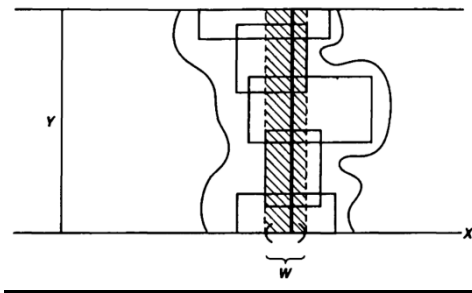


(Munkres §27) [notation change below: N]

Tube Lem fix a in X

for all open $N \subset X \times Y$ s.t. $\{a\} \times Y \subset N$,
there is open $W \subset X$ s.t. $\{a\} \times Y \subset W \times Y \subset N$



Pf recall: product top $X \times Y$ gen by $U \times V$
for open $U \subset X$ and open $V \subset Y$

so N is a union of such sets $U \times V$:

say, $\{U_i \times V_i\}_i$

then $\{a\} \times Y \subset N = \bigcup_i (U_i \times V_i)$

so by compactness, there is a finite set I s.t.

$\{a\} \times Y \subset \bigcup_{i \in I} (U_i \times V_i)$

now set $W = \bigcap_{i \in I} U_i$

check:

- W is open in X , since I is finite
- a in W , since a in U_i for all i
- $W \times Y \subset N$, since...tricky

pick (x, y) in $W \times Y$: want (x, y) in N

know (a, y) in $U_i \times V_i$ for some i in I [why?]

then (x, y) in $W \times V_i \subset U_i \times V_i \subset N \quad \square$

(Munkres §30, 36) second half of the course:

topological manifolds, esp. 2-dim'l manifolds

idea: a space X “locally” homeo to \mathbb{R}^n for given n

Ex the analytic circle $S^1 \subset \mathbb{R}^2$ is
not homeo to \mathbb{R} ,
but is a union of subspaces that are

Ex more generally, the unit n -sphere
 $\{x \in \mathbb{R}^{n+1} \mid \sum_i x_i^2 = 1\}$

Ex nice graphs, no “weird” intersections:
 $\{(x, y, z) \mid z = 1/(x^2 + y^2)\}$

Ex $[0, 1]$ should not be a manifold per se
(maybe: a “manifold with boundary” ...)
but $\{0, 1\}$ should be;
even discrete sets like $\{1/n \mid n = 1, 2, \dots\}$

Rem why “manifold”?

Riemann (June 10, 1854), trans. Clifford (1873),
“On the hypotheses which underlie geometry”:

“Mannigfaltigkeit” to convey a “manifold-ness”
of directions, lengths, etc.

“Df” 1 X is a manifold iff, for all x in X ,
there is an open nbd $U \subset X$ of x s.t.
 U is homeo to \mathbb{R}^n for some $n \geq 0$

Objection 1 take the line with two origins:

$A = (\mathbb{R} \times \{a, b\})/\sim$, where $(x, a) \sim (x, b)$ for $x \neq 0$

each “0” in A has a nbd homeo to $(-1, 1)$,
hence to \mathbb{R} [just exclude the other “0”]

want convergent seq’s in manifolds to have
unique limits

let’s say that X is locally Euclidean iff
 X satisfies the “local” condition in Df 1

“Df” 2 a manifold is a Hausdorff, locally Euclid.
topological space

Objection 2 locally \mathbb{R}^n can still be globally blah

1) take X an uncountable discrete set
each x has open nbd $\{x\}$ homeo to \mathbb{R}^0
but $|X|$ can be unimaginably large

2) the “long line”:

suppose S is a totally ordered set
there is a lexicographic total order on $S \times [0, 1)$:
 $(s, x) < (s, y)$ for arbitrary s and $x < y$
 $(s, x) < (t, y)$ for $s < t$ and arbitrary x, y

we define the S-line to be

$(S \times [0, 1)) - \{\text{any lex'ically min or max elts}\}$

we give it the order topology gen by intervals:
 $\{(s, z) \mid x < z < y\}$ for arbitrary s and $x < y$
 $\{(u, z) \mid s \leq u \leq t,$ for $s < t$ and arbitrary x, y
 $x < z$ if $s = u,$
 $z < y$ if $u = t\}$

Thm (Hartogs) there is a minimal uncountable
well-ordered set $(\omega_1, <)$

i.e. ω_1 is an uncountable set, $<$ a total order, s.t.
any strict subset of ω_1 is countable
any nonempty subset of ω_1 has a least elt
[apparently does not require the axiom of choice]

the ω_1 -line is VERY LONG

Thm the ω_1 -line is locally homeo to \mathbb{R} ,
and path-connected, but not homeo to \mathbb{R}

Pf see Munkres 158–159, #12

[so here is a way to rule out the long line:]

Df we say X is second-countable iff
the top. of X is gen by a countable basis

Rem there's also a notion of first-countable,
but Hausdorff + locally Euclid. implies it

Df a manifold is a Hausdorff,
second-countable, locally Euclid. space

Rem if X is second-countable
then $\{\text{conn. comp's of } X\}$ is countable
ruling out uncountable discrete sets

but second-countability is stronger:
the ω_1 -line is not 2nd-countable
yet has only one conn. comp.

[we mainly picture manifolds as locally Euclidean
subsets of \mathbb{R}^N for some large N :]

Df an embedding is a cts injective map

Thm if a manifold is compact, then it admits
an embedding into \mathbb{R}^N for some $N > 0$

Pf Munkres §36

in gen'l, a manifold embedded in \mathbb{R}^N
is locally homeo to \mathbb{R}^n for various $n \leq N$
[that is, n is usually smaller than N]

we say X is an n -manifold iff it is covered by
open nbds all homeo to \mathbb{R}^n for the same n

in this case, we say n is the dimension of X

Q why not define
manifolds via compact manifolds,
compact manifolds via embedding?

Intrinsic vs Extrinsic

goal of Riemann, Poincaré, Weyl, etc.:
study manifolds without fixing a larger “container”

the universe should be a 4-manifold
(with extra metric structure)

but we don't want to presume it's contained in
some larger \mathbb{R}^N that we can't detect

[how to prove the embedding thm?
uses a notion called “partitions of unity”]

[we may or may not get to it]