Chapter 3:-

-: Random Variables and Probability Distributions :-

3.1 X: Discrete variable because the number of automobile accidents is countable.
Y: Continuous Variable because the length of time is measurable.
M: Continuous Variable because the amount of milk is continuous.
N: Discrete variable because the number of eggs is countable.
P: Discrete variable because the number of building permits is countable.
Q: Continuous Variable because the weight of grain is measurable.

3.2 Solution:-
The random variable X represents the number of automobiles with paint blemishes (a stain or spot). Since 5 automobiles contain 2 that have slight paint blemishes so the values of random variable X can be 0, 1, 2.
If agency receives 3 automobiles, the sample space is.

Sample Space	X
NNN	0
NNB	1
NBN	1
BNN	1
NBB	2
BBN	2
BNB	2

using the letter B for blemished, N for non-blemished.

33 Solution:-

The elements of the Sample Space S for the 3 tosses of coin are

S = {HHH, HHT, HTH, THH, TTT, TTH, THT, HTT}

The random variable W represents the number of heads minus the number of tails ie $W = H - T$.

Now assigning the values of random variable W to each sample point.

Sample Space	H	T	W = H − T
HHH	3	0	3
HHT	2	1	1
HTH	2	1	1
THH	2	1	1
TTT	0	3	−3
TTH	1	2	−1
THT	1	2	−1
HTT	1	2	−1

34 Solution:-

In this experiment we are required to keep on flipping the coin until 3 heads in succession occurs. Listing only those elements of the Sample space that require 6 or less tosses.

S = {HHH, THHH, TTHHH, TTTHHH, THTHHH, HTHHH, HHTHHH ....}

By Definition 3.2 : "If sample space contains a finite no. of possibilities or an unending sequence with as many elements as there are whole numbers it is called a Discrete Sample Space".

So the above Sample Space 'S' is discrete.

Solution:-

The function $f(x)$ can serve as probability distribution of the discrete random variable X only if $\sum_x f(x) = 1$.

a) It is given that $f(x) = c(x^2+4)$, $x = 0,1,2,3$ is probability distribution so

$$\sum_{x=0}^{3} f(x) = 1$$

$$\sum_{x=0}^{3} c(x^2+4) = 1$$

$$c\sum_{x=0}^{3} (x^2+4) = 1$$

$$c\left[(0^2+4) + (1^2+4) + (2^2+4) + (3^2+4)\right] = 1$$

$$c\left[4 + 5 + 8 + 13\right] = 1$$

$$c(30) = 1 \Rightarrow \boxed{c = \frac{1}{30}}$$

b) It is given that $f(x) = c\binom{2}{x}\binom{3}{3-x}$, $x = 0,1,2$

is probability distribution, so

$$\sum_{x=0}^{2} f(x) = 1$$

$$c\sum_{x=0}^{2}\left[\binom{2}{x}\binom{3}{3-x}\right] = 1$$

$$c\left[\binom{2}{0}\binom{3}{3-0} + \binom{2}{1}\binom{3}{3-1} + \binom{2}{2}\binom{3}{3-2}\right] = 1$$

$$c\left[{}^2C_0 \cdot {}^3C_3 + {}^2C_1\,{}^3C_2 + {}^2C_2\,{}^3C_1\right] = 1$$

$$c\left[1 + 6 + 3\right] = 1$$

$$c(10) = 1$$

$$\Rightarrow \boxed{c = 1/10}$$

3.6 Solution:-

a) $P(X \geq 200) = \int_{200}^{\infty} f(x)\,dx$

$= \int_{200}^{\infty} \frac{20{,}000}{(x+100)^3}\,dx$

$= 20{,}000 \int_{200}^{\infty} (x+100)^{-3}\,dx = 20{,}000 \left| \frac{(x+100)^{-3+1}}{-3+1} \right|_{200}^{\infty}$

$= \frac{20{,}000}{-2} \left| (\infty+100)^{-2} - (200+100)^{-2} \right|$

$= -10{,}000 \left| 0 - (300)^{-2} \right|$

$= + \frac{10{,}000}{(300)^2} = \frac{10{,}000}{90{,}000} = \frac{1}{9}$

b) $P(80 \leq x \leq 120) = \int_{80}^{120} f(x)\,dx$

$= \int_{80}^{120} \frac{20{,}000}{(x+100)^3}\,dx = 20{,}000 \int_{80}^{120} (x+100)^{-3}\,dx$

$= 20{,}000 \left| \frac{(x+100)^{-3+1}}{-3+1} \right|_{80}^{120} = -10{,}000 \left| (x+100)^{-2} \right|_{80}^{120}$

$= -10{,}000 \left[(120+100)^{-2} - (80+100)^{-2} \right]$

$= -10{,}000 \left[(220)^{-2} - (180)^{-2} \right] = -10{,}000 \left[\frac{1}{(220)^2} - \frac{1}{(180)^2} \right]$

$= -10{,}000 \left[0.0000206612 - 0.000030864 \right]$

$= +0.1020.$

3.7 Solution:-

$P(X < 120) = P(X < 120/100) = P(X < 1.2)$

because time was measured in units of 100 hours.

$$P(X<1.2) = \int_0^{1.2} f(x)\,dx$$

$$= \int_0^1 f(x)\,dx + \int_1^{1.2} f(x)\,dx.$$

$$= \int_0^1 x\,dx + \int_1^{1.2} (2-x)\,dx$$

$$= \left|\frac{x^2}{2}\right|_0^1 + \left\{ 2|x|_1^{1.2} - \left|\frac{x^2}{2}\right|_1^{1.2} \right\}$$

$$= \frac{1}{2}(1^2-0^2) + \left\{ 2(1.2-1) - \frac{1}{2}\left((1.2)^2 - 1^2\right) \right\}$$

$$= \frac{1}{2} + \left\{ 2(0.2) - 0.5(0.44) \right\}$$

$$= 0.68$$

b) $P(50 < X < 100) = P(50/100 < X < 100/100)$

$$= P(0.5 < X < 1)$$

$$= \int_{0.5}^1 f(x)\,dx = \int_{0.5}^1 x\,dx = \left|\frac{x^2}{2}\right|_{0.5}^1$$

$$= \frac{1}{2}\left(1^2 - (0.5)^2\right) = 0.5(0.75) = 0.375.$$

3.8 **Solution:-**

As the head is twice as likely to occur as a tail so the $P(H) = \frac{2}{3}$ and $P(T) = \frac{1}{3}$.

Using the solution of Q3.3.

Sample Space	W	P(ω)
HHH	3	$P(HHH) = 2/3 \cdot 2/3 \cdot 2/3 = 8/27$
HHT	1	$P(HHT) = 2/3 \cdot 2/3 \cdot 1/3 = 4/27$
HTH	1	$P(HTH) = 2/3 \cdot 1/3 \cdot 2/3 = 4/27$
THH	1	$P(THH) = 1/3 \cdot 2/3 \cdot 2/3 = 4/27$
TTT	-3	$P(TTT) = 1/3 \cdot 1/3 \cdot 1/3 = 1/27$
TTH	-1	$P(TTH) = 1/3 \cdot 2/3 \cdot 2/3 = 2/27$
THT	-1	$P(THT) = 1/3 \cdot 2/3 \cdot 1/3 = 2/27$
HTT	-1	$P(HTT) = 2/3 \cdot 1/3 \cdot 1/3 = 2/27$

The required Probability distribution is:

W	$P(W)$
-3	$1/27$
-1	$2/27 + 2/27 + 2/27 = 6/27 = 2/9$
1	$4/27 + 4/27 + 4/27 = 12/27 = 4/9$
3	$8/27$

3.9 Solution:-

$$f(x) = \begin{cases} \frac{2}{5}(x+2), & 0<x<1 \\ 0, & \text{elsewhere} \end{cases}$$

a) $P(0<X<1) = 1$

$$\int_0^1 \frac{2}{5}(x+2)\,dx = 1$$

$$\frac{2}{5}\int_0^1 (x+2)\,dx = 1 \;;\quad \frac{2}{5}\left[\frac{x^2}{2} + 2x\right]_0^1 = 1$$

$$\frac{2}{5}\left[\left\{\frac{1^2}{2} + 2(1)\right\} - \left\{\frac{0^2}{2} + 2(0)\right\}\right] = 1$$

$$\frac{2}{5}\left[(0.5+2) - (0+0)\right] = 1 \;;\quad 0.4(2.5) = 1$$

$$\Rightarrow \quad 1 = 1$$

As Required.

(b) $P\left(\frac{1}{4}<X<\frac{1}{2}\right) = P(0.25<X<0.5)$

$$= \int_{0.25}^{0.5} f(x)\,dx = \int_{0.25}^{0.5} \frac{2}{5}(x+2)\,dx = \frac{2}{5}\int_{0.25}^{0.5}(x+2)\,dx$$

$$= \frac{2}{5}\left[\frac{x^2}{2} + 2x\right]_{0.25}^{0.5} = \frac{2}{5}\left[\left\{\frac{0.5^2}{2} + 2(0.5)\right\} - \left\{\frac{0.25^2}{2} + 2(0.25)\right\}\right]$$

$$= 0.4\left[1.125 - 0.53125\right] = 0.2375.$$

3.10 Solution:-

when a single die is rolled once the probability of each outcome is 1/6 so The formula for the prob. dist. of random variable X is

$f(x) = 1/6$.

The probability distribution is:

X	f(x)
1	1/6
2	1/6
3	1/6
4	1/6
5	1/6
6	1/6

11 **Solution:-**

7 television sets contains 2 defective sets. If hotel makes purchase of 3 sets and X denotes The number of defective sets then X can assume values 0, 1, 2. The probability distribution of X is:

X	P(X)
0	$\frac{{}^2C_0\,{}^5C_3}{{}^7C_3} = \frac{1\times 10}{35} = \frac{\cancel{10}^{2}}{\cancel{35}_{7}} = \frac{2}{7}$
~~2~~ 1	$\frac{{}^2C_1\,{}^5C_2}{{}^7C_3} = \frac{2\times 10}{35} = \frac{\cancel{20}^{4}}{\cancel{35}_{7}} = \frac{4}{7}$
2	$\frac{{}^2C_2\,{}^5C_0}{{}^7C_3} = \frac{1\times 5}{35} = \frac{\cancel{5}^{1}}{\cancel{35}_{7}} = \frac{1}{7}$.

3.12 a) **Solution:-**

$$F(t) = \begin{cases} 0, & t < 1 \\ 1/4, & 1 \le t < 3 \\ 1/2, & 3 \le t < 5 \\ 3/4, & 5 \le t < 7 \\ 1, & t \ge 7 \end{cases}$$

$P(T=5) = F(5) - F(4) = \frac{3}{4} - \frac{1}{2} = \frac{6-4}{8} = \frac{\cancel{2}}{\cancel{8}_{4}} = \frac{1}{4}$.

b) $P(T>3) = 1 - P(T \leq 3)$
$= 1 - F(3)$
$= 1 - \frac{1}{2} = \frac{2-1}{2} = \frac{1}{2}$

c) $P(1.4 < T < 6) = F(6) - F(1.4)$
$= \frac{3}{4} - \frac{1}{4} = \frac{2}{4} = \frac{1}{2}$

d) $P(T \leq 5 \mid T \geq 2) = \frac{P(T \leq 5 \text{ and } T \geq 2)}{P(T \geq 2)} = \frac{P(2 \leq T \leq 5)}{1 - P(T<2)}$
$= \frac{F(5) - F(2)}{1 - F(2)} = \frac{3/4 - 1/4}{1 - 1/4} = \frac{2/4}{3/4} = \frac{2}{3}$

3.13) Solution:-

x	$f(x)$	$F(x)$
0	0.41	0.41
1	0.37	$0.41+0.37=0.78$
2	0.16	$0.78+0.16=0.94$
3	0.05	$0.94+0.05=0.99$
4	0.01	$0.99+0.01=1$

The cummulatwie distribution function is:

$$F(x) = \begin{cases} 0, & x<0 \\ 0.41, & 0 \leq x < 1 \\ 0.78, & 1 \leq x < 2 \\ 0.94, & 2 \leq x < 3 \\ 0.99, & 3 \leq x < 4 \\ 1, & x \geq 1 \end{cases}$$

3.14 Solution:-
we are required to find the probability of waiting time less than 12 minutes i.e $P(x<12)$. Since time is measured in hours

1 hour = 60 minutes

So 12 minutes = $^{12}/_{60}$ hours = 0.2 hours

$\Rightarrow P(X<12) = P(X<0.2)$

$$F(x) = \begin{cases} 0 & , x<0 \\ 1-e^{-8x} & , x\geq 0 \end{cases}$$

a) Using $F(x)$: $P(X<0.2) = F(0.2)$

$= 1-e^{-8(0.2)}$

$= 1-e^{-1.6}$

$= 1-0.2019$

$= 0.798$

b) Using $f(x)$

Since $\frac{d}{dx}F(x) = f(x)$

So $\frac{d}{dx}(1-e^{-8x}) = \frac{d}{dx}(1) - \frac{d}{dx}(e^{-8x})$

$= 0 - e^{-8x}(-8) = 0 + 8e^{-8x}$

$\Rightarrow f(x) = 8e^{-8x} \quad , x\geq 0$

Now $P(X<0.2) = \int_0^{0.2} f(x)\,dx$

$= \int_0^{0.2} 8e^{-8x}\,dx$

$= 8\left|\frac{e^{-8x}}{-8}\right|_0^{0.2}$

$= -(e^{-8(0.2)} - e^{-8(0)})$

$= -(e^{-1.6} - e^{-0})$

$= -e^{-1.6} + 1$

$= 1-e^{-1.6}$

$= 1-0.2019$

$= 0.798$

3.15 Solution:-

The cummulative distribution function of the random variable X in Exercise 3-11 is:

x	$P(x)$	$F(x)$
0	$2/7$	$2/7$
1	$4/7$	$2/7+4/7=6/7$
2	$1/7$	$6/7+1/7=7/7=1$

$$F(x)=\begin{cases}0 & , x<0\\ 2/7 & , 0\leq x<1\\ 6/7 & , 1\leq x<2\\ 1 & , x\geq 2\end{cases}$$

Using $F(x)$

a) $P(X=1) = F(1)-F(0) = 6/7 - 2/7 = 4/7$

b) $P(0<X\leq 2) = F(2)-F(0) = 1 - 2/7 = 5/7$

3.17 Given $f(x)=\frac{1}{2}$, $1\leq x\leq 3$

Solution:-

a) Area under the curve is equal to 1 iff $\int_{-\infty}^{\infty} f(x)dx=1$.

Here $\int_1^3 f(x)\,dx = 1$

$\int_1^3 \frac{1}{2}dx = 1$; $\frac{1}{2}\int_1^3 dx = 1$; $\frac{1}{2}|x|_1^3 = 1$

$\frac{1}{2}(3-1)=1$; $\frac{1}{2}(2)=1$; $1=1$.

Thus the area under the curve is equal to 1.

b) $P(2<X<2.5) = \int_2^{2.5} f(x)\,dx = \int_2^{2.5}\frac{1}{2}dx = \frac{1}{2}\int_2^{2.5} dx$.

$= \frac{1}{2}|x|_2^{2.5} = \frac{1}{2}(2.5-2) = \frac{1}{2}(0.5) = 0.25$

$$P(X \leq 1.6) = \int_1^{1.6} f(x)\,dx = \int_1^{1.6} \frac{1}{2}\,dx = \frac{1}{2}\left|x\right|_1^{1.6} = \frac{1}{2}(1.6-1)$$

$$= \frac{1}{2}(0.6) = 0.3$$

3.18 **Solution:-**

Given $f(x) = \frac{2}{27}(1+x), \quad 2 \leq x \leq 5.$

a) $$P(X<4) = \int_2^4 f(x)\,dx = \int_2^4 \frac{2}{27}(1+x)\,dx = \frac{2}{27}\int_2^4 (1+x)\,dx$$

$$= \frac{2}{27}\left|x + \frac{x^2}{2}\right|_2^4 = \frac{2}{27}\left[\left\{4+\frac{4^2}{2}\right\} - \left\{2+\frac{2^2}{2}\right\}\right]$$

$$= \frac{2}{27}\left[12-4\right] = \frac{16}{27}$$

b) $$P(3 \leq X < 4) = \int_3^4 f(x)\,dx = \int_3^4 \frac{2}{27}(1+x)\,dx = \frac{2}{27}\left|x+\frac{x^2}{2}\right|_3^4$$

$$= \frac{2}{27}\left[\left\{4+\frac{4^2}{2}\right\} - \left\{3+\frac{3^2}{2}\right\}\right] = \frac{2}{27}\left[12 - 7.5\right]$$

$$= \frac{2}{27} \times 4.5 = \frac{9}{27} = \frac{1}{3}$$

3.19 **Solution:-**

In Exercise 3.17, $f(x) = \frac{1}{2}, \quad 1 \leq x \leq 3$

For $F(x)$; $F(x) = \int_{-\infty}^{x} f(t)\,dt$

Here, $F(x) = \int_1^x \frac{1}{2}\,dt$

$$= \frac{1}{2}\left|t\right|_1^x = \frac{1}{2}(x-1).$$

Now $P(2 \leq X < 2.5)$ using $F(x)$ is:

$P(2 \leq X < 2.5) = F(2.5) - F(2)$

$$= \left(\frac{2.5-1}{2}\right) - \left(\frac{2-1}{2}\right) = \frac{1.5}{2} - \frac{1}{2} = \frac{0.5}{2} = 0.25$$

3.20 **Solution:-**

$$f(x) = \frac{2}{27}(1+x), \quad 2 \le x \le 5 \qquad \text{(Exercise 3.18)}$$

$$F(x) = \int_2^x \frac{2}{27}(1+t)\,dt$$

$$= \frac{2}{27}\int_2^x (1+t)\,dt = \frac{2}{27}\left| t + \frac{t^2}{2} \right|_2^x$$

$$= \frac{2}{27}\left[\left\{x + \frac{x^2}{2}\right\} - \left\{2 + \frac{2^2}{2}\right\}\right] = \frac{2}{27}\left[\frac{2x+x^2}{2} - 4\right]$$

$$= \frac{\cancel{2}}{27}\left[\frac{2x+x^2-8}{\cancel{2}}\right] = \frac{x^2+2x-8}{27}.$$

Using $F(x)$, $P(3 \le X < 4)$ is:

$$P(3 \le X < 4) = F(4) - F(3)$$

$$= \frac{4^2+2(4)-8}{27} - \frac{3^2+2(3)-8}{27}$$

$$= \frac{16}{27} - \frac{7}{27} = \frac{9}{27} = \frac{1}{3}$$

3.21 **Solution:-**

$$f(x) = \begin{cases} k\sqrt{x}, & 0 < x < 1 \\ 0, & \text{elsewhere} \end{cases}$$

a) Since X is continuous random variable so

$$\int_{-\infty}^{\infty} f(x)\,dx = 1.$$

$$\Rightarrow \int_0^1 k\sqrt{x}\,dx = 1$$

$$k\int_0^1 x^{1/2}\,dx = 1$$

$$k\left|\frac{x^{3/2}}{3/2}\right|_0^1 = 1$$

$$\frac{2}{3}k\left(1^{3/2} - 0^{3/2}\right) = 1 \Rightarrow \frac{2}{3}k = 1 \Rightarrow \boxed{k = \frac{3}{2}}$$

Now $f(x) = \begin{cases} \frac{3}{2}\sqrt{x}, & 0<x<1 \\ 0, & \text{elsewhere} \end{cases}$

b) $F(x) = \int_{-\infty}^{x} f(t)\,dt$

$\Rightarrow F(x) = \int_0^x \frac{3}{2}\sqrt{t}\,dt = \int_0^x \frac{3}{2} t^{1/2}\,dt = \frac{3}{2}\left| \frac{t^{1/2+1}}{1/2+1} \right|_0^x$

$= \frac{3}{2}\left| \frac{t^{3/2}}{3/2} \right|_0^x = \frac{2}{3} \times \frac{3}{2}\left(x^{3/2} - 0^{3/2}\right)$

$\Rightarrow F(x) = x^{3/2}$.

Using $F(x)$, $P(0.3<x<0.6) = F(0.6) - F(0.3)$

$= (0.6)^{3/2} - (0.3)^{3/2}$

$= (0.6)^{1.5} - (0.3)^{1.5}$

$= 0.30044$

3.22 Solution:-

In a deck of 52 cards there are 13 spades. Since 3 cards are drawn then the random variable, X, representing the no. of spades can take values 0, 1, 2 and 3. The probability distribution of X is:

x	$P(x)$
0	$\frac{^{13}C_0\, ^{39}C_3}{^{52}C_3} = \frac{9139}{22100} = 0.414$
1	$\frac{^{13}C_1\, ^{39}C_2}{^{52}C_3} = \frac{13 \times 741}{22100} = \frac{9633}{22100} = 0.436$
2	$\frac{^{13}C_2\, ^{39}C_1}{^{52}C_3} = \frac{78 \times 39}{22100} = \frac{3042}{22100} = 0.138$
3	$\frac{^{13}C_3\, ^{39}C_0}{^{52}C_3} = \frac{286 \times 1}{22100} = 0.013$

Note: $F(x) = P(X \leq x)$

For discrete: $P(a<X<b)$ or $P(a \leq X \leq b)$ or $P(a < X \leq b) = F(b) - F(a)$

but $P(a \leq X < b) = P(X<b) - P(X<a)$

For continuous: $P(a<X<b) = P(a \leq X \leq b) = F(b) - F(a)$

3.23 Solution:-

The prob. dist. of w is: (Exercise 3.8)

w	$P(w)$	$F(w)$
-3	$1/27$	$1/27$
-1	$6/27$	$1/27 + 6/27 = 7/27$
1	$12/27$	$7/27 + 12/27 = 19/27$
3	$8/27$	$19/27 + 8/27 = 27/27 = 1$

The cummulative distribution function of w is:

$$F(w) = \begin{cases} 0 & , w < -3 \\ 1/27 & , -3 \leq w < -1 \\ 7/27 & , -1 \leq w < 1 \\ 19/27 & , 1 \leq w < 3 \\ 1 & , w \geq 3 \end{cases}$$

a) Using $F(w)$, $P(w>0) = 1 - P(w \leq 0)$

$= 1 - F(0)$

$= 1 - 7/27 = 20/27$

b) $P(-1 \leq w < 3) = P(w<3) - P(w<-1)$

$= 19/27 - 1/27$

$= \frac{18}{27}$

$= \frac{2}{3}$

3.24 Solution:-

Total no. of CDs $= 5+2+3 = 10$

No. of jazz CDs $= 5$

Selected CDs $= 4$

let x represents the no. of jazz CDs then the prob. dist of X can be represented by means of formula: $f(x) = \frac{{}^5C_x \, {}^5C_{4-x}}{{}^{10}C_4}$

where, x can take values $0, 1, 2, 3, 4$.

The probability distribution of X is:

x	$f(x) = \frac{{}^5C_x \, {}^5C_{4-x}}{{}^{10}C_4}$
0	$\frac{{}^5C_0 \, {}^5C_4}{{}^{10}C_4} = \frac{5}{210} = \frac{1}{42}$
1	$\frac{{}^5C_1 \, {}^5C_3}{{}^{10}C_4} = \frac{5\times 10}{210} = \frac{50}{210} = \frac{5}{21}$
2	$\frac{{}^5C_2 \, {}^5C_2}{{}^{10}C_4} = \frac{10\times 10}{210} = \frac{100}{210} = \frac{10}{21}$
3	$\frac{{}^5C_3 \, {}^5C_1}{{}^{10}C_4} = \frac{10\times 5}{210} = \frac{5}{21}$
4	$\frac{{}^5C_4 \, {}^5C_0}{{}^{10}C_4} = \frac{5\times 1}{210} = \frac{5}{210}$

3.25 **Solution:-**

1 dime = 10 cents and
1 nickel = 5 cents.

If a box contains 4 dimes and 2 nickels and 3 coins are selected then the random variable T representing the total of 3 coins can assume values 30, 25, 20

Sample Space: $S = \{\underset{10\;10\;10}{DDD}, \underset{10\;10\;5}{DDN}, \underset{10\;5\;5}{DNN}\}$ (without replacement)

The Probability distribution of T is:

T	P(T)
30	$\frac{{}^2C_0 \, {}^4C_3}{{}^6C_3} = \frac{4}{20} = \frac{1}{5}$
25	$\frac{{}^2C_1 \, {}^4C_2}{{}^6C_3} = \frac{2\times 6}{20} = \frac{12}{20} = \frac{3}{5}$
20	$\frac{{}^2C_2 \, {}^4C_1}{{}^6C_3} = \frac{1\times 4}{20} = \frac{1}{5}$

3.26 Since the ball drawn is replaced before the next ball is drawn so we cannot use combination to find probabilities.
Here, the random variable X represents the no. of green balls and box contains 2 green and 4 black balls. If 3 balls are drawn $X = 0, 1, 2, 3$
The probability distribution of X is:

x	Sample Space	$P(x)$
0	BBB	$\frac{4}{6}\cdot\frac{4}{6}\cdot\frac{4}{6} = \frac{64}{216} = \frac{8}{27}$
1	BBG	$\frac{4}{6}\cdot\frac{4}{6}\cdot\frac{2}{6} = \frac{32}{216} = \frac{4}{27}$
1	BGB	$\frac{4}{6}\cdot\frac{2}{6}\cdot\frac{4}{6} = \frac{32}{216} = \frac{4}{27}$
1	GBB	$\frac{4}{6}\cdot\frac{2}{6}\cdot\frac{4}{6} = \frac{32}{216} = \frac{4}{27}$
2	BGG	$\frac{4}{6}\cdot\frac{2}{6}\cdot\frac{2}{6} = \frac{16}{216} = \frac{2}{27}$
2	GGB	$\frac{2}{6}\cdot\frac{2}{6}\cdot\frac{4}{6} = \frac{16}{216} = \frac{2}{27}$
2	GBG	$\frac{2}{6}\cdot\frac{4}{6}\cdot\frac{2}{6} = \frac{16}{216} = \frac{2}{27}$
3	GGG	$\frac{2}{6}\cdot\frac{2}{6}\cdot\frac{2}{6} = \frac{8}{216} = \frac{1}{27}$

B denotes black ball and G denotes green ball

$$P(B) = \frac{n(B)}{\text{Total balls}} = \frac{4}{6} = 2$$

(Since ball is replaced before the next draw so x can take 3 value).

$$P(G) = \frac{n(G)}{\text{Total balls}} = \frac{2}{6}$$

3.27 **Solution:-**

$$f(x) = \begin{cases} \frac{1}{2000}\exp(-x/2000), & x \geq 0 \\ 0, & x < 0 \end{cases}$$

a) $F(x) = \int_{-\infty}^{x} f(t)\,dt$

$$= \int_0^x \frac{1}{2000} e^{-t/2000}\,dt$$

$$= \frac{1}{2000}\left|\frac{e^{-\frac{1}{2000}t}}{-1/2000}\right|_0^x = \frac{-2000}{2000}\left[e^{-x/2000} - e^{-0}\right]$$

$$= -e^{-x/2000} + 1 = 1 - e^{-x/2000}.$$

(b) P(component lasts more than 1000 hours) = $P(X>1000)$

$P(X>1000) = 1 - P(X \le 1000)$

$= 1 - F(1000)$

Since $F(x) = 1 - e^{-x/2000}$

$\Rightarrow F(1000) = 1 - e^{-1000/2000} = 1 - e^{-0.5}$

$= 1 - 0.607$

$= 0.393$

So $P(X>1000) = 1 - F(1000) = 1 - 0.393 = 0.607$

c) P(component fails before 2000 hours) = $P(X<2000)$

$= F(2000)$

$= 1 - e^{-2000/2000}$

$= 1 - e^{-1} = 0.632$

3.28 **Solution:-**

a)
$$f(x) = \begin{cases} \frac{2}{5}, & 23.75 \le x \le 26.25 \\ 0, & \text{elsewhere} \end{cases}$$

$f(x)$ is valid density function, if

$$\int_{-\infty}^{\infty} f(x)\,dx = 1.$$

$$\int_{23.75}^{26.25} \frac{2}{5}\,dx = 1$$

$$\frac{2}{5}\int_{23.75}^{26.25} 1.\,dx = 1.$$

$$\frac{2}{5}\left|x\right|_{23.75}^{26.25} = 1.$$

$$\frac{2}{5}(26.25 - 23.75) = 1.$$

$$0.4 \times 2.5 = 1$$

$$1 = 1$$

$\Rightarrow f(x)$ is valid density function.

b) P(weight smaller than 24 ounces) = $P(X<24)$

$$= \int_{23.75}^{24} f(x)\,dx$$

$$= \int_{23.75}^{24} \frac{2}{5}\,dx = \frac{2}{5}\Big|x\Big|_{23.75}^{24} = \frac{2}{5}(24-23.75)$$

$$= 0.4 \times 0.25$$
$$= 0.1$$

c) P(weight exceeding 26 ounces) = $P(X>26)$

$$= \int_{26}^{26.25} f(x)\,dx$$

$$= \frac{2}{5}\int_{26}^{26.25} 1 \,.\, dx = \frac{2}{5}\Big|x\Big|_{26}^{26.25}$$

$$= \frac{2}{5}(26.25-26) = 0.4 \times 0.25 = 0.1$$

3.29 Solution:

$$f(x) = \begin{cases} 3x^{-4} & , x>1 \\ 0 & , \text{elsewhere} \end{cases}$$

a) This is a valid density function if

$$\int_{-\infty}^{\infty} f(x)\,dx = 1.$$

$$\int_{1}^{\infty} 3x^{-4}\,dx = 1$$

$$3\int_{1}^{\infty} x^{-4}\,dx = 1$$

$$3\left|\frac{x^{-4+1}}{-4+1}\right|_{1}^{\infty} = 1$$

$$\frac{-3}{3}\left|\infty^{-3} - 1^{-3}\right| = 1$$

$$-(0-1) = 1 \Rightarrow +1 = 1$$

So $f(x)$ is valid density function.

$$F(x) = \int_{-\infty}^{x} f(t)\,dt$$

$$= \int_{1}^{x} 3t^{-4}\,dt$$

$$= 3\left|\frac{t^{-4+1}}{-4+1}\right|_{1}^{x} = \frac{-3}{3}\left| x^{-3} - 1^{-3}\right| = -\left(\frac{1}{x^3} - 1\right)$$

$$F(x) = 1 - \frac{1}{x^3}$$

c) P(particle exceeds 4 micrometers) = $P(X>4)$

$$= 1 - P(X<4)$$

$$= 1 - F(4)$$

$$= 1 - \left[1 - \frac{1}{4^3}\right]$$

$$= \cancel{1} - \cancel{1} + \frac{1}{4^3}$$

$$= 0.0156.$$

3.30 Solution:-

$$f(x) = \begin{cases} k(3-x^2), & -1 \le x \le 1 \\ 0, & \text{elsewhere} \end{cases}$$

Since $f(x)$ is a valid density function so $\int_{-1}^{1} f(x)\,dx$ must be equal to 1.

i.e $\int_{-1}^{1} k(3-x^2)\,dx = 1$

$$k\int_{-1}^{1}(3-x^2)\,dx = 1$$

$$k\left|3x - \frac{x^3}{3}\right|_{-1}^{1} = 1$$

$$k\left[\left\{3(1) - \frac{1^3}{3}\right\} - \left\{3(-1) - \frac{(-1)^3}{3}\right\}\right] = 1$$

$$k\left[\left(3 - \frac{1}{3}\right) - \left(-3 + \frac{1}{3}\right)\right] = 1.$$

$$k\left[\frac{8}{3} - \left(\frac{-8}{3}\right)\right] = 1$$

$$k\left[\frac{8}{3} + \frac{8}{3}\right] = 1$$

$$k\left[\frac{16}{3}\right] = 1$$

$$k(16/3) = 1$$

$$\Rightarrow k = 3/16$$

So, $f(x) = \begin{cases} \frac{3}{16}(3-x^2), & -1 \le x \le 1 \\ 0, & \text{elsewhere} \end{cases}$

b) $P(\text{random error} < 1/2) = P(X < 1/2)$

$$= \int_{-1}^{0.5} \frac{3}{16}(3-x^2)\,dx$$

$$= \frac{3}{16}\left| 3x - \frac{x^3}{3} \right|_{-1}^{0.5}$$

$$= \frac{3}{16}\left[\left\{3(0.5) - \frac{(0.5)^3}{3}\right\} - \left\{3(-1) - \frac{(-1)^3}{3}\right\}\right]$$

$$= \frac{3}{16}\left[1.458 - (-2.667)\right]$$

$$= \cancel{0.275}\ 0.7734$$

c) $P(\text{magnitude of error i.e } |x| \text{ exceeds } 0.8) = P(|x| > 0.8)$

$= P(X < -0.8) + P(X > 0.8)$ (use CDF or find prob directly as in (b))

$= P(X < -0.8) + (1 - P(X < 0.8))$

$= F(-0.8) + 1 - F(0.8)$ — ①

For finding $F(x)$

$$F(x) = \int_{-1}^{x} \frac{3}{16}(3-t^2)\,dt = \frac{3}{16}\left| 3t - \frac{t^3}{3} \right|_{-1}^{x}$$

$$= \frac{3}{16}\left[\left\{3x - \frac{x^3}{3}\right\} - \left\{3(-1) - \frac{(-1)^3}{3}\right\}\right]$$

$$= \frac{3}{16}\left[\left\{3x - \frac{x^3}{3}\right\} - \left\{-3 + \frac{1}{3}\right\}\right]$$

$$= \frac{3}{16}\left[3x - \frac{x^3}{3} + \frac{8}{3}\right]$$

$$= \frac{\cancel{3}}{16}\left[\frac{9x - x^3 + 8}{\cancel{3}}\right]$$

$$= \frac{-x^3 + 9x + 8}{16}$$

So $F(0.8) = \frac{-(0.8)^3 + 9(0.8) + 8}{16} = \frac{14.688}{16}$

$= 0.918$

$F(-0.8) = \frac{-(-0.8)^3 + 9(-0.8) + 8}{16} = \frac{(0.8)^3 - 9(0.8) + 8}{16}$

$= \frac{1.312}{16} = 0.082$

Substituting $F(0.8)$ and $F(-0.8)$ in eq ①.

$\Rightarrow P(|x| > 0.8) = F(-0.8) + 1 - F(0.8)$

$= 0.082 + 1 - 0.918$

$= 0.164$

3.31 **Solution:-**

$$f(y) = \begin{cases} \frac{1}{4} e^{-y/4}, & y \geq 0 \\ 0, & \text{elsewhere} \end{cases}$$

a) $P(Y > 6) = \int_6^{\infty} \frac{1}{4} e^{-y/4} \, dy$

$= \frac{1}{4} \left| \frac{e^{-y/4}}{-1/4} \right|_6^{\infty}$

$= -\frac{4}{4} \left| e^{-\infty/4} - e^{-6/4} \right|$

$= -(e^{-\infty} - e^{-1.5}) = -(0 - e^{-1.5}) = +e^{-1.5}$

$= 0.223$

b) $P(Y \leq 1) = \int_0^1 \frac{1}{4} e^{-y/4} dy$

$= \frac{1}{4} \left| \frac{e^{-y/4}}{-1/4} \right|_0^1 = -\frac{4}{4}\left(e^{-1/4} - e^{-0/4}\right)$

$= -1\left(e^{-1/4} - 1\right) = 1 - e^{-1/4} = 1 - e^{-0.25}$

$= 0.221$

3.32 Solution:-

$$f(y) = \begin{cases} 5(1-y)^4, & 0 < y \leq 1 \\ 0, & \text{elsewhere} \end{cases}$$

a) The above is a valid density function if

$\int_0^1 f(y)\,dy = 1$

$\int_0^1 5(1-y)^4 dy = 1$

$5 \left| \frac{(1-y)^5}{-5} \right|_0^1 = 1$

$- \left| (1-y)^5 \right|_0^1 = 1$

$-\left[(1-1)^5 - (1-0)^5\right] = 1$

$-[0 - 1] = 1$

$\Rightarrow 1 = 1$

Hence, $f(y)$ is a valid density function.

b) $P(X < 10\%) = P(X < 0.10)$

$= \int_0^{0.10} f(y)\,dy$

$= \int_0^{0.10} 5(1-y)^4 dy = 5 \left| \frac{(1-y)^5}{-5} \right|_0^{0.10}$

$= -\left[(1-0.10)^5 - (1-0)^5\right]$

$= -(0.59049) + 1 = 0.40951$

c) $P(X > 50\%) = P(X > 0.5) = \int_{0.5}^{1} 5(1-y)^4 dy = 5 \left| \frac{(1-y)^5}{-5} \right|_{0.5}^{1}$

$= -\left[(1-1)^5 - (1-0.5)^5 \right] = -\left[0 - (0.5)^5 \right] = +(0.5)^5$

$= 0.03125$

3.33 **Solution:-**

(a) For the valid density function

$\int_0^1 f(y)\,dy = 1$

$\int_0^1 k y^4 (1-y)^3 dy = 1$

$k \int_0^1 y^4 \left(1^3 - y^3 - 3y(1-y)\right) dy = 1 \quad \because (a-b)^3 = a^3 - b^3 - 3ab(a-b)$

$k \int_0^1 (y^4 - y^7 - 3y^5 + 3y^6)\,dy = 1$

$k \left| \frac{y^5}{5} - \frac{y^8}{8} - \frac{3y^6}{6} + \frac{3y^7}{7} \right|_0^1 = 1$

$k \left[\left(\frac{1}{5} - \frac{1}{8} - \frac{1}{2} + \frac{3}{7} \right) - (0 - 0 - 0 + 0) \right] = 1$

$k \left[\frac{56 - 35 - 140 + 120}{280} \right] = 1$

$k \left(\frac{1}{280} \right) = 1 \Rightarrow \boxed{k = 280}$

b) $P(Y \leq 50\%) = P(Y \leq 0.5) = \int_0^{0.5} 280 y^4 (1-y)^3 dy$

$= 280 \int_0^{0.5} (y^4 - y^7 - 3y^5 + 3y^6)\,dy$

$= 280 \left| \frac{y^5}{5} - \frac{y^8}{8} - \frac{3}{6} y^6 + \frac{3}{7} y^7 \right|_0^{0.5}$

$= 280 \left[\left\{ \frac{(0.5)^5}{5} - \frac{(0.5)^8}{8} - \frac{3}{6}(0.5)^6 + \frac{3}{7}(0.5)^7 \right\} - \{0 - 0 - 0 + 0\} \right]$

$= 280(0.001297) = 0.3633$

c) $P(Y \geq 80\%) = P(Y \geq 0.8) = \int_{0.8}^{1} 280 y^4 (1-y)^3 dy$

$= 280 \int_{0.8}^{1} (y^4 - y^7 - 3y^5 + 3y^6) dy$

$= 280 \left| \frac{y^5}{5} - \frac{y^8}{8} - \frac{3y^6}{6} + \frac{3y^7}{7} \right|_{0.8}^{1}$

$= 280 \left[\left\{ \frac{1}{5} - \frac{1}{8} - \frac{1}{2} + \frac{3}{7} \right\} - \left\{ \frac{(0.8)^5}{5} - \frac{(0.8)^8}{8} - \frac{(0.8)^6}{2} + \frac{3(0.8)^7}{7} \right\} \right]$

$= 280 \left[0.003571 - 0.003370 \right]$

$= 280 (0.000200577)$

$= 0.05616$

3.34 **Solution:-**

a) let the random variable Y represents the no. of random tubes selected then $y = 0, 1, 2, 3, 4, 5$

The probability that random tube meets length specification is 0.99, then the prob. that random tube doesnot meet specification is $1 - 0.99 = 0.01$

If exactly $Y = y$ tubes meet length specifications then $5-y$ will not. The probability for one such situation is $(0.99)^y (0.01)^{5-y}$. Since 5 tubes can be selected in ${}^5C_y$ ways, so the prob function of Y is:

$$f(y) = {}^5C_y (0.99)^y (0.01)^{5-y}, \quad y = 0, 1, 2, 3, 4, 5$$

b) If 3 are outside specification in 5 tubes then 2 tubes will meet the specification

Required prob $= f(2) = {}^5C_2 (0.99)^2 (0.01)^{5-2}$

$= 10 \times (0.99)^2 \times (0.01)^3$

$= 0.000009801$

Since the prob. is very small so it doesnot support the given hypothesis.

3.35 **Solution:-**

$$f(x) = \frac{e^{-6}6^x}{x!} \quad \text{for } x = 0,1,2 \ldots$$

a) $P(X>8) = 1 - P(X \leq 8)$

$= 1 - [P(X=0) + P(X=1) + P(X=2) + P(X=3) + P(X=4) + P(X=5) + P(X=6) + P(X=7) + P(X=8)]$

$= 1 - \left[\frac{e^{-6}6^0}{0!} + \frac{e^{-6}6^1}{1!} + \frac{e^{-6}6^2}{2!} + \frac{e^{-6}6^3}{3!} + \frac{e^{-6}6^4}{4!} + \frac{e^{-6}6^5}{5!} + \frac{e^{-6}6^6}{6!} + \frac{e^{-6}6^7}{7!} + \frac{e^{-6} \cdot 6^8}{8!}\right]$

$= 1 - e^{-6}\left[1 + 6 + \frac{6^2}{2} + \frac{6^3}{6} + \frac{6^4}{24} + \frac{6^5}{120} + \frac{6^6}{720} + \frac{6^7}{5040} + \frac{6^8}{40320}\right]$

$= 1 - 0.00247(341.8) = 0.1558$

b) $P(x=2) = \frac{e^{-6}6^2}{2!} = \frac{e^{-6} \times 36}{2} = 0.0446$

3.36 **Solution:-**

$$f(x) = \begin{cases} 2(1-x), & 0<x<1 \\ 0, & \text{otherwise} \end{cases}$$

a) $P(X \leq 1/3) = \int_0^{1/3} 2(1-x)\,dx$

$= 2\left|x - \frac{x^2}{2}\right|_0^{1/3}$

$= 2\left[\left\{\frac{1}{3} - \frac{(1/3)^2}{2}\right\} - \left\{0 - \frac{0^2}{2}\right\}\right]$

$= 2\left[(0.33 - 0.056) - 0\right]$

$= 0.556$

b) $P(X>0.5) = \int_{0.5}^{1} 2(1-x)\,dx = 2\int_{0.5}^{1}(1-x)\,dx$

$= 2\left|x - \frac{x^2}{2}\right|_{0.5}^{1} = 2\left[\left\{1 - \frac{1^2}{2}\right\} - \left\{0.5 - \frac{0.5^2}{2}\right\}\right]$

$= 2\left[0.5 - 0.375\right] = 0.25$

(c) $P(X<0.75 \text{, Given that } X \geq 0.5) = P(X<0.75 \mid X \geq 0.5)$

$$= \frac{P(X<0.75 \text{ and } X \geq 0.5)}{P(X \geq 0.5)} \quad \text{Since } P(A|B) = \frac{P(A \cap B)}{P(B)}$$

$$= \frac{P(0.5 \leq X < 0.75)}{P(X \geq 0.5)}$$

where $P(0.5 \leq X < 0.75) = \int_{0.5}^{0.75} 2(1-x)\,dx$

$$= 2\left|x - \frac{x^2}{2}\right|_{0.5}^{0.75}$$

$$= 2\left[\left(0.75 - \frac{0.75^2}{2}\right) - \left(0.5 - \frac{0.5^2}{2}\right)\right]$$

$$= 2\left[0.46875 - 0.375\right] = 0.1875.$$

From part (b), $P(X>0.5) = 0.25$

$$\Rightarrow \frac{P(0.5 \leq X < 0.75)}{P(X>0.5)} = \frac{0.1875}{0.25} = 0.75$$