Last time defn of a metric $d : X \times X$ to $[0, \infty)$

for all x, y, z in X:

1) $d(x, y) = 0$ if and only if $x = y$
2) $d(x, y) = d(y, x)$
3) $d(x, y) + d(y, z) \geq d(x, z)$

Warmup do the following define a metric?

$\eta(x, x) = 0$, and $\eta(x, y) = 1$ for $x \neq y$

1) and 2) easy

3) [how many cases to check?]

if $x = z$: $\eta(x, y) + \eta(y, z) \geq 0 = \eta(x, z)$

if $x \neq z$: either $y \neq x$ or $y \neq z$

so $\eta(x, y) + \eta(y, z) \geq 1 = \eta(x, z)$

what are the balls $B_d(x, \delta)$? [$\{x\}$ for all x, δ]

so this metric generates the discrete topology
so we call it the discrete metric

Ex [draw $B_e(\mathbf{0}, 1)$, $B_\rho(\mathbf{0}, 1)$, $B_e(\mathbf{0}, \sqrt{2})$]

euclidean metric:

$e(x, y) = \mathrm{sqrt}((x_1 - y_1)^2 + \ldots + (x_n - y_n)^2)$

square metric:

$\rho(x, y) = \max(|x_1 - y_1|, \ldots, |x_n - y_n|)$

$B_e(0, 1)$ sub $B_\rho(0, 1)$ sub $B_e(0, \sqrt{2})$

Lem for all x and δ,

1) $B_e(x, \delta)$ sub $B_\rho(x, \delta)$
2) $B_\rho(x, \delta)$ sub $B_e(x, \sqrt{n} \cdot \delta)$

Pf of 2) want: $\rho(x, y) < \delta$ implies $e(x, y) < \sqrt{n} \cdot \delta$

enough: $e(x, y) \leq \sqrt{n} \cdot \rho(x, y)$

[note: reversed positions of e and ρ]

$e(x, y) \leq \sqrt{n \max_i (x_i - y_i)^2} = \sqrt{n} \cdot \rho(x, y)$

Thm let d, resp. d', be a metric on X, generating a topology T, resp. T'

suppose that for all x in X and $\varepsilon > 0$,

we have $\delta > 0$ s.t. $B_{d'}(x, \delta)$ sub $B_d(x, \varepsilon)$

then T' is finer than T

Pf pick U in T

want U in T', i.e.,

for all x in U, some δ s.t. $B_{d'}(x, \delta)$ sub U

since U in T, get ε s.t. $B_d(x, \varepsilon)$ sub U

by hypothesis, get δ s.t. $B_{d'}(x, \delta)$ sub $B_d(x, \varepsilon)$

Cor Euclidean and square metrics induce the same topology on $\mathbb{R}^n$

[the analytic topology]

Rem d, d' are called equivalent

iff

there are fixed $A, B > 0$ s.t. for all x, y,

$d(x, y) \leq A\, d'(x, y)$,

$d'(x, y) \leq B\, d(x, y)$ [uniformly]

if two metrics are equivalent,
then they generate the same topology

converse is false: see PS2, #9–10

Q what about infinite-dim'l space?

$R^\omega = \{(x_1, x_2, \dots) \mid x_i \in R \text{ for all } i\}$
$R^\infty = \{(x_1, x_2, \dots) \mid x_i \neq 0 \text{ for only fin. many } i\}$

Euclidean and square metrics don't work on R^ω

Q do they still work on R^∞? [yes]
Q are there other metrics on R^ω? [yes]
see PS2, #11

(Munkres §15, 19) given $\{X_i\}_{i \in I}$,

$\prod_{i \in I} X_i$
$:= \{\text{sequences } (x_i)_{i \in I} \mid x_i \in X_i \text{ for all } i\}$

Q what if $X_i = \varnothing$ for some i?
[then $\prod_i X_i = \varnothing$]

Q is R^n of this form?
R^ω?
R^∞?
[yes, yes, no]

Q if each X_i has a topology, do we get a natural topology on $\prod_{i \in I} X_i$?

Df the box topology on $\prod_{i \in I} X_i$:
generated by the basis
$\{\prod_i U_i \mid U_i \text{ is open in } X_i \text{ for all } i\}$

Df the product topology on $\prod_{i \in I} X_i$:
generated by the basis
$\{\prod_i U_i \mid U_i \text{ is open in } X_i \text{ for all } i,$
$U_i \neq X_i \text{ for fin many } i\}$

Rem if the indexing set I is finite
then box = product

Q take $I = \{1, 2, \ldots, n\}$ and $X_i = \mathbb{R}$ for all i
then $\prod_i X_i = \mathbb{R}^n$
what is the box/product topology here?

basis is the collection of boxes

$$(a_1, b_1) \times (a_2, b_2) \times \ldots \times (a_n, b_n)$$

contains square balls, so finer than analytic top
conversely, all such boxes are analytically open

Q any open set in $\mathbb{R}^n$ not of the form
$U_1 \times U_2 \times \ldots \times U_n$?

[draw]

Q take $I = \mathbb{Z}_+$ and $X_i = \mathbb{R}$ for all i
then $\prod_i X_i = \mathbb{R}^\omega$
do we have box = product here?

$(-1, 1) \times (-1, 1) \times \ldots$ is box-open,
but not product-open

the box top can be finer than the product top

Universal Property of Product Topology

for all i' let $pr_{i'} : \prod_i X_i \to X_{i'}$ be

the projection map $pr_{i'}((x_i)_i) = x_{i'}$

the product top is the coarsest top on $\prod_i X_i$
s.t. pr_i is continuous for all i