ECE405C - ASSIGNMENT # 1:

Q1: 1) I'll use a truth table.

We recall from MATH135 that OR can be expressed with AND and NOT: $P \vee Q = \neg(\neg P \wedge \neg Q)$

A	B	$A \oplus B$	$(A \wedge \neg B) \vee (\neg A \wedge B)$	$\neg(\neg(A \wedge \neg B) \wedge \neg(\neg A \wedge B))$
T	F	T	$T \vee F = T$	T
T	T	F	$F \vee F = F$	F
F	F	F	$F \vee F = F$	F
F	T	T	$F \vee T = T$	T

DeMorgan's Law.

$\neg(A \wedge B) = \neg A \vee \neg B$

$\neg(A \vee B) = \neg A \wedge \neg B$

$(A \wedge \neg B) \vee (\neg A \wedge B) = \neg(\neg((A \wedge \neg B) \vee (\neg A \wedge B)))$ global double negation

$\neg(\neg(A \wedge \neg B) \wedge \neg(\neg A \wedge B))$ by DeMorgans Law

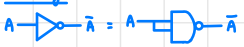
2) $\text{NAND}(A, A) = \neg(A \wedge A) = \neg A$ since $A \wedge A = A$

FANOUT lets us copy A to connect it to both NAND inputs.

↳ First: $\text{FANOUT}(A) \rightarrow (A, A)$

↳ Then: $\text{NAND}(A, A) = \neg A$

Visually:



A	$\neg A$	A NAND A
0	1	1
1	0	0

Q2: $\hat{H}_1 := \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, $\hat{H}_n = \hat{H}_1 \otimes \hat{H}_{n-1} = \frac{1}{\sqrt{2}}\begin{pmatrix} \hat{H}_{n-1} & \hat{H}_{n-1} \\ \hat{H}_{n-1} & -\hat{H}_{n-1} \end{pmatrix}$, $n \geq 2$, $\hat{H}_n \in \mathbb{R}^{2^n \times 2^n}$

1) $\hat{H}_2 = \hat{H}_1 \otimes \hat{H}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \otimes \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{2} \cdot \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$

2) The recursive definition as we've defined it makes it clear that orthonormality is preserved at every step, that is $\hat{H}_n$ is normalized $\forall\, n \in \mathbb{Z}^{1+}$. $(\hat{H}_n^T \hat{H}_n = I_{2^n}\ \forall\, n \in \mathbb{Z}^{1+})$.

I'll prove this inductively to make it clear:

Base case: $n = 1 \rightarrow \hat{H}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ has orthonormal colums by inspection $(\hat{H}_1^T \hat{H}_1 = I_2)$.

Inductive hypothesis: Assume for some $n-1 \cdot$ $\hat{H}_{n-1}^T \hat{H}_{n-1} = I_{2^{n-1}}$, where $n \geq 2$.

using the given tensor product identity $\hat{H}_n = \hat{H}_1 \otimes \hat{H}_{n-1}$:

$$\hat{H}_n^T \hat{H}_n = (\hat{H}_1 \otimes \hat{H}_{n-1})^T (\hat{H}_1 \otimes \hat{H}_{n-1})$$

$= (\hat{H}_1^T \otimes \hat{H}_{n-1}^T)(\hat{H}_1 \otimes \hat{H}_{n-1})$ since $(A \otimes B)^T = A^T \otimes B^T$

$= (\hat{H}_1^T \hat{H}_1) \otimes (\hat{H}_{n-1}^T \hat{H}_{n-1})$ since $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$

$= I_2 \otimes I_{2^{n-1}}$ by the base case and inductive hypothesis

$= I_{2^n}$ since $I_a \otimes I_b = I_{ab}$

$\therefore$ P(1) holds and $P(n-1) \Rightarrow P(n)\ \forall\, n \geq 2$. By POMI, P(n) holds $\forall\, n \geq 1$.

Hence, $\hat{H}_n^T \hat{H}_n = I_{2^n}$, so $\hat{H}_n$ is normalized and $\hat{H}_n^{-1} = \hat{H}_n^T$.

Intrestingly, $\hat{H}_n^{-1} = \hat{H}_n$ as well since $\hat{H}_n$ is symmetric $(\hat{H}_n^T = \hat{H}_n)$; I'll briefly justify below:

Base case: $\hat{H}_1$ is symmetric by inspection

Induction: $(A \otimes B)^T = A^T \otimes B^T$. If $\hat{H}_{n-1}^T = \hat{H}_{n-1}$ and $\hat{H}_1^T = \hat{H}_1$, then

$$\hat{H}_n^T = (\hat{H}_1 \otimes \hat{H}_{n-1})^T = \hat{H}_1^T \otimes \hat{H}_{n-1}^T = \hat{H}_1 \otimes \hat{H}_{n-1} = \hat{H}_n.$$

Q3. 1) $\langle \phi_0 | \phi_1 \rangle = \begin{pmatrix} 1/\sqrt{2} & 0 & 0 & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 0 \\ 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{pmatrix} = 0$

2) <u>Show that $\hat{U} \otimes \hat{I} |\phi_0\rangle = \hat{I} \otimes \hat{U}^T |\phi_0\rangle$:</u>

Let $|\phi_0\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)$ and define arbitrary $\hat{U} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. $\Rightarrow \hat{U}^T = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$

LHS:

$(\hat{U} \otimes \hat{I})|\phi_0\rangle = \frac{1}{\sqrt{2}}\left((\hat{U}\otimes\hat{I})|00\rangle + (\hat{U}\otimes\hat{I})|11\rangle\right)$

$= \frac{1}{\sqrt{2}}\left((\hat{U}|0\rangle) \otimes |0\rangle + (\hat{U}|1\rangle) \otimes |1\rangle\right)$ since $(A\otimes B)(C \otimes D) = (AC) \otimes (BD)$

$= \frac{1}{\sqrt{2}}\left(a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle\right)$ Since: $\hat{U}|0\rangle = a|0\rangle + c|1\rangle$ and $\hat{U}|1\rangle = b|0\rangle + d|1\rangle$

RHS:

$(\hat{I} \otimes \hat{U}^T)|\phi_0\rangle = \frac{1}{\sqrt{2}}\left((\hat{I}\otimes\hat{U}^T)|00\rangle + (\hat{I}\otimes\hat{U}^T)|11\rangle\right)$

$= \frac{1}{\sqrt{2}}\left(|0\rangle \otimes (\hat{U}^T|0\rangle) + |1\rangle \otimes (\hat{U}^T|1\rangle)\right)$ since $(A\otimes B)(C \otimes D) = (AC) \otimes (BD)$

$= \frac{1}{\sqrt{2}}\left(a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle\right)$ Since: $\hat{U}^T|0\rangle = a|0\rangle + b|1\rangle$ and $\hat{U}^T|1\rangle = c|0\rangle + d|1\rangle$

Since LHS = RHS, we've shown directly the vectors are equal.

Q4: 1) $B = \{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}^*$ $\Rightarrow$ first qubit is top rail, second is bottom rail.

As for "deriving" the matrix the most natural way to do this to me is determine where the operator maps the basis states, then construct the matrix. A two-qubit gate is just a linear map on $\mathcal{H}_4$. For a matrix M, the j-th column is $M|e_j\rangle$ (the output when you feed in the j^{th} basis vector).

For the <u>first circuit</u>, the bottom is control and the top is the target, so if top qubit = 1, flip bottom.

↳ The map is thus $|c,t\rangle \mapsto |c, t\oplus c\rangle$. Explicitly; $|00\rangle \mapsto |00\rangle$, $|01\rangle \mapsto |01\rangle$, $|10\rangle \mapsto |11\rangle$, $|11\rangle \mapsto |10\rangle$.

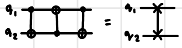
↳ <u>In the basis order above</u>*; col 1 = image of $|00\rangle = |00\rangle = (1\ 0\ 0\ 0)^T$
col 2 = image of $|01\rangle = |01\rangle = (0\ 1\ 0\ 0)^T$
col 3 = image of $|10\rangle = |11\rangle = (0\ 0\ 0\ 1)^T$
col 4 = image of $|11\rangle = |10\rangle = (0\ 0\ 1\ 0)^T$

$\Rightarrow CNOT_{1\to 2} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$

For the second circuit, our map is then $|t,c\rangle \mapsto |t\oplus c, c\rangle$: $|00\rangle \mapsto |00\rangle$, $|01\rangle \mapsto |11\rangle$, $|10\rangle \mapsto |10\rangle$, $|11\rangle \mapsto |01\rangle$.

↳ <u>In the basis order above</u>*; col 1 = image of $|00\rangle = |00\rangle = (1\ 0\ 0\ 0)^T$
col 2 = image of $|01\rangle = |11\rangle = (0\ 0\ 0\ 1)^T$
col 3 = image of $|10\rangle = |10\rangle = (0\ 0\ 1\ 0)^T$
col 4 = image of $|11\rangle = |01\rangle = (0\ 1\ 0\ 0)^T$

$\Rightarrow CNOT_{2\to 1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$

2) 

$\Rightarrow SWAP_{1,2} = CNOT_{1\to 2}\, CNOT_{2\to 1}\, CNOT_{1\to 2}$ (this was also a problem in ECE405A).

<u>For verification, consider a sanity check WLOG on bits a and b:</u>

$(a,b) \xrightarrow{CNOT_{1\to2}} (a, a\oplus b) \xrightarrow{CNOT_{2\to1}} (a\oplus(a\oplus b), a\oplus b) = (b, a\oplus b) \xrightarrow{CNOT_{1\to2}} (b, (a\oplus b)\oplus b) = (b,a)$ SWAPPED!

↳ note that multiplying out the matricies would be another valid way to show the maps are equivalent:

FROM ECE405A

b) The sequence is equivelant to the SWAP gate. In the two-qubit computational basis:

$$\underbrace{\begin{bmatrix} 1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0 \end{bmatrix}}_{CNOT_{12}} \underbrace{\begin{bmatrix} 1&0&0&0\\0&0&0&1\\0&0&1&0\\0&1&0&0 \end{bmatrix}}_{CNOT_{21}} \underbrace{\begin{bmatrix} 1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0 \end{bmatrix}}_{CNOT_{12}} = \begin{bmatrix} 1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0 \end{bmatrix} \begin{bmatrix} 1&0&0&0\\0&0&1&0\\0&0&0&1\\0&1&0&0 \end{bmatrix} = \underbrace{\begin{bmatrix} 1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1 \end{bmatrix}}_{\text{SWAP MATRIX!}}$$

3) We construct a block-diagonal matrix, like what Prof. Kim showed in class on Monday Jan 12th, 2026. If U acts on n qubits and we add 1 control qubit, the controlled-U gate acts on n+1 qubits, so the matrix size is $2^{n+1} \times 2^{n+1}$.

We'll assume the first qubit is the control and the remaining n qubits are the target register.

In the computational basis ordered as: $\left(|0\rangle \otimes x,\ |1\rangle \otimes x\ \ \forall x \in \{0,1\}^n\right)$,

the matrix is block-diagonal: $C\text{-}U = \begin{pmatrix} I_{2^n} & 0 \\ 0 & U \end{pmatrix}$.

For the simpler case described in the question $(n=1)$, using the basis $(|00\rangle, |01\rangle, |10\rangle, |11\rangle)$, with the first qubit as control:

$C\text{-}U = \begin{pmatrix} I_2 & 0 \\ 0 & U \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & U_{00} & U_{01} \\ 0 & 0 & U_{10} & U_{11} \end{pmatrix}$

Q5: For the basis order $|x_2\rangle|x_1\rangle|x_0\rangle$ (with x_2 as MSB), we have:

$$B = \left(|000\rangle, |001\rangle, |010\rangle, |011\rangle, |100\rangle, |101\rangle, |110\rangle, |111\rangle\right)$$

For T_{210}, the target qubit x_k flips iff $x_i = x_j = 1$, so it's an identity on all states except for swapping the two basis states that differ in the target bit, with controls fixed at 11.
(Swaps $|110\rangle$ and $|111\rangle$)

$$T_{210} = \begin{bmatrix} 1&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0 \\ 0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&1&0 \end{bmatrix}$$

For T_{201}, we apply similar logic, but the position of the controls are now x_2 and x_0, so we swap the two basis states that differ in the target bit x_1, with controls fixed at 11.
(Swaps $|101\rangle$ and $|111\rangle$)

$$T_{201} = \begin{bmatrix} 1&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0 \\ 0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&1&0&0 \end{bmatrix}$$

Finally for T_{102} by the same logic, we're swapping the basis states $|011\rangle$ and $|111\rangle$.

$$T_{102} = \begin{bmatrix} 1&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0 \\ 0&0&0&0&0&0&0&1 \\ 0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&1&0 \\ 0&0&0&1&0&0&0&0 \end{bmatrix}$$

Q6: **Claim:** There is no operator $\hat{U}$ s.t. for all qubit states $|\psi\rangle$,

$$\hat{U}(|\psi\rangle|0\rangle) = |\psi\rangle|\psi\rangle$$

Proof:

Assume for the sake of contradiction that such a $\hat{U}$ exists.
In particular, it must clone $|0\rangle$ and $|1\rangle$:

$\hat{U}(|0\rangle|0\rangle) = |0\rangle|0\rangle, \quad \hat{U}(|1\rangle|0\rangle) = |1\rangle|1\rangle.$

Now consider the superposition state $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$.

If $\hat{U}$ clones all states, then: $\hat{U}(|+\rangle|0\rangle) = |+\rangle|+\rangle$.

BUT! $\hat{U}$ is linear, so $\hat{U}(|+\rangle|0\rangle) = \hat{U}\left(\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes |0\rangle\right) = \frac{1}{\sqrt{2}}\left(\hat{U}(|0\rangle|0\rangle) + \hat{U}(|1\rangle|0\rangle)\right)$

If we substitute the cloning action on $|1\rangle$ and $|0\rangle$:

$\frac{1}{\sqrt{2}}\left(\hat{U}(|0\rangle|0\rangle) + \hat{U}(|1\rangle|0\rangle)\right) = \frac{1}{\sqrt{2}}\left(|0\rangle|0\rangle + |1\rangle|1\rangle\right).$

Thus, linearity implies $\hat{U}(|+\rangle|0\rangle) = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right).$

However, $|+\rangle|+\rangle = \left(\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\right) \otimes \left(\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\right) = \frac{1}{2}\left(|00\rangle + |01\rangle + |10\rangle + |11\rangle\right) \neq \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)$, <u>a contradiction.</u>

$\Rightarrow$ No such $\hat{U}$ exists.

$\Rightarrow$ No universal quantum cloning operator exists.

Q7: 1) $|\psi\rangle = \frac{1}{\sqrt{3}}|00\rangle + \frac{1}{\sqrt{3}}|10\rangle + \frac{1}{\sqrt{3}}|11\rangle$

The terms with first bit 1 are $|10\rangle$ and $|11\rangle$.

Probability is the sum of squared magnitudes of all basis states consistent with that measurement outcome.

$$\Rightarrow P = \left|\frac{1}{\sqrt{3}}\right|^2 + \left|\frac{1}{\sqrt{3}}\right|^2 = \frac{2}{3}$$

2) Let $|\psi\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right), \quad M = X_1 Z_2 = X \otimes Z$

$\langle M\rangle = \langle\psi|(X\otimes Z)|\psi\rangle$

$\langle M\rangle = \langle\psi|(X\otimes Z)\left(\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)\right)$

$\langle M\rangle = \langle\psi|\frac{1}{\sqrt{2}}(X\otimes Z)(|00\rangle + |11\rangle)$ recall: $X|0\rangle = |1\rangle$, $X|1\rangle = |0\rangle$, $Z|0\rangle = |0\rangle$, $Z|1\rangle = -|1\rangle$

$\langle M\rangle = \langle\psi|\frac{1}{\sqrt{2}}\left(|10\rangle - |01\rangle\right)$

$\langle M\rangle = \frac{1}{\sqrt{2}}\left(\langle 00| + \langle 11|\right) \cdot \frac{1}{\sqrt{2}}\left(|10\rangle - |01\rangle\right)$

$\langle M\rangle = \frac{1}{2}\left(\overset{0}{\cancel{\langle 00|10\rangle}} - \overset{0}{\cancel{\langle 00|01\rangle}} + \overset{0}{\cancel{\langle 11|10\rangle}} - \overset{0}{\cancel{\langle 11|01\rangle}}\right)$

$$\langle M\rangle = \frac{1}{2}(0) = 0$$

Q8: 1) Macbook Pro (2020), Apple M1, 16GB UNIFIED MEMORY (LPDDR4X)

2) MacOS Tahoe 26.2 (25C56)

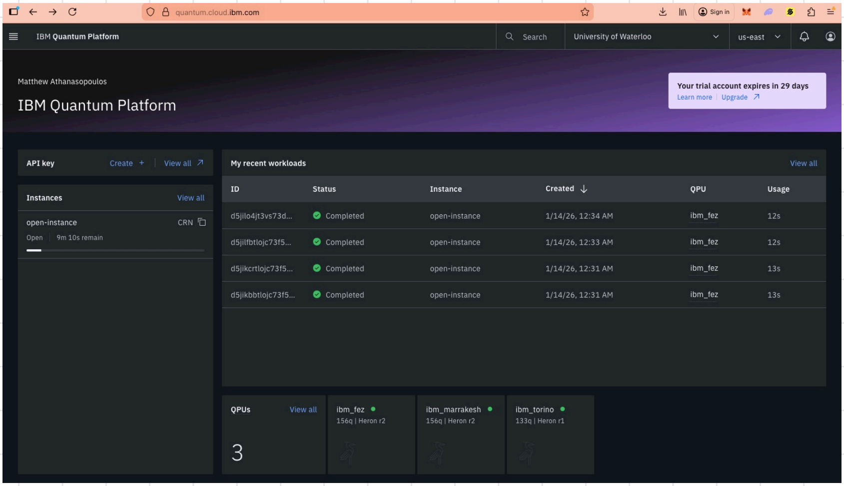
3) ≤ 3.12 ✓ ⇒ cpython-3.11.11-macos-aarch64-none, using uv for package management.

↳ jupyterlab == 4.5.2 (like jupyter notebook but more features)

⇒ NeoVim (Astro!) v0.11.4, LuaJIT 2.1.1753364724

⇒ Visual Studio Code v1.107.1

4) <u>Quantum Cloud Dashboard:</u>



5) Will try out other resources as well!