Last time p : E to X is a covering map iff,

for all x in X, have open U ni x s.t.

1) p^{-1}(U) is homeo to a union of disjoint copies of U
2) p restricts to a homeo from each copy onto U

here E is called a covering space or cover of X

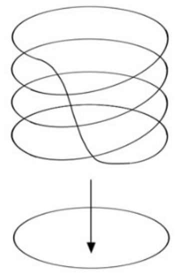
Ex consider p(x) = (cos(2πx), sin(2πx))

p : (–1, 1) to S^1 is not a covering map [why?]
but p : R to S^1 is a covering map

Ex identity maps Id : X → X are always covering maps

Ex more generally:

for any n > 0, a covering p_n : S^1 to S^1 s.t. the fiber p_n^{-1}(x) has cardinality n for all x



https://ahilado.wordpress.com/2017/04/14/covering-spaces/

we say that the covering is of degree n, or n-fold

Ex $p_m \times p_n : S^1 \times S^1$ to $S^1 \times S^1$ is a covering:

of what degree? [mn]

more generally:

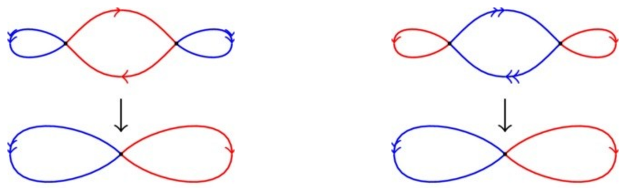
if $p : E$ to X and $p' : E'$ to X' are covering maps, then so is $p \times p'$

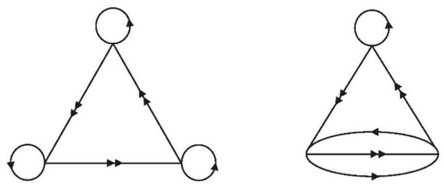
Ex consider the relation ~ on S^2 given by $(x, y, z) \sim (-x, -y, -z)$

$S^2/\sim$ is called the real projective plane, or RP^2

S^2 to RP^2 is a 2-fold covering map

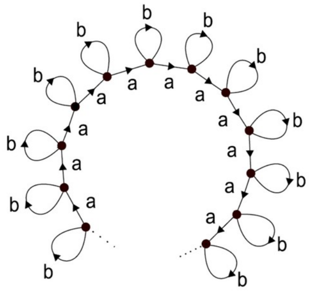
Ex coverings of the figure-eight:



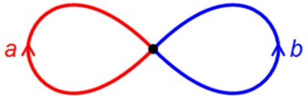


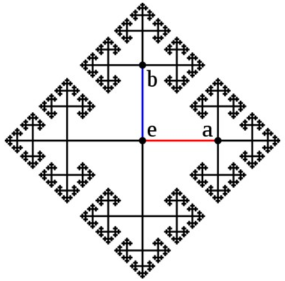
(1) https://www.homepages.ucl.ac.uk/~ucahjde/tg/html/cov-01.html
(2) https://groupoids.org.uk/images/fig10-3.jpg

weirder:



https://www.math.cmu.edu/~nkomarov/NK-NormalSubFreeGrp.pdf





https://math.stackexchange.com/a/3762676

Thm if $p : E$ to X is a covering, then:

I) for any path $\gamma : [0, 1]$ to X and e in E s.t. $p(e) = \gamma(0)$,

a unique $\Gamma : [0, 1]$ to R s.t. $\Gamma(0) = e$ and $\gamma = p \circ \Gamma$, called a lift of γ

II) for any path homotopy $h : [0, 1]^2$ to X s.t. $h(–, 0) = \gamma$, and lift Γ of γ,

a unique path-homotopy $H : [0, 1]^2$ to R s.t. $H(–, 0) = \Gamma$ and $h = p \circ H$, called a lift of h

Pf Lemmas 54.1 and 54.2 in Munkres

finally, return to:

<u>Thm</u> $\Phi : Z$ to $\pi_1(S^1, o)$ def by $\Phi(n) = [\omega_n]$ is an isomorphism

where $o = (1, 0)$,
$\omega_n(s) = (\cos(2\pi ns), \sin(2\pi ns))$ (n in Z)

previously: Φ is a homomorphism
today: Φ is bijective

<u>Φ surjective</u> for all loops γ at o have n s.t. $[\gamma] = [\omega_n]$

observe: $p^{-1}(o) = Z$

so for any a in Z, I) gives Γ s.t.
$\Gamma(0) = a$ and $\gamma = p \circ \Gamma$
let $b = \Gamma(1)$
then b in $p^{-1}(o) = Z$

now, by PS6, #2, have $[\Gamma] = [\acute{\omega}_{a, b}]$
so by PS5, #1, have $[\gamma] = [\omega_{b - a}]$

<u>Φ injective</u> suppose that $[\omega_m] = [\omega_n]$

observe: $\acute{\omega}_{0, m}$ lifts ω_m
so for any h from ω_m to ω_n, II) gives H s.t.
$H(–, 0) = \acute{\omega}_{0, m}$ and $h = p \circ H$
let $\Gamma = H(–, 1)$
then Γ lifts $\omega_n = h(–, 1)$

but by I), there is a unique lift of ω_n starting at 0

so $\Gamma = \acute{\omega}_{0, n}$

so H goes from $\acute{\omega}_{0, m}$ to $\acute{\omega}_{0, n}$

so $m = n$