Warmup let $D : F[x]$ to $F[x]$ be $D(p) = dp/dx$

what are some D-stable linear subspaces?

if W is D-stable and q in W,
then $D(q)$, $D^2(q)$, $D^3(q)$, … in W

e.g., for all n,
$P_n = \{p \mid p = 0 \text{ or } \deg(p) \leq n\}$ is D-stable

non-example:
$\{p \mid p(3) = 0\}$ is not D-stable [why?]

Q does D have eigenvectors in $F[x]$?

no: cannot solve $D(p) = \lambda p$ for polynomial p

Df a formal power series over F ($= \mathbb{R}, \mathbb{C}$) is an infinite sum $f(x) = \sum_{k \geq 0} a_k x^k$ with a_i in F for all i

$F[[x]]$ = {formal power series $f(x)$}

the formal derivative on $F[[x]]$ is
the F-linear operator D def by

$D(\sum_k a_k x^k) = \sum_k k a_k x^{k-1}$

Q does D have eigenvectors in $F[[x]]$?

yes: $\exp(\alpha x)$ where $\exp(x) = \sum_k (1/k!) x^k$

Q is $F[[x]]$ iso to a familiar v.s.? [yes: $F^{\mathbf{N}}$]

(Axler §5C) recap: fix T : V to V

- if V = R^n, then T might have no eigenvals

[example? any rotation of θ ≠ 0, π radians]

- if V = F[x], then T might have no eigenvals

[multiplication by x gives another example]

- if V = {**0**}, then T has no eigenvals by defn

[stated last time:]

Thm if F = C and V is fin. dim. and not {**0**}
then T must have <u>some</u> eigenval

what does this fact say in terms of matrices?

Lem if V is fin. dim. and T has an eigenline
then T has a matrix of the form

$$\begin{matrix} * & * & \dots & * \\ 0 & * & \dots & * \\ \dots & \dots & & \dots \\ 0 & * & \dots & * \end{matrix}$$

Pf suppose v is the eigenvector for T
set v_1 = v
extend to a basis (v_i)_i for V

Cor if F = C and n > 0
then any n × n matrix is conjugate to
one of this form

[something stronger holds:]

<u>Thm</u> if $F = C$ and V is fin. dim., then any T has an upper-triangular matrix:

$$\begin{matrix} * & * & \dots & * \\ 0 & * & \dots & * \\ 0 & \dots & \dots & \dots \\ 0 & \dots & 0 & * \end{matrix}$$

idea: induct on dim V
if dim V = 0 then done
else T has some eigenvector v with eigenvalue λ
[what next?] Tv = λv means v in ker(T – λ)
so dim ker(T – λ) > 0
so dim im(T – λ) < dim V
so want to apply inductive hypothesis to im(T – λ)

<u>Stability Lem</u> for any T : V to V and p(z) in C[z], im(p(T)) is T-stable

<u>Pf</u> if w in im(p(T))
then w = p(T) v for some v in V
so Tw = T(p(T) v) = p(T)(T v) in im(p(T))

to see the last equality:
zp(z) = p(z)z as polynomials
so Tp(T) = p(T)T as operators on V
[even though general operators don't commute!]

<u>Pf of Thm</u> let n = dim V; can assume n > 0

suppose Tv = λv where v ≠ **0**
let W = im(T – λ)

by lem, W is T-stable and $\dim W < n$
by inductive hypothesis,
have ordered basis for W making $T|_W$ triangular:
say, $(w_1, \ldots, w_m)$

extend ordered basis from W to V:
say $(w_1, \ldots, w_m, v_1, \ldots, v_\ell)$
claim that T is triangular wrt this extended basis[!]
suffices to check Tv_i's: for all i,

$$Tv_i = (T - \lambda)v_i + \lambda v_i \text{ in } W + Fv_i$$

[draw matrix] □

<u>Cor</u> any square matrix is conjugate to an upper-triangular matrix

<u>Cor</u> let $f : \mathrm{Mat}_2 \to F$ be a function def by a polynomial in matrix coords

i.e. $f = p(x_{11}, x_{12}, x_{21}, x_{22})$

$$\begin{matrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{matrix}$$

if f is conj-invariant
then f is a polynomial in tr and det

<u>Pf Sketch</u> $\mathrm{Tri}_2 = \{$upper-triangular matrices$\}$
$\mathrm{Diag}_2 = \{$diagonal matrices$\}$

by thm, f is uniq. determ. by $f|_{\mathrm{Tri}_2}$
observe: $f|_{\mathrm{Tri}_2} = q(x_{11}, x_{12}, x_{22})$ for some q
$\mathrm{tr}|_{\mathrm{Tri}_2} = x_{11} + x_{22}$
$\det|_{\mathrm{Tri}_2} = x_{11} x_{22}$
want $f|_{\mathrm{Tri}_2}$ to be a poly in $x_{11} + x_{22}$, $x_{11} x_{22}$

Claim 1) q is indep of x12

so q is uniq. determ. by q|_{Diag_2}

Claim 2) q|_{Diag_2} invariant for x11 ↔ x22

to finish, use <u>Viète’s Thm</u>:

any poly in X, Y invariant under X ↔ Y
is a poly in X + Y and XY
[look up “elementary symmetric functions”]

shows q|_{Diag_2} is a poly in x11 + x22, x11 x22

1) $\begin{matrix} a & \\ & 1/a \end{matrix} \quad \begin{matrix} x11 & x12 \\ & x22 \end{matrix} \quad \begin{matrix} 1/a & \\ & a \end{matrix} \quad = \quad \begin{matrix} x11 & aa\ x12 \\ & x22 \end{matrix}$

so q(x11, x12, x22) = q(x11, a^2 x12, x22) for all a

[exercise:] forces q to be indep of x12

2) $\begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} \quad \begin{matrix} x11 & \\ & x22 \end{matrix} \quad \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} \quad = \quad \begin{matrix} x22 & \\ & x11 \end{matrix}$

forces q|_{Diag_2} invariant under x11 ↔ x22 □

return to the triangularity thm:

if F = C and V is fin. dim.
then T = T′ + T″, where
T′ has a diagonal matrix
T″ has a nilp. upper-triangular matrix
[in particular, it has 0’s on the diagonal]

$$\begin{matrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{matrix} \quad = \quad \begin{matrix} * & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{matrix} \quad + \quad \begin{matrix} 0 & * & * \\ 0 & 0 & * \\ 0 & 0 & 0 \end{matrix}$$

<u>Q</u> how block-diagonal can we make T?

next week:

<u>Thm</u> if W fin. dim. and $S : W$ to W nilpotent
then S has a matrix where
the only nonzero entries are 1's
on the "super-diagonal"

problem: in general, $T = T' + T''$

<u>Q</u> basis where T' is super-diagonal,
T'' is diagonal
simultaneously?