Review for vector spaces V and W, have

a vector space $W \otimes V = \mathrm{Bil}(W^\vee, V^\vee)$
called their tensor product
[whose elts are called tensors]

and a bilinear map $W \times V$ to $W \otimes V$ denoted

(w, v) mapsto $w \otimes v$,

explicitly, $w \otimes v : W^\vee \times V^\vee$ to F is def by
$(w \otimes v)(\psi, \theta) = \psi(w)\theta(v)$

Rem the elts of $W \otimes V$ taking the form $w \otimes v$
are called pure tensors
all other elts are called mixed tensors

today: be more concrete, assuming V, W fin. dim'l

Q what is $\dim W \otimes V$ in terms of
dim W and dim V? [pause]

A $\dim W \otimes V = \dim \mathrm{Bil}(W^\vee, V^\vee)$
$= (\dim W^\vee)(\dim V^\vee)$
$= (\dim W)(\dim V)$

Q how to get a basis for $W \otimes V$
of this size? [pause]
in terms of bases of W and V? [pause]

A fix ordered bases $v_1, \ldots, v_n$ for V
$w_1, \ldots, w_m$ for W
consider the elts $w_j \otimes v_i$

Thm $(w_j \otimes v_i)_{j,i}$ is a basis for $W \otimes V$

Pf this set has size $mn = \dim W \otimes V$

so, enough to show it is linearly indep.

suppose $\sum_{j,i} c_{j,i} (w_j \otimes v_i) = \mathbf{0}$

for some $c_{j,i}$ in F

[idea: need to use def of $w_j \otimes v_i$ somehow]

our chosen bases of V and W define dual bases

$\theta_1, \dots, \theta_n$ for $V^\vee$

$\psi_1, \dots, \psi_m$ for $W^\vee$

[recall what dual means]

then the def of $w_j \otimes v_i$ becomes:

$$(w_j \otimes v_i)(\psi_\ell, \theta_k) = \begin{cases} 1 & \text{if } (j, i) = (\ell, k) \\ 0 & \text{else} \end{cases}$$

so evaluating $\sum_{j,i} c_{j,i} (w_j \otimes v_i)$

(as a bilinear functional) on (ψ_ℓ, θ_k)

gives $c_{\ell,k} (w_\ell \otimes v_k)(\psi_\ell, \theta_k) = c_{\ell,k}$

so if $\sum_{j,i} c_{j,i} (w_j \otimes v_i) = \mathbf{0}_{W \otimes V}$

then $c_{\ell,k} = 0$ for all ℓ, k

hence the set of $w_j \otimes v_i$'s is linearly indep. □

Q suppose $v = \sum_i a_i v_i$

$w = \sum_j b_j w_j$

what is the expansion of $w \otimes v$ wrt the basis

$(w_j \otimes v_i)_{j,i}$?

simpler questions:

given v, v' in V and w, w' in W and c in F,

how to simplify $(w + w') \otimes v$?

$w \otimes (v + v')$?

$(cw) \otimes v$?

$w \otimes (cv)$?

A [mentioned last time that we would prove:]

Lem the map $B : W \times V$ to $W \otimes V$ def by $B(w, v) = w \otimes v$ is bilinear

i.e., $B(w, –) : V$ to $W \otimes V$

and $B(–, v) : W$ to $W \otimes V$

are linear maps for any w in W, v in V

equivalently: for all v, v', w, w', and c,

1) $(w + w') \otimes v = w \otimes v + w' \otimes v$

2) $w \otimes (v + v') = w \otimes v + w \otimes v'$

3) $(cw) \otimes v = c(w \otimes v) = w \otimes (cv)$

Pf of 1) for any (ψ, θ) in $W^\vee \times V^\vee$:

$((w + w') \otimes v)(\psi, \theta)$ [pause: next?]

$= \psi(w + w')\theta(v)$

$= (\psi(w) + \psi(w'))\theta(v)$

$= \psi(w)\theta(v) + \psi(w')\theta(v)$

$= (w \otimes v)(\psi, \theta) + (w' \otimes v)(\psi, \theta)$

$= (w \otimes v + w' \otimes v)(\psi, \theta)$

hence $(w + w') \otimes v = w \otimes v + w' \otimes v$

Pf of 2) similar

Pf of 3) “left to the reader”

Cor given $v = \sum_i a_i v_i$,
$w = \sum_j b_j w_j$

$w \otimes v = \sum_{j, i} (b_j w_j) \otimes (a_i v_i)$
$= \sum_{j, i} b_j a_i (w_j \otimes v_i)$

Q what is the "simplest" mixed tensor, i.e., tensor not of the form $w \otimes v$?
[pause]
if $\dim V = 1$ or $\dim W = 1$? no dice

take $V = W = F^2$ and (e_1, e_2) the standard basis

by the thm, $F^2 \otimes F^2$ has the basis
$(e_1 \otimes e_1, e_1 \otimes e_2, e_2 \otimes e_1, e_2 \otimes e_2)$

any pure tensor will look like

$$(b_1 e_1 + b_2 e_2) \otimes (c_1 e_1 + c_2 e_2)$$
$$= b_1 c_1 (e_1 \otimes e_1) + b_1 c_2 (e_1 \otimes e_2) + b_2 c_1 (e_2 \otimes e_1) + b_2 c_2 (e_2 \otimes e_2)$$

the mixed tensors are the ones that cannot be written this way for any b_1, b_2, c_1, c_2
[pause: example?]
e.g., $e_1 \otimes e_1 + e_2 \otimes e_2$

Rem so far we've discussed the $\otimes$ operation on vectors

remember that $\otimes$ also denotes a separate operation on vector spaces

Properties of Tensor Products of Vector Spaces

recall that $W \oplus V$ denotes the vector space formed by $W \times V$
[e.g., $F^m \oplus F^n = F^{m + n}$]

[below, equality signs should be isomphm signs]

1) $W \otimes (V \oplus V') = W \otimes V \oplus W \otimes V'$
2) $(W \oplus W') \otimes V = W \otimes V \oplus W' \otimes V$
3) $(W \otimes V) \otimes U = W \otimes (V \otimes U)$

seem obvious but take more work:
iso's, not equalities, so we must give actual maps

Q what's the left-to-right linear iso in 1)?

$W \otimes (V \oplus V')$ spanned by pure tensors $w \otimes (v, v')$
so enough to say where they go [pause: where?]

A $w \otimes (v, v') \mapsto (w \otimes v, w \otimes v')$

(Axler §9B, 9D) using the associativity 3), form iterated tensor products:

$$V_1 \otimes V_2 \otimes \ldots \otimes V_r$$

Df a map $\mu : V_1 \times V_2 \times \ldots \times V_r \to U$ is multilinear iff,

for any index i, and choice of w_j in V_j for all $j \neq i$, the map $V_i \to U$ given by
$v \mapsto \mu(\ldots, w_{i-1}, v, w_{i+1}, \ldots)$ is linear

$U = F$: we say μ is a <u>multilinear functional</u>

$U = F$ and $V = V_i$ for all i: it's an <u>r-linear form</u>

let $\mathrm{Mult}(V_1, \ldots, V_r) = \{$multilinear functionals on $V_1 \times V_2 \times \ldots \times V_r\}$

<u>Thm</u> just as $V_1 \otimes V_2$ satisfies
$\mathrm{Bil}(V_1, V_2) = (V_1 \otimes V_2)^\vee$,

so $V_1 \otimes \ldots \otimes V_r$ satisfies
$\mathrm{Mult}(V_1, \ldots, V_r) = (V_1 \otimes \ldots \otimes V_r)^\vee$

[why care?] on Wed:
determinants as multilinear forms