

Today (modified) Jordan Separation Thm:

Thm if  $C \subset S^2$  is a simple closed curve  
then  $S^2 - C$  is not path connected

Pf assume that  $S^2 - C$  is path connected

fix a homeo  $C \simeq S^1$

let  $a, b, A_1, A_2 \subset C$  correspond to

$(1, 0), (-1, 0), \{y \geq 0\}, \{y \leq 0\} \subset S^1$

let  $U_1 = S^2 - A_1$  and  $U_2 = S^2 - A_2$  [draw]

then  $U_1 \cap U_2 = S^2 - A_1 - A_2 = S^2 - C$

so it is path connected

but  $U_1 = (U_1 \cap U_2) \cup A_2$ , etc.

so  $U_1, U_2$  are also path connected

finally,  $U_1 \cup U_2 = S^2 - \{a, b\}$  [draw]

fix  $x \in U_1 \cap U_2 = S^2 - C$

Seifert–van Kampen gives a surjective hom

$\pi_1(U_1, x) * \pi_1(U_2, x) \rightarrow \pi_1(S^2 - \{a, b\}, x)$

induced by the inclusion maps  $i_j : U_j \rightarrow S^2$

will show that  $(i_1)_*, (i_2)_*$  are trivial hom's

but  $\pi_1(S^2 - \{a, b\}) \simeq \pi_1(\mathbb{R}^2 - \{(0, 0)\}) \simeq \mathbb{Z}$

so contradiction

enough to show that  $(i_1)_*$  is trivial

pick a loop  $\gamma : [0, 1] \rightarrow U_1$  based at  $x$   
 lift to cts  $\beta : S^1 \rightarrow U_1$  s.t.  $\beta((1, 0)) = x$   
 get cts  $i_1 \circ \beta : S^1 \rightarrow S^2 - \{a, b\}$  [draw]

note that  $a, b$  both lie in  $A_1$

Lem 1    suppose  $K$  is compact,  
                   $f : K \rightarrow S^2 - \{a, b\}$  is cts

if  $a, b$  lie in the same path comp of  $S^2 - f(K)$ ,  
 then  $f$  is nulhomotopic

Lem 2    for any  $X$  and cts  $f : S^1 \rightarrow X$ , TFAE:

- 1)         $f$  is nulhomotopic
- 2)         $f$  extends to cts map on the closed disk
- 3)         $f_*$  is trivial

[Lem 2 is the more memorable and important one]

Pf of Lem 1                fix a homeo  $S^2 - \{b\} \cong \mathbb{R}^2$   
                                       s.t.  $a$  corresponds to  $\mathbf{0} = (0, 0)$

then  $f : K \rightarrow S^2 - \{a, b\}$  corresponds to  
 some  $g : L \rightarrow \mathbb{R}^2 - \{\mathbf{0}\}$   
 since  $L$  compact, have  $r > 0$  s.t.  $L \subset B(\mathbf{0}, r)$

fix a path from  $a$  to  $b$  in  $S^2 - K$ ,  
 it corresponds to a path from  $\mathbf{0}$  “to infy” in  $\mathbb{R}^2 - L$

fix  $p$  in  $\mathbb{R}^2 - B(\mathbf{0}, r)$   
 fix a path  $\alpha$  from  $\mathbf{0}$  to  $p$  in  $\mathbb{R}^2 - L$  [draw]

let  $h : L \times [0, 1] \rightarrow \mathbb{R}^2 - \{\mathbf{0}\}$  be def by

$$h(x, t) = g(x) - \alpha(t) \text{ [why well-defined?]}$$

let  $H : L \times [0, 1] \rightarrow \mathbb{R}^2 - \{\mathbf{0}\}$  be def by

$$H(x, t) = tg(x) - p \text{ [why well-defined?]}$$

$h$  is a homotopy from  $g(x)$  to  $g(x) - p$

$H$  is a homotopy from a const map to  $g(x) - p$

so  $g$  is nulhomotopic

so  $f$  is too