last time, we saw:

- given inner prod. spaces $(V, < , >)$, $(W, \{ , \})$,
  $T : V$ to W defines an adjoint $T^* : W$ to V
  characterized by $\{Tv, w\} = <v, T^*w>$

- if V, W are finite-dim'l
  then get orthonormal bases for V and W
  i.e., the basis vectors e_i satisfy

  $<e_i, e_i> = 1$

  $<e_j, e_i> = 0$ if $j \neq i$

let M be the matrix of T wrt the orthonormal bases

<u>Q</u> what is the matrix of T^*?

[let them cook a bit]

<u>A</u> WLOG can take $V = F^n$ and $W = F^m$
under their dot / skew-dot products
and choose the bases to be the std bases

$F = R$: for all v and w, we require

$\{Tv, w\} = (Tv)^t w = v^t T^t w$

$\{v, T^*w\} = v^t T^*w$

taking $v = e_j$ and $w = e_i$ shows

$T^t_{\{j, i\}} = T^*_{\{j, i\}}$ for all j, i

so $T^* = T^t$

$F = C$: for all v and w,

$\{Tv, w\} = (Tv)^t \bar{w} = v^t T^t \bar{w}$

$\{v, T^*w\} = v^t \overline{(T^*w)} = v^t \overline{(T^*)} \bar{w}$

so $T^t_{\{j, i\}} = \overline{(T^*)}_{\{j, i\}}$ for all j, i

so $T^* = \overline{(T^t)}$ [that is:]

Thm wrt orthonormal bases for both V and W, T : V to W and T* : W to V have matrices that are mutual conjugate transposes [we may write M* = (M^t)¯]

[works for both R and C: conjugation does nothing in the case of R]

[we deduce:]

Properties of Adjoints

- (aS + T)* = a¯S* + T* [for all a in F and S, T]
- Id* = Id
- (T*)* = T
- (S ○ T)* = T* ○ S*
- T* is invertible iff T is, and in this case, (T*)^{-1} = (T^{-1})*

[backing up a bit to §6B–6C, we revisit]

Orthogonal Complements

Df given linear U sub V, its orthogonal complement (wrt < , >) is

U⊥ = {v in V | <v, u> = 0 for all u in U}

- by definiteness of < , >, U⊥ cap U = {**0**}
- [we saw last time:] by Gram–Schmidt, if V is finite-dim'l, then V = U + U⊥

if V is finite-dim'l, then we also have:

- (U_1 + U_2)⊥ = [what?] (U_1)⊥ cap (U_2)⊥ [and why?]
- (U⊥)⊥ = U

[notice similarity to properties of adjoints...]

Q how do adjoints interact with complements?

Thm 1) $\mathrm{im}(T)^\perp = \ker(T^*)$ and $\ker(T^*)^\perp = \mathrm{im}(T)$
2) $\ker(T)^\perp = \mathrm{im}(T^*)$ and $\mathrm{im}(T^*)^\perp = \ker(T)$

Pf 2) follows from 1) by swapping T and T*

to show $\mathrm{im}(T)^\perp = \ker(T^*)$:

w in $\ker(T^*)$ iff $T^*w = \mathbf{0}\_V$
iff $\langle v, T^*w \rangle = 0$ for all v in V
iff $\langle Tv , w \rangle = 0$ for all v in V
iff w in $\mathrm{im}(T)^\perp$

taking $(\ )^\perp$ of both sides, we get $\ker(T^*)^\perp = \mathrm{im}(T)$

Cor if V, W are finite-dim'l, then direct sums:

$V = \ker(T) + \mathrm{im}(T^*)$
$W = \mathrm{im}(T) + \ker(T^*)$

(Axler §7B) now consider a linear op $T : V$ to V

Df [a linear op] T is self-adjoint iff $T^* = T$,
i.e., $\langle Tv', v \rangle = \langle v', Tv \rangle$ for all v, v'

if M is the matrix of T wrt an orthonormal basis,
then $T^* = T$ iff $M^* = M$
[where M* denotes the conjugate tranpose]

Prop if T is self-adjoint (over either R or C)
then every eigenval of T is real
[is the converse true? no]

<u>Pf</u> let v be an eigenvec with eigenval λ

[what is $\langle Tv, v\rangle$? pause]

$\langle Tv, v\rangle = \langle \lambda v, v\rangle = \lambda\langle v, v\rangle$

but also $\langle v, Tv\rangle = \langle v, \lambda v\rangle = \bar{\lambda}\langle v, v\rangle$

since $v \neq \mathbf{0}$, we know $\langle v, v\rangle \neq 0$ by definiteness

so $\lambda = \bar{\lambda}$

[here is a slightly weaker notion:]

<u>Df</u> a linear op T is normal iff

$T^* \circ T = T \circ T^*$, i.e., they commute

[thus, any self-adjoint operator is normal]

<u>Ex</u> let $M : F^2$ to F^2 be

$M = \begin{matrix} 1 & -1 \\ 1 & 1 \end{matrix}$ so that $M^* = \begin{matrix} 1 & 1 \\ -1 & 1 \end{matrix}$

then $M^* \neq M$, yet $M^*M = \begin{matrix} 2 & 0 \\ 0 & 2 \end{matrix} = MM^*$

<u>Prop</u> T is normal iff $\|Tv\| = \|T^*v\|$ for all v in V

<u>Pf</u> T is normal

iff $T^* \circ T - T \circ T^*$ is zero

iff $\langle (T^* \circ T - T \circ T^*)v, v\rangle = 0$ for all v

iff $\langle T^*Tv, v\rangle = \langle TT^*v, v\rangle$ for all v

iff $\langle Tv, Tv\rangle = \langle T^*v, T^*v\rangle$ for all v

<u>Thm</u> if $T : V$ to V is normal, then:

1) $\ker(T^*) = \ker(T)$

2) $\mathrm{im}(T^*) = \mathrm{im}(T)$

3) $T - \lambda$ is normal for all λ in F

<u>Pf</u> 1) follows from the prop

2) from $\mathrm{im}(T^*) = \ker(T)^\perp = \ker(T^*)^\perp = \mathrm{im}(T)$

3) from $(T - \lambda) \circ (T - \lambda)^*$

$$
\begin{aligned}
&= (T - \lambda) \circ (T^* - \bar{\lambda}) \\
&= T \circ T^* - \bar{\lambda} T - \lambda T^* + |\lambda|^2 \\
&= T^* \circ T - \bar{\lambda} T - \lambda T^* + |\lambda|^2 \\
&= (T^* - \bar{\lambda}) \circ (T - \lambda) \\
&= (T - \lambda)^* \circ (T - \lambda)
\end{aligned}
$$

<u>Cor</u> if $T : V$ to V is normal

then V is a direct sum $\ker(T) + \mathrm{im}(T)$

<u>Pf</u> V is a direct sum $\ker(T) + \mathrm{im}(T^*)$

but $\mathrm{im}(T^*) = \mathrm{im}(T)$

<u>Cor</u> if $T : V$ to V is normal

then for all v in V and λ in F, we have

$Tv = \lambda v$ if and only if $T^*v = \bar{\lambda} v$

<u>Pf</u> $\ker(T^* - \bar{\lambda}) = \ker((T - \lambda)^*) = \ker(T - \lambda)$

[our goal next time: the] <u>Spectral Thm</u>

if V is finite-dim'l and $T : V$ to V is normal

then T is diagonalizable