MATH 250: TOPOLOGY I PROBLEM SET #1

FALL 2025

Due Wednesday, September 3. Please attempt all of the problems. Six of them will be graded. You may consult books, papers, and websites as long as you cite all sources and write up your solutions in your own words.

Problem 1. Let $f : X \to Y$ be an arbitrary map between sets.

(1) Let $\{X_\alpha\}_\alpha$ be an arbitrary collection of subsets of X. Show that

$$f\left(\bigcup_\alpha X_\alpha\right) = \bigcup_\alpha f(X_\alpha) \quad \text{and} \quad f\left(\bigcap_\alpha X_\alpha\right) \subseteq \bigcap_\alpha f(X_\alpha).$$

(2) In the setup of (1), give an example where

$$f\left(\bigcap_\alpha X_\alpha\right) \neq \bigcap_\alpha f(X_\alpha).$$

(3) Let $\{Y_\beta\}_\beta$ be an arbitrary collection of subsets of Y. Show that

$$f^{-1}\left(\bigcup_\beta Y_\beta\right) = \bigcup_\beta f^{-1}(Y_\beta) \quad \text{and} \quad f^{-1}\left(\bigcap_\beta Y_\beta\right) = \bigcap_\beta f^{-1}(Y_\beta).$$

Problem 2. Let X be a set, and let $\mathcal{T}$ be a collection of subsets of X such that if $U, V \in \mathcal{T}$, then $U \cap V \in \mathcal{T}$. Prove that for any $k > 0$ and $U_1, \ldots, U_k \in \mathcal{T}$, we have $U_1 \cap \cdots \cap U_k \in \mathcal{T}$.

Problem 3. We say that $U \subseteq \mathbf{Z}$ is *evenly-spaced* if and only if it is a (possibly empty) union of sets of the form

$$a\mathbf{Z} + b := \{aq + b \mid q \in \mathbf{Z}\}$$

for various $a, b \in \mathbf{Z}$ with $a \neq 0$. Prove that the collection of evenly-spaced sets is a topology on $\mathbf{Z}$. *Hint:* Use Problem 2 to check the axiom about finite intersections more efficiently.

Problem 4 (Munkres 83, #1)**.** Let X be a topological space, and let A be a subset of X. Suppose that for each $x \in A$, there is an open set U containing x such that $U \subseteq A$. Show that A is also open.

Problem 5 (Munkres 83, #3)**.** Let X be any set. Show that the collection

$$\{\emptyset\} \cup \{U \subseteq X \mid X - U \text{ countable}\}$$

always forms a topology on X. Does

$$\{\emptyset, X\} \cup \{U \subseteq X \mid X - U \text{ is infinite}\}$$

always form a topology on X?

Problem 6 (Munkres 83, #4(b)–(c)).

(1) Let $\{\mathcal{T}_\alpha\}_\alpha$ be a family of topologies on X. Show that there exist a unique *smallest* topology on X that contains each $\mathcal{T}_\alpha$ as a subset, and a unique *largest* topology that is contained in each $\mathcal{T}_\alpha$ as a subset.

(2) Suppose that $X = \{a, b, c\}$ and

$$\mathcal{T}_1 = \{\emptyset, X, \{a\}, \{a, b\}\} \quad \text{and} \quad \mathcal{T}_2 = \{\emptyset, X, \{a\}, \{b, c\}\}.$$

Find the smallest topology containing $\mathcal{T}_1$ and $\mathcal{T}_2$ as subsets, and the largest topology that is contained in both $\mathcal{T}_1$ and $\mathcal{T}_2$ as a subset.

Problem 7 (Munkres 83, #8(a)). Using Munkres Lemma 13.2, show that the countable collection

$$\mathcal{B} = \{(a, b) \mid a < b \text{ and } a, b \text{ are rational}\}$$

forms a basis for the analytic topology on $\mathbf{R}$.

Problem 8. Endow $\mathbf{R}$ with the analytic topology. Give an example of a <u>continuous</u>, <u>non-constant</u> map $f : \mathbf{R} \to \mathbf{R}$ and an open set $U \subseteq \mathbf{R}$ such that $f(U)$ is *not* open. *Hint:* There is a solution where f is a quadratic polynomial. You may assume that polynomial maps are continuous.

Problem 9. Let X, Y be topological spaces, and let $f : X \to Y$ be a continuous bijection. Show that if $f(U)$ is open in Y for every open set U in X, then f is a homeomorphism.

Problem 10. Recall the notion of a *group* from the initial reading. Show that:

(1) $\mathbf{R}$ forms a group under the law of addition.
(2) $\mathbf{R}$ <u>does not</u> form a group under the law of multiplication.
(3) The set of positive real numbers $\mathbf{R}_+$ forms a group under multiplication.
(4) The set of positive integers $\mathbf{Z}_+$ <u>does not</u> form a group under multiplication.