MATH 250: TOPOLOGY I PROBLEM SET #1

FALL 2025

Due Wednesday, September 3. Please attempt all of the problems. Six of them will be graded. You may consult books, papers, and websites as long as you cite all sources and write up your solutions in your own words.

Problem 1. Let $f : X \to Y$ be an arbitrary map between sets.

(1) Let $\{X_\alpha\}_\alpha$ be an arbitrary collection of subsets of X. Show that

$$f\left(\bigcup_\alpha X_\alpha\right) = \bigcup_\alpha f(X_\alpha) \quad \text{and} \quad f\left(\bigcap_\alpha X_\alpha\right) \subseteq \bigcap_\alpha f(X_\alpha).$$

(2) In the setup of (1), give an example where

$$f\left(\bigcap_\alpha X_\alpha\right) \neq \bigcap_\alpha f(X_\alpha).$$

(3) Let $\{Y_\beta\}_\beta$ be an arbitrary collection of subsets of Y. Show that

$$f^{-1}\left(\bigcup_\beta Y_\beta\right) = \bigcup_\beta f^{-1}(Y_\beta) \quad \text{and} \quad f^{-1}\left(\bigcap_\beta Y_\beta\right) = \bigcap_\beta f^{-1}(Y_\beta).$$

Problem 2 (Munkres 83, #1)**.** Let X be a topological space, and let A be a subset of X. Suppose that for each $x \in A$, there is an open set U containing x such that $U \subseteq A$. Show that A is also open.

Problem 3 (Munkres 83, #3)**.** Let X be any set. Show that the collection

$$\{\emptyset\} \cup \{U \subseteq X \mid X - U \text{ countable}\}$$

always forms a topology on X. Does

$$\{\emptyset, X\} \cup \{U \subseteq X \mid X - U \text{ is infinite}\}$$

always form a topology on X?

Problem 4 (Munkres 83, #4(b)–(c))**.**

(1) Let $\{\mathcal{T}_\alpha\}_\alpha$ be a family of topologies on X. Show that there exist a unique *smallest* topology on X that contains each $\mathcal{T}_\alpha$ as a subset, and a unique *largest* topology that is contained in each $\mathcal{T}_\alpha$ as a subset.

(2) Suppose that $X = \{a, b, c\}$ and

$$\mathcal{T}_1 = \{\emptyset, X, \{a\}, \{a, b\}\} \quad \text{and} \quad \mathcal{T}_2 = \{\emptyset, X, \{a\}, \{b, c\}\}.$$

Find the smallest topology containing $\mathcal{T}_1$ and $\mathcal{T}_2$ as subsets, and the largest topology that is contained in both $\mathcal{T}_1$ and $\mathcal{T}_2$ as a subset.

Problem 5 (Munkres 83, #8(a))**.** Using Munkres Lemma 13.2, show that the countable collection

$$\mathcal{B} = \{(a, b) \mid a < b \text{ and } a, b \text{ are rational}\}$$

forms a basis for the analytic topology on $\mathbf{R}$.

Problem 6. Let $a\mathbf{Z} + b = \{aq + b \mid q \in \mathbf{Z}\}$, for any integers a and b. Let

$$\mathcal{B} = \{a\mathbf{Z} + b \mid a, b \in \mathbf{Z} \text{ with } a \neq 0\}.$$

Show that $\mathcal{B}$ forms a basis for some topology on $\mathbf{Z}$. In class, we refer to the topology it generates as the *evenly-spaced topology*.

Problem 7. Endow $\mathbf{R}$ with the analytic topology. Give an example of a <u>continuous</u>, <u>non-constant</u> map $f : \mathbf{R} \to \mathbf{R}$ and an open set $U \subseteq \mathbf{R}$ such that $f(U)$ is *not* open. *Hint:* There is a solution where f is a quadratic polynomial. You may assume that polynomial maps are continuous.

Problem 8. Let X, Y be topological spaces, and let $f : X \to Y$ be a continuous bijection. Show that if $f(U)$ is open in Y for every open set U in X, then f is a homeomorphism.

Problem 9. Recall the notion of a *group* from the initial reading. Show that:

(1) $\mathbf{R}$ forms a group under the law of addition.
(2) $\mathbf{R}$ <u>does not</u> form a group under the law of multiplication.
(3) The set of positive real numbers $\mathbf{R}_+$ forms a group under multiplication.
(4) The set of positive integers $\mathbf{Z}_+$ <u>does not</u> form a group under multiplication.

Problem 10. For part (3), recall or look up the notion of a *subgroup*.

(1) Show that for any set X, the set of bijections from X to itself forms a group under the law of composition (*i.e.*, $g \circ f$ defined by $(g \circ f)(x) = g(f(x))$). This group is usually denoted $\mathrm{Sym}(X)$.
(2) Give two elements $f, g \in \mathrm{Sym}(\{a, b, c\})$ such that $g \circ f \neq f \circ g$.
(3) Suppose that X is endowed with a topology. Show that the set of homeomorphisms from X to itself forms a subgroup of $\mathrm{Sym}(X)$. This subgroup is usually denoted $\mathrm{Homeo}(X)$.