2D Wave Propagation in an Elastic Plate

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1 Introduction

1.1 Motivation

This study aims to simulate how a 2D elastic plate or laminae (preferably ductile) behaves when tapped at its center. Such a scenario can visualize:

- Acoustic systems (e.g., tabla or drums),
- Localized stress wave transmission (e.g., in Kevlar armor),
- General impact response of flat mechanical structures.

To simplify visualization, the plate is treated as a 2D structure (length $\times$ breadth, no depth), without damping or external forces. Only pure wave propagation is simulated.

1.2 Physical Analogy

Think of a trampoline being struck at its center. The surface momentarily deforms and sends waves radially outward. Our simulation replicates this behavior computationally.

2 Mathematical Modeling

2.1 Governing Equation

The governing equation for 2D wave propagation is:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$$

Where:

- $u(x, y, t)$: Transverse displacement,
- c: Wave speed, depending on material stiffness.

2.2 Boundary Conditions

Dirichlet boundary conditions are used:

$$u(x, y, t) = 0 \quad \text{on all four edges}$$

This represents a clamped plate with fixed edges.

2.3 Considerations

- The mode of impact here is not considered.
- Any and all types of impacts will be considered as point impact for ease of visualization and calculation.
- Any and all external factors will not be considered in the system. which effectively aims to visualize the specimen only in space. And no dampening of the disturbance will be entertained.

2.4 Material Assumption

The material is assumed to be elastic and stiff. No specific parameters are used beyond the wave speed constant c.

3 Solution Approach

3.1 Finite Difference Method (FDM)

The domain is discretized as:

$$x = i \cdot \Delta x, \quad y = j \cdot \Delta y, \quad t = n \cdot \Delta t$$

Approximating the second derivatives using central difference formulas:

$$\frac{\partial^2 u}{\partial t^2} \approx \frac{u_{i,j}^{n+1} - 2u_{i,j}^n + u_{i,j}^{n-1}}{\Delta t^2}$$

$$\frac{\partial^2 u}{\partial x^2} \approx \frac{u_{i+1,j}^n - 2u_{i,j}^n + u_{i-1,j}^n}{\Delta x^2}, \quad \frac{\partial^2 u}{\partial y^2} \approx \frac{u_{i,j+1}^n - 2u_{i,j}^n + u_{i,j-1}^n}{\Delta y^2}$$

Combined, the update equation becomes:

$$u_{i,j}^{n+1} = 2u_{i,j}^n - u_{i,j}^{n-1} + c^2 \Delta t^2 \left(\frac{u_{i+1,j}^n - 2u_{i,j}^n + u_{i-1,j}^n}{\Delta x^2} + \frac{u_{i,j+1}^n - 2u_{i,j}^n + u_{i,j-1}^n}{\Delta y^2} \right)$$

3.2 Initial and Boundary Conditions

- $u(x, y, 0) = f(x, y)$: Initial displacement
- $\frac{\partial u}{\partial t}\big|_{t=0} = g(x, y)$: Initial velocity
- $u = 0$: Boundary condition on all edges

4 MATLAB Implementation

```
% 2D Wave Equation using Finite Difference Method (FDM)

% Grid parameters
Nx = 50;             % Number of grid points in x
Ny = 50;             % Number of grid points in y
Lx = 1;              % Length of plate in x
Ly = 1;              % Length of plate in y
dx = Lx / (Nx - 1);
dy = Ly / (Ny - 1);

% Time parameters
c = 1;               % Wave speed
dt = 0.001;          % Time step
T = 10;               % Total simulation time
```

```
Nt = round(T / dt);  % Number of time steps

% CFL condition check
CFL = c * dt / dx;
if CFL > 1/sqrt(2)
    warning('CFL condition not satisfied. Reduce dt or increase dx.');
end

% Initialize u: displacement grids
u_now = zeros(Nx, Ny);      % At time n
u_old = zeros(Nx, Ny);      % At time n-1
u_new = zeros(Nx, Ny);      % At time n+1

% Initial condition: small Gaussian in the center
[X, Y] = meshgrid(linspace(0, Lx, Nx), linspace(0, Ly, Ny));
u_now = exp(-100 * ((X - 0.5).^2 + (Y - 0.5).^2));
u_old = u_now;  % Assume zero initial velocity

% Precompute constants
coeff_x = (c * dt / dx)^2;
coeff_y = (c * dt / dy)^2;

% Simulation loop
for n = 1:Nt
    for i = 2:Nx-1
        for j = 2:Ny-1
            u_new(i,j) = 2*u_now(i,j) - u_old(i,j) + ...
                coeff_x * (u_now(i+1,j) - 2*u_now(i,j) + u_now(i-1,j)) + ...
                coeff_y * (u_now(i,j+1) - 2*u_now(i,j) + u_now(i,j-1));
        end
    end

    % Boundary conditions (Dirichlet: fixed edges)
    u_new(1,:) = 0; u_new(Nx,:) = 0;
    u_new(:,1) = 0; u_new(:,Ny) = 0;

    % Visualization
    if mod(n,10) == 0
        surf(X, Y, u_new', 'EdgeColor', 'none');
        title(sprintf('2D Wave at time %.3f s', n*dt));
        xlabel('x'); ylabel('y'); zlabel('Displacement');
        axis([0 1 0 1 -1 1]);
        drawnow;
    end
```

```
    % Update time steps
    u_old = u_now;
    u_now = u_new;
end
```

5 Results and Discussion

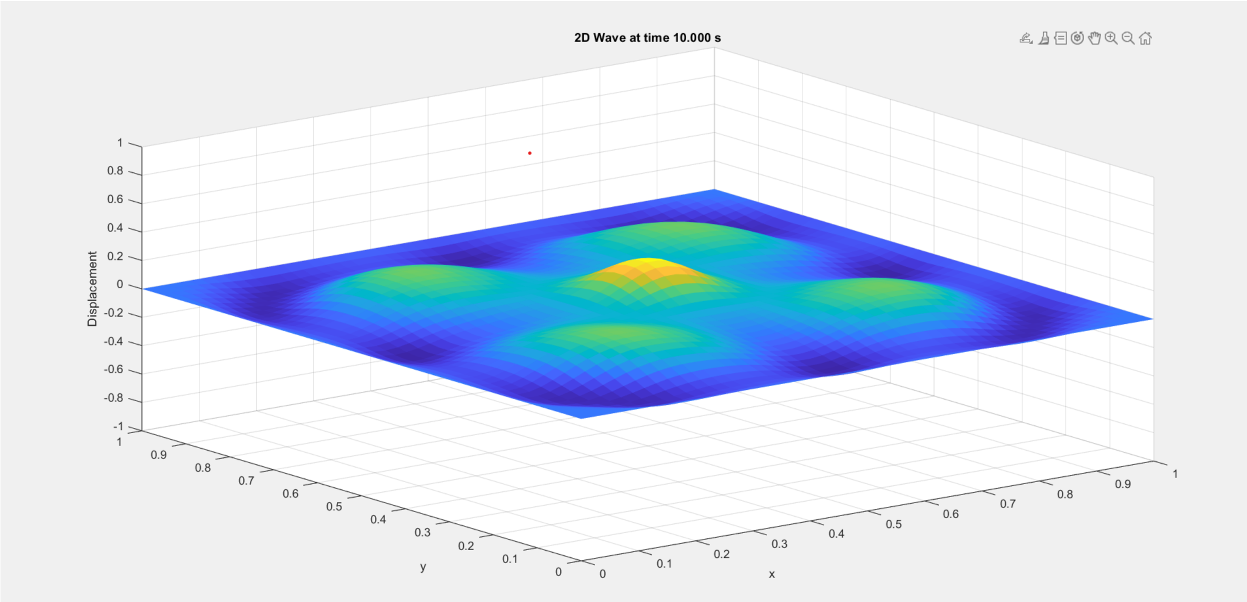

Figure 1: Wave Visualization

- The simulation successfully demonstrates wave propagation on a 2D elastic plate under the influence of a central disturbance. As seen in Figure **??**, a symmetric circular wavefront emerges from the center of the domain, spreading outward with time.

- The zero-displacement boundary condition ensures that waves reflect upon reaching the edges, leading to noticeable interference patterns as the simulation progresses. This behavior is consistent with physical expectations for clamped boundaries.

- The amplitude and speed of the wave depend on the wave speed parameter c and the spatial-temporal discretization. No damping was applied, so the waves continue oscillating with the same amplitude. In practical systems, damping would reduce these amplitudes over time.

- This simulation can be extended to model real-world scenarios such as vibrations in thin plates, surface wave propagation in sensors, or even basic drum membrane behavior.

6 Conclusion and Future Work

The 2D wave equation was successfully solved using FDM. The response of the elastic plate to an initial central tap was visualized, showing realistic propagation behavior.

Future extensions could include:

- Adding damping or material losses,
- Simulating anisotropic materials,
- Introducing nonlinear effects or real-world geometries.

References

1. S. S. Rao, *Mechanical Vibrations*, 6th Ed., Pearson Education, 2018.