Last time: a homotopy equivalence btw X and Y is a pair of maps

$f : X \to Y$ and $g : Y \to X$

s.t. $g \circ f \sim id_X$ and $f \circ g \sim id_Y$

Ex: suppose X is contractible
pick x_0 in X s.t. $id_X \sim$ (const map at x)
set $Y = \{x_0\}$

$f : X \to Y$ $f(x) = x_0$, the constant map
$g : Y \to X$ $g(x_0) = x_0$, the inclusion

then $(g \circ f)(x) = x_0$, so $g \circ f \sim id_X$
while $f \circ g = id_Y \sim id_Y$

so X is homotopy equivalent to $\{x_0\}$

Thm: if $f : X \to Y$ and $g : Y \to X$ form a homotopy equivalence

then $f_* : \pi_1(X, x) \to \pi_1(Y, f(x))$,
$g_* : \pi_1(Y, y) \to \pi_1(X, g(y))$

are isomorphisms for any x in X and y in Y

Rem: these examples are a specific kind of homotopy equivalence called a deformation retract

Pf of Thm: suppose $f : X \to Y$ and $g : Y \to X$ s.t. $g \circ f \sim id_X$ and $f \circ g \sim id_Y$

will show that $f_* : \pi_1(X, x) \to \pi_1(Y, f(x))$ is an iso for any x in X [argument for g_* is similar]

1) if $f : X \to Y$ and $g : Y \to Z$ are cts maps then $(g \circ f)_* = g_* \circ f_*$ [from last time]

2) if $\varphi : G \to H$ and $\psi : H \to K$ are maps s.t. $\psi \circ \varphi$ is bijective then φ is injective and ψ is surjective

3) if α is a path in X from x_0 to x_1 then $\breve{\alpha} : \pi_1(X, x_0) \to \pi_1(X, x_1)$ def by

$$\breve{\alpha}([\gamma]) = [\bar{\alpha} * \gamma * \alpha]$$

is an isomorphism

by 1),
$$g_* \circ f_* = (g \circ f)_*, \qquad f_* \circ g_* = (f \circ g)_*$$

so by 2), just need $(g \circ f)_*$ and $(f \circ g)_*$ to be isos

pick homotopies j from $g \circ f$ to id_X,
k from $f \circ g$ to id_Y

can use 3) to show: if $f, f' : A \to X$ are cts, h a homotopy from f to f', a in A,

then $f'_* = \breve{\alpha}_h \circ f_* : \pi_1(A, a) \to \pi_1(X, f'(a))$

where $\alpha_h(s) = h(a, s)$, a path from $f(a)$ to $f'(a)$

apply to j and k:

$$(g \circ f)_* = \breve{\alpha}_j \circ id_{X, *} = \breve{\alpha}_j$$
$$(f \circ g)_* = \breve{\alpha}_k \circ id_{Y, *} = \breve{\alpha}_k$$

but by 3), $\breve{\alpha}_j$ and $\breve{\alpha}_k$ are isos