(Munkres ≈ §71) last time, we saw:

$$\pi_1(\text{figure-eight}) = \mathbf{Z} * \mathbf{Z}$$

it was a corollary of Seifert–van Kampen:
given open $U_1, U_2 \subseteq X$ s.t.

- $X = U_1 \cup U_2$,
- U_1 and U_2 are path-connected,
- $U_1 \cap U_2$ is path-connected,
- $x \in U_1 \cap U_2$,

there is a surjective homomorphism

$$\pi_1(U_1, x) * \pi_1(U_2, x) \to \pi_1(X, x)$$

whose kernel depends on $\pi_1(U_1 \cap U_2, x)$

[how did it work?] let A_1, A_2 be copies of S^1 with resp. basepoints a_1, a_2

$$\text{figure-eight} = (A_1 \sqcup A_2)/(a_1 \sim a_2)$$

[how are the U_j's related to the A_j's?]
U_j is a small open nbd of the image of A_j

<u>Df</u> for general $(A_1, a_1), (A_2, a_2)$

$$A_1 \vee A_2 = (A_1 \sqcup A_2)/(a_1 \sim a_2)$$

is called the wedge sum of A_1 and A_2

$A_1 \vee A_2$ has a “natural” basept:
the common image of a_1, a_2

then wedge sum becomes an <u>associative</u> binary operation on spaces with basepoints

<u>Q</u> let V_n denote the wedge sum of n copies of S^1 [with any basepts] [thus V_2 is the figure-eight]

what is $\pi_1(V_n)$? $Z * Z * \ldots * Z$ with n copies
[note that $*$ is also an associative operation]
how to prove? induction

<u>Rem</u> as this line of thinking suggests:

Seifert–van Kampen can be restated, more generally, for $X = U_1 \cup \ldots \cup U_n$

here, it turns out that: [and we omit proofs]

- to get surjectivity of

$$\pi_1(U_1, x) * \ldots * \pi_1(U_n, x) \to \pi_1(X, x)$$

we just need path-connectedness of $U_i \cap U_j$ for all i, j (including i = j)

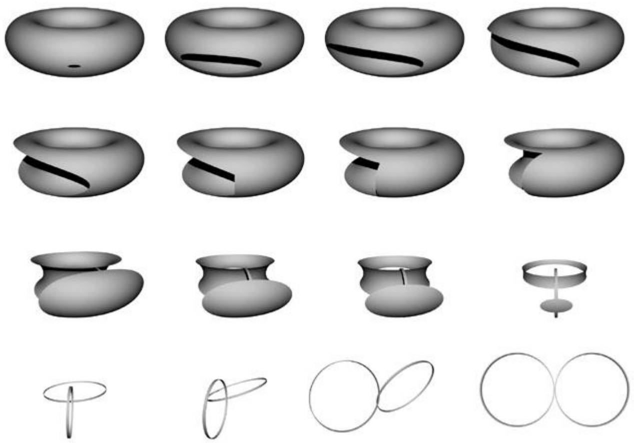
- to describe the kernel easily, we need path-connectedness of $U_i \cap U_j \cap U_k$ for all i, j, k

it is the smallest normal subgp of the free product containing, for all ordered pairs (i, j), $\mathrm{im}(\pi_1(U_i \cap U_j) \to \pi_1(U_i) \to \pi_1(\text{product}))$

(Munkres §72) a cool trick about the torus T:

Prop T – disk deformation retracts onto the figure-eight

Pf https://www.technomagi.com/josh/



Cor $\pi_1(T - \text{disk}) = \mathbb{Z} * \mathbb{Z}$

compare:

$$\begin{aligned} \pi_1(T) &= \pi_1(S^1) \times \pi_1(S^1) \\ &= \mathbb{Z} \times \mathbb{Z} \\ &= \langle a, b \mid [a, b] \rangle \\ &\qquad \text{where } [a, b] = aba^{-1}b^{-1} \end{aligned}$$

$$\pi_1(T - \text{disk}) = \langle a, b \rangle$$

somehow, puncturing T corresponds to adjoining [a, b] to the set of relations

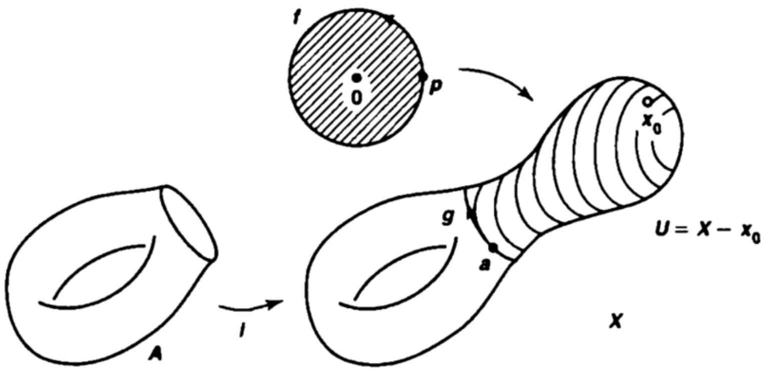
[ask: is there a way to systematize this?]

let D^2 be the closed unit disk with boundary S^1
fix a basept p in S^1

<u>Thm</u> let X be Hausdorff
let $i : A$ to X be inclusion of
a closed path-connected subspace

suppose there is cts $\zeta : D^2$ to X s.t.
ζ maps $\mathrm{Int}(D^2)$ bijectively onto $X - A$
ζ maps S^1 into A
let $a = \zeta(p)$ and $\eta = \zeta|_{S^1}$
then:

1) $i_* : \pi_1(A, a)$ to $\pi_1(X, a)$ is surjective
2) $\ker(i_*) = \mathrm{im}(\eta_* : \pi_1(S^1, p)$ to $\pi_1(A, a))$



<u>Pf Sketch</u> let $x = \zeta(\mathbf{0})$ in X and $U = X - \{x\}$

since $D^2 - \{\mathbf{0}\}$ deformation retracts onto S^1
we can show U deformation retracts onto A

remains to show:
1) $\pi_1(U, a)$ to $\pi_1(X, a)$ is surjective
2) its kernel is $\mathrm{im}(\pi_1(D^2 - \mathbf{0}, p)$ to $\pi_1(U, a))$

[want to use Seifert–van Kampen somewhere]
let $V = X - A$ = bijective image of D^2 under ζ
now see:

$X = U \cup V$,
U is path-connected bc A is,
V is path-connected bc D^2 is,
$U \cap V = (X - \{x\}) \cap V = V - \{x\}$
is also path-connected

so for any b in $U \cap V$:

surjective $\pi_1(U, b) * \pi_1(V, b)$ to $\pi_1(X, b)$

with kernel described in terms of $\pi_1(U \cap V, b)$
but V is also simply-connected bc D^2 is

so we get surjective $\pi_1(U, b)$ to $\pi_1(X, b)$
with kernel the minimal normal subgp containing
$\mathrm{im}(\pi_1(U \cap V, b)$ to $\pi_1(U, b))$

to finish: pick a path from a in A to b in $U \cap V$
translate results above into results about a, $[\gamma]$ □

<u>Gluing Diagrams for Surfaces</u>

[how to wield this thm efficiently?]
[draw pictures]

https://divisbyzero.com/2020/04/08/make-a-real-projective-plane-boys-surface-out-of-paper/

take X to be the quotient space of $[0, 1]^2$
resulting from the edge identifications

take A to be the image of the boundary square

take ζ to be a homeo from D^2 onto $[0, 1]^2$

in the torus and Klein-bottle cases

take a to be the path following “>”

take b to be the path following “>>”

then

$$\pi_1(X) = \langle a, b \mid R \rangle$$

where R is read off of a loop traversal of A