Character Formulas from Lusztig Varieties and Affine Springer Fibers

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This talk is about...

1. Braids
2. Lusztig Varieties
3. Springer Fibers
4. Affine Springer Fibers

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1 Braids

The braid group $Br_n =$

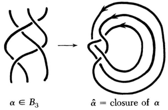
$$\left\langle \sigma_1, \ldots, \sigma_{n-1}, \;\middle|\; \begin{array}{l} \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \\ \sigma_i \sigma_j = \sigma_j \sigma_i \text{ when } |i - j| > 1 \end{array} \right\rangle$$

appears in knot theory and representation theory.

σ_i

A *link* is a collection of circles (tamely) embedded in $\mathbf{R}^3$. Knot theory is about isotopy invariants of links.

(Alexander) Every link is the *closure* of some braid.



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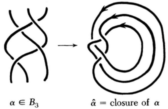

Let $G = \mathrm{GL}_n$ and B its upper-triangular subgroup.

$$V_n(q) = \{\text{functions } G(\mathbf{F}_q)/B(\mathbf{F}_q) \to \mathbf{C}\},$$
$$H_n(q) = \mathrm{End}_{G(\mathbf{F}_q)}(V_n(q)).$$

(Iwahori) $\quad H_n(q) \simeq \dfrac{\mathbf{C}Br_n}{\langle \sigma_i^2 - (q-1)\sigma_i - q\rangle}.$

To explain, recall Bruhat: $G = \coprod_{w \in S_n} B\dot{w}B$.

Then $\mathbf{C}Br_n \curvearrowright V_n(q)$ via

$$\sigma_i \cdot \mathbf{1}_{xB(\mathbf{F}_q)} = \sum_{yB \xrightarrow{i} xB} \mathbf{1}_{yB(\mathbf{F}_q)},$$

where $yB \xrightarrow{i} xB$ means $By^{-1}xB = B\dot{w}_{(i,i+1)}B$.

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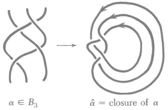

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Motivates a *Hecke algebra* $H_n(\mathsf{q})$ over $\mathbf{C}[\mathsf{q}^{\pm 1}]$.

Ocneanu used functions $Br_n \to H_n(\mathsf{q}) \to \mathbf{C}(\mathsf{q})[a^{\pm 1}]$ to construct a link invariant

$$\text{HOMFLYPT} : \{\text{links}\}/\text{isotopy} \to \mathbf{C}(\mathsf{q})[\mathsf{a}^{\pm 1}].$$

Jones computed it for torus knots. Remarkably, the values encode q-Catalan (and q-Kirkman) numbers.

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2 Lusztig Varieties

Suppose that β is *positive*:

$$\beta = \sigma_{i_1} \cdots \sigma_{i_\ell}.$$

(Deligne) The variety $O(\beta) =$

$$\left\{ (g_0 B, g_1 B, \ldots, g_\ell B) \,\middle|\, g_{j-1} B \xrightarrow{i_j} g_j B \text{ for all } j \right\}$$

only depends on β, up to isomorphisms that keep $g_0 B$ and $g_\ell B$ fixed.

For any positive β, β', we have

$$O(\beta\beta') \simeq O(\beta) \times_{G/B} O(\beta'),$$

where $\times_{G/B}$ means the variety of pairs $(\vec{g}B, \vec{g}'B)$ such that $g_\ell B = g'_0 B$.

A literal geometric representation of positive braids.

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$$\mathcal{B}(\beta)_x = \{\vec{g}B \in O(\beta) \mid g_\ell B = x g_0 B\}.$$

(Shende–Treumann–Zaslow) Up to a monomial in q,

$$\frac{|\mathcal{B}(\beta)_1(\mathbf{F}_q)|}{|\mathrm{PGL}_n(\mathbf{F}_q)|}$$

is the "highest" a-degree of HOMFLYPT$(\hat{\beta})$ at $\mathsf{q} \to q$.

Example Let $n = 2$ and $\beta = \sigma_1^3 \in Br_2$.

$$O(\beta) \simeq \{\vec{g} \in (\mathbf{P}^1)^4 \mid g_0 \neq g_1 \neq g_2 \neq g_3\},$$
$$\mathcal{B}(\beta)_1 \simeq \{\vec{g} \in (\mathbf{P}^1)^3 \mid g_1, g_2, g_3 \text{ pairwise distinct}\}.$$

PGL_2 acts simply transitively on the latter.

Indeed, HOMFLYPT$(\widehat{\sigma_1^3}) = a^2(\mathsf{q} + \mathsf{q}^{-1}) - a^4$.

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3 Springer Fibers

How to access other a-degrees? Observe that

$$\mathcal{B}_x := \mathcal{B}(1)_x = \{gB \mid gB = xgB\}$$

is the usual Springer fiber over x, whose cohomology defines a character of S_n: namely,

$$\Psi_x(w) := \sum_i \mathsf{q}^i \operatorname{tr}(w \mid \mathrm{H}^{2i}(\mathcal{B}_x)).$$

Thm 1 (T) Let $\mathcal{U} \subseteq G$ be the the unipotent variety,

$$\Psi_\beta(w) = \sum_{u \in \mathcal{U}(\mathbf{F}_q)} \frac{|\mathcal{B}(\beta)_u(\mathbf{F}_q)|}{|\mathrm{PGL}_n(\mathbf{F}_q)|} \Psi_u(w)|_{\mathsf{q} \to q}.$$

Then $(\chi_{(n-k,1,\ldots,1)}, \Psi_\beta)_{S_n}$ sees the kth a-degree.

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Think of $\beta \mapsto \Psi_\beta$ as a function

$$Br_n \to H_n(q) \to \{\text{characters of } S_n\}.$$

Example Again, let $n = 2$ and $\beta = \sigma_1^3 \in Br_2$.

$$\Psi_u = \begin{cases} 1 + \mathsf{q}\,\mathsf{sgn} & u = 1, \\ 1 & u \neq 1. \end{cases}$$

$$\Psi_\beta = q^2 + 1 + q\,\mathsf{sgn}.$$

Recall $\text{HOMFLYPT}(\widehat{\sigma_1^3}) = a^2(\mathsf{q} + \mathsf{q}^{-1}) - a^4$.

Thm 2 (T) The cohomology of $\mathcal{U}(\beta) \times_{\mathcal{U}} \mathcal{U}(1)$, where

$$\mathcal{U}(\beta) = \{(u, \vec{g}B) \mid u \in \mathcal{U} \text{ and } \vec{g}B \in \mathcal{B}(\beta)_u\},$$

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The *full twist* $\pi = (\sigma_1 \cdots \sigma_{n-1})^n$:

Thm 3 (T) Suppose $\beta^m = \pi^d$ for some $d, m > 0$.

Then up to a monomial, $\Psi_\beta(w)$ equals

$$\frac{\mathsf{sgn}(w)}{\det(1 - qw \mid \mathfrak{h})} \sum_{\lambda \vdash n} q^{c(\lambda)d/m} D_\lambda(e^{2\pi i d/m}) \chi_\lambda(w)$$

where:

- $\mathfrak{h}$ is the *reflection representation.*
- $c(\lambda)$ is the sum of *contents* of λ.
- $D_\lambda(t) = K_{\lambda,(1^n)}(t)$ is the *fake degree* of λ.

Recovers Jones's HOMFLYPT formula for torus knots.

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Thm 3 generalizes to any reductive G, once we replace:

- S_n with the Weyl group W.
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If $\gcd(d, m) = 1$ and m is the Coxeter number of W, then the formula simplifies:

$$\left.\text{(monomial)} \cdot \frac{\det(1 - q^d w \mid \mathfrak{h})}{\det(1 - qw \mid \mathfrak{h})}\right\} =: \Pi_q^{(d)}.$$

$\Pi_q^{(d)}$ is the character of a *rational parking space*.

$(\mathsf{triv}, \Pi_q^{(d)})_W$ is a *rational q-Catalan number*.

Example If $W = S_n$, then $(\mathsf{triv}, \Pi_q^{(d)})_W = \frac{[n+d-1]_q!}{[n]_q![d]_q!}$.

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4 Affine Springer Fibers

Rational parking spaces appear in a loop or *affine* analogue of Springer theory.

finite Springer	affine Springer
G	$G(\!(z)\!)$
G/B	$G(\!(z)\!)/I$
W	$\widetilde{W} = W \ltimes X^\vee$

Above:

- $G(\!(z)\!)$ is the loop group $G(\!(z)\!)(R) := G(R(\!(z)\!))$.
- I is the preimage of B in $G[\![z]\!]$.
- $X^\vee$ is the cocharacter lattice of B.

We now study Springer fibers over the Lie algebras, not the groups, and over $\mathbf{C}$, not $\mathbf{F}_q$.

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$$x : \quad \mathcal{B}_x = \{gB \in G/B \mid g^{-1}xg \in \mathrm{Lie}(B)\},$$

$$\gamma = \gamma(z) : \quad \mathcal{B}_\gamma^{\mathrm{aff}} = \{gI \in G(\!(z)\!)/I \mid g^{-1}\gamma g \in \mathrm{Lie}(I)\}.$$

The table hides key differences:

In the finite case, $\mathcal{B}_x$ is most interesting for x nilpotent.

In the affine case, $\mathcal{B}_\gamma^{\mathrm{aff}}$ is *terribly infinite* for $\gamma = \gamma(z)$ nilpotent, but interesting for $\gamma(z)$ regular semisimple.

Example If $G = \mathrm{SL}_2$ and $\gamma(z) = \begin{pmatrix} 0 & 1 \\ z^3 & 0 \end{pmatrix}$, then

$$\mathcal{B}_\gamma^{\mathrm{aff}} \simeq \mathbf{P}^1 \sqcup_{\mathrm{pt}} \mathbf{P}^1.$$

Dream Braid Lusztig varieties see the *finite* part of affine Springer representations.

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$$\mathcal{B}_\gamma^{\mathrm{aff}} \simeq \mathbf{P}^1 \sqcup_{\mathrm{pt}} \mathbf{P}^1.$$

Dream Braid Lusztig varieties see the *finite* part of affine Springer representations.

Fix $\nu = d/m > 0$ in lowest terms. Let $\mathbf{C}^\times \curvearrowright \mathfrak{g}(\!(z)\!)$:

$$c \cdot_\nu \gamma(z) = c^{2d\rho^\vee} \gamma(c^{2m} z) c^{-2d\rho^\vee},$$

where $2\rho^\vee = \sum_{\alpha \in \Phi^+} \alpha^\vee$.

Let $\mathfrak{g}(\!(z)\!)_{\nu,k}$ be the weight-$2k$ eigenspace.

Lemma If γ is an eigenvector for $\cdot_\nu$, then the induced action on $G(\!(z)\!)/I$ preserves $\mathcal{B}^{\mathrm{aff}}_\gamma$.

Lemma $\mathfrak{g}(\!(z)\!)_{\nu,0}$ is the Lie algebra of a connected reductive group L_ν. Moreover,

$$(G(\!(z)\!)/I)^{\mathbf{C}^\times} = \coprod_{w \in W(L_\nu)\backslash \widetilde{W}} L_\nu \dot{w} I/I.$$

$$x : \quad \mathcal{B}_x = \{gB \in G/B \mid g^{-1}xg \in \mathrm{Lie}(B)\},$$

$$\gamma = \gamma(z) : \quad \mathcal{B}^{\mathrm{aff}}_\gamma = \{gI \in G(\!(z)\!)/I \mid g^{-1}\gamma g \in \mathrm{Lie}(I)\}.$$

The table hides key differences:

In the finite case, $\mathcal{B}_x$ is most interesting for x nilpotent.

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In the previous SL_2 example, $\gamma \in \mathfrak{g}((z))_{3/2,\,3}$.

$$\text{Springer}: \quad \widetilde{W} \curvearrowright \mathrm{H}^*(\mathcal{B}_\gamma^{\mathrm{aff}}), \mathrm{H}^*_{\mathbf{C}^\times}(\mathcal{B}_\gamma^{\mathrm{aff}}).$$

(Sommers) If m is the Coxeter number, then:

- L_ν is a torus, and $L_\nu \dot{w} I = \dot{w} I$.
- $(\mathcal{B}_\gamma^{\mathrm{aff}})^{\mathbf{C}^\times}$ is a finite subset of the $\dot{w} I$.
- Writing $\mathrm{H}^*_{\mathbf{C}^\times}(\mathrm{pt}) = \mathbf{C}[\epsilon]$, we have

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where $P_{\nu,w} := L_\nu \cap \dot{w} I \dot{w}^{-1}$.

(Oblomkov–Yun) They are smooth. Finitely many are nonempty.

For such $\mathrm{Hess}_{\gamma,w}$, the codimension in $L_\nu/P_{\nu,w}$ is the number of affine roots $\alpha + k$ such that:

- $\langle \alpha, \nu\rho^\vee \rangle + k = \nu$.
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Conj (T) For general ν, the representation

$$W \curvearrowright \mathrm{H}^*_{c,\mathbf{C}^\times}(\mathcal{B}_\gamma^{\mathrm{aff}})|_{\epsilon \to 1}$$

contains a summand whose character is the $q \to 1$ limit of our earlier formula:

$$\frac{\mathsf{sgn}(w)}{\det(1 - qw \mid \mathfrak{h})} \sum_{\chi \in \mathrm{Irr}(W)} q^{c(\chi)\nu} D_\chi(e^{2\pi i\nu}) \chi(w).$$

Moreover, the Oblomkov–Yun filtration restores q.

Thank you for listening.