Warmup let $D : F[x]$ to $F[x]$ be $D(p) = dp/dx$

what are some D-stable linear subspaces?

if W is D-stable and q in W,
then $D(q)$, $D^2(q)$, $D^3(q)$, … in W

e.g., for all n,
$P_n = \{p \mid p = 0 \text{ or } \deg(p) \leq n\}$ is D-stable

non-example:
$\{p \mid p(3) = 0\}$ is not D-stable [why?]

Q does D have eigenvectors in $F[x]$?

no: cannot solve $D(p) = \lambda p$ for polynomial p

Df a formal power series over F ($= \mathbb{R}, \mathbb{C}$) is an infinite sum $f(x) = \sum_{k \geq 0} a_k x^k$ with a_i in F for all i

$F[[x]] = \{\text{formal power series } f(x)\}$

the formal derivative on $F[[x]]$ is the F-linear operator D def by

$D(\sum_k a_k x^k) = \sum_k k a_k x^{k-1}$

Q does D have eigenvectors in $F[[x]]$?

yes: $\exp(\alpha x)$ where $\exp(x) = \sum_k (1/k!) x^k$

Q is $F[[x]]$ iso to a familiar v.s.? [yes: $F^{\mathbf{N}}$]

(Axler §5B–5C)   recap: fix T : V to V

- if V = F[x], then T might have no eigenvals

[multiplication by x gives another example]

- if V = R^n, then T might have no eigenvals

[example? any rotation of θ ≠ 0, π radians]

[stated last time:]

Thm   if F = C and V is fin. dim. and not {**0**},
then T must have some eigenval

last time: proved thm assuming a lemma about polynomials in T

today: prove lemma in a sharper form

Lem   for any F and V fin. dim. and v ≠ **0**:

1) there is some p(z) s.t.
p(z) nonconst and p(T) v = **0**

2) if m = minimal deg among such p,
then dim ker(p(T)) ≥ m for such p

Pf of 1)   set n = dim V
v, Tv, …, T^nv must be lin. dep.
so there exist a_0, a_1, …, a_n in C s.t.
(a_0 + a_1T + … + a_nT^n) v = **0**
if a_1 = … = a_n = 0 then v = **0**

Pf of 2)   v, Tv, …, T^(m – 1)v must be lin. indep.

but p(T)(T^i v) = T^i(p(T) v) = T^i(**0**) = **0** for all i
so dim ker(p(T)) ≥ dim span(v, Tv, …, T^(m – 1)v)

[bootstrap:]

<u>Thm</u> for any F and V of dim n:
there is some poly $p(z)$ of deg $\leq n$ s.t.
$p(T)$ = zero op

<u>Pf</u> if $n = 0$ or 1, then done
induct on n:

pick $v \neq \mathbf{0}$
by lemma, can pick $f(z)$ of minimal deg s.t.
$f(z)$ nonconst and $f(T)\, v = \mathbf{0}$
also by lemma, $\dim \ker(f(T)) \geq \deg f(z) > 0$
so $\dim \operatorname{im}(f(T)) = n - \dim \ker(f(T)) < n$

want to apply inductive hypothesis to $\operatorname{im}(f(T))$

set $W = \operatorname{im}(f(T))$. key observation:

W is T-stable [! why?], so get an op $T|\_W$

pick $g(z)$ of deg $\leq \dim W$ s.t. $g(T|\_W)$ = zero
set $p(z) = g(z)f(z)$
then for all v in V: $p(T)\, v = g(T)\, (f(T)\, v) = \mathbf{0}$
and also $\deg p = \deg f + \deg g$
$\leq \dim \ker(f(T)) + \dim W$
$= \dim V$ □

<u>Df</u> the minimal polynomial is
the monic poly $p(z)$ of minimal degree s.t.
$p(T)$ = zero op

we denote it by minpoly_T(z) in C[z]

(monic = the coeff of the highest power of z is 1)

<u>Rem</u> thm shows minpoly_T(z) exists
<u>and</u> deg minpoly_T(z) ≤ dim V
[both < and = can occur:]

<u>Ex</u> if T = id_V, then minpoly_T(z) = z – 1

<u>Ex</u> if T = zero op, then minpoly_T(z) = 1

<u>Ex</u> what if T has a diagonal matrix?

let {λ_1, …, λ_k} be the nonzero entries
<u>without</u> repetition

then minpoly_T(z) = (z – λ_1)…(z – λ_k) [why?]

<u>Ex</u> what if T has a matrix

$$\begin{matrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{matrix} \quad ?$$

minpoly_T(z) = (z – λ)^2 [not z – λ]

what if it is instead

$$\begin{matrix} \lambda & 1 & 0 \\ 0 & \lambda & 2 \\ 0 & 0 & \lambda \end{matrix} \quad ?$$

minpoly_T(z) = (z – λ)^3 [why?]

<u>Thm</u> if T has an upper-triangular matrix
with diagonal entries λ_1, …, λ_n,
<u>including</u> any repetition,
then (T – λ_1)…(T – λ_n) = zero

Rem this gives a new proof that

deg minpoly $\leq$ dim V

in cases where T has a triangular form

Pf let $p(z) = (z - \lambda_1)\dots(z - \lambda_n)$

key : the $(T - \lambda_j)$'s commute

so $p(T) = (T - \lambda_{j_1})\dots(T - \lambda_{j_n})$

for any index reordering $j_1, \dots, j_n$

let $e_1, \dots, e_n$ = ordered basis for triangularity

$(T - \lambda_1) e_1 = \mathbf{0}$,

$(T - \lambda_2) e_2$ in span(e_1),

…,

$(T - \lambda_j) e_j$ in span$(e_1, e_2, \dots, e_{j-1})$

so $p(T) e_j$
$$\begin{aligned} &= (\text{stuff})(T - \lambda_1)\dots(T - \lambda_j) e_j \\ &= (\text{stuff})(T - \lambda_1)(T - \lambda_2) e_2 \\ &= (\text{stuff})(T - \lambda_1) e_1 \\ &= \mathbf{0} \quad \square \end{aligned}$$

when does T have a triangular matrix? next time:

Thm if F = C and V is fin. dim.

then any linear op T : V to V has

an upper-triangular matrix