TOPICS IN REPRESENTATION THEORY: SYMPLECTIC REFLECTION ALGEBRAS

INSTRUCTOR: IVAN LOSEV

Class info:
MATH 7364-01, CRN: 14815.
Time: MR, 4:00-5.30pm; first meeting Sep. 6.
Location: Forsyth building 242.

Instructor info: Prof. Ivan Loseu
e-mail: i.loseu@neu.edu
office: 519LA, *office hours*: M 10.30-11.20, 2.30-3.50 and by appointment.
Prerequisites, grades, literature: see below.

Description.
Symplectic reflection algebras were introduced by Etingof and Ginzburg about 10 years ago based on the previous work of Cherednik, Dunkl, Crawley-Boevey and Holland, and others. They happen to be connected to various parts of Mathematics: Representation theory (representations of quivers, classical and quantum Nakajima quiver varieties, Hecke algebras and categorical Kac-Moody actions), Algebraic geometry (resolutions of quotient singularities, geometry of plane curves) Combinatorics (Macdonald polynomials), Deformation theory, Integrable systems (systems of Calogero-Moser type), Knot theory (invariants of toric knots). The main goal of this course is to demonstrate some of these connections.

(Approximate) program.
1) Kleinian singularities and their deformations (a warm-up).
1.1) Kleinian singularities and finite subgroups of $\mathrm{SL}_2(\mathbb{C})$.
1.2) Notion of a deformation. Algebras of Crawley-Boevey and Holland.
1.3) Representations of quivers and categorical quotients. McKay correspondence revisited.
1.4) CBH algebras and deformed preprojective algebras.

2) Construction of SRA (deformation theory).
2.1) Formal deformations and Hochschild cohomology.
2.2) Cohomology of smash-products. Symplectic reflection algebras.
2.3) Proof of flatness of SRA. Symplectic reflection groups.

3) Algebraic properties of SRA, incl. centers and spherical subalgebras (algebra).
3.1) Double centralizer property.
3.2) Commutativity of the spherical subalgebra.
3.3) Satake isomorphism.

4) Calogero-Moser systems and Cherednik algebras (integrable systems).
4.1) Classical Calogero-Moser system, Hamiltonian reduction and Calogero-Moser space.
4.2) Quantization as deformation. Quantum CM system.
4.3) Dunkl operators and first integrals for quantum CM system.

5) Quotient singularities and SRA for wreath-products (algebraic geometry).
5.1) Nakajima quiver varieties.
5.2) Quotient singularities vs quiver varieties.

5.3) Deformation theory of symplectic varieties.
5.4) Procesi bundles and SRA.

6) Categories $\mathcal{O}$ for Cherednik algebras (finally, representation theory).
6.1) Categories $\mathcal{O}$ and their properties.
6.2) KZ functor.
6.3) Hecke algebras.
6.4) Induction and restriction functors.
6.5) Categorical Kac-Moody actions.

Literature.

There is basically one textbook on the subject and several review texts, all available online. The textbook is: P. Etingof. *Lectures on Calogero-Moser systems.*
http://arxiv.org/abs/math/0606233

There are also notes from MIT class, 2009:
P. Etingof, X. Ma. *Lecture notes on Cherednik algebras.*
http://arxiv.org/abs/1001.0432

One of the review texts:
I. Gordon. *Symplectic reflection algebras.*
http://arxiv.org/abs/0712.1568

I plan to post lecture notes. In the notes some additional references (to original papers) will be provided.

Prerequisites.

Algebra: groups, fields and algebras, the structure theory of semisimple finite dimensional algebras, representations of finite groups, etc.

Algebraic geometry: algebraic varieties, correspondence btw. commutative algebras and varieties, algebra homomorphisms and morphisms of varieties. Dominant and finite morphisms. For parts 4 and 5 we will also need smoothness, tangent spaces, cotangent bundles. Finally, in part 5 we will need Cech and De Rham cohomology in the algebraic setting.

Symplectic geometry: for parts 4 and 5 we will need symplectic manifolds, incl. cotangent bundles, symplectic forms, Hamiltonian vector fields, etc. Also for part 4 we will need basics of classical Hamiltonian mechanics. Most of this was covered in the Spring by B. Webster and will be covered in the Fall by J. Weitsman.

Lie groups and Lie algebras: for parts 4 and 5 it will be useful to understand the correspondence between Lie groups and their Lie algebras, incl. the correspondence between their representations etc. Also throughout the course, a familiarity with Weyl groups, roots, etc. will be helpful.

Category theory: basic notions of the category theory (categories, objects, functors, morphisms of functors, abelian categories, exact functors, projective objects etc.) are required for part 6.

Grades are based on the homework. The homework is divided into two parts: exercises and problems. For each class, there will be a problem set to be handed in class and also posted online containing all problems and exercises for the class. You are responsible for all exercises that constitute 50% of the grade. You also will get an individual assignment consisting of 5 problems. Grade cut-offs and due dates are to be determined.

LECTURES ON SYMPLECTIC REFLECTION ALGEBRAS

IVAN LOSEV

1. Kleinian singularities

Kleinian singularities are remarkable singular affine surfaces (varieties of dimension 2). They arise as quotients of $\mathbb{C}^2$ by finite subgroups of $\mathrm{SL}_2(\mathbb{C})$. Our main interest is not in these singularities themselves but in their (not necessarily commutative) deformations.

This lecture is organized as follows. First, 1.1, we present Kleinian singularities as surfaces in $\mathbb{C}^3$. Next, we recall the classification of finite subgroups of $\mathrm{SL}_2(\mathbb{C})$ in 1.2. In 1.3 we present the simplest version of the so called *McKay correspondence* that relates finite subgroups of $\mathrm{SL}_2(\mathbb{C})$ to Dynkin diagrams. Then we relate the singularities and the subgroups as promised in the previous paragraph. One advantage of this realization, is that the singularities acquire a natural *grading*. We discuss graded algebras in 1.5.

After that we start to proceed to our next topic and discuss the general notion of a deformation. Finally, we sketch a purely algebro-geometric way to connect the Kleinian singularities to Dynkin diagrams, 1.7.

For more information on Kleinian singularities (and, in particular, their relation to simple Lie algebras) see [Sl], Section 6, in particular.

1.1. Singularities.

There are some remarkable singular affine algebraic varieties of dimension 2. They have many names (Kleinian singularities, rational double points, du Val singularities) and also many nice properties (e.g., these are only normal Gorenstein singularities in dimension 2). They can be described very explicitly, as surfaces in $\mathbb{C}^3$ given by a single equation on the variables x_1, x_2, x_3. They split into two families and three exceptional types. Here are the equations

(A_r) $x_1^{r+1} + x_2x_3 = 0$, $r \geqslant 1$.
(D_r) $x_1^{r-1} + x_1x_2^2 + x_3^2$, $r \geqslant 4$.
(E_6) $x_1^4 + x_2^3 + x_3^2 = 0$.
(E_7) $x_1^3x_2 + x_2^3 + x_3^2 = 0$.
(E_8) $x_1^5 + x_2^3 + x_3^2 = 0$.

Of course, A_r, D_r, E_6, E_7, E_8 are precisely the simply laced Dynkin diagrams. In a way, this and three subsequent lectures are to explain relationship between the singularities and the diagrams.

1.2. Finite subgroups of $\mathrm{SL}_2(\mathbb{C})$.

It turns out that the Kleinian singularities can be realized as quotients of $\mathbb{C}^2$ by finite subgroups of $\mathrm{SL}_2(\mathbb{C})$. Let us recall their classification.

Any finite subgroup in $\mathrm{SL}_2(\mathbb{C})$ admits an invariant hermitian product on $\mathbb{C}^2$ and so is conjugate to a subgroup in SU_2. Recall that there is a covering $\mathrm{SU}_2 \twoheadrightarrow \mathrm{SU}_2/\{\pm E\} \cong \mathrm{SO}_3(\mathbb{R})$ given by the adjoint representation of SU_2. So the first step in classifying finite subgroups of SU_2 is to classify those in $\mathrm{SO}_3(\mathbb{R})$.

Inside $\mathrm{SO}_3(\mathbb{R})$ we have the following finite subgroups:

(1) The cyclic group of order n, its generator is a rotation by the angle of $2\pi/n$.

(2) The dihedral group of order $2n$ with $n \geqslant 2$ realized as the group of rotation symmetries of a regular n-gon on the plane inside of the 3D space. Of course, a regular 2-gon is just a segment.
(3) The group of rotational symmetries of the regular tetrahedron isomorphic to the alternating group A_4.
(4) The group of rotational symmetries of the regular cube/octahedron isomorphic to the symmetric group S_4.
(5) The group of rotational symmetries of the regular dodecahedron/icosahedron isomorphic to A_5.

Problem 1.1. *Prove that these group form a complete list of finite subgroups of* $\mathrm{SO}_3(\mathbb{R})$. *You may use the following strategy. Let G be a finite subgroup of* $\mathrm{SO}_3(\mathbb{R})$. *Consider its action on the unit sphere. Show that any non-unit element of G fixes a unique pair of opposite points and that the stabilizer of each point P is cyclic of some order, say, n_P. Choose representatives $P_1, \ldots, P_k$ of orbits with non-trivial stabilizers, one in each orbit. Show that*

$$2\left(1-\frac{1}{n}\right)=\sum_{i=1}^{k}\left(1-\frac{1}{n_{P_i}}\right).$$

Analyze the possibilities for the numbers $n, n_{P_1}, \ldots, n_{P_k}$ and deduce the classification.

Now the classification of finite subgroups in $\mathrm{SL}_2(\mathbb{C})$ up to conjugacy is as follows.

(A_r) The cyclic group of order $r+1$, i.e., $\{\begin{pmatrix}\epsilon & 0\\ 0 & \epsilon^{-1}\end{pmatrix} | \epsilon^{r+1}=1\}$.
(D_r) The dihedral group of order $4(r-2)$ with $r \geqslant 4$, i.e., $\{\begin{pmatrix}\epsilon & 0\\ 0 & \epsilon^{-1}\end{pmatrix}, \begin{pmatrix}0 & \epsilon\\ -\epsilon^{-1} & 0\end{pmatrix} | \epsilon^{2(r-2)} = 1\}$.
(E_6) The double cover of $A_4 \subset \mathrm{SO}_3(\mathbb{R})$.
(E_7) The double cover of $S_4 \subset \mathrm{SO}_3(\mathbb{R})$.
(E_8) The double cover of $A_5 \subset \mathrm{SO}_3(\mathbb{R})$.

Problem 1.2. *Use the result of Problem 1.1 to deduce this classification.*

1.3. **McKay correspondence I.** Again, we have labeled the finite subgroups of $\mathrm{SL}_2(\mathbb{C})$ by simply laced Dynkin diagrams. To get a diagram from a subgroup one can use the recipe called the *McKay correspondence* to be described now.

Pick a finite subgroup $\Gamma \subset \mathrm{SL}_2(\mathbb{C})$. Let $N_0, \ldots, N_r$ denote the irreducible representations of Γ and suppose that N_0 is the trivial representation. Set $m_{ij} = \dim \mathrm{Hom}_\Gamma(N_i \otimes \mathbb{C}^2, N_j)$, where $\mathbb{C}^2$ denotes the representation of Γ coming from the inclusion $\Gamma \subset \mathrm{SL}_2(\mathbb{C})$. The representation $\mathbb{C}^2$ is self-dual because it admits an invariant non-degenerate skew-symmetric bilinear form. It follows that $m_{ij} = \dim \mathrm{Hom}_\Gamma(N_i \otimes \mathbb{C}^2, N_j) = \dim \mathrm{Hom}_\Gamma(N_i, N_j \otimes \mathbb{C}^2) = \dim \mathrm{Hom}_\Gamma(N_j \otimes \mathbb{C}^2, N_i) = m_{ji}$.

Consider the graph with vertices $0, \ldots, r$ and the number of edges between i and j equal to m_{ij}. It is called the *McKay graph* of Γ.

To state the result we need to recall various things regarding root systems.

Fix a simply laced Dynkin diagram with vertices labeled by $1, \ldots, r$. To the diagram one assigns a root system in the Euclidian space $\mathbb{R}^n$ together with its subset, a simple root system, say $\alpha_1, \ldots, \alpha_r$, that constitute a basis in $\mathbb{R}^n$. The vertices i, j are connected if and only if $(\alpha_i, \alpha_j) \neq 0$ (in which case one necessarily has $(\alpha_i, \alpha_j) = -1$ – we only consider simply laced diagrams). In the root system, there is unique maximal root, say δ, that can be, in the simply laced case, characterized by the property that $(\alpha_i, \delta) \geqslant 0$ for all i. Set $\alpha_0 := -\delta$. Then the *extended Dynkin diagram*, by definition, has vertices $0, \ldots, r$ and the vertices are connected according to the same rule as for the usual Dynkin diagram.

The following result is the most elementary form of the so called McKay correspondence.

Proposition 1.1. *The McKay graph Γ is the extended Dynkin diagram corresponding to the finite Dynkin diagram labeling Γ. Moreover, 0 is the extending vertex. Finally, one has $\sum_{i=0}^{r} \dim N_i \cdot \alpha_i = 0$.*

We are not going to prove Proposition 1.1 in all cases. We are going to consider the easiest case as an example and propose two more cases in the form of problems.

The example we are going to consider is, of course, the cyclic group, $\Gamma = \{\begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon^{-1} \end{pmatrix} | \epsilon^{r+1} = 1\}$. Since the group is abelian, all irreducible representations are 1-dimensional and are given by characters. Let N_i, denote the representation corresponding to the character given by $\begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon^{-1} \end{pmatrix} \mapsto \epsilon^i$ so that $N_i = N_j$ if and only if $i-j$ is divisible by $r+1$. Of course, $\mathbb{C}^2 = N_{-1} \oplus N_1$. So $\dim \mathrm{Hom}_\Gamma(N_i \otimes \mathbb{C}^2, N_j) = 1$ if $i - j = \pm 1 \mod r+1$ and $\dim \mathrm{Hom}_\Gamma(N_i \otimes \mathbb{C}^2, N_j) = 0$, else. Therefore the McKay graph is cyclic with $r+1$ vertices.

On the other hand, the corresponding root system can be realized in the space $\{(x_1, \ldots, x_{r+1}) \in \mathbb{R}^{r+1} | \sum_{i=1}^{r+1} x_i = 0\}$ with the scalar product restricted from the standard one on $\mathbb{R}^{r+1}$. The simple roots are $\alpha_i = e_i - e_{i+1}, i = 1, \ldots, r$, where $e_1, \ldots, e_{r+1}$ denote the tautological basis elements in $\mathbb{R}^{r+1}$. The whole root system consists of the elements of the form $\{e_i - e_j, i \neq j\}$. Further, $\alpha_0 = e_{r+1} - e_1$. From here we see that the extended Dynkin diagram is again the cyclic graph with $r+1$ vertices, and $\sum_{i=0}^{r} \alpha_i = 0$.

Problem 1.3. *Check Proposition 1.1 for the group of type D_r.*

Problem 1.4. *This problem discusses the group of type E_6.*

1) We start with a construction. Take the group Q_8 of unit quaternions. It has elements $\{\pm 1, \pm i, \pm j, \pm k\}$. Show that the cyclic group $\mathbb{Z}_3$ acts on Q_8 by automorphisms in such a way that the generator ω acts as follows: $\omega(-1) = -1, \omega(i) = j, \omega(j) = k, \omega(k) = i$. Embed the semi-direct product $\Gamma := \mathbb{Z}_3 \rtimes Q_8$ into $\mathrm{SL}_2(\mathbb{C})$. Further, show that $\Gamma/\{\pm 1\} \cong A_4$.

2) Show that Γ has 3 one-dimensional, 3 two-dimensional and 1 three-dimensional irreducible representations.

3) Prove Proposition 1.1 in this case.

1.4. **Quotients.** Our goal here is to show that the Kleinian singularities are actually quotients of $\mathbb{C}^2$ by the action of finite subgroups of $\mathrm{SL}_2(\mathbb{C})$.

First, let us recall some general properties of quotients under finite group actions. Let X be an affine algebraic variety and Γ be a group acting on X by automorphisms. Then Γ also acts on the algebra $\mathbb{C}[X]$ of regular functions on X: for $g \in \Gamma, f \in \mathbb{C}[X]$ and $x \in X$ we have $g.f(x) := f(g^{-1}x)$. The Γ-invariant elements in $\mathbb{C}[X]$ form a subalgebra denoted by $\mathbb{C}[X]^\Gamma$.

Now suppose that Γ is finite. We will need the following three classical results from Invariant theory, see [PV],[Sp] for references.

A) The algebra $\mathbb{C}[X]^\Gamma$ is finitely generated. In particular, we can form the corresponding algebraic variety to be denoted by X/Γ.

B) The inclusion $\mathbb{C}[X]^\Gamma \subset \mathbb{C}[X]$ gives rise to a morphism $\pi : X \to X/\Gamma$ of algebraic varieties called the quotient morphism. Each fiber of this morphism is a single Γ-orbit. This means that the variety X/Γ parameterizes Γ-orbits on X.

C) The algebra $\mathbb{C}[X]$ is finite over $\mathbb{C}[X]^\Gamma$. Equivalently, the morphism π is finite.

Also it is easy to see that X/Γ is irreducible provided X is. Property C) implies that the dimensions of X and X/Γ coincide.

Let us return to the case of finite subgroups of $\mathrm{SL}_2(\mathbb{C})$. The group $\mathrm{SL}_2(\mathbb{C})$ acts naturally on $\mathbb{C}^2$. Let Γ be a finite subgroup of $\mathrm{SL}_2(\mathbb{C})$.

Proposition 1.2. *The variety $\mathbb{C}^2/\Gamma$ is isomorphic to the Kleinian singularity of the same type as Γ.*

Again, let us show this in the simplest example of a cyclic group. The algebra of regular functions on $\mathbb{C}^2$ is nothing else but the polynomial algebra $\mathbb{C}[x,y]$. A generator $g = \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon^{-1} \end{pmatrix}$ acts on a monomial $x^n y^m$ as follows: $g.x^n y^m = \epsilon^{m-n} x^n y^m$. From here we see that the invariant subalgebra $\mathbb{C}[x,y]^\Gamma$ is spanned by all monomials $x^n y^m$, where $n-m$ is divisible by $r+1$. Any such monomial can be written in the form $x_1^a x_2^b x_3^c$, where $x_1 := xy, x_2 := x^{r+1}, x_3 := y^{r+1}$. So $\mathbb{C}[x,y]^\Gamma$ is generated by x_1, x_2, x_3. So $\mathbb{C}^2/\Gamma$ is realized as an (automatically, irreducible) subvariety in $\mathbb{C}^3$. On the other hand, by C), the dimension of $\mathbb{C}^2/\Gamma$ equals 2. Therefore $\mathbb{C}^2/\Gamma$ is a divisor and hence can be defined by a single equation. Clearly, the elements x_1, x_2, x_3 satisfy the relation $x_1^{r+1} = x_2 x_3$. The polynomial $x_2x_3 - x_1^{r+1}$ is easily seen to be irreducible. So $\mathbb{C}[x,y]^\Gamma = \mathbb{C}[x_1,x_2,x_3]/(x_1^{r+1} - x_2x_3)$, and we are done.

Problem 1.5. *Prove Proposition 1.2 for the dihedral groups.*

In the sequel we will basically need only a quotient realization of Kleinian singularities.

1.5. **Graded algebras.** Our goal here is to introduce a natural grading on $\mathbb{C}[x,y]^\Gamma$.

We start with a general definition of a graded algebra. Let A be an associative algebra with unit. Let $\mathbb{Z}_{\geqslant 0}$ denote the set of non-negative integers. A $\mathbb{Z}_{\geqslant 0}$-grading on A is a decomposition $A = \bigoplus_{n \geqslant 0} A^n$ into the direct sum of subspaces subject to the following conditions:

- $A^n A^m \subset A^{n+m}$.
- $1 \in A^0$.

In the case when A^0 is spanned by 1, we will say that the grading is positive. Analogously, one can define the notions of a $\mathbb{Z}$-grading, $\mathbb{Z}^n$-grading, etc. Of course, by a graded algebra one means an algebra equipped with a grading. Informally, the definition of a graded algebra is given in such a way that we have a notion of a "homogeneous element of degree n".

For example, consider the polynomial algebra $A := \mathbb{C}[x_1,\ldots,x_n]$. It is positively graded, the space A^m, by definition, is spanned by all monomials of degree m. Another example is the tensor algebra $T(V)$ of a vector space V. By definition, $T(V) = \bigoplus_{n=0}^\infty V^{\otimes n}$ and we set $T(V)^n := V^{\otimes n}$. With a chosen basis in V, say $x_1,\ldots,x_n$, we can identify $T(V)$ with the free algebra $\mathbb{C}\langle x_1,\ldots,x_n\rangle$ generated by $x_1,\ldots,x_n$. For a basis in $\mathbb{C}\langle x_1,\ldots,x_n\rangle$ we can take all noncommutative monomials in the elements x_i.

More examples of graded algebras can be obtained by taking quotients of graded algebras by homogeneous ideals. By definition, a subspace I in a graded algebra is called *homogeneous* if $I = \bigoplus_n I^n$, where $I^n := A^n \cap I$. Then, of course, $A/I = \bigoplus_n A^n/I^n$ and we set $(A/I)^n := A^n/I^n$. It is straightforward to check that if I is, in addition, a two-sided ideal, then $A/I = \bigoplus_n (A/I)^n$ is a grading. Of course, a two-sided ideal generated by homogeneous elements is homogeneous.

For example, $\mathbb{C}[x_1,\ldots,x_n]$ is the quotient of $\mathbb{C}\langle x_1,\ldots,x_n\rangle$ by the relations $x_ix_j - x_jx_i = 0$. Another example is provided by the exterior algebra $\Lambda(V)$ that is the quotient of $T(V)$ by the relations $u\otimes v + v\otimes u = 0, u, v \in V$.

Or we can realize graded algebras as subalgebras. Let $A=\bigoplus_{n\geqslant 0}A^n$ be a graded algebra and B be its subalgebra. We say that B is a graded subalgebra if $B=\bigoplus_{n\geqslant 0}B^n$, where $B^n:=A^n\cap B$. Then, of course, the decomposition $B=\bigoplus_{n\geqslant 0}B^n$ is a grading. For example, take $A=\mathbb{C}[x,y]$ and $B:=\mathbb{C}[x,y]^\Gamma$. Since Γ preserves the graded components of A, the subalgebra B is graded. In more elementary terms, an element in B is homogeneous of degree n if it is so in A.

1.6. **Definition of deformation.** In this course, we are not much interested in quotient singularities themselves (or their algebras of functions). Rather we want to study their both commutative and non-commutative deformations. In this lecture we start our discussion of deformations giving a general algebraic definition.

Let B be a commutative algebra and let $\mathfrak{m}$ be a maximal ideal of B. Further, let A be an associative algebra such that B is embedded into the center of A.

Definition 1.3. Let A_0 be an associative algebra. We say that A is a deformation of A_0 over B if $A_0\cong A/A\mathfrak{m}$ and A is flat as a B-module.

Recall that a B-module M is called *flat* if the functor $M\otimes_B\bullet$ is exact (in general, this functor is only right exact). For example, if $B=\mathbb{C}[x]$, the polynomial algebra in one variable, then M is flat if and only if the polynomial $x-\alpha$ is a not a zero divisor in M for ant $\alpha\in\mathbb{C}$.

If the algebra B in the previous definition is the algebra $\mathbb{C}[X]$ of regular functions on some affine variety, then to any point $x\in X$ we can assign the quotient $A_x:=A/A\mathfrak{m}_x$, where $\mathfrak{m}_x$ is the maximal ideal of x in $\mathbb{C}[X]$. So we get a family A_x of associative algebras parameterized by points of X. The flatness assumption informally means that A_x depends continuously on x.

For example, let $B=\mathbb{C}[c_0,\ldots,c_{r-1}]$. Then we can set $A=B[x_1,x_2,x_3]/(x_2x_3-x_1^{r+1}-\sum_{i=0}^{r-1}c_ix_1^i)$. If we take $\mathfrak{m}=(c_0,\ldots,c_{r-1})$, then we see that $A_0=\mathbb{C}[x_1,x_2,x_3]/(x_2x_3-x_1^{r+1})$ – the singularity of type A_r. We will not check flatness – usually to do this is not trivial. But modulo the flatness condition, A is a deformation of A_0.

1.7. **Resolutions.** Let us explain an algebro-geometric way to attach a Dynkin diagram to a singularity X from our list. Namely, recall that by a resolution of X one means a smooth (but non-affine) variety $\tilde{X}$ equipped with a projective birational morphism $\tilde{X}\to X$. Since we are in dimension 2, there is a *minimal* resolution $\pi:\tilde{X}_{min}\to X$, the minimality condition means that any other resolution factors through X.

The point 0 in $\mathbb{C}^3$ belongs to X. The fiber $\pi^{-1}(0)$ happens to be a union of projective lines, say $P_1,\ldots,P_r$. It turns out that the intersection matrix of the P_i's is the negative of the Cartan matrix of a Dynkin diagram, and this is precisely the diagram of the type of a singularity.

References

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[Sl] P. Slodowy. *Simple singularities and simple algebraic groups.* Lect. Notes Math., v.815. Springer, Berlin/Heidelberg/New York, 1980.

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LECTURES ON SYMPLECTIC REFLECTION ALGEBRAS

IVAN LOSEV

2. Algebras of Crawley-Boevey and Holland

This lecture consists of three different pieces. The first two are related to the deformation theory of Kleinian singularities and can be characterized as a general one and a specific one.

In the first lecture we have seen that a Kleinian singularity can be presented as the quotient $\mathbb{C}^2/\Gamma$, where Γ is a non-trivial finite subgroup of $\mathrm{SL}_2(\mathbb{C})$. This allows to equip the algebra $\mathbb{C}[\mathbb{C}^2/\Gamma] = \mathbb{C}[x,y]^\Gamma$ with a positive grading.

Also we have given a general definition of a deformation. This definition is too general to be convenient. For graded algebras there is another notion – of a filtered deformations. This is explored in 2.1. Then we relate filtered deformations to the usual ones, via a *Rees algebra* construction, 2.2.

Then we proceed to describing some deformations of Kleinian singularities discovered by Crawley-Boevey and Holland in [CBH]. Their idea was to deform, first, not the algebras $\mathbb{C}[x,y]^\Gamma$ but closely related algebras $\mathbb{C}[x,y]\#\Gamma$ (known as "orbifolds" or "smash-products") and then produce deformations of $\mathbb{C}[x,y]^\Gamma$ from those of $\mathbb{C}[x,y]\#\Gamma$. We treat the algebras $\mathbb{C}[x,y]\#\Gamma$ in 2.3 and their filtered deformations (to be called CBH algebras) in 2.4.

After that we switch to an entirely different topic providing some generalities for the next lecture. Namely, we recall basic definitions regarding quivers and their representations.

2.1. **Filtered deformations.** Let us proceed to the definition of a filtered algebra. Let $\mathcal{A}$ be an associative algebra with unit. An increasing $\mathbb{Z}_{\geqslant 0}$-filtration (to be simply called a filtration below) on $\mathcal{A}$ is a collection of subspaces $\mathcal{A}^{\leqslant n}, n \in \mathbb{Z}_{\geqslant 0}$ with the following properties:

- $\mathcal{A}^{\leqslant n} \subset \mathcal{A}^{\leqslant m}$ whenever $n < m$.
- $\bigcup_n \mathcal{A}^n = \mathcal{A}$.
- $\mathcal{A}^{\leqslant n}\mathcal{A}^{\leqslant m} \subset \mathcal{A}^{\leqslant n+m}$ for any $n, m \in \mathbb{Z}_{\geqslant 0}$.
- $1 \in \mathcal{A}_{\leqslant 0}$.

Again, we say that a filtration is positive if $\mathcal{A}^{\leqslant 0}$ is spanned by 1. Roughly speaking, in a filtered algebra we have a notion of an "element of degree n" but cannot speak about homogeneous elements.

Every graded algebra is automatically filtered: just set $A^{\leqslant n} := \bigoplus_{i=0}^n A^i$. On the other hand, any quotient of a filtered (for example, graded) algebra inherits a natural filtration. More precisely, let $\varphi : \mathcal{A} \twoheadrightarrow \mathcal{A}'$ be an epimorphism of algebras. Then we can set $\mathcal{A}'^{\leqslant n} := \varphi(\mathcal{A}^{\leqslant n})$.

Exercise 2.1. *Check that $\mathcal{A}'^{\leqslant n}$ is an algebra filtration.*

For example, consider the Weyl algebra $\mathrm{W}_2 := \mathbb{C}\langle x,y\rangle/(xy - yx - 1)$. This algebra is spanned by monomials in x and y and, the filtration subspace $\mathrm{W}_2^{\leqslant n}$ is spanned by all monomials of degree $\leqslant n$.

The following problem presents a basis of W_2.

Problem 2.2. *Show that the monomials $x^iy^j, i,j \geqslant 0$, form a basis of* W_2. *For this, construct a representation of* W_2 *on* $\mathbb{C}[x]$.

It turns out that from a filtered algebra we can cook a graded one by taking the *associated graded algebra*. Namely, set $A^n := \mathcal{A}^{\leqslant n}/\mathcal{A}^{\leqslant n-1}$ (where we assume that $\mathcal{A}^{\leqslant -1} = 0$). Then $A := \bigoplus_{n=0}^{+\infty} A^n$ becomes a graded associative algebra. The product is defined as follows. Pick $a \in \mathcal{A}^{\leqslant n}/\mathcal{A}^{\leqslant n-1}, b \in \mathcal{A}^{\leqslant m}/\mathcal{A}^{\leqslant m-1}$ and lift them to elements $\bar{a} \in \mathcal{A}^{\leqslant n}, \bar{b} \in \mathcal{A}^{\leqslant m}$. Then $\bar{a}\bar{b}$ is an element in $\mathcal{A}^{\leqslant n+m}$. Moreover, since $\mathcal{A}^{\leqslant n-1}\mathcal{A}^{\leqslant m}, \mathcal{A}^{\leqslant n}\mathcal{A}^{\leqslant m-1} \subset \mathcal{A}^{\leqslant n+m-1}$, the class of $\bar{a}\bar{b}$ in $A^{n+m} = \mathcal{A}^{\leqslant n+m}/\mathcal{A}^{\leqslant n+m-1}$ does not depend on the choice of $\bar{a}, \bar{b}$. By definition, ab is that class.

Exercise 2.3. *Check that this product is associative and has a unit.*

The associated graded algebra of $\mathcal{A}$ is denoted by $\operatorname{gr}\mathcal{A}$.

Now let us give a definition of a filtered deformation.

Definition 2.1. Let A be a $\mathbb{Z}_{\geqslant 0}$-graded algebra. We say that a filtered algebra $\mathcal{A}$ is a *filtered deformation* of A if $\operatorname{gr}\mathcal{A} \cong A$.

For example, let A be a graded algebra and let I be a two-sided ideal. For an element $a \in A$ we write $\operatorname{gr} a$ for the top degree summand of a. We write $\operatorname{gr} I$ for the set $\{\operatorname{gr} a | a \in I\}$. Recall that A/I is equipped with a natural filtration.

Exercise 2.4. *Show that* $\operatorname{gr} I$ *is a two-sided ideal of* A *and identify* $\operatorname{gr} A/I$ *with* $A/\operatorname{gr} I$.

For example, if I is generated by elements $a_1, \ldots, a_k$, then $\operatorname{gr} a_1, \ldots, \operatorname{gr} a_k \in \operatorname{gr} I$. But it is not necessary that $\operatorname{gr} I$ is generated by $\operatorname{gr} a_1, \ldots, \operatorname{gr} a_k$. However, with a wise choice of generators, this is so in many example.

For example, take $A = \mathbb{C}\langle x, y\rangle$ and let I be the ideal generated by $xy - yx - 1$ so that $A/I = \mathrm{W}_2$. Then of course $\operatorname{gr} I$ contains the element $xy - yx$ and so we have a natural epimorphism $\mathbb{C}[x,y] \twoheadrightarrow \operatorname{gr}\mathrm{W}_2$. This epimorphism maps x^iy^j to x^iy^j. Recall however that the elements x^iy^j form a basis of W_2. It follows that the epimorphism is an isomorphism, equivalently, that $\operatorname{gr}\mathrm{W}_2 = \mathbb{C}[x,y]$.

2.2. **Rees algebras.** The two notions of deformations are related as follows: given a filtered deformation $\mathcal{A}$ of A one can produce a deformation of A over $\mathbb{C}[h]$ in the sense of the general definition of the previous lecture, where h is an independent variable, whose fiber at $h = 0$ is A, while the fiber at any nonzero h is $\mathcal{A}$. This is achieved by using an object called the *Rees algebra* of $\mathcal{A}$.

By definition, the Rees algebra $R_h(\mathcal{A})$ is the subspace $\bigoplus_{i=0}^{+\infty} \mathcal{A}^{\leqslant i}h^i \subset \mathcal{A}[h]$. This subspace contains $1, h$ and is closed under multiplication (of polynomials with coefficients in $\mathcal{A}$). Also $R_h(\mathcal{A})$ is graded as a subalgebra of $\mathcal{A}[h]$ so that $R_h(\mathcal{A})^i := \mathcal{A}^{\leqslant i}h^i$.

Problem 2.5. *Establish natural isomorphisms* $R_h(\mathcal{A})/hR_h(\mathcal{A}) \cong \operatorname{gr}\mathcal{A}, R_h(\mathcal{A})/(h-\alpha)R_h(\mathcal{A}) \cong \mathcal{A}$, *where* $\alpha \in \mathbb{C} \setminus \{0\}$. *Also check that* $R_h(\mathcal{A})$ *is flat over* $\mathbb{C}[h]$.

2.3. **Orbifolds.** Deformations of the algebra $\mathbb{C}[x,y]^\Gamma$ are not so easy to define and study. One approach could be to take the defining equation, say $x_1^{r+1} - x_2x_3 = 0$ and modify it adding terms of lower degree: $P(x_1) - x_2x_3 = 0$, where $P(x_1) = x_1^{r+1} + a_rx_1^r + \ldots + a_0$ (we can assume that $a_r = 0$). We do get commutative deformations in this way and actually can extend this approach to non-commutative deformations as well, see [H],[Sm]. In the other types the commutative deformations are also not hard to write down, but non-commutative ones are considerably harder. We will make some remarks about that in the end of the lecture.

The problem in studying deformations of $\mathbb{C}[x,y]^\Gamma$ ideologically comes from the fact that the variety $\mathbb{C}^2/\Gamma$ is not smooth. We are not going to explain here why smoothness is related to a nice deformation theory, however, we will see several manifestations of this principle later. Fortunately, one can replace $\mathbb{C}[x,y]^\Gamma$ with a closely related algebra, the smash-product $\mathbb{C}[x,y]\#\Gamma$, that is smooth in some precise sense although is no longer commutative. Then one can deform $\mathbb{C}[x,y]\#\Gamma$ in a way "compatible" with Γ and cook a deformation of $\mathbb{C}[x,y]^\Gamma$ out of it. This was proposed by Crawley-Boevey and Holland in [CBH] and this is an approach we are going to take.

Let A be an associative algebra with unit equipped with an action of a finite group Γ by automorphisms. Let us define the algebra $A\#\Gamma$. As a vector space, it coincides with the tensor product $A\otimes\mathbb{C}\Gamma$, where $\mathbb{C}\Gamma$ stands for the group algebra of Γ. It is enough to define the product on the elements of the form $f\otimes\gamma$, $f\in A,\gamma\in\Gamma$, then we can extend it by linearity. We set

$$f_1\otimes\gamma_1\cdot f_2\otimes\gamma_2 := f_1\gamma_1(f_2)\otimes\gamma_1\gamma_2,$$

where $\gamma_1(f_2)$ stands for the image of f_2 under the action of γ_1.

The algebra $\mathbb{C}[x,y]\#\Gamma$ carries a natural grading: with Γ in degree 0 and x,y in degree 1.

One can easily recover the algebra A^Γ from $A\#\Gamma$. Namely, consider the idempotent e in $\mathbb{C}\Gamma$ corresponding to the trivial representation, $e=\frac{1}{\Gamma}\sum_{\gamma\in\Gamma}\gamma$ (recall that being an idempotent means that $e^2=e$). The algebra $\mathbb{C}\Gamma$ is embedded into $A\#\Gamma$ via $\gamma\mapsto 1\otimes\gamma$. So we can view e as an element of $A\#\Gamma$. The subspace $e(A\#\Gamma)e$ is closed under the multiplication. It is not a subalgebra in the sense that it does not contain 1. But it has its own unit, e. It is called a *spherical subalgebra* of $A\#\Gamma$.

Lemma 2.2. *The map $a\mapsto ae$ identifies A^Γ with the spherical subalgebra $e(A\#\Gamma)e$ and is an isomorphism of unital algebras.*

Proof. First of all, we remark that A^Γ embedded into $A\#\Gamma$ via $a\mapsto a\otimes 1$ commutes with $\mathbb{C}\Gamma$. It follows that the map $a\mapsto ae=ea$ is an algebra homomorphism. Then we have the equalities $(A\#\Gamma)e=A\otimes e, e(A\otimes e)=A^\Gamma\otimes e$ in $A\#\Gamma=A\otimes\mathbb{C}\Gamma$ (check them; use that $\mathbb{C}\Gamma e=\mathbb{C}e$ and $\gamma ae=\gamma(a)e$). This implies the claim. □

Exercise 2.6. *In the notation of the previous proof, check that A^Γ coincides with the center of $A\#\Gamma$.*

2.4. **Definition of CBH algebras.** Although the algebra $\mathbb{C}[x,y]\#\Gamma$ is no longer commutative, it has an advantage over $\mathbb{C}[x,y]^\Gamma$: a presentation of the former via generators and relations becomes kind of simpler. Namely,

$$\mathbb{C}[x,y]\#\Gamma := \mathbb{C}\langle x,y\rangle\#\Gamma/(xy-yx). \tag{1}$$

So to get a filtered deformation of $\mathbb{C}[x,y]\#\Gamma$ we can just correct a relation $xy-yx=0$ by a smaller degree (i.e., degree 0 or 1) term, and get something of the form $xy-yx=c$. The quotient comes equipped with a standard quotient filtration explained above. The degree 0 part of $\mathbb{C}\langle x,y\rangle\#\Gamma$ is $\mathbb{C}\Gamma$ and the degree 1 part is $\mathrm{Span}(x,y)\otimes\mathbb{C}\Gamma$. The following problem shows that we are forced to take c in the center of $\mathbb{C}\Gamma$ (that is the same as the subalgebra $(\mathbb{C}\Gamma)^\Gamma$ of Γ-invariants under the adjoint action).

Problem 2.7. *Show that if* $\operatorname{gr}\mathbb{C}\langle x,y\rangle\#\Gamma/(xy-yx-c)=\mathbb{C}[x,y]\#\Gamma$*, then* c *lies in the center of* $\mathbb{C}\Gamma$ *(that is equal to* $(\mathbb{C}\Gamma)^\Gamma$*, where the invariants are taken with respect to the adjoint action).*

For $c\in(\mathbb{C}\Gamma)^\Gamma$ we set $H_c:=\mathbb{C}\langle x,y\rangle\#\Gamma/(xy-yx-c)$. This is an algebra introduced by Crawley-Boevey and Holland. They checked that $\operatorname{gr}H_c=\mathbb{C}[x,y]\#\Gamma$. We are not going to show this right now, in fact, we will obtain a more general result later. We would like to remark that the claim that $\operatorname{gr}H_c=\mathbb{C}[x,y]\#\Gamma$ is equivalent to saying that the elements $x^iy^j\gamma$, where $i+j\leqslant n$ form a basis in $H_c^{\leqslant n}$.

Now we can take the spherical subalgebra $eH_ce\subset H_c$. This algebra is filtered, we just restrict the filtration from H_c, i.e., $(eH_ce)^{\leqslant n}:=eH_ce\cap H_c^{\leqslant n}$. Equivalently, $(eH_ce)^{\leqslant n}=eH_c^{\leqslant n}e$.

Exercise 2.8. *Deduce* $\operatorname{gr}eH_ce=\mathbb{C}[x,y]^\Gamma$ *from* $\operatorname{gr}H_c=\mathbb{C}[x,y]\#\Gamma$ *(i.e., show that taking the spherical subalgebra commutes with taking the associated graded).*

So we get a family of filtered deformations of $\mathbb{C}[x,y]^\Gamma$ indexed by the space $(\mathbb{C}\Gamma)^\Gamma$.

In fact, it is expected (and proved for Γ of types A,D) that all filtered deformations of $\mathbb{C}[x,y]^\Gamma$ are of this form. It is a general fact (to be proved later in this course) that the algebra eH_ce is commutative if and only if $c_1=0$. It is known (for this one uses results of Slodowy, [Sl], together with some other considerations that will be explained later) that all commutative filtered deformations of $\mathbb{C}^2/\Gamma$ are of the form eH_ce. To prove that all non-commutative deformations are of the form eH_ce in type A is relatively easy. Type D is due to Boddington, [B], this is much harder and more technical, see also [L].

The following problem describes the algebras eH_ce in type A more explicitly.

Problem 2.9. *Let* Γ *be the group* $\mathbb{Z}/(r+1)\mathbb{Z}$*. We write* $x,y\in H_c$ *for the images of* $x,y\in\mathbb{C}\langle x,y\rangle\#\Gamma$*.*

1) Show that the algebra H_c *is* $\mathbb{Z}$*-graded with* Γ *in degree* 0*,* x *in degree* 1 *and* y *in degree* -1*.*

2) We can write c *as* $\sum_{\gamma\in\Gamma}c_\gamma\gamma$*. Produce element* $h\in(H_c)_{\leqslant 2}$ *that commutes with* Γ *and satisfies* $[h,x]=c_1x,[h,y]=-c_1y$ *(such an element is defined uniquely up to adding a constant provided* $c_1\neq 0$*).*

3) Set $x_1:=eh,x_2:=ex^n,x_3:=ey^n$*. Check that there are polynomials* P,Q *in one variable of degree* $r+1$ *such that* $x_2x_3=P(x_1),x_3x_2=Q(x_1)$ *in* eH_ce*. How are these polynomials related? Express their coefficients via the coefficients* c_γ*.*

4) Use $\operatorname{gr}eH_ce=\mathbb{C}[x,y]^\Gamma$ *to show that* $eH_ce=\mathbb{C}\langle x_1,x_2,x_3\rangle/([x_1,x_2]=(r+1)c_1x_2,[x_1,x_3]=-(r+1)c_1x_3,x_2x_3=P(x_1),x_3x_2=Q(x_1))$*.*

2.5. **Quivers and their representations.** Formally, a quiver Q is a collection of the following data: two sets, Q_0 (vertices) and Q_1 (arrows) and two maps $Q_1\to Q_0$, head h and tail t that to each arrow assign its target and source vertices. Mostly, people consider the case when both Q_0 and Q_1 are finite. A representation of a quiver Q is a collection of vector spaces $V_i,i\in Q_0$, and of linear maps $x_a:V_{t(a)}\to V_{h(a)},a\in Q_1$.

For example, we can consider a quiver with a single vertex, say 1, and a single arrow, a, with $t(a)=h(a)=1$. We can draw this quiver as an oriented loop. This is a so called Jordan quiver, the reason is that its representation is a vector space together with its linear endomorphism.

The dimension of a representation (V_i,x_a) is the vector $(\dim V_i)_{i\in Q_0}$. In these lectures, we will only consider finite dimensional representations. Usually, for V_i we take the coordinate vector space $\mathbb{C}^{v_i}$. Then the set of representations of given dimension naturally becomes a vector space: for example, the sum of the representations $(x_a)_{a\in Q_1}, (x'_a)_{a\in Q_1}$ is, by definition, $(x_a + x'_a)_{a\in Q_1}$. The space of representations of Q of dimension v (where recall v is a vector $(v_i)_{i\in Q_0}$) is denoted by $\mathrm{Rep}(Q, v)$. As a vector space, it is naturally identified with $\bigoplus_{a\in Q_1} \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^{v_{t(a)}}, \mathbb{C}^{v_{h(a)}})$.

On $\mathrm{Rep}(Q, v)$ we have a natural action of the group $\mathrm{GL}(v) := \prod_{i\in Q_0} \mathrm{GL}(\mathbb{C}^{v_i})$ induced by its natural action on $\bigoplus_i \mathbb{C}^{v_i}$ by the change of bases. In more detail, an element $g = (g_i)_{i\in Q_0}, g_i \in \mathrm{GL}(\mathbb{C}^{v_i})$ maps on an element $(x_a)_{a\in Q_1}$ to $(x'_a)_{a\in Q_1}$ with $x'_a = g_{h(a)} x_a g^{-1}_{t(a)}$. The elements of the same $\mathrm{GL}(v)$-orbit can be thought as a single representation but written in different basis. A basic problem in the study of quiver representations is therefore to describe the orbits.

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LECTURES ON SYMPLECTIC REFLECTION ALGEBRAS

IVAN LOSEV

3. McKay correspondence upgraded

3.1. **Categorical quotients.** Let us start by reminding a few general definitions regarding algebraic groups. By an algebraic group one means a group that is also an algebraic variety in such a way that the group structure maps (the product and the inverse) are morphisms of algebraic varieties. For example, $\mathrm{GL}_n(\mathbb{C})$ is an algebraic group. Homomorphisms of algebraic groups as well as their actions are also supposed to be morphisms of algebraic varieties. Hence one has a notion of a representation of an algebraic group (a homomorphism $G \to \mathrm{GL}(V)$), such representations are usually called *rational*.

An algebraic group is called *reductive* if any its rational representation is completely reducible. Any finite group is reductive. In fact, any general linear group and any product of those are reductive as well, this was first proved by Weyl.

Let us return to representation of quivers. We have mentioned that a basic problem is to describe the orbits of $\mathrm{GL}(v)$ on $\mathrm{Rep}(Q, v)$, i.e., to describe a *quotient* of $\mathrm{Rep}(Q, v)$ by the $\mathrm{GL}(v)$-action. We can try to find an algebraic variety parameterizing the orbits. The field of Mathematics that tries to construct quotients for algebraic group actions is called (Geometric) Invariant theory. Basically, it offers two approaches, both work for reductive groups. A simpler one produces so called *categorical quotients*. We will consider it right below. Another one, *GIT quotients*, will be considered later.

Let X be an affine algebraic variety (in fact, one can take any affine scheme of finite type) and G be a reductive algebraic group acting on X. Let us try to produce a quotient as an affine algebraic variety, say Y. Then we should have a morphism $\varphi : X \to Y$ that is G-invariant, i.e., constant on the orbits. Consider the corresponding homomorphism $\varphi^* : \mathbb{C}[Y] \to \mathbb{C}[X]$. The condition that φ is G-invariant is equivalent to the condition that the pull-back of any function from $\mathbb{C}[Y]$ is G-invariant, i.e., $\varphi^*(\mathbb{C}[Y]) \subset \mathbb{C}[X]^G$. As in the case of finite groups, there is a general theorem again due to Hilbert saying that the algebra $\mathbb{C}[X]^G$ is finitely generated. So we can form an affine variety with algebra of functions equal to $\mathbb{C}[X]^G$. This variety is called the *categorical quotient* (for the G-action on X) and is denoted by $X/\!/G$. The reason for the name is a universality property of the natural morphism $\pi : X \to X/\!/G$ induced by the inclusion $\mathbb{C}[X]^G \subset \mathbb{C}[X]$ (and called the quotient morphism): for any other G-invariant morphism $\varphi : X \to Y$ of affine varieties there is a unique morphism $\psi : X/\!/G \to Y$ such that $\varphi = \psi \circ \pi$. In other words, $X/\!/G$ provides a finest parametrization of orbits that we can get using an affine variety (in fact, one can drop the condition of being affine).

However, sometimes such parametrization is pretty useless. For example, let $X = \mathbb{C}^n$ and let the one-dimensional torus $\mathbb{C}^\times := \mathrm{GL}_1(\mathbb{C}) = \{z \in \mathbb{C} | z \neq 0\}$ act on X by $t.(x_1, \ldots, x_n) = (tx_1, \ldots, tx_n)$.

Exercise 3.1. *Prove that there are no non-constant invariant polynomials for this action.*

So $X/\!/G$ in our example is just a point.

Still the quotient morphism has some remarkable properties proved by Hilbert.

Theorem 3.1. *Let X be an affine algebraic variety, G be a reductive algebraic group acting on X and $\pi : X \to X//G$ denote the quotient morphism. Then the following is true.*

- *π is surjective.*
- *Every fiber of π contains a single closed orbit.*
- *Let $X_1 \subset X$ be a closed subvariety (or, more generally, a closed subscheme). Then $\pi(X_1) \subset X//G$ is closed and a natural morphism $X_1//G \to \pi(X_1)$ is an isomorphism.*

Exercise 3.2. *Use the theorem to show that the closure of any orbit contains a unique closed orbit.*

So the categorical quotient parameterizes the closed orbits for the action of G on X.

Let us provide an example. Consider the adjoint action of $G := \mathrm{GL}_n(\mathbb{C})$ on the space $X := \mathrm{Mat}_n(\mathbb{C})$ of $n \times n$-matrices by conjugations.

Problem 3.3. *Show that the algebra of invariants $\mathbb{C}[X]^G$ is generated by the coefficients of the characteristic polynomial of a matrix and is isomorphic to the algebra of polynomials in n variables. A hint: consider the restriction to the subspace of diagonal matrices.*

So any fiber of the quotient morphism consists of all matrices with a prescribed collection of eigenvalues.

Problem 3.4. *Show that every fiber indeed contains a single closed orbit and that this orbit consists of diagonalizable matrices.*

3.2. **Parametrization of representations.** An interesting special case is the construction of the representation varieties that parameterize (or try to parameterize) representations of a given associative algebra up to an isomorphism.

Take a finitely presented unital associative algebra $\mathcal{A}$. It can be presented as the quotient of a free algebra $\mathbb{C}\langle x_1, \ldots, x_n\rangle$ by some relations $f_1, \ldots, f_m$. Therefore representations of $\mathcal{A}$ in $\mathbb{C}^N$ are in one-to-one correspondence with the N-tuples $(M_1, \ldots, M_n)$ of matrices subject to $f_i(M_1, \ldots, M_N) = 0, i = 1, 2, \ldots, m$. So the representations are parameterized by the points of the subvariety in $\mathrm{Mat}_N(\mathbb{C})^n$ defined by the polynomials above. Denote this subvariety by $\mathrm{Rep}(\mathcal{A}, N)$.

Two representations $(M_1, \ldots, M_n), (M_1', \ldots, M_n')$ are isomorphic if and only if they lie in the same $G := \mathrm{GL}_N(\mathbb{C})$-orbit for the action given by $g.(M_1, \ldots, M_n) = (gM_1g^{-1}, \ldots, gM_ng^{-1})$. We can parameterize the closed orbits for this action by points of the categorical quotient $\mathrm{Rep}(\mathcal{A}, N)//G$. The question is to find some representation theoretic description of representations with closed orbits. It turns out that the description is pretty elegant.

Theorem 3.2. *The G-orbit of a representation is closed if and only if the representation is semisimple (=completely reducible).*

Exercise 3.5. *Show the "only if" part: if the orbit is closed, then the representation is semisimple.*

The proof of the other implication is harder, it is based on the following general result from Invariant theory called the *Hilbert-Mumford* criterium (proved in the full generality by Mumford; a relatively elementary proof can be found in [B]).

Theorem 3.3. *Let a reductive group G act on an affine variety X. Let $x, y \in X$ be such that Gy is the closed orbit in $\overline{Gx}$. Then there is a one-parameter subgroup (=algebraic group homomorphism) $\gamma : \mathbb{C}^\times \to G$ such that $\lim_{t\to 0} \gamma(t).x \in Gy$.*

Problem 3.6. *Deduce the if part of Theorem 3.2 from the Hilbert-Mumford criterium.*

3.3. **McKay correspondence upgraded: an overview.** Let $\Gamma \subset \mathrm{SL}_2(\mathbb{C})$ be a finite subgroup. To this subgroup we can assign an un-oriented graph $\underline{Q}$ with vertices $0, \ldots, r$, where $N_0, \ldots, N_r$ are the irreducible representations of Γ with N_0 being the trivial representation. The number of edges between i and j is $m_{ij} := \dim \mathrm{Hom}_\Gamma(\mathbb{C}^2 \otimes N_i, N_j)$. This is of course not yet a quiver, as the latter has to be oriented. For what follows it will be convenient to *choose* an orientation on $\underline{Q}$ making it into a quiver. Then we consider the double quiver: for each arrow a in $\underline{Q}_1$ we add a new arrow a^* with opposite orientation. So the number of arrows from i to j is m_{ij}. Also recall the dimension vector $\delta = (\delta_i)_{i \in Q_0}$ with $\delta_i = \dim N_i$.

A variety that we are going to produce from Q is as follows. We consider the representation space

$$\mathrm{Rep}(Q, \delta) \cong \bigoplus_{a \in Q_1} \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^{\delta_{t(a)}}, \mathbb{C}^{\delta_{h(a)}}) = \bigoplus_{i,j=0}^{r} \mathrm{Hom}(\mathbb{C}^{\delta_i}, \mathbb{C}^{\delta_j})^{m_{ij}}$$

that come equipped with an action of $\mathrm{GL}(\delta)$ given by $(g_i)_{i \in Q_0}(x_a)_{a \in Q_1} = (g_{h(a)} x_a g_{t(a)}^{-1})$. Let $\mathfrak{gl}(\delta)$ denote the Lie algebra of $\mathrm{GL}(\delta)$, i.e., $\mathfrak{gl}(\delta) := \prod_{i \in Q_0} \mathfrak{gl}_{\delta_i}$, where $\mathfrak{gl}_{\delta_i}$ is the Lie algebra of all matrices. We have a quadratic map $\mu : \mathrm{Rep}(Q, \delta) \to \mathfrak{gl}(\delta)$, $\mu = (\mu_i)_{i \in Q_0}$, where

$$\mu_i\left((x_a)_{a \in Q_1}\right) = \sum_{a \in \underline{Q}_1, t(a) = i} x_{a^*} x_a - \sum_{a \in \underline{Q}_1, h(a) = i} x_a x_{a^*}.$$

For example, consider Γ of type A_r. Then we can orient $\underline{Q}$ counterclockwise, let a_i denote the arrow from i to $i-1$ and let a_i^* be the opposite arrow. Then $\mu_i = x_{a_i^*} x_{a_i} - x_{a_{i+1}} x^*_{a_{i+1}}$.

The variety we are interested in is $\mu^{-1}(0)//G$. The following is what we mean by the upgraded McKay correspondence.

Theorem 3.4. *We have an isomorphism of varieties* $\mathbb{C}^2/\Gamma \cong \mu^{-1}(0)// \mathrm{GL}(\delta)$.

The proof of the theorem consists of several steps. To describe them we need some more notation. Let A be an algebra acted on by Γ. Recall the smash-product $A\#\Gamma$ which contains $\mathbb{C}\Gamma$ as a subalgebra. We have a distinguished representation of that subalgebra in itself (by left multiplications). We consider the variety $\mathrm{Rep}_\Gamma(A\#\Gamma, \mathbb{C}\Gamma)$ of all representations of $A\#\Gamma$ that restrict to $\mathbb{C}\Gamma$ as that fixed representation. Similarly to the previous section, this set carries a natural structure of an algebraic variety. Since the representations are fixed on $\mathbb{C}\Gamma$, they are isomorphic if and only if they lie in the same orbit under a natural action of the group $\mathrm{GL}(\mathbb{C}\Gamma)^\Gamma$ of all Γ-equivariant elements of $\mathrm{GL}(\mathbb{C}\Gamma)$.

Here are the steps we use to prove Theorem 3.4:

Step 1: Identify the isomorphism classes of semisimple representations of $\mathrm{Rep}_\Gamma(\mathbb{C}[x,y]\#\Gamma, \mathbb{C}\Gamma)$ with $\mathbb{C}^2/\Gamma$ (via taking "central character").

Step 2: Identify $\mathrm{Rep}_\Gamma(\mathbb{C}\langle x,y\rangle\#\Gamma, \mathbb{C}\Gamma)$ with $\mathrm{Rep}(Q,\delta)$ so that the action of $\mathrm{GL}(\mathbb{C}\Gamma)^\Gamma$ on the former becomes the action of $G = \mathrm{GL}(\delta)$ on the latter. In fact, such an identification is not unique so we will need to choose a nice one.

Step 3: $\mathrm{Rep}_\Gamma(\mathbb{C}[x,y]\#\Gamma, \mathbb{C}\Gamma)$ is a subvariety (more precisely, a subscheme) in $\mathrm{Rep}_\Gamma(\mathbb{C}\langle x,y\rangle\#\Gamma, \mathbb{C}\Gamma)$ given by the condition that $[x,y] = xy - yx$ acts by 0. To prove Theorem 3.4 it remains to show that, under a suitable identification of $\mathrm{Rep}_\Gamma(\mathbb{C}\langle x,y\rangle\#\Gamma, \mathbb{C}\Gamma)$ with $\mathrm{Rep}(Q,\delta)$, the assignment that maps a representation φ to $\varphi(xy - yx)$ becomes μ (this includes the claim that these can be viewed as maps to the same space). To prove the coincidence is the most delicate and technical step.

Then Theorem 3.4 will just follow from the fact that the categorical quotient parameterizes the semisimple representations (well, not quite, we still need to establish a few things: that $\mu^{-1}(0)//G$ is a reduced scheme, that the bijection $\mathbb{C}^2/\Gamma \cong \mu^{-1}(0)//G$ that we are going to get is a morphism of varieties). We will make some remarks on these issues below.

3.4. **Step 1.** Recall that $\mathbb{C}[x, y]^\Gamma$ is embedded (and actually coincides with) the center of $\mathbb{C}[x, y]\#\Gamma$. According to the Schur lemma, any central element, say a, has to act by a scalar, say a_V, on an irreducible representation V. The map $a \mapsto a_V : \mathbb{C}[x, y]^\Gamma \to \mathbb{C}$ is easily seen to be an algebra homomorphism. So it defines a point, say p_V, in $\mathbb{C}^2/\Gamma$.

Lemma 3.5. *For each nonzero $p \in \mathbb{C}^2/\Gamma$ there is a unique (up to an isomorphism) irreducible representation V of $\mathbb{C}[x, y]\#\Gamma$ with $p_V = p$ and, moreover, V is isomorphic to $\mathbb{C}\Gamma$ as a representation of Γ.*

Proof. Consider the subalgebra $\mathbb{C}[x, y] \subset \mathbb{C}[x, y]\#\Gamma$. This algebra is commutative and so any its irreducible representation is 1-dimensional. So there is a point $\tilde{p} \in \mathbb{C}^2$ and a vector $v \in V$ such that $a.v = \tilde{p}(a)v$ for any $a \in \mathbb{C}[x, y]$ (in other words, v is an eigenvector for $\mathbb{C}[x, y]$ with "eigenvalue" $\tilde{p}$). Since $\mathbb{C}[x, y]^\Gamma$ (viewed as the center of $\mathbb{C}[x, y]\#\Gamma$) is contained in $\mathbb{C}[x, y]$ we see that $\tilde{p}|_{\mathbb{C}[x,y]^\Gamma} = p$ or, equivalently, $\tilde{p}$ lies in the Γ-orbit corresponding to p.

Next we remark that the subalgebra $\mathbb{C}[x, y] \subset \mathbb{C}[x, y]\#\Gamma$ is normalized by the adjoint action of Γ. It follows that the vector $gv, g \in \Gamma$, is again an eigenvector for $\mathbb{C}[x, y]$, whose eigenvalue is the point $g\tilde{p}$. The following exercise shows that all points $g\tilde{p}$ are distinct.

Exercise 3.7. *Show that the stabilizer in Γ of any nonzero point in $\mathbb{C}^2$ is trivial.*

The span of the vectors $g\tilde{p}, g \in \Gamma$, is clearly Γ-stable and also $\mathbb{C}[x, y]$-stable. So it is a submodule hence needs to coincide with V. Also the vectors $g\tilde{p}$ are linearly independent because their eigenvalues are pairwise different. This completes the proof. $\square$

Now let us consider the irreducible representations V with $p_V = 0$.

Lemma 3.6. *Suppose that V is an irreducible representation of $\mathbb{C}[x, y]\#\Gamma$ such that $p_V = 0$. Then x, y act on V by 0 and the restriction of V on $\mathbb{C}\Gamma$ is irreducible.*

Proof. Similarly to the first paragraph of the proof of Lemma 3.5, all eigenvalues of $\mathbb{C}[x, y]$ on V are zero. So the common kernel V_0 of x, y in V is nonzero. But V_0 is Γ-stable and hence a $\mathbb{C}[x, y]\#\Gamma$-submodule. So $V = V_0$ and V_0 is an irreducible representation of Γ. $\square$

Lemmas 3.5,3.6 allow to describe the semisimple representations in $\mathrm{Rep}_\Gamma(\mathbb{C}[x, y]\#\Gamma, \mathbb{C}\Gamma)$ (up to an isomorphism) completely. Such a representation is either an irreducible representation V with $p_V = 0$ or is isomorphic to $\mathbb{C}\Gamma$ with zero action of x and y. It follows that there is a bijection of the isomorphism classes of semisimple representations in $\mathrm{Rep}_\Gamma(\mathbb{C}[x, y]\#\Gamma, \mathbb{C}\Gamma)$ and $\mathbb{C}^2/\Gamma$.

3.5. **Step 2.** Let us discuss an alternative way to look at the representations from $\mathrm{Rep}_\Gamma(\mathbb{C}\langle x, y\rangle\#\Gamma, \mathbb{C}\Gamma)$. On any such representation, a Γ-action is already given and so we only need to specify the actions of x, y. To give such an action is the same as to give a linear map $\mathbb{C}^2 \otimes \mathbb{C}\Gamma \to \mathbb{C}\Gamma$ (where we view x, y as a basis in $\mathbb{C}^2$).

Exercise 3.8. *Such map extends to an action of $\mathbb{C}\langle x, y\rangle\#\Gamma\mathbb{C}\langle x, y\rangle\#\Gamma$ if and only if it is Γ-equivariant.*

So the representations we are interested in are parameterized by the points in the vector space $\mathrm{Hom}_\Gamma(\mathbb{C}^2 \otimes \mathbb{C}\Gamma, \mathbb{C}\Gamma)$. The Γ-module $\mathbb{C}\Gamma$ is decomposed as $\bigoplus_{i=0}^r N_i \otimes N_i^*$, where Γ acts on the left factors and so, choosing bases in the spaces $N_i^*, i = 0, \ldots, r$, we identify $\mathbb{C}\Gamma$ with $\bigoplus_{i=0}^r N_i^{\oplus \delta_i}$. We set $M_{ij} := \mathrm{Hom}_\Gamma(\mathbb{C}^2 \otimes N_i, N_j)$ so that $m_{ij} := \dim M_{ij}$.

Exercise 3.9. *Show that*

$$\mathrm{Hom}_\Gamma(\mathbb{C}^2 \otimes \mathbb{C}\Gamma, \mathbb{C}\Gamma) = \bigoplus_{i,j=0}^{r} M_{ij} \otimes \mathrm{Hom}_{\mathbb{C}}(N_i^*, N_j^*) = \bigoplus_{i,j=0}^{r} \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^{\delta_i}, \mathbb{C}^{\delta_j})^{m_{ij}}.$$

We remark that the first equality here is a completely canonical isomorphism, while the second one is not: it requires choosing bases in, first, the spaces $N_i^*, i = 0, \ldots, r$, and, second, the spaces M_{ij}. We recall that, with one exception, the latter are always one or zero dimensional and so a basis vector is defined uniquely up to proportionality. The exception is $\Gamma = \mathbb{Z}/2\mathbb{Z}$, here M_{12} is two dimensional. We will ignore this exception.

The previous exercise implies that the space of representations of $\mathbb{C}\langle x, y\rangle \# \Gamma$ is nothing else but $\mathrm{Rep}(Q, \delta)$.

If we decompose $\mathbb{C}\Gamma$ as $\bigoplus_{i=0}^r N_i^{\delta_i}$, then (using the Schur lemma) we see that $\mathrm{GL}(\mathbb{C}\Gamma)^\Gamma = \mathrm{GL}(\delta) := \prod_{i=0}^r \mathrm{GL}(\delta_i)$ and, under our identification of $\mathrm{Rep}_\Gamma(\mathbb{C}\langle x, y\rangle \# \Gamma, \mathbb{C}\Gamma)$ with $\mathrm{Rep}(Q, \delta)$, the $\mathrm{GL}(\mathbb{C}\Gamma)^\Gamma$-action on the former becomes the $\mathrm{GL}(\delta)$-action on the latter.

So our conclusion is that the semisimple representations in $\mathrm{Rep}_\Gamma(\mathbb{C}\langle x, y\rangle \# \Gamma, \mathbb{C}\Gamma)$ are parameterized up to an isomorphism by the points of $\mathrm{Rep}(Q, \delta) /\!/ \mathrm{GL}(\delta)$.

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4. Deformed preprojective algebras

Recall that we have introduced the double McKay quiver Q, its representation space $\operatorname{Rep}(Q,\delta)$ acted on by the group $\operatorname{GL}(\delta)$ and also a somewhat mysterious quadratic map $\mu : \operatorname{Rep}(Q,\delta) \to \mathfrak{gl}(\delta)$. We have claimed that $\mathbb{C}^2/\Gamma \cong \mu^{-1}(0)/\!/\operatorname{GL}(\delta)$.

For this we have realized $\mathbb{C}^2/\Gamma$ as a "moduli space" for certain representations of $\mathbb{C}[x,y]\#\Gamma$. Namely, we considered the variety $\operatorname{Rep}_\Gamma(\mathbb{C}[x,y]\#\Gamma, \mathbb{C}\Gamma)$ of all representations of $\mathbb{C}[x,y]\#\Gamma$ whose restriction to $\mathbb{C}\Gamma$ is the representation by left multiplications. A few remarks about this choice are in order.

First, it is not necessary to <u>fix</u> an isomorphism of a representation of $\mathbb{C}[x,y]\#\Gamma$ with $\mathbb{C}\Gamma$, it is enough to consider all representations of $\mathbb{C}[x,y]\#\Gamma$ whose restriction to $\mathbb{C}\Gamma$ is isomorphic to the regular representation. In this case the symmetry group becomes bigger, $\operatorname{GL}(\mathbb{C}\Gamma)$, and is no longer identified with $\operatorname{GL}(\delta)$, but the "moduli space" (=the space parameterizing the representations) remains the same.

Second, let us explain why we choose to consider the representations in $\mathbb{C}\Gamma$ and not in some other Γ-module V. The reason is that if $\mathbb{C}\Gamma \not\hookrightarrow_\Gamma V$, then x, y act by 0 act on all simple $\mathbb{C}[x,y]\#\Gamma$-modules entering V (this follows from the classification of the irreducible $\mathbb{C}[x,y]\#\Gamma$-modules performed last time) and so we do not get an interesting moduli space.

To prove an isomorphism $\mathbb{C}^2/\Gamma \cong \mu^{-1}(0)/\!/\operatorname{GL}(\delta)$, it remains to do two steps: to show that $\operatorname{Rep}_\Gamma(\mathbb{C}\langle x,y\rangle\#\Gamma, \mathbb{C}\Gamma)$ with $\operatorname{Rep}(Q,\delta)$ and to show that the condition $\varphi(xy-yx)=0$ on $\varphi \in \operatorname{Rep}_\Gamma(\mathbb{C}\langle x,y\rangle\#\Gamma, \mathbb{C}\Gamma)$ is equivalent to $\mu(\varphi)=0$ (we do not prove the promised equality of $\varphi \mapsto \varphi(xy-yx)$ and μ, it is a subtle question, in what sense this equality holds).

4.1. **Step 2.**

We need an alternative way to look at the representations from $\operatorname{Rep}_\Gamma(\mathbb{C}\langle x,y\rangle\#\Gamma, \mathbb{C}\Gamma)$. On any such representation, a Γ-action is already given and so we only need to specify the actions of x, y. To give such an action is the same as to give a linear map $\mathbb{C}^2 \otimes \mathbb{C}\Gamma \to \mathbb{C}\Gamma$ (where we view x, y as a basis in $\mathbb{C}^2$).

Exercise 4.1. *A map $\mathbb{C}^2 \otimes \mathbb{C}\Gamma \to \mathbb{C}\Gamma$ extends to an action of $\mathbb{C}\langle x,y\rangle\#\Gamma$ if and only if it is Γ-equivariant.*

So the representations we are interested in are parameterized by the points in the vector space $\operatorname{Hom}_\Gamma(\mathbb{C}^2\otimes\mathbb{C}\Gamma, \mathbb{C}\Gamma)$. The Γ-module $\mathbb{C}\Gamma$ is decomposed as $\bigoplus_{i=0}^r N_i \otimes N_i^*$, where Γ acts on the left factors and so, choosing bases in the spaces $N_i^*, i = 0,\ldots,r$, we identify $\mathbb{C}\Gamma$ with $\bigoplus_{i=0}^r N_i^{\oplus\delta_i}$. We set $M_{ij} := \operatorname{Hom}_\Gamma(\mathbb{C}^2 \otimes N_i, N_j)$ so that $m_{ij} := \dim M_{ij}$.

Exercise 4.2. *Show that*

$$\mathrm{Hom}_\Gamma(\mathbb{C}^2\otimes\mathbb{C}\Gamma,\mathbb{C}\Gamma)=\bigoplus_{i,j=0}^r M_{ij}\otimes\mathrm{Hom}_{\mathbb{C}}(N_i^*,N_j^*)$$
$$=\bigoplus_{i,j=0}^r \mathrm{Hom}_{\mathbb{C}}(N_i^*,N_j^*)^{\oplus m_{ij}}=\bigoplus_{i,j=0}^r \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^{\delta_i},\mathbb{C}^{\delta_j})^{m_{ij}}.$$

Note that the first equality is canonical, the second depends on the choice of a basis in M_{ij}, while the third depends on the choice of bases in N_i^*.

We recall that, with one exception, the space M_{ij} are always one or zero dimensional and so a basis vector is defined uniquely up to proportionality. The exception is $\Gamma=\mathbb{Z}/2\mathbb{Z}$, here M_{12} is two dimensional. We will ignore this exception.

The previous exercise implies that the space of representations of $\mathbb{C}\langle x,y\rangle\#\Gamma$ is nothing else but $\mathrm{Rep}(Q,\delta)$.

If we decompose $\mathbb{C}\Gamma$ as $\bigoplus_{i=0}^r N_i^{\delta_i}$, then (using the Schur lemma) we see that $\mathrm{GL}(\mathbb{C}\Gamma)^\Gamma=\prod_{i=0}^r\mathrm{GL}(N_i^*)=\mathrm{GL}(\delta):=\prod_{i=0}^r\mathrm{GL}(\delta_i)$ (where the first equality is canonical, while the second is not) and, under our identification of $\mathrm{Rep}_\Gamma(\mathbb{C}\langle x,y\rangle\#\Gamma,\mathbb{C}\Gamma)$ with $\mathrm{Rep}(Q,\delta)$, the $\mathrm{GL}(\mathbb{C}\Gamma)^\Gamma$-action on the former becomes the $\mathrm{GL}(\delta)$-action on the latter.

So our conclusion is that the semisimple representations in $\mathrm{Rep}_\Gamma(\mathbb{C}\langle x,y\rangle\#\Gamma,\mathbb{C}\Gamma)$ are parameterized up to an isomorphism by the points of $\mathrm{Rep}(Q,\delta)/\!/\,\mathrm{GL}(\delta)$.

That's what we need on Step 2. But actually, in order to accomplish Step 3 of our original program, we will need some ramifications of Step 2.

4.2. **Path algebras.** As in the case of a group or a Lie algebra, a representation of a quiver Q is the same as a module over a certain algebra. This algebra is called the *path algebra* of Q, is denoted by $\mathbb{C}Q$ and is constructed as follows. For a basis in $\mathbb{C}Q$ we will take all paths in Q, i.e., all sequences of arrows $p=(a_1,\ldots,a_k)$ such that $h(a_1)=t(a_2),\ldots,h(a_{k-1})=t(a_k)$. We set $t(p):=t(a_1),h(p)=h(a_k)$ and we say that p has length k. We also include empty paths $\epsilon_i,i\in Q_0$, with $t(\epsilon_i)=h(\epsilon_i):=i$. By definition, the product p_1p_2 of two pathes p_1,p_2 is zero if $h(p_2)\neq t(p_1)$ and is the concatenation of p_1 and p_2 else. For example, the path algebra of the Jordan quiver is the polynomial algebra in one variable. Another example: take the Dynkin quiver of type A_2, i.e, the quiver with two vertices, 1 and 2, and a single arrow a with $t(a)=1,h(a)=2$. The path algebra is three dimensional, its basis is ϵ_1,ϵ_2,a and the only nonzero products are $\epsilon_1^2=\epsilon_1,\epsilon_2^2=\epsilon_2,a\epsilon_1=a,\epsilon_2a=a$.

Exercise 4.3. *Show that $\mathbb{C}Q$ is associative and $\sum_{i\in Q_0}\epsilon_i$ is a unit in $\mathbb{C}Q$. Further, show that, as a unital associative algebra, $\mathbb{C}Q$ is generated by $\epsilon_i,i\in Q_0$, and $a\in Q_1$ subject to the relations $\epsilon_i\epsilon_j=\delta_{ij}\epsilon_i,\sum_{i\in Q_0}\epsilon_i=1,\epsilon_ia=\delta_{ih(a)}a,a\epsilon_i=\delta_{it(a)}a$.*

If (V_i,x_a) is a representation of Q, then the space $V=\bigoplus_i V_i$ is equipped with a unique $\mathbb{C}Q$-module structure such that ϵ_i acts by the projection to the summand V_i and a acts by x_a. Conversely, to a $\mathbb{C}Q$-module U one assigns a representation of Q with $V_i=\epsilon_iU$.

Finally, let us remark that the algebra $\mathbb{C}Q$ is graded, $\mathbb{C}Q=\bigoplus_{i=0}^{+\infty}(\mathbb{C}Q)^i$, with $(\mathbb{C}Q)^i$ being the linear span of all pathes with length i.

4.3. $\mathbb{C}Q$ **vs** $\mathbb{C}\langle x,y\rangle\#\Gamma$**.** Now we are going to relate $\mathbb{C}Q$ for the (doubled) McKay quiver Q to $\mathbb{C}\langle x,y\rangle\#\Gamma$. For this we will realize both as tensor algebras.

Namely, if we have an associative algebra A and its *bimodule* M, we can form the tensor products $M^{\otimes n} := M \otimes_A M \otimes_A \ldots \otimes_A M$ and hence also the tensor algebra $T_A(M) = \bigoplus_n M^{\otimes n}$. This algebra has a usual universal property.

To represent $\mathbb{C}\langle x, y\rangle \# \Gamma$ in this form we will take $A = \mathbb{C}\Gamma$ and $M := \mathbb{C}^2 \otimes \mathbb{C}\Gamma$, where the left A-action comes from the Γ-module structure on $\mathbb{C}^2 \otimes \mathbb{C}\Gamma$, while the right action is by right multiplications on the second factor.

To represent $\mathbb{C}Q$ in this form we take $A = (\mathbb{C}Q)^0 \cong \mathbb{C}^{Q_0}$ and $M = (\mathbb{C}Q)^1$, the span of all arrows with the bimodule structure coming from Exercise 4.3.

Exercise 4.4. *Use the universal properties of all algebras involved to show that* $\mathbb{C}\langle x, y\rangle \# \Gamma \cong T_{\mathbb{C}\Gamma}(\mathbb{C}^2 \otimes \mathbb{C}\Gamma)$ *and* $\mathbb{C}Q \cong T_{(\mathbb{C}Q)^0}(\mathbb{C}Q)^1$.

A relation between the algebras $\mathbb{C}\langle x, y\rangle \# \Gamma$ and $\mathbb{C}Q$ is as follows: there is an idempotent $f \in \mathbb{C}\Gamma$ such that $\mathbb{C}Q = f\mathbb{C}\langle x, y\rangle \# \Gamma f$. To prove this we will need to examine an interplay between the constructions of spherical subalgebras and of tensor algebras.

Now suppose that A is an arbitrary associative algebra and that $e \in A$ is an idempotent. Then we can form the spherical subalgebra eAe. The space Ae has commuting actions of A on the left and eAe on the right. We have a functor $\pi : A\text{-Mod} \to eAe\text{-Mod}$ that sends M to eM. On the other hand, consider a functor $\pi^!$ in the opposite direction given by $\pi^!(N) = Ae \otimes_{eAe} N$.

Exercise 4.5.
- *Show that* π *is an exact functor, that* π *can be written as* $M \mapsto eA \otimes_A M$, *and that* $\pi^!$ *is left adjoint to* π.
- *Suppose that* $AeA = A$. *Check the that if* $\pi(M) = 0$, *then* $M = 0$. *Further check that the natural homomorphism* $Ae \otimes_{eAe} eM \to M$ *is surjective. Finally, show that* $Ae \otimes_{eAe} eM \to M$ *is injective by applying* π.
- *Deduce that* $Ae \otimes_{eAe} eA = A$ *as bimodules.*

The previous exercise shows that the categories of modules for A and for eAe are equivalent. In this case one says that the algebras A and eAe are Morita equivalent (or that e induces a Morita equivalence).

Our goal now is to find an idempotent $f \in \mathbb{C}\langle x, y\rangle \# \Gamma$ such that $\mathbb{C}Q \cong f\mathbb{C}\langle x, y\rangle \# \Gamma f$. Recall that $\mathbb{C}\Gamma = \bigoplus_{i \in Q_0} \operatorname{End}(N_i^*)$. Pick a primitive idempotent (a.k.a. diagonal matrix unit) $f_i \in \operatorname{End}(N_i)$ and set $f := \sum_{i \in Q_0} f_i$. Then, obviously, $\mathbb{C}\Gamma f \mathbb{C}\Gamma = \mathbb{C}\Gamma$ and $f\mathbb{C}\Gamma f = \mathbb{C}^{Q_0}$. Let us compute $f(\mathbb{C}^2 \otimes \mathbb{C}\Gamma)f$. For this we will need to understand the structure of the bimodule $\mathbb{C}^2 \otimes \mathbb{C}\Gamma$ over the algebra $\mathbb{C}\Gamma = \bigoplus_{i \in Q_0} \operatorname{End}(N_i^*)$. We have $\mathbb{C}^2 \otimes \mathbb{C}\Gamma = \bigoplus_{i \in Q_0} \mathbb{C}^2 \otimes N_i \otimes N_i^*$, where the left action of Γ is on the first two factors, while the right action is on the third factor. Further, $\mathbb{C}^2 \otimes N_i = \bigoplus_j M_{ij} \otimes N_j$, where Γ acts trivially on the first factor. So $\mathbb{C}^2 \otimes \mathbb{C}\Gamma = \bigoplus_{ij} M_{ij} \otimes N_j \otimes N_i^* = \bigoplus_{ij} M_{ij} \operatorname{Hom}(N_j^*, N_i^*)$. The space $\operatorname{Hom}(N_j^*, N_i^*)$ has a natural left action of $\operatorname{End}(N_i^*)$ and a natural right action of $\operatorname{End}(N_j^*)$ that gives the structure of a $\bigoplus_{i \in Q_0} \operatorname{End}(N_i^*)$-bimodule on $\mathbb{C}^2 \otimes \mathbb{C}\Gamma$. So $f_{j'} \operatorname{Hom}(N_i^*, N_j^*) f_{i'} = \delta_{ii'}\delta_{jj'}\mathbb{C}$ and we get $f(\mathbb{C}^2 \otimes \mathbb{C}\Gamma)f = \bigoplus_{ij} M_{ij} = (\mathbb{C}Q)^1$.

The following exercise implies that $f(\mathbb{C}\langle x, y\rangle \# \Gamma)f = \mathbb{C}Q$.

Exercise 4.6. *Suppose* e *is an idempotent in* A *such that* $AeA = A$. *Show that the functor* $M \mapsto eMe$ *is an equivalence between the categories of* A *and* eAe*-bimodules intertwining the tensor products (meaning that* $e(M \otimes_A N)e = eMe \otimes_{eAe} eNe$*). Deduce that* $eT_A(M)e$ *is naturally identified* $T_{eAe}(eMe)$.

Now we can revisit the identification $\mathrm{Rep}_\Gamma(\mathbb{C}\langle x,y\rangle\#\Gamma, \mathbb{C}\Gamma)\cong \mathrm{Rep}(Q,\delta)$. The latter is the representation space of $\mathbb{C}Q$ in $\bigoplus_{i\in Q_0} N_i^*$. The isomorphism is induced by the map $M\mapsto fM$, where $f\in \mathrm{Rep}_\Gamma(\mathbb{C}\langle x,y\rangle\#\Gamma,\mathbb{C}\Gamma)$.

4.4. **Deformed preprojective algebras.** The identification of $\mathrm{Rep}_\Gamma(\mathbb{C}\langle x,y\rangle\#\Gamma,\mathbb{C}\Gamma)$ with $\mathrm{Rep}(Q,\delta)$ (as well as the isomorphism $f(\mathbb{C}\langle x,y\rangle\#\Gamma)f\cong \mathbb{C}Q$) depended on the choice of bases in the spaces N_i^* and also on the choice of bases in the spaces M_{ij}. We are going to prove that the condition that $\varphi(xy-yx)=0$ is equivalent to $\mu(\varphi)=0$ (under a suitable choice of a basis in M_{ij}).

In fact, we will prove a stronger result. Namely, recall the algebra $H_c=\mathbb{C}\langle x,y\rangle\#\Gamma/(xy-yx-c)_{\mathbb{C}\langle x,y\rangle\#\Gamma}$, where $c\in(\mathbb{C}\Gamma)^\Gamma$. Here and below the superscript $\mathbb{C}\langle x,y\rangle\#\Gamma$ after the brackets means that we take the two-sided ideal in $\mathbb{C}\langle x,y\rangle\#\Gamma$ generated by the element(s) in brackets. The algebra fH_cf is a quotient of $f(\mathbb{C}\langle x,y\rangle\#\Gamma)f=\mathbb{C}Q$ by the ideal $f(xy-yx-c)_{\mathbb{C}\langle x,y\rangle\#\Gamma}f$ and the question is how to describe the ideal explicitly. The answer is due to Crawley-Boevey and Holland and is as follows.

For $\lambda=(\lambda_i)_{i\in Q_0}$, define the *deformed preprojective algebra* Π^λ by as the quotient of $\mathbb{C}Q$ by the relations

$$\sum_{a\in \underline{Q}_1, t(a)=i} a^*a-\sum_{a\in\underline{Q}_1,h(a)=i} aa^*-\lambda_i\epsilon_i=0, \tag{1}$$

one for each $i\in Q_0$. Below we will write $[a^*,a]_i$ for $\sum_{a\in\underline{Q}_1,t(a)=i}a^*a-\sum_{a\in\underline{Q}_1,h(a)=i}aa^*$.

Theorem 4.1. *With a suitable choice with of bases in M_{ij}, the ideal $f(xy-yx-c)f$ is generated by the $[a^*,a]_i-\lambda_i\epsilon_i$, where $\lambda_i:=\mathrm{tr}_{N_i}(c)$.*

In particular, $xy-yx$ acts trivially on $M\in\mathrm{Rep}_\Gamma(\mathbb{C}\langle x,y\rangle\#\Gamma,\mathbb{C}\Gamma)$ if and only if M is annihilated by the ideal $(xy-yx)_{\mathbb{C}\langle x,y\rangle\#\Gamma}$ if and only if $fM\in\mathrm{Rep}(Q,\delta)$ is annihilated by $f(xy-yx)_{\mathbb{C}\langle x,y\rangle\#\Gamma}f$, i.e. by all elements $[a^*,a]_i\in\mathbb{C}Q$. By that element just acts by the operator $\mu_i\in\mathrm{End}(\mathbb{C}^{\delta_i})$.

4.5. **CBH lemma.** To prove Theorem 4.1 we will need a lemma from [CBH]. First, we need a concrete form of isomorphisms between $\mathrm{Hom}_\Gamma(\mathbb{C}^2\otimes N_i,N_j)$ and $\mathrm{Hom}_\Gamma(N_i,\mathbb{C}^2\otimes N_j)$. We view x,y as a basis in $\mathbb{C}^2$, identifying $\mathbb{C}^2$ with its dual via the symplectic form ω given by $\omega(y,x)=1=-\omega(x,y)$. Further, let $\zeta:=x\otimes y-y\otimes x\in\mathbb{C}^2\otimes\mathbb{C}^2$. We can view ω as a map $\mathbb{C}^2\otimes\mathbb{C}^2\to\mathbb{C}$, and ζ as a map $\mathbb{C}\to\mathbb{C}^2\otimes\mathbb{C}^2$. Both maps are Γ-equivariant if we view $\mathbb{C}$ as the trivial module.

Let M,M' be Γ-modules. Take $\psi\in\mathrm{Hom}_\Gamma(M,\mathbb{C}^2\otimes M')$. It gives rise to a map $1_{\mathbb{C}^2}\otimes\psi:\mathbb{C}^2\otimes M\to\mathbb{C}^2\otimes\mathbb{C}^2\otimes M'$. Then define $\psi^\heartsuit:=(\omega\otimes 1_M)\circ(1_{\mathbb{C}^2}\otimes\psi)\in\mathrm{Hom}_\Gamma(\mathbb{C}^2\otimes M,M')$. Conversely, we can map $\varphi\in\mathrm{Hom}_\Gamma(\mathbb{C}^2\otimes M,M')$ to $(1_{\mathbb{C}^2}\otimes\varphi)\circ(\zeta\otimes 1_M)\in\mathrm{Hom}(M,\mathbb{C}^2\otimes M')$.

Exercise 4.7. *Check that the maps* $\mathrm{Hom}(M,\mathbb{C}^2\otimes M')\to\mathrm{Hom}(\mathbb{C}^2\otimes M,M'),\psi\mapsto(\omega\otimes 1_M)\circ(1_{\mathbb{C}^2}\otimes\psi)$ *and* $\mathrm{Hom}(\mathbb{C}^2\otimes M,M')\to\mathrm{Hom}(M,\mathbb{C}^2\otimes M'),\varphi\mapsto(1_{\mathbb{C}^2}\otimes\varphi)\circ(\zeta\otimes 1_M)$ *are inverse to each other.*

The following claim is [CBH, Lemma 3.2].

Lemma 4.2. *To each $a\in\underline{Q}_1$ one can associate $\eta_a\in\mathrm{Hom}_\Gamma(N_{t(a)},\mathbb{C}^2\otimes N_{h(a)}),\theta_a\in\mathrm{Hom}_\Gamma(N_{h(a)},\mathbb{C}^2\otimes N_{t(a)})$ that combine to form bases in the spaces $\mathrm{Hom}_\Gamma(N_i,\mathbb{C}^2\otimes N_j)$ are all i,j and satisfy*

$$\sum_{a\in\underline{Q}_1,t(a)=i}(1_{\mathbb{C}^2}\otimes\theta_a)\eta_a-\sum_{a\in\underline{Q}_1,h(a)=i}(1_{\mathbb{C}^2}\otimes\eta_a)\theta_a=\delta_i(\zeta\otimes 1_{N_i}), \tag{2}$$

(the equality of maps $N_i \to \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes N_i$) for all i.

There are two rather different cases: Γ is cyclic or Γ is non-cyclic. The difference is that in the second case $\underline{Q}$ is a tree, while in the first case $\underline{Q}$ is not.

Problem 4.8. *Prove the CBH lemma in the cyclic case, assuming that the orientation on $\underline{Q}$ is also cyclic. Hint: for x, y we can take Γ-eigenvectors.*

The case when Γ is non-cyclic will be considered in the next lecture.

References

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4. Deformed preprojective algebras, cont'd

4.1. **Recap.** Recall that in the previous lecture we have identified $\mathbb{C}\langle x,y\rangle\#\Gamma$ with $T_{\mathbb{C}\Gamma}(\mathbb{C}^2\otimes\mathbb{C}\Gamma)$, and $\mathbb{C}Q$ with $T_{(\mathbb{C}Q)^0}(\mathbb{C}Q)^1$. Also recall that $f\mathbb{C}\langle x,y\rangle\#\Gamma f\cong\mathbb{C}Q$, where $f=\bigoplus_{i=0}^r f_i\in\mathbb{C}\Gamma=\bigoplus_{i=0}^r \operatorname{End}(N_i^*)$ with f_i being a primitive idempotent in $\operatorname{End}(N_i)^*$. Under this identification, $f_i\in\mathbb{C}Q$ becomes the path ϵ_i.

Further, to $i\in Q_0$ we have assigned an element $[a^*,a]_i\in\epsilon_i(\mathbb{C}Q)^2\epsilon_i$ by the formula

$$[a^*,a]_i=\bigoplus_{a\in\underline{Q}_1,t(a)=i}a^*a-\sum_{a\in\underline{Q}_1,h(a)=i}aa^*.$$

Also to $c\in(\mathbb{C}\Gamma)^\Gamma$ we assign $\lambda=(\lambda_i)_{i\in Q_0}$ by $\lambda_i=\operatorname{tr}_{N_i}c$. The main result we are going to prove is a theorem of Crawley-Boevey and Holland.

Theorem 4.1. *The ideal $f(xy-yx-c)_{\mathbb{C}\langle x,y\rangle\#\Gamma}f$ is generated by the elements $[a^*,a]_i-\lambda_i\epsilon_i, i\in Q_0$.*

A key step in the proof is the following lemma again due to Crawley-Boevey and Holland.

Lemma 4.2. *To each $a\in\underline{Q}_1$ one can associate $\eta_a\in\operatorname{Hom}_\Gamma(N_{t(a)},\mathbb{C}^2\otimes N_{h(a)}),\theta_a\in\operatorname{Hom}_\Gamma(N_{h(a)},\mathbb{C}^2\otimes N_{t(a)})$ that combine to form bases in the spaces $\operatorname{Hom}_\Gamma(N_i,\mathbb{C}^2\otimes N_j)$ are all i,j and satisfy*

$$\sum_{a\in\underline{Q}_1,t(a)=i}(1_{\mathbb{C}^2}\otimes\theta_a)\eta_a-\sum_{a\in\underline{Q}_1,h(a)=i}(1_{\mathbb{C}^2}\otimes\eta_a)\theta_a=\delta_i(\zeta\otimes 1_{N_i}),\tag{1}$$

(the equality of maps $N_i\to\mathbb{C}^2\otimes\mathbb{C}^2\otimes N_i$) for all i.

To prove the lemma we have introduced explicit mutually inverse isomorphisms of the spaces $\operatorname{Hom}_\Gamma(M,\mathbb{C}^2\otimes M'),\operatorname{Hom}_\Gamma(\mathbb{C}^2\otimes M,M')$. Namely, we map $\psi\in\operatorname{Hom}_\Gamma(M,\mathbb{C}^2\otimes M')$ to $\psi^\heartsuit:=(\omega\otimes 1_{M'})\circ(1_{\mathbb{C}^2}\otimes\psi)\in\operatorname{Hom}_\Gamma(\mathbb{C}^2\otimes M,M')$, and we map $\varphi\in\operatorname{Hom}_\Gamma(\mathbb{C}^2\otimes M,M')$ to $(1_{\mathbb{C}^2}\otimes\varphi)\circ(\zeta\otimes 1_M)\in\operatorname{Hom}(M,\mathbb{C}^2\otimes M')$. Here ω is the skew-symmetric form on $\mathbb{C}^2$ given by $\omega(y,x)=1=-\omega(x,y)$ (and viewed as a map $\mathbb{C}^2\otimes\mathbb{C}^2\to\mathbb{C}$) and $\zeta=x\otimes y-y\otimes x$ (viewed as a map $\mathbb{C}\to\mathbb{C}^2\otimes\mathbb{C}^2$).

4.2. **Proof of the CBH lemma.** From now on we concentrate on the non-cyclic case. A special feature of this case is that Γ is a tree.

The spaces $\operatorname{Hom}_\Gamma(N_i,\mathbb{C}^2\otimes N_j)$ when $i=t(a),j=h(a)$ or vice versa are 1-dimensional. For a moment, choose arbitrary nonzero η_a,θ_a, they are defined up to a nonzero scalar multiple. Then $\theta_a^\heartsuit\eta_a$ is a nonzero endomorphism of $N_{t(a)}$, while $\eta_a^\heartsuit\theta_a$ is a nonzero endomorphism of $N_{h(a)}$. Multiplying θ_a by a nonzero scalar k, we also multiply those two endomorphisms by k. We claim that there are nonzero scalars $d_i, i\in Q_0$, with the property that (after rescaling the θ_i's) we get

$$\theta_a^\heartsuit\eta_a=d_{h(a)}1_{N_{t(a)}},\eta_a^\heartsuit\theta_a=-d_{t(a)}1_{N_{h(a)}}.\tag{2}$$

This is a consequence of $\underline{Q}$ being a tree. Namely, we fix all η_a and some vertex i. Pick d_i. This fixes θ_a for all a with $h(a) = i$ (from $\theta_a^\heartsuit \eta_a = d_{h(a)} 1_{N_{t(a)}}$) or $t(a) = i$ (from $\eta_a^\heartsuit \theta_a = -d_{t(a)} 1_{N_{h(a)}}$) and so also d_j for all vertices j connected to i (for example, if $h(a) = i$, then $d_{t(a)}$ is determined from $\eta_a^\heartsuit \theta_a = -d_{t(a)} 1_{N_{h(a)}}$, where we now know the left hand side). Then we proceed with i replaced by one of these j's. Since our graph is a tree, we see that every vertex appears only ones, and when our argument finishes, we get all d_i and all θ_a fixed.

The map $\eta_a \theta_a^\heartsuit : \mathbb{C}^2 \otimes N_{h(a)} \to \mathbb{C}^2 \otimes N_{h(a)}$ therefore equals $d_{h(a)} \pi_{N_{t(a)}}$, where $\pi_{N_{t(a)}}$ is the projection to the summand $N_{t(a)}$ in $\mathbb{C}^2 \otimes N_{h(a)}$. Similarly, $\theta_a \eta_a^\heartsuit = -d_{t(a)} \pi_{N_{h(a)}}$. The modules $N_{t(a)}$ with $h(a) = i$, and $N_{h(a)}$ with $t(a) = i$ is a complete list of the simple summands of $\mathbb{C}^2 \otimes N_i$. So for $i \in Q_0$ we have

$$\sum_{a \in \underline{Q}_1, h(a)=i} \eta_a \theta_a^\heartsuit - \sum_{a \in \underline{Q}_1, t(a)=i} \theta_a \eta_a^\heartsuit = d_i 1_{\mathbb{C}^2 \otimes N_i}. \tag{3}$$

We want to tensor the previous equation with $1_{\mathbb{C}^2}$ (on the left) and compose with $\zeta \otimes 1$ (on the right) so that we get an equality of maps $N_i \to \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes N_i$. We have $(1_{\mathbb{C}^2} \otimes \theta_a \eta_a^\heartsuit) \circ (\zeta \otimes 1) = (1_{\mathbb{C}^2} \otimes \theta_a)(1_{\mathbb{C}^2} \otimes \eta_a^\heartsuit)(\zeta \otimes 1_{N_i}) = (1_{\mathbb{C}^2} \otimes \theta_a)\eta_a$. So what we get is

$$\sum_{a \in \underline{Q}_1, h(a)=i} (1_{\mathbb{C}^2} \otimes \eta_a)\theta_a - \sum_{a \in \underline{Q}_1, t(a)=i} (1_{\mathbb{C}^2} \otimes \theta_a)\eta_a = d_i \zeta \otimes 1_{N_i}.$$

To show that we can take $-\delta_i$ for d_i, we compose both sides of the equality on the left with $\omega \otimes 1$ (to get a map $N_i \to N_i$). We have $(\omega \otimes 1_{N_i})(1_{\mathbb{C}^2} \otimes \theta_a)\eta_a = \theta_a^\heartsuit \eta_a = d_{h(a)} 1_{N_i}$ and similarly $(\omega \otimes 1_{N_i})(1_{\mathbb{C}^2} \otimes \eta_a)\theta_a = -d_{t(a)} 1_{N_i}$. So on the left hand side we get $-\sum_j d_j 1_{N_i}$, where we sum over all j connected to i. On the right hand side we get $-\omega(\zeta) d_i 1_{N_i} = -2d_i 1_{N_i}$. So $2d_i - \sum_j d_j = 0$ for each $i \in Q_0$. This is a linear system whose matrix is precisely the Cartan matrix of the extended Dynkin diagram. The space of solutions of this equation is 1-dimensional and is generated by δ. Rescaling the maps θ_a, we achieve $d_i = -\delta_i$.

4.3. **Proof of Theorem 4.1.** We need to interpret (1) so that it becomes an equality in $\mathbb{C}Q = T_{(\mathbb{C}Q)^0}(\mathbb{C}Q)^1 = f\mathbb{C}\langle x, y\rangle \# \Gamma f$.

$\operatorname{Hom}_\Gamma(N_i, \mathbb{C}^2 \otimes N_j)$ is just $\operatorname{Hom}_\Gamma(\mathbb{C}\Gamma f_i, \mathbb{C}^2 \otimes \mathbb{C}\Gamma f_j) = f_j(\mathbb{C}^2 \otimes \mathbb{C}\Gamma) f_i = \epsilon_j (\mathbb{C}Q)^1 \epsilon_i$. We take η_a for a, and θ_a for a^*. Then, similarly, $\operatorname{Hom}_\Gamma(N_i, \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes N_i) = \operatorname{Hom}_\Gamma(\mathbb{C}\Gamma f_i, \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}\Gamma f_j) = f_j(\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}\Gamma) f_i = \epsilon_i (\mathbb{C}Q)^2 \epsilon_i$, and $(1 \otimes \theta_a)\eta_a \in \operatorname{Hom}_\Gamma(N_i, \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes N_i)$ is nothing else but $a^* a$.

The elements $xy - yx$ and c of $\mathbb{C}\langle x, y\rangle \# \Gamma$ are Γ-invariant and hence commute with $\mathbb{C}\Gamma$ and in particular, with the idempotents f_i. Under our identifications, the element $(xy - yx) f_i = f_i (xy - yx) f_i$ is nothing else but $\zeta \otimes 1_{N_i}$. So the CBH Lemma just says that

$$\sum_{a \in \underline{Q}_1, t(a)=i} a^* a - \sum_{a \in \underline{Q}_1, h(a)=i} a a^* = \delta_i (xy - yx) f_i$$

We remark that $\delta_i c f_i$ is precisely $\lambda_i f_i$, so

$$\sum_{a \in \underline{Q}_1, t(a)=i} a^* a - \sum_{a \in \underline{Q}_1, h(a)=i} a a^* - \lambda_i f_i = \delta_i (xy - yx - c) f_i$$

Now we notice that the left hand sides of the previous equality all lie in $f(xy - yx - c)_{\mathbb{C}\langle x,y\rangle \# \Gamma} f$. On the other hand, let us show that $f(xy - yx - c)_{\mathbb{C}\langle x,y\rangle \# \Gamma} f$ coincides with the ideal in $\mathbb{C}Q$ generated by $(xy - yx - c) f$. Recall that $\mathbb{C}\Gamma f \mathbb{C}\Gamma = \mathbb{C}\Gamma$ and so there are elements $r_j, s_j \in \mathbb{C}\Gamma$ with $\sum_j r_j f s_j = 1$. So $\sum_j r_j (xy - yx - c) f s_j = \sum_j (xy - yx - c) r_j f s_j = xy - yx - c$.

So $xy - yx - c$ lies in the ideal of $\mathbb{C}\langle x,y\rangle\#\Gamma$ generated by $(xy-yx-c)f$. It follows that $(xy-yx-c)_{\mathbb{C}\langle x,y\rangle\#\Gamma} \cap f(\mathbb{C}\langle x,y\rangle\#\Gamma)f = ((xy-yx-c)f)_{\mathbb{C}Q}$.

4.4. **Remarks and ramifications.**

4.4.1. *Orientation.* Formally, the algebra Π^λ (and the map $\mu : \mathrm{Rep}(Q,\delta) \to \mathfrak{gl}(\delta)$) depends on the orientation. Pick an arrow $b \in \underline{Q}_1$ and switch its orientation, let b_1 denote the same arrow with the inverted orientation. Let Π'^λ, μ' be constructed from this new orientation. The map $\epsilon_i \mapsto \epsilon_i, a \mapsto a, a^* \mapsto a^*, a \in \underline{Q}_1 \setminus \{b\}, b \mapsto b_1^*, b^* \mapsto -b_1$ extends to an automorphism of $\mathbb{C}Q$ that gives rise to an algebra isomorphism $\Pi^\lambda \mapsto \Pi'^\lambda$. Also this automorphism gives rise to a linear $\mathrm{GL}(\delta)$-equivariant automorphism of $\mathrm{Rep}(Q,\delta)$ that intertwines the maps μ and μ'. So our constructions do not depend on the choice of an orientation up to distinguished isomorphisms.

4.4.2. *Identification of* $\mathrm{Rep}_\Gamma(\mathbb{C}[x,y]\#\Gamma, \mathbb{C}\Gamma)/\!/\,\mathrm{GL}(\mathbb{C}\Gamma)^\Gamma$ *with* $\mathbb{C}^2/\Gamma$. We have identified

$$\mathrm{Rep}_\Gamma(\mathbb{C}[x,y]\#\Gamma, \mathbb{C}\Gamma)/\!/\,\mathrm{GL}(\mathbb{C}\Gamma)^\Gamma$$

(viewed as a set of isomorphism classes of semisimple representations) with $\mathbb{C}^2/\Gamma$. Let us show that this is an identification of algebraic varieties. Recall the spherical subalgebra $\mathbb{C}[x,y]^\Gamma \cong e(\mathbb{C}[x,y]\#\Gamma)e \subset \mathbb{C}[x,y]\#\Gamma$. An element $\varphi \in \mathrm{Rep}_\Gamma(\mathbb{C}[x,y]\#\Gamma,\mathbb{C}\Gamma)$ restricts to a representation of $e(\mathbb{C}[x,y]\#\Gamma)e$ in $e\mathbb{C}\Gamma = \mathbb{C}$. The latter is nothing else but a point of $\mathbb{C}^2/\Gamma$ and so we get another map $\xi : \mathrm{Rep}_\Gamma(\mathbb{C}[x,y]\#\Gamma,\mathbb{C}\Gamma) \to \mathbb{C}^2/\Gamma$ that is clearly $\mathrm{GL}(\mathbb{C}\Gamma)^\Gamma$-equivariant and so descends to $\mathrm{Rep}_\Gamma(\mathbb{C}[x,y]\#\Gamma,\mathbb{C}\Gamma)/\!/\,\mathrm{GL}(\mathbb{C}\Gamma)^\Gamma \to \mathbb{C}^2/\Gamma$. It is clear from the construction that it coincides with our previous map, given by taking the central character. Our new map is a morphism of algebraic varieties: the pull-back $\xi^*(F)$ of $F \in \mathbb{C}[x,y]^\Gamma$ evaluated on a representation φ is just a matrix coefficient of φ evaluated on eFe (or F) and so $\xi^*(F)$ is a polynomial on $\mathrm{Rep}_\Gamma(\mathbb{C}[x,y]\#\Gamma,\mathbb{C}\Gamma)$.

4.4.3. *Scheme structure on* $\mu^{-1}(0)/\!/G$. Before we have viewed $\mu^{-1}(0)/\!/G$ as a variety, but in fact, it is an affine scheme with the following algebra of functions: $(\mathbb{C}[R]/\mathbb{C}[R]\mu^*(\mathfrak{g}))^G$, where we write G for $\mathrm{GL}(\delta)$, $\mathfrak{g}$ for $\mathfrak{gl}(\delta)$ and R for $\mathrm{Rep}(Q,\delta)$. Here $\mu^* : \mathfrak{g} \to \mathbb{C}[R]$ is a pull-back map induced by $\mu : R \to \mathfrak{g}$ (we identify $\mathfrak{g}$ with $\mathfrak{g}^*$ via the trace pairings on all $\mathfrak{gl}_{\delta_i}(\mathbb{C})$).

It turns out however, that even the subscheme $\mu^{-1}(0) \subset R$ (not only the quotient $\mu^{-1}(0)/\!/G$) is reduced (and is a complete intersection). This follows from the claim that every irreducible component of $\mu^{-1}(0)$ contains a G-orbit with trivial stabilizer (and also from some standard properties of moment maps). The claim about the existence of an orbit follows from the representation theory of quivers.

Problem 4.1. *Check the claims of the previous paragraph by hand in the case of the cyclic quiver Q.*

4.4.4. $\mathrm{Rep}_\Gamma(H_c,\mathbb{C}\Gamma)$. We have identified $\mu^{-1}(0)/\!/\,\mathrm{GL}(\delta)$ with $\mathrm{Rep}_\Gamma(H_c,\mathbb{C}\Gamma)/\!/\,\mathrm{GL}(\mathbb{C}\Gamma)^\Gamma$ using Theorem 4.1 with $c = 0$. Let us see what happens for arbitrary c.

First of all, we claim that H_c has no representations in $\mathbb{C}\Gamma$ if $c_1 \neq 0$. Indeed, $xy - yx$ acts in the same way as c. In particular, $c \in (\mathbb{C}\Gamma)^\Gamma$ has trace 0 on any H_c-module. But the trace of c on $\mathbb{C}\Gamma$ is $|\Gamma| c_1$. From now on, we consider the case $c_1 = 0$.

Thanks to Theorem 4.1, $\mu^{-1}(\lambda)/\!/\,\mathrm{GL}(\delta)$ is identified with $\mathrm{Rep}_\Gamma(H_c,\mathbb{C}\Gamma)/\!/\,\mathrm{GL}(\mathbb{C}\Gamma)^\Gamma$. Similarly to a remark above, we have a morphism $\mathrm{Rep}_\Gamma(H_c,\mathbb{C}\Gamma)/\!/\mathbb{C}\Gamma \to \mathrm{Rep}(eH_ce,\mathbb{C})$. We will see below that if $c_1 = 0$, then the algebra eH_ce is commutative. So $\mathrm{Rep}(eH_ce,\mathbb{C}) = \mathrm{Spec}(eH_ce)$. Also one can show that the morphism $\mathrm{Rep}_\Gamma(H_c,\mathbb{C}\Gamma)/\!/\mathbb{C}\Gamma \to \mathrm{Rep}(eH_ce,\mathbb{C})$ is an isomorphism. We are not going to do this, but we will prove that $\mu^{-1}(\lambda)/\!/ \operatorname{GL}(\delta) \cong \operatorname{Spec}(eH_{c'}e)$ (perhaps for a different c'). Also we will see that the map $z \mapsto ez$ from the center $Z(H_c)$ of H_c to eH_ce is an isomorphism of algebras.

4.4.5. *Deformations of* $\mathbb{C}^2/\Gamma$. Finally, let us remark that the algebras $\mathbb{C}[\mu^{-1}(\lambda)/\!/G]$ with $\sum_{i=0}^r \delta_i\lambda_i = 0$ (this is equivalent to $c_1 = 0$) form an r-parametric deformation of $\mathbb{C}^2/\Gamma$. Of course, all deformations obtained in this way are commutative. We will see below that one can produce non-commutative using a related construction called a *quantum Hamiltonian reduction.*

5. Symplectic quotient singularities

5.1. **Quotient singularities.** We are interested in studying deformations of the algebras of the form $S(V)^\Gamma$, where V is a vector space, $S(V)$ is its symmetric algebra, and Γ is a finite subgroup of $\operatorname{GL}(V)$. This algebra is the algebra of polynomial functions on the quotient V^*/Γ. In fact, to get some non-trivial theory we will need to restrict the class of groups we are dealing with. Before we considered $\Gamma \subset \operatorname{SL}_2(\mathbb{C})$ so our first guess would be that we need $\Gamma \subset \operatorname{SL}(V)$. However, this class is still too large. We will consider the case when V is a symplectic vector space, i.e., possesses a non-degenerate skew-symmetric form, say ω, and Γ preserves this form, i.e., lies in the symplectic group $\operatorname{Sp}(V)$.

5.2. **Poisson brackets.** As usual, one of the reasons why we make this restriction is that this situation is easier. But there is a reason for that too. Roughly speaking, a (non-commutative) deformation of a commutative algebra gives rise to a new structure on this algebra, a *Poisson bracket*, and, for $\Gamma \subset \operatorname{Sp}(V)$, the algebra $S(V)^\Gamma$ already comes equipped with such a bracket.

Let us start with a general definition of a (Poisson bracket). Let A be a commutative associative unital algebra. A *bracket* on A is a skew-symmetric $\mathbb{C}$-bilinear map $\{\cdot,\cdot\} : A\times A \to A$ satisfying the following two axioms, known as the Leibniz identity:

(L) $\{a, bc\} = \{a, b\}c + \{a, c\}b$,

for all $a, b, c \in A$. We remark, that thanks to $\{\cdot,\cdot\}$ being skew-symmetric, we have also $\{ab, c\} = a\{b, c\} + b\{a, c\}$. A bracket is called *Poisson* if it satisfies the Jacobi identity

(J) $\{a, \{b, c\}\} + \{b, \{c, a\}\} + \{c, \{a, b\}\} = 0$,

An algebra equipped with a Poisson bracket is called a Poisson algebra.

Exercise 5.1. *Let A be a commutative associative unital algebra.*

1. *Let A be equipped with a bracket $\{\cdot,\cdot\}$. Show that $\{1, a\} = 0$ for all $a \in A$.*
2. *Show that if $a_1, \ldots, a_k$ are generators of A, then there is at most one bracket $\{\cdot,\cdot\}$ with given $\{a_i, a_j\}$. Show that this bracket satisfies the Jacobi identity for all a, b, c, if it does so for all a_i, a_j, a_k.*
3. *Finally, prove that if $A = \mathbb{C}[a_1, \ldots, a_k]$, then a bracket exists for any values of $\{a_i, a_j\}$ as long as $\{a_i, a_j\} = -\{a_j, a_i\}$.*

Let us proceed to examples. Let V, ω and Γ be as above. Define $\{\cdot,\cdot\}$ on $S(V)$ by setting $\{u, v\} := \omega(u, v)$ for $u, v \in V$ and extending this bracket to the whole $S(V)$ in a unique possible way. By the previous exercise we get a Poisson bracket, since $\{\{u, v\}, w\} = 0$ for all $u, v, w \in V$.

Exercise 5.2. *We can choose a basis $x_1, \ldots, x_n, y_1, \ldots, y_n$ in V so that $\omega(x_i, x_j) = \omega(y_i, y_j) = 0, \omega(y_i, x_j) = \delta_{ij}$. Let us identify $S(V)$ with $\mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n]$. Then $\{\cdot, \cdot\}$ is given by the formula*

$$\{f, g\} = \sum_{i=1}^{n} \frac{\partial f}{\partial y_i} \frac{\partial g}{\partial x_i} - \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial y_i}.$$

Now, since $\Gamma \subset \mathrm{Sp}(V)$ we have $\omega(\gamma u, \gamma v) = \omega(u, v)$ and hence $\{\gamma u, \gamma v\} = \{u, v\}$. We deduce that γ leaves the bracket on $S(V)$ invariant, i.e., $\{\gamma f, \gamma g\} = \gamma\{f, g\}$ for all $\gamma \in \Gamma, f, g \in S(V)$. In particular, the subalgebra of invariants $S(V)^\Gamma$ is closed under $\{\cdot, \cdot\}$ and so it becomes a Poisson algebra.

Let us make a remark regarding a compatibility between the brackets and gradings. Assume that A is graded, $A = \bigoplus_{n=0}^{\infty} A^n$. We say that $\{\cdot, \cdot\}$ has degree $-d$ if $\{A^i, A^j\} \subset A^{i+j-d}$. For example, the Poisson bracket on $S(V)$ (and hence also on $S(V)^\Gamma$ has degree -2).

Finally, let us discuss a connection between Poisson brackets and (filtered) deformations. Let $\mathcal{A}$ be a filtered (associative unital) algebra. Assume that $A := \operatorname{gr} \mathcal{A}$ is commutative. This means that $[\mathcal{A}^{\leqslant i}, \mathcal{A}^{\leqslant j}] \subset \mathcal{A}^{i+j-1}$. Let us pick a positive integer d such that $[\mathcal{A}^{\leqslant i}, \mathcal{A}^{\leqslant j}] \subset \mathcal{A}^{i+j-d}$. We can define a bracket of degree $-d$ on A in a way similar to the definition of the product. Namely, pick $a \in A^i, b \in A^j$ and lift them to elements $\bar{a} \in \mathcal{A}^{\leqslant i}, \bar{b} \in \mathcal{A}^{\leqslant j}$. Then set $\{a, b\} := [\bar{a}, \bar{b}] + \mathcal{A}^{\leqslant i+j-d-1}$, this is an element of A^{i+j-d}.

Exercise 5.3. *Check that $\{\cdot, \cdot\}$ on $A = \operatorname{gr} \mathcal{A}$ is well-defined and is indeed a Poisson bracket.*

LECTURES ON SYMPLECTIC REFLECTION ALGEBRAS

IVAN LOSEV

6. Symplectic reflection algebras

6.1. **Definition of SRA.** Let V be a finite dimensional complex vector space equipped with a non-degenerate skew-symmetric form ω. Then there is a distinguished filtered deformation of the symmetric algebra $S(V)$. This is the Weyl algebra $W(V) = T(V)/(u\otimes v - v\otimes u - \omega(u,v))$.

Exercise 6.1. *Show that $W(V)$ is a filtered deformation of $S(V)$ (the case $\dim V = 2$ was considered above). Moreover, check that the Poisson bracket on $S(V)$ induced from $W(V)$ coincides with the initial bracket.*

Let Γ be a finite subgroup of $\mathrm{Sp}(V)$. We want to study filtered deformations of $S(V)^\Gamma$ that are compatible with the Poisson bracket on $S(V)^\Gamma$. For non-commutative deformations, this means that the bracket on $S(V)^\Gamma$ induced by the deformation has to coincide with (or be proportional to) the initial bracket on $S(V)^\Gamma$. One can state a compatibility condition for commutative deformations as well (they itself have to be Poisson algebras) but we are not going to do this.

As before, we are going to produce deformations of $S(V)\#\Gamma$ first (and then pass to spherical subalgebras to get deformations of $S(V)^\Gamma$, we will recall how this is done below). As in the case when $\dim V = 2$, we have $S(V)\#\Gamma = T(V)\#\Gamma/(u\otimes v - v\otimes u \,|\, u, v\in V)$. So we can take a linear map $\kappa : \bigwedge^2 V \to (T(V)\#\Gamma)^{\leqslant 1}$ and form the quotient $H_\kappa = T(V)\#\Gamma/(u\otimes v - v\otimes u - \kappa(u,v))$. Of course, $H_0 = S(V)\#\Gamma$.

Exercise 6.2. *Show that if $\operatorname{gr} H_\kappa = S(V)\#\Gamma$, then κ is a Γ-equivariant map (where Γ acts on $\mathbb{C}\Gamma$ via the adjoint representation). Furthermore, show that if $-1_V \in \Gamma$, then the image of κ lies in $\mathbb{C}\Gamma$.*

In general, it is still a good idea to consider only $\kappa : \bigwedge^2 V \to \mathbb{C}\Gamma$. This is motivated, in part, by our compatibility condition of Poisson brackets: we want filtered deformations $\mathcal{A}$ of $S(V)^\Gamma$ with $[\mathcal{A}^{\leqslant i}, \mathcal{A}^{\leqslant j}] \subset \mathcal{A}^{i+j-2}$. Of course, H_κ deforms $S(V)\#\Gamma$, not $S(V)^\Gamma$, but it is still reasonable to require that $[u,v] \in H_\kappa^{\leqslant 0}$ (that should be equal to $\mathbb{C}\Gamma$). So below we only consider Γ-equivariant κ with image in $\mathbb{C}\Gamma$. We can write κ as $\sum_{\gamma\in\Gamma}\kappa_\gamma\gamma$, where $\kappa_\gamma \in \bigwedge^2 V^*$. It turns out that for $\operatorname{gr} H_\kappa = S(V)\#\Gamma$ some κ_γ must vanish.

We map V to H_κ via $V \hookrightarrow T(V)\#\Gamma \twoheadrightarrow H_\kappa$, this is an embedding whenever $\operatorname{gr} H_\kappa = S(V)\#\Gamma$. In H_κ we must have $[[u,v],w] + [[v,w],u] + [[w,u],v] = 0$ for all $u,v,w\in V$, equivalently,

$$[\kappa(u,v),w] + [\kappa(v,w),u] + [\kappa(w,u),v] = 0. \tag{1}$$

Exercise 6.3. *We have $[\kappa(u,v),w] = \sum_{\gamma\in\Gamma}\kappa_\gamma(u,v)(\gamma(w)-w)\gamma$.*

Since $\operatorname{gr} H_\kappa = S(V)\#\Gamma$, the map $V\otimes\mathbb{C}\Gamma \to H_\kappa$ induced by the embedding $V\otimes\mathbb{C}\Gamma \to T(V)\#\Gamma$ is injective. It follows that the equalities

$$\kappa_\gamma(u,v)(\gamma(w)-w) + \kappa_\gamma(v,w)(\gamma(u)-u) + \kappa_\gamma(w,u)(\gamma(v)-v) = 0 \tag{2}$$

hold for any $\gamma \in \Gamma$. Let us show that if $\mathrm{rk}(\gamma - 1_V) > 2$, then $\kappa_\gamma = 0$. Indeed, if u, v, w are such that $\gamma(u) - u, \gamma(v) - v, \gamma(w) - w$ are linearly independent, then we must have $\kappa_\gamma(u, v) = \kappa_\gamma(v, w) = \kappa_\gamma(w, u) = 0$. But the linear independence definitely holds for vectors in general position provided $\mathrm{rk}(\gamma - 1_V) > 2$, so κ is 0.

Exercise 6.4. *Show that* $\mathrm{im}(\gamma - 1_V) \oplus \ker(\gamma - 1_V) = V$ *for any* $\gamma \in \Gamma$. *Further, show that these space are orthogonal with respect to* ω *and, in particular, the restrictions of* ω *to these subspaces are non-degenerate.*

Now consider the case when $\mathrm{rk}(\gamma - 1_V) = 2$. Suppose that in (2) $\gamma(w) = w$. Then $\kappa_\gamma(w, v)(\gamma(u) - u) = \kappa_\gamma(w, u)(\gamma(v) - v)$. If $\gamma(u) - u, \gamma(v) - v$ are linearly independent (that is true for u, v in general position), then $\kappa_\gamma(w, v) = 0$. It follows that κ_γ is proportional, say with coefficient c_γ, to the form ω_γ defined by $\omega_\gamma = \omega$ on $\mathrm{im}(\gamma - 1_V)$ and $\ker(\gamma - 1_V) \subset \ker \omega_\gamma$.

Exercise 6.5. *Show that* $\omega_\gamma(u, v)(\gamma(w) - w) + \omega_\gamma(v, w)(\gamma(u) - u) + \omega_\gamma(w, u)(\gamma(v) - v) = 0$.

Finally, let us consider κ_1. This is a Γ-invariant skew-symmetric form on V. We say that V is *symplectically irreducible* if there is no Γ-invariant symplectic subspace, i.e. a subspace with non-degenerate restriction of ω.

Exercise 6.6. *Show that a symplectically irreducible* Γ*-module* V *is either irreducible, or is the sum* $U \oplus U^*$*, where* U *is irreducible and not symplectic. Deduce that the space of* Γ*-invariant skew-symmetric forms on* V *is one-dimensional and so is generated by* ω.

So we have $\kappa(u, v) = t\omega(u, v) + \sum_{s \in S} c_s \omega_s(u, v)s$, where S is the set of all $s \in \Gamma$ with $\mathrm{rk}(\gamma - 1_V) = 2$, such s are called *symplectic reflections*, we will later explain a reason for this name. We remark that S is a union of conjugacy classes.

We should have $\kappa(\gamma u, \gamma v) = \gamma \kappa(u, v)\gamma^{-1}$, equivalently,

$$\sum_{s \in S} c_s \omega_s(\gamma u, \gamma v)s = \sum_{s \in S} c_s \omega_s(u, v)\gamma s \gamma^{-1} = \sum_{s \in S} c_{\gamma^{-1}s\gamma} \omega_{\gamma^{-1}s\gamma}(u, v)s.$$

It is straightforward to see that $\omega_{\gamma^{-1}s\gamma}(u, v) = \omega_s(\gamma u, \gamma v)$ – both forms are just projections of ω to $\mathrm{im}(\gamma^{-1}s\gamma - 1_V)$. So the condition that κ is Γ-equivariant translates to $c_s = c_{\gamma^{-1}s\gamma}$ for all $\gamma \in \Gamma, s \in S$, i.e., the map $s \mapsto c_s$ has to be Γ-invariant.

Now we are ready to give the definition of a Symplectic reflection algebra due to Etingof and Ginzburg, [EG]. Take $\Gamma \subset \mathrm{Sp}(V)$ (we do not require now that V is symplectically irreducible). Pick a complex number t and a conjugation invariant function $c \mapsto c_s : S \to \mathbb{C}$. Then set

$$H_{t,c} = T(V)\#\Gamma / \left(u \otimes v - v \otimes u - t\omega(u, v) - \sum_{s \in S} c_s \omega_s(u, v)s \right).$$

This is a SRA. We will see below that $\mathrm{gr}\, H_{t,c}$ is indeed $S(V)\#\Gamma$.

6.2. **Symplectic reflection groups.** In the definition of a SRA only symplectic reflections matter. More precisely, consider the subgroup Γ' of Γ generated by the symplectic reflections. This is a normal subgroup. Let $H'_{t,c}$ be the SRA defined for Γ'. Then $H_{t,c}$ is naturally identified with the algebra $H'_{t,c}\#_{\Gamma'}\Gamma$ defined as follows: as a vector space $H'_{t,c}\#_{\Gamma'}\Gamma$ is $H'_{t,c} \otimes_{\mathbb{C}\Gamma'} \mathbb{C}\Gamma$, where we view $\mathbb{C}\Gamma$ as a $\mathbb{C}\Gamma'$-module with respect to the left action of Γ' and $H'_{t,c}$ with respect to the right action. The algebra structure is introduced by the analogy with the usual smash-product.

This is why people usually consider the SRA only for *symplectic reflection groups*, i.e., groups generated by symplectic reflections. The full classification of such groups is known, see [C]. We will need only two classes of such groups.

First, we remark that any non-unit element of Γ is a symplectic reflection provided $\dim V = 2$. So any Kleinian group is a symplectic reflection group. This example has a higher dimensional generalization: wreath-product groups. Namely, let $L = \mathbb{C}^2$, with a symplectic form ω_L. Set $V := L^{\oplus n}, \omega = \omega_L^{\oplus n}$. Pick a Kleinian group Γ_1. For $\Gamma(= \Gamma_n)$ we take the semi-direct product $\mathfrak{S}_n \ltimes \Gamma_1^n$. It acts on V as follows. Let $\gamma_{(i)}$ denote the element $\gamma \in \Gamma_1$ in the ith copy of Γ_1. It acts as γ on the ith copy of L and as 1 on the other copies. The subgroup $\mathfrak{S}_n \subset \Gamma$ acts by permuting the copies of L. It is clear that this group preserves ω.

Let us describe the symplectic reflections in Γ_n. First of all, any $\gamma_{(i)}$ is a symplectic reflection. Clearly, $\gamma_{(i)}$ and $\gamma'_{(j)}$ are conjugate if and only if γ and γ' are conjugate in Γ_1. So we have r conjugacy classes $S_1, \ldots, S_r$ of symplectic reflection in Γ_n, one per non-trivial conjugacy class in Γ_1. For $n > 1$, there is yet another class of symplectic reflections: it contains a transposition s_{ij} from $\mathfrak{S}_n$. This class, S_0, consists of the elements of the form $s_{ij}\gamma_{(i)}\gamma_{(j)}^{-1}$, where s_{ij} is the transposition in $\mathfrak{S}_n$ permuting the elements with indexes i and j.

Exercise 6.7. *Prove that $S_0, \ldots, S_r$ exhaust the conjugacy classes of symplectic reflections in Γ_n.*

In any case, we see that Γ_n is generated by symplectic reflections. Also it is clear that V is symplectically irreducible.

Another family we are going to consider is obtained from complex reflection groups. Namely, recall that by a *complex reflection* in the group $\mathrm{GL}(\mathfrak{h})$, where $\mathfrak{h}$ is a finite dimensional vector space, we mean an element s of finite order such that $\mathrm{rk}(s - 1_{\mathfrak{h}}) = 1$. Complex reflection groups (=finite groups generated by complex reflections) were classified in [ST]. They include all real reflection groups and, in particular, all Weyl groups.

Problem 6.8. *The goal of this problem is to construct a family of complex reflection groups, that includes Weyl groups of types B, D. Namely, fix $n, \ell \geqslant 1$ and a divisor r of ℓ. Consider all $n \times n$-matrices with the following properties: each column and each row contains a single non-zero element that is a root of 1 of order ℓ, and the product of the nonzero elements is a root of 1 of order r. Show that this is a complex reflection group. This group is denoted by $G(\ell, r, n)$. In particular, $B_n = G(2, 1, n)$ and $D_n = G(2, 2, n)$.*

In fact, there are only finitely many complex reflection groups different from $G(\ell, r, n)$.

Problem 6.9. *This problem concerns one exceptional complex reflection group, G_4. Take the Kleinian group Γ of type E_6. It has three two-dimensional irreducible representations, $\mathbb{C}^2$ and two other. Prove that the other two are dual to each other and Γ acts on them as a complex reflection group.*

Now let us explain how to produce a symplectic reflection group from a complex reflection group W (this construction is a kind of a justification of the name "symplectic reflection"). Set $V := \mathfrak{h} \oplus \mathfrak{h}^*$. We have a natural symplectic form on V: $\omega(\alpha, \beta) = \omega(a, b) = 0$ for $\alpha, \beta \in \mathfrak{h}^*, a, b \in \mathfrak{h}$, while $\omega(a, \alpha) = \langle a, \alpha \rangle$. For Γ we take the image of W in $\mathrm{GL}(\mathfrak{h}) \times \mathrm{GL}(\mathfrak{h}^*)$, i.e., $w(a, \alpha) = (wa, w\alpha)$ for $w \in W, a \in \mathfrak{h}, \alpha \in \mathfrak{h}^*$. Clearly, Γ preserves ω. An element $s \in \Gamma$ is a symplectic reflection if and only if the same element is a complex reflection in W.

The SRA corresponding to W has a special name, a Rational Cherednik algebra, shortly RCA (in the case when W is a Weyl group, this algebra is a "rational degeneration" of a certain algebra, a double affine Hecke algebra, introduced by Cherednik).

Problem 6.10. *Show that the relations for a RCA can be written as follows. For a complex reflection s let $\alpha_s \in \operatorname{im}(s - 1_{\mathfrak{h}^*}), \alpha_s^\vee \in \operatorname{im}(s - 1_{\mathfrak{h}})$ be such that $\langle \alpha_s, \alpha_s^\vee \rangle = 2$ (this is motivated by the Weyl group case). Show that the relations for the RCA can be written as*

$$[x, x'] = 0, [y, y'] = 0, [y, x] = t\langle y, x\rangle - \sum_{s \in S} c_s \langle \alpha_s, y\rangle \langle \alpha_s^\vee, x\rangle s, \quad x, x' \in \mathfrak{h}^*, y, y' \in \mathfrak{h}.$$

Note that here the coefficients c_s are not quite the same as in the presentation of an SRA. How are the coefficients related?

We remark that the two classes intersect. The intersection is the family $G(\ell, 1, n) = (\mathbb{Z}/\ell\mathbb{Z})_n$. Formally, here we can also take $\ell = 1$, the corresponding group is S_n. This case is somewhat degenerate, in particular, this group is not symplectically irreducible (when $\mathfrak{h} = \mathbb{C}^n$).

6.3. **Universal deformation.** In our proof of $\operatorname{gr} H_{t,c} = S(V)\#\Gamma$, we will use a different formalism. Namely, we will consider the universal SRA H obtained as follows. Let $S_1, \ldots, S_m$ be all classes of symplectic reflections in Γ. Pick independent variables $c_1, \ldots, c_m$, one for each conjugacy class S_i, and also an independent variable t. Consider the vector space P with basis $t, c_1, \ldots, c_m$. Then H will be the algebra over $S(P)$ defined by the same generators and relations as the usual SRA, i.e.,

$$H = S(P) \otimes T(V)\#\Gamma/(u \otimes v - v \otimes u - t\omega(u, v) - \sum_{i=1}^m c_i \sum_{s \in S_i} \omega_s(u, v)s).$$

An advantage of this setting is that the algebra H is graded: with Γ in degree 0, V in degree 1, and the parameter space P in degree 2. All algebras $H_{t',c'}$ for numerical t', c' have the form $\mathbb{C} \otimes_{S(P)} H$, where the epimorphism $S(P) \twoheadrightarrow \mathbb{C}$ given by $t \mapsto t', c_i \mapsto c_i'$. Also the filtration on $H_{t',c'}$ is induced from the grading on H.

Now let us define the notion of a *graded deformation* (this is a special case of the general notion of a deformation). Let $A = \bigoplus_{i=0}^\infty A^i$ be a graded commutative algebra with $A^0 = \mathbb{C}$ and $\mathfrak{m}$ be the augmentation ideal $\bigoplus_{i=1}^\infty A^i$. Let B be a graded A-algebra (i.e., we have a graded algebra homomorphism $A \to B$). Set $B_0 = B/B\mathfrak{m}$. We say that B is a graded deformation of B_0 (over A), if B is a free A-module (this is, in fact, equivalent to flatness in this setting).

Exercise 6.11. *Suppose B is a graded deformation of B_0 over A. Show that a basis in B can be obtained as follows. Let ι be any graded section of the projection $B \twoheadrightarrow B_0$. Then for a basis of B (viewed as an A-module), we can take $\iota(B_0)$.*

We will see that H is a graded deformation of A. Moreover, we will see that (under the restriction that V is symplectically irreducible) H satisfies a certain universality property, a precise statement will be given below.

To prove this claim we will use a usual apparatus in the deformation theory, Hochschild cohomology. We will introduce them today. In the next lecture we will explain their connection to the deformation theory and then compute the relevant cohomology of $S(V)\#\Gamma$. With this computation, to deduce the claims of the previous paragraph will be relatively easy. Our main reference for Hochschild cohomology will be [E].

6.4. **Hochshild cohomology.** Let A be an associative algebra with unit and let M be an A-bimodule. By a Hochschild n-cochain with coefficients in M we mean a $\mathbb{C}$-linear map $A^{\otimes n} \to M$. The space of such cochains is denoted by $C^n(A, M)$. There is a map $d : C^n(A, M) \to C^{n+1}(A, M)$ given by

$$\begin{aligned} df(a_1 \otimes a_2 \otimes \ldots \otimes a_{n+1}) &= f(a_1 \otimes \ldots \otimes a_n)a_{n+1} - f(a_1 \otimes \ldots \otimes a_{n-1} \otimes a_n a_{n+1}) + \\ &+ f(a_1 \otimes \ldots \otimes a_{n-1}a_n \otimes a_{n+1}) - \ldots + (-1)^n f(a_1 a_2 \otimes a_3 \ldots \otimes a_{n+1}) + \\ &+ (-1)^{n+1} a_1 f(a_2 \otimes a_3 \otimes \ldots \otimes a_{n+1}). \end{aligned}$$

Exercise 6.12. *Check that $d^2 = 0$.*

So the cochains $C^n(A, M)$ form a complex $C^0(A, M) \xrightarrow{d} C^1(A, M) \xrightarrow{d} C^2(A, M) \xrightarrow{d} \ldots$ called (not surprisingly) the Hochschild complex. Its cohomology, denoted by $\mathrm{HH}^i(A, M)$ (or simply $\mathrm{HH}^i(A)$ if $A = M$), are called the Hochshild cohomology.

Exercise 6.13. *Show that $\mathrm{HH}^0(A, M)$ coincides with the center of M, i.e., the space of all elements $m \in M$ such that $am = ma$. Show that Hochshild 1-cocycles are the derivations of M, i.e., the maps $A \to M$ that satisfy the Leibniz identity, while the Hochshild 1-coboundaries are inner derivations, i.e., maps $A \to M$ of the form $a \mapsto am - ma$ for some $m \in M$. So $\mathrm{HH}^1(A, M)$ is the quotient of the two, the so called space of outer derivations.*

The groups that are relevant for the deformation theory are mostly $\mathrm{HH}^2(A)$ and $\mathrm{HH}^3(A)$.

The definition of the Hochschild cohomology may look strange. However, this is a special case of a more general construction, the Ext groups. Let us recall some generalities.

Let B be an associative algebra. A B-module P is called projective, if it is a direct summand of a free B-module. An alternative characterization: P is projective if the functor $\mathrm{Hom}_B(P, \bullet)$ is exact. For a B-module M we have a projective (for example, free) resolution $\to P_i \to P_{i-1} \to \ldots \to P_0$. Then given another B-module N, we can form the Ext groups $\mathrm{Ext}^i(M, N)$, by definition $\mathrm{Ext}^i(M, N)$ is the ith cohomology group of the complex $\mathrm{Hom}(P_0, N) \to \mathrm{Hom}(P_1, N) \to \ldots$. This definition is independent of the choice of a resolution.

Lemma 6.1. $\mathrm{HH}^i(A, M) = \mathrm{Ext}^i_{A-Bimod}(A, M)$.

Proof. Consider a free resolution of the A-bimodule A (called the standard resolution). It is given by $P_i = A^{\otimes i+2}$, where we view $A^{\otimes i+2}$ as a bimodule using external actions: $b(a_1 \otimes \ldots \otimes a_{i+2})c = ba_1 \otimes a_2 \otimes \ldots a_{i+1} \otimes a_{i+2}c$, for a basis of the A-bimodule $A^{\otimes i+2}$ we can take the basis of $1 \otimes A^{\otimes i} \otimes 1$ over $\mathbb{C}$. The differential is defined by

$$d(a_1 \otimes \ldots \otimes a_{i+2}) := a_1 \otimes a_2 \otimes \ldots \otimes a_{i+1}a_{i+2} - a_1 \otimes a_2 \otimes \ldots \otimes a_i a_{i+1} \otimes a_{i+2} + \ldots.$$

To see that we indeed get a resolution one considers the homotopy $h : A^{\otimes i+1} \to A^{\otimes i+2}$ given by $h(x) = 1 \otimes x$.

We can identify $C^n(A, M)$ with $\mathrm{Hom}_{A-Bimod}(A^{\otimes n+2}, M)$ by sending $\varphi \in \mathrm{Hom}_{A-Bimod}(A^{\otimes n+2}, M)$ to the map $\Phi : A^{\otimes n} \to M$ given by $\Phi(x) = \varphi(1 \otimes x \otimes 1)$. It is easy to see that the Hochschild differential becomes identified with one induced from the standard resolution. $\square$

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LECTURES ON SYMPLECTIC REFLECTION ALGEBRAS

IVAN LOSEV

7. Hochschild cohomology and deformations

In the previous lecture we have introduced Hochschild cohomology. These were cohomology of the complex of the spaces $C^n(A,M)=\operatorname{Hom}_{\mathbb{C}}(A^{\otimes n},M)$ with differentials

$$\begin{aligned}
df(a_1\otimes a_2\otimes\ldots\otimes a_{n+1}) &= f(a_1\otimes\ldots\otimes a_n)a_{n+1}-f(a_1\otimes\ldots\otimes a_{n-1}\otimes a_na_{n+1})+\\
&+f(a_1\otimes\ldots\otimes a_{n-1}a_n\otimes a_{n+1})-\ldots+(-1)^nf(a_1a_2\otimes a_3\ldots\otimes a_{n+1})+\\
&+(-1)^{n+1}a_1f(a_2\otimes a_3\otimes\ldots\otimes a_{n+1}).
\end{aligned}$$

We also have seen that $\operatorname{HH}^n(A,M)=\operatorname{Ext}^n(A,M)$.

Today we will see that that if A_0 is a $\mathbb{Z}_{\geqslant 0}$-graded algebra, then the space $\operatorname{HH}^n(A_0):=\operatorname{HH}^n(A_0,A_0)$ can be basically $\mathbb{Z}$-graded. Then we will see that the graded *1st order* deformations are described by the group $\operatorname{HH}^2(A_0)^{-2}$. Next, we will show that if $\operatorname{HH}^3(A_0)^i=0$ for all $i\leqslant -4$, then all deformations are *unobstructed*. From here, modulo $\operatorname{HH}^2(A_0)^i=0$ for $i<-2$, we will deduce the existence of some *universal* graded deformation of A over the algebra $S\left((\operatorname{HH}^2(A_0)^{-2})^*\right)$. After that we will compute the relevant cohomology groups for $S(V)\#\Gamma$, where Γ is a finite subgroup of $\operatorname{Sp}(V)$. We will see that $\operatorname{HH}^3(S(V)\#\Gamma)^i=0$ for $i\leqslant -4$ and $\operatorname{HH}^3(S(V)\#\Gamma)^i=0$ for $i\leqslant -3$, while $\operatorname{HH}^2(S(V)\#\Gamma)^{-2}$ has dimension $m+1$, where m is the number of conjugacy classes of symplectic reflections in Γ, and $\operatorname{HH}^2(S(V)\#\Gamma)^i=0$ for $i<-2$. Then to prove that the universal SRA H is the universal deformation will be relatively easy. This will be done in the next lecture.

7.1. **Hochschild cohomology of graded algebras.** Let A be a $\mathbb{Z}$-graded algebra and M be a $\mathbb{Z}$-graded A-bimodule (meaning that the structure map $A\otimes M\otimes A\to M$ preserves the gradings, where we set $(A\otimes M\otimes A)^n:=\bigoplus_{i+j+k=m}A^i\otimes M^j\otimes A^k$). The space $A^{\otimes n}$ is also naturally graded. Set

$$C^n(A,M)^m:=\{f:A^{\otimes n}\to M|f\left((A^{\otimes n})^i\right)\subset M^{i+m},\forall i\}.$$

The formula for the differential implies that $d(C^n(A,M)^m)\subset C^{n+1}(A,M)^m$. Of course, in general, $C^n(A,M)$ does not need to coincide $\bigoplus_{m\in\mathbb{Z}}C^n(A,M)^m$ (basically, the former space is very large if A is infinite dimensional; it looks rather like the direct product of its graded components). We will modify the definition of $C^n(A,M)$ so that it coincides with $\bigoplus_{m\in\mathbb{Z}}C^n(A,M)^m$ and also modify the definition of $\operatorname{HH}^n(A,M)$ accordingly. The coincidence with the Ext groups is still true, if one modifies the definition of the Ext's in a similar way (considering graded resolutions and taking only homomorphisms sitting in finite number of degrees, the standard resolution of A is definitely graded).

7.2. $\operatorname{HH}^2(A)$ **and 1st order deformations.** Pick a finite dimensional vector space P and consider the graded algebra $S(P)/(P^2)=\mathbb{C}\oplus P$. Let A_0 be a $\mathbb{Z}_{\geqslant 0}$-graded algebra. By a 1st order (graded) deformation of A we mean a graded deformation of A_0 over $S(P)/(P^2)$. If A_1 is such a deformation, then it has the ideal $P\otimes A_0$ that squares to 0 and $A_1/P\otimes A_0=$

A_0. Definitely, if A is a graded deformation of A_0 over $S(P)$, then A/AP^2 is a 1st order deformation of A_0. So understanding 1st order deformation is the 1st step in understanding deformations over $S(P)$.

In what follows we will always assume that the degree of P is 2.

We denote the product on A_0 as usual, ab. We also use the same notation for the maps $A_0 \otimes (P \otimes A_0), (P \otimes A_0) \otimes A_0 \to P \otimes A_0$ induced by the product on A_0. Let μ^1 denote the product on A_1. We can represent A_1 as a direct sum of vector spaces $A_0 \oplus P \otimes A_0$ if we choose a graded section of the projection $A_1 \twoheadrightarrow A_0$. The values $\mu^1(a,b)$ are determined uniquely from μ_0 if one of the arguments a, b lies in $P \otimes A_0$: it is zero if both lie in $P \otimes A_0$. Also $\mu^1(a, p \otimes b') = p\mu^1(a, b') = p \otimes ab$ because μ^1 coincides with the product on A_0 modulo P. So μ^1 is recovered from its values on $A_0 \otimes A_0$. Let us write $\mu^1(a,b) = \mu_0(a,b) + \mu_1(a,b)$, where $\mu_1 : A_0 \otimes A_0 \to P$ (that depends on the choice of σ). In other words, $\mu_1 \in P \otimes C^2(A_0, A_0)$.

We want A_1 to be associative. As the following exercise shows, this is equivalent to $d\mu_1 = 0$, where d is the Hochschild differential.

Exercise 7.1. *Show that* $\mu^1(\mu^1(a,b), c) = \mu^1(a, \mu^1(b,c))$ *for* $a, b, c \in A_0$ *is equivalent to* $\mu_1(a,b)c + \mu_1(ab, c) = a\mu_1(b,c) + \mu_1(a, bc)$, *i.e., to* $d\mu_1 = 0$.

Let $Z^2(A_0)$ denote the space of Hochschild 2-cocycles. We have $\mu_1 \in P \otimes Z^2(A_0)$. But also the algebra A_1 has to be graded: $\mu^1(A_1^i, A_1^j) \subset A_0^{i+j}$ or equivalently, $\mu_1(A_0^i, A_0^j) \subset (P \otimes A_0)^{i+j} = P \otimes A_0^{i+j-2}$. So $\mu_1 \in P \otimes Z^2(A_0)^{-2}$.

It is natural to identify certain graded deformations. Namely, we say that graded $S(P)/(P^2)$-algebras A_1, A_1' are equivalent if there is an isomorphism $\sigma : A_1 \to A_1'$ of graded $S(P)/(P^2)$-algebras that is the identity modulo P. If we view A_1, A_1' as the space $A_0 \oplus P \otimes A_0$ with products μ, μ', then σ is the identity on $P \otimes A_0$ and on A_0 is given by $a \mapsto a + \sigma_1(a)$, where $\sigma_1 \in P \otimes C^1(A_0)$. Of course, this isomorphism preserves the gradings if and only if $\sigma_1 \in P \otimes C^1(A_0)^{-2}$.

Exercise 7.2. *The equality* $\sigma \circ \mu'^1 = \mu^1 \circ \sigma$ *is equivalent to* $\mu_1'(a,b) + \sigma(ab) = \mu_1(a,b) + a\sigma(b) + \sigma(a)b$, *i.e., to* $\mu' = \mu + d\sigma$.

So the classes in $Z^2(A_0)^{-2}$ correspond to equivalent deformations if and only if they are cohomologous. In other words, the equivalence classes of the 1st order graded deformations over $S(P)/(P^2)$ are parameterized by $P \otimes \mathrm{HH}^2(A_0)^{-2}$.

Now suppose that $\mathrm{HH}^2(A_0)^{-2}$ is finite dimensional. Then $P \otimes \mathrm{HH}^2(A_0)^{-2} = \mathrm{Hom}(\mathrm{HH}^2(A_0)^{-2*}, P)$. We can consider the space $P_{un} := \mathrm{HH}^2(A_0)^{-2*}$. Take the deformation $A_{1,un}$ over $S(P_{un})/(P_{un}^2)$ corresponding to $1 \in \mathrm{Hom}(\mathrm{HH}^2(A_0)^{-2*}, P_{un})$. By construction, this deformation has the following universal property: for any other 1st order graded deformation A_1 over $S(P)/(P^2)$, there is a unique linear map $P_{un} \to P$ such that $A_1 \sim S(P) \otimes_{S(P_{un})} A_{1,un}$.

7.3. $\mathrm{HH}^3(A_0)$ **and obstructions.** Of course, a graded deformation of A_0 over $S(P)$ also produces graded deformations over all $S(P)/(P^k)$. So we can try to produce a deformation step by step, lifting a deformation A_{k-1} over $S(P)/(P^k)$ (with $k \geqslant 2$) to a deformation A_k over $S(P)/(P^{k+1})$. The kernel of $A_k \twoheadrightarrow A_{k-1}$ is the ideal $S^k(P) \otimes A_0$ annihilated by the multiplication by P. We can decompose A_{k-1} and A_k as

$$A_{k-1} = A_0 \oplus P \otimes A_0 \oplus \ldots \oplus S^{k-1}(P) \otimes A_0, \quad A_k = A_0 \oplus P \otimes A_0 \oplus \ldots \oplus S^k(P) \otimes A_0.$$

Due to $S(P)$-linearity, the products μ^{k-1} on A_{k-1} and μ^k on A_k are recovered from their restrictions to $A_0 \otimes A_0$. There we can represent them in the form $a \cdot b = \sum_{i=0}^{k-1} \mu_i(a,b)$ in

A_{k-1} and $a \cdot b = \sum_{i=0}^{k} \mu_i(a,b)$ in A_k. Here μ_i is a graded map $A_0 \otimes A_0 \to S^i(P) \otimes A_0$, we know $\mu_0, \ldots, \mu_{k-1}$ and want to determine all possible μ_k.

The product on A_k has to be associative. We only need to check that the terms of $(a \cdot b) \cdot c$ and of $a \cdot (b \cdot c)$ that lie in $S^k(P) \otimes A$ coincides. We arrive at the equality

$$\sum_{i=0}^{k} \mu_i(\mu_{k-i}(a,b),c) = \sum_{i=0}^{k} \mu_i(a, \mu_{k-i}(b,c))$$

Let us move all terms containing μ_k to the left and all other (known) terms to the right. We get

$$\mu_k(a,b)c + \mu_k(ab,c) - \mu_k(a,bc) - a\mu_k(b,c) = \sum_{i=1}^{k-1} \left(\mu_i(a,\mu_{k-i}(b,c)) - \mu_i(\mu_{k-i}(a,b),c)\right). \tag{1}$$

The left hand side is $d\mu_k(a,b,c)$. It turns out that the right hand side is always a cocycle. This is a quite unpleasant computation, where one needs to use the fact that the products $\bigoplus_{i=0}^{j} \mu_j(a,b)$ are associative for all $j < k$ (and so $d\mu_j$ equals to the expression similar to the r.h.s. of (1)).

Exercise 7.3. *Show that the r.h.s. of (1) is a cocycle when $k = 2$.*

Problem 7.4. *Show that the r.h.s. of (1) is a cocycle for general k.*

In order for μ_k to exist the r.h.s., that belongs to $C^3(A,A)^{-2k}$, has to be a coboundary. This is automatically true when $\mathrm{HH}^3(A)^{-2k} = 0$.

Now suppose that, for whatever reasons, μ_k exists. Then it is defined up to adding $\nu \in C^2(A,A)^{-2k}$. As in the case of $k = 1$, we can define an equivalence relation on extensions of A_{k-1}: A_k and A_k' are equivalent if there is a graded $S(P)$-linear algebra isomorphism $\sigma : A_k \to A_k'$ that is the identity on A_{k-1}. Such an isomorphism is given by $\sum_{i=0}^{k} a_i \mapsto \sum_{i=0}^{k} a_i + \sigma_k(a_0)$, where $a_i \in S^i(P) \otimes A_0$ and $\sigma_k \in S^k(P) \otimes C^1(A_0,A_0)^{-2k}$. Similarly to the above, the condition that σ is an algebra isomorphism is equivalent to $\mu_k' - \mu_k = d\sigma_k$. We conclude that the equivalence classes of extensions A_k of A_{k-1} are parameterized by $\mathrm{HH}^2(A_0)^{-2k}$. In particular, if $\mathrm{HH}^2(A_0)^{-2k} = 0$, such extension is unique (and it exists provided $\mathrm{HH}^3(A_0)^{-2k} = 0$).

Finally, let us explain how to get a graded deformation A out of consecutive extensions $\ldots \twoheadrightarrow A_k \twoheadrightarrow A_{k-1} \twoheadrightarrow \ldots \twoheadrightarrow A_0$. The first idea is perhaps to take the inverse limit, but this is not what we want as the inverse limit has no grading (take the algebra of formal power series, for example). The correct answer is that one needs to take the inverse limit component wise. Namely, we have the inverse system (A_k^i) for each $i \geqslant 0$. We remark that this system stabilizers: the natural epimorphism $A_{k+1}^i \twoheadrightarrow A_k^i$ is an isomorphism for $2k > i$. This is because the kernel of $A_{k+1} \twoheadrightarrow A_k$ is generated by $S^{k+1}(P)$. Since the degrees for A are $\geqslant 0$ and the degree of $S^{k+1}(P)$ is $2(k+1)$, we see that the smallest degree in the kernel of $A_{k+1} \twoheadrightarrow A_k$ is $2(k+1)$, hence our claim. We set $A^i := \varprojlim_k A_k^i$. Then for A we take $\bigoplus_{i=0}^{+\infty} A$.

Exercise 7.5. *Equip A with a graded $S(P)$-algebra structure so that we have epimorphisms $A \twoheadrightarrow A_k$ for all k. Further, check that A is a free graded $S(P)$-module.*

7.4. **Universal deformation.** Let us assume that $\dim \mathrm{HH}^2(A_0)^{-2} < \infty, \mathrm{HH}^2(A_0)^i = 0$ for $i < -2$ and $\mathrm{HH}^3(A)^i = 0$ for $i < -3$. Consider the 1st order deformation $A_{1,un}$ constructed above. Thanks to our assumptions, we see by induction that there are uniquely determined (up to an equivalence) deformations $A_{un,k}$ such that $A_{un,k}$ lifts $A_{un,k-1}$. As explained above, we get a graded deformation A_{un} of A_0 over $S(P_{un}), P_{un} = \mathrm{HH}^2(A)^{-2*}$. Let A be a graded deformation of A_0 over $S(P)$.

Proposition 7.1. *There is a unique linear map $P_{un} \to P$ such that the deformations $S(P) \otimes_{S(P_{un})} A_{un}$ and A are equivalent (meaning that there is a graded $S(P)$-algebra isomorphism $S(P) \otimes_{S(P_{un})} A_{un} \xrightarrow{\sim} A$ that is the identity modulo P).*

Proof. Set $A' := S(P) \otimes_{S(P_{un})} A$. We will prove by induction on k that A'_k and A_k are equivalent, then it will imply that A' and A are so. We have already seen that there is a unique map $P_{un} \to P$ that makes A'_1 and A_1 equivalent. Since $\mathrm{HH}^2(A_0)^i = 0$ for $i < -2$, the equivalence class of A_k is determined by that of A_1. But as A_1 and A'_1 have the same equivalence classes, A'_k and A_k are equivalent. $\square$

Of course, the previous proposition is not yet a universality property, for that one wants an equivalence to be unique. The next exercise explains when this is so (in fact, in the case of $A_0 = S(V)\#\Gamma$ to see the uniqueness is easier).

Exercise 7.6. *Show that if $\mathrm{HH}^1(A_0)^i = 0$ for $i \leqslant -2$, then an equivalence in the proposition is unique.*

Problem 7.7. *Let A be a graded deformation of A_0 over $S(P)$. Describe the set of all auto-equivalences of A in terms of the groups $\mathrm{HH}^1(A_0)^i$ with $i \leqslant -2$.*

7.5. **Computations for $S(V)\#\Gamma$.** Now we set $A_0 = S(V)\#\Gamma$, where V is symplectically irreducible. We want to prove that $\mathrm{HH}^2(A_0)^i = 0$ for $i < -2$, that $\mathrm{HH}^3(A_0)^{-2} = 0$ for $i < -3$ and that $\dim \mathrm{HH}^2(A_0)^{-2} = m + 1$, where m is the number of classes of complex reflections. First, we are going to relate the computation of HH for $S(V)\#\Gamma$ for computations for $S(V)$.

Let B be an associative graded algebra acted on by Γ (so that Γ preserves the grading). Then we introduce a B-bimodule $B\gamma$ as follows. As a graded left B-module, $B\gamma = B$, but the right action is given by $(m\gamma)b = (m\gamma(b))\gamma$. We remark that Γ naturally acts on the direct sum $\bigoplus_{\gamma\in\Gamma} B\gamma$: $\gamma'(b\gamma) = \gamma'(b)\gamma'\gamma\gamma'^{-1}$. Of course, the B-bimodule $\bigoplus_{\gamma\in\Gamma} B\gamma$ is nothing else but $B\#\Gamma$, and the Γ-action on the direct sum is the adjoint Γ-action on $B\#\Gamma$. So we get a Γ action on $\bigoplus_{\gamma\in\Gamma} \mathrm{HH}^i(B, B\gamma)$ (induced by the Γ-actions on B and $\bigoplus_{\gamma\in\Gamma} B\gamma$) and it makes sense to speak about Γ-invariants.

Lemma 7.2. *We have an isomorphism $\mathrm{HH}^j(B\#\Gamma, B\#\Gamma) \cong \left(\bigoplus_{\gamma\in\Gamma} \mathrm{HH}^j(B, B\gamma)\right)^\Gamma$ of graded vector spaces.*

Proof. Let B-Bimod^Γ denote the category of Γ-equivariant B-bimodules, i.e., B-bimodules M equipped with a Γ-action making the structure map $B \otimes M \otimes B \to M$ equivariant. We have a forgetful functor $B\#\Gamma$-Bimod $\to B$-Bimod^Γ that remembers the adjoint action of Γ. This functor has left adjoint, $M \mapsto M\#\Gamma$, where, as a vector space, $M\#\Gamma = M \otimes \mathbb{C}\Gamma$, and the left and right multiplications by elements of $B\#\Gamma$ are defined similarly to the multiplication on $B\#\Gamma$ itself:

$$b \otimes \gamma \cdot m \otimes \gamma' = b\gamma(m) \otimes \gamma\gamma', \quad m \otimes \gamma' \cdot b \otimes \gamma = m\gamma'(b) \otimes \gamma'\gamma.$$

The functor is exact. For any $N \in B\#\Gamma$-Bimod we have an isomorphism of functors $M \mapsto \mathrm{Hom}_{B\#\Gamma-Bimod}(M\#\Gamma, N)$ and $M \mapsto \mathrm{Hom}_{B-\mathrm{Bimod}^\Gamma}(M, N) = \mathrm{Hom}_{B-\mathrm{Bimod}}(M, N)^\Gamma$. Being the left adjoint functor of an exact functor, $\bullet\#\Gamma$ maps projective objects to projective objects. Any Γ-equivariant B-bimodule has a Γ-equivariant projective resolution (say, by free modules). By some basic Homological algebra, it follows that $\mathrm{Ext}^i_{B\#\Gamma-\mathrm{Bimod}}(M\#\Gamma, N)$ is naturally identified with $\mathrm{Ext}^i_{B-Bimod^\Gamma}(M,N) = \mathrm{Ext}^i_{B-Bimod}(M,N)^\Gamma$. Plugging $M = B, N = B\#\Gamma$, and using the coincidence of $\mathrm{HH}^\bullet$ with $\mathrm{Ext}^\bullet$, we complete the proof of the lemma. □

Now let $B = S(V)$. There is an advantage of the $S(V)$-bimodules $S(V)\gamma$ over the $S(V)\#\Gamma$-bimodule $S(V)\#\Gamma$: in a suitable basis $x_1,\ldots,x_n$ (depending on γ), the element γ is diagonalizable, $\gamma = \mathrm{diag}(\gamma_1,\ldots,\gamma_n)$. Then $S(V)\gamma = \bigotimes_{i=1}^n \mathbb{C}[x_i]\gamma_i$, where we view γ_i as an element of a suitable cyclic group acting on a 1-dimensional space.

Here is the following general fact from Homological algebra that is an analog of the Künneth formula from Topology:

$$\mathrm{Ext}^\bullet_{B_1\otimes B_2-Bimod}(B_1\otimes B_2, M_1\otimes M_2) = \mathrm{Ext}^\bullet_{B_1-Bimod}(B_1, M_1)\otimes \mathrm{Ext}^\bullet_{B_2-Bimod}(B_2,M_2),$$

where the homological degree on the l.h.s. corresponds to the sum of the homological degrees on the r.h.s. This isomorphism is compatible with gradings if the algebras and bimodules under consideration are graded. We deduce that

$$\mathrm{HH}^\bullet(S(V), S(V)g) = \bigotimes_{i=1}^n \mathrm{HH}^\bullet(\mathbb{C}[x_i], \mathbb{C}[x_i]\gamma_i).$$

So our next goal is to compute $\mathrm{HH}^\bullet(\mathbb{C}[x],\mathbb{C}[x]\gamma)$, where now γ is an element of some cyclic group acting on $\mathbb{C}[x]$ so that for $m \in C[x]\gamma$ we have $m\cdot x := \gamma m x$. For a graded space M and $n\in\mathbb{Z}$ we write $M[n]$ for the same vector space but with shifted grading: $M[n]^m := M^{n+m}$.

Lemma 7.3. (i) *We have* $\mathrm{HH}^i(\mathbb{C}[x],\mathbb{C}[x]\gamma) = 0$ *for any* γ *and* $i \geqslant 2$.
(ii) *We have* $\mathrm{HH}^0(\mathbb{C}[x],\mathbb{C}[x]) = \mathbb{C}[x]$ *with the usual grading, and* $\mathrm{HH}^1(\mathbb{C}[x],\mathbb{C}[x]) = \mathbb{C}[x][1]$.
(iii) *Suppose* γ *is nontrivial. Then* $\mathrm{HH}^0(\mathbb{C}[x],\mathbb{C}[x]\gamma) = 0$ *and* $\mathrm{HH}^1(\mathbb{C}[x],\mathbb{C}[x]\gamma) = \mathbb{C}[1]$.

Proof. An advantage of dealing with computing the Hochschild cohomology of the algebras $\mathbb{C}[X]$, where X is a smooth affine variety, is that we can take a Koszul type resolution of the $\mathbb{C}[X]$-bimodule $\mathbb{C}[X]$. In the case of $\mathbb{C}[x]$ this resolution is: $\mathbb{C}[x]\otimes\mathbb{C}[x]\to\mathbb{C}[x]\otimes\mathbb{C}[x]$, where the map given by $a\otimes b\to ax\otimes b - a\otimes xb$. Since the length of the resolution is 1, we immediately get (i).

Exercise 7.8. *Use the Koszul resolution to check that* $\mathrm{HH}^\bullet(\mathbb{C}[x],\mathbb{C}[x]\gamma)$ *is the cohomology of the complex* $\mathbb{C}[x]\gamma\to\mathbb{C}[x]\gamma$*, where the map is a left* $\mathbb{C}[x]$*-module homomorphism given by* $1\gamma\mapsto(x-\gamma(x))\gamma$*, where we shift the grading on the target module by* 1 *(so that* 1 *there has degree* -1*).*

So when $\gamma = 1$, the map is 0 and we get (ii). For $\gamma\neq 1$, the map is injective, its cokernel is $\mathbb{C}$ sitting in degree -1. This is (iii). □

Now return to our situation, where we need to compute certain graded components of $\left(\bigoplus_{\gamma\in\Gamma}\mathrm{HH}^\bullet(S(V),S(V)\gamma)\right)^\Gamma$. We have

$$\mathrm{HH}^i(S(V),S(V)\gamma) = \bigoplus\bigotimes_k \mathrm{HH}^{i_k}(\mathbb{C}[x_k],\mathbb{C}[x_k]\gamma_k)^{j_k},$$

where the summation is taken over all n-tuples $(i_1,\ldots,i_n)$ of 0 and 1, and $(j_1,\ldots,j_n)$ of integers $\geqslant -1$ such that $i_1+\ldots+i_n = i, j_1+\ldots+j_n = j$. By the previous lemma we have $\bigotimes_k \mathrm{HH}^{i_k}(\mathbb{C}[x_k],\mathbb{C}[x_k]\gamma_k)^{j_k} = 0$ if for $\gamma_k\neq 1$ we have $i_k\neq -1, j_j\neq -1$.

Exercise 7.9. *Show that* $\gamma \in \Gamma$ *has even number of eigenvalues different from* 1. *Deduce that* $\mathrm{HH}^2(S(V), S(V)\gamma)^i = 0$ *for* $i < -2$ *and* $\mathrm{HH}^3(S(V), S(V)\gamma)^i = 0$ *for* $i < -3$.

It remains to show that $\dim \left(\bigoplus_{\gamma\in\Gamma} \mathrm{HH}^\bullet(S(V), S(V)\gamma)^{-2}\right)^\Gamma = m + 1$. This is achieved in the following exercise.

Exercise 7.10.
(1) *Suppose that* γ *is not a symplectic reflection. Prove that any element in* $\mathrm{HH}^\bullet(S(V), S(V)\gamma)$ *has homological degree at least* 4.
(2) *Let* γ *be a symplectic reflection. Show that* $\mathrm{HH}^2(S(V), S(V)\gamma)^{-2}$ *is one dimensional.*
(3) *Let* $S = S_1 \sqcup S_2 \sqcup \ldots \sqcup S_m$. *Show that* $\dim \left(\bigoplus_{\gamma\in S_i} \mathrm{HH}^2(S(V), S(V)\gamma)^{-2}\right)^\Gamma = 1$.
(4) *Show that, as a* Γ*-module,* $\mathrm{HH}^2(S(V), S(V))^{-2}$ *is* $\bigwedge^2 V$. *Deduce that*
$$\dim \left(\mathrm{HH}^2(S(V), S(V))^{-2}\right)^\Gamma = 1.$$

LECTURES ON SYMPLECTIC REFLECTION ALGEBRAS

IVAN LOSEV

8. Spherical SRA

Let V be a finite dimensional vector space equipped with a symplectic form ω and Γ be a finite subgroup in $\mathrm{Sp}(V)$. Let S denote the subset of all symplectic reflections $s \in \Gamma$, i.e., all elements with $\mathrm{rk}(s-1_V)=2$ and let $S = S_1 \sqcup S_2 \sqcup \ldots \sqcup S_m$ be the decomposition of S into conjugacy classes. To each $s \in S$ we assign the form $\omega_s \in \bigwedge^2 V^*$ by

$$\omega_s(u,v) = \begin{cases} \omega(u,v), u,v \in \mathrm{im}(s-1_V), \\ 0, u \in \ker(s-1_V). \end{cases}$$

In Lecture 6 we have introduced the universal SRA. We pick independent variables $t, c_1, \ldots, c_m$ and consider the vector space P with basis $t, c_1, \ldots, c_m$. Then we define the algebra H by

$$H = S(P) \otimes T(V)\#\Gamma/(u\otimes v - v \otimes u - t\omega(u,v) - \sum_{i=1}^m c_i \sum_{s\in S_i} \omega_s(u,v)s| \quad u,v \in V).$$

Our goal is to prove that H is a graded deformation of $S(V)\#\Gamma$, meaning H is free as an $S(P)$-module.

In Lecture 7 we have computed some Hochschild cohomology of $S(V)\#\Gamma$. Namely, we have seen that $\mathrm{HH}^2(S(V)\#\Gamma)^i = 0$ for $i \leqslant -3$, and $\mathrm{HH}^3(S(V)\#\Gamma)^i = 0$ for $i \leqslant -4$. This implies that there is a universal graded deformation H_{un} of $S(V)\#\Gamma$ over $S(P_{un})$, where $P_{un} := \left(\mathrm{HH}^2(S(V)\#\Gamma)^{-2}\right)^*$. We have computed P_{un} in the case when V is symplectically irreducible as Γ-module, we have found that $\dim P_{un} = m+1$. In fact, in general, $\dim P_{un} = m + \dim(\bigwedge^2 V)^\Gamma$, this is proved completely analogously to the last exercise in the previous lecture. The universality means that for any other graded deformation $\tilde{H}$ of $S(V)\#\Gamma$ over $S(\tilde{P})$ there is a unique linear map $P_{un} \to \tilde{P}$ such that there is an $S(\tilde{P})$-linear isomorphism $S(\tilde{P}) \otimes_{S(P_{un})} H_{un} \xrightarrow{\sim} \tilde{H}$ that is the identity modulo $\tilde{P}$.

The first thing we will do in this lecture: we will identify H_{un} and H in the case when V is symplectically irreducible. This will easily imply that H is a deformation, in general. We will also see that the isomorphism in the end of the previous paragraph is unique that makes H_{un} a universal object in the categorical sense.

Our original goal was to study the deformations of the invariant subalgebra $S(V)^\Gamma$. We get a deformation eHe over $S(P)$ (it is almost for sure is not universal in the categorical sense; in general it is unknown whether this exhausts all reasonable deformations). We will study an interplay between H and eHe. This interplay is provided by the bimodules He and eH. We will see that eHe and H are mutual centralizers of each other in these bimodules (the double centralizer theorem).

Finally, we will discuss what is known about Morita equivalence between the specializations $eH_{t,c}e$ and $H_{t,c}$ to numerical parameters.

8.1. **Universal SRA as a universal deformation.**

Theorem 8.1. *Suppose that V is symplectically irreducible. Then H is a deformation of $S(V)\#\Gamma$, and there is an isomorphism $P_{un} \cong P$ making H and H_{un} equivalent in the sense explained above.*

Proof. We will check that H_{un} is given by the same generators and relations as H.

Let us deal with generators first. Let π denote the natural projection $H_{un} \twoheadrightarrow SV\#\Gamma$. Since P_{un} has degree 2, π identifies the degree 0 component of H_{un} with $(S(V)\#\Gamma)^0 = \mathbb{C}\Gamma$ and the degree 1 component with $(S(V)\#\Gamma)^1 = V \otimes \mathbb{C}\Gamma$. In particular, there is a natural inclusion of V into H_{un}. The $S(P_{un})$-subalgebra generated by V and $\mathbb{C}\Gamma$ is graded and surjects onto $S(V)\#\Gamma$. It follows from the next exercise that this subalgebra coincides with H_{un}.

Exercise 8.1. *Let $M = \bigoplus_{i=0}^{\infty} M$ be a graded module over $S(P)$, where P is a vector space. Let M_0 be a graded $S(P)$-submodule such that $M_0 \twoheadrightarrow M/PM$. Show that $M_0 = M$.*

So we get a natural epimorphism $S(P_{un}) \otimes T(V)\#\Gamma \twoheadrightarrow H_{un}$.

Let us proceed to relations, i.e., to describing the kernel of the epimorphism above. For $u, v \in V \subset H_{un}$, the element $[u, v]$ has degree 2 and lies in $\ker \pi$. But the degree 2 component of $\ker \pi$ is $P_{un} \otimes \mathbb{C}\Gamma$. So there is a map $\kappa : \bigwedge^2 V \to P_{un} \otimes \mathbb{C}\Gamma$ such that $[u, v] = \kappa(u, v)$. Let $\tilde{H}_{un} := S(P_{un}) \otimes T(V)\#\Gamma/(u \otimes v - v \otimes u - \kappa(u, v))$. Then $\tilde{H}_{un}/P_{un}\tilde{H}_{un} = S(V)\#\Gamma$, while there is an epimorphism $\tilde{H}_{un} \twoheadrightarrow H_{un}$. Since H_{un} is a free graded $S(P_{un})$-module, the following exercise implies $\tilde{H}_{un} = H_{un}$.

Exercise 8.2. *Let M_1, M_2 be two non-negatively graded $S(P)$-modules, where P is a vector space. Suppose that M_2 is a graded free module. Consider an epimorphism $M_1 \twoheadrightarrow M_2$ that induces an isomorphism $M_1/PM_1 \xrightarrow{\sim} M_2/PM_2$. Show that this epimorphism is an isomorphism.*

We claim that there are $t', c'_1, \ldots, c'_m \in P_{un}$ such that $\kappa(u, v) = t'\omega(u, v) + \sum_{i=1}^m c'_i \sum_{s \in S_i} \omega_s(u, v)s$. This follows from our computations in Lecture 6 (for example, using passing to a numerical specialization of H_{un}). Also we remark that $t', c'_1, \ldots, c'_m$ is a basis in P_{un} – here finally we will use that H_{un} is a universal deformations, the arguments above worked for any deformation. It is enough to show that $t', c'_1 \ldots, c'_m$ span P_{un} because $\dim P_{un} = m + 1$. Let P'_{un} be the subspace spanned by $t', c'_1, \ldots, c'_m$. Deformations $S(P_{un}) \otimes_{S(P_{un})} H_{un}$ and H_{un} are equivalent for any linear map $P_{un} \to P_{un}$ that is the identity on P'_{un} (they are just algebras given by exactly the same relations). But a linear map $P_{un} \to P_{un}$ with $S(P_{un}) \otimes_{S(P_{un})} \sim H_{un}$ and H_{un} has to be unique and hence $P'_{un} = P_{un}$. $\square$

Exercise 8.3. *Use the theorem to deduce that H is a graded deformation of $S(V)\#\Gamma$ even if V is not symplectically irreducible.*

We remark that any graded deformation H' of $S(V)\#\Gamma$ over $S(P')$ has no nontrivial self-equivalences. This is because any such self-equivalence is forced to be the identity on $\mathbb{C}\Gamma$ and V (thanks to $\deg P' = 2$). But the $S(P')$-algebra H' is generated by $\mathbb{C}\Gamma$ and V. So the equivalence is the identity and H_{un} is a universal object in the categorical sense.

8.2. **Algebra eHe and bimodule eHe.**

Let e be the idempotent $\frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \gamma \in \mathbb{C}\Gamma$. We can view e as an element of H or of its numerical specialization $H_{t,c}$. Then we get spherical subalgebras $eHe \subset H, eH_{t,c}e \subset H_{t,c}$.

Exercise 8.4. *Prove that eHe is a graded deformation of $S(V)^\Gamma$ over $S(P)$. Also prove that the specialization of eHe at $t, c_1, \ldots, c_m \in \mathbb{C}$ coincides with $eH_{t,c}e$ so that $\operatorname{gr} eH_{t,c}e = S(V)^\Gamma$.*

We can consider the $S(P)$-module He and also its specializations $H_{t,c}e$. The space He has commuting actions of H on the left and eHe on the right and so becomes an H-eHe-bimodule.

Lemma 8.2. *The right eHe-module He is finitely generated.*

Proof. We know that $S(V) = He/PHe$ is finitely generated over $S(V)^\Gamma = eHe/PeHe$. We can choose finitely many homogeneous generators $m_1, \ldots, m_n$. Then lift them to homogeneous elements $\tilde{m}_1, \ldots, \tilde{m}_n$ of He. Exercise 8.1 implies that $\tilde{m}_1, \ldots, \tilde{m}_n$ generate the right eHe-module He. □

Similarly, eH is a finitely generated left eHe-module.

8.3. **Double centralizer property.** We are going to prove that the algebras $H_{t,c}, eH_{t,c}e$ are mutual centralizers in the bimodule $H_{t,c}e$. One statement here is easy.

Exercise 8.5. *The homomorphism $eH_{t,c}e^{opp} \to \mathrm{End}_{H_{t,c}}(H_{t,c}e)$ is an isomorphism.*

The following theorem is due to Etingof and Ginzburg, [EG].

Theorem 8.3. *The homomorphism $H_{t,c} \to \mathrm{End}_{eH_{t,c}e^{opp}}(H_{t,c}e)$ is an isomorphism.*

Proof. The proof is organized as follows. We start with the case $t = 0, c = 0$. We first prove the injectivity, which is easier, and then the surjectivity, which is harder. After that the general case is done basically by passing to associated graded.

Step 1. We claim that the natural map $S(V)\#\Gamma \to \mathrm{End}_{S(V)^\Gamma}(S(V))$ is injective. In what follows we will identify $S(V)$ with $\mathbb{C}[V]$ using the identification of V and V^* coming from the symplectic form.

Let V^0 denote the subset in V consisting of all points with trivial stabilizers. This subset is open and, since Γ acts faithfully – only the unit element acts as 1_V, we have $V^0 \neq \varnothing$. Let $\sum_\gamma f_\gamma \gamma$ lie in the kernel of $\mathbb{C}[V]\#\Gamma \to \mathrm{End}_{\mathbb{C}[V]^\Gamma}(\mathbb{C}[V])$. This means that $\sum_{\gamma\in\Gamma} f_\gamma \gamma(g) = 0$ for any $g \in \mathbb{C}[V]$. Pick $v \in V^0$. For any complex numbers z_γ there is $g \in \mathbb{C}[V]$ such that $g(\gamma^{-1}v) = z_\gamma$. It follows that $\sum_\gamma f_\gamma(v) z_\gamma = 0$ and so $f_\gamma(v) = 0$. Since V^0 is open and non-empty, we deduce that $f_\gamma = 0$.

Step 2. To prove that the homomorphism $\mathbb{C}[V]\#\Gamma \to \mathrm{End}_{\mathbb{C}[V]^\Gamma}(\mathbb{C}[V])$ is surjective we will need the following lemma.

Lemma 8.4. *Let X be a smooth affine variety equipped with a free action of a finite group Γ. Then the homomorphism $\mathbb{C}[X]\#\Gamma \to \mathrm{End}_{\mathbb{C}[X]^\Gamma}(\mathbb{C}[X])$ is an isomorphism.*

Proof of Lemma. We remark that both $\mathbb{C}[X]\#\Gamma$ and $\mathrm{End}_{\mathbb{C}[X]^\Gamma}(\mathbb{C}[X])$ are locally free $\mathbb{C}[X]^G$-modules of rank $|\Gamma|^2$. To prove this we only need to check that $\mathbb{C}[X]$ is a locally free $\mathbb{C}[X]^G$-module of rank $|\Gamma|$. Pick a point $x \in X$ and let $\pi : X \to X/\Gamma$ denote the quotient morphism. Then there are $g_\gamma \in \mathbb{C}[X], \gamma \in \Gamma$, such that the matrix $(g_\gamma(\gamma' x))_{\gamma,\gamma'\in\Gamma}$ is non-degenerate. These elements form a basis of the $\mathbb{C}[X]^\Gamma$-module $\mathbb{C}[X]$ after an appropriate localization. This implies that $\mathbb{C}[X]$ is locally free of rank $\mathbb{C}[X]^\Gamma$.

Now to show that the homomorphism $\mathbb{C}[X]\#\Gamma \to \mathrm{End}_{\mathbb{C}[X]^\Gamma}(\mathbb{C}[X])$ is bijective it is enough to show that it is injective fiberwise, i.e., the induced homomorphism $A_y\#\Gamma \to \mathrm{End}(A_y)$ is injective for any $y \in X/\Gamma$, where $A_y = \mathbb{C}[X]/\mathbb{C}[X]\mathfrak{m}_y$, $\mathfrak{m}_y$ being the maximal ideal of y in $\mathbb{C}[X]^\Gamma$. But the algebra A_y is just the algebra of functions on $\pi^{-1}(y)$, a free Γ-orbit. The injectivity is checked as in the proof of step 1. □

Recall that $\Gamma \subset \mathrm{Sp}(V)$. In particular, for any $\gamma \in \Gamma$ the fixed point subspace V^γ has codimension at least 2. So $\operatorname{codim}_V V \setminus V^0 \geqslant 2$. For every point $v \in V^0$ we can find $f_v \in \mathbb{C}[V]^\Gamma$ such that $f_v(v) \neq 0, f_v(V \setminus V^0) = 0$. Let $V^0_v := \{u \in V | f_v(u) \neq 0\}$, this is a Γ-stable affine open subset of V^0. For convenience, we can choose a finite covering $V^0 = \bigcup_i V_i$ by subsets of the form V^0_v, let f_i denote the corresponding polynomial, so that $\mathbb{C}[V_i] = \mathbb{C}[V]_f$ and $\mathbb{C}[V_i]^\Gamma = \mathbb{C}[V]^\Gamma_f$. By general Commutative algebra, $\operatorname{End}_{\mathbb{C}[V_i]^\Gamma}(\mathbb{C}[V_i])$ is just the localization of $\operatorname{End}_{\mathbb{C}[V]^\Gamma}(\mathbb{C}[V])$ by f_i. In particular, we have a homomorphism $\iota_i : \operatorname{End}_{\mathbb{C}[V]^\Gamma}(\mathbb{C}[V]) \to \operatorname{End}_{\mathbb{C}[V_i]^\Gamma}(\mathbb{C}[V_i])$. It is injective: if we have $\iota_i(\varphi)(f/f_i^k) = 0$, then $\iota_i(\varphi)(f) = 0$ for any $f \in \mathbb{C}[V]$.

Pick $\varphi \in \operatorname{End}_{\mathbb{C}[V]^\Gamma}(\mathbb{C}[V])$. Now, by Lemma applied to $X = V_i$, we have $\iota_i(\varphi) = \sum_{\gamma\in\Gamma} f^i_\gamma \gamma$ for some (uniquely determined) elements $f^i_\gamma \in \mathbb{C}[V_i]$.

Set $V_{ij} = V_i \cap V_j$. We claim that $f^i_\gamma|_{V_{ij}} = f^j_\gamma|_{V_{ij}}$. We have, again injective, homomorphisms $\operatorname{End}_{\mathbb{C}[V]^\Gamma}(\mathbb{C}[V]), \operatorname{End}_{\mathbb{C}[V_i]^\Gamma}(\mathbb{C}[V_i]), \operatorname{End}_{\mathbb{C}[V_j]^\Gamma}(\mathbb{C}[V_j]) \to \operatorname{End}_{\mathbb{C}[V_{ij}]^\Gamma}(\mathbb{C}[V_{ij}])$, denote them by $\iota_{ij}, \iota'_j, \iota'_i$, respectively. Of course, $\iota_{ij} = \iota'_i \circ \iota_j = \iota'_j \circ \iota_i$. Clearly, ι'_i sends $\sum_{\gamma\in\Gamma} f^j_\gamma \gamma$ to the same element (where we now view the f^j_γ's as elements of $\mathbb{C}[V_{ij}]$ not of $\mathbb{C}[V_j]$) and the similar claim holds for ι'_j. But, by the lemma above, the natural homomorphism $\mathbb{C}[V_{ij}]\#\Gamma \to \operatorname{End}_{\mathbb{C}[V_{ij}]^\Gamma}(\mathbb{C}[V_{ij}])$ is injective. It follows that $f^i_\gamma = f^j_\gamma$ in $\mathbb{C}[V_{ij}]$.

So the functions f^i_γ glue to a regular function f_γ on V^0. But recall that $\operatorname{codim} V \setminus V^0 \geqslant 2$. It follows that f_γ is regular on the whole V. The element $\sum_\gamma f_\gamma \gamma \in \mathbb{C}[V]\#\Gamma$ produces the endomorphism φ. This follows, for example, from the injectivity of $\operatorname{End}_{\mathbb{C}[V]^\Gamma}(\mathbb{C}[V]) \to \operatorname{End}_{\mathbb{C}[V_i]^\Gamma}(\mathbb{C}[V_i])$ and the construction of f_γ.

Step 3. Let us equip the algebra $\operatorname{End}_{eH_{t,c}e^{opp}}(H_{t,c}e)$ with a filtration.

Exercise 8.6. *Let $\mathcal{A}$ be a $\mathbb{Z}_{\geqslant 0}$-filtered algebra and M be its module. Equip M with a filtration compatible with that on $\mathcal{A}$ in such a way that $\operatorname{gr} M$ is finitely generated $\operatorname{gr}\mathcal{A}$-module. We set $\operatorname{End}_{\mathcal{A}}(M)^{\leqslant n} := \{\psi \in \operatorname{End}_{\mathcal{A}}(M) | \psi(M^{\leqslant m}) \subset M^{\leqslant n+m}, \quad \forall m\}$.*

1. *Show that this is a $\mathbb{Z}$-filtration and that $\operatorname{End}_{\mathcal{A}}(M)^{\leqslant n} = 0$ for $n \ll 0$.*
2. *Construct a natural homomorphism $\operatorname{gr}\operatorname{End}_{\mathcal{A}}(M) \to \operatorname{End}_{\operatorname{gr}\mathcal{A}}(\operatorname{gr} M)$ of graded algebras.*
3. *Show that this homomorphism is injective.*

Exercise 8.7. *Let us retain the conventions of the previous exercise. Let $\mathcal{B}$ be another $\mathbb{Z}_{\geqslant 0}$-filtered algebra such that M becomes a filtered $\mathcal{A}\otimes\mathcal{B}$-module. Show that there is a homomorphism $\mathcal{B} \to \operatorname{End}_{\mathcal{A}}(M)$ of filtered algebras. Moreover, show that the composite homomorphism $\operatorname{gr}\mathcal{B} \to \operatorname{gr}\operatorname{End}_{\mathcal{A}}(M) \to \operatorname{End}_{\operatorname{gr} A}(\operatorname{gr} M)$ coincides with the homomorphism induced by the action of $\operatorname{gr}\mathcal{A}\otimes\operatorname{gr}\mathcal{B}$ on $\operatorname{gr} M$.*

The right $eH_{t,c}e$-module $H_{t,c}e$ satisfies the conditions of the exercise. So we get a monomorphism $\operatorname{gr}\operatorname{End}_{eH_{t,c}e^{opp}}(H_{t,c}e) \to \operatorname{End}_{\operatorname{gr} eH_{t,c}e^{opp}}(\operatorname{gr} H_{t,c}e) = \operatorname{End}_{S(V)^\Gamma}(S(V))$ of graded algebras. Clearly, $H_{t,c}e$ is filtered as an $H_{t,c}\otimes eH_{t,c}e^{opp}$-module. So we get the induced homomorphism $S(V)\#\Gamma = \operatorname{gr} H_{t,c} \to \operatorname{gr}\operatorname{End}_{eH_{t,c}e^{opp}}(H_{t,c}e)$. The composite homomorphism $S(V)\#\Gamma \to \operatorname{End}_{S(V)^\Gamma}(S(V))$ is the same as one from Steps 1,2 and so is an isomorphism. We deduce that $\operatorname{gr} H_{t,c} \to \operatorname{gr}\operatorname{End}_{eH_{t,c}e^{opp}}(H_{t,c}e)$ is an isomorphism. According to the following exercise, the homomorphism $H_{t,c} \to \operatorname{End}_{eH_{t,c}e^{opp}}(H_{t,c}e)$ is an isomorphism.

Exercise 8.8. *Let M, N be $\mathbb{Z}$-filtered vector spaces such that $M^{\leqslant n} = 0$ for $n \ll 0$. Let $\varphi : M \to N$ be a filtration preserving linear map. Show that if $\operatorname{gr}\varphi : \operatorname{gr} M \to \operatorname{gr} N$ is an isomorphism, then φ is an isomorphism.*

$\square$

Problem 8.9. *Show that H, eHe also satisfy the double centralizer property.*

8.4. **Spherical parameters.** The double centralizer property for $eH_{t,c}e$ and $H_{t,c}$ is not to be confused with the Morita equivalence condition: $H_{t,c}eH_{t,c} = H_{t,c}$, the latter is far more restrictive. For example, the irreducible modules for $S(V)\#\Gamma$ with zero action of V are precisely the Γ-irreducibles. All such modules but the trivial one are annihilated by e.

The parameter (t, c) such that $H_{t,c}eH_{t,c} = H_{t,c}$ is called *spherical*. Let us explain what is known about spherical parameters when $t = 1$. The case when $t = 0$ will be mentioned in the next lecture. This dichotomy is justified by the next exercise.

Exercise 8.10. *Let $a \in \mathbb{C}^\times$. Establish a natural isomorphism between $H_{t,c}$ and $H_{at,ac}$.*

In the case when $\dim V = 2$ the description of spherical parameters was obtained by Crawley-Boevey and Holland in [CBH]. Namely, recall that (in the notation of lectures 1-4) to t, c we can assign the $r + 1$-tuple $(\lambda_i)_{i=0}^r$ by

$$\lambda_i = \operatorname{tr}_{N_i}(t\omega(u, v) + \sum_{i=1}^{m} c_i \sum_{s \in S_i} \omega_s(u, v)s)$$

Then the parameter (t, c) is spherical (no matter whether $t = 0$ or not) if and only if $\sum_{i=1}^r \lambda_i\alpha_i \neq 0$ for any root $\alpha = \sum_{i=1}^n \alpha_i\epsilon_i$ of the corresponding finite root system (we use the convention that $\epsilon_1, \ldots, \epsilon_r$ correspond to simple roots).

The answer is known (and easy to state) also for the Rational Cherednik algebra of type A – corresponding to the group $\mathfrak{S}_n$ and the double of its reflection representation. In this case, c is a single complex number. It is known, see [BEG], that $(1, c)$ is spherical if and only if $c \neq \frac{r}{d}$, where $d = 2, 3, \ldots, n$ and r is an integer with $-d < r < 0$.

Dunkl and Griffeth, [DG], obtained the description of the spherical parameters for the complex reflection groups $G(\ell, 1, n)$. The answer is too complicated to be reproduced here.

Finally, let us mention that there is a conjecture of Etingof on the structure of the spherical parameters for all groups of the form $\Gamma_n = \mathfrak{S}_n \ltimes \Gamma_1^n$, [E], that generalizes results of Dunkl and Griffeth. At the moment it is unclear how to prove that conjecture.

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LECTURES ON SYMPLECTIC REFLECTION ALGEBRAS

IVAN LOSEV

9. Commutativity and centers

9.1. **Commutativity theorem: statement and scheme of proof.** It is a natural question to ask when the algebra $eH_{t,c}e$ is commutative. This happens to have a very elegant answer that was already stated in Lecture 5 in the case when $\dim V = 2$.

Theorem 9.1 ([EG]). *The algebra $eH_{t,c}e$ is commutative if and only if $t = 0$.*

We will give a proof in the case when V is symplectically irreducible. The proof will be in three steps.

Step 1. We have the bracket $\{\cdot,\cdot\}_{t,c}$ of degree -2 on $S(V)^\Gamma$ induced by the filtered deformation $eH_{t,c}e$. We will see that this bracket depends on t and c linearly. The reason why we are interested in considering the bracket $\{\cdot,\cdot\}_{t,c}$ is that this bracket is zero when $eH_{t,c}e$ is commutative.

Step 2. So we have a linear map from P^* to the space of brackets of degree -2 on $S(V)^\Gamma$. We will see that such a bracket on $S(V)^\Gamma$ is unique up to proportionality provided V is symplectically irreducible. Thanks to this we will have a hyperplane $P_0 \subset P^*$ such that for $p = (t,c) \in P_0$ we have $\{\cdot,\cdot\}_p = 0$.

A priori it may be that the algebra eH_pe is not commutative although $\{\cdot,\cdot\}_p = 0$. In this case, $eH_{t,c}e$ still gives rise to a non-zero bracket on $S(V)^\Gamma$ but the degree of that bracket will be $-d$ with $d > 2$. We will see however that all brackets on $S(V)^\Gamma$ of that degree are 0. So for $(t,c) \in P_0$ the algebra $eH_{t,c}e$ is indeed commutative.

Step 3. It remains to show that P_0 is given by $t = 0$. For this it is enough to produce a representation of $H_{t,c}$ isomorphic to $\mathbb{C}\Gamma$ as a Γ-module. Indeed, if N is such a representation, then we have $0 = \mathrm{tr}_N[u,v] = \mathrm{tr}_N(\omega(u,v)t + \sum_{s\in S} c_s\omega_s(u,v)s) = t\dim N\omega(u,v)$. In fact, we will show that for N one can take $H_{t,c}e/\mathfrak{m}$, where $\mathfrak{m}$ is a generic maximal ideal of $eH_{t,c}e$.

9.2. **Step 1.** We write p for $(t,c) \in P^*$, $\mathcal{A}$ for eHe, and $\mathcal{A}_p$ for $eH_{t,c}e$.

Lemma 9.2. (1) *We have $[a,b] \in P\mathcal{A}$ for $a,b \in \mathcal{A}$.*
(2) *$[\mathcal{A}^i, \mathcal{A}^j] \subset P\mathcal{A}^{i+j-2}$,*
(3) *and $[\mathcal{A}_p^{\leqslant i}, \mathcal{A}_p^{\leqslant j}] \subset \mathcal{A}_p^{i+j-2}$.*

Proof. (1) follows from $\mathcal{A}/P\mathcal{A} \cong S(V)^\Gamma$ because $S(V)^\Gamma$ is commutative. (2) follows from (1) and the condition that the degree of P is 2. (3) follows from (2). □

So we do have a bracket $\{\cdot,\cdot\}_p$ of degree -2 on $S(V)^\Gamma$. Moreover, it is obtained as follows: take homogeneous $a_0, b_0 \in S(V)^\Gamma$. Then let a,b be homogeneous elements in eHe lifting a_0, b_0. Then $[a,b] \in P\mathcal{A}$. In particular, we can consider the projection $\{a_0,b_0\}$ of $[a,b]$ to $P \otimes S(V)^\Gamma = P\mathcal{A}/P^2\mathcal{A}$ (recall that $\mathcal{A} = S(P) \otimes S(V)^\Gamma$ as an $S(P)$-module). Then $\{a,b\}_p$ is a specialization of $\{a,b\}$ at $p : P \to \mathbb{C}$. This implies the claim on linearity.

9.3. **Step 2.** We claim that every bracket on $S(V)^\Gamma$ uniquely lifts to a Γ-invariant bracket on $S(V)$. To prove this we need to explore a geometric nature of brackets.

Let X be an affine algebraic variety. Suppose that $\mathbb{C}[X]$ is equipped with a bracket $\{\cdot,\cdot\}$. Pick a smooth point $x \in X$. Then we can define a bivector $\mathcal{P}_x \in \bigwedge^2 T_x X$ by setting $\langle \mathcal{P}_x, df \wedge dg\rangle = \{f,g\}(x)$. It is easy to see that this is well-defined. A bit more subtle observation (that is consequence of the fact that the tangent and cotangent sheaves on a smooth variety are locally free) is that the bivectors $\mathcal{P}_x$ glue together to form a section $\mathcal{P}$ of $\bigwedge^2 TX^{reg}$, where X^{reg} denotes the smooth locus of X. Now assume that the variety X is normal so that, in particular, $\mathbb{C}[X] = \mathbb{C}[X^{reg}]$. Then a bivector $\mathcal{P} \in \Gamma(X^{reg}, \bigwedge^2 TX^{reg})$ gives rise to a bracket on $\mathbb{C}[X^{reg}] = \mathbb{C}[X]$ – by $\{f,g\} = \langle \mathcal{P}, df \wedge dg\rangle$.

Another fact about brackets that we need is that a bracket can be pulled back by an étale morphism. Recall that a morphism $\varphi : Y \to X$ of smooth varieties is called étale at $y \in Y$ if $d_y\varphi$ is an isomorphism. We say that φ is étale if it is étale at any point. So we can identify $T_yY \cong T_{\varphi(x)}X$ and therefore also $\bigwedge^2 T_yY$ with $\bigwedge^2 T_{\varphi(y)}X$. A more subtle claim again, is that one can pull-back $\mathcal{P} \in \Gamma(X, \bigwedge^2 TX)$ to get a well-defined element $\varphi^*(\mathcal{P}) \in \Gamma(Y, \bigwedge^2 TY)$ (this follows from $TY = \varphi^*(TX)$).

Finally, we need to characterize $(V/\Gamma)^{reg}$ and find the locus, where the quotient morphism $\pi : V \to V/\Gamma$ is étale. This is explained in the following lemma, where we assume that Γ is just some finite subgroup of $\mathrm{GL}(V)$. Recall the notation $V^0 = \{v \in V | \Gamma_v = 0\}$.

Lemma 9.3. *We have $V^0/\Gamma \subset (V/\Gamma)^{reg}$. The morphism π is étale at all points of V^0.*

We are not going to prove the lemma. It is clear if we work in the complex analytic, not algebraic category. It also fixes a gap in the proof of a technical lemma of Step 2 in the proof of the double centralizer property. Finally, let us remark that if Γ contains no *complex* reflections (this is always the case for $\Gamma \subset \mathrm{Sp}(V)$), then the inclusions in the lemma are actually equalities.

Now we are ready to prove Step 2. Let $\{\cdot,\cdot\}$ be a bracket on $S(V)^\Gamma \cong \mathbb{C}[V]^\Gamma$ and let $\mathcal{P}$ be a corresponding bivector on $V^0/\Gamma \subset (V/\Gamma)^{reg}$. Then we get a bivector $\pi^*(\mathcal{P})$ on V^0 and hence a bracket $\{\cdot,\cdot\}'$ on $\mathbb{C}[V^0] = \mathbb{C}[V]$. The bivector and hence the bracket are Γ-equivariant by construction. Also by construction, the restriction of $\{\cdot,\cdot\}'$ to $\mathbb{C}[V]^\Gamma$ coincides with $\{\cdot,\cdot\}$ and $\{\cdot,\cdot\}'$ is a unique Γ-equivariant (the latter is not necessary) bracket with these properties. From here it follows that the degree of $\{\cdot,\cdot\}'$ is the same as that of $\{\cdot,\cdot\}$ (if $\{\cdot,\cdot\}'$ has components of other degrees, then they restrict to 0 on $\mathbb{C}[V]^\Gamma$).

So now the question is: describe Γ-equivariant brackets of degree $\leqslant -2$ on $\mathbb{C}[V]$. If the degree of $\{\cdot,\cdot\}'$ is less than -2, then this bracket vanishes on V^*, the degree 1 component of $\mathbb{C}[V]$. So $\{\cdot,\cdot\}'$ is identically 0. Similarly, the bracket of degree -2 just comes from a skew-symmetric form on $V^* \cong V$. This form is Γ-invariant. If V is symplectically irreducible, then there is a unique such form up to proportionality.

Let us remark that for some p we do have $\{\cdot,\cdot\}_p \neq 0$. Indeed, consider the case when $c = 0, t = 1$. Then $H_{t,c} = T(V)\#\Gamma/(u\otimes v - v\otimes u - \omega(u,v)) = W(V)\#\Gamma$ so that $eH_{t,c}e \cong W(V)^\Gamma$. As we have seen, the bracket on $S(V)$ induced by $W(V)$ coincides with the standard bracket. It follows that the bracket on $S(V)^\Gamma$ induced by $W(V)^\Gamma$ also coincides with the standard bracket on hence is nonzero.

So we have a hyperplane $P_0 \subset P^*$ such that $\{\cdot,\cdot\}_p = 0$ for all $p \in P_0$ and hence $\mathcal{A}_p$ is commutative.

Exercise 9.1. *Show that $\{\cdot,\cdot\}_{t,c} = t\{\cdot,\cdot\}$, where $\{\cdot,\cdot\}$ is the standard bracket on $S(V)^\Gamma$.*

Exercise 9.2. *Prove the commutativity theorem in the case when V is not necessarily symplectically irreducible.*

9.4. **Step 3.** According to the next exercise $\mathcal{A}_p$ is always finitely generated.

Exercise 9.3. *Let $\mathcal{A}$ be a $\mathbb{Z}_{\geqslant 0}$-filtered algebra. If $\operatorname{gr}\mathcal{A}$ is finitely generated, then so is $\mathcal{A}$.*

So for $p \in P_0$ one can consider $C_p := \operatorname{Spec}(\mathcal{A}_p)$ (a problem below implies that $\mathcal{A}_p$ has no zero divisors so that C_p is an irreducible variety). Recall that we have commuting actions of H_p and $\mathcal{A}_p$ on H_pe. As we have seen, it is enough to prove the following claim. Let $p \in P_0$, then for a general point $x \in C_p$ the H_p-module $H_pe/\mathfrak{m}_x$ is isomorphic to $\mathbb{C}\Gamma$ as a Γ-module. This amounts to checking that for any Γ-irreducible L the dimension of the fiber of $M_p^L := \operatorname{Hom}_\Gamma(L, H_pe)$ at a general point $x \in C_p$ equals $\dim L$ (here the action of $\mathcal{A}_p$ on $\operatorname{Hom}_\Gamma(L, H_pe)$ is induced from the action of $\mathcal{A}_p$ on H_pe from the right). We remark that this holds for $p = 0$: for $v \in V_0$ we have $\mathbb{C}[V]_{\pi(v)} = \mathbb{C}[\Gamma v] \cong \mathbb{C}\Gamma$.

Pick a nonzero parameter $p \in P_0$. To prove our claim we will need to include C_p and V/Γ into a single variety. For this let R be the one-dimensional subspace of P^* spanned by p. Consider the algebras $H_R := \mathbb{C}[R] \otimes_{S(P)} H, \mathcal{A}_R := eH_Re$. Then $\mathcal{A}_R$ is a deformation of $S(V)^\Gamma$ over $\mathbb{C}[R]$ and hence $\mathcal{A}_R$ is finitely generated. Set $C_R := \operatorname{Spec}(\mathcal{A}_R)$. This is an algebraic variety equipped with a $\mathbb{C}^\times$-action coming from the grading on $\mathcal{A}_R$ and also a $\mathbb{C}^\times$-equivariant morphism $C_R \to R$, whose zero fiber is V/Γ, while a nonzero fiber is naturally identified with C_p (the fiber over p is literally C_p, and all other fibers are translated to C_p using the $\mathbb{C}^\times$-action). The $\mathcal{A}_R$-module H_Re is flat over $\mathbb{C}[R]$. Consider the module $M_R^L := \operatorname{Hom}_\Gamma(L, H_Re)$. Let $\mathfrak{p}$ denote the ideal of $\mathcal{A}_R$ generated by R^*. Since $\mathcal{A}_R/\mathfrak{p} \cong \mathbb{C}[V]^\Gamma$, the ideal $\mathfrak{p}$ is prime. So we can consider the localization $\mathcal{A}_{R,\mathfrak{p}}$, a local ring of dimension 1 with a local parameter r, a basis element in R^*, and residue field $\mathbb{C}(V/\Gamma)$. We know that the localization $M_{R,\mathfrak{p}}^L$ is flat over $\mathbb{C}[r]$ and the fiber at $r = 0$ has dimension $\dim L$ over $\mathbb{C}(V/\Gamma)$. What we need to prove is that the dimension of the localization $\mathbb{C}(C_R) \otimes_{\mathcal{A}_R} M_R^L$ equals $\dim L$ (this implies existence of a required point x on a general fiber of $C_R \to R$ and, therefore, thanks to $\mathbb{C}^\times$-equivariance, on an arbitrary nonzero fiber). Thanks to flatness, $\dim_{\mathbb{C}(C_R)} \mathbb{C}(C_R) \otimes_{\mathcal{A}_R} M_R^L = \dim_{\mathbb{C}(V/\Gamma)} M_{R,\mathfrak{p}}^L/(r)$. The latter coincides with $\dim L$.

This completes the proof of the commutativity theorem.

9.5. **Satake isomorphism.**

Theorem 9.4 ([EG]). *Let Z_c be the center of $H_{0,c}$. The map $z \mapsto ez$ is an isomorphism of Z_c and $eH_{0,c}e$.*

Proof. We are going to find an inverse homomorphism. We write p for $(0, c)$. Recall that the action of H_p on H_pe gives rise to an isomorphism $H_p \xrightarrow{\sim} \operatorname{End}_{\mathcal{A}_p}(H_pe)$. Since the algebra $\mathcal{A}_p$ is commutative the map $m \mapsto mb$ is an endomorphism of the $\mathcal{A}_p$-module H_pe for any $b \in \mathcal{A}_p$. Such an endomorphism commutes with any other. Let $\hat{b}$ denote a unique element of $H_p = \operatorname{End}_{\mathcal{A}_p}(H_pe)$ such that $\hat{b}m = mb$. Clearly, $b \mapsto \hat{b}$ is an algebra homomorphism $\mathcal{A}_p \to Z_p$. The claim that this homomorphism is inverse to $z \mapsto ez$ means $\widehat{ez} = z, e\hat{b} = b$. We have $\widehat{ez}m = mez = mz = zm$ and so $\widehat{ez} = z$. On the other hand, $me\hat{b} = m\hat{b} = \hat{b}m = mb$. Plugging $m = e$, we get $e\hat{b} = eb = b$. $\square$

Problem 9.4. *Let $p \in P_0$. Equip Z_p with a filtration restricted from H_p. Show that $\operatorname{gr} Z_p = S(V)^\Gamma$. Deduce that H_p is a finitely generated module over Z_p.*

Problem 9.5. *Now let $p \notin P_0$. Show that the center of H_p coincides with $\mathbb{C}$ as follows:*

(1) *Let z lie in the center of H_p. Show that $\operatorname{gr} z \in \operatorname{gr} H_p = S(V)\#\Gamma$ actually lies in $S(V)^\Gamma$.*
(2) *Show that $\operatorname{gr} z$ lies in the Poisson center of $S(V)^\Gamma$, meaning that $\{\operatorname{gr} z, S(V)^\Gamma\} = 0$.*
(3) *Show that the Poisson center of $S(V)^\Gamma$ coincides with $\mathbb{C}$.*

Problem 9.6. *In this problem we are going to equip Z_c with a structure of a Poisson algebra. Fix c and consider $H_{t,c}$ as an algebra over $\mathbb{C}[t]$ by making t an independent variable.*

(1) *Let $a, b \in Z_c$. Lift $a, b \in H_c = H_{t,c}/(t)$ to elements $\tilde{a}, \tilde{b} \in H_{t,c}$. Show that $[\tilde{a}, \tilde{b}] \in tH_{t,c}$ and that the element $\frac{1}{t}[\tilde{a}, \tilde{b}]$ modulo t depends only on a, b and lies in Z_c. Let $\{a, b\}$ be that element. Show that $\{\cdot, \cdot\}$ is the Poisson bracket.*
(2) *Show that $\{Z_c^{\leqslant i}, Z_c^{\leqslant j}\} \subset Z_c^{i+j-2}$. Show that the induced bracket on $\operatorname{gr} Z_c = S(V)^\Gamma$ is a nonzero multiple of the standard bracket. Can you identify the scalar factor?*

Problem 9.7. *Show that the scheme C_p is irreducible and normal (and, well, Cohen-Macaulay and Gorenstein, if you know what these words mean).*

9.6. **Further algebraic properties.** Perhaps, the first question about the structure of an irreducible normal (and also Cohen-Macaulay and Gorenstein) algebraic variety you can ask is whether it is smooth.

First of all, one can describe the smooth points $x \in C_p$ in terms of the representation theory of the algebra $H_p/H_p\mathfrak{m}_x$.

Theorem 9.5 ([EG])**.** *The following are equivalent.*

(1) $x \in C_p^{reg}$.
(2) $H_p/H_p\mathfrak{m}_x \cong \operatorname{End}(\mathbb{C}\Gamma)$ *(a Γ-equivariant algebra isomorphism).*
(3) *Any simple $H_p/H_p\mathfrak{m}_x$-module is isomorphic to $\mathbb{C}\Gamma$ as a Γ-module.*

Problem 9.8. *Show that if C_p is smooth, then H_pe is a locally free H_p-module.*

The proof is based on properties of PI (polynomial identity) rings and we are not going to provide it.

Let us explain what is known about smoothness of the varieties C_p. First, of all existence of p such that C_p is smooth is a very restrictive assumption on Γ.

If $\Gamma = \Gamma_n$ is a wreath-product $\mathfrak{S}_n \ltimes \Gamma_1^n$, where $\Gamma_1 \subset \mathrm{SL}_2(\mathbb{C})$, then C_p is smooth if and only if p lies outside the union of explicitly described hyperplanes. This follows from the interpretation of C_p as affine quiver varieties to be covered later in this course.

In the class of complex reflection groups, the answer is also known. Besides the groups $G(\ell, 1, n)$ that belong also to the previous list, there is only one complex reflection group such that there is a smooth C_p. It is the group G_4 that appeared in one of the problems of Lecture 6. This was proved by Bellamy in [B].

The situation with the groups that do not belong to one of these families is unknown. Recently, Bellamy and Schedler, [BS], found an example of such a group that does admit smooth C_p.

This question has to do with sphericity of parameters discussed last time. Namely, C_p is smooth iff p is spherical (meaning that $H_peH_p = H_p$). Indeed, the smoothness of $C_p = \operatorname{Spec}(\mathcal{A}_p)$ is equivalent to $\mathcal{A}_p$ having finite global dimension. The following problem implies that H_p has finite global dimension (for an arbitrary $p \in P^*$).

Problem 9.9. *Let $\mathcal{A}$ be a filtered algebra. Show that if $\operatorname{gr} \mathcal{A}$ has finite global dimension, then $\mathcal{A}$ does.*

As we have discussed earlier, $S(V)\#\Gamma$ has finite global dimension (equal $\dim V$) and so the global dimension of H_p is finite. Now the global dimension is an invariant of the category of modules, and $H_p e H_p = H_p$ means that the categories of modules for H_p and eH_pe are equivalent. So if p is spherical, then C_p is smooth.

Problem 9.10. *Conversely, prove that if C_p is smooth, then p is spherical (we deal here with $p \in P_0$).*

In fact, $p \in P^*$ is spherical iff the global dimension of eH_pe is finite even if the latter is not commutative. This is a result of Bezrukavnikov, [E2].

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LECTURES ON SYMPLECTIC REFLECTION ALGEBRAS

IVAN LOSEV

10. Moment maps in algebraic setting

10.1. **Symplectic algebraic varieties.** An affine algebraic variety X is said to be *Poisson* if $\mathbb{C}[X]$ is equipped with a Poisson bracket.

Exercise 10.1. *Let A be a commutative algebra and B be a localization of A. Let A be equipped with a bracket. Show that there is a unique bracket on B such that the natural homomorphism $A \to B$ respects the bracket.*

Thanks to this exercise, the *sheaf* $\mathcal{O}_X$ of regular functions on X acquires a bracket (i.e., we have brackets on all algebras of sections and the restriction homomorphisms are compatible with the bracket). We say that an arbitrary (=not necessarily affine) variety X is Poisson if the sheaf $\mathcal{O}_X$ comes equipped with a Poisson bracket.

Recall that on a variety X such that $\mathcal{O}_X$ is equipped with a bracket we have a bivector (=a bivector field) $P \in \Gamma(X^{reg}, \bigwedge^2 TX^{reg})$. This gives rise to a map $v_x : T^*_x X \to T_x X$ for $x \in X^{reg}, \alpha \mapsto P_x(\alpha, \cdot)$. We say that P is nondegenerate in x if this map is an isomorphism. In this case, we can use this map to get a 2-form $\omega_x \in \bigwedge^2 T^*_x X$: $\omega_x(v_x(\alpha), v_x(\beta)) = P_x(\alpha, \beta) = \langle \alpha, v_x(\beta)\rangle = -\langle v_x(\alpha), \beta\rangle$.

Now suppose X is smooth. Suppose that P is non-degenerate (=non-degenerate at all points). So we have a non-degenerate form ω on X. The condition that P is Poisson is equivalent to $d\omega = 0$. A non-degenerate closed form ω is called *symplectic* (and X is called a *symplectic variety*).

The most important for us class of symplectic varieties is cotangent bundles. Let X_0 be a smooth algebraic variety, set $X := T^*X_0$. A symplectic form ω on X is introduced as follows. First, let us introduce a canonical 1-form α. We need to say how α_x pairs with a tangent vector for any $x \in X$. A point X can be thought as a pair (x_0, β), where $x_0 \in X_0$ and $\beta \in T^*_{x_0} X_0$. Consider the projection $\pi : X \twoheadrightarrow X_0$ (defined by $\pi(x) = x_0$). For $x = (x_0, \beta)$ we define α_x by $\langle \alpha_x, v\rangle = \langle \beta, d_x\pi(v)\rangle$.

We can write α in "coordinates". If we worked in the C^∞- or analytic setting, we could use the usual coordinates. However, we cannot do this because we want to show that α is an algebraic form. So we will use an algebro-geometric substitute for coordinate charts: étale coordinates. Namely, we can introduce étale coordinates in a neighborhood of each point $x_0 \in X_0$. Let us choose functions $x^1, \ldots, x^n$ with a property that $d_{x_0}x^1, \ldots d_{x_0}x^n$ form a basis in $T^*_{x_0} X_0$. Then $dx^1, \ldots, dx^n$ are linearly independent at any point from some neighborhood X_0^0 of x_0. So the map $\varphi : X_0^0 \to \mathbb{C}^n$ given by $(x^1, \ldots, x^n)$ is étale and we call $x^1, \ldots, x^n$ étale coordinates. Then we can get étale coordinates $y_1, \ldots, y_n$ on $T^*X_0^0$ as follows: by definition $y^i(x_0, \beta)$ is the coefficient of $d_{x_0}x^i$ in β, i.e., $\beta = \sum_{i=1}^n y_i(x_0, \beta) d_{x_0}x^i$ (and we view $x^1, \ldots, x^n$ as functions on $T^*X_0^0$ via pull-back). Then, on $T^*X_0^0$, α is given by $\sum_{i=1}^n y_i dx^i$.

There is an important remark about α: it is canonical. In particular, if we have a group action on X_0, it naturally lifts to T^*X_0: $g(x_0, \beta) = (gx_0, g_{*x_0}\beta)$, where g_{*x_0} is the isomorphism

$T^*_{x_0}X_0 \to T^*_{gx_0}X_0$ induced by g. The coordinate free definition of α implies that α is invariant under any such group action on T^*X_0.

Now set $\omega = -d\alpha$ so that, in the étale coordinates, $\omega = \sum_{i=1}^n dx^i \wedge dy_i$. We immediately see that ω is a symplectic form. Also let us point out that if X_0 is a vector space, then ω is a constant form (=skew-symmetric bilinear form) on the double vector space $X_0 \oplus X_0^*$. The remark in the previous paragraph applies to ω as well.

10.2. **Hamiltonian vector fields.** Let X be a Poisson variety and f be a local section of $\mathcal{O}_X$. Then we can form the vector field $v(f) = P(df, \cdot)$ (defined in the domain of definition of f). This is called the *Hamiltonian vector field* (or the *skew gradient*) of f. Clearly, v is linear, and satisfies the Leibniz identity $v(fg) = gv(f) + fv(g)$. Further, we have

$$L_{v(f)}g = -\langle v(f), dg\rangle = \{f, g\}. \tag{1}$$

Here and below we write L_ξ for the Lie derivative of ξ so that $L_\xi f = -\partial_\xi f$. Recall that in the C^∞-situation, the Lie derivative is defined as follows. We pick a flow $g(t)$ produced by the vector field ξ and then for a tensor field τ define $L_\xi\tau = \frac{d}{dt}g(t)_*\tau|_{t=0}$. In particular, if τ is a function f, then we get $L_\xi(f) = \frac{d}{dt}f(g(-t))|_{t=0} = -\partial_\xi f$. If τ is a vector field, then $L_\xi\tau = [\xi, \tau]$, where, by convention, the bracket on the vector fields is introduced by $L_{[\xi,\eta]}f = [L_\xi, L_\eta]f$. Finally, if τ is a form, then we have the Cartan formula:

$$L_\xi\tau = -d\iota_\xi\tau - \iota_\xi d\tau, \tag{2}$$

where ι_ξ stands for the contraction with ξ (as the first argument): $\iota_\xi\tau(\ldots) = \tau(\xi, \ldots)$. In particular, if both ξ and τ are algebraic, then so is $L_\xi\tau$, and we can <u>define</u> $L_\xi\tau$ using the formulas above.

Using (1) and the Jacobi identity for $\{\cdot,\cdot\}$, we deduce that the map $f \mapsto v(f)$ is a Lie algebra homomorphism. Also we remark that every Hamiltonian vector field is Poisson, i.e.,

$$L_{v(f)}P = 0 \tag{3}$$

(this is yet another way to state the Jaconi identity for $\{\cdot,\cdot\}$).

If X is symplectic, we can rewrite the definition of the Hamiltonian vector field as

$$\iota_{v(f)}\omega = df. \tag{4}$$

Also we have

$$\omega(v(f), v(g)) = \{f, g\} \tag{5}$$

and

$$L_{v(f)}\omega = 0. \tag{6}$$

So in this case $f \mapsto v(f)$ is a Lie algebra homomorphism between $\mathbb{C}[X]$ and the algebra $\mathrm{SVect}(X)$ of symplectic vector fields on X.

Consider the case of $X = T^*X_0$, where, for simplicity, we assume that X_0 is affine. Then $\mathbb{C}[X] = S_{\mathbb{C}[X_0]}(\mathrm{Vect}(X_0))$. As a function on $\mathbb{C}[X]$ the vector field ξ is given by

$$\xi(x_0, \beta) = \langle \beta, \xi_{x_0}\rangle. \tag{7}$$

Let us compute the vector fields $v(f), f \in \mathbb{C}[X_0]$, and $v(\xi), \xi \in \mathrm{Vect}(X_0)$. We claim that $v(f) = -df$, viewed as a vertical vector field on T^*X_0, its value on the fiber $T^*_{x_0}X_0$ is constant $-d_{x_0}f$. To avoid confusion below we will write Df for the vector field df. Further, to a vector field ξ on X_0 we can assign a vector field $\tilde{\xi}$ on X by requiring $L_{\tilde{\xi}}\eta = [\xi, \eta], L_{\tilde{\xi}}g = L_\xi g$ for $\eta \in \mathrm{Vect}(X_0), g \in \mathbb{C}[X_0]$. We claim that $v(\xi) = \tilde{\xi}$. The vector field $\tilde{\xi}$ has the following meaning. Assume that we are in the C^∞-setting. Then to ξ we can assign its flow $g(t)$ (of diffeomorphisms of X_0). Then we can canonically lift this flow to T^*X_0. The vector field $\tilde{\xi}$ is associated to the lifted flow. In particular, from this description one sees that $d_{(x_0,\beta)}\tilde{\xi} = \xi_{x_0}$.

Applying (2) to $\tau = \alpha$ and a vector field η on T^*X_0, and using $-d\alpha = \omega$, we get $L_\eta\alpha = -d\iota_\eta\alpha + \iota_\eta\omega$ and so

$$\iota_\eta\omega = L_\eta\alpha + \iota_\eta\omega. \tag{8}$$

If $\eta = -Df$, then $\iota_\eta\alpha = 0$ (α vanishes on all vertical vector fields by the coordinate free construction). So we get $\iota_{-Df}\omega = L_{-Df}\alpha$. Again, the construction of α implies that $L_{-Df}\alpha = \partial_{Df}\alpha = df$ (in local coordinates we have $\partial_{Df}\alpha = \sum_{i=1}^n \partial_{Df}y_i dx^i = \sum_{i=1}^n \partial_{x_i} f dx^i = df$). So $\iota_{-Df}\omega = df = \iota_{v(f)}\omega$ so $v(f) = -Df$.

Now let us check that $v(\xi) = \tilde{\xi}$. We claim that $L_{\tilde{\xi}}\alpha = 0$. In the C^∞-setting, this follows from the observation that α is preserved by any diffeomorphism of T^*X_0 lifted from X_0. Since all formulas in the algebraic setting are the same as in the C^∞ one, we get our claim. Also we remark that by the construction of $\tilde{\xi}$, we have $d\pi(\tilde{\xi}) = \xi$ and therefore, thanks to (7), $\iota_{\tilde{\xi}}\alpha = \xi$ (as functions on T^*X_0). So we have $\iota_{\tilde{\xi}}\omega = d\iota_{\tilde{\xi}}\alpha$. But $\iota_{\tilde{\xi}}\alpha$ is ξ, by the description of the function ξ above.

Exercise 10.2. *Show that the Poisson bracket on $\mathbb{C}[X]$ can be characterized as follows: we have $\{f,g\} = 0, \{\xi, f\} = L_\xi f, \{\xi,\eta\} = [\xi,\eta]$ for $f,g \in \mathbb{C}[X_0], \xi,\eta \in \mathrm{Vect}(X_0)$. Deduce that, with respect to the standard grading on $\mathbb{C}[X] = S_{\mathbb{C}[X_0]}(\mathrm{Vect}(X_0))$, the bracket has degree -1.*

The construction of Hamiltonian vector fields is of importance in Classical Mechanics. Namely, we can consider a mechanical system on a Poisson variety X whose velocity vector is $-v(H)$ so that $\frac{d}{dt}f(x(t)) = (L_{v(H)}f)(x(t))$. In this case, the function H is interpreted as the *Hamiltonian* (=the full, i.e., "kinetic + potential", energy) of this system. The condition on a function F to be a first integral (=preserved quantity) of this system is $L_{v(H)}F = 0$, i.e., $\{H,F\} = 0$. In particular, H itself is the first integral (the energy conservation law).

Let us consider a very classical example. Let X_0 (a configuration space) be an open subset in $\mathbb{C}^n$ with coordinates $x^1,\ldots,x^n$. Consider the mechanical system with potential $V = V(x^1,\ldots,x^n)$, its evolution is given by $\ddot{x}^i = -\frac{\partial V}{\partial x^i}$. Introduce new variables $y_i = \dot{x}^i$ and the Hamiltonian $H = \frac{1}{2}\sum_{i=1}^n y_i^2 + V$. Then we can rewrite the system as $\dot{x}^i = y_i = \frac{\partial H}{\partial y_i} = -\{H, x_i\}, \dot{y}_i = -\frac{\partial H}{\partial x^i} = -\{H, y_i\}$. So H becomes the Hamiltonian of our system (considered on the *phase space* T^*X_0).

10.3. **Moment maps.** Now let X be a smooth variety equipped with an action of an algebraic group G. Let $\mathfrak{g}$ be the Lie algebra of G. To the G-action one assigns a Lie algebra homomorphism $\mathfrak{g} \to \mathrm{Vect}(X), \xi \mapsto \xi_X$. In the C^∞-setting, ξ_X is the vector field associated to the flow $\exp(t\xi)$. The definition in the algebraic setting is a bit more subtle. If X is affine, then this homomorphism can be described as follows. We have the induced action of G on $\mathbb{C}[X]$. Every function lies in a finite dimensional G-stable subspace. So we have a representation of $\mathfrak{g}$ in $\mathbb{C}[X]$ and this representation is by derivations, let ξ_X be the derivation corresponding to ξ. An important special case: if X is a vector space and the G-action is linear, then $\xi_{X,x} = \xi x$, the image of x under the operator corresponding to ξ. We write ξ_x for the value of this field at $x \in X$, and $\mathfrak{g}x$ for $\{\xi_x | \xi \in \mathfrak{g}\}$, of course, $\mathfrak{g}x = T_x(Gx)$. In the non-affine case one needs to use some structural results regarding algebraic group actions.

Now assume that X is symplectic with form ω and that G preserves ω. Then $L_{\xi_X}\omega = 0$ so we have a homomorphism $\mathfrak{g} \to \mathrm{SVect}(X)$. This homomorphism is obviously G-equivariant.

Also we have a homomorphism $\mathbb{C}[X] \to \mathrm{SVect}(X)$ given by $f \mapsto v(f)$, it is also G-equivariant. We say that the action is *Hamiltonian*, if there is a G-equivariant Lie algebra homomorphism $\xi \mapsto H_\xi, \mathfrak{g} \to \mathbb{C}[X]$ such that $v(H_\xi) = \xi_X$.

Exercise 10.3. *Show that a G-equivariant map $\xi \mapsto H_\xi$ with $v(H_\xi) = \xi_X$ is automatically a Lie algebra homomorphism.*

The map $\xi \mapsto H_\xi$ is called *a comoment map*. By the *moment map* we mean the dual map, $\mu : X \to \mathfrak{g}^*$, given by $\langle \mu(x), \xi \rangle = H_\xi(x)$ (this map is dual to the homomorphism $\mathbb{C}[\mathfrak{g}^*] = S(\mathfrak{g}) \to \mathbb{C}[X]$ induced by $\xi \mapsto H_\xi$). The map μ is G-equivariant and satisfies $\langle d_x\mu, \xi \rangle = \omega(\xi_X, \cdot)$ (the equality of elements of $T^*_x X$, both sides are just $d_x H_\xi$).

We remark that the (co)moment map is not determined uniquely.

Exercise 10.4. *Let μ, μ' be two moment maps, and X be connected. Then $\mu - \mu'$ is a constant function equal to some G-invariant element of $\mathfrak{g}^{*G}$.*

The following exercise describes some properties of the kernel and the image of $d_x\mu$.

Exercise 10.5. *Prove that $\ker d_x\mu = (\mathfrak{g}x)^\perp$ and $\operatorname{im} d_x\mu = \mathfrak{g}_x^\perp$, where in the first equality the superscript $\perp$ stands for the skew-orthogonal complement with respect to ω_x, and in the second case for the annihilator in the dual space; we write $\mathfrak{g}_x$ for the Lie algebra of stabilizer G_x. Deduce that $d_x\mu$ is surjective if and only if G_x is discrete.*

Let us consider the example of cotangent bundles. Let G act on X_0. Then this action canonically lifts to a G-action on $X = T^*X_0$ preserving α and $\omega = -d\alpha$. We claim that the assignment $H_\xi = \xi_{X_0}$ is a comoment map. Indeed, we have $\xi_X = \tilde{\xi}_{X_0}$ (the easiest way to see this is to use the C^∞-description) and, as we have already seen, $v(\xi_{X_0}) = \tilde{\xi}_{X_0}$.

Exercise 10.6. *Let $\mu : T^*X_0 \to \mathfrak{g}^*$ be the moment map. Show that $\mu^{-1}(0)$ is the union of conormal bundles to the G-orbits in X_0.*

Problem 10.7. *Let G act on a vector space V with finitely many orbits. Show that G acts on V^* with finitely many orbits and exhibit a bijection between the two sets of orbits.*

We will still need a further specialization that we have already met in Lecture 3. Take a quiver $\underline{Q} = (Q_0, \underline{Q}_1)$ and consider the double quiver $Q = (Q_0, Q_1)$, where, for each arrow $a \in \underline{Q}_1$, we add an opposite arrow a^*. Fix a dimension vector $v = (v_i)_{i \in Q_0}$ and consider the representation space $R_0 = \mathrm{Rep}(\underline{Q}, v)$. This space has a natural action of $G = \mathrm{GL}(v) (= \prod_{i \in Q_0} \mathrm{GL}(v_i))$. For each arrow a, we identify the space $\mathrm{Hom}(\mathbb{C}^{v_{h(a)}}, \mathbb{C}^{v_{t(a)}})$ with $\mathrm{Hom}(\mathbb{C}^{v_{h(a)}}, \mathbb{C}^{v_{t(a)}})^*$ by means of the trace form, $\langle A, B \rangle := \mathrm{tr}(AB)$. Also we identify the Lie algebra $\mathfrak{g} = \mathfrak{gl}(v)$ with its dual in a similar way. So $R := \mathrm{Rep}(Q, v)$ becomes identified with $R_0 \oplus R_0^* = T^*R_0$ and we can view the moment map μ as a morphism $R \to \mathfrak{g}$.

Proposition 10.1. *We have $\mu = (\mu_i)_{i \in Q_0}$, where*

$$\mu_i(x_a, x_{a^*}) = \sum_{a \in \underline{Q}_1, h(a) = i} x_a x_{a^*} - \sum_{a \in \underline{Q}_1, t(a) = i} x_{a^*} x_a.$$

This is different by the sign from what we had before.

Proof. We start with a few general properties concerning products of varieties/groups and restrictions to subgroups. Most of these properties follow from the definitions in a straightforward way.

(i) If $G_1 \times G_2$ acts on X_0, then $\mu_{G_1 \times G_2}(x) = (\mu_{G_1}(x), \mu_{G_2}(x))$.
(ii) If G acts acts on $X_0 \times X_0'$, then $\mu_G(x, x') = \mu_G(x) + \mu_G(x')$ (because $\xi_{X_0 \times X_0'} = (\xi_{X_0}, \xi_{X_0'})$).
(iii) Finally, if H is a subgroup of G, then $\mu_H(x) = \rho \circ \mu_G(x)$, where $\rho : \mathfrak{g}^* \to \mathfrak{h}^*$ is the restriction map.
(iv) Let V_0 be a vector space. The moment maps for $V = V_0 \oplus V_0^*$ viewed as T^*V_0 and as $T^*V_0^*$ are negative of each other (because, first, the forms are negatives of each other, and, second, we have chosen unique moment maps that are homogeneous quadratic).

The variety R_0 is the direct product of the Hom spaces. Using (ii) (and an easy part of (i) when one of the groups acts trivially) we reduce the proof to the case when $\underline{Q}_1$ has a single arrow a. Here we have two cases. First, $a : i \to j$ is not a loop and the group acting is $\mathrm{GL}(v_i) \times \mathrm{GL}(v_i)$. Second, $a : i \to i$ is a loop and the group acting is $\mathrm{GL}(v_i)$.

Let us consider the first case. By (i), we can compute μ_j and μ_i separately. Consider μ_j. We need to show that for, $A \in \mathrm{Hom}(\mathbb{C}^{v_i}, \mathbb{C}^{v_j}), B \in \mathrm{Hom}(\mathbb{C}^{v_j}, \mathbb{C}^{v_i})$, we have $\mu(A, B) = AB$. We have $\xi_A = \xi A$. So $\mathrm{tr}(\mu(A, B)\xi) = \langle B, \xi A \rangle = \mathrm{tr}(B\xi A) = \mathrm{tr}(AB\xi)$ and hence $\mu(A, B) = AB$. Using (iv) we deduce that $\mu_i(A, B) = -BA$.

To get the case of a loop from the previous case we notice that the action of $\mathrm{GL}(v_i)$ on $\mathrm{Hom}(\mathbb{C}^{v_i}, \mathbb{C}^{v_i})$ is obtained by embedding $\mathrm{GL}(v_i)$ diagonally to $\mathrm{GL}(v_i) \times \mathrm{GL}(v_i)$. Under our identification of $\mathfrak{gl}(v_i) \cong \mathfrak{gl}(v_i)^*$, the map ρ just sends (X, Y) to $X + Y$. It remains to apply (iii). $\square$

Problem 10.8. *Let V be a symplectic vector space with form ω and let G act on V via a homomorphism $G \to \mathrm{Sp}(V)$. Show that this action is Hamiltonian with $H_\xi(v) = \frac{1}{2}\omega(\xi v, v)$.*

The importance of moment maps in Mechanics comes from the observation that the functions H_ξ are the first integrals of any G-invariant Hamiltonian system. So all trajectories are contained in fibers of μ.

Problem 10.9. *This problem discusses symplectic forms on coadjoint orbits. Let G be an algebraic group. Pick $\alpha \in \mathfrak{g}^*$.*

(1) *Equip $T_\alpha G\alpha$ with a form ω_α by setting $\omega_\alpha(\xi_\alpha, \eta_\alpha) = \langle \alpha, [\xi, \eta] \rangle$. Prove that this is well-defined.*
(2) *Show that ω_α extends to a unique G-invariant form on $G\alpha$ (the Kirillov-Kostant form) and that this form is symplectic. Further, show that the G-action on $G\alpha$ is Hamiltonian with moment map being the inclusion.*
(3) *Let X be a homogeneous space for G equipped with a symplectic form ω such that the G-action is Hamiltonian with moment map μ. Show that the image of μ is a single orbit, say $G\alpha$, that μ is a locally trivial covering, and that ω is obtained as the pull-back of the Kirillov-Kostant form.*

LECTURES ON SYMPLECTIC REFLECTION ALGEBRAS

IVAN LOSEV

11. Calogero-Moser system and Hamiltonian reduction

11.1. **Calogero-Moser system.** The Calogero-Moser system is the system of n distinct points of the same mass, say 1, on the line (we work over $\mathbb{C}$ so we consider the complex line) with coordinates $x^1,\ldots,x^n$ that interact with pairwise potentials of the form $\frac{c}{(x^i-x^j)^2}$, where c is some nonzero constant. We can rescale and assume that $c=-1$. The total potential is $V=-\sum_{i<j}\frac{1}{(x_i-x_j)^2}$.

We want to consider the corresponding Hamiltonian system but we first need to decide what will be the symplectic variety to "accommodate" the system. Our original configuration space is $(\mathbb{C}^n)^{Reg}:=\{(x^1,\ldots,x^n)|x^i\neq x^j,\forall i\neq j\}$. So we could take the variety $X:=T^*(\mathbb{C}^n)^{Reg}$ and consider the Hamiltonian $H=\frac{1}{2}\sum_{i=1}^n y_i^2-\sum_{i<j}\frac{1}{(x_i-x_j)^2}$. However, there is a better choice. Our points are indistinguishable and so we can view $(x^1,\ldots,x^n)$ as an unordered n-tuple. So the configuration space is $(\mathbb{C}^n)^{Reg}/\mathfrak{S}_n$ and we consider its cotangent bundle $X:=T^*(\mathbb{C}^n)^{Reg}/\mathfrak{S}_n$. Also $\mathfrak{S}_n$ acts naturally on $T^*(\mathbb{C}^n)^{Reg}$, the action is induced from $(\mathbb{C}^n)^{Reg}$ and hence preserves the symplectic form. As the following exercise shows $T^*((\mathbb{C}^n)^{Reg}/\mathfrak{S}_n)=(T^*(\mathbb{C}^n)^{Reg})/\mathfrak{S}_n$. So we can view H (that is an $\mathfrak{S}_n$-invariant function) as a function on X. This is a Hamiltonian of our system. Below we will write C^{Reg} instead X. We will see that it is still not the best possible phase space for our system: we can actually embed C^{Reg} to some symplectic affine variety C (Calogero-Moser space) to "avoid collisions".

Exercise 11.1. *Let X_0 be a smooth algebraic variety equipped with a free action of a finite group Γ. Show that $T^*(X_0/\Gamma)$ is naturally identified with $(T^*X_0)/\Gamma$ (an isomorphism of symplectic varieties).*

Let us explain what kind of results regarding the Calogero-Moser (CM) system we want to get. First of all, we want to describe the trajectories, as explicitly as possible. Second, we want to produce 1st integrals subject to certain conditions. Namely, we want 1st integrals $H_1,\ldots,H_n$ with $H_2=H$ such that $\{H_i,H_j\}=0$ for all i,j and $d_xH_1,\ldots,d_xH_n$ being linearly independent for a general point $x\in X$. By some general results of Algebraic geometry (generic smoothness) the last condition is equivalent to $H_1,\ldots,H_n$ being algebraically independent. As the following exercise shows, here n is the maximal possible number for which such first integral may exist.

Exercise 11.2. *Let $f_1,\ldots,f_m$ be functions on a symplectic variety X such that $\{f_i,f_j\}=0$ for all i,j. Show that the dimension of the span of $d_xf_1,\ldots,d_xf_m$ has dimension not exceeding $\frac{1}{2}\dim X$.*

Systems admitting such collection of functions $H_1,\ldots,H_n$ (with Hamiltonian H being one of them) are called *completely integrable*. The reason is the Arnold-Liouville theorem that roughly states that (well, under some additional assumptions, and in C^∞-setting) such systems can be explicitly integrated. As we can explicitly integrate the system under consideration without using that theorem, we want to skip the details.

11.2. **Trajectories and 1st integrals of CM system.** We will follow an approach by Kazhdan, Kostant and Sternberg, [KKS], based on Hamiltonian reduction. A key observation is as follows. Pick a point $p \in T^*(\mathbb{C}^n)^{Reg}, p = (x^1, \ldots, x^n, y_1, \ldots, y_n)$ and construct two matrices, X_p, Y_p from p:

$$X_p = \operatorname{diag}(x^1, x^2, \ldots, x^n), Y_p = (y_{ij})_{i,j=1}^n, y_{ii} := y_i, y_{ij} := \frac{1}{x^i - x^j}, i \neq j.$$

The first indication that it is a reasonable thing to consider is that $H(p) = \frac{1}{2}\operatorname{tr}(Y_p^2)$. Also we notice that $[X_p, Y_p]$ has 0's on the diagonal, and 1's elsewhere (the anti-unit matrix). So $[X_p, Y_p] + E$ has only 1's and so has rank 1 and trace n. Set $O := \{A \in \operatorname{Mat}_n(\mathbb{C}) | \operatorname{tr} A = 0, \operatorname{rk}(A + E) = 1\}$, this is a single conjugacy class. Consider the adjoint $G := \operatorname{GL}(n)$-action on $R := \operatorname{Mat}_n(\mathbb{C})^2$. This is an action of the type considered in the last lecture – the action on the representation space of a double quiver. So $\mu : R \to \mathfrak{g}, \mu(X, Y) := [X, Y]$ is the moment map for this action. So we get a map $\iota : T^*(\mathbb{C}^n)^{Reg} \hookrightarrow \mu^{-1}(O), p \mapsto (X_p, Y_p)$. The image is definitely contained in the subset $\mu^{-1}(O)^{Reg}$ consisting of all $(X, Y) \in \mu^{-1}(O)$ such that X has distinct eigenvalues. We remark that $\mu^{-1}(O)^{Reg}$ is a principal open subset of $\mu^{-1}(O)$, it is the non-vanishing locus of a G-invariant polynomial.

Exercise 11.3. *Show that the action of G on $\mu^{-1}(O)^{Reg}$ is free. Also check that $\operatorname{im}\iota$ intersects any orbit and that elements $\iota(p), \iota(p')$ are G-conjugate if and only if p and p' are $\mathfrak{S}_n$-conjugate.*

Problem 11.4. *Show that the action of G on $\mu^{-1}(O)$ is free.*

This problem implies that all orbits of G in $\mu^{-1}(O)$ are closed and hence the categorical quotient $\mu^{-1}(O)//G$ coincides with the naive quotient. Also we see that $\mu^{-1}(O)^{Reg}//G$ is a principal open subset in $\mu^{-1}(O)//G$ and coincides with the naive quotient set-theoretically.

Thanks to 11.3, we can identify C^{Reg} with the naive orbit space $\mu^{-1}(O)^{Reg}/G$ (below we will see that our identification is an isomorphism of varieties). The following is our main result concerning the integration of Calogero-Moser systems.

Theorem 11.1 ([KKS])**.** (1) *Let $p = p(0)$ be a point in $C^{Reg} = \mu^{-1}(O)^{Reg}/G$ and let $p(t)$ be its trajectory. The pairs $(X_{p(t)}, Y_{p(t)})$ and $(X_p - tY_p, Y_p)$ lie in the same G-orbit.*
(2) *The functions $H_k(p) = \operatorname{tr}(Y_p^k), k = 1, \ldots, n$ on C^{Reg} commute w.r.t. $\{\cdot, \cdot\}$ and are linearly independent thus making the CM system completely integrable.*

For this we will equip $C := \mu^{-1}(O)//G$ with a symplectic form. Then for $H \in \mathbb{C}[R]^G$ we can consider the induced function $\underline{H} \in \mathbb{C}[C]$. We will see that the trajectories for $\underline{H}$ are obtained by projecting those for H. Finally, we will show that the embedding $C^{Reg} \hookrightarrow C$ respects the symplectic forms.

The Poisson structure on $\mu^{-1}(O)/G$ is obtained by the procedure called Hamiltonian reduction. We will explain this procedure in the next two sections, first on the algebraic level and then on the geometric one.

11.3. **Hamiltonian reduction, algebraically.** Let A be a Poisson algebra and G be an algebraic group acting on A by Poisson algebra automorphisms and rationally. Since the action is rational, it differentiates to a natural Lie algebra homomorphism $\mathfrak{g} \to \operatorname{Der}(A), \xi \mapsto \xi_A$. We also require that the G-action on A is Hamiltonian, i.e., we have a G-equivariant

Lie algebra homomorphism $\xi \mapsto H_\xi : \mathfrak{g} \to A$ such that $\xi_A = \{H_\xi, \cdot\}$. Let μ^* denote the corresponding homomorphism $S(\mathfrak{g}) \to A$. Then the ideal $A\mu^*(\mathfrak{g})$ is G-stable. By the Hamiltonian reduction $A///_0G$ we mean the algebra $[A/A\mu^*(\mathfrak{g})]^G$.

The point of considering this algebra is that it comes equipped with a natural Poisson bracket. Namely, for $a+A\mu^*(\mathfrak{g}), b+A\mu^*(\mathfrak{g}) \in [A/A\mu^*(\mathfrak{g})]^G$, we set $\{a+A\mu^*(\mathfrak{g}), b+A\mu^*(\mathfrak{g})\} := \{a,b\}+A\mu^*(\mathfrak{g})$. We, first, need to check that this is well-defined, i.e., $\{A\mu^*(\mathfrak{g}), b+A\mu^*(\mathfrak{g})\} \subset A\mu^*(\mathfrak{g})$ as long as $b + A\mu^*(\mathfrak{g}) \in [A/A\mu^*(\mathfrak{g})]^G$. The latter is equivalent to $gb - b \in A\mu^*(\mathfrak{g})$ and implies $\xi_A b \in A\mu^*(\mathfrak{g})$ for all $\xi \in \mathfrak{g}$. But $\xi_A b = \{H_\xi, b\}$. So $\{A\mu^*(\mathfrak{g}), b\} \subset \{A, b\}\mu^*(\mathfrak{g}) + A\{\mu^*(\mathfrak{g}), b\}$. Both summands are contained in $A\mu^*(\mathfrak{g})$, the second because our choice of b. Also $\{A\mu^*(\mathfrak{g}), A\mu^*(\mathfrak{g})\} \subset A\mu^*(\mathfrak{g})$ because $\{\mu^*(\mathfrak{g}), \mu^*(\mathfrak{g})\} \subset \mu^*(\mathfrak{g})$. This completes the proof of the fact that the bracket on $A///_0G$ is well-defined. Since the bracket $\{a+A\mu^*(\mathfrak{g}), b+A\mu^*(\mathfrak{g})\}$ is well-defined, it is G-invariant. Indeed, $g\{a,b\} = \{ga, gb\}$ coincides with $\{a,b\}$ modulo $A\mu^*(\mathfrak{g})$. So we do get a well-defined bracket on $[A/A\mu^*(\mathfrak{g})]^G$.

This construction is not sufficient for our purposes and we will need its generalization. Let us notice that $S(\mathfrak{g})$ comes equipped with a natural Poisson bracket given by $\{\xi, \eta\} := [\xi, \eta]$ on $\mathfrak{g}$. By a Poisson ideal in a Poisson algebra B we mean an ideal I such that $\{B, I\} \subset I$.

Exercise 11.5. *Prove that a G-invariant ideal in $S(\mathfrak{g})$ is automatically Poisson. Also show that the converse is true provided G is connected.*

For example, the ideal in $\mathbb{C}[\mathfrak{g}^*] = S(\mathfrak{g})$ of any orbit in $\mathfrak{g}^*$ is Poisson. Given a Poisson ideal $I \subset S(\mathfrak{g})$ we can form the reduction $A///_IG := [A/A\mu^*(I)]^G$.

Exercise 11.6. *Equip the algebra $A///_IG$ with a natural Poisson bracket.*

11.4. **Hamiltonian reduction, geometrically.** Let us describe the geometric side of the picture. Let X be an affine symplectic variety with form ω acted on by G in a Hamiltonian way, let μ be the moment map. Let Y be a Poisson subscheme of $\mathfrak{g}^*$ (a subscheme given by a Poisson ideal I). Then the scheme-theoretic preimage $\mu^{-1}(Y)$ has algebra of functions $\mathbb{C}[X]/\mathbb{C}[X]\mu^*(I)$. So $X///_YG := \mu^{-1}(Y)//G$ is the spectrum of the Hamiltonian reduction $\mathbb{C}[X]///_IG$. Below we always assume that G is reductive.

We will need some conditions for $X///_YG$ to be a variety and to be a smooth variety.

Proposition 11.2. (1) *Let Y be a reduced locally complete intersection in $\mathfrak{g}^*$ (i.e., smooth). Suppose that every irreducible component of $\mu^{-1}(Y)$ contains a point without stabilizer. Then $\mu^{-1}(Y)$ is reduced and hence $X///_YG$ is a variety.*

(2) *Suppose that Y is smooth and the action of G on $\mu^{-1}(Y)$ is free. Then $X///_YG$ is smooth.*

Proof. Recall that $d_x\mu$ is a surjection provided G_x is finite. The assumptions of (1) imply that $\operatorname{codim}_X \mu^{-1}(Y) = \operatorname{codim}_{\mathfrak{g}^*} Y$ and hence $\mu^{-1}(Y)$ is a locally complete intersection. This scheme is generically reduced, and hence, by the Serre criterium, is reduced.

Similarly, under the assumptions of (2), $\mu^{-1}(Y)$ is smooth. Now (2) follows from the general facts about free actions on smooth varieties that are stated as a lemma below. $\square$

Lemma 11.3. *Let Z be a smooth affine algebraic variety equipped with a free action of a reductive group G. Then the quotient $Z//G$ is smooth and parameterizes G-orbits (and so coincides with the naive quotient as a set) and the morphism $\pi : Z \to Z//G$ is smooth (i.e., is a submersion at any point).*

The proof is based on an easy observation that, for any point $z \in Z$, one can choose a smooth subvariety $Z_0 \subset Z$ containing z and transversal to Gz. Then, at least in the analytic topology, there is a neighborhood Z_0^0 of $z \in Z_0$ such that the natural map $G \times Z_0^0 \to Z$ is an open embedding whose image is some G-stable neighborhood of Gz. In the algebraic category, one cannot deal with analytic topology, but can do étale one. A general statement here: the étale slice theorem of Luna can be found in [PV]. This is quite technical and we are not going to elaborate on this.

Part (2) of the previous lemma has a standard corollary that we will need in the sequel.

Corollary 11.4. *In the notation of the previous lemma,*

(1) $\mathrm{Vect}(Z//G)$ *is the quotient of* $\mathrm{Vect}(Z)^G$ *by the vertical vector fields, i.e., vector fields tangent to the fibers of* π. *This identification is provided by the map* $\pi_* : \mathrm{Vect}(Z)^G \twoheadrightarrow \mathrm{Vect}(Z//G)$.

(2) *The pull-back* π^* *identifies the space* $\Gamma(Z//G, \bigwedge^i T^*Z//G)$ *with the subspace in* $\Gamma(Z, \bigwedge^i T^*Z)^G$ *consisting of all forms that vanish on a vertical vector field.*

Let us describe the assumptions under which $X///_Y G$ is symplectic and describe the corresponding symplectic form. Assume that in (2) of the previous lemma, Y is a single orbit. Since this orbit is closed, it is of a semisimple element (under a usual identification of $\mathfrak{g}$ with $\mathfrak{g}^*$). In particular, the stabilizer G_α of $\alpha \in Y$ is a reductive subgroup. The inclusion $\mu^{-1}(\alpha) \hookrightarrow \mu^{-1}(Y)$ induced an identification $\mu^{-1}(\alpha)//G_\alpha \xrightarrow{\sim} \mu^{-1}(Y)//G$. The latter identification follows from the observation that $\mu^{-1}(Y)$ is a G-homogeneous bundle over Y with fiber $\mu^{-1}(\alpha)$ (formally, this means that the natural morphism $G \times \mu^{-1}(\alpha) \to \mu^{-1}(Y)$ is the quotient for the diagonal action of G_α; this formal description implies the identification $\mu^{-1}(\alpha)//G_\alpha \xrightarrow{\sim} \mu^{-1}(Y)//G$).

Proposition 11.5. *We retain the assumptions of the previous paragraph, in particular, assume that* Y *is a single orbit. Let* $\pi : \mu^{-1}(\alpha) \twoheadrightarrow X///_Y G$ *be the quotient morphism, and* $\iota : \mu^{-1}(\alpha) \hookrightarrow X$ *be the inclusion. There is a unique* 2*-form* $\underline{\omega}$ *on* $\mu^{-1}(\alpha)//G_\alpha$ *such that* $\pi^*(\underline{\omega}) = \iota^*(\omega)$. *This form is symplectic.*

Proof. Step 1. To establish the existence and uniqueness of $\underline{\omega}$, we use (2) of Corollary 11.4 with G_α instead of G and $Z := \mu^{-1}(\alpha)$. We need to show that $\omega_x(u, v) = 0$, when v is tangent to $\mu^{-1}(\alpha)$, while u is tangent to $G_\alpha x$. The first condition means that $d_x\mu(v) = 0$. The second condition means that $u = \xi_x$ for some $\xi \in \mathfrak{g}_\alpha$. So $\omega_x(u, v) = \omega_x(\xi_x, v) = \langle d\mu_x(v), \xi\rangle = 0$.

The form $\underline{\omega}$ is closed because $\pi^*\underline{\omega} = \iota^*\omega$ is. It remains to check that $\underline{\omega}$ is non-degenerate and that the Poisson structure induced by $\underline{\omega}$ on $\mu^{-1}(Y)//G$ agrees with the initial one.

Step 2. To prove that $\underline{\omega}$ is non-degenerate is equivalent to $\{\xi_x | \xi \in \mathfrak{g}_\alpha\} = \ker d_x\mu \cap \ker d_x\mu^\perp$. As we have seen in the previous lecture, $\ker d_x\mu = (\mathfrak{g}x)^\perp$. So what we need to prove is that if $\xi \in \mathfrak{g}$ is such that $\omega(\eta_x, \xi_x) = 0$ for all $\eta \in \mathfrak{g}$, then $\xi \in \mathfrak{g}_\alpha$. But again, $\omega(\eta_x, \xi_x) = \langle d_x\mu(\xi_x), \eta\rangle$. Since μ is a G-equivariant map, we see that $d_x\mu(\xi_x) = \xi_{\mu(x)} = \xi_\alpha = 0$. So our condition is that $\langle \xi_\alpha, \eta\rangle = 0$. Obviously, it holds for all η if and only if $\xi_\alpha = 0$, equivalently, $\xi \in \mathfrak{g}_\alpha$. □

In the next lecture we will see that the Poisson structure on $X///_Y G$ induced from ω coincides with one obtained algebraically.

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LECTURES ON SYMPLECTIC REFLECTION ALGEBRAS

IVAN LOSEV

12. Calogero-Moser systems and quantum mechanics

12.1. **Hamiltonian reduction, finished.** Let us recall the set-up of the end of the previous lecture. We have a symplectic affine variety X with form ω. We equip this variety with a Hamiltonian action of a reductive algebraic group G, let $\mu : X \to \mathfrak{g}^*$ be a moment map. Further, we choose a closed orbit $Y \subset \mathfrak{g}^*$. We assume that G acts freely on $\mu^{-1}(Y)$. Then, as we have seen, $X///_Y G := \mu^{-1}(Y)//G$ is a smooth symplectic variety. A symplectic form $\underline{\omega}$ on $X///_Y G$ can be described as follows. We pick $\alpha \in Y$. Then $\mu^{-1}(Y)//G$ is naturally identified with $\mu^{-1}(\alpha)//G_\alpha$. Then the symplectic form $\underline{\omega}$ is defined as a unique form satisfying $\pi^*\underline{\omega} = \iota^*\omega$, where $\pi : \mu^{-1}(\alpha) \twoheadrightarrow \mu^{-1}(\alpha)//G_\alpha$ is the quotient morphism, and $\iota : \mu^{-1}(\alpha) \hookrightarrow X$ is the inclusion.

Recall that $X///_Y G$ comes also with a Poisson structure, the bracket is defined directly on the algebra of functions of this variety. This description can be translated into a more geometric language as follows. It is enough to specify the Hamiltonian vector fields for the functions on $X///_Y G$.

Suppose $F \in \mathbb{C}[X]$ is such that its restriction $\underline{F}$ to $\mu^{-1}(Y)$ is G-invariant. Then, tracking the definition of the bracket on $\mathbb{C}[X///_Y G] = [\mathbb{C}[X]/\mathbb{C}[X]I]^G$, where $I \subset S(\mathfrak{g})$ is the ideal of Y, we see that $v(F)$ preserves I, equivalently, is tangent to $\mu^{-1}(Y)$. The restriction of $v(F)$ to $\mu^{-1}(Y)$ is G-invariant and the induced (see (1) of Corollary 11.4 of the previous lecture) vector field on $X///_Y G$ is $v(\underline{F})$.

Next, let $x \in \mu^{-1}(\alpha)$. We claim that $v_x(F)$ is tangent to $\mu^{-1}(\alpha)$. This is equivalent to $d_x\mu(v_x(F)) = 0$. But $\langle d_x\mu(v_x(F)), \xi\rangle = \omega_x(\xi_x, v_x(F)) = L_{\xi_X}F(x)$. The latter is zero because the restriction of F to $\mu^{-1}(Y)$ is G-invariant. Of course, the induced vector field on $\mu^{-1}(\alpha)//G_\alpha$ is still $v(\underline{F})$.

Now we are in position to prove that the bracket induced by $\underline{\omega}$ is the same as that of the reduction. This boils down to $\iota_{v(\underline{F})}\underline{\omega} = d\underline{F}$. Thanks to Step 4, the left hand side is the form on the reduction induced by $\iota^*(\iota_{v(F)}\omega)$ as in (2) of Corollary 11.4. The right hand side is induced by $\iota^*(dF)$. Since $\iota_{v(F)}\omega = dF$, we are done.

12.2. **CM system via reduction.** Let us return to the CM system. Let $G = \mathrm{PGL}_n(\mathbb{C})$. We have the Hamiltonian $H = \frac{1}{2}\operatorname{tr}(Y^2)$ on $R = T^*\operatorname{Mat}_n(\mathbb{C})$ and the induced Hamiltonian $\underline{H}$ on $C = \mu^{-1}(\alpha)///G_\alpha$, where α is the anti-unit matrix. The system we are interested in is obtained by restricting $\underline{H}$ to the open subset $C^{Reg} := T^*(\mathbb{C}^n)^{Reg}/\mathfrak{S}_n \hookrightarrow C$. Thanks to the previous section, the trajectories for $\underline{H}$ are obtained by projecting those for H.

Exercise 12.1. *The trajectories for H are of the form $(X - tY, Y)$.*

So what remains to prove is that the map $C^{Reg} \to C$ induced by $\iota : T^*(\mathbb{C}^n)^{Reg} \to \mu^{-1}(O)^{Reg}$ is an open inclusion of algebraic varieties that preserves the symplectic forms. First let us check the "open inclusion part". The map ι is a morphism and therefore so is the composition $\pi_{G_\alpha} \circ \iota : T^*(\mathbb{C}^n)^{Reg} \to \mu^{-1}(O)^{Reg}//G$, where $\pi_{G_\alpha} : \mu^{-1}(\alpha) \twoheadrightarrow \mu^{-1}(\alpha)//G_\alpha$. The latter is a principal open subvariety of C. The composition is $\mathfrak{S}_n$-invariant by Exercise 11.3 and so descends to $\underline{\iota}: C^{Reg} \to \mu^{-1}(O)^{Reg}//G$. Again, by Exercise 11.3, this morphism is bijective. Now we can use a general fact that a bijective morphism into a smooth (or even normal) variety is an isomorphism. We will also check that our morphism is iso below.

Now let us show that our morphism is compatible with the symplectic forms. Let ω_R be the symplectic form on R, it is given by $\sum_{i,j=1}^n dx_{ij} \wedge dy_{ji}$, where x_{ij}, y_{ij} are the matrix entries for X, Y. Let $\underline{\omega}_R$ be the form on the reduction. Let $\omega = \sum_{i=1}^n dx^i \wedge dy_i$ be the symplectic form on $T^*(\mathbb{C}^n)^{Reg}$ and $\underline{\omega}$ be the induced formula on C^{Reg} so that $\omega = \pi_{\mathfrak{S}_n}^* \underline{\omega}$. We need $\underline{\iota}^* \underline{\omega}_R = \underline{\omega}$. First, we claim that $\iota^* \omega_R = \omega$. This is a direct computation that uses the explicit form of ι: $\iota^*\omega = \sum_{i=1}^n d\iota^*(x_{ij}) \wedge d\iota^*(y_{ij}) = \sum_{i=1}^n dx^i \wedge dy_i + \sum_{i\neq j} 0 \wedge d\frac{1}{x^j - x^i}$. We remark that $\iota^*\omega_R = \iota^* \pi_{G_\alpha}^* \underline{\omega}_R$ because the image of ι lies in $\mu^{-1}(\alpha)$. So

$$\pi_{\mathfrak{S}_n}^* \underline{\omega} = \omega = \iota^* \omega_R = \iota^* \pi_{G_\alpha}^* \underline{\omega}_R = \pi_{\mathfrak{S}_n}^* \underline{\iota}^* \underline{\omega}_R,$$

the last equality follows from $\pi_{G_\alpha} \circ \iota = \underline{\iota} \circ \pi_{\mathfrak{S}_n}$. So we see that $\pi_{\mathfrak{S}_n}^* \underline{\omega} = \pi_{\mathfrak{S}_n}^*(\underline{\iota}^* \underline{\omega}_R)$ and therefore $\underline{\omega} = \underline{\iota}^* \underline{\omega}_R$.

Actually the last equality shows that $\underline{\iota}$ is étale: the kernel of $d_p \underline{\iota}$ is forced to lie in the kernel of $\underline{\iota}^* \underline{\omega}_R$ but the latter is non-degenerate. An étale bijective morphism of (smooth) varieties has to be an isomorphism.

Problem 12.2. *Prove (2) of the main theorem of the previous lecture.*

12.3. **Alternative realization of CM space.** We will need to realize C as a different Hamiltonian reduction. Namely, consider the quiver $\underline{Q}$ with two vertices ∞ and 0 and two arrows, $a: 0 \to 0, e: \infty \to 0$. Let Q be the double quiver. We consider the representation space $\tilde{R}$ for Q with dimension vector $n\epsilon_0 + \epsilon_\infty$. Of course, $\operatorname{Rep}(\underline{Q}, v) = \operatorname{Mat}_n(\mathbb{C}) \oplus \mathbb{C}^n$, and $\tilde{R} = T^* \operatorname{Rep}(\underline{Q}, v) = \operatorname{Mat}_n(\mathbb{C})^{\oplus 2} \oplus \mathbb{C}^n \oplus \mathbb{C}^{n*}$. We write an element of $\tilde{R}$ as (X, Y, i, j) with $i \in \mathbb{C}^n, j \in \mathbb{C}^{n*}$. On $\tilde{R}$ we have a natural action of $\tilde{G} := \operatorname{GL}_n(\mathbb{C})$ on $\tilde{R}$ (while before we considered the action of $\operatorname{PGL}_n(\mathbb{C})$ on $\operatorname{Mat}_n(\mathbb{C})^{\oplus 2}$). The moment map $\tilde{\mu}: \tilde{R} \to \operatorname{Mat}_n(\mathbb{C})$ for the $\tilde{G}$-action on $\tilde{R}$ is given by $\tilde{\mu}(X, Y, i, j) = [X, Y] + ij$.

Proposition 12.1. *The reduction $\tilde{R}///_{-E}\tilde{G}$ is naturally identified with $C = R///_O G$.*

Proof. We can consider the natural projection $\rho: \tilde{R} \to R, \rho(X, Y, i, j) = (X, Y)$. We have $\tilde{\mu}(X, Y, i, j) = \mu \circ \rho(X, Y, i, j) + ij$. So for $(X, Y, i, j) \in \tilde{\mu}^{-1}(-E)$ we have $\mu(\rho(X, Y, i, j)) + E = -ij$. The trace of the left hand side is n and hence $ij \neq 0$. Clearly, any operator of rank 1 has the form $-ij$. It follows that $\rho(\tilde{\mu}^{-1}(E)) \subset \mu^{-1}(O)$. Moreover, $ij = i'j' (\neq 0)$ if and only if $i' = ti, j' = t^{-1}j$, where t is a (uniquely determined) element of $\mathbb{C}^\times$. So the restriction of ρ to $\tilde{\mu}^{-1}(-E)$ is a principal bundle over $\mu^{-1}(O)$ for the group $\mathbb{C}^\times$ (that acts as the center of $\tilde{G}$). It follows that $\mu^{-1}(O) = \tilde{\mu}^{-1}(-E)//\mathbb{C}^\times$, this identification is G-equivariant. This leads to an identification of $\tilde{\mu}^{-1}(-E)//\tilde{G}$ and $\mu^{-1}(O)//G$. □

Problem 12.3. *Check that the symplectic forms on $\tilde{\mu}^{-1}(-E)//\tilde{G}$ and $\mu^{-1}(O)//G$ are the same.*

A reason why one wants to consider $\tilde{\mu}: R \to \operatorname{Mat}_n(\mathbb{C})$ instead of $\mu: R \to \mathfrak{pgl}_n(\mathbb{C}) = \mathfrak{sl}_n(\mathbb{C})$ is that the former map is flat and has reduced fibers. The latter is definitely not flat (fibers have different dimensions) and it is a big problem to determine whether the fibers are reduced (it is enough to check the reducedness of the zero fiber).

12.4. **Quantum Mechanics.** In Classical Mechanics, observables (i.e., physical quantities that can be measured on trajectories of our system) form a Poisson algebra. The equation of motion is the Hamilton equation, $\dot{f} = \{H, f\}$.

In the traditional formalism of Quantum Mechanics, observables are self-adjoint operators on Hilbert spaces, and the equation of motion is given by the Heisenberg equation $\dot{F} = \frac{1}{\hbar}[H, F]$, where the Hamiltonian H is also such an operator. Here $\hbar$ is a normalized Plank constant, a purely imaginary number with very small absolute value.

Classical and quantum systems should correspond to each other: relatively large objects like insects or planets should obey classical laws, while quantum effects only appear for small objects, such as electrons. So one should be able to pass from a quantum system to classical (by taking the "quasi-classical" limit $\hbar \to 0$) and vice versa (quantization). One of the problems with the traditional formalism of Quantum Mechanics is that it is very different from the classical set-up that makes even taking the quasi-classical limit a non-trivial procedure.

There is a different and significantly simplified approach to Quantum Mechanics based on the deformation theory: we need to choose the simplest formalism, where the Heisenberg equation still makes sense and where the quasi-classical limit is easy. Namely, for an algebra of observables we take a (flat) deformation $A_\hbar$ of a commutative algebra A over $\mathbb{C}[\hbar]$, where we view $\hbar$ as an independent variable. We also require $A_\hbar$ to be separated with respect to the $\hbar$-adic topology (this definitely holds when $A_\hbar$ is a free $\mathbb{C}[\hbar]$-module). Since $A_\hbar$ is flat and commutative modulo $\hbar$, the expression $\frac{1}{\hbar}[\tilde{a}, \tilde{b}]$ makes sense for all $\tilde{a},\ \tilde{b} \in A_\hbar$. So we can consider the Heisenberg equation. Also, modulo $\hbar$, the expression $\frac{1}{\hbar}[\tilde{a}, \tilde{b}]$ depends only on the classes of $\tilde{a}, \tilde{b}$ modulo $\hbar$. This defines a bracket on A and the bracket is Poisson. So when we set $\hbar = 0$, the Heisenberg equation becomes the Hamilton equation.

A drawback of this approach is that it is unclear what a trajectory of a point should be (it's also somewhat tricky in the original formalism). Still, the notion of a first integral makes sense and one can define a completely integrable system. More precisely, let A be an algebra of functions on an affine symplectic variety of dimension $2n$ and let $A_\hbar$ be a deformation of A as above that induces the bracket on A coming from the symplectic form. Pick a Hamiltonian H. By a completely integrable system we mean a collection $H_1, \ldots, H_n \in A_\hbar$ of pairwise commuting elements of $A_\hbar$ including H such that the classes of $H_1, \ldots, H_n$ modulo $\hbar$ are algebraically independent. The same definition works when the Poisson bracket on $\operatorname{Spec}(A)$ is non-degenerate generically.

Let us introduce a basic class of deformations that we will need in the sequel. First of all, we have seen that the Weyl algebra $W(V)$ is a filtered deformation of $S(V)$ (compatible with Poisson brackets). So its homogenized version $W_\hbar(V) = T(V)[\hbar]/(u \otimes v - v \otimes u - \hbar\omega(u, v))$ is a deformation of $S(V)$ in the above sense.

This example can be globalized as follows. Let X_0 be a smooth affine algebraic variety. We consider the algebra $D_\hbar(X_0)$ of homogenized differential operators that is the quotient of $T(\mathbb{C}[X_0] \oplus \operatorname{Vect}(X_0))[\hbar]$ by the relations

$$f \otimes g = fg, f \otimes \xi = f\xi, \xi \otimes f = f\xi + \hbar L_\xi f, \xi \otimes \eta - \eta \otimes \xi = \hbar[\xi, \eta], f, g \in \mathbb{C}[X_0], \xi, \eta \in \operatorname{Vect}(X_0).$$

When we specialize $\hbar = 1$ we get the usual algebra of differential operators $D(X_0)$. Also if we specialize $\hbar = 0$, we recover $\mathbb{C}[T^*X_0]$.

Exercise 12.4. *Show that the algebra $D_\hbar(X_0)$ is a deformation of $\mathbb{C}[T^*X_0]$ compatible with the usual bracket there. Hint: how does the sheaf $D_\hbar(X_0)$ on X_0 behave under étale base changes?*

In fact, one can filter the algebra $D(X_0)$ by the order of a differential operator and then $D_\hbar(X_0)$ becomes the Rees algebra of $D(X_0)$.

We remark that usually one imposes one more assumption on $\mathcal{A}_\hbar$: that it is complete in the $\hbar$-adic topology. This is to reflect the physical fact that $\hbar$ is very small. We are not going to do this so far. If we have a deformation $\mathcal{A}_\hbar$ satisfying our assumptions, then its $\hbar$-adic completion $\mathcal{A}'_\hbar := \varprojlim_n \mathcal{A}_\hbar/\hbar^n\mathcal{A}_\hbar$ is still flat and separated and, in addition, $\hbar$-adically complete (the condition that $\mathcal{A}_\hbar$ is separated precisely means that a natural homomorphism $\mathcal{A}_\hbar \to \mathcal{A}'_\hbar$ is an embedding). In fact, in some cases one can recover $\mathcal{A}_\hbar$ from $\mathcal{A}'_\hbar$.

Exercise 12.5. *Let $\mathcal{A}_\hbar$ be a $\mathbb{Z}_{\geqslant 0}$-graded $\mathbb{C}[\hbar]$-algebra with $\hbar$ being of positive degree. Let $\mathcal{A}'_\hbar$ be the $\hbar$-adic completion of $\mathcal{A}_\hbar$. Explain how to recover $\mathcal{A}_\hbar$ back from $\mathcal{A}'_\hbar$ using some natural action of $\mathbb{C}^\times$ on $\mathcal{A}'_\hbar$.*

12.5. **Quantum Calogero-Moser system.** We will consider a quantum CM system associated to an arbitrary real reflection group W. Let $\mathfrak{h}$ be the complexification of the reflection representation of W, so $\mathfrak{h}$ is an irreducible W-module that comes equipped with a W-invariant symmetric form $(\cdot,\cdot)$. Let S be the set of reflections in W. To each reflection $s \in W$ we assign $\alpha_s \in \mathfrak{h}^*$, a nonzero vector with $s\alpha_s = -\alpha_s$. Further we pick a conjugation invariant function $c_\bullet : S \to \mathbb{C}$. To the function c we associate the following potential:

$$V = -\sum_{s\in S} \frac{c_s(c_s+\hbar)(\alpha_s,\alpha_s)}{\alpha_s^2}$$

This potential can be viewed as an element of $\mathbb{C}[\mathfrak{h}^{Reg}][\hbar]$, where $\mathfrak{h}^{Reg} = \{x \in \mathfrak{h} | \langle \alpha_s, x\rangle \neq 0, \forall s \in S\}$, this is precisely the locus, where W acts freely. If we consider W of type A and specialize $\hbar = 0$, then we get $V = -\sum_{i\neq j} \frac{2c^2}{(x^i-x^j)^2}$, because there is only one conjugacy class of reflections. So V is a usual CM potential.

A quantum mechanical system with potential V has Hamiltonian $H = \Delta + V$ with $\Delta := \sum_{i=1}^{\dim \mathfrak{h}} y_i^2$, where vectors y_i form an orthonormal basis in $\mathfrak{h}$, is the Laplace operator. We view H as an element of $D_\hbar(\mathfrak{h}^{Reg})$, where $\mathfrak{h}$ is viewed a subspace in $\mathrm{Vect}(\mathfrak{h}^{Reg})$ consisting of constant vector fields. This is a so called *Olshanetsky-Perelomov* Hamiltonian.

Exercise 12.6. *Let X_0 be a smooth affine variety acted freely by a finite group Γ. Equip $D_\hbar(X_0)$ with a natural Γ-action by $\mathbb{C}[\hbar]$-algebra automorphisms and then identify $D_\hbar(X_0)^\Gamma$ with $D_\hbar(X_0/\Gamma)$.*

So we get a W-action on $D_\hbar(\mathfrak{h}^{Reg})$ by algebra automorphisms. We remark that both Δ and V are W-invariant and so H is W-invariant too. In the sequel, for $\mathcal{A}_\hbar$ we will take $D_\hbar(\mathfrak{h}^{Reg})^W$, by the previous exercise, this is the same as $D_\hbar(\mathfrak{h}^{Reg}/W)$.

LECTURES ON SYMPLECTIC REFLECTION ALGEBRAS

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13. Quantum CM systems and Rational Cherednik algebras

13.1. **Setting.** We pick an irreducible real reflection group W. Let $\mathfrak{h}$ be the complexification of the reflection representation of W, so $\mathfrak{h}$ is an irreducible W-module that comes equipped with a W-invariant symmetric form $(\cdot,\cdot)$. We set $n=\dim\mathfrak{h}$. Let S be the set of reflections in W. To each reflection $s\in W$ we assign a nonzero vector $\alpha_s\in\mathfrak{h}^*$, with $s\alpha_s=-\alpha_s$. Further we pick a function $c_\bullet: S\rightarrow\mathbb{C}$ that is constant on conjugacy classes. To the function c we can associate two objects, a quantum generalized Calogero-Moser system and a Rational Cherednik algebra.

First, let us recall the algebra of quantum observables for the system. Consider the algebra $D_\hbar(\mathfrak{h})$ that is the quotient of $T(\mathfrak{h}\oplus\mathfrak{h}^*)[\hbar]$ by the relations

$$[x,x']=0,[a,a']=0,[a,x]=\hbar\langle a,x\rangle,\quad x,x'\in\mathfrak{h}^*,a,a'\in\mathfrak{h}.$$

In this algebra we have the Vandermond element $\delta=\prod_{s\in S}\alpha_s$ and we can invert it getting the algebra $D_\hbar(\mathfrak{h}^{Reg})=D_\hbar(\mathfrak{h})[\delta^{-1}]$. We remark that we have $[a,\delta^{-1}]=-\hbar\frac{\partial_a\delta}{\delta^2}$ and so the localization does make sense. Then on $D_\hbar(\mathfrak{h})$ we have an action of W by automorphisms that fixes $\hbar$ and on $\mathfrak{h},\mathfrak{h}^*$ is given as before. The element δ is W-sign-invariant and so the W-action extends to $D_\hbar(\mathfrak{h}^{Reg})$. The algebra of quantum observables will be $\mathcal{A}_\hbar:=D_\hbar(\mathfrak{h}^{Reg})^W$.

We set $\Delta=\sum_{i=1}^n a_i^2\in D_\hbar(\mathfrak{h})$, where the vectors $a_1,\ldots,a_n$ form an orthonormal basis. The Hamiltonian of interest, the Olshanetsky-Perelomov Hamiltonian, is given by

$$H=\Delta-\sum_{s\in S}\frac{c_s(c_s+\hbar)(\alpha_s,\alpha_s)}{\alpha_s^2}\in D_\hbar(\mathfrak{h}^{Reg}).$$

Further, we remark that Δ is W-invariant because so is $(\cdot,\cdot)$. Also

$$w.\frac{(\alpha_s,\alpha_s)}{\alpha_s^2}=\frac{(\alpha_{wsw^{-1}},\alpha_{wsw^{-1}})}{\alpha_{wsw^{-1}}^2}.$$

So the sum

$$\sum_{s\in S}\frac{c_s(c_s+\hbar)(\alpha_s,\alpha_s)}{\alpha_s^2}$$

is also W-invariant and $H\in\mathcal{A}_\hbar$.

Our goal is to find pair-wise commuting elements $H_2,\ldots,H_{n+1}\in\mathcal{A}_\hbar$ that are algebraically independent modulo $\hbar$ such that $H=H_2$. The construction is based on Rational Cherednik algebras.

Recall that the Rational Cherednik algebra associated to $(\mathfrak{h},W)$ is the SRA for $\mathfrak{h}\oplus\mathfrak{h}^*,W$. We will need parameters c_s as before and the parameter t will be the independent variable $\hbar$, so we get an algebra over $\mathbb{C}[\hbar]$. The corresponding Cherednik algebra $H_{\hbar,c}$ is the quotient of $T(\mathfrak{h}\oplus\mathfrak{h}^*)[\hbar]\#W$ modulo the following relations, see Problem 6.10,

$$[x,x']=0=[y,y'],[y,x]=\hbar\langle y,x\rangle-\sum_{s\in S}c_s\langle\alpha_s,y\rangle\langle\alpha_s^\vee,x\rangle s,\quad x,x'\in\mathfrak{h}^*,y,y'\in\mathfrak{h},$$

where $\alpha_s^\vee$ is a vector in $\mathfrak{h}$ with $s\alpha_s^\vee = -\alpha_s^\vee$ and $\langle \alpha_s^\vee, \alpha_s\rangle = 2$.

The main idea of constructing the elements $H_2, \ldots, H_{n+1}$ is as follows. Suppose that all c_s are 0. Then $H = \Delta$. Of course, in $D_h(\mathfrak{h})$ all elements from $S(\mathfrak{h}) \subset D_\hbar(W)$ commute with H. Only the W-invariant part $S(\mathfrak{h})^W$ is still present in $\mathcal{A}_\hbar$. We want to use the same strategy for the general c_s's. One shouldn't expect an embedding $S(\mathfrak{h}) \to D_\hbar(\mathfrak{h}^{Reg})$ such that $S(\mathfrak{h})^W$ commutes with H. Instead we will see that there is an embedding $S(\mathfrak{h}) \hookrightarrow D_\hbar(\mathfrak{h}^{Reg})\#W$ that works. Also there is a natural embedding $\mathbb{C}[\mathfrak{h}]\#W \hookrightarrow D_\hbar(\mathfrak{h}^{Reg})\#W$. The two embeddings will combine to a monomorphism $H_{\hbar,c} \to D_\hbar(\mathfrak{h}^{Reg})\#W$.

The required homomorphism will be constructed in two steps. We first construct a homomorphism $\Theta : H_{\hbar,c} \to D(\mathfrak{h}^{Reg})\#W$ that is essentially due to Dunkl. Then we will produce an an automorphism φ of $D_\hbar(\mathfrak{h}^{Reg})\#W$ such that $\varphi \circ \Theta$ has required properties.

13.2. **Dunkl operators.** For $a \in \mathfrak{h}$ we define an element $D_a \in D_\hbar(\mathfrak{h}^{Reg})\#W$ by

$$D_a = a - \sum_{s\in S} \frac{c_s\langle \alpha_s, a\rangle}{\alpha_s}(1-s). \tag{1}$$

Clearly, D_a is linear in a.

Exercise 13.1. *Prove that $wD_aw^{-1} = D_{wa}$ for all $w \in W, a \in \mathfrak{h}$.*

We will see that a map $x \mapsto x, w \mapsto w, a \mapsto D_a$ extends to a homomorphism $H_{\hbar,c} \to D_\hbar(\mathfrak{h}^{Reg})\#W$. An important tool for this is a natural action of $D_\hbar(\mathfrak{h}^{Reg})\#W$ on $\mathbb{C}[\mathfrak{h}^{Reg}][\hbar]$.

The algebra $D_\hbar(\mathfrak{h}^{Reg})\#W$ acts on $\mathbb{C}[\mathfrak{h}^{Reg}][\hbar]$: the action of W is induced from the W-action on $\mathfrak{h}^{Reg}$, $f \in \mathbb{C}[\mathfrak{h}^{Reg}] \subset D_\hbar(\mathfrak{h}^{Reg})$ acts by the multiplication by f, and, finally, $a \in \mathfrak{h} \subset D_\hbar(\mathfrak{h}^{Reg})$ acts by $\hbar\partial_a$.

Lemma 13.1. *The representation of $D_\hbar(\mathfrak{h}^{Reg})\#W$ in $\mathbb{C}[\mathfrak{h}^{Reg}][\hbar]$ is faithful.*

Proof. The algebra $D_\hbar(\mathfrak{h}^{Reg})$ is a free module over $\mathbb{C}[\hbar]$, it is freely spanned by the vector space $S(\mathfrak{h}) \otimes \mathbb{C}W \otimes \mathbb{C}[\mathfrak{h}^{Reg}]$. Also this algebra is graded "by the order of a differential operator": with $\mathbb{C}W$ and $\mathbb{C}[\mathfrak{h}^{Reg}]$ in degree 0, and $\hbar, \mathfrak{h}$ in degree 1. The action on $\mathbb{C}[\mathfrak{h}^{Reg}][\hbar]$ is compatible with the grading and so the kernel is a graded ideal. A homogeneous element in $D_\hbar(\mathfrak{h}^{Reg})$ is 0 if its specialization at $\hbar = 1$ is 0. So it is enough to check that the representation of $D(\mathfrak{h}^{Reg})\#W$ in $\mathbb{C}[\mathfrak{h}^{Reg}]$ is faithful.

Let $d = \sum_{w\in W} d_w w$ with $d_w \in D(\mathfrak{h}^{Reg})$ lie in the kernel. Pick a point $x \in \mathfrak{h}^{Reg}$ and consider the completion $\mathbb{C}[\mathfrak{h}]^\wedge_{Wx}$ of $\mathbb{C}[\mathfrak{h}^{Reg}]$ (or of $\mathbb{C}[\mathfrak{h}]$) at the ideal of Wx. The actions of both $D(\mathfrak{h}^{Reg})$ and W extend from $\mathbb{C}[\mathfrak{h}^{Reg}]$ to $\mathbb{C}[\mathfrak{h}]^\wedge_{Wx}$. The action is continuous with respect to the inverse image topology on the completion. So d still acts by 0 on $\mathbb{C}[\mathfrak{h}]^\wedge_{Wx}$. On the other hand, $\mathbb{C}[\mathfrak{h}]^\wedge_{Wx} = \bigoplus_{w\in W} \mathbb{C}[\mathfrak{h}]^\wedge_{wx}$, where the latter is the completion at wx, a formal power series algebra. Each d_w preserves the summands, while W permutes them transitively. So each separate d_w acts by 0 on $\mathbb{C}[\mathfrak{h}]^\wedge_x$. We can represent d_w in the form $\sum_\alpha f_\alpha \partial^\alpha$, where $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $\partial^\alpha = a_1^{\alpha_1} \ldots a_n^{\alpha_n}$. Let $x^1, \ldots, x^n$ be the dual basis to $a_1, \ldots, a_n$. We can prove that $f_\alpha = 0$ by induction on $|\alpha| = \sum_i \alpha_i$ starting from $|\alpha| = 0$. For this we consider the action of d_w on the monomials in $x^1, \ldots, x^n$ of degree $|\alpha|$. The details are left to a reader. $\square$

We remark that, although D_a does not lie in $D_\hbar(\mathfrak{h})\#W$, it does preserve the subspace $\mathbb{C}[\mathfrak{h}][\hbar] \subset \mathbb{C}[\mathfrak{h}^{Reg}][\hbar]$. This is because $f - s(f)$ is divisible by α_s for any $f \in \mathbb{C}[\mathfrak{h}]$.

Proposition 13.2. *The map $x \mapsto x, w \mapsto w, y \mapsto D_y, x \in \mathfrak{h}^*, w \in W, y \in \mathfrak{h}$ defines a $\mathbb{C}[\hbar]$-algebra homomorphism $H_{\hbar,c} \to D_\hbar(\mathfrak{h}^{Reg})\#W$.*

Proof. We need to show that the defining relations in the Cherednik algebra also hold in $D_\hbar(\mathfrak{h}^{Reg})\#W$. Obviously, two elements from $\mathfrak{h}^*$ in $D_\hbar(\mathfrak{h}^{Reg})\#W$ commute. Thanks to the previous exercise, the commutation relations between $\mathfrak{h}$ and W hold (for $\mathfrak{h}^*$ and W this is obvious). It remains to prove that

$$[D_y, x] = \langle y, x\rangle\hbar - \sum_{s\in S} c_s\langle y, \alpha_s\rangle\langle x, \alpha_s^\vee\rangle s. \tag{2}$$

$$[D_y, D_{y'}] = 0. \tag{3}$$

(2) reduces to $[\alpha_s^{-1}(1-s), x] = \langle x, \alpha_s\rangle s$. We can write x in the form $t\alpha_s + x_0$, where $sx_0 = x_0$. Then in the left hand side we get $\frac{1}{\alpha_s}(-s\alpha_s + \alpha_s s)t = 2st$ and in the right hand side we get the same.

The proof of (3) is more tricky. Thanks to Lemma 13.1, it is enough to check that $[D_y, D_{y'}]$ acts by 0 on $\mathbb{C}[\mathfrak{h}^{Reg}][\hbar]$. First of all, we claim that $[[D_y, D_{y'}], x] = 0$. To check this we use the Jacobi identity: $[[D_y, D_{y'}], x] = [D_y, [D_{y'}, x]] - [D_{y'}, [D_y, x]]$. By (2), we have

$$[D_y, [D_{y'}, x]] = [D_y, \hbar\langle y', x\rangle - \sum_{s\in S} c_s\langle y', \alpha_s\rangle\langle x, \alpha_s^\vee\rangle s] = \sum_{s\in S} c_s\langle y', \alpha_s\rangle\langle x, \alpha_s^\vee\rangle [s, D_y]. \tag{4}$$

But $[s, D_y] = sD_y - D_y s = (sD_y s^{-1} - D_y)s = (D_{sy} - D_y)s = -\langle\alpha_s, y\rangle D_{\alpha_s^\vee} s$. So we see that

$$[D_y, [D_{y'}, x]] = -\sum_{s\in S} c_s\langle y', \alpha_s\rangle\langle x, \alpha_s^\vee\rangle\langle y, \alpha_s\rangle D_{\alpha_s^\vee}.$$

This expression is symmetric in y and y' and so $[[D_y, D_{y'}], x] = [D_y, [D_{y'}, x]] - [D_{y'}, [D_y, x]] = 0$.

Also we remark that $D_y 1 = 0$ (the equality in $\mathbb{C}[\mathfrak{h}^{Reg}][\hbar]$) and hence $[D_y, D_{y'}]1 = 0$. But, since $[[D_y, D_{y'}], x] = 0$, we see that, on $\mathbb{C}[\mathfrak{h}^{Reg}][\hbar]$, the bracket $[D_y, D_{y'}]$ commutes with $\mathbb{C}[\mathfrak{h}^{Reg}][\hbar]$ acting by multiplications. So $[D_y, D_{y'}]$ is zero on $\mathbb{C}[\mathfrak{h}^{Reg}][\hbar]$ and therefore is zero as an element of $D_\hbar(\mathfrak{h}^{Reg})\#W$. □

Since D_y preserves $\mathbb{C}[\mathfrak{h}][\hbar]$ we get a representation of $H_{\hbar,c}$ on $\mathbb{C}[\mathfrak{h}][\hbar]$ (called the polynomial representation). This and more general representations, of the form $\mathbb{C}[\mathfrak{h}][\hbar]\otimes E$, where E is an irreducible W-module, play an important role in the representation theory of $H_{\hbar,c}$, these are analogs of Verma modules.

We remark that everything explained in this section works not only for real but also for complex reflection groups. One only needs to modify the definition of the Dunkl operator as follows. By α_s we denote an element in $\mathfrak{h}^*$ with non-unit eigenvalue, say λ_s. Then we define D_a by

$$D_a = a - \sum_{s\in S}\frac{2c_s\langle\alpha_s, a\rangle}{(1-\lambda_s)\alpha_s}(1-s).$$

Exercise 13.2. *Prove an analog of Proposition 13.2 for complex reflection groups.*

Exercise 13.3. *Let W be a complex reflection group.*

(1) *Show that* $\operatorname{ad} f$ *is a locally nilpotent operator on $H_{\hbar,c}$ for any $f\in\mathbb{C}[\mathfrak{h}]^W$.*
(2) *Deduce that the localization $H_{\hbar,c}[\delta^{-1}]$ exists.*
(3) *Show that the Dunkl homomorphism $H_{\hbar,c}\to D_\hbar(\mathfrak{h}^{Reg})\#W$ factors through a unique homomorphism $H_{\hbar,c}[\delta^{-1}]\to D_\hbar(\mathfrak{h}^{Reg})\#W$.*
(4) *Show that the homomorphism $H_{\hbar,c}[\delta^{-1}]\to D_\hbar(\mathfrak{h}^{Reg})\#W$ is an isomorphism.*

13.3. **OP Hamiltonian via Dunkl operators.** Let Θ denote the Dunkl homomorphism $H_{\hbar,c} \to D_\hbar(\mathfrak{h}^{Reg})\#W$.

Exercise 13.4. *Prove that Θ is injective modulo $\hbar$ and hence is injective.*

We denote the induced homomorphism $eH_{\hbar,c}e \to eD_\hbar(\mathfrak{h}^{Reg})\#We = D_\hbar(\mathfrak{h}^{Reg})^W$ also by Θ. We remark that there is a natural algebra homomorphism $S(\mathfrak{h}) \to H_{\hbar,c}$ that is injective even modulo $\hbar$. This is because $H_{\hbar,c}$ is a free $\mathbb{C}[\hbar]$-module with spanned by $S(\mathfrak{h}\oplus\mathfrak{h}^*)\#W = S(\mathfrak{h}^*)\otimes\mathbb{C}W\otimes S(\mathfrak{h})$. So we have a homomorphism $S(\mathfrak{h})^W \to eH_{\hbar,c}e \to D_\hbar(\mathfrak{h}^{Reg})^W$ that is injective modulo $\hbar$.

Lemma 13.3. *We have $\Theta(\Delta) = \overline{H} := \Delta - \sum_{s\in S} c_s \frac{(\alpha_s,\alpha_s)}{\alpha_s}\alpha_s^\vee$.*

Proof. It follows from Lemma 13.1 that the algebra $D_\hbar(\mathfrak{h}^{Reg})^W = eD_\hbar(\mathfrak{h}^{Reg})\#We$ acts on $\mathbb{C}[\mathfrak{h}^{Reg}]^W[\hbar] = e\mathbb{C}[\mathfrak{h}^{Reg}][\hbar]$. So it is enough to compute $\sum_{i=1}^n D_{a_i}^2 f$, where $f\in\mathbb{C}[\mathfrak{h}^{Reg}]^W$. But $D_{a_i}f = a_i f$ because $s(f)=f$ for all $s\in S$. Now

$$D_{a_i}a_i f = a_i^2 f - \sum_{s\in S} c_s\frac{\langle\alpha_s,a\rangle}{\alpha_s}(1-s)a_i f = a_i^2 f - \sum_{s\in S} c_s\frac{\langle\alpha_s,a_i\rangle}{\alpha_s}(a_i - s(a_i))sf =$$
$$(a_i^2 - \sum_{s\in S} c_s\frac{\langle\alpha_s,a_i\rangle^2}{\alpha_s}\alpha_s^\vee)f.$$

To get the statement of the lemma, we need to sum the previous equalities for $i=1,\ldots,n$ and notice that $\sum_{i=1}^n\langle\alpha_s,a_i\rangle^2 = (\alpha_s,\alpha_s)$ because $a_1,\ldots,a_n$ is an orthonormal basis. □

We will define a $\mathbb{C}[\hbar]$-algebra automorphism φ of $D_\hbar(\mathfrak{h}^{Reg})\#W$.

Exercise 13.5. *Define φ to be the identity on $\mathbb{C}[\mathfrak{h}^{Reg}]\#W$ and $\varphi(a) = a + \sum_{s\in S} c_s\frac{\langle a,\alpha_s\rangle}{\alpha_s}$. Show that φ extends to a $\mathbb{C}[\hbar]$-linear automorphism of $D_\hbar(\mathfrak{h}^{Reg})\#W$.*

Lemma 13.4. *We have $\varphi(\overline{H}) = H$.*

Proof. We will prove that $\overline{H} = \varphi^{-1}(H)$, this is easier because

$$\varphi^{-1}(H) = \varphi^{-1}(\Delta) - \sum_{s\in S}\frac{c_s(c_s+\hbar)(\alpha_s,\alpha_s)}{\alpha_s^2}.$$

We have

$$\varphi^{-1}(a_i^2) = (a_i - \sum_{s\in S} c_s\frac{\langle\alpha_s,a_i\rangle}{\alpha_s})^2 = a_i^2 - \sum_{s\in S} c_s\langle\alpha_s,a_i\rangle(a_i\alpha_s^{-1}+\alpha_s^{-1}a_i) + \sum_{s,s'\in S} c_s c_{s'}\frac{\langle\alpha_s,a_i\rangle\langle\alpha_{s'},a_i\rangle}{\alpha_s\alpha_{s'}}$$
$$= a_i^2 - 2\sum_{s\in S} c_s\langle\alpha_s,a_i\rangle\alpha_s^{-1}a_i + \hbar\sum_{s\in S} c_s\frac{\langle\alpha_s,a_i\rangle^2}{\alpha_s^2} + \sum_{s,s'\in S} c_s c_{s'}\frac{\langle\alpha_s,a_i\rangle\langle\alpha_{s'},a_i\rangle}{\alpha_s\alpha_{s'}}.$$

Summing over all i, we get

$$\varphi^{-1}(\Delta) = \Delta - \sum_{s\in S}\frac{c_s}{\alpha_s}\sum_{i=1} 2\langle\alpha_s,a_i\rangle a_i + \hbar\sum_{s\in S} c_s\alpha_s^{-2}\sum_{i=1}^n\langle\alpha_s,a_i\rangle^2 +$$
$$\sum_{s,s'} c_s c_{s'}\alpha_s^{-1}\alpha_{s'}^{-1}\sum_{i=1}^n\langle\alpha_s,a_i\rangle\langle\alpha_{s'},a_i\rangle = \Delta - \sum_{s\in S}\frac{c_s}{\alpha_s}(\alpha_s,\alpha_s)\alpha_s^\vee +$$
$$\sum_{s\in S}\frac{c_s(c_s+\hbar)(\alpha_s,\alpha_s)}{\alpha_s^2} + \sum_{s\neq s'} c_s c_{s'}\frac{(\alpha_s,\alpha_{s'})}{\alpha_s\alpha_{s'}}.$$

To show that $\varphi^{-1}(H) = \overline{H}$ it remains to show that the last summand, say P, in the previous sum is 0. We remark that P is W-invariant and so δP is W-sign-invariant. Also δP is a polynomial of degree $\deg \delta - 2 = |S| - 2$. However, if $s(F) = -F$ for $F \in \mathbb{C}[\mathfrak{h}]$, then F is divisible by α_s. It follows that δP is divisible by δ, which is nonsense. $\square$

Summarizing, for the free generators $F_2 = \Delta, \dots, F_{n+1}$ of $S(\mathfrak{h})^W$, the elements $H_i := \varphi(\Theta(F_i))$ form a completely integrable system (modulo $\hbar$, they generate a subalgebra isomorphic to $S(\mathfrak{h})^W$).

13.4. **Future directions.** We have found first integrals for the classical CM system of type A in two different ways: using Hamiltonian reduction and using the Dunkl homomorphism. One can ask how these two ways are related. The second way allowed us to deal with the quantum system as well. So another question is whether we can extend a Hamiltonian reduction procedure to deal with quantum systems.

These are basically questions that we will be dealing with in the next four lectures. We remark that we have already seen some kind of the answer to the first question in our study of Kleinian singularities. Namely, we have seen that the spherical SRA $eH_{0,c}e$ for the Kleinian group can be realized as a Hamiltonian reduction of the space $\mathrm{Rep}(Q, \delta)$ under the action of $GL(\delta)$ at a suitable point $\lambda \in \mathfrak{gl}(\delta)^{*\,\mathrm{GL}(\delta)}$, where Q is the double McKay quiver and δ is the imaginary root.

We will see that the algebra $eH_{t,c}e$ for $\Gamma = \mathfrak{S}_n \ltimes \Gamma_1^n$ can be realized as a *quantum* Hamiltonian reduction of the representation space of a suitable quiver.

Problem 13.6. *Let* $\Gamma = \mathfrak{S}_n$ *and* $\mathfrak{h} = \mathbb{C}^n$ *(and not the reflection representation, this is a minor technicality). The goal of this problem will be to relate the CM space* C *to* $\mathrm{Spec}(eH_{0,c}e)$*. We are going to produce a morphism* $\mathrm{Rep}_\Gamma(H_{0,c}, \mathbb{C}\Gamma)/\!/\,\mathrm{GL}(\mathbb{C}\Gamma)^\Gamma \to C$*, to show that this is an isomorphism. Then we prove that the natural morphism* $\mathrm{Rep}_\Gamma(H_{0,c}, \mathbb{C}\Gamma)/\!/\,\mathrm{GL}(\mathbb{C}\Gamma)^\Gamma \to \mathrm{Spec}(eH_{0,c}e)$ *is an isomorphism.*

Let $y_1, \dots, y_n$ *be the tautological basis in* $\mathbb{C}^n = \mathfrak{h}$ *and* $x_1, \dots, x_n$ *be the dual basis in* $\mathfrak{h}^*$*. The elements* x_n, y_n *still act on* $N^{\mathfrak{S}_{n-1}} \cong \mathbb{C}^n$*. Show that* $[x_n, y_n] \in O = \{A|\operatorname{tr} A = 0, \mathrm{rk}(A+E) = 1\}$ *for a suitable choice of* c*. Deduce that we have a morphism* $\mathrm{Rep}_\Gamma(H_{0,c}, \mathbb{C}\Gamma) \to \mu^{-1}(O)$*. Show that it descends to a morphism* $\mathrm{Rep}_\Gamma(H_{0,c}, \mathbb{C}\Gamma)/\!/\,\mathrm{GL}(\mathbb{C}\Gamma)^\Gamma \to C$*. Show that the latter is finite and birational. Deduce that it is an isomorphism.*

Show that a natural morphism $\mathrm{Rep}_\Gamma(H_{0,c}, \mathbb{C}\Gamma)/\!/\,\mathrm{GL}(\mathbb{C}\Gamma)^\Gamma \to \mathrm{Spec}(eH_{0,c}e)$ *(how is it constructed, by the way?) is also finite and birational. Deduce that it is an isomorphism.*

LECTURES ON SYMPLECTIC REFLECTION ALGEBRAS

IVAN LOSEV

14. Quantum Hamiltonian reduction and SRA for wreath-products

14.1. **Quantum comoment maps.** Let $\mathcal{A}_\hbar$ be an associative unital algebra over $\mathbb{C}[\hbar]$ that is flat and separated in the $\hbar$-adic topology and such that $\mathcal{A}_\hbar/(\hbar)$ is commutative. As we have seen in Lecture 12, we have a natural Poisson bracket on $A := \mathcal{A}_\hbar/(\hbar)$: it is induced by $\frac{1}{\hbar}[\cdot,\cdot]$.

We suppose that an algebraic group G acts on $\mathcal{A}_\hbar$ by $\mathbb{C}[\hbar]$-algebra automorphisms. We will use two different settings.

(S1) $\mathcal{A}_\hbar$ is complete in the $\hbar$-adic topology and the action of G is pro-rational, i.e., the induced action of G on every quotient $\mathcal{A}_\hbar/(\hbar^k)$ is rational.

(S2) $\mathcal{A}_\hbar$ is graded, $\mathcal{A}_\hbar = \bigoplus_{i=0}^{+\infty} \mathcal{A}_\hbar^i$, with $\hbar$ of some positive degree, say d, and G preserves the grading and acts rationally.

We remark that in the second case the Poisson bracket on A is of degree $-d$.

We will mostly use (S2) but still occasionally need (S1).

In both cases we have the induced representation of $\mathfrak{g}$ on $\mathcal{A}_\hbar$ and this representation is by derivations. Let $\xi_{\mathcal{A}}$ denote the derivation corresponding to ξ. Of course, the map $\xi \mapsto \xi_{\mathcal{A}}$ is G-equivariant.

By a quantum comoment map for the action of G on $\mathcal{A}_\hbar$ we mean a linear map $\Phi : \mathfrak{g} \to \mathcal{A}_\hbar$ that is G-equivariant and satisfies $\frac{1}{\hbar}[\Phi(\xi),\cdot] = \xi_{\mathcal{A}}$. We remark that a quantum comoment map is not recovered uniquely. For example, under the assumption that G is connected, if Φ is a quantum comoment map and Ψ is a map from $\mathfrak{g}/[\mathfrak{g},\mathfrak{g}]$ to the center of $\mathcal{A}_\hbar$, then $\Phi+\Psi$ is also a quantum comoment map and all quantum comoment maps are obtained in this way. Also we remark that modulo $\hbar$, the map Φ is a classical comoment map. Finally, in the setting (S2) we always assume that Φ lands in degree d.

Exercise 14.1. *Prove that* $\Phi([\xi,\eta]) = \frac{1}{\hbar}[\Phi(\xi),\Phi(\eta)]$ *for any* $\xi,\eta \in \mathfrak{g}$.

Let us explain two (related) examples of quantum comoment maps. First, let V be a symplectic vector space and consider the Weyl algebra $W_\hbar(V)$ (with its usual grading, where the degree of $\hbar$ is 2). The group $G = \mathrm{Sp}(V)$ acts by automorphisms of $W_\hbar(V)$. We are going to establish a quantum comoment map for this action.

The degree 1 component $W_\hbar(V)^1$ is naturally identified with V. Now for $a \in W_\hbar(V)^2$ the map $\frac{1}{\hbar}[a,\cdot] : W_\hbar(V) \to W_\hbar(V)$ is degree preserving. In particular, $W_\hbar(V)^2$ is a Lie subalgebra with respect to the bracket $\frac{1}{\hbar}[\cdot,\cdot]$ and V is a module over this algebra. The symplectic form on V can also be described as $\frac{1}{\hbar}[\cdot,\cdot]$. From the Jacobi identity for $W_\hbar(V)$ applied to elements from $W_\hbar(V)^2, W_\hbar(V)^1, W_\hbar(V)^1$, we see that the action of $W_\hbar(V)^2$ on $W_\hbar(V)^1$ annihilates the symplectic form and so we get a Lie algebra homomorphism $W_\hbar(V)^2 \to \mathfrak{sp}(V)$. Definitely, $\hbar \in W_\hbar(V)^2$ lies in the kernel. We claim that the kernel is spanned by this element. Indeed, any element of the kernel lies in the center of $W_\hbar(V)$ because that algebra is spanned by V.

Exercise 14.2. *Prove that the center of* $W_\hbar(V)$ *coincides with* $\mathbb{C}[\hbar]$.

As a vector space, $W_\hbar(V)^2 = S^2(V) \oplus \mathbb{C}\hbar$. So the dimensions of $W_\hbar(V)^2/\mathbb{C}\hbar$ and $\mathfrak{sp}(V)$ coincide. Therefore the homomorphism $W_\hbar(V)^2 \to \mathfrak{sp}(V)$ is surjective. So $W_\hbar(V)^2$ is an extension of $\mathfrak{sp}(V)$ by $\mathbb{C}$ that is forced to split (because $\mathfrak{sp}(V)$ is simple) and actually in a unique, and hence $\mathrm{Sp}(V)$-equivariant, way. For Φ we take this splitting, it is a quantum comoment map. The latter follows from the observation that ξ_W acts on $V = W_\hbar(V)^1$ as the operator ξ.

We remark that if G acts on V by linear symplectomorphisms, then the induced action of G on $W_\hbar(V)$ also admits a quantum comoment map, the composition of the induced homomorphism $\mathfrak{g} \to \mathfrak{sp}(V)$ with Φ constructed above.

Let us proceed to our second example. Let X_0 be a smooth affine variety acted on by an algebraic group G. Then we can form the algebra $D_\hbar(X_0)$ of homogenized differential operators. This algebra is graded ($\mathbb{C}[X_0]$ has degree 0, while $\hbar$ and $\mathrm{Vect}(X_0)$ have degree 1), and G satisfying the assumptions of (S2).

Exercise 14.3. *Describe the map $\xi \mapsto \xi_{\mathcal{A}}$ for $\mathcal{A}_\hbar = D_\hbar(X_0)$ and show that $\xi \mapsto \xi_{X_0}$ is a quantum comoment map.*

Now consider the special case when X_0 is a vector space. Then $D_\hbar(X_0)$ is naturally identified with $W_\hbar(X_0 \oplus X_0^*)$. We have two quantum comoment maps, Φ_W and Φ_D.

Problem 14.4. *Describe the difference $\Phi_D - \Phi_W$.*

In these examples we only used setting (S2). We can get setting (S1) if we pass to the $\hbar$-adic completions. This useful for the reason that many constructions from commutative algebra, like localization or completion, do not work with (S2) but do with (S1).

Exercise 14.5. *Let $\mathcal{A}_\hbar$ be an associative unital algebra over $\mathbb{C}[\hbar]$, flat over $\mathbb{C}[\hbar]$, complete and separated in the $\hbar$-adic topology, and such that $A := \mathcal{A}_\hbar/(\hbar)$ is commutative. Let S be a mutliplicatively closed subset of A that does not contain 0 and let π_k denote the projection $\mathcal{A}_\hbar/(\hbar^k) \twoheadrightarrow A$. Show that $\pi_k^{-1}(S)$ satisfies the Ore condition: i.e., for all $a \in \mathcal{A}_\hbar/(\hbar^k), s \in \pi_k^{-1}(S)$, there are $a' \in \mathcal{A}_\hbar/(\hbar^k), s' \in \pi_k^{-1}(S)$ such that $as' = a's$. Show that there are natural epimorphisms $\mathcal{A}_\hbar/(\hbar^{k+1})[\pi_{k+1}(S)^{-1}] \twoheadrightarrow \mathcal{A}_\hbar/(\hbar^k)[\pi_k(S)^{-1}]$ and prove that $\mathcal{A}_\hbar[S^{-1}] := \varprojlim_k \mathcal{A}_\hbar/(\hbar^k)[\pi_k(S)^{-1}]$ is flat over $\mathbb{C}[[\hbar]]$.*

14.2. **Quantum Hamiltonian reduction.** Let $\mathcal{A}_\hbar, G, \Phi$ be as in the previous section. We can consider the quantum Hamiltonian reduction $\mathcal{A}_\hbar /\!/\!/_0 G := [\mathcal{A}_\hbar/\mathcal{A}_\hbar\Phi(\mathfrak{g})]^G$. The latter space is an associative unital $\mathbb{C}[\hbar]$-algebra with product given by $(a+\mathcal{A}_\hbar\Phi(\mathfrak{g}))\cdot(b+\mathcal{A}_\hbar\Phi(\mathfrak{g})) = ab + \mathcal{A}_\hbar\Phi(\mathfrak{g})$. We remark that in setting (S2), this algebra is naturally graded, the grading is induced from $\mathcal{A}_\hbar$.

As in the Poisson case, this construction can be generalized to the reduction at ideals. Namely, thanks to Exercise 14.1, the map $\Phi : \mathfrak{g} \to A_\hbar$ extends to a G-equivariant (graded for (S2)) algebra homomorphism $U_\hbar(\mathfrak{g}) \to \mathcal{A}_\hbar$, where $U_\hbar(\mathfrak{g})$ is a homogenized universal enveloping algebra defined as follows

$$U_\hbar(\mathfrak{g}) = T(\mathfrak{g})[\hbar]/(\xi \otimes \eta - \eta \otimes \xi - [\xi, \eta]\hbar).$$

Here the grading on $U_\hbar(\mathfrak{g})$ is defined so that $\deg \mathfrak{g} = \deg \hbar = d$.

We pick a graded G-stable two-sided ideal $\mathcal{I} \subset U_\hbar(\mathfrak{g})$ that is $\hbar$-saturated in the sense that $\hbar x \in \mathcal{I}$ implies $x \in \mathcal{I}$ (equivalently, the quotient $U_\hbar(\mathfrak{g})/\mathcal{I}$ is flat over $\mathbb{C}[\hbar]$). Then we set $\mathcal{A}_\hbar /\!/\!/_{\mathcal{I}} G := [\mathcal{A}_\hbar/\mathcal{A}_\hbar\Phi(\mathcal{I})]^G$. This is an associative algebra with respect to a product analogous to the above.

Exercise 14.6. *Show that the product on $\mathcal{A}_\hbar///_{\mathcal{I}}G$ is well-defined.*

For example, for $\mathcal{I}$ we can take $\mathfrak{g}U_\hbar(\mathfrak{g})$ so that $U_\hbar(\mathfrak{g})/\mathcal{I} = \mathbb{C}[\hbar]$. Another option, for $\lambda \in \mathfrak{g}^{*G}$ we can consider the ideal in $U_\hbar(\mathfrak{g})$ generated by the ideal $\{\xi - \hbar\langle\lambda, \xi\rangle | \xi \in \mathfrak{g}\}$. The corresponding reduction will be denoted by $\mathcal{A}_\hbar///_{\lambda\hbar}G$. Finally, and this will be our favorite choice, we can consider the ideal $\mathcal{I} = [\mathfrak{g}, \mathfrak{g}]U_\hbar(\mathfrak{g})$. We have $U_\hbar(\mathfrak{g})/\mathcal{I} = S(\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}])[\hbar]$. The corresponding reduction will be denoted by $\mathcal{A}_\hbar///G$. This is an algebra over $S(\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}])[\hbar]$. A map $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}] \to \mathcal{A}_\hbar///G$ is induced by Φ (we mod out $[\mathfrak{g}, \mathfrak{g}]$).

Exercise 14.7. *Check that the image of $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$ in $[\mathcal{A}_\hbar/\mathcal{A}_\hbar\Phi([\mathfrak{g}, \mathfrak{g}])]$ consists of G-invariant elements that commute with $[\mathcal{A}_\hbar/\mathcal{A}_\hbar\Phi([\mathfrak{g}, \mathfrak{g}])]^G$.*

The reduction $\mathcal{A}_\hbar///_{\lambda\hbar}G$ is the specialization $\mathbb{C}[\hbar]\otimes_{S(\mathfrak{g}/[\mathfrak{g},\mathfrak{g}])[\hbar]}\mathcal{A}_\hbar///G$ for the homomorphism $S(\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}])[\hbar] \to \mathbb{C}[\hbar]$ given by $\xi \mapsto \langle\lambda, \xi\rangle\hbar$.

Problem 14.8. *Let G be a reductive group acting freely on a smooth affine variety X_0. Identify $D_\hbar(X_0)///_0G$ with $D_\hbar(X_0//G)$.*

Below we will denote $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$ by $\mathfrak{z}$.

14.3. **Sufficient condition for flatness.** From now on, we assume that the group G is reductive, in particular, $\mathfrak{g} = \mathfrak{z} \oplus [\mathfrak{g}, \mathfrak{g}]$. Under this assumption, we have

$$[\mathcal{A}_\hbar/\mathcal{A}_\hbar\Phi([\mathfrak{g}, \mathfrak{g}])]^G/(\mathfrak{z}, \hbar) = [\mathcal{A}_\hbar/(\mathcal{A}_\hbar\Phi([\mathfrak{g}, \mathfrak{g}]) + (\mathfrak{z}, \hbar))]^G = [A/\mu^*(\mathfrak{g})A]^G = A///_0G$$

We want to find conditions for $\mathcal{A}_\hbar///G$ to be flat over $S(\mathfrak{z})[\hbar]$. We will assume that G is reductive and that $A = \mathcal{A}_\hbar/(\hbar)$ is a finitely generated algebra. Let X denote the corresponding scheme and let $\mu : X \to \mathfrak{g}^*$ be the moment map (that comes from the comoment map given by Φ modulo $\hbar$).

Proposition 14.1. *Let $\mathcal{A}_\hbar$ be as in (S2) and assume, in addition, that the grading is positive, i.e., $\mathcal{A}_\hbar^0 = \mathbb{C}$. Suppose that $\operatorname{codim}_X \mu^{-1}(0) = \dim \mathfrak{g}$. Then $\mathcal{A}_\hbar///G$ is flat over $S(\mathfrak{z})[\hbar]$.*

Proof. Recall that a sequence of elements $f_1, \ldots, f_k \in A$ is called *regular* if, for each i, the element f_i is not a zero divisor in $A/(f_1, \ldots, f_{i-1})$. This is equivalent to the condition that the subscheme defined by $f_1, \ldots, f_k$ has codimension k.

The proof of the proposition is based on the following property of regular sequences, see, for example, [E, Chapter 17]. Assume that A is a $\mathbb{Z}_{\geqslant 0}$-graded algebra with $A^0 = \mathbb{C}$. Suppose that $f_1, \ldots, f_k$ is a regular sequence of homogeneous elements and $g_1, \ldots, g_k \in A$ are such that $\sum_{i=1}^k f_i g_i = 0$. Then there are elements $g_{ij} \in A$ with $g_{ij} = -g_{ji}$ with the property that $g_i = \sum_{i=1}^k g_{ij} f_j$ (obviously, for this choice of the elements g_i the sum $\sum_{i=1}^k f_i g_i$ vanishes).

Let $x_1, \ldots, x_m$ be a basis in $[\mathfrak{g}, \mathfrak{g}]$, and $z_1, \ldots, z_k$ be a basis in $\mathfrak{z}$. First, we want to show that the ideal $A_\hbar\Phi([\mathfrak{g}, \mathfrak{g}])$ is $\hbar$-saturated. Let $a \in \mathcal{A}_\hbar$ be such that $\hbar a \in \mathcal{A}_\hbar\Phi([\mathfrak{g}, \mathfrak{g}])$. We need to check that $\hbar a \in \hbar\mathcal{A}_\hbar\Phi([\mathfrak{g}, \mathfrak{g}])$. We can write a as $\sum_{i=1}^m G_i\Phi(x_i)$, where we can assume that at least one G_i is not divisible by $\hbar$. Let g_i be the class of G_i modulo $\hbar$. We have $\sum_{i=1}^m g_i\mu^*(x_i) = 0$. Since $\mu^*(x_1), \ldots, \mu^*(x_m)$ form a regular sequence in A, we see that there are elements $g_{ij} \in A$ with $g_{ij} = -g_{ji}$ and $g_i = \sum_j g_{ij}\mu^*(x_j)$. Lift the elements g_{ij} to $G_{ij} \in \mathcal{A}_\hbar$ with $G_{ij} = -G_{ji}$. We deduce that $G_i = \sum_{j=1}^m G_{ij}\Phi(x_j) + \hbar G_i'$ for some $G_i' \in \mathcal{A}_\hbar$. We can rewrite the sum $\sum_{i=1}^m G_i\Phi(x_i)$ as

$$\sum_{i,j=1}^m G_{ij}\Phi(x_j)\Phi(x_i)+\hbar\sum_{i=1}^m G'_i\Phi(x_i)=\sum_{i<j} G_{ij}[\Phi(x_j),\Phi(x_i)]+\hbar\sum_{i=1}^m G'_i\Phi(x_i)=$$
$$\hbar\sum_{i<j} G_{ij}\Phi([x_j,x_i])+\hbar\sum_{i=1}^m G'_i\Phi(x_i).$$

This shows that $\hbar a\in\hbar\mathcal{A}_\hbar\Phi([\mathfrak{g},\mathfrak{g}])$.

So far, we have only used that $\mu^*(x_1),\ldots,\mu^*(x_m)$ form a regular sequence. Since $\mu^*(x_1),\ldots,\mu^*(x_m),\mu^*(z_1),\ldots,\mu^*(z_k)$ form a regular sequence in A, we see that $\mu^*(z_1),\ldots,\mu^*(z_k)$ form a regular sequence in $A/A\mu^*([\mathfrak{g},\mathfrak{g}])$. Therefore $\mu^*(z_i)$ is a nonzero divisor in

$$[A/A\mu^*([\mathfrak{g},\mathfrak{g}])]/(\mu^*(z_1),\ldots,\mu^*(z_{i-1})).$$

Now the claim of the proposition follows from the following general fact that is left as an exercise:

Let $M=\bigoplus_{i=0}^{+\infty}M_i$ with $\dim M_i<\infty$ be a graded module over the polynomial ring $\mathbb{C}[y_1,\ldots,y_l]$ (where all y_i's are supposed to have positive degrees). Then the following two conditions are equivalent:

(i) M is a graded free module.
(ii) y_i is a nonzero divisor in $M/(y_1,\ldots,y_{i-1})M$.

We apply this to $M=\mathcal{A}_\hbar/\mathcal{A}_\hbar\Phi([\mathfrak{g},\mathfrak{g}])$ and $y_1=\hbar,y_2=\mu^*(z_1),\ldots,y_l=\mu^*(z_k)$. □

In particular, if the assumption of Proposition 14.1 holds, then all reductions $\mathcal{A}///_{\lambda\hbar}G$ are deformations of $A///_0G$ over $\mathbb{C}[\hbar]$.

14.4. **Spherical SRA as quantum Hamiltonian reductions.** We have already seen some connections between spherical subalgebras in SRA and Hamiltonian reductions: in the cases when a group Γ was a Kleinian subgroup $\Gamma_1\subset\mathrm{SL}_2(\mathbb{C})$ and a symmetric group $\mathfrak{S}_n$ acting on the double $\mathbb{C}^{2n}$ of its permutation representation $\mathbb{C}^n$. The Hamiltonian reduction in both cases was of similar nature: a space R being reduced was the representation space $\mathrm{Rep}(Q,v)$ of some double quiver Q and a group G was the product of several general linear groups. More precisely, in the case of a Kleinian group, Q was the double McKay quiver, v was the indecomposable imaginary root δ, and G was $\mathrm{GL}(\delta)$. In the case of a symmetric group, Q has two vertices, 0 and ∞, two loops at 0 and two arrows between 0 and ∞ going in opposite directions. The dimension vector v in this case equals $n\epsilon_0+\epsilon_\infty$, where $\epsilon_0,\epsilon_\infty$ are coordinate vectors at the corresponding vertices. Finally, we took $G=\mathrm{GL}(n)$.

It is natural to expect that a connection should extend to the case of $\Gamma=\Gamma_n=\mathfrak{S}_n\ltimes\Gamma_1^n$.. This is indeed so. For Q we take the double Q of the following quiver: we take the (undoubled) McKay quiver with an additional vertex ∞ and an additional arrow $\infty\to 0$. We set $v=n\delta+\epsilon_\infty$ and for G take $\mathrm{GL}(n\delta)$.

Theorem 14.2 (Gan-Ginzburg, [GG]). *In the above notation, we have the following.*

(i) *The fiber $\mu^{-1}(0)$ is reduced and has codimension* $\dim G$ *in R.*
(ii) *There is a $\mathbb{C}^\times$-equivariant isomorphism of schemes $R///_0G\cong\mathbb{C}^{2n}/\Gamma_n$.*

The theorem will be proved in the next lecture.

Here the $\mathbb{C}^\times$-action on $\mathbb{C}^{2n}/\Gamma_n$ is induced from the dilations action on $\mathbb{C}^{2n}$ and the action on $R///_0G$ is induced from the dilations action on R.

Thanks to (i) and Proposition 14.1, $\mathrm{W}_\hbar(R)///G$ is a graded deformation of $\mathbb{C}[R///_0G]$ over $S(\mathfrak{z})[\hbar]$. The dimension of $\mathfrak{z}$ coincides with the number of irreducible Γ_1-modules (provided $n>1$). So $\dim \mathfrak{z}\oplus\mathbb{C}\hbar$ coincides with the dimension of the parameter space P of the universal SRA.

Theorem 14.3. *There is a graded algebra isomorphism $eHe \to \mathrm{W}_\hbar(V)///G$ that maps P to $\mathfrak{z}\oplus\mathbb{C}\hbar$ and $t\in P$ to $\hbar$.*

It is possible to write an explicit formula for the isomorphism $P\to\mathfrak{z}\oplus\mathbb{C}\hbar$, we may return to this in a subsequent lecture.

Here is a brief history of Theorem 14.3. It was first proved by Holland in the case of Kleinian groups (strictly speaking not for our Q but for the double of the McKay quiver, but this difference is not essential in this case). Then Etingof and Ginzburg proved a somewhat weaker version for the symmetric groups. This result was refined by Gan and Ginzburg. Then Oblomkov proved an analog of the Etingof-Ginzburg result for cyclic Γ_1. His result was refined by Gordon. Finally, Etingof, Gan, Ginzburg and Oblomkov gave a proof in the remaining cases.

An alternative proof was given by the author in [L] (the reader is referred to that paper for references). This is a proof that we are going to explain.

14.5. **Outline of proof.** A problem with studying deformations of $\mathbb{C}^{2n}/\Gamma_n$ is that this variety is not smooth. In particular, there seems to be no deformation of $\mathbb{C}^{2n}/\Gamma_n$ with a categorical universality property. However, and we have already seen this, it is possible to relate deformations of $\mathbb{C}^{2n}/\Gamma_n$ to deformations of something smooth, namely the smash-product $\mathbb{C}[\mathbb{C}^{2n}]\#\Gamma_n$: the deformation eHe of $\mathbb{C}^{2n}/\Gamma_n$, by the very definition, can be "lifted" to a deformation H of $\mathbb{C}[\mathbb{C}^{2n}]\#\Gamma_n$, which now has a universality property.

One can try to consider a purely algebro-geometric resolution of $\mathbb{C}^{2n}/\Gamma_n$ and ask about its deformations. We are very fortunate here: there is a (non-unique) *symplectic resolution* $\widetilde{\mathbb{C}^{2n}/\Gamma_n}$ of $\mathbb{C}^{2n}/\Gamma_n$ and it also can be obtained by a suitable version of Hamiltonian reduction. Thanks to this, we can lift $\mathrm{W}_\hbar(V)///G$ to a deformation of the resolution (that will be a sheaf, not a single algebra). This deformation will be, in fact, universal, but we will not need that.

Then one needs to relate the deformations of two different kind of resolutions, $\mathbb{C}[\mathbb{C}^{2n}]\#\Gamma_n$ and $\widetilde{\mathbb{C}^{2n}/\Gamma_n}$. This will be done using a so called *Procesi bundle*, a vector bundle on $\widetilde{\mathbb{C}^{2n}/\Gamma_n}$ whose endomorphisms are $\mathbb{C}[\mathbb{C}^{2n}]\#\Gamma_n$.

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LECTURES ON SYMPLECTIC REFLECTION ALGEBRAS

IVAN LOSEV

15. Quotient singularities as quiver varieties

15.1. **Main theorem.** We fix $n \geqslant 1$ and a Kleinian group $\Gamma_1 \subset \mathrm{SL}_2(\mathbb{C})$. We form the wreath-product group $\Gamma_n = \mathfrak{S}_n \ltimes \Gamma_1^n$, it naturally acts on $\mathbb{C}^{2n} = (\mathbb{C}^2)^{\oplus n}$. We are going to describe the quotient singularity $\mathbb{C}^{2n}/\Gamma_n$ as a quiver variety, i.e., as a Hamiltonian reduction of the representation space of an appropriate double quiver.

Recall that from Γ_1 we can produce its McKay quiver $\underline{Q}^{MK}$ that is of affine type, its vertices numbered by $0, \ldots, r$ are in one-to-one correspondence with Γ_1-irreps, where 0 corresponds to the trivial representation. We take the quiver $\underline{Q}^{CM}$ obtained from $\underline{Q}^{MK}$ by adding an additional vertex ∞ and one arrow from ∞ to 0. Then we take the double quiver Q^{CM} of $\underline{Q}^{CM}$.

Consider the representation space $R := \mathrm{Rep}(Q^{CM}, v)$, where $v = n\delta + \epsilon_\infty$, δ being the indecomposable imaginary root (supported on the vertices $0, \ldots, r$) and ϵ_∞ is the coordinate vector at ∞. We consider the group $G := \mathrm{GL}(n\delta)$, it acts on R in a Hamiltonian way with moment map μ constructed in Lecture 10. We remark that we can consider the larger group, $\bar{G} := G \times \mathbb{C}^\times$, where $\mathbb{C}^\times$ acts on the one-dimensional space at ∞; this group still acts on R. However, the one-dimensional torus $(x \,\mathrm{id}_{\mathbb{C}^{v_i}})_{i \in \underline{Q}_0^{CM}}$ acts trivially on R. Moreover, the moment map $\bar{\mu}$ for $\bar{G}$ is recovered from μ as follows: $\bar{\mu}(r) = (\mu(r), -\sum_{i=0}^r \operatorname{tr} \mu(r)_i)$. So the reductions with respect to G and with respect to $\bar{G}$ are the same.

Theorem 15.1. *[Gan-Ginzburg, [GG]] The fiber $\mu^{-1}(0)$ is reduced and has codimension $\dim G$ in R.*

We first show that the codimension of $\mu^{-1}(0)$ is $\dim G$. For this we recall (Lecture 10) that $\mu^{-1}(0)$ is the union of cotangent bundles to orbits in $R_0 := \mathrm{Rep}(\underline{Q}, v)$. The codimension of any conormal bundle is $\dim R_0$. The codimension of the union of the conormal bundles is therefore $\dim R_0 + m$, where m is "the maximal number of parameters describing G-orbits in R_0". This will be defined precisely and computed below.

Then we will show that the fiber $\mu^{-1}(0)$ is reduced. For this, as we have seen in Lecture 11, it is enough to prove that each component of $\mu^{-1}(0)$ admits a free G-orbit. To achieve this, we will need an explicit description of the components. In particular, we will see that there are $n+1$ of them.

Theorem 15.2. *We have a $\mathbb{C}^\times$-equivariant isomorphism $\mu^{-1}(0)/\!/G \cong \mathbb{C}^{2n}/\Gamma_n$.*

This is a special case of [CB2, Theorem 1.1].

15.2. **Theorems on quiver representations.** First of all, let us discuss the number of parameters needed to describe representations of a quiver $\underline{Q}$ with given dimension v up to an isomorphism. Here $\underline{Q}$ is an arbitrary quiver.

We will need a stratification of $\mathrm{Rep}(\underline{Q}, v)$ by dimensions of indecomposable summands (recall that each representation has a decomposition into the direct sum of indecomposables, the multiplicities of the summands do not depend on the choice of a decomposition, this is a special case of the Krull-Schmidt theorem). Let $I(\alpha^1,\ldots,\alpha^n)$ denote the subset of $\operatorname{Rep}(\underline{Q},v)$ of all representations, whose decomposition into indecomposables contains summands of dimensions $\alpha^1,\ldots,\alpha^n$. A choice of a decomposition of the graded vector space of dimension v into the summands of dimensions $\alpha^1,\ldots,\alpha^n$ gives rise to an embedding $\prod_{i=1}^n I(\alpha^i)\hookrightarrow I(\alpha^1,\ldots,\alpha^m)$ and to a surjection $\mathrm{GL}(v)\times\prod_{i=1}^n I(\alpha^i)\twoheadrightarrow I(\alpha^1,\ldots,\alpha^m)$ that descends to a surjection

$$\mathrm{GL}(v)\times_{\prod_{i=1}^n \mathrm{GL}(\alpha^i)}\prod_{i=1}^n I(\alpha^i)\twoheadrightarrow I(\alpha^1,\ldots,\alpha^m). \tag{1}$$

Using this (and the classical algebro-geometric result that the image of a constructible subset under a morphism is constructible), one can prove by induction that $I(\alpha^1,\ldots,\alpha^n)$ is a constructible set (i.e., is a union of finitely many locally closed subvarieties) and that these subvarieties can be chosen $\mathrm{GL}(v)$-stable.

We are now ready to define $m(\alpha)$, the number of parameters needed to describe indecomposable representations of dimension α. Let Z be an irreducible algebraic variety acted on by a connected algebraic group G. For $i\geqslant 0$ consider $Z_i:=\{z\in Z|\dim Gz=i\}$, this is a locally closed subvariety. We set $m(Z):=\max_i \dim Z_i-i$. We remark that $m(Z)=0$ is equivalent to Z having only finitely many G-orbits. The definition of $m(Z)$ extends to the case when Z is a G-stable constructible subset in some G-variety. Now we set $m(\alpha)=m(I(\alpha))$. Similarly, we can define the number $m(\alpha^1,\ldots,\alpha^n):=m(I(\alpha^1,\ldots,\alpha^n))$.

Lemma 15.3. *We have $m(\alpha^1,\ldots,\alpha^n)=\sum_{i=1}^n m(\alpha^i)$.*

Proof. The inequality $m(\alpha^1,\ldots,\alpha^n)\leqslant\sum_{i=1}^n m(\alpha^i)$ is an easy consequence of (1). Let us prove the opposite inequality. We may assume that $m(\alpha^1),\ldots,m(\alpha^k)>0, m(\alpha^{k+1})=\ldots=m(\alpha^n)=0$. Let $I^0(\alpha^i), i=1,\ldots,k$ be irreducible $\mathrm{GL}(\alpha^i)$-stable locally closed subvarieties in $I(\alpha^i)$ such that $m(I^0(\alpha^i))>0$. We still have a surjection

$$\mathrm{GL}(v)\times_{\prod_{i=1}^k \mathrm{GL}(\alpha^i)\times \mathrm{GL}(\alpha^{k+1}+\ldots+\alpha^n)}(\prod_{i=1}^k I^0(\alpha^i)\times I(\alpha^{k+1}+\ldots+\alpha^n))\twoheadrightarrow I(\alpha^1,\ldots,\alpha^n).$$

It is easy to see that the stabilizer in $\mathrm{GL}(v)$ of a generic element of $\prod_{i=1}^k I^0(\alpha^i)\times I(\alpha^{k+1},\ldots,\alpha^n)$ is contained in $\prod_{i=1}^k \mathrm{GL}(\alpha^i)\times\mathrm{GL}(\alpha^{k+1}+\ldots+\alpha^n)$. So the surjection above generically has finite fibers. It follows that $m(\alpha^1,\ldots,\alpha^n)=m(\alpha^1)+\ldots+m(\alpha^k)+m(\alpha^{k+1},\ldots,\alpha^n)=m(\alpha^1)+\ldots+m(\alpha^k)$. ☐

Example 15.4. Let us consider the case of a quiver with one vertex and a single loop. Here $I(\alpha^1,\ldots,\alpha^n)$ consists of matrices whose Jordan normal form has n blocks of sizes $\alpha^1,\ldots,\alpha^n$. Clearly, $m(\alpha^1,\ldots,\alpha^n)=n$.

There is a formula for $m(\alpha)$ found by Kac. Consider the quadratic function $(v,v)=\sum_{i\in\underline{Q}_0}v_i^2-\sum_{a\in\underline{Q}_1}v_{h(a)}v_{t(a)}$. A nonzero element $\alpha\in\mathbb{Z}_{\geqslant 0}^{Q_0}$ is called a *root* if $(\alpha,\alpha)\leqslant 1$. Then set $p(v)=1-(v,v)$.

Theorem 15.5. (1) *$I(\alpha)\neq\varnothing$ if and only if α is a root and $m(\alpha)=p(\alpha)$.*

(2) *there is a decomposition $I(\alpha)=\bigsqcup_{i=0}^N I^i(\alpha)$ into irreducible locally closed G-stable subvarieties such that $m(I^0(\alpha))=p(\alpha), m(I^i(\alpha))<p(\alpha)$.*

The first part is a well-known theorem of Kac. A reference for the second one can be found in the proof of [GG, Theorem 3.2.3].

Now let us describe the doubled setting. Let Q be the double of $\underline{Q}$ and $R_0 = \mathrm{Rep}(\underline{Q}, v), R = \mathrm{Rep}(Q, v) = T^*R_0$. We have the moment map $\mu : R \to \mathfrak{g} := \mathfrak{gl}(v)$. From the description of $\mu^{-1}(0)$ recalled above, we see that $\dim \mu^{-1}(0) = \dim R_0 + m(R_0)$. Indeed, let $\rho : R \twoheadrightarrow R_0$ be the projection. Let $R_{0i} := \{r \in R_0 | \dim Gr = i\}$. Then $\rho^{-1}(R_{0,i}) \cap \mu^{-1}(0)$ surjects to $\rho^{-1}(R_{0,i})$ with fibers of dimensions $\dim R_0 - i$.

A one-dimensional subtorus of G acts trivially, so $\mathrm{im}\,\mu \subset \mathfrak{sl}(v) := \{(A_i)_{i\in Q_0} | \sum_i \mathrm{tr}(A_i) = 0\}$ and $\mathrm{codim}_R\, \mu^{-1}(0) \leqslant \dim \mathfrak{g} - 1$. The equality $\mathrm{codim}_R\, \mu^{-1}(0) = \dim \mathfrak{g} - 1$ is equivalent to

$$m(R_0) = \dim R_0 - \dim \mathfrak{g} + 1 = \sum_{a\in \underline{Q}_1} v_{t(a)}v_{h(a)} - \sum_{i\in Q_0} v_i^2 + 1 = p(v).$$

On the other hand, from the discussion above, we see that $m(R_0) = \max \sum_{i=1}^n p(\alpha^i)$, where the max is taken over all decompositions $v = \alpha^1 + \ldots + \alpha^n$ into the sum of roots.

Theorem 15.6. *The following conditions are equivalent.*

(1) $\mathrm{codim}_R\, \mu^{-1}(0) = \dim \mathfrak{g} - 1$ *(this includes the claim that fiber is non-empty).*
(2) $p(v) \geqslant \sum_{i=1}^n p(\alpha^i)$ *for all decompositions* $v = \sum_{i=1}^n \alpha^i$ *into the sum of roots* α^i.

Both $\mathbb{C}[R], \mathbb{C}[\mathfrak{sl}(v)]$ are positively graded and μ is homogeneous, we now can apply a graded analog of [E, Theorem 18.16] to see that μ is flat. Being flat, μ is open, and, being in addition $\mathbb{C}^\times$-equivariant, it is surjective.

Now let us explain why we need part 2 of Theorem 15.5. Assume the equivalent conditions of Theorem 15.6 hold. It follows from Theorem 15.5 and the proof of Lemma 15.3 that one can decompose $I(\alpha^1, \ldots, \alpha^n)$ into the union of locally closed irreducible G-stable subvarieties $\bigsqcup_{j\geqslant 0} I^j(\alpha^1, \ldots, \alpha^n)$ such that $m(I^0(\alpha^1, \ldots, \alpha^n)) = \sum_{i=1}^n p(\alpha^i) > m(I^j(\alpha^1, \ldots, \alpha^n))$ for $j > 0$. Consider the subvariety $\rho^{-1}(I^j(\alpha^1, \ldots, \alpha^n)) \cap \mu^{-1}(0)$. Being a vector bundle over an irreducible variety, the intersection is irreducible. Its dimension is $\leqslant \dim R_0 + \sum_{i=1}^n p(\alpha^i)$ with equality achieved only if $j = 0$. Each irreducible component of $\mu^{-1}(0)$ contains exactly one dense $\rho^{-1}(I^j(\alpha^1, \ldots, \alpha^n)) \cap \mu^{-1}(0)$. We see that the irreducible components of $\mu^{-1}(0)$ are in one-to-one correspondence with decompositions $v = \sum_{i=1}^n \alpha^i$ such that $p(v) = \sum_{i=1}^n p(\alpha^i)$.

Below it will be sometimes convenient to deal with preprojective algebras. Recall that the preprojective algebra for Q is the quotient of the path algebra $\mathbb{C}Q$ by the relations

$$\sum_{a\in \underline{Q}_1, h(a)=i} aa^* - \sum_{a\in \underline{Q}_1, t(a)=i} a^*a = 0,$$

one for each $i \in Q_0$. Of course, $\mathrm{Rep}(\Pi^0(Q), v) = \mu^{-1}(0)$.

15.3. **Codimension.** Now we return to the case when $\underline{Q} = \overline{Q^{CM}}$. Consider the decomposition $n\delta + \epsilon_\infty = \sum_{i=0}^m \alpha^i$ into the sum of roots, where $\alpha^0_\infty = 1$ and $\alpha^i_\infty = 0$ for $i > 0$. So α^i is a root in the corresponding affine root system.

Let p^{MK} denote the p-function for the McKay quiver. We have $p(\alpha^i) = p^{MK}(\alpha^i)$. The latter is zero when α is a real root, and 1 when α^i is a multiple of δ. Further, we have $p(n\delta + \epsilon_\infty) = p^{MK}(n\delta) - 1 + n = 1 - 1 + n = n$.

Now we prove $p(n\delta + \epsilon_\infty) \geqslant \sum_{i=0}^m p(\alpha^i)$ and that the equality holds in exactly one of the following situations: $\alpha^0 = k\delta + \epsilon_\infty, \alpha^1 = \ldots = \alpha^{n-k} = \delta$ for some $k = 0, \ldots, n$.

We have $p(\alpha^0)=\alpha^0_0+p^{MK}(\alpha^0-\epsilon_\infty)-1$. We have $p^{MK}(\alpha^0-\epsilon_\infty)\leqslant 1$ with equality only if $\alpha^0-\epsilon_\infty=k\delta$. So either $p(\alpha^0)<\alpha^0_0$ or $p(\alpha^0)=\alpha^0_0$ for $\alpha^0=k\delta+\epsilon$. We also have $p(\alpha^i)\leqslant\alpha^i_0$ with equality only if $\alpha^i=\delta$. Since $\sum_{i=0}^m\alpha^i_0=n$, we are done.

This already proves the claim about codimension. Also this proves that the total number of irreducible components is $n+1$.

15.4. **Points without stabilizer.** We will describe the $n+1$ components of $\mu^{-1}(0)\subset \operatorname{Rep}(Q,n\delta+\epsilon_\infty)$ explicitly and in each we produce a point with a trivial stabilizer. But first we need to determine simple representations in $\mu^{-1}(0)$ for some other dimension vectors.

Lemma 15.7. *Let v be a dimension vector for Q^{MK}.*

1. *If $v<\delta$ (i.e., $v\neq\delta$ and all coordinates of $\delta-v$ are non-negative), then the only semi-simple representation in* $\operatorname{Rep}(\Pi^0(Q^{MK}),v)$ *is* 0.
2. *If $v=\delta$, then* $\operatorname{Rep}(\Pi^0(Q^{MK}),v)$ *is irreducible and a generic representation is simple.*

Proof. It is enough to prove the claim for the simple representations. The dimension of all components of $\operatorname{Rep}(\Pi^0(Q^{MK}),v)$ is $\sum_{a\in Q_1^{MK}}v_{t(a)}v_{h(a)}$. If there is a non-zero simple representation, then, due to $\mathbb{C}^\times$-equivariance, there is a one-parameter family of such, each with G-orbit of dimension $\sum_{i=0}^r v_i^2-1$. So we see that $0\leqslant\dim\mu^{-1}(0)-\dim G=-(v,v)<0$, contradiction.

Let us now consider the case of $v=\delta$. Then there is only one component of $\operatorname{Rep}(\Pi^0(Q^{MK}),\delta)$ of dimension $\sum_a\delta_{t(a)}\delta_{h(a)}+1$. This is proved by analogy with the previous section. Since $\operatorname{Rep}(\Pi^0(Q^{MK}),\delta)/\!/\operatorname{GL}(\delta)\cong\mathbb{C}^2/\Gamma_1$, we see that there are infinitely many isomorphism classes of semi-simple representations. On the other hand, by (1), any reducible nonzero semisimple representation is 0. So any representation lying in the complement of the zero fiber of $\operatorname{Rep}(\Pi^0(Q^{MK}),\delta)\to\operatorname{Rep}(\Pi^0(Q^{MK}),\delta)/\!/\operatorname{GL}(\delta)$ is simple. $\square$

Take pairwise distinct simple representations $x_1,\ldots,x_n$ of $\operatorname{Rep}(\Pi^0(Q^{MK}),\delta)$. Pick a decomposition of $\bigoplus_{i=0}^r\mathbb{C}^{n\delta_i}$ into $(\bigoplus\mathbb{C}^{\delta_i})^{\oplus n}$. Then $x:=\bigoplus_{i=1}^n x_i$ is in $\operatorname{Rep}(\Pi^0(Q^{MK}),n\delta)$. The stabilizer of x in G is isomorphic to $(\mathbb{C}^\times)^n\hookrightarrow\operatorname{GL}(\delta)^{\times n}\hookrightarrow\operatorname{GL}(n\delta)$. It acts on $\mathbb{C}^n$ (the space of maps corresponding to the arrow from ∞ to 0) faithfully by diagonal matrices, let $e_1,\ldots,e_n$ be an eigenbasis. Consider the locally closed subvariety $\mathcal{M}_k:=\{(x_1,\ldots,x_n,i,j)\}$, where $x_1,\ldots,x_n$ are as above, $i\in\mathbb{C}^n$ a vector that is the span of $e_1,\ldots,e_k$ with nonzero coefficients, $j\in\mathbb{C}^{n*},j(e_1)=\ldots=j(e_k)=0,j(e_{k+1}),\ldots,j(e_n)\neq0$. In particular, we see that $ij=0$ and so $\mathcal{M}_k\subset\mu^{-1}(0)$. The stabilizer of (i,j) in $\mathbb{C}^{\times n}$ is trivial and so the stabilizer of any point in $\mathcal{M}_k$ is trivial. We claim that $\overline{G\mathcal{M}_k}$ are different irreducible components of $\mu^{-1}(0)$. It is easy to see that $G\mathcal{M}_k\cap G\mathcal{M}_{k'}=\varnothing$ for $k\neq k'$ (just consider the (i,j) components). Clearly, $\mathcal{M}_k$ is stable under $\operatorname{GL}(\delta)^{\times n}$ and the action of this group is free. The dimension of the quotient is the number of parameters for the x_ℓ's and this number is $2n$. The map

$$\operatorname{GL}(n\delta)\times_{\operatorname{GL}(\delta)^{\times n}}\mathcal{M}_k\to\mu^{-1}(0),(g,m)\mapsto gm$$

has finite fibers (that are orbits for a natural action of $\mathfrak{S}_k\times\mathfrak{S}_{n-k}$). So $\dim\overline{G\mathcal{M}_k}=\dim G+2n=\dim\mu^{-1}(0)$. Our claim is proved and this finishes the proof of Theorem 15.1.

15.5. **Sketch of proof of Theorem 15.2.** In fact, one can construct a morphism $\mathbb{C}^{2n}/\Gamma_n=(\mathbb{C}^2/\Gamma_1)^n/\mathfrak{S}_n\to\mu^{-1}(0)/\!/G$ and then prove that this is an isomorphism.

Recall that $\mathbb{C}^2/\Gamma_1=\mu_1^{-1}(0)/\!/\operatorname{GL}(\delta)$, where $\mu_1:\operatorname{Rep}(Q^{MK},\delta)\to\mathfrak{gl}(\delta)$ is the moment map. We have a map $[\mu_1^{-1}(0)/\!/\operatorname{GL}(\delta)]^n\to\mu^{-1}(0)/\!/G$ induced by $(x_1,\ldots,x_n)\in\mu_1^{-1}(0)^n\mapsto$

$(x_1 \oplus \ldots \oplus x_n, 0, 0)$. Since permuting the summands does not change the G-orbit, this morphism descends to $\psi : [\mu_1^{-1}(0)/\!/ \mathrm{GL}(\delta)]^n/\mathfrak{S}_n \to \mu^{-1}(0)/\!/G$.

We claim that this morphism is bijective. This amounts to showing that every semisimple representation of in $\mathrm{Rep}(Q, n\delta+\epsilon_\infty)$ decomposes into the sum $x_1 \oplus \ldots \oplus x_n \oplus (0,0)$, where $x_k \in \mu_1^{-1}(0)$ (and then $x_1, \ldots, x_n$ are defined uniquely up to isomorphisms and a permutation). This is a consequence of the following theorem of Crawley-Boevey describing the possible dimension vectors of simple representations in $\mu^{-1}(0)$ together with our computations in Section 3.

Theorem 15.8. *Let Q be a double quiver of $\underline{Q}$, v be its dimension vector. Then the following statements are equivalent.*

(1) *There is a simple representation in* $\mathrm{Rep}(\Pi^0(Q), v)$.
(2) $p(v) > \sum_{i=1}^m p(\alpha^i)$ *for any proper decomposition of v into the sum of roots.*

By the construction ψ is $\mathbb{C}^\times$-equivariant. The $\mathbb{C}^\times$-actions on both varieties contract everything to 0. Since the preimage of 0 under ψ is a single point, we deduce that ψ is finite, this is a geometric version of the graded Nakayama lemma.

The variety $\mathbb{C}^{2n}/\Gamma_n$ is normal. There is a general result of Crawley-Boevey, [CB3], saying that $\mu^{-1}(0)/\!/ \mathrm{GL}(v)$ is normal for any double quiver Q and any dimension vector v. So in our case the variety $\mu^{-1}(0)/\!/G$ is normal, and this completes the proof.

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LECTURES ON SYMPLECTIC REFLECTION ALGEBRAS

IVAN LOSEV

Correction to Section 15.4

Unfortunately, the argument in Section 15.4 of the lecture that shows that all $n+1$ components of $\mu^{-1}(0)$ contain a point with a trivial stabilizer is incorrect. The reason is that only $\mathcal{M}_0, \mathcal{M}_n$ are subvarieties in $\mu^{-1}(0)$, while the other $n-1$ varieties $\mathcal{M}_i$ even do not intersect $\mu^{-1}(0)$. A correct argument is below.

Lemma 0.1. *A generic representation in* $\operatorname{Rep}(\underline{Q}^{MK}, \delta)$ *is indecomposable and its stabilizer in* $\operatorname{GL}(\delta)$ *is* $\mathbb{C}^\times$.

Proof. All subsets $I(\alpha^1, \ldots, \alpha^n) \subset \operatorname{Rep}(\underline{Q}^{MK}, \delta)$ with $n > 1$ contain finitely many orbits. Let us decompose $I(\delta)$ into locally closed irreducible G-stable subvarieties, $I(\delta) = I^0(\delta) \sqcup I^1(\delta) \sqcup \ldots \sqcup I^k(\delta)$ with $m(I^0(\delta)) = 1$ and $m(I^\ell(\delta)) = 0$ for $\delta > 0$. This means that we may assume that all $I^j(\delta)$ are single G-orbits. The dimension of every orbit does not exceed $\dim \operatorname{GL}(\delta) - 1$. We have $\dim \underline{R} = \dim \operatorname{GL}(\delta)$. So we see that $I^0(\delta)$ is dense in $\underline{R}$. Also a dimension count shows that a generic orbit in $I^0(\delta)$ has to have dimension $\dim \operatorname{GL}(\delta) - 1$. The stabilizer of every representation is connected (it is an open subset in the space of all endomorphisms of the representation). So we see that the stabilizer of a generic representation in $I^0(\delta)$ is forced to coincides with $\mathbb{C}^\times$, the kernel of the $\operatorname{GL}(\delta)$-action. $\square$

Recall from Section 15.3, that the components of $\mu^{-1}(0)$ are the closures of the conormal bundles to some locally closed subsets $I^k \subset I(k\delta + \epsilon_\infty, \delta, \ldots, \delta) \subset \operatorname{Rep}(\underline{Q}^{CM}, n\delta + \epsilon_\infty)$ with $m(I^k) = n$. We will now present such subsets. Namely, let $\underline{x}_1, \ldots, \underline{x}_n$ be pairwise distinct indecomposable elements from $\operatorname{Rep}(\underline{Q}^{MK}, \delta)$ with stabilizer $\mathbb{C}^\times$. Such representations exist thanks to the previous lemma. Choose a decomposition $\bigoplus_{i \in \underline{Q}_0^{MK}} \mathbb{C}^{n\delta_i} = (\bigoplus_{i \in \underline{Q}_0^{MK}} \mathbb{C}^{\delta_i})^{\oplus n}$. Then we can view $\underline{x} := \bigoplus_{j=1}^n \underline{x}_j$ as an element of $\operatorname{Rep}(\underline{Q}^{MK}, n\delta)$. Also we have the induced decomposition of the space $\mathbb{C}^n$ sitting at the vertex 0 into the direct sum of one-dimensional subspaces. Let $e_1, \ldots, e_n$ be a basis compatible with this decomposition. To get an element of $\operatorname{Rep}(\underline{Q}^{CM}, n\delta + \epsilon_\infty)$ from an element of $\operatorname{Rep}(\underline{Q}^{MK}, n\delta)$ we need to add an element of $\mathbb{C}^n$. Set $I^k := \{\underline{x}, i := \sum_{\ell=1}^k i_\ell e_\ell | i_1 \ldots i_k \neq 0\}$, where $\underline{x}$ is as above. The stabilizer of $\underline{x}$ in G is $\mathbb{C}^{\times n} \hookrightarrow \operatorname{GL}(\delta)^{\times n} \hookrightarrow G = \operatorname{GL}(n\delta)$. So the stabilizer of $(\underline{x}, i) \in I^k$ is $\mathbb{C}^{\times(n-k)}$, the last $n-k$ copies of $\mathbb{C}^\times$ in $\mathbb{C}^{\times n}$. We need to show that the stabilizer in $\mathbb{C}^{\times(n-k)}$ of a generic point of the fiber in $(\underline{x}, i)$ of the conormal bundle to $G(\underline{x}, i)$ is trivial.

The space $\mathfrak{g}(\underline{x}, i) = T_x G(\underline{x}, i)$ admits an epimorphism onto $\mathfrak{g}\underline{x}$ with kernel $\mathfrak{g}_{\underline{x}} i$. Clearly, $\mathfrak{g}_{\underline{x}} i = \operatorname{Span}(e_1, \ldots, e_k)$. So the conormal space to the orbit $G(\underline{x}, i)$ naturally surjects onto $(\mathbb{C}^n / \operatorname{Span}(e_1, \ldots, e_k))^*$. The action of $(\mathbb{C}^\times)^{n-k}$ on the latter space is faithful and so it is faithful on the whole conormal space implying, in particular, that the stabilizer of a generic point is trivial.

LECTURES ON SYMPLECTIC REFLECTION ALGEBRAS

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16. Symplectic resolutions of $\mu^{-1}(0)//G$ and their deformations

16.1. **GIT quotients.** We will need to produce a resolution of singularities for $\mathbb{C}^{2n}/\Gamma_n = \mu^{-1}(0)//G$. Such resolutions may be constructed as GIT quotients for the action of G on $\mu^{-1}(0)$. In this section we are going to recall a few basics about such quotients.

Let X be an affine algebraic variety acted on by a reductive group G. The categorical quotient $X//G$ parameterizes closed orbits for the G-action. It is insufficient for many applications (e.g., in constructing moduli spaces in Algebraic geometry) as one needs to parameterize other types of orbits. One thing one can try to do is to choose some G-stable affine open subsets $X_1,\ldots,X_n$ of X, take their categorical quotients $X_i//G$ and then glue the quotients together (the quotients $X_i//G$ and $X_j//G$ are glued along $(X_i\cap X_j)//G$). However, the result generally will be a nasty space (not an algebraic variety). There is however one setting, where we get a nice variety that often appears to be even better than the usual categorical quotient.

Namely, fix a character θ of G. We say that $x\in X$ is *θ-semi-stable* if there is $f\in \mathbb{C}[X]^{G,n\theta} := \{f\in\mathbb{C}[X] | g.f = \theta(g)^n f\}$ for some $n>0$ and $f(x)\neq 0$. A not semi-stable point is called *unstable*. A semistable point with finite stabilizer is usually called *stable*. The locus of θ-semistable points is denoted by $X^{\theta-ss}$. By definition, it can be covered by principal open subsets X_f, where f is as above. We can take the categorical quotients $X_f//G$ and glue them together in a natural way. The result of gluing is the Proj of the graded algebra $\bigoplus_{n=0}^{+\infty}\mathbb{C}[X]^{G,n\theta}$ (the product is restricted from $\mathbb{C}[X]$, the degree n component, by definition, is $\mathbb{C}[X]^{G,n\theta}$) denoted by $X//^\theta G$. By the construction, there is a natural projective morphism $X//^\theta G\to X//G$. We remark that for $\theta=0$, all points are semi-stable, and $X//^\theta G = X//G$. We also remark that, for $m>0$, the sets of semistable points for θ and for $m\theta$ coincide, and $X//^\theta G = X//^{m\theta}G$.

For example, we can consider the action of $\mathbb{C}^\times$ on $\mathbb{C}^2$ by $t.(x_1,x_2)=(tx_1,tx_2)$. If $\theta(t)=t^{-1}$, then $\mathbb{C}[X]^{G,n\theta}=\mathbb{C}[x_1,x_2]^n$ and $X//^\theta G=\mathbb{P}^1$. If $\theta(t)=t$, then $\mathbb{C}[X]^{G,n\theta}=\{0\}$, and $X^{\theta-ss}=\varnothing$.

Exercise 16.1. *Consider the G-action on $X\times\mathbb{C}$ given by $g(x,z)=(gx,\theta(g)z)$. Show that $x\in X^{ss}$ iff $\overline{G(x,1)}$ doesn't intersect $X\times\{0\}$.*

16.2. **Case of quivers.** As an application of the previous exercise we will compute semi-stable point in $\operatorname{Rep}(Q,v)$, where Q is a quiver, under the action of $\operatorname{GL}(v)$. Any character θ is given by a Q_0-tuple $(\theta_i)_{i\in Q_0}, \theta((g_i)_{i\in Q_0})=\prod_{i\in Q_0}\det(g_i)^{\theta_i}$. We remark that if $\theta\cdot v := \sum_{i\in Q_0}\theta_i v_i\neq 0$, then $\mathbb{C}[\operatorname{Rep}(Q,v)]^{\operatorname{GL}(v),\theta}=0$ (the semi-invariant space can only be nonzero if the character vanishes on the kernel of the action). Below we only consider θ with this property.

Proposition 16.1. *A representation of Q with dimension v is θ-semistable iff it does not contain a subrepresentation with dimension vector v' subject to $\theta\cdot v'>0$.*

Proof. Set $U := \bigoplus_{i\in Q_0}\mathbb{C}^{v_i}$, and let x be a representation of Q in this space. Let U' be a subrepresentation of dimension v' with $\theta\cdot v' > 0$. Choose a complement U''. Consider the one-parametric subgroup $\gamma : \mathbb{C}^\times \to \mathrm{GL}(v)$ defined by $\gamma(t) = (t1_{U'}, 1_{U''})$. Clearly $\lim_{t\to 0}\gamma(t)x$ exists. Also, since $\theta\cdot v' > 0$, we see that $\lim_{t\to 0}\theta(\gamma(t)) = 0$. It follows that $\lim_{t\to 0}(\gamma(t)x, \theta(\gamma(t)))$ exists and lies in $\mathrm{Rep}(Q, v)\times\{0\}$. By the previous exercise, x is not semi-stable.

Conversely, assume x is not semi-stable. According to the Hilbert-Mumford criterium (see Lecture 3), there is a one-parameter subgroup $\gamma : \mathbb{C}^\times \to \mathrm{GL}(v)$ such that $\lim_{t\to 0}(\gamma(t)x, \theta(\gamma(t)))$ exists and lies in $\mathrm{Rep}(Q, v)\times\{0\}$. Let U^j be the eigen-subspace for $\gamma(t)$ in U with eigen-character $t\mapsto t^j$, let v^j be the dimension of U^j. The equality $\lim_{t\to 0}\theta(\gamma(t)) = 0$ is equivalent to $\sum_j j\theta\cdot v^j > 0$. Equivalently, $\sum_k \theta\cdot(v^k + v^{k+1} + \ldots) > 0$. On the other hand, since $\lim_{t\to 0}\gamma(t)x$ exists, $\bigoplus_{j\geqslant k} U^j$ is a subrepresentation. We are done. $\square$

16.3. **Nakajima quiver varieties.** Fix a quiver $\underline{Q}$, and $v, w\in\mathbb{Z}_{\geqslant 0}^{\underline{Q}_0}$. Also fix vector spaces $V_i, W_i, i\in\underline{Q}_0, \dim V_i = v_i, \dim W_i = w_i$. Then consider the space

$$R_0 = \mathrm{Rep}(\underline{Q}, v, w) := \bigoplus_{a\in\underline{Q}_1}\mathrm{Hom}(V_{t(a)}, V_{h(a)}) \oplus \bigoplus_{i\in\underline{Q}_0}\mathrm{Hom}(W_i, V_i),$$

its double $R = T^*R_0$ and the action of $G := \mathrm{GL}(v)$ on these spaces. Before we considered the case when $\underline{Q}$ is the McKay quiver, $v = n\delta$, and $w = \epsilon_0$. As before, $\mathrm{Rep}(\underline{Q}, v, w) = \mathrm{Rep}(\underline{Q}^w, v+\epsilon_\infty)$, where $\underline{Q}^w$ is the quiver obtained from $\underline{Q}$ by adding the new vertex ∞ together with w_i arrows from ∞ to i. We extend a character θ of G to that $\mathrm{GL}(v+\epsilon_\infty) = G\times\mathbb{C}^\times$ by $\bar\theta(g, t) = \theta(g)t^{-v\cdot\theta}$.

Consider the moment map $\mu : R\to\mathfrak{g}$ and form the Hamiltonian reduction $\mathcal{M}^\theta(Q, v, w) := \mu^{-1}(0)/\!/^\theta G$. The action of $\mathbb{C}^\times$ on R by dilations descends to $\mathcal{M}^\theta(Q, v, w)$.

Nakajima proved (this is not a very deep result; it follows from an alternative point of view on the reductions $\mu^{-1}(0)/\!/^\theta G$ – via so called *hyper-Kähler reduction*) that the action of G on $\mu^{-1}(0)^{ss}$ is free iff θ is *generic* in the sense that $\theta\cdot v'\neq 0$ for all $v'\leqslant v$. In particular, θ with $\theta_i = -1$ for all i is generic. So, for generic θ, $\mathcal{M}^\theta(Q, v, w)$ is a smooth symplectic variety with form, say, ω. This form is compatible with the $\mathbb{C}^\times$-action: it gets rescaled, $t.\omega = t^2\omega$.

Exercise 16.2. *Prove that for $\theta = (-1)_{i\in Q_0}$, the subset R^{ss} consists of all elements*

$$(x_a, x_{a^*}, y_i, z_i)_{a\in\underline{Q}_1, i\in\underline{Q}_0}$$

(here $x_a\in\mathrm{Hom}(V_{t(a)}, V_{h(a)}), x_{a^*}\in\mathrm{Hom}(V_{h(a)}, V_{t(a)}), y_i\in\mathrm{Hom}(W_i, V_i), z_i\in\mathrm{Hom}(V_i, W_i)$) such that there are no proper subspaces $V_i'\subset V_i$ that are stable under all x_a, x_{a^*} and such that $\operatorname{im} y_i\subset V_i'$. Deduce that the action of $\mathrm{GL}(v)$ on $\mathrm{Rep}(Q, v, w)^{\theta-ss}$ is free.

By the definition of $\mathcal{M}^\theta(Q, v, w)$ we have a projective morphism $\rho : \mathcal{M}^\theta(Q, v, w)\to\mathcal{M}^0(Q, v, w)$, where the target scheme is affine. This morphism is $\mathbb{C}^\times$-equivariant and Poisson. Often, this morphism happens to be a resolution of singularities. For example, this is so when $\mu^{-1}(0)$ contains a free closed orbit and θ is generic. In the case of interest for us (affine $\underline{Q}, v = n\delta, w = \epsilon_0$) this does not hold. However, ρ is still a resolution of singularities. This is true for an arbitrary generic θ but we will only prove the statement for the choice of θ above. We will need to use a general fact about the varieties $\mathcal{M}^\theta(Q, v, w)$: they are connected. This follows from the claim that $\mathcal{M}^\theta(Q, v, w)$ is diffeomorphic to $\mu^{-1}(\lambda)/\!/G$ for a generic $\lambda\in\mathfrak{g}^{*G}$ (due to Nakajima, this again follows from the hyper-Kähler reduction approach) and a theorem of Crawley-Boevey that $\mu^{-1}(\lambda)$ is connected (see a problem below proving this in our case).

Proposition 16.2. *Suppose $\underline{Q}$ is an affine quiver, $v = n\delta, w = \epsilon_0, \theta = (-1)_{i\in\underline{Q}_0}$. Then ρ is a resolution of singularities.*

Proof. Since the G-action on R^{ss} is free, we see that

$$\dim \mathcal{M}^\theta(Q, v, w) = \dim \mu^{-1}(0)^{ss} - \dim G = \dim R - 2\dim G = 2n = \dim \mathbb{C}^{2n}/\Gamma_n = \\ = \dim \mathcal{M}^0(Q, v, w).$$

The only thing that we need to prove is

(*) for a generic point $p \in \mathcal{M}^0(Q, v, w)$, the fiber $\rho^{-1}(p)$ is a single point.

We claim that R^{ss} intersects not more than one component of $\mu^{-1}(0)$. Indeed, any free orbit in $\mu^{-1}(0)$ lies in a single component (because μ is a submersion at any point of a free orbit). The variety $\mathcal{M}^\theta(Q, v, w)$ is connected and hence so is $R^{ss} \cap \mu^{-1}(0)$. Since $R^{ss} \cap \mu^{-1}(0)$ is connected, we see that it lies in a single component of $\mu^{-1}(0)$ and is dense there for dimension reasons (if non-empty). Further, we claim that the component is of the form $\overline{\{(x, i, 0)\}}$ with $i \in \mathbb{C}^n$ and x being a generic semisimple representation in $\operatorname{Rep}(Q^{MK}, n\delta)$. Indeed, it is easy to show that all points $\{(x, i, 0)\}$ with x, i generic satisfy the conditions of Exercise 16.2. The claim (*) boils down to showing that, for fixed generic $x \in \operatorname{Rep}(\Pi^0(Q^{MK}), \delta)$, all points $\{(x, i, 0)\} \in R^{ss}$ lie in the same G-orbit. Equivalently, we need to show that all i such that $(x, i, 0) \in R^{ss}$ lie in the same G_x-orbit. As we have seen in Lecture 15, $G_x = (\mathbb{C}^\times)^n$ acting on $\mathbb{C}^n$ in the usual way. We know that G acts on R^{ss} freely and so G_x acts on the set of i with $(x, i, 0) \in R^{ss}$ freely. But there is only one free orbit of $(\mathbb{C}^\times)^n$ in $\mathbb{C}^n$. We are done. □

Problem 16.3. *Prove that a generic fiber $\mu^{-1}(\lambda), \lambda \in \mathfrak{g}^{*G}$, is smooth and connected. You may use the following strategy:*

1. *Show that all G-orbits in $\mu^{-1}(\lambda)$ are free. Deduce that $\mu^{-1}(\lambda)$ is smooth.*
2. *Show that $\mu^{-1}(\mathbb{C}\lambda)$ is normal.*
3. *The affine version of Zariski's main theorem says that the morphism $\mu^{-1}(\mathbb{C}\lambda) \to \mathbb{C}\lambda$ decomposes into a composition of a morphism $\mu^{-1}(\mathbb{C}\lambda) \to Y$ with connected general fibers and a finite morphism $Y \to \mathbb{C}\lambda$. Use this to prove that $\mu^{-1}(\lambda)$ is connected.*

Example 16.3. Consider the example, when $\underline{Q}$ is a Jordan quiver. Here $\mu^{-1}(0)^{ss}$ consists of all triples (X, Y, i) with $X, Y \in \operatorname{Mat}_n(\mathbb{C}), i \in \mathbb{C}^n$ such that $[X, Y] = 0$ and i is a cyclic vector for (X, Y): $\mathbb{C}[X, Y]i = \mathbb{C}^n$. To such a triple we can assign an ideal of codimension n in $\mathbb{C}[x, y]$ consisting of all polynomials $f \in \mathbb{C}[x, y]$ such that $f(X, Y) = 0$. The scheme $\mu^{-1}(0)/\!/^\theta G$ is therefore the Hilbert scheme $\operatorname{Hilb}_n(\mathbb{C}^2)$ parameterizing such ideals in $\mathbb{C}[x, y]$.

16.4. **Deformations of GIT Hamiltonian reductions.** Now suppose that V is a symplectic vector space and a connected reductive group G acts on V via a homomorphism $G \to \operatorname{Sp}(V)$. Let $\mu : V \to \mathfrak{g}^*$ be the moment map and $\Phi : \mathfrak{g} \to W_\hbar(V)$ be the quantum comoment map, where $W_\hbar(V)$ is the Weyl algebra. Let θ be a character of G such that G acts freely on $\mu^{-1}(0)^{ss}$. Recall the notation $\mathfrak{z} = \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$.

Recall that under some additional assumption the universal quantum Hamiltonian reduction $W_\hbar(V)/\!/\!/G := [W_\hbar(V)/W_\hbar(V)\Phi([\mathfrak{g}, \mathfrak{g}])]^G$ is a graded flat deformation of $V/\!/\!/_0 G$ (more precisely, of the corresponding algebra of functions) over the algebra $\mathbb{C}[\mathfrak{z}^*, \hbar]$.

We are going to produce a deformation $\mathcal{D}$ of $X := V/\!/\!/^\theta G$. This variety is not affine, it cannot be defined using a single algebra of functions. Rather, it is defined by its sheaf of regular functions $\mathcal{O}_X$. Let us recall some details on this sheaf, as this will serve a motivation for the definition of $\mathcal{D}$.

The variety X is covered by the open subsets $V_f///G$ with $f \in \mathbb{C}[V]^{G,n\theta}$, where V_f is the principal open subset associated to f. By definition, the space $H^0(V_f///G, \mathcal{O}_X)$ of sections of $\mathcal{O}_X$ is $\mathbb{C}[V_f]///_0G$. For $f_1 \in \mathbb{C}[V]^{G,n_1\theta}, f_2 \in \mathbb{C}[V]^{G,n_2\theta}$ we have $(V_{f_1}///_0G) \cap (V_{f_2}///_0G) = V_{f_1f_2}///_0G$ and the restriction map $H^0(V_{f_1}///G, \mathcal{O}_X) \to H^0(V_{f_1f_2}///G, \mathcal{O}_X)$ is induced from the inclusion $\mathbb{C}[V_{f_1}] = \mathbb{C}[V][f_1^{-1}] \hookrightarrow \mathbb{C}[V_{f_1f_2} = \mathbb{C}[V][f_1^{-1}f_2^{-1}]$. We also remark that X can be covered by the subsets $V_f///_0G$, where f is homogeneous (with respect to the action of $\mathbb{C}^\times$ by dilations). This is because all homogeneous component of $f \in \mathbb{C}[V]^{G,n\theta}$ are again in $\mathbb{C}[V]^{G,n\theta}$.

The deformation $\mathcal{D}$ will be a $\mathbb{C}^\times$-equivariant sheaf of algebras $\mathcal{D}$ over $\mathbb{C}[[\mathfrak{z}^*, \hbar]]$, flat, complete and separated in the $(\mathfrak{z}^*, \hbar)$-adic topology. The precise meaning of the word "deformation" in this setting is $\mathcal{D}/(\mathfrak{z}, \hbar) = \mathcal{O}_X$. The deformation is produced using a suitable (="sheaf") version of the quantum Hamiltonian reduction.

Namely, for $f \in \mathbb{C}[V]^{G,n\theta}$ we can consider the localization $W_\hbar(V)[f^{-1}]$ that is $\hbar$-adically complete and separated and is a flat deformation of $\mathbb{C}[V][f^{-1}]$, see Exercise 14.4. The group G still acts on $W_\hbar(V)[f^{-1}]$ and Φ is a quantum comomoment map. Consider the left ideal $W_\hbar(V)[f^{-1}]\Phi([\mathfrak{g},\mathfrak{g}]) = [\mathrm{W}_\hbar(V)\Phi([\mathfrak{g},\mathfrak{g}])]\,[f^{-1}]$.

Problem 16.4. *Show that the algebra $W_\hbar(V)[f^{-1}]$ is Noetherian. Deduce from here that any left ideal is closed in the $\hbar$-adic topology.*

Also we claim that $\mathcal{M}_\hbar := \mathrm{W}_\hbar(V)/W_\hbar(V)[f^{-1}]\Phi([\mathfrak{g},\mathfrak{g}])$ is $\hbar$-flat. The proof is a modification of that of Proposition 14.1. The argument there does not apply directly in our setting: a result on regular sequences used there works for positively graded or local rings. However, one can reduce to the case of local rings as follows. Let K be the kernel of the $\hbar$-action on $\mathcal{M}_\hbar[f^{-1}]$. If K is nonzero, then so is its stalk at some point $x \in \mu^{-1}(0)^{ss}$. On the other hand, we can localize at x, and we get $(\mathcal{M}_\hbar)_x = W_\hbar(V)_x/W_\hbar(V)_x\Phi([\mathfrak{g},\mathfrak{g}])$. Now the argument of the proposition works and we see that $(\mathcal{M}_\hbar)_x$ is flat over $\mathbb{C}[[\hbar]]$. So $K_x = 0$, a contradiction.

So we get a a flat deformation $\mathrm{W}_\hbar(V)[f^{-1}]///G = \mathcal{M}_\hbar^G$ of

$$\mathbb{C}[V][f^{-1}]///G := (\mathbb{C}[V][f^{-1}]/\mathbb{C}[V][f^{-1}]\mu^*([\mathfrak{g},\mathfrak{g}]))^G.$$

The algebra $\mathrm{W}_\hbar(V)[f^{-1}]///G$ is complete and separated in the $\hbar$-adic topology. This follows from Problem 16.4. For two elements $f_1 \in \mathbb{C}[V]^{n_1\theta}, f_2 \in \mathbb{C}[V]^{n_2\theta}$ we have a natural homomorphism $W_\hbar(V)[f_1^{-1}]///G \to W_\hbar(V)[f_1^{-1}f_2^{-1}]///G$ induced by $W_\hbar(V)[f_1^{-1}] \hookrightarrow \mathrm{W}_\hbar(V)[f_1^{-1}f_2^{-1}]$.

Exercise 16.5. *Reducing modulo $\hbar^k$ for all k, show that the data of $W_\hbar(V)[f^{-1}]///G$ constitutes a sheaf on $V///^\theta G := \mu^{-1}(\mathfrak{z}^*)//^\theta G$ with respect to the covering $V_f///G$ of $V///^\theta G$. Recall that this means the following: if we have a covering of $V_f///G$ by $V_{f_i}///G, i = 1, \ldots, n$ and sections $a_1, \ldots, a_n$ of $W_\hbar(V)[f_i^{-1}]///G$ that agree on intersections, then they glue together to a unique element $W_\hbar(V)[f^{-1}]///G$.*

So we can glue the algebras $W_\hbar(V)[f^{-1}]///G$ to a sheaf on $X///^\theta G$ denoted by $W_\hbar(V)///^\theta G$. This sheaf is $\mathbb{C}^\times$-equivariant: if $f \in \mathbb{C}[V]^{G,n\theta}$ is homogeneous, the algebra $W_\hbar(V)[f^{-1}]///G$ has a natural $\mathbb{C}^\times$-action. Since these actions are compatible with the restriction homomorphisms, they give rise to a $\mathbb{C}^\times$-action on the sheaf $W_\hbar(V)///^\theta G$. By the construction, $W_\hbar(V)///^\theta G/(\hbar) = \mathcal{O}_{V///^\theta G}$.

The sheaf $W_\hbar(V)///^\theta G$ is basically a sheaf we need, but we want to make it $(\mathfrak{z}, \hbar)$-adically complete and leaving on $X = \mu^{-1}(0)//^\theta G$, rather than on $V///^\theta G$. This is a pretty formal procedure, we pull-back $W_\hbar(V)///^\theta G$ to $\mu^{-1}(0)//^\theta G \subset V///^\theta G$ <u>sheaf-theoretically</u> and then complete the resulted sheaf with respect to the $(\mathfrak{z}, \hbar)$-adic topology, to get the required sheaf

$\mathcal{D}$. In more pedestrian terms, the sections of $\mathcal{D}$ on $V_f///_0G$ is the $(\mathfrak{z},\hbar)$-adic completion of $W_\hbar(V)[f^{-1}]///G$. Since the natural morphism $V///^\theta G \to \mathfrak{z}^*$ is flat near 0, we see that $\mathcal{D}$ is flat over $\mathbb{C}[[\mathfrak{z}^*,\hbar]]$.

LECTURES ON SYMPLECTIC REFLECTION ALGEBRAS

IVAN LOSEV

17. Procesi bundles and their deformations

17.1. **Recap.** Let V denote a symplectic vector space and let G be a reductive group acting on V via a homomorphism $G \to \mathrm{Sp}(V)$. This gives rise to the moment map $\mu : V \to \mathfrak{g}^*$, a G-action on the homogenized Weyl algebra $W_\hbar(V)$, and a quantum comoment map $\Phi : \mathfrak{g} \to W_\hbar(V)$ that equals to μ^* modulo $\hbar$. Let $\mathfrak{z}$ denote the quotient $\mathfrak{g}/[\mathfrak{g},\mathfrak{g}]$. Pick a a character θ of G.

We make the following assumptions:

(i) Every irreducible component of $\mu^{-1}(0)$ contains a free G-orbit.
(ii) The action of G on $\mu^{-1}(0)^{\theta-ss}$ is free.

Under assumption (i), the quantum Hamiltonian reduction

$$W_\hbar(V)///G = [W_\hbar(V)/W_\hbar(V)\Phi([\mathfrak{g},\mathfrak{g}])]^G$$

is a graded deformation of $\mathbb{C}[V///_0G]$ over $\mathbb{C}[\mathfrak{z}^*,\hbar]$. Under assumption (ii), the variety $X := \mu^{-1}(0)//^\theta G$ is smooth and symplectic. Furthermore, it comes with a deformation $\mathcal{D}$ of $\mathcal{O}_X$, where $\mathcal{D}$ is a sheaf of $\mathbb{C}[[\mathfrak{z}^*,\hbar]]$-algebras. This deformation is flat (over $\mathbb{C}[[\mathfrak{z}^*,\hbar]]$) and complete and separated in the $(\mathfrak{z},\hbar)$-adic topology. Below we will abbreviate this as "FCS deformation over $\mathbb{C}[[\mathfrak{z}^*,\hbar]]$". Moreover, $\mathcal{D}$ is $\mathbb{C}^\times$-equivariant.

Let us recall how the deformation $\mathcal{D}$ is constructed. It is enough to specify the sections and the restriction homomorphisms on some base of topology on X. For a base, we can take $X_f := V_f///_0G$, where $f \in \mathbb{C}[V]^{G,n\theta}$ and V_f stands for the principal open subset of V defined by f. By definition, the algebra of section $\mathcal{D}(X_f)$ of $\mathcal{D}$ on X_f is the $(\mathfrak{z},\hbar)$-adic completion of $W_\hbar(V)[f^{-1}]//G$, where $W_\hbar(V)[f^{-1}]$ is the completed localization of $W_\hbar(V)$ with respect to f, see Exercise 14.4. For $f_1 \in \mathbb{C}[V]^{G,n_1\theta}, \mathbb{C}[V]^{G,n_2\theta}$, we have a natural homomorphism of algebras $W_\hbar(V)[f_1^{-1}] \to W_\hbar(V)[f_1^{-1}f_2^{-1}]$ that gives rise to a homomorphism $W_\hbar(V)[f_1^{-1}]///G \to W_\hbar(V)[f_1^{-1}f_2^{-1}]///G$ that, in its turn, produces a homomorphism $\mathcal{D}(X_{f_1}) \to \mathcal{D}(X_{f_1f_2})$. This is a restriction homomorphism we need.

To establish the $\mathbb{C}^\times$-equivariant structure on $\mathcal{D}$, it is sufficient to construct a $\mathbb{C}^\times$-action on $\mathcal{D}(X_f)$ with homogeneous $f \in \mathbb{C}[V]^{G,n\theta}$ (the corresponding subsets X_f also cover X) and make sure that the restriction homomorphisms are $\mathbb{C}^\times$-equivariant. We take the $\mathbb{C}^\times$-action on $\mathcal{D}(X_f)$ induced from the $\mathbb{C}^\times$-action on $W_\hbar(V)[f^{-1}]$, such action is compatible with restriction homomorphisms.

Finally, let us remark that we have a natural $\mathbb{C}^\times$-equivariant and $\mathbb{C}[\mathfrak{z}^*,\hbar]$-equivariant homomorphism $W_\hbar(V)///G \to \mathcal{D}(X)$. Indeed, the homomorphisms $W_\hbar(V) \to W_\hbar(V)[f^{-1}]$ give rise to homomorphisms $W_\hbar(V)///G \to \mathcal{D}(X_f)$ that agree on intersections and hence glue to a homomorphism $W_\hbar(V)///G \to \mathcal{D}(X)$.

17.2. **Deformations of sheaves.** Let X be an algebraic variety (or scheme), $\mathcal{F}_0$ be a coherent sheaf on X and let $\mathcal{D}$ be a FCS deformation of X over the formal power series ring $\mathbb{C}[[x_1,\ldots,x_n]]$. We are interested in the following two questions. First, is there a deformation

$\mathcal{F}$ of $\mathcal{F}_0$ to a flat (and then, in fact, automatically complete and separated) right $\mathcal{D}$-module and if so, is such deformation unique? Second, how are the global sections of $\mathcal{F}$ and $\mathcal{F}_0$ are related, in particular, whether the former is a deformation of the latter? To ensure affirmative answers to our questions, we will need to impose some cohomology vanishing assumptions.

Here is an answer to the first question.

Lemma 17.1. *Let X be an algebraic variety, $\mathcal{F}_0$ be a coherent sheaf on X with $H^i(X, \mathcal{F}_0) = 0$ for $i > 0$. Let $\mathcal{F}$ be a sheaf of $\mathcal{D}$-modules, flat over $\mathbb{C}[[x_1, \ldots, x_n]]$, complete and separated in the $(x_1, \ldots, x_n)$-adic topology and such that $\mathcal{F}/(x_1, \ldots, x_n) = \mathcal{F}_0$. Then $H^p(X, \mathcal{F}) = 0$ for $p > 0$ and $\mathcal{F}(X)$ is a FCS deformation of $\mathcal{F}_0(X)$.*

Proof. Before we prove the lemma in the general case, we need to understand the case when X is affine. Here the cohomology vanishing for $\mathcal{F}_0$ is automatic. Given a finitely generated $\mathcal{D}(X)$-module F we can sheafify it to a sheaf $\mathcal{F}$ over X (so that $\mathcal{F}(X_f) = \mathcal{D}(X)[f^{-1}] \otimes_{\mathcal{D}(X)} F$ for any $f \in \mathbb{C}[X]$). Similarly to the case of usual sheaves, the sheafification functor is an equivalence of categories (the target category is that of finite generated sheaves of modules over $\mathcal{D}$), a quasi-inverse equivalence is provided by taking global sections. And, as in the usual algebro-geometric story, the Čech cohomology of $\mathcal{F}$ are derived functors to the functor of taking global sections. Since the latter is exact, the higher Čech cohomology vanish.

Let us return to the general case. Thanks to the previous paragraph, we can compute the Čech cohomology using open affine coverings. Set $\mathcal{F}_i = \mathcal{F}/(x_{i+1}, \ldots, x_n)\mathcal{F}$. We will prove by induction on i that $H^p(X, \mathcal{F}_{i+1}) = 0$ for $p > 0$ and that $\mathcal{F}_{i+1}(X)$ is a flat deformation of $\mathcal{F}_i(X)$. Our base is $i = -1$, where the second claim is vacuous.

Choose a covering of X by open affine subsets, $X = \bigcup_j X_j$. The intersection of two affine open subsets is affine. So, for $X_{j_1,\ldots,j_m} = \bigcap_{k=1}^m X_{j_k}$, we have $\mathcal{F}_{i+1}(X_{j_1,\ldots,j_m})/x_{i+1}\mathcal{F}_{i+1}(X_{j_1,\ldots,j_m}) = \mathcal{F}_i(X_{j_1,\ldots,j_m})$.

Take a cocycle $a_\bullet \in \bigoplus_{j_1,\ldots,j_m} \mathcal{F}_{i+1}(X_{j_1,\ldots,j_m})$. It is a cocycle also modulo x_{i+1}. So, since $H^m(X, \mathcal{F}_i) = 0$, there are $b^0_\bullet \in \bigoplus_{j_1,\ldots,j_{m-1}} \mathcal{F}_{i+1}(X_{j_1,\ldots,j_{m-1}})$, $a^1_\bullet \in \mathcal{F}_{i+1}(X_{j_1,\ldots,j_m})$ such that $a_\bullet = db^0_\bullet + x_{i+1}a^1_\bullet$. We have $0 = da_\bullet = d^2 b_\bullet + d(x_{i+1}a^1_\bullet) = x_{i+1}da^1_\bullet$. Since $\mathcal{F}_{i+1}$ is flat over $\mathbb{C}[[x_{i+1}]]$, we see that $da^1_\bullet = 0$. We can repeat the same procedure with $a^1_\bullet$, getting elements $b^1_\bullet, a^2_\bullet$, etc. So $a_\bullet = d(b^0 + x_{i+1}b^1 + \ldots)$ (the element in the right hand side is well-defined because $\mathcal{F}_{i+1}$ is complete and separated in the x_{i+1}-adic topology). This proves $H^m(X, \mathcal{F}_{i+1}) = 0$.

Being a $\mathbb{C}[[x_{i+1}]]$-submodule in $\bigoplus_j \mathcal{F}_{i+1}(X_j)$, the $\mathbb{C}[[x_{i+1}]]$-module $\mathcal{F}_{i+1}(X)$ is flat and separated in the x_{i+1}-adic topology. So it remains to prove that $\mathcal{F}_{i+1}(X)/x_{i+1}\mathcal{F}_{i+1}(X) = \mathcal{F}_i(X)$ (by induction, this will also show that $\mathcal{F}_{i+1}(X)$ is FCS deformation of $\mathcal{F}_0(X)$ over $\mathbb{C}[[x_1, \ldots, x_{i+1}]]$). Since $\mathcal{F}_{i+1}$ is a deformation of $\mathcal{F}_i$ (as a sheaf), we only need to prove that the natural map $\mathcal{F}_{i+1}(X) \to \mathcal{F}_i(X)$ is surjective.

Pick a global section $s \in \mathcal{F}_i(X)$. The restriction s_i of s to X_j lifts to $\bar{s}_j \in \mathcal{F}_{i+1}(X_j)$. Then $d\bar{s}_\bullet$ is a cocycle that vanishes modulo x_{i+1}. So there is a cocycle $(c_\bullet) \in \bigoplus_j \mathcal{F}_{i+1}(X_j)$ such that $d\bar{s}_\bullet = x_{i+1}c_\bullet$. Since $H^1(X, \mathcal{F}_{i+1}) = 0$, we see that $c_\bullet$ is a coboundary, $c_\bullet = dc'_\bullet$. Therefore $d(\bar{s}_\bullet - x_{i+1}c'_\bullet) = 0$ meaning that $(\bar{s} - x_{i+1}c')_\bullet$ is a global section of $\mathcal{F}_{i+1}$ that projects to s. $\square$

Now let us give some answer to the first question: on existence and uniqueness of a deformation.

Lemma 17.2. *Let $\mathcal{F}_0$ be a locally free coherent sheaf on X such that $H^i(X, \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{F}_0)) = 0$. Let $\mathcal{D}$ be a FCS deformation of $\mathcal{O}_X$ over $\mathbb{C}[[x_1, \ldots, x_n]]$. Then the following is true.*

1. *There is a unique $\mathbb{C}[[x_1, \ldots, x_n]]$-flat deformation $\mathcal{F}$ of $\mathcal{F}_0$ to a right $\mathcal{D}$-module.*
2. *We have $H^i(X, \mathcal{E}nd_{\mathcal{D}^{opp}}(\mathcal{F})) = 0$ and $\operatorname{End}_{\mathcal{D}^{opp}}(\mathcal{F})$ is a FCS deformation of $\operatorname{End}_{\mathcal{O}_X}(\mathcal{F}_0)$.*

Here $\mathcal{E}nd$ stands for the sheaf of endomorphisms and End for the algebra of endomorphisms, the global sections of the sheaf of endomorphisms (and also the space of endomorphisms in the corresponding category).

We are not going to prove Lemma 17.2, we will just make some remarks.

First of all, the condition that $\mathcal{F}_0$ is locally free implies that $H^i(X, \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{F}_0)) = \operatorname{Ext}^i_{\mathcal{O}_X}(\mathcal{F}_0, \mathcal{F}_0)$. Now part (a) (in the case when $n = 1$, the general case is obtained by induction) can be deduced from the following problem.

Problem 17.1. *Let X be an algebraic variety, $\mathcal{F}_0$ be a coherent sheaf on X and $\mathcal{D}$ be a FCS (=flat, complete and separated) deformation of $\mathcal{O}_X$ over $\mathbb{C}[[\hbar]]$.*

1. *Show that the category of finitely generated modules (i.e., sheaves) over $\mathcal{D}/(\hbar^n)$ has enough injective objects. How are the injectives for different n related?*
2. *Show that if $\operatorname{Ext}^2(\mathcal{F}_0, \mathcal{F}_0) = 0$, then there exists a flat deformation of $\mathcal{F}_0$ to a right module $\mathcal{F}_n$ over $\mathcal{D}/(\hbar^{n+1})$. Moreover, show that these deformations may be chosen in a compatible way and so give rise to a FCS deformation $\mathcal{F}$ of $\mathcal{F}_0$ to a right module over $\mathcal{D}$.*
3. *Finally, show that if $\operatorname{Ext}^1(\mathcal{F}_0, \mathcal{F}_0) = 0$, then all the deformations above are unique.*

Also the condition that $\mathcal{F}_0$ is locally free shows that $\mathcal{E}nd_{\mathcal{D}^{opp}}(\mathcal{F})$ is a deformation of $\mathcal{E}nd_{\mathcal{O}_X}(\mathcal{F}_0)$ and so part (b) follows from part (a).

Finally, let us mention that a straightforward analog of Lemma 17.2 holds when a $\mathbb{C}^\times$-action is present ($\mathbb{C}^\times$ acts on X and both $\mathcal{F}_0$ and $\mathcal{D}$ are $\mathbb{C}^\times$-equivariant; the resulting sheaf $\mathcal{F}$ is also then equivariant). To see it one either adjusts the previous problem to the $\mathbb{C}^\times$-equivariant setting or just uses the uniqueness of $\mathcal{F}$. In more detail, for $t \in \mathbb{C}^\times$ we can define the push-forward $t_*\mathcal{F}$ with respect to the action of t. This sheaf is also a deformation so that $t_*\mathcal{F} \cong \mathcal{F}$. This yet does not imply that $\mathcal{F}$ is equivariant, one needs to show that isomorphisms $t_*\mathcal{F} \cong \mathcal{F}$ are compatible with the product on $\mathbb{C}^\times$. The latter can be deduced from Hilbert's theorem 90.

17.3. **Global sections.** We want to understand a relationship between $\mathcal{D}$ and the algebra $W_\hbar(V)///G$. We can take the algebra $\mathcal{D}(X)$ of global sections of $\mathcal{D}$. This algebra carries the action of $\mathbb{C}^\times$ and is complete in the $(\mathfrak{z}, \hbar)$-adic topology. We recall that there is a natural morphism $W_\hbar(V)///G \to \mathcal{D}(X)$. This morphism is $\mathbb{C}^\times$-equivariant and $\mathbb{C}[\mathfrak{z}^*, \hbar]$-linear.

Theorem 17.3. *Suppose that $V///_0G$ is a normal scheme and that the natural morphism $\rho : X \to V///_0G$ is a resolution of singularities. Then $W_\hbar(V)///G$ is dense in $\mathcal{D}(X)$ and coincides with the subspace of $\mathbb{C}^\times$-finite elements in $\mathcal{D}(X)$.*

Proof. Step 1. We claim that $\mathbb{C}[X] = \mathbb{C}[V///_0G]$ (the isomorphism is ρ^*) and $H^i(X, \mathcal{O}_X) = 0$ for $i > 0$. The former follows from the claim that $\mathbb{C}[X]$ is a finite birational extension of $\mathbb{C}[V///_0G]$. The cohomology vanishing follows from the observation that, being symplectic, X has trivial canonical bundle by applying the Grauert-Riemenschneider vanishing theorem.

Step 2. Thanks to Step 1, we can apply Lemma 17.1 to $\mathcal{F}_0 = \mathcal{O}_X$ and $\mathcal{F} = \mathcal{D}$. We get that $\mathcal{D}(X)$ is a FCS deformation of $\mathbb{C}[X]$.

Step 3. The homomorphism $W_\hbar(V)///G \to \mathcal{D}(X)$ extends to the $(\mathfrak{z},\hbar)$-adic completion $(W_\hbar(V)///G)^\wedge$ of $W_\hbar(V)///G$. The resulting homomorphism is the identity modulo $(\mathfrak{z},\hbar)$ and both algebras in consideration are complete and flat. The claim that $(W_\hbar(V)///G)^\wedge \xrightarrow{\sim} \mathcal{D}(X)$ now follows from the next exercise.

Exercise 17.2. *Let V_1, V_2 are $\mathbb{C}[[\mathfrak{z}^*,\hbar]]$-modules that are flat, complete and separated. Let $\iota : V_1 \to V_2$ be a $\mathbb{C}[[\mathfrak{z}^*,\hbar]]$-module homomorphism that is an isomorphism modulo $(\mathfrak{z},\hbar)$. Show that ι is an isomorphism.*

Step 4. It remains to show that $\mathrm{W}_\hbar(V)///G$ coincides with the $\mathbb{C}^\times$-finite part of $(\mathrm{W}_\hbar(V)///G)^\wedge$. This is an easy consequence of the observation that $\mathrm{W}_\hbar(V)///G$ is positively graded, see the following exercise.

Exercise 17.3. *Let A_0 be a $\mathbb{Z}_{\geqslant 0}$-graded vector space and A be its FCS deformation over $\mathbb{C}[[x_1,\ldots,x_n]]$. Equip A with a $\mathbb{C}^\times$-action such that $t.(x_i a) = t^2 x_i t.a$ and the projection $A \twoheadrightarrow A_0$ is $\mathbb{C}^\times$-equivariant (where the action of $\mathbb{C}^\times$ on the ith component of $\mathcal{A}_0$ is by $t \mapsto t^i$). Show that the $\mathbb{C}^\times$-finite part of A is a graded deformation of A_0 over $\mathbb{C}[x_1,\ldots,x_n]$.*

$\square$

The previous theorem can be informally stated as: the deformation $W_\hbar(V)///G$ of $V///_0G$ can be lifted to its resolution of singularities X.

17.4. **Procesi bundles.** We return to the setting of Lecture 15: $V = T^*\operatorname{Rep}(\underline{Q}^{MK}, n\delta, \epsilon_0), G = \mathrm{GL}(n\delta)$. All assumptions made above (the assumptions (i) and (ii) in Section 1 and the assumptions of Theorem 17.3) hold.

Let X be a $\mathbb{C}^\times$-equivariant symplectic resolution of $V///_0G = \mathbb{C}^{2n}/\Gamma_n$ (where the $\mathbb{C}^\times$-action and the form ω on X are related via $t.\omega = t^2\omega, t \in \mathbb{C}^\times$). For example, we can take $X = V///_0^\theta G$ for generic θ, conjecturally there are no other possibilities.

By a *Procesi bundle* on X we mean a $\mathbb{C}^\times$-equivariant vector bundle $\mathcal{P}$ satisfying the following conditions.

(P1) $\mathrm{End}_{\mathcal{O}_X}(\mathcal{P})$ is an algebra over $\mathbb{C}[X] = \mathbb{C}[\mathbb{C}^{2n}]^{\Gamma_n}$ that comes equipped with a natural $\mathbb{C}^\times$-action. We require the $\mathbb{C}[\mathbb{C}^{2n}]^{\Gamma_n}$-algebras $\mathrm{End}_{\mathcal{O}_X}(\mathcal{P})$ and $\mathbb{C}[\mathbb{C}^{2n}]\#\Gamma_n$ to be $\mathbb{C}^\times$-equivariantly isomorphic.

(P2) $H^i(X, \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{P})) = 0$ for $i > 0$.

There is yet one more condition. To state it we remark that an endomorphism algebra of a vector bundle acts on every fiber. In particular, we have a fiberwise action of Γ_n on $\mathcal{P}$.

Exercise 17.4. *Show that any fiber of a Procesi bundle is isomorphic to $\mathbb{C}\Gamma_n$ as a Γ_n-module.*

In particular, the bundle $\mathcal{P}^{\Gamma_n}$ of Γ_n-invariants has rank 1. Clearly, (P1) and (P2) are preserved by multiplication by a line bundle. We impose a normalization condition:

(P3) $\mathcal{P}^{\Gamma_n} = \mathcal{O}_X$.

The only result we need about Procesi bundles is the following.

Theorem 17.4 ([BK]). *A Procesi bundle exists.*

We notice that a Procesi bundle is never unique. This follows, for example, from the following problem.

Problem 17.5. *Show that the dual of a Procesi bundle is again a Procesi bundle.*

A more subtle statement is that a Procesi bundle cannot be isomorphic to its dual (as a $\mathbb{C}^\times$-equivariant bundle).

Let us discuss origins of Procesi bundles and motivations to consider them. This discussion will not be used below.

There is a case when it is easy to construct Procesi bundles: $n = 1$. Here a Procesi bundle can be obtained as a tautological bundle on $X = V///^\theta G$. Namely, consider the G-equivariant vector bundle N on V that is trivial with fiber $\bigoplus_{i \in \underline{Q}_0^{MK}} (\mathbb{C}^{\delta_i})^{\oplus \delta_i}$ as a usual vector bundle and is equipped with a natural fiberwise action of G. We restrict N to $\mu^{-1}(0)^{ss}$ and then push it forward to X getting a vector bundle of rank $|\Gamma_1|$. The conditions (P1),(P2) were checked in [KV]. Their motivation was to interpret the McKay correspondence as an equivalence of derived categories. Namely, in the general case of Γ_n, a Procesi bundle $\mathcal{P}$ gives rise to an equivalence $\mathrm{RHom}(\mathcal{P}, \bullet) : D^b(\mathrm{Coh}(X)) \to D^b(\mathbb{C}[\mathbb{C}^{2n}]\#\Gamma_n - \mathrm{mod})$ of triangulated categories.

The first construction of $\mathcal{P}$ in the case when $\Gamma_n = \mathfrak{S}_n$ is due to Haiman, [H1], see also review [H2]. Motivation of Haiman to consider Procesi bundles had an entirely different origins: algebraic combinatorics, more precisely the study of Macdonald polynomials. These are certain symmetric polynomials depending on the independent variables q and t, parameterized by partitions λ and defined "axiomatically" – as only polynomials subject to certain conditions. They are linear combinations of Schur polynomials with coefficients in $\mathbb{Z}[q^{\pm 1}, t^{\pm 1}]$. The Macdonald positivity conjecture says that the coefficients are actually in $\mathbb{Z}_{\geqslant 0}[q^{\pm 1}, t^{\pm 1}]$. This conjecture was first proved by Haiman using the Procesi bundle on $\mathrm{Hilb}_n(\mathbb{C}^2)$.

On $\mathrm{Hilb}_n(\mathbb{C}^2)$ we have an action of $(\mathbb{C}^\times)^2$ (that can be viewed as induced from the action on $\mathbb{C}^2$). The dilation actions considered above is obtained by embedding $\mathbb{C}^\times$ to $(\mathbb{C}^\times)^2$ diagonally. There are finitely many fixed points for the $(\mathbb{C}^\times)^2$-action on $\mathrm{Hilb}_n(\mathbb{C}^2)$, they are precisely the monomial ideals and so are parameterized by partitions λ of n, let x_λ be the point corresponding to λ. Haiman's Procesi bundle happens to be $(\mathbb{C}^\times)^2$-equivariant. So the fiber $\mathcal{P}_\lambda$ of $\mathcal{P}$ at x_λ is a bigraded $\mathfrak{S}_n$-module. Its Frobenius character is a linear combination of Schur polynomials with coefficients in $\mathbb{Z}_{\geqslant 0}[q^{\pm 1}, t^{\pm 1}]$. Haiman proved that these are Macdonald polynomials thus proving the positivity conjecture.

Haiman's proof does not use any very advanced machinery but is very complicated. Several other proofs of the positivity conjecture including combinatorial ones are available. Some of them are based on alternative constructions of Procesi bundles. There is a proof due to Gordon, [Go], based on a construction of a Procesi bundle due to Ginzburg, [Gi]. Also there is a proof by Bezrukavnikov and Finkelberg, [BF], based on the construction of a Procesi bundle from [BK]. This proof generalizes to the case when $\Gamma_n = \mathfrak{S}_n \ltimes (\mathbb{Z}/\ell\mathbb{Z})$ and instead of the usual Macdonald polynomials one considers so called *wreath* Macdonald polynomials.

17.5. **Isomorphism of quantum Hamiltonian reduction and spherical SRA.** Recall that our goal was to establish an isomorphism of $W_\hbar(V)///G$ and eHe. Recall that $W_\hbar(V)///G$ is a graded algebra over $\mathbb{C}[\mathfrak{z}^*, \hbar]$, where $\mathfrak{z}$ and $\hbar$ have degree 2. Similarly, eHe is a graded algebra over P, where P is a vector space with basis $\hbar, c_0, c_1, \ldots, c_r$, all these elements are in degree 2. We have $W_\hbar(V)///G/(\mathfrak{z}, \hbar) = eHe/(P) = \mathbb{C}[\mathbb{C}^{2n}]^{\Gamma_n}$. So we want a graded algebra isomorphism $eHe \to W_\hbar(V)///G$ that maps P to $\mathfrak{z} \oplus \mathbb{C}\hbar$ and induces the identity on $\mathbb{C}[\mathbb{C}^{2n}]^{\Gamma_n}$.

We will prove a weaker statement. Namely, following [L], we will see that there is a map $\nu : P \to \mathfrak{z} \oplus \mathbb{C}\hbar$ such that $\mathbb{C}[\mathfrak{z}, \hbar] \otimes_{S(P)} eHe \cong W_\hbar(V)///G$. A crucial tool here is a deformation of the Procesi bundle.

By Lemma 17.2(1), there is a unique deformation $\widehat{\mathcal{P}}$ of $\mathcal{P}$ to a right $\mathcal{D}$-module. By part (2) of the same lemma, $\operatorname{End}_{\mathcal{D}^{opp}}(\widehat{\mathcal{P}})$ is a FCS deformation of $\mathbb{C}[\mathbb{C}^{2n}]\#\Gamma_n$ over $\mathbb{C}[[\mathfrak{z}^*,\hbar]]$. Exercise 17.3 shows that the subalgebra $H' \subset \operatorname{End}_{\mathcal{D}^{opp}}(\widehat{\mathcal{P}})$ of $\mathbb{C}^\times$ finite elements is a graded deformation of $\mathbb{C}[\mathbb{C}^{2n}]\#\Gamma_n$. But H is a universal graded deformation of $\mathbb{C}[\mathbb{C}^{2n}]\#\Gamma_n$ (at least in the case when $\Gamma_n \neq \mathfrak{S}_n$, in the case of the symmetric group one needs to modify the argument slightly). So we get a linear map $\nu_{\mathcal{P}} : P \to \mathfrak{z}^* \oplus \mathbb{C}\hbar$ such that $\mathbb{C}[\mathfrak{z}^*,\hbar]\otimes_{S(P)} H \cong H'$.

We remark that we still have an inclusion $\Gamma_n \subset H'$. We claim that $eH'e$ is naturally identified with $W_\hbar(V)///G$. To see this we remark that we have a decomposition $\widehat{\mathcal{P}} = e\widehat{\mathcal{P}} \oplus (1-e)\widehat{\mathcal{P}}$ and that $e\widehat{\mathcal{P}}$ is a deformation of $e\mathcal{P} = \mathcal{O}_X$. From the uniqueness of a deformation, we see that $e\widehat{\mathcal{P}} = \mathcal{D}$. Next, on one hand, $\operatorname{End}_{\mathcal{D}^{opp}}(e\widehat{\mathcal{P}}) = e\operatorname{End}_{\mathcal{D}^{opp}}(\widehat{\mathcal{P}})e$ and on the other hand, $\operatorname{End}_{\mathcal{D}^{opp}}(e\widehat{\mathcal{P}}) = \mathcal{D}(X)$. Taking finite elements and using Theorem 17.3, we arrive at $eH'e = W_\hbar(V)///G$. So $\mathbb{C}[\mathfrak{z},\hbar]\otimes_{S(P)} eHe \cong W_\hbar(V)///G$.

One can get an explicit formula for ν that also shows that it is an isomorphism. We need some more notation to write it down.

Let $N_0,\ldots,N_r$ be the irreducible Γ_1-modules with N_0 being the trivial module. Let $S_1^0,\ldots,S_r^0$ denote the non-trivial conjugacy classes in Γ_1 so that S_i corresponds to S_i^0. Form the element $c = \hbar + \sum_{i=1}^r c_i \sum_{\gamma\in S_i^0}\gamma \in P\otimes\mathbb{C}\Gamma_1$. Let $\check{\epsilon}_i, i=0,\ldots,r$, denote the element $\operatorname{tr}_i \in \mathfrak{z}^*$. This is a basis, let $\epsilon_0,\ldots,\epsilon_r$ be the dual basis in $\mathfrak{z}$. Then for ν we can take the inverse of the following map $\mathfrak{z}\oplus\mathbb{C}\hbar \to P$:

$$\hbar \mapsto \hbar, \epsilon_i \mapsto \operatorname{tr}_{N_i} c/|\Gamma_1|, \epsilon_0 \mapsto \operatorname{tr}_{N_0} c/|\Gamma_1| - \frac{1}{2}(c_0+\hbar). \tag{1}$$

We remark that we have already seen similar formulas in Lecture 5.

Let us say a few words about the proof. Consider the group W_{fin} of the finite root system with Dynkin diagram $\underline{Q}^{MK}\setminus\{0\}$. We have a linear action of $W_{fin}\times\mathbb{Z}/2\mathbb{Z}$ on $\mathfrak{z}\oplus\mathbb{C}\hbar$ (where $\hbar$ is invariant). There is a reduction procedure from Γ_n to $\Gamma_1, \mathfrak{S}_2$. It is based on considering completions of the algebras in consideration. The reduction shows that ν is different from the map described by (1) by the action of an element from $W_{fin}\times\mathbb{Z}/2\mathbb{Z}$. But then one shows that the group of graded automorphisms of eHe that preserve P and act as identity on $\mathbb{C}[\mathbb{C}^{2n}]^{\Gamma_n}$ is naturally identified with $W_{fin}\times\mathbb{Z}/2\mathbb{Z}$. So one can twist ν_P by an element of that group and get the required isomorphism.

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LECTURES ON SYMPLECTIC REFLECTION ALGEBRAS

IVAN LOSEV

18. Category O for rational Cherednik algebra

We begin to study the representation theory of Rational Chrednik algebras. Perhaps, the class of representations we want to start with is finite dimensional irreducible ones. Questions that we may want to ask about them is their classification and computation of dimensions.

It is unreasonable however to restrict our attention to finite dimensional representations only. The situation is similar to the representation theory of semisimple Lie algebras $\mathfrak{g}$: to classify and, especially, to compute characters of finite dimensional irreducibles one also needs to consider certain infinite dimensional representations – Verma modules. It is then reasonable to include all these modules (finite dimensional ones and Vermas) into a single category, known as the BGG (=Bernstein-Gelfand-Gelfand) category $\mathcal{O}$: the category of all finitely generated $U(\mathfrak{g})$-modules, where maximal nilpotent subalgebra acts locally nilpotently.

A similar construction can be done for a Rational Cherednik algebra (at $t \neq 0$, which reduces to $t = 1$). However, the corresponding category is, in a way, more complicated then the BGG categories $\mathcal{O}$: finite dimensional representations are no longer completely reducible, and their classification and character formulas are only known in special cases.

Our plan regarding categories $\mathcal{O}$ for RCA is as follows. Today we will give a definition, explain the highest weight structure and some related constructions. In the next four lectures we will study some functors: Bezrukavnikov-Etingof's functors, KZ functors, and, finally, Kac-Moody categorification functors.

18.1. **Some structural results on H_c.** Let us recall the definition of H_c. Let W be a complex reflection group with reflection representation $\mathfrak{h}$. Let S denote the subset of symplectic reflections in W. Pick a conjugation invariant function $c : S \to \mathbb{C}$. For $s \in S$ let $\alpha_s, \alpha_s^\vee$ denote eigenvectors for s in $\mathfrak{h}^*$ and $\mathfrak{h}$, respectively, with non-unit eigenvalues, λ_s and λ_s^{-1}, respectively. Both vectors are defined up to a non-zero constant factor, and we partially fix the ambiguity by requiring $\langle \alpha_s, \alpha_s^\vee \rangle = 2$. Then the algebra H_c (denoted before by $H_{1,c}$) is defined as the quotient of $T(\mathfrak{h} \oplus \mathfrak{h}^*)\#W$ by the relations

$$[x, x'] = [y, y'] = 0, [y, x] = \langle y, x\rangle - \sum_{s\in S} c_s \langle \alpha_s, y\rangle\langle \alpha_s^\vee, x\rangle s, \quad x, x' \in \mathfrak{h}^*, y, y' \in \mathfrak{h}.$$

From the definition, we have natural homomorphisms $\mathbb{C}[\mathfrak{h}] = S(\mathfrak{h}^*), S(\mathfrak{h}), \mathbb{C}W \to H_c$. Recall that $\operatorname{gr} H_c = S(\mathfrak{h}\oplus\mathfrak{h}^*)\#W$, where the associated graded is taken with respect to the filtration defined by $\deg W = 0, \deg \mathfrak{h} \oplus \mathfrak{h}^* = 1$.

Exercise 18.1. *Use $\operatorname{gr} H_c = S(\mathfrak{h} \oplus \mathfrak{h}^*)\#W$ to show that H_c is Noetherian.*

The homomorphisms $S(\mathfrak{h}), S(\mathfrak{h}^*), \mathbb{C}W \hookrightarrow S(\mathfrak{h}\oplus\mathfrak{h}^*)\#W$ define a vector space isomorphism $S(\mathfrak{h}^*) \otimes \mathbb{C}W \otimes S(\mathfrak{h}) \xrightarrow{\sim} S(\mathfrak{h} \oplus \mathfrak{h}^*)\#W$ given by $a \otimes b \otimes c \mapsto abc$. Hence the natural map $S(\mathfrak{h}^*) \otimes \mathbb{C}W \otimes S(\mathfrak{h}) \to H_c$ is also an isomorphism of vector spaces (the map $S(\mathfrak{h}^*) \otimes \mathbb{C}W \otimes$

$S(\mathfrak{h}) \xrightarrow{\sim} S(\mathfrak{h}\oplus\mathfrak{h}^*)\#W$ is the associated graded of the map $S(\mathfrak{h}^*)\otimes\mathbb{C}W\otimes S(\mathfrak{h})\to H_c$). We will call this isomorphism a *triangular decomposition.* One should compare this with the triangular decomposition $U(\mathfrak{g}) = U(\mathfrak{n}^-)\otimes U(\mathfrak{h})\otimes U(\mathfrak{n})$ for semisimple Lie algebras.

We will need a certain element $h\in H_c$ called the (deformed) Euler element. It is given by

$$h = \sum_{i=1}^n x_i y_i + \frac{1}{2}\dim\mathfrak{h} - \sum_{s\in S}\frac{2c_s}{1-\lambda_s}s. \tag{1}$$

Here y_i is a basis in $\mathfrak{h}$, x_i is the dual basis of $\mathfrak{h}^*$, and $\lambda_s\in\mathbb{C}$ is given by $s\alpha_s = \lambda_s\alpha_s$. The reason why this element is interesting (and is called Euler) is explained in the following exercise.

Exercise 18.2. *We have* $[h,x]=x, [h,w]=0, [h,y]=-y$ *for all* $x\in\mathfrak{h}^*, w\in W, y\in\mathfrak{h}$.

18.2. **Category $\mathcal{O}$.** By definition, the category $\mathcal{O}(=\mathcal{O}_c)$ (formally introduced in [GGOR]) is the full subcategory in the category of H_c-modules consisting of all modules M satisfying the following two conditions:

- M is finitely generated.
- $\mathfrak{h}$ acts on M by locally nilpotent endomorphisms.

Since H_c is Noetherian any submodule in a finitely generated module is finitely generated. It follows that $\mathcal{O}$ is an abelian category (and even a Serre subcategory in the category of all H_c-modules – it is closed under extensions).

Let us produce some examples of modules in $\mathcal{O}$. First of all, the actions of both x's and y's on any finite dimensional module are locally nilpotent, this follows from Exercise 18.2. So any finite dimensional H_c-module is on $\mathcal{O}$.

Another example is analogs of Verma modules constructed as follows. Pick a W-module E. We can view it as a $S(\mathfrak{h})\#W$-module by making $\mathfrak{h}$ act by 0. Then we set $\Delta(E) = H_c\otimes_{S(\mathfrak{h})\#W}E$. The module $\Delta(E)$ with irreducible E is called a Verma (or standard) module.

Thanks to the triangular decomposition, we have an isomorphism $H_c\cong S(\mathfrak{h}^*)\otimes(S(\mathfrak{h})\#W)$ of right $S(\mathfrak{h})\#W$-module, we get a W-equivariant isomorphism $\Delta(E) = S(\mathfrak{h}^*)\otimes E$. We have already seen such module in a special case $E=\text{triv}$: it appeared as the Dunkl operator representation on $S(\mathfrak{h}^*) = \mathbb{C}[\mathfrak{h}]$, where $y\in\mathfrak{h}$ acts by the Dunkl operator

$$D_y = \partial_y - \sum_{s\in S}\frac{2c_s\langle\alpha_s,y\rangle}{(1-\lambda_s)\alpha_s}(1-s).$$

Problem 18.3. *Write an action of y on $\Delta(E)$ via a 1st order differential operator with poles.*

To check that $\Delta(E)\in\mathcal{O}$ we need to check that y acts locally nilpotently. For this (and future purposes) we will need to examine the action of h on $\Delta(E)$. On $E\subset\Delta(E)$ the element h acts by the scalar operator

$$c_E := \frac{1}{2}\dim\mathfrak{h} - \sum_{s\in S}\frac{2}{1-\lambda_s}c_s s|_E$$

(it is scalar because it is W-equivariant, and E is irreducible). Since $[h,x]=x$, we see that the eigenvalue of h on $S^n(\mathfrak{h}^*)\otimes E$ is c_E+n. Since $[h,y]=-y$, we see that y maps $S^n(\mathfrak{h}^*)\otimes E$ to $S^{n-1}(\mathfrak{h}^*)\otimes E$ (this also can be seen directly) and so y is locally nilpotent.

Exercise 18.4. *Show that* $\operatorname{Hom}_{\mathcal{O}}(\Delta(E),M) = \operatorname{Hom}_W(E,M)$*, where* $M^{\mathfrak{h}} = \{m\in M|\mathfrak{h}m = 0\}$.

Exercise 18.5. *Show that each $\Delta(E)$ has a unique irreducible quotient, denoted $L(E)$. Show that the natural inclusion $E \hookrightarrow \Delta(E)$ gives rise to an inclusion $E \hookrightarrow L(E)$. Further, show that $L(E)$ form a complete list of irreducible objects in $\mathcal{O}$.*

18.3. **Highest weight structure.** In the usual BGG category we have various triangularity properties (say, for Hom's between Verma modules or for simple constituents of Vermas). In fact, the analogous properties hold in the Cherednik category $\mathcal{O}$ as well.

We define a partial order on the set of W-irreps as follows: $E < E'$ if $c_E - c_{E'} \in \mathbb{Z}_{>0}$. The reason for introducing this ordering is as follows: if $L(E)$ appears as a composition factor in a Jordan-Hölder series of $\Delta(E')$, then either $E < E'$ or $E = E'$ and $L(E)$ is the irreducible quotient of $\Delta(E')$. Indeed, as we have seen, the eigenvalues of h in $\Delta(E')$ are $c_{E'} + n$, where $n = 0$ corresponds to the "top copy" $E' \subset \Delta(E')$. Since $L(E)$ is a composition factor of $\Delta(E')$, any eigenvalue of h in $L(E)$ is also an eigenvalue in $\Delta(E')$, i.e., of the form $c_{E'} + n$. But Exercise 18.5 implies that c_E is one of eigenvalues of h in $L(E)$. Also the eigenvalues of h in the radical of $\Delta(E)$ (the kernel of the projection $\Delta(E) \twoheadrightarrow L(E)$) are of the form $c_{E'} + n$ for $n > 0$. This implies our claim.

Exercise 18.6. *Prove that h acts locally finitely on any object in $\mathcal{O}$ and that any object in $\mathcal{O}$ has finite length. Deduce that all generalized subspaces for h are finite dimensional and that any module in $\mathcal{O}$ is finitely generated over $S(\mathfrak{h}^*)$.*

For $M \in \mathcal{O}$ we will write M_a for the generalized eigenspace for h with eigenvalue a.

Let us now give a definition of a highest weight category that formalizes various upper-triangularity properties. Let $\mathcal{O}$ be an abelian $\mathbb{C}$-linear (i.e., all Hom's are vector spaces over $\mathbb{C}$) category with finitely many simples, all objects of finite length, and sufficiently many projectives. In other words, $\mathcal{O}$ has to be equivalent to the category of finite dimensional representations of a finite dimensional algebra.

A highest weight structure on $\mathcal{O}$ is some additional data satisfying certain axioms. Let Λ be a labeling set of simples, we write $L(\lambda)$ for the simple corresponding to $\lambda \in \Lambda$ and $P(\lambda)$ for its projective cover. A *highest weight structure* on $\mathcal{O}$ is a poset structure on Λ together with so called standard objects $\Delta(\lambda)$, one for each $\lambda \in \Lambda$. These data should satisfy the following axioms.

(HW1) $\mathrm{Hom}_{\mathcal{O}}(\Delta(\lambda), \Delta(\mu)) \neq 0 \Rightarrow \lambda \leqslant \mu$.
(HW2) $\mathrm{End}_{\mathcal{O}}(\Delta(\lambda)) = \mathbb{C}$.
(HW3) There is an epimorphism $P(\lambda) \twoheadrightarrow \Delta(\lambda)$ whose kernel admits a filtration with successive quotients of the form $\Delta(\mu)$ for $\mu > \lambda$.

It is a classical fact that the BGG category $\mathcal{O}$ is a highest weight category (when we restrict to infinitesimal blocks), see, e.g., [H]. It turns out that the Cherednik category $\mathcal{O}$ is also highest weight. We have already seen that there are finitely many simples and all objects have finite length.

Exercise 18.7. *Show that (HW1) and (HW2) hold for $\mathcal{O}_c$.*

It remains to prove that there are enough projectives and that they satisfy (HW3). The proof is in several steps.

Step 0. Pick $m \in \mathbb{Z}_{\geqslant 1}$ and consider the induced module $\Delta_m(E) := H_c \otimes_{S(\mathfrak{h})\# W} (E \otimes S(\mathfrak{h})/(\mathfrak{h}^m))$. Obviously $\mathrm{Hom}_{\mathcal{O}}(\Delta_m(E), M) = \mathrm{Hom}_W(E, M^{\mathfrak{h}^m})$. Further, we have

$$\mathrm{Hom}_{H_c}(\varprojlim_m \Delta_m(E), M) = \varinjlim_m \mathrm{Hom}_W(E, M^{\mathfrak{h}^m}) = \mathrm{Hom}_W(E, \varinjlim_m M^{\mathfrak{h}^m}) = \mathrm{Hom}_W(E, M)$$

The latter defines an exact functor, and so $\varprojlim_m \Delta_m(E)$ is "kind of projective". A problem is that this object does not lie in the category $\mathcal{O}$. We will produce a projective in $\mathcal{O}$ by taking a direct summand of a "graded version" of the inverse limit.

Step 1. Consider the grading on H, where $\deg x = 1, \deg w = 0, \deg y = -1$ (the graded components are eigenspaces for $[h, \cdot]$ and the degrees are eigenvalues). Then $\Delta_m(E)$ is naturally graded with E in degree 0. We are going to establish a natural decomposition of $\Delta_m(E)$ into graded direct summands.

Let now $M = \bigoplus M^i$ be an arbitrary $\mathbb{Z}$-graded H_c-module. Since h is in degree 0, it preserves all M^i. Set $W_a(M) := \bigoplus_i M^i_{a+i}$, it is a graded submodule of M, and $M = \bigoplus_{a\in\mathbb{C}} W_a(M)$. Set $\tilde{\Delta}_m(E) = W_{c_E}(\Delta_m(E))$. Similarly, for an ungraded H_c-module M, we can consider its summands $W_{a+\mathbb{Z}}(M)$ (all generalized eigenspaces of h with e-values in $a+\mathbb{Z}$).

Of course, $\tilde{\Delta}_1(E) = \Delta(E)$ and we have a natural epimorphism $\tilde{\Delta}_{m+1}(E) \twoheadrightarrow \tilde{\Delta}_m(E)$ of graded H_c-modules. We claim that this epimorphism is iso for all sufficiently large m.

Step 2. We have an exact sequence

$$0 \to S^m(\mathfrak{h}) \otimes E \to S(\mathfrak{h})/(\mathfrak{h}^{m+1}) \otimes E \to S(\mathfrak{h})/(\mathfrak{h}^m) \otimes E \to 0.$$

Since H_c is isomorphic to $S(\mathfrak{h}^*) \otimes (S(\mathfrak{h})\#W)$ as a right $S(\mathfrak{h})\#W$-module, the induction functor $H_c \otimes_{S(\mathfrak{h})\#W} \bullet$ is exact. This yields an exact sequence of graded H_c-modules.

$$0 \to \Delta(S^m\mathfrak{h} \otimes E) \to \Delta_{m+1}(E) \to \Delta_m(E) \to 0. \tag{2}$$

The kernel $\Delta(S^m\mathfrak{h} \otimes E)$ decomposes into the direct sum of various $\Delta(E')$. The subspace $S^n(\mathfrak{h}^*) \otimes E' \subset \Delta(E')$ has degree $n - m$ and the eigenvalue of h equal to $c_{E'} + n$ (the latter follows from computations in the previous section) so will belong to the summand $W_{c_{E'}+m}(\Delta_{m+1}(E))$. For m large enough, we have $c_E \neq c_{E'} + m$ for all possible E'. Applying the exact functor W_{c_E} to exact sequence (2) we get $W_{c_E}(\Delta_{m+1}(E)) \xrightarrow{\sim} W_{c_E}(\Delta_m(E))$.

Step 3. Let $\tilde{\Delta}(E)$ denote the module $\tilde{\Delta}_m(E)$ for m sufficiently large. Let us prove that this object is projective. More precisely, we will show that $\operatorname{Hom}(\tilde{\Delta}(E), M) = \operatorname{Hom}_W(E, M_{c_E})$. Since $M \mapsto M_{c_E}$ is an exact functor, this will show that $\tilde{\Delta}(E)$ is projective.

Pick a module $M \in \mathcal{O}$. Clearly, $\operatorname{Hom}(\tilde{\Delta}(E), M) = \operatorname{Hom}(\tilde{\Delta}(E), W_{c_E+\mathbb{Z}}(M))$. So it is enough to assume that $W_{c_E+\mathbb{Z}}(M) = M$. The module M can be graded, for M^i take the generalized eigenspace of h with eigenvalue $c_E + i$.

The Hom space in $\mathcal{O}$ between two graded modules is naturally graded. Of course, $\operatorname{Hom}(\Delta_m(E), M) = \bigoplus_i \operatorname{Hom}_{\mathcal{O}}(W_{c_E-i}(\Delta_m(E)), M)$. From the definition of $W_\bullet$, the construction of grading on M and the observation that any H_c-linear homomorphism preserves the eigenspaces for h, it follows that $\operatorname{Hom}_{\mathcal{O}}(W_{c_E-i}(\Delta_m(E)), M) = \operatorname{Hom}_{\mathcal{O}}(\Delta_m(E), M)^i$. In particular, $\operatorname{Hom}_{\mathcal{O}}(\tilde{\Delta}(E), M) = \operatorname{Hom}_{\mathcal{O}}(\Delta_m(E), M)^0$ for $m \gg 0$.

The identification $\operatorname{Hom}(\Delta_m(E), M) \cong \operatorname{Hom}_W(E, M^{\mathfrak{h}^m})$ preserves the gradings, where the grading on the right hand side is induced by that on M. But also, since M is finitely generated over $\mathbb{C}[\mathfrak{h}]$, the grading on M is bounded from below. Since $\mathfrak{h}^m$ decreases degrees by m, we see that, for any given k, $(M^k)^{\mathfrak{h}^m} = M^k$ for $m \gg 0$. So $\operatorname{Hom}_{\mathcal{O}}(\Delta_m(E), M)^0 = \operatorname{Hom}_W(E, M^0) = \operatorname{Hom}_W(E, M_{c_E})$ for $m \gg 0$. This proves that $\operatorname{Hom}(\tilde{\Delta}(E), M) = \operatorname{Hom}_W(E, M_{c_E})$.

Step 4. Let us show that the module $\tilde{\Delta}(E)$ has a filtration required by (HW3). Indeed, it is filtered with successive quotients being $W_{c_E}(\Delta(S^m\mathfrak{h} \otimes E))$. The letter module splits into the sum of certain $\Delta(E')$. The top subspace $E' \subset \Delta(E') \subset W_{c_E}(\Delta(S^m\mathfrak{h} \otimes E))$ is in degree $-m$ and the eigenvalue is $c_{E'}$. So we should have $c_{E'} = c_E - m$ and hence $E < E'$.

Step 5. We are still not done. The module $\tilde{\Delta}(E)$ has a required filtration but, in general, it is not indecomposable. The claim that there is an indecomposable summand $P(E)$ of $\tilde{\Delta}(E)$ with required filtration is a consequence of the following exercise.

Exercise 18.8. *Show that if an exact sequence* $0 \to \Delta(E) \to M \to \Delta(E') \to 0$ *does not split, then* $E' < E$. *Deduce that if* $M_1 \oplus M_2$ *is* Δ*-filtered (i.e. admits a filtration whose quotients are* Δ*'s), then* M_1 *and* M_2 *are* Δ*-filtered.*

Indeed, for $P(E)$ we can take a unique indecomposable summand of $\tilde{\Delta}(E)$ that has $\Delta(E)$ (occurring in $\tilde{\Delta}(E)$ with multiplicity 1) as a filtration quotient. This is well-defined because the multiplicity of $\Delta(E)$ in a Δ-filtered object is an additive function.

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LECTURES ON SYMPLECTIC REFLECTION ALGEBRAS

IVAN LOSEV

19. KZ functor I: double centralizer property

The notation $W, S, c, \alpha_s, \alpha_s^\vee$ has the same meaning as in the previous lecture. We have the category $\mathcal{O} = \mathcal{O}_c$. Sometimes we will write $\mathcal{O}_c(\mathfrak{h})$ to indicate that it is $\mathfrak{h}$ that acts locally nilpotently (we can also consider the analogous category $\mathcal{O}_c(\mathfrak{h}^*)$).

A KZ functor introduced in [GGOR] is one of the most powerful tools to study the category $\mathcal{O}_c$. It is an exact functor $\mathcal{O}_c \to \mathcal{H}_c(W)$-mod, where $\mathcal{H}_c(W)$ is the *Hecke algebra* of the complex reflection group W corresponding to the parameter c. This functor induces an equivalence of $\mathcal{O}_c/\mathcal{O}_c^{tor}$ and $\mathcal{H}_c(W)$-mod, where $\mathcal{O}_c^{tor}$ is the Serre subcategory of $\mathcal{O}_c$ consisting of all objects M in $\mathcal{O}_c$ that are *torsion* for $S(\mathfrak{h}^*)$ (i.e., any element of M is annihilated by some element of $S(\mathfrak{h}^*)$). Finally, this functor has a very nice property, it is fully faithful on projective objects (sometimes this is also called the double centralizer property).

19.1. **Localization functor.** The first ingredient to construct the KZ functor is the *localization* functor π. Consider the element $\delta \in S(\mathfrak{h}^*), \delta := \prod_{s\in S}\alpha_s$. This is a W-semiinvariant element. Recall, Exercise 13.3, that the localization $H_c[\delta^{-1}]$ makes sense and is isomorphic (via the Dunkl operator homomorphism) to $D(\mathfrak{h}^{Reg})\#W$, where $\mathfrak{h}^{Reg}$ is the principal open subset defined by δ (coinciding with the locus, where the W-action is free). So we have the localization functor $\pi : H_c\text{-mod} \to D(\mathfrak{h}^{Reg})\#W, M \mapsto D(\mathfrak{h}^{Reg})\#W \otimes_{H_c} M$. As a $\mathbb{C}[\mathfrak{h}^{Reg}]$-module, $\pi(M) = \mathbb{C}[\mathfrak{h}^{Reg}] \otimes_{\mathbb{C}[\mathfrak{h}]} M$. So, clearly, π is an exact functor.

Recall that any object $M \in \mathcal{O}$ is finitely generated over $\mathbb{C}[\mathfrak{h}]$. So $\mathrm{Loc}(M) = 0$ if and only if M is annihilated by some power of δ, i.e., is supported (as a coherent sheaf on $\mathfrak{h}$) outside $\mathfrak{h}^{Reg}$. We claim that these are all torsion modules in $\mathcal{O}$. Let us prove this. Our first observation is that $D(\mathfrak{h}^{Reg})\#W$-module $\pi(M)$ is finitely generated over $\mathbb{C}[\mathfrak{h}^{Reg}]$.

Proposition 19.1. *Let X be a smooth variety and let M be a coherent sheaf that is also a D_X-module (here D_X stands for the sheaf of differential operators on X). Then M is a vector bundle.*

Proof. It is enough to prove that the restriction of M to a formal neighborhood of any point in X is free. Consider the corresponding algebra of differential operators $D := D(\mathbb{C}[[x_1,\ldots,x_n]])$ (as a vector space, D is $\mathbb{C}[[x_1,\ldots,x_n]][\partial_1,\ldots,\partial_n]$ with obvious commutation relations). We claim that any D-module M that is finitely generated over $\mathbb{C}[[x_1,\ldots,x_n]]$ is isomorphic to the sum of several copies of $\mathbb{C}[[x_1,\ldots,x_n]]$. The module M is complete and separated in the $(x_1,\ldots,x_n)$-adic topology. We claim that any such module (not necessarily finitely generated) is the sum of several copies of $\mathbb{C}[[x_1,\ldots,x_n]]$.

The operator $p_i := \sum_{j=0}^\infty \frac{(-1)^j}{j!}x_i^j\partial_i^j$ is well-defined. It is straight-forward to check that the operators p_i pairwise commute. Also $\partial_i p_i = \sum_{j=0}^\infty \frac{(-1)^j}{j!}(jx_i^{j-1}\partial_i^j + x_i^j\partial_i^{j+1}) = 0$. Let M^∂ denote the image of $p_1\ldots p_n$. The vectors in M^∂ are annihilated by all ∂_i and they generate M modulo $(x_1,\ldots,x_n)$ (because each p_i is the identity modulo $(x_1,\ldots,x_n)$). So M^∂ generates M as a $\mathbb{C}[[x_1,\ldots,x_n]]$-module. Let us check that the natural map $\mathbb{C}[[x_1,\ldots,x_n]]\otimes M^\partial \to M$

is an isomorphism. Let $f = \sum_\alpha x^\alpha m_\alpha$ be in the kernel, here we write x^α for $x_1^{\alpha_1}\ldots x_n^{\alpha}$. We remark that the homomorphism $\mathbb{C}[[x_1,\ldots,x_n]]\otimes M^\partial \to M$ is that of D-modules. So the kernel is also a D-module. In particular, $x_i\partial_i f$ also lies in the kernel. But $x_i\partial_i x^\alpha m_\alpha = \alpha_i x^\alpha m_\alpha$. Also the kernel is closed in the $(x_1,\ldots,x_n)$-adic topology. Since with any element $\sum_\alpha x^\alpha m_\alpha$ it also contains $\sum_\alpha \alpha_i x^\alpha m_\alpha$, we see that the kernel is spanned by the monomials $x^\alpha m_\alpha$. Applying ∂^α to $x^\alpha m_\alpha$, we see that m_α lies in the kernel. But our map is tautologically injective on M^∂. So the kernel is zero. □

In particular, if the module $M\in\mathcal{O}$ is torsion, then $\pi(M)$ is torsion and hence is 0. So any torsion module in $\mathcal{O}$ is annihilated by δ^m for $m\gg 0$.

Problem 19.1. *Let M_1, M_2 be D_X-modules that are coherent sheaves. Show that*

$$\dim \operatorname{Hom}_{D_X}(M_1,M_2)<\infty.$$

19.2. **Double centralizer property.** The main result about π for today is the *double centralizer property*: the claim that π is fully faithful on projectives.

Theorem 19.2. *The functor π is fully faithful on projective objects, i.e., if P_1, P_2 are projective in $\mathcal{O}$, then the natural homomorphism $\operatorname{Hom}_{H_c}(P_1,P_2)\to \operatorname{Hom}_{D(\mathfrak{h}^{Reg})\#W}(\pi(P_1),\pi(P_2))$ is an isomorphism.*

Presently, we are in position to prove the injectivity. The surjectivity is more subtle and will be proved after some preparation.

Recall that any projective projective in $\mathcal{O}$ is Δ-filtered, i.e., admits a filtration whose successive quotients are Verma modules $\Delta(E)$. The injectivity follows from the next lemma.

Lemma 19.3. *Let $M, N\in\mathcal{O}$ and suppose that N is Δ-filtered. Then the natural homomorphism $\operatorname{Hom}_{\mathcal{O}}(M,N)\to \operatorname{Hom}_{D(\mathfrak{h}^{Reg})\#W}(\pi(M),\pi(N))$ is injective.*

Proof. Any $\Delta(E)$ is free over $S(\mathfrak{h}^*)$ and hence so is any Δ-filtered object. Being free, N is torsion-free. The equality $\pi(\varphi)=0$ for $\varphi\in \operatorname{Hom}_{\mathcal{O}}(M,N)$ is equivalent to $\pi(\operatorname{im}\varphi)=0$. But δ is a nonzero divisor on N. □

19.3. **Naive duality.** We want some conditions on M that will insure that $\operatorname{Hom}_{\mathcal{O}}(M,N)\to \operatorname{Hom}_{D(\mathfrak{h}^{Reg})\#W}(\pi(M),\pi(N))$ is surjective. This condition is kind of dual to that of Lemma 19.3. This indicates that we need to establish some kind of duality on the categories $\mathcal{O}$. Such construction is well-known for the BGG category $\mathcal{O}$ and is obtained in a similar fashion in the Cherednik case.

For a parameter $c: S\to\mathbb{C}$ define $c^\vee: S\to \mathbb{C}$ via $c^\vee(s)=c(s^{-1})$ (so that if W is a real reflection group, then $c^\vee=c$). Obviously, $(c^\vee)^\vee=c$. We will write $\mathcal{O}(\mathfrak{h}^*)$ for the category defined similarly to $\mathcal{O}$ but for $\mathfrak{h}^*$ instead of $\mathfrak{h}$. Our goal is to define an involutive contravariant equivalence $\bullet^\vee:\mathcal{O}_c\to\mathcal{O}_{c^\vee}(\mathfrak{h}^*)$.

We start by establishing an algebra isomorphism $\sigma: H_c\to H_{c^\vee}^{opp}$. On the generators, it is given by $\sigma(x)=x, \sigma(y)=-y, \sigma(w)=w^{-1}$. It is straightforward to see that $[\sigma(x),\sigma(y)] = -\sigma([x,y])$ and so σ does lift to an algebra homomorphism. Also it is clear that $\sigma^2=1$.

Now let us define the functor $\bullet^\vee$. Pick $M\in\mathcal{O}$. It admits a decomposition $M=\bigoplus_{a\in\mathbb{C}} M_a$ into generalized eigenspaces for h. Then $M^*=\operatorname{Hom}(M,\mathbb{C})$ has a natural structure of a right H-module. Consider the restricted dual $M^{(*)}=\bigoplus_a M_a^*\subset M^*$. It is stable with respect to $\mathfrak{h}, W, \mathfrak{h}^*$ and so is an H-submodule. We can view $M^{(*)}$ as a left $H_{c^\vee}$-module using σ. This is the module that we are going to denote by $M^\vee$. Let us check that this module belongs to $\mathcal{O}(\mathfrak{h}^*)$. First of all, we need to check the action of $\mathfrak{h}^*$ on $M^{(*)}$ is locally nilpotent. Since

$\mathfrak{h}^* M_{a-1} \subset M_a$, it follows from $M_a^* \mathfrak{h}^* \subset M_{a-1}^*$. Using the claim that the h-eigen-values on M are bounded from below (i.e., for any eigenvalue a, the number $a - m$ is no longer an eigenvalue for $m \gg 0$), we see that $\mathfrak{h}^*$ acts locally nilpotently on $M^\vee$. The claim that $M^\vee$ is finitely generated follows from the next exercise, because one can check that the subspaces $M_a^* \subset M^\vee$ are the generalized eigenspaces for h.

Exercise 19.2. *Let M be an H_c-module with locally nilpotent action of $\mathfrak{h}$. Show that M is finitely generated iff the action of h on M is locally finite and all generalized eigen-subspaces are finite dimensional.*

It is follows from $\sigma^2 = 1$ (and the fact that taking the restricted dual is involutive) that $(M^\vee)^\vee = M$.

One application of the duality is to define so called *costandard objects* using it. Namely, we set $\nabla_c(E) := \Delta_{c^\vee}(E^*)^\vee$. This definition and the construction of duality imply that $\mathrm{Hom}_{\mathcal{O}}(M, \nabla(E)) = \mathrm{Hom}_W(M/\mathfrak{h}M, E)$. It follows that $\nabla(E)$ has simple socle equal to $L(E)$.

Problem 19.3. *Show that* $\mathrm{Ext}^i(\Delta(E), \nabla(E')) = \mathbb{C}$ *if $E = E', i = 0$, and 0 else. Moreover, show that if* $\mathrm{Ext}^1(\Delta(E), M) = 0$ *for all E, then M is ∇-filtered, i.e., admits a filtration with successive quotients $\nabla(E')$.*

Now let us study a relationship between the subcategory $\mathcal{O}^{tor}$ (of all modules in $\mathcal{O}$ that are torsion over $S(\mathfrak{h}^*)$) and the duality $\bullet^\vee$.

Lemma 19.4. *If $M \in \mathcal{O}_c^{tor}$, then $M^\vee \in \mathcal{O}(\mathfrak{h}^*)_{c^\vee}^{tor}$.*

Proof. As in the previous lecture, we may assume that all eigenvalues of h on M are congruent modulo $\mathbb{Z}$, in which case we can turn M into a graded module. Since grading is bounded from below, we can shift it and assume that M is positively graded, $M = \bigoplus_{i \geqslant 0} M^i$. Then $M^\vee$ is naturally negatively graded, $M^\vee = \bigoplus_{i \leqslant 0} (M^\vee)^i$ with $(M^\vee)^i = (M^{-i})^*$. The condition that M is $S(\mathfrak{h})$-torsion is equivalent to $\lim_{i \to +\infty} i^{1-n} \dim M_i = 0$, where $n = \dim \mathfrak{h}$. The condition that $M^\vee$ is $S(\mathfrak{h}^*)$-torsion is equivalent to $\lim_{i \to -\infty} i^{1-n} \dim (M^\vee)^i = 0$. Since $\dim (M^\vee)^i = \dim M^{-i}$, we are done. □

Lemma 19.5. *Let M be ∇-filtered. Then* $\mathrm{Hom}_{\mathcal{O}}(M, N) \twoheadrightarrow \mathrm{Hom}_{D(\mathfrak{h}^{Reg})\#W}(\pi(M), \pi(N))$.

Proof. By the construction,

$$\mathrm{Hom}_{D(\mathfrak{h}^{Reg})\#W}(\pi(M), M') = \mathrm{Hom}_{H_c}(M, M')$$

and so what we need to show is that

$$\mathrm{Hom}_{H_c}(M, N) \to \mathrm{Hom}_{D(\mathfrak{h}^{Reg})\#W}(\pi(M), \pi(N)) = \mathrm{Hom}_{H_c}(M, \pi(N))$$

is surjective. This will follow if we check that $\mathrm{Hom}_{H_c}(M, \pi(N)/N) = 0$. For any vector $v \in \pi(N)$ there is $m > 0$ such that $\delta^m v \in N$. So $\pi(N)/N$ is torsion. By Lemma 19.4, any ∇-filtered object has no torsion quotients. So $\mathrm{Hom}_{H_c}(M, \pi(N)/N) = 0$. □

An object in $\mathcal{O}$ is called *tilting* if it is both Δ- and ∇-filtered (the filtrations are different, in general).

Corollary 19.6. *If M, N are tilting, then* $\mathrm{Hom}_{\mathcal{O}}(M, N) \xrightarrow{\sim} \mathrm{Hom}_{D(\mathfrak{h}^{Reg})\#W}(\pi(M), \pi(N))$.

19.4. **Homological/Ringel duality.** Generally, projectives are not tilting. Indeed, any simple module occurs in the head of some projective, while only torsion-free simple can occur in the head of a tilting. Projectives and tiltings are again kind of dual (but not via the naive duality, it sends tiltings to tiltings, and projectives to injectives). To make this statement precise we note that, according to Problem 19.3, a Δ-filtered object M is tilting iff $\operatorname{Ext}^1(\Delta(E), M) = 0$ for all E.

Problem 19.4. *A Δ-filtered object M is projective iff $\operatorname{Ext}^1(M, \Delta(E)) = 0$ for all E.*

So what we need is an exact contravariant duality between the categories of Δ-filtered objects (these are *exact categories*, in particular, the notions of an exact sequence and so of Ext^1 make sense), maybe with different parameters, that is compatible with the localization functors. We note that such duality cannot be exact on the whole category $\mathcal{O}_c$ (then it would map projectives to injectives, not to tiltings). Such duality exists for all highest weight categories and is known as a *Ringel* duality. In our particular case, this duality is a standard homological duality that makes sense for all rings.

Namely, let A be an associative algebra with unit. Then we have the functor $\operatorname{Hom}_A(\bullet, A)$ from the category of left A-modules to the category of right A-modules. This functor behaves well on free or, more generally, projective A-modules (in particular, it is involutive) has no other good properties (for example, it often sends a module to 0). To remedy this, one considers the derived functor $\operatorname{RHom}(\bullet, A)$ from the category of finitely generated left A-modules to the derived category of right A-modules. By definition, this functor maps an A-module M to the complex $\operatorname{Hom}_A(F_0, A) \to \operatorname{Hom}_A(F_1, A) \to \ldots$, where $\ldots \to F_1 \to F_0$ is a free resolution of M and the A^{opp}-action on the complex is given by the action of A on A from the right. In particular, if A has finite homological dimension, then the functor induces a contravariant equivalence $D : D^b(A) \to D^b(A^{opp})$, where $D^b(A)$ stands for the bounded (=of complexes of finite length) derived category of the category of left A-modules. We remark that $D^2 = \mathrm{id}$.

The algebra H_c has finite homological dimension, by Problem 9.6. We can view right H_c-modules as left $H_{c^\vee}$-modules and so we get a contravariant equivalence $D : D^b(H_c) \to D^b(H_{c^\vee})$ with $D^2 = 1$. By the construction, D commutes with the localization and so $D \circ \pi_c = \pi_{c^\vee} \circ D$.

Let us compute $D(\Delta_c(E))$.

Lemma 19.7. *We have $H^i(D(\Delta_c(E))) = 0$ if $i \neq n$ and $H^n(D(\Delta_c(E))) = \Delta_{c^\vee}(E')$ for certain E'.*

Proof. Of course, $H^i(D(\Delta_c(E))) = \operatorname{Ext}^i_{H_c}(\Delta_c(E), H_c)$. We have $\Delta_c(E) = H_c \otimes_{S(\mathfrak{h})\#W} (E)$. Since $H_c \otimes_{S(\mathfrak{h})\#W} \bullet$ is an exact functor, and its right adjoint functor (the restriction from H_c to $S(\mathfrak{h})\#W$) is also exact, we see that $\operatorname{Ext}^i_{H_c}(\Delta_c(E), H_c) = \operatorname{Ext}^i_{S(\mathfrak{h})\#W}(E, H_c)$. By the triangular decomposition (written in the opposite order), H_c is a free left $S(\mathfrak{h})\#W$-module. So $\operatorname{Ext}^i_{S(\mathfrak{h})\#W}(E, H_c) = \operatorname{Ext}^i_{S(\mathfrak{h})\#W}(E, S(\mathfrak{h})\#W) \otimes_{S(\mathfrak{h})\#W} H_c$. Since a $S(\mathfrak{h})\#W$-linear map is the same as a W-equivariant and $S(\mathfrak{h})$-linear map, we see that

$$\operatorname{Ext}^i_{S(\mathfrak{h})\#W}(E, S(\mathfrak{h})\#W) = \operatorname{Ext}^i_{S(\mathfrak{h})}(E, S(\mathfrak{h})\#W)^W = \left(\operatorname{Hom}(E, \mathbb{C}W) \otimes \operatorname{Ext}^i_{S(\mathfrak{h})}(\mathbb{C}, S(\mathfrak{h}))\right)^W$$

The $S(\mathfrak{h})$-module $\operatorname{Ext}^i_{S(\mathfrak{h})}(\mathbb{C}, S(\mathfrak{h}))$ is 0 if $i \neq n$ and is $\bigwedge^n \mathfrak{h}^*$ if $i = n$. To see this one uses the Koszul resolution of the $S(\mathfrak{h})$-module $\mathbb{C}$. We conclude that $\operatorname{Ext}^i_{H_c}(\Delta_c(E), H_c) = 0$ for $i \neq n$ and $\operatorname{Ext}^i_{H_c}(\Delta_c(E), H_c) = E'' \otimes_{S(\mathfrak{h})\#W} H_c$, where $E'' = [\operatorname{Hom}(E, \mathbb{C}W) \otimes \bigwedge^n \mathfrak{h}^*]^W$ is an irreducible right W-module. So we have $H^i(D(\Delta_c(E))) = 0$ if $i \neq n$ and $H^n(D(\Delta_c(E))) =$

$\Delta_{c^\vee}(E')$ for an irreducible left W-module E' (equal to $\bigwedge^n \mathfrak{h}^* \otimes E^*$, but we will not need this). $\square$

From now on, we replace D with $D[n]$, so that $D(\Delta_c(E))$ is concentrated in the homological degree 0. Since an abelian category is a full subcategory of its derived category (of all complexes with cohomology only in degree 0), we conclude that we have a contravariant equivalence $D : \mathcal{O}_c^\Delta \to \mathcal{O}_{c^\vee}^\Delta$ (the superscript Δ stands for the category of Δ-filtered objects) that maps standards to standards. By problems 1 and 2, this equivalence has to map tiltings to projectives and vice versa.

Now we are in position to finish the proof of Theorem 19.2.

Proof of Theorem 19.2. Let P_1, P_2 be projectives in $\mathcal{O}_c$. Then, since $\pi \circ D_{H_c} = D_{D(\mathfrak{h}^{Reg})\#W} \circ \pi$, we have the following commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{O}_c}(P_1, P_2) & \longrightarrow & \mathrm{Hom}_{\mathcal{O}_{c^\vee}}(D(P_2), D(P_1)) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{D(\mathfrak{h}^{Reg})\#W}(\pi(P_1), \pi(P_2)) & \longrightarrow & \mathrm{Hom}_{D(\mathfrak{h}^{Reg})\#W}(\pi\, D(P_2), \pi\, D(P_1)) \end{array}$$

The horizontal arrows in this diagram are induced by D, which is an equivalence of derived categories, and so are isomorphisms. The right vertical arrow is an isomorphism by Corollary 19.6. So the left vertical arrow is an isomorphism. $\square$

19.5. **Quotient property.** We claim that the functor π admits a right adjoint π^* that maps a finitely generated $D(\mathfrak{h}^{Reg})\#W$-module N to the sum of all its submodules lying in $\mathcal{O}$. The natural isomorphism $\mathrm{Hom}_{H_c}(M, \pi^*(N)) \cong \mathrm{Hom}_{H_c}(\pi(M), N) = \mathrm{Hom}_{D(\mathfrak{h}^{Reg})\#W}(\pi(M), N)$ is clear, the only thing we need to check is that $\pi^*(N)$ is finitely generated. By Problem 19.1, $\dim \mathrm{Hom}_{D(\mathfrak{h}^{Reg})\#W}(\pi(M), N) < \infty$. Applying this to the projective $P(E)$, we see that $\dim \mathrm{Hom}_{H_c}(P(E), \pi^*(N)) < \infty$. So $\pi^*(N)$ has finite length and hence is in $\mathcal{O}$.

Since $\pi(M)/M$ is torsion, we see that $\pi^*\,\pi(M)/M$ is torsion and so $\pi\,\pi^*\,\pi(M) = \pi(M)$. So for $N \in \mathrm{im}\,\pi$ we have $\pi\pi^*(N) = N$. It follows that $\mathrm{im}\,\pi$ is isomorphic to the quotient category $\mathcal{O}/\mathcal{O}^{tor}$ and π is the quotient functor (the quotient functor is one satisfying a universality property for all functors that kills a given Serre subcategory). We will describe the image of π in the following lecture.

Since $\mathcal{O}$ is equivalent to the category of finite dimensional representations of some finite dimensional algebra, say A, the functor π can be described as follows. Let e be an idempotent in A corresponding to the projective that covers the torsion-free simples. Then $\mathrm{im}\,\pi$ is the category of modules over eAe and the functor π is $M \mapsto eM$.

The following problem shows that π is fully faithful on injectives and has left adjoint.

Problem 19.5. (1) *Show that the double centralizer property is equivalent to* $\mathrm{Ext}^1(M, P) = 0$ *for any projective* P *and* $M \in \mathcal{O}^{tor}$.
(2) *Use the naive duality to show that* π *is fully faithful on injectives.*
(3) *Show that* π *has left adjoint* $\pi^!$ *and that* $\pi \circ \pi^!$ *is the identity on the image of* π.

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LECTURES ON SYMPLECTIC REFLECTION ALGEBRAS

IVAN LOSEV

20. KZ functor, II: image

20.1. **Some general facts about D_X-modules.** Let X be a smooth algebraic variety and let D_X denote the sheaf of differential operators on X. Consider the subcategory $\operatorname{Loc}(X) \subset D_X$-mod consisting of all D_X-modules that are coherent sheaves on X. This is a Serre subcategory in D_X-mod. As we have seen in the previous lecture, all objects L in $\operatorname{Loc}(X)$ are vector bundles. To specify a D_X-module structure on L means to provide a map $\operatorname{Vect}_X \otimes L \to L$ or equivalently $\nabla : L \to L \otimes \Omega^1$, where Ω^1 is the sheaf of 1-forms. The compatibility condition between $\mathcal{O}_X$- and Vect_X-actions on L mean that ∇ is a connection. The condition that the map $\operatorname{Vect}_X \otimes L \to L$ defines a Lie algebra action means that the connection ∇ is flat. So $\operatorname{Loc}(X)$ is a category of vector bundles equipped with a flat connection (to be called, shortly, *flat bundles*).

There is an exact functor $M \mapsto M^\nabla$ of "taking flat sections" (i.e., sections σ with $\nabla\sigma = 0$) from $\operatorname{Loc}(X)$ to the category $\pi_1(X)$-mod of finite dimensional representations of the fundamental group $\pi_1(X)$. This functor is constructed as follows. One can show that any point $x \in X$ has a neighborhood U (in the usual complex-analytic topology), where a connection ∇ trivializes (we have seen a formal version of this statement in the previous lecture), this version follows from the existence and uniqueness of solutions of analytic differential equations. This means that $L_U \cong \underline{L} \otimes \mathcal{O}_D^{an}$, where $\underline{L}$ is a vector space, and under this identification $\nabla(\ell \otimes f) = \ell \otimes df$. The space L is uniquely recovered as the space of flat sections of L on U. So for all points $x' \in U$ we can canonically identify the fibers L_x and $L_{x'}$ (via their identification with $\underline{L}$). It follows that for two arbitrary points x, x' and a curve γ starting at x and ending at x' we can define a linear transformation $\mu_\gamma : L_x \to L_{x'}$ by identifying the fibers along the curve. Homotopic curves define the same transformation (that's again some not very difficult fact from DE). In particular, for $x = x'$ we have a representation of $\pi_1(X)$ in L_x (the monodromy representation). We take this representation for L^∇.

A similar construction, of course, can be done for complex analytic flat vector bundles. In that setting, the functor $M \mapsto M^\nabla$ is an equivalence of categories (take a representation V of $\pi_1(X)$ form a trivial $\pi_1(X)$-equivariant bundle on the universal cover $\tilde{X}$ of X with fiber V and the trivial connection and then push it forward to X). This construction is not algebraic and so does not produce an equivalence in the algebraic category that we need (it is unclear whether the bundle we get is algebraic, and two non-isomorphic algebraic flat bundles can become isomorphic as analytic flat bundles; for example, for $X = \mathbb{C}$, we have non-isomorphic connections $\nabla_1 = d, \nabla_1 = d + dz$ on $\mathcal{O}_X$, while the fundamental group is trivial).

This can be fixed if we restrict to a special class of flat bundles, those with *regular singularities*. Namely, consider a smooth proper variety $\bar{X}$ containing X as an open subset (that always exists but is not unique, in general). Let $Y_1, \ldots, Y_r$ be all divisors in $\bar{X} \setminus X$. Pick a general point $x_i \in Y_i$ and consider its small analytic neighborhood U. Let $z = 0$ be the equation of Y_i in this neighborhood. Pick a transversal line to Y_i passing through x_i, we can view z as a coordinate on this line. We can restrict the bundle and the connection to the line getting a flat bundle $\tilde{L}$ on a punctured disk $D^\times = D \setminus \{0\}$. Shrinking the disk, we may assume that that $\tilde{L}$ is a free module over $\mathcal{O}^{an}_{D^\times}$, let L_0 be a subspace of $\tilde{L}$ with $\tilde{L} = L_0 \otimes \mathcal{O}^{an}_{D^\times}$. Then we can write the connection as $\nabla = d + \sum_{i=-j}^{+\infty} z^i dz A_i$ with $A_i \in \operatorname{End}_{\mathbb{C}}(L_0)$ meaning that $\nabla(f \otimes \ell) = df \otimes l + f \sum_{i=-j}^{+\infty} z^i dz A_i \ell$. We say that the connection ∇ is regular along Y_i if there is L_0 such that $A_i = 0$ for $i < -1$. We say that ∇ is *regular* if it is regular along any divisor Y_i. According to Deligne, this is independent of the choice of a compactification $\bar{X}$.

Let $\operatorname{Loc}_{rs}(X)$ denote the category of all flat bundles with regular singularities. Then, according to Deligne, the functor $M \mapsto M^\nabla$ is an equivalence $\operatorname{Loc}_{rs}(X) \xrightarrow{\sim} \pi_1(X)$-mod. Also we would like to remark that $\operatorname{Loc}_{rs}(X)$ is a Serre subcategory in $\operatorname{Loc}(X)$.

Let us explain how to compute μ_γ for a loop γ around 0 in D. Fixing L_0, we have $\nabla = d + A_{-1}\frac{dz}{z} + \ldots$, where $\ldots$ means the regular part of the connection form that does not contribute to the monodromy (again, some fact from complex analysis) and so we may assume that it equals to 0. By the definition, μ_γ it is constructed as follows: we consider the differential equation $df + \frac{f}{z} dz \cdot A_{-1} = 0$, it has a multiple valued (matrix-valued) solution $f = z^{-A} = \exp(-A \ln(z))$. If we parameterize γ as usual: $\gamma(t) = \exp(2\pi\sqrt{-1}t)$, then $f(\exp(2\pi\sqrt{-1}t) = \exp(-2\pi\sqrt{-1}A_{-1}t)$. Then $\mu_\gamma = f(1) = \exp(-2\pi\sqrt{-1}A_{-1})$.

For example, the monodromy of the connection (=differential equation) $d - \frac{dz}{2z}$ on the trivial bundle of rank 1 on $\mathbb{C}$ is given by $\mu_\gamma = -1$ (and a multi-valued flat section is $\sqrt{z}$).

20.2. **Cherednik case.** Let us return back to the situation of interest. We have a complex reflection group W, its reflection representation $\mathfrak{h}$ and a parameter $c : S \to \mathbb{C}$, where S is the subset of W consisting of all complex reflections. This gives rise to the rational Cherednik algebra H_c and its category $\mathcal{O}$. In the last lecture we have introduced the localization functor $\pi : \mathcal{O} \to \operatorname{Loc}^W(\mathfrak{h}^{Reg})$, where the latter stands for the category of W-equivariant flat bundles on $\mathfrak{h}^{Reg}$.

We claim that $\operatorname{im} \pi \subset \operatorname{Loc}^W_{rs}(\mathfrak{h}^{Reg})$. Since $\operatorname{Loc}^W_{rs}(\mathfrak{h}^{Reg})$ is a Serre subcategory in $\operatorname{Loc}^W(\mathfrak{h}^{Reg})$, if is enough to show that the connection on $\pi(\Delta(E))$ has regular singularities (any simple is the quotient of $\Delta(E)$ and any other object has finite Jordan-Hölder series). We will do this in a simple example when $\dim \mathfrak{h} = 1$ and W is a cyclic group.

The algebra in this case is the quotient of $\mathbb{C}\langle x, y\rangle \# \mathbb{Z}_\ell$, where we write $\mathbb{Z}_\ell = \mathbb{Z}/\ell\mathbb{Z}$, by $[y, x] = 1 - 2\sum_{i=1}^{\ell-1} c_i \gamma^i$, where γ is the generator of $\mathbb{Z}_\ell$ acting on $\mathfrak{h}$ by $\eta := \exp(2\pi\sqrt{-1}/\ell)$. Let E_i be the irreducible $\mathbb{Z}_\ell$-module, where s acts by η^i. Then $\Delta(E_i) = \mathbb{C}[x]$, where x acts by multiplication by x, s acts on x^j by η^{i-j}. Finally, y acts by the Dunkl operator $\partial_x + \sum_{j=1}^{\ell-1} \frac{2c_j}{(1-\eta^{-j})x}(s^j - 1)$. We trivialize $\pi(\Delta_c(E_i)) = \mathcal{O}_{\mathfrak{h}^{Reg}} \otimes E_i$. The Dunkl operator annihilates E_i and so ∂_x acts on E_i by

$$-\sum_{j=1}^{\ell-1} \frac{2c_j}{(1-\eta^{-j})x}(\eta^{ij} - 1).$$

Therefore the connection is given by

$$\nabla = d - \sum_{j=1}^{\ell-1} \frac{2c_j}{(1-\eta^{-j})}(\eta^{ij} - 1)\frac{dx}{x}.$$

We compactify $\mathfrak{h}^{Reg} = \mathbb{C}^\times$ by embedding it into $\mathbb{P}^1$ in a standard way. The 1-form in the connection has poles of order 1 at both 0 and ∞. So it has regular singularities.

Problem 20.1. *Write the connection on* $\pi(\Delta(E))$ *and show that it lies in* $\mathrm{Loc}_{rs}(\mathfrak{h}^{Reg})$ *in the general case.*

To apply the flat sections functor we want a usual category of local systems rather than the equivariant one. It turns out that we can replace $\mathrm{Loc}^W(\mathfrak{h}^{Reg})$ with $\mathrm{Loc}(\mathfrak{h}^{Reg}/W)$. Namely, recall that the W-action on $\mathfrak{h}^{Reg}$ is free.

Problem 20.2. *Let* $\rho : \mathfrak{h}^{Reg} \to \mathfrak{h}^{Reg}/W$ *be the quotient morphism. Show that the functors* $M \mapsto \rho_*(M)^W$ *and* ρ^* *define mutually (quasi)-inverse equivalences between the following pairs of categories:*

(1) $\mathcal{O}_{\mathfrak{h}^{Reg}}$-$\mathrm{mod}^W$ *and* $\mathcal{O}_{\mathfrak{h}^{Reg}/W}$-mod.
(2) $\mathrm{Coh}^W(\mathfrak{h}^{Reg})$ *and* $\mathrm{Coh}(\mathfrak{h}^{Reg}/W)$.
(3) $D_{\mathfrak{h}^{Reg}}\#W$-mod *and* $D_{\mathfrak{h}^{Reg}/W}$-mod.
(4) $\mathrm{Loc}^W(\mathfrak{h}^{Reg})$ *and* $\mathrm{Loc}(\mathfrak{h}^{Reg}/W)$.

It is not really difficult to see (and quite easy to believe) that under the identification $\mathrm{Loc}^W(\mathfrak{h}^{Reg}) \cong \mathrm{Loc}(\mathfrak{h}^{Reg}/W)$ we have $\mathrm{Loc}^W_{rs}(\mathfrak{h}^{Reg}) = \mathrm{Loc}_{rs}(\mathfrak{h}^{Reg}/W)$ (a hint: $\frac{dt^r}{t^r} = r\frac{dt}{t}$).

Let $\tilde{\mathrm{KZ}}$ denote the composition of $\pi : \mathcal{O}_c \to \mathrm{Loc}_{rs}(\mathfrak{h}^{Reg}/W)$ and the flat section functor $\mathrm{Loc}_{rs}(\mathfrak{h}^{Reg}/W) \to \pi_1(\mathfrak{h}^{Reg}/W)-\mathrm{mod}_{f.d.}$. At this point we know that $\tilde{KZ}$ is an equivalence of $\mathcal{O}/\mathcal{O}^{tor}$ and $\operatorname{im}\tilde{\mathrm{KZ}} \subset \pi_1(\mathfrak{h}^{Reg}/W)$-$\mathrm{mod}_{f.d.}$ and this equivalence is fully faithful on projectives.

20.3. **Braid groups.** Our goal is to describe $\pi_1(\mathfrak{h}^{Reg}/W)$. The most classical case here is $W = \mathfrak{S}_n$, where the fundamental group is the usual braid group B_n given by the generators $T_1, \ldots, T_{n-1}$ subject to the relations: $T_iT_j = T_jT_i, |i-j| > 1$, and $T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}$. The generators T_i are obtained as follows. Pick a point p close to the hyperplane $x_i = x_{i+1}$. Then take a small semi-circle γ_i connecting p and s_ip. Then T_i corresponds to the image of γ_i under the quotient morphism.

This has a classical generalization to the case of an arbitrary Coxeter group W: we pick a fundamental chamber for W, let $\alpha_1, \ldots, \alpha_m$ be the equations of its walls. Then $B_W := \pi_1(\mathfrak{h}^{Reg}/W)$ is given by the generators $T_1, \ldots, T_m$ subject to the relations of the form $T_iT_jT_i \ldots = T_jT_iT_j \ldots$, where the number of factors in each part is the integer m_{ij} such that the angle between α_i, α_j is $\pi - \frac{\pi}{m_{ij}}$ (in particular, if α_i, α_j are orthogonal, then T_i and T_j commute). For example, in type B_n, the group B_W is generated by the $T_0, \ldots, T_{n-1}$ such that $T_1, \ldots, T_{n-1}$ satisfy the braid relations in type A, T_0 commutes with T_i for $i > 0$, and $T_0T_1T_0T_1 = T_1T_0T_1T_0$.

This can be generalized to the case of arbitrary complex reflection groups. We are not going to provide the construction of B_W in this case. Let us only mention that the group B_W is generated by elements T_i corresponding to some hyperplanes $\ker \alpha_s$. In the case of $W = G(\ell, 1, n) = \mathfrak{S}_n \ltimes \mathbb{Z}_\ell^n$ (this is the case of most interest for us) the group B_W is the same as for $\ell = 2$ (and the hyperplanes that give rise to the generators $T_0, \ldots, T_{n-1}$ are $x_1 = 0, x_1 = x_2, \ldots, x_{n-1} = x_n$). The generator T_i corresponds to an arc of angle $\frac{2\pi}{\ell}$ around $x_1 = 0$. In particular (and this is very easy to see), for $n = 1$, the braid group is just $\mathbb{Z}$.

20.4. **Hecke algebras.** It turns out that all $\mathbb{C}B_W$-modules of the form $\tilde{\mathrm{KZ}}(M)$ for $M \in \mathcal{O}_c$ factor through a certain quotient of $\mathbb{C}B_W$ depending on c, the Hecke algebra $\mathcal{H}_c(W)$. To motivate the definition we compute $\tilde{\mathrm{KZ}}(\Delta(E_i))$ for the cyclic group. The corresponding representation of B_W is in the fiber of $L := \pi(\Delta(E_i))$ at some point, i.e., in the one-dimensional space E_i.

In our example, a connection on $\mathfrak{h}^{Reg} = \mathbb{C}^\times$ is on the trivial bundle of rank 1 and has $A_{-1} = -k_i$, where

$$k_i := \sum_{j=1}^{\ell-1} \frac{2c_j}{1-\eta^{-j}}(\eta^{ij}-1). \tag{1}$$

We notice that $k_0 = 0$.

So the monodromy will be given by $\exp(2\pi\sqrt{-1}k_i)$. However, we need not the monodromy for this flat bundle, but rather the monodromy for the induced flat bundle on $\mathbb{C}^\times/\mathbb{Z}_\ell$. The fiber of the induced bundle in $\rho(1) \in \mathbb{C}^\times/\mathbb{Z}_\ell$ can be identified with each of the fibers of L at the points η^j, the induced identification $L_1 \to L_{\eta^j}$ is given by s^j, i.e., by the multiplication with η^{ij}. A preimage of the parameterized unit circle γ in $\mathbb{C}^\times/\mathbb{Z}_\ell$ is the curve $\exp(2\pi\sqrt{-1}t/\ell)$ that connects 1 to η. So $\mu_\gamma = \eta^{-i}\exp(2\pi\sqrt{-1}k_i/\ell) = q_i$, where

$$q_i = \exp(2\pi\sqrt{-1}(k_i - i)/\ell) \tag{2}$$

So if T is the generator of $B_{\mathbb{Z}_\ell} = \mathbb{Z}$ corresponding to γ, we see that the action of $\mathbb{C}\mathbb{Z}$ on $\pi(\Delta_c(E_i))$ is zero on the element $\prod_{i=0}^{\ell-1}(T-q_i)$.

For us, this is a motivation of a Hecke algebra $\mathcal{H}_c(W)$ in the general case. Namely, for each hyperplane H of the form $\ker\alpha_s$ the point-wise stabilizer W_H of H in W is a cyclic group of order say ℓ_H. Pick a generator, s, in this group that acts on $\mathfrak{h}/\ker\alpha_s$ by η. Then define the number $k_{H,i}$ similarly to (1) where $c_j := c_{s^j}$. Then define $q_{H,i}$ by (2). Denote by $\underline{q}$ the collection of numbers $q_{H,i}$ (the number of a priori different q's is the same as the number of c's). Then define $\mathcal{H}_{\underline{q}}(W)$ as the quotient of $\mathbb{C}B_W$ by the relations

$$(T_H - 1)\prod_{j=1}^{\ell_H-1}(T_H - q_{H,j}). \tag{3}$$

We write $\mathcal{H}_c(W)$ for $\mathcal{H}_{\underline{q}}(W)$, where $\underline{q}$ is defined by c as above.

Let us consider the case when $W = \mathfrak{S}_n$. Here we have only one conjugacy class of hyperplanes H with $\ell_H = 2$. We have $k_{H,1} = -2c$ and $q_{H,1} = -\exp(2\pi\sqrt{-1}c)$. If we set $q := -q_{H,1}$ and rescale all generators T_i by -1, then we will recover the classical Hecke algebra $\mathcal{H}_q(n)$ of type A: the quotient of $\mathbb{C}B_n$ by the relations $(T_i - q)(T_i + 1) = 0$. For other Weyl groups and constant functions c we recover the usual Iwahori-Hecke algebras (with the same additional relations) that appear in Lie theory (say in the study of the representations of finite reductive groups). For non-constant c we recover their straightforward generalization (Hecke algebras with unequal parameters).

We will need the case when $W = G(\ell, 1, n)$. In this case we have two conjugacy classes of H: the class containing $x_1 = x_2$ and the class containing $x_1 = 0$. Let $q_0 = 1, \ldots, q_{n-1}$ be the parameters corresponding to the second class. Then the Hecke algebra $\mathcal{H}_c(W)$ is quotient of $\mathbb{C}B_W$ by the relations $(T_i - q)(T_i + 1) = 0$ for $i = 1, \ldots, n-1$ and $\prod_{i=0}^{\ell-1}(T_0 - q_i)$. If we drop the last relation, we will get the affine Hecke algebra $\mathcal{H}_q^{aff}(n)$ of type A. The Hecke algebra $\mathcal{H}_c(W) = \mathcal{H}_q^{aff}(n)/\prod_{j=0}^{\ell-1}(T_0 - q_j)$ is called the cyclotomic Hecke algebra of type A (and level ℓ).

Of course, $\mathcal{H}_0(W) = \mathbb{C}W$. Conjecturally, $\dim\mathcal{H}_c(W) = |W|$ for all c and all W. This is known in both examples considered above. We will assume this as a hypothesis.

20.5. **Image of $\tilde{\mathrm{KZ}}$.**

Theorem 20.1. (1) *For any $M \in \mathcal{O}_c$ the action of $\mathbb{C}B_W$ on $\tilde{\mathrm{KZ}}(M)$ factors through $\mathcal{H}_c(W)$.*

(2) *The induced functor* $\mathrm{KZ} : \mathcal{O}_c \to \mathcal{H}_c(W)$-mod *is essentially surjective (so giving rise to an equivalence $\mathcal{O}_c/\mathcal{O}_c^{tor} \cong \mathcal{H}_c(W)$*-mod*).*

In the proof of (1) we will need the following lemma.

Lemma 20.2. *If c is generic (=away from countably many hyperplanes), then $\mathcal{O}_c$ is a semisimple category with simples $\Delta_c(E)$.*

Proof. We have seen that if $L(E')$ is a composition factor of $\Delta(E)$ different from the irreducible quotient $L(E)$ or if there is a nontrivial extension of $\Delta(E')$ by $\Delta(E)$, then $c_{E'} - c_E \in \mathbb{Z}_{>0}$. So to prove the statements of the lemma it is enough to show that, for c generic, $c_{E'} - c_E$ cannot be a nonzero integer. Recall that

$$c_E := \frac{1}{2}\dim\mathfrak{h} - \sum_{s\in S}\frac{2}{1-\lambda_s}c_s s|_E.$$

When $c = 0$, we have $c_E = 0$ for all E. Since c_E is an affine function in c, our claim follows. □

Proof of Theorem 20.1. It is enough to prove (1) when M is projective. As we have seen any indecomposable projective is a direct summand of the module $\Delta_m(E) := H_c \otimes_{S(\mathfrak{h})\#W} E \otimes S(\mathfrak{h})/(\mathfrak{h}^m)$, so it is enough to prove (1) for those modules. The connection on $\pi(\Delta_m(E))$ and hence the operators corresponding to the generators T_H on $\tilde{\mathrm{KZ}}(\Delta_m(E))$ depend continuously on c. So it is enough to prove (1) when c is generic. Here $\Delta_m(E)$ splits into the direct sum of Verma modules and so it is enough to prove the claim for those. This is done by a direct computation similar to what we have done before (the result will be reproved later independently).

Problem 20.3. *Show that the action of $\mathbb{C}B_W$ on $\tilde{KZ}(\Delta_c(M))$ factors through $\mathcal{H}_c(W)$.*

Let us proceed to (2). Since the functor KZ is exact it is isomorphic to a functor $\mathrm{Hom}_{\mathcal{O}}(P_{KZ}, \bullet)$ for some projective P_{KZ}, where the action of $\mathcal{H} := \mathcal{H}_c(W)$ on KZ(?) comes from a homomorphism $\phi : A := \mathcal{H}^{opp} \to B := \mathrm{End}_{\mathcal{O}}(P_{KZ})^{opp}$. What we need to show is that ϕ is an isomorphism.

We will do it in two steps. In this lecture we will show that ϕ is surjective. In the next lecture, we will compute the dimension of B and see that it equals $|W|$. According to our hypothesis, $\dim A = |W|$. This will imply that ϕ is an isomorphism.

We claim that the surjectivity of ϕ is equivalent to the claim that $\mathrm{im}\,\mathrm{KZ}$ is closed under taking subobjects. Indeed, the latter is equivalent to a similar statement on the pull-back functor ϕ^*, i.e., to the claim that any A-submodule in a B-module is a B-submodule. In particular, $\mathrm{im}\,\phi$ is an A-submodule in B containing 1 so it has to be B. Therefore ϕ is surjective.

Since $\tilde{\mathrm{KZ}}$ is obtained by composing the localization functor π with a category equivalence, to check that $\mathrm{im}\,\mathrm{KZ}$ is closed under taking subobjects, it suffices to show a similar claim for $\mathrm{im}\,\pi$.

So let us take $M \in \mathcal{O}_c$ and take a subobject $N \subset M[\delta^{-1}]$. We may assume that M is torsion free by modding out the torsion part. So $M \subset M[\delta^{-1}]$. The subobject $M \cap N$ of M satisfies $\pi(M) = N$. Indeed, for any $n \in N$ there is $k \in \mathbb{Z}_{>0}$ such that $\delta^k n \in M$ and hence $\delta^k n \in M \cap N$. We conclude that $(M \cap N)[\delta^{-1}] = N$. □

References

[GGOR] V. Ginzburg, N. Guay, E. Opdam and R. Rouquier, *On the category $\mathcal{O}$ for rational Cherednik algebras*, Invent. Math., **154** (2003), 617-651.

SRA. Lec 21.

0) Reminder.

1) Compl. of proof of Thm

2) Induction & Restrictions for $\mathcal{H}_q(W)$

3) Isomorphism of completions for RCA

0) $M\in \mathcal{O}_c \rightsquigarrow$ Sheaf $\pi(M)\in Coh(\mathfrak{h}^{W,Reg}) \rightsquigarrow M'\in Coh(\mathfrak{h}^{Reg}/W)$

- accounts for $S(\mathfrak{h}^*)$ & $\mathbb{C}W$-actions,

$\mathfrak{h}$-action $\xrightarrow{\text{flat}}$ conn. ∇ on $M' \rightsquigarrow B_W := \pi_1(\mathfrak{h}^{Reg}/W, x) \curvearrowright M'_x$ (nearby fibers of M are canon. ident.) $B_W = \langle T_H,\ H := \ker \alpha_s\rangle$ / rel-ns

$H \rightsquigarrow W_H$ w. $\ell_H = |W_H|$, $c: S\to W \rightsquigarrow q_{H,i}\in \mathbb{C}^\times\setminus\{0\}$, $q_{H_0}=1$

$\rightsquigarrow \mathcal{H}_c(W) = \mathbb{C}B_W/(\prod_{j=0}^{\ell_H-1}(T_H - q_{H,j}))$ $\quad \mathcal{H}_0(W)=\mathbb{C}W$, Hypoth. $\dim \mathcal{H}_c(W)=|W|$

$\mathcal{H}_c(W)\curvearrowright M_x$. $\quad \dim M_x$ = gen. rk of M as $S(\mathfrak{h}^*)$-module

$KZ: M\mapsto M_x$, $KZ = \mathrm{Hom}_{\mathcal{O}}(P_{KZ},\bullet)$, P_{KZ}-proj. w.

$\varphi: \mathcal{H}_c(W)\longrightarrow End_{\mathcal O}(P_{KZ})$. Have seen φ is surjective

Remains $\dim \underbrace{End_{\mathcal O}(P_{KZ})}_{B:=} = |W|$.

1) mult of $P(E)$ in $P_{KZ} = \dim_{\mathcal O}(P_{KZ}, L(E)) = \dim KZ(L(E))$

$$\dim B = \sum_{E,E'} \dim KZ(L(E))\, KZ(L(E'))\, \dim Hom_{\mathcal O}(P(E),P(E'))$$

$$\dim Hom_{\mathcal O}(P(E),P(E')) = [P(E'):L(E)] = \sum_{E''}[P(E'):\Delta(E'')][\Delta(E''):L(E)]$$

Δ-filt

$$\dim B = \sum_{E''}\Big(\underbrace{\sum_E \dim KZ(L(E))[\Delta(E''):L(E)]}_{\substack{= \dim KZ(\Delta(E''))\text{ b/c }KZ\text{ is exact}\\ = \dim E''}}\Big)\Big(\sum_{E'}\dim KZ(L(E'))[P(E'):\Delta(E'')]\Big)$$

claim: $[\nabla(E''):L(E')]$

Reason: M – Δ-filt.

$[M:\Delta(E'')] = Hom_{\mathcal O}(M, \nabla(E''))$ (Prob. 19.2)

$M = P(E')$: $= Hom_{\mathcal O}(P(E'),\nabla(E'')) = [\nabla(E''):L(E')]$

So 2nd $(\dots) = \dim KZ(\nabla(E''))$

Claim: gen. rk $\nabla(E'')$ = gen. rk $\Delta(E'')$

Reason: $M = \bigoplus_i M^i$ -graded / $\mathbb{C}[x_1..x_n]$, then gen. rk $M = \lim_{i\to+\infty} i^{1-n}\dim M_i$

Then compare w. Lem 19.4.

So $\dim B = \sum_{E''} (\dim E'')^2 = |W|$.

2) Classic: $H \subset G$ – fin. gr-ps: Res^H_G: G-Rep $\to$ H-Rep,

Ind_H^G: H-Rep $\to$ G-Rep: $N \mapsto \mathbb{C}G \otimes_{\mathbb{C}H} N$

Coind_H^G: H-Rep $\to$ G-Rep $N \mapsto \mathrm{Hom}_{\mathbb{C}H}(\mathbb{C}G, N)$

So Res^H_G, Ind_H^G – exact biadjoint functors

Goal: version for Hecke algebras:

$b \in \mathfrak{h} \rightsquigarrow \underline{W} := W_b \subset W$ – also gen. by refl-ns

refl-n rep-n $\mathfrak{h}_{\underline{W}}$ = unique $\underline{W}$-stab. compl. in $\mathfrak{h}^{\underline{W}}$.

compl. refl-ns in $\underline{W} = S \cap \underline{W}$: $c: S \to \mathbb{C} \rightsquigarrow c: S \cap \underline{W} \to \mathbb{C}$

$\rightsquigarrow \mathcal{H}_c(\underline{W})$

Lem: $\mathcal{H}_c(\underline{W}) \hookrightarrow \mathcal{H}_c(W)$

Sketch of proof: $B_{\underline{W}} \hookrightarrow B_W$: $\exists$ neigh-d of Wb = disc $\times$ neigh-d of 0 in $\mathfrak{h}_{\underline{W}}/\underline{W}$ so loop in $\mathfrak{h}_{\underline{W}}^{reg}/\underline{W} \rightsquigarrow$ loop in $\mathfrak{h}^{reg}/W$: $T_{\underline{H}} \mapsto T_H \rightsquigarrow \mathcal{H}_c(\underline{W}) \to \mathcal{H}_c(W)$ – injective $\square$

Fact (enhancement of hypoth.) $\mathcal{H}_c(W)$ is free left/right module / $\mathcal{H}_c(\underline{W})$

(Will elaborate for $W = G(\ell, 1, n)$ later)

Lem: $\rightsquigarrow$ $\mathrm{Res}^{\underline{W}}_W : \mathcal{H}_c(W)\text{-mod} \to \mathcal{H}_c(\underline{W})$-mod, $\mathrm{Ind}^W_{\underline{W}}$, $\mathrm{Coind}^W_{\underline{W}}$: $\mathcal{H}_c(\underline{W})$-mod $\to \mathcal{H}_c(W)$-mod

Hypothesis: $\mathcal{H}_c(W)$ is symmetric i.e. $\mathcal{H}_c(W) \underset{\mathcal{H}_c(W)\text{-bimod}}{\simeq} \mathcal{H}_c(W)^* \Rightarrow \mathrm{Ind}^W_{\underline{W}} \simeq \mathrm{Coind}^W_{\underline{W}}$

– holds for Weyl groups & all $G(\ell, r, n)$

– will elaborate for $G(\ell, 1, n)$ later

Goal: similar functors for H_c: $\mathrm{Res}^{\underline{W}}_W : \mathcal{O}_c(W, \mathfrak{h}) \to \mathcal{O}_c(\underline{W}, \underline{\mathfrak{h}})$

$\mathrm{Ind}^W_{\underline{W}} : \mathcal{O}_c(\underline{W}, \underline{\mathfrak{h}}) \longrightarrow \mathcal{O}_c(W, \mathfrak{h})$

– exact, biadjoint & satisfying $KZ \circ \mathrm{Res}^{\underline{W}}_W = \mathrm{Res}^{\underline{W}}_W \circ KZ \underset{adj}{\Leftrightarrow} KZ \circ \mathrm{Ind}^W_{\underline{W}} = \mathrm{Ind}^W_{\underline{W}} \circ KZ$.

Cherednik Hecke

Problem: $H_c(\underline{W}, \mathfrak{h})$ is not subalg. in $H_c(W, \mathfrak{h})$ (and this wouldn't help anyway)

Fix: use isomorphism of completions (Bezrukavnikov-Etingof)

3) $b\in\mathfrak{h} \rightsquigarrow b:\mathbb{C}[\mathfrak{h}]\to\mathbb{C} \rightsquigarrow$ restr to $b: \mathbb{C}[\mathfrak{h}/W], \mathbb{C}[\mathfrak{h}/\underline{W}]\to\mathbb{C}$ $(\underline{W}=W_b)$

$\mathbb{C}[\mathfrak{h}]^{\wedge b}$, $\mathbb{C}[\mathfrak{h}/W]^{\wedge b}$, $\mathbb{C}[\mathfrak{h}/\underline{W}]^{\wedge b}$. since $\mathfrak{h}/\underline{W}\to\mathfrak{h}/W$ is etale at $\underline{W}b$:

$\mathbb{C}[\mathfrak{h}/W]^{\wedge b} = \mathbb{C}[\mathfrak{h}/\underline{W}]^{\wedge b}$

Q: How about $\mathbb{C}[\mathfrak{h}]\#W$ & $\mathbb{C}[\mathfrak{h}]\#\underline{W}$ or more precisely

$(\mathbb{C}[\mathfrak{h}]\#W)^{\wedge b} := \mathbb{C}[\mathfrak{h}/W]^{\wedge b}\otimes_{\mathbb{C}[\mathfrak{h}/W]}\mathbb{C}[\mathfrak{h}]\#W = \mathbb{C}[\mathfrak{h}]^{\wedge Wb}\#W \leftarrow (\bigoplus_{v\in Wb}\mathbb{C}[\mathfrak{h}]^{\wedge v})\#W$

and $(\mathbb{C}[\mathfrak{h}]\#\underline{W})^{\wedge b} = \mathbb{C}[\mathfrak{h}/\underline{W}]^{\wedge b}\otimes_{\mathbb{C}[\mathfrak{h}/\underline{W}]}\mathbb{C}[\mathfrak{h}]\#\underline{W} = \mathbb{C}[\mathfrak{h}]^{\wedge b}\#\underline{W}$ - not as unital subalg

Rather: $(\mathbb{C}[\mathfrak{h}]\#W)^{\wedge b}\simeq \mathrm{Mat}_{|W/\underline{W}|}(\mathbb{C}[\mathfrak{h}]^{\wedge b}\#\underline{W})$

More invariant: $H\subset G$ - fin. groups, $A\supset\mathbb{C}H$ - ass. alg. w. 1 $\rightsquigarrow \mathrm{Hom}_H(\mathbb{C}G, A)$ ~~free right A-module~~ = ~~left H-module~~

$= \{\varphi:\mathbb{C}G\to A \mid \varphi(hg)=h\varphi(g)\}$

free right A-module trivialized by choice of el-ts in all Hg.

$\rightsquigarrow \mathcal{Z}(G,H,A) := \mathrm{End}_{A^{opp}}(\mathrm{Hom}_H(\mathbb{C}G,A)) \simeq \mathrm{Mat}_{|G/H|}(A)$

Lem: $(\mathbb{C}[\mathfrak{h}]\#W)^{\wedge b}\simeq \mathcal{Z}(W,\underline{W},\mathbb{C}[\mathfrak{h}]^{\wedge b}\#\underline{W})$

Proof: Need compat. homom-sms $\mathbb{C}[\mathfrak{h}], \mathbb{C}W\to\mathcal{Z}(W,\underline{W},\mathbb{C}[\mathfrak{h}]\#\underline{W})$

General: $G\to\mathcal{Z}(G,H,A)$ - need $G \curvearrowright \mathrm{Hom}_H(\mathbb{C}G,A)$ commuting w. A

$g.\varphi(h)=\varphi(hg)$: need $f\in\mathbb{C}[\mathfrak{h}]$, $\varphi\in\mathrm{Hom}_H(\mathbb{C}G,A)\rightsquigarrow f\cdot\varphi$

$(f\cdot\varphi)(w) = ({}^{w}f)\cdot\varphi(w)$

Problem ~~Exercise~~*: gives $\mathbb{C}[\mathfrak{h}]\#W\longrightarrow\mathcal{Z}(W,\underline{W},\mathbb{C}[\mathfrak{h}]\#\underline{W})$ that lifts to iso

$(\mathbb{C}[\mathfrak{h}]\#W)^{\wedge b}\xrightarrow{\sim}\mathcal{Z}(W,\underline{W},\mathbb{C}[\mathfrak{h}]^{\wedge b}\#\underline{W})$.

Finally on the level of RCA: $H_c = H_c(W,\mathfrak{h})$, $\underline{H}_c = H_c(\underline{W},\mathfrak{h})$

$H_c^{\wedge b} := \mathbb{C}[\mathfrak{h}/W]^{\wedge b}\otimes_{\mathbb{C}[\mathfrak{h}/W]} H_c$ - algebra w. multiplication extended from H_c by continuity: reason $\mathfrak{m}_b\subset\mathbb{C}[\mathfrak{h}/W]$ - max ideal $\Rightarrow$

$[y,\mathfrak{m}_b^k]\subset\mathfrak{m}_b^{k-1}$ $\forall y\in\mathfrak{h}$. $\mathbb{C}[\mathfrak{h}]\#W$

$\underline{H}_c^{\wedge b} := \mathbb{C}[\mathfrak{h}/\underline{W}]^{\wedge b}\otimes_{\mathbb{C}[\mathfrak{h}/\underline{W}]}\underline{H}_c$.

Thm (Bezrukavnikov-Etingof): $\exists!$ iso $\theta: H_c^{\wedge b}\xrightarrow{\sim}\mathcal{Z}(W,\underline{W},\underline{H}_c^{\wedge b})$

~~given on~~ restricting to isom $(\mathbb{C}[\mathfrak{h}]\#W)^{\wedge b}\xrightarrow{\sim}\mathcal{Z}(W,\underline{W},\mathbb{C}[\mathfrak{h}]^{\wedge b}\#\underline{W})$

and $y\longmapsto\theta(y)$ s.t. $[\theta(y)\varphi](w) = (wy)\varphi(w) + \sum_{s\in S\setminus\underline{W}} \frac{2c_s}{1-\lambda_s}\frac{\langle\alpha_s, wy\rangle}{\alpha_s}\cdot$

$\cdot(\varphi(sw)-\varphi(w))$

Problem: prove thm.

Addit. summand in $\theta(y)$ is really a part of Dunkl operator "outside of $\underline{W}$" – that's how they obtain the formula

Rem: $\underline{H}^{\wedge_b} \xrightarrow{\sim} D(\mathfrak{h}^{\underline{W}})^{\wedge_b} \hat{\otimes} \underline{H}_c(\underline{W}, \mathfrak{h}_{\underline{W}})^{\wedge_0}$

allow some infinite sums

$\underline{H}^{\wedge_b} \downarrow \wr$ $\underline{H}^{\wedge_0}$ – shift by b possible b/c b is $\underline{W}$-equiv.

To produce $Res^W_{\underline{W}}$, $Ind^W_{\underline{W}}$ (functors depending on choice of b) introduce intermediate category $O_c^{\wedge_b} = \{$modules / $H_c^{\wedge_b}$ fin. gen over $\mathbb{C}[\mathfrak{h}]^{\wedge W b}\}$

Next time: equivalence $\iota: O_c^{\wedge_b} \xrightarrow{\sim} O_c(\underline{W}, \mathfrak{h}_{\underline{W}})$.

completion functor: $\bullet^{\wedge_b}: O_c \to O_c^{\wedge_b}$: $M \mapsto \mathbb{C}[\mathfrak{h}/W]^{\wedge_b} \otimes_{\mathbb{C}[\mathfrak{h}/W]} M$

Its right adjoint: $E_0: O_c^{\wedge_b} \to O_c$: $N \mapsto$ gen. e-space of $\mathfrak{h}$ w. e-value 0 in N.

$Res^W_{\underline{W}} = \iota \circ (\bullet^{\wedge_b})$, $Ind^W_{\underline{W}} = E_0 \circ \iota^{-1}$ – not clear so far why image lies in O_c (why fin. generated)

SRA Lecture 22
Ind & Res: continued

0) Reminder.

1) Equivalence $\mathcal{O}_c^{\wedge_b} \xrightarrow{\sim} \underline{\mathcal{O}}_c^{+}$

2) Exactness of Ind

3) Properties of Res & Ind

0) Reminder: $b\in\mathfrak{h} \rightsquigarrow \underline{W}=W_b$, $H_c^{\wedge_b} = \mathbb{C}[\mathfrak{h}/W]^{\wedge_b}\otimes_{\mathbb{C}[\mathfrak{h}/W]} H_c$

$\underline{H}_c = H_c(\mathfrak{h},\underline{W})$, $\underline{H}_c^{\wedge_b} = \mathbb{C}[\mathfrak{h}/\underline{W}]^{\wedge_b}\otimes_{\mathbb{C}[\mathfrak{h}/\underline{W}]}\underline{H}_c$.

$\rightsquigarrow Z(W,\underline{W},\underline{H}_c^{\wedge_b}) = End_{\underline{H}_c^{\wedge_b}}(Hom_{\underline{W}}(\mathbb{C}W, \underline{H}_c^{\wedge_b}))$

Have iso $H_c^{\wedge_b}\xrightarrow[\theta]{\sim} Z(W,\underline{W},\underline{H}_c^{\wedge_b})$

$[\theta(w)\varphi](w') = \varphi(w'w)$ $\quad \varphi\in Hom_{\underline{W}}(\mathbb{C}W,\underline{H}_c^{\wedge_b})$, $w'\in W$, $w\in W$,

$[\theta(x)\varphi](w') = w'x\cdot\varphi(w')$ $\quad x\in W$

$[\theta(y)\varphi](w')$ = some formula: $w'y\cdot\varphi(w')$ + correction

$\mathcal{O}_c^{\wedge_b} := \{M\in H_c^{\wedge_b}\text{-mod} \mid M \text{ is fin. gen}/\mathbb{C}[\mathfrak{h}/W]^{\wedge_b}\}$

Functors: $\bullet^{\wedge_b}: \mathcal{O}_c\longrightarrow\mathcal{O}_c^{\wedge_b}: M\longmapsto \mathbb{C}[\mathfrak{h}/W]^{\wedge_b}\otimes_{\mathbb{C}[\mathfrak{h}/W]}M$ - exact

$E_0: \mathcal{O}_c^{\wedge_b}\longrightarrow\tilde{\mathcal{O}}_c = \{M\in H_c\text{-Mod} \mid \mathfrak{h} \text{ acts loc. nilp}\}: N\longmapsto$ gen e-space for $\mathfrak{h}$ w. e-value 0.

+ will construct $\iota: \mathcal{O}_c^{\wedge_b}\xrightarrow{\sim}\underline{\mathcal{O}}_c^{+} := \mathcal{O}_c(\mathfrak{h}_{\underline{W}},\underline{W})$

$\rightsquigarrow Res_{\underline{W}}^{W}: \mathcal{O}_c\longrightarrow\underline{\mathcal{O}}_c^{+}: M\longmapsto\iota(M^{\wedge_b})$

$Ind_{\underline{W}}^{W}: \underline{\mathcal{O}}_c^{+}\longrightarrow\tilde{\mathcal{O}}_c: N\longmapsto E_0(\iota^{-1}(N))$

$$\mathcal{O}_c^{\wedge_b}\xrightarrow{\sim}\mathcal{O}^{\wedge_b}_{Z(W,\underline{W},\underline{H}_c^{\wedge_b})}\xrightarrow{\sim}\underline{\mathcal{O}}_c^{\wedge_b}\xrightarrow{\sim}\underline{\mathcal{O}}_c^{+\wedge_0}\xrightarrow{\sim}\underline{\mathcal{O}}_c^{+}$$

1) a) Equiv. $H_c^{\wedge_b}\text{-mod} \xrightarrow{\sim} Z(W,\underline{W},\underline{H}_c^{\wedge_b})\text{-mod}: \theta_*$.

maps $\mathcal{O}_c^{\wedge_b}$ to $\{M\in Z(W,\underline{W},\underline{H}_c^{\wedge_b})\text{-mod} \mid M \text{ is fin. gen}/\mathbb{C}[\mathfrak{h}/\underline{W}]^{\wedge_b}\} = \mathcal{O}^{\wedge_b}_{Z(W,\underline{W},\underline{H}_c^{\wedge_b})}$

b) Equiv. $Z(W,\underline{W},\underline{H}_c^{\wedge_b})\text{-mod} \xrightarrow{\sim} \underline{H}_c^{\wedge_b}\text{-mod}$:

$\underbrace{e(\underline{W})}_{\text{matrix unit}}\in \underbrace{Z(W,\underline{W},\underline{H}_c^{\wedge_b})}_{\text{matrix alg.}/\underline{H}_c^{\wedge_b}}$, $\quad e(\underline{W})\varphi(w) = \begin{cases}\varphi(w), & w\in\underline{W}\\ 0, & \text{else}\end{cases}$ $\quad M\longmapsto e(\underline{W})M$

c) Equivalence $\underline{\mathcal{O}}_c^{\wedge_b} \xrightarrow{\sim} \underline{\mathcal{O}}_c^{+\wedge_0}$

$\underline{H}_c \xrightarrow{\sim} \mathcal{D}(\mathfrak{h}^W)\otimes_{\mathbb{C}} \underline{H}_c^+ \to \underline{H}_c^{\wedge_b} \xrightarrow{\sim} \mathcal{D}(\mathfrak{h}^W)^{\wedge_b}\hat{\otimes}\, \underline{H}_c^{+\wedge_0}$

allow some infinite sums

$M\in \underline{\mathcal{O}}_c^{\wedge_b}$ is compl. & sep. in $\underline{m}_b$-adic topol., where $\underline{m}_b$ is max. ideal of b in $\mathbb{C}[\mathfrak{h}/\underline{W}] \Rightarrow$ compl. & sep in $\hat{\underline{m}}_b$-adic topol., $\hat{\underline{m}}_b \in \mathbb{C}[\mathfrak{h}^{\underline{W}}]$

$\mathbb{C}[\mathfrak{h}/\underline{W}] = \mathbb{C}[\mathfrak{h}^{\underline{W}}]\otimes\mathbb{C}[\mathfrak{h}_{\underline{W}}/\underline{W}]$

$\Downarrow$ Prop. 19.1

$$M \simeq \mathbb{C}[\mathfrak{h}^W]^{\wedge_b}\otimes M'$$

M' = Annih. of $\mathfrak{h}^W$, obj. in $\underline{\mathcal{O}}^{+\wedge_0}$.

Equiv. $M\mapsto M'$ (inverse $M'\mapsto \mathbb{C}[\mathfrak{h}^W]^{\wedge_b}\hat{\otimes} M'$)

d) Equiv. $\underline{\mathcal{O}}_c^{+\wedge_0}\xrightarrow{\sim}\underline{\mathcal{O}}_c^+$

$\longleftarrow$: $N\mapsto N^{\wedge_0}$

$\longrightarrow$: $M\mapsto$ finite vectors for Euler element.

Problem 1: Check these two are (quasi) inverse to each other.

Corollary of existence of i: All Hom spaces in $\mathcal{O}_c^{\wedge_b}$ are fin. dim $\Longrightarrow$ $E_0(M)$ has fin. length $\forall\, M\in\mathcal{O}_c^{\wedge_b}$ (b/c $[\mathrm{Hom}_{H_c}(M', E_0(M)) = \mathrm{Hom}_{H_c}(M'^{\wedge_b}, M)]$) $\Rightarrow E_0: \mathcal{O}_c^{\wedge_b}\to\mathcal{O}_c$ is right adjoint to $\bullet^{\wedge_b}$.

2) Exactness of Ind $\Leftrightarrow$ of E_0 | Recall $M\mapsto M^\vee$ = fin. vectors in $M^* = \mathrm{Hom}(M,\mathbb{C})$

- $\bullet^*: M\to \mathrm{Hom}(M,\mathbb{C})$: $\mathcal{O}_c(\mathfrak{h},W)\xrightarrow{\sim}\mathcal{O}_{c^\vee}(\mathfrak{h}^*,W)^{\wedge_0}$
- $\bullet^*: N\to\mathrm{Hom}_{cont}(N,\mathbb{C}) = \{\varphi: N\to\mathbb{C},\ \varphi((\mathfrak{h})^n N)=0,\ n\gg0\}$: $\mathcal{O}_{c^\vee}(\mathfrak{h}^*,W)^{\wedge_0}\xrightarrow{\sim}\mathcal{O}_c(\mathfrak{h},W)$
- $\bullet^*: N\to\mathrm{Hom}_{cont}(N,\mathbb{C})$: $\mathcal{O}_c(\mathfrak{h},W)^{\wedge_b}\xrightarrow{\sim}\mathcal{O}_{c^\vee}(\mathfrak{h}^*,W)_b =$

= full subcat. of $H_{c^\vee}$-mod that are

- finite length
- ~~fin. gen all Ann's of m_b^n are fin dim~~
- $\mathbb{C}[\mathfrak{h}/W]$ acts w. gen. e-value b.

} will see that this implies fin gen / $\mathbb{C}[\mathfrak{h}^*]$ – insert (*) here

- $\bullet^*: M\to\mathrm{Hom}(M,\mathbb{C})$: $\mathcal{O}_{c^\vee}(\mathfrak{h}^*,W)_b\xrightarrow{\sim}\mathcal{O}_c(\mathfrak{h},W)^{\wedge_b}$] $\longleftarrow$ cor. of (*)

Observation: $*$ intertwines E_b & $\bullet^{\wedge_b}$: $(M^{\wedge_b})^* = E_b(M^*)$ $(M \in \mathcal{O}_c(\mathfrak{h}, W))$.
$(M^{\wedge_{0,y}})^* = E_0(M^*)$, $M \in \mathcal{O}_{c^\vee}(\mathfrak{h}^*, W)_b$.
So $E_0(N)^* = (N^*)^{\wedge_{0,y}}$ $(N \in \mathcal{O}_c^{\wedge_b} \Rightarrow N^* \in \mathcal{O}_{c^\vee}(\mathfrak{h}^*, W)_b)$.

(*) ~~Problem Establish equivalence~~ $\cancel{\mathcal{O}_{c^\vee}(\mathfrak{h}^*, W)_b \xrightarrow{\sim} \mathcal{O}_{c^\vee}(\mathfrak{h}^*_{\underline{W}}, \underline{W})}$ ~~using~~ $\cancel{\mathcal{O}}$.
~~Cor.~~ $\cancel{M \in \mathcal{O}_{c^\vee}(\mathfrak{h}^*, W)_b}$ ~~has fin length~~;
all simples are fin. gen / $\mathbb{C}[\mathfrak{h}^*] \# W$ (by spaces of "sing. vectors")
$\Rightarrow$ all objects in $\mathcal{O}_{c^\vee}(\mathfrak{h}^*, W)_b$ are fin. gen / $\mathbb{C}[\mathfrak{h}^*]$.

~~So~~ $\cancel{\bullet^{\wedge_b}}$: Functor $N \mapsto (N^*)^{\wedge_{0,y}}$ is exact $\Longrightarrow E_0(N)$ is exact.

3) Properties of Res & Ind

3.1) Behavior on K_0

Formal character: $M = \bigoplus_\alpha M_\alpha$ - gen. e-space for Euler element h (W-module)
$\rightsquigarrow \operatorname{ch} M = \sum_\alpha [M_\alpha]_W \cdot e^\alpha$ $\leftarrow$ class in $K_0(W\text{-mod})$

Problem 2: • $\operatorname{ch} M_1 = \operatorname{ch} M_2 \Longleftrightarrow [M_1] = [M_2]$ - classes in $K_0(\mathcal{O}_c)$.
• $\operatorname{ch} \Delta(E) = \operatorname{ch} \nabla(E)$.

Identify $K_0(\mathcal{O}_c)$ w. $K_0(W\text{-mod})$ via $[\Delta_c(E)] \longmapsto [E]_W$.

Prop (Bezrukavnikov-Etingof). $[\operatorname{Res}_{\underline{W}}^W] = [\operatorname{Res}_{\underline{W}}^W]_W$, $[\operatorname{Ind}_{\underline{W}}^W] = [\operatorname{Ind}_{\underline{W}}^W]_W$.

Proof: Res: $\Delta_c(E) \xmapsto[\wedge_b]{} \mathbb{C}[\mathfrak{h}]^{\wedge_{Wb}} \otimes E \xmapsto[e(\underline{W})]{} \mathbb{C}[\mathfrak{h}]^{\wedge_b} \otimes E \longmapsto$
$\mathbb{C}[\mathfrak{h}_{\underline{W}}]^{\wedge_0} \otimes E \longmapsto \mathbb{C}[\mathfrak{h}_{\underline{W}}] \otimes E$. - as $\mathbb{C}[\mathfrak{h}_{\underline{W}}] \# \underline{W}$-module

Problem 3: let $M \in \mathcal{O}_c$ be such that $M \simeq \mathbb{C}[\mathfrak{h}] \otimes E$ $(E \in W\text{-mod})$, then $[M] = [\Delta_c(E)]$ and M is Δ-filtered
$\Rightarrow [\operatorname{Res}_{\underline{W}}^W] = [\operatorname{Res}_{\underline{W}}^W]_W$.

For Ind's: define $(\cdot,\cdot)$ on $K_0(\mathcal{O}_c)$ by $([M],[N]) = \sum_i (-1)^i \dim \operatorname{Ext}^i(M,N)$
- well-def.; $([\Delta(E)], [\Delta(E')]) = ([\Delta(E)], [\nabla(E')]) = \delta_{E,E'}$.

So $K_0(\mathcal{O}_c) \xrightarrow{\sim} K_0(W\text{-mod})$ preserves $(\cdot,\cdot)$; $\operatorname{Ind}_{\underline{W}}^W$ is exact & right adj. to $\operatorname{Res}_{\underline{W}}^W \Longrightarrow [\operatorname{Ind}_{\underline{W}}^W]$ is adj-t to $[\operatorname{Res}_{\underline{W}}^W]$. Since $[\operatorname{Ind}_{\underline{W}}^W]_W$ is adj-t to $[\operatorname{Res}_{\underline{W}}^W]_W$, we are done by above $\square$

Rem: Functors $\mathrm{Ind}_{\underline{W}}^{W}$, $\mathrm{Res}_{\underline{W}}^{W}$ are defined using b: different choices of b (w. $W_b = \underline{W}$) give isomorphic functors ($\exists$ flat connection)

3.2) Ind/Res & KZ

Thm (Shan) $\mathrm{Ind}_{\underline{W}}^{W} \circ \underline{KZ} = KZ \circ \mathrm{Ind}_{\underline{W}}^{W}$, $\mathrm{Res}_{\underline{W}}^{W} \circ KZ = \underline{KZ} \circ \mathrm{Res}_{\underline{W}}^{W}$.

Will use: $\mathrm{End}(\mathrm{Ind}_{\underline{W}}^{W})$ same for $\mathcal{H}_c(W)$-mod & $\mathcal{O}_c$ - b/c

KZ is fully faithful on projectives

3.3) $\mathrm{Ind}_{\underline{W}}^{W}$ & $\mathrm{Res}_{\underline{W}}^{W}$ on $\mathcal{O}_c$ are biadjoint (cor. of 3.2 – Shan)

3.4) $\mathrm{Ind}_{\underline{W}}^{W}$ & $\mathrm{Res}_{\underline{W}}^{W}$ preserve Δ-filt. objects (I.L.)

Next time: categorical actions on $\mathcal{O}_c$ for $G(\ell,1,n)$

$\mathcal{O}_c = \bigoplus_{n=0}^{+\infty} \mathcal{O}_c(G(\ell,1,n))$

~~Recall $q \in \mathbb{C}^\times$ - q not root of 1 ~ action of $\mathfrak{sl}_\infty$~~

~~q-primit. e-th root of 1~~

Functors: direct summands of $\bigoplus_n \mathrm{Res}_n^{n-1}: \mathcal{O}_c(G(\ell,1,\bullet)) \to \mathcal{O}_c(G(\ell,1,\bullet-1))$

$\bigoplus_n \mathrm{Ind}_n^{n-1}: \mathcal{O}_c(G(\ell,1,\bullet)) \to \mathcal{O}_c(G(\ell,1,\bullet+1))$

SRA, Lec 23.

Categorical Kac-Moody actions.

1) Cyclotomic Hecke algebras
2) Categorical $\hat{\mathfrak{sl}}_e$-actions
3) Category $\mathcal{O}$.

1.1) Affine HA: Braid group of type $G(\ell,1,n)$: $B^{aff}(n) = \langle T_0, \ldots T_{n-1}\rangle$ mod rel-ns:
$T_iT_j = T_jT_i$, $T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}$, $i>0$, $T_0T_1T_0T_1 = T_1T_0T_1T_0$
$q \in \mathbb{C}^\times \rightsquigarrow \mathcal{H}_q^{aff}(n) = \mathbb{C}B^{aff}(n)/((T_i - q)(T_i+1), i=1,\ldots n-1)$.

Alt. presentation: $X_1 = T_0$, $X_2 = q^{-1}T_1X_1T_1$, $X_3 = q^{-1}T_2X_2T_2, \ldots X_n = q^{-1}T_{n-1}X_{n-1}T_{n-1}$
Rel-ns: $X_iX_j = X_jX_i$ ($i=1, j=2$: $T_0T_1T_0T_1 = T_1T_0T_1T_0$), $T_iX_iT_i = qX_{i+1}$
$T_iX_j = T_jX_i$ ($i-j \neq 0,1$) + X_i is invertible
~~Fact~~ So $\mathcal{H}_q^{aff}(n) = \mathbb{C}\langle T_1, \ldots T_{n-1}, X_1^{\pm1}, \ldots X_n^{\pm1}\rangle$/rel-ns for T_i's + rel-ns for X_i's & T_i's above

Important formula: $p \in \mathbb{C}[X_1^{\pm1}, \ldots X_n^{\pm1}] \rightsquigarrow s_i(p)(X_1, \ldots X_n) = p(X_1, \ldots X_{i+1}, X_i, \ldots X_n)$

(1) $$T_ip - s_i(p)T_i = (q-1)\frac{p - s_i(p)}{1 - X_iX_{i+1}^{-1}}$$

Cor: $\mathbb{C}[X_1^{\pm1}, \ldots X_n^{\pm1}]^{\mathfrak{S}_n}$ is central

Basis: ~~Prop~~ $w \in \mathfrak{S}_n$, $w = s_{i_1} \ldots s_{i_k}$ - reduced expression (= min k)
$\rightsquigarrow T_w = T_{i_1} \ldots T_{i_k}$ - well-defined

Prop: El-ts $X_1^{m_1} \ldots X_n^{m_n} T_w$, ($m_1, \ldots m_n \in \mathbb{Z}$, $w \in \mathfrak{S}_n$) - basis in $\mathcal{H}_q^{aff}(n)$
(same true for $T_wX_1^{m_1} \ldots X_n^{m_n}$)

Sketch of proof: i): prove that el-ts span $\mathcal{H}_q^{aff}(n)$
ii) $\mathcal{H}_q^{aff}(n) \curvearrowright \mathbb{C}[X_1^{\pm1}, \ldots X_n^{\pm1}]$ (using (1) w. $T_i \cdot 1 = q$)
el-ts act by lin. indep. operators $\square$

Cor: $\mathcal{H}_q^{aff}(n-1) \hookrightarrow \mathcal{H}_q^{aff}(n)$ ($T_i \mapsto T_i$, $X_i \mapsto X_i$)

1.2) Cyclotomic HA: $\underline{q} = (q_0, \ldots q_{\ell-1}) \in (\mathbb{C}^\times)^\ell \rightsquigarrow \mathcal{H}_q^{\underline{q}}(n) = \mathcal{H}_q^{aff}(n)/\prod_{i=0}^{\ell-1}(X_1 - q_i)$
Thm (Ariki-Koike) $X_1^{m_1} \ldots X_n^{m_n} T_w$ ($0 \le m_i \le \ell-1$, $\forall i$, $w \in \mathfrak{S}_n$)
is basis in $\mathcal{H}_q^{\underline{q}}(n)$

Cor: $\mathcal{H}_q^{\underline{q}}(n-1) \hookrightarrow \mathcal{H}_q^{\underline{q}}(n)$

Trace: $\tau: H_q^{\underline{q}}(n) \longrightarrow \mathbb{C}$ $\quad \tau(X_1^{m_1} \dots X_n^{m_n} T_w) = \delta_{m_1,0} \dots \delta_{m_n,0}\, \delta_{w,1}$.

Thm (Malle, Mathas) $(A,B) := \tau(AB)$ is symm. non-deg. form

Cor: $A \mapsto (A,\cdot)$ is iso $\mathcal{H}_q^{\underline{q}}(n) \xrightarrow[\text{Bimod}]{\sim} \mathcal{H}_q^{\underline{q}}(n)^*$ ($\mathcal{H}_q^{\underline{q}}(n)$ is symmetr. alg.).

1.3) Res & its decomp-n. (due to Ariki)

From now on: q = primit. $\sqrt[e]{1}$, $q_i = q^{s_i}$; $s_i \in \mathbb{Z}$, $\mathcal{H}_q^{\underline{s}}(n) := \mathcal{H}_q^{\underline{q}}(n)$

$\mathcal{M} \in \mathcal{H}_q^{\underline{s}}(n)$-mod: want study $\mathcal{M}$ induct-ly -restr. to $\mathcal{H}_q^{\underline{s}}(n-1)$

$[X_n, \mathcal{H}_q^{\underline{s}}(n-1)] = 0 \Rightarrow X_n \circlearrowright \mathcal{M}$ by $\mathcal{H}_q^{\underline{s}}(n-1)$-lin. endom-s

$(1) \Longrightarrow$ e-vals $\in \{\sqrt[e]{1}\}$ (proved by ind. on n)

$i \in \mathbb{Z}/e\mathbb{Z} \rightsquigarrow E_i(\mathcal{M})$ = gen e-space for X_n w. e-val q^i

E_i - exact endof-r of $\mathcal{C} = \bigoplus_{n=0}^{+\infty} \mathcal{C}_n$, $\mathcal{C}_n = \mathcal{H}_q^{\underline{s}}(n)$-mod ($E_i|_{\mathcal{C}_0} = 0$).

$\bigoplus_{i \in \mathbb{Z}/e\mathbb{Z}} E_i = E \left(= \bigoplus_{n=0}^{+\infty} \mathrm{Res}_n^{n-1}\right)$

Rem: Have $X \in \mathrm{End}(E)$ - coming from X_n.

$T \in \mathrm{End}(E^2)$, $E^2 = \bigoplus \mathrm{Res}_n^{n-2}$, coming from T_{n-1}.

$X \rightsquigarrow 1_E X, X 1_E \in \mathrm{End}(E^2)$ ($1_E X$ acts by X_n, $X 1_E$ by X_{n-1})

and sim. $1_E T, T 1_E \in \mathrm{End}(E^3)$ satisf.

$(1_E X)(X 1_E) = (X 1_E)(1_E X)$ in $\mathrm{End}(E^2)$

$T(X 1_E) T = q(1_E X)$ in $\mathrm{End}(E^2)$

$(T-q)(T+1) = 0$ — . — . —.

$(T 1_E)(1_E T)(T 1_E) = (1_E T)(T 1_E)(1_E T)$ in $\mathrm{End}(E^3)$

$\rightsquigarrow$ ~~[illegible]~~ $\mathcal{H}_q^{\mathrm{aff}}(m) \longrightarrow \mathrm{End}(E^m)$ - alg. homom.

1.4) Ind & its decomp

$F = \bigoplus_{n=0}^{+\infty} \mathrm{Ind}_n^{n+1}$ - left & right adj. to E

$E = \bigoplus E_i \rightsquigarrow$ right adjunction $F = \bigoplus_{i \in \mathbb{Z}/e\mathbb{Z}} F_i$

Claim: F_i is also left adj-t of E_i:

Proof: $Z = X_1 \dots X_n \in$ center of $\mathcal{H}_q^{aff}(n)$, $M \in \mathcal{H}_q^{\underline{q}}(n) \rightsquigarrow M_\alpha := \{m \in M \mid (Z-\alpha)^N m = 0,\ m \gg 0\} \rightsquigarrow M = \bigoplus_\alpha M_\alpha$ - decomp. into $\mathcal{H}_q^{\underline{q}}(n)$-mod $\rightsquigarrow \mathcal{C} = \bigoplus_\alpha \mathcal{C}_\alpha$

$\mathcal{C}_\alpha = \{M \mid M = M_\alpha\}$; ~~$E_i : M_\alpha \Rightarrow M_{q^{-i}}$~~ $E_i(M_\alpha) = E(M)_{q^{-i}\alpha}$

$\Rightarrow E_i : \mathcal{C}_\alpha \to \mathcal{C}_{q^{-i}\alpha}$; right adjointness $F_i : \mathcal{C}_\alpha \to \mathcal{C}_{q^i\alpha}$, since $\bigoplus F_i$ is left adj to $\bigoplus E_i$, see that F_i is left adj. to E_i.

1.5) K_0.

$K_0(W\text{-mod})$: $G(\ell,1,n)$-irreps $\xleftrightarrow{\sim}$ ℓ-multipartitions $\lambda = (\lambda^{(0)}, \dots, \lambda^{(\ell-1)})$ of n

$\lambda^{(i)} \rightsquigarrow$ rep-n $S_{\lambda^{(i)}}$ of $\mathfrak{S}_{|\lambda^{(i)}|} \rightsquigarrow$ rep-n $S_{\lambda^{(i)}}(i)$ of $G(\ell,1,|\lambda^{(i)}|)$

w. $\mathfrak{S}_{|\lambda^{(i)}|}$ act. as before, $\eta \in$ copy of $\mathbb{Z}_\ell$ - by η^i.

$\rightsquigarrow S_\lambda = \mathrm{Ind}^{G(\ell,1,n)}_{\prod G(\ell,1,|\lambda^{(i)}|)} \boxtimes_{i=0}^{\ell-1} S_{\lambda^{(i)}}(i)$

Tits deform argument: for $q, \underline{q}$ generic $K_0(\mathcal{H}_q^{\underline{q}}(n)\text{-mod}) = K_0(G(\ell,1,n)\text{-mod})$

$\rightsquigarrow$ irrep. S_λ of $\mathcal{H}_q^{\underline{q}}(n)$-mod $\rightsquigarrow$ ~~cl~~ well-def. class $[S_\lambda]$ for any $q, \underline{q}$.

$[F][S_\lambda] = \bigoplus_\mu [S_\mu]$ – μ obt. from λ by adding a box.

$[E][S_\lambda] = \bigoplus_{\mu'} [S_{\mu'}]$ μ' – – – – – – – removing – – –

~~A box~~ $\alpha \in \mathbb{Z}/e\mathbb{Z}$: a box in jth diagram is an α-box if

$y - x + s_j \equiv \alpha$ mod e (x = # of row, y = # column)

first prove analog for $q, \underline{q}$ generic and then specialize

$[F_i][S_\lambda] = \bigoplus_\mu [S_\mu]$ – – – – – – – – – i-box

$[E_i][S_\lambda] = \bigoplus_\mu [S_\mu]$ – – – – – – – – – – –

Can define $\hat{\mathfrak{sl}}_e$-action on space w. basis of all $\lambda = (\lambda^{(0)}, \dots, \lambda^{(\ell-1)})$ by these formulas - get level ℓ Fock space w. multicharge $\underline{s}$. $K_0(\mathcal{H}_q^{\underline{s}}\text{-mod})$ is quotient of this - irred rep-n $\underbrace{\text{w.}}_{L(\Lambda)}$ highest weight $\Lambda = \sum_{i=0}^{\ell-1} \Lambda_{s_i}$ ← fund. weight.

2) $\mathcal{C}$ – abelian, artinian cat w. enough projectives equipped w. functors E_i, F_i $(i \in \mathbb{Z}_e)$ and $X \in End(E)$, $T \in End(E^2)$ $(E = \bigoplus_{i \in \mathbb{Z}/e\mathbb{Z}} E_i)$

These data is categorical $\hat{\mathfrak{sl}}_e$-action if:

(1) E_i is biadj. to F_i

(2) E_i, F_i define $\hat{\mathfrak{sl}}_e$-action on $K_0(\mathcal{C})$, integrable

(3) class of $\overset{\text{any}}{\text{simple}}$ is a weight vector.

(4) X, T satisfy Hecke rel-ns (w. $q = e^{2\pi\sqrt{-1}\,?}$) + E_i - e-functor $\overbrace{\text{w.}}^{\text{for } T}$ e-value q^i

Rem: (3) holds for $\mathcal{C} = \bigoplus_{n=0}^{+\infty} \mathcal{H}_q^S(n)$-mod (from action of center in $\mathcal{H}_q^{aff}(n)$)

Other examples: • category $\mathcal{O}$ for $\mathfrak{gl}_n$ and its ramifications

• Cherednik cat-y $\mathcal{O}$ (below)

3) Have $\mathcal{O}(n) = \mathcal{O}_c(\mathbb{C}^n, G(\ell, 1, n))$ w. c giving $q, \underline{q}$ as before

$\mathcal{O} = \bigoplus_{n=0}^{+\infty} \mathcal{O}(n)$ w. endofunctors ${}^{\mathcal{O}}E, {}^{\mathcal{O}}F$

Have $KZ: \mathcal{O} \to \mathcal{C}$ intertw. E, F + f. faith. on $\mathcal{O}$-proj. $\rightsquigarrow$

$\mathrm{End}({}^{\mathcal{O}}E) = \mathrm{End}(E)$, $\mathrm{End}({}^{\mathcal{O}}F) = \mathrm{End}(F) \rightsquigarrow X \in \mathrm{End}(E)$

$+ \rightsquigarrow T \in \mathrm{End}(E^2)$

$\rightsquigarrow E_i, F_i$

$[KZ(\Delta_c(\lambda))] = [S(\lambda)] \rightsquigarrow$ so action of $[E_i], [F_i]$ on $[\Delta_c(\lambda)]$ as before

So $K_0(\mathcal{O}) =$ Fock space and $[KZ] =$ projection from Fock space to $L(\Lambda)$