(Munkres §51) recall: for cts $f, g : S$ to X,

a <u>homotopy</u> from f to g is a cts $h : S \times [0, 1]$ to X

s.t. for all s in S, $h(s, 0) = f(s)$
$h(s, 1) = g(s)$

<u>Ex</u> take $X = R$
take $f : S$ to R defined by $f(s) = 0$ for all s
[draw picture]
which g allow a homotopy from f to g?

need, for all s, $h(s, 0) = 0$
$h(s, 1) = g(s)$

$h(s, t) = t*g(s)$ works [draw]

<u>Ex</u> keep $X = R$
take $f(s) = 2025$ for all s

$h(s, t) = (1 – t)*2025 + t*g(s)$ works

<u>Ex</u> take X to be any convex subsp of R^n
take $\mathbf{x}$ any point in X

$h(s, t) = (1 – t)*\mathbf{x} + t*g(s)$ works
[where * now means scalar multiplication]

<u>Df</u> $f, g : S$ to X are <u>homotopic</u> iff
there is some homotopy from f to g

here we write $f \sim g$

Thm ~ is an equivalence relation

Pf reflexive: take $h(s, t) = f(s)$ for all s, t

symmetric:

if $h(s, t)$ is a homotopy from f to g

then $h(s, 1 - t)$ is a homotopy from g to f

transitive:

suppose h is a homotopy from f_1 to f_2,

j is a homotopy from f_2 to f_3

need cts $k : S \times [0, 1]$ to X from f_1 to f_3

s.t. for all s, $k(s, 0) = f_1(s)$

$k(s, 1) = f_3(s)$

take $k(s, t) = h(s, 2t)$ for $t \leq 1/2$

$k(s, t) = j(s, 2t - 1)$ for $t \geq 1/2$

note that $t = 1/2$ gives $h(s, 1) = f_2(s) = j(s, 0)$

cts by pasting lemma (Munkres Thm 18.3)

Df g is nulhomotopic iff there's a homotopy from some const map to g

Df X is contractible iff id_X is nulhomotopic

Thm any cts map into a convex subsp of R^n is nulhomotopic

Cor convex subsp's of R^n are contractible

Q take $S = S^1$
take $X = R^2 - \{0\}$ [not convex]

take $f(x, y) = (x + 2, y)$ and $g(x, y) = (1, 0)$
still homotopic? [yes]

Q keep $S = S^1$ and $X = R^2 - \{0\}$

take $f(x, y) = (x, y)$ and $g(x, y) = (1, 0)$
homotopic? [no: why?]

Q take $S = [0, 1]$ and $X = R^2 - \{0\}$

take $f(s) = (\cos(2\pi s), \sin(2\pi s))$ and $g(s) = (1, 0)$
homotopic? [yes: why?]

now: consider paths $\gamma, \gamma' : [0, 1]$ to X
with the same start and end

a path homotopy from γ to γ' is a homotopy h

s.t. for all t in $[0, 1]$, $h(0, t) = \gamma(0) = \gamma'(0)$
$h(1, t) = \gamma(1) = \gamma'(1)$

γ, γ' with the same start & end are path homotopic
iff there is a path homotopy from γ to γ'

here we write $\gamma \sim_p \gamma'$

Thm fix a start point x & end point y

then $\sim_p$ is an equiv relation on paths from x to y