

## 1

$G = \mathrm{GL}_n$

$\mathcal{U} = \{\text{unipotent } n \times n \text{ matrices}\} \subseteq G$

Ex for  $n = 2$ , have  $\mathcal{U} \simeq \left\{ \begin{pmatrix} 1+x & y \\ z & 1-x \end{pmatrix} \middle| x^2 + yz = 0 \right\}$

## 2

$B \subseteq G$  upper-triangular

Bruhat, Chevalley understand  $G$  via  $B$ :

$$G = \bigsqcup_{w \in S_n} B \dot{w} B$$

can we understand  $\mathcal{U}$  via  $U := B \cap \mathcal{U}$ ?

## 3

$U_- \subseteq B_- \subseteq G$  lower-triangular

Fine-Herstein '58, Steinberg '65

$$|\mathcal{U}(\mathbb{F}_q)| = q^{n(n-1)} = |U(\mathbb{F}_q)|^2 = |UU_-(\mathbb{F}_q)|$$

Kawanaka '75 for any  $w \in S_n$ ,

$$|\overbrace{(\mathcal{U} \cap B \dot{w} B)(\mathbb{F}_q)}^{\mathcal{U}_w}| = |\overbrace{(UU_- \cap B \dot{w} B)(\mathbb{F}_q)}^{\mathcal{V}_w}|$$

## 4

Ex for any  $n$ , have  $\mathcal{U}_{\mathrm{id}} = U = \mathcal{V}_{\mathrm{id}}$

Ex for  $n = 3$  and  $\dot{w} = \begin{pmatrix} & 1 & \\ 1 & & \end{pmatrix}$ , not even homeo over  $\mathbb{C}$

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## 5

diagonal  $T \curvearrowright \mathcal{U}_w, \mathcal{V}_w$  by conjugation

Thm (T)  $\text{gr}_*^W H_{c,T}^*(\mathcal{U}_w(\mathbb{C})) \simeq \text{gr}_*^W H_{c,T}^*(\mathcal{V}_w(\mathbb{C}))$

where  $W$  is the weight filtration on  $H_{c,T}^*$

almost implies (Kawanaka) via results of Katz

## 6

Conj (T) the  $T$ -equivariant map

$$UU_- \rightarrow \mathcal{U} \quad \text{given by } xy \mapsto xyx^{-1}$$

restricts to a homotopy equivalence  $\mathcal{V}_w(\mathbb{C}) \rightarrow \mathcal{U}_w(\mathbb{C})$

## 7

for  $\beta = \sigma_{w_1} \cdots \sigma_{w_k} \in Br_n^+$ ,

$$\begin{array}{ccc} X_\beta^C & \rightarrow & X_\beta := B\dot{w}_1 B \times^B B\dot{w}_2 B \times^B \cdots \times^B B\dot{w}_k B \\ \downarrow & & \downarrow \\ C & \rightarrow & G \end{array}$$

$\mathcal{U}_w = X_{\sigma_w}^{\mathcal{U}}$  and  $\mathcal{V}_w = X_{\sigma_w \pi}^1$ , where  $\pi$  is the full twist

## 8

Thm (T) for any  $\beta \in Br_n^+$ ,

$$|X_\beta^{\mathcal{U}}(\mathbb{F}_q)| = |X_{\beta \pi}^1(\mathbb{F}_q)|,$$

$$\text{gr}_*^W H_{c,T}^*(X_\beta^{\mathcal{U}}(\mathbb{C})) \simeq \text{gr}_*^W H_{c,T}^*(X_{\beta \pi}^1(\mathbb{C}))$$

**9**

Kálmán '09 writing  $P$  for HOMFLYPT,

$$P(\widehat{\beta})[a^{|\beta|-n+1}] = P(\widehat{\beta\pi})[a^{|\beta|+n-1}]$$

Gorsky–Hogancamp–Mellit–Nakagane '19

true with KhR superpolynomial  $\mathbb{P}$  in place of  $P$

**10**

Springer resolution  $\tilde{\mathcal{U}} = \{(u, gB) \in \mathcal{U} \times G/B \mid ugB = gB\}$

$$\begin{array}{ccccccc} X_\beta^1 \times G/B & \rightarrow & \tilde{X}_\beta^{\mathcal{U}} & \rightarrow & X_\beta^{\mathcal{U}} & \rightarrow & X_\beta \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \{1\} \times G/B & \rightarrow & \tilde{\mathcal{U}} & \rightarrow & \mathcal{U} & \rightarrow & G \end{array}$$

**11**

Thm (T) if  $\beta \in Br_n^+$ , then  $S_n \curvearrowright \text{gr}_*^W H_{c,T}^*(\tilde{X}_\beta^{\mathcal{U}})$  and

$$\mathbb{P}(\widehat{\beta}) \propto (\Lambda^*(\mathbb{V}), \text{gr}_*^W H_{c,T}^*(\tilde{X}_\beta^{\mathcal{U}}(\mathbb{C})))_{S_n}, \quad \text{where } \mathbb{V} = \mathbb{C}^{n-1}$$

$$\begin{aligned} a \text{ tracks } \Lambda^* \text{-grading} &\implies \mathbb{P}(\widehat{\beta})[a^{\text{lo}}] \propto \text{gr}_*^W H_{c,T}^*(X_\beta^{\mathcal{U}}(\mathbb{C})) \\ &\implies \mathbb{P}(\widehat{\beta})[a^{\text{hi}}] \propto \text{gr}_*^W H_{c,T}^*(X_\beta^1(\mathbb{C})) \end{aligned}$$

**12**

Jordan type:  $\mathcal{U} = \bigcup_{\lambda \vdash n} \mathcal{U}_\lambda$

$$\tilde{X}_\beta^{\mathcal{U}} = \bigcup_{\lambda} (X_\beta^{\mathcal{U}_\lambda} \times \tilde{\mathcal{U}}_{u_\lambda}) \quad \text{for fixed } u_\lambda \in \mathcal{U}_\lambda$$

$$P(\widehat{\beta})[a^{\text{lo}+2k}] = \sum_{\lambda} \underbrace{(\Lambda^k(\mathbb{V}), [H^*(\tilde{\mathcal{U}}_{u_\lambda}(\mathbb{C}))]_q)_{S_n}}_{\text{known}} |X_\beta^{\mathcal{U}_\lambda}(\mathbb{F}_q)|$$

## 13

just as  $P$  arises from traces on Hecke algebras,

$$\begin{aligned} \mathrm{D}_{mix,G}^b \mathsf{Perv}(\mathcal{U}) &\simeq \mathrm{D}^b \mathsf{Mod}(\mathbb{C} S_n \ltimes \mathrm{Sym}) \quad (\text{Rider}) \\ &\simeq \mathsf{hTr}(\mathrm{Hecke}(S_n)) \quad (\text{Gorsky--Wedrich}) \end{aligned}$$