set of regular semisimple elements of $\mathfrak{g}_1$, and $\overline{\check{\mathcal{O}}_{m,t}} = \mathfrak{g}_1$.

3.1. Nearby cycles for type DIII and split type BDI

In this subsection we describe the nearby cycle sheaves when (G, K) is of type DIII or of split type BDI.

Let $I = Z_K(\mathfrak{a})/Z_K(\mathfrak{a})^0$ and $\chi \in \hat{I}$. Let

$$P_\chi \in \mathrm{Perv}_K(\mathcal{N}_1)$$

be the nearby cycle sheaf defined in [6] (see also [13, §3.2]). Recall that the full support constituents of $\mathfrak{F}P_\chi$ are character sheaves, where

$$\mathfrak{F} : \mathrm{Perv}_K(\mathcal{N}_1) \subset \mathrm{Perv}_K(\mathfrak{g}_1) \to \mathrm{Perv}_K(\mathfrak{g}_1)$$

is the Fourier transform functor. We have ([6, Theorem 3.6], see also [13, §3.2])

$$\mathfrak{F}P_\chi \cong \mathrm{IC}(\mathfrak{g}_1^{rs}, \mathcal{M}_\chi)$$

where $\mathcal{M}_\chi$ is the K-equivariant local system on $\mathfrak{g}_1^{rs}$ given by the representation

$$M_\chi = \mathbb{C}[\widetilde{B}_{W_\mathfrak{a}}] \otimes_{\mathbb{C}[\widetilde{B}^{\chi,0}_{W_\mathfrak{a}}]} (\mathbb{C}_\chi \otimes \mathcal{H}_{W^0_{\mathfrak{a},\chi}})$$

of $\widetilde{B}_{W_\mathfrak{a}} := \pi_1^K(\mathfrak{g}_1^{rs})$, the equivariant fundamental group of $\mathfrak{g}_1^{rs}$. Here $\widetilde{B}_{W_\mathfrak{a}} \cong B_{W_\mathfrak{a}} \ltimes I$, $W^0_{\mathfrak{a},\chi}$ is a Coxeter subgroup of $W_{\mathfrak{a},\chi}$, and $\mathcal{H}_{W^0_{\mathfrak{a},\chi}}$ is a Hecke algebra associated to $W^0_{\mathfrak{a},\chi}$ with specified Hecke relations (see [6, §3.3]).

3.1.1. Type DIII Suppose that $(G = Spin_{2n}, K)$ is of type DIII. We have

$$I = 1 \text{ (resp. } \mathbb{Z}/2\mathbb{Z}) \text{ if } n \text{ is odd (resp. even)}, \quad W_\mathfrak{a} = W_{[n/2]}.$$

Note that in this case we have $\pi_1^K(\mathfrak{g}_1^{rs}) = B_{W_{[n/2]}} \times I$ as $B_{W_{[n/2]}}$ acts trivially on I. Let χ_0 denote the trivial character of I, and when n is even, let χ_1 denote the nontrivial character of I. We have $W^0_{\mathfrak{a},\chi_0} \cong W_\mathfrak{a}$ and $W^0_{\mathfrak{a},\chi_1} \cong S_n$. By [6, Theorem 3.6] (see also [13, §3.2]), we have $\mathcal{H}_{W^0_{\mathfrak{a},\chi_0}} \cong \mathcal{H}_{W_{[n/2]},1,-1}$ and $\mathcal{H}_{W^0_{\mathfrak{a},\chi_1}} \cong \mathbb{C}[S_n]$. It follows that

$$M_{\chi_0} = \mathcal{H}_{W_{[n/2]},1,-1}, \quad M_{\chi_1} = \mathcal{H}_{W_{n/2},1,1} \otimes \mathbb{C}_{\chi_1} \cong \mathbb{C}[W_{n/2}] \otimes \mathbb{C}_{\chi_1}.$$

For each $\tau \in \mathcal{P}([n/2])$ (resp. $\sigma \in \mathcal{P}_2(n/2)$), recall the simple module L_ρ (resp. L_τ) of $\mathcal{H}_{W_{[n/2]},1,-1}$ (resp. $\mathcal{H}_{W_{[n/2]},1,1}$). Let $\mathcal{L}_\tau$ (resp. $\mathcal{L}_\sigma \otimes \mathbb{C}_{\chi_1}$) denote the K-equivariant local system on $\mathfrak{g}_1^{rs}$ corresponding to the irreducible representation L_τ (resp. $L_\sigma \otimes \mathbb{C}_{\chi_1}$) of $\pi_1^K(\mathfrak{g}_1^{rs}) = B_{W_{[n/2]}} \times I$ where $B_{W_{[n/2]}}$ acts on L_τ (resp. L_σ), and I acts via χ_0 (resp. χ_1). It follows that

$$\begin{aligned} &\{\mathrm{IC}(\mathfrak{g}_1^{rs}, \mathcal{L}_\tau) \mid \tau \in \mathcal{P}([n/2])\} \subset \mathrm{Char}^{\mathrm{f}}_K(\mathfrak{g}_1)_{\kappa_0}, \\ &\{\mathrm{IC}(\mathfrak{g}_1^{rs}, \mathcal{L}_\sigma \otimes \mathbb{C}_{\chi_1}) \mid \sigma \in \mathcal{P}_2(n/2)\} \subset \mathrm{Char}^{\mathrm{f}}_K(\mathfrak{g}_1)_{\kappa_1} \text{ (when } n \text{ is even).} \end{aligned} \tag{3.2}$$

We in fact have equality in the above two equations, which will follow once we determine all character sheaves.

3.1.2. Split type BDI Suppose that θ is split, i.e.,

$$(G, K) = (Spin_N, K^{q+t,q}), \ |t| \leq 1.$$

Recall that

$$\begin{gathered} I = \langle \gamma_1, \ldots, \gamma_n \rangle \cong (\mathbb{Z}/2\mathbb{Z})^n, \ \gamma_i = \check{\alpha}_i(-1), \ i = 1, \ldots, n, \\ W_\mathfrak{a} = W_n \text{ (resp. } W'_n \text{) if } N \text{ is odd (resp. even)} \end{gathered} \tag{3.3}$$

where $\check{\alpha}_1, \ldots, \check{\alpha}_n$, are the simple coroots in $\check{R}(G, A)$ with respect to the θ-split maximal torus $A = Z_G(\mathfrak{a})$.

Suppose that $N = 2n+1$ (resp. $N = 2n$). We choose a set of simple roots as $\alpha_i = e_i - e_{i+1}$, $i = 1, \ldots, n-1$, and $\alpha_n = e_n$ (resp. $\alpha_n = e_{n-1} + e_n$). We define $\chi_m \in \hat{I}$, $0 \leq m \leq n$, by

$$\chi_m(\gamma_i) = 1,\ i \neq m,\ \chi_m(\gamma_m) = -1.$$

Let us write

$$\begin{gathered} \widetilde{B}_{W_n} = \widetilde{B}_{W_\mathfrak{a}} \cong B_{W_n} \ltimes (\mathbb{Z}/2\mathbb{Z})^n \\ (\text{resp. } \widetilde{B}_{W_n'} = \widetilde{B}_{W_\mathfrak{a}} \cong B_{W_n'} \ltimes (\mathbb{Z}/2\mathbb{Z})^n). \end{gathered} \tag{3.4}$$

The following proposition can be checked directly or be derived from [15].

Proposition 3.1. (i) *Suppose that* $(G, K) = (Spin_{2n+1}, K^{q+t,q})$, $|t| = 1$. *We have*

$$\left\{\mathfrak{F}(P_\chi) \mid \chi \in \hat{I}/W_n\right\} = \{\mathrm{IC}(\mathfrak{g}_1^{rs}, \mathcal{M}_{\chi_m}),\ m \in [0, [\tfrac{n}{2}]]\} \cup \{\mathrm{IC}(\mathfrak{g}_1^{rs}, \mathcal{M}_{\chi_n})\}$$

with

$$\begin{aligned} M_{\chi_m} &\cong \mathbb{C}[\widetilde{B}_{W_n}] \otimes_{\mathbb{C}[\widetilde{B}_{W_n}^{\chi_m,0}]} \big(\mathbb{C}_{\chi_m} \otimes (\mathcal{H}_{W_m,-1} \otimes \mathcal{H}_{W_{n-m},-1})\big) \ \ \textit{if } m \leq [\tfrac{n}{2}] \\ M_{\chi_n} &\cong \mathbb{C}[\widetilde{B}_{W_n}] \otimes_{\mathbb{C}[\widetilde{B}_{W_n}^{\chi_n}]} (\mathbb{C}_{\chi_n} \otimes \mathcal{H}_{S_n,-1}). \end{aligned}$$

(ii) *Suppose that* $(G, K) = (Spin_{2n}, K^{n,n})$, *and* $n \geq 3$. *We have*

$$\begin{aligned} &\left\{\mathfrak{F}(P_\chi) \mid \chi \in \hat{I}/W_n'\right\} \\ &\qquad = \left\{\mathrm{IC}(\mathfrak{g}_1^{rs}, \mathcal{M}_{\chi_m}),\ m \in [0, [\frac{n}{2}]] \cup \{n\} \cup (\textit{if } n \textit{ even})\{n-1\}\right\} \end{aligned}$$

with

$$\begin{aligned} M_{\chi_m} &\cong \mathbb{C}[\widetilde{B}_{W_n'}] \otimes_{\mathbb{C}[\widetilde{B}_{W_n'}^{\chi_m,0}]} \Big(\mathbb{C}_{\chi_m} \otimes (\mathcal{H}_{W_m',-1} \otimes \mathcal{H}_{W_{n-m}',-1})\Big) \ \ \textit{if } m \leq [\tfrac{n}{2}] \\ M_{\chi_m} &\cong \mathbb{C}[\widetilde{B}_{W_n'}] \otimes_{\mathbb{C}[\widetilde{B}_{W_n'}^{\chi_m,0}]} (\mathbb{C}_{\chi_m} \otimes \mathcal{H}_{S_n,-1}) \ \ \textit{if } m = n-1, n. \end{aligned}$$

Let us write $\Theta_{n,t}$ ($|t| \leq 1$) for the set of simple $\mathbb{C}[\widetilde{B}_{W_\mathfrak{a}}]$-modules that appear as composition factors of M_χ, $\chi \in \widehat{I}$. We have $\Theta_{n,t} = \Theta_{n,t}^{\kappa_0} \sqcup \Theta_{n,t}^{\kappa_1}$, where $\Theta_{n,t}^{\kappa_0}$ (resp. $\Theta_{n,t}^{\kappa_1}$) denote the subset of modules such that ϵ acts by 1 (resp. -1) via the natural map $\ker\pi \to I$. Note that $\epsilon = \gamma_n$ if N is odd, and $\epsilon = \gamma_{n-1}\gamma_n$ if N is even. Moreover, $\Theta_{n,1} = \Theta_{n,-1}$.

For each $\phi \in \Theta_{n,t}$, we write $\mathcal{T}_\phi$ for the corresponding local system on $\mathfrak{g}_1^{rs}$. It follows from Proposition 3.1 that $\mathrm{IC}(\mathfrak{g}_1^{rs}, \mathcal{T}_\phi) \in \mathrm{Char}_K^{\mathrm{f}}(\mathfrak{g}_1)$. Suppose that N is odd. We have $W_{\mathfrak{a},\chi_m} = W^0_{\mathfrak{a},\chi_m}$ for $0 \le m < n/2$ and $W_{\mathfrak{a},\chi_{n/2}}/W^0_{\mathfrak{a},\chi_{n/2}} \cong \mathbb{Z}/2\mathbb{Z}$. Suppose that N is even. We have

$$\begin{aligned}
&W_{\mathfrak{a},\chi_m} = W^0_{\mathfrak{a},\chi_m} \text{ if } m = 0, \text{ or if } m = n \text{ and } n \text{ is odd},\\
&W_{\mathfrak{a},\chi_m}/W^0_{\mathfrak{a},\chi_m} \cong \mathbb{Z}/2\mathbb{Z}, \text{ if } 1 \le m < n/2 \text{ or if } m = n-1, n \text{ and } n \text{ is even},\\
&W_{\mathfrak{a},\chi_{\frac{n}{2}}}/W^0_{\mathfrak{a},\chi_{\frac{n}{2}}} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}.
\end{aligned}$$

Applying the discussion in §2.2, we conclude that the IC sheaves in the following proposition are full support character sheaves. The fact that these are all full support character sheaves will follow once we determine all character sheaves.

Proposition 3.2. *Suppose that $(G, K) = (Spin_N, K^{q+t,q})$, where $|t| \le 1$. We have*

$$\mathrm{Char}_K^{\mathrm{f}}(\mathfrak{g}_1)_{\kappa_i} = \{\mathrm{IC}(\mathfrak{g}_1^{rs}, \mathcal{T}_\rho) \mid \rho \in \Theta^{\kappa_i}_{n,|t|}\},\ i = 0, 1,$$

$$\begin{aligned}
\Theta^{\kappa_0}_{n,1} = \{&V_{\rho_1\otimes\rho_2,\chi_k} \mid \rho_1 \in \mathrm{Irr}\,\mathcal{H}_{W_k,-1},\ \rho_2 \in \mathrm{Irr}\,\mathcal{H}_{W_{n-k},-1},\\
&\qquad k \in [0, \tfrac{n}{2}], \rho_1 \ncong \rho_2\}\\
&\cup \{V^\delta_{\rho\otimes\rho,\chi_{n/2}} \mid \rho \in \mathrm{Irr}\,\mathcal{H}_{W_{n/2},-1},\ \delta = \mathrm{I}, \mathrm{II}\}\\
&\qquad \textit{where } V_{\rho_1\otimes\rho_2,\chi_{n/2}} \cong V_{\rho_2\otimes\rho_1,\chi_{n/2}},\\
\Theta^{\kappa_0}_{n,0} = \{&V^\delta_{\rho_1\otimes\rho_2,\chi_m} \mid \rho_1 \in \mathrm{Irr}\,\mathcal{H}_{W'_m,-1},\ \rho_2 \in \mathrm{Irr}\,\mathcal{H}_{W'_{n-m},-1},\\
&\qquad m \in [1, \tfrac{n}{2}], \rho_1 \ncong \rho_2,\ \delta = \mathrm{I}, \mathrm{II}\}\\
&\cup \{V^\delta_{\rho\otimes\rho,\chi_{n/2}} \mid \rho \in \mathrm{Irr}\,\mathcal{H}_{W'_{n/2},-1},\ \delta = \mathrm{I}, \mathrm{II}, \mathrm{III}, \mathrm{IV}\}\\
&\cup \{V_{\rho,\chi_0} \mid \rho \in \mathrm{Irr}\,\mathcal{H}_{W'_n,-1}\},\\
&\qquad \textit{where } V^\delta_{\rho_1\otimes\rho_2,\chi_{n/2}} \cong V^\delta_{\rho_2\otimes\rho_1,\chi_{n/2}},
\end{aligned} \tag{3.5}$$

$$\begin{aligned}
&\Theta^{\kappa_1}_{n,1} = \{V^\delta_{\rho,\chi_n} \mid \rho \in \mathrm{Irr}\,\mathcal{H}_{S_n,-1},\ \delta = \mathrm{I}, \mathrm{II}\},\\
&\Theta^{\kappa_1}_{n,0} = \{V_{\rho,\chi_n} \mid \rho \in \mathrm{Irr}\,\mathcal{H}_{S_n,-1}\} \textit{ if } n \textit{ is odd},\\
&\Theta^{\kappa_1}_{n,0} = \{V^\delta_{\rho,\chi_i} \mid \rho \in \mathrm{Irr}\,\mathcal{H}_{S_n,-1},\ \delta = \mathrm{I}, \mathrm{II}, i = n-1, n\} \textit{ if } n \textit{ is even}.
\end{aligned} \tag{3.6}$$

3.2. Generalised nearby cycles

In this subsection we assume that (G, K) is of type BDI and $\check{\mathcal{O}} = \check{\mathcal{O}}_{1_+^m 1_-^m \sqcup \mu_t}$, $|t| \ge 2$. Let us write $I_{\check{\mathcal{O}}} := I_{m,t}$.

Lemma 3.3. (i) *We have*

$$I_{m,t} \cong A_K(\mathcal{O}_{\mu_t}) \text{ if } m = 0,\ I_{m,t} \cong \Gamma_{|t|} \times (\mathbb{Z}/2\mathbb{Z})^{m-1} \text{ if } m \geq 1,$$

where $\Gamma_{|t|}$ is a central extension of $(\mathbb{Z}/2\mathbb{Z})^{|t|-1}$ by $\mathbb{Z}/2\mathbb{Z}$. Moreover, when $m \geq 1$, $(\widehat{I}_{m,t})_{\kappa_1}$ consists of 2^{m-1} (resp. 2^m) irreducible representations of dimension $2^{(|t|-1)/2}$ (resp. $2^{(|t|-2)/2}$) if t is odd (resp. even).

(ii) *Suppose that $m \geq 1$. The action of $W_{\breve{\mathcal{O}}} = W_m$ on $(\widehat{I_{\breve{\mathcal{O}}}})_{\kappa_1}$ is transitive (resp. has two orbits) if $\eta_{m,t} \in \{1, 2\}$ (resp. $\eta_{m,t} = 4$), where $\eta_{m,t}$ is defined in (3.1). More precisely, let E (resp. E_1, E_2) be the irreducible representation(s) in $(\widehat{I_{\breve{\mathcal{O}}}})_{\kappa_1}$ such that the factor $(\mathbb{Z}/2\mathbb{Z})^{m-1}$ acts trivially. We have $(\widehat{I_{\breve{\mathcal{O}}}})_{\kappa_1} = W_m E$ (resp. $(\widehat{I_{\breve{\mathcal{O}}}})_{\kappa_1} = W_m E_1 \sqcup W_m E_2$).*

Moreover, $\mathrm{Stab}_{W_m}(E) \cong S_m$ *if* $\eta_{m,t} = 1$, $\mathrm{Stab}_{W_m}(E) \cong S_m \rtimes \langle \tau \rangle$ *if* $\eta_{m,t} = 2$, *and* $\mathrm{Stab}_{W_m}(E_i) \cong S_m \rtimes \langle \tau \rangle$ *if* $\eta_{m,t} = 4$, $i = 1, 2$, *where* $\tau^2 = 1$.

Proof. The lemma holds when $m = 0$ since we have $\breve{\mathcal{O}} = \mathcal{O}$. Suppose that $m \geq 1$. Let $x \in \breve{\mathcal{O}}$. We have an orthogonal decomposition

$$V = \oplus_{i=1}^{[\frac{|t|+1}{2}]} U_i \oplus_{i=1}^{[\frac{|t|}{2}]} W_i \oplus \oplus_{i=1}^{m} V_i$$

into x-stable subspaces such that $\dim U_i = 4i - 3$, $\dim W_i = 4i - 1$ and $\dim V_i = 2$. Moreover, there exist an orthonormal basis $e_j^{i,1} \in V^{sgn_t}, j \in [1, 2i-1], f_j^{i,1} \in V^{-sgn_t}, j \in [1, 2i-2]$ of U_i, an orthonormal basis $e_j^{i,2} \in V^{sgn_t}, j \in [1, 2i], f_j^{i,2} \in V^{-sgn_t}, j \in [1, 2i-1]$ of W_i and an orthonormal basis $e^{i,3} \in V^+$, $f^{i,3} \in V^-$ of V_i.

We define

$$x_i = e_1^{i,1} \cdots e_{2i-1}^{i,1} f_1^{i,1} \cdots f_{2i-2}^{i,1} \in C_V,\ i \in [1, [\tfrac{|t|+1}{2}]]$$
$$y_i = e_1^{i,2} \cdots e_{2i}^{i,2} f_1^{i,2} \cdots f_{2i-1}^{i,2} e^{1,3} f^{1,3} \in C_V,\ i \in [1, [\tfrac{|t|}{2}]]$$

and $z_i = e^{i,3} f^{i,3} \in C_V$, $i \in [1, m]$.

Let $\Gamma_{|t|}$ be the subgroup of $C_V^\times$ consisting of the products of an even number of x_i, y_i's (combined). Then $A_K(x) \cong \Gamma_{|t|} \times \langle \gamma_i, i \in [1, m-1] \rangle$, where $\gamma_i = z_i z_{i+1}$ and $\langle \gamma_i, i \in [1, m-1] \rangle \cong (\mathbb{Z}/2\mathbb{Z})^{m-1}$. Moreover, $\Gamma_{|t|}$ is a central extension of $(\mathbb{Z}/2\mathbb{Z})^{|t|-1}$ by $\mathbb{Z}/2\mathbb{Z} = \{1, \epsilon\}$ and $\mathbb{C}[\Gamma_{|t|}]/(\epsilon + 1)$ is isomorphic to the positive part of the Clifford algebra in $|t|$ variables, $C_{|t|}^+$. Part (i) of the lemma follows.

To prove part (ii) of the lemma, we note that when $|t|$ is even, the two non-isomorphic irreducible representations of $C_{|t|}^+$ are distinguished by the action of

$\gamma = x_1 \cdots x_{\frac{|t|}{2}} y_1 \cdots y_{\frac{|t|}{2}}$. Let $\rho \in (\widehat{I_{\check{\mathcal{O}}}})_{\kappa_1}$. For simplicity, let us continue to write γ_j, γ in place of $\rho(\gamma_j), \rho(\gamma)$. One checks that the action of $W_m = \langle s_1, \ldots, s_m \rangle$ (where s_m is the special simple reflection) are as follows

$$\begin{aligned} &s_i(\gamma_j) = \gamma_j,\ j \neq i-1, i+1,\ \ s_i(\gamma_j) = \gamma_i\gamma_j,\ j = i-1, i+1,\ i \in [1, m-1],\\ &s_m\gamma_j = \gamma_j,\ j \neq m-1,\ s_m(\gamma_{m-1}) = -\gamma_{m-1}. \end{aligned}$$

Moreover, when $t \equiv 0 \bmod 4$, $s_i\gamma = \gamma$, $i \neq m$, $s_m\gamma = -\gamma$, and when $t \equiv 2 \bmod 4$, $s_i\gamma = \gamma$, $i \neq 1, m$, $s_1\gamma = \gamma\gamma_1$, $s_m\gamma = -\gamma$.

It follows that $\mathrm{Stab}_{W_m} E, \mathrm{Stab}_{W_m} E_i \subset \mathrm{Stab}_{W_m} \chi = \langle s_1, \ldots, s_{m-1} \rangle \rtimes \langle \tau \rangle$, where

$$\chi : \langle \epsilon, \gamma_1, \ldots, \gamma_{m-1} \rangle \to \mathbb{C}^*,\ \chi(\epsilon) = -1\, \chi(\gamma_i) = 1$$

and

$$\tau = \prod_{i=1}^{[\frac{m+1}{2}]} s_{e_i + e_{m+1-i}}.$$

Here $s_{e_i+e_{m+1-i}} = s_{e_i} s_{e_i - e_{m+1-i}} s_{e_i}$ if $i \neq (m+1)/2$,

$$\begin{aligned} s_{e_i} &= s_i s_{i+1} \cdots s_{m-1} s_m s_{m-1} \cdots s_{i+1} s_i,\\ s_{e_i - e_{m+1-i}} &= s_i s_{i+1} \cdots s_{m-i-1} s_{m-i} s_{m-i-1} \cdots s_{i+1} s_i. \end{aligned}$$

Furthermore, when $t \equiv 0 \bmod 4$, $\tau\gamma = -\gamma$ (resp. γ) if m is odd (resp. even), and when $t \equiv 2 \bmod 4$, $\tau\gamma = \gamma \prod_{j=1}^{m-1} \gamma_j$ (resp. $-\gamma \prod_{j=1}^{m-1} \gamma_j$) if m is odd (resp. even). One then readily verifies that $\mathrm{Stab}_{W_m} E$ and $\mathrm{Stab}_{W_m} E_i$ are as desired and that E_1 and E_2 are not in the same W_m-orbit. This completes the proof of the lemma. □

Suppose that $m \geq 1$ and $|t| \geq 2$. Let $\mathcal{E}$ (resp. $\mathcal{E}_1, \mathcal{E}_2$) be the K-equivariant local system(s) on $X_x = K.x$, $x \in \check{\mathcal{O}}$, corresponding to the irreducible representation(s) E (resp. E_1, E_2) in $(\widehat{I_{\check{\mathcal{O}}}})_{\kappa_1}$ when $\eta_{m,t} \in \{1,2\}$ (resp. when $\eta_{m,t} = 4$). As in [14], we form the nearby cycle sheaf $P_{\check{\mathcal{O}},\mathcal{E}} \in \mathrm{Perv}_K(\mathcal{N}_1)$ (resp. $P_{\check{\mathcal{O}},\mathcal{E}_i} \in \mathrm{Perv}_K(\mathcal{N}_1)$, $i = 1, 2$).

Let us write $\widetilde{B}_{m,t} = \pi_1^K(\check{\mathcal{O}}_{m,t}) = B_{W_m} \ltimes I_{m,t}$. We use the notations from §2.2.

Proposition 3.4. *We have* $\mathfrak{F}P_{\check{\mathcal{O}},\mathcal{E}} \cong \mathrm{IC}(\check{\mathcal{O}}, \mathcal{M}_{\mathcal{E}})$ *(resp.* $\mathfrak{F}P_{\check{\mathcal{O}},\mathcal{E}_i} \cong \mathrm{IC}(\check{\mathcal{O}}, \mathcal{M}_{\mathcal{E}_i})$*,* $i = 1, 2$*), where* $\mathcal{M}_{\mathcal{E}}$ *(resp.* $\mathcal{M}_{\mathcal{E}_i}$*) corresponds to the following representation of* $\pi_1^K(\check{\mathcal{O}})$

$$M_E = \mathbb{C}[\widetilde{B}_{m,t}] \otimes_{\mathbb{C}[\widetilde{B}_{m,t}^{E,0}]} (\mathcal{H}_{S_m,-1} \otimes E)$$

$$\textit{(resp. } M_{E_i} = \mathbb{C}[\widetilde{B}_{m,t}] \otimes_{\mathbb{C}[\widetilde{B}_{m,t}^{E_i,0}]} (\mathcal{H}_{S_m,-1} \otimes E_i)\textit{).}$$

Proof. By Lemma 3.3, $W^0_{\check{\mathcal{O}},E} \cong W^0_{\check{\mathcal{O}},E_i} \cong S_m$. Moreover, $W_{\check{\mathcal{O}},E}/W^0_{\check{\mathcal{O}},E} = 1$ (resp. $\mathbb{Z}/2\mathbb{Z}$) if $\eta_{m,t} = 1$ (resp. 2), and $W_{\check{\mathcal{O}},E_i}/W^0_{\check{\mathcal{O}},E_i} \cong \mathbb{Z}/2\mathbb{Z}$. The proposition follows from [14, Theorem 3.2] and Proposition 3.1 provided that we check Assumption 3.1 in [14] holds. Let $e \in \mathcal{O}_{m,t}$ and $a \in (\mathfrak{a}^{\phi})^{rs}$. As in [14], it suffices to show that for any $e' \in Z_{\mathfrak{g}_1}(a) \cap \mathcal{O}'$, where $\mathcal{O}' \subset \bar{\mathcal{O}}$ and $\mathcal{O}' \neq \mathcal{O}$, and $a' \in (\mathfrak{a}^{\phi_{e'}})^{rs}$, $Z_K(a,a',e')/Z_K(a,a',e')^0$ does not afford the character κ_1. We have

$$Z_G(a) \cong \left(\{(v, g_1, \ldots, g_m) \in \times(\mathbb{C}^*)^{m+1} | v^2 = g_1 \cdots g_m\} \times Spin_{t^2}\right) / \langle \nu, \epsilon_t \rangle$$
$$Z_K(a) \cong \left(K_1 \times Spin_{\frac{t^2+t}{2}} \times Spin_{\frac{t^2-t}{2}}\right) / \langle \nu, \epsilon_1, \epsilon_2 \rangle$$

where $\nu = (-1, 1, \ldots, 1)$, ϵ_t is the nontrivial element in the kernel of $Spin_{t^2} \to SO_{t^2}$ and $K_1 = \{(v, g_1, \ldots, g_l) \in \mathbb{C}^* \times \{\pm 1\} \times \cdots \times \{\pm 1\} | v^2 = g_1 \cdots g_l\}$. The action of $Z_K(a)$ on $Z_{\mathfrak{g}_1}(a)$ is isomorphic to the action of K' on $\mathfrak{g}'_1$ for the symmetric pair $(G', K') = (Spin_{t^2}, K^{\frac{t^2+t}{2}, \frac{t^2-t}{2}})$. We are reduced to considering the symmetric pair (G', K') and which $\check{\mathcal{O}}'$ can afford the non-trivial character κ'_1 for any $\mathcal{O}' \subset \bar{\mathcal{O}}_{\mu_t}$. Suppose that $\mathcal{O}' = (\lambda_1)_+^{p_1} (\lambda_1)_-^{q_1} \cdots (\lambda_s)_+^{p_s} (\lambda_s)_-^{q_s}$. The same argument as in Lemma 2.1 shows that $\check{\mathcal{O}}'$ can afford κ'_1 only if $|p_i - q_i| \leq 1$. Let $n_0 = |\{i | p_i = q_i\}|$, $n_1 = |\{i | p_i = q_i + 1\}|$, $n_{-1} = |\{i | p_i = q_i - 1\}|$. Then we have $n_1 - n_{-1} = t$. Thus $n_1 \geq t$ and it follows that

$$\sum (p_i + q_i) \lambda_i \geq 1 + 3 + \cdots + 2t - 1 = t^2.$$

Here we have used the fact that if $p_i = q_i + 1$, then λ_i is odd. We conclude that $\mathcal{O}' = \mathcal{O}_{\mu_t}$. This completes the proof of the proposition. □

Let us write $\Theta^{\kappa_1}_{m,t}$ ($|t| \geq 2$) for the set of simple $\mathbb{C}[\widetilde{B}_{m,t}]$-modules that appear as composition factors of M_E, or M_{E_i}, $i = 1, 2$. For each $\phi \in \Theta^{\kappa_1}_{m,t}$, we write $\mathcal{T}_\phi$ for the corresponding local system on $\check{\mathcal{O}}_{m,t}$. We have the following corollary of Proposition 3.4. Note that the proof of the proposition implies that for the symmetric pair $(Spin_{t^2}, K^{\frac{t^2+t}{2}, \frac{t^2-t}{2}})$, the only $\check{\mathcal{O}}$ that can afford the non-trivial character κ_1 is $\check{\mathcal{O}}_{\mu_t} = \mathcal{O}_{\mu_t}$.

Corollary 3.5. *Suppose that $(G, K) = (Spin_N, K^{q+t,q})$, $|t| \geq 2$, and $m = \frac{N-t^2}{2}$. We have*

$$\left\{\mathrm{IC}(\check{\mathcal{O}}_{1_+^m 1_-^m \sqcup \mu_t}, \mathcal{T}_\phi) \mid \phi \in \Theta^{\kappa_1}_{m,t}\right\} \subset \mathrm{Char}_K(\mathfrak{g}_1)_{\kappa_1},$$
$$\Theta^{\kappa_1}_{0,t} = \widehat{A_K(\mu_t)}_{\kappa_1}$$

$$\Theta_{m,t}^{\kappa_1} = \{V_{\rho,E}^{\delta} \mid \rho \in \operatorname{Irr}\mathcal{H}_{S_m,-1},\ \delta = \mathrm{I},\mathrm{II}\} \textit{ if } m \geq 1 \textit{ and } \eta_{m,t} = 2$$
$$\Theta_{m,t}^{\kappa_1} = \{V_{\rho,E_i}^{\delta} \mid \rho \in \operatorname{Irr}\mathcal{H}_{S_m,-1},\ \delta = \mathrm{I},\mathrm{II},\ i = 1,2\} \textit{ if } m \geq 1 \textit{ and } \eta_{m,t} = 4$$
$$\Theta_{m,t}^{\kappa_1} = \{V_{\rho,E} \mid \rho \in \operatorname{Irr}\mathcal{H}_{S_m,-1}\} \textit{ if } m \geq 1 \textit{ and } \eta_{m,t} = 1.$$

4. Character sheaves

In this section we give explicit descriptions of the character sheaves following the strategy given in [13].

4.1. Supports of character sheaves and equivariant fundamental groups

In this subsection we describe the supports of character sheaves. As in [13], we define a set $\underline{\mathcal{N}_1}^{\mathrm{cs},\kappa_i}$ of nilpotent K-orbits in $\mathcal{N}_1$ such that for $\mathcal{O} \in \underline{\mathcal{N}_1}^{\mathrm{cs},\kappa_i}$ the corresponding $\check{\mathcal{O}}$ are the supports of character sheaves in $\mathrm{Char}_K(\mathfrak{g}_1)_{\kappa_i}$, $i = 0, 1$. The fact that these are indeed the supports of character sheaves will follow once we construct all character sheaves.

4.1.1. Type BDI Assume that (G, K) is of type BDI. We first recall the Richardson orbits associated to θ-stable Borel groups from [13, §3.5, §4].

Definition 4.1. *Let $\Sigma_b^{p,q} \subset \Sigma^{p,q}$ consist of signed Young diagrams of the form (see (2.1b))*

$$\lambda = (2\mu_1 + 1)_{\epsilon_1}(2\mu_2 + 1)_{\epsilon_2} \cdots (2\mu_k + 1)_{\epsilon_k}$$

where $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_k \geq 0$, $\epsilon_i \in \{0, 1\}$ and $\epsilon_i = \epsilon_j$ if $\mu_i = \mu_j$, such that the following conditions hold

(a) *when N is odd, $\epsilon_1 \equiv \min(p, q) \bmod 2$ and $\epsilon_{2i} + \mu_{2i} \equiv \epsilon_{2i+1} + \mu_{2i+1} \bmod 2$ for $i \geq 1$;*
(b) *when N is even, $\epsilon_{2i-1} + \mu_{2i-1} \equiv \epsilon_{2i} + \mu_{2i} \bmod 2$ for $i \geq 1$.*

Note that $\Sigma_b^{p,q} \cap \Sigma_3 = \emptyset$. Let

$$\Sigma_b = \sqcup_{p,q}\Sigma_b^{p,q},\ \Sigma_{b,i}^{p,q} = \Sigma_b^{p,q} \cap \Sigma_i,\ \Sigma_{b,i} = \Sigma_b \cap \Sigma_i,\ i = 1, 2.$$

The set of Richardson orbits attached to θ-stable Borel subgroups is the following ([12])

$$\{\mathcal{O}_\lambda \mid \lambda \in \Sigma_{b,1}\} \sqcup \{\mathcal{O}_\lambda^\delta \mid \lambda \in \Sigma_{b,2}, \delta = \mathrm{I}, \mathrm{II}\}.$$

Definition 4.2. *The set $\underline{\mathcal{N}_1}^{cs,\kappa_0}$ consists of the following orbits:*

$$\mathcal{O}_{m,k,\mu} = \mathcal{O}_{1_+^m 1_-^m 2_+^k 2_-^k \sqcup \mu}, \quad m \equiv q \bmod 2 \text{ if } N \text{ is even}, \\ \mu \in \Sigma_b^{p-m-2k,q-m-2k} \cup \{\emptyset\}.$$

The set $\underline{\mathcal{N}_1}^{cs,\kappa_1}$ consists of the following orbits:

$$\mathcal{O}_{m,t} = \mathcal{O}_{1_+^m 1_-^m 2_+^k 2_-^k \sqcup \mu_t}, \quad k = (N - t^2 - 2m)/4,$$

where μ_t is defined in (2.7).

Here $\lambda \sqcup \mu$ denotes the signed Young diagram obtained by joining λ and μ together, i.e., the rows of $\lambda \sqcup \mu$ are the rows of λ and μ rearranged according to the lengths of the rows. Moreover, $\mathcal{O}_{m,k,\mu}$ denotes $\mathcal{O}^\delta_{m,k,\mu}$ if $m = 0$ and $\mu \in \Sigma_{b,2} \cup \{\emptyset\}$, and $\mathcal{O}_{m,t}$ denotes $\mathcal{O}^\delta_{m,t}$ if $m = 0$ and $|t| \leq 1$.

Note that $\underline{\mathcal{N}_1}^{\mathrm{cs},\kappa_0} \cap \underline{\mathcal{N}_1}^{\mathrm{cs},\kappa_1} \neq \emptyset$ if and only if $|t| \leq 1$, that is, θ is split. In the latter case $\underline{\mathcal{N}_1}^{\mathrm{cs},\kappa_1} \subset \underline{\mathcal{N}_1}^{\mathrm{cs},\kappa_0}$.

In what follows we describe the equivariant fundamental groups of the above $\check{\mathcal{O}}$'s. The character sheaves in $\mathrm{Char}_K(\mathfrak{g}_1)_{\kappa_0}$ are given by local systems supported on $\check{\mathcal{O}}_{m,k,\mu}$ corresponding to irreducible representations of the equivariant fundamental group $\pi_1^K(\check{\mathcal{O}}_{m,k,\mu})$ for which the actions of $\pi_1^K(\check{\mathcal{O}}_{m,k,\mu})$ factor through $\pi_1^{\bar{K}}(\check{\mathcal{O}}_{m,k,\mu})$. Thus it suffices to write down the equivariant fundamental groups $\pi_1^{\bar{K}}(\check{\mathcal{O}}_{m,k,\mu})$. Using entirely similar argument as in [13, §6.1], we conclude that

$$\pi_1^{\bar{K}}(\check{\mathcal{O}}_{m,k,\mu}) \cong \begin{cases} {}_0\widetilde{B}_{W'_m} \times \widetilde{B}^1_{W_k} & \text{if } \mu = \emptyset \\ \widetilde{B}^1_{W_m} \times \widetilde{B}^1_{W_k} \times (\mathbb{Z}/2\mathbb{Z})^{r_\mu} & \text{if } \mu \neq \emptyset \text{ and } \mu \in \Sigma_{b,1} \\ {}_0\widetilde{B}_{W_m} \times \widetilde{B}^1_{W_k} \times (\mathbb{Z}/2\mathbb{Z})^{r_\mu} & \text{if } \mu \neq \emptyset \text{ and } \mu \in \Sigma_{b,2} \end{cases} \tag{4.1}$$

$$\pi_1^K(\check{\mathcal{O}}_{m,t}) \cong \widetilde{B}^1_{W_k} \times \widetilde{B}_{m,t}, \; k = (N - t^2 - 2m)/4 \tag{4.2}$$

where ${}_0\widetilde{B}_{W_m} = (\mathbb{Z}/2\mathbb{Z})^{m-1} \rtimes B_{W_m}$, ${}_0\widetilde{B}_{W'_m} = (\mathbb{Z}/2\mathbb{Z})^{m-1} \rtimes B_{W'_m}$, $\widetilde{B}^1_{W_m} = (\mathbb{Z}/2\mathbb{Z})^m \rtimes^1 B_{W_m}$, $\widetilde{B}_{m,0} = \widetilde{B}_{W'_m}$, $\widetilde{B}_{m,1} = \widetilde{B}_{W_m}$, $\widetilde{B}_{m,t} \cong B_{W_m} \ltimes I_{m,t}$ ($|t| \geq 2$, see §3.2), and r_μ are defined in (2.3). Moreover, ${}_0\widetilde{B}_{W_0} = {}_0\widetilde{B}_{W'_0} = B_{W_0} = B_{W'_0} = \widetilde{B}^1_{W_0} = \{1\}$ and the superscript 1 in the notation of $\widetilde{B}^1_{W_m}$ indicates the difference of the action of the braid group on the 2-group with that in $\widetilde{B}_{W_m}$ of (3.4). Moreover, $\widetilde{B}^1_{W_m}$ can be identified with $\widetilde{B}_{W_m}$ of [13].

4.1.2. Type DIII Assume that (G, K) is of type DIII. We first recall the Richardson orbits associated to θ-stable Borel groups from [13, §3.5, §4].

Definition 4.3. *Let $\Lambda_b^{n,n} \subset \Lambda^{n,n}$ (see (2.6)) consist of signed Young diagrams such that $p_i = q_i \leq 1$ when λ_i is odd, and $p_i \cdot q_i = 0$ when λ_i is even.*

The set of Richardson orbits attached to θ-stable Borel subgroups is $\{\mathcal{O}_\lambda \mid \lambda \in \Lambda_b^{n,n}\}$ ([12]). Note that by Corollary 2.3, $\mathrm{Char}_K(\mathfrak{g}_1)_{\kappa_1} = \emptyset$ when n is odd.

Definition 4.4. *We define*

$$\begin{aligned}\underline{\mathcal{N}_1}^{cs,\kappa_0} &= \{\mathcal{O}_{k,\mu} = \mathcal{O}_{1_+^{2k}1_-^{2k}\sqcup\mu} \mid 0 \leq k \leq [n/2],\ \mu \in \Lambda_b^{n-2k,n-2k} \cup \{\emptyset\}\},\\ \underline{\mathcal{N}_1}^{cs,\kappa_1} &= \{\mathcal{O}_{n/2,\emptyset} = \mathcal{O}_{1_+^n 1_-^n}\} \text{ when } n \text{ is even.}\end{aligned}$$

Note that $\underline{\mathcal{N}_1}^{\mathrm{cs},\kappa_1} \subset \underline{\mathcal{N}_1}^{\mathrm{cs},\kappa_0}$. The equivariant fundamental groups are as follows

$$\pi_1^{\bar{K}}(\check{\mathcal{O}}_{k,\mu}) \cong B_{W_k} \text{ when } \mu \neq \emptyset, \quad \pi_1^{K}(\check{\mathcal{O}}_{n/2,\emptyset}) \cong B_{W_{n/2}} \times \mathbb{Z}/2\mathbb{Z}. \tag{4.3}$$

4.2. The set $\Pi_{\mathcal{O}}$ associated to Richardson orbits in type BDI

In this subsection we assume that (G, K) is of type BDI. To write down irreducible representations of the equivariant fundamental groups that give rise to character sheaves, we define a subset $\Pi_{\mathcal{O}_\mu} \subset \widehat{A_K(\mathcal{O}_\mu)}_{\kappa_0} \cong \widehat{A_{\bar{K}}(\mathcal{O}_\mu)}$ for the Richardson orbits $\mathcal{O}_\mu$ following [13, §4], where $A_K(\mathcal{O}_\mu) := A_K(x_\mu)$, $A_{\bar{K}}(\mathcal{O}_\mu) := A_{\bar{K}}(x_\mu)$, $x_\mu \in \mathcal{O}_\mu$. Let $\mu \in \Sigma_b$ and let $\mathcal{O}_\mu$ be the corresponding Richardson orbit. We can write μ as follows

$$\mu = (2\mu_1+1)_{\epsilon_1}^{m_1}(2\mu_2+1)_{\epsilon_2}^{m_2}\cdots(2\mu_s+1)_{\epsilon_s}^{m_s},$$

where $\mu_1 > \mu_2 > \cdots > \mu_s \geq 0$, $m_i > 0$ and $\epsilon_i \in \{0,1\}$, $1 \leq i \leq s$. We define

$$\begin{aligned}J_\mu^1 &= \{i = 1,\ldots,s \mid \mu_i \equiv 0 \bmod 2, \epsilon_i = 0\}\\ &\quad \cup \{i = 1,\ldots,s \mid \mu_i \equiv 1 \bmod 2, \epsilon_i = 1\}\\ J_\mu^2 &= \{i = 1,\ldots,s \mid \mu_i \equiv 0 \bmod 2, \epsilon_i = 1\}\\ &\quad \cup \{i = 1,\ldots,s \mid \mu_i \equiv 1 \bmod 2, \epsilon_i = 0\}.\end{aligned}$$

Let $x \in \mathcal{O}_\mu$. Then

$$A_{\bar{K}}(\mathcal{O}_\mu) \cong S\left(\prod_{i\in J_\mu^1} O_{m_i}\right) \times S\left(\prod_{i\in J_\mu^2} O_{m_i}\right) / \left(SO_{m_1} \times \cdots \times SO_{m_s}\right).$$

We define $\Omega_{\mathcal{O}_\mu} = \Omega_\mu \subset \{1,\ldots,s\}$ to be the set of $j \in [1,s]$ such that

1. $\sum_{a=j}^{s} m_a$ is even,
2. if $j \geq 2$, then either $\mu_{j-1} \geq \mu_j + 2$ or $\epsilon_{j-1} = \epsilon_j$.

We set $l_\mu = l_{\mathcal{O}_\mu} := |\Omega_{\mathcal{O}_\mu}|$. Suppose that $\Omega_{\mathcal{O}_\mu} = \{j_1, \ldots, j_l\}$, $j_1 < \cdots < j_l$, $l = l_{\mathcal{O}}$, and we write $j_{l+1} = s+1$. Note that $j_1 = 1$ if and only if N is even. Thus $l_{\mathcal{O}} \geq 1$ when N is even.

For $1 \leq i \leq s-1$, let δ_i denote the generator of $S(O_{m_i} \times O_{m_{i+1}})/(SO_{m_i} \times SO_{m_{i+1}}) \cong \mathbb{Z}/2\mathbb{Z}$. Suppose that $\mu \in \Sigma_{b,1}$. Then $J_\mu^1 \neq \emptyset$ and $J_\mu^2 \neq \emptyset$. Let $i_1 < i_2 < \ldots < i_t$ be the set of j's in $[1, s-1]$ such that $\mu_j + \epsilon_j \neq \mu_{j+1} + \epsilon_{j+1}$. Then $A_{\bar{K}}(\mathcal{O}_\mu)$ is generated by δ_j, $j \in [i_a + 1, i_{a+1} - 1]$, $0 \leq a \leq t$, $\tau_{i_a} = \prod_{b=i_a}^{i_{a+1}} \delta_b$, $a = 1, \ldots, t-1$. Here we have written $i_0 = 0$ and $i_{t+1} = s$. Note that $i_a + 1 \in \Omega_\mu$, $a = 1, \ldots, t$. Suppose that $\mu \in \Sigma_{b,2}$. Then either $J_\mu^1 = \emptyset$ or $J_\mu^2 = \emptyset$, and $A_{\bar{K}}(\mathcal{O}_\mu)$ is generated by δ_i, $1 \leq i \leq s-1$.

The subset $\Pi_{\mathcal{O}_\mu} = \Pi_\mu \subset \widehat{A_K(\mathcal{O}_\mu)}_{\kappa_0}$ is defined as follows (via $\widehat{A_K(\mathcal{O}_\mu)}_{\kappa_0} \cong \widehat{A_{\bar{K}}(\mathcal{O}_\mu)}$)

$$\Pi_{\mathcal{O}_\mu} = \{\chi \in \widehat{A_{\bar{K}}(\mathcal{O}_\mu)} \mid \chi(\delta_r) = 1 \text{ if } r+1 \notin \Omega_{\mathcal{O}_\mu}\}. \tag{4.4}$$

Note that this is well-defined since $i_a + 1 \in \Omega_{\mathcal{O}_\mu}$, $a = 1, \ldots, t$ when $\mu \in \Sigma_{b,1}$. In particular, we see that

$$|\Pi_{\mathcal{O}_\mu}| = \begin{cases} 2^{l_\mu - 1} \text{ (resp. } 2^{l_\mu}) & \text{if } N \text{ is odd and } \mu \in \Sigma_{b,1} \text{ (resp. } \Sigma_{b,2}) \\ 2^{l_\mu - 2} \text{ (resp. } 2^{l_\mu - 1}) & \text{if } N \text{ is even and } \mu \in \Sigma_{b,1} \text{ (resp. } \Sigma_{b,2}). \end{cases} \tag{4.5}$$

4.3. Character sheaves

To describe the character sheaves for type BDI, we first write down representations of the fundamental groups in (4.1) and (4.2).

Recall the set $\Theta_{n,1}^{\kappa_0}$ (resp. $\Theta_{n,0}^{\kappa_0}$) of irreducible representations of $\widetilde{B}_{W_n}$ (resp. $\widetilde{B}_{W_n'}$) defined in §3.1. We can regard it as a set of simple $\mathbb{C}[{}_0\widetilde{B}_{W_n}]$-modules (resp. $\mathbb{C}[{}_0\widetilde{B}_{W_n'}]$-modules) via the natural projection $\widetilde{B}_{W_n} \to {}_0\widetilde{B}_{W_n}$ (resp. $\widetilde{B}_{W_n'} \to {}_0\widetilde{B}_{W_n'}$) where $\epsilon \mapsto 1$.

Let $\Theta_{n,1}^{\kappa_0,1}$ and $\Theta_{n,0}^{\kappa_0,1}$ denote the following sets of non-isomorphic simple $\mathbb{C}[\widetilde{B}_{W_n}^1]$-modules

$$\begin{aligned} \Theta_{n,1}^{\kappa_0,1} &= \{V_{\rho_1 \otimes \rho_2, \tilde{\chi}_k} \mid \rho_1 \in \operatorname{Irr} \mathcal{H}_{W_k,-1},\ \rho_2 \in \operatorname{Irr} \mathcal{H}_{W_{n-k},-1},\ k \in [0,n]\} \\ \Theta_{n,0}^{\kappa_0,1} &= \{V_{\rho_1 \otimes \rho_2, \tilde{\chi}_k} \mid \rho_1 \in \operatorname{Irr} \mathcal{H}_{W_k,-1,1},\ \rho_2 \in \operatorname{Irr} \mathcal{H}_{W_{n-k},-1,1},\ k \in [0,n]\} \end{aligned} \tag{4.6}$$

where $\tilde{\chi}_k$ is a set of representatives of B_{W_n}-orbits on $(\widehat{\mathbb{Z}/2\mathbb{Z}})^n$ such that $(W_n)_{\tilde{\chi}_k} \cong W_k \times W_{n-k}$.

Let $\Theta^{\kappa_0,2}_{n,1}$ and $\Theta^{\kappa_0,2}_{n,0}$ denote the following set of non-isomorphic simple $\mathbb{C}[{}_0\widetilde{B}_{W_n}]$-modules

$$
\begin{aligned}
(4.7)\qquad \Theta^{\kappa_0,2}_{n,1} &= \Theta^{\kappa_0}_{n,1} \text{ (see (3.5))}\\
\Theta^{\kappa_0,2}_{n,0} &= \{V_{\rho_1\otimes\rho_2,\chi_k} \mid \rho_1 \in \operatorname{Irr}\mathcal{H}_{W_k,-1,1},\ \rho_2 \in \operatorname{Irr}\mathcal{H}_{W_{n-k},-1,1},\\
&\qquad k \in [0,n/2], \rho_1 \ncong \rho_2\}\\
&\quad \cup \{V^{\delta}_{\rho\otimes\rho,\chi_{n/2}} \mid \rho \in \operatorname{Irr}\mathcal{H}_{W_{n/2},-1,1},\ \delta = \mathrm{I}, \mathrm{II}\}\\
&\text{where } V_{\rho_1\otimes\rho_2,\chi_{n/2}} \cong V_{\rho_2\otimes\rho_1,\chi_n/2}.
\end{aligned}
$$

Let us write

$$\varepsilon_t = 1 \text{ (resp. 0) if } t \text{ is odd (resp. even)}.$$

Suppose that $\mu \in \Sigma_{b,1}$ (resp. $\Sigma_{b,2}$, $\emptyset$). For each $\psi \in \Theta^{\kappa_0,1}_{m,\varepsilon_t}$ (resp. $\Theta^{\kappa_0,2}_{m,\varepsilon_t}$, $\Theta^{\kappa_0}_{m,0}$), $\tau \in \mathcal{P}(k)$, and $\phi \in \Pi_{\mathcal{O}_\mu}$ (see (4.4)), let $L_\psi \boxtimes L_\tau \boxtimes \phi$ denote the representation of $\pi_1^K(\check{\mathcal{O}}_{m,k,\mu})$ such that the action of $\pi_1^K(\check{\mathcal{O}}_{m,k,\mu})$ factors through $\pi_1^{\bar{K}}(\check{\mathcal{O}}_{m,k,\mu})$, where $\widetilde{B}^1_{W_m}$ (resp. ${}_0\widetilde{B}_{W_m}$, ${}_0\widetilde{B}_{W'_m}$) acts via ψ, $\widetilde{B}_{W_k}$ acts via the B_{W_k}-representation L_τ, and $(\mathbb{Z}/2\mathbb{Z})^{r_\mu}$ acts via ϕ. Let $\mathcal{T}_{\psi,\tau,\phi}$ denote the corresponding irreducible K-equivariant local system on $\check{\mathcal{O}}_{m,k,\mu}$. We write $\mathcal{T}_{\psi,\tau}$ (resp. $\mathcal{T}_{\tau,\phi}$) instead of $\mathcal{T}_{\psi,\tau,\phi}$ when $\mu = \emptyset$ (resp. $m = 0$) etc.

Recall the set $\Theta^{\kappa_1}_{m,t}$ of irreducible representations of $\widetilde{B}_{m,t}$ defined in §3.1 and §3.2. For each $\sigma \in \mathcal{P}_2(k)$ and $\rho \in \Theta^{\kappa_1}_{m,t}$, let $\mathcal{F}_{\rho,\sigma}$ denote the irreducible K-equivariant local system on $\check{\mathcal{O}}_{1^m_+1^m_-2^k_+2^k_-\sqcup\mu_t}$ corresponding to the irreducible representation $L_\sigma \boxtimes \rho$ of $\widetilde{B}^1_{W_k} \times \widetilde{B}_{m,t}$ such that $\widetilde{B}^1_{W_k}$ acts via the B_{W_K}-representation L_σ, and $\widetilde{B}_{m,t}$ acts via ρ.

Theorem 4.5. *Suppose that $(G,K) = (Spin_N, K^{q+t,q})$. Then*

$$
\begin{aligned}
\operatorname{Char}_K(\mathfrak{g}_1)_{\kappa_0} \;=\;& \{\mathrm{IC}(\check{\mathcal{O}}_{1^m_+1^m_-2^k_+2^k_-\sqcup\mu}, \mathcal{T}_{\psi,\tau,\phi}) \mid \mu \in \Sigma_{b,1},\\
&\qquad \psi \in \Theta^{\kappa_0,1}_{m,\varepsilon_t},\ \tau \in \mathcal{P}(k),\ \phi \in \Pi_{\mathcal{O}_\mu}\}\\
\sqcup\;& \Big\{\mathrm{IC}(\check{\mathcal{O}}_{1^m_+1^m_-2^k_+2^k_-\sqcup\mu}, \mathcal{T}_{\psi,\tau,\phi}) \mid m>0,\ \mu \in \Sigma_{b,2},\\
&\qquad \psi \in \Theta^{\kappa_0,2}_{m,\varepsilon_t},\ \tau \in \mathcal{P}(k),\ \phi \in \Pi_{\mathcal{O}_\mu}\Big\}\\
\sqcup\;& \Big\{\mathrm{IC}(\check{\mathcal{O}}^{\delta}_{2^k_+2^k_-\sqcup\mu}, \mathcal{T}_{\tau,\phi}) \mid \delta = \mathrm{I}, \mathrm{II},\\
&\qquad \mu \in \Sigma_{b,2},\ \tau \in \mathcal{P}(k),\ \phi \in \Pi_{\mathcal{O}_\mu}\Big\}
\end{aligned}
$$

$$\begin{aligned}
&\sqcup \quad \Big\{\mathrm{IC}(\check{\mathcal{O}}_{1_+^m1_-^m2_+^k2_-^k}, \mathcal{T}_{\psi,\tau}) \,|\, m>0,\\
&\qquad\qquad\qquad \psi\in\Theta^{\kappa_0}_{m,0}, \tau\in\mathcal{P}(k)\Big\} \ (t=0)\\
&\sqcup \quad \Big\{\mathrm{IC}(\check{\mathcal{O}}^{\delta}_{2_+^{n/2}2_-^{n/2}}, \mathcal{T}_{\tau}) \,|\, \delta=\mathrm{I},\mathrm{II},\mathrm{III},\mathrm{IV},\ \tau\in\mathcal{P}(\tfrac{n}{2})\Big\}\\
&\qquad\qquad\qquad (t=0 \textit{ and } n \textit{ even}).
\end{aligned}$$

$$\begin{aligned}
\mathrm{Char}_K(\mathfrak{g}_1)_{\kappa_1} \ = \ & \Big\{\mathrm{IC}(\check{\mathcal{O}}_{1_+^m1_-^m2_+^k2_-^k\sqcup\mu_t}, \mathcal{F}_{\rho,\sigma}) \mid m>0, \sigma\in\mathcal{P}_2(k),\ \rho\in\Theta^{\kappa_1}_{m,t}\Big\}\\
\sqcup \ & \Big\{\mathrm{IC}(\check{\mathcal{O}}_{2_+^k2_-^k\sqcup\mu_t}, \mathcal{F}_{\rho,\sigma}) \mid \sigma\in\mathcal{P}_2(k),\\
& \qquad \rho\in\Theta^{\kappa_1}_{0,t}=\widehat{A_K(\mathcal{O}_{\mu_t})}_{\kappa_1}\Big\} \ (|t|\geq 2)\\
\sqcup \ & \Big\{\mathrm{IC}(\check{\mathcal{O}}^{\delta}_{2_+^k2_-^k\sqcup 1_{sgn_t}}, \mathcal{F}_{\sigma}) \mid \sigma\in\mathcal{P}_2(k),\ \delta=\mathrm{I},\mathrm{II}\Big\} \ (|t|=1)\\
\sqcup \ & \Big\{\mathrm{IC}(\check{\mathcal{O}}^{\delta}_{2_+^k2_-^k}, \mathcal{F}_{\sigma}) \mid \sigma\in\mathcal{P}_2(k),\\
& \qquad \delta=\mathrm{I},\mathrm{II},\mathrm{III},\mathrm{IV}\} \ (t=0 \textit{ and } n \textit{ even}).
\end{aligned}$$

Theorem 4.6. *Suppose that (G,K) is of type* DIII. *Then*

$$\begin{aligned}
\mathrm{Char}_K(\mathfrak{g}_1)_{\kappa_0} &= \Big\{\mathrm{IC}(\check{\mathcal{O}}_{1_+^{2k}1_-^{2k}\sqcup\mu}, \mathcal{L}_\tau) \,|\, \mu\in\Lambda_b^{n-2k,n-2k},\ \tau\in\mathcal{P}(k)\Big\}\\
\mathrm{Char}_K(\mathfrak{g}_1)_{\kappa_1} &= \{\mathrm{IC}(\mathfrak{g}_1^{rs}, \mathcal{L}_\sigma\otimes\mathbb{C}_{\chi_1}) \mid \sigma\in\mathcal{P}_2(n/2)\}\,.
\end{aligned}$$

Here $\mathcal{L}_\tau$ denotes the irreducible local system given by the irreducible representation L_τ, where $\pi_1^K(\check{\mathcal{O}}_{k,\mu})$ acts through the action of $\pi_1^{\bar{K}}(\check{\mathcal{O}}_{k,\mu}) = B_{W_k}$, and $\mathcal{L}_\sigma\otimes\mathbb{C}_{\chi_1}$ is defined in §3.1.1. Theorem 4.6 follows from [13, Theorem 6.2], Corollary 2.3, and (3.2). We prove Theorem 4.5 in the next section.

Corollary 4.7. *Theorem 1.1 from the introduction holds. Moreover, we have*

$$\begin{aligned}
\sum_n |\,\mathrm{Char}^{\mathrm{cusp}}_{K^{n\pm1,n}}(\mathfrak{g}_1)_{\kappa_0}|x^n &= \frac{1}{2}\prod_{s\geq1}(1+x^{2s})^2(1+x^s)^2\\
&\quad +\frac{3}{2}\prod_{s\geq1}(1+x^{4s})(1+x^{2s})\\
\sum_{n \textit{ odd}} |\,\mathrm{Char}^{\mathrm{cusp}}_{K^{n,n}}(\mathfrak{g}_1)_{\kappa_0}|x^n &= x\prod_{s\geq1}(1+x^{4s})^4(1+x^{2s})^4\\
\sum_{n \textit{ even}} |\,\mathrm{Char}^{\mathrm{cusp}}_{K^{n,n}}(\mathfrak{g}_1)_{\kappa_0}|x^n &= \frac{1}{4}\prod_{s\geq1}(1+x^{4s-2})^4(1+x^{2s})^4\\
&\quad +\frac{3}{2}\prod_{s\geq1}(1+x^{4s-2})(1+x^{2s})
\end{aligned}$$

$$|\operatorname{Char}^{\mathrm{cusp}}_{K^{m+\frac{t^2+t}{2},m+\frac{t^2-t}{2}}}(\mathfrak{g}_1)_{\kappa_1}| = \textit{coefficient of } x^m \textit{ in } \eta_{m,t}\prod_{s\geq 1}(1+x^s).$$

Corollary 4.8. (i) *Assume that $(G,K)=(Spin_N, K^{p,q})$ and either p or q is even. We have*

$$\begin{aligned}\operatorname{Char}^{\mathrm{n}}_K(\mathfrak{g}_1)_{\kappa_0} &= \{\mathrm{IC}(\mathcal{O}_\mu,\mathcal{E}_\phi)\,|\,\mu\in\Sigma^{p,q}_{b,1},\ \phi\in\Pi_{\mathcal{O}_\mu}\}\\ &\quad\cup\{\mathrm{IC}(\mathcal{O}^\delta_\mu,\mathcal{E}_\phi)\,|\,\mu\in\Sigma^{p,q}_{b,2},\ \phi\in\Pi_{\mathcal{O}_\mu},\ \delta=\mathrm{I},\mathrm{II}\}\end{aligned}$$

where $\Pi_{\mathcal{O}_\mu}$ is defined in (4.4). Moreover,

$$|\operatorname{Char}^{\mathrm{n}}_{K^{q+t,q}}(\mathfrak{g}_1)_{\kappa_0}| = \textit{coefficient of } x^q \textit{ in}$$
$$\begin{cases}\dfrac{1}{2(1+x^t)}\displaystyle\prod_{s\geq1}\frac{(1+x^{2s-1})^2}{(1-x^{2s})^2}+\frac{3(1+x^t)}{2(1+x^{2t})}\prod_{s\geq1}\frac{1+x^{4s-2}}{(1-x^{2s})^2} & \textit{if } t \textit{ is odd}\\ \dfrac{1}{2(1+x^t)}\displaystyle\prod_{s\geq1}\frac{(1+x^{2s})^2}{(1-x^{2s})^2}+\frac{3(1+x^t)}{2(1+x^{2t})}\prod_{s\geq1}\frac{1+x^{4s}}{(1-x^{2s})^2} & \textit{if } t,q \textit{ both even.}\end{cases}$$

(ii) *We have*

$$\operatorname{Char}^{\mathrm{n}}_{K^{\frac{t^2+t}{2},\frac{t^2-t}{2}}}(\mathfrak{g}_1)_{\kappa_1} = \left\{\mathrm{IC}(\mathcal{O}_{\mu_t},\mathcal{E}_\phi)\mid\phi\in\widehat{A_K(\mathcal{O}_{\mu_t})_{\kappa_1}}\right\}.$$

Moreover, $|\operatorname{Char}^{\mathrm{n}}_{K^{\frac{t^2+t}{2},\frac{t^2-t}{2}}}(\mathfrak{g}_1)_{\kappa_1}| = \eta_{0,t}$.

5. Proof of the main theorem 4.5 and corollaries

In this section we assume that (G,K) is of type BDI unless otherwise stated. We prove Theorem 4.5 and its corollaries. To prove Theorem 4.5, as in [13], we show that the sheaves in the theorem are indeed character sheaves by constructing them using parabolic induction. We then conclude by showing that the number of the sheaves we have constructed coincides with the number of character sheaves.

5.1. Parabolic induction

In this subsection we study parabolic inductions of character sheaves in $\operatorname{Char}_{(L^\theta)^0}(\mathfrak{l}_1)$ for a family of θ-stable Levi subgroups L contained in θ-stable parabolic subgroups.

We first consider sheaves in $\operatorname{Char}_K(\mathfrak{g}_1)_{\kappa_0}$, identified with $\operatorname{Char}_{\bar{K}}(\mathfrak{g}_1)$. We will work in the setting of $(\bar{G},\bar{K})=(SO_N, SO_p\times SO_q)$. We follow [13, §7.2,

§7.3]. Recall the family of θ-stable parabolic subgroups $P_{m,k,\mu}$ together with their θ-stable Levi subgroups $L_{m,k,\mu}$ such that $\bar{K}.(\mathfrak{p}_{m,k,\mu})_1 = \overline{\check{\mathcal{O}}}_{m,k,\mu}$, $\mathcal{O}_{m,k,\mu} \in \underline{\mathcal{N}_1^{\mathrm{cs},\kappa_0}}$,

$$L_{m,k,\mu} \cong SO_{2m+\varepsilon_N} \times GL_{2k} \times GL_1^{n-m-2k}$$

and

$$(L^\theta_{m,k,\mu})^0 \cong SO_{m+\varepsilon_N} \times SO_m \times GL_k \times GL_k \times GL_1^{n-m-2k}$$

where $\varepsilon_N = 1$ (resp. 0) if N is odd (resp. even). Note that when $m = 0$ and $\mu \in \Sigma_2$ (resp. $\mu = \emptyset$), the $\tilde{K}$-conjugacy class of $P_{0,k,\mu}$ in the partial flag variety $G/P_{0,k,\mu}$ decomposes into 2 (resp. 4) $\bar{K}$-conjugacy classes.

Assume that either $m \neq 0$ or $k \neq 0$. In what follows we write $L = L_{m,k,\mu}$, $P = P_{m,k,\mu}$, and $\overline{\check{\mathcal{O}}} = \bar{K}.\mathfrak{p}_1$ etc. We describe the equivariant fundamental groups $\pi_1^{P_{\bar{K}}}(\mathfrak{p}_1^r)$, $\pi_1^{\bar{K}}(\check{\mathcal{O}})$ and $\pi_1^{(L^\theta)^0}(\mathfrak{l}_1^{rs})$, where $P_{\bar{K}} = P \cap \bar{K}$, $\mathfrak{p}_1^r = \mathfrak{p}_1 \cap \check{\mathcal{O}}$, and $\mathfrak{l}_1^{rs}$ is the set of regular semisimple elements of $\mathfrak{l}_1$ with respect to the symmetric pair $(L, (L^\theta)^0)$. Recall that $\bar{I} \cong (\mathbb{Z}/2\mathbb{Z})^{m-1}$ for the symmetric pair $(SO_{2m+\varepsilon_N}, SO_{m+\varepsilon_N} \times SO_m)$ (see (3.3)). Moreover $I = 1$ and $W_\mathfrak{a} = W_k$ for the symmetric pair $(GL_{2k}, GL_k \times GL_k)$ (see [13]). It follows that we have

$$\pi_1^{(L^\theta)^0}(\mathfrak{l}_1^{rs}) \cong {}_0\widetilde{B}_{W_m} \text{ (resp. } {}_0\widetilde{B}_{W_m'}) \times B_{W_k}, \qquad N \text{ odd (resp. even)}.$$

Consider the the map $\pi : \bar{K} \times^{P_{\bar{K}}} \mathfrak{p}_1 \to \overline{\check{\mathcal{O}}}$. Let $x \in \check{\mathcal{O}}$. Using similar argument as in [13, Lemma 7.1], one can show that $\pi^{-1}(x)$ has 2^{l_μ} irreducible components, and the $A_K(x)$-action on the set of irreducible components is transitive if N is odd, and has two orbits if N is even. Moreover, let $s_\mu = r_\mu - l_\mu + 1$ (resp. $r_\mu - l_\mu$) if N is odd and $\mu \in \Sigma_1$ (resp. Σ_2), and $s_\mu = r_\mu - l_\mu + 2$ (resp. $r_\mu - l_\mu + 1$) if N is even and $\mu \in \Sigma_1$ (resp. Σ_2). Then (cf. [13, (7.2b), (7.2c)])

$$\pi_1^{P_{\bar{K}}}(\mathfrak{p}_1^r) \cong \pi_1^{\bar{K}}(\check{\mathcal{O}}) \cong {}_0\widetilde{B}_{W_m'} \times \widetilde{B}^1_{W_k} \qquad \text{if } \mu = \emptyset \tag{5.1a}$$

$$\pi_1^{P_{\bar{K}}}(\mathfrak{p}_1^r) \cong \begin{cases} {}_0\widetilde{B}_{W_m} \times \widetilde{B}^1_{W_k} \times (\mathbb{Z}/2\mathbb{Z})^{s_\mu} & N \text{ odd} \\ {}_0\widetilde{B}^0_{W_m} \times \widetilde{B}^1_{W_k} \times (\mathbb{Z}/2\mathbb{Z})^{s_\mu} & N \text{ even} \end{cases} \tag{5.1b}$$

where ${}_0\widetilde{B}^0_{W_m} = (\mathbb{Z}/2\mathbb{Z})^{m-1} \rtimes B^0_{W_m}$ and $B^0_{W_m}$ is the subgroup of B_{W_m} defined as the kernel of the map $B_{W_m} \to \mathbb{Z}/2\mathbb{Z}$ sending a word in the braid generators $\sigma_1, \ldots, \sigma_m$ (σ_m is the special one) to 0 (resp. 1) if the braid generator σ_m appears even (resp. odd) number of times. Furthermore, the factor $(\mathbb{Z}/2\mathbb{Z})^{s_\mu - l_\mu}$ is given by $\{a \in A_{\bar{K}'}(\mathcal{O}_\mu) \,|\, \phi(a) = 1 \text{ for all } \phi \in \Pi_{\mathcal{O}_\mu}\}$.

We consider the parabolic induction of full support character sheaves on $\mathfrak{l}_1$. For each irreducible representation ψ of $\pi_1^{(L^\theta)^0}(\mathfrak{l}_1^{rs})$, we write ψ also for the irreducible representation of $\pi_1^{P_{\bar{K}}}(\mathfrak{p}_1^r)$ obtained via pull-back from the surjective map $\Phi:\pi_1^{P_{\bar{K}}}(\mathfrak{p}_1^r)\to\pi_1^{(L^\theta)^0}(\mathfrak{l}_1^{rs})$. Note that when N even and $\mu\neq\emptyset$, the map Φ gives rise to a surjective map $\zeta:{}_0\widetilde{B}^0_{W_m}\twoheadrightarrow{}_0\widetilde{B}_{W'_m}$. As in [13], one can check that $\Theta^{\kappa_0,1}_{n,|t|}$ (resp. $\Theta^{\kappa_0,2}_{n,|t|}$) coincides with the set of non-isomorphic simple $\mathbb{C}[\widetilde{B}^1_{W_n}]$-modules (resp. $\mathbb{C}[{}_0\widetilde{B}_{W_n}]$-modules) that arise as a direct summand of $\mathbb{C}[\widetilde{B}^1_{W_n}]\otimes_{\mathbb{C}[{}_0\widetilde{B}_{W_n}]}\psi$ (resp. $\mathbb{C}[{}_0\widetilde{B}_{W_n}]\otimes_{\mathbb{C}[{}_0\widetilde{B}^0_{W_n}]}\psi$), $\psi\in\Theta^{\kappa_0}_{n,|t|}$, $|t|\leq 1$. The following claims then follow from the description of $\pi_1^{\bar{K}}(\check{\mathcal{O}})$ in (4.1), the description of $\pi_1^{P_{\bar{K}}}(\mathfrak{p}_1^r)$ in (5.1a)–(5.1b).

Let $\mathcal{L}_\rho\boxtimes\mathcal{L}_\tau$, $\rho\in\Theta^{\kappa_0}_{m,1}$ (resp. $\Theta^{\kappa_0}_{m,0}$), and $\tau\in\mathcal{P}(k)$, be the irreducible $(L^\theta)^0$-equivariant local system on $\mathfrak{l}_1^{rs}$ given by the irreducible representation $\rho\boxtimes L_\tau$ of $\pi_1^{(L^\theta)^0}(\mathfrak{l}_1^{rs})\cong{}_0\widetilde{B}_{W_m}\times B_{W_k}$ (resp. ${}_0\widetilde{B}_{W'_m}\times B_{W_k}$), where ${}_0\widetilde{B}_{W_m}$ (resp. ${}_0\widetilde{B}_{W'_m}$) acts via ρ and B_{W_k} acts via L_τ. Then

$$\begin{gathered}\bigoplus_{\phi\in\Pi_{\mathcal{O}_\mu}}\mathrm{IC}(\check{\mathcal{O}},\oplus_{\psi\in\Theta^{\kappa_0,i}_{m,1}}\mathcal{T}_{\psi,\tau,\phi})[-]\ \bigoplus\ \mathrm{Ind}^{\mathfrak{g}_1}_{\mathfrak{l}_1\subset\mathfrak{p}_1}\mathrm{IC}(\mathfrak{l}_1^{rs},\oplus_{\rho\in\Theta^{\kappa_0}_{m,1}}\mathcal{L}_\rho\boxtimes\mathcal{L}_\tau),\\ \mu\in\Sigma_i, i=1,2,\\ \mathrm{IC}(\check{\mathcal{O}}_{m,k,\emptyset},\mathcal{T}_{\rho,\tau})[-]\ \bigoplus\ \mathrm{Ind}^{\mathfrak{g}_1}_{\mathfrak{l}_1\subset\mathfrak{p}_1}\mathrm{IC}(\mathfrak{l}_1^{rs},\mathcal{L}_\rho\boxtimes\mathcal{L}_\tau),\ \text{when } m>0,\\ \mathrm{IC}(\check{\mathcal{O}}^\delta_{0,n/2,\emptyset},\mathcal{T}_\tau)[-]\ \bigoplus\ \mathrm{Ind}^{\mathfrak{g}_1}_{\mathfrak{l}_1^\delta\subset\mathfrak{p}_1^\delta}\mathrm{IC}((\mathfrak{l}_1^\delta)^{rs},\mathcal{L}_\tau),\ \delta=\mathrm{I},\mathrm{II},\mathrm{III},\mathrm{IV}.\end{gathered}$$

We consider next the sheaves in $\mathrm{Char}_K(\mathfrak{g}_1)_{\kappa_1}$. Let P be a θ-stable subgroup such that $\pi(P)$ stabilises the flag $0\subset V_{2k}\subset V_{2k}^\perp\subset V$, where $V_{2k}=\mathrm{span}\{e_i,f_i,i=1,\dots,k\}$. Let $L\subset P$ be the θ-stable Levi subgroup such that $\pi(L)\cong GL(V_{2k})\oplus SO(V_{2k}^\perp/V_{2k})$. We have

$$\pi(L^\theta)^0\cong GL_k\times GL_k\times SO_{p-2k}\times SO_{q-2k}.$$

Consider the stratum $\check{\mathcal{O}}^{\mathfrak{l}_1}:=\check{\mathcal{O}}^{\mathfrak{l}_1}_{1^k_+1^k_-\boxtimes 1^m_+1^m_-\sqcup\mu_t}$ in $\mathfrak{l}_1$. We have

$$\pi_1^{L^\theta}(\check{\mathcal{O}}^{\mathfrak{l}_1})\cong B_{W_k}\times\mathbb{Z}/2\mathbb{Z}\times\widetilde{B}_{m,t}.$$

For each $\sigma\in\mathcal{P}_2(k)$ and $\rho\in\Theta^{\kappa_1}_{m,t}$, let $\mathcal{L}_{\rho,\sigma}$ denote the local system corresponding to the $\pi_1^{L^\theta}(\check{\mathcal{O}}^{\mathfrak{l}_1})$-representation where B_{W_k} acts via L_σ of W_k, $\mathbb{Z}/2\mathbb{Z}$ acts via the non-trivial character and $\widetilde{B}_{m,t}$ acts via ρ. Consider the IC sheaf $\mathrm{IC}(\overline{\check{\mathcal{O}}^{\mathfrak{l}_1}},\mathcal{L}_{\rho,\sigma})$. This is a character sheave on $\mathfrak{l}_1$. Note that the image of $K\times^{P_K}(\overline{\check{\mathcal{O}}^{\mathfrak{l}_1}}+(\mathfrak{n}_P)_1)$ under the map $\pi:K\times^{P_K}\mathfrak{p}_1\to K.\mathfrak{p}_1$ is $\overline{\check{\mathcal{O}}_{1^m_+1^m_-2^k_+2^k_-\sqcup\mu_t}}$. The same argument as before shows that

$$\mathrm{IC}(\check{\mathcal{O}}_{1^m_+1^m_-2^k_+2^k_-\sqcup\mu_t},\mathcal{F}_{\rho,\sigma})\ \bigoplus\ \mathrm{Ind}^{\mathfrak{g}_1}_{\mathfrak{l}_1\subset\mathfrak{p}_1}\mathrm{IC}(\check{\mathcal{O}}^{\mathfrak{l}_1},\mathcal{L}_{\rho,\sigma}).$$

Since Fourier transform commutes with parabolic induction, we conclude that the IC sheaves in Theorem 4.5 are character sheaves. Since they are pairwise non-isomorphic, it remains to check that the number of the IC sheaves equals the number of character sheaves. This will be done in the next subsections.

5.2. The number of character sheaves

In this subsection, we determine the numbers $|\operatorname{Char}_K(\mathfrak{g}_1)_{\kappa_0}|$ and $|\operatorname{Char}_K(\mathfrak{g}_1)_{\kappa_1}|$ of character sheaves.

Proposition 5.1. *Suppose that $(G,K)=(Spin_N, K^{q+t,q})$. We have that*

$$\begin{aligned}(5.2)\quad |\operatorname{Char}_K(\mathfrak{g}_1)_{\kappa_0}| = \ &\textit{coefficient of } x^q \textit{ in}\\ &\frac{1}{2(1+x^t)}\prod_{s\geq 1}\frac{1+x^s}{(1-x^s)^3}+\frac{3(1+x^t)}{2(1+x^{2t})}\prod_{s\geq 1}\frac{(1+x^{2s})^2}{(1-x^{2s})^3}\\ &+\frac{9}{4}\delta_{t,0}\prod_{s\geq 1}\frac{1}{1-x^{2s}}\end{aligned}$$

$$(5.3)\quad |\operatorname{Char}_K(\mathfrak{g}_1)_{\kappa_1}| = \textit{coefficient of } x^{N-t^2} \textit{ in } \eta_{\frac{N-t^2}{2},t}\prod_{s\geq 1}\frac{1}{(1-x^{4s})(1-x^{2s})}.$$

Lemma 5.2. *We have*

$$\sum_{\lambda\in\Sigma_2^{q+t,q}\cup\Sigma_3^{q+t,q}} 2^{r_\lambda} = \textit{ coefficient of } x^q \textit{ in } \frac{1+x^t}{1+x^{2t}}\prod_{s\geq 1}\frac{(1+x^{2s})^2}{(1-x^{2s})^3}.$$

Proof. We mimic the proof of [13, Proposition B.2, Corollary B.3] by Dennis Stanton and use the notations there. First, the 2-variable generating function of $\sum_{\lambda\in\Sigma_2\cup\Sigma_3} 2^{r_\lambda}$ is

$$(5.4)\qquad \begin{aligned}&\prod_{m\geq 0}\frac{1+u^{2m+2}v^{2m+1}}{1-u^{2m+2}v^{2m+1}}\prod_{m\geq 0}\frac{1+u^{2m}v^{2m+1}}{1-u^{2m}v^{2m+1}}\prod_{m\geq 1}\frac{1}{1-u^{2m}v^{2m}}\\ &+\prod_{m\geq 0}\frac{1+u^{2m+1}v^{2m+2}}{1-u^{2m+1}v^{2m+2}}\prod_{m\geq 0}\frac{1+u^{2m+1}v^{2m}}{1-u^{2m+1}v^{2m}}\prod_{m\geq 1}\frac{1}{1-u^{2m}v^{2m}}.\end{aligned}$$

By the q-binomial theorem, the first term of (5.4) equals

$$\frac{1}{(u^2v^2;u^2v^2)_\infty}\sum_n\frac{(-1;u^2v^2)_n}{(u^2v^2;u^2v^2)_n}(u^2v)^n\sum_n\frac{(-1;u^2v^2)_n}{(u^2v^2;u^2v^2)_n}v^n.$$

Picking out the terms of the form $u^{m+t}v^m$ we get

$$\frac{1}{(u^2v^2;u^2v^2)_\infty}\sum_n \frac{(-1;u^2v^2)_{n+t}}{(u^2v^2;u^2v^2)_n}\frac{(-1;u^2v^2)_{n+t}}{(u^2v^2;u^2v^2)_n}u^{2n+2t}v^{2n+t}.$$

Setting $u=v=x$, we get $\dfrac{2x^{3t}}{1+x^{4t}}\dfrac{(-x^4;x^4)_\infty^2}{(x^4;x^4)_\infty^3}$. Similarly the second term of (5.4) gives us $\dfrac{2x^t}{1+x^{4t}}\dfrac{(-x^4;x^4)_\infty^2}{(x^4;x^4)_\infty^3}$. Thus we have

$$\sum_q\left(2\sum_{\lambda\in\Sigma_2^{q+t,q}\cup\Sigma_3^{q+t,q}}2^{r_\lambda}\right)x^{2q+t}=\frac{2(x^{3t}+x^t)}{1+x^{4t}}\frac{(-x^4;x^4)_\infty^2}{(x^4;x^4)_\infty^3}.$$

Dividing both sides by $2x^t$ and replacing x^2 by x we get

$$\sum_q\left(\sum_{\lambda\in\Sigma_2^{q+t,q}\cup\Sigma_3^{q+t,q}}2^{r_\lambda}\right)x^{q}=\frac{2(x^{t}+1)}{1+x^{2t}}\frac{(-x^2;x^2)_\infty^2}{(x^2;x^2)_\infty^3}$$

and the lemma follows. □

Proof of Proposition 5.1. Note that $|\operatorname{Char}_K(\mathfrak{g}_1)_{\kappa_i}|=|\mathcal{A}_K(\mathfrak{g}_1)_{\kappa_i}|$, $i=0,1$, $|\mathcal{A}_K(\mathfrak{g}_1)_{\kappa_0}|=|\mathcal{A}_{\bar K}(\mathfrak{g}_1)|$ and that $|A_{\bar K}(x_\lambda)|=|A_{\widetilde K}(x_\lambda)|/2$, if $\lambda\in\Sigma_1$, $|A_{\bar K}(x_\lambda)|=|A_{\widetilde K}(x_\lambda)|$, if $\lambda\notin\Sigma_1$. Thus

$$\begin{aligned}|\operatorname{Char}_{\bar K}(\mathfrak{g}_1)|&=\sum_{\lambda\in\Sigma_1^{q+t,q}}2^{r_\lambda}+2\sum_{\lambda\in\Sigma_2^{q+t,q}}2^{r_\lambda}+4\sum_{\lambda\in\Sigma_3^{q+t,q}}2^{r_\lambda}\\|\operatorname{Char}_{\widetilde K}(\mathfrak{g}_1)|&=\sum_{\lambda\in\Sigma_1^{q+t,q}}2^{r_\lambda+1}+\sum_{\lambda\in\Sigma_2^{q+t,q}}2^{r_\lambda}+2\sum_{\lambda\in\Sigma_3^{q+t,q}}2^{r_\lambda}.\end{aligned}$$

It follows that

$$|\operatorname{Char}_{\bar K}(\mathfrak{g}_1)|=\frac12|\operatorname{Char}_{\widetilde K}(\mathfrak{g}_1)|+\frac32\sum_{\lambda\in\Sigma_2^{q+t,q}\cup\Sigma_3^{q+t,q}}2^{r_\lambda}+\frac32\mathbf{p}\left(\frac N4\right)\delta_{t,0}.$$

Hence (5.2) of the proposition follows from Lemma 5.2 and the following equation from [13, Corollary 8.6]

$$|\operatorname{Char}_{\widetilde K}(\mathfrak{g}_1)|=\text{ coefficient of } x^q \text{ in } \frac{1}{1+x^t}\prod_{s\geq1}\frac{1+x^s}{(1-x^s)^3}+\frac{3\delta_{t,0}}{2}\prod_{s\geq1}\frac{1}{1-x^{2s}}.$$

To prove (5.3), following [9, §14], we write $P_{N,t}$ for the set of partitions of N into distinct odd parts (and no even parts) such that the number of parts of the form $4m+1$ minus the number of parts of the form $4m+3$ equals t. Then we have $|P_{N,t}| = \mathbf{p}\left(\frac{N-(2t^2-t)}{4}\right)$. Let λ be a signed Young diagram such that $p_i \leq 1, q_i \leq 1$ for all odd λ_i. Recall the sets J_1, J_2 defined in (2.4) and (2.5). Then λ_i, $i \in J_1$ form a partition in P_{N_1,t_1} for some $N_1 \leq N, t_1 \in \mathbb{Z}$ and λ_i, $i \in J_2$ form a partition in P_{N_2,t_2} for some $N_2 \leq N, t_2 \in \mathbb{Z}$. Moreover, $t_1 - t_2 = t$. Thus it follows from Lemma 2.1 that

$$\begin{aligned}|\mathcal{A}_K(\mathfrak{g}_1)_{\kappa_1}| &= \eta_{\frac{N-t^2}{2},t} \sum_{\substack{N_1+N_2\leq N\\ t_1\in\mathbb{Z},t_2\in\mathbb{Z}\\ t_1-t_2=t}} \mathbf{p}\left(\frac{N_1-(2t_1^2-t_1)}{4}\right)\mathbf{p}\left(\frac{N_2-(2t_2^2-t_2)}{4}\right)\\ &\qquad\qquad\qquad\qquad\cdot\mathbf{p}\left(\frac{N-N_1-N_2}{4}\right)\\ &= \text{coefficient of } x^{N-t^2} \text{ in } \eta_{\frac{N-t^2}{2},t}\prod_{s\geq 1}\frac{1}{(1-x^{4s})(1-x^{2s})}.\end{aligned}$$

Here we have used the Jacobi triple product formula to deduce that

$$\sum_{t_1-t_2=t} x^{2t_1^2-t_1+2t_2^2-t_2} = \sum_{t_2} x^{(2t_2+t)^2-(2t_2+t)+t^2} = x^{t^2}\prod_{s\geq 1}(1-x^{4s})(1+x^{2s}). \qquad \square$$

Let

$$T^0_{p,q} = |\operatorname{Char}_{K^{p,q}}(\mathfrak{g}_1)_{\kappa_0}| \text{ and } T^0_N = \sum_{p+q=N} T^0_{p,q}. \tag{5.5}$$

Corollary 5.3. *We have*

$$\begin{aligned}\sum_{N=0}^{\infty} T^0_N x^N &= \frac{1}{4}\prod_{s\geq 1}\frac{(1+x^{2s-1})^2}{(1-x^{4s})(1-x^{2s-1})^2} + \frac{3}{2}\prod_{s\geq 1}\frac{1+x^{2s-1}}{(1-x^{4s})(1-x^{2s-1})}\\ &\quad + \frac{9}{4}\prod_{s\geq 1}\frac{1}{1-x^{4s}}\\ \sum_{n=0}^{\infty} T^0_{2n+1}x^n &= \prod_{s\geq 1}\frac{(1+x^{2s})^4(1+x^s)^3}{(1-x^s)} + 3\prod_{s\geq 1}\frac{(1+x^{4s})^2(1+x^{2s})(1+x^s)}{(1-x^s)}\\ \sum_{n=0}^{\infty} T^0_{2n}x^n &= \frac{1}{4}\prod_{s\geq 1}\frac{(1+x^{2s-1})^4(1+x^s)^3}{(1-x^s)}\\ &\quad + \frac{3}{2}\prod_{s\geq 1}\frac{(1+x^{4s-2})^2(1+x^{2s})(1+x^s)}{(1-x^s)} + \frac{9}{4}\prod_{s\geq 1}\frac{1}{1-x^{2s}}.\end{aligned}$$

Proof. Using the Ramanujan's ${}_1\psi_1$ formula as in [3, Appendix B], we see that

$$\sum_{k=-\infty}^{\infty} \frac{x^k}{1+x^{2k}} = \frac{1}{2}\prod_{s\geq 1}\frac{(1+x^{2s-1})^2(1-x^{2s})^2}{(1-x^{2s-1})^2(1+x^{2s})^2}$$

$$\sum_{k=-\infty}^{\infty} \frac{x^k(1+x^{2k})}{1+x^{4k}} = {}_1\psi_1(-1,-x^4;x^4;x) = \prod_{s\geq 1}\frac{(1+x^{2s-1})(1-x^{4s})^2}{(1-x^{2s-1})(1+x^{4s})^2}.$$

Moreover, it follows from

$$\begin{aligned}\sum_{k=-\infty,\, k \text{ odd}}^{\infty} \frac{x^k(1+x^{2k})}{1+x^{4k}} &= \frac{2x^3}{1+x^4}{}_1\psi_1(-x^{-4},-x^4;x^8;x^2)\\ &= 2x\prod_{s\geq 1}\frac{(1+x^{4s-2})(1-x^{8s})^2}{(1-x^{4s-2})(1+x^{8s-4})^2}\\ \sum_{k=-\infty,\, k \text{ even}}^{\infty} \frac{x^k(1+x^{2k})}{1+x^{4k}} &= {}_1\psi_1(-1,-x^8;x^8;x^2)\\ &= \prod_{s\geq 1}\frac{(1+x^{4s-2})(1-x^{8s})^2}{(1-x^{4s-2})(1+x^{8s})^2}\end{aligned}$$

that

$$\begin{aligned}\prod_{s\geq 1}\frac{1+x^{2s-1}}{1-x^{2s-1}} &= 2x\prod_{s\geq 1}(1+x^{8s})^2(1+x^{4s})(1+x^{2s})^2\\ &\quad+\prod_{s\geq 1}(1+x^{8s-4})^2(1+x^{4s})(1+x^{2s})^2.\end{aligned}$$

Recall that ([13, Appendix C.3])

$$\prod_{s\geq 1}\frac{(1+x^{2s-1})^2}{(1-x^{2s-1})^2} = 4x\prod_{s\geq 1}(1+x^{4s})^4(1+x^{2s})^4+\prod_{s\geq 1}(1+x^{4s-2})^4(1+x^{2s})^4. \tag{5.6}$$

The corollary follows from Proposition 5.1. □

5.3. Comparing the number of sheaves

The number of sheaves in Theorem 4.5 (ii) equals the coefficient of x^{N-t^2} in

$$\begin{aligned}&\eta_{\frac{N-t^2}{2},t}\prod_{s\geq 1}\frac{1}{(1-x^{4s})^2}\prod_{s\geq 1}(1+x^{2s})\\ &=\eta_{\frac{N-t^2}{2},t}\prod_{s\geq 1}\frac{1}{(1-x^{4s})(1-x^{2s})}\overset{(5.3)}{=}|\operatorname{Char}_K(\mathfrak{g}_1)_{\kappa_1}|\end{aligned}$$

since $\eta_{\frac{N-t^2}{2},t}=\eta_{\frac{N-4k-t^2}{2},t}$ (see (3.1)). Thus part (ii) of Theorem 4.5 follows.

Let us write $T'_{p,q}$ for the number of sheaves in Theorem 4.5 (i) and let $T'_N=\sum_{p+q=N}T'_{p,q}$. It suffices to show that $T'_N=T^0_N$ (see (5.5)). Since $0\leq T'_N\leq T^0_N$ by definition, it will then follow that $T'_{p,q}=T^0_{p,q}$.

Let $f^i_{m,t}=|\Theta^{\kappa_0,i}_{m,t}|$, $i=1,2$, $t=0,1$, and $f_{m,0}=|\Theta^{\kappa_0}_{m,0}|$, see (3.5), (4.6) and (4.7). We have

$$
\begin{aligned}
T'_{p,q}=&\sum_k\sum_{m\geq 1}\sum_{\mu\in\Sigma_{b,1}^{p-2k-m,q-2k-m}}\mathbf{p}(k)f^1_{m,1}2^{l_\mu-1}\\
&+\sum_k\sum_{m\geq 1}\sum_{\mu\in\Sigma_{b,2}^{p-2k-m,q-2k-m}}\mathbf{p}(k)f^2_{m,1}2^{l_\mu}\\
&+\sum_k\sum_{\mu\in\Sigma_{b,1}^{p-2k,q-2k}}\mathbf{p}(k)2^{l_\mu-1}+2\sum_k\sum_{\mu\in\Sigma_{b,2}^{p-2k,q-2k}}\mathbf{p}(k)2^{l_\mu}\\
&\qquad\text{if } p+q \text{ is odd},\\
T'_{p,q}=&\sum_k\sum_{m\geq 1}\sum_{\mu\in\Sigma_{b,1}^{p-2k-m,q-2k-m}}\mathbf{p}(k)f^1_{m,0}2^{l_\mu-2}\\
&+\sum_k\sum_{m\geq 1}\sum_{\mu\in\Sigma_{b,2}^{p-2k-m,q-2k-m}}\mathbf{p}(k)f^2_{m,0}2^{l_\mu-1}\\
&+\sum_k\sum_{\mu\in\Sigma_{b,1}^{p-2k,q-2k}}\mathbf{p}(k)2^{l_\mu-2}+2\sum_k\sum_{\mu\in\Sigma_{b,2}^{p-2k,q-2k}}\mathbf{p}(k)2^{l_\mu-1}\\
&+\delta_{p,q}\sum_k\sum_{m\geq 1}\mathbf{p}(k)f_{m,0}+4\delta_{p,q}\mathbf{p}(n/2)\text{ if } p+q \text{ is even}.
\end{aligned}
\tag{5.7}
$$

Let

$$
\begin{aligned}
&b^i_{p,q}=\sum_{\mu\in\Sigma^{p,q}_{b,i}}|\Pi_{\mathcal{O}_\mu}|,\ b^i_N=\sum_{p+q=N}b^i_{p,q},\ \ i=1,2,\\
&\tilde{b}_{p,q}=2b^1_{p,q}+b^2_{p,q},\tilde{b}_N=\sum_{p+q=N}\tilde{b}_{p,q}.
\end{aligned}
\tag{5.8}
$$

Then $\tilde{b}_{p,q}=|\operatorname{Char}^{\mathrm{n}}_{\widetilde{K}^{p,q}}(\mathfrak{g}_1)|$, $\tilde{b}_{2n+\varepsilon}=\sum_{p+q=2n+\varepsilon}\sum_{\mu\in\Sigma_b^{p,q}}2^{l_\mu-1+\varepsilon}$, $\varepsilon=0,1$. Moreover, by [13] we have

$$
\begin{aligned}
&\sum_{n=0}^\infty\tilde{b}_{2n+1}x^n=2\prod_{s\geq 1}(1+x^s)^2(1+x^{2s})^2,\\
&\sum_{n=0}^\infty\tilde{b}_{2n}x^n=\frac{1}{2}\prod_{s\geq 1}(1+x^s)^2(1+x^{2s-1})^2.
\end{aligned}
\tag{5.9}
$$

Lemma 5.4. *We have*

$$\sum_{p=0}^{\infty} b^2_{2n+1}x^n = 2\prod_{s\geq 1}(1+x^s)^2(1+x^{4s}), \tag{5.10}$$

$$\sum_{p=0}^{\infty} b^2_{2n}x^n = \prod_{s\geq 1}(1+x^s)^2(1+x^{4s-2}).$$

Proof. We follow the proof in [3, Appendix B]. Let $\mathcal{P}^{odd}(N)$ denote the set of partitions of N into odd parts. We write a partition $\lambda \in \mathcal{P}^{odd}(N)$ as $\lambda = (2\mu_1+1)+(2\mu_2+1)+\cdots+(2\mu_s+1)$ where $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_s \geq 0$. Note that $s \equiv N \mod 2$. We set

$$wt_\lambda = 2^{\#\{1\leq j\leq (s-1)/2\,|\,\mu_{2j-1}\geq \mu_{2j}+2\}} \text{ when } N \text{ is odd;}$$

$$wt_\lambda = 2^{\#\{1\leq j\leq s/2-1\,|\,\mu_{2j}\geq \mu_{2j+1}+2\}} \text{ when } N \text{ is even.}$$

We have

$$b^2_N = \sum_{\lambda\in\mathcal{P}^{odd}(N)} wt_\lambda.$$

The first equation is equivalent to

$$\sum_{n=0}^{\infty} b^2_{2n+1}x^{2n+1} = x(-x^8;x^8)_\infty(-x^2;x^2)^2_\infty.$$

This can be seen as follows. Suppose λ with odd parts has an odd number of parts, say $2k+1$. Consider the columns of λ, which have possible sizes $1,2,\ldots,2k+1$. The part $2k+1$ occurs an odd number of times, the generating function is $\frac{x^{2k+1}}{1-x^{4k+2}}$. The part $2k$ occurs an even number of times, the generating function is $\frac{1}{1-x^{4k}}$. The part $2k-1$ occurs an even number of times, the weighted generating function is $1+x^{4k-2}+\frac{2x^{8k-4}}{1-x^{4k-2}} = \frac{1+x^{8k-4}}{1-x^{4k-2}}$. This continues down to part size 1, to obtain the generating function

$$\begin{aligned}
\sum_{n=0}^{\infty} b^2_{2n+1}x^{2n+1} &= \sum_{k=0}^{\infty}\frac{x^{2k+1}}{1-x^{4k+2}}\frac{1}{\prod_{j=1}^{k}(1-x^{4j})}\prod_{i=1,\text{odd}}^{2k-1}\frac{1+x^{4i}}{1-x^{2i}}\\
&= \frac{x}{1-x^2}\sum_{k=0}^{\infty}\frac{(-\mathbf{i}x^2;x^4)_k(\mathbf{i}x^2;x^4)_k}{(x^6;x^4)_k(x^4;x^4)_k}x^{2k}\\
&= \frac{x}{1-x^2}\frac{(\mathbf{i}x^4;x^4)_\infty(-\mathbf{i}x^4;x^4)_\infty}{(x^6;x^4)_\infty(x^2;x^4)_\infty}
\end{aligned}$$

$$= x(-x^8;x^8)_\infty(-x^2;x^2)^2_\infty,$$

where in the second last equality we have applied the q-Gauss theorem [3] with $q \to x^4$ and $a = -\mathbf{i}x^2$, $b = \mathbf{i}x^2$, $c = x^6$. The second equation is equivalent to $\sum_{n=0}^{\infty} b^2_{2n}x^{2n} = \frac{1}{2}(-x^4;x^8)_\infty(-x^2;x^2)^2_\infty$. Assume λ is a partition into odd parts with $2k$ parts. We argue as above, this time the even parts have weights. Thus

$$\begin{aligned}\sum_{n=0}^{\infty} b^2_{2n}x^{2n} &= \sum_{k=0}^{\infty} \frac{x^{2k}}{1-x^{4k}} \frac{1}{\prod_{j=1}^{k}(1-x^{4j-2})} \prod_{i=1,\text{even}}^{2k-2} \frac{1+x^{4i}}{1-x^{2i}} \\ &= \frac{1}{2}\sum_{k=0}^{\infty} \frac{(-\mathbf{i};x^4)_k(\mathbf{i};x^4)_k}{(x^2;x^4)_k(x^4;x^4)_k} x^{2k} \\ &= \frac{1}{2}\frac{(-\mathbf{i}x^2;x^4)_\infty(\mathbf{i}x^2;x^4)_\infty}{(x^2;x^4)_\infty(x^2;x^4)_\infty} = \frac{1}{2}(-x^4;x^8)_\infty(-x^2;x^2)^2_\infty. \qquad \square\end{aligned}$$

By (2.9a) and (2.9b) we have

$$\sum_m f^1_{m,1}x^m = \prod_{s\geq 1}(1+x^{2s})^2(1+x^s)^2,$$

$$\sum_m f^1_{m,0}x^m = \prod_{s\geq 1}(1+x^{2s-1})^2(1+x^s)^2$$

$$\sum_m f^2_{m,1}x^m = \frac{1}{2}\prod_{s\geq 1}(1+x^{2s})^2(1+x^s)^2 + \frac{3}{2}\prod_{s\geq 1}(1+x^{4s})(1+x^{2s}) \tag{5.11}$$

$$\sum_m f_{m,0}x^m = \frac{1}{4}\prod_{s\geq 1}(1+x^{2s-1})^2(1+x^s)^2 + \frac{3}{2}\prod_{s\geq 1}(1+x^{4s-2})(1+x^{2s}) \tag{5.12}$$

$$\sum_m f^2_{m,0}x^m = \frac{1}{2}\prod_{s\geq 1}(1+x^{2s-1})^2(1+x^s)^2 + \frac{3}{2}\prod_{s\geq 1}(1+x^{4s-2})(1+x^{2s}).$$

Let us write $F(x) = \prod_{s\geq 1}\frac{1}{1-x^{2s}} = \sum \mathbf{p}(k)x^{2k}$. It follows from (5.7), (5.8), (5.9), (5.10) and the above equations that

$$\begin{aligned}\sum_{n\geq 0} T'_{2n+1}x^n = \Bigg(&\left(\sum_m f^1_{m,1}x^m\right)\left(\frac{1}{2}\sum_n \tilde{b}_{2n+1}x^n\right) \\ &+\frac{3}{2}\prod_{s\geq 1}(1+x^{4s})(1+x^{2s})\left(\sum_n b^2_{2n+1}x^n\right)\Bigg) F(x)\end{aligned}$$

$$= \prod_{s\geq 1}\frac{(1+x^{2s})^4(1+x^s)^4}{1-x^{2s}} + 3\prod_{s\geq 1}\frac{(1+x^{4s})^2(1+x^{2s})(1+x^s)^2}{1-x^{2s}}$$

$$\sum_{n\geq 0} T'_{2n}x^{2n} = \Bigg(\Big(\sum_m f^1_{m,0}x^m\Big)\Big(\frac{1}{2}\sum_n \tilde{b}_{2n}x^n\Big) + \frac{3}{2}\prod_{s\geq 1}(1+x^{4s-2})(1+x^{2s})\Big(\sum_n b^2_{2n}x^n\Big) + \frac{9}{4}\Bigg)F(x)$$

$$= \frac{1}{4}\prod_{s\geq 1}\frac{(1+x^{2s-1})^4(1+x^s)^4}{1-x^{2s}} + \frac{3}{2}\prod_{s\geq 1}\frac{(1+x^{4s-2})^2(1+x^{2s})(1+x^s)^2}{1-x^{2s}} + \frac{9}{4}\prod_{s\geq 1}\frac{1}{1-x^{2s}}.$$

We conclude that $T'_N = T^0_N$ in view of Corollary 5.3. This completes the proof of the theorem.

5.4. Proofs of Corollary 4.7 and Corollary 4.8

Proof of Corollary 4.7. In view of [14, Corollary 6.7] and its proof, it suffices to consider the sheaves in $\mathrm{Char}_K(\mathfrak{g}_1)_{\kappa_1}$. Suppose that (G,K) is of type DIII and $n\geq 4$. We show that all character sheaves can be obtained from parabolic induction. Suppose that $n=2n_0$. Let $\{e_i, i=1,\ldots,n\}$, $\{f_i, i=1,\ldots,n\}$ be a basis of V^+, V^- respectively, such that $(e_i,f_j)=\delta_{i+j,n+1}$. Consider the θ-stable parabolic subgroup P such that $\pi(P)$ stabilises the flag $0\subset V_n := \mathrm{span}\{e_i,f_i,i\in[1,n_0]\}\subset V$. Let $L\subset P$ be the θ-stable Levi subgroup such that $\pi(L)\cong GL_{V_n}\cong GL_{2n_0}$. We have $\pi(L^\theta)\cong GL_{V_n\cap V^+}\times GL_{V_n\cap V^-}\cong GL_{n_0}\times GL_{n_0}$. Moreover, $\pi_1^{L^\theta}(\mathfrak{l}_1)\cong B_{W_{n_0}}\times\mathbb{Z}/2\mathbb{Z}$. For each $\sigma\in\mathcal{P}_2(n_0)$, consider the IC sheaf $\mathrm{IC}(\mathfrak{l}_1^{rs},\mathcal{T}_{\sigma,\chi_1})$ where $\mathcal{T}_{\sigma,\chi_1}$ corresponds to the $\pi_1^{L^\theta}(\mathfrak{l}_1)$-representation $L_\sigma\otimes\chi_1$ where $B_{W_{n_0}}$ acts via the irreducible representation L_σ of W_{n_0} and $\mathbb{Z}/2\mathbb{Z}$ acts via the nontrivial character χ_1. This is a character sheave in $\mathrm{Char}_{L^\theta}(\mathfrak{l}_1)$ by [14, Theorem 5.1]. As before one checks that $\mathrm{IC}(\mathfrak{g}_1^{rs},\mathcal{L}_\sigma\otimes\mathbb{C}_{\chi_1})\bigoplus\mathrm{Ind}_{\mathfrak{l}_1\subset\mathfrak{p}_1}^{\mathfrak{g}_1}\mathrm{IC}(\mathfrak{l}_1^{rs},\mathcal{T}_{\sigma,\chi_1})$. Suppose that (G,K) is of BDI. By the proof of Theorem 4.5, $\mathrm{IC}(\mathcal{O}_{\mu_t},\mathcal{E}_\phi)$, $\phi\in\widehat{A_K(\mathcal{O}_\mu)}$ is a cuspidal character sheaf and $\mathrm{Char}_K^{\mathrm{cusp}}(\mathfrak{g}_1)_{\kappa_1}$ is a subset of the set in Theorem 1.1 (ii). Thus the only possible θ-stable Levi subgroups L contained in proper θ-stable parabolic subgroups P with $\mathrm{Char}_{L^\theta}^{cusp}(\mathfrak{l}_1)_{\kappa_1}\neq\emptyset$ are such that $\pi(L)\cong GL_2^k\times Spin(N-4k)$, $k>0$, $N-4k\geq t^2$. But for such L we have $K.\mathfrak{p}_1\subsetneq\overline{\check{\mathcal{O}}_{m,t}}$. Theorem 1.1 then follows.

It remains to check the number of cuspidal character sheaves. The claims on $|\,\mathrm{Char}_K^{\mathrm{cusp}}(\mathfrak{g}_1)_{\kappa_0}|$ follow from (5.11), (5.12) and (5.6). The claim on $|\,\mathrm{Char}_K^{\mathrm{cusp}}(\mathfrak{g}_1)_{\kappa_1}|$ follows from the definitions of $\Theta_{m,t}^{\kappa_1}$, $\eta_{m,t}$ and the fact that $\sum_n |\,\mathrm{Irr}\,\mathcal{H}_{S_n,-1}|x^n = \prod_{s\geq 1}(1+x^s)$. □

Proof of Corollary 4.8. The fact that the sheaves in the corollary are precisely the nilpotent support character sheaves follows from Theorem 4.5. Recall that (see [13])

$$\sum_{q=0}^{\infty}\tilde{b}_{q+2k+1,q}x^q = \frac{1}{1+x^{2k+1}}\prod_{s\geq 1}\frac{(1+x^{2s-1})^2}{(1-x^{2s})^2}, \tag{5.13a}$$

$$\sum_{q=0}^{\infty}\tilde{b}_{2q+2k,2q}x^{2q} = \frac{1}{1+x^{2k}}\prod_{s\geq 1}\frac{(1+x^{2s})^2}{(1-x^{2s})^2}. \tag{5.13b}$$

Applying Proposition 5.1 and entirely similar argument as in §5.3 (using (5.7) and (5.13a)), we obtain that

(5.14)

$$b^2_{q+t,q} = \text{ Coefficient of } x^q \text{ in } \begin{cases} \dfrac{(1+x^t)}{(1+x^{2t})}\displaystyle\prod_{s\geq 1}\frac{1+x^{4s-2}}{(1-x^{2s})^2} & \text{if } t \text{ is odd} \\ \dfrac{(1+x^t)}{(1+x^{2t})}\displaystyle\prod_{s\geq 1}\frac{1+x^{4s}}{(1-x^{2s})^2} & \text{if } t,\ q \text{ both even.} \end{cases}$$

The claims on $|\,\mathrm{Char}_K^{\mathrm{n}}(\mathfrak{g}_1)_{\kappa_0}|$ then follow from (4.5), (5.8) and (5.13a).

The claim on $|\,\mathrm{Char}_K^{\mathrm{n}}(\mathfrak{g}_1)_{\kappa_1}|$ follows from Lemma 2.1 and the definition of $\eta_{0,t}$. □

6. Examples

In this section, we list the character sheaves for small rank spin groups. We use the notations from previous sections. We write $\theta = \theta^{p,q} : G \to G$ for the type BDI involutions such that $G^\theta = K^{p,q}$. We can and will assume that $p \geq q$.

Type B_1 We have $Spin_3 \cong SL_2$ and $\pi_1^K(\mathfrak{g}_1^{rs}) \cong \mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$. The character sheaves are

$$\mathrm{Char}_{K^{2,1}}(\mathfrak{g}_1)_{\kappa_0} = \{\mathrm{IC}(\mathcal{O}_{3^\delta_+},\mathbb{C}),\ \delta=\mathrm{I},\mathrm{II},\ \mathbb{C}_{\mathfrak{g}_1}[2]\}$$

$$\mathrm{Char}_{K^{2,1}}(\mathfrak{g}_1)_{\kappa_1} = \{\mathrm{IC}(\mathfrak{g}_1^{rs},\mathcal{T}_{\mathbf{1}\otimes\mathrm{sgn}}),\ \mathrm{IC}(\mathfrak{g}_1^{rs},\mathcal{T}_{\mathrm{sgn}\otimes\mathrm{sgn}})\}.$$

Here and henceforth we write $\mathbf{1}$ (resp. sgn) for the trivial (resp. nontrivial) character of $\mathbb{Z}/2\mathbb{Z}$ and also for the irreducible representation of $\mathbb{Z}$ obtained by pulling back $\mathbf{1}$ (resp. sgn) via the quotient $\mathbb{Z} \to W_{\mathfrak{a}} = S_2 \cong \mathbb{Z}/2\mathbb{Z}$.

Type D_2 We have $Spin_4 \cong SL_2 \times SL_2$.

Let $\theta = \theta^{3,1}$. The character sheaves are

$$\mathrm{Char}_{K^{3,1}}(\mathfrak{g}_1)_{\kappa_0} = \{\mathrm{IC}(\mathfrak{g}_1^{rs}, \mathcal{T}_{\mathrm{sgn}}),\ \mathbb{C}_{\mathfrak{g}_1}[3]\},\quad \mathrm{Char}_{K^{3,1}}(\mathfrak{g}_1)_{\kappa_1} = \{\mathrm{IC}(\mathcal{O}_{3_+1_+}, \mathcal{E})\}$$

where $\mathcal{E}$ is the nontrivial local system on $\mathcal{O}_{3_+1_+}$.

Let $\theta = \theta^{2,2}$. We have $I \cong Z_G = \langle\epsilon\rangle \times \langle\omega\rangle \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, where $\omega = v_1v_2v_3v_4$ and $\{v_i\}$ is an orthonormal basis of V. Moreover, $\pi_1^K(\mathfrak{g}_1^{rs}) \cong \mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$. Let $\chi_i : I \to \mathbb{G}_m$ be defined by $\chi_1(\epsilon) = -1$, $\chi_1(\omega) = 1$, $\chi_2(\epsilon) = -1$, $\chi_2(\omega) = -1$, $\chi_3(\epsilon) = 1$, and $\chi_3(\omega) = -1$. The character sheaves are

$$\begin{aligned}
\mathrm{Char}_{K^{2,2}}(\mathfrak{g}_1)_{\kappa_0} &= \{\mathrm{IC}(\mathcal{O}^{\delta}_{3_+1_-}, \mathbb{C}),\ \mathrm{IC}(\mathcal{O}^{\delta}_{3_-1_+}, \mathbb{C}) \mid \delta = \mathrm{I}, \mathrm{II}\}\\
&\sqcup\ \{\mathrm{IC}\,\check{\mathcal{O}}^{\delta}_{2_+2_-}, \mathbb{C}\} \mid \delta = \mathrm{I}, \mathrm{II}, \mathrm{III}, \mathrm{IV}\}\\
&\sqcup\ \{\mathrm{IC}(\mathfrak{g}_1^{rs}, \mathcal{T}_{\mathbf{1}\otimes\mathbf{1}\otimes\chi_3}),\ \mathrm{IC}(\mathfrak{g}_1^{rs}, \mathcal{T}_{\mathrm{sgn}\otimes\mathbf{1}\otimes\chi_3}),\ \mathrm{IC}(\mathfrak{g}_1^{rs}, \mathcal{T}_{\mathbf{1}\otimes\mathrm{sgn}\otimes\chi_3}),\\
&\qquad \mathrm{IC}(\mathfrak{g}_1^{rs}, \mathcal{T}_{\mathrm{sgn}\otimes\mathrm{sgn}\otimes\chi_3}),\ \mathbb{C}_{\mathfrak{g}_1}[4]\},\\
\mathrm{Char}_{K^{2,2}}(\mathfrak{g}_1)_{\kappa_1} &= \{\mathrm{IC}\,\check{\mathcal{O}}^{\delta}_{2_+2_-}, \mathcal{F}_\sigma\} \mid \sigma \in \mathcal{P}_2(1), \delta = \mathrm{I}, \mathrm{II}, \mathrm{III}, \mathrm{IV}\}\\
&\sqcup\ \{\mathrm{IC}(\mathfrak{g}_1^{rs}, \mathcal{T}_{\mathbf{1}\otimes\mathbf{1}\otimes\chi_1}), \mathrm{IC}(\mathfrak{g}_1^{rs}, \mathcal{T}_{\mathrm{sgn}\otimes\mathbf{1}\otimes\chi_1}),\ \mathrm{IC}(\mathfrak{g}_1^{rs}, \mathcal{T}_{\mathbf{1}\otimes\mathbf{1}\otimes\chi_2}),\\
&\qquad\qquad \mathrm{IC}(\mathfrak{g}_1^{rs}, \mathcal{T}_{\mathbf{1}\otimes\mathrm{sgn}\otimes\chi_2})\}.
\end{aligned}$$

Let θ be of type DIII. The character sheaves are

$$\begin{aligned}
&\mathrm{Char}_K(\mathfrak{g}_1)_{\kappa_0} = \{\mathrm{IC}(\mathcal{O}_{2^2_+}, \mathbb{C}),\ \mathrm{IC}(\mathcal{O}_{2^2_-}, \mathbb{C}),\ \mathbb{C}_{\mathfrak{g}_1}[2]\},\\
&\mathrm{Char}_K(\mathfrak{g}_1)_{\kappa_1} = \{\mathrm{IC}(\mathfrak{g}_1^{rs}, \mathcal{L}_\sigma \otimes \mathbb{C}_{\chi_1} \mid \sigma \in \mathcal{P}_2(1)\}.
\end{aligned}$$

Type B_2 We have $Spin_5 \cong Sp_4$.

Let $\theta = \theta^{4,1}$. The character sheaves are

$$\mathrm{Char}_{K^{4,1}}(\mathfrak{g}_1)_{\kappa_0} = \{\mathrm{IC}(\mathcal{O}_{3_+1^2_+}, \mathbb{C}),\ \mathbb{C}_{\mathfrak{g}_1}[4]\}.$$

Let $\theta = \theta^{3,2}$. The character sheaves are

$$\begin{aligned}
\mathrm{Char}_{K^{3,2}}(\mathfrak{g}_1)_{\kappa_0} &= \{\mathrm{IC}(\mathcal{O}^{\delta}_{5_+}, \mathbb{C}),\ \mathrm{IC}(\mathcal{O}^{\delta}_{3_-1^2_+}, \mathbb{C}),\ \delta = \mathrm{I}, \mathrm{II}\} \sqcup \{\mathrm{IC}(\check{\mathcal{O}}_{3_+1_+1_-}, \mathbb{C})\}\\
&\sqcup\ \{\mathrm{IC}(\check{\mathcal{O}}^{\delta}_{2_+2_-1_+}, \mathbb{C}),\ \delta = \mathrm{I}, \mathrm{II}\} \sqcup \{\mathrm{IC}(\mathfrak{g}_1^{rs}, \mathcal{T}_\psi) \mid \psi \in \Theta^{\kappa_0}_{2,1}\}
\end{aligned}$$

$$\begin{aligned}\mathrm{Char}_{K^{3,2}}(\mathfrak{g}_1)_{\kappa_1} &= \{\mathrm{IC}(\check{\mathcal{O}}^\delta_{2_+2_-1_+},\mathcal{F}_\sigma),\ \delta=\mathrm{I},\mathrm{II},\ \sigma\in\mathcal{P}_2(1)\}\\ &\sqcup\ \{\mathrm{IC}(\mathfrak{g}_1^{rs},\mathcal{F}_\rho)\mid\rho\in\Theta^{\kappa_1}_{2,1}\}\end{aligned}$$

where $\Theta^{\kappa_0}_{2,1}=\{V_{\rho,\chi_0}\mid\rho\in\mathrm{Irr}\,\mathcal{H}_{W_2,-1}\}\sqcup\{V^\delta_{\mathbf{1}\otimes\mathbf{1},\chi_1},\ \delta=\mathrm{I},\mathrm{II}\}$ and $\Theta^{\kappa_1}_{2,1}=\{V^\delta_{\mathbf{1},\chi_2},\ \delta=\mathrm{I},\mathrm{II}\}$. In particular $|\Theta^{\kappa_0}_{2,1}|=4$ and $|\Theta^{\kappa_1}_{2,1}|=2$.

Type D_3 We have $Spin_6\cong SL_4$.

Let $\theta=\theta^{5,1}$. The character sheaves are

$$\mathrm{Char}_{K^{5,1}}(\mathfrak{g}_1)_{\kappa_0}=\{\mathrm{IC}(\mathfrak{g}_1^{rs},\mathcal{T}_{\mathrm{sgn}}),\ \mathbb{C}_{\mathfrak{g}_1}[5]\}.$$

Let $\theta=\theta^{4,2}$. The character sheaves are

$$\begin{aligned}\mathrm{Char}_{K^{4,2}}(\mathfrak{g}_1)_{\kappa_0} &= \{\mathrm{IC}(\mathcal{O}^\delta_{5_+1_+},\mathbb{C}),\ \mathrm{IC}(\mathcal{O}^\delta_{3^2_+},\mathbb{C}),\ \mathrm{IC}(\mathcal{O}^\delta_{3_-1^3_+},\mathbb{C}),\ \delta=\mathrm{I},\mathrm{II}\}\\ &\sqcup\{\mathrm{IC}(\check{\mathcal{O}}^\delta_{2_+2_-1^2_+},\mathbb{C}),\ \delta=\mathrm{I},\mathrm{II}\}\sqcup\{\mathrm{IC}(\mathfrak{g}_1^{rs},\mathcal{T}_\psi)\mid\psi\in\Theta^{\kappa_0,2}_{2,0}\},\\ \mathrm{Char}_{K^{4,2}}(\mathfrak{g}_1)_{\kappa_1} &= \{\mathrm{IC}(\check{\mathcal{O}}_{3_+1^2_+1_-},\mathcal{F}_\rho),\ \rho\in\Theta^{\kappa_1}_{2,2}\}.\end{aligned}$$

where $|\Theta^{\kappa_0,2}_{2,0}|=7$ and $|\Theta^{\kappa_1}_{2,2}|=4$.

Let $\theta=\theta^{3,3}$. The character sheaves are

$$\begin{aligned}\mathrm{Char}_{K^{3,3}}(\mathfrak{g}_1)_{\kappa_0} &= \{\mathrm{IC}(\mathcal{O}_{2_+2_-1_+1_-},\mathbb{C})\}\sqcup\{\mathrm{IC}(\check{\mathcal{O}}_{3_+1_+1^2_-},\mathbb{C}),\ \mathrm{IC}(\check{\mathcal{O}}_{3_+1_+1^2_-},\mathcal{T}_{\mathrm{sgn}})\\ &\sqcup\ \{\mathrm{IC}(\check{\mathcal{O}}_{3_-1^2_+1_-},\mathbb{C}),\ \mathrm{IC}(\check{\mathcal{O}}_{3_-1^2_+1_-},\mathcal{T}_{\mathrm{sgn}})\}\\ &\sqcup\ \{\mathrm{IC}(\mathfrak{g}_1^{rs},\mathcal{T}_\psi)\mid\psi\in\Theta^{\kappa_0}_{3,0}\},\\ \mathrm{Char}_{K^{3,3}}(\mathfrak{g}_1)_{\kappa_1} &= \{\mathrm{IC}(\mathcal{O}_{2_+2_-1_+1_-},\mathcal{F}_\sigma),\ \sigma\in\mathcal{P}_2(1)\}\sqcup\{\mathrm{IC}(\mathfrak{g}_1^{rs},\mathcal{F}_\rho)\mid\rho\in\Theta^{\kappa_1}_{3,0}\}.\end{aligned}$$

where $|\Theta^{\kappa_0}_{3,0}|=4$ and $|\Theta^{\kappa_1}_{3,0}|=2$.

Let θ be of type DIII. The character sheaves are

$$\begin{aligned}\mathrm{Char}_K(\mathfrak{g}_1)=\mathrm{Char}_K(\mathfrak{g}_1)_{\kappa_0}=\{&\mathrm{IC}(3_+3_-,\mathbb{C}),\ \mathrm{IC}(\mathcal{O}_{2^2_+1_+1_-},\mathbb{C}),\\ &\mathrm{IC}(\mathcal{O}_{2^2_+1_+1_-},\mathbb{C}),\ \mathbb{C}_{\mathfrak{g}_1}[4]\}.\end{aligned}$$

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