3/31/14

Representation theory of Lie algebras and geometry of semisimple groups

V. Ginzburg

Historical perspective

Fin. dim reps of compact groups (H. Weyl)
1920s – 1930s

Rep. Analysis ⇝ Algebra ⇝ Geom.

Inf. dim rep.
Analysis (1940–mid 1950s)
von Neumann (Quantum mechanics)
4-dim real vect. $\underbrace{x_1^2+x_2^2+x_3^2}_{\text{spatial coord}} - \underset{\text{time}}{t^2}$
Lorentz group $SO(3,1)$ noncompact semisimple

Algebra (mid 50's – 1980s)
Harish-Chandra, Bernstein-Gelfand-Gelfand

Geometric part (1980s –
Beilinson-Bernstein (1980) - proved KL conj — first serious application of AG to RT
Lusztig
Nakajima
Geometric Langlands

Let G be a simple noncompact Lie group.
Then any finite dimensional rep cannot be made unitary

Pf Let $G \xrightarrow{\rho} GL(V)$ fin. dim. unitary

$G \to U(V) \to GL(V)$

$\mathrm{Lie}\, G \longrightarrow \mathrm{Lie}\, U(n) =$ skew Hermitian operators

Define a hermitian form on Lie G

$$(x,y)_\rho = -\mathrm{tr}(x\cdot y|_V) \qquad x,y \in \mathrm{Lie}\, G$$

This form is positive semidefinite.

$$(x,x)_\rho = -\mathrm{tr}(x^2) \geq 0 \quad \text{since } x \text{ is skew hermitian}$$

G simple $\Rightarrow \rho$ faithful $\Rightarrow \mathrm{Lie}\, G \longrightarrow \mathrm{Lie}\, U(n)$ is faithful

So $(x,x)_\rho = 0 \iff x=0$

So $(x,y)_\rho$ pos. definite $\Rightarrow$ pos. def. Killing form $\Rightarrow$ G compact. □

Fix an inf. dim Hilbert space V
$U(V)$ = unitary group
G = a real Lie group

Defn A hom $G \longrightarrow U(V)$ is called a unitary rep if $\forall v \in V$, $g \longmapsto gv$, $G \longrightarrow V$ is a cts function.

Typical examples X loc. comp. top., $G \curvearrowright X$, μ is a G-inv measure
$G \curvearrowright L^2(X,\mu)$ is a unitary rep

From now on G is a real ss group

$K \subset G$ a maximal compact subgroup.

Eg 1) $G = SL_2(\mathbb{R}) \supset K = SO_2(\mathbb{R})$

2) $G = SL_n(\mathbb{C}) \supset K = SU_n$

<u>Def</u> $v \in V$ is called K-finite if v is contained in a K-stable fin-dim subspace of V

V^{fin} = subspace of K-finite vectors (not nec. closed)

<u>Lemma</u> 1) $V^{fin} = \bigoplus_{\rho \in \mathrm{Irr} K} V_\rho$

2) V^{fin} is dense in V.

V_ρ = ρ-isotypic components i.e. V_ρ = direct sum of some number (∞) of copies of ρ.

<u>Pf</u> $\underset{\substack{\text{group alg} \\ \text{of cts func}}}{C(K)} \ni e_\rho$ central idempotent corresponding to ρ

$$V_\rho = e_\rho \cdot V = \left\{ \int_K e_\rho(k)\, kv\, dk \;\middle|\; v \in V \right\}$$

$\rho \not\cong \rho' \Rightarrow V_\rho \perp V_{\rho'}$

Choose $f_n \in C(K)$ st. $f_n \longrightarrow \delta \Rightarrow \int f_n(k)\, kv\, dk \longrightarrow v \quad \forall v \in V$.

Peter–Weyl $\Longrightarrow \forall \varepsilon > 0$, $\exists \rho_1, \ldots, \rho_\ell \in \mathrm{Irr} K$ st. $\left| \left(\sum_{i=1}^{\ell} e_{\rho_i} \right) * f - f \right| < \varepsilon$

$\Rightarrow$ finite direct sums of V_ρ's are dense in V.

<u>Ex</u> H = upper half plane

$SL_2(\mathbb{R}) \curvearrowright H$ by $z \longrightarrow \frac{az+b}{cz+d}$ on right

Isotropy group of $+i$ is $\cong SO_2(\mathbb{R}) = K$

$H = K \backslash SL_2\mathbb{R}$

Let C be a compact Riemann surface of genus > 1.

Universal covering of C is $\simeq H$.

$C \cong H/\Gamma \qquad \Gamma = \pi_1(C)$ a discrete subgroup of $SL_2(\mathbb{R})$

$= K \backslash SL_2(\mathbb{R}) / \Gamma$

$G = SL_2(\mathbb{R}) \curvearrowright SL_2(\mathbb{R})/\Gamma$

$e^{2\pi i n\theta}$-weight space of $K = SO_2(\mathbb{R})$ $=: {}^nC^\infty(SL_2(\mathbb{R})/\Gamma) = C^\infty$-sections of a line bundle on $C = K \backslash SL_2\mathbb{R}/\Gamma$

<u>Defn</u> $G \longrightarrow U(V)$ is a unitary rep

$v \in V$ is called <u>smooth</u> if the fn $g \longrightarrow gv$ is a C^∞-fn on G.

V^∞ = space of smooth vectors in V.

Let Lie $G = \mathfrak{g}$

$\forall v \in V^\infty$, $x \in \mathfrak{g}$ acts on v, so xv makes sense, so V^∞ is a $\mathfrak{g}$-rep.

<u>Lemma</u> If $v_0 \in V^\infty \cap V^{fin} \Longrightarrow xv_0$ is K-finite $\forall x \in \mathfrak{g}$

<u>Pf</u> Suppose $V_0 \ni v_0$ is a K-stable fin. dim subspace.

$a: \mathfrak{g} \otimes V_0 \longrightarrow V$, $x \otimes v \longmapsto xv$

$K \curvearrowright \mathfrak{g}$ by adjoint action. So $\underbrace{\mathfrak{g} \otimes V_0}_{K\text{-rep}} \longrightarrow V$ is K-equivariant

$kxk^{-1} \cdot kv = k(xv)$.

$\mathfrak{g} \otimes V_0$ is fin. dim $\Longrightarrow$ Im a is fin dim K-stable, containing xv_0. □

<u>Defn</u> A $(\mathfrak{g}, K)$-module is a vector space M with an action of $\mathfrak{g}$ and with a rep $K \xrightarrow{\rho} GL(M)$ s.t. $\forall x \in \mathfrak{g}$, $k \in K$, $\mathrm{Ad}\, k(x)\, m = \rho(k) x \rho(k)^{-1} m$

<u>Cor</u> $V^\infty \cap V^{fin}$ is a $(\mathfrak{g}, K)$-module.

$\mathfrak{g}$-rep $\Longleftrightarrow$ $\mathcal{U}\mathfrak{g}$-modules where $\mathcal{U}\mathfrak{g}$ = enveloping alg of $\mathfrak{g}$

Let $Z\mathfrak{g}$ = Center of $\mathcal{U}\mathfrak{g}$

<u>Defn</u> A $(\mathfrak{g}, K)$-module M is called a Harish-Chandra if

1) $\dim M_\rho < \infty$ $\forall \rho \in \mathrm{Irr}\, K$

2) $Z(\mathfrak{g})$-action on M is locally finite, i.e. $Z(\mathfrak{g}) \cdot m$ finite dim

Reduction from Analysis to Alg.

<u>Thm (HC)</u> 1) Let V be a unitary irrep of G.

Then $V^\infty \cap V^{fin}$ is a <u>simple</u> HC-module and, moreover $V^\infty \cap V^{fin}$ is dense in V.

2) Let V_1, V_2 be unitary irreps. s.t. $V_1^\infty \cap V_1^{fin} \underset{(\mathfrak{g},K)\text{-mod}}{\simeq} V_2^\infty \cap V_2^{fin} \Longrightarrow V_1 \simeq V_2$

One nasty point: given HC module, don't know if you can endow w/ metric to make unitary rep. Unnatural question. (mess for Symplectic grps)

Pf of HC thm

<u>Lemma 2</u> $\forall \rho \in \mathrm{Irr}\, K$, $V^\infty \cap V_\rho$ is dense in V_ρ, so $V^\infty \cap V^{fin}$ is dense in V.

<u>Pf</u> $f \in C_c^\infty(G)$, $f \cdot v := \int_G f(g)\, gv\, dg \in V^\infty$ $\forall v \in V$

$\underset{\hat{G}}{h} \longrightarrow h(fv)$ is a smooth f-n on G.

$e_\rho \in C^\infty(K)$ the central projector

$e_\rho * f \in C_c^\infty(G) \quad \forall f \in C_c^\infty(G)$

$(e_\rho * f)v \in V^\infty \cap V_\rho$

$f_n \longrightarrow \delta$

$e_\rho * f_n \longrightarrow e_\rho$ therefore $\{(e_\rho * f)v \mid v \in V,\ f \in C_c^\infty(G)\}$ is dense in V_ρ

<u>Schur Lemma</u> Let $z \in Z_{\mathfrak{g}}$

If V is a unitary <u>irrep</u> then $z|_{V^\infty}$ is a scalar operator.

<u>Sketch of Proof</u>

$*: \mathcal{U}\mathfrak{g} \longrightarrow \mathcal{U}\mathfrak{g}$ anti-involution $x \longrightarrow -x$ on $\mathfrak{g}$.

$Z(\mathfrak{g})^* = Z(\mathfrak{g})$

One proves that the action of x in V gives a skew-adjoint op

Similarly the act of $z \in Z(\mathfrak{g})$ is an unbounded normal op. $\longleftarrow$ this is why you need <u>unitary rep</u>

<u>Spectral</u> $\forall$ measurable subset $S \subset \mathbb{C}$, $P_S^2 = P_S : H \longrightarrow H$ spectrum is contained in S.

If spectrum not 1 elt, project to nontriv subspace stable under $\mathfrak{g}$. Now rest of argument the same □

$\underset{\text{real ss. group}}{G} \supset \underset{\text{max compact subgp}}{K}$ $\qquad$ $\mathrm{Lie}\, G = \mathfrak{g} \supset \mathrm{Lie}\, K = \mathfrak{k}$ $\qquad$ 4/4/14

$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}, \quad \mathfrak{p}^\perp = \mathfrak{k}$

$\mathrm{Killing}_{\mathfrak{k}} = -\sum x_i^2$

$\mathrm{Killing}_{\mathfrak{g}} = -\sum x_i^2 + \underset{\text{on } \mathfrak{p}}{\sum y_j^2}$

$\mathrm{Killing}_{\mathfrak{g}} - 2\,\mathrm{Killing}_{\mathfrak{k}} = \sum x_i^2 + \sum y_j^2 > 0.$

$\mathfrak{g}$ = left inv v-fields on G. $\qquad \Delta_{\mathfrak{g}} = -\sum x_i^2 + \sum y_j^2$

$\mathcal{U}\mathfrak{g} \longrightarrow \mathcal{D}(G)^G$ = left inv diff operators

$\Delta_{\mathfrak{g}}$, $\Delta_{\mathfrak{k}} \in \mathrm{Center}(\mathcal{U}\mathfrak{k})$ 2nd order diff operators

Central $\qquad$ Casimir elts

$\mathrm{symbol}(\Delta_{\mathfrak{g}} - 2\Delta_{\mathfrak{k}})$ positive def

<u>Cor</u> $\Delta_{\mathfrak{g}} - 2\Delta_{\mathfrak{k}}$ is elliptic

Now let V unitary irrep of G, $\quad V^\infty$ smooth vectors

V^{fin} K-finite

<u>Lemma</u> Let $v \in V^\infty \cap V^{fin}$, $w \in V$

$f_{v,w}(g) = (gv, w) \qquad f_{v,w} \in C^\infty(G)$. Then $f_{v,w}$ is real analytic function

Pf Can assume $v \in V_\rho$ for $\rho \in \mathrm{Irr}K$.

$$\Delta_K\big|_{V_\rho} = c_\rho \cdot \mathrm{Id} \quad \text{for scalar } c_\rho \in \mathbb{C}$$

$$\Delta_{\mathfrak{g}}\big|_{V^\infty} = c_V \cdot \mathrm{Id}$$

$\mathcal{U}\mathfrak{g} \ni u \rightsquigarrow \partial_u \in \mathcal{D}(G)$ corresponding left inv diff op.

$\partial_u f_{v,w} = f_{uv,w}$

Compute $\partial_{\Delta_{\mathfrak{g}}} f_{v,w} = c_V f_{v,w}$ $\qquad \partial_{\Delta_K} f_{v,w} = c_\rho \cdot f_{v,w}$

$$\left(\partial_{\Delta_{\mathfrak{g}}} - 2\partial_{\Delta_K}\right) f_{v,w} = (c_V - 2c_\rho) f_{v,w} \rightsquigarrow f_{v,w} \text{ solution of } 2^{\text{nd}} \text{ order elliptic PDE.}$$

Since any soln of an elliptic PDE is real analytic, $f_{v,w}$ is real analytic. □

Prop Have a bijection:

$$\left\{ \begin{matrix} (\mathfrak{g},K)\text{-submodules} \\ \text{in } V^\infty \cap V^{\text{fin}} \end{matrix} \right\} \xleftrightarrow{\sim} \left\{ G\text{-subreps of } V \right\}$$

$$M \longrightarrow \overline{M}$$

$$W^\infty \cap W^{\text{fin}} \longleftarrow W \subset V$$

Pf Know $W^\infty \cap W^{\text{fin}}$ dense in W so $\leftarrow \rightarrow$ is id.

Must check that $\overline{M}$ is G-stable

Fix $w \in \overline{M}^\perp$, WTS: $\forall v \in M, \forall g \in G \Rightarrow gv \perp w$

Equivalently, $f_{v,w}(g) = 0 \quad \forall g, v$

$f_{v,w}(1) = 0$ $\qquad$ Taylor coefficients:

$u \in \mathcal{U}\mathfrak{g}$ $\qquad (\partial_u f_{v,w})(1) = f_{uv,w}(1) = (uv, w) = 0$ □

$\Rightarrow f_{v,w} = 0$ since it is real analytic.

Cor If V is unit. irrep of $G \Rightarrow V^\infty \cap V^{\text{fin}}$ is a simple $(\mathfrak{g},K)$-module.

Recall defn A HC-module is a $(\mathfrak{g},K)$-mod M st. 1) $\dim M_\rho < \infty \ \forall \rho \in \mathrm{Irr}K$
2.) $\dim Z(\mathfrak{g}) \cdot m < \infty \quad \forall m \in M$

Thm Let M be a $(\mathfrak{g},K)$-module s.t. 1) M is f.g. over $\mathcal{U}\mathfrak{g}$
2) $Z(\mathfrak{g})$ acts on M by scalars.

Then $\dim M_\rho < \infty \quad \forall \rho \in \mathrm{Irr}K$.

Cor: If V is a unitary irrep of G

$V^{\text{fin}} \subset V^\infty$

Pf: Know $V^\infty \cap V_\rho$ dense in V_ρ. $\dim V_\rho < \infty \Rightarrow V^\infty \cap V_\rho = V_\rho$

Prop If M is a simple $(\mathfrak{g},K)$-module $\Rightarrow Z(\mathfrak{g})$ acts on M by scalars.

$\mathfrak{g}$ any fin dim Lie alg

$\mathcal{U}\mathfrak{g} = T\mathfrak{g}/(x\otimes y - y\otimes x - [x,y])$

Have increasing filtration $\mathcal{U}_i\mathfrak{g} = \text{Image } T^{\leq i}\mathfrak{g}$ $\quad i=0,1,2,\dots$

<u>PBW</u> thm $\quad \operatorname{gr}\mathcal{U}\mathfrak{g} = \bigoplus \mathcal{U}_i\mathfrak{g}/\mathcal{U}_{i-1}\mathfrak{g} \simeq S\mathfrak{g}$

$Z(\mathfrak{g}) \hookrightarrow \mathcal{U}\mathfrak{g}$

<u>Goal</u>: $\operatorname{gr} Z(\mathfrak{g}) \simeq (S\mathfrak{g})^G$ if G is connected.

G acts on $T\mathfrak{g}, \mathcal{U}\mathfrak{g}, S\mathfrak{g}$

$$(S\mathfrak{g})^G = (S\mathfrak{g})^{\operatorname{ad}\mathfrak{g}}$$

G acts on $T\mathfrak{g}$, $\mathcal{U}\mathfrak{g}$, $S\mathfrak{g}$

<u>Claim</u>: $u \in \mathcal{U}\mathfrak{g}$ is central $\iff \operatorname{ad}x\,(u) = 0 \quad \forall x \in \mathfrak{g}$

Hence $\operatorname{ad}x\,(u) = 0 \iff u$ commutes with x, $\forall x \in \mathfrak{g}$

$$\mathcal{U}(\mathfrak{g})^G = \mathcal{U}(\mathfrak{g})^{\mathfrak{g}} = Z(\mathfrak{g})$$

$$\operatorname{gr} Z(\mathfrak{g}) = \operatorname{gr}\left(\mathcal{U}(\mathfrak{g})^G\right) \xhookrightarrow[\text{PBW}]{} (S\mathfrak{g})^G$$

Symmetrization map $\quad \sigma : S\mathfrak{g} \longrightarrow \mathcal{U}\mathfrak{g}$

$$x_1 \cdots x_n \longmapsto \frac{1}{n}\sum_{s\in S_n} x_{s(1)}\cdots x_{s(n)}$$

$x_i \in \mathfrak{g}$

① σ is a G-equivariant linear map.

② σ is a linear bijection. $\qquad S^{\leq i}\mathfrak{g} \xrightarrow{\sigma} \mathcal{U}^{\leq i}\mathfrak{g}$ $\quad$ identity on gr

$\sigma : (S\mathfrak{g})^G \xrightarrow{\sim} (\mathcal{U}\mathfrak{g})^G = Z(\mathfrak{g})$ $\qquad \operatorname{gr}(\sigma) : (S\mathfrak{g})^G \xrightarrow{\sim} \operatorname{gr} Z(\mathfrak{g})$ $\qquad \operatorname{gr}\mathcal{U}\mathfrak{g} \xrightarrow{\sim} S\mathfrak{g}$

$S\mathfrak{g} = \mathbb{C}[\mathfrak{g}^*]$, $\quad \operatorname{gr} Z(\mathfrak{g}) \simeq \mathbb{C}[\mathfrak{g}^*]^G$ $\qquad \longleftarrow$ must be isomorphism. $\qquad \square$

<u>Thm</u> Let M be a finitely generated $(\mathfrak{g}, K)$-module s.t. $Z(\mathfrak{g})$ acts on M locally finite. Then $\dim M_\rho < \infty$ $\quad \forall \rho \in \operatorname{Irr} K$.

<u>Pf</u> Choose $M_0 \subset M$ a fin dim subspace s.t. $M = \mathcal{U}\mathfrak{g}\cdot M_0$

Using local finiteness of K-act and $Z\mathfrak{g}$ act can choose M_0 to be K-stable and $Z(\mathfrak{g})$-stable

$M_i = (\mathcal{U}_i\mathfrak{g})(M_0)$ $\qquad M_0 \subset M_1 \subset \cdots$

ascending filtration by fin dim subspaces.

$\operatorname{gr}M$ is a fin gen module over $\operatorname{gr}\mathcal{U}\mathfrak{g} = \mathbb{C}[\mathfrak{g}^*] \underset{\text{Killing form}}{\simeq} \mathbb{C}[\mathfrak{g}]$

M_0 is K-stable $\Rightarrow (\mathcal{U}_i\mathfrak{g})M_0$ is K-stable.

$\Rightarrow M_i$ is $\mathfrak{k}$-stable

$\Rightarrow \mathfrak{k} \in \mathcal{U}_1\mathfrak{g}$ annihilates $\qquad \Longrightarrow \quad \operatorname{supp}(\operatorname{gr}M) \subset \mathfrak{k}^\perp = \mathfrak{p}$

$\operatorname{gr}M$ is annihilated by $\mathfrak{k}$

M_0 is $Z(\mathfrak{g})$-stable $\Longrightarrow$ $(\mathcal{U}_i\mathfrak{g})M_0$ is $Z(\mathfrak{g})$-stable $\Rightarrow$ $gr(Z(\mathfrak{g})\cap \mathcal{U}\mathfrak{g}_+)$ annihilates $gr\, M$

$$gr(Z(\mathfrak{g})\cap \mathcal{U}(\mathfrak{g})_+) = \mathbb{C}[\mathfrak{g}^*]^G_+$$

<u>Lemma</u> $\mathrm{Var}(\mathbb{C}[\mathfrak{g}]^G_+) = \mathcal{N}$ = set of nilpotent elements of $\mathfrak{g}$

$= \{x\in \mathfrak{g} \mid \mathrm{ad}\, x : \mathfrak{g}\to\mathfrak{g}$ is nilp$\}$

<u>Pf</u>: $x \longmapsto \mathrm{Tr}(\mathrm{ad}\, x)^n$ is a G-inv fn $\forall n=1,2,\dots$

$\mathrm{Var}(\quad) \subset \{x \mid \mathrm{Tr}(\mathrm{ad}\, x)^n = 0 \quad \forall n\geq 1\}$ ∎

$\mathcal{N}$ is a $\mathrm{Ad}\, G$-stable subset of $\mathfrak{g}$

$\mathfrak{g} = \mathfrak{k}\oplus\mathfrak{p}$ $\qquad$ $\mathfrak{k}$ & $\mathfrak{p}$ are $\mathrm{Ad}\, K$-stable

$\mathcal{N}\cap\mathfrak{p}$ is $\mathrm{Ad}\, K$-stable.

<u>Geometric result (Kostant)</u>

$\dim(\mathbb{C}[\mathcal{N}\cap\mathfrak{p}]_\rho) < \infty \qquad \forall \rho \in \mathrm{Irr}\, K$

$\dim M_\rho = \dim((gr\, M)_\rho)$

$gr\, M$ is f.g. $\mathbb{C}[\mathfrak{g}]$-module annihilated by $\mathfrak{k}$ and $\mathbb{C}[\mathfrak{g}]^G_+$

i.e. $\mathbb{C}[\mathfrak{g}]$-action on $gr\, M$ factors through $\mathbb{C}[\mathfrak{p}\cap\mathcal{N}]$-action.

<u>End of proof of Thm</u>

4/7/14

Changing notation: $\mathfrak{g}$ <u>complex</u> ss Lie alg
$\cup$
$\mathfrak{h}$ Cartan

$R \subset \mathfrak{h}^*$ set of roots $\qquad R = R^+ \sqcup R^-$

$Q = \mathbb{Z}$-span of R inside $\mathfrak{h}^*$ root lattice
$\cup$
$Q^\pm = \mathbb{Z}_{\geq 0} R^\pm$ semigroups

$\mathfrak{g} = \mathfrak{h}\oplus\bigoplus_{\alpha\in R}\mathfrak{g}_\alpha$ root decomposition

$\mathfrak{n}^\pm = \bigoplus_{\alpha\in R^\pm}\mathfrak{g}_\alpha$ nilp Lie subalg $\qquad \mathfrak{g} = \mathfrak{n}^+\oplus\mathfrak{h}\oplus\mathfrak{n}^- \qquad \mathcal{U}\mathfrak{g} = \mathcal{U}^-\otimes\mathcal{U}\mathfrak{h}\otimes\mathcal{U}^+$

$\mathcal{U}^\pm := \mathcal{U}\mathfrak{n}^\pm$

$\mathfrak{h} \overset{\mathrm{ad}}{\curvearrowright} \mathfrak{g} \qquad \mathfrak{h} \overset{\mathrm{ad}}{\curvearrowright} \mathcal{U}\mathfrak{g}$ ss. $\qquad \mathcal{U}\mathfrak{g} = \bigoplus_{\lambda\in Q}\mathcal{U}_\lambda\mathfrak{g}, \quad \mathcal{U}_\lambda\mathfrak{g} = \{u\in\mathcal{U}\mathfrak{g} \mid \mathrm{ad}(h)u = \lambda(h)u\}$

Choose a basis $x_1,\dots,x_n$ of $\mathfrak{g}$ $\quad x_i\in\mathfrak{g}_{\alpha_i}$

$$\mathcal{U}_\lambda(\mathfrak{g}) = \mathbb{C}\text{-span of } \begin{pmatrix} x_1^{k_1}x_2^{k_2}\cdots x_n^{k_n} \\ k_i\in\mathbb{Z} \\ \sum k_i\alpha_i = \lambda \end{pmatrix}$$

$\mathcal{U}^\pm_\lambda = \mathcal{U}_\lambda\mathfrak{g}\cap\mathcal{U}^\pm \qquad \mathcal{U}^\pm = \bigoplus_\lambda \mathcal{U}^\pm_\lambda$ is a grading

$\dim \mathcal{U}^+_\lambda = p(\lambda) = (\#\text{ ways of writing } \lambda = \alpha_1+\dots+\alpha_k) < \infty$

Let M be a $\mathfrak{g}$-module, $\lambda \in \mathfrak{h}^*$

$M_\lambda := \{ m \in M \mid (h-\lambda(h))^N m = 0 \text{ for some } N = N(m) \}$ generalized λ-weight space.

So if $\mathcal{U}\mathfrak{h}$-action on M is loc. finite $\Rightarrow$ $M = \bigoplus_{\lambda \in \mathfrak{h}^*} M_\lambda$

$\operatorname{Spec} M = \{ \lambda \in \mathfrak{h}^* \mid M_\lambda \neq 0 \}$

<u>Verma modules</u>

$\mathfrak{n} := \mathfrak{n}^+$ $\quad \mathfrak{b} = \mathfrak{h} + \mathfrak{n}$ a Borel in $\mathfrak{g}$

$\lambda \in \mathfrak{h}^*$ $\quad \lambda : \mathfrak{b} \longrightarrow \mathbb{C}$, $\quad \underset{\mathfrak{h}}{\overset{\cap}{h}} + \underset{\mathfrak{n}}{\overset{\cap}{n}} \longmapsto \lambda(h)$ $\quad \mathfrak{n} = [\mathfrak{b}, \mathfrak{b}]$

extends to an alg hom

$\lambda : \mathcal{U}\mathfrak{b} \longrightarrow \mathbb{C}$ $\qquad \mathbb{C}_\lambda$ is a 1-dim $\mathcal{U}\mathfrak{b}$-module on which $\mathcal{U}\mathfrak{b}$ acts via λ.

The Verma module with highest weight $\lambda \in \mathfrak{h}^*$

$$\Delta(\lambda) = \mathcal{U}\mathfrak{g} \otimes_{\mathcal{U}\mathfrak{b}} \mathbb{C}_\lambda = \mathcal{U}\mathfrak{g} \Big/ \left(\mathcal{U}\mathfrak{g} \cdot \mathfrak{n} + \mathcal{U}\mathfrak{g} \cdot \{ (h - \lambda(h)) \}_{h \in \mathfrak{h}} \right)$$

$\cup$

$1_\lambda := 1$ mod ideal.

<u>Properties</u>:

① $\operatorname{Hom}_{\mathcal{U}\mathfrak{g}}(\Delta(\lambda), M) = \operatorname{Hom}_{\mathcal{U}\mathfrak{b}}(\mathbb{C}_\lambda, M) = \operatorname{Hom}_{\mathcal{U}^+}(\text{triv}, \operatorname{Hom}_{\mathcal{U}\mathfrak{h}}(\lambda, M))$

$= \{ m \in M \mid hm = \lambda(h) m, \ nm = 0 \ \forall n \in \mathfrak{n} \}$

If $\mathfrak{h} \curvearrowright M$ is ss $\Rightarrow$ $M_\lambda^{\mathfrak{n}}$

②
$\mathcal{U}\mathfrak{g} = \mathcal{U}\mathfrak{n}^- \otimes \underbrace{\mathcal{U}\mathfrak{h} \otimes \mathcal{U}\mathfrak{n}} = \mathcal{U}\mathfrak{n}^- \otimes \mathcal{U}\mathfrak{b}$

$\Delta(\lambda) = \mathcal{U}\mathfrak{g} \otimes_{\mathcal{U}\mathfrak{b}} \mathbb{C}_\lambda = \mathcal{U}\mathfrak{n}^-$ $\Rightarrow$ $\Delta(\lambda)$ is a free $\mathcal{U}\mathfrak{n}^-$-module with generator 1_λ

③ $\mathcal{U}\mathfrak{n}^- \xrightarrow{\sim} \Delta(\lambda)$ $\qquad$ $\mathfrak{h}$-action on $\Delta(\lambda)$ is semisimple

$\cup$ $\qquad \Rightarrow$

$\mathcal{U}_\mu \mathfrak{n}^- \longrightarrow \Delta_{\mu+\lambda}(\lambda)$ $\qquad \operatorname{Spec} \Delta(\lambda) = \lambda - Q^+$

$\mu \in Q^- = -Q^+$

④ $\Delta_\lambda(\lambda) = \mathbb{C} \cdot 1_\lambda$

⑤ The module $\Delta(\lambda)$ has a unique simple quotient $L(\lambda)$

<u>Pf</u> Let $M \subsetneq \Delta(\lambda)$ proper $\mathcal{U}\mathfrak{g}$-submodule

$\Rightarrow M = \bigoplus_\mu M_\mu$ $\qquad$ If $M_\lambda \neq 0 \Rightarrow M_\lambda = \mathbb{C} 1_\lambda \Rightarrow M = \Delta(\lambda)$, contradicting assumptions

Therefore $M_\lambda = 0$ $\qquad \left(\sum_{M \subsetneq \Delta(\lambda)} M \right)_\lambda = \sum M_\lambda = 0 \Rightarrow L(\lambda)_\lambda = \mathbb{C} \cdot 1_\lambda \neq 0 \Rightarrow L(\lambda) \neq 0$

$L(\lambda) \overset{\text{def}}{=} \Delta(\lambda) \Big/ \sum_{M \subsetneq \Delta(\lambda)} M$, $\qquad \square$

Prop: Let α be a simple positive root, $\lambda \in \mathfrak{h}^*$ s.t. $\langle \lambda, \check{\alpha} \rangle =: n \in \mathbb{Z}_{\geq 0}$

Then $\exists\ \Delta(\lambda-(n+1)\alpha) \longrightarrow \Delta(\lambda)$ a nonzero $\mathcal{U}\mathfrak{g}$-module hom.

Pf: Let $\underbrace{e_1, \dots, e_r}_{\mathfrak{n}}, \underbrace{f_1, \dots, f_r}_{\mathfrak{n}^-}, \underbrace{h_1, \dots, h_r}_{\mathfrak{h}} \in \mathfrak{g}$ Chevalley generators.

$\langle e_i, f_i, h_i \rangle = \mathfrak{sl}_2 \qquad [e_i, f_j] = 0 \quad \forall i \neq j \qquad \Delta(\lambda) \ni 1_\lambda$

$h_i 1_\lambda = \lambda(h_i) 1_\lambda = \langle \lambda, \alpha_i^\vee \rangle \cdot 1_\lambda$

Let $\langle e_1, f_1, h_1 \rangle \longleftrightarrow \alpha = \alpha_1$

$\Delta(\lambda) \ni u := f_1^{n+1} 1_\lambda$

__Claim__ $e_i u = 0$ (in $\Delta(\lambda)$) $\forall i = 1, \dots, r$

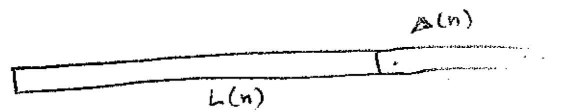
Special case $\mathfrak{g} = \mathfrak{sl}_2 = \langle e, f, h \rangle \qquad \mathfrak{n}^- = \langle f \rangle, \quad \lambda \in \mathbb{C}$

$\Delta(\lambda) = \mathbb{C}[f] \cdot 1_\lambda$ as an $\mathcal{U}\mathfrak{n}^- = \mathbb{C}[f]$-module.

$$\underset{\lambda}{1_\lambda} \underset{e}{\overset{f}{\rightleftarrows}} \underset{\lambda-2}{f} \underset{e}{\overset{f}{\rightleftarrows}} \underset{\lambda-4}{f^2} \quad \cdots \quad f^n \to f^{n+1} \quad \cdots$$

$e = 0$

If $\lambda = n \in \mathbb{Z}_{\geq 0}$



__Proof of claim__ For $i = 1$, by the $\mathfrak{sl}_2$-computation $e_1 u = 0$

If $i \neq 1$, then f_1 commutes with e_i. Then $e_i u = e_i f_1^{n+1} 1_\lambda = f_1^{n+1} e_i 1_\lambda = 0$ $\square$

Can check $h_i u = h_i f_1^{n+1} 1_\lambda = \left(\lambda(h_i) - (n+1)\alpha(h_i)\right) u$

Therefore $1_{\lambda-(n+1)\alpha} \longmapsto u$ extends to hom $\Delta(\lambda-(n+1)\alpha) \longrightarrow \Delta(\lambda)$

Let $\rho \in \mathfrak{h}^*$ half-sum of positive roots

$\rho := \frac{1}{2} \sum_{\alpha > 0} \alpha$

__Then__ $\langle \rho, \alpha^\vee \rangle = 1 \quad \forall$ simple coroots

Let s_α be the reflection corresponding to α.

$s_\alpha(\lambda) = \lambda - \langle \lambda, \alpha^\vee \rangle \alpha$

$s_\alpha(\rho) = \rho - \alpha \quad \forall$ simple α

Let W = Weyl group $\quad W \curvearrowright \mathfrak{h}^*$

Define the dot-action of W on $\mathfrak{h}^*$: $\quad w \cdot \lambda := w(\lambda + \rho) - \rho$

__Rem__ $-\rho$ is fixed point under $\cdot$ action

$$s_\alpha \cdot \lambda = s_\alpha(\lambda+\rho) - \rho = \lambda - \langle\lambda,\alpha^\vee\rangle\alpha + (\rho-\alpha) - \rho$$
$$= \lambda - \left(\langle\lambda,\alpha^\vee\rangle + 1\right)\alpha$$

Say that $\lambda \in \mathfrak{h}^*$ is <u>integral</u> if $\langle\lambda,\alpha^\vee\rangle \in \mathbb{Z}$ $\forall \alpha \in R$

<u>dominant</u> if $\langle\lambda,\alpha^\vee\rangle \geq 0$ $\forall \alpha \in R^+$

<u>Cor</u> If λ is a dominant integral weight and α is a simple root $\Rightarrow$

$$\exists\, \Delta(s_\alpha\cdot\lambda) \longrightarrow \Delta(\lambda)$$

- $Z(\mathfrak{g})$ acts on $\Delta(\lambda)$ by a scalar

<u>Pf</u> $Z(\mathfrak{g})$ commutes with $\mathfrak{h} \subset \mathcal{U}\mathfrak{g}$

$$Z(\mathfrak{g})\,\Delta_\lambda(\lambda) \subset \Delta_\lambda(\lambda)$$

$\forall z \in Z(\mathfrak{g})$, $\exists\, c \in \mathbb{C}$ s.t. $z\big|_{\Delta_\lambda(\lambda)} = c\cdot \mathrm{Id}$

Therefore $z\cdot(u\cdot 1_\lambda) = u\cdot(z\cdot 1_\lambda) = c\,u 1_\lambda$ $\forall u \in \mathcal{U}\mathfrak{g}$. $\square$

Have Killing form on $\mathfrak{h}$ $\rightsquigarrow$ $(-,-)$ on $\mathfrak{h}^*$

$C \in Z(\mathfrak{g})$ the Casimir.

<u>Lemma</u>

$$C\big|_{\Delta(\lambda)} = |\lambda+\rho|^2 - |\rho|^2$$

<u>Pf</u>

If x_i is any basis of $\mathfrak{g}$, x^i the dual basis (wrt Killing form)

$$C = \sum x_i x^i \in \mathcal{U}\mathfrak{g}$$

Choose $e_\alpha \in \mathfrak{g}_\alpha$ s.t. $\langle e_\alpha, e_{-\alpha}\rangle = 1$

$\{h_i\}$ orthonormal basis in $\mathfrak{h}$

$$C = \sum h_i^2 + \sum_{\alpha\in R} e_\alpha e_{-\alpha} = \sum h_i^2 + \sum_{\alpha>0} e_\alpha e_{-\alpha} + \sum_{\alpha<0} e_\alpha e_{-\alpha} = \sum h_i^2 + 2\underbrace{\sum_{\alpha>0} e_{-\alpha} e_\alpha}_{\cdot 1_\lambda = 0} + \sum_{\alpha>0}[e_\alpha, e_{-\alpha}]$$

$$C\cdot 1_\lambda = \sum_i \lambda(h_i)^2 + \sum_{\alpha>0}\lambda(t_\alpha) \qquad [e_\alpha, e_{-\alpha}] = t_\alpha$$

<u>Fact</u>: $\mathfrak{h}^* \xrightarrow{\sim} \mathfrak{h}$ $\quad\Big|\quad$ $\rho \longleftrightarrow \frac{1}{2}\sum t_\alpha$

$$\sum_{\alpha>0}\alpha \mapsto \sum_{\alpha>0} t_\alpha$$

$$C\cdot 1_\lambda = |\lambda|^2 + 2(\lambda,\rho) = |\lambda+\rho|^2 - |\rho|^2 \qquad \square$$

<u>Cor</u>: $\Delta(\lambda)$ has finite length.

<u>Pf</u> $\Delta(\lambda) \supsetneq M_1 \supsetneq M_2 \supsetneq \cdots$

$\mathrm{Spec}\, M_1 \supset \mathrm{Spec}\, M_2 \supset \cdots$

$\Rightarrow$ Can choose subsequence st. $\mathrm{Spec}\, M_{i_1} \supsetneq \mathrm{Spec}\, M_{i_2} \supsetneq \cdots$

M_{i_j} has a non-zero weight vector v_{i_j} st. $\mathfrak{n} v_{i_j} = 0$ with weight μ_{i_j}

$\Longrightarrow$ the sequence μ_{i_j} contains infinitely many different terms.

$C|_{M_{i_j}} = |\lambda+\rho|^2 - |\rho|^2$. On the other hand, $|\mu_{i_j}+\rho|^2 - |\rho^2| = |\lambda+\rho|^2 - |\rho|^2$ $\square$

4/11/14

let E be a $\mathcal{U}\mathfrak{b}$-module

$\Delta(E) = \mathcal{U}\mathfrak{g} \otimes_{\mathcal{U}\mathfrak{b}} E$

Since $\mathcal{U}\mathfrak{g}$ is free over $\mathcal{U}\mathfrak{b}$, $\mathcal{U}\mathfrak{g} \otimes_{\mathcal{U}\mathfrak{b}} -$ is exact

$\Delta(E/E') \simeq \Delta(E)/\Delta(E')$ $\quad$ $\mathcal{U}\mathfrak{g} = \mathcal{U}^- \otimes \mathcal{U}\mathfrak{b}$ $\quad$ $\Delta(E)$ is free over $\mathcal{U}^-$ $\quad \Delta(E) = \mathcal{U}^- \otimes_{\mathbb{C}} E$

$\mathfrak{n} \subset \mathfrak{b}$ $\quad \mathcal{U}\mathfrak{b} \longrightarrow \mathcal{U}(\mathfrak{b}/\mathfrak{n}) = \mathcal{U}\mathfrak{b}/\mathcal{U}\mathfrak{b}\cdot\mathfrak{n} \cong \mathcal{U}\mathfrak{h}$

$\Rightarrow \mathcal{U}\mathfrak{h}$ is an $\mathcal{U}\mathfrak{b}$-module

<u>Universal Verma</u>

$\Delta = \Delta(\mathcal{U}\mathfrak{h}) = \mathcal{U}\mathfrak{g} \otimes_{\mathcal{U}\mathfrak{b}} \mathcal{U}\mathfrak{h} = \mathcal{U}\mathfrak{g} \otimes_{\mathcal{U}\mathfrak{b}} (\mathcal{U}\mathfrak{b}/\mathcal{U}\mathfrak{b}\cdot\mathfrak{n}) = \mathcal{U}\mathfrak{g}/\mathcal{U}\mathfrak{g}\cdot\mathfrak{n}$

- Δ is an $(\mathcal{U}\mathfrak{g}, \mathcal{U}\mathfrak{h})$ bimodule
- $\mathcal{U}(\mathfrak{b}_-) \xrightarrow{\sim} \Delta \quad : u \mapsto u(1)$ an isom of $\mathcal{U}\mathfrak{b}^-$-modules.
- There is an adjoint $\mathcal{U}\mathfrak{h}$-action on Δ. $\quad \mathrm{ad}(h)\, m = hm - mh$.

  $\Rightarrow$ ad-action is locally finite since the ad action on $\mathcal{U}\mathfrak{b}^-$ is.

<u>Lemma</u> $\mathrm{End}_{(\mathcal{U}\mathfrak{g}, \mathcal{U}\mathfrak{h})} \Delta \xleftarrow{\text{right action}} \mathcal{U}\mathfrak{h}$ is an algebra isom.

<u>Pf</u> $f: \Delta \longrightarrow \Delta$ bimodule endom.

$\Rightarrow f$ commutes with ad-action

In particular $f(\Delta_{0\text{-ad}}) \subset \Delta_{0\text{-ad}}$ $\quad (\mathcal{U}\mathfrak{b}^-)_{0\text{-ad}} \xrightarrow{\sim} \Delta_{0\text{-ad}}$.
wt space

$(\mathcal{U}\mathfrak{b}^-)_{0\text{-ad}} = \mathcal{U}(\mathfrak{n}^- \oplus \mathfrak{h})_{0\text{-ad}} = (\mathcal{U}(\mathfrak{n}^-) \otimes \mathcal{U}\mathfrak{h})_{0\text{-ad}} = 1 \otimes \mathcal{U}\mathfrak{h} \longrightarrow \mathcal{U}\mathfrak{h}\cdot 1 = 1\cdot\mathcal{U}\mathfrak{h}$

$1 \in \Delta_{0\text{-ad}} \Rightarrow f(1) = 1\cdot u$ for some $u \in \mathcal{U}\mathfrak{h}$

$\Rightarrow \forall x \in \mathcal{U}\mathfrak{g}$ have $f(x\cdot 1) = x f(1) = x\cdot u$

$\Rightarrow f$ is equal to the multiplication by $u \in \mathcal{U}\mathfrak{h}$ on the right. $\square$

$\lambda : \mathfrak{h} \longrightarrow \mathbb{C}$ $\qquad \mathbb{C}_\lambda \qquad \Delta(\lambda) = \mathcal{U}\mathfrak{g} \otimes_{\mathcal{U}\mathfrak{b}} \mathbb{C}_\lambda = \left(\mathcal{U}\mathfrak{g} \otimes_{\mathcal{U}\mathfrak{b}} \mathcal{U}\mathfrak{h}\right) \otimes_{\mathcal{U}\mathfrak{h}} \mathbb{C}_\lambda$

$\mathcal{U}\mathfrak{h} = S\mathfrak{h} = \mathbb{C}[\mathfrak{h}^*]$

<u>Def</u> The HC-hom $\quad \Theta : Z(\mathfrak{g}) \xrightarrow[\text{act}]{\text{left}} \mathrm{End}_{(\mathcal{U}\mathfrak{g}, \mathcal{U}\mathfrak{h})} \Delta \xleftarrow{\sim} \mathcal{U}\mathfrak{h} = \mathbb{C}[\mathfrak{h}^*]$

this Θ is algebra hom. $\qquad z \qquad z \cdot 1 = 1 \cdot \Theta(z)$

$(z_1 z_2) 1 = 1 \; \Theta(z_1 z_2)$

$z_1 (z_2 \cdot 1) = z_1 (1 \cdot \Theta(z_2)) = (1 \; \Theta(z_2))\Theta(z_1) \implies \Theta(z_1 z_2) = \Theta(z_1)\Theta(z_2)$

$\lambda \in \mathfrak{h}^* \qquad z|_{\Delta(\lambda)} = \Theta(z)(\lambda)\, \mathrm{Id}_{\Delta(\lambda)}$

Recall dot-action of W = Weyl group $\qquad w : \lambda \mapsto w(\lambda+\rho) - \rho = w \cdot \lambda$

<u>Thm</u> The HC-hom yields an alg isom

$$\Theta : Z(\mathfrak{g}) \xrightarrow{\sim} \mathbb{C}[\mathfrak{h}^*]^{W\cdot} \qquad \text{(invariance wrt dot-action)}$$

<u>An alternative construction of Θ</u>

$\mathcal{U}\mathfrak{g} = \bigoplus_{\lambda \in Q} \mathcal{U}_\lambda \mathfrak{g} \qquad \mathcal{U}_0 = \mathcal{U}_0 \mathfrak{g} = (\mathcal{U}\mathfrak{g})^{\mathfrak{h}} \hookleftarrow Z(\mathfrak{g})$

$\qquad\qquad\qquad\qquad\qquad\qquad \cup \; \mathcal{U}\mathfrak{h}$

<u>Lemma</u> $\mathcal{U}_0 \cap (\mathcal{U}\mathfrak{g}) \cdot \mathfrak{n}$ is a two-sided ideal in $\mathcal{U}_0$

and $\mathcal{U}_0 / (\mathcal{U}_0 \cap \mathcal{U}\mathfrak{g} \cdot \mathfrak{n}) \xleftarrow{\sim} \mathcal{U}\mathfrak{h}$

<u>Pf of lemma</u>

$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha \qquad \mathcal{U}\mathfrak{g} = \mathcal{U}^- \otimes \mathcal{U}\mathfrak{h} \otimes \mathcal{U}^+$

$$\mathcal{U}_\lambda \mathfrak{g} = \bigoplus_{\substack{\mu, \nu \in Q^+ \\ \mu - \nu = \lambda}} \mathcal{U}^-_{\nu} \otimes \mathcal{U}\mathfrak{h} \otimes \mathcal{U}^+_\mu$$

$$\mathcal{U}_0 \mathfrak{g} = \bigoplus_{\mu \in Q^+} \mathcal{U}^-_{-\mu} \otimes \mathcal{U}\mathfrak{h} \otimes \mathcal{U}^+_\mu$$

Hence $\mathcal{U}_0 \cap \mathcal{U}\mathfrak{g} \cdot \mathfrak{n} = (\mathfrak{n}^- \cdot \mathcal{U}\mathfrak{g}) \cap \mathcal{U}_0 \longleftarrow$ right ideal

$\mathcal{U}_0 / (\mathcal{U}_0 \cap \mathcal{U}\mathfrak{g} \cdot \mathfrak{n}) = \mathcal{U}\mathfrak{h}$ $\qquad$ (mod out everything with $\mathfrak{n}$ on right and hence $\mathfrak{n}^-$ on left) $\square$

Can define the HC-hom.

$\Theta : Z(\mathfrak{g}) \hookrightarrow \mathcal{U}_0 \mathfrak{g} \longrightarrow \mathcal{U}_0 / (\mathcal{U}_0 \cap \mathcal{U}\mathfrak{g} \cdot \mathfrak{n}) \xleftarrow{\sim} \mathcal{U}\mathfrak{h}$ $\qquad$ algebra map

<u>Comment:</u> why are constructions the same?

$f \in \mathrm{End}_{(\mathcal{U}\mathfrak{g}, \mathcal{U}\mathfrak{h})} \Delta = \mathcal{U}\mathfrak{h}$

$$\Delta = \mathcal{U}\mathfrak{g}/\mathcal{U}\mathfrak{g}\cdot\mathfrak{n} \hookleftarrow \mathcal{U}_0/(\mathcal{U}_0 \cap \mathcal{U}\mathfrak{g}\cdot\mathfrak{n})$$

$f(1)$

Pf of HC thm

Step 1 $\quad \operatorname{Im}\Theta \subset \mathbb{C}[\mathfrak{h}^*]^{W\cdot}$

P^+ = set of dominant integral wts $= \{\lambda \in \mathfrak{h}^* \mid \lambda(\alpha^\vee) = \langle\lambda, \check{\alpha}\rangle \in \mathbb{Z}_{\geq 0} \ \forall \alpha \in R^+\}$

Let $\lambda \in P^+$

α simple root $\quad z \in Z(\mathfrak{g})$, $\quad \exists \quad \Delta(s_\alpha\cdot\lambda) \longrightarrow \Delta(\lambda)$ non-zero $\mathcal{U}\mathfrak{g}$-map

z

$$\Theta(z)(s_\alpha\cdot\lambda) = \Theta(z)(\lambda) \Leftarrow \left(z\big|_{\Delta(s_\alpha\cdot\lambda)} = \Theta(z)(s_\alpha\cdot\lambda) \ \Big|\ z\big|_{\Delta(\lambda)} = \Theta(z)(\lambda) \right)$$

P^+ is Zariski dense in $\mathfrak{h}^*$, so

$\Theta(z)(s_\alpha\cdot\lambda) = \Theta(z)(\lambda)$ for <u>every</u> $\lambda \in \mathfrak{h}^*$ $\quad$ Q.E.D.

Have PBW filtration on $\mathcal{U}\mathfrak{g}$

$\mathcal{U}_0\mathfrak{g} \subseteq \mathcal{U}_1\mathfrak{g} \subseteq \dots$

$\operatorname{gr}\mathcal{U}\mathfrak{g} \simeq \mathbb{C}[\mathfrak{g}^*]$

Get induced filtration on $Z(\mathfrak{g})$

$\operatorname{gr} Z(\mathfrak{g}) \simeq \mathbb{C}[\mathfrak{g}^*]^{\mathfrak{g}}$

The natural grading $\mathbb{C}[\mathfrak{h}^*] = \bigoplus_i \mathbb{C}^i[\mathfrak{h}^*]$ does not induce a grading on $\mathbb{C}[\mathfrak{h}^*]^{W\cdot}$

But it does induce a filtration $\mathbb{C}^{\leq i}[\mathfrak{h}^*]^{W\cdot}$

$$\operatorname{gr}(\mathbb{C}[\mathfrak{h}^*]^{W\cdot}) = \mathbb{C}[\mathfrak{h}^*]^W$$

<u>Claim</u> $\quad \Theta : Z\mathfrak{g} \longrightarrow \mathbb{C}[\mathfrak{h}^*]^{W\cdot}$ respects the filtrations

<u>Pf</u> follows from the 2nd construction of Θ.

4/14/14

Triangular decomp $\quad \mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}$

$$\mathfrak{g}^* = \underbrace{(\mathfrak{n}^-)^* \oplus \mathfrak{h}^*}_{\mathfrak{n}^\perp} \oplus \mathfrak{n}^*$$

$\Rightarrow \mathfrak{h}^* \hookrightarrow \mathfrak{g}^*$

We conclude the proof of HC.

Step 2 : $\Theta : Z(\mathfrak{g}) \longrightarrow \mathbb{C}[\mathfrak{h}^*]^{W\cdot}$ and $\mathbb{C}[\mathfrak{g}^*]^{\mathfrak{g}} \simeq \operatorname{gr}(Z(\mathfrak{g})) \xrightarrow{\operatorname{gr}(\Theta)} \operatorname{gr}(\mathbb{C}[\mathfrak{h}^*]^{W\cdot}) \cong \mathbb{C}[\mathfrak{h}^*]^W$

The composition $\mathbb{C}[\mathfrak{g}^*]^{\mathfrak{g}} \longrightarrow \mathbb{C}[\mathfrak{h}^*]^W$ is restriction. This has to be checked.

Hence Chevalley rest thm says that this is an alg isom.

$\operatorname{gr}(\Theta)$ isom $\Rightarrow \Theta$ is an isom. $\square$

Suppose A is f.d. comm $\mathbb{C}$-alg. Then the set of max. ideal of A is finite.
$$\overset{\shortparallel}{\{I_1, \dots, I_n\}}$$

$$\Rightarrow \quad A/_{\mathrm{Rad}(A)} = \underbrace{\mathbb{C} \oplus \cdots \oplus \mathbb{C}}_{n \text{ times}}$$

<u>Lem</u> Any fin gen. A-mod M has a canon decomp $M = \bigoplus_i M_i$
where $\mathrm{supp}(M_i) = \{I_i\}$

Moreover if $\mathrm{supp}(M) = I$ and $\mathrm{supp}(N) = J$ where $I \neq J$, then $\mathrm{Hom}_A(M, N) = 0$

Let Z be a fin. gen commutative alg

$I \in \mathrm{Max}\, Z$, $\quad M$ is a Z-mod

define $\quad M_I = \{I\text{-torsion } \text{elts}\}$ $\quad$ submodule of M.

$$= \{m \in M \mid \exists k_m \text{ st. } I^{k_m} \cdot m = 0\}$$

Let $Z\text{-Mod}_f$ = category of Z-mods annihilated by an ideal of finite codim

$Z\text{-Mod}_I = \left(\text{cat of mods } M \text{ st. } M = M_I\right) \subseteq Z\text{-Mod}_f$

then $\displaystyle Z\text{-Mod}_f = \bigoplus_{I \in \mathrm{Max} Z} Z\text{-Mod}_I$

Take $Z = Z(\mathfrak{g}) \overset{HC}{\cong} \mathbb{C}[\mathfrak{h}^*]^{W_\circ}$

Then $\mathrm{Max}\, \mathbb{C}[\mathfrak{h}^*]^{W_\circ} = \mathfrak{h}^*/_{W_\circ} = W_\circ$ - orbits in $\mathfrak{h}^*$

Hence $\displaystyle Z(\mathfrak{g})\text{-Mod}_f = \bigoplus_{\chi \in \mathfrak{h}^*/W_\circ} Z(\mathfrak{g})\text{-Mod}_\chi$ $\qquad$ <u>Spectral decomp</u>

Let E be a fin. dim $\mathcal{U}\mathfrak{b}$-module $\Rightarrow \Delta(E) = \mathcal{U}\mathfrak{g} \otimes_{\mathcal{U}\mathfrak{b}} E$

Then there are some properties:

1) $\mathrm{Hom}_{\mathcal{U}\mathfrak{g}}(\Delta(E), M) \cong \mathrm{Hom}_{\mathcal{U}\mathfrak{b}}(E, M)$

2) $\Delta(E) \cong \mathcal{U}^- \otimes E$ is a $\mathcal{U}^-$-module

3) Let $E = E^0 \supset E^1 \supset \cdots \supset E^n$ be a $\mathfrak{b}$-stable complete flag (Lie thm)

E^i/E^{i+1} is 1-dim and on it $\mathfrak{n}$ ⟳ by 0, $\mathfrak{h}$ by λ $\Leftarrow$

$\Rightarrow E^i/E^{i+1} \cong \mathbb{C}_\lambda$

Applying ind to flag $\Rightarrow \Delta(E) \supset \Delta(E^1) \supset \cdots \supset \Delta(E^n)$
and $\Delta(E^i)/\Delta(E^{i+1}) \cong \Delta(\mathbb{C}_\lambda) \rightsquigarrow$ Verma

In this situation we say $\Delta(E)$ has a Verma flag.

4) $\mathcal{U}\mathfrak{h}$-action on $\Delta(E)$ is loc-finite and $\mathrm{Spec}\, \Delta(E) = \mathrm{Spec}\, E - Q^+$

Defn Category $\mathcal{O}$ (Bernstein–Gelf–Gelf) is the category of $\mathcal{U}\mathfrak{g}$-modules M s.t.

i) M is finitely generated over $\mathcal{U}\mathfrak{g}$

ii) $\mathfrak{h}$-action on M is diagonal

iii) $\mathcal{U}\mathfrak{n}$ is locally nilpotent

1) $\mathcal{O}$ is abelian category ($\mathcal{U}\mathfrak{g}$ is noetherian) full subcat of $\mathcal{U}\mathfrak{g}$-mod.

2) $\Delta(\lambda) \in \mathcal{O}$

3) Any object $M \in \mathcal{O}$ is a quotient of $\Delta(E)$ for some f.dim $\mathfrak{h}$-ss $\mathcal{U}\mathfrak{b}$-module E

Pf Let $m_1, \dots, m_n$ be $\mathcal{U}\mathfrak{g}$-generators of M

Can assume each m_i is a weight vector of $\mathfrak{h}$

$E :=$ $\mathcal{U}\mathfrak{b}$-submodule generated by $m_1, \dots, m_n$.

Then E is $\mathcal{U}\mathfrak{b}$-stable f.dim subspace of M.

$$\Delta(E) \longrightarrow M$$

$\mathcal{U}\mathfrak{g} = \mathcal{U}^- \cdot \mathcal{U}\mathfrak{b}$ $\qquad M = \mathcal{U}\mathfrak{g}\langle m_i \rangle = \mathcal{U}^- E \twoheadleftarrow \Delta(E)$

4) Let M be a f.generated $\mathcal{U}\mathfrak{g}$-module, $\mathfrak{h}$-ss

Then $M \in \mathcal{O} \Leftrightarrow \operatorname{Spec} M$ is bounded.

i.e. $\exists$ finite subset $S \subset \mathfrak{h}^*$ $\quad \operatorname{Spec} M \subset S - Q^+$

Define a partial order on $\mathfrak{h}^*$ by $\lambda \geq \mu$ if $\lambda - \mu \in Q^+$

Pf "$\Leftarrow$" $\quad \mathfrak{g}_\alpha(M_\mu) \subset M_{\alpha+\mu}$

If $\mu \leq \lambda$ $\quad \operatorname{Spec}(\mathfrak{n}^n M_\mu) = \mu + nQ^+ \not\subset \lambda - Q^+ \Rightarrow \mathfrak{n}^n \cdot M_\mu = 0$

"$\Rightarrow$" If $M \in \mathcal{O} \Rightarrow M$ is a quotient of $\Delta(E)$

$\operatorname{Spec} M \subset \operatorname{Spec} \Delta(E) = \operatorname{Spec} E - Q^+$

5) $M \in \mathcal{O} \Rightarrow M_\mu$ is finite dim $\forall \mu \in \mathfrak{h}^*$ (since this holds for $\Delta(E)$)

6) M is finitely generated over $\mathcal{U}^-$

7) $M, N \in \mathcal{O}$ $\quad \dim \operatorname{Hom}(M, N) < \infty$

$\exists\, \Delta(E) \twoheadrightarrow M$ $\qquad \operatorname{Hom}(M,N) \hookrightarrow \operatorname{Hom}_{\mathcal{U}\mathfrak{g}}(\Delta(E), N) = \operatorname{Hom}_{\mathcal{U}\mathfrak{b}}(E, N)$ finite dim

8) Any $M \in \mathcal{O}$ is annihilated by an ideal $I \subset Z(\mathfrak{g})$ of finite codim $\qquad \subseteq \operatorname{Hom}_{\mathcal{U}\mathfrak{h}}(E, N) \simeq \bigoplus_{\mu \in \operatorname{Spec} E} \operatorname{Hom}(E_\mu, N_\mu)$

Pf $Z(\mathfrak{g}) \longrightarrow \operatorname{End}_{\mathcal{O}} M \longleftarrow$ f.dim alg $\Rightarrow$ Ker of this map has finite codim

9) $\mathcal{O} = \bigoplus_{\chi \in \mathfrak{h}^*/W} \mathcal{O}_\chi$

10) Fix $\chi = W \cdot \lambda$ for some $\lambda \in \mathfrak{h}^*$

Then i) any $M \in \mathcal{O}_\chi$ has a simple subquotient

ii) The simple objects of categ $\mathcal{O}$ are $\{L(w(\lambda+\rho)-\rho),\ w \in W\}$

In particular, $\mathcal{O}_\chi$ has fin. many simple objects.

Pf Let $M \in \mathcal{O}_\chi \Rightarrow \operatorname{Spec} M$ is bounded from above

let $\lambda \in \operatorname{Spec} M$ be a max. weight, $m \in M_\lambda$

$\Rightarrow \mathfrak{n} \cdot m = 0 \Rightarrow \exists f : \Delta(\lambda) \longrightarrow M,\ f \neq 0$

$\Delta(\lambda) \twoheadrightarrow \operatorname{Im}(f) \twoheadrightarrow L(\lambda) \Rightarrow L(\lambda)$ is a simple subquotient of M.

$Z(\mathfrak{g})$ acts on $\Delta(\mu)$ via evaluation at μ

$\Rightarrow \Delta(\mu) \in \mathcal{O}_\chi \iff \mu \in W \cdot \lambda$

11) Any object $M \in \mathcal{O}$ has finite length

Pf Since $\mathcal{O} = \bigoplus \mathcal{O}_\chi$ can assume $M \in \mathcal{O}_\chi$

Suppose $M = M^1 \supsetneq M^2 \supsetneq M^3 \cdots$ an infinite chain of submodules $M^i/M^{i+1} \in \mathcal{O}_\chi$

M^i/M^{i+1} has simple subquotient $L(\lambda_i)$, $W\lambda_i = \chi$

$\Rightarrow \exists \lambda_i$ st. $L(\lambda_i)$ occurs as a simple subquotient of M^i/M^{i+1} for infinitely many i's.

$\Rightarrow \lambda_i$ occurs as a weight of M inf. many times $\Rightarrow M_{\lambda_i}$ is inf. dim.

12) • If λ is a dominant weight then $\Delta(\lambda)$ is projective in $\mathcal{O}$.

• If $-\lambda$ is ——— " ——— $\Delta(\lambda)$ is simple

Lemma If λ is dominant $\Rightarrow w(\lambda+\rho)-\rho < \lambda$ for any $w \neq 1$

If $-\lambda$ is dominant $\Rightarrow w(\lambda+\rho)-\rho > \lambda$ for $w \neq 1$.

Pf To prove $\Delta(\lambda)$ is projective $\iff \operatorname{Hom}_{\mathcal{O}}(\Delta(\lambda), -) = \operatorname{Hom}_{\mathcal{U}\mathfrak{b}}(\mathbb{C}_\lambda, -)$ is exact

$\Delta(\lambda) \in \mathcal{O}_\chi \quad \chi = W \cdot \lambda$

Claim $\operatorname{Spec} M \subset \lambda - Q^+ \quad \forall M \in \mathcal{O}_\chi$

The only composition factors of M are $L(w(\lambda+\rho)-\rho)$ $\quad \operatorname{Spec} M \subset \bigcup_w \operatorname{Spec} L(w(\lambda+\rho)-\rho) \subset \lambda - Q^+$

Claim implies that $\forall M \in \mathcal{O}_\chi$, $m \in M_\lambda$, $\mathfrak{n} m = 0$

$$\mathrm{Hom}_{\mathcal{U}\mathfrak{b}}(\mathbb{C}_\lambda, M) = \mathrm{Hom}_{\mathfrak{h}}(\mathbb{C}_\lambda, M)^{\mathfrak{n}} = M_\lambda^{\mathfrak{n}} = M_\lambda \quad \leftarrow \text{exact functor}$$

$\square$

Proof of (ii) similar

Projective objects in $\mathcal{O}$

Let V be a finite dim $\mathfrak{g}$-rep

$F_V : \mathcal{U}\mathfrak{g}\text{-mod} \longrightarrow \mathcal{U}\mathfrak{g}\text{-mod}$ $\quad M \longrightarrow M \otimes_{\mathbb{C}} V$ is an exact functor

$\mathrm{Hom}_{\mathfrak{g}}(M \otimes V, N) = \mathrm{Hom}_{\mathfrak{g}}(M, N \otimes V^*)$ $\quad$ i.e. F_{V^*} is the left and right adjoint of F_V

$F_V : \mathcal{O} \longrightarrow \mathcal{O}$

Prop Let V be a finite dim irrep with highest weight μ and lowest weight ν.

Then
- $\Delta(\lambda) \otimes V$ has a Verma flag
- $\Delta(\lambda) \otimes V \twoheadrightarrow \Delta(\lambda+\nu)$ "top" of flag
- $\Delta(\lambda+\mu) \hookrightarrow \Delta(\lambda) \otimes V$ "bottom"

Pf Tensor identity

For any $\mathcal{U}\mathfrak{b}$-module E

$$\Delta(E \otimes V) = \Delta(E) \otimes V$$

$$\Delta(\lambda) \otimes V = \Delta(\underbrace{\mathbb{C}_\lambda \otimes V}_{E})$$

$V = V^1 \supset V^2 \supset \cdots \supset V^n = \mathbb{C}_\mu$

complete $\mathfrak{b}$-stable flag by Lie thm.

$V^n = \mathbb{C}_\mu \qquad V^1/V^2 = \mathbb{C}_\nu$ $\square$

$\mathbb{C}_\lambda \otimes V = \mathbb{C}_\lambda \otimes V^1 \supset \cdots \supset \mathbb{C}_\lambda \otimes \mathbb{C}_\mu = \mathbb{C}_{\lambda+\mu}$

4/18/14

Reminder Category $\mathcal{O} = \bigoplus_{\chi \in \mathfrak{h}^*/W} \mathcal{O}_\chi$

$\mathfrak{h}^*/W \simeq \mathrm{Max}\, Z(\mathfrak{g})$
$\uparrow$ HC isom

If $\chi = W \cdot \lambda$ ← dot action $\quad$ then simples of $\mathcal{O}_\chi$ are $L(w(\lambda+\rho)-\rho)$ $\quad w \in W$

Thm For any simple $L(\mu)$ $\exists$ an indecomposable projective $P \in \mathcal{O}$ st. $P \twoheadrightarrow L(\mu)$.

Proof 1) If λ is a dominant weight then $\Delta(\lambda)$ is projective

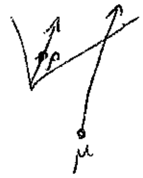
2) Thus the functor $M \mapsto \mathrm{Hom}_{\mathcal{O}}(V \otimes \Delta(\lambda), M) = \mathrm{Hom}_{\mathcal{O}}(\Delta(\lambda), V^* \otimes M)$ is exact

(λ dominant) $\Rightarrow$ $V \otimes \Delta(\lambda)$ is projective

Fix an arbitrary $\mu \in \mathfrak{h}^*$

Can choose a very dominant ν s.t. $\mu+\nu$ is also dominant

($e.g.$ $\nu = N\cdot\rho$ $N \gg 0$)



Let V be a finite dim irrep with lowest weight $-\nu$.

$\Delta(\mu+\nu)$ projective

$$P' = \underbrace{V\otimes\Delta(\mu+\nu)}_{\text{proj}} \twoheadrightarrow \Delta(\mu+\nu-\nu) = \Delta(\mu) \twoheadrightarrow L(\mu)$$

Choose P an indecomposable summand of P' st. $\mathrm{Hom}(P, L(\mu)) \neq 0$

<u>Abstract nonsense</u>

Let $\mathcal{C}$ be a k-linear abelian cat

- all Hom-spaces are fin. dim'l $/k$
- all objects have finite length

Let $P\in\mathcal{C}$ be a projective generator, i.e. P is projective and $\mathrm{Hom}(P,M)\neq 0$ $\forall M\neq 0$.

$\boxed{A := \mathrm{End}_{\mathcal{C}}(P)}$ f.dim algebra   For any $M\in\mathcal{C}$, $\mathrm{Hom}(P,M)$ is a right A-module.

$M \longmapsto \mathrm{Hom}(P,M)$ exact functor $F : \mathcal{C} \longrightarrow$ (finite dimensional right A-modules $=$ mod-A)

<u>Prop:</u> F is an equivalence

<u>Sketch of Proof</u> Need to show F is essentially surjective and fully faithful.

- $\forall M\in\mathcal{C}$, $M\neq 0$, $\exists n>0$ & $P^n \twoheadrightarrow M$
- $P^m \xrightarrow{f} P^n \longrightarrow M \longrightarrow 0$ apply F: $A^m \xrightarrow{F(f)} A^n \longrightarrow F(M) \longrightarrow 0$

A-mod map

$F(f) =$ right mult by $m\times n$ matrix (a_{ij})

$a_{ij}\in \mathrm{End}\,P$ defines f

Working from (a_{ij}), we define f.

Then $M \Rightarrow F(M) = \mathrm{coker}((a_{ij}))$

shows essential surj.

$F(M)\longrightarrow F(N)$ in mod-A lifts to

$$\begin{tikzcd} P^m \arrow[r] \arrow[d, dashed] & P^n \arrow[r] \arrow[d, dashed] & M \arrow[r] \arrow[d] & 0 \\ P^{m'} \arrow[r] & P^{n'} \arrow[r] & N \arrow[r] & 0 \end{tikzcd}$$

$\square$

<u>Cor</u> $\forall \chi\in\mathfrak{h}^*/W$ $\exists$ f.d. algebra A_χ s.t. $\mathcal{O}_\chi \simeq$ mod-A_χ

<u>Pf</u> In $\mathcal{O}_\chi$ all Homs are f.d. and all objects finite length.

Need to construct a projective generator. $\chi = W\cdot\lambda$

$\forall w \in W \quad \exists$ projective $\quad P_w \twoheadrightarrow L(w(\lambda+\rho)-\rho)$

Put $\quad P := \bigoplus_{w \in W} P_w$

q.e.d.

<u>Cor</u> Let P be an indec projective s.t. $P \twoheadrightarrow L$ simple

P' ———— " ———— L'

Any $L \longrightarrow L'$ can be lifted uniquely to
$$\begin{array}{ccc} P & \longrightarrow & P' \\ \downarrow & & \downarrow \\ L & \longrightarrow & L' \end{array}$$

- For any μ $\exists!$ (up to isom) indecom. proj. $P(\mu) \longrightarrow L(\mu)$
- $\dim \operatorname{Hom}(P(\mu), P(\lambda)) = \delta_{\mu\lambda} = \begin{cases} 1 & \lambda = \mu \\ 0 & \lambda \neq \mu \end{cases}$
- $\forall M \in \mathcal{O} \quad [M : L(\mu)] = \dim \operatorname{Hom}(P(\mu), M)$

$M = M^0 \supset M^1 \supset \cdots \supset M^n$ composition series; apply $\operatorname{Hom}(P(\mu), -)$

In particular $[P(\mu) : L(\mu)] = \operatorname{Hom}(P(\mu), L(\mu)) = 1.$

<u>Remark</u> $-\rho \in \mathfrak{h}^*$ is a fixed point of $\circ$-action of W.

$W \circ (-\rho) = \{-\rho\}$

$L(-\rho)$

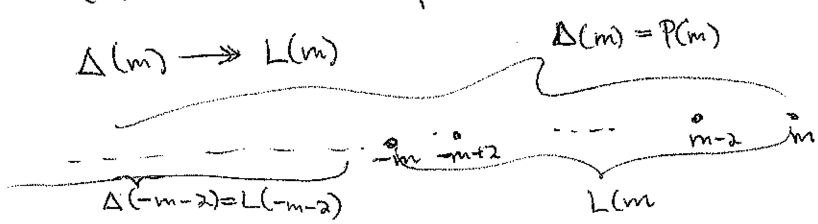
The composition series of $\Delta(-\rho)$ consists of 1-term, ie, $\Delta(-\rho) = L(-\rho) \overset{\text{show later}}{=} P(-\rho)$

<u>Ex</u> $\mathfrak{g} \simeq \mathfrak{sl}_2 \qquad \mathfrak{h} = \mathbb{C} \qquad W = \mathbb{Z}/(2)$

Take $\lambda = m \in \mathbb{Z}_{\geq 0}$

$L(m) =$ fin dim irrep of highest weight m.

$\Delta(m) \twoheadrightarrow L(m)$



$\rho = 1 \qquad -1 \in W \quad \circ\text{-acts} \quad \lambda \longrightarrow -\lambda-2$

$-m-2$ is antidominant

$\Delta(-m-2) = L(-m-2) \neq P(-m-2)$ ~~indecomposable by decomp series~~

Projective gets bigger as you get more anti-dominant

$L(-1) = \Delta(-1) = P(-1)$

$L(m+1) \otimes \Delta(-1)$ is a projective in $\mathcal{O}$

has a Verma flag $\quad M^1 \supseteq M^2 \supset \cdots M^{\text{last}}$

$M^{\text{last}} = \Delta(m+1-1) \qquad M^i/M^{i+1} = \Delta(m+1-2i-1) \qquad i = 0, \ldots, m+1$

last one $\Delta(-m-2)$

$$P(m) = \Delta(m) = \begin{pmatrix} L(m) \\ \\ L(-m-2) \end{pmatrix} \quad \text{composition series}$$

$$P(-m-2) = \begin{pmatrix} \Delta(-m-2) \\ \Delta(m) \end{pmatrix} = \begin{pmatrix} L(-m-2) \\ L(m) \\ L(-m-2) \end{pmatrix}$$

$$P := P(m) \oplus P(-m-2)$$

\begin{tikzcd} \bullet \arrow[r, bend left, "u"] & \bullet \arrow[l, bend left, "v"] \end{tikzcd}

$A = \operatorname{End} P =$ quotient of the path alg of a quiv of 2-vert

$$= kQ/(vu)$$

$\boxed{\text{relations } v \circ u = 0}$

$$\operatorname{End} P(-m-2) = \frac{k[x]}{(x^2)}$$

For general $\mathfrak{g}$

Let μ be an anti-dominant weight

$P(\mu-\rho) \simeq V \otimes \Delta(-\rho) \twoheadrightarrow L(\mu-\rho)$, V is a fd irrep with lowest weight μ.

eliminate parts not in $\mathcal{O}_{W\cdot\mu}$ — only weights in same $W\cdot$-orbit of μ are

Verma flag for $P(\mu-\rho)$ are $\Delta(w\cdot\mu-\rho)$ $w \in W$

Verma components of $P(\mu-\rho) \xleftrightarrow{\sim} \left\{ \begin{matrix} \text{extremal} \\ \text{weights of } V \end{matrix} \right\} = \{ w\cdot\mu-\rho, w \in W \}$

<u>Thm</u> (Soergel)

$$\operatorname{End}_{\mathcal{O}} P(\mu-\rho) \simeq \mathbb{C}[\mathfrak{h}] / (\mathbb{C}^{>0}[\mathfrak{h}]^W) \overset{\text{Borel}}{\simeq} H^*(G/B)$$

Other proof: Bernstein (Traces in categories)

$\boxed{\text{Duality in } \mathcal{O} : \text{ next topic}}$

<u>Koszul duality in $\mathcal{O}$</u>

$$\operatorname{Hom}(P, P^*) = \operatorname{Ext}^\bullet(\mathbb{C}_{G/B}, \mathbb{C}_{G/B}) = H^*(G/B)$$

Koszul ← uses Soergel

<u>Digression</u> 4/21/14

$\mathfrak{b}_- = \mathfrak{h} \oplus \mathfrak{n}_-$

<u>Consider</u> $\mathcal{U}\mathfrak{b}_-$-modules which are ss as $\mathfrak{h}$-modules

$$M = \bigoplus_{\mu \in \mathfrak{h}^*} M_\mu$$

If M f.gen over $\mathcal{U}\mathfrak{n}^- = \mathcal{U}^-$ $\Rightarrow$ Spec M bounded from above

① If M is f.g. over $\mathcal{U}^-$ on $M \neq 0 \Rightarrow M_{\mathfrak{n}_-} := M/\mathfrak{n}_- M \neq 0$

Pf Let $\lambda \in \mathrm{Spec}\, M$ be a max elem

$$M_\lambda \neq 0 \qquad \lambda \notin \mathrm{Spec}(\mathfrak{n}_- M) \Rightarrow \lambda \in \mathrm{Spec}(M/\mathfrak{n}_- M) \Rightarrow (M/\mathfrak{n}_- M)_\lambda \neq 0.$$

② $H_\bullet(\mathfrak{n}_-, M) = H_\bullet(\Lambda^\bullet \mathfrak{n}_- \underset{\mathbb{C}}{\otimes} M, d)$ ← Chevalley–Eilenberg

$\left(\cong \mathrm{Tor}_\bullet^{\mathcal{U}^-}(\mathbb{C}, M)\right)$

Ad $\mathfrak{h}$ action on $\Lambda \mathfrak{n}_-$ is ss.

$\mathfrak{h}$-action on M is ss $\Longrightarrow$ —— " —— $H_i(\mathfrak{n}_-, M)$ is ss.

③ Say that an $\mathcal{U}\mathfrak{b}_-$-module M is <u>free</u> if $M = \mathcal{U}^- \underset{\mathbb{C}}{\otimes} E \qquad E = \bigoplus_\mu E_\mu$ is a ss $\mathfrak{h}$-mod

<u>Lemma</u> Let M be $\left[\mathcal{U}\mathfrak{b}_-\text{-module } \mathfrak{h}\text{-ss f.gen over } \mathcal{U}^-\right.$

M is free $\Longleftrightarrow H_i(\mathfrak{n}_-, M) = 0 \quad \forall i > 0 \quad [\Leftrightarrow H_1 = 0]$

<u>Pf</u> "$\Rightarrow$" follows from $H_i(\mathfrak{n}_-, \mathcal{U}^-) = 0 \quad \forall i > 0.$

"$\Leftarrow$" $H_0(\mathfrak{n}_-, M) = M/\mathfrak{n}_- M = M_{\mathfrak{n}_-}$

Choose $\mathfrak{h}$-splitting

$M \xrightarrow{\ \ } M/\mathfrak{n}_- M$ i.e. $E \subset M$ $\mathfrak{h}$-submodule $E \xrightarrow{\sim} M/\mathfrak{n}_- E$

$\mathrm{Ker} \to \mathcal{U}^- \underset{\mathbb{C}}{\otimes} E \longrightarrow M \to \mathrm{Coker}$ $\quad \mathcal{U}\mathfrak{b}_-$-map $\qquad (\ )_{\mathfrak{n}_-}$ is right exact

So $(\mathrm{Coker})_{\mathfrak{n}_-} = 0 \Rightarrow \mathrm{Coker} = 0$

$$H_1(\mathfrak{n}_-, M) \longrightarrow (\mathrm{Ker})_{\mathfrak{n}_-} \longrightarrow (\mathcal{U}^- \underset{\mathbb{C}}{\otimes} E)_{\mathfrak{n}_-} \longrightarrow M_{\mathfrak{n}_-}$$

with $H_1(\mathfrak{n}_-, M) = 0$, $(\mathcal{U}^- \otimes E)_{\mathfrak{n}_-} = E \xrightarrow{\sim} M_{\mathfrak{n}_-}$

q.e.d.

Restricted duality

If M is a left $\mathcal{U}\mathfrak{b}_-$-module, $M^* = \mathrm{Hom}_{\mathbb{C}}(M, \mathbb{C})$ is a right $\mathcal{U}\mathfrak{b}_-$-module

$$M = \bigoplus_\mu M_\mu \qquad M^\vee = \bigoplus_\mu M_\mu^*$$

If M is f.g. over $\mathcal{U}^-$, $\quad M = \{M_{\leq\lambda}\}_{\lambda \in Q} \qquad \lambda > \mu \Rightarrow M_{\leq\lambda} \supset M_{\leq\mu}$

$M_{\leq\lambda}$ is $\mathcal{U}^-$-stable, $M/M_{\leq\lambda}$ is f.dim

$M_{\leq\lambda} \supset M_{\leq\mu} \quad (\mu < \lambda)$

$$M/M_{\leq\lambda} \twoheadleftarrow M/M_{\leq\mu} \qquad (M/M_{\leq\lambda})^* \hookrightarrow (M/M_{\leq\mu})^*$$

$$M^\vee = \varinjlim_\lambda \left(M/M_{\leq\lambda}\right)^* \qquad \boxed{M^{\vee\vee} \cong M}$$

If M is free, then $\forall$ module M' and any $\mathfrak{h}$-module map $M/\mathfrak{n}_-M \longrightarrow M'/\mathfrak{n}_-M'$

has a lift to a $\mathcal{U}\mathfrak{b}_-$-module map $M \longrightarrow M'$.

M is cofree then $\forall$ M' any $(M')^{\mathfrak{n}_-} \longrightarrow M^{\mathfrak{n}_-}$ $\mathfrak{h}$-map

can be extended to a $\mathcal{U}\mathfrak{b}_-$-module map $M' \longrightarrow M$

M is free $\Rightarrow$ $M^\vee$ is cofree.

Eg. $(\mathcal{U}^-)^\vee$ $\quad \delta : 1 \longrightarrow 1$

Cartan anti-involution on $\mathfrak{g}$, $\mathcal{U}\mathfrak{g}$

$\mathfrak{g} \ni a \longrightarrow a^t$ $\quad$ (for $\mathfrak{g} = \mathfrak{gl}_n$ this is transposition)

$a^t = a$ for $a \in \mathfrak{h}$

$\mathfrak{g}_\alpha \longrightarrow \mathfrak{g}_{-\alpha}$ $\quad \forall$ root α

Assume $\dim M_\mu < \infty$

$\forall$ $\mathfrak{g}$-module $M = \bigoplus M_\mu$ $\quad$ $\mathfrak{h}$-semisimple $\quad M^\vee = \bigoplus_\mu M_\mu^*$ $\quad$ becomes a left $\mathcal{U}\mathfrak{g}$-module via the Cartan-anti-invol

- weights unchanged.

① $\operatorname{Spec} M^\vee = \operatorname{Spec} M$

② $(M^\vee)^\vee \cong M$

③ L simple $\Rightarrow$ $L^\vee$ is simple

④ $L(\lambda)^\vee$ simple, λ is the maximal element in $\operatorname{Spec} L(\lambda)^\vee$

$\dim \underset{\substack{\psi \\ \varphi}}{L(\lambda)^\vee_\lambda} = 1$ $\qquad \mathfrak{n}\cdot\varphi = 0$ $\qquad \exists \quad \Delta(\lambda) \xrightarrow{\neq 0} L(\lambda)^\vee \ : \ 1_\lambda \longmapsto \varphi$

This map is surjective $\Rightarrow L(\lambda)^\vee$ is a simple quotient of $\Delta(\lambda)$

$\Rightarrow L(\lambda)^\vee \simeq L(\lambda)$

Prop 1) ~~Restricted~~ $M \in \mathcal{O} \Rightarrow M^\vee \in \mathcal{O}$

2) $M \longmapsto M^\vee$ exact contravariant anti-equivalence

$\operatorname{Hom}_{\mathcal{O}}(M, N) \simeq \operatorname{Hom}_{\mathcal{O}}(N^\vee, M^\vee)$

3) $M^\vee$ and M has same simple composition factors

Pf $\quad M \rightsquigarrow$ composition series $\rightsquigarrow M^\vee$ composition series still in $\mathcal{O}$. (f.g.)

Cor 1) $P \in \mathcal{O}$ is projective $\Rightarrow$ $P^\vee$ is injective

2) $\mathcal{O}$ has enough injectives

Def $\nabla(\lambda) = \Delta(\lambda)^\vee$

- $\nabla_\lambda(\lambda)$ is 1-dim
- $\nabla(\lambda)$ is co free as a $\mathcal{U}^+$-module with $\nabla_\lambda(\lambda)$ as a co-generator
- $\nabla(\lambda)$ has the unique simple submodule which is $L(\lambda)$

Prop: There is a can. isom. for $M \in \mathcal{O}$

$$\mathrm{Ext}^\bullet_{\mathcal{O}}(M, \nabla(\lambda)) \simeq H_\bullet(n_-, M)^*(\lambda)$$

← λ weight space

Pf
- Both sides are contravariant functors
- " — " — δ-functors
- Must check 1) They agree in degree 0
  2) Any object has a resolution by objects which are acyclic by both factors

$$\mathrm{Hom}^\cdot_{\mathcal{O}}(M, \nabla(\lambda)) = \mathrm{Hom}_{\mathcal{O}}(\Delta(\lambda), M^\vee) = (M^\vee)^{n}_\lambda = (M/n_-M)^*_\lambda = H_0(n_-, M)^*_\lambda$$

let P be a projective in $\mathcal{O}$. $\mathrm{Ext}^{>0}_{\mathcal{O}}(P, \Delta(\lambda)) = 0$

Claim: $H^{>0}(n_-, P) = 0$

Assume P is indecomposable.

$$\underset{\text{f.dim}}{V} \otimes \Delta(\lambda) = \Delta(V \otimes \mathbb{C}_\lambda) \quad \text{as } \mathcal{U}^-\text{-mod} = \mathcal{U}^- \otimes (V \otimes \mathbb{C}_\lambda) \quad \text{free so } H^{>0} = 0.$$

Cor $\dim \mathrm{Ext}^i_{\mathcal{O}}(\Delta(\mu), \nabla(\lambda)) = \begin{cases} 1 & \text{if } \mu = \lambda,\ i = 0 \\ 0 & \text{otherwise} \end{cases}$

Pf By Prop, LHS $= H_i(n_-, \Delta(\mu))^*_\lambda = \begin{cases} (\mathbb{C} \cdot 1_\mu)^*_\lambda & \text{if } i = 0 \\ 0 & \text{if } i \neq 0 \end{cases}$ □

$\mathrm{Hom}(\Delta(\lambda), \nabla(\lambda))$ is 1-dim

$$\Delta(\lambda) \twoheadrightarrow L(\lambda) \hookrightarrow \nabla(\lambda)$$

Thm (i) For $M \in \mathcal{O}$ TFAE:
1) M has a Verma flag
2) $H_{>0}(n_-, M) = 0$
3) $\mathrm{Ext}^{>0}_{\mathcal{O}}(M, \nabla(\lambda)) = 0 \quad \forall \lambda$

(ii) If i) holds, $[M : \Delta(\mu)] = \dim \mathrm{Hom}_{\mathcal{O}}(M, \nabla(\mu))$

Pf Know $(2) \Leftrightarrow (3)$

1) $\Rightarrow$ 2) Induct on the length of the Verma flag

length $= 1 \Rightarrow M =$ Verma $\Rightarrow M$ is free over $\mathcal{U}^- \Rightarrow H_{>0}(n_-, M) = 0$

2) $\Rightarrow$ 1) 2) $\Rightarrow H_1(n_-, M) = 0 \Rightarrow M$ is free i.e. $M \simeq \mathcal{U}^- \otimes E$

st. $E = \bigoplus_\lambda E_\lambda$ E is f. dim.

Let λ maximal elt in Spec E

$\mathrm{Spec}\, M = \mathrm{Spec}\, E - Q^+$

λ is max elt of Spec M moreover $M_\lambda = 1 \otimes E_\lambda$

deduce that $n M_\lambda = 0$

Have a map $f : \Delta(\lambda) \otimes M_\lambda \longrightarrow M$ of $\mathcal{U}\mathfrak{g}$-mod.

$$\begin{array}{ccc} \Delta(\lambda)\otimes M_\lambda & \longrightarrow & M \\ \cup & & \cup \\ M_\lambda & \xrightarrow{Id} & M_\lambda \end{array}$$

As a map of $\mathcal{U}^-$-modules $\mathcal{U}^- \otimes E_\lambda \longrightarrow \mathcal{U}^- \otimes E$ induced by $E_\lambda \hookrightarrow E$

See that f is injective

$M / \Delta(\lambda) \otimes M_\lambda = \mathcal{U}^- \otimes (E/E_\lambda)$ still satisfies assumptions of 2). Repeat the procedure.

$\square$ (i)

<u>Pf of (ii)</u> $\mathrm{Hom}(M, \nabla(\lambda)) = (M_{n_-})^*_\lambda = E^*_\lambda$ dim of $\mathrm{Hom}(M, \nabla(\lambda)) = \dim E_\lambda$

$= [M : \Delta(\lambda)]$

4/25/14 All weights λ are assumed to be integral, $\langle \lambda, \check{\alpha} \rangle \in \mathbb{Z}$ $\forall$ coroot $\check{\alpha}$

<u>lemma</u> 1) Any morphism of Verma modules is injective

2) $\Delta(\mu)$ is simple iff μ is antidominant

3) Let ν int. weight $\mu :=$ antidominant elt in $W \cdot \nu$

Then $\Delta(\mu) \subset \Delta(\nu)$ is the unique simple submodule of $\Delta(\nu)$

<u>Pf</u> 1) $f : \Delta(\mu) \longrightarrow \Delta(\nu)$ this is in particular an $\mathcal{U}^-$-module map $\mathcal{U}^- \longrightarrow \mathcal{U}^-$

$\Rightarrow \exists a \in \mathcal{U}^-$ st. $f(u) = u \cdot a$ $\forall u \in \mathcal{U}^-$

$\mathcal{U}^-$ has no zero divisors $\Rightarrow$ $u \mapsto ua$ injective

2) Know "$\Leftarrow$"

"$\Rightarrow$" suppose μ not anti-dominant

$\Rightarrow \exists$ simple reflection s_α st. $s_\alpha \cdot \lambda < \lambda$

$\Rightarrow \exists\ \Delta(s_\alpha \cdot \lambda) \xrightarrow{\neq} \Delta(\lambda)$ $\Rightarrow$ Image is a proper submodule in $\Delta(\lambda)$

$\Rightarrow \Delta(\lambda)$ is not simple

3) Let $L(\mu')$ be a simple submodule in $\Delta(\nu)$, so $L(\mu') \hookrightarrow \Delta(\nu)$ can be lifted to $\Delta(\mu') \longrightarrow \Delta(\nu)$ with image $L(\mu')$ is injective by 1)

$\Rightarrow$ $\Delta(\mu') \simeq L(\mu')$ $\Rightarrow$ μ' is anti-dominant by 2)

$\Rightarrow$ $\mu' = \mu$

$\Rightarrow$ $[\Delta(\nu) : L(\mu)] \geq 1$. $\square$

Last time we discussed duality in $\mathcal{O}$

$M \longrightarrow M^\vee$

• λ is dominant $\Rightarrow$ $\Delta(\lambda)$ is projective $\Rightarrow$ $\nabla(\lambda)$ injective

<u>Thm</u> M has a Verma flag iff $\mathrm{Ext}^{>0}(M, \nabla(\mu)) = 0$ $\forall \mu$

<u>Cor</u> If M has a Verma flag then so does any direct summand of M.

<u>Cor</u> Any projective in $\mathcal{O}$ has a Verma flag

<u>BGG reciprocity</u> $[P(\lambda), \Delta(\mu)] = [\Delta(\mu) : L(\lambda)]$

<u>Comment:</u> for a finite category $[P_\alpha : L_\beta] = c_{\alpha\beta}$ Cartan matrix

$C = \|c_{\alpha\beta}\|$

For category $\mathcal{O}$: $C = B \cdot B^t$

<u>Pf of BGG</u>: $[\Delta(\mu) : L(\lambda)] = [\nabla(\mu) : L(\lambda)] = \dim \mathrm{Hom}(P(\lambda), \nabla(\mu)) \underset{\text{since } P(\lambda) \text{ has Verma flag}}{=} [P(\lambda) : \Delta(\mu)]$ QED.

Let λ be a regular integral antidominant

$\Rightarrow$ $\lambda = -\rho + \mu$ ~~where~~ for some μ anti-dominant

<u>Prop</u> 1) $[P(\lambda) : \Delta(w \cdot \lambda)] = 1$ $\forall w \in W$

2) $P(\lambda)^\vee \simeq P(\lambda)$ both injective and projective, tilting (has Verma and dual Verma flag)

<u>Pf</u> Recall that $P(\lambda) = P(-\rho + \mu)$ is constructed as follows.

$\Delta(-\rho)$ is a projective

Let V a f.dim irrep with lowest weight μ.

$$V \otimes \Delta(-\rho) \twoheadrightarrow L(-\rho + \mu) = L(\lambda)$$

$\mathrm{pr} : \mathcal{O} \longrightarrow \mathcal{O}_\chi$ $\quad \chi := W \cdot \lambda$

Then $\mathrm{pr}(V \otimes \Delta(-\rho))$ is a projective in $\mathcal{O}_\chi$

$V\otimes\Delta(-\rho)$ has a Verma flag with subquotients $\Delta(\nu-\rho)$, $\nu\in\operatorname{Spec}V$

~~$\Delta(\nu-\rho)$ has Verma flag~~ such that $\nu-\rho\in W\cdot\lambda$

$$\underset{\{w(\lambda+\rho)-\rho\}=\{w(\mu)-\rho\}}{\overset{\parallel}{W\cdot\lambda}}$$

$[\operatorname{pr}(V\otimes\Delta(-\rho)):\Delta(w\cdot\lambda)]=1$

$P(\lambda)$ is an indecomp. summand of $\operatorname{pr}(-)$

$[P(\lambda):\Delta(w\cdot\lambda)]\leq 1$

By BGG, $[P(\lambda):\Delta(w\cdot\lambda)]=[\Delta(w\cdot\lambda):L(\lambda)]\geqslant 1$ by first lemma.

hence $=1$ $\square$

<u>Cor</u> $\dim\operatorname{End}_G P(\lambda)=\#W$

<u>Pf</u> $\dim\operatorname{Hom}(P(\lambda),M)=[M:L(\lambda)]$

For $M=P(\lambda)$, $\dim\operatorname{End}=[P(\lambda):L(\lambda)]=\sum_w \underset{1}{\underset{\parallel}{[P(\lambda):\Delta(w\cdot\lambda)]}}\underset{1}{\underset{\parallel}{[\Delta(w\cdot\lambda):L(\lambda)]}}$

$=\#W.$ QED

Construction of a morphism $P(\lambda)\to P(\lambda)^\vee$ not the naive map that kills almost everything

$$\begin{tikzcd}
& P(\lambda)^\vee \arrow[rr, dashed, leftrightarrow, "\exists"] & & P(\lambda) \\
L(\lambda) \arrow[ur, "\text{injective hull}"] \arrow[rr, hook, "\text{the unique simple submodule}"'] \arrow[urrr, bend right=60, "\text{injective}"'] & & \Delta(w_0\lambda) \arrow[ur] &
\end{tikzcd}
\qquad P(\lambda)=\begin{pmatrix}\Delta(\lambda)\\ \vdots\\ \Delta(w_0\cdot\lambda)\end{pmatrix}$$

w_0 the largest elt of W

From cat $\mathcal{O}$ to HC-modules

$G\supset K$ ← max cpct

G is a connected, simply connected <u>complex</u> s.s. grp.

$\mathfrak{g}=\operatorname{Lie}G=\mathfrak{k}\oplus i\mathfrak{k}$ ← complex

$\cup$

$\mathfrak{k}=\operatorname{Lie}K$ ← real

E.g. $G=SL_n(\mathbb{C})$, $K=$ ~~SO~~ SU_n

$\mathfrak{sl}_n(\mathbb{C})=\underset{\mathfrak{k}}{\underset{\parallel}{\text{skew herm}}}+\sqrt{-1}\cdot\text{skew herm}$

$\exists$ a real anti-involution $\mathfrak{g}\to\mathfrak{g}:x\mapsto x^*$

s.t. $\mathfrak{k}=\{x\in\mathfrak{g}\mid x^*=-x\}$

$$\begin{array}{ccc}
\mathbb{C}\otimes_{\mathbb{R}}\mathfrak{g} & \xleftarrow{\sim} & \mathfrak{g}\oplus\mathfrak{g}^{op}\\
\cup & & \uparrow\\
\mathbb{C}\otimes_{\mathbb{R}}\mathfrak{k} & \xleftarrow{\sim} & \mathfrak{g}\ \text{anti-diag}
\end{array}$$

$-1\otimes x^*-i\otimes ix^*\longleftarrow\!\shortmid (x,0)$

$-1\otimes x+i\otimes ix\longleftarrow\!\shortmid (0,x)$

extend $*$ to a $\mathbb{C}$-linear anti-inv

$(x,0)\mapsto(-1\otimes x^*-i\otimes ix^*)^*=-1\otimes x-i\otimes(-i(x^*)^*)$

$=-1\otimes x+i\otimes ix\longleftarrow\!\shortmid(0,x)$

So $*:(x,y)\mapsto(y,x)$

$\mathbb{C}\otimes k \longleftrightarrow \{(x,y)\in \mathfrak{g}\oplus\mathfrak{g}^{op} \mid (x,y)^* = -(x,y)\} = \{(x,-x),\ x\in\mathfrak{g}\}$

Recall that a $(\mathfrak{g},K)$-module is a rep of $\mathfrak{g}$,
and a compatible (locally finite) K-action

$\Leftrightarrow$ $\mathbb{C}\otimes\mathfrak{g}$-reps $=$ $\mathfrak{g}\oplus\mathfrak{g}^{op}$-rep
which is locally finite under the action of $\mathfrak{g} \overset{\text{anti-diag}}{\hookrightarrow} \mathfrak{g}\oplus\mathfrak{g}^{op}$

$\Leftrightarrow$ $\mathcal{U}(\mathfrak{g}\oplus\mathfrak{g}^{op})$-modules $=$ $\mathcal{U}\mathfrak{g}$-bimodules,
adjoint $\mathfrak{g}$-action $xm-mx$ locally finite $= \mathcal{HC}$ category of HC-modules.

E.g. $\mathcal{U}\mathfrak{g}\in\mathcal{HC}$ | $\mathcal{U} := \mathcal{U}\mathfrak{g}$

Let V be a finite dim $\mathfrak{g}$-rep.

$P_V = V\otimes\mathcal{U}\mathfrak{g}$ left $\mathfrak{g}$ action $\mathfrak{g}\ni x$ $v\otimes u \longrightarrow (xv)\otimes u + v\otimes xu$
right action $x: v\otimes u \longrightarrow v\otimes(ux)$

$\mathrm{ad}x(v\otimes u) = xv\otimes u + v\otimes \mathrm{ad}x(u)$ is loc. finite action

$\Rightarrow P_V \in \mathcal{HC}$. $\quad P_V \hookleftarrow V := V\otimes 1$

<u>Lemma</u> 1) $\mathrm{Hom}_{\mathcal{U}\text{-bimod}}(P_V, M) = \mathrm{Hom}_{\mathfrak{g}}(V, M^{ad})$

The functor $M \longmapsto \mathrm{Hom}_{\mathfrak{g}}(V, M^{ad})$ is exact on $\mathcal{HC}$
since $\mathfrak{g}$ is a ss Lie alg. So P_V is projective in $\mathcal{HC}$

2) We have $P_V = V\mathcal{U} = \mathcal{U}V$

<u>Pf</u> 1) $f\in \mathrm{Hom}_{\mathcal{U}\text{-bimod}}(P_V, M) \rightsquigarrow f_V := f|_{V\otimes 1} \in \mathrm{Hom}_{\mathfrak{g}}(V, M^{ad})$

$[\, f: v\otimes u \longmapsto f_V(v)\otimes u\,] \leftsquigarrow f_V \in \mathrm{Hom}_{\mathfrak{g}}(V, M^{ad})$

2) It is clear that $V\mathcal{U} = P_V$. It suffices to show that $\mathcal{U}V$ is right $\mathcal{U}$-stable.

$u(v\otimes 1)x = u(v\otimes x) = u[x(v\otimes 1) - (xv)\otimes 1] = ux(v\otimes 1) - u((xv)\otimes 1) \in \mathcal{U}V.$ $\square$

4/28/14

$\mathcal{HC} = \begin{cases} \text{• finitely generated } \mathcal{U}\otimes\mathcal{U}^{op}\text{-modules} \\ \text{• locally finite wrt. ad} \end{cases}$ $\qquad \boxed{\mathcal{U} = \mathcal{U}\mathfrak{g}}$

$Z = \mathrm{Center}(\mathcal{U}\mathfrak{g})$

$P_V = V\otimes\mathcal{U} \in \mathcal{HC}$

$Z(\mathcal{U}\otimes\mathcal{U}^{op}) \cong Z\otimes Z$

If M is a $\mathcal{U}$-bimod $\qquad Z\otimes Z \longrightarrow \mathrm{End}_{\mathcal{U}\text{-bimod}} M \hookrightarrow \mathrm{End}_{\mathrm{ad}\,\mathfrak{g}} M^{ad}$
$M\in\mathcal{HC}$

If $M \in \mathcal{HC}$

$$M^{ad} = \bigoplus_{E \in \mathrm{Irr}\,\mathfrak{g}} M^E \qquad M^E := E\text{-isotypic component of } \mathrm{ad}\,\mathfrak{g}$$

$$\mathrm{End}_{\mathrm{ad}\,\mathfrak{g}} M^{ad} = \prod_{E \in \mathrm{Irr}\,\mathfrak{g}} \mathrm{End}_{\mathfrak{g}}(M^E)$$

Let I_χ a max ideal in Z.

Notation $\mathcal{U}_\chi = \mathcal{U}/(I_\chi)$

$$\mathcal{HC}_\chi = \left\{\text{f.gen } (\mathcal{U}, \mathcal{U}_\chi)\text{-bimod with } \overset{ad}{\text{locally finite}}\right\} = \left\{ M \in \mathcal{HC} \mid M \cdot I_\chi = 0 \right\}$$

$$P_{\chi,V} = \underset{\mathcal{HC}_\chi}{\underset{\cap}{V \otimes \mathcal{U}_\chi}} = (V \otimes \mathcal{U})/I_\chi = P_V/I_\chi$$

$$\mathrm{Hom}_{(\mathcal{U},\mathcal{U}_\chi)\text{-bimod}}(P_{\chi,V}, M) = \mathrm{Hom}_{\mathfrak{g}}(V, M^{ad}) \qquad \text{for any } M \in \mathcal{HC}_\chi$$

Thm (Kostant) For any $E \in \mathrm{Irr}\,\mathfrak{g}$

$\mathrm{Hom}_{\mathfrak{g}}(E, \mathcal{U}^{ad})$ is a free Z-module of rank $= \dim E_0$ ($:=$ zero weight multiplicity of E)

Cor $[\mathcal{U}_\chi^{ad} : E] = \dim E_0$

more generally $[P_{\chi,V}^{ad} : E] = \dim (V^* \otimes E)_0$

Pf $\mathrm{Hom}(E, P_{\chi,V}^{ad}) = \mathrm{Hom}(E, V \otimes \mathcal{U}_\chi^{ad}) = \mathrm{Hom}(E \otimes V^*, \mathcal{U}_\chi^{ad})$ $\square$

Thm (Kostant) Let M be a finitely generated left $\mathcal{U}$-mod.
V a finite dim $\mathfrak{g}$-rep.

Then, if the Z-action on M is loc. finite then so is the Z-action on $V \otimes M$.

Not obvious! $\mathcal{U} \xrightarrow{\text{cup}} \mathcal{U} \otimes \mathcal{U} \curvearrowright V \otimes M$

$$\begin{array}{l} \cup \\ Z \ni z \longrightarrow \sum z_i' \otimes z_i'' \quad z_i', z_i'' \text{ not central} \end{array}$$

Pf: Step 1: M is annihilated by an ideal $\mathcal{J} \subset Z$ of finite codim

For a $m \in M$ $\quad \mathrm{Ann}_Z\, m$ is an ideal of Z of finite codim.

$M = \sum_{i=1}^n \mathcal{U} \cdot m_i \qquad \mathcal{J} = \bigcap_{i=1}^n \mathrm{Ann}_Z(m_i)$ ideal of finite codim, annihilates M

Step 2: $V \otimes M$ has a chain of submodules $V\otimes M \supset F^1 \supset F^2 \supset \dots \supset F^n$ of submodules

s.t. $F^i/F^{i+1} \twoheadleftarrow V \otimes \mathcal{U}_{\chi_i}$

Pf: For M we have the spectral decomposition. $M = \bigoplus_{\chi} M^{(\chi)}$ s.t. $I_\chi^n M^{(\chi)} = 0$

Can assume $I_\chi^n M = 0$ for some n, χ.

$$M \supset I_\chi M \supset I_\chi^2 M \supset \dots \supset I_\chi^n M = 0.$$

$I_\chi^k M / I_\chi^{k+1} M$ is annihilated by I_χ so treat as a f.gen $\mathcal{U}_\chi$-module

$\| $

$\sum_{j=1}^N \mathcal{U}_\chi \cdot m_j$ has a finite filtration with $\mathcal{U}_\chi$-cyclic quotients

$V \otimes_{\mathbb{C}} -$ is exact.

Step 3: It suffices to show that the Z-action on each F^i/F^{i+1} is loc. finite.

Thus it suffices to show that this is the case for $V \otimes \mathcal{U}_\chi = P_{\chi,V}$

$Z \otimes Z \supset \underbrace{Z \otimes 1}_{\text{want to show that this acts locally finitely}}$

$(Z\otimes 1)(P_{\chi,V}^E) \subset P_{\chi,V}^E$

By 1st theorem

$\mathrm{Ann}_{Z\otimes 1}(P_{\chi,V}^V)$ has finite codim. □

Lemma 1) Any $M \in \mathcal{HC}$ is a quotient of P_V for some V f.d.

$M \in \mathcal{HC}_\chi$ —— " —— $P_{\chi,V}$ —— " ——

2) Any $M \in \mathcal{HC}$ is finitely generated as either left or right $\mathcal{U}$-mod

3) $M, N \longrightarrow M \otimes_{\mathcal{U}} N$ gives $\mathcal{HC}$ a monoidal structure.

Pf 1) $M = \sum_{i=1}^n \mathcal{U} m_i \mathcal{U}$ ad-action is loc. finite

$\Rightarrow \exists$ fin-dim ad-stable $V \subset M$ s.t. $m_i \in V \ \forall i$

$M = \mathcal{U} V \mathcal{U}$

$f_V : V \xhookrightarrow[\mathfrak{g}\text{-map}]{} M^{ad} \Rightarrow f : P_V \xrightarrow[\mathcal{U}\text{-bimod}]{} M$

$P_V = V\mathcal{U} = \mathcal{U}V$

$\mathcal{U} f_V(V) \mathcal{U} = M$

$f(P_V) = M$

$M = \mathcal{U} f_V(V) = f_V(V)\mathcal{U}$ □ 1), 2)

$$P_V \underset{\mathcal{U}}{\otimes} P_{V'} = (V\otimes \mathcal{U}) \underset{\mathcal{U}}{\otimes} (V'\otimes\mathcal{U}) = V\otimes V' \otimes \mathcal{U} = P_{V\otimes V'} \in \mathcal{HC}$$

$\underset{\mathcal{U}}{\otimes}-$ right exact.

$\forall M, M' \in \mathcal{HC}$

$P_V \twoheadrightarrow M \quad P_{V'} \twoheadrightarrow M' \quad \Longrightarrow \quad P_V \underset{\mathcal{U}}{\otimes} P_{V'} \twoheadrightarrow M \underset{\mathcal{U}}{\otimes} M'$ etc. $\square$ 3)

Prop: For $M \in \mathcal{HC}$ TFAE:

1) $[M^{ad} : E] < \infty \quad \forall E \in \mathrm{Irr}\, \mathfrak{g}$
2) $\mathrm{Ann}_{Z\otimes Z} M$ is an ideal of fin. codim in $Z\otimes Z$
3) The $(1\otimes Z)$-action on M is locally finite.

Pf 1) $\Rightarrow$ 2) as in the pf of the previous thm of Kostant

2) $\Rightarrow$ 3) clear

3) $\Rightarrow$ 1) Similarly to the argument of Step 2 of thm

$\exists$ finite chain $M \supset M^1 \supset M^2 \supset \dots \supset M^n = 0$

s.t. $\left(M^i/M^{i+1}\right) I_{\chi_i} = 0$ for some $I_{\chi_i} \in \mathrm{Max}\, Z$.

So M^i/M^{i+1} as $(\mathcal{U}, \mathcal{U}_{\chi_i})$-mod $\Rightarrow \exists P_{\chi_i, V_i} \twoheadrightarrow M^i/M^{i+1}$ | $[P_{\chi_i,V_i} : E] < \infty$

$\square$

$\mathcal{O} \supset \mathcal{O}_{int} = \left\{ M \in \mathcal{O} \,\middle|\, M_\mu = 0 \text{ for any nonintegral weight } \mu \in \mathfrak{h}^* \right\}$

If λ is integral $\Rightarrow \Delta(\lambda) \in \mathcal{O}_{int}$

$\underline{\text{Fix } \lambda}$

Functor $\mathcal{U}\text{-bimodules} \xrightarrow{\Phi} \mathcal{U}\text{-modules}$

$$M \longmapsto M \underset{\mathcal{U}}{\otimes} \Delta(\lambda) = M \underset{\mathcal{U}}{\otimes} \left(\mathcal{U}/\mathcal{U}\mathfrak{n} + \mathcal{U}(h-\lambda(h))\right) = \left(M/M\cdot\mathfrak{n}\right) \Big/ \left(m\cdot h = m\cdot\lambda(h)\right)$$

Lem If λ is integral then $\Phi(\mathcal{HC}) \subset \mathcal{O}_{int}$

Pf $M \in \mathcal{HC}$

Must check that $M/M\mathfrak{n} + M(h-\lambda(h))$ is loc. $\mathfrak{n}$-nilp and $\mathfrak{h}$-ss.

$$\forall x \in \mathfrak{g} \qquad x \cdot m = \underbrace{(x\cdot - \cdot x)}_{\text{ad-}\mathfrak{n} \text{ loc. nilp}} + \underbrace{\cdot x}_{\text{zero}}$$

Thm Let λ be a dominant integral weight, $\chi := W\cdot\lambda$

Then $\Phi_\lambda : \mathcal{HC}_\chi \xrightarrow{\sim} \mathcal{O}_{int}$ is an equiv

Abstract nonsense lemma. Let $\Phi : \mathcal{C} \to \mathcal{C}'$ be a right exact functor between ab categories. Let $\mathcal{P} \subset \mathrm{Ob}\,\mathcal{C}$ some set of projective objects s.t.

(i) Any $M \in \mathcal{C}$ is a quotient of some $P \in \mathcal{P}$

(ii) $\Phi(P)$ is projective and any object of $\mathcal{C}'$ is a quotient of $\Phi(P)$ for some $P \in \mathcal{P}$

(iii) $\mathrm{Hom}(P, P') \xrightarrow[\sim]{\Phi} \mathrm{Hom}(\Phi(P), \Phi(P'))$ is an isom $\forall P, P' \in \mathcal{P}$.

Then Φ is an equivalence.

Pf of thm. $\Phi = \cdot \underset{\mathcal{U}}{\otimes} \Delta(\lambda)$ is right exact.

$\Phi(P_{\chi,V}) = (V \otimes \mathcal{U}_\chi) \underset{\mathcal{U}}{\otimes} \Delta(\lambda) = V \otimes \Delta(\lambda)$

Take $\mathcal{P} = \{ P_{\chi,V},\ V \text{ f.dim } \mathfrak{g}\text{-rep} \}$, then i) is OK.

λ dom $\Rightarrow$ $\Delta(\lambda)$ is projective $\Rightarrow$ $V \otimes \Delta(\lambda)$ is projective $\Rightarrow$ ii)

(same proof as before where we use $-\mu$)

Duflo thm. Let $\mu \in \mathfrak{h}^*$, $\chi := W \cdot \mu$

$\mathrm{Ann}_{\mathcal{U}}(\Delta(\mu)) = \mathcal{U} \cdot I_\chi$

5/2/14

Lemma. Let V be a finite dim rep, λ dominant integral

$\dim \mathrm{Hom}(\Delta(\lambda), V \otimes \Delta(\lambda)) = \dim(V_0)$

Pf. λ dominant $\Rightarrow$ $\Delta(\lambda)$ is projective

$\dim \mathrm{Hom}(\Delta(\lambda), M) = [M : L(\lambda)] \quad \forall M \in \mathcal{O}$

$\mathrm{LHS} = [V \otimes \Delta(\lambda) : L(\lambda)] = \sum_{\mu \in \mathrm{Spec}\,V} [\Delta(\lambda+\mu) : L(\lambda)] \dim V_\mu$

$$= \sum_{\substack{\mu \in \mathrm{Spec}\,V \\ \lambda+\mu = w\cdot\lambda \\ w \in W}} [\Delta(w\cdot\lambda) : L(\lambda)] \dim V_\mu$$

$V \otimes \Delta(\lambda)$ has a Verma flag with subquotient $\Delta(\lambda+\mu)$, $\mu \in \mathrm{Spec}\,V$ with multiplicity $\dim V_\mu$

Note that if $w\cdot\lambda \neq \lambda \Rightarrow w\cdot\lambda \in \lambda - (Q^+ \setminus \{0\}) \Rightarrow \lambda \notin \mathrm{Spec}\,\Delta(w\cdot\lambda)$

$\Rightarrow [\Delta(w\cdot\lambda) : L(\lambda)] = 0$

$$= \sum_{\mu=0} \underbrace{[\Delta(\lambda) : L(\lambda)]}_{=1} \dim V_0 = \dim V_0$$ $\square$

Fix λ integral dominant, $\chi = W\cdot\lambda$

$\mathcal{HC}_\chi = \{ M \in \mathcal{HC} \mid M \cdot I_\chi = 0 \} = \{ M \in \mathcal{HC} \mid M \text{ is } (\mathcal{U}, \mathcal{U}_\chi)\text{-module} \}$

$\phi_\lambda : \mathcal{HC}_\chi \longrightarrow \mathcal{O}_{int}$

$$M \longmapsto M \underset{\mathcal{U}_\chi}{\otimes} \Delta(\lambda)$$

<u>Thm</u> This is an equiv.

Last time reduced the pf to showing that $\mathrm{Hom}_{\mathcal{U}\text{-bimod}}(P_{\chi,V_1}, P_{\chi,V_2}) \to \mathrm{Hom}_{\mathcal{U}\text{-mod}}(P_{\chi,V_1}\otimes\Delta(\lambda), P_{\chi,V_2}\otimes\Delta(\lambda))$

isom $\forall$ f-diml V_1, V_2.

$P_{\chi,V} := V \otimes \mathcal{U}_\chi$ $\qquad P_{\chi,V} \underset{\mathcal{U}_\chi}{\otimes} \Delta(\lambda) = V \underset{\mathbb{C}}{\otimes} \Delta(\lambda)$

$V := V_1^* \otimes V_2$

By adjunction $\mathrm{Hom}_{\mathbb{C}}(V \underset{\mathbb{C}}{\otimes} ?_1, ?_2) = \mathrm{Hom}_{\mathbb{C}}(?_1, V^* \otimes ?_2)$

$\Rightarrow$ Suffices to prove $\mathrm{Hom}_{\mathcal{U}\text{-bimod}}(\mathcal{U}_\chi, V\otimes\mathcal{U}_\chi) \xrightarrow[?\uparrow \text{ isom}]{} \mathrm{Hom}_{\mathcal{U}\text{-mod}}(\Delta(\lambda), V\otimes\Delta(\lambda))$

$$\mathrm{LHS} = (V\otimes\mathcal{U}_\chi)^{ad} = \mathrm{Hom}_{\mathfrak{g}}(V^*, \mathcal{U}_\chi^{ad})$$

$$\dim \mathrm{Hom}_{\mathfrak{g}}(V^*, \mathcal{U}_\chi^{ad}) \underset{\text{by Kostant}}{=} \dim (V^*)_0 = \dim V_0$$

Thus $\dim \mathrm{LHS} = \dim V_0$. $\quad$ Previous lemma $\Rightarrow$ $\dim \mathrm{RHS} = \dim V_0$.

It suffices to show ϕ_λ is injective, i.e. if $0 \neq f : \mathcal{U}_\chi \longrightarrow V\otimes\mathcal{U}_\chi$

then $\phi_\lambda(f) : \Delta(\lambda) \longrightarrow V\otimes\Delta(\lambda)$ is nonzero.

Suppose $\phi_\lambda(f) = 0$. $\qquad f(1) = \sum v_i \otimes u_i$ $\qquad \{v_i\}$ is a $\mathbb{C}$-basis of V, $u_i \in \mathcal{U}_\chi$

$u\cdot 1 = 1\cdot u \quad \forall u \in \mathcal{U}$

$u(\sum v_i \otimes u_i) = \sum v_i \otimes u_i u$ $\quad$ an equation in $V\otimes\mathcal{U}_\chi$

$\Rightarrow$ if 1_λ is generator of $\Delta(\lambda)$, then $u(\sum v_i\otimes u_i)1_\lambda = \sum v_i \otimes (u_i u 1_\lambda)$ an equation in $V\otimes\Delta(\lambda)$

$$u(\phi_\lambda(f)(1_\lambda)) = u(\sum v_i \otimes (u_i 1_\lambda)) = \sum v_i \otimes (u_i u 1_\lambda)$$

$\phi_\lambda(f) = 0 \Rightarrow \quad = 0 = \sum v_i \otimes (u_i u 1_\lambda) \Rightarrow u_i u 1_\lambda = 0 \quad \forall i$

$\Rightarrow u_i \Delta(\lambda) = 0 \Rightarrow u_i \in \mathrm{Ann}\,\Delta(\lambda)$

$\overset{\text{Duflo}}{\Rightarrow} u_i \in \mathcal{U}\cdot I_\chi \Rightarrow u_i = 0$ in $\mathcal{U}_\chi$

$\Rightarrow f(1) = \sum v_i \otimes u_i = 0$ in $V\otimes\mathcal{U}_\chi \Rightarrow f(u\cdot 1) = 0 \;\forall u \Rightarrow f = 0.$ $\square$

<u>Cor</u> For $M \in \mathcal{HC}$ the TFAE

1) M is annihilated by an ideal in $Z\otimes Z$ of finite codim
2) M has finite length

Pf 2) $\Rightarrow$ 1) $M \supset M^1 \supset \cdots \supset M^n = 0$ a comp series

(Reminder, $Z \otimes Z$ countable so Schur lemma applies, center acts by scalars on irrep)

Then $\exists (\psi_i, \chi_i) \in \mathrm{Max}(Z \otimes Z)$

$$I_{\psi_i} \left(M^i/M^{i+1}\right) I_{\chi_i} = 0$$

$\Rightarrow M$ is annihilated by $\prod_i I_{\psi_i} \otimes \prod_i I_{\chi_i}$ which is an ideal of f codim

1) $\Rightarrow$ 2) 1) $\Rightarrow$ Spectral decomp $\Rightarrow \exists\, M = M^0 \supset M^1 \supset \cdots \supset M^n$ chain of subobjects s.t. $\left(M^i/M^{i+1}\right) I_{\chi_i} = 0$ for some $\chi_i \in \mathrm{Max}\, Z$.

i.e. the $(\mathcal{U}, \mathcal{U})$-action on M^i/M^{i+1} descends to $(\mathcal{U}, \mathcal{U}_\chi)$-action

Reduced to showing that M^i/M^{i+1} has finite length.

Want to show that any object of $\mathcal{HC}_\chi$ has finite length.

By the equivalence, have $\mathcal{HC}_\chi \simeq \mathcal{O}_{int} \leftarrow$ fin length.

Cor The iso classes of simple objects in $\mathcal{HC}$ are parametrized by

$$\left\{ (\mu, \lambda) \in \mathfrak{h}^* \times \mathfrak{h}^* \;\middle|\; \begin{array}{l} \lambda - \mu \text{ is integral} \\ \lambda \text{ is dominant} \end{array} \right\}$$

Pf Let M be a simple object of $\mathcal{HC}$.

$\Rightarrow$ the $(\mathcal{U}, \mathcal{U})$-action descends $(\mathcal{U}_\psi, \mathcal{U}_\chi)$-action

Assume χ integral (general case is just technically more complicated)

$$\text{Iso classes} = \coprod_{\chi \in \mathrm{Max}\, Z} \text{Simples in } \mathcal{HC}_\chi$$

$$= \coprod_{\lambda \text{ dominant}} \text{Simples in } \mathcal{HC}_{W \cdot \lambda}$$

$$\text{Simples in } \mathcal{HC}_{W\cdot\lambda} \overset{\Phi}{\simeq} \text{Simples in } \mathcal{O}_{int} = \{ L(\mu),\ \mu \text{ is integral} \}$$

<u>Construction of the inverse functor</u>

$\Phi : M \longmapsto M \otimes_{\mathcal{U}} \Delta(\lambda)$ is right exact, want to construct a right adjoint Φ^*

$$\mathcal{HC}_\chi \underset{\Phi^*}{\overset{\Phi}{\rightleftarrows}} \mathcal{O}_{int}$$

$$\mathrm{Hom}_{\mathcal{U}\text{-mod}}\left(M \otimes_{\mathcal{U}} \Delta(\lambda), N\right) = \mathrm{Hom}_{(\mathcal{U},\mathcal{U})\text{bimod}}\left(M, \mathrm{Hom}_{\mathbb{C}}(\Delta(\lambda), N)\right)$$

$$= \mathrm{Hom}_{\mathcal{U}\text{-bimod}}\left(M, \mathrm{Hom}^{ad\ l.f.}(\Delta(\lambda), N)\right)$$

(locally fin)

$$\mathrm{Hom}^{l.f.}_{\mathbb{C}}(N_1, N_2) = \{ f : N_1 \longrightarrow N_2 \}$$

s.t. $[x_1, [x_2, \ldots, [x_n, f]]$ span f.d. vector space

$$[x, f](n) = x f(n) - f(xn) \quad \forall x \in \mathfrak{g}$$

Let V be a finite dim rep of $\mathfrak{g}$ (G ss, simply connected)

V exponentiates to an alg G-rep

$$V \otimes \mathbb{C}[G] = \mathrm{Maps}^{alg}(G, V) \supset \mathrm{Maps}_{\mathbb{Z}}(G, V)$$

$$V \longrightarrow (f_v : g \mapsto gv) \qquad f_v(gh) = f_{h \cdot v}(g)$$

$$v \longmapsto$$

$$a \in \mathrm{Hom}^{l.f.}(\Delta(\lambda), N) \rightsquigarrow f_a : G \longrightarrow \mathrm{Hom}_{\mathbb{C}}(\Delta(\lambda), N)$$

inf dim'l, but $G \cdot f_a$ lines in f.d. subspace by l.f.

~~$f_a : G \times \Delta(\lambda) \to N$~~

$$\mathrm{Map}_r(G, \mathrm{Hom}(\Delta(\lambda), N))$$

$$\| $$

$$\mathrm{Hom}_{\mathfrak{g}}(\Delta(\lambda), \mathrm{Map}(G, N))$$

left action

right translation

$$\|$$

$$\mathrm{Hom}_{\mathfrak{b}}(\mathbb{C}_\lambda, \mathrm{Map}(G, N)) = \{ f : G \longrightarrow N, \ f(gb) = \lambda(b)\, b\, f(g) \}$$

$$N \in \mathcal{O} \rightsquigarrow \phi^*(N) = \mathrm{Coind}^G_{\mathcal{U}\mathfrak{b}}(\mathbb{C}_\lambda \otimes N) = \lim_{\substack{N' \subset N \\ \text{f.d.} \\ \mathfrak{b}\text{-stable}}} \mathrm{Coind}^G_{\mathcal{U}\mathfrak{b}}(\mathbb{C}_\lambda \otimes N')$$

5/5/14

$\mathfrak{g} \supset \mathfrak{b} = \mathfrak{h} + \mathfrak{n}$ Borel

$G \supset B$ Borel subgrp

Let N be a fin dim $\mathfrak{b}$-rep s.t. $\mathfrak{h}$-acts semisimply with integral weights

Then the action of $\mathfrak{b}$ can be exponentiated to a B-action

$$\mathrm{Coind}^G_B N = \left\{ \begin{array}{l} \text{alg maps } f : G \longrightarrow N \\ f(gb^{-1}) = b \cdot f(g) \quad \forall b \in B \end{array} \right.$$

G acts on Coind by left translations

If $N = \varinjlim N_\nu$ $\quad N_\nu$ are f.dim $\mathfrak{b}$-submodules as before

$$\mathrm{Coind}^G_B N := \varinjlim \mathrm{Coind}^G_B N_\nu$$

Let N', N are $\mathfrak{g}$-reps $\qquad \mathfrak{g} \overset{ad}{\curvearrowright} \mathrm{Hom}_{\mathbb{C}}(N', N) \ni F$

$$\mathrm{ad}\, x\, (F) = x \circ F - F \circ x$$

$\mathrm{Hom}^{l.f.}(N', N)$ = subspace $\subset \mathrm{Hom}_{\mathbb{C}}(N', N)$ of ad-locally finite elements

$\mathrm{Hom}^{l.f.}$ is a $(\mathcal{U}, \mathcal{U})$-stable subspace in $\mathrm{Hom}_{\mathbb{C}}(N', N)$.

Let λ be an integral weight, $N \in \mathcal{O}_{int}$

Define a linear map $\mathrm{Hom}^{l.f.}(\Delta(\lambda), N) \longrightarrow \mathrm{Coind}_B^G(\mathbb{C}_\lambda \otimes N)$

$$\underset{F}{\overset{\cup}{}} \longrightarrow [f_F : g \longmapsto \mathrm{Ad}\, g^{-1}(F)\, 1_\lambda]$$

<u>Claim</u> This map is a linear bijection (verified last time)

Want to define $(\mathcal{U}, \mathcal{U})$-action on Coind st. above map becomes bimod map.

i.e. $\forall x \in \mathfrak{g}$

$$x f_F = f_{x \circ F} \qquad f_F\, x = f_{F \circ x}$$

In particular $x f_F - f_F x = f_{[x,F]}$

Thus we must have $\boxed{x \cdot f_F - f_F \cdot x = -L_x(f_F)}$ $\Leftarrow$

$$f_{[x,F]}(g) = \frac{d}{d\varepsilon}\left(f_{\mathrm{Ad}(e^{\varepsilon x})(F)}(g)\right)\Big|_{\varepsilon=0}$$

$$f_{\mathrm{Ad}\, e^{\varepsilon x}(F)}(g) = \left(g^{-1} e^{\varepsilon x} F e^{-\varepsilon x} g\right) 1_\lambda$$
$$= (e^{-\varepsilon x} g)^{-1} F (e^{-\varepsilon x} g)\, 1_\lambda$$
$$= f_F(e^{-\varepsilon x} g)\, 1_\lambda$$

$$f_{xF}(g) = g^{-1} x F g\, 1_\lambda = g^{-1} x g g^{-1} F g\, 1_\lambda = \mathrm{Ad}\, g^{-1}(x)\, \mathrm{Ad}\, g^{-1}(F)\, 1_\lambda = \underset{\mathfrak{g}}{\underset{\cap}{\mathrm{Ad}\, g^{-1}(x)}} \cdot \underset{N}{\underset{\cap}{\underbrace{f_F(g)}}}$$

$$\boxed{(x f_F)(g) = \mathrm{Ad}\, g^{-1}(x)\, f_F(g)}$$

Thus we define a left, resp right $\mathfrak{g}$-action on Coind: $(xf)(g) := \mathrm{Ad}\, g^{-1}(x)\, f(g)$

$(fx)(g) := x \cdot f + L_x f$

Check that left and right actions commute.

<u>Cor</u> We have a $(\mathcal{U}, \mathcal{U})$-mod isom $\mathrm{Hom}^{l.f.}(\Delta(\lambda), N) \simeq \mathrm{Coind}_B^G(\mathbb{C}_\lambda \otimes N)$

Thus the functor $(\chi := W \cdot \lambda)$

$$\Psi : \mathcal{O}_{int} \longrightarrow \mathcal{HC}_\chi \;:\; N \longmapsto \mathrm{Hom}^{l.f.}(\Delta(\lambda), N) = \mathrm{Coind}$$

is a ~~left~~ right adjoint of the functor $\phi_\lambda : M \longrightarrow M \otimes_{\mathcal{U}_\chi} \Delta(\lambda)$

Cor (of theorem) For λ dominant, $\Psi = \phi_\lambda^{-1}$

Applications

For any $\mathfrak{g}$-rep N, $\mathcal{U} \xrightarrow[\text{on } N]{\varphi \text{ action}} \mathrm{End}^{l.f.}(N)$

In particular
$$\begin{array}{ccc} \mathcal{U}^{op} & \longrightarrow & \mathrm{End}^{l.f.}\Delta(\lambda) \\ \downarrow & \nearrow & \\ \mathcal{U}_\chi^{op} & & \end{array}$$

Dufflo thm $\Rightarrow$ $\mathcal{U}_\chi^{op} \hookrightarrow \mathrm{End}^{l.f.}\Delta(\lambda)$ injective

<u>Cor</u> For λ dominant $\mathcal{U}_\chi^{op} \xrightarrow{\sim} \mathrm{End}^{l.f.}\Delta(\lambda)$

<u>Pf</u> $\phi_\lambda(\mathcal{U}_\chi) = \mathcal{U}_\chi \otimes_{\mathcal{U}_\chi} \Delta(\lambda) = \Delta(\lambda)$

Hence $\mathcal{U}_\chi^{op} = \phi_\lambda^{-1}(\Delta(\lambda)) = \mathrm{End}^{l.f.}\Delta(\lambda)$

Let M be a left $\mathcal{U}$-module

$\mathrm{Ann}\, M = \mathrm{Ker}(\mathcal{U} \xrightarrow{\text{action}} \mathrm{End}_{\mathbb{C}} M) = \{u \in \mathcal{U} \mid um = 0 \quad \forall m \in M\}$

is a two-sided ideal in $\mathcal{U}$.

If M is a $(\mathcal{U},\mathcal{U})$-bimodule

$\mathrm{LAnn}\, M = \mathrm{Ann}_{\mathcal{U}\otimes 1} M.$

<u>Lemma</u> (λ dominant, $W\cdot\lambda =: \chi$ from now on) For any $M \in \mathcal{HC}_\chi$, $\mathrm{LAnn}\, M = \mathrm{Ann}\, \phi_\lambda(M)$

<u>Pf</u> $\mathrm{LAnn}\, M \subset \mathrm{Ann}\, \phi_\lambda(M)$ obvious: $\phi_\lambda(M) = M \otimes_{\mathcal{U}_\chi} \Delta(\lambda)$ ✓

For opposite inclusion it suffices to check $\mathrm{LAnn}\ \mathrm{Coind}_B^G(\mathbb{C}_\lambda \otimes N) \supseteq \mathrm{Ann}\, N$

$$(xf)(g) = \mathrm{Ad}\, g^{-1}(x) f(g) \quad \forall x \in \mathfrak{g} \qquad \left\{ \begin{array}{l} f: G \longrightarrow N \\ f(gb^{-1}) = \lambda(b)\, b\, f(g) \end{array} \right\}$$

$\Rightarrow$ for any $u \in \mathcal{U}$ $\quad (uf)(g) = \mathrm{Ad}\, g^{-1}(u)\, f(g)$

$\mathrm{Ann}\ N$ is a two-sided ideal in $\mathcal{U}$.

$u \in \mathrm{Ann}\, N \Rightarrow \mathrm{Ad}\, g^{-1}(u) \in \mathrm{Ann}\, N \Rightarrow \mathrm{Ad}\, g^{-1}(u)\, f(g) = 0 \Rightarrow uf = 0$ ✓ □

<u>Prop</u> The assignment $\mathcal{J} \longrightarrow \mathcal{J}\Delta(\lambda)$ is a bijection

$$\left\{ \begin{array}{l} \text{2-sided ideal} \\ \text{in } \mathcal{U}_\chi \end{array} \right\} \xrightarrow{\sim} \left\{ \begin{array}{l} \text{submodules} \\ \text{in } \Delta(\lambda) \end{array} \right\}$$

<u>Pf</u> $\mathcal{U}_\chi \in \mathcal{HC}_\chi$

a two-sided ideal in $\mathcal{U}_\chi$ $\equiv$ sub objects of $\mathcal{U}_\chi$

$j : \mathcal{J} \hookrightarrow \mathcal{U}_\chi$, Apply Φ_λ get $\mathcal{J} \otimes_{\mathcal{U}_\chi} \Delta(\lambda) \xrightarrow{\Phi_\lambda(j)} \Delta(\lambda)$, image is $\mathcal{J}\Delta(\lambda)$

Since Φ_λ is an equivalence, it is <u>exact</u> so $\Phi_\lambda(j)$ is injective

$\Phi_\lambda(\mathcal{U}_\chi/\mathcal{J}) = \Delta(\lambda)/\mathcal{J}\Delta(\lambda)$ also by exactness. $\Rightarrow \Phi_\lambda(\mathcal{J}) = \mathcal{J}\Delta(\lambda)$ $\square$

<u>Thm (Duflo)</u> If $\mathcal{J} = \text{Ann}\, M$ for a simple left $\mathcal{U}$-module M.
Then $\exists\, \lambda \in \mathfrak{h}^*$ s.t. $\mathcal{J} = \text{Ann}\, L(\lambda)$

<u>Defn</u>: Let $\mathcal{J} \subsetneq \mathcal{U}$ a proper two-sided ideal.

1) $\mathcal{J}$ is called primitive if $\exists\, M$ a simple $\mathcal{U}$-module st. $\mathcal{J} = \text{Ann}\, M$
2) $\mathcal{J}$ is <u>prime</u> if: given two-sided ideals $\mathcal{J}_1, \mathcal{J}_2$ st. $\mathcal{J}_1\mathcal{J}_2 \subset \mathcal{J} \Rightarrow$ either $\mathcal{J}_1 \subset \mathcal{J}$ or $\mathcal{J}_2 \subset \mathcal{J}$

<u>Lemma</u> If $\mathcal{J} \subset \mathcal{U}$ is primitive then $\exists\, \chi$ st. $I_\chi \subset \mathcal{J}$ and $\mathcal{J}/(I_\chi)$ is a primitive ideal in $\mathcal{U}_\chi$.

<u>Pf</u> If $\mathcal{J} = \text{Ann}\, M$, M simple. By Schur lemma $\exists\, \chi$
$I_\chi \cdot M = 0 \implies I_\chi \subset \mathcal{J}$ so $\mathcal{U}$-action on M descends to $\mathcal{U}_\chi$.

<u>Thm'</u> For a two-sided proper ideal $\mathcal{J} \subsetneq \mathcal{U}_\chi$ TFAE:
1) $\mathcal{J}$ is prime
2) $\mathcal{J}$ is primitive
3) $\mathcal{J}$ is $\text{Ann}\, L(\lambda)$ for some λ s.t. $\chi = W \cdot \lambda$

$$\begin{array}{ccc} \text{Primitive } \mathcal{U} & & \mathcal{J} \\ \downarrow & & \downarrow \\ \text{Max } Z(\mathfrak{g}) = \mathfrak{h}^*/W. & & \mathcal{J} \cap Z(\mathfrak{g}) = I_\chi \end{array}$$

Thm' says fibers are finite and bounded by $\# W$ (might be smaller)

<u>Pf of Thm'</u> 3) $\Rightarrow$ 2) is clear.
2) $\Rightarrow$ 1) Let $\mathcal{J} = \text{Ann}\, M$, M is simple $\mathcal{J}_1\mathcal{J}_2 \subset \mathcal{J}$. Then $\mathcal{J}_2 M$ submodule in M.
either $\mathcal{J}_2 M = 0 \Rightarrow \mathcal{J}_2 \subset \mathcal{J}$ or $\mathcal{J}_2 M = M \Rightarrow \mathcal{J}_1 M = 0$

1) $\Rightarrow$ 3) $\mathcal{J} = \text{LAnn}(\mathcal{U}_\chi/\mathcal{J}) = \text{Ann}\, \Phi_\lambda(\mathcal{U}_\chi/\mathcal{J})$

$$\Phi_\lambda(\mathcal{U}_\chi/\mathcal{J}) = N^0 \supset N^1 \supset \cdots \supset N^k \supset N^{k+1} = 0 \quad \text{composition series}$$

$\mathcal{J}_i := \text{Ann}\left(N^i/N^{i+1}\right)$ $\mathcal{J} \subset \mathcal{J}_i$ $\forall i$

Claim $\exists\, i$ st. $\mathcal{I} = \mathcal{I}_i$

$$\mathcal{I}_k \cdot \mathcal{I}_{k-1} \cdot \mathcal{I}_{k-2} \cdots \mathcal{I}_0 N^0 = 0 \Rightarrow \mathcal{I}_k \cdots \mathcal{I}_0 \subset \overset{\text{prime}}{\mathcal{I}}$$

$\square$

5/9/14

$\mathfrak{g}$ ss Lie alg$/\mathbb{C}$

G conn simply conn. grp with $\operatorname{Lie} G = \mathfrak{g}$

$\mathbb{C}[\mathfrak{g}] \supset \mathbb{C}[\mathfrak{g}]^G$

Thm 1 (Kostant) $\exists$ Ad G-stable homogeneous subspace $H \subset \mathbb{C}[\mathfrak{g}]$ s.t. the multiplication map $\mathbb{C}[\mathfrak{g}]^G \otimes H \longrightarrow \mathbb{C}[\mathfrak{g}]$ is a vector space isom. in particular, it is a G-equivariant isom of $\mathbb{C}[\mathfrak{g}]^G$-modules

Chevalley: Fix a Cartan

$$\begin{tikzcd} \mathfrak{h} \arrow[r, hook, "i"] \arrow[d, "W = \text{Weyl gp}"'] & \mathfrak{g} \arrow[dl, "\theta"'] \arrow[d] \\ \mathfrak{h}/W \arrow[r, phantom, "="] & \operatorname{Spec}(\mathbb{C}[\mathfrak{g}]^G) \end{tikzcd}$$

Thm (Chevalley) $\mathbb{C}[\mathfrak{h}]$ is a free $\mathbb{C}[\mathfrak{h}]^W$-module

Choose a homogeneous basis $\bar{\varphi}_1, \dots, \bar{\varphi}_r$ of $\mathbb{C}[\mathfrak{h}]$ as $\mathbb{C}[\mathfrak{h}]^W$

Step 2 Lemma $\mathbb{C}[\mathfrak{g}]$ is a free $\mathbb{C}[\mathfrak{g}]^G$-module

Choose homogeneous $\varphi_1, \dots, \varphi_r \in \mathbb{C}[\mathfrak{g}]$ s.t. $i^* \varphi_k = \bar{\varphi}_k \quad \forall k$

$\mathfrak{h}^\perp \subset \mathfrak{g}^*$. Choose a $\mathbb{C}$-basis $\{g_\ell\}$ of $\operatorname{Sym} \mathfrak{h}^\perp$

Claim $\{\varphi_k \cdot g_\ell\}$ is a basis of $\mathbb{C}[\mathfrak{g}]$ as a $\mathbb{C}[\mathfrak{g}]^G$-module.

Pf of Claim $F \in \mathbb{C}[\mathfrak{g}]$ a homog poly of deg d.

Want to find $F_{k,\ell} \in \mathbb{C}[\mathfrak{g}]^G$ s.t. $F = \sum_{k,\ell} F_{k,\ell}\, \varphi_k\, g_\ell$

$i^* F = \sum \bar{\varphi}_j \cdot f_j$ uniquely for some $f_j \in \mathbb{C}[\mathfrak{h}]^W$

Lift f_j to a G-invariant polynomial $\tilde{f}_j$ on $\mathfrak{g}$.

$$i^*\left(\sum \varphi_j \tilde{f}_j\right) = \sum \bar{\varphi}_j \cdot f_j = i^* F \Rightarrow i^*\left(F - \sum \varphi_j \tilde{f}_j\right) = 0$$

$\underbrace{F - \sum \varphi_j \tilde{f}_j}_{\deg d} \in$ Ideal generated by a linear basis $g_1, \dots, g_m$ of $\mathfrak{h}^\perp \subset \mathfrak{g}^*$

$= F_1 g_1 + \dots + F_m g_m \quad \deg F_j = d-1$

Therefore $\exists$ a homogeneous $E \subset \mathbb{C}[\mathfrak{g}]$ st. $\mathbb{C}[\mathfrak{g}]^G \otimes E \longrightarrow \mathbb{C}[\mathfrak{g}]$ is isom.

where $E = \mathbb{C}$-linear span of $\{\varphi_k g_\ell\}$.

$\mathbb{C}[\mathfrak{g}]^G \supset \mathbb{C}[\mathfrak{g}]^G_+$ = G-invariant poly without constant term.

$I_0 = \mathbb{C}[\mathfrak{g}]\cdot\mathbb{C}[\mathfrak{g}]^G_+$ an ideal in $\mathbb{C}[\mathfrak{g}]$

I_0 is a G-stable homogeneous ideal

For each $\mathbb{C}[\mathfrak{g}]^d = I_0^d \oplus H^d$ $\quad$ H^d is a G-stable complement

$d=0,1,\dots$

Put $H := \bigoplus_{d\geq 0} H^d \subset \mathbb{C}[\mathfrak{g}]$ a graded G-stable subsp

<u>Claim</u> $\mathbb{C}[\mathfrak{g}]^G \otimes H \xrightarrow{\text{mult}} \mathbb{C}[\mathfrak{g}] = \mathbb{C}[\mathfrak{g}]^G \otimes E$. is a $\mathbb{C}[\mathfrak{g}]^G$-module isom.

Apply $\mathbb{C}[\mathfrak{g}]^G/\mathbb{C}[\mathfrak{g}]^G_+ \underset{\mathbb{C}[\mathfrak{g}]^G}{\otimes} (-)$ get $\mathbb{C}$-lin $H \xrightarrow{\sim} E \Leftarrow$

$\mathbb{C}[\mathfrak{g}]^G = \mathbb{C} \oplus \mathbb{C}[\mathfrak{g}]^G_+$

$\mathbb{C}[\mathfrak{g}]^G \otimes E = E \oplus \underbrace{\mathbb{C}[\mathfrak{g}]^G_+ \cdot E}_{I_0}$

Have a $\mathbb{C}$alg $A = \mathbb{C}[\mathfrak{g}]^G \supset \mathfrak{a}$ morphism of A-modules $\underset{(\mathbb{C}[\mathfrak{g}]^G \otimes H)}{\overset{\shortparallel}{M}} \longrightarrow N = \mathbb{C}[\mathfrak{g}]$

Know $A/\mathfrak{a} \underset{A}{\otimes} M \longrightarrow A/\mathfrak{a} \underset{A}{\otimes} N$ a vector space isom.

Want to conclude "by Nakayama" that $M \longrightarrow N$ is an isom.

This statement holds whenever $A = \bigoplus_{i\geq 0} A_i$ graded alg $\quad \mathfrak{a} := \bigoplus_{i>0} A_i$

$M = \bigoplus_{i\geq 0} M_i$, $N = \bigoplus_{i\geq 0} N_i$ over A - graded mod

$\dim A_i$, $\dim M_i$, $\dim N_i < \infty$ $\forall i$ $\qquad$ □ Thm 1

<u>Remk</u> $A = \bigoplus_{i\geq 0} A_i \supset B = \bigoplus_{i\geq 0} B_i \qquad B_+ = \bigoplus_{i>0} B_i$

$A \cdot B_+ \subset A$ an ideal $\qquad A = AB_+ \oplus H \qquad B \otimes H \to A$ gives Tor-sequence.

$$\begin{tikzcd} \mathfrak{g} \arrow[dr, "\theta"'] & \mathfrak{h} \arrow[l, "i"'] \arrow[d] \\ & \mathfrak{h}/W \end{tikzcd}$$

Chevalley restr

<u>Lemma</u> If $s, s' \in \mathfrak{g}$ are <u>semisimple</u> elts st. $\theta(s) = \theta(s')$

$\Rightarrow s' \in G\cdot s$

Concretely $\theta(x) = \theta(x')$ means $f(x) = f(x')$ $\forall f \in \mathbb{C}[\mathfrak{g}]^G$

<u>Pf</u> $\exists\, h, h' \in \mathfrak{h}$ $\quad s \in G\cdot h$, $s' \in G\cdot h'$ $\Rightarrow f(h') = f(s') = f(s) = f(h)$

$\Rightarrow f(h') = f(h)$ $\forall f \in \mathbb{C}[\mathfrak{h}]^W$

$h' \in W\cdot h \Longrightarrow h' \in G\cdot h$ done. <u>QED</u>

$\mathcal{N}$:= set of nilpotent elements of $\mathfrak{g}$

Scheme structure on $\mathcal{N}$

Approach 1 $\qquad \mathbb{C}[\mathfrak{g}]^G_+$

$\mathcal{N} = \mathrm{Var}(\mathbb{C}[\mathfrak{g}] \cdot \mathbb{C}[\mathfrak{g}]^G_+) \overset{\text{closed}}{\subset} \mathfrak{g}$

Approach 2

Choose Borel $\mathfrak{g} \supset \mathfrak{b} \supset \mathfrak{n}$ $\leftarrow$ nilrad $(\mathfrak{b})$

$B \overset{ad}{\curvearrowright} \mathfrak{b} \qquad \mathfrak{n} = [\mathfrak{b}, \mathfrak{b}]$ is B-stable.

$$\begin{tikzcd} (g,n) \arrow[d, mapsto] & G \overset{B}{\times} \mathfrak{n} \arrow[r] \arrow[d, "\mu"] & G/B \\ \mathrm{Ad}\, g(n) & \mathfrak{g} & \end{tikzcd}$$

vector bundle with fiber $\mathfrak{n}$

Set theoretically $\mathcal{N} = \mathrm{Im}\, \mu$.

Give $\mathcal{N}$ the scheme-theoretic image structure. (μ proper)

Cor (of the second approach)

$\dim \mathcal{N} \le \dim (G \underset{B}{\times} \mathfrak{n}) = \dim G/B + \dim \mathfrak{n} = 2(\#\text{positive roots}) = \#\text{roots}$

Thm $\mathbb{C}[\mathfrak{h}]^W \cong \mathbb{C}[f_1, \dots, f_r]$ where $r = \mathrm{rank}\, \mathfrak{g} = \dim \mathfrak{h}$

If $\varphi_1, \dots, \varphi_r \in \mathbb{C}[\mathfrak{g}]^G$ are st. $i^* \varphi_j = f_j$ then Chevalley $\mathbb{C}[\mathfrak{g}]^G \cong \mathbb{C}[\varphi_1, \dots, \varphi_r]$

$\mathbb{C}[\mathfrak{g}] \cdot \mathbb{C}[\mathfrak{g}]^G_+ = \mathbb{C}[\mathfrak{g}](\varphi_1, \dots, \varphi_r)$

First approach $\mathcal{N}$ is cut out by r equations.

$\Rightarrow \dim \mathcal{N} \ge \dim \mathfrak{g} - r = 2(\#\text{pos roots})$

So we get $\dim \mathcal{N} = 2(\#\text{pos roots}) \Longrightarrow$

Cor $\mathcal{N}$ is a complete intersection in $\mathfrak{g}$

Fix s semisimple element in $\mathfrak{g}$

$\mathfrak{g}_s$ = centralizer of s in $\mathfrak{g}$ is a reductive subalg

$G_s = \{g \in G \mid \mathrm{Ad}\, g(s) = s\}$ is a connected reductive group.

$\mathcal{N}_s$ = nilpotent elements of $\mathfrak{g}_s$

$\theta : \mathfrak{g} \longrightarrow \mathfrak{h}/W$

Prop Let $h \in \mathfrak{h}$, $\bar{h} := W \cdot h \in \mathfrak{h}/W$. $\qquad \theta^{-1}(\bar{h}) = \mathrm{Ad}\, G(h + \mathcal{N}_h)$

Pf 1) $G(h+\mathcal{N}_h) \subset \theta^{-1}(\bar{h})$

Clearly $\theta^{-1}(\bar{h})$ is G-stable. Must check $h+\mathcal{N}_h \subset \theta^{-1}(\bar{h})$

i.e. $\forall f \in \mathbb{C}[\mathfrak{g}]^G$ $\quad f|_{h+\mathcal{N}_h}$ is constant $= f(h)$

Consider $\mathbb{C}\cdot h+\mathcal{N}_h \subset \mathfrak{g}$.

May assume that f is homogeneous of deg d.

$\cancel{f|_{\mathbb{C}h+\mathcal{N}_h} = \sum t}$ $\quad f(\underset{\mathbb{C}\cdot h+\mathcal{N}_h}{\underset{\cap}{t\cdot h+n}}) = \sum_{k=0}^{d} t^{d-k} f_k(n)$ where f_k is a homog poly of deg k on $\mathcal{N}_h$

Since f is G-invariant, $f|_{\mathfrak{g}_h}$ is G_h-invariant $\Rightarrow f_j$ is a G_h-invariant poly on $\mathcal{N}_h$

$\Rightarrow f_j|_{\mathcal{N}_h} = 0 \quad \forall j>0.$ □

5/12/14

$\mathbb{C}[\mathfrak{h}]^W = \mathbb{C}[\mathfrak{h}/W] = \mathbb{C}[\varphi_1, \dots, \varphi_r]$ free poly alg

$\varphi_1, \dots, \varphi_r$ coordinates on $\mathfrak{h}/W \simeq \mathbb{C}^r$

$\mathbb{C}[\mathfrak{g}]^G \xrightarrow{\sim} \mathbb{C}[\mathfrak{h}]^W$ $\qquad \mathbb{C}[\mathfrak{g}]^G \simeq \mathbb{C}[\theta_1, \dots, \theta_r]$

$\theta_1, \dots, \theta_r \in \mathbb{C}[\mathfrak{g}]^G$ $\qquad \theta_i|_{\mathfrak{h}} = \varphi_i$ $\qquad \theta : \mathfrak{g} \longrightarrow \mathfrak{h}/W, \quad x \mapsto (\theta_1(x), \dots, \theta_r(x))$

$h \in \mathfrak{h}$ $\quad \bar{h} \in \mathfrak{h}/W$

Prop $\theta^{-1}(\bar{h}) = G(h+\mathcal{N}_h)$ $\qquad \mathfrak{g}_h =$ centralizer of h in $\mathfrak{g}$

$\mathcal{N}_h =$ nil cone in $\mathfrak{g}_h$

Last time proved the inclusion $G(h+\mathcal{N}_h) \subset \theta^{-1}(\bar{h})$

amounts to $f|_{h+\mathcal{N}_h} = f(h) \quad \forall f \in \mathbb{C}[\mathfrak{g}]^G$

Opposite inclusion

If $x \in \theta^{-1}(\bar{h})$ then x is G-conjugate to a point in $h+\mathcal{N}_h$

The Jordan decomp $x = s+n$, $n \in \mathcal{N}_s$ $\quad$ ~~s can be conjugate to an element $h' \in \mathfrak{h}$~~

$\cancel{s = Adg(h')}$

$\cancel{\Rightarrow x = s+n = Adg(h'+n')}$ ~~where~~ $\cancel{n' \in \mathcal{N}_{h'}}$

$\cancel{f(s) = f(s+n) = f(x) = f(Adg(h'+n')) = f(h')}$ ~~using prev. direction~~

$\cancel{\Rightarrow s}$

$f(s) = f(s+n) = f(x) = f(h)$ $\quad$ since $x \in \theta^{-1}(\bar{h})$

$\Rightarrow \exists g \in G$ st. $h = Adg(s) \Rightarrow Adg(x) = Adg(s+n) = Adg(s) + Adg(n)$

$= h+n'$ where $n' \in \mathcal{N}_h$. □

Fact $\rho : \mathfrak{h} \longrightarrow \mathfrak{h}/W$, $\quad \text{Jacobian}(\rho) = \prod_{\alpha \text{ positive roots}} \alpha$

$\det \|\partial_i \varphi_j\| = \prod_{\alpha>0} \alpha$

If $t_1, \dots, t_r$ are coordinates on $\mathfrak{h}$ $\quad d\varphi_1 \wedge d\varphi_2 \wedge \dots \wedge d\varphi_r = \left(\prod_{\alpha>0} \alpha\right) dt_1 \wedge \dots \wedge dt_r$

$\Downarrow$

$\mathfrak{h}^r = \{h \in \mathfrak{h} \mid \alpha(h) \neq 0 \ \forall \text{ root } \alpha\}$ $\quad d\varphi_1, \dots, d\varphi_r$ are linearly indep at $h \in \mathfrak{h}^r$

regular elts

$\mathfrak{g}^{rs} = \{x \in \mathfrak{g} \mid \mathfrak{g}_x \text{ is a Cartan subalg}\} \cong G \cdot \mathfrak{h}^r$

Cor If $x \in \mathfrak{g}^{rs}$ then $\underbrace{d\theta_1, \dots, d\theta_r}_{\text{1-forms}}$ are linearly independent at x.

i.e. the differentials of $\theta : \mathfrak{g} \longrightarrow \mathfrak{h}/W$ is surjective at any $x \in \mathfrak{g}^{rs}$

Cor Let $h \in \mathfrak{h}^r$, $\bar{h} \in \mathfrak{h}/W$

1) $\theta^{-1}(\bar{h}) = G \cdot h$

2) $\theta^{-1}(\bar{h})$ is smooth, reduced

3) Action map of G yields $G/G_h \xrightarrow{\sim} G \cdot h \simeq \theta^{-1}(\bar{h})$ of alg varieties

Pf 1) h regular $\Rightarrow \mathfrak{g}_h = \mathfrak{h} \Rightarrow \mathcal{N}_h = 0$

$\Rightarrow \theta^{-1}(\bar{h}) = G(h + \mathcal{N}_h) = G \cdot h$

2) $d\theta$ is surjective at any pt of $G \cdot h$

$\Rightarrow$ the fiber of θ over $\bar{h}$ is smooth and reduced.

3) $g \longmapsto \mathrm{Ad}\, g(h)$, $G/G_h \xrightarrow{\text{bijection}} \theta^{-1}(\bar{h})$ of smooth alg var $\Rightarrow$ isom of scheme (ZMT)

$\mathcal{O}_x \xleftarrow{f^*} \mathcal{O}_{f(x)}$ local ring

Enough check df isom $T_e(G/G_h) = \mathfrak{g}/\mathfrak{h} \xrightarrow{\sim} \mathrm{Ker}(d\theta) = \mathfrak{h}^\perp$

Let J a max ideal in $\mathbb{C}[\mathfrak{g}]^G$

$J = (\theta_1 - z_1, \dots, \theta_r - z_r)$ for some $z_1, \dots, z_r \in \mathbb{C}$

$h \longmapsto \bar{h} \longmapsto (z_1 = \varphi_1(h), \dots, z_r = \varphi_r(h)) \in \mathbb{C}^r$

$J_{\bar{h}} = J$

By defn $\mathbb{C}[\theta^{-1}(\bar{h})] := \mathbb{C}[\mathfrak{g}]/(J_h)$

Cor For any $h \in \mathfrak{h}$, V fin dim irrep of G

$[\mathbb{C}[\mathfrak{g}]/(J_h) : V] = \dim V_0$

Pf Step 1 Assume $h \in \mathfrak{h}^r$

$$\mathbb{C}[\theta^{-1}(h)] \simeq \mathbb{C}[G/G_h] := \mathbb{C}[G]^{G_h} = \Big(\bigoplus_{V \in \mathrm{Irr}\, G} V \otimes V^*\Big)^{G_h} = \bigoplus_V V \otimes (V^*)^{G_h = \mathfrak{h}}$$

$$= \bigoplus_V V \otimes (V^*)_0$$

$$[\mathbb{C}[\theta^{-1}(h)] : V] = \dim (V^*)_0 = \dim V_0$$

Step 2 Show that $[\mathbb{C}[\mathfrak{g}]/(\mathcal{J}_h) : V]$ is independent of h. $= [H : V]$

$\exists$ G stable subspace $H \subset \mathbb{C}[\mathfrak{g}]$

st. $\mathbb{C}[\mathfrak{g}] \simeq \mathbb{C}[\mathfrak{g}]^G \otimes_{\mathbb{C}} H \Rightarrow \mathbb{C}[\mathfrak{g}]/(\mathcal{J}_h) \simeq \left(\frac{\mathbb{C}[\mathfrak{g}]^G}{\mathcal{J}_h}\right) \otimes H \simeq \mathbb{C} \otimes H = H$ ☐

Special case: $h = 0$ $\theta^{-1}(0) = \{\theta_1 = 0, \dots, \theta_r = 0\}$
$= \mathcal{N}$

$\Rightarrow \mathbb{C}[\mathcal{N}] := \mathbb{C}[\mathfrak{g}]/(\mathcal{J}_0)$ Thus $[\mathbb{C}[\mathcal{N}] : V] = \dim V_0$

not single orbit

Rem $\mathbb{C}[\mathcal{N}] = \bigoplus_i \mathbb{C}^i[\mathcal{N}]$ $[\mathbb{C}^i[\mathcal{N}] : V]$ is complicated (IC)

$\mathcal{U} = \mathcal{U}\mathfrak{g} \supset Z\mathfrak{g}$ center of $\mathcal{U}\mathfrak{g}$

Prop $\forall V \in \mathrm{Irr}\, G$, $\mathrm{Hom}_G(V, \mathcal{U}^{ad})$ is a free $Z\mathfrak{g}$-module of rank $\dim V_0$

Pf We'll prove the following: $\exists$ Ad G-stable subspace $\tilde{H} \subset \mathcal{U}\mathfrak{g}$ st.
the multiplication map yields an isom $Z\mathfrak{g} \otimes \tilde{H} \xrightarrow{m} \mathcal{U}$
G-equiv isom $Z\mathfrak{g}$-modules, moreover $[\tilde{H}, V] = \dim V_0$

Pf $\mathbb{C}[\mathfrak{g}] = \mathrm{Sym}\,\mathfrak{g}$ Thm from last time: $\mathrm{Sym}\,\mathfrak{g} \simeq (\mathrm{Sym}\,\mathfrak{g})^G \otimes H$

H homogeneous, G-stable

$H = \bigoplus_{i \geq 0} H^i$ $H^i \subset \mathrm{Sym}^i \mathfrak{g}$

PBW filtration $\mathcal{U}_0 \subset \mathcal{U}_1 \subset \cdots$ on $\mathcal{U}$

$$\begin{tikzcd} \mathcal{U}_i \arrow[r, two heads] & \mathcal{U}_i/\mathcal{U}_{i-1} \arrow[r, phantom, "\simeq"] & \mathrm{Sym}^i \mathfrak{g} \\ H^i \arrow[u, dashed] \arrow[urr, hook] & & \end{tikzcd}$$

Choose a G-equiv linear map that makes diagram commute (G semisimple)

$\tilde{H}^i := \mathrm{image}(H^i \longrightarrow \mathcal{U}_i)$ Put $\tilde{H} := \bigoplus_{i \geq 0} \tilde{H}^i$

To complete the pf of Prop, $\mathrm{gr}(Z\mathfrak{g} \otimes \tilde{H}) \xrightarrow{\mathrm{gr}(m)} \mathrm{gr}\,\mathcal{U}$

$$(\mathrm{Sym}\,\mathfrak{g})^G \otimes H = \mathrm{gr}(Z\mathfrak{g}) \otimes \mathrm{gr}\,\tilde{H} \xrightarrow[\sim]{\text{mult}} \mathrm{Sym}\,\mathfrak{g}$$

By last time ☐

<u>Rem</u> $U\mathfrak{g}$ is free $Z\mathfrak{g}$-module.

max ideal $I_\chi \subset Z\mathfrak{g}$ $\qquad$ $\mathcal{U}\cdot I_\chi = I_\chi \cdot \mathcal{U}$ two-sided ideal

$$\mathcal{U}_\chi = \mathcal{U}/\mathcal{U}\cdot I_\chi$$

PBW filtration on $\mathcal{U}$ induces $(\mathcal{U}\cdot I_\chi)_i := (\mathcal{U}\cdot I_\chi) \cap \mathcal{U}_i$

$$\underbrace{\mathrm{gr}(\mathcal{U}\cdot I_\chi)}_{\subset} \supsetneq \underbrace{(\mathrm{gr}\,\mathcal{U})(\mathrm{gr}\, I_\chi)}_{\subset}$$

Sym $\mathfrak{g}$

<u>Basic warning:</u> $A = \mathbb{C}\langle x,y\rangle / \langle xy - yx = 1\rangle$ $\qquad$ $\deg x = \deg y = 1$

Take $J := (x,y)$ two-sided ideal $= A$.

$\mathrm{gr}\, J = \mathrm{gr}\, A = \mathbb{C}[x,y]$ $\qquad$ $(\mathrm{gr}(x), \mathrm{gr}(y)) = (x,y)$

<u>Cor</u> $\mathrm{gr}(\mathcal{U}\cdot I_\chi) = (J_0)$

$$\mathrm{gr}\,\mathcal{U}_\chi = \mathrm{gr}\,\mathcal{U}/\mathrm{gr}(\mathcal{U}\cdot I_\chi) = \mathbb{C}[\mathfrak{g}]/(J_0) = \mathbb{C}[\mathcal{N}]$$

<u>Pf of Cor</u> Let $I \subset Z\mathfrak{g}$ an ideal.

PBW filtration $I_i = I \cap \mathcal{U}_i$

$\mathrm{gr}\, I$ is an ideal in $\mathbb{C}[\mathfrak{g}]^G$

$\mathcal{U}\cdot I = I\cdot\mathcal{U} = I(Z\mathfrak{g} \otimes \tilde{H}) = I \otimes \tilde{H}$

$(\mathcal{U}I)_i = \sum_{j\le i} I_j \cdot \tilde{H}_{\le i-j}$ $\qquad$ $\mathrm{gr}(\mathcal{U}\cdot I) \simeq \mathrm{gr}\, I \otimes \mathrm{gr}\,\tilde{H} = \mathrm{gr}\, I \otimes H$

$I = I_\chi$ $\qquad$ $\mathrm{gr}\, I_\chi = J_0$ $\qquad$ $\mathrm{gr}(\mathcal{U}\cdot I_\chi) = J_0 \otimes H = J_0 \cdot H$

5/16/14

$\mathcal{N} \subset \mathfrak{g}$, set of nilpotent elts

$J_0 := \mathbb{C}[\mathfrak{g}]^G_+$

$J_{\mathcal{N}} := \mathbb{C}[\mathfrak{g}]\cdot J_0$ $\quad$ so $\mathbb{C}[\mathcal{N}] := \mathbb{C}[\mathfrak{g}]/(J_{\mathcal{N}})$

Another approach: $\tilde{\mathcal{N}} = G \times_B \mathfrak{n}$

B is a Borel and $\mathfrak{b} = \mathrm{Lie}\, B$, $\mathfrak{n} = [\mathfrak{b},\mathfrak{b}]$ is nilradical

$$\begin{array}{c} \tilde{\mathcal{N}} \\ \downarrow \text{ vector bundle} \\ G/B \end{array} \quad \& \quad \begin{array}{l} \mu : \tilde{\mathcal{N}} \longrightarrow \mathfrak{g} \\ (g, n) \longmapsto \mathrm{Ad}\, g(n) \end{array}$$

$\mathcal{N} = \mu(\tilde{\mathcal{N}})$ as a set

$\mu^* : \mathbb{C}[\mathfrak{g}] \longrightarrow H^0(\tilde{\mathcal{N}}, \mathcal{O}_{\tilde{\mathcal{N}}})$ $(:= \mathbb{C}[\tilde{\mathcal{N}}])$ $\qquad$ G-equivariant

<u>Thm</u> (Kostant) μ^* is surjective and $(\ker \mu^*) = \mathcal{I}_{\mathcal{N}}$

<u>Equivalently</u>, μ^* induces an isom $\mathbb{C}[\mathcal{N}] := \mathbb{C}[\mathfrak{g}]/\mathcal{I}_{\mathcal{N}} \cong \mathbb{C}[\tilde{\mathcal{N}}]$

<u>Rmk</u> We have natural gradings $\mathbb{C}[\mathfrak{g}] = \bigoplus_{i \geq 0} \mathbb{C}^i[\mathfrak{g}]$.

There is a natural $\mathbb{C}^*$-action on $\tilde{\mathcal{N}}$ by rescaling fibers of $\tilde{\mathcal{N}} \to G/B$

This gives a grading on $\mathbb{C}[\tilde{\mathcal{N}}]$

$\mathbb{C}^i[\tilde{\mathcal{N}}] := \{ f \in \mathbb{C}[\tilde{\mathcal{N}}] \mid f(\lambda x) = \lambda^i f(x) \quad \forall \lambda \in \mathbb{C}^* \}$

μ^* respects grading

$\mathbb{C}[G \times_B \mathfrak{n}] =$ "$\mathrm{Coind}_B^G \mathbb{C}[\mathfrak{n}]$", i.e. $\forall i$, $\mathbb{C}^i[G \times_B \mathfrak{n}] = \mathrm{Coind}_B^G \mathbb{C}^i[\mathfrak{n}]$

$$= \{ F : G \to \mathbb{C}^i[\mathfrak{n}] \mid F(gb^{-1}) = bF(g) \ \forall b \in B \}$$

$$\begin{aligned} \mu^* : \mathbb{C}^i[\mathfrak{g}] &\longrightarrow \mathrm{Coind}_B^G \mathbb{C}^i[\mathfrak{n}] \\ f &\longmapsto [(\mu^* f)(g, n) = f(\mathrm{Ad}\, g(n))] \end{aligned}$$

Reformulate in terms of symmetric algebras:

$$\begin{aligned} \mathbb{C}[\mathfrak{n}] &= \mathrm{Sym}\, \mathfrak{n}^* \\ &= \mathrm{Sym}(\mathfrak{g}^*/\mathfrak{n}^\perp) \\ &\cong \mathrm{Sym}(\mathfrak{g}/\mathfrak{b}) \cong \mathrm{Sym}\,\mathfrak{g}/(\mathrm{Sym}\,\mathfrak{g})\mathfrak{b} \end{aligned}$$

Now use Killing form $\mathfrak{g}^* \simeq \mathfrak{g}$

So we get a map $\mathrm{Sym}^i \mathfrak{g} \longrightarrow \mathrm{Coind}_B^G \left(\mathrm{Sym}^i \mathfrak{g} / (\mathrm{Sym}^{i-1} \mathfrak{g})\mathfrak{b} \right)$

$f \longmapsto$ "project" i.e. $(g, n) \mapsto f(\mathrm{Ad}\, g\, \bar{n})$

<u>Duflo's thm:</u> $\lambda \in \mathfrak{h}^*$ & $\chi \in W \cdot \lambda \in \mathfrak{h}^*/W$ $\leftrightarrow I_\chi \in \mathrm{Max}(Z\mathfrak{g})$, $\Delta(\lambda) =$ Verma module

<u>Thm</u> $\mathrm{Ann}_{\mathcal{U}\mathfrak{g}} \Delta(\lambda) = \mathcal{U}\mathfrak{g} \cdot I_\chi$

* $\mathrm{Ann}_{\mathcal{U}(\mathfrak{g})} \Delta(\lambda)$ is a 2-sided ideal in $\mathcal{U}\mathfrak{g} := I$

$I_\chi := I \cap Z\mathfrak{g}$

Duflo thm $\quad \mathcal{U}\mathfrak{g}(I \cap Z\mathfrak{g}) = I$

(we know already that LHS $\overset{j}{\subset}$ RHS). We have PBW filtrations on $\mathcal{U}\mathfrak{g}$ & I, & take gr

$$\mathrm{gr}(\mathcal{U}\mathfrak{g} \cdot I_\chi) \xhookrightarrow{\mathrm{gr}(j)} \mathrm{gr}\, I \quad \leftarrow \text{enough to show this is surjective}$$

Last time: we computed LHS: we showed $\mathrm{gr}(\mathcal{U}\mathfrak{g}\cdot I_\chi) = (\mathrm{Sym}\,\mathfrak{g})\mathcal{J}_0$

Equivalently $\quad \mathrm{gr}(\mathcal{U}/\mathcal{U}\cdot I_\chi) \xrightarrow{\mu^*} \mathrm{gr}(\mathcal{U}/I) \overset{?}{\cong} \mathbb{C}[\tilde{\mathcal{N}}]$

$$\mathrm{gr}(\mathcal{U}/\mathcal{U}\cdot I_\chi) = \mathrm{Sym}\,\mathfrak{g}/\mathrm{Sym}\,\mathfrak{g}\cdot\mathcal{J}_0 = \mathbb{C}[\mathcal{N}] \quad \text{(Kostant)}$$

Proof (Kostant $\Rightarrow$ Duflo) Suppose I any ideal in $\mathcal{U}$ st, it is (1) a 2-sided ideal & (2) $I \cdot 1_\lambda = 0$ $\quad [1_\lambda = \text{generator of } \Delta(\lambda)]$

Claim $\quad I \subseteq \mathcal{U}\mathfrak{g}\cdot I_\chi$

$$\Delta(\lambda) = \mathcal{U}/(\mathcal{U}\cdot\mathfrak{n} + \mathcal{U}(h - \lambda(h)))$$

(1') I is a left ideal st. $\mathrm{Ad}\,g(I) \subset I$ $\forall g \in G$

(2') $I \subset \mathcal{U}\mathfrak{n} + \mathcal{U}(h-\lambda(h))$

$1'+2'$: $\bigcap \mathrm{Ad}\,g\,(\mathcal{U}\mathfrak{n} + \mathcal{U}(h-\lambda(h)))$

On I satisfying (1) + (2) we have a PBW filtration $I_i = \mathcal{U}_i \cap I$

We have a map $\quad I \xhookrightarrow{\theta} \mathrm{Coind}_B^G(\mathcal{U}\mathfrak{n} + \mathcal{U}(h-\lambda(h)))$

$\quad u \longmapsto [g \mapsto \mathrm{Ad}g(u)] \leftarrow$ Graded map

$$\mathrm{gr}\,\theta : \mathrm{gr}\, I \longrightarrow \mathrm{Coind}_B^G\, \mathrm{gr}(\mathcal{U}\mathfrak{n} + \mathcal{U}(h-\lambda(h)))$$

$$= \mathrm{Coind}_B^G\,[(\mathrm{Sym}\,\mathfrak{g})\cdot\mathfrak{n} + (\mathrm{Sym}\,\mathfrak{g})\cdot\mathfrak{h}] = \mathrm{Coind}_B^G\,[(\mathrm{Sym}\,\mathfrak{g})\cdot\mathfrak{b}]$$

$\mathbb{C}[\tilde{N}] = \mathrm{Coind}_B^G\left(\mathrm{Sym}\,\mathfrak{g}/(\mathrm{Sym}\,\mathfrak{g})\mathfrak{b}\right)$ and $\mathrm{Sym}\,\mathfrak{g} \xrightarrow{\mu^*} \mathbb{C}[\tilde{N}]$

so, $\quad (\ker \mu^*) = \mathrm{Coind}_B^G\,[(\mathrm{Sym}\,\mathfrak{g})\cdot\mathfrak{b}]$. $\leftarrow$ kernel of Kostant map ∎

(Dixmier : Enveloping algebras)

<u>Pfs of Kostant's theorems</u>

$$\mathbb{C}[\mathfrak{g}]/(J_0) =: \mathbb{C}[\mathcal{N}] \xrightarrow[\overset{\cong}{?}]{\mu^*} \Gamma(\tilde{\mathcal{N}}, \mathcal{O}_{\tilde{\mathcal{N}}}) =: \mathbb{C}[\tilde{\mathcal{N}}]$$

If we assume the thm, then $\tilde{\mathcal{N}} = G \times_B \mathfrak{n}$ is smooth & connected

$\Rightarrow \mathbb{C}[\tilde{\mathcal{N}}]$ has no zerodivisors (& no nilpotents)

$\Rightarrow \mathcal{N}$ is reduced & irreducible

Smoothness $\Rightarrow$ $\mathbb{C}[\tilde{\mathcal{N}}]$ is integrally closed $\Rightarrow$ so is $\mathbb{C}[\mathcal{N}]$, so $\mathcal{N}$ is normal [since $\mathcal{N}$ is affine)

<u>Prop</u>

1) N is a finite union of G-orbits

Pf idea a) Use Jacobson–Morozov thm

b) (Richardson): Reduce to $\mathfrak{gl}_n$: $\mathfrak{g} \hookrightarrow \mathfrak{gl}(V)$
$\mathcal{N} \hookrightarrow \mathcal{N}(\mathfrak{gl}(V))$

If $\mathfrak{g} \hookrightarrow \mathfrak{g}'$ then any G'-orbit of $\mathfrak{g}'$ intersects $\mathrm{Im}(\mathfrak{g})$ in finitely many orbits.

2) the map $\tilde{\mathcal{N}} \to \mathcal{N}$ is surjective

$\tilde{\mathcal{N}}$ irred $\Rightarrow \mathcal{N}$ irred $\Rightarrow \exists!$ open dense orbit; $\mathcal{N}^{reg} \subset \mathcal{N}$

3) N is a complete intersection.

<u>Proof of Kostant</u>: 1^{st} approach

(1) Prove that $\mathcal{N}$ is reduced at any point of $\mathcal{N}^{reg}$ $\Rightarrow$ $\mathcal{N}$ is generically reduced.

(2) Any complete intersection is Cohen-Macaulay

(3) Generically reduced + Cohen-Macaulay $\Rightarrow$ reduced

(4) $\mathcal{N}^{reg}$ is codim 2

(5) If $\mathcal{N}$ Cohen Macaulay & smooth in codimension 2 $\Rightarrow$ normal

(6) $\mu: \mu^{-1}(\mathcal{N}^{reg}) \longrightarrow \mathcal{N}^{reg}$ is an isomorphism

$\mathbb{C}[\mathcal{N}] \xrightarrow{\mu^*} \mathbb{C}[\tilde{\mathcal{N}}] \ni f$ surjective?

Consider $f\big|_{\mu^{-1}(\mathcal{N}^{reg})} = \mu^* \varphi$ where $\varphi \in \Gamma(\mathcal{N}^{reg}, \mathcal{O}_{\mathcal{N}^{reg}})$

Codim 2 + normal $\Rightarrow$ φ extends to all of $\mathcal{N}$ as $\varphi' \in \mathbb{C}[\mathcal{N}]$

& $\mu^*\varphi' = \mu^*\varphi$ on the open dense set $\mathcal{N}^{reg}$.

<u>Proof</u> (Approach 2)

$$\begin{tikzcd} & \tilde{\mathfrak{g}} \arrow[dl, "\mu"'] \arrow[d, "\mu\times\nu"] \arrow[dr, "\nu"] & \\ \mathfrak{g} \arrow[dr, "\theta"'] & Z \arrow[l] \arrow[r] & \mathfrak{h} \arrow[dl] \\ & \mathfrak{h}/W & \end{tikzcd}$$

$$Z := \mathfrak{g} \times_{\mathfrak{h}/W} \mathfrak{h} \subset \mathfrak{g} \times \mathfrak{h}$$

$$I_Z \subset \mathbb{C}[\mathfrak{g}] \otimes \mathbb{C}[\mathfrak{h}]$$
$$= \left(f\otimes 1 - 1 \otimes f|_{\mathfrak{h}} \right) \quad \forall f \in \mathbb{C}[\mathfrak{g}]^G$$

$$\tilde{\mathfrak{g}} := G \times_B \mathfrak{b}$$

Show: $(\mu\times\nu)^*$ induces an isom of fns ; Z affine & $\tilde{\mathfrak{g}}$ non-affine.

[restricting to $\{0\} \in \mathfrak{h}/W$ gives Kostant's picture.]

5/30/14

<u>Kostant thm</u> $\mu^* : \mathbb{C}[\mathcal{N}] \longrightarrow \mathbb{C}[\tilde{\mathcal{N}}] = \Gamma(\tilde{\mathcal{N}}, \mathcal{O}_{\tilde{\mathcal{N}}})$ is an isom.

$$\tilde{\mathcal{N}} = G \times_B \mathfrak{n}$$
$$\mu \downarrow$$
$$\mathcal{N} \subset \mathfrak{g} \quad \text{nilp variety}$$

$\mathfrak{n}$ = nilradical of $\mathfrak{b} \subset \mathfrak{g}$

① μ is surjective, $\tilde{\mathcal{N}}$ smooth connected $\Big\} \Rightarrow \mathcal{N}$ irred.

$\Rightarrow \exists$ unique open dense G-orbit $\mathcal{N}^{reg} \subset \mathcal{N}$

② $\mu : \mu^{-1}(\mathcal{N}^{reg}) \longrightarrow \mathcal{N}^{reg}$ is a bijection

Let e_i be the simple root vectors $\subset \mathfrak{n} = \mathbb{C}$-span of $\{e_\alpha, \alpha \in R_+\}$

Put $e = \sum e_i$ $\qquad e \in \mathcal{N}^{reg}$

$\exists!\ \mathfrak{b}'$ is the unique Borel st. $\mathfrak{b}' \ni e$. $\quad \mathfrak{b}' = \mathfrak{b}$

$$\mu^{-1}(e) = \left\{ (gB, x) \,\middle|\, \begin{array}{l} x \in \mathfrak{b} \\ \mathrm{Ad}\, g(x) = e \end{array} \right\} \Leftrightarrow \mathrm{Ad}\, g(\mathfrak{b}) \ni e$$
$$\Leftrightarrow g \in B.$$

③ Any G-conj class in $\mathfrak{g}$ has even dimension

For any $a \in \mathfrak{g}$ $\qquad \Omega_a : \mathfrak{g}\times\mathfrak{g} \longrightarrow \mathbb{C}$ $\qquad \Omega_a$ is skew-symmetric
$$x, y \longmapsto \langle a, [x,y] \rangle$$

<u>Claim</u> $\mathrm{Rad}\,\Omega_a := \{ x \in \mathfrak{g} \mid \Omega_a(x,y) = 0 \ \forall y \} = \mathfrak{g}_a$ $\leftarrow$ centralizer

$$\forall y \quad 0 = \langle a, [x,y] \rangle = \pm \langle [a,x], y \rangle \Leftrightarrow [a,x] = 0 \Leftrightarrow x \in \mathfrak{g}_a$$

Claim $\Rightarrow$ Ω_a descends to a non-degenerate 2-form on $\mathfrak{g}/\mathfrak{g}_a$

Ad G-action on $\mathfrak{g}$

$$\mathrm{Tangent}_a(\mathrm{Ad}\,G(a)) = \frac{\mathrm{Lie}\,G}{\mathrm{Lie}\,G_a} = \mathfrak{g}/\mathfrak{g}_a$$

$\Rightarrow$ $\dim \mathrm{Ad}\,G(a) = $ even

④ $\mathrm{Codim}_{\mathcal{N}}(\mathcal{N} \setminus \mathcal{N}^{reg}) \geq 2$ since $\mathcal{N} \setminus \mathcal{N}^{reg}$ = finite union of even dim G-orbits

⑤ $\mathcal{N}$ is complete intersection $\Rightarrow$ Cohen-Macaulay.

~~Recall~~: Let X be an irred alg variety.

<u>Defn</u> X is said to be CM if $\exists$ smooth Y and $\pi: X \to Y$ a finite morphism s.t. $\pi_* \mathcal{O}_X$ is a <u>locally-free</u> $\mathcal{O}_Y$-module.

<u>Prop</u>: Let X be Cohen-Macaulay and let $X^\circ \subset X$ be a smooth Zariski open subset (X is irreducible). Then i) X is reduced

ii) If $\dim(X \setminus X^\circ) \leq \dim X - 2$ then X is normal

In particular, if X affine, then $\mathbb{C}[X] \to \mathbb{C}[X^\circ]$ is an isomorphism.

<u>Key fact</u>: $\mathcal{N}^{reg}$ is smooth.

<u>Key fact $\Rightarrow$ Thm</u>:

$$\left.\begin{array}{l} \mathcal{N}^{reg} \text{ smooth} \\ \mathcal{N} \ \ \mathrm{CM} \end{array}\right\} \overset{\text{Prop}}{\Longrightarrow} \mathcal{N} \text{ is reduced}$$

$\dim(\mathcal{N} \setminus \mathcal{N}^{reg}) \leq \dim \mathcal{N} - 2 \Rightarrow \mathcal{N}$ is normal

and $\mathbb{C}[\mathcal{N}] \xrightarrow{\sim} \mathbb{C}[\mathcal{N}^{reg}]$.

$$\left.\begin{array}{l} \mathcal{N} \text{ reduced and irreducible} \\ \mu^{-1}(\mathcal{N}^{reg}) \longrightarrow \mathcal{N}^{reg} \end{array}\right\} \Rightarrow \mu \text{ is birational, i.e. } \mu^*: \mathbb{C}(\mathcal{N}) \xrightarrow[\text{isom.}]{\sim} \mathbb{C}(\mu^{-1}(\mathcal{N})) \quad \mathbb{C}(\mathcal{N}) \supset \mathbb{C}[\mathcal{N}]$$

$\Rightarrow \mu^*$ is injective

$\tilde{f} \in \mathbb{C}[\tilde{\mathcal{N}}] \Rightarrow \exists\, g \in \mathbb{C}[\mathcal{N}^{reg}]$ s.t. $\tilde{f}|_{\mu^{-1}(\mathcal{N}^{reg})} = \mu^* g$

$\Rightarrow \exists f \in \mathbb{C}[\mathcal{N}]$ st. $f|_{\mathcal{N}^{reg}} = g \Rightarrow \mu^* f = \tilde{f} \Rightarrow \mu^*$ surjective. ∎ Thm

<u>Pf of Prop</u> (i) X° smooth in X.

$X^{Bad} = X \setminus X^\circ$ $\quad \pi: X \to Y$ $\quad Y^{Bad} = \pi(X^{Bad})$

$Y^\circ = Y \setminus Y^{Bad}$ Zariski open in Y

$\pi^{-1}(Y^\circ) \subset X^\circ$ $\quad 0 \to \mathcal{J} \to \mathcal{O}_X \twoheadrightarrow \mathcal{O}_{X^{red}}$

$$0 \to \pi_* \mathcal{J} \to \underset{\substack{\| \\ \text{locally free}}}{\pi_* \mathcal{O}_X} \to \pi_* \mathcal{O}_{X^{red}}$$

isom over $Y^\circ \subset Y$

$\Rightarrow \quad \pi_*(\mathcal{T})\big|_{Y^o} = 0 \quad \Rightarrow \quad \pi_*(\mathcal{T}) = 0$ since no subset of $\mathcal{O}_Y$ supported on proper closed subset

locally free $\mathcal{O}_Y$-mod

(ii) Assume X affine. Have $g \in \mathbb{C}[X^o] \Rightarrow$ can view g as a section of $\pi_*\mathcal{O}_X\big|_{Y^o}$

Since $\pi_*\mathcal{O}_X \overset{\text{locally}}{\cong} \mathcal{O}_Y^n \Rightarrow$ any section of $\pi_*\mathcal{O}_X|_{Y^o}$ extends to Y.

(alg. version of Hartog's thm)

Y smooth, locally $\mathbb{C}^n$, polynomial sections extend outside codim 2.

Direct pf that $\mathcal{N}$ is CM.

$\mathfrak{g} \longrightarrow \mathfrak{g}/\mathfrak{h}$ Let $\varphi_1, \dots, \varphi_s$ be a basis of $(\mathfrak{g}/\mathfrak{h})^*$

$\mathbb{C}[\mathfrak{g}]^G = \mathbb{C}[p_1, \dots, p_r]$

<u>Claim</u> $\mathbb{C}[\mathfrak{g}]$ is a free module of finite rk over $\mathbb{C}[\varphi_1, \dots, \varphi_s, p_1, \dots, p_r]$, i.e.

$\mathbb{C}[\varphi_i, p_j] \hookrightarrow \mathbb{C}[\mathfrak{g}]$ gives a finite map $\pi : \mathfrak{g} \longrightarrow \mathbb{C}^{s+r}$

st. $\pi_*\mathcal{O}_{\mathfrak{g}}$ is a free $\mathcal{O}_{\mathbb{C}^{s+r}}$-module

$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{g}/\mathfrak{h}$

$\mathbb{C}[\mathfrak{g}] = \mathbb{C}[\mathfrak{h}] \otimes \mathbb{C}[\varphi_1, \dots, \varphi_s]$

<u>Chevalley</u> $\Rightarrow$ $\mathbb{C}[\mathfrak{h}]$ is a free module over $\mathbb{C}[\mathfrak{h}]^W$ with basis $\overline{\Psi}_1, \dots, \overline{\Psi}_m$

Let Ψ_i be a lift of $\overline{\Psi}_i$ so $\Psi_i \in \mathbb{C}[\mathfrak{g}]$ $\Psi_i|_{\mathfrak{h}} = \overline{\Psi}_i$

Then one shows that $\Psi_1, \dots, \Psi_m$ is a free basis of $\mathbb{C}[\mathfrak{g}]$ as a $\mathbb{C}[\varphi_1, \dots, \varphi_s, p_1, \dots, p_r]$-mod

$\mathbb{C}[\mathcal{N}] = \mathbb{C}[\mathfrak{g}]/(\mathcal{J}_o)$ $\quad \mathcal{J}_o = (p_1, \dots, p_r)$

$\Rightarrow$ $\mathbb{C}[\mathcal{N}]$ is a free-module of finite rank over $\mathbb{C}[\varphi_1, \dots, \varphi_s]$

$\mathcal{N} \xrightarrow{\pi} \mathbb{C}^s$ has been found.

Roughly speaking $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{g}/\mathfrak{h}$

$\mathcal{N}$ same dim, sufficiently "good position"

<u>Pf ~~that~~ of Key Fact</u> $\mathcal{N}^{reg}$ is smooth.

$\mathcal{N} = \{ p_1 = 0, \dots, p_r = 0 \}$

$\mathcal{N}^{reg}$. We must show that $dp_1, \dots, dp_r$ are linearly independent at any point $a \in \mathcal{N}^{reg}$

$(dp_1 \wedge \dots \wedge dp_r) \neq 0 \quad \forall a \in \mathcal{N}^{reg}$

<u>Define</u> $* : \Omega^i(\mathfrak{g}) \longrightarrow \Omega^{\dim \mathfrak{g} - i}(\mathfrak{g})$

$$(*\alpha^i) \wedge \beta^i = (\alpha^i, \beta^i) \cdot \mathrm{vol}_{\mathfrak{g}}$$ $(-,-)$ induced by Killing form

For each $a \in \mathfrak{g}$ $\quad \Omega_a : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathbb{C}$

$$x, y \longmapsto \langle a, [x,y] \rangle$$

Have a well-defined 2-form Ω on $\mathfrak{g}$

$\mathrm{rk}\, \Omega_a = \dim \left(\mathfrak{g} / \mathfrak{g}_a \right)$ $\quad$ | $\dim \mathfrak{g}_a \geqslant r := \dim \mathfrak{h}$

$\leq \underset{= 2q}{\dim \mathfrak{g}} - r$ $\quad$ + equality $\Longleftrightarrow$ a is regular

$\dim \mathfrak{g} = r + 2q$ $\qquad q :=$ # positive roots

$\mathrm{rk}\, \Omega_a = 2q \Longleftrightarrow a$ is regular, i.e. $\dim \mathfrak{g}_a = r$

$(\Lambda^q \Omega)_a \neq 0$ iff a is regular

<u>Prop</u> $\quad *(dp_1 \wedge \dots \wedge dp_r) = \Lambda^q \Omega$ $\qquad$ (Prop $\Rightarrow$ Key point)

<u>Pf of Prop</u> $\quad$ Suffices to check equality on $\mathfrak{g}^{rs}$.

Ad G - invariance, so also suffices to check at $\mathfrak{h}^{reg}$

$h \in \mathfrak{h}^{reg} \hookrightarrow \mathfrak{g}$ $\qquad *(dp_1 \wedge \dots \wedge dp_r)_h \overset{?}{=} (\Lambda^q \Omega)_h$

$\qquad (dp_1 \wedge \dots \wedge dp_r)_h \overset{?}{=} (* \Lambda^q \Omega)_h$

$\mathfrak{g} = T_h \mathfrak{g} \supset \mathfrak{h}$

$\quad \| $

$\mathfrak{h} + \underbrace{T\, \mathrm{Ad}\, G(h)}_{\mathfrak{g}/\mathfrak{h} \text{ via Killing form}}$ $\qquad dp_i \big|_{T_h \mathrm{Ad}\, G(h)} = 0$

$x \in (\mathrm{Rad}\, \Omega)^{\perp} \Rightarrow$ contraction with x annihilates $*(\Lambda^q \Omega)$

$\mathrm{Rad}(\Omega_h) = \mathfrak{g}_h = \mathfrak{h}$ $\qquad \mathrm{Rad}(\Omega_h)^{\perp}_{\text{Killing}} = \mathfrak{h}^{\perp} \underset{\text{check}}{=} T_h(\mathrm{Ad}\, G(h))$

So we compare two sides as diff forms on $\mathfrak{h}$.

So $\quad dp_1 \wedge \dots \wedge dp_r \big|_{\mathfrak{h}} = f_1(h) \cdot dh_1 \wedge \dots \wedge dh_r$ $\qquad$ where $h_1, \dots, h_r$ are coordinates on $\mathfrak{h}$

$\qquad f_1 \in \mathbb{C}[\mathfrak{h}]$

$(* \Lambda^q \Omega)\big|_{\mathfrak{h}} = f_2(h)\, dh_1 \wedge \dots \wedge dh_r$

Want to show that $f_2 = \underset{\neq 0}{\mathrm{const}} \cdot f_1$.

$$f_1 = \prod_{\alpha \in R_+} \alpha \qquad (\text{Jacobian of } \mathfrak{h} \longrightarrow \mathfrak{h}/W)$$

Ω is a G-equivariant 2-form on $\mathfrak{g}$

$*(\Lambda^q \Omega)$ ——— " ———

$*(\Lambda^q \Omega)|_{\mathfrak{h}}$ is W-equiv $\Rightarrow$ $w(f_2) = \operatorname{sign}(w) f_2$ sign $\Rightarrow$ (reflection across root hyperplanes)

$\Rightarrow$ f_2 is divisible by $\prod_{\alpha \in R_+} \alpha$.

Ω degree 1 (linear), then $\Lambda^q \Omega$ degree q $\Rightarrow$ f_2 homog poly degree $q = \# R_+$

Hence $f_2 = \text{const} \cdot \prod_{\alpha \in R_+} \alpha$ $\square$

6/2/14

$\tilde{\mathcal{N}} = G \times_B \mathfrak{n} \hookrightarrow \tilde{\mathfrak{g}} = G \times_B \mathfrak{b} \xrightarrow{p} G/B$ $\Rightarrow$ $\tilde{\mathfrak{g}}$ is smooth connected.

$\mu : \tilde{\mathfrak{g}} \longrightarrow \mathfrak{g}$

$(gB, x) \longmapsto \operatorname{Ad} g(x)$

$$\begin{tikzcd} \tilde{\mathfrak{g}} \arrow[r, hook, "p\times\mu", "\text{closed imbed}"'] \arrow[rr, bend left, "\mu"] & G/B \times \mathfrak{g} \arrow[r, "\text{proj}"] & \mathfrak{g} \end{tikzcd} \Rightarrow \boxed{\mu \text{ is proper}}$$

$\nu : \tilde{\mathfrak{g}} \longrightarrow \mathfrak{h}$ $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$

$(gB, x) \longmapsto x \bmod \mathfrak{n}$

For $h \in \mathfrak{h}$ $\quad \underbrace{\nu^{-1}(h)}_{\text{smooth}} = G \times_B (h + \mathfrak{n})$

ν is a smooth morphism (flat with smooth fibers)

$$\begin{tikzcd} & \tilde{\mathfrak{g}} \arrow[dl, "\mu"'] \arrow[dr, "\nu"] & \\ \mathfrak{g} \arrow[dr, "\theta"'] & & \mathfrak{h} \arrow[dl, "\rho"] \\ & \mathfrak{h}/W & \end{tikzcd}$$

This diagram commutes

$Z := \mathfrak{g} \times_{\mathfrak{h}/W} \mathfrak{h} = \{ (x, h) \in \mathfrak{g} \times \mathfrak{h} \mid P(x) = P|_{\mathfrak{h}}(h),\ \forall P \in \mathbb{C}[\mathfrak{g}]^G \}$

$\mu \times \nu$ factors through a map $\eta : \tilde{\mathfrak{g}} \longrightarrow Z$

Version of Kostant Thm $\quad \eta^* : \mathbb{C}[Z] \xrightarrow{\sim} \Gamma(\tilde{\mathfrak{g}}, \mathcal{O}_{\tilde{\mathfrak{g}}})$, $\quad \mathbb{C}[Z] = \mathbb{C}[\mathfrak{g}] \otimes_{\mathbb{C}[\mathfrak{h}]^W} \mathbb{C}[\mathfrak{h}]$

$\mathfrak{g}^{rs}$ = reg semisimple elements

$\mathfrak{h}^r = \mathfrak{h} \setminus$ root hyperplanes

$\mathfrak{g}^{rs} = \operatorname{Ad} G(\mathfrak{h}^r)$

$x \in \mathfrak{g}^{rs} \Longleftrightarrow \theta(x) \in \mathfrak{h}^r/W$

$Z^{rs} = \mathfrak{g}^{rs} \times_{\mathfrak{h}/W} \mathfrak{h} = \mathfrak{g}^{rs} \times_{\mathfrak{h}^r/W} \mathfrak{h}^r$

$\tilde{\mathfrak{g}}^{rs} = \mu^{-1}(\mathfrak{g}^{rs}) \subset \tilde{\mathfrak{g}}$

Lemma i) $\mu : \tilde{\mathfrak{g}}^{rs} \longrightarrow \mathfrak{g}^{rs}$ #W-sheeted covering

ii) $\eta : \tilde{\mathfrak{g}}^{rs} \longrightarrow Z^{rs}$ ~~#W-sheeted~~ is an isom.

Pf

$$\begin{array}{ccc} \tilde{\mathfrak{g}}^{rs} & & \mathfrak{h}^{r} \\ W \downarrow & & \downarrow W \\ \mathfrak{g}^{rs} & & \mathfrak{h}^{r}/W \end{array}$$

$$\tilde{\mathfrak{g}}^{rs} = (G \times_B \mathfrak{b})^{rs} = G \times_B (\mathfrak{b} \cap \mathfrak{g}^{rs})$$

$$\mathfrak{b} \cap \mathfrak{g}^{rs} = \mathrm{Ad}\,B(\mathfrak{h}^r) = B/H \times \mathfrak{h}^r$$

Cartan torus

$$\tilde{\mathfrak{g}}^{rs} = G \times_B (B/H \times \mathfrak{h}^r) = G/H \times \mathfrak{h}^r \begin{array}{l} \xrightarrow{\mu : (gH, h) \mapsto \mathrm{Ad}g(h)} \mathfrak{g} \\ \xrightarrow[\nu : (gH,h) \mapsto h]{} \mathfrak{h} \end{array}$$

Cor 1) Z^{rs} is smooth

2) Z^{rs} is dense in Z.

Lemma 2 The map $\eta : \tilde{\mathfrak{g}} \longrightarrow Z$ is proper & surjective

Pf $\mu : \tilde{\mathfrak{g}} \longrightarrow \mathfrak{g}$ is proper $\Rightarrow$ $\eta = \mu \times \nu$ is proper

$\eta(\tilde{\mathfrak{g}}^{rs}) \twoheadrightarrow Z^{rs}$ dense in Z, so $\eta(\tilde{\mathfrak{g}}) = Z$.

Lemma 3 Z is an irreducible complete intersection in $\mathfrak{g} \times \mathfrak{h}$ in particular, Z is CM.

Pf $Z = \mathrm{Im}(\tilde{\mathfrak{g}}) \Rightarrow Z$ is irred

Let $\mathbb{C}[\mathfrak{g}]^G = \mathbb{C}[P_1, \dots, P_r]$

$Z = \{ (x,h) \mid P_i(x) = P_i|_{\mathfrak{h}}(h) \}$

Z given by r-eqns in $\mathfrak{g} \times \mathfrak{h}$, i.e. in a $(\dim \mathfrak{g} + r)$-dim vector space

$\Rightarrow \dim Z \geq (\dim \mathfrak{g} + r) - r = \dim \mathfrak{g}$

But $Z = \mathrm{Im}(\tilde{\mathfrak{g}}) \Rightarrow \dim Z \leq \dim \tilde{\mathfrak{g}} = \dim G/B + \dim \mathfrak{b} = \dim G = \dim \mathfrak{g}$. □

Direct proof that Z is CM:

$Z \xrightarrow{pr_1} \mathfrak{g}$

$\mathbb{C}[Z]$ is a free $\mathbb{C}[\mathfrak{g}]$-module of finite rk.

$\| \qquad\qquad \Uparrow$

$\mathbb{C}[\mathfrak{g}] \otimes_{\mathbb{C}[\mathfrak{h}]^W} \mathbb{C}[\mathfrak{h}]$ $\quad$ $\mathbb{C}[\mathfrak{h}]$ free over $\mathbb{C}[\mathfrak{h}]^W$

$$\begin{array}{ccc} Z & \longrightarrow & \mathfrak{h} \\ \downarrow & \square & \downarrow \\ \mathfrak{g} & \longrightarrow & \mathfrak{h}/W \end{array}$$

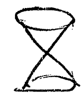
Ex $\mathfrak{g} = \mathfrak{sl}_2$ $\quad \mathfrak{h} = \mathbb{C}$ with coord u

$\mathbb{C}[\mathfrak{g}]^G = \mathbb{C}[\det]$

$\mathbb{C}[\mathfrak{g}] = \mathbb{C}[x,y,z] \qquad \det = x^2+y^2+z^2$

quadratic cone in $\mathbb{C}^4$

$$Z = \{ (x,y,z) \in \mathbb{C}^3,\ u \in \mathbb{C} \mid x^2+y^2+z^2 = u^2 \}$$



$\mathfrak{sl}_2^r = \mathfrak{sl}_2 \setminus \{0\}$

$Z^r = Z \setminus \{0\}$ is smooth!

$\operatorname{Codim}(\mathfrak{g} \setminus \mathfrak{g}^r) = 3 \qquad \operatorname{codim}(Z \setminus Z^r) = 3$

$$Z^{rs} \subset Z^{good} \subset Z^r \subset Z$$

$$\mathfrak{h}^{good} := \mathfrak{h}^r \cup \bigcup_{\alpha \in R} \Big(\mathfrak{h}_\alpha \setminus \bigcup_{\beta \neq \alpha} \mathfrak{h}_\beta \Big) = \mathfrak{h} \setminus \Big(\bigcup_{\alpha \neq \beta} \mathfrak{h}_\alpha \cap \mathfrak{h}_\beta \Big)$$

$\operatorname{codim}(\mathfrak{h} \setminus \mathfrak{h}^{good}) = 2.$

$$Z^{good} := pr_1^{-1}(\mathfrak{g}^r) \cap pr_2^{-1}(\mathfrak{h}^{good})$$

$$\begin{tikzcd} & Z = \mathfrak{g} \times_{\mathfrak{h}/W} \mathfrak{h} \arrow[dl, "pr_1"'] \arrow[dr, "pr_2"] & \\ \mathfrak{g} & & \mathfrak{h} \end{tikzcd}$$

Lemma 4 (i) $\operatorname{Codim}(Z \setminus Z^{good}) \geq 2$

(ii) Z^{good} is smooth.

Pf (i) follows from ~~$\operatorname{Codim}(\mathfrak{g} \setminus \mathfrak{g}^r) \geq 2$~~, $\operatorname{codim}(\mathfrak{h} \setminus \mathfrak{h}^{good}) \geq 2$.

$$Z \longrightarrow \mathfrak{h} \supset \mathfrak{h}^{good} = \mathfrak{h}^{rs} \cup \bigcup_\alpha \mathfrak{h}_\alpha$$

$$\mathfrak{h}_\alpha^\circ = \mathfrak{h}_\alpha \setminus \bigcup_{\beta \neq \alpha} \mathfrak{h}_\beta$$

$Z^{good} \subset pr_2^{-1}(\mathfrak{h}^{good}), \qquad Z^{good} \subset \underset{\substack{\uparrow \\ \text{smooth}}}{Z^{rs}} \cup \bigcup_{\alpha \in R} pr_2^{-1}(\mathfrak{h}_\alpha^\circ)$

Let $z \in Z$ st. $pr_2(z) = h \in \mathfrak{h}_\alpha^\circ$

Smoothness at z reduces to $\mathfrak{sl}_2$. $\square$

Thus Z is an irred CM variety, Z^{good} is a smooth Zariski open subset s.t.

$\operatorname{codim}(Z \setminus Z^{good}) \geq 2$

Cor Z is reduced and normal

Pf of Thm

$\eta : \tilde{\mathfrak{g}} \longrightarrow Z$

Claim $\eta_* \mathcal{O}_{\tilde{\mathfrak{g}}} \simeq \mathcal{O}_Z$

(Claim $\Rightarrow$ Thm $\Gamma(Z, \eta_* \mathcal{O}_{\tilde{\mathfrak{g}}}) = \Gamma(\tilde{\mathfrak{g}}, \mathcal{O}_{\tilde{\mathfrak{g}}}) = \Gamma(Z, \mathcal{O}_Z) = \mathbb{C}[Z]$.)

- $\tilde{\mathfrak{g}} \xrightarrow{\eta} Z$ is birational
- η is proper $\Rightarrow$ $\eta_* \mathcal{O}_{\tilde{\mathfrak{g}}}$ is coherent $\mathcal{O}_Z$-module.
- Z is normal. $\eta_* \mathcal{O}_{\tilde{\mathfrak{g}}}$ f.d. $\mathcal{O}_Z$-subalg of Fraction Field (Z), i.e. integral over $\mathcal{O}_Z$ $\Rightarrow$ equality.

$\square$ Claim

Thm The map η restricts to an isomorphism $\tilde{\mathfrak{g}}^r \xrightarrow{\sim} Z^r$, where $\tilde{\mathfrak{g}}^r = \mu^{-1}(\mathfrak{g}^r)$

Pf i) $\mu : \tilde{\mathfrak{g}}^r \longrightarrow \mathfrak{g}^r$, $\tilde{\mathfrak{g}}^r = \mu^{-1}(\mathfrak{g}^r)$, has finite fibers.

μ is proper $\Rightarrow$ μ is a finite morphism.

$\Rightarrow$ $\eta : \tilde{\mathfrak{g}}^r \longrightarrow Z^r$ is a finite morphism.

ii) η is birational

iii) Z^r is normal.

$\Rightarrow$ $\eta|_{\tilde{\mathfrak{g}}^r}$ is an isom.

Cor Z^r is smooth.

2 thms from today $\Rightarrow$ Kostant's theorem.

Prop $\mathcal{N}^{reg}$ is contained in the smooth locus of $\mathcal{N}$

Pf

$$\begin{tikzcd} \tilde{\mathfrak{g}}^r \arrow[r, "\sim"] \arrow[r, swap, "\eta"] \arrow[rr, bend left, "\nu"] & Z^r \arrow[r, "pr_2"] \arrow[d] \arrow[dr, phantom, "\square"] & \mathfrak{h} \arrow[d] \\ & \mathfrak{g}^r \arrow[r, "\theta"] & \mathfrak{h}/W \end{tikzcd}$$

- ν is a smooth morphism

$\Rightarrow pr_2 : Z^r \longrightarrow \mathfrak{h}$ is a smooth morphism

- $Z \longrightarrow \mathfrak{h}$ obtained from $\theta : \mathfrak{g} \to \mathfrak{h}/W$ by flat finite base change.

$\Rightarrow$ $\theta : \mathfrak{g}^r \longrightarrow \mathfrak{h}/W$ is smooth

$\Rightarrow$ 0-fiber is smooth, 0-fiber $= \mathcal{N}^{reg}$

Last class next time:

another pf of Kostant (topological, HT)