UT-Dallas 2025

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$G = \mathrm{GL}_n(\mathbb{F}_q)$ $\qquad$ q prime power

B upper-triangular subgroup

U unipotent upper-triangular subgroup

$\mathcal{U}$ set of <u>all</u> unipotent elts of G

<u>Thm</u> (Steinberg < 1965) $\quad |\mathcal{U}| = |U|^2 \qquad (= q^{n(n-1)})$

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T diagonal subgroup

$N_G(T)$ monomial matrices, $\quad W := N_G(T)/T \simeq S_n$

Bruhat decomposition $G = \bigsqcup_{w \in W} BwB$

<u>Thm</u> (Kawanaka 1975, v1)

$$|\mathcal{U} \cap BwB| = |UU_- \cap BwB|$$

where $U_- = w_\circ U w_\circ$ is opposite to U

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<u>Ex</u> $(n = 2) \quad w_\circ = \left(\begin{smallmatrix} & 1 \\ 1 & \end{smallmatrix}\right)$

$$\begin{aligned} \mathcal{U} \cap B &= UU_- \cap B && = U \\ \mathcal{U} \cap Bw_\circ B &= UU_- \cap Bw_\circ B && = \mathcal{U} \setminus U \end{aligned}$$

<u>Ex</u> $(n = 3) \quad w_\circ = \begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix}$

$$\mathcal{U} \cap Bw_\circ B \simeq U \times \{(a,b,c,d) \in (\mathbb{F}_q^\times)^2 \times \mathbb{F}_q^2 \mid (1 + \tfrac{1}{ab})^3 = \tfrac{cd}{ab}\}$$

$$UU_- \cap Bw_\circ B \simeq U \times \{(a,b,c,d) \in (\mathbb{F}_q^\times)^2 \times \mathbb{F}_q^2 \mid 1 + ab = abcd\}$$

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Fact 1 everything extends to a finite reductive group G:

$$\mathrm{SL}_n(\mathbb{F}_q), \quad \mathrm{PGL}_n(\mathbb{F}_q), \quad \mathrm{Sp}_{2n}(\mathbb{F}_q), \quad \ldots$$

B becomes a Borel subgrp, $U := [B, B]$
$W := N_G(T)/T$ is called the Weyl group

Fact 2 Kawanaka proved an even more general thm (v2)

$$|\mathcal{U} \cap v^{-1} P_J v \cap BwB| = |U_{v^{-1}} (w_{J\circ} v)^{-1} U^-_{w_{J\circ} v} w_{J\circ} v \cap BwB|$$

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Goal clarify (& go even further) using Hecke algebras

Hecke algebra: $\mathcal{H} = \{G\text{-invariant functions } X \times X \to \mathbf{C}\}$
where $X := G/B$ is the flag variety, under

$$(\varphi * \psi)(yB, xB) := \sum_{zB} \varphi(yB, zB)\psi(zB, xB)$$

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$\mathcal{H}$ "is" a deformation of $\mathbf{C}W$:
$\mathcal{H} = \mathbf{C}\langle 1_w \mid w \in W\rangle$ where $1_w(yB, xB) = \begin{cases} 1 & y^{-1}x \in BwB \\ 0 & \text{else} \end{cases}$

write Kawanaka's thm (v2) as $\mathrm{LH}^v_{J,w} = \mathrm{RH}^v_{J,w}$:

Thm (T–Williams) $\sum_w \mathrm{LH}^v_{J,w}\, 1_w, \sum_w \mathrm{RH}^v_{J,w}\, 1_w \in Z(\mathcal{H})$

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in fact, both arise from the horocycle correspondence

$$G \xleftarrow{\mathrm{pr}} G \times X \xrightarrow{\mathrm{act}} X \times X \quad \text{where } \mathrm{act}(yB, z) = (yB, zyB)$$

Harish-Chandra transform: $\mathsf{hc} = \mathrm{act}_! \, \mathrm{pr}^* : Cl(G) \to Z(\mathcal{H})$

explicitly given by $\mathsf{hc}(\varphi)(yB, xB) = \displaystyle\sum_{\substack{g \in G \\ gyB = xB}} \varphi(g)$

Thm (Kawanaka, v1) $\quad \mathsf{hc}(1_{\mathcal{U}}) = 1_{w_\circ} * 1_{w_\circ}$

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fix system of simple refl's $S \subseteq W$ and $J \subseteq S$

defines parabolic $P_J = U_J \rtimes L_J \supseteq B$

$$\text{parabolic induction} \quad \mathsf{ind}_J^S : Cl(L_J) \to Cl(P_J) \to Cl(G),$$

$$\text{relative norm} \quad \mathsf{n}_J^S : Z(\mathcal{H}(L_J)) \to Z(\mathcal{H}(G))$$

n_J^S defined by $\mathsf{n}_J^S(\alpha) = \displaystyle\sum_{v \in W^J} q^{-\ell(v)} \, 1_v^{-1} * \alpha * 1_v$

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Thm (T)

(1) $\quad \sum_w \mathrm{LH}^v_{J,w} \, 1_w \, \propto \, \mathsf{hc}_G \, \mathsf{ind}_J^S(1_{\mathcal{U}(L_J)})$

(2) $\quad \sum_w \mathrm{RH}^v_{J,w} \, 1_w \, \propto \, \mathsf{n}_J^S(1_{w_{J\circ}} * 1_{w_{J\circ}}) = \mathsf{n}_J^S \, \mathsf{hc}_{L_J}(1_{\mathcal{U}(L_J)})$

Conj (T)
$$\begin{tikzcd} Cl(L_J) \arrow[r, "\mathsf{ind}"] \arrow[d, "\mathsf{hc}"'] & Cl(G) \arrow[d, "\mathsf{hc}"] \\ Z(\mathcal{H}(L_J)) \arrow[r, "\mathsf{n}"] & Z(\mathcal{H}(G)) \end{tikzcd}$$
commutes

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<u>Tie-In 1</u> observe that $\mathsf{hc}(1_{\{z\}}) = 1_{\mathrm{id}}$ for $z \in Z(L_J)$

<u>Thm</u> ($\approx$ Lusztig) if $J = \emptyset$, meaning $L_J = T$, then

$$\mathsf{hc}\,\mathsf{ind}_J^S(1_{\{z\}}) = \mathsf{n}_J^S(1_{\mathrm{id}}) \quad \text{for all } z \in T$$

$$\begin{aligned} z \text{ generic:} &\quad \textstyle\sum_w |\{g \in BwB \mid g \sim z\}| 1_w \\ z = 1: &\quad \textstyle\sum_w |\{(g, xB) \in BwB \times X \mid g \in xUx^{-1}\}| 1_w \end{aligned}$$

observed by Gu (2021) with $1_{w_\circ} * 1_{w_\circ}$ replacing 1_w

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<u>Tie-In 2</u> a <u>trace</u> on $\mathcal{H}$ is a linear function $\tau : \mathcal{H} \to \mathbf{C}$ s.t. $\tau(\alpha\beta) = \tau(\beta\alpha)$

$$\underline{\text{standard trace}} \quad \tau(1_w) = \begin{cases} 1 & w = \mathrm{id} \\ 0 & \text{else} \end{cases}$$

$Z(\mathcal{H}) \xrightarrow{\sim} \{\text{traces on } \mathcal{H}\}$: $\zeta \mapsto \tau[\zeta]$ def by $\tau[\zeta](\beta) = \tau(\zeta\beta)$
so can recast results in terms of traces

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recall $\mathcal{H}(\mathrm{GL}_n) \simeq \mathbf{C}Br_n/\langle \cdots \rangle$
Ocneanu: traces $\mu_n : \mathcal{H}(\mathrm{GL}_n) \to \mathbf{C}[a^{\pm 1}]$ s.t.

- μ_n "deforms" the standard trace
- if $\beta \in Br_n$ with link closure $\hat{\beta}$, then

$$\mathbb{P}(\beta) := (-a)^{\mathrm{wr}(\beta)} \mu_n(\beta) \quad \text{only depends on } \hat{\beta}$$

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Thm (Kálmán 2009) $\mathbb{P}_{\mathrm{hi}}(\beta\delta^2) = \mathbb{P}_{\mathrm{lo}}(\beta)$, where $\delta \mapsto 1_{w_\circ}$

Thm (T 2021) if $\beta = \sigma_{s_1} \cdots \sigma_{s_\ell}$, then

$$\mathbb{P}(\beta) \propto \sum_{u \in \mathcal{U}} |M(\beta)_u| \tilde{H}_{\mathrm{Jordan}(u)}(a, q),$$

where $\tilde{H}_\mu(a, q) = \sum_k (-a^2)^k \langle s_{k,1^{n-k}}, \tilde{H}_\mu(q)\rangle$ and

$$M(\beta)_u = \{[g_i]_i \in Bs_1B \times^B \cdots \times^B Bs_\ell B \mid g_1 \cdots g_\ell = u\}$$

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Cor (T) $\mathbb{P}_{\mathrm{lo}}(\beta) \propto \sum_u |M(\beta)_u|$, $\mathbb{P}_{\mathrm{hi}}(\beta\delta^2) \propto |M(\beta\delta^2)_1|$

Cor (T) Kawanaka v1 $\iff$ Kálmán via formula

$$\begin{array}{rcl} \mathbb{P} & \rightsquigarrow & \text{triply-graded KhR homology} \\ \text{Kawanaka v1} & \rightsquigarrow & \mathrm{H}^*_{c,B}(\mathcal{U} \cap BwB) \simeq \mathrm{H}^*_{c,B}(UU_- \cap BwB) \end{array}$$