(Munkres §26)

<u>Df</u> an <u>open cover</u> of X is a collection of open sets of X whose union is X

a <u>subcover</u> is a subcollection that remains a cover

<u>Df</u> X is <u>compact</u> iff every open cover of X contains a finite subcover

<u>Facts</u>

1) (Heine–Borel) [0, 1] is compact
2) images of a compact space under cts maps are compact
3) if X, Y are compact, then X × Y is compact

[compare to corresponding statements about connectedness]

<u>Cor</u>

1) S^1 is compact as a quotient space of [0, 1]
2) [0, 1]^n, (S^1)^n, etc. are compact for all n

[compare to statements about connectedness]

<u>Q</u> when is a subspace A sub X compact?

observe:

- subspace top is {A cap V | V open in X}
- {A cap V_i}_i covers A iff A sub bigcup_i V_i

Df — an open cover in X of A is a collection of open sets of X whose union contains A
[subcovers defined as before]

so A is a compact subspace iff every open cover in X of A contains a finite subcover

[now, a result familiar from analysis:]

Thm — if X is compact and A sub X is closed, then A is compact

Pf — suppose {V_i}_i is an open cover of A
then {X – A} cup {V_i}_i covers X
the cover of X has a finite subcover
so the cover of A has a finite subcover

[recall Hausdorff condition]

Thm — if X is Hausdorff, A sub X is compact, and x in X – A,
then there exist disjoint open U, V sub X s.t. A sub U and x in V

Pf — for all a in A, pick disjoint open U_a, V_a s.t. a in U_a and x in V_a

{U_a}_{a in A} is a cover in X of A
pick a finite subcover {U_a}_{a in I}
U = bigcup_{a in I} U_a
V = bigcap_{a in I} V_a

<u>Cor</u> if X is Hausdorff and A sub X compact, then A is closed

<u>Pf</u> for all x in $X - A$, pick disj open U_x, V_x s.t. A sub U_x and x in V_x

then $X - A = \bigcup_{x \in X - A} V_x$
so $X - A$ is open

<u>Cor</u> if X is compact, Y is Hausdorff, and $f : X \to Y$ is a cts bijection, then f is a homeomorphism

<u>Ex</u> the cts bijection $f : [0, 1) \to S^1$ def by
$f(t) = (\cos(2\pi t), \sin(2\pi t))$
is not a homeo! [[0, 1) is not compact!]

<u>Pf</u> want to show f^{-1} cts
know $(f^{-1})^{-1}(U) = f(U)$

so enough to show:
if U sub X is open, then $f(U)$ is open
[that is, f is an <u>open map</u>]

since f is bijective, $f(X - A) = Y - f(A)$ for any A
so enough to show:
if A sub X is closed, then $f(A)$ is closed

indeed:
- X compact + A closed implies A compact
- A compact implies $f(A)$ compact
- Y Hausdorff + $f(A)$ compact implies A closed