4.

Notes on frame operators, the Welch bounds, and equiangular lines.

4.1.

Let V be an inner product space of dimension n, and let $(e_1, \dots, e_n)$ be an orthonormal basis for V. What does projection onto e_i look like? It is precisely the linear operator on V that sends $v \mapsto \langle v, e_i \rangle e_i$. We can reconstruct v as the sum of these projections:

$$v = \langle v, e_1 \rangle e_1 + \cdots + \langle v, e_n \rangle e_n.$$

In particular we have the Bessel identity

$$\|v\|^2 = |\langle v, e_1 \rangle|^2 + \cdots + |\langle v, e_n \rangle|^2.$$

If we replace $(e_i)_i$ with a more general list, can we still reconstruct v from its inner products with the vectors in the list?

Definition 4.1. Let $f = (f_1, \dots, f_m)$ be an arbitrary list of vectors in V. The associated *frame operator* on V is the linear operator $\Phi_f : V \to V$ defined by

$$\Phi_f v = \langle v, f_1 \rangle f_1 + \cdots + \langle v, f_m \rangle f_m.$$

We often, but not always, assume that the f_i are unit vectors, so that $\langle v, f_i \rangle f_i$ is the projection of v onto f_i. We say that f is a *tight frame* if and only if

$$\Phi_f v = \lambda \cdot v \quad \text{for some fixed real } \lambda > 0 \text{ and all } v \in V.$$

Example 4.2. Every orthonormal basis is a tight frame. Can we find a tight frame that is not an orthonormal basis? Sure: Take a list composed of several orthonormal bases. Another example where $V = \mathbf{R}^2$ and $\langle -, - \rangle$ is the dot product: Take

$$(f_1, f_2, f_3, f_4) = \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right).$$

What is the frame operator? $\Phi_f \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$ and $\Phi_f \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$.

Example 4.3. Another example in $\mathbf{R}^2$: Take the vertices of an equilateral triangle centered at the origin:

$$(f_1, f_2, f_3) = \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1/2 \\ \sqrt{3}/2 \end{pmatrix}, \begin{pmatrix} -1/2 \\ -\sqrt{3}/2 \end{pmatrix} \right).$$

What is the frame operator? $\Phi_f \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3/2 \\ 0 \end{pmatrix}$ and $\Phi_f \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 3/2 \end{pmatrix}$.

4.2.

We can think of the list f as a linear map $A_f : F^m \to V$: namely, $A_f(e_i) = f_i$, where $(e_i)_i$ is now the standard basis of F^m. The adjoint $A_f^* : V \to F^m$, with respect to the inner product $\langle -, - \rangle$ on V and the skew dot product on F^m, must satisfy

$$e_i \cdot \overline{A_f^* v} = \langle A_f e_i, v \rangle = \langle f_i, v \rangle \quad \text{for all } v \in V \text{ and } i.$$

This tells us that $\overline{A_f^* v} = \langle f_1, v \rangle e_1 + \cdots + \langle f_m, v \rangle e_m$, from which

$$A_f^* v = \langle v, f_1 \rangle e_1 + \cdots + \langle v, f_m \rangle e_m.$$

It also shows us that the frame operator is precisely $\Phi_f = A_f \circ A_f^* : V \to V$.

In particular, Φ_f is self-adjoint(!). So by the spectral theorem, there is always a basis of V consisting of orthonormal eigenvectors for Φ_f. Moreover, the eigenvalues of Φ_f must be real and nonnegative: say, $\lambda_1 \geq \cdots \geq \lambda_n \geq 0$. In particular:

Lemma 4.4. *$f = (f_1, \ldots, f_m)$ forms a tight frame if and only if the eigenvalues of Φ_f are all equal and positive.*

4.3.

What else can we say about the eigenvalues? Note that $A_f^* A_f$ is easier to understand:

$$A_f^* A_f e_i = \langle f_i, f_1 \rangle e_1 + \cdots + \langle f_i, f_m \rangle e_m,$$

so the matrix of $A_f^* A_f$ in the standard basis is given by

$$(A_f^* A_f)_{j,i} = \langle f_i, f_j \rangle.$$

Using the identity $\operatorname{tr}(AB) = \operatorname{tr}(BA)$, we deduce that

$$\overbrace{\lambda_1 + \cdots \lambda_n}^{\operatorname{tr}(\Phi_f)} = \operatorname{tr}(A_f A_f^*) = \operatorname{tr}(A_f^* A_f) = \sum_i \|f_i\|^2.$$

Moreover, since Φ_f is diagonal in the eigenvector basis, Φ_f^2 is also diagonal in that basis. The eigenvalues of Φ_f^2 are the squares of the eigenvalues of Φ_f. Therefore,

$$\overbrace{\lambda_1^2 + \cdots \lambda_n^2}^{\operatorname{tr}(\Phi_f^2)} = \operatorname{tr}((A_f A_f^*)^2) = \operatorname{tr}((A_f^* A_f)^2) = \sum_{i,j} \langle f_i, f_j \rangle \langle f_j, f_i \rangle = \sum_{i,j} |\langle f_i, f_j \rangle|^2.$$

Theorem 4.5. *For any list of vectors $f_1, \ldots, f_m \in V$,*

$$\sum_{i,j} |\langle f_i, f_j \rangle|^2 \geq \frac{1}{n} \left(\sum_i \|f_i\|^2 \right)^2.$$

Equality holds if and only if the f_i form a tight frame.

Proof. We must show that $n\,\mathrm{tr}(\Phi_f^2) \geq \mathrm{tr}(\Phi_f)^2$. Let $\vec{u} = (1,\dots,1)$ and $\vec{v} = (\lambda_1,\dots,\lambda_n)$ in F^n. By Cauchy–Schwarz,

$$\begin{aligned} n(\lambda_1^2 + \cdots + \lambda_n^2) &= \|\vec{u}\|^2\|\vec{v}\|^2 \\ &\geq |\langle \vec{u}, \vec{v}\rangle|^2 \\ &= (\lambda_1 + \cdots + \lambda_n)^2. \end{aligned}$$

Equality holds above if and only if the λ_i are all equal. □

Corollary 4.6 (First Welch Bound)**.** *For any list of unit vectors $f_1,\dots,f_m \in V$,*

$$\sum_{i\neq j} |\langle f_i, f_j\rangle|^2 \geq \frac{m^2}{n} - m.$$

Equality holds if and only if the f_i form a tight frame.

Remark 4.7. More generally, the kth Welch bound (proved in 1974) states that above,

$$\sum_{i\neq j} |\langle f_i, f_j\rangle|^{2k} \geq \frac{m^2}{\binom{n+k-1}{k}} - m.$$

It can be proved via a similar argument, but applied to symmetric tensors in $V^{\otimes k}$. See "Geometry of the Welch Bounds" by Datta–Howard–Cochran.

4.4.

Note that by a pigeonhole-type argument,

$$\max_{i\neq j} |\langle f_i, f_j\rangle|^2 \geq \frac{1}{m(m-1)} \sum_{i,j} |\langle f_i, f_j\rangle|^2,$$

with equality if and only if the inner products $\langle f_i, f_j\rangle$ are all equal.

Definition 4.8. A collection of unit vectors $f_1,\dots,f_m$ is *equiangular* if and only if $\langle f_i, f_j\rangle$ is the same for all $i \neq j$.

Corollary 4.9. *For any list of unit vectors $f_1,\dots,f_m \in V$,*

$$\max_{i\neq j} |\langle f_i, f_j\rangle|^2 \geq \frac{m-n}{n(m-1)}.$$

Equality holds if and only if the f_i form an equiangular tight frame.

Example 4.10. The vectors f_1, f_2, f_3 in Example 4.3 are unit vectors. We compute

$$A_f^* A_f = \begin{pmatrix} 1 & -1/2 & -1/2 \\ -1/2 & 1 & -1/2 \\ -1/2 & -1/2 & 1 \end{pmatrix}$$

We get $\max_{i\neq j} |\langle f_i, f_j\rangle|^2 = \frac{1}{4} = \frac{m-2}{2(3-1)}$, confirming that these vectors form an equiangular tight frame.

4.5.

Intuitively, an equiangular tight frame is a collection of unit vectors whose pairwise inner products are as small as possible. What is the maximum size of an equiangular tight frame?

Theorem 4.11 (Sustik–Tropp–Dhillon–Heath Jr., 2007)**.** *If $m > n \geq 2$ and there is an equiangular tight frame of m unit vectors in $\mathbf{C}^n$, then one of the following must hold:*

(1) $m = n + 1$ and the vectors form the vertices of a regular m-simplex.
(2) $m = 2n$. Then n is odd and $2n - 1$ is a sum of two perfect squares.
(3) $m \neq n + 1, 2n$. Then $\frac{n(m-1)}{m-n}$ and $\frac{(m-n)(m-1)}{n}$ are both odd perfect squares.

A related question: What is the maximum size s_n of an arbitrary equiangular set of lines in $\mathbf{R}^n$? (Here, a line is a pair of opposing unit vectors. We compute angles by choosing the smallest inner products.) The numbers s_n form sequence A002853 in the OEIS:

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	$\cdots$
s_n	1	3	6	6	10	16	28	28	28	28	28	28	28	28	36	$\cdots$

The value $s_2 = 3$ is achieved using the lines through the vertices of an equilateral triangle: *i.e.*, Example 4.3. The value $s_3 = 6$ is achieved using the lines through the opposing vertices of an icosahedron. The value $s_7 = 28$ can be achieved as follows. Take all $\binom{8}{2} = \frac{8!}{2!6!} = 28$ images of the unit vector

$$\frac{1}{\sqrt{24}}(-3, -3, 1, 1, 1, 1, 1, 1) \in \mathbf{R}^8.$$

They are all orthogonal to the vector $(1, 1, \ldots, 1)$, so we may regard them inside the 7-dimensional subspace $V = \{(a_1, \ldots, a_8) \in \mathbf{R}^8 \mid a_1 + \cdots + a_8 = 0\}$. The inner product on $\mathbf{R}^8$ restricts to an inner product on V.