



Zeta Functions as Knot Invariants

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O. Kivinen, M. Q. Trinh. The Hilb-vs-Quot Conjecture.
J. reine angew. Math. (Crelle), (2025). 44 pp.

1 The Riemann Hypothesis

(Euler ~1730s) Proof of the infinitude of primes:

If there were finitely many, then we'd have

$$\prod_{\text{prime } p > 0} \left(1 + \frac{1}{p} + \frac{1}{p^2} + \cdots\right) = \sum_{n=1}^{\infty} \frac{1}{n}.$$

But the series diverges—a contradiction.

He also implicitly studied the *zeta function*

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

For $s > 1$, we have $\zeta(s) = \prod_p \left(1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \cdots\right).$

What if we allow s to be complex?

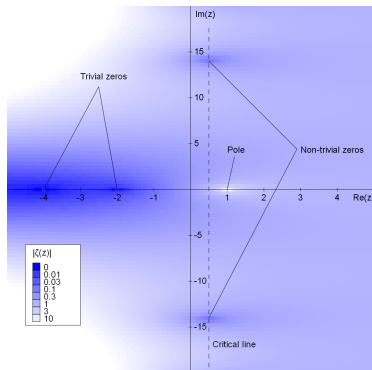
(Riemann 1859) A unique \mathbf{C} -valued function ζ that is

- *holomorphic* (complex-differentiable) when $s \neq 1$.
- given by $\sum_{n=1}^{\infty} \frac{1}{n^s}$ when $\operatorname{Re}(s) > 1$.

He checked that $\zeta(n) = 0$ for $n = -2, -4, -6, \dots$ by relating these zeros to poles of the *gamma function*.

He speculated from examples that all other zeros of ζ live on the *critical line* $\operatorname{Re}(s) = \frac{1}{2}$.

Location of zeros \leftrightarrow distribution of prime numbers.



[Wikipedia](#)

(Hardy 1914) Infinitely many zeros on the line.

(Pratt–Robles–Zaharescu–Zeindler 2020) Among zeros s with $0 < \operatorname{Re}(s) < 1$, over five-twelfths on the line.

(Dedekind ~1860s) Generalize the formula

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

by replacing \mathbf{Z} with other *rings* R .

Thus R is a set with operations $+$ and \cdot resembling addition and multiplication.

$$\zeta_R(s) = \sum_{\substack{\text{ideals } I \subseteq R \\ |R/I| < \infty}} \frac{1}{|R/I|^s}$$

An *ideal* I is the collection of all finite linear combos $c_1 \cdot x_1 + \cdots + c_k \cdot x_k$ for some fixed $x_1, x_2, \dots \in R$.

The *quotient* R/I is the set of translates $y + I \subseteq R$.

Note For ζ_R to make sense, the number of I such that $|R/I| = n$ must be finite for each $n > 0$.

Ex Every ideal of \mathbf{Z} takes the form

$$n\mathbf{Z} = \{\text{multiples of } n\} \quad \text{for some integer } n \geq 0.$$

For instance, the ideal generated by 30 and 2025 is

$$\{c_1 \cdot 30 + c_2 \cdot 2025 \mid c_1, c_2 \in \mathbf{Z}\} = 15\mathbf{Z}.$$

We see that

$$\zeta_{\mathbf{Z}}(s) = \sum_{\substack{n\mathbf{Z} \subseteq \mathbf{Z} \\ n > 0}} \frac{1}{|\mathbf{Z}/n\mathbf{Z}|^s} = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

In other rings, ideals may not simplify so nicely.

Why care?

(Hilbert–Polyá ~1910s) The values e^{it} such that

$$\zeta(\tfrac{1}{2} + it) = 0 \quad \text{and} \quad 0 < \operatorname{Re}(\tfrac{1}{2} + it) < 1$$

behave like the eigenvalues of a generic unitary matrix.

$$\text{RH} \iff t \text{ always real}$$

$$\iff e^{it} \text{ always on the unit circle.}$$

(Weil ~1940s) There is a class of rings R ,
coming from *algebraic geometry* over $\mathbf{F}_p := \mathbf{Z}/p\mathbf{Z}$,
where analogous facts for ζ_R might be provable.

(Grothendieck–Deligne ~1960s–70s) Yes.

2 Weil's Rosetta Stone Algebraic geometry studies
varieties: shapes cut out by polynomial equations.

For simplicity, we'll stick to (*affine*) *hypersurfaces*

$$V_f = \{\vec{a} = (a_0, a_1, \dots, a_d) \in \bar{\mathbf{F}}_p^{d+1} \mid f(\vec{a}) = 0\},$$

cut out by a single polynomial $f \in \mathbf{F}_p[x_0, x_1, \dots, x_d]$.

V_f is *smooth* at a point \vec{a} when $\frac{\partial f}{\partial x_i}(\vec{a}) \neq 0$ for some i .
Else, *singular* at \vec{a} .

Ex 1-dim. hypersurfaces are plane curves. Consider

$$f(x, y) = y^2 - x^3 - c \quad \text{for constant } c \in \mathbf{F}_p.$$

For which c is V_f smooth everywhere?

The *ring of polynomial functions* on V_f is

$$R_f := \mathbf{F}_p[x_0, x_1, \dots, x_d]/(f).$$

In a letter to his sister, Weil described a dictionary:

Z	R_f	V_f
$n\mathbf{Z}$	ideals	(closed) subvarieties
$p\mathbf{Z}$	maximal ideals	(closed) points

The first and last columns: a Rosetta stone between number theory and algebraic geometry.

Conj (Weil 1948) Assume V_f is smooth everywhere.

Then zeros of $\zeta_{R_f}(s)$ have $\operatorname{Re}(s) \in \{\frac{1}{2}, \frac{3}{2}, \dots, \frac{2d-1}{2}\}$.

Thm (Weil) True for $f = c_0 x_0^{n_0} + \dots + c_d x_d^{n_d} - c$.

Set $\zeta_f(s) := \zeta_{R_f}(s)$ for convenience.

(Grothendieck ~1964) $\zeta_f(s)$ is a rational function in

$$\mathbf{q} := p^{-s}.$$

In fact: polynomials $\phi_0, \phi_1, \dots, \phi_{2d}$ such that

$$\zeta_f(s) = \frac{\phi_1(\mathbf{q}) \cdots \phi_{2d-1}(\mathbf{q})}{\phi_0(\mathbf{q}) \cdot \phi_2(\mathbf{q}) \cdots \phi_{2d}(\mathbf{q})}.$$

ϕ_k is the charpoly of a certain operator F on a certain vector space $H^k(V_f)$.

Reduces Weil's conjecture to a "Hilbert–Polyá" claim:

Conj The F -eigenvalues on $H^k(V_f)$ all have complex absolute value* $p^{k/2}$.

* With respect to any embedding into \mathbf{C} .

(Deligne 1974) True for all (smooth) f .

In fact, Weil conjectured—and Deligne proved—results for all smooth varieties, not just hypersurfaces.

Ex Taking $d = 1$ and $f(x, y) = y^2 - x^3 - c$:

$$\phi_0(t) = 1 - p\mathbf{q}$$

$$\phi_1(t) = 1 - \textcolor{red}{a}_p \mathbf{q} + p\mathbf{q}^2 \quad \text{for some integer } \textcolor{red}{a}_p,$$

giving $\zeta_f(s) = \frac{1 - \textcolor{red}{a}_p p^{-s} + p^{1-2s}}{1 - p^{1-s}}$. It turns out:

- $|a_p| \leq 2p^{1/2}$.
- So the two roots of $\phi_1(\mathbf{q})$ satisfy $|\mathbf{q}| = p^{-1/2}$.
- So the zeros of $\zeta_f(s)$ satisfy $\operatorname{Re}(s) = \frac{1}{2}$.

What if V_f has singularities?

Simplest case: V_f has a unique singularity at the origin $(0, \dots, 0) \in \bar{\mathbf{F}}_p^N$. It turns out that here,

$$\zeta_f(s) = \zeta_f^\circ(s) \cdot \hat{\zeta}_f(s),$$

where:

- ζ_f° satisfies Weil's Riemann Hypothesis.
- $\hat{\zeta}_f$ is the analogue of ζ_f with the power-series ring

$$\hat{R}_f := \mathbf{F}_p[[x_0, x_1, \dots, x_d]]/(f)$$

in place of R_f .

Does $\hat{\zeta}_f(s) = \sum_{\substack{I \subseteq \hat{R}_f \\ |\hat{R}_f/I| < \infty}} \frac{1}{|\hat{R}_f/I|^s}$ satisfy a RH?

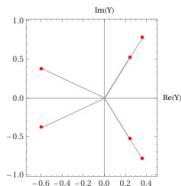
Ex If $f = y^2 - x^3$, then $\hat{\zeta}_f(s) = \frac{1 + pq^2}{1 - q}$.

Roots are $q = \pm p^{-1/2}$.

Ex If $f = y^3 - x^4$, then

$$\hat{\zeta}_f(s) = \frac{1 + pq^2 + p^2q^3 + p^2q^4 + p^3q^6}{1 - q}.$$

Two roots on the circle $|q| = p^{-1/2}$. The rest not on any circle $|q| = p^{-k/2}$.



WolframAlpha

3 From Curves to Knots Fix $f \in \mathbf{Z}[x, y]$ cutting out a plane curve through the origin.

It turns out we have $P_f(t, q) \in \mathbf{Z}\left[t, q, \frac{1}{1-q}\right]$ such that

$$\hat{\zeta}_{f \bmod p}(s) = \frac{P_f(p, p^{-s})}{1 - p^{-s}} \quad \text{for almost all } p.$$

The polynomials P_f are remarkably ubiquitous.

(Gorsky–Mazin 2013)

If $f = y^n - x^{n+1}$, then $P_f(1, 1) = \frac{(2n)!}{(n+1)!n!}$, the n th Catalan number.

For instance, if $f = y^3 - x^4$, then

$$P_f(t, q) = 1 + tq^2 + t^2q^3 + t^2q^4 + t^3q^6,$$

$$P_f(1, 1) = 5.$$

The P_f also arise from *knot/link invariants*.

A *knot* is a (tame) embedding of S^1 into \mathbf{R}^3 or S^3 .



A *link* is similar, but can have multiple circles.



Two links are *isotopic* when they fit into a continuous family of embeddings.



Chmutov–Duzhin–Mostovoy

A *complex* plane curve $V_f \subseteq \mathbf{C}^2$ through $(0,0)$ defines a link (for any small ϵ):

$$L_f := V_f \cap S^3, \quad \text{where } S^3 = \{|x|^2 + |y|^2 = \epsilon\}.$$

Ex If $f = y^n - x^m$, then L_f is the (m,n) *torus link*. It's a knot when m and n are coprime.



Wolfram Language

Ex If $f = y^4 - 2x^3y^2 - 4x^5y + x^6 - x^7$, then L_f is the closure of the braid



Cherednik–Danilenko

Conj (Oblomkov–Shende ~2010)

$$P_f(1, q^2) = \lim_{a \rightarrow 0} \left[(q/a)^\mu \mathbb{P}_{L_f}(a, q) \right],$$

where $\mu \in \mathbf{Z}$ and \mathbb{P} is the *HOMFLYPT invariant*, discovered in 1986 and defined by the rules:

$$(1) \quad a \mathbb{P}_{\nearrow \nwarrow} - a^{-1} \mathbb{P}_{\nwarrow \nearrow} = (q - q^{-1}) \mathbb{P}_{\searrow \swarrow}$$

$$(2) \quad \mathbb{P}_{\bigcirc} = 1$$

Surprising, since P_f is *intrinsic* to f , while \mathbb{P} is defined *diagrammatically*.

Full statement incorporates a by upgrading P_f .

(Maulik 2012) True for all plane curves.

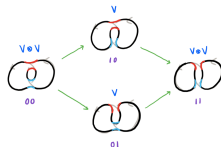
Proof sketch Blow up the singularity; control P_f via wall-crossing and L_f via skein algebra.

Conj (Oblomkov–Rasmussen–Shende ~2013)

$$P_f(t^2, q^2) = \lim_{a \rightarrow 0} \left[(q/a)^\mu \mathbf{P}_{L_f}(a, t, q) \right],$$

where \mathbf{P} is a refinement of \mathbb{P} , discovered in the 2000s by Khovanov–Rozansky.

\mathbf{P} is defined by *categorifying* (1)–(2). Unknown how to categorify Maulik’s proof.



Melissa Zhang

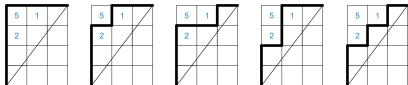
(Kivinen–T 2025) True for $f = y^3 - x^m$ with $3 \nmid m$.

Cor (Kivinen–T) New closed formula for $\mathbf{P}_{\text{torus}(m,3)}$.

Proof Sketch

1 Combinatorial recursions for $\mathbf{P}_{\text{torus}(m,n)}(\mathbf{a}, \mathbf{t}, \mathbf{q})$ due to Mellit and Hogancamp–Mellit.

2 When m and n are coprime, get a formula summing over $m \times n$ Dyck paths:



Here, $\hat{R}_{f \bmod p} \simeq \mathbf{F}_p[[u^m, u^n]]$.

We relate Dyck paths to \hat{R}_f -submodules $M \subseteq \mathbf{F}_p[[u]]$.

3 We relate $\sum_M \frac{1}{|\mathbf{F}_p[[u]]/M|^s}$ to $\sum_I \frac{1}{|\hat{R}_{f \bmod p}/I|^s}$.

Uses *Serre duality*. For now, requires $\min(m, n) \leq 3$.

Big Picture

I'm interested in special functions that appear in

- *algebraic geometry* (e.g., zeta functions)
- *knot theory* (e.g., HOMFLYPT polynomials)
- *combinatorics* (e.g., Dyck-path statistics)

One modern way to study such special functions is called *representation theory*.

T (2021) If L comes from a positive n -strand braid, then $\mathbf{P}_L(\mathbf{a}, \mathbf{t}, \mathbf{q})$ can be recovered from a *representation* of S_n on the cohomology of an explicit variety \mathcal{Z}_L .

So if $L = L_f$, then \mathbf{P}_{L_f} relates to both V_f and \mathcal{Z}_{L_f} .

Any direct relationship between these varieties?

4 Cherednik's New Hypothesis

Recall: For $f = y^3 - x^4$ and prime p , the roots of

$$P_f(p, q) = 1 + pq^2 + p^2q^3 + p^2q^4 + p^3q^6$$

do not all satisfy $|q| = p^{-1/2}$.

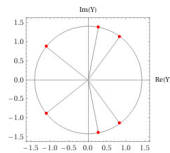
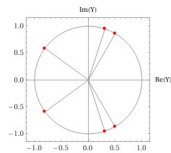
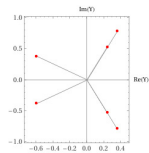
Conj (Cherednik 2018) For any plane curve f :

$$0 < t \leq \frac{1}{2} \implies \text{all roots of } P_f(t, q) \text{ satisfy } |q| = t^{-1/2}.$$

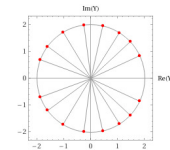
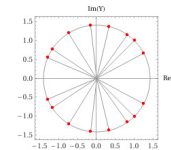
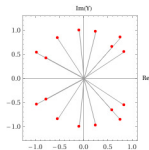
Would imply *arithmetic* constraints on $\mathbf{P}_L(a, t, q)$.

Dream (Cherednik) Related to the Lee–Yang theorem on zeros of *partition functions* in statistical mechanics.

$$f = y^3 - x^4 \quad t \in \{2, 1, \frac{1}{2}\}:$$



$$f = y^4 - 2x^3y^2 - 4x^5y + x^6 - x^7 \quad t \in \{1, \frac{1}{2}, \frac{1}{4}\}:$$



Thank you for listening.