(Munkres §61) separation theorems:
easy to state, hard to prove

Df a simple closed curve in R^2 (or S^2) is a subspace homeomorphic to S^1

Jordan Separation Thm

if C sub R^2 is a simple closed curve,
then R^2 – C is disconnected

[stronger:] Jordan Curve Thm

if C sub R^2 is a simple closed curve
then R^2 – C has exactly two connected comp's

today we prove:

Thm if C sub S^2 is a simple closed curve
then S^2 – C is not path connected

Rem this implies Jordan Separation:

- S^2 – C is open in S^2, so it remains locally path connected (§25);
thus its path comp's are its conn comp's (Theorem 25.5)

- pick p notin C, homeo f : S^2 – {p} to R^2;
then bijection between comp's of S^2 – C and comp's of R^2 – f(C) (Lemma 61.1)

Pf assume that $S^2 - C$ is path connected

fix a homeo $C \simeq S^1$

let a, b, A_1, A_2 sub C correspond to

$(1, 0)$, $(-1, 0)$, $\{y \geq 0\}$, $\{y \leq 0\}$ sub S^1

let $U_1 = S^2 - A_1$ and $U_2 = S^2 - A_2$ [draw]

then $U_1 \cap U_2 = S^2 - A_1 - A_2 = S^2 - C$

so it is path connected

but $U_1 = (U_1 \cap U_2) \cup A_2$, etc.

so U_1, U_2 are also path connected

finally, $U_1 \cup U_2 = S^2 - \{a, b\}$ [draw]

fix x in $U_1 \cap U_2 = S^2 - C$

Seifert–van Kampen gives a surjective hom

$\pi_1(U_1, x) * \pi_1(U_2, x)$ to $\pi_1(S^2 - \{a, b\}, x)$

induced by $(i_1)_*$, $(i_2)_*$ for the inclusion maps

$i_j : U_j$ to S^2

will show that $(i_1)_*$, $(i_2)_*$ are trivial hom's

but $\pi_1(S^2 - \{a, b\}) \simeq \pi_1(R^2 - \{(0, 0)\}) \simeq Z$

so contradiction

enough to show that $(i_1)_*$ is trivial

pick a loop $\gamma : [0, 1]$ to U_1 based at x
lift to cts β : S^1 to U_1 s.t. $\beta((1, 0)) = x$
get cts $i_1 \circ \beta$: S^1 to S^2 – {a, b} [draw]

note that a, b both lie in A_1

<u>Lem 1</u> suppose K is compact,
f : K to S^2 – {a, b} is cts

if a, b lie in the same path comp of S^2 – f(K),
then f is nulhomotopic

<u>Lem 2</u> for any X and cts f : S^1 to X, TFAE:
1) f is nulhomotopic
2) f extends to cts map on the closed disk
3) f_* is trivial

<u>Pf of Lem 1</u> fix a homeo S^2 – {b} ≃ R^2
s.t. a corresponds to **0** = (0, 0)

then f : K to S^2 – {a, b} corresponds to
some g : L to R^2 – {**0**}
since L compact, have r > 0 s.t. L sub B(**0**, r)

fix a path from a to b in S^2 – K,
it corresponds to a path from **0** “to infty” in R^2 – L
fix p in R^2 – B(**0**, r)
fix a path α from **0** to p in R^2 – L [draw]

let h : L × [0, 1] to R^2 – {**0**} be def by

h(x, t) = g(x) – α(t) [why well-defined?]

let $H : L \times [0, 1]$ to $R^2 - \{\mathbf{0}\}$ be def by

$$H(x, t) = tg(x) - p$$ [why well-defined?]

h is a homotopy from $g(x)$ to $g(x) - p$

H is a homotopy from a const map to $g(x) - p$