


Simple braids tend toward positive entropy

Luke Robitaille 

*Massachusetts Institute of Technology,
77 Massachusetts Avenue, Cambridge, MA 02139, USA
lrobitai@mit.edu*

Minh-Tâm Quang Trinh 

*Department of Mathematics, Yale University,
P. O. Box 208283, New Haven, CT 06520, USA
minh-tam.trinh@yale.edu*

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ABSTRACT

A simple braid is a positive braid that can be drawn so that any two strands cross at most once. We prove that as $n \rightarrow \infty$, the proportion of simple braids on n strands that have positive topological entropy tends toward 100%. Notably, such braids are either pseudo-Anosov or reducible with a pseudo-Anosov component. Our proof involves a method of reduction from simple braids to non-simple three-strand braids that may be of independent interest.

Keywords: Braid group; topological entropy; simple braid; pseudo-Anosov.

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1. Introduction

1.1.

Let Br_n be the braid group on n strands. A braid $\beta \in Br_n$ is *simple* if, in some planar diagram for β , the crossings are all positive and any two strands cross at most once. The subset of simple braids $E_n \subseteq Br_n$ forms a generating set: The most natural one for the Garside theory of Br_n [5].

At the same time, Br_n can be identified with the mapping class group of a disk with n marked points rel its boundary. By [1], every self-map of a compact topological surface can be assigned a nonnegative real number called its *topological entropy*, or *entropy* for short, roughly measuring the growth rate of its mixing of open covers. The entropy of a mapping class is defined to be the infimum of the

*Corresponding author.

entropies of the maps it represents. The goal of this paper is to prove the following theorem.

Theorem 1. *The proportion of simple braids on n strands that have positive topological entropy tends to 100% as n tends to infinity.*

In fact, we give a more precise version: Theorem 17. The idea of the proof is to reduce from studying simple braids on n strands to studying non-simple braids on three strands, whose positive entropy can be detected via the quotient homomorphism $Br_3 \rightarrow \mathrm{SL}_2(\mathbf{Z})$. The reduction step, as well as the combinatorics that ensures that sufficiently many of the resulting three-strand braids have positive entropy, may be of independent interest.

1.2.

One motivation for Theorem 1 is the study of a different, but closely related, property of braids. Recall that under the Nielsen–Thurston classification, a mapping class is either *periodic*, *reducible*, or *pseudo-Anosov*. These options amount to the possible dynamics for its action on simple closed curves [13]. The entropy and Nielsen–Thurston type of a mapping class constrain each other: Namely, its entropy is zero if and only if it is either periodic or reducible with solely periodic components [2].

Caruso and Wiest showed that if $n \geq 3$, then in the Cayley graph of (Br_n, E_n) , the proportion of pseudo-Anosov braids in the ball of radius ℓ tends to 100% as ℓ tends to infinity [3, 4]. This confirmed a folklore expectation dating to the work of Thurston. Mahler and Sisto showed similar results, phrased in terms of random walks in non-elementary subgroups [10, 11].

The following sharpening of Theorem 1, which we leave to future work, would be directly complementary to the work of Caruso and Wiest.

Conjecture 2. *The proportion of simple braids on n strands that are pseudo-Anosov tends to 100% as n tends to infinity.*

1.3.

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2. Topological Entropy

In this section, we collect the only properties of topological entropy that we actually need.

2.1.

Let S be an compact topological surface, possibly with boundary, and $I \subset S$ a finite set of points in its interior. Let $M = \text{Mod}(S, I, \partial S)$ be the mapping class group of (S, I) rel the boundary ∂S . Explicitly, $M = \pi_0(\text{Homeo}^+(S, I, \partial S))$, where $\text{Homeo}^+(S, I, \partial S)$ is the group of self-homeomorphisms of S that stabilize I and fix ∂S , endowed with the compact-open topology [6, Sec. 2.1].

2.2.

We define the *entropy* of a map $f : S \rightarrow S$ to be its topological entropy $h(f)$ in the sense of [1]. We define the *entropy* of a mapping class $\phi \in M$ to be

$$h(\phi) = \inf_{f \in \phi} h(f),$$

where the notation $f \in \phi$ means f is a representative of ϕ .

Lemma 3. *The function $h : M \rightarrow \mathbf{R}_{\geq 0}$ has the following properties:*

- (1) *h is constant along conjugacy classes.*
- (2) *For any $\phi \in M$ and integer $k > 0$, we have $h(\phi^k) \leq kh(\phi)$.*

Proof. Parts (1) and (2), respectively, follow from Theorems 1 and 2 in *ibid.* \square

2.3.

Suppose that $I' \subseteq I$. By construction, a mapping class $\phi \in M$ can be lifted to $M' := \text{Mod}(S, I', \partial S)$ if and only if some, or equivalently any, representative of ϕ stabilizes I' . We deduce that the following lemma.

Lemma 4. *If $\phi \in M$ lifts to $\phi' \in M'$, then $h(\phi) \geq h(\phi')$.*

2.4.

Let D be a closed disk, and $I \subset D$ a finite set of points in its interior. Let

$$\text{Br}_I = \pi_1(\text{Conf}^{|I|}(D), I),$$

where $\text{Conf}^n(D)$ denotes the configuration space of n unordered points in D . As explained in [6, Sec. 9.1.3], there is an explicit isomorphism

$$\beta \mapsto \phi(\beta) : \text{Br}_I \xrightarrow{\sim} \text{Mod}(D, I, \partial D).$$

At the same time, we can identify Br_I with the usual braid group on $|I|$ strands, up to fixing an ordering of I .

We define the *entropy* of a braid β to be that of the corresponding mapping class: $h(\beta) = h(\phi(\beta))$. Now we can rewrite Lemma 4 in terms of braids. For any $\beta \in Br_I$ and $I' \subseteq I$, we say that I' is *stable* under β if we can delete strands from β to obtain an element of $Br_{I'}$. In this case, we denote the new braid by $\beta|_{I'}$. Since $\phi(\beta)$ lifts to $\phi(\beta|_{I'})$, Lemma 4 says that

$$h(\beta) \geq h(\beta|_{I'}). \tag{2.1}$$

2.5.

For any integer $N > 0$ and $g \in \text{Mat}_N(\mathbf{C}[t^{\pm 1}])$, the characteristic polynomial of g is a polynomial of degree N with coefficients in $\mathbf{C}[t^{\pm 1}]$. For any complex number $z \neq 0$, let $\text{Spec}(g(z))$ be the eigenvalue spectrum of $g(z) := g|_{t \rightarrow z}$, viewed as an unordered multiset of N complex numbers. The *spectral radius* of g , which we will denote $\text{rad}(g)$, is the maximum value of $|\lambda|$ as we run over z on the unit circle and $\lambda \in \text{Spec}(g(z))$. The following result linking spectral radius to entropy was shown by Fried [7] and Kolev [8] independently.

Theorem 5 (Fried, Kolev). *Let $\rho : Br_n \rightarrow \text{GL}_n(\mathbf{Z}[t^{\pm 1}])$ be the unreduced Burau representation of Br_n , as in [8, Sec. 2]. Then $\log \text{rad}(\rho(\beta)) \leq h(\beta)$ for all $\beta \in Br_n$.*

Viewed as a $\mathbf{Z}[t^{\pm 1}][Br_n]$ -module, the unreduced Burau representation is the direct sum of a trivial $\mathbf{Z}[t^{\pm 1}]$ -module of rank 1 and a free $\mathbf{Z}[t^{\pm 1}]$ -module of rank $n - 1$ [14, Sec. 1.3]. The latter defines the *reduced Burau representation*.

Corollary 6. *Theorem 5 also holds with the reduced Burau representation in place of the unreduced one.*

Proof. Let $\bar{\rho} : Br_n \rightarrow \text{GL}_{n-1}(\mathbf{Z}[t^{\pm 1}])$ be the reduced version. Then $\log \text{rad}(\rho(\beta)) = \max(0, \log \text{rad}(\bar{\rho}(\beta)))$ for all β . Since $h(\beta) \geq 0$, the claim follows. □

Corollary 7. *Let $\gamma : Br_3 \rightarrow \text{SL}_2(\mathbf{Z})$ be the reduced Burau representation at $n = 3$ and $t = -1$. Then $|\text{tr}(\gamma(\beta))| > 2$ implies $h(\beta) > 0$ for all $\beta \in Br_3$.*

Proof. If $|\text{tr}(\gamma(\beta))| > 2$, then $\gamma(\beta)$ must have an eigenvalue greater than 1. □

3. Simple Braids

3.1.

Following Artin, the braid group on n strands has the presentation

$$Br_n = \left\langle \sigma_1, \dots, \sigma_{n-1} \left| \begin{array}{ll} \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} & i = 1, \dots, n-2, \\ \sigma_i \sigma_j = \sigma_j \sigma_i & |i - j| > 1 \end{array} \right. \right\rangle,$$

where σ_i represents the positive simple twist of the i th and $(i + 1)$ th strands. The *writhe* of a braid on n strands is its image under the quotient map $\ell : Br_n \rightarrow \mathbf{Z}$ that sends $\ell(\sigma_i) = 1$ for all i .

3.2.

Let S_n be the symmetric group on n letters. Let s_i be the simple transposition that swaps i and $i + 1$. There is a quotient homomorphism $Br_n \rightarrow S_n$ given by $\sigma_i \mapsto s_i$.

The set of simple braids $E_n \subseteq Br_n$ is the image of a right inverse to this quotient map. Indeed, every permutation $w \in S_n$ can be written as

$$w = (s_{i_1} \cdots s_1)(s_{i_2} \cdots s_2) \cdots (s_{i_{n-1}} \cdots s_{n-1}),$$

for some uniquely determined i_1, i_2, \dots, i_n such that $j - 1 \leq i_j \leq n - 1$. Let

$$\sigma_w = (\sigma_{i_1} \cdots \sigma_1)(\sigma_{i_2} \cdots \sigma_2) \cdots (\sigma_{i_{n-1}} \cdots \sigma_{n-1}).$$

Then $w \mapsto \sigma_w$ is a right inverse of the quotient map $Br_n \rightarrow S_n$, and furthermore, $E_n = \{\sigma_w \mid w \in S_n\}$ [5].

3.3.

For any $w \in S_3$ and integer $N > 0$, let

$$P_w(N) = \{\vec{w} = (w_1, \dots, w_N) \in S_3^N \mid w_1 \cdots w_N = w\}.$$

For any $\vec{w} \in S_3^N$, let $\sigma_{\vec{w}} = \sigma_{w_1} \cdots \sigma_{w_N}$. We now show that many three-strand braids of the form $\sigma_{\vec{w}}$ for some $\vec{w} \in P_w(N)$ are braids of positive entropy.

Lemma 8. *Suppose that there exist 3-strand braids $\beta_1, \dots, \beta_k \in Br_3$ such that*

- (1) *They are all positive, i.e. can be written without negative powers of the σ_i .*
- (2) *They are all pure, i.e. map to the identity of S_n .*
- (3) *There is no matrix $g \in \text{SL}_2(\mathbf{Z})$ such that*

$$|\text{tr}(g\gamma(\beta_i))| \leq 2 \quad \text{for all } i.$$

Let $L = \max_i \ell(\beta_i)$, the maximum writhe among the β_i . Then

$$\frac{|\{\vec{w} \in P_w(N) \mid h(\sigma_{\vec{w}}) > 0\}|}{|P_w(N)|} \geq 6^{-L},$$

for any integer $N \geq L + 1$.

Proof. Observe that $|P_w(N)| = 6^{N-1}$. So by Corollary 7, it suffices to exhibit at least 6^{N-L-1} elements $\vec{w} \in P_w(N)$ such that $|\text{tr}(\gamma(\sigma_{\vec{w}}))| > 2$.

Pick the first $N - L - 1$ entries of \vec{w} freely. Then pick the $(N - L)$ th entry to ensure that the product of the first $N - L$ entries of \vec{w} equals w . By condition (3), there is some index i such that

$$|\text{tr}(\gamma(\sigma_{w_1} \cdots \sigma_{w_{N-L}})\gamma(\beta_i))| > 2.$$

Using condition (1), we can write $\beta_i = \sigma_{w_{N-L+1}} \cdots \sigma_{w_{N-L+k}}$ for some $k \leq L$ and $w_{N-L+1}, \dots, w_{N-L+k} \in S_3$. For j such that $k < j \leq L$, we set $w_{N-L+j} = 1$.

Finally, by condition (2), the product of all of the entries in the resulting tuple \vec{w} equals w . \square

Lemma 9. *In the setup of Lemma 8, it is possible to choose the braids $\beta_i \in Br_3$ so that $k = 5$ and $L = 6$. Explicitly,*

$$(\beta_i)_{i=1}^4 = (1, \sigma_1^2 \sigma_2^2, \sigma_2^2 \sigma_1^2, \sigma_1^2 \sigma_2^2 \sigma_1^2, \sigma_2 \sigma_1^4 \sigma_2).$$

Proof. Conditions (1) and (2) on the β_i are immediate; it remains to check condition (3). Without loss of generality, we can normalize the reduced Burau representation so that the homomorphism γ in Corollary 7 takes the following form:

$$\gamma(\sigma_1) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \gamma(\sigma_2) = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}.$$

We compute that

$$(\gamma(\beta_i))_{i=1}^5 = \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -3 & 2 \\ -2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ -2 & -3 \end{pmatrix}, \begin{pmatrix} -3 & -4 \\ -2 & -3 \end{pmatrix}, \begin{pmatrix} -3 & 4 \\ 2 & -3 \end{pmatrix} \right).$$

So we must show that there cannot exist $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbf{Z})$ such that

$$|a + d| \leq 2, \tag{3.1}$$

$$|-3a - 2b + 2c + d|, |a - 2b + 2c - 3d| \leq 2, \tag{3.2}$$

$$|3a + 2b + 4c + 3d|, |-3a + 2b + 4c - 3d| \leq 2. \tag{3.3}$$

(The structure of our argument will clarify why we group the inequalities in this way.) In what follows, set $f = -b + c$. First, by (3.2),

$$4|a - d| \leq |-3a + 2f + d| + |a + 2f - 3d| \leq 4,$$

from which we deduce

$$|a - d| \leq 1. \tag{3.4}$$

Next, by (3.1), $2|a| \leq |a - d| + |a + d| \leq 3$, from which $a \in \{-1, 0, 1\}$.

If $a = 0$, then by (3.4), $d \in \{-1, 0, 1\}$. If $d = 0$, then (3.2) says $|f| \leq 1$, from which $f \in \{-1, 0, 1\}$. This contradicts the fact that $-bc = ad - bc = 1$. If $d = 1$, then (3.2) says $|2f + 1|, |2f - 3| \leq 2$. This forces $2f = 1$, contradicting the fact that $f \in \mathbf{Z}$. The argument when $d = -1$ is similar, but with flipped signs.

If $a = 1$, then by (3.1) and (3.4), $d \in \{0, 1\}$. If $d = 0$, then the same argument as for $(a, d) = (0, 1)$ shows $2f = 1$, again contradicting $f \in \mathbf{Z}$. If $d = 1$, then (3.2) says $|f - 1| \leq 1$, from which $f \in \{0, 1, 2\}$. Since we also need $bc = 0$, we deduce that $(b, c) \in \{(-2, 0), (-1, 0), (0, 0), (0, 1), (0, 2)\}$. Each option contradicts (3.3).

Finally, the argument when $a = -1$ is similar to that when $a = 1$. For instance, in the $d = -1$ subcase, we get $(b, c) \in \{(0, -2), (0, -1), (0, 0), (1, 0), (2, 0)\}$, again contradicting (3.3). \square

3.4.

Next, we show how to pass from simple braids on many strands to non-simple braids on three strands of equal or lower entropy. For any integer $N > 0$, let

$$C(N) = \{w \in S_{3N} \mid w \text{ is a single cycle of length } 3N\}.$$

Let $P_{\text{cyc}}(N) = P_{s_1 s_2}(N) \sqcup P_{s_2 s_1}(N)$. We will define a map

$$\vec{p}: C(N) \rightarrow P_{\text{cyc}}(N). \quad (3.5)$$

First, for any $c \in C(N)$ and $j \in \mathbf{Z}$, let $a_j = a_j(c)$ be the image of 1 under the permutation of $\{1, 2, \dots, 3N\}$ given by c^j . In other words, c is the cycle $1 = a_0 \mapsto a_1 \mapsto \dots \mapsto a_{3N} = 1$. Next, for $1 \leq i \leq N$, let $p_i(c) \in S_3$ be the permutation of $\{1, 2, 3\}$ defined in three stages as follows:

- (1) The order-preserving bijection $\{1, 2, 3\} \xrightarrow{\sim} \{a_{i-1}, a_{i-1+N}, a_{i-1+2N}\}$. (It sends 1 to the smallest element of the range and 3 to the largest.)
- (2) The bijection $\{a_{i-1}, a_{i-1+N}, a_{i-1+2N}\} \xrightarrow{\sim} \{a_i, a_{i+N}, a_{i+2N}\}$ that sends $a_j \mapsto a_{j+1}$.
- (3) The order-preserving bijection $\{a_i, a_{i+N}, a_{i+2N}\} \xrightarrow{\sim} \{1, 2, 3\}$.

Below, we take the convention that S_3 acts on $\{1, 2, 3\}$ from the right.

Lemma 10. *For all $c \in C(N)$, the product $p_1(c) \cdots p_N(c)$ is a three-cycle, so (3.5) can be defined using $\vec{p}(c) := (p_1(c), \dots, p_N(c))$. Moreover,*

$$|\{c \in C(N) \mid \vec{p}(c) = \vec{w}\}| = \frac{|C(N)|}{|P_{\text{cyc}}(N)|} \quad \text{for any } \vec{w} \in P_{\text{cyc}}(N).$$

That is, the fibers of \vec{p} are equinumerous.

Proof. In the notation of the discussion above, $p_1(c) \cdots p_N(c)$ is the permutation of $\{1, 2, 3\}$ defined in stages as follows:

- (1) The order-preserving bijection $\{1, 2, 3\} \xrightarrow{\sim} \{1, a_N, a_{2N}\}$.
- (2) The permutation of $\{1, a_N, a_{2N}\}$ that sends $a_j \mapsto a_{j+N}$. (Recall that $1 = a_0 = a_{3N}$.)
- (3) The order-preserving bijection $\{1, a_N, a_{2N}\} \xrightarrow{\sim} \{1, 2, 3\}$.

Altogether, $p_1(c), \dots, p_N(c)$ is the three-cycle that sends $1 \mapsto 2$, respectively, $1 \mapsto 3$, when $a_N < a_{2N}$, respectively, $a_{2N} < a_N$. This proves the first assertion of the lemma.

Let $Q(N)$ be the set of ordered tuples $\vec{\xi} = (\xi_i)_{i=1}^N$ such that the ξ_i form a partition of $\{1, \dots, 3N\}$ into 3-element subsets, and such that $1 \in \xi_N$. For all $c \in C(N)$, let $\vec{q}(c) \in Q(N)$ be defined by

$$q_i(c) = \{a_i(c), a_{i+N}(c), a_{i+2N}(c)\}.$$

We claim that the map $\vec{p} \times \vec{q} : C(N) \rightarrow P_{\text{cyc}}(N) \times Q(N)$ is a bijection, which will prove that the fibers of \vec{p} are equinumerous.

It suffices to construct the inverse map. Given $\vec{w} \in P_{\text{cyc}}(N)$ and $\vec{\xi} \in Q(N)$, we use induction to construct the sequence a_1, \dots, a_{3N} that determines the corresponding element $c \in C(N)$. For the base case: If $\xi_N = \{1, x, y\}$, then $w_1 \cdots w_N$ determines whether we set $(a_N, a_{2N}) = (x, y)$ or $(a_N, a_{2N}) = (y, x)$. In general, once a_i, a_{i+N}, a_{i+2N} are fixed, the permutation w_i determines how the elements of ξ_{i-1} are assigned to $a_{i-1}, a_{i-1+N}, a_{i-1+2N}$. \square

Lemma 11. *For all $c \in C(N)$, we have*

$$h(\sigma_c) \geq \frac{1}{N} h(\sigma_{\vec{p}(c)}).$$

Proof. We use the setup and language of Sec. 2. Let $I \subseteq D$ be a finite set of $3N$ points, and order them from 1 to $3N$, so that we can identify Br_{3N} with Br_I . If c is the $3N$ -cycle $1 = a_0 \mapsto a_1 \mapsto \cdots \mapsto a_{3N} = 1$, then c^N contains the three-cycle $1 \mapsto a_N \mapsto a_{2N} \mapsto 1$. Thus $I' := \{1, a_N, a_{2N}\}$ is stable under σ_c^N . In fact, if we identify $Br_{I'}$ with Br_3 via the order-preserving bijection $\{1, 2, 3\} \xrightarrow{\sim} \{1, a_N, a_{2N}\}$, then $\sigma_c^N|_{I'}$ can be identified with $\sigma_{\vec{p}(c)}$ up to conjugacy. Now $Nh(\sigma_c) \geq h(\sigma_c^N) \geq h(\sigma_c^N|_{I'}) = h(\sigma_{\vec{p}(c)})$ by Lemma 3 and display (2.1). \square

3.5.

Now we combine Lemmas 8–11.

Lemma 12. *For any $N \geq 7$, we have*

$$\frac{|\{c \in C(N) \mid h(\sigma_c) > 0\}|}{|C(N)|} \geq 6^{-6}.$$

Proof. We have

$$\begin{aligned} |\{c \in C(N) \mid h(\sigma_c) > 0\}| &\geq |\{c \in C(N) \mid h(\sigma_{\vec{p}(c)}) > 0\}| \quad \text{by Lemma 11} \\ &= \frac{|\{\vec{w} \in P_{\text{cyc}}(N) \mid h(\sigma_{\vec{w}}) > 0\}| |C(N)|}{|P_{\text{cyc}}(N)|} \quad \text{by Lemma 10.} \end{aligned}$$

But $|\{\vec{w} \in P_{\text{cyc}}(N) \mid h(\sigma_{\vec{w}}) > 0\}| \geq 6^{-6} |P_{\text{cyc}}(N)|$ by Lemmas 8 and 9. \square

For any n and $w \in S_n$, we will call a cycle of w *relevant* if its length is divisible by 3 and at least $3 \cdot 7 = 21$, and *irrelevant* otherwise. We apply the same name to the corresponding orbit, *i.e.*, to the underlying unordered subset of $\{1, \dots, n\}$. We define an equivalence relation on S_n as follows: $w \approx w'$ means that w and w' have the same irrelevant cycles and the same relevant orbits.

We arrive at the following result, reducing the proof of Theorem 1 to exhibiting sufficiently many elements of S_n with sufficiently many relevant cycles.

Proposition 13. *Let $D \subseteq S_n$ be an equivalence class for the relation \approx in which the elements each have r relevant cycles. Then*

$$\frac{|\{w \in D \mid h(\sigma_w) > 0\}|}{|D|} \geq 1 - (1 - 6^{-6})^r.$$

Proof. Let \mathcal{O} be the collection of relevant orbits arising from elements of D . By restricting any element of D to its behavior on these orbits, we get a bijection

$$D \xrightarrow{\sim} \{(c_O)_{O \in \mathcal{O}} \mid c_O \text{ is an } |O|\text{-cycle with underlying orbit } O\}.$$

Moreover, if $w \mapsto (c_O)_O$, then $h(\sigma_w) \geq \max_O h(\sigma_{c_O(w)})$ by (2.1). So it remains to bound the proportion of tuples $(c_O)_O$ that have $h(\sigma_{c_O}) = 0$ for all O .

For any $O \in \mathcal{O}$, we must have $|O| = 3N$ for some $N \geq 7$. Fix an ordering of the elements of O , so that we can identify the possibilities for c_O with elements $c \in C(N)$. Then, by Lemmas 3(1) and 12, the proportion of possibilities for c_O with $h(\sigma_{c_O}) = 0$ is at most $1 - 6^{-6}$. Applying this argument to each of the r relevant orbits, we see that the proportion of tuples $(c_O)_O$ that have $h(\sigma_{c_O}) = 0$ for all O is at most $(1 - 6^{-6})^r$. \square

4. Permutations with Many Long Cycles

4.1.

For any integers $n, \ell, r > 0$, let

$$S_n(\ell, r) = \left\{ w \in S_n \mid \begin{array}{l} w \text{ has at least } r \text{ cycles of length divisible by } 3 \\ \text{and length at least } 3\ell \end{array} \right\}.$$

The goal of this section is to prove the following proposition.

Proposition 14. *For fixed $\ell, r > 0$ and $n \gg_{\ell, r} 0$, we have*

$$\frac{|S_n(\ell, r)|}{|S_n|} = 1 - o\left(r \left(\frac{n}{3\ell \cdot 2^r}\right)^{-\frac{1}{6\ell \cdot 2^r}}\right),$$

where the little- o constant is independent of ℓ, r .

Proof. By Lemma 16, we know that the proportion of elements of S_n that have no cycles of length $j \cdot 3 \cdot 2^i$ with j odd is $O((n/(3 \cdot 2^i))^{-\frac{1}{6 \cdot 2^i}})$.

Let $i_0 = \lceil \log_2(\ell) \rceil$. Then the proportion of elements that have at least one cycle of length $j \cdot 3 \cdot 2^i$ with j odd, for each i such that $i_0 + 1 \leq i \leq i_0 + r$, is

$$1 - o\left(r \left(\frac{n}{3\ell \cdot 2^r}\right)^{-\frac{1}{6\ell \cdot 2^r}}\right) \quad \text{for } n \gg_{\ell, r} 0.$$

In any such element, these r cycles must be pairwise distinct because their lengths are. Moreover, their lengths are divisible by 3 and at least 3ℓ . \square

4.2.

For any integers $n, k > 0$, let

$$X_{n,k} = \{w \in S_n \mid w \text{ has no cycles of length } k, 3k, 5k, \dots\}.$$

Lemma 15. *We have*

$$\sum_{n \geq 0} |X_{n,k}| \frac{x^n}{n!} = (1-x)^{-1} (1-x^k)^{\frac{1}{k}} (1-x^{2k})^{-\frac{1}{2k}}. \quad (4.1)$$

Proof. For each integer $m > 0$, fix an indeterminate t_m , and for each $w \in S_n$, let $\lambda_m(w)$ be the number of m -cycles in w . Display (5.30) of [12] says that

$$\sum_{n \geq 0} \sum_{w \in S_n} t_1^{\lambda_1(w)} \dots t_n^{\lambda_n(w)} \frac{x^n}{n!} = \exp \left(\sum_{m \geq 1} t_m \frac{x^m}{m} \right),$$

where $\exp(X) = \sum_{n \geq 0} \frac{X^n}{n!}$ as a formal series. Now set $t_m = 0$ whenever $m = jk$ with j odd and $t_m = 1$ for all other m . The left-hand side simplifies to that of (4.1), while the right-hand side simplifies to

$$\exp \left(\sum_{m \geq 1} \frac{x^m}{m} - \sum_{j \geq 1} \frac{x^{jk}}{jk} + \sum_{i \geq 1} \frac{x^{2ik}}{2ik} \right).$$

To finish, use $\exp(\sum_{m \geq 1} \frac{X^m}{m}) = (1-X)^{-1}$. □

Lemma 16. *For fixed $k > 0$ and $n \gg_k 0$, we have*

$$\frac{|X_{n,k}|}{|S_n|} = O \left(\left(\frac{n}{k} \right)^{-\frac{1}{2k}} \right),$$

where the big- O constant is independent of k .

Proof. We will study the right-hand side of (4.1). First, we expand

$$(1-x)^{-1} (1-x^k)^{\frac{1}{k}} = \frac{1-x^k}{1-x} \sum_{i \geq 0} \binom{\frac{1}{k}-1}{i} (-1)^i x^{ik}, \quad (4.2)$$

$$(1-x^{2k})^{-\frac{1}{2k}} = \sum_{i \geq 0} \binom{-\frac{1}{2k}}{i} (-1)^i x^{2ik}. \quad (4.3)$$

The right-hand side of (4.2) simplifies to a power series with nonnegative coefficients. As for (4.3): Observe that for any $\alpha \in (0, 1]$ and integer $i \geq 0$, we have

$$\begin{aligned} \binom{-\alpha}{i} (-1)^i &= \frac{\alpha(\alpha+1) \cdots (\alpha+i-1)}{i!} \leq 2 \cdot \frac{\alpha(\alpha+1) \cdots (\alpha+2i-1)}{(2i)!} \\ &= 2 \binom{-\alpha}{2i}, \end{aligned}$$

where we handle $i = 0$ separately to prove the inequality. Therefore, for each $i \geq 0$, the coefficient of x^{2ik} on the right-hand side of (4.3) is nonnegative and bounded above by $2\binom{-1/(2k)}{2i}$. That is, the coefficients in the series expansion of $(1 - x^{2k})^{-\frac{1}{2k}}$ are nonnegative and bounded above by the respective coefficients in the series expansion of $2(1 - x^k)^{-\frac{1}{2k}}$.

Altogether, by Lemma 15, $|X_{n,k}|/|S_n|$ is bounded above by the coefficient of x^n in the series expansion

$$\begin{aligned} 2(1-x)^{-1}(1-x^k)^{\frac{1}{k}}(1-x^k)^{-\frac{1}{2k}} &= 2(1-x)^{-1}(1-x^k)^{\frac{1}{2k}} \\ &= 2\left(\frac{1-x^k}{1-x}\right) \sum_{i \geq 0} \binom{\frac{1}{2k}-1}{i} (-1)^i x^{ik} \\ &= 2 \sum_{n \geq 0} \binom{\frac{1}{2k}-1}{\lfloor \frac{n}{k} \rfloor} (-1)^{\lfloor \frac{n}{k} \rfloor} x^n. \end{aligned}$$

Finally, for any $\alpha \in \mathbf{R} \setminus \mathbf{Z}_{\geq 0}$, it is known [9, Theorem 2] that

$$\left| \binom{\alpha}{m} \right| \sim \frac{1}{|\Gamma(-\alpha)m^{1+\alpha}|} \quad \text{as } m \rightarrow \infty.$$

Note that taking $\alpha = \frac{1}{2k} - 1$ gives $\frac{1}{2} \leq -\alpha \leq 1$. On this interval, $\Gamma(-\alpha) \geq 1$, so we're done. \square

5. Conclusion

What follows is a quantitative refinement of Theorem 1.

Theorem 17. *For any $0 < \epsilon < 1$ and integer $r > \log_{1-6^{-6}}(\epsilon)$, we can pick N large enough that for all $n \geq N$, we have*

$$(1 - (1 - 6^{-6})^r) \cdot \frac{|S_n(7, r)|}{|S_n|} > 1 - \epsilon,$$

in the notation of Sec. 4. For such n , the proportion of simple braids on n strands that have positive topological entropy is greater than $1 - \epsilon$.

Proof. Proposition 14 implies the first claim. Proposition 13 gives

$$\frac{|\{w \in S_n(7, r) \mid h(\sigma_w) > 0\}|}{|S_n(7, r)|} \geq 1 - (1 - 6^{-6})^r,$$


from which

$$\frac{|\{w \in S_n \mid h(\sigma_w) > 0\}|}{|S_n|} \geq \frac{|S_n(7, r)|}{|S_n|} \cdot \frac{|\{w \in S_n(7, r) \mid h(\sigma_w) > 0\}|}{|S_n(7, r)|} > 1 - \epsilon,$$

proving the second claim. \square

ORCID

Luke Robitaille  <https://orcid.org/0000-0003-3281-6875>

Minh-Tâm Quang Trinh  <https://orcid.org/0000-0002-3013-8911>

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