Character Formulas from Lusztig Varieties and Affine Springer Fibers

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Yale University

This talk is about...

1. Braids
2. Lusztig Varieties
3. Springer Fibers
4. Affine Springer Fibers

1 Braids The braid group $Br_n =$

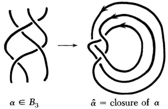
$$\left\langle \sigma_1, \ldots, \sigma_{n-1}, \;\middle|\; \begin{array}{l} \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}, \\ \sigma_i\sigma_j = \sigma_j\sigma_i \text{ when } |i-j| > 1 \end{array} \right\rangle$$

appears in knot theory and representation theory.

σ_i

A *link* is a collection of circles (tamely) embedded in $\mathbf{R}^3$. Knot theory is about isotopy invariants of links.

(Alexander) Every link is the *closure* of some braid.



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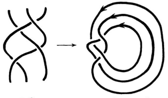

Let $G = \mathrm{SL}_n$ and B its upper-triangular subgroup.

Let $R(q) = \{\mathbf{Z}\text{-valued functions on } G(\mathbf{F}_q)/B(\mathbf{F}_q)\}$.

(Iwahori) There is a surjective homomorphism

$$\mathbf{Z}Br_n \twoheadrightarrow H_n(q) := \mathrm{End}_{G(\mathbf{F}_q)}(R(q)).$$

To describe it, recall the Bruhat decomposition

$$G = \bigsqcup_{w \in S_n} B\dot{w}B.$$

Let $h_w \curvearrowright R(q)$ be the *Hecke operator*

$$h_w(\mathbf{1}_{xB(\mathbf{F}_q)}) = \sum_{y^{-1}x \in B\dot{w}B} \mathbf{1}_{yB(\mathbf{F}_q)}.$$

Iwahori sends $\sigma_i \mapsto h_{s_i}$, where $s_i = (i, i+1)$.

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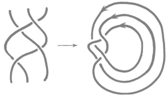

$\alpha \in B_3$ $\quad$ $\hat{\alpha}$ = closure of α

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The *Hecke algebra*

$$H_n := \frac{\mathbf{Z}[\mathsf{q}]Br_n}{\langle \sigma_i^2 - (\mathsf{q}-1)\sigma_i - \mathsf{q} \rangle}$$

is a generic version: $H_n|_{\mathsf{q}\to q} = H_n(q)$.

Jones–Ocneanu used traces $H_n \xrightarrow{\mu_n} \mathbf{Q}(\mathsf{q})[a^{\pm 1}]$ to construct the HOMFLYPT *link invariant*.

If $L = \hat{\beta}$ for some n and $\beta \in Br_n$, then

$$\text{HOMFLYPT}(\hat{\beta}) = (-a)^{e(\beta)}\mu_n(\beta),$$

where $e : Br_n \to \mathbf{Z}$ is the *writhe* map $\sigma_i \mapsto 1$.

Surprisingly, special values of HOMFLYPT are famous polynomials in combinatorics: q-Catalan numbers, q-Kirkman numbers, *etc.*

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2 Lusztig Varieties

We can geometrize Iwahori.

Fix a *positive* braid $\beta = \sigma_{i_1} \cdots \sigma_{i_\ell}$.

(Deligne) The variety

$$O(\beta) = \left\{ (g_0 B, g_1 B, \ldots, g_\ell B) \;\middle|\; \begin{array}{c} g_{j-1}^{-1} g_j \in B \dot{s}_{i_j} B \\ \text{for } j = 1, \ldots, \ell \end{array} \right\}$$

only depends on β, not $(i_1, i_2, \ldots, i_\ell)$, up to isomorphisms that fix $g_0 B$ and $g_\ell B$.

In fact, if we fix $\bar{g}_0, \bar{g}_\ell$ such that $\bar{g}_0^{-1} \bar{g}_\ell \in B \dot{w} B$, then

$$\left| \left\{ \vec{g}B \in O(\beta)(\mathbf{F}_q) \;\middle|\; \begin{array}{l} g_0 B = \bar{g}_0 B, \\ g_\ell B = \bar{g}_\ell B \end{array} \right\} \right|$$

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For any $x \in G(\mathbf{F}_q)$, form the *braid Lusztig variety*

$$\mathcal{B}(\beta)_x = \{\vec{g}B \in O(\beta) \mid g_\ell B = x g_0 B\}.$$

(Shende–Treumann–Zaslow) Up to a monomial in q,

$$\frac{|\mathcal{B}(\beta)_1(\mathbf{F}_q)|}{|\mathrm{PGL}_n(\mathbf{F}_q)|}$$

is the "highest" a-degree of $\textsc{homflypt}(\hat{\beta})$ at $\mathsf{q} \to q$.

Example Let $n = 2$ and $\beta = \sigma_1^3 \in Br_2$.

$$O(\beta) \simeq \{\vec{g} \in (\mathbf{P}^1)^4 \mid g_0 \neq g_1 \neq g_2 \neq g_3\},$$
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PGL_2 acts simply transitively on the latter.

Indeed, $\textsc{homflypt}(\widehat{\sigma_1^3}) = a^2(\mathsf{q} + \mathsf{q}^{-1}) - a^4$.

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3 Springer Fibers

How to access other a-degrees?

One way uses Springer theory. Observe that

$$\mathcal{B}_x := \mathcal{B}(1)_x = \{gB \mid gB = xgB\}$$

is the usual Springer fiber over z, whose cohomology defines a character of S_n: namely,

$$Q_x(w) := \sum_i q^i \mathrm{tr}(w \mid \mathrm{H}^{2i}(\mathcal{B}_x)).$$

Most interesting over the unipotent variety $\mathcal{U} \subseteq G$.

Thm 1 (T) Let

$$Q_\beta(w) = \frac{1}{|\mathrm{PGL}_n(\mathbf{F}_q)|} \sum_{u \in \mathcal{U}(\mathbf{F}_q)} |\mathcal{B}(\beta)_u(\mathbf{F}_q)| Q_u(w).$$

Then $(\chi_{(n-k,1,\ldots,1)}, Q_\beta)_{S_n}$ sees the kth a-degree.

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$$|\mathcal{B}(\beta)_u| = \begin{cases} |\mathrm{PGL}_2| & u = 1, \\ q^3 & u \neq 1, \end{cases}$$

$$Q_u = \begin{cases} 1 + q\,\mathsf{sgn} & u = 1, \\ 1 & u \neq 1. \end{cases}$$

Moreover, $\mathrm{PGL}_2(\mathbf{F}_q) \curvearrowright \mathcal{U}(\mathbf{F}_q) - \{1\}$ transitively, with stabilizer of size q.

$$\begin{aligned} Q_\beta &= \frac{|\mathrm{PGL}_2|}{|\mathrm{PGL}_2|} \cdot (1 + q\,\mathsf{sgn}) + \frac{q^3}{q} \cdot 1 \\ &= 1 + q^2 + q\,\mathsf{sgn}. \end{aligned}$$

Thm 2 (T) The cohomology of $\mathcal{U}(\beta) \times_{\mathcal{U}} \mathcal{U}(1)$ sees finer invariants of $\hat{\beta}$, where

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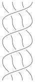
The *full twist* $\pi = (\sigma_1 \cdots \sigma_{n-1})^n$ generates $Z(Br_n)$.



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Then up to a monomial, $Q_\beta(w)$ equals

$$\frac{\mathsf{sgn}(w)}{\det(1 - qw \mid \mathfrak{h})} \sum_{\lambda \vdash n} q^{c(\lambda)d/m} D_\lambda(e^{2\pi i d/m}) \chi_\lambda(w)$$

where:

- $\mathfrak{h}$ is the *reflection representation*.
- $c(\lambda)$ is the sum of *contents* of λ.
- $D_\lambda(t) = K_{\lambda,(1^n)}(t)$ is the *fake degree* of λ.

Proof uses the character theory of S_n and H_n.

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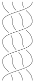

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Thm 3 generalizes to any reductive G.

Replace S_n with the Weyl group W.

Replace c with $c(\chi) = \sum_{t \text{ refl.}} \frac{\chi(t)}{\chi(1)}$ and fake degrees with *generic degrees*:

$$D_\chi(t) \in \mathbf{Q}[t] \quad \text{such that } R(q) = \bigoplus_{\chi \in \mathrm{Irr}(W)} \chi_q^{\oplus D_\chi(q)}.$$

When $m = n$ and $\gcd(d, n) = 1$, the formula simplifies:

$$\left.(\text{monomial}) \cdot \frac{\det(1 - q^d w \mid \mathfrak{h})}{\det(1 - qw \mid \mathfrak{h})}\right\} =: \Pi_q^{(d)}.$$

$\Pi_q^{(d)}$ is the character of a *rational parking space*.

$(\mathsf{triv}, \Pi_q^{(d)})_W$ is a *rational q-Catalan number*.

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4 Affine Springer Fibers

Now work over $\mathbf{C}$.

Rational parking spaces form modules over *rational Cherednik algebras* = *rational DAHAs*:

$$\mathfrak{H}_W = \frac{\mathbf{C}W \ltimes (\mathrm{Sym}(\mathfrak{h}) \otimes \mathrm{Sym}(\mathfrak{h}^\vee))}{\langle \text{ relations } \rangle}.$$

finite Springer	affine Springer
G	$G(\!(z)\!)$
G/B	$G(\!(z)\!)/I$
W	$\widetilde{W} = W \ltimes X^\vee$
$\mathbf{C}W \ltimes \mathrm{Sym}(\mathfrak{h})$	$\mathbf{C}\widetilde{W} \ltimes \mathrm{Sym}(\mathfrak{h})$ or $\mathfrak{H}_W$

Above:

- $G(\!(z)\!)$ is the loop group $G(\!(z)\!)(R) := G(R(\!(z)\!))$.
- I is the preimage of B in $G[\![z]\!]$.
- $X^\vee$ is the cocharacter lattice of B.

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$$\mathfrak{H}_W = \frac{\mathbf{C}W \ltimes (\mathrm{Sym}(\mathfrak{h}) \otimes \mathrm{Sym}(\mathfrak{h}^\vee))}{\langle \text{relations} \rangle}.$$

finite Springer	affine Springer
G	$G(\!(z)\!)$
G/B	$G(\!(z)\!)/I$
W	$\widetilde{W} = W \ltimes X^\vee$
$\mathbf{C}W \ltimes \mathrm{Sym}(\mathfrak{h})$	$\mathbf{C}\widetilde{W} \ltimes \mathrm{Sym}(\mathfrak{h})$ or $\mathfrak{H}_W$

Above:

- $G(\!(z)\!)$ is the loop group $G(\!(z)\!)(R) := G(R(\!(z)\!))$.
- I is the preimage of B in $G[\![z]\!]$.
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4 Affine Springer Fibers

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Henceforth, we consider Springer fibers over the Lie algebras $\mathfrak{g}$ and $\mathfrak{g}(\!(z)\!)$, not the groups.

$$\begin{aligned} x : &\quad \mathcal{B}_x = \{gB \in G/B \mid g^{-1}xg \in \mathfrak{b}\},\\ \gamma = \gamma(z) : &\quad \mathcal{B}_\gamma^{\mathrm{aff}} = \{gI \in G(\!(z)\!)/I \mid g^{-1}\gamma g \in \mathfrak{I}\}. \end{aligned}$$

The table hides some key differences:

In the finite case, $\mathcal{B}_x$ is interesting for x nilpotent, and eas(ier) for x regular semisimple.

In the affine case, $\mathcal{B}_\gamma^{\mathrm{aff}}$ is *terribly infinite* for $\gamma = \gamma(z)$ nilpotent, but interesting for $\gamma(z)$ regular semisimple.

Example If $G = \mathrm{SL}_2$ and $\gamma(z) = \begin{pmatrix} 0 & 1 \\ z^3 & 0 \end{pmatrix}$, then $\mathcal{B}_\gamma^{\mathrm{aff}} \simeq \mathbf{P}^1 \sqcup_{\mathrm{pt}} \mathbf{P}^1$.

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Fixing $\nu = d/m > 0$ in lowest terms, $\mathbf{C}^\times \curvearrowright \mathfrak{g}(\!(z)\!)$:

$$c \cdot_\nu \gamma(z) = c^{2d\rho^\vee} \gamma(c^{2m} z) c^{-2d\rho^\vee},$$

where $2\rho^\vee = \sum_{\alpha \in \Phi^+} \alpha^\vee$.

Let $\mathfrak{g}(\!(z)\!)_{\nu,k}$ be the weight-$2k$ eigenspace.

Lemma If γ is an eigenvector for $\cdot_\nu$, then the induced action on $G(\!(z)\!)/I$ fixes $\mathcal{B}_\gamma^{\mathrm{aff}}$.

Lemma $\mathfrak{g}(\!(z)\!)_{\nu,0}$ is the Lie algebra of a connected reductive group L_ν. Moreover,

$$(G(\!(z)\!)/I)^{\mathbf{C}^\times} = \bigsqcup_{w \in W_\nu \backslash \widetilde{W}} L_\nu w I/I,$$

where W_ν is the Weyl group of L_ν.

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Let $\mathfrak{g}((z))_{\nu,k}$ be the weight-$2k$ eigenspace.

Lemma If γ is an eigenvector for $\cdot_\nu$, then the induced action on $G((z))/I$ fixes $\mathcal{B}_\gamma^{\mathrm{aff}}$.

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$$(G((z))/I)^{\mathbf{C}^\times} = \coprod_{w \in W_\nu \backslash \widetilde{W}} L_\nu w I/I,$$

where W_ν is the Weyl group of L_ν.

Henceforth, $\gamma \in \mathfrak{g}((z))_{\nu,d}$.

By Springer, $\widetilde{W} \curvearrowright \mathrm{H}_c^*(\mathcal{B}_\gamma^{\mathrm{aff}}), \mathrm{H}_{c,\mathbf{C}^\times}^*(\mathcal{B}_\gamma^{\mathrm{aff}})$.

(Sommers) If m is the Coxeter number of W, then:

- L_ν is a torus, so $(\mathcal{B}_\gamma^{\mathrm{aff}})^{\mathbf{C}^\times} \hookrightarrow \widetilde{W}$.
- Writing $\mathrm{H}_{\mathbf{C}^\times}^*(\mathrm{pt}) = \mathbf{C}[\epsilon]$, we have

$$\mathrm{H}_c^*(\mathcal{B}_\gamma^{\mathrm{aff}}) \simeq \mathrm{H}_{c,\mathbf{C}^\times}^*(\mathcal{B}_\gamma^{\mathrm{aff}})|_{\epsilon \to 1} \simeq \mathrm{H}^0((\mathcal{B}_\gamma^{\mathrm{aff}})^{\mathbf{C}^\times}).$$

- For $w \in W$, we have

$$\mathrm{tr}(w \mid \mathrm{H}_c^*(\mathcal{B}_\gamma^{\mathrm{aff}})) = \boxed{\lim_{q \to 1} \frac{\det(1 - q^d w \mid \mathfrak{h})}{\det(1 - qw \mid \mathfrak{h})}}.$$

Example In the previous SL_2 example, $\gamma \in \mathfrak{g}((z))_{\nu,3}$.

Recall $\mathcal{B}_\gamma^{\mathrm{aff}} = \mathbf{P}^1 \sqcup_{\mathrm{pt}} \mathbf{P}^1$. It turns out $|(\mathcal{B}_\gamma^{\mathrm{aff}})^{\mathbf{C}^\times}| = 3$.

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$$(\mathcal{B}^{\mathrm{aff}}_\gamma)^{\mathbf{C}^\times} = \coprod_{w \in W_\nu \backslash \widetilde{W}} \mathrm{Hess}_{\gamma,w},$$

a disjoint union of *partial Hessenberg varieties*

$$\mathrm{Hess}_{\gamma,w} = \{gP_{\nu,w} \in L_\nu / P_{\nu,w} \mid g^{-1}\gamma g \in \mathfrak{P}_{\nu,w}\},$$

where $P_{\nu,w} := L_\nu \cap \dot{w} I \dot{w}^{-1}$ and $\mathfrak{P}_{\nu,w} = \mathrm{Lie}(P_{\nu,w})$.

They are smooth. They can be empty.

If $\mathrm{Hess}_{\gamma,w} \neq \emptyset$, then its codim in $L_\nu/P_{\nu,w}$ is

$$\left| \left\{ \begin{array}{l|l} \text{hyperplanes } H & H(\xi) = \langle \alpha, \xi \rangle + k, \\ \text{in } X^\vee \otimes \mathbf{R} \text{ between} & \langle \alpha, \nu\rho^\vee \rangle = \nu, \\ \nu\rho^\vee \text{ and } w \cdot \frac{1}{n}\rho^\vee & \alpha \in \Phi,\ k \in \mathbf{Z} \end{array} \right\} \right|.$$

Proof uses *Moy–Prasad theory*.

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Conj (T) For general ν, the representation

$$W \curvearrowright \mathrm{H}^*_{c,\mathbf{C}^\times}(\mathcal{B}^{\mathrm{aff}}_\gamma)|_{\epsilon \to 1}$$

contains a summand whose character is the $q \to 1$ limit of our earlier formula:

$$\frac{\mathsf{sgn}(w)}{\det(1 - qw \mid \mathfrak{h})} \sum_{\chi \in \mathrm{Irr}(W)} q^{c(\chi)\nu} D_\chi(e^{2\pi i \nu})\chi(w)\,.$$

Dream For certain choices $\gamma \leftrightsquigarrow \beta$,

$$\mathcal{B}^{\mathrm{aff}}_\gamma \quad \text{and} \quad [(\mathcal{U}(\beta) \times_{\mathcal{U}} \tilde{\mathcal{U}})/G]$$

have the "same" Springer theory.

Thank you for listening.