Last time — a homotopy equivalence btw $X$ and $Y$ is a pair of maps

$f : X \to Y$ and $g : Y \to X$

s.t. $g \circ f \sim id_X$ and $f \circ g \sim id_Y$

Ex — suppose $X$ is contractible
pick $x_0$ in $X$ s.t. $id_X \sim$ (const map at $x$)
set $Y = \{x_0\}$

$f : X \to Y$ — $f(x) = x_0$, the constant map
$g : Y \to X$ — $g(x_0) = x_0$, the inclusion

then — $(g \circ f)(x) = x_0$, so $g \circ f \sim id_X$
while — $f \circ g = id_Y \sim id_Y$

so $X$ is homotopy equivalent to $\{x_0\}$

Thm — if $f : X \to Y$ and $g : Y \to X$ form a homotopy equivalence

then — $f_* : \pi_1(X, x) \to \pi_1(Y, f(x))$,
$g_* : \pi_1(Y, y) \to \pi_1(X, g(y))$

are isomorphisms for any $x$ in $X$ and $y$ in $Y$

Rem — these examples are a specific kind of homotopy equivalence called a deformation retract

Pf of Thm — suppose $f : X \to Y$ and $g : Y \to X$ s.t. $g \circ f \sim id_X$ and $f \circ g \sim id_Y$

will show that $f_* : \pi_1(X, x) \to \pi_1(Y, f(x))$
is an iso for any $x$ in $X$ [argument for $g_*$ is similar]

1) if $f : X \to Y$ and $g : Y \to Z$ are cts maps
then $(g \circ f)_* = g_* \circ f_*$ [from last time]

2) if $\varphi : G \to H$ and $\psi : H \to K$ are maps
s.t. $\psi \circ \varphi$ is bijective
then $\varphi$ is injective and $\psi$ is surjective

3) if $\alpha$ is a path in $X$ from $x_0$ to $x_1$
then $\breve{\alpha} : \pi_1(X, x_0) \to \pi_1(X, x_1)$ def by

$$\breve{\alpha}([\gamma]) = [\bar{\alpha} * \gamma * \alpha]$$

is an isomorphism

by 1), $g_* \circ f_* = (g \circ f)_*$,
$f_* \circ g_* = (f \circ g)_*$

so by 2), just need $(g \circ f)_*$ and $(f \circ g)_*$ to be isos

pick homotopies $j$ from $g \circ f$ to $id_X$,
$k$ from $f \circ g$ to $id_Y$

can use 3) to show: if $f, f' : A \to X$ are cts,
$h$ a homotopy from $f$ to $f'$,
$a$ in $A$,

then $f'_* = \breve{\alpha}_h \circ f_* : \pi_1(A, a) \to \pi_1(X, f'(a))$

where $\alpha_h(s) = h(a, s)$, a path from $f(a)$ to $f'(a)$

apply to j and k:

$$(g \circ f)_* = \breve{\alpha}_j \circ id_{X, *} = \breve{\alpha}_j$$
$$(f \circ g)_* = \breve{\alpha}_k \circ id_{Y, *} = \breve{\alpha}_k$$

but by 3), $\breve{\alpha}_j$ and $\breve{\alpha}_k$ are isos