<u>Last Time</u> fix top spaces X, Y and x in X

<u>Thm</u> if $f : X$ to Y is a homotopy equiv., then
$f_* : \pi_1(X, x)$ to $\pi_1(Y, f(x))$ is an iso

<u>Lem 1</u> if $\varphi : G$ to G' and $\psi : G'$ to G'' are maps
s.t. $\psi \circ \varphi$ is bijective
then φ is injective and ψ is surjective

<u>Lem 2</u> let $h : X \times [0, 1]$ to Y be a homotopy
set $f_0(s) = h(s, 0)$,
$f_1(s) = h(s, 1)$,
$\alpha(t) = h(0, t)$ [starting pt at time t]

then we have
$f_{1, *} = \breve{\alpha} \circ f_{0, *} : \pi_1(X, x)$ to $\pi_1(Y, f_1(x))$

here, α is a path in Y and $\breve{\alpha}([\gamma]) = [\alpha'] * [\gamma] * [\alpha]$
thus $\breve{\alpha}$ is an <u>automorphism</u> of $\pi_1(Y, f_1(x))$

<u>Pf of Thm from Lem's</u>

have $g : Y$ to X s.t. $g \circ f$ is homotopic to id_X
$f \circ g$ is homotopic to id_Y

now look at $(f \circ g \circ f)_* = f_* \circ g_* \circ f_*$
$: \pi_1(X, x)$ to $\pi_1(Y, f_1(x))$

by lem 2,
$$g_* \circ f_* = (g \circ f)_* = \breve{\alpha}_X \circ id_{X, *} = \breve{\alpha}_X$$
$$f_* \circ g_* = (f \circ g)_* = \breve{\alpha}_Y \circ id_{Y, *} = \breve{\alpha}_Y$$
for some paths α_X, α_Y in X, Y, respectively

so, by lem 1, f_* is both injective and surjective
i.e., f_* is bijective □

to finish off, some more discussion of retracts:
recall that

- a function-theoretic <u>retract</u> is
  a map r with a right inverse
- a <u>retract</u> of a space X onto a subspace A is
  a map $r : X$ to A s.t. $r(a) = a$ for all a in A

  here, $r \circ i = id_A$, where $i : A$ to X is inclusion

<u>Df</u> a deformation retract of X onto A is
a homotopy $h : X \times [0, 1]$ to X s.t.

$h(–, 0) = id_X$
$h(–, 1)$ has image A,
$h(–, t)|_A = id_A$ for all t

sometimes we also say that A is, itself,
the <u>deformation retract</u> of X
[how does this relate to homotopy equivalences?]

<u>Lem</u> if h is a deformation retract of X onto A,
$r : X$ to A is given by $r(x) = h(x, 1)$,
$i : A$ to X is the inclusion,
then r and i form a homotopy equiv.

<u>Pf</u> $r \circ i = id_A$
h is a homotopy from id_X to $i \circ r$

<u>Ex</u> let $X = R^2 – \{(0, 0)\}$ and $A = S^1$
here r is radial projection
the point: can choose h so that, at any t,
the map $h(–, t)$ restricts to id_{S^1}

so a deformation retract is just a special kind of homotopy equivalence

(Munkres §68–69) next: Seifert–van Kampen
first: more group theory

Df for any set X, let

$$X^{\pm} = X \cup \{x^{-1} \mid x \in X\}$$

where x^{-1} is just a formal symbol indexed by x

- a (signed) word in X is a finite sequence of elts of $X^{\pm}$
- a word is reduced iff no consecutive elts look like "x, x^{-1}" or "x^{-1}, x"

an elementary reduction in a word w is the operation of deleting such consec elts from w

a reduction of w is a reduced[!] word obtained from w by successive elementary reductions

Ex let $X = \{g, h, k\}$ and [from Terry Tao]

$$w = g^{-1}k^{-1}gk\,k^{-1}\,g^{-1}h^{-1}gh\,k$$

elementary reductions give $g^{-1}k^{-1}h^{-1}ghk$

Thm every word in X has a unique reduction

Pf existence: words have finite length

uniqueness: induct on the length $|w|$

if w is empty, then done

else let w to u_1 to u_2 to … to u_m

w to v_1 to v_2 to … to v_n

be two chains of elementary reductions

with u_m and v_n reduced

<u>Claim</u> either $u_1 = v_1$

or there is a word w' obtained from both

by a single elementary reduction

<u>Pf of Claim</u> either the elementary reductions

from w to u_1, v_1 overlap

or they do not:

check each case separately

<u>Claim Implies Thm</u>

if $u_1 = v_1$, then $u_m = v_n$ by the inductive hyp.

because $|u_1| = |v_1| < |w|$

otherwise, let w'' be a reduction of w'

then w'' is also a reduction of both u_1 and v_1

so $u_m = w'' = v_n$, again by the inductive hyp.

<u>Df</u> let $v|w$ denote concatenation of v and w

the free group generated by X is

$$F_X = \{\text{reduced words in } X\}$$

under the group law $v \bullet w = \text{reduction}(v|w)$

we call this <u>concatenation</u> as well, and drop the $\bullet$

associativity:

$\quad$ reduct(reduct(u|v)|w)

$\quad$ = reduct(u|v|w)

$\quad$ = reduct(u|reduct(v|w))

[what is the id elt?] id_{F_X} = empty word

[inverses should be clear]

Universal Property of Free Groups

for any group G, there is a bijection

{set-theoretic maps X to G} = {hom.'s F_X to G}

f : X to G goes to φ_f : F_X to G def by

$\quad$ φ_f(x1^e1, x2^e2, …) = f(x1)^e1*f(x2)^e2*…)

[F_X is the "freest", or "most universal", way to build a group from an arbitrary set X]

if X is finite, and we only care about n = |X|, then we write F_n in place of F_X

Ex $\quad$ F_1 is a copy of Z

Groups via Generators and Relations

Df $\quad$ for any S sub G, the subgroup of G generated by S is both:

1) the image of the hom. F_S to G corresponding to the inclusion of S

2) the unique minimal subgroup of G containing S

if it is G, then we say S is <u>generating set</u> for G
in this case, the map

F_S to G

is surjective [but usually F_S is much much larger]
[how to measure the shrinkage?]

recall: the <u>kernel</u> of a homomorphism φ: G to K is

ker(φ) = {g in G | φ(g) = e_K}

Fact a subgroup H sub G is a kernel iff
H is <u>normal</u>: i.e., gHg^{-1} = H for all g

[where gHg^{-1} = {ghg^{-1} | h in H}]

<u>Df</u> for any R sub G and generating set S:

R is a set of relations for G wrt S iff ker(F_S to G)
is the minimal normal subgroup of G containing R

in this case, we write G = <S | R>
and say G is <u>gen'd by S modulo the relations R</u>

<u>Ex</u> up to iso, a unique group of size 2:
G = {e, s} s.t. e*e = s*s = e
s*e = e*s = s
S = {s} generates G

F_S to G sends powers of s to powers of s
ker(F_S to G) = {even powers of s}
altogether, G = <s | s^2>