(Munkres §54) return to:

Thm $\pi_1(S^1) \simeq \mathbb{Z}$

let $o = (1, 0)$ in S^1

for all n in $\mathbb{Z}$, let $\omega_n : [0, 1]$ to S^1 be defined by

$$\omega_n(s) = (\cos(2\pi ns), \sin(2\pi ns))$$

will show that $\Phi : \mathbb{Z}$ to $\pi_1(S^1, o)$ defined by

$$\Phi(n) = [\omega_n]$$

is an isomorphism: 1) homomorphism
2) bijective

Step 1 must show that $\Phi(m + n) = \Phi(m) + \Phi(n)$,
meaning $[\omega_{m + n}] = [\omega_m * \omega_n]$

key idea: let $p : \mathbb{R}$ to S^1 be

$$p(x) = (\cos(2\pi x), \sin(2\pi x))$$

Lem for any a, b in $\mathbb{Z}$ s.t. $b - a = n$, we have

$$\omega_n = p \circ \acute{\omega}_{a, b}$$

where $\acute{\omega}_{a, b} : [0, 1]$ to $\mathbb{R}$ is defined by

$$\acute{\omega}_{a, b}(s) = (1 - s)a + sb$$

[draw]

Lem for any a, b, c in Z, we have

$$[\acute{\omega}_{a, b} * \acute{\omega}_{b, c}] = [\acute{\omega}_{a, c}]$$

[draw]

therefore,

$$[(p \circ \acute{\omega}_{a, b}) * (p \circ \acute{\omega}_{b, c})]$$
$$= [p \circ (\acute{\omega}_{a, b} * \acute{\omega}_{b, c})]$$ [Munkres 327]
$$= [p \circ \acute{\omega}_{a, c}]$$ [PS5, #1]

giving $[\omega_m * \omega_n] = [\omega_{m + n}]$

Step 2 must show: Φ surjective,
Φ injective

Assumptions suppose we know that

I) for any path $\gamma : [0, 1]$ to S^1 and a in R s.t. $p(a) = \gamma(0)$,

there is a unique $\Gamma : [0, 1]$ to R s.t. $\Gamma(0) = a$ and $\gamma = p \circ \Gamma$, which we call a lift of γ

II) for any path homotopy $h : [0, 1]^2$ to S^1 s.t. $h(-, 0) = \gamma$, and lift Γ of γ,

there is a path homotopy $H : [0, 1]^2$ to R s.t. $H(-, 0) = \Gamma$ and $h = p \circ H$

these are called the homotopy lifting properties of the map $p : R$ to S^1

Φ surjective for all loops γ at o
have n s.t. $[\gamma] = [\omega_n]$

observe: $p^{-1}(o) = Z$

so for any a in Z, I) gives Γ s.t.
$\Gamma(0) = a$ and $\gamma = p \circ \Gamma$
let $b = \Gamma(1)$
then b in $p^{-1}(o) = Z$

now, by PS6, #2, have $[\Gamma] = [\acute{\omega}_{a, b}]$
so by PS5, #1, have $[\gamma] = [\omega_{b - a}]$

Φ injective suppose that $[\omega_m] = [\omega_n]$

observe: $\acute{\omega}_{0, m}$ lifts ω_m

so for any h from ω_m to ω_n, II) gives H s.t.
$H(–, 0) = \acute{\omega}_{0, m}$ and $h = p \circ H$
let $\Gamma = H(–, 1)$
then Γ lifts $\omega_n = h(–, 1)$

but by I), there is a unique lift of ω_n starting at 0
so $\Gamma = \acute{\omega}_{0, n}$
so H goes from $\acute{\omega}_{0, m}$ to $\acute{\omega}_{0, n}$
so $m = n$

(Munkres §53) main idea of covering spaces:

maps that generalize the homotopy lifting
properties of the map $p : R$ to S^1

Df for any cts map $p : E$ to X,
open U sub X,
we say that U is evenly covered by p iff:

1) $p^{-1}(U)$ is homeomorphic to a nonempty(!) disjoint union of copies of U
2) p restricts to a homeomorphism from each copy onto U

Ex let $p : (–1, 1)$ to $[0, 1)$ be squaring

if $0 < a < b$, then (a, b) is evenly covered
but $[0, b)$ is never evenly covered

Ex let $p : (–1, 1)$ to S^1 be
$p(x) = (\cos(2\pi x), \sin(2\pi x))$

open nbd's of 0 are not evenly covered

Df we say that $p : E$ to X is a covering map iff, for all x in X, there's an open nbd of x evenly covered by p

here, we say that E is the covering space

Ex although $p : (–1, 1)$ to S^1 defined by

$$p(x) = (\cos(2\pi x), \sin(2\pi x))$$

is not a covering map, $p : \mathbb{R}$ to S^1 defined by the same formula is a covering map