Zeta Functions as Knot Invariants

Minh-Tâm Quang Trinh

Howard University

P. Galashin, T. Lam, M. Q. Trinh, N. Williams. Rational noncrossing Coxeter–Catalan combinatorics. *Proc. London Math. Soc.*, **129**(4) (2024), e12643, 50 pp.

O. Kivinen, M. Q. Trinh. The Hilb-vs-Quot conjecture. *J. reine angew. Math. (Crelle)*, (2025). 44 pp.

M. Q. Trinh. From the Hecke category to the unipotent locus. 88 pp. arXiv:2106.07444

1 The Riemann Hypothesis

(Euler ~1730s) Proof of the infinitude of primes:

If there were finitely many, then we'd have

$$\prod_{\text{prime } p > 0} \left(1 + \frac{1}{p} + \frac{1}{p^2} + \cdots\right) = \sum_{n=1}^{\infty} \frac{1}{n}.$$

But the series diverges—a contradiction.

He also implicitly studied the *zeta function*

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

For $s > 1$, we have $\zeta(s) = \displaystyle\prod_p \left(1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \cdots\right)$.

What if we allow s to be complex?

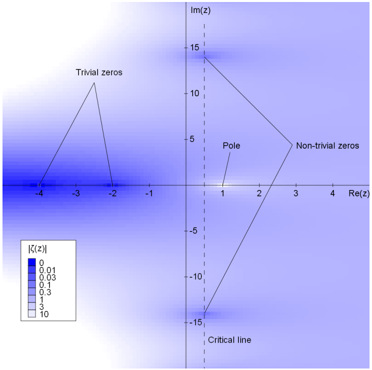
(Riemann 1859) A unique $\mathbf{C}$-valued function ζ that is

- *holomorphic* (complex-differentiable) when $s \neq 1$.
- given by $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^s}$ when $\mathrm{Re}(s) > 1$.

He checked that $\zeta(n) = 0$ for $n = -2, -4, -6, \ldots$ by relating these zeros to poles of the *gamma function*.

He speculated from examples that all other zeros of ζ live on the *critical line* $\mathrm{Re}(s) = \frac{1}{2}$.

Location of zeros $\leftrightsquigarrow$ distribution of prime numbers.



Wikipedia

(Hardy 1914) Infinitely many zeros on the line.

(Pratt–Robles–Zaharescu–Zeindler 2020) Among zeros s with $0 < \mathrm{Re}(s) < 1$, over five-twelfths on the line.

(Dedekind ~1860s) Generalize the formula

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

by replacing $\mathbf{Z}$ with other *rings* R.

Thus R is a set with operations $+$ and $\cdot$ resembling addition and multiplication.

$$\zeta_R(s) = \sum_{\substack{\text{ideals } I \subseteq R \\ |R/I| < \infty}} \frac{1}{|R/I|^s}$$

An *ideal* I is the collection of all finite linear combos $c_1 \cdot x_1 + \cdots + c_k \cdot x_k$ for some fixed $x_1, x_2, \ldots \in R$.

The *quotient* R/I is the set of translates $y + I \subseteq R$.

Note For ζ_R to make sense, the number of I such that $|R/I| = n$ must be finite for each $n > 0$.

Ex Every ideal of $\mathbf{Z}$ takes the form

$$n\mathbf{Z} = \{\text{multiples of } n\} \quad \text{for some integer } n \geq 0.$$

For instance, the ideal generated by 30 and 2025 is

$$\{c_1 \cdot 30 + c_2 \cdot 2025 \mid c_1, c_2 \in \mathbf{Z}\} = 15\mathbf{Z}.$$

We see that

$$\zeta_{\mathbf{Z}}(s) = \sum_{\substack{n\mathbf{Z} \subseteq \mathbf{Z} \\ n > 0}} \frac{1}{|\mathbf{Z}/n\mathbf{Z}|^s} = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

In other rings, ideals may not simplify so nicely.

Why care?

(Hilbert–Polyá ~1910s) The values e^{it} such that

$$\zeta(\tfrac{1}{2} + it) = 0 \quad \text{and} \quad 0 < \mathrm{Re}(\tfrac{1}{2} + it) < 1$$

behave like the eigenvalues of a generic unitary matrix.

$$\begin{aligned} \mathrm{RH} &\iff t \text{ always real} \\ &\iff e^{it} \text{ always on the unit circle.} \end{aligned}$$

(Weil ~1940s) There is a class of rings R,

coming from *algebraic geometry* over $\mathbf{Z}/p\mathbf{Z}$,

where analogous facts for ζ_R might be provable.

(Grothendieck–Deligne ~1960s–70s) Yes.

2 Weil's Rosetta Stone

Algebraic geometry studies *varieties*: shapes cut out by polynomial equations.

For simplicity, we'll stick to *(affine) hypersurfaces*

$$V_f = \{\vec{a} = (a_0, a_1, \ldots, a_d) \mid f(\vec{a}) = 0\},$$

cut out by a single polynomial $f \in \mathbf{Z}[x_0, x_1, \ldots, x_d]$.

V_f is *smooth* at $\vec{a}$ mod p when $\frac{\partial f}{\partial x_i}(\vec{a}) \not\equiv 0 \pmod{p}$ for some i. Else, *singular*.

Ex For $d = 1$, hypersurfaces are plane curves, like

$$f(x, y) = y^2 - x^3 - c \quad \text{for constant } c.$$

For which c is V_f smooth everywhere mod p?

The *ring of polynomial functions* on V_f mod p is

$$R_{f,p} := \frac{\mathbf{F}_p[x_0, \ldots, x_d]}{\mathbf{F}_p[x_0, \ldots, x_d] \cdot f}, \quad \text{where } \mathbf{F}_p = \mathbf{Z}/p\mathbf{Z}.$$

In a letter to his sister, Weil described a dictionary:

$\mathbf{Z}$	$R_{f,p}$	V_f mod p
$n\mathbf{Z}$	ideals	subvarieties
$p\mathbf{Z}$	maximal ideals	points

The first and last columns: a Rosetta stone between number theory and algebraic geometry.

Conj (Weil 1948) Assume V_f is smooth everywhere. Then zeros of $\zeta_{R_{f,p}}(s)$ have $\mathrm{Re}(s) \in \{\frac{1}{2}, \frac{3}{2}, \ldots, \frac{2d-1}{2}\}$.

Thm (Weil) True for $f = c_0 x_0^{n_0} + \cdots + c_d x_d^{n_d} - c$.

Set $\zeta_{f,p}(s) := \zeta_{R_{f,p}}(s)$ for convenience.

(Grothendieck ~1964) $\zeta_{f,p}(s)$ is a rational function in

$$\mathsf{q} := p^{-s}.$$

In fact: polynomials $\phi_0, \phi_1, \ldots, \phi_{2d-1}$ such that

$$\zeta_{f,p}(s) = \frac{\phi_1(\mathsf{q}) \cdot \phi_3(\mathsf{q}) \cdots \phi_{2d-1}(\mathsf{q})}{\phi_0(\mathsf{q}) \cdot \phi_2(\mathsf{q}) \cdots \phi_{2d-2}(\mathsf{q})}.$$

ϕ_k is the charpoly of a certain operator on a certain vector space, describing the *étale topology* of V_f.

Conj For all k, the roots of $\phi_k(\mathsf{q})$ live on the circle

$$|\mathsf{q}| = p^{k/2}.$$

$\implies$ Weil's Riemann Hypothesis.

(Deligne 1974) True for all f (smooth mod p).

In fact, Weil conjectured—and Deligne proved—results for all varieties, not just hypersurfaces.

Ex Taking $d = 1$ and $f(x, y) = y^2 - x^3 - c$:

$$\phi_0(t) = 1 - p\mathsf{q}$$
$$\phi_1(t) = 1 - a_p\mathsf{q} + p\mathsf{q}^2 \quad \text{for some integer } a_p,$$

giving $\zeta_{f,p}(s) = \dfrac{1 - a_p p^{-s} + p^{1-2s}}{1 - p^{1-s}}$. It turns out:

- $|a_p| \leq 2p^{1/2}$.
- So the two roots of $\phi_1(\mathsf{q})$ satisfy $|\mathsf{q}| = p^{-1/2}$.
- So the zeros of $\zeta_{f,p}(s)$ satisfy $\mathrm{Re}(s) = \frac{1}{2}$.

What if V_f has singularities?

Simplest case: Unique singularity at $(0, \ldots, 0)$. It turns out that here,

$$\zeta_{f,p}(s) = \zeta^\circ_{f,p}(s) \cdot \hat{\zeta}_{f,p}(s),$$

where:

- $\zeta^\circ_{f,p}$ satisfies Weil's Riemann Hypothesis.
- $\hat{\zeta}_{f,p}$ is analogous to $\zeta_{f,p}$, with the power-series ring

$$\hat{R}_{f,p} := \mathbf{F}_p[\![x_0, x_1, \ldots, x_d]\!]/(f)$$

in place of $R_{f,p}$.

Does $\hat{\zeta}_{f,p}(s) = \displaystyle\sum_{\substack{I \subseteq \hat{R}_{f,p} \\ |\hat{R}_{f,p}/I| < \infty}} \frac{1}{|\hat{R}_{f,p}/I|^s}$ satisfy a RH?

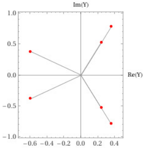
Ex For $f = y^2 - x^3$,

$$\hat{\zeta}_{f,p}(s) = \frac{1 - p^{1-2s}}{1 - p^{-s}} = \frac{1 + p\mathsf{q}^2}{1 - \mathsf{q}}.$$

Ex For $f = y^3 - x^4$,

$$\hat{\zeta}_{f,p}(s) = \frac{1 + p\mathsf{q}^2 + p^2\mathsf{q}^3 + p^2\mathsf{q}^4 + p^3\mathsf{q}^6}{1 - \mathsf{q}}.$$

Here, not all roots satisfy $|\mathsf{q}| = p^{-1/2}$.



WolframAlpha

3 From Curves to Knots

For general $f(x, y)$, it turns out there's $P_f(\mathsf{t}, \mathsf{q}) \in \mathbf{Z}\left[\mathsf{t}, \mathsf{q}, \frac{1}{1-\mathsf{q}}\right]$ such that

$$\hat{\zeta}_{f,p}(s) = \frac{P_f(p, p^{-s})}{1 - p^{-s}} \quad \text{for almost all } p.$$

These polynomials have many surprising features.

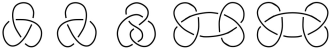
(Gorsky–Mazin 2013) If $f = y^n - x^{n+1}$, then $P_f(1, 1) = \frac{(2n)!}{(n+1)!n!}$, the nth Catalan number.

Ex If $f = y^3 - x^4$, then

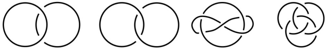
$$P_f(\mathsf{t}, \mathsf{q}) = 1 + \mathsf{t}\mathsf{q}^2 + \mathsf{t}^2\mathsf{q}^3 + \mathsf{t}^2\mathsf{q}^4 + \mathsf{t}^3\mathsf{q}^6,$$
$$P_f(1, 1) = 5.$$

The P_f also arise from *knot/link invariants*.

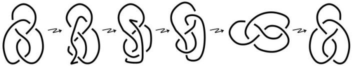
A *knot* is an embedding of a circle into $\mathbf{R}^3$ or S^3.



A *link* is a generalization allowing multiple circles.



Two links are *isotopic* when we can deform one into the other without self-intersections.



Chmutov–Duzhin–Mostovoy

Let $S^3_\epsilon = \{(x, y) \in \mathbf{C}^2 \mid |x|^2 + |y|^2 = \epsilon\}$. The subset

$$L_f = \{(x, y) \in S^3_\epsilon \mid f(x, y) = 0\}$$

is a link in S^3_ϵ when $\epsilon > 0$ is small enough.

Ex If $f = y^n - x^m$, then L_f is the (m, n) *torus link*. It's a knot when m and n are coprime.

Wolfram Language

Ex If $f = y^4 - 2x^3y^2 - 4x^5y + x^6 - x^7$, then L_f is the closure of the braid

Cherednik–Danilenko

Conj (Oblomkov–Shende ~2010)

$$P_f(1, \mathsf{q}^2) = \lim_{\mathsf{a}\to 0} \left[(\mathsf{q}/\mathsf{a})^{\mu}\, \mathbb{P}_{L_f}(\mathsf{a}, \mathsf{q})\right],$$

where $\mu \in \mathbf{Z}$ and $\mathbb{P}$ is the *HOMFLYPT invariant*, discovered in 1986 and defined by the *skein rules*:

(1) $$\mathsf{a}\mathbb{P}_{\times} - \mathsf{a}^{-1}\mathbb{P}_{\times} = (\mathsf{q} - \mathsf{q}^{-1})\mathbb{P}_{)(}$$

(2) $$\mathbb{P}_{\bigcirc} = 1$$

Full statement incorporates a, by upgrading P_f.

(Maulik 2012) True for all plane curves.

Proof sketch Blow up the singularity repeatedly.

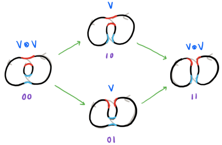
Control P_f via *wall crossing* and L_f via skein algebra.

Conj (Oblomkov–Rasmussen–Shende ~2013)

$$P_f(\mathsf{t}^2, \mathsf{q}^2) = \lim_{\mathsf{a}\to 0} \left[(\mathsf{q}/\mathsf{a})^{\mu}\, \mathbf{P}_{L_f}(\mathsf{a}, \mathsf{t}, \mathsf{q})\right],$$

where $\mathbf{P}$ is a refinement of $\mathbb{P}$, discovered in the 2000s by Khovanov–Rozansky.

$\mathbf{P}$ is defined by *categorifying* (1)–(2). Unknown how to categorify Maulik's proof.



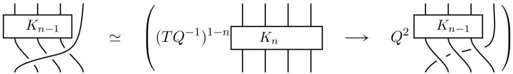
Melissa Zhang

(Kivinen–T 2025) True for $f = y^3 - x^m$ with $3 \nmid m$.

Cor (Kivinen–T) New closed formula for $\mathbf{P}_{\mathrm{torus}(m,3)}$.

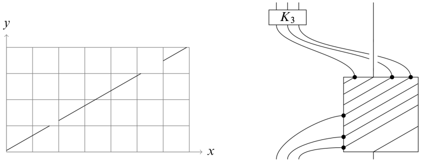
Proof Sketch

1 Recursions that compute $\mathbf{P}_{\mathrm{torus}(m,n)}(\mathsf{a},\mathsf{t},\mathsf{q})$, due to Elias–Hogancamp–Mellit.



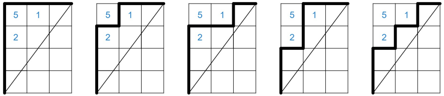
Elias–Hogancamp

Encoding of braids into grid diagrams, due to Mellit.



Mellit

2 For m, n coprime, yields a sum over $\frac{(m+n-1)!}{m!n!}$ many *rational Dyck paths*.



At the same time, $\hat{R}_{f,p} \simeq \mathbf{F}_p[\![u^m, u^n]\!]$.

We relate Dyck paths to $\hat{R}_{f,p}$-*submodules* $M \subseteq \mathbf{F}_p[\![u]\!]$.

3 We then relate

$$\sum_M \frac{1}{|\mathbf{F}_p[\![u]\!]/M|^s} \quad \text{and} \quad \sum_I \frac{1}{|\hat{R}_{f,p}/I|^s}$$

Uses *Serre duality*. For now, requires $\min(m,n) \leq 3$.

Big Picture I study special functions that appear in

- *algebraic geometry*
- *knot theory*, especially *q-algebra*
- *combinatorics*

I use *representation theory* to decompose them into simpler functions, like in Fourier analysis.

Hikita (2016) Dyck-path formula $\rightsquigarrow$ decomposition of *diagonal harmonics* into *LLT polys*.

T (2021) Generalizations of $\mathbb{P}$, $\mathbf{P}$ to Coxeter groups, explicitly decomposing into *irreducible characters*.

Galashin–Lam–T–Williams (2024) Ideas from T (2021) solve conjectures about Coxeter groups from 2012.

4 Cherednik's New Hypothesis

Recall: For $f = y^3 - x^4$ and <u>prime</u> p, the roots of

$$P_f(p, \mathsf{q}) = 1 + p\mathsf{q}^2 + p^2\mathsf{q}^3 + p^2\mathsf{q}^4 + p^3\mathsf{q}^6$$

do not all satisfy $|\mathsf{q}| = p^{-1/2}$.

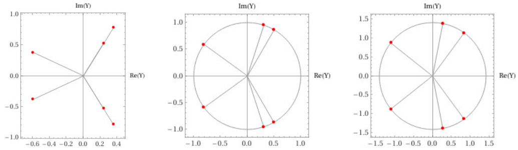
Conj (Cherednik 2018) For any plane curve f:

$0 < t \leq \frac{1}{2} \implies$ all roots of $P_f(t, \mathsf{q})$ satisfy $|\mathsf{q}| = t^{-1/2}$.

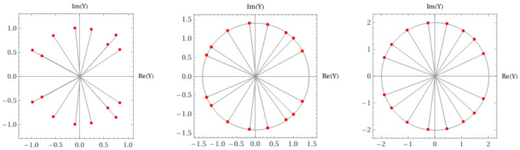
Would imply *arithmetic* constraints on $\mathbf{P}_L(\mathsf{a}, \mathsf{t}, \mathsf{q})$.

Dream (Cherednik) Related to the Lee–Yang theorem on zeros of *partition functions* in statistical mechanics.

$f = y^3 - x^4 \qquad t \in \{2,\ 1,\ \tfrac{1}{2}\}$:



$f = y^4 - 2x^3y^2 - 4x^5y + x^6 - x^7 \qquad t \in \{1,\ \tfrac{1}{2},\ \tfrac{1}{4}\}$:



Thank you for listening.