

Last time $(f : R \text{ to } R^2, g : R^2 \text{ to } R) \text{ def by}$
 $f(x) = (x, 0) \text{ and } g(x, y) = x$
is a homotopy equivalence

note: $g \circ f = \text{id}_R$, stronger than $g \circ f \sim \text{id}_R$

Rem a homotopy equivalence (f, g) where
 f is an inclusion and $g \circ f = \text{id}$ is called
a deformation retraction

Rem in general, if $g \circ f = \text{id}$, then
 f is injective and g is surjective

Ex there is a deformation retraction
from $R^2 - \{(0, 0)\}$ onto S^1 [draw]

Prob if $X \sim Y$ via (f, g) and $Y \sim Z$ via (j, k)
then $X \sim Z$ via $(j \circ f, g \circ k)$ [diagram]

need to prove: $j \circ f \sim \text{id}_X$ and $g \circ k \sim \text{id}_Z$

Cor \sim is an equiv relation on spaces

[Pf $X \sim X$ via $(\text{id}_X, \text{id}_X)$
 $X \sim Y$ via (f, g) iff $Y \sim X$ via (g, f)
 $X \sim Y$ and $Y \sim Z$ imply $X \sim Z$ by prob]

Rem if $X \sim Y$, then
1) X is connected iff Y is
2) X is path connected iff Y is

but X compact need not imply Y compact

(Munkres §51) a path in X is a cts map
 $\gamma : [0, 1]$ to X

$\gamma(0)$ is the starting pt and $\gamma(1)$ is the ending pt

Df given paths $\gamma, \gamma' : [0, 1]$ to X
s.t. $\gamma(0) = \gamma'(0)$ and $\gamma(1) = \gamma'(1)$

a path homotopy from γ to γ' is a homotopy

$$\varphi : [0, 1] \times [0, 1] \text{ to } X \text{ from } \gamma \text{ to } \gamma'$$

s.t. $\varphi(0, t) = \gamma(0) = \gamma'(0)$ and $\varphi(1, t) = \gamma(1) = \gamma'(1)$
for all times t

in this case, we write $\gamma \sim_p \gamma'$

Prob for any space X , points x, y in X ,
paths $\gamma_1, \gamma_2, \gamma_3$ from x to y ,

given path homotopies φ from γ_1 to γ_2 ,
 ψ from γ_2 to γ_3 ,

find path homotopy from γ_1 to γ_3 using φ, ψ

Cor \sim_p is an equiv rel on paths from x to y

[why interesting?]

Df for paths β, γ s.t. $\beta(1) = \gamma(0)$ [draw], set

$$\begin{aligned} (\beta * \gamma)(s) &= \beta(2s) & \text{for } s \leq 1/2 \\ &= \gamma(2s - 1) & \text{for } s \geq 1/2 \end{aligned}$$

$\beta * \gamma$ is a path called the concatenation of β and γ

Thm write $[\gamma]$ for “path homotopy class of γ ”

then $[\beta * \gamma]$ only depends on $[\beta]$ and $[\gamma]$

that is:

if $\beta \sim_p \beta'$ and $\gamma \sim_p \gamma'$, then $\beta * \gamma \sim_p \beta' * \gamma'$

(Munkres §52) a loop based at x is a path γ
s.t. $\gamma(0) = x = \gamma(1)$

Ex the constant loop at x is def by
 $e_x(s) = x$ for all s

Prob show that if X is contractible,
then any loop based at x is $\sim_p e_x$

Df the fundamental group of X at x is

$$\pi_1(X, x) = \{\text{loops in } X \text{ based at } x\} / \sim_p$$

under the operation $[\beta] * [\gamma] := [\beta * \gamma]$

the id element is $[e_x]$

[Rem in general, there is a group $\pi_n(X, x)$
called the n th homotopy group]

Cor $\pi_1(\mathbb{R}, x)$ and $\pi_1(\mathbb{R}^2, x)$ are trivial
[by previous prob]

next time: $\pi_1(S^1, x) \cong \mathbb{Z}$