# CENTRAL ELEMENTS, CELL DECOMPOSITIONS, AND PARTIAL SPRINGER RESOLUTIONS

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ABSTRACT. For any finite Weyl group $W$ and parabolic subgroup $W_J$, arising from a finite reductive group $G$ and parabolic subgroup $P_J$, we prove identities relating the partial Springer resolutions of type $J$ to central elements in the Hecke algebra, given by sums of terms $q^{-\ell(v)}T_{v^{-1}}T_v$ as $v$ runs over minimal- or maximal-length representatives of the right cosets of $W_J$ in $W$. We thereby obtain formulas for Hecke traces arising from these central elements, generalizing work of Lascoux and Wan–Wang beyond type $A$, and cell decompositions of new braid varieties involving $J$, generalizing work of Shende–Treumann–Zaslow. From the latter, we construct noncrossing sets that interpolate between Catalan and parking objects, generalizing our work with Galashin–Lam, and new formulas for arbitrary $a$-degrees of the HOMFLYPT polynomials of positive braid closures.

## 1. INTRODUCTION

1.1. Fix a finite Coxeter system $(W, S)$ and a subset $J \subseteq S$ generating a subgroup $W_J \subseteq W$. Let $H_W$ and $H_{W_J}$ be the Hecke algebras over $\mathbf{Z}[\boldsymbol{q}^{\pm 1}]$ corresponding to $W$ and $W_J$. We take the convention where the Hecke operators $T_s \in H_W$, for $s \in S$, obey the relations $T_s^2 = (\boldsymbol{q} - 1)T_s + \boldsymbol{q}$. We identify $H_{W_J}$ with the subalgebra of $H_W$ generated by the elements $T_s$ with $s \in J$.

Under this embedding, the center $Z(H_{W_J})$ need not embed into the center $Z(H_W)$. Nonetheless, Hoefsmit–Scott constructed an injective, linear *relative norm* map

$$N_J^S : Z(H_{W_J}) \to Z(H_W),$$

that they, and L. K. Jones, used to study induction from $H_{W_J}$ to $H_W$ [Jon90]. To define $N_J^S$, recall that each right coset of $W_J$ in $W$ contains a unique representative of minimal Bruhat length. Let $W^J$ be the set of such representatives. Then

$$N_J^S(\alpha) = \sum_{v \in W^J} \boldsymbol{q}^{-\ell(v)} T_{v^{-1}} \alpha T_v \quad \text{for all } \alpha \in Z(H_{W_J}),$$

where $T_v$ and $\ell(v)$ denote the Hecke operator for and Bruhat length of $v$.

When $W$ is crystallographic, we can interpret it as the Weyl group of a split finite reductive group $G$. We can then interpret the above algebras geometrically, as convolution algebras of functions on the flag variety of $G$ or its square. The main observation of this paper is that in the geometric framework, the relative norm $N_J^S$ is related to the two partial Springer resolutions for $J$, defined in (1.1).

From this relationship, we obtain applications to traces on $H_W$, generalizing work of Lascoux [Las06] and Wan–Wang [WW15]; cell decompositions of *partial* *braid Steinberg varieties*, generalizing work of Shende–Treumann–Zaslow [STZ17]; and the rational parking combinatorics of $(W, S)$, generalizing our prior work with Galashin–Lam [GLTW24].

1.2. Let $\mathbf{F}$ be a finite field of order $q$. Let $\mathbf{G}$ be a connected reductive algebraic group over $\bar{\mathbf{F}}$, with an $\mathbf{F}$-form given by a Frobenius map $F : \mathbf{G} \to \mathbf{G}$. We assume that the characteristic of $\mathbf{F}$ is a good prime for $\mathbf{G}$ [Car93, 28].

Fix an $F$-stable maximal torus in an $F$-stable Borel: $\mathbf{T} \subseteq \mathbf{B} \subseteq \mathbf{G}$. Let $\mathbf{W} = N_{\mathbf{G}}(\mathbf{T})/\mathbf{T}$. We now take $W$ to be the finite Coxeter group $\mathbf{W}^F$. Similarly, we write $G$, $B$, *etc.* for the groups formed by the $F$-fixed points of $\mathbf{G}$, $\mathbf{B}$, *etc.*

The $G$-invariant, $\mathbf{Z}[q^{\pm 1}]$-valued functions on $(G/B)^2$ form a convolution algebra $H_B^G$. If $G$ is split, meaning $W = \mathbf{W}$, then $H_B^G$ is the specialization at $\boldsymbol{q} \to q$ of the algebra $H_W$ presented earlier. Explicitly, $T_w$ specializes to the indicator function on the set of pairs $(hB, gB)$ such that $Bh^{-1}gB = BwB$. In Section 2, we review the presentation of $H_B^G$ for general $G$. *In the rest of this introduction, we assume that $G$ is split, for simplicity.*

We take $S$ to be the system of simple reflections arising from $\mathbf{B}$. Let $\mathbf{P}_J = \mathbf{B}W_J\mathbf{B}$, a parabolic subgroup of $\mathbf{G}$. Let $\mathbf{U}_J$ be its unipotent radical and $\mathbf{V}_J$ the variety of all unipotent elements in $\mathbf{P}_J$. If $J = \emptyset$, then $\mathbf{P}_J = \mathbf{B}$ and $\mathbf{U}_J = \mathbf{V}_J$; otherwise, $\mathbf{V}_J$ is larger than $\mathbf{U}_J$. At the level of points, the two *partial Springer resolutions* of type $J$ are defined by

$$\begin{aligned} \mathbf{Spr}_J^+ &= \{(u, y\mathbf{P}_J) \in \mathbf{G} \times \mathbf{G}/\mathbf{P}_J \mid u \in y\mathbf{V}_J y^{-1}\}, \\ \mathbf{Spr}_J^- &= \{(u, y\mathbf{P}_J) \in \mathbf{G} \times \mathbf{G}/\mathbf{P}_J \mid u \in y\mathbf{U}_J y^{-1}\}. \end{aligned} \tag{1.1}$$

The $+$ case is a partial resolution of singularities of the unipotent variety $\mathbf{V} \subseteq \mathbf{G}$, while the $-$ case is a resolution of the closure of the Richardson orbit for $J$.

It will be convenient to set $\mathbf{E}_J^\pm := \mathbf{G}/\mathbf{B} \times \mathbf{Spr}_J^\pm$ and $E_J^\pm = (\mathbf{E}_J^\pm)^F$. There is a natural left $G$-action on $E_J^\pm$, under which the map $f : E_J^\pm \to (G/B)^2$ defined by

$$f(hB, u, yP_J) = (hB, uhB)$$

is equivariant. For any set $E$ equipped with a $G$-action and an equivariant map $f : E \to (G/B)^2$, we write $f_!\delta_E \in H_B^G$ to denote the function whose value at a point in $(G/B)^2$ is the size of its preimage in $E$.

Let $w_\circ$ and $w_{J\circ}$ respectively denote the longest elements of $W$ and $W_J$. For convenience, we set $\ell_S = \ell(w_\circ)$ and $\ell_J = \ell(w_{J\circ})$. Recall that $w_\circ, w_{J\circ}$ are involutions, and that $T_{w_{J\circ}}^2$ is central in $H_{W_J}$ [BMR98]. We can now state the split case of our main result, proven for general $G$ in Section 3.

**Theorem 1.1.** *For any $J \subseteq S$, we have*

$$\begin{aligned} f_!\delta_{E_J^-} &= q^{\ell_S - \ell_J} N_J^S(1)|_{\boldsymbol{q}\to q}, \\ f_!\delta_{E_J^+} &= q^{\ell_S - \ell_J} N_J^S(T_{w_{J\circ}}^2)|_{\boldsymbol{q}\to q}. \end{aligned}$$

Let $W^{J,-} = W^J$, and by analogy, let $W^{J,+}$ of *maximal-length* representatives for the right cosets of $W_J$ in $W$, so that multiplication by $w_{J\circ}$ interchanges $W^{J,-}$ with $W^{J,+}$. Then the identities above can be rewritten as:

$$\begin{aligned} f_! \delta_{E_J^-} &= q^{\ell_S - \ell_J} \sum_{w \in W^{J,-}} q^{-\ell(v)} T_{v^{-1}} T_v, \\ f_! \delta_{E_J^+} &= q^{\ell_S} \sum_{w \in W^{J,+}} q^{-\ell(v)} T_{v^{-1}} T_v. \end{aligned}$$

We emphasize that the $+$ case is deeper than the $-$ case. The $-$ case only uses standard results about Bruhat decomposition. Under the assumption that $G$ is split, we can refine it to an algebro-geometric statement: essentially, that $\mathbf{E}_J^-$ can be partitioned into fiber bundles over appropriate varieties. See Proposition 3.3 for details. By contrast, the $+$ case relies on a difficult theorem of Kawanaka [Kaw75]. We expect that it can only be refined to a statement at the level of cohomology. This issue seems related to the sheafification of Kawanaka's work discussed in [Tri22].

1.3. Recall that a *trace* on an algebra is a linear map that vanishes on commutators. We write $R(H_W)$ to denote the vector space of $\mathbf{Q}(\boldsymbol{q})$-valued traces on $H_W$. Our first application of Theorem 1.1 is to identify certain elements of $R(H_W)$ arising from $N_J^S(1)$ and $N_J^S(T_{w_{J\circ}}^2)$.

Let $e \in W$ be the identity. Let $\tau : H_W \to \mathbf{Z}[\boldsymbol{q}^{\pm 1}]$ be the trace given by $\tau(T_e) = 1$ and $\tau(T_w) = 0$ for all $w \neq e$. Then any central element $\zeta \in Z(H_W)$ gives rise to a trace $\tau[\zeta] : H_W \to \mathbf{Z}[\boldsymbol{q}^{\pm 1}] \subseteq \mathbf{Q}(\boldsymbol{q})$: namely,

$$\tau[\zeta](\beta) = \tau(\beta\zeta).$$

These traces are best understood when $W$ is a symmetric group.

Let $S_n$, the symmetric group on $n$ letters, and let $\Lambda_n$ be the vector space of symmetric functions over $\mathbf{Q}(\boldsymbol{q})$ of degree $n$ in variables $X = (X_1, X_2, \ldots, X_n)$. Then $R(H_{S_n})$ is isomorphic to $\Lambda_n$, as both of these vector spaces have bases indexed by the integer partitions of $n$. Let $ch_{\boldsymbol{q}} : R(H_{S_n}) \xrightarrow{\sim} \Lambda_n$ be the ***q**-deformed Frobenius characteristic* isomorphism that sends the irreducible character $\chi_{\boldsymbol{q}}^\lambda$ to the Schur function $s_\lambda(X)$.

For $W = S_n$, we can take $S = \{s_1, \ldots, s_{n-1}\}$, where $s_i$ transposes $i$ and $i+1$. With this choice, there is a bijective correspondence between subsets $J \subseteq S$ and integer partitions $\lambda \vdash n$. Let $e_\lambda(X)$ and $h_\lambda(X)$ respectively denote the elementary and complete homogeneous symmetric functions indexed by $\lambda$ in $\Lambda_n$. Wan–Wang [WW15], recasting work of Lascoux [Las06], show that

$$\begin{aligned} ch_{\boldsymbol{q}}(\tau[N_J^S(1)]) &= (\boldsymbol{q}-1)^n e_\lambda\left(\frac{X}{\boldsymbol{q}-1}\right), \\ ch_{\boldsymbol{q}}(\tau[N_J^S(T_{w_{J\circ}}^2)]) &= \boldsymbol{q}^{\ell_J} (\boldsymbol{q}-1)^n h_\lambda\left(\frac{X}{\boldsymbol{q}-1}\right). \end{aligned} \tag{1.2}$$

Using these formulas, they show that the maps $N_J^S$ give rise to a ring structure on the direct sum of the centers $Z(\mathbf{Q}(\boldsymbol{q}) \otimes H_{S_n})$, isomorphic to the ring of symmetric functions over $\mathbf{Q}(\boldsymbol{q})$. We will generalize the formulas to any crystallographic $W$.

Recall that Springer constructed a $W$-action on the étale cohomologies of the fibers of his resolution of the unipotent variety, now called *Springer fibers*. In [Tri21], the first author used this action to construct a trace on $H_W$ valued in $\mathbf{Q}(\boldsymbol{q})$-linear traces on $W$, or equivalently, a *bitrace*

$$\tau_G : \mathbf{Q}W \otimes H_W \to \mathbf{Q}(\boldsymbol{q}),$$

which refines the Markov traces on $H_W$ studied by Gomi [Gom06] and Webster–Williamson [WW11]. These, in turn, were motivated by the traces used by Ocneanu to construct the HOMFLYPT link polynomial [Jon87].

In this paper, we use a normalization for $\tau_G$ characterized by the formula

$$\tau_G(z \otimes T_w)|_{\boldsymbol{q} \to q} = \frac{1}{|G|} \sum_{\substack{u \in G \\ \text{unipotent}}} |O(w)_u| \chi_u(z) \quad \text{for all } z, w \in W,$$

where $\chi_u$ is the total Springer character for $u$, reviewed in §4.2, and $O(w)_u$ is the set of pairs $(hB, gB)$ such that $h^{-1}gB = BwB$ and $gB = uhB$. Let $e_{J,-}$, *resp.* $e_{J,+}$, denote the antisymmetrizer, *resp.* symmetrizer, in $\mathbf{Q}W_J$, reviewed in §4.3. By combining Theorem 1.1 with work of Borho–MacPherson [BM83], we show:

**Theorem 1.2.** *For any $J \subseteq S$, we have*

$$\begin{aligned} \tau[N_J^S(1)] &= (\boldsymbol{q} - 1)^{\mathrm{rk}(G)}\, \tau_G(e_{J,-} \otimes -), \\ \tau[N_J^S(T_{w_{J\circ}}^2)] &= \boldsymbol{q}^{\ell_J} (\boldsymbol{q} - 1)^{\mathrm{rk}(G)}\, \tau_G(e_{J,+} \otimes -) \end{aligned}$$

*as traces on $H_W$.*

For $G = \mathrm{GL}_n(\mathbf{F})$, we will show that Theorem 1.2 recovers (1.2), by explaining why $ch_{\boldsymbol{q}}$ sends $\tau_G(e_{J,-} \otimes -)$ to $e_\lambda(\frac{X}{\boldsymbol{q}-1})$ and $\tau_G(e_{J,+} \otimes -)$ to $h_\lambda(\frac{X}{\boldsymbol{q}-1})$.

1.4. Our second application of Theorem 1.1 is to provide point-counting formulas for new kinds of braid varieties, by means of Deodhar-type decompositions. In what follows, we write $h\mathbf{B} \xrightarrow{w} g\mathbf{B}$ to mean $\mathbf{B}h^{-1}g\mathbf{B} = \mathbf{B}w\mathbf{B}$.

Let $\vec{s} = (s^{(1)}, s^{(2)}, \ldots, s^{(\ell)})$ be a word in $S$. Recall that in [Deo85], Deodhar showed how to partition a certain *Richardson variety* for $\vec{s}$ into strata of the form $\mathbf{A}^{\mathbf{d}} \times \mathbf{G}_m^{\mathbf{e}}$, now called *Deodhar cells*. As in [GLTW24], we will work with a variant definition depending on an element $v \in W$:

$$\mathbf{R}^{(v)}(\vec{s}) = \{\vec{g}\mathbf{B} = (g_i\mathbf{B})_i \in (\mathbf{G}/\mathbf{B})^\ell \mid vw_\circ\mathbf{B} \xrightarrow{s^{(1)}} g_1\mathbf{B} \xrightarrow{s^{(2)}} \cdots \xrightarrow{s^{(\ell)}} g_\ell\mathbf{B} \xleftarrow{vw_\circ} \mathbf{B}\}.$$

The geometry of Theorem 1.1 shows how to relate the disjoint union of the varieties $\mathbf{R}^{(v)}(\vec{s})$ for $v \in W^{J,\pm}$ to the variety

$$\mathbf{Z}_J^{\mp}(\vec{s}) = \{(\vec{g}\mathbf{B}, u, y\mathbf{P}_J) \in (\mathbf{G}/\mathbf{B})^\ell \times \mathbf{Spr}_J^{\pm} \mid u^{-1}g_\ell\mathbf{B} \xrightarrow{s^{(1)}} g_1\mathbf{B} \xrightarrow{s^{(2)}} \cdots \xrightarrow{s^{(\ell)}} g_\ell\mathbf{B}\}.$$

Note the sign flip above in the preceding statement. It arises because the element $w_\circ$ in the formula for $\mathbf{R}^{(v)}(\vec{s})$ interchanges $W^{J,-}$ with $W^{J,+}$.

Note that $\mathbf{Z}^+_\emptyset(\vec{s})$ and $\mathbf{Z}^-_\emptyset(\vec{s})$ coincide, and match the *braid Steinberg variety* of $\vec{s}$ introduced in [Tri21]. At the other extreme, $\mathbf{Z}^+_S(\vec{s})$ and $\mathbf{Z}^-_S(\vec{s})$ are the varieties respectively denoted $\mathcal{U}(\vec{s})$ and $\mathcal{X}(\vec{s})$ in *ibid.* The latter was studied even earlier by Shende–Treumann–Zaslow [STZ17], who described a partition of it into subvarieties resembling Deodhar's cell decompositions.

To sketch Deodhar's results, recall that a *subword* of $\vec{s}$ is a sequence $\vec{\omega}$ of the same length with $\omega^{(i)} \in \{e, s^{(i)}\}$ for all $i$. We set $\omega_{(i)} := \omega^{(1)} \cdots \omega^{(i)}$. For any $v \in \mathbf{W}$, a *$v$-distinguished subword* of $\vec{s}$ is a subword $\vec{\omega}$ such that

$$v\omega_{(i)} \leq v\omega_{(i-1)}s^{(i)} \quad \text{for all } i.$$

Let $\mathcal{D}^{(v)}(\vec{s})$ be the set of $v$-distinguished subwords $\vec{\omega}$ of $\vec{s}$ for which $\omega_{(\ell)} = e$. Then the Deodhar cells of $\mathbf{R}^{(v)}(\vec{s})$ are indexed by $\mathcal{D}^{(v)}(\vec{s})$. The Deodhar cell for a given element $\vec{\omega}$ is isomorphic to $\mathbf{A}^{\mathsf{d}_{\vec{\omega}}} \times \mathbf{G}_m^{\mathsf{e}_{\vec{\omega}}}$ for certain disjoint subsets $\mathsf{d}_{\vec{\omega}}, \mathsf{e}_{\vec{\omega}} \subseteq \{1, 2, \ldots, \ell\}$, reviewed in Section 5. In this way, we can count $R^{(v)}(\vec{s}) := \mathbf{R}^{(v)}(\vec{s})^F$:

$$|R^{(v)}(\vec{s})| = \sum_{\vec{\omega} \in \mathcal{D}^{(v)}(\vec{s})} q^{|\mathsf{d}_{\vec{\omega}}|}(q-1)^{|\mathsf{e}_{\vec{\omega}}|}.$$

At the same time, a formula from [GLTW24] expresses $|R^{(v)}(\vec{s})|$ in terms of $\tau$, while our work in Section 4 expresses the size of $Z^\pm_J(\vec{s}) := \mathbf{Z}^\pm_J(\vec{s})^F$ in terms of $\tau_G$. By combining Theorem 1.2 with these formulas, we deduce:

**Corollary 1.3.** *For any word $\vec{s}$, we have*

$$\begin{aligned}\frac{1}{q^{\ell_J}(q-1)^{\mathrm{rk}(G)}} \sum_{v \in W^{J,-}} \sum_{\vec{\omega} \in \mathcal{D}^{(v)}(\vec{s})} q^{|\mathsf{d}_{\vec{\omega}}|}(q-1)^{|\mathsf{e}_{\vec{\omega}}|} &= \frac{|Z^+_J(\vec{s})|}{|G|},\\ \frac{1}{(q-1)^{\mathrm{rk}(G)}} \sum_{v \in W^{J,+}} \sum_{\vec{\omega} \in \mathcal{D}^{(v)}(\vec{s})} q^{|\mathsf{d}_{\vec{\omega}}|}(q-1)^{|\mathsf{e}_{\vec{\omega}}|} &= \frac{|Z^-_J(\vec{s})|}{|G|}.\end{aligned}$$

*(Note the sign flip between the left and right sides of each identity.)*

We will explain in Section 5 that when $J = S$, the $-$ case of Corollary 1.3 recovers [STZ17, Prop. 6.31].

1.5. Our third application of Theorem 1.1, by way of Theorem 1.2, is to construct noncrossing sets of interest in the Catalan combinatorics of $(W, S)$. *In the rest of this introduction, we assume that $W$ is irreducible, with Coxeter number $h$.*

Let $d_1, \ldots, d_{|S|}$ be the fundamental degrees of the action of $W$ on its (irreducible) reflection representation. For each $i$, let $e_i = d_i - 1$. For any positive integer $p$ coprime to $h$, the *rational Catalan number* of $(W, p)$ is

$$\mathrm{Cat}_{W,p} := \prod_i \frac{p + e_i}{d_i},$$

while the *rational parking number* of $(W, p)$ is $p^{|S|}$. These numbers enumerate disparate families of combinatorial objects. Most are constructed from root-theoretic data generalizing nonnesting partitions and parking functions, respectively. The collective study of these families and the bijections between them is the "nonnesting" side of rational Catalan/parking combinatorics. In [GLTW24], we instead sought, and constructed, "noncrossing" families: those depending on a chosen ordering of $S$, or *Coxeter word*.

For any word $\vec{s}$ in $S$, let $\mathcal{M}^{(v)}(\vec{s}) \subseteq \mathcal{D}^{(v)}(\vec{s})$ be the subset of elements $\vec{\omega}$ such that $|\mathbf{e}_{\vec{\omega}}|$ attains its minimum value $|S|$. Let $\vec{c}$ be a Coxeter word for $(W, S)$. The main results of [GLTW24] are the identities

$$|\mathcal{M}^{(e)}(\vec{c}^p)| = \mathrm{Cat}_{W,p} \quad \text{and} \quad \sum_{v \in W} |\mathcal{M}^{(v)}(\vec{c}^p)| = p^{|S|},$$

proved by way of $\boldsymbol{q}$-deformed identities involving $\mathcal{D}^{(v)}(\vec{c}^p)$ and taking $\boldsymbol{q} \to 1$.

In Section 6, we prove an identity that interpolates between the two above. Let $d_1^J, \dots, d_{|J|}^J$ be the fundamental degrees of $W_J$. Let $e_1^J, \dots, e_{|J|}^J$ be the exponents of the $W_J$-action on the reflection representation of $W$. We define the *rational parabolic parking numbers* of $(W, p, J)$ to be

$$\mathrm{Park}_{W,p}^{J,\pm} = \prod_i \frac{p \pm e_i^J}{d_i^J}.$$

Then $\mathrm{Park}_{W,p}^{S,+} = \mathrm{Cat}_{W,p}$ and $\mathrm{Park}_{W,p}^{\emptyset,+} = \mathrm{Park}_{W,p}^{\emptyset,-} = p^{|S|}$. We relate these numbers to $\tau_G$ via a result from [Tri21], which describes $\tau_G(- \otimes T_{\vec{c}^p})$ as the graded character of a *rational parking space* for $(W, p)$, in the sense of [ARR15] and [ALW16]. By combining Corollary 1.3 with this description, we will show:

**Corollary 1.4.** *For any Coxeter word $\vec{c}$ and integer $p > 0$ coprime to $h$, we have*

$$\sum_{v \in W^{J,\pm}} |\mathcal{M}^{(v)}(\vec{c}^p)| = \mathrm{Park}_{W,p}^{J,\mp}.$$

*(Note the sign flip.) That is, $\coprod_{v \in W^{J,\pm}} \mathcal{M}^{(v)}(\vec{c}^p)$ is a $\vec{c}$-noncrossing set enumerated by the $\mp$-rational parabolic parking number of $(W, p, J)$.*

1.6. In Section 7, we explain how Theorem 1.2 and Corollary 1.4 resemble results about Markov traces and rational Kirkman numbers that follow from work of Bezrukavnikov–Tolmachov [BT22].

First, we observe that $W^{J,-}$, *resp.* $W^{J,+}$, consists of those $w \in W$ whose *(left) ascent set* $\mathsf{Asc}(w) \subseteq S$, *resp. descent set* $\mathsf{Des}(w) \subseteq S$, contains $J$. Hence, $N_J^S(1)$ and $\boldsymbol{q}^{-\ell_J} N_J^S(T_{w_{J\circ}}^2)$ respectively decompose as sums, over supersets $I \supseteq J$, of elements

$$\zeta_I^+ := \sum_{\mathsf{Asc}(v) = I} \boldsymbol{q}^{-\ell(v)} T_{v^{-1}} T_v \quad \text{and} \quad \zeta_I^- := \sum_{\mathsf{Des}(v) = I} \boldsymbol{q}^{-\ell(v)} T_{v^{-1}} T_v.$$

Note that $\zeta_S^+ = \zeta_\emptyset^- = 1$ and $\zeta_\emptyset^+ = \zeta_S^- = \Pi_S$. By an inclusion-exclusion argument, the elements $\zeta_I^\pm$ are again central in $H_W$.

**Question 1.5.** For general $W$ and $I$, is there a more familiar description of the traces on $H_W$ of the form $\tau[\zeta_I^{\pm}]$?

We now take $W = S_n$ and $S = \{s_1, \ldots, s_{n-1}\}$. The HOMFLYPT Markov trace on $H_{S_n}$ can be written as a $\mathbf{Q}(\boldsymbol{q})[a^{\pm 1}]$-valued trace. For $0 \leq k \leq n-1$, let $\mu^{(k)} : H_W \to \mathbf{Q}(\boldsymbol{q})$ be the coefficient of the $k$th highest power of $a$. Then [BT22, Cor. 6.1.2] can be recast as the identity

$$\tau[\zeta_{I_k}^-] = (\boldsymbol{q}-1)^{n-1}\mu^{(k)}, \quad \text{where } I_k = \{s_1, s_2, \ldots, s_{n-1-k}\}. \tag{1.3}$$

Let $e_{\Lambda^k} \in \mathbf{Q}W$ be the Young symmetrizer of the hook partition $(n-k, 1, \ldots, 1) \vdash n$, which indexes the $k$th exterior power of the reflection representation of $S_n$. By combining (1.3) with the result in [Tri21] relating the Markov trace to $\tau_G$, we get the following analogue of Theorem 1.2:

**Theorem 1.6.** *For $G$ split semisimple of type $A_{n-1}$, and any integer $k$, we have*

$$\tau[\zeta_{I_k}^-] = (\boldsymbol{q}-1)^{n-1}\tau_G(e_{\Lambda^k} \otimes -)$$

*as traces on $H_{S_n}$.*

For general $W$ and $0 \leq k \leq |S|-1$, we can use the rational parking space for $(W, p)$ mentioned earlier to define rational generalizations $\mathrm{Kirk}_{W,p}^{(k)}$ of the Kirkman numbers studied in [ARR15]. For $W = S_n$, the preceding corollary implies the following analogue of Corollary 1.4:

**Corollary 1.7.** *Take $W = S_n$ and $S = \{s_1, \ldots, s_{n-1}\}$. Then for any Coxeter word $\vec{c}$, any integer $p > 0$ coprime to $n$, and any $k$, we have*

$$\sum_{\mathsf{Des}(v) = I_k} |\mathcal{M}^{(v)}(\vec{c}^{\,p})| = \mathrm{Kirk}_{W,p}^{(k)}.$$

*That is, $\coprod_{\mathsf{Des}(v)=I_k} \mathcal{M}^{(v)}(\vec{c}^{\,p})$ is a $\vec{c}$-noncrossing set enumerated by the $k$th rational Kirkman number of $(W, p)$.*

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## 2. Geometry of the Hecke Algebra

2.1. In this section, we review the general definition of the convolution algebra $H_B^G$ without assuming $G$ to be split, following [Car95, §3.3]. At the end, we explain how to adapt $N_J^S$ to this generality. Along the way, we review Bruhat decomposition and related facts about Borel subgroups. Our geometric setup is essentially Kawanaka's in [Kaw75]. See also [Car93, Car95].

We keep $\mathbf{F}$, $q$, $\mathbf{G}$, $\mathbf{B}$, $\mathbf{T}$, $\mathbf{W}$ as in §1.2. Let $S_{\mathbf{B}}$ be the system of simple reflections of $\mathbf{W}$ arising from $\mathbf{B}$, and let $\ell_{\mathbf{B}}$ be the Bruhat length function on $\mathbf{W}$ defined by $S_{\mathbf{B}}$.

2.2. **Bruhat Decomposition.** Note that $w\mathbf{B}$ and $\mathbf{B}w$ are well-defined for any $w \in \mathbf{W}$. Bruhat decomposition says that as we run over all $w$, the double cosets $\mathbf{B}w\mathbf{B}$ are pairwise disjoint and partition $\mathbf{G}$.

Let $\mathbf{U}$ be the unipotent radical of $\mathbf{B}$, so that $\mathbf{B} = \mathbf{T} \ltimes \mathbf{U}$. Note that $w\mathbf{U}w^{-1}$ is well-defined for all $w \in \mathbf{W}$. Let

$$\begin{aligned} \mathbf{U}_w &= \mathbf{U} \cap w\mathbf{U}w^{-1}, \\ \mathbf{U}_w^- &= \mathbf{U} \cap (ww_\circ)\mathbf{U}(ww_\circ)^{-1}. \end{aligned}$$

Note that $\mathbf{U}_w, \mathbf{U}_w^-$ are stable under the conjugation action of $\mathbf{T}$ on $\mathbf{U}$. The following results are proved in [Car93, §2.5]:

**Lemma 2.1.** *For all $w \in \mathbf{W}$:*

(1) *If $\ell_{\mathbf{B}}(wv) = \ell_{\mathbf{B}}(w) + \ell_{\mathbf{B}}(v)$, then $\mathbf{U}_{wv}^- = \mathbf{U}_w^-\mathbf{U}_v^-$, and $\mathbf{U}_w^- \cap \mathbf{U}_v^- = \{1\}$.*
(2) *$\mathbf{U} = \mathbf{U}_w\mathbf{U}_w^- = \mathbf{U}_w^-\mathbf{U}_w$, and $\mathbf{U}_w \cap \mathbf{U}_w^- = \{1\}$.*
(3) *$\mathbf{B}w\mathbf{B} = \mathbf{U}_w^- w\mathbf{B}$, and the map $\mathbf{U}_w^- \to \mathbf{U}_w^- w\mathbf{B}/\mathbf{B}$ is an isomorphism.*
(4) *As an algebraic variety (but not group), $\mathbf{U}_w^-$ is the product of the root subgroups inverted by $w$, hence an affine space of dimension $\ell_{\mathbf{B}}(w)$.*

2.3. **Bott–Samelson Varieties.** The double cosets of $\mathbf{B}$ in $\mathbf{G}$ are in bijection with the set of diagonal $\mathbf{G}$-orbits on $(\mathbf{G}/\mathbf{B})^2$. As in the introduction, we write $h\mathbf{B} \xrightarrow{w} g\mathbf{B}$ to mean $\mathbf{B}h^{-1}g\mathbf{B} = \mathbf{B}w\mathbf{B}$. Such pairs $(h\mathbf{B}, g\mathbf{B})$ form the points of the $\mathbf{G}$-orbit of $(\mathbf{G}/\mathbf{B})^2$ corresponding to $w$, which we will denote by $\mathbf{O}(w)$.

More generally, for any sequence of elements $\vec{w} = (w^{(1)}, w^{(2)}, \dots, w^{(k)})$ in $\mathbf{W}$, let $\mathbf{O}(\vec{w})$ be the subvariety of $(\mathbf{G}/\mathbf{B})^{1+k}$ defined on points by

$$\mathbf{O}(\vec{w}) = \{\vec{g}\mathbf{B} = (g_0\mathbf{B}, g_1\mathbf{B}, \dots, g_k\mathbf{B}) \mid g_0\mathbf{B} \xrightarrow{w^{(1)}} g_1\mathbf{B} \xrightarrow{w^{(2)}} \cdots \xrightarrow{w^{(k)}} g_m\mathbf{B}\}.$$

The Zariski closure of $\mathbf{O}(\vec{w})$ is sometimes called the *Bott–Samelson variety* of $\vec{w}$. For this reason, $\mathbf{O}(\vec{w})$ is sometimes called the *open Bott–Samelson variety*.

For any subset $I \subseteq \{1, \dots, k\}$, we write $pr_I : \mathbf{O}(\vec{w}) \to (\mathbf{G}/\mathbf{B})^I$ to denote the map that sends $pr_I(\vec{g}\mathbf{B}) = (g_i\mathbf{B})_{i\in I}$. When writing out $\vec{w}$, *resp.* $I$, explicitly, we will omit the parentheses, *resp.* brackets, where convenient.

Lemma 2.1(1) implies that if $\ell_{\mathbf{B}}(wv) = \ell_{\mathbf{B}}(w) + \ell_{\mathbf{B}}(v)$, then $pr_{0,2}$ induces an explicit isomorphism $\mathbf{O}(w, v) \xrightarrow{\sim} \mathbf{O}(wv)$. By induction, any variety of the form $\mathbf{O}(\vec{w})$ is explicitly isomorphic to one of the form $\mathbf{O}(\vec{s})$, where $\vec{s}$ is a word in $S_{\mathbf{B}}$.

2.4. **Frobenius Maps.** For algebraic varieties over $\bar{\mathbf{F}}$ equipped with Frobenius maps, we use italics to denote the corresponding sets of Frobenius-fixed points.

As in §1.2, we fix a Frobenius map $F : \mathbf{G} \to \mathbf{G}$ arising from an $\mathbf{F}$-form, such that $\mathbf{B}$ and $\mathbf{T}$ are $F$-stable. Then $\mathbf{W}$ and $S_{\mathbf{B}}$ are also $F$-stable. The group $W := \mathbf{W}^F$ is again a Coxeter group, which can be identified with $N_G(T)/T$.

*Remark* 2.2. When $\mathbf{G}$ is almost-simple, the options for $G$ and $W$ are listed in [Car95, §1.5–1.6]. Notably, $W$ is crystallographic except when it has factors of type ${}^2F_4$.

There is a system of simple reflections for $W$, which we will denote $S$, indexed by the $F$-orbits on $S_{\mathbf{B}}$: Each element $s \in S$ is the product of all the elements in the given $F$-orbit, which pairwise commute and form a reduced word in $S_{\mathbf{B}}$ in any order. Let $\ell$ be the Bruhat length function on $W$ defined by $S$.

By Lang's theorem, $g\mathbf{B}$ is $F$-stable if and only if $g \in G$, and in this case, $gB = (x\mathbf{B})^F$. Similarly, $\mathbf{B}w\mathbf{B}$ is $F$-stable if and only if $w \in W$, and in this case, $BwB = (\mathbf{B}w\mathbf{B})^F$. Thus, the double cosets $BwB$ for $w \in W$ partition $G$, while the $G$-orbits on $(G/B)^2$ are the sets $O(w)$ for $w \in W$. As explained in [Car93], parts (1)–(3) of Lemma 2.1 have exact analogues with $\mathbf{W}$ replaced by $W$. See also [Kaw75, §1].

**Lemma 2.3.** *For all $w \in W$:*

(1) *If $\ell(wv) = \ell(w) + \ell(v)$, then $U_{wv}^- = U_w^- U_v^-$, and $U_w^- \cap U_v^- = \{1\}$.*
(2) *$U = U_w U_w^- = U_w^- U_w$, and $U_w \cap U_w^- = \{1\}$.*
(3) *$BwB = U_w^- wB$, and the map $U_w^- \to U_w^- wB/B$ is a bijection.*

The one point where caution is needed concerns the sizes of $U_w$ and $U_w^-$, as they use $\ell_{\mathbf{B}}(w)$, not $\ell(w)$, in general [Car93, 74].

**Lemma 2.4.** *For all $w \in W$, we have $|U_w^-| = q^{\dim(\mathbf{U}_w^-)} = q^{\ell_{\mathbf{B}}(w)}$.*

2.5. **Operations on Functions.** For any finite set $X$ equipped with the action of a finite group $G$, we write $\mathcal{C}_G(X)$ to denote the free module of $\mathbf{Z}$-valued, $G$-invariant functions on $X$. For any $G$-stable subset $Y \subseteq X$, we write $\delta_Y \in \mathcal{C}_G(X)$ to denote the indicator function on $Y$.

For a $G$-equivariant map $f : Y \to X$, the *pullback* of functions along $f$ is the linear map $f^* : \mathcal{C}_G(X) \to \mathcal{C}_G(Y)$ given by $f^*(\varphi)(y) = \varphi(f(y))$. The *pushforward*, or *integral*, of functions along $f$ is the linear map $f_! : \mathcal{C}_G(Y) \to \mathcal{C}_G(X)$ given by

$$f_!(\psi)(x) = \sum_{y \in f^{-1}(x)} \psi(y).$$

When $f$ can be understood from context, we omit $f_!$ from our notation.

Let $*$ denote the *convolution product* on $\mathcal{C}(X \times X)$ defined in terms of the three projection maps $pr_{i,j} : X^3 \to X^2$ by

$$\varphi_1 * \varphi_2 = pr_{1,3,!}(pr_{1,2}^*(\varphi_1) \cdot pr_{2,3}^*(\varphi_2)),$$

where $\cdot$ denotes pointwise multiplication. Explicitly,

$$(\varphi_1 * \varphi_2)(y, x) = \sum_{z \in X} \varphi_1(y, z)\varphi_2(z, x).$$

The indicator function of the diagonal $X \subseteq X^2$ is the identity element for this operation. If $X$ is equipped with a $G$-action, and $G$ acts on $X^2$ diagonally, then $*$ restricts to an operation on $\mathcal{C}_G(X \times X)$ with the same identity element.

Iwahori proved that the ring formed by $\mathcal{C}_G(G/B \times G/B)$ under convolution is freely generated by the elements $\delta_w := \delta_{O(w)}$ for $w \in W$ modulo the following relations for all $w \in W$ and $s \in S$:

$$\delta_s * \delta_w = \begin{cases} \delta_{sw} & \ell(sw) > \ell(w), \\ |U_s^-|\,\delta_{sw} + (|U_s^-| - 1)\,\delta_w & \ell(sw) < \ell(w) \end{cases}$$

See [Car95, §3.3] or [Kaw75, Thm. 2.6]. We define $H_B^G$ to be the $\mathbf{Z}[\frac{1}{q}]$-algebra

$$H_B^G = \mathcal{C}_G(G/B \times G/B)[\tfrac{1}{q}].$$

If $G$ is *split*, meaning $W = \mathbf{W}$, then $\ell_{\mathbf{B}}(s) = \ell(s) = 1$ and $|U_s^-| = q$ for all $s \in S$. This is the case on which the introduction focused. Here, $W$ is crystallographic, and $H_B^G$ is a specialization of the $\mathbf{Z}[\boldsymbol{q}^{\pm 1}]$-algebra $H_W$ freely generated by elements $T_w$ for $w \in W$ modulo the following relations for all $w \in W$ and $s \in S$:

$$T_s T_w = \begin{cases} T_{sw} & \ell(sw) = \ell(w) + 1, \\ \boldsymbol{q} T_{sw} + (\boldsymbol{q} - 1) T_w & \ell(sw) = \ell(w) - 1 \end{cases}$$

2.6. **Parabolic Subgroups.** Fix an $F$-stable subset $J_{\mathbf{B}} \subseteq S_{\mathbf{B}}$, corresponding to a subset $J \subseteq S$. Let $\mathbf{W}_J \subseteq \mathbf{W}$, *resp.* $W_J \subseteq W$, be the subgroup generated by $J_{\mathbf{B}}$, *resp.* $J$. Then $\mathbf{W}_J$ is $F$-stable and $W_J = \mathbf{W}_J^F$.

Let $\mathbf{P}_J = \mathbf{B}\mathbf{W}_J\mathbf{B} \subseteq \mathbf{G}$. We can write $\mathbf{P}_J = \mathbf{L}_J \ltimes \mathbf{U}_J$, where $\mathbf{L}_J$ is reductive with Weyl group $\mathbf{W}_J$ and $\mathbf{U}_J$ the unipotent radical of $\mathbf{P}_J$. These subgroups are $F$-stable, and on $F$-fixed points, we have $P_J = L_J \ltimes U_J$.

By construction, $\mathbf{B}_J = \mathbf{L}_J \cap \mathbf{B}$ is a Borel subgroup of $\mathbf{L}_J$. The inclusion $L_J \subseteq P_J$ descends to an $L_J$-equivariant bijection $L_J/B_J \simeq P_J/B$, which in turn yields an isomorphism of algebras

$$\mathcal{C}_{L_J}(L_J/B_J \times L_J/B_J) \simeq \mathcal{C}_{L_J}(P_J/B \times P_J/B).$$

Once we adjoin $\frac{1}{q}$, the left-hand side becomes $H_{B_J}^{L_J}$, and the right-hand side becomes the subalgebra of $H_B^G$ generated by the elements $\delta_w$ with $w \in W_J$. Henceforth, we identify these $\mathbf{Z}[\frac{1}{q}]$-algebras with each other.

As in the introduction, let $W^{J,-} \subseteq W$ be the set of minimal-length right coset representatives for $W_J$. By Lemma 2.1, the split case of the following definition recovers the $\boldsymbol{q} \to q$ specialization of the relative norm map in §1.1.

**Definition 2.5.** The *relative norm* map $N_J^S : H_{B_J}^{L_J} \to H_B^G$ is defined by

$$N_J^S(\alpha) = \sum_{v \in W^{J,-}} \frac{1}{|U_v^-|}\, \delta_{v^{-1}} * \alpha * \delta_v.$$

We have implicitly used Lemma 2.4 to ensure that $|U_v^-|$ is a power of $q$.

2.7. Let $w_\circ$ and $w_{J\circ}$ respectively denote the longest elements of $W$ and $W_J$ with respect to $S$. Then $U = U_{w_\circ}$ and $U_J = U_{w_{J\circ}}$. The following fact will be useful:

**Lemma 2.6.** *For any $J \subseteq S$ and $v \in W^{J,-}$, we have*

$$U_J \cap U_v = U_{w_{J\circ} v} \quad \text{and} \quad U_J \cap U_v^- = U_v^-.$$

*In particular,* $U_J = U_{w_{J\circ}v}U_v^- = U_v^- U_{w_{J\circ}v}$ *and* $U_{w_{J\circ}v} \cap U_v^- = \{1\}$*. In the split case, the analogous identities hold with* $\mathbf{U}_J$*,* $\mathbf{U}_v$*,* etc. *in place of* $U_J$*,* $U_v$*,* etc..

*Proof.* To show $U_J \cap U_v = U_{w_{J\circ}v}$: In general, if $w, v \in W$ satisfy $\ell(wv) = \ell(w) + \ell(v)$, then $U_{wv}^- = U_w^- U_v^-$ and $U_w^- \cap U_v^- = \{1\}$ by Lemma 2.3(1), which implies that $U_{wv} = U_w \cap U_v$ by Lemma 2.3(2).

To show $U_J \cap U_v^- = U_v^-$, meaning $U_v^- \subseteq U_J$: In general, if $w \in W_J$ and $v \in W^{J,-}$, then the $F$-orbits of root subgroups of $\mathbf{U}_J$ inverted by $wv$ are precisely those inverted by $w$. Taking $w = e$ gives the result.

In the split case, $\ell_{\mathbf{B}} = \ell$, and thus, $v$ minimizes $\ell_{\mathbf{B}}$ in $\mathbf{W}_J v$. So we can repeat all the arguments above with the varieties in place of the sets. □

## 3. Partial Springer Resolutions

3.1. Recall the partial Springer resolutions $\mathbf{Spr}_J^{\pm} \subseteq \mathbf{G} \times \mathbf{G}/\mathbf{P}_J$ and the varieties $\mathbf{E}_J^{\pm} = \mathbf{G}/\mathbf{B} \times \mathbf{Spr}_J^{\pm}$ from §1.2. The latter are stable under the left $\mathbf{G}$-action on $\mathbf{G}/\mathbf{B} \times \mathbf{G} \times \mathbf{G}/\mathbf{P}_J$ defined by

$$g \cdot (h\mathbf{B}, u, y\mathbf{P}_J) = (gh\mathbf{B}, gug^{-1}, gy\mathbf{P}_J). \tag{3.1}$$

Let $f : \mathbf{G}/\mathbf{B} \times \mathbf{G} \times \mathbf{G}/\mathbf{P}_J \to (\mathbf{G}/\mathbf{B})^2$ be the $\mathbf{G}$-equivariant map defined by

$$f(h\mathbf{B}, u, y\mathbf{P}_J) = (h\mathbf{B}, uh\mathbf{B}).$$

On $F$-fixed points, it restricts to $G$-equivariant maps $f : E_J^{\pm} \to (G/B)^2$. These recover the maps $f$ in §1.2. The goal of this section is to prove the identities

$$\begin{aligned} f_! \delta_{E_J^-} &= |U_J| N_J^S(1), \\ f_! \delta_{E_J^+} &= |U_J| N_J^S(\delta_{w_{J\circ}}^2), \end{aligned} \tag{3.2}$$

where $N_J^S$ is now given by Definition 2.5. They recover Theorem 1.1 in the split case.

3.2. **Reduction to Strata.** Observe that $\mathbf{E}_J^{\pm}$ is a union of $\mathbf{G}$-stable strata $\mathbf{E}_{J,v}^{\pm}$ for $\mathbf{W}_J v \in \mathbf{W}_J \backslash \mathbf{W}$, where on points,

$$\mathbf{E}_{J,v}^{\pm} = \{(h\mathbf{B}, u, y\mathbf{P}_J) \in \mathbf{G}/\mathbf{B} \times \mathbf{Spr}_J^{\pm} \mid \mathbf{P}_J y^{-1} h\mathbf{B} = \mathbf{P}_J v\mathbf{B}\}.$$

From §2.4, we see that $\mathbf{P}_J v\mathbf{B}$ is $F$-stable if and only if $v \in W$, and in this case, $P_J vB = (\mathbf{P}_J v\mathbf{B})^F$. Therefore, $E_J^{\pm}$ is the union of its $G$-stable subsets $E_{J,v}^{\pm}$ as $v$ runs over a full set of right coset representatives for $W_J$: for instance, $W^{J,-}$. As Lemma 2.6 shows that $U_J \simeq U_{w_{J\circ}v} \times U_v^-$, we reduce (3.2) to:

**Theorem 3.1.** *If* $v \in W^{J,-}$*, then:*

(1) $f_! \delta_{E_{J,v}^-} = |U_{w_{J\circ}v}|\, \delta_{v^{-1}} * \delta_v$.
(2) $f_! \delta_{E_{J,v}^+} = |U_{w_{J\circ}v}|\, \delta_{v^{-1}} * \delta_{w_{J\circ}}^2 * \delta_{w_{J\circ}v}$.

3.3. **Reduction to Borel Cosets.** Let $\check{\mathbf{E}}^{\pm}_{J,v} \subseteq \mathbf{G}/\mathbf{B} \times \mathbf{G} \times \mathbf{G}/\mathbf{B}$ be the subvariety defined on points by

$$\check{\mathbf{E}}^{\pm}_{J,v} = \{(h\mathbf{B}, u, y\mathbf{B}) \mid (u, y\mathbf{B}) \in \mathbf{Spr}^{\pm}_J \text{ and } y\mathbf{B} \xrightarrow{v} h\mathbf{B}\}$$

The forgetful map $\mathbf{G}/\mathbf{B} \to \mathbf{G}/\mathbf{P}_J$ induces a map $\check{\mathbf{E}}^{\pm}_{J,v} \to \mathbf{E}^{\pm}_{J,v}$.

**Lemma 3.2.** *If $v \in W^{J,-}$, then $\check{E}^{\pm}_{J,v} \to E^{\pm}_{J,v}$ is a bijection. In the split case, this bijection arises from an isomorphism $\check{\mathbf{E}}^{\pm}_{J,v} \to \mathbf{E}^{\pm}_{J,v}$.*

*Proof.* The first claim is just the fact that if $v$ minimizes $\ell$ in $W_J v$, then there are compatible bijections from $U_v^-$ to the Schubert cells $BvB/B$ and $BvP_J/P_J$.

For the second claim: As in the proof of Lemma 2.6, $v$ minimizes $\ell_{\mathbf{B}}$ in $\mathbf{W}_J v$. So we can repeat the argument above, but with the varieties $\mathbf{U}_v^-$, $\mathbf{B}$, $\mathbf{P}_J$ in place of the sets $U_v^-$, $B$, $P_J$, and isomorphisms in place of bijections. ☐

The varieties $\check{\mathbf{E}}^{\pm}_J$ are stable under the $\mathbf{G}$-action on $\mathbf{G}/\mathbf{B} \times \mathbf{G} \times \mathbf{G}/\mathbf{B}$ analogous to (3.1). Let $\check{f} : \mathbf{G}/\mathbf{B} \times \mathbf{G} \times \mathbf{G}/\mathbf{B} \to (\mathbf{G}/\mathbf{B})^3$ be the equivariant map defined by

$$\check{f}(h\mathbf{B}, u, y\mathbf{B}) = (h\mathbf{B}, y\mathbf{B}, uh\mathbf{B}).$$

The proofs of the two parts of Theorem 3.1 will use $\check{f}$ in different ways.

3.4. **Proof of (1).** In the notation of Section 2,

$$pr_{0,2,!}\delta_{O(v^{-1},v)} = \delta_{v^{-1}} * \delta_v.$$

This suggests comparing $\mathbf{E}^-_{J,v}$ to a bundle over $\mathbf{O}(v^{-1}, v)$. It turns out that $\check{\mathbf{E}}^-_{J,v}$ is the bundle we seek.

Observe that if $(h\mathbf{B}, u, y\mathbf{B})$ is a point of $\check{\mathbf{E}}^-_{J,v}$, then $\mathbf{B}y^{-1}uh\mathbf{B} = \mathbf{B}y^{-1}h\mathbf{B} = \mathbf{B}v\mathbf{B}$. Therefore, $\check{f}$ restricts to a map from $\check{\mathbf{E}}^-_{J,v}$ into $\mathbf{O}(v^{-1}, v)$, giving an equivariant commutative diagram:

$$\begin{tikzcd}
\check{\mathbf{E}}^-_{J,v} \arrow[r] \arrow[d, "\check{f}"'] & \mathbf{E}^-_{J,v} \arrow[ddl, bend left, "f"] \\
\mathbf{O}(v^{-1}, v) \arrow[d, "pr_{0,2}"'] & \\
(\mathbf{G}/\mathbf{B})^2 &
\end{tikzcd}$$

By Lemma 3.2 and this diagram, we reduce case (1) of Theorem 3.1 to:

**Proposition 3.3.** *If $v \in W^{J,-}$, then*

$$\check{f}_! \delta_{\check{E}^-_{J,v}} = |U_{w_J \circ v}|\, \delta_{O(v^{-1},v)}$$

*in $\mathcal{C}_G(O(v^{-1}, v))$. In the split case, this identity arises from $\check{\mathbf{E}}^-_{J,v} \xrightarrow{\check{f}} \mathbf{O}(v^{-1}, v)$ being a smooth fiber bundle with fiber $\mathbf{U}_{w_J \circ v}$ above $(\mathbf{B}, v\mathbf{B})$.*

*Proof.* For the first claim: Recall that the $G$-action on pairs $(yB, hB) \in O(v)$ is transitive. So by equivariance of $\check{f}$ and homogeneity, it suffices to compute $\check{f}$ over any fixed choice of such $yB$ and $hB$.

We take $(yB, hB) = (B, vB)$. Over this pair, the fiber of $\check{E}_J^-$ consists of $(vB, u, B)$ with $u \in U_J$, the fiber of $O(v^{-1}, v)$ consists of $(vB, B, gB)$ with $gB \in BvB/B$, and $\check{f}$ is given by $u \mapsto uvB$. Therefore, under the bijections $U_J \simeq U_{w_{J\circ}v} \times U_v^-$ of Lemma 2.6 and $BvB/B \simeq U_v^-$ of Lemma 2.3(3), $\check{f}$ corresponds to the projection $U_{w_{J\circ}v} \times U_v^- \to U_v^-$. This proves the claim.

For the second claim: As in the proof of Lemma 2.6, we observe that $v$ minimizes $\ell_{\mathbf{B}}$ in $\mathbf{W}_J v$. So we can repeat the arguments above with the varieties $\mathbf{G}$, $\mathbf{O}(v)$, *etc.* in place of the sets $G$, $O(v)$, *etc.*, and Lemma 2.1 in place of Lemma 2.3. □

3.5. **Proof of (2).** In the notation of Section 2,

$$pr_{0,4,!}\delta_{O(v^{-1}, w_{J\circ}, w_{J\circ}, v)} = \delta_{v^{-1}} * \delta_{w_{J\circ}}^2 * \delta_v.$$

This suggests comparing $\mathbf{E}_{J,v}^+$ to a bundle over $\mathbf{O}(v^{-1}, w_{J\circ}, w_{J\circ}, v)$. But unlike the situation in case (1), there is no obvious map from $\check{\mathbf{E}}_{J,v}^+$ into the latter variety.

We do know that $\check{f}$ restricts to a map from $\check{\mathbf{E}}_{J,v}^+$ into $\mathbf{O}(v^{-1}) \times \mathbf{G}/\mathbf{B}$, giving an equivariant commutative diagram:

$$\begin{tikzcd}
\check{\mathbf{E}}^+_{J,v} \arrow[r] \arrow[d, "\check{f}"'] & \mathbf{E}^+_{J,v} \arrow[ddl, bend left, "f"] \\
\mathbf{O}(v^{-1}) \times \mathbf{G}/\mathbf{B} \arrow[d, "pr_0 \times \mathrm{id}"'] & \\
(\mathbf{G}/\mathbf{B})^2 &
\end{tikzcd}$$

At the same time, we have a map

$$\mathbf{O}(v^{-1}, w_{J\circ}, w_{J\circ}, v) \xrightarrow{pr_{0,1,4}} \mathbf{O}(v^{-1}) \times \mathbf{G}/\mathbf{B}.$$

So by Lemma 3.2 and the discussion above, we reduce case (2) of Theorem 3.1 to:

**Proposition 3.4.** *If $v \in W^{J,-}$, then*

$$\check{f}_!\delta_{E_{J,v}^+} = |U_{w_{J\circ}v}|\, pr_{0,1,4,!}\delta_{O(v^{-1}, w_{J\circ}, w_{J\circ}, v)}$$

*in $\mathcal{C}_G(O(v^{-1}) \times G/B)$.*

*Proof.* For any $w \in W$, let

$$O(v^{-1}) \times_w G/B = \{(hB, yB, gB) \in O(v^{-1}) \times G/B \mid Bh^{-1}gB = BwB\},$$

the preimage of $O(w)$ along $pr_0 \times \mathrm{id}$. Recall that the $G$-action on $O(w)$ is transitive. So by equivariance and homogeneity, the fibers of $\check{E}_{J,v}^+$ and $O(v^{-1}, w_{J\circ}, w_{J\circ}, v)$ have constant size over $O(v^{-1}) \times_w G/B$. So it suffices to compare them over any fixed choice of $(hB, gB) \in O(w)$, for each $w \in W$. Moreover, to do this, it suffices to fix $hB$ and average over $gB \in hBwB/B$.

We take $hB = B$. Then we must compare the preimages of

$$\{(B, yB, gB) \in O(v^{-1}) \times_w G/B\} \tag{3.3}$$

in $\check{E}_J^+$ and $O(v^{-1}, w_{J\circ}, w_{J\circ}, v)$. Since $v \in W^{J,-}$, we can trade the latter set and the map $pr_{0,1,4}$ for the set $O(v^{-1}, w_{J\circ}, w_{J\circ}v)$ and the map $pr_{0,1,3}$.

The preimage of (3.3) in $\check{E}_J^+$ consists of $(B, u, yB)$ such that $u \in yV_Jy^{-1}$ and $u \in BwB$. Hence it has size

$$|yV_Jy^{-1} \cap BwB|. \tag{3.4}$$

The preimage of (3.3) in $O(v^{-1}, w_{J\circ}, w_{J\circ}v)$ consists of $(B, yB, zB, gB)$ such that

$$yB \xleftarrow{w_{J\circ}} zB \xrightarrow{w_{J\circ}v} gB$$

and $gB \in BwB/B$. Observe that $yB \in Bv^{-1}B/B$, so homogeneity under left multiplication by $B$ lets us count the preimage for a given $yB$ by averaging over the preimages for all $yB \in Bv^{-1}B/B$. Since $v \in W^{J,-}$, Lemma 2.3(1) shows that the union of these preimages is parametrized by $(zB, gB)$ such that

$$B \xleftarrow{w_{J\circ}v} zB \xrightarrow{w_{J\circ}v} gB \tag{3.5}$$

and $gB \in BwB/B$. It also shows that there is a bijection from $U^-_{(w_{J\circ}v)^{-1}} \times U^-_{w_{J\circ}v}$ to the set of pairs $(zB, gB)$ satisfying (3.5), given by

$$(u, u') \mapsto (u(w_{J\circ}v)^{-1}B, u(w_{J\circ}v)^{-1}u'w_{J\circ}vB).$$

So the set of $(zB, gB)$ satisfying (3.5) and $gB \in BwB/B$ is parametrized by

$$(U^-_{(w_{J\circ}v)^{-1}}(w_{J\circ}v)^{-1}U^-_{w_{J\circ}v}w_{J_\circ}v) \cap BwB.$$

Since $U^-_{(w_{J\circ}v)^{-1}} \subseteq B$, this last set can be identified with

$$U^-_{(w_{J\circ}v)^{-1}} \times ((w_{J\circ}v)^{-1}U^-_{w_{J\circ}v}w_{J_\circ}v \cap BwB).$$

By Lemma 2.3(3), we have $|U^-_{v^{-1}}|$ many choices for $yB \in Bv^{-1}B/B$, and since $v \in W^{J,-}$, we also have $|U^-_{(w_{J\circ}v)^{-1}}| = |U^-_{w_{J\circ}}||U^-_{v^{-1}}|$. Altogether, we conclude that the size of the preimage of (3.3) in $O(v^{-1}, w_{J\circ}, w_{J\circ}v)$ is

$$|U^-_{w_{J\circ}}||(w_{J\circ}v)^{-1}U^-_{w_{J\circ}v}w_{J_\circ}v \cap BwB|. \tag{3.6}$$

Finally, we compare (3.4) and (3.6). Theorem 4.1 of [Kaw75] says

$$|yV_Jy^{-1} \cap BwB| = |U_{v^{-1}}||(w_{J\circ}v)^{-1}U^-_{w_{J\circ}v}w_{J_\circ}v \cap BwB|.$$

Again using $|U^-_{(w_{J\circ}v)^{-1}}| = |U^-_{w_{J\circ}}||U^-_{v^{-1}}|$, we see that $|U_{v^{-1}}| = |U_{(w_{J\circ}v)^{-1}}||U^-_{w_{J\circ}}| = |U_{w_{J\circ}v}||U^-_{w_{J\circ}}|$, giving the desired identity. $\square$

*Remark* 3.5. The asymmetry of the variety $\mathbf{O}(v^{-1}) \times \mathbf{G}/\mathbf{B}$ may seem defective. To make the geometry more symmetrical, one might try to replace the diagram

$$\mathbf{E}_J^+ \xrightarrow{\check{f}} \mathbf{O}(v^{-1}) \times \mathbf{G}/\mathbf{B} \xleftarrow{pr_{0,1,4}} \mathbf{O}(v^{-1}, w_{J\circ}, w_{J\circ}, v)$$

with the diagram

$$\mathbf{E}_J^+ \xrightarrow{\check{f}'} \mathbf{O}(v^{-1}) \times \mathbf{O}(v) \xleftarrow{pr_{0,1,3,4}} \mathbf{O}(v^{-1}, w_{J\circ}, w_{J\circ}, v)$$

in which $\check{f}'(h\mathbf{B}, u, x\mathbf{B}) = (h\mathbf{B}, x\mathbf{B}, ux\mathbf{B}, uh\mathbf{B})$. Then one would hope that

$$\check{f}'_! \delta_{E_{J,v}^+} = |U_J| \, pr_{0,1,3,4,!} \delta_{O(v^{-1}, w_{J\circ}, w_{J\circ}, v)}$$

in $\mathcal{C}_G(O(v^{-1}) \times O(v))$. However, Kawanaka's work does not seem to establish this stronger identity.

## 4. Traces on the Hecke Algebra

4.1. The goal of this section is to prove a form of Theorem 1.2 for general $G$, and also, to prove that it recovers (1.2) when $G = \mathrm{GL}_n$.

We keep the general setup of Section 2. In order to work with étale cohomology, we fix a prime $\ell$ invertible in $\mathbf{F}$. The notation $\mathrm{H}_c^*(-, \bar{\mathbf{Q}}_\ell)$ will always mean compactly-supported étale cohomology with coefficients in the constant $\bar{\mathbf{Q}}_\ell$-sheaf.

4.2. **Springer Fibers.** A reference for this subsection is [Sho88]. Henceforth, let $\mathbf{V} = \mathbf{V}_\emptyset$ and

$$\mathbf{Spr} = \mathbf{Spr}_\emptyset^+ = \mathbf{Spr}_\emptyset^- \subseteq \mathbf{V} \times \mathbf{G}/\mathbf{B}.$$

By the *Springer resolution*, we mean either $\mathbf{Spr}$ or the projection map from $\mathbf{Spr}$ onto $\mathbf{V}$. For any $u \in \mathbf{V}$, the *Springer fiber* over $u$ is the (reduced) fiber of this map over $u$, viewed as a subvariety $\mathbf{Spr}_u$ of $\mathbf{G}/\mathbf{B}$. On points,

$$\mathbf{Spr}_u = \{y\mathbf{B} \in \mathbf{G}/\mathbf{B} \mid u \in y\mathbf{U}y^{-1}\}.$$

Springer showed that this is a projective variety with no odd cohomology. For $u \in V := \mathbf{V}^F$, he constructed an action of $W$ on $\mathrm{H}_c^*(\mathbf{Spr}_u)$, essentially from twisting the Frobenius $F$ by elements of $W$. Let $\chi_u : \mathbf{Q}W \to \bar{\mathbf{Q}}_\ell$ be the trace defined by

$$\chi_u(w) = \mathrm{tr}(Fw \mid \mathrm{H}_c^*(\mathbf{Spr}_u)).$$

Later, other authors found other constructions of this action, and generalizations beyond $\mathbf{F}$, that instead use perverse sheaves, Fourier transforms, or, in the complex case, purely topological arguments. The various constructions actually lead to two separate Springer actions, differing by a sign twist. We will use the one where the sign representation of $W$ only occurs in the top cohomology of $\mathbf{Spr}_1$.

As reviewed in [Sho88, §15], it is now known $\chi_u$ arises from the specialization at $\boldsymbol{q} \to q$ of a $\mathbf{Z}[\boldsymbol{q}]$-valued trace on $\mathbf{Z}W$. In particular, $\chi_u(w) \in \mathbf{Z}$ for all $w \in W$.

4.3. **Partial Springer Fibers.** For all $J \subseteq S$, the *symmetrizer* and *antisymmetrizer* in $\mathbf{Q}W_J$ are respectively defined by

$$e_{J,+} = \frac{1}{|W_J|} \sum_{w \in W_J} w \quad \text{and} \quad e_{J,-} = \frac{1}{|W_J|} \sum_{w \in W_J} (-1)^{\ell(w)} w.$$

These are central elements of $\mathbf{Q}W_J$, such that $\mathbf{Q}W_J e_{J,+}$ and $\mathbf{Q}W_J e_{J,-}$ respectively afford the trivial and sign representations of $W_J$.

Borho–MacPherson related $e_{J,-}$ and $e_{J,+}$ to the *partial Springer fibers*

$$\begin{aligned}\mathbf{Spr}_{J,u}^- &= \{y\mathbf{P}_J \in \mathbf{G}/\mathbf{P}_J \mid u \in y\mathbf{U}_J y^{-1}\},\\ \mathbf{Spr}_{J,u}^+ &= \{y\mathbf{P}_J \in \mathbf{G}/\mathbf{P}_J \mid u \in y\mathbf{V}_J y^{-1}\}.\end{aligned}$$

By §2.4, the set of $F$-fixed points $Spr_{J,u}^-$, *resp.* $Spr_{J,u}^+$, is the set of $yP_J \in G/P_J$ such that $u \in yU_J y^{-1}$, *resp.* $u \in yV_J y^{-1}$. For our choice of Springer action, the main result of [BM83] implies that for all $J \subseteq S$ and $u \in V$, we have

$$\begin{aligned}\chi_u(e_{J,-}) &= q^{\ell_J} |Spr_{J,u}^-|,\\ \chi_u(e_{J,+}) &= |Spr_{J,u}^+|.\end{aligned} \tag{4.1}$$

Two subtleties are worth noting. First, the Springer action used by Borho–MacPherson is the *sign twist* of ours. Second, the factor of $q^{\ell_J}$ in the $-$ case arises from the grading shift in case (b) of [BM83, §3.4], via the fact that the sign twist introduces a Poincaré–Verdier dual, as explained in [AHJR14].

4.4. **The Bitrace.** As in §1.3, let $O(w)_u$ be the subset of $O(w)$ of pairs taking the form $(hB, uhB)$. Let $\tau_G : \mathbf{Q}W \otimes H_B^G \to \mathbf{Q}$ be defined by

$$\tau_G(z \otimes \delta_w) = \frac{1}{|G|} \sum_{u \in V} |O(w)_u| \chi_u(z).$$

The results of [Tri21] show that this is, indeed, a bitrace, meaning $\tau_G(z \otimes -)$ and $\tau_G(- \otimes \delta_w)$ are traces for all $z, w \in W$. In the split case, it recovers the $\boldsymbol{q} \to q$ specialization of the trace denoted $\tau_G$ in the introduction.

**Lemma 4.1.** *For all $J \subseteq S$ and $w \in W$, we have*

$$\tau_G(e_{J,\pm} \otimes \delta_w) = \frac{1}{|G|} \sum_{(hB,gB) \in O(w)} f_! \delta_{E_J^\pm}(hB, gB),$$

*where $E_J^\pm$ and $f$ are defined as in Section 3.*

*Proof.* Applying (4.1) to the formula for $\tau_G$. Then observe that

$$\begin{aligned}\coprod_{u \in V} O(w)_u \times Spr_{J,u}^\pm &= \{(hB, u, yP_J) \in E_J^\pm \mid (hB, uhB) \in O(w)\}\\ &= \coprod_{(hB,gB) \in O(w)} f^{-1}(hB, gB).\end{aligned}$$

□

4.5. **Traces from Relative Norms.** As in §1.3, let $\tau : H_B^G \to \mathbf{Z}[\frac{1}{q}]$ be the trace given by $\tau(\delta_e) = 1$ and $\tau(\delta_w) = 0$ for all $w \neq e$, and for any central element $\zeta \in Z(H_B^G)$, let $\tau[\zeta] : H_B^G \to \mathbf{Z}[\frac{1}{q}]$ be the trace given by $\tau[\zeta](\beta) = \tau(\beta * \zeta)$.

**Lemma 4.2.** *For all $J \subseteq S$ and $w \in W$ and $\alpha \in Z(H_{B_J}^{L_J})$, we have*

$$\tau[N_J^S(\alpha)](\delta_w) = \frac{|B|}{|G|} \sum_{(hB,gB)\in O(w)} N_J^S(\iota(\alpha))(hB, gB),$$

*where $\iota$ is the additive anti-involution of $H_{B_J}^{L_J}$ given by $\iota(\delta_w) = \delta_{w^{-1}}$.*

*Proof.* For any $\beta \in H_B^G$ and $xB \in G/B$, we have $\tau(\beta) = \beta(xB, xB)$. Moreover, $|G/B| = |G|/|B|$. So for any $\zeta \in Z(H_B^G)$, we have

$$\tau[\zeta](\beta) = \frac{|B|}{|G|} \sum_{xB\in G/B} (\beta * \zeta)(xB, xB).$$

Next, for any $w, v, z \in W$, observe that there is a bijection

$$\begin{aligned}&\{(x_0B, x_1B, x_2B, x_3B, x_4B) \in O(w, v^{-1}, z, v) \mid x_0B = x_4B\}\\ &\quad\xrightarrow{\sim} \{(g_0B, g_1B, g_2B, g_3B) \in O(v^{-1}, z^{-1}, v) \mid g_0B \xrightarrow{w} g_3B\}\end{aligned}$$

given by $(g_0B, g_1B, g_2B, g_3B) = (x_4B, x_3B, x_2B, x_1B)$. This shows the identity

$$\sum_{gB\in G/B} (\delta_w * \delta_{v^{-1}} * \delta_z * \delta_v)(gB, gB) = \sum_{(hB,gB)\in O(w)} (\delta_{v^{-1}} * \delta_{z^{-1}} * \delta_v)(hB, gB).$$

By expanding $\alpha$ in the basis $(\delta_z)_{z\in W_J}$ for $H_{B_J}^{L_J}$, and summing over all $v \in W^{J,-}$, we deduce that

$$\sum_{xB\in G/B} (\beta * N_J^S(\alpha))(xB, xB) = \sum_{(hB,gB)\in O(w)} N_J^S(\iota(\alpha))(hB, gB),$$

concluding the proof. □

The split case of the following result recovers Theorem 1.2, as it works for infinitely many finite fields $\mathbf{F}$, allowing us to upgrade $q$ to $\boldsymbol{q}$.

**Theorem 4.3.** *For any $J \subseteq S$, we have*

$$\begin{aligned}\tau[N_J^S(1)] &= |B_J|\, \tau_G(e_{J,-} \otimes -),\\ \tau[N_J^S(\delta_{w_{J\circ}}^2)] &= |B_J|\, \tau_G(e_{J,+} \otimes -)\end{aligned}$$

*as traces on $H_W$.*

*Proof.* Combine (3.2) with Lemmas 4.1–4.2, noting that 1 and $\delta_{w_{J\circ}}^2$ are invariant under $\iota$. Then observe that $B = B_J \ltimes U_J$, from which $|B|/|U_J| = |B_J|$. □

4.6. **Recovering Lascoux–Wan–Wang.** In this subsection, we assume that $\mathbf{G} = \mathbf{GL}_n$ and $F$ is the standard Frobenius that raises each matrix coordinate to its $q$th power. Then $G = \mathrm{GL}_n(\mathbf{F})$ and $W = \mathbf{W} = S_n$.

As in §1.3, we take $S = \{s_1, \dots, s_{n-1}\}$, where $s_i \in S_n$ is the transposition swapping $i$ and $i+1$. Then the correspondence between partitions $\lambda \vdash n$ and subsets $J \subseteq S$ is given by sending $\lambda$ to

$$J_\lambda = \{\}$$

**Proposition 4.4.** *For all $J \subseteq S$, we have*

$$ch_{\boldsymbol{q}}(\tau_G(e_{J,+} \otimes -)) = \boldsymbol{q}^{-\ell_J} e_\lambda(\frac{X}{\boldsymbol{q}-1}),$$
$$ch_{\boldsymbol{q}}(\tau_G(e_{J,+} \otimes -)) = h_\lambda(\frac{X}{\boldsymbol{q}-1}).$$

## 5. Braid Varieties and Cell Decompositions

5.1.

## 6. Parking Numbers

6.1.

## 7. Markov Traces and Kirkman Numbers

7.1.

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