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15. Verify that $y = cx$ is a solution of the differential equation $xy' = y$ for every value of the parameter c. Find at least two solutions of the initial-value problem

$$xy' = y, \qquad y(0) = 0.$$

Observe that the piecewise-defined function

$$y = \begin{cases} 0, & x < 0 \\ x, & x \geq 0 \end{cases}$$

satisfies the condition $y(0) = 0$. Is it a solution of the initial-value problem?

16. (a) Consider the differential equation

$$\frac{dy}{dx} = 1 + y^2.$$

Determine a region of the xy-plane for which the equation has a unique solution through a point (x_0, y_0) in the region.

(b) Formally show that $y = \tan x$ satisfies the differential equation and the condition $y(0) = 0$.

(c) Explain why $y = \tan x$ is not a solution of the initial-value problem

$$\frac{dy}{dx} = 1 + y^2, \qquad y(0) = 0,$$

on the interval $(-2, 2)$.

(d) Explain why $y = \tan x$ is a solution of the initial-value problem in part (c) on the interval $(-1, 1)$.

In Problems 17–20 determine whether Theorem 2.1 guarantees that the differential equation $y' = \sqrt{y^2 - 9}$ possesses a unique solution through the given point.

17. $(1, 4)$

18. $(5, 3)$

19. $(2, -3)$

20. $(-1, 1)$

2.2 SEPARABLE VARIABLES

Note to the student: In solving a differential equation you will often have to utilize, say, integration by parts, partial fractions, or possibly a substitution. It will be worth a few minutes of your time to review some techniques of integration.

We begin our study of the methodology of solving first-order equations with the simplest of all differential equations.

If $g(x)$ is a given continuous function, then the first-order equation

$$\frac{dy}{dx} = g(x) \tag{1}$$

can be solved by integration. The solution of (1) is

$$y = \int g(x)\,dx + c.$$

EXAMPLE 1

Solve (a) $\dfrac{dy}{dx} = 1 + e^{2x}$ and (b) $\dfrac{dy}{dx} = \sin x$.

Solution As illustrated above, both equations can be solved by integration.

(a) $y = \int (1 + e^{2x})\,dx = x + \frac{1}{2}e^{2x} + c$

(b) $y = \int \sin x\,dx = -\cos x + c$ ∎

Equation (1), as well as its method of solution, is just a special case of the following:

DEFINITION 2.1 Separable Equation

A differential equation of the form

$$\frac{dy}{dx} = \frac{g(x)}{h(y)}$$

is said to be separable or to have **separable variables.**

Observe that a separable equation can be written as

$$h(y)\frac{dy}{dx} = g(x). \tag{2}$$

It is seen immediately that (2) reduces to (1) when $h(y) = 1$.

Now if $y = f(x)$ denotes a solution of (2), we must have

$$h(f(x))f'(x) = g(x),$$

and therefore
$$\int h(f(x))f'(x)\,dx = \int g(x)\,dx + c. \tag{3}$$

But $dy = f'(x)\,dx$, so (3) is the same as
$$\int h(y)\,dy = \int g(x)\,dx + c. \tag{4}$$

Method of Solution

Equation (4) indicates the procedure for solving separable differential equations. A one-parameter family of solutions, usually given implicitly, is obtained by integrating both sides of $h(y)\,dy = g(x)\,dx$.

Note There is no need to use two constants in the integration of a separable equation since
$$\int h(y)\,dy + c_1 = \int g(x)\,dx + c_2$$
$$\int h(y)\,dy = \int g(x)\,dx + c_2 - c_1 = \int g(x)\,dx + c,$$
where c is completely arbitrary. In many instances throughout the following chapters, we shall not hesitate to relabel constants in a manner that may prove convenient for a given equation. For example, multiples of constants or combinations of constants can sometimes be replaced by one constant.

EXAMPLE 2

Solve $$(1 + x)\,dy - y\,dx = 0.$$

Solution Dividing by $(1 + x)y$, we can write $dy/y = dx/(1 + x)$, from which it follows that
$$\int \frac{dy}{y} = \int \frac{dx}{1 + x}.$$
$$\ln|y| = \ln|1 + x| + c_1$$
$$y = e^{\ln|1+x| + c_1}$$
$$= e^{\ln|1+x|} \cdot e^{c_1}$$
$$= |1 + x|e^{c_1}.$$
$$= \pm e^{c_1}(1 + x)$$

$|1 + x| = 1 + x,\ x \geq -1$
$|1 + x| = -(1 + x),\ x < -1$

Relabeling $\pm e^{c_1}$ as c then gives $y = c(1 + x)$.

Alternative Solution Since each integral results in a logarithm, a judicious choice for the constant of integration is $\ln|c|$ rather than c:

$$\ln|y| = \ln|1 + x| + \ln|c|$$

or

$$\ln|y| = \ln|c(1 + x)|$$

so that

$$y = c(1 + x).$$

Even if not *all* the indefinite integrals are logarithms, it may still be advantageous to use $\ln|c|$. However, no firm rule can be given. ■

EXAMPLE 3

Solve the initial-value problem

$$\frac{dy}{dx} = -\frac{x}{y}, \quad y(4) = 3.$$

Solution From $y\,dy = -x\,dx$ we get

$$\int y\,dy = -\int x\,dx \quad \text{and} \quad \frac{y^2}{2} = -\frac{x^2}{2} + c_1.$$

This solution can be written as $x^2 + y^2 = c^2$ by replacing the constant $2c_1$ by c^2. The solution represents a family of concentric circles.

Now when $x = 4$, $y = 3$ so that $16 + 9 = 25 = c^2$. Thus the initial-value problem determines $x^2 + y^2 = 25$. In view of Theorem 2.1, we can conclude that it is the only circle of the family passing through the point (4, 3). See Figure 2.5. ■

Figure 2.5

EXAMPLE 4

Solve

$$xe^{-y}\sin x\,dx - y\,dy = 0.$$

Solution After multiplying by e^y, we get

$$x \sin x\,dx = ye^y\,dy.$$

Integration by parts on both sides of the equality gives

$$-x\cos x + \sin x = ye^y - e^y + c.$$ ■

EXAMPLE 5

Solve

$$xy^4\,dx + (y^2 + 2)e^{-3x}\,dy = 0. \qquad (5)$$

Solution By multiplying the given equation by e^{3x} and dividing by y^4, we obtain

$$xe^{3x}\,dx + \frac{y^2 + 2}{y^4}\,dy = 0 \quad \text{or} \quad xe^{3x}\,dx + (y^{-2} + 2y^{-4})\,dy = 0. \qquad (6)$$

Using integration by parts on the first term yields

$$\frac{1}{3}xe^{3x} - \frac{1}{9}e^{3x} - y^{-1} - \frac{2}{3}y^{-3} = c_1.$$

The one-parameter family of solutions can also be written as

$$e^{3x}(3x - 1) = \frac{9}{y} + \frac{6}{y^3} + c, \tag{7}$$

where the constant $9c_1$ is rewritten as c. ■

Two points are worth mentioning at this time. First, unless it is important or convenient there is no need to try to solve an expression representing a family of solutions for y explicitly in terms of x. Equation (7) shows that this task may present more problems than just the drudgery of symbol pushing. As a consequence it is often the case that the interval over which a solution is valid is not apparent. Second, some care should be exercised when separating variables to make certain that divisors are not zero. A constant solution may sometimes get lost in the shuffle of solving the problem. In Example 5 observe that $y = 0$ is a perfectly good solution of (5) but is not a member of the set of solutions defined by (7).

EXAMPLE 6

Solve the initial-value problem

$$\frac{dy}{dx} = y^2 - 4, \qquad y(0) = -2.$$

Solution We put the equation into the form

$$\frac{dy}{y^2 - 4} = dx \tag{8}$$

and use partial fractions on the left side. We have

$$\left[\frac{-\frac{1}{4}}{y + 2} + \frac{\frac{1}{4}}{y - 2}\right] dy = dx \tag{9}$$

so that

$$-\frac{1}{4}\ln|y + 2| + \frac{1}{4}\ln|y - 2| = x + c_1. \tag{10}$$

Thus

$$\ln\left|\frac{y - 2}{y + 2}\right| = 4x + c_2$$

and

$$\frac{y - 2}{y + 2} = ce^{4x},$$

where we have replaced $4c_1$ by c_2 and e^{c_2} by c. Finally, solving the last equation for y, we get

$$y = 2\frac{1 + ce^{4x}}{1 - ce^{4x}}. \tag{11}$$

Substituting $x = 0$, $y = -2$ leads to the dilemma

$$-2 = 2\frac{1 + c}{1 - c}$$

$$-1 + c = 1 + c \quad \text{or} \quad -1 = 1.$$

Let us consider the differential equation a little more carefully. The fact is, the equation

$$\frac{dy}{dx} = (y + 2)(y - 2)$$

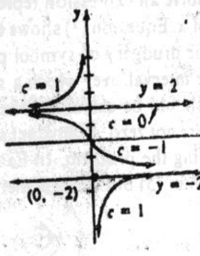

Figure 2.6

is satisfied by two constant functions, namely, $y = -2$ and $y = 2$. Inspection of equations (8), (9), and (10) clearly indicates we must preclude $y = -2$ and $y = 2$ at those steps in our solution. But it is interesting to observe that we can subsequently recover the solution $y = 2$ by setting $c = 0$ in equation (11). However, there is no finite value of c that will ever yield the solution $y = -2$. This latter constant function is the only solution to the original initial-value problem. See Figure 2.6. ■

If, in Example 6, we had used $\ln|c|$ for the constant of integration, then the form of the one-parameter family of solutions would be

$$y = 2\frac{c + e^{4x}}{c - e^{4x}}. \tag{12}$$

Note that (12) reduces to $y = -2$ when $c = 0$, but now there is no finite value of c that will give the constant solution $y = 2$.

If an initial condition leads to a particular solution by finding a specific value of the parameter c in a family of solutions for a first-order differential equation, it is a natural inclination of most students (and instructors) to relax and be content. In Section 2.1 we saw, however, that a solution of an initial-value problem may not be unique. For example, the problem

$$\frac{dy}{dx} = xy^{1/2}, \quad y(0) = 0, \tag{13}$$

has at least two solutions, namely, $y = 0$ and $y = x^4/16$. We are now in a position to solve the equation. Separating variables

$$y^{-1/2}\,dy = x\,dx$$

and integrating give

$$2y^{1/2} = \frac{x^2}{2} + c_1 \quad \text{or} \quad y = \left(\frac{x^2}{4} + c\right)^2.$$

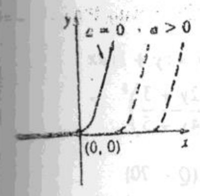

Figure 2.7

When $x = 0$, $y = 0$, so necessarily $c = 0$. Therefore $y = x^4/16$. The solution $y = 0$ was lost by dividing by $y^{1/2}$. In addition, the initial-value problem (13) possesses infinitely more solutions, since for any choice of the parameter $a \geq 0$ the piecewise-defined function

$$y = \begin{cases} 0, & x < a \\ \dfrac{(x^2 - a^2)^2}{16}, & x \geq a \end{cases}$$

satisfies both the differential equation and the initial condition. See Figure 2.7.

Remark We saw in some of the preceding examples that the constant in the one-parameter family of solutions for a first-order differential equation can be relabeled when convenient. Also, it can easily happen that two individuals solving the same equation arrive at dissimilar answers. For example, by separation of variables we can show that one-parameter families of solutions for $(1 + y^2)\,dx + (1 + x^2)\,dy = 0$ are

$$\arctan x + \arctan y = c \quad \text{or} \quad \arctan x + \arctan y = \arctan c \quad \text{or} \quad \frac{x + y}{1 - xy} = c.$$

As you work your way through the next several sections, bear in mind that families of solutions may be *equivalent* in the sense that one family may be obtained from another by either relabeling the constant or applying algebra and trigonometry.

2.2 EXERCISES

Answers to odd-numbered problems begin on page A-2.

In Problems 1–40 solve the given differential equation by separation of variables.

1. $\dfrac{dy}{dx} = \sin 5x$

2. $\dfrac{dy}{dx} = (x + 1)^2$

3. $dx + e^{3x}\,dy = 0$

4. $dx - x^2\,dy = 0$

5. $(x + 1)\dfrac{dy}{dx} = x + 6$

6. $e^x\dfrac{dy}{dx} = 2x$

7. $xy' = 4y$

8. $\dfrac{dy}{dx} + 2xy = 0$

9. $\dfrac{dy}{dx} = \dfrac{y^3}{x^2}$

10. $\dfrac{dy}{dx} = \dfrac{y + 1}{x}$

11. $\dfrac{dx}{dy} = \dfrac{x^2y^2}{1 + x}$

12. $\dfrac{dx}{dy} = \dfrac{1 + 2y^2}{y \sin x}$

13. $\dfrac{dy}{dx} = e^{3x + 2y}$

14. $e^x y\dfrac{dy}{dx} = e^{-y} + e^{-2x-y}$

15. $(4y + yx^2)\,dy - (2x + xy^2)\,dx = 0$
16. $(1 + x^2 + y^2 + x^2y^2)\,dy = y^2\,dx$
17. $2y(x + 1)\,dy = x\,dx$
18. $x^2y^2\,dy = (y + 1)\,dx$
19. $y \ln x \dfrac{dx}{dy} = \left(\dfrac{y + 1}{x}\right)^2$
20. $\dfrac{dy}{dx} = \left(\dfrac{2y + 3}{4x + 5}\right)^2$
21. $\dfrac{dS}{dr} = kS$
22. $\dfrac{dQ}{dt} = k(Q - 70)$
23. $\dfrac{dP}{dt} = P - P^2$
24. $\dfrac{dN}{dt} + N = Nte^{t+2}$
25. $\sec^2 x\,dy + \csc y\,dx = 0$
26. $\sin 3x\,dx + 2y\cos^3 3x\,dy = 0$
27. $e^y \sin 2x\,dx + \cos x(e^{2y} - y)\,dy = 0$
28. $\sec x\,dy = x \cot y\,dx$
29. $(e^y + 1)^2 e^{-y}\,dx + (e^x + 1)^3 e^{-x}\,dy = 0$
30. $\dfrac{y}{x}\dfrac{dy}{dx} = (1 + x^2)^{-1/2}(1 + y^2)^{1/2}$
31. $(y - yx^2)\dfrac{dy}{dx} = (y + 1)^2$
32. $2\dfrac{dy}{dx} - \dfrac{1}{y} = \dfrac{2x}{y}$
33. $\dfrac{dy}{dx} = \dfrac{xy + 3x - y - 3}{xy - 2x + 4y - 8}$
34. $\dfrac{dy}{dx} = \dfrac{xy + 2y - x - 2}{xy - 3y + x - 3}$?
35. $\dfrac{dy}{dx} = \sin x(\cos 2y - \cos^2 y)$
36. $\sec y \dfrac{dy}{dx} + \sin(x - y) = \sin(x + y)$
37. $x\sqrt{1 - y^2}\,dx = dy$
38. $y(4 - x^2)^{1/2}\,dy = (4 + y^2)^{1/2}\,dx$
39. $(e^x + e^{-x})\dfrac{dy}{dx} = y^2$
40. $(x + \sqrt{x})\dfrac{dy}{dx} = y + \sqrt{y}$

In Problems 41–48 solve the given differential equation subject to the indicated initial condition.

41. $(e^{-y} + 1)\sin x\,dx = (1 + \cos x)\,dy, \quad y(0) = 0$?
42. $(1 + x^4)\,dy + x(1 + 4y^2)\,dx = 0, \quad y(1) = 0$
43. $y\,dy = 4x(y^2 + 1)^{1/2}\,dx, \quad y(0) = 1$
44. $\dfrac{dy}{dt} + ty = y, \quad y(1) = 3$
45. $\dfrac{dx}{dy} = 4(x^2 + 1), \quad x\left(\dfrac{\pi}{4}\right) = 1$
46. $\dfrac{dy}{dx} = \dfrac{y^2 - 1}{x^2 - 1}, \quad y(2) = 2$
47. $x^2y' = y - xy, \quad y(-1) = -1$
48. $y' + 2y = 1, \quad y(0) = \dfrac{5}{2}$?

In Problems 49 and 50 find a solution of the given differential equation that passes through the indicated points.

49. $\dfrac{dy}{dx} - y^2 = -9$ (a) $(0, 0)$ (b) $(0, 3)$ (c) $\left(\dfrac{1}{3}, 1\right)$

50. $x\dfrac{dy}{dx} = y^2 - y$ (a) $(0, 1)$ (b) $(0, 0)$ (c) $\left(\dfrac{1}{2}, \dfrac{1}{2}\right)$

51. Find a singular solution for the equation in Problem 37.

52. Find a singular solution for the equation in Problem 39.

Often a radical change in the solution of a differential equation corresponds to a very small change in either the initial condition or the equation itself. In Problems 53–56 compare the solutions of the given initial-value problems.

53. $\dfrac{dy}{dx} = (y - 1)^2, \quad y(0) = 1$

54. $\dfrac{dy}{dx} = (y - 1)^2, \quad y(0) = 1.01$

55. $\dfrac{dy}{dx} = (y - 1)^2 + 0.01, \quad y(0) = 1$

56. $\dfrac{dy}{dx} = (y - 1)^2 - 0.01, \quad y(0) = 1$

A differential equation of the form $dy/dx = f(ax + by + c)$, $b \neq 0$, can always be reduced to an equation with separable variables by means of the substitution $u = ax + by + c$. Use this procedure to solve Problems 57–62.

57. $\dfrac{dy}{dx} = (x + y + 1)^2$

58. $\dfrac{dy}{dx} = \dfrac{1 - x - y}{x + y}$

59. $\dfrac{dy}{dx} = \tan^2(x + y)$

60. $\dfrac{dy}{dx} = \sin(x + y)$

61. $\dfrac{dy}{dx} = 2 + \sqrt{y - 2x + 3}$

62. $\dfrac{dy}{dx} = 1 + e^{y - x + 5}$

2.3 HOMOGENEOUS EQUATIONS

Before considering the concept of a homogeneous first-order differential equation and its method of solution, we need to examine closely the nature of a homogeneous function. We begin with the definition of this concept.

DEFINITION 2.2 Homogeneous Function

If a function f has the property that

$$f(tx, ty) = t^n f(x, y) \tag{1}$$

for some real number n, then f is said to be a **homogeneous function of degree n**.

EXAMPLE 1

(a) $f(x, y) = x^2 - 3xy + 5y^2$

$$\begin{aligned} f(tx, ty) &= (tx)^2 - 3(tx)(ty) + 5(ty)^2 \\ &= t^2x^2 - 3t^2xy + 5t^2y^2 \\ &= t^2[x^2 - 3xy + 5y^2] = t^2 f(x, y) \end{aligned}$$

The function is homogeneous of degree two.

(b) $f(x, y) = \sqrt[3]{x^2 + y^2}$

$$f(tx, ty) = \sqrt[3]{t^2x^2 + t^2y^2} = t^{2/3}\sqrt[3]{x^2 + y^2} = t^{2/3} f(x, y)$$

The function is homogeneous of degree 2/3.

(c) $f(x, y) = x^3 + y^3 + 1$

$$f(tx, ty) = t^3x^3 + t^3y^3 + 1 \neq t^3 f(x, y)$$

since $t^3 f(x, y) = t^3x^3 + t^3y^3 + t^3$. The function is not homogeneous.

(d) $f(x, y) = \dfrac{x}{2y} + 4$

$$f(tx, ty) = \frac{tx}{2ty} + 4 = \frac{x}{2y} + 4 = t^0 f(x, y)$$

The function is homogeneous of degree zero. ■

As parts (c) and (d) of Example 1 show, a constant added to a function destroys homogeneity, unless the function is homogeneous of degree zero. Also, in many instances a homogeneous function can be recognized by examining the *total degree* of each term.

EXAMPLE 2

(a) $f(x, y) = \overbrace{6\underbrace{x}_{\text{degree 1}}\underbrace{y^3}_{\text{degree 3}}}^{\text{degree 4}} - \overbrace{\underbrace{x^2}_{\text{degree 2}}\underbrace{y^2}_{\text{degree 2}}}^{\text{degree 4}}$

The function is homogeneous of degree four.

(b) $f(x, y) = \underbrace{x^2}_{\text{degree 2}} - \underbrace{y}_{\text{degree 1}}$ degrees not the same

The function is not homogeneous since the degrees of the two terms are different. ■

If $f(x, y)$ is a homogeneous function of degree n, notice that we can write

$$f(x, y) = x^n f\left(1, \frac{y}{x}\right) \quad \text{and} \quad f(x, y) = y^n f\left(\frac{x}{y}, 1\right), \tag{2}$$

where $f(1, y/x)$ and $f(x/y, 1)$ are both homogeneous of degree zero.

EXAMPLE 3

We see that $f(x, y) = x^2 + 3xy + y^2$ is homogeneous of degree two. Thus

$$f(x, y) = x^2\left[1 + 3\left(\frac{y}{x}\right) + \left(\frac{y}{x}\right)^2\right] = x^2 f\left(1, \frac{y}{x}\right)$$

$$f(x, y) = y^2\left[\left(\frac{x}{y}\right)^2 + 3\left(\frac{x}{y}\right) + 1\right] = y^2 f\left(\frac{x}{y}, 1\right).$$

A homogeneous first-order differential equation is defined in terms of homogeneous functions.

DEFINITION 2.3 Homogeneous Equation

A differential equation of the form

$$M(x, y)\,dx + N(x, y)\,dy = 0 \tag{3}$$

is said to be homogeneous if both coefficients M and N are homogeneous functions of the same degree.

In other words, $M(x, y)\,dx + N(x, y)\,dy = 0$ is homogeneous if

$$M(tx, ty) = t^n M(x, y) \quad \text{and} \quad N(tx, ty) = t^n N(x, y).$$

Method of Solution

A homogeneous differential equation $M(x, y)\,dx + N(x, y)\,dy = 0$ can be solved by means of an algebraic substitution. Specifically, *either* substitution $y = ux$ or $x = vy$, where u and v are new dependent variables, *will reduce the equation to a separable first-order differential equation.* To see this, let us substitute $y = ux$ and its differential $dy = u\,dx + x\,du$ into (3):

$$M(x, ux)\,dx + N(x, ux)[u\,dx + x\,du] = 0.$$

Now, by the homogeneity property given in (2), we can write

$$x^n M(1, u)\,dx + x^n N(1, u)[u\,dx + x\,du] = 0$$

or

$$[M(1, u) + uN(1, u)]\,dx + xN(1, u)\,du = 0,$$

which gives $$\frac{dx}{x} + \frac{N(1, u)\,du}{M(1, u) + uN(1, u)} = 0.$$

We hasten to point out that the preceeding formula should not be memorized; rather, the *procedure should be worked through each time*. The proof that the substitution $x = vy$ in (3) also leads to a separable equation is left as an exercise. See Problem 45.

EXAMPLE 4

Solve $$(x^2 + y^2)\,dx + (x^2 - xy)\,dy = 0.$$

Solution Both $M(x, y)$ and $N(x, y)$ are homogeneous of degree two. If we let $y = ux$, it follows that

$$(x^2 + u^2x^2)\,dx + (x^2 - ux^2)[u\,dx + x\,du] = 0$$

$$x^2(1 + u)\,dx + x^3(1 - u)\,du = 0$$

$$\frac{1 - u}{1 + u}\,du + \frac{dx}{x} = 0$$

$$\left[-1 + \frac{2}{1 + u}\right] du + \frac{dx}{x} = 0 \qquad \boxed{\text{long division}}$$

After integration the last line gives

$$-u + 2\ln|1 + u| + \ln|x| = \ln|c|$$

$$-\frac{y}{x} + 2\ln\left|1 + \frac{y}{x}\right| + \ln|x| = \ln|c|.$$

Using the properties of logarithms, we can write the preceding solution as

$$\ln\left|\frac{(x + y)^2}{cx}\right| = \frac{y}{x}.$$

The definition of a logarithm then yields

$$(x + y)^2 = cxe^{y/x}.$$ ■

EXAMPLE 5

Solve $$(2\sqrt{xy} - y)\,dx - x\,dy = 0.$$

Solution The coefficients $M(x, y)$ and $N(x, y)$ are homogeneous of degree one. If $y = ux$, the differential equation becomes, after simplifying,

$$\frac{du}{2u - 2u^{1/2}} + \frac{dx}{x} = 0.$$

The integral of the first term can be evaluated by the further substitution $t = u^{1/2}$. The result is

$$\frac{dt}{t-1} + \frac{dx}{x} = 0.$$

Integrating yields

$$\ln|t-1| + \ln|x| = \ln|c|$$

$$\ln\left|\sqrt{\frac{y}{x}} - 1\right| + \ln|x| = \ln|c| \quad \boxed{t = u^{1/2} \text{ and } u = y/x}$$

$$x\left(\sqrt{\frac{y}{x}} - 1\right) = c$$

$$\sqrt{xy} - x = c. \qquad \blacksquare$$

By now you may be asking: when should the substitution $x = vy$ be used? Although it can be used for every homogeneous differential equation, in practice we try $x = vy$ whenever the function $M(x, y)$ is simpler than $N(x, y)$. In solving $(x^2 + y^2)\,dx + (x^2 - xy)\,dy = 0$, for example, we know there is no appreciable difference between M and N, so either $y = ux$ or $x = vy$ could be used. Also, it could happen that after using one substitution, we encounter integrals that are difficult or impossible to evaluate in closed form; switching substitutions may result in an easier problem.

EXAMPLE 6

Solve $$2x^3y\,dx + (x^4 + y^4)\,dy = 0.$$

Solution Each coefficient is a homogeneous function of degree four. Since the coefficient of dx is slightly simpler than the coefficient of dy, we try $x = vy$. After substituting, we simplify the equation

$$2v^3y^4[v\,dy + y\,dv] + (v^4y^4 + y^4)\,dy = 0$$

to

$$\frac{2v^3\,dv}{3v^4 + 1} + \frac{dy}{y} = 0.$$

Integration gives

$$\frac{1}{6}\ln(3v^4 + 1) + \ln|y| = \ln|c_1| \quad \text{or} \quad 3x^4y^2 + y^6 = c,$$

where $c = c_1^6$. Now had the substitution $y = ux$ been used, then

$$\frac{dx}{x} + \frac{u^4 + 1}{u^5 + 3u}\,du = 0.$$

You are urged to reflect on how to evaluate the integral of the second term in the last equation. $\blacksquare$

A homogeneous differential equation can always be expressed in the alternative form

$$\frac{dy}{dx} = F\left(\frac{y}{x}\right).$$

To see this suppose we write the equation $M(x, y)\,dx + N(x, y)\,dy = 0$ as $dy/dx = f(x, y)$, where

$$f(x, y) = -\frac{M(x, y)}{N(x, y)}.$$

The function $f(x, y)$ must necessarily be homogeneous of degree zero when M and N are homogeneous of degree n. From (2), it follows that

$$f(x, y) = -\frac{x^n M\left(1, \frac{y}{x}\right)}{x^n N\left(1, \frac{y}{x}\right)} = -\frac{M\left(1, \frac{y}{x}\right)}{N\left(1, \frac{y}{x}\right)}.$$

The last ratio is recognized as a function of the form $F(y/x)$. We leave it as an exercise to demonstrate that a homogeneous differential equation can also be written as $dy/dx = G(x/y)$. See Problem 47.

EXAMPLE 7

Solve the initial-value problem

$$x\frac{dy}{dx} = y + xe^{y/x}, \qquad y(1) = 1.$$

Solution By writing the equation as

$$\frac{dy}{dx} = \frac{y}{x} + e^{y/x},$$

we see that the function to the right of the equality is homogeneous of degree zero. From the form of this function we are prompted to use $u = y/x$. After differentiating $y = ux$ by the product rule and substituting, we find

$$u + x\frac{du}{dx} = u + e^u \quad \text{or} \quad e^{-u}\,du = \frac{dx}{x}.$$

Integrating and substituting $u = y/x$ give

$$-e^{-u} + c = \ln|x|$$

$$-e^{-y/x} + c = \ln|x|.$$

Since $y = 1$ when $x = 1$, we get $-e^{-1} + c = 0$ or $c = e^{-1}$. Therefore the solution to the initial-value problem is

$$e^{-1} - e^{-y/x} = \ln|x|.$$ ■

2.3 EXERCISES

Answers to odd-numbered problems begin on page A-3.

In Problems 1-10 determine whether the given function is homogeneous. If so, state the degree of homogeneity.

1. $x^3 + 2xy^2 - y^4/x$
2. $\sqrt{x+y}(4x + 3y)$
3. $\dfrac{x^3y - x^2y^2}{(x+8y)^2}$
4. $\dfrac{x}{y^2 + \sqrt{x^4 + y^4}}$
5. $\cos \dfrac{x^2}{x+y}$
6. $\sin \dfrac{x}{x+y}$
7. $\ln x^2 - 2 \ln y$
8. $\dfrac{\ln x^3}{\ln y^3}$
9. $(x^{-1} + y^{-1})^2$
10. $(x + y + 1)^2$

In Problems 11–30 solve the given differential equation by using an appropriate substitution.

11. $(x - y)\,dx + x\,dy = 0$
12. $(x + y)\,dx + x\,dy = 0$
13. $x\,dx + (y - 2x)\,dy = 0$
14. $y\,dx = 2(x + y)\,dy$
15. $(y^2 + yx)\,dx - x^2\,dy = 0$
16. $(y^2 + yx)\,dx + x^2\,dy = 0$
17. $\dfrac{dy}{dx} = \dfrac{y - x}{y + x}$
18. $\dfrac{dy}{dx} = \dfrac{x + 3y}{3x + y}$
19. $-y\,dx + (x + \sqrt{xy})\,dy = 0$
20. $x\dfrac{dy}{dx} - y = \sqrt{x^2 + y^2}$
21. $2x^2y\,dx = (3x^3 + y^3)\,dy$
22. $(x^4 + y^4)\,dx - 2x^3y\,dy = 0$
23. $\dfrac{dy}{dx} = \dfrac{y}{x} + \dfrac{x}{y}$
24. $\dfrac{dy}{dx} = \dfrac{y}{x} + \dfrac{x^2}{y^2} + 1$
25. $y\dfrac{dx}{dy} = x + 4ye^{-2x/y}$
26. $(x^2e^{-y/x} + y^2)\,dx = xy\,dy$
27. $\left(y + x\cot\dfrac{y}{x}\right)dx - x\,dy = 0.$
28. $\dfrac{dy}{dx} = \dfrac{y}{x}\ln\dfrac{y}{x}$
29. $(x^2 + xy - y^2)\,dx + xy\,dy = 0$
30. $(x^2 + xy + 3y^2)\,dx - (x^2 + 2xy)\,dy = 0$

In Problems 31–44 solve the given differential equation subject to the indicated initial condition.

31. $xy^2\dfrac{dy}{dx} = y^3 - x^3, \quad y(1) = 2$
32. $(x^2 + 2y^2)\,dx = xy\,dy, \quad y(-1) = 1$
33. $2x^2\dfrac{dy}{dx} = 3xy + y^2, \quad y(1) = -2$

34. $xy\,dx - x^2\,dy = y\sqrt{x^2 + y^2}\,dy, \quad y(0) = 1$

35. $(x + ye^{y/x})\,dx - xe^{y/x}\,dy = 0, \quad y(1) = 0$

36. $y\,dx + \left(y\cos\frac{x}{y} - x\right)dy = 0, \quad y(0) = 2$?

37. $(y^2 + 3xy)\,dx = (4x^2 + xy)\,dy, \quad y(1) = 1$

38. $y^3\,dx = 2x^3\,dy - 2x^2y\,dx, \quad y(1) = \sqrt{2}$

39. $(x + \sqrt{xy})\frac{dy}{dx} + x - y = x^{-1/2}y^{3/2}, \quad y(1) = 1$

40. $y\,dx + x(\ln x - \ln y - 1)\,dy = 0, \quad y(1) = e$

41. $y^2\,dx + (x^2 + xy + y^2)\,dy = 0, \quad y(0) = 1$

42. $(\sqrt{x} + \sqrt{y})^2\,dx = x\,dy, \quad y(1) = 0$

43. $(x + \sqrt{y^2 - xy})\frac{dy}{dx} = y, \quad y(\tfrac{1}{2}) = 1$

44. $\frac{dy}{dx} - \frac{y}{x} = \cosh\frac{y}{x}, \quad y(1) = 0$

45. Suppose $M(x, y)\,dx + N(x, y)\,dy = 0$ is a homogeneous equation. Show that the substitution $x = vy$ reduces the equation to one with separable variables.

46. Suppose $M(x, y)\,dx + N(x, y)\,dy = 0$ is a homogeneous equation. Show that the substitutions $x = r\cos\theta$, $y = r\sin\theta$ reduce the equation to one with separable variables.

47. Suppose $M(x, y)\,dx + N(x, y)\,dy = 0$ is a homogeneous equation. Show that the equation has the alternative form $dy/dx = G(x/y)$.

48. If $f(x, y)$ is a homogeneous function of degree n, show that

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf.$$

2.4 EXACT EQUATIONS

Although the simple equation

$$y\,dx + x\,dy = 0$$

is both separable and homogeneous, we should recognize that it is also equivalent to the differential of the product of x and y; that is,

$$y\,dx + x\,dy = d(xy) = 0.$$

By integrating we immediately obtain the implicit solution $xy = c$.

From calculus you might remember that if $z = f(x, y)$ is a function having continuous first partial derivatives in a region R of the xy-plane, then its *total differential* is

$$dz = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy. \tag{1}$$

Now if $f(x, y) = c$, it follows from (1) that

$$\frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy = 0. \tag{2}$$

In other words, given a family of curves $f(x, y) = c$, we can generate a first-order differential equation by computing the total differential.

EXAMPLE 1

If $x^2 - 5xy + y^3 = c$, then (2) gives

$$(2x - 5y)\,dx + (-5x + 3y^2)\,dy = 0 \quad \text{or} \quad \frac{dy}{dx} = \frac{5y - 2x}{-5x + 3y^2}.$$

For our purposes, it is more important to turn the problem around; namely, given an equation such as

$$\frac{dy}{dx} = \frac{5y - 2x}{-5x + 3y^2}, \tag{3}$$

can we identify the equation as being equivalent to the statement

$$d(x^2 - 5xy + y^3) = 0?$$

Notice that equation (3) is neither separable nor homogeneous.

DEFINITION 2.4 Exact Equation

A differential expression

$$M(x, y)\,dx + N(x, y)\,dy$$

is an **exact differential** in a region R of the xy-plane if it corresponds to the total differential of some function $f(x, y)$. A differential equation of the form

$$M(x, y)\,dx + N(x, y)\,dy = 0$$

is said to be an **exact equation** if the expression on the left side is an exact differential.

EXAMPLE 2

The equation $x^2y^3\,dx + x^3y^2\,dy = 0$ is exact since it is recognized that

$$d(\tfrac{1}{3}x^3y^3) = x^2y^3\,dx + x^3y^2\,dy.$$

The following theorem is a test for an exact differential.

THEOREM 2.2 Criterion for an Exact Differential

Let $M(x, y)$ and $N(x, y)$ be continuous and have continuous first partial derivatives in a rectangular region R defined by $a < x < b$, $c < y < d$. Then a necessary and sufficient condition that

$$M(x, y)\,dx + N(x, y)\,dy$$

be an exact differential is

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}. \tag{4}$$

Proof of the Necessity For simplicity let us assume that $M(x, y)$ and $N(x, y)$ have continuous first partial derivatives for all (x, y). Now if the expression $M(x, y)\,dx + N(x, y)\,dy$ is exact, there exists some function f for which

$$M(x, y)\,dx + N(x, y)\,dy = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy$$

for all (x, y) in R. Therefore

$$M(x, y) = \frac{\partial f}{\partial x}, \qquad N(x, y) = \frac{\partial f}{\partial y},$$

and $$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y\,\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial N}{\partial x}.$$

The equality of the mixed partials is a consequence of the continuity of the first partial derivatives of $M(x, y)$ and $N(x, y)$. $\square$

The sufficiency part of Theorem 2.2 consists of showing that there exists a function f for which $\partial f/\partial x = M(x, y)$ and $\partial f/\partial y = N(x, y)$ whenever (4) holds. The construction of the function f actually reflects a basic procedure for solving exact equations.

Method of Solution

Given the equation

$$M(x, y)\,dx + N(x, y)\,dy = 0, \tag{5}$$

first show that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

Then assume that

$$\frac{\partial f}{\partial x} = M(x, y)$$

so we can find f by integrating $M(x, y)$ with respect to x, while holding y constant. We write

$$f(x, y) = \int M(x, y)\,dx + g(y), \tag{6}$$

where the arbitrary function $g(y)$ is the "constant" of integration. Now differentiate (6) with respect to y and assume $\partial f/\partial y = N(x, y)$:

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\int M(x, y)\,dx + g'(y) = N(x, y).$$

This gives

$$g'(y) = N(x, y) - \frac{\partial}{\partial y}\int M(x, y)\,dx. \tag{7}$$

Finally, integrate (7) with respect to y and substitute the result in (6). The solution of the equation is $f(x, y) = c$.

Note Some observations are in order. First, it is important to realize that the expression $N(x, y) - (\partial/\partial y)\int M(x, y)\,dx$ in (7) is independent of x, since

$$\frac{\partial}{\partial x}\left[N(x, y) - \frac{\partial}{\partial y}\int M(x, y)\,dx\right] = \frac{\partial N}{\partial x} - \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}\int M(x, y)\,dx\right)$$

$$= \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 0.$$

Second, we could just as well start the foregoing procedure with the assumption that $\partial f/\partial y = N(x, y)$. After integrating N with respect to y and then differentiating that result, we find the analogues of (6) and (7) would be, respectively,

$$f(x, y) = \int N(x, y)\,dy + h(x) \quad \text{and} \quad h'(x) = M(x, y) - \frac{\partial}{\partial x}\int N(x, y)\,dy.$$

In either case *none of these formulas should be memorized.* Also, when testing an equation for exactness, make sure it is of form (5). Often a differential equation is written $G(x, y)\,dx = H(x, y)\,dy$. In this case write the equation as $G(x, y)\,dx - H(x, y)\,dy = 0$ and then identify $M(x, y) = G(x, y)$ and $N(x, y) = -H(x, y)$.

EXAMPLE 3

Solve
$$2xy\,dx + (x^2 - 1)\,dy = 0.$$

Solution With $M(x, y) = 2xy$ and $N(x, y) = x^2 - 1$ we have

$$\frac{\partial M}{\partial y} = 2x = \frac{\partial N}{\partial x}.$$

Thus the equation is exact and so, by Theorem 2.2, there exists a function $f(x, y)$ such that

$$\frac{\partial f}{\partial x} = 2xy \quad \text{and} \quad \frac{\partial f}{\partial y} = x^2 - 1.$$

From the first of these equations we obtain, after integrating,

$$f(x, y) = x^2 y + g(y).$$

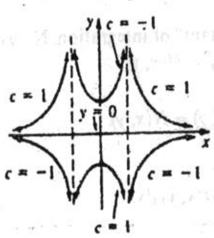

Figure 2.8

Taking the partial derivative of the last expression with respect to y and setting the result equal to $N(x, y)$ give

$$\frac{\partial f}{\partial y} = x^2 + g'(y) = x^2 - 1. \qquad \boxed{N(x,y)}$$

It follows that
$$g'(y) = -1 \quad \text{and} \quad g(y) = -y.$$

The constant of integration need not be included in the preceding line since the solution is $f(x, y) = c$. Some of the family of curves $x^2y - y = c$ are given in Figure 2.8.

Note The solution of the equation is *not* $f(x, y) = x^2y - y$. Rather it is $f(x, y) = c$ or $f(x, y) = 0$ if a constant is used in the integration of $g'(y)$. Observe that the equation could also be solved by separation of variables. ∎

EXAMPLE 4

Solve
$$(e^{2y} - y\cos xy)\,dx + (2xe^{2y} - x\cos xy + 2y)\,dy = 0.$$

Solution The equation is neither separable nor homogeneous but is exact since

$$\frac{\partial M}{\partial y} = 2e^{2y} + xy\sin xy - \cos xy = \frac{\partial N}{\partial x}.$$

Hence a function $f(x, y)$ exists for which

$$M(x, y) = \frac{\partial f}{\partial x} \quad \text{and} \quad N(x, y) = \frac{\partial f}{\partial y}.$$

Now for variety we shall start with the assumption that $\partial f/\partial y = N(x, y)$: that is,

$$\frac{\partial f}{\partial y} = 2xe^{2y} - x\cos xy + 2y$$

$$f(x, y) = 2x\int e^{2y}\,dy - x\int \cos xy\,dy + 2\int y\,dy.$$

Remember, the reason x can come out in front of the symbol $\int$ is that in the integration with respect to y, x is treated as an ordinary constant. It follows that

$$f(x, y) = xe^{2y} - \sin xy + y^2 + h(x)$$

$$\frac{\partial f}{\partial x} = e^{2y} - y\cos xy + h'(x) = e^{2y} - y\cos xy,$$

so that $\quad h'(x) = 0 \quad$ and $\quad h(x) = c.$

Hence a one-parameter family of solutions is given by

$$xe^{2y} - \sin xy + y^2 + c = 0.$$ ■

EXAMPLE 5

Solve the initial-value problem

$$(\cos x \sin x - xy^2)\,dx + y(1 - x^2)\,dy = 0, \qquad y(0) = 2.$$

Solution The equation is exact since

$$\frac{\partial M}{\partial y} = -2xy = \frac{\partial N}{\partial x}.$$

Now

$$\frac{\partial f}{\partial y} = y(1 - x^2)$$

$$f(x, y) = \frac{y^2}{2}(1 - x^2) + h(x)$$

$$\frac{\partial f}{\partial x} = -xy^2 + h'(x) = \cos x \sin x - xy^2.$$

The last equation implies

$$h'(x) = \cos x \sin x$$

$$h(x) = -\int (\cos x)(-\sin x\,dx) = -\frac{1}{2}\cos^2 x.$$

Thus
$$\frac{y^2}{2}(1-x^2)-\frac{1}{2}\cos^2 x = c_1$$

or
$$y^2(1-x^2)-\cos^2 x = c,$$

where $2c_1$ has been replaced by c. The initial condition $y = 2$ when $x = 0$ demands that $4(1) - \cos^2(0) = c$ or that $c = 3$. Thus a solution of the problem is

$$y^2(1-x^2)-\cos^2 x = 3.$$

Integrating Factor

It is sometimes possible to convert a nonexact differential equation into an exact equation by multiplying it by a function $\mu(x, y)$ called an **integrating factor.** However, the resulting exact equation

$$\mu M(x, y)\,dx + \mu N(x, y)\,dy = 0$$

may not be equivalent to the original equation in the sense that a solution of one is also a solution of the other. It is possible for a solution to be lost or gained as a result of the multiplication.

EXAMPLE 6

Solve $(x + y)\,dx + x \ln x\,dy = 0$, using $\mu(x, y) = \frac{1}{x}$, on $(0, \infty)$.

Solution Let $M(x, y) = x + y$ and $N(x, y) = x \ln x$ so that $\partial M/\partial y = 1$ and $\partial N/\partial x = 1 + \ln x$. The equation is not exact. However, if we multiply the equation by $\mu(x, y) = 1/x$, we obtain

$$\left(1+\frac{y}{x}\right)dx + \ln x\,dy = 0.$$

From this latter form we make the identifications:

$$M(x, y) = 1 + \frac{y}{x}, \qquad N(x, y) = \ln x, \qquad \frac{\partial M}{\partial y} = \frac{1}{x} = \frac{\partial N}{\partial x}.$$

Therefore the second differential equation is exact. It follows that

$$\frac{\partial f}{\partial x} = 1 + \frac{y}{x} = M(x, y)$$

$$f(x, y) = x + y \ln x + g(y)$$

$$\frac{\partial f}{\partial y} = 0 + \ln x + g'(y) = \ln x$$

and so
$$g'(y) = 0 \quad \text{and} \quad g(y) = c.$$

Hence $f(x, y) = x + y \ln x + c$. It is readily verified that

$$x + y \ln x + c = 0$$

is a solution of both equations on $(0, \infty)$. ■

2.4 EXERCISES

Answers to odd-numbered problems begin on page A-3.

In Problems 1–24 determine whether the given equation is exact. If exact, solve.

1. $(2x - 1)\,dx + (3y + 7)\,dy = 0$
2. $(2x + y)\,dx - (x + 6y)\,dy = 0$
3. $(5x + 4y)\,dx + (4x - 8y^3)\,dy = 0$
4. $(\sin y - y \sin x)\,dx + (\cos x + x \cos y - y)\,dy = 0$
5. $(2y^2x - 3)\,dx + (2yx^2 + 4)\,dy = 0$
6. $\left(2y - \dfrac{1}{x} + \cos 3x\right)\dfrac{dy}{dx} + \dfrac{y}{x^2} - 4x^3 + 3y \sin 3x = 0$
7. $(x + y)(x - y)\,dx + x(x - 2y)\,dy = 0$
8. $\left(1 + \ln x + \dfrac{y}{x}\right)dx = (1 - \ln x)\,dy$
9. $(y^3 - y^2 \sin x - x)\,dx + (3xy^2 + 2y \cos x)\,dy = 0$
10. $(x^3 + y^3)\,dx + 3xy^2\,dy = 0$
11. $(y \ln y - e^{-xy})\,dx + \left(\dfrac{1}{y} + x \ln y\right)dy = 0$
12. $\dfrac{2x}{y}\,dx - \dfrac{x^2}{y^2}\,dy = 0$
13. $x\dfrac{dy}{dx} = 2xe^x - y + 6x^2$
14. $(3x^2y + e^y)\,dx + (x^3 + xe^y - 2y)\,dy = 0$
15. $\left(1 - \dfrac{3}{x} + y\right)dx + \left(1 - \dfrac{3}{y} + x\right)dy = 0$
16. $(e^y + 2xy \cosh x)y' + xy^2 \sinh x + y^2 \cosh x = 0$
17. $\left(x^2y^3 - \dfrac{1}{1 + 9x^2}\right)\dfrac{dx}{dy} + x^3y^2 = 0$
18. $(5y - 2x)y' - 2y = 0$
19. $(\tan x - \sin x \sin y)\,dx + \cos x \cos y\,dy = 0$
20. $(3x \cos 3x + \sin 3x - 3)\,dx + (2y + 5)\,dy = 0$
21. $(1 - 2x^2 - 2y)\dfrac{dy}{dx} = 4x^3 + 4xy$
22. $(2y \sin x \cos x - y + 2y^2e^{xy^2})\,dx = (x - \sin^2 x - 4xye^{xy^2})\,dy$

23. $(4x^3y - 15x^2 - y)\,dx + (x^4 + 3y^2 - x)\,dy = 0$

24. $\left(\frac{1}{x} + \frac{1}{x^2} - \frac{y}{x^2 + y^2}\right)dx + \left(ye^y + \frac{x}{x^2 + y^2}\right)dy = 0$

In Problems 25–30 solve the given differential equation subject to the indicated initial condition.

25. $(x + y)^2\,dx + (2xy + x^2 - 1)\,dy = 0, \quad y(1) = 1$

26. $(e^x + y)\,dx + (2 + x + ye^y)\,dy = 0, \quad y(0) = 1$

27. $(4y + 2x - 5)\,dx + (6y + 4x - 1)\,dy = 0, \quad y(-1) = 2$

28. $\left(\frac{3y^2 - x^2}{y^5}\right)\frac{dy}{dx} + \frac{x}{2y^4} = 0, \quad y(1) = 1$

29. $(y^2 \cos x - 3x^2y - 2x)\,dx + (2y \sin x - x^3 + \ln y)\,dy = 0, \quad y(0) = e$

30. $\left(\frac{1}{1 + y^2} + \cos x - 2xy\right)\frac{dy}{dx} = y(y + \sin x), \quad y(0) = 1$

In Problems 31–34 find the value of k so that the given differential equation is exact.

31. $(y^3 + kxy^4 - 2x)\,dx + (3xy^2 + 20x^2y^3)\,dy = 0$

32. $(2x - y \sin xy + ky^4)\,dx - (20xy^3 + x \sin xy)\,dy = 0$

33. $(2xy^2 + ye^x)\,dx + (2x^2y + ke^x - 1)\,dy = 0$

34. $(6xy^3 + \cos y)\,dx + (kx^2y^2 - x \sin y)\,dy = 0$

35. Determine a function $M(x, y)$ so that the following differential equation is exact:

$$M(x, y)\,dx + \left(xe^{xy} + 2xy + \frac{1}{x}\right)dy = 0.$$

36. Determine a function $N(x, y)$ so that the following differential equation is exact:

$$\left(y^{1/2}x^{-1/2} + \frac{x}{x^2 + y}\right)dx + N(x, y)\,dy = 0.$$

In Problems 37–42 solve the given differential equation by verifying that the indicated function $\mu(x, y)$ is an integrating factor.

37. $6xy\,dx + (4y + 9x^2)\,dy = 0, \quad \mu(x, y) = y^2$

38. $-y^2\,dx + (x^2 + xy)\,dy = 0, \quad \mu(x, y) = 1/x^2y$

39. $(-xy \sin x + 2y \cos x)\,dx + 2x \cos x\,dy = 0, \quad \mu(x, y) = xy$

40. $y(x + y + 1)\,dx + (x + 2y)\,dy = 0, \quad \mu(x, y) = e^x$

41. $(2y^2 + 3x)\,dx + 2xy\,dy = 0, \quad \mu(x, y) = x$

42. $(x^2 + 2xy - y^2)\,dx + (y^2 + 2xy - x^2)\,dy = 0, \quad \mu(x, y) = (x + y)^{-2}$

43. Show that any separable first-order differential equation in the form $h(y)\,dy - g(x)\,dx = 0$ is also exact.

or
$$\frac{d\mu}{dx} = \mu P(x).$$

This is a separable equation from which we can determine $\mu(x)$. We have

$$\frac{d\mu}{\mu} = P(x)\,dx$$

$$\ln|\mu| = \int P(x)\,dx \tag{5}$$

so that
$$\mu(x) = e^{\int P(x)\,dx}. \tag{6}$$

The function $\mu(x)$ defined in (6) is an **integrating factor** for the linear equation. Note that we need not use a constant of integration in (5) since (3) is unaffected by a constant multiple. Also, $\mu(x) \neq 0$ for every x in I and is continuous and differentiable.

It is interesting to observe that equation (3) is still an exact differential equation even when $f(x) = 0$. In fact, $f(x)$ plays no part in determining $\mu(x)$ since we see from (4) that $(\partial/\partial y)\mu(x)f(x) = 0$. Thus both

$$e^{\int P(x)\,dx}\,dy + e^{\int P(x)\,dx}[P(x)y - f(x)]\,dx$$

and
$$e^{\int P(x)\,dx}\,dy + e^{\int P(x)\,dx}P(x)y\,dx$$

are exact differentials. We now write (3) in the form

$$e^{\int P(x)\,dx}\,dy + e^{\int P(x)\,dx}P(x)y\,dx = e^{\int P(x)\,dx}f(x)\,dx$$

and recognize that we can write this equation as

$$d[e^{\int P(x)\,dx}y] = e^{\int P(x)\,dx}f(x)\,dx.$$

Integrating the last equation gives

$$e^{\int P(x)\,dx}y = \int e^{\int P(x)\,dx}f(x)\,dx + c$$

or
$$y = e^{-\int P(x)\,dx}\int e^{\int P(x)\,dx}f(x)\,dx + ce^{-\int P(x)\,dx}. \tag{7}$$

In other words, if (1) has a solution, it must be of form (7). Conversely, it is a straightforward matter to verify that (7) constitutes a one-parameter family of solutions of equation (1).

Summary of the Method

No attempt should be made to memorize the formula given in (7). The procedure should be followed each time, so for convenience we summarize the results.

Solving a Linear First-Order Equation

(i) To solve a linear first-order equation first put it into the form (1); that is, make the coefficient of dy/dx unity.

(ii) Identify $P(x)$ and find the integrating factor

$$e^{\int P(x)\,dx}.$$

(iii) Multiply the equation obtained in step (i) by the integrating factor:

$$e^{\int P(x)\,dx}\frac{dy}{dx} + P(x)e^{\int P(x)\,dx}y = e^{\int P(x)\,dx}f(x).$$

(iv) The left side of the equation in step (iii) is the derivative of the product of the integrating factor and the dependent variable y; that is,

$$\frac{d}{dx}\left[e^{\int P(x)\,dx}y\right] = e^{\int P(x)\,dx}f(x).$$

(v) Integrate both sides of the equation found in step (iv).

EXAMPLE 1

Solve $$x\frac{dy}{dx} - 4y = x^6e^x.$$

Solution Write the equation as

$$\frac{dy}{dx} - \frac{4}{x}y = x^5e^x \tag{8}$$

by dividing by x. Since $P(x) = -4/x$, the integrating factor is

$$e^{-4\int dx/x} = e^{-4\ln|x|} = e^{\ln x^{-4}} = x^{-4}.$$

Here we have used the basic identity $b^{\log_b N} = N$, $N > 0$. Now we multiply (8) by this term

$$x^{-4}\frac{dy}{dx} - 4x^{-5}y = xe^x \tag{9}$$

and obtain

$$\frac{d}{dx}[x^{-4}y] = xe^x.^* \tag{10}$$

---

* You should perform the indicated differentiations a few times in order to be convinced that all equations, such as (8), (9), and (10), are formally equivalent.

It follows from integration by parts that

$$x^{-4}y = xe^x - e^x + c$$

or $$y = x^5e^x - x^4e^x + cx^4.$$ ■

EXAMPLE 2

Solve $$\frac{dy}{dx} - 3y = 0.$$

Solution The equation is already in form (1). Hence the integrating factor is

$$e^{\int(-3)dx} = e^{-3x}.$$

Therefore $$e^{-3x}\frac{dy}{dx} - 3e^{-3x}y = 0$$

$$\frac{d}{dx}[e^{-3x}y] = 0$$

$$e^{-3x}y = c$$

and so $$y = ce^{3x}.$$ ■

General Solution

If it is assumed that $P(x)$ and $f(x)$ are continuous on an interval I and x_0 is any point in the interval, then it follows from Theorem 2.1 that there exists only one solution of the initial-value problem

$$\frac{dy}{dx} + P(x)y = f(x), \quad y(x_0) = y_0. \tag{11}$$

But we saw earlier that (1) possesses a family of solutions and that every solution of the equation on the interval I is of form (7). Thus, obtaining the solution of (11) is a simple matter of finding an appropriate value of c in (7). Consequently, we are justified in calling (7) the **general solution** of the differential equation. In retrospect, you should recall that in several instances we found singular solutions of nonlinear equations. This cannot happen in the case of a linear equation when proper attention is paid to solving the equation over a common interval on which $P(x)$ and $f(x)$ are continuous.

EXAMPLE 3

Find the general solution of

$$(x^2 + 9)\frac{dy}{dx} + xy = 0.$$

Solution We write $$\frac{dy}{dx} + \frac{x}{x^2+9}y = 0.$$

The function $P(x) = x/(x^2+9)$ is continuous on $(-\infty, \infty)$. Now the integrating factor for the equation is

$$e^{\int x\,dx/(x^2+9)} = e^{\frac{1}{2}\int 2x\,dx/(x^2+9)} = e^{\frac{1}{2}\ln(x^2+9)} = \sqrt{x^2+9}$$

so that $$\sqrt{x^2+9}\,\frac{dy}{dx} + \frac{x}{\sqrt{x^2+9}}\,y = 0$$

$$\frac{d}{dx}\left[\sqrt{x^2+9}\,y\right] = 0$$

$$\sqrt{x^2+9}\,y = c.$$

Hence the general solution on the interval is

$$y = \frac{c}{\sqrt{x^2+9}}.$$ ■

EXAMPLE 4

Solve the initial-value problem

$$\frac{dy}{dx} + 2xy = x, \qquad y(0) = -3.$$

Solution The functions $P(x) = 2x$ and $f(x) = x$ are continuous on $(-\infty, \infty)$. The integrating factor is

$$e^{2\int x\,dx} = e^{x^2}$$

so that $$e^{x^2}\frac{dy}{dx} + 2xe^{x^2}y = xe^{x^2}$$

$$\frac{d}{dx}\left[e^{x^2}y\right] = xe^{x^2}$$

$$e^{x^2}y = \int xe^{x^2}\,dx = \frac{1}{2}e^{x^2} + c.$$

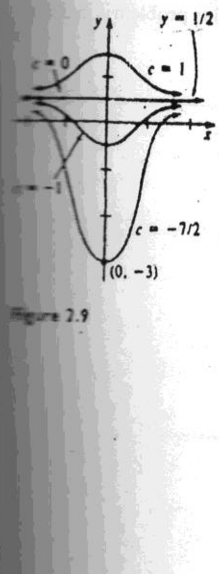

Figure 2.9

Thus the general solution of the differential equation is

$$y = \frac{1}{2} + ce^{-x^2}.$$

The condition $y(0) = -3$ gives $c = -7/2$, and hence the solution of the initial-value problem on the interval is

$$y = \frac{1}{2} - \frac{7}{2}e^{-x^2}.$$

See Figure 2.9.

EXAMPLE 5

Solve the initial-value problem

$$x\frac{dy}{dx} + y = 2x, \quad y(1) = 0.$$

Solution Write the given equation as

$$\frac{dy}{dx} + \frac{1}{x}y = 2$$

and observe that $P(x) = 1/x$ is continuous on any interval not containing the origin. In view of the initial condition, we solve the problem on the interval $(0, \infty)$.

The integrating factor is

$$e^{\int dx/x} = e^{\ln|x|} = x$$

and so

$$\frac{d}{dx}[xy] = 2x$$

gives

$$xy = x^2 + c.$$

The general solution of the equation is

$$y = x + \frac{c}{x}. \tag{12}$$

But $y(1) = 0$ implies $c = -1$. Hence we obtain

$$y = x - \frac{1}{x}, \quad 0 < x < \infty. \tag{13}$$

Considered as a one-parameter family of curves, the graph of (12) is given in Figure 2.10. The solution (13) of the initial-value problem is indicated by the colored portion of the graph.

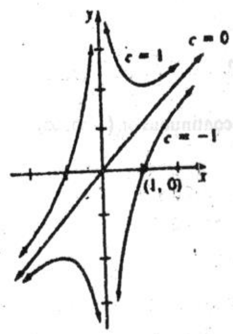

Figure 2.10

EXAMPLE 6

Solve the initial-value problem

$$\frac{dy}{dx} = \frac{1}{x + y^2}, \quad y(-2) = 0.$$

Solution The given differential equation is not separable, homogeneous, exact, or linear in the variable y. However, if we take the reciprocal, then

$$\frac{dx}{dy} = x + y^2 \quad \text{or} \quad \frac{dx}{dy} - x = y^2.$$

This latter equation is *linear in* x, so the corresponding integrating factor is $e^{-\int dy} = e^{-y}$. Hence,

$$\frac{d}{dy}[e^{-y}x] = y^2e^{-y}$$

$$e^{-y}x = \int y^2e^{-y}\,dy.$$

Integrating by parts twice then gives

$$e^{-y}x = -y^2e^{-y} - 2ye^{-y} - 2e^{-y} + c$$

$$x = -y^2 - 2y - 2 + ce^{y}.$$

When $x = -2$, $y = 0$, we find $c = 0$ and so

$$x = -y^2 - 2y - 2. \qquad \blacksquare$$

The next example illustrates a way of solving (1) when the function f is discontinuous.

EXAMPLE 7

Find a continuous solution satisfying

$$\frac{dy}{dx} + y = f(x), \qquad \text{where} \quad f(x) = \begin{cases} 1, & 0 \le x \le 1 \\ 0, & x > 1 \end{cases}$$

and the initial condition $y(0) = 0$.

Solution From Figure 2.11 we see that f is discontinuous at $x = 1$. Consequently, we solve the problem in two parts. For $0 \le x \le 1$ we have

$$\frac{dy}{dx} + y = 1$$

$$\frac{d}{dx}[e^x y] = e^x$$

$$y = 1 + c_1e^{-x}.$$

Since $y(0) = 0$, we must have $c_1 = -1$, and therefore

$$y = 1 - e^{-x}, \qquad 0 \le x \le 1.$$

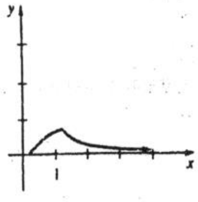

Figure 2.12

For $x > 1$ we then have $\dfrac{dy}{dx} + y = 0,$

which leads to $y = c_2 e^{-x}.$

Hence we can write $y = \begin{cases} 1 - e^{-x}, & 0 \le x \le 1 \\ c_2 e^{-x}, & x > 1. \end{cases}$

Now in order that y be a continuous function, we certainly want $\lim_{x \to 1^+} y(x) = y(1)$. This latter requirement is equivalent to $c_2 e^{-1} = 1 - e^{-1}$, or $c_2 = e - 1$. As Figure 2.12 shows, the function

$$y = \begin{cases} 1 - e^{-x}, & 0 \le x \le 1 \\ (e - 1)e^{-x}, & x > 1 \end{cases}$$

is continuous but not differentiable at $x = 1$. ■

Remark Formula (7), representing the general solution of (1), actually consists of the sum of two solutions. We define

$$y = y_c + y_p, \tag{14}$$

where

$$y_c = ce^{-\int P(x)\,dx} \quad \text{and} \quad y_p = e^{-\int P(x)\,dx} \int e^{\int P(x)\,dx} f(x)\,dx.$$

The function y_c is readily shown to be the general solution of $y' + P(x)y = 0$, whereas y_p is a particular solution of $y' + P(x)y = f(x)$. As we shall see in Chapter 4, the additivity property of solutions (14) to form a general solution is an intrinsic property of linear equations of any order.

2.5 EXERCISES

Answers to odd-numbered problems begin on page A-4.

In Problems 1–40 find the general solution of the given differential equation. State an interval on which the general solution is defined.

1. $\dfrac{dy}{dx} = 5y$
2. $\dfrac{dy}{dx} + 2y = 0$
3. $3\dfrac{dy}{dx} + 12y = 4$
4. $x\dfrac{dy}{dx} + 2y = 3$
5. $\dfrac{dy}{dx} + y = e^{3x}$
6. $\dfrac{dy}{dx} = y + e^{x}$
7. $y' + 3x^2 y = x^2$
8. $y' + 2xy = x^3$
9. $x^2 y' + xy = 1$
10. $y' = 2y + x^2 + 5$
11. $(x + 4y^2)\,dy + 2y\,dx = 0$
12. $\dfrac{dx}{dy} = x + y$

13. $x\,dy = (x \sin x - y)\,dx$

14. $(1 + x^2)\,dy + (xy + x^3 + x)\,dx = 0$

15. $(1 + e^x)\dfrac{dy}{dx} + e^x y = 0$

16. $(1 - x^3)\dfrac{dy}{dx} = 3x^2 y$

17. $\cos x \dfrac{dy}{dx} + y \sin x = 1$

18. $\dfrac{dy}{dx} + y \cot x = 2 \cos x$

19. $x\dfrac{dy}{dx} + 4y = x^3 - x$

20. $(1 + x)y' - xy = x + x^2$

21. $x^2 y' + x(x + 2)y = e^x$

22. $xy' + (1 + x)y = e^{-x} \sin 2x$

23. $\cos^2 x \sin x\,dy + (y \cos^3 x - 1)\,dx = 0$

24. $(1 - \cos x)\,dy + (2y \sin x - \tan x)\,dx = 0$

25. $y\,dx + (xy + 2x - ye^y)\,dy = 0$

26. $(x^2 + x)\,dy = (x^5 + 3xy + 3y)\,dx$

27. $x\dfrac{dy}{dx} + (3x + 1)y = e^{-3x}$

28. $(x + 1)\dfrac{dy}{dx} + (x + 2)y = 2xe^{-x}$

29. $y\,dx - 4(x + y^6)\,dy = 0$

30. $xy' + 2y = e^x + \ln x$

31. $\dfrac{dy}{dx} + y = \dfrac{1 - e^{-2x}}{e^x + e^{-x}}$

32. $\dfrac{dy}{dx} - y = \sinh x$

33. $y\,dx + (x + 2xy^2 - 2y)\,dy = 0$

34. $y\,dx = (ye^y - 2x)\,dy$

35. $\dfrac{dr}{d\theta} + r \sec \theta = \cos \theta$

36. $\dfrac{dP}{dt} + 2tP = P + 4t - 2$

37. $(x + 2)^2 \dfrac{dy}{dx} = 5 - 8y - 4xy$

38. $(x^2 - 1)\dfrac{dy}{dx} + 2y = (x + 1)^2$

39. $y' = (10 - y) \cosh x$

40. $dx = (3e^y - 2x)\,dy$

In Problems 41–54 solve the given differential equation subject to the indicated initial condition.

41. $\dfrac{dy}{dx} + 5y = 20, \quad y(0) = 2$

42. $y' = 2y + x(e^{3x} - e^{2x}), \quad y(0) = 2$

43. $L\dfrac{di}{dt} + Ri = E; \quad L, R,$ and E constants, $i(0) = i_0$

44. $y\dfrac{dx}{dy} - x = 2y^2, \quad y(1) = 5$

45. $y' + (\tan x)y = \cos^2 x, \quad y(0) = -1$

46. $\dfrac{dQ}{dx} = 5x^4 Q, \quad Q(0) = -7$

47. $\dfrac{dT}{dt} = k(T - 50); \quad k$ a constant, $T(0) = 200$

48. $x\,dy + (xy + 2y - 2e^{-x})\,dx = 0, \quad y(1) = 0$

49. $(x + 1)\dfrac{dy}{dx} + y = \ln x, \quad y(1) = 10$

50. $xy' + y = e^x, \quad y(1) = 2$

51. $x(x - 2)y' + 2y = 0, \quad y(3) = 6$

52. $\sin x\dfrac{dy}{dx} + (\cos x)y = 0, \quad y\left(-\dfrac{\pi}{2}\right) = 1$

53. $\dfrac{dy}{dx} = \dfrac{y}{y - x}, \quad y(5) = 2$

54. $\cos^2 x\dfrac{dy}{dx} + y = 1, \quad y(0) = -3$

In Problems 55–58 find a continuous solution satisfying each differential equation and the given initial condition.

55. $\dfrac{dy}{dx} + 2y = f(x), \quad f(x) = \begin{cases} 1, & 0 \le x \le 3 \\ 0, & x > 3 \end{cases}, \quad y(0) = 0$

56. $\dfrac{dy}{dx} + y = f(x), \quad f(x) = \begin{cases} 1, & 0 \le x \le 1 \\ -1, & x > 1 \end{cases}, \quad y(0) = 1$

57. $\dfrac{dy}{dx} + 2xy = f(x), \quad f(x) = \begin{cases} x, & 0 \le x < 1 \\ 0, & x \ge 1 \end{cases}, \quad y(0) = 2$

58. $(1 + x^2)\dfrac{dy}{dx} + 2xy = f(x), \quad f(x) = \begin{cases} x, & 0 \le x < 1 \\ -x, & x \ge 1 \end{cases}, \quad y(0) = 0$

2.6 EQUATIONS OF BERNOULLI, RICATTI, AND CLAIRAUT*

In this section we are not going to study any one particular type of differenti[al] equation. Rather, we are going to consider three classical equations that [in] some instances can be transformed into equations we have already studi[ed].

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*Jakob Bernoulli (1654–1705) The Bernoullis were a Swiss family of scholars whose contributions to mathematics, physics, astronomy, and history spanned the sixteenth to the twentieth centuries. Jakob, the elder of the two sons of the patriarch Jacques Bernoulli, made many contributions to the then-new fields of calculus and probability. Originally the second of the two major divisions of calculus was called *calculus summatorius*. In 1696, a[t] Jakob Bernoulli's suggestion, its name was changed to *calculus integralis* or, as we know i[t] today, integral calculus.

Jacobo Francesco Ricatti (1676–1754) An Italian count, Ricatti was also a mathema[ti]cian and phi[l]osopher.

Alexis Claude Clairaut (1713–1765) Born in Paris in 1713, Clairaut was a child prodig[y] who wrote his first book on mathematics at the age of eleven. He was among the first t[o] discover singular solutions of differential equations. Like many mathematicians of his er[a] Clairaut was also a physicist and astronomer.

A family of curves can be **self-orthogonal** in the sense that a member of the orthogonal trajectories is also a member of the original family. In Problems 39 and 40 show that the given family of curves is self-orthogonal.

39. parabolas $y^2 = c_1(2x + c_1)$

40. confocal conics $\dfrac{x^2}{c_1 + 1} + \dfrac{y^2}{c_1} = 1$

41. Verify that the orthogonal trajectories of the family of curves given by the parametric equations $x = c_1e^t \cos t$, $y = c_1e^t \sin t$ are

$$x = c_2e^{-t} \cos t, \; y = c_2e^{-t} \sin t.$$

[*Hint:* $dy/dx = (dy/dt)/(dx/dt)$.]

42. Show that two polar curves $r = f_1(\theta)$ and $r = f_2(\theta)$ are orthogonal at a point of intersection if and only if

$$(\tan \psi_1)_{\theta_1}(\tan \psi_2)_{\theta_2} = -1.$$

3.2 APPLICATIONS OF LINEAR EQUATIONS

Growth and Decay

The initial-value problem

$$\frac{dx}{dt} = kx, \qquad x(t_0) = x_0, \tag{1}$$

where k is a constant of proportionality, occurs in many physical theories involving either growth or decay. For example, in biology it is often observed that the rate at which certain bacteria grow is proportional to the number of bacteria present at any time. Over short intervals of time, the population of small animals, such as rodents, can be predicted fairly accurately by the solution of (1). In physics an initial-value problem such as (1) provides a model for approximating the remaining amount of a substance that is disintegrating, or decaying, through radioactivity. The differential equation in (1) could also determine the temperature of a cooling body. In chemistry the amount of a substance remaining during certain reactions is also described by (1).

The constant of proportionality k in (1) is either positive or negative and can be determined from the solution of the problem using a subsequent measurement of x at a time $t_1 > t_0$.

EXAMPLE 1

A culture initially has N_0 number of bacteria. At $t = 1$ hour the number of bacteria is measured to be $(3/2)N_0$. If the rate of growth is proportional to the number of bacteria present, determine the time necessary for the number of bacteria to triple.

Solution We first solve the differential equation

$$\frac{dN}{dt} = kN \tag{2}$$

subject to $N(0) = N_0$. Then we use the empirical condition $N(1) = (3/2)N_0$ to determine the constant of proportionality k.

Now (2) is both separable and linear. When it is put into the form

$$\frac{dN}{dt} - kN = 0,$$

we can see by inspection that the integrating factor is e^{-kt}. Multiplying both sides of the equation by this term gives immediately

$$\frac{d}{dt}[e^{-kt}N] = 0.$$

Integrating both sides of the last equation yields

$$e^{-kt}N = c \quad \text{or} \quad N(t) = ce^{kt}.$$

At $t = 0$ it follows that $N_0 = ce^0 = c$ and so $N(t) = N_0e^{kt}$. At $t = 1$ we have

$$\frac{3}{2}N_0 = N_0e^k \quad \text{or} \quad e^k = \frac{3}{2},$$

from which we get to four decimal places

$$k = \ln\left(\frac{3}{2}\right) = 0.4055.$$

Thus

$$N(t) = N_0e^{0.4055t}.$$

To find the time at which the bacteria have tripled we solve

$$3N_0 = N_0e^{0.4055t}$$

for t. It follows from this equation that $0.4055t = \ln 3$ and so

$$t = \frac{\ln 3}{0.4055} \approx 2.71 \text{ hours}.$$

See Figure 3.7.

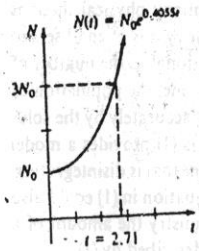


Figure 3.7

Note We can write the function $N(t)$ obtained in Example 1 in an alternative form. From the laws of exponents,

$$N(t) = N_0(e^k)^t = N_0\left(\frac{3}{2}\right)^t$$

since $e^k = 3/2$. This latter solution provides a convenient method for computing $N(t)$ for small positive integral values of t. It also clearly shows the influence of the subsequent experimental observation at $t = 1$ on the solution

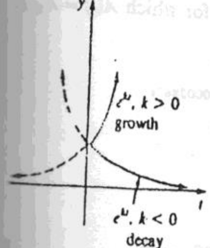


Figure 3.8

for all time. We notice, too, that the actual number of bacteria present at time $t = 0$ is quite irrelevant in finding the time required to triple the number in the culture. The necessary time to triple, say, 100 or 10,000 bacteria is still approximately 2.71 hours.

As shown in Figure 3.8, the exponential function e^{kt} increases as t increases for $k > 0$, and decreases as t increases for $k < 0$. Thus problems describing growth, such as population, bacteria, or even capital, are characterized by a positive value of k, whereas problems involving decay, as in radioactive disintegration, yield a negative k value.

Half-Life

In physics the **half-life** is a measure of the stability of a radioactive substance. The half-life is simply the time it takes for one-half of the atoms in an initial amount A_0 to disintegrate, or transmute, into the atoms of another element. The longer the half-life of a substance, the more stable it is. For example, the half-life of highly radioactive radium, Ra-226, is about 1700 years. In 1700 years one-half of a given quantity of Ra-226 is transmuted into radon, Rn-222. The most commonly occurring uranium isotope, U-238, has a half-life of approximately 4,500,000,000 years. In about 4.5 billion years, one-half of a quantity of U-238 is transmuted into lead, Pb-206.

EXAMPLE 2

A breeder reactor converts the relatively stable uranium 238 into the isotope plutonium 239. After 15 years it is determined that 0.043% of the initial amount A_0 of the plutonium has disintegrated. Find the half-life of this isotope if the rate of disintegration is proportional to the amount remaining.

Solution Let $A(t)$ denote the amount of plutonium remaining at any time. As in Example 1, the solution of the initial-value problem

$$\frac{dA}{dt} = kA, \qquad A(0) = A_0,$$

is

$$A(t) = A_0 e^{kt}.$$

If 0.043% of the atoms of A_0 have disintegrated, then 99.957% of the substance remains. To find k we use $0.99957A_0 = A(15)$; that is,

$$0.99957A_0 = A_0 e^{15k}.$$

Solving for k then gives $15k = \ln(0.99957)$, or

$$k = \frac{\ln(0.99957)}{15} = -0.00002867.$$

Hence

$$A(t) = A_0 e^{-0.00002867t}.$$

Now the half-life is the corresponding value of time for which $A(t) = A_0/2$. Solving for t gives

$$\frac{A_0}{2} = A_0 e^{-0.00002867t} \quad \text{or} \quad \frac{1}{2} = e^{-0.00002867t}$$

$$-0.00002867t = \ln\left(\frac{1}{2}\right) = -\ln 2$$

$$t = \frac{\ln 2}{0.00002867} \approx 24{,}180 \text{ years.}$$

Carbon Dating

About 1950 the chemist Willard Libby devised a method of using radioactive carbon as a means of determining the approximate ages of fossils. The theory of carbon dating is based on the fact that the isotope carbon 14 is produced in the atmosphere by the action of cosmic radiation on nitrogen. The ratio of the amount of C-14 to ordinary carbon in the atmosphere appears to be a constant, and as a consequence the proportionate amount of the isotope present in all living organisms is the same as that in the atmosphere. When an organism dies, the absorption of C-14, by either breathing or eating, ceases. Thus, by comparing the proportionate amount of C-14 present, say, in a fossil with the constant ratio found in the atmosphere, it is possible to obtain a reasonable estimation of its age. The method is based on the knowledge that the half-life of the radioactive C-14 is approximately 5600 years. For his work Libby won the Nobel Prize for chemistry in 1960. Libby's method has been used to date wooden furniture in Egyptian tombs and the woven flax wrappings of the Dead Sea scrolls.

EXAMPLE 3

A fossilized bone is found to contain 1/1000 the original amount of C-14. Determine the age of the fossil.

Solution The starting point is again $A(t) = A_0e^{kt}$. To determine the value of k we use the fact that $A_0/2 = A(5600)$, or $A_0/2 = A_0e^{5600k}$. Hence we have

$$5600k = \ln\left(\frac{1}{2}\right) = -\ln 2$$

$$k = -\frac{\ln 2}{5600} = -0.00012378.$$

Therefore $\quad A(t) = A_0e^{-0.00012378t}$

When $A(t) = A_0/1000$, we have

$$\frac{A_0}{1000} = A_0 e^{-0.00012378t}$$

so that

$$-0.00012378t = \ln\left(\frac{1}{1000}\right) = -\ln 1000$$

$$t = \frac{\ln 1000}{0.00012378} \approx 55{,}800 \text{ years.}$$

∎

The date found in Example 3 is really at the border of accuracy for this method. The usual carbon 14 technique is limited to about 9 half-lives of the isotope, or about 50,000 years. One reason is that the chemical analysis needed to obtain an accurate measurement of the remaining C-14 becomes somewhat formidable around the point of $A_0/1000$. Also, this analysis demands the destruction of a rather large sample of the specimen. If this measurement is accomplished indirectly, based on the actual radioactivity of the specimen, then it is very difficult to distinguish between the radiation from the fossil and the normal background radiation. But in recent developments, the use of a particle accelerator has enabled scientists to separate the C-14 from the stable C-12 directly. By computing the precise value of the ratio of C-14 to C-12, the accuracy of this method can be extended to 70,000–100,000 years. Other isotopic techniques such as using potassium 40 and argon 40 can give dates of several million years. Nonisotopic methods based on the use of amino acids are also sometimes possible.

Cooling

Newton's law of cooling states that the rate at which the temperature $T(t)$ changes in a cooling body is proportional to the difference between the temperature in the body and the constant temperature T_m of the surrounding medium—that is,

$$\frac{dT}{dt} = k(T - T_m), \tag{3}$$

where k is a constant of proportionality.

EXAMPLE 4

When a cake is removed from a baking oven, its temperature is measured at 300°F. Three minutes later its temperature is 200°F. How long will it take to cool off to a room temperature of 70°F?

Solution In (3) we make the identification $T_m = 70$. We must then solve the initial-value problem

$$\frac{dT}{dt} = k(T - 70), \qquad T(0) = 300, \tag{4}$$

and determine the value of k so that $T(3) = 200$.

Equation (4) is both linear and separable. Separating variables yields

$$\frac{dT}{T - 70} = k\,dt$$

$$\ln|T - 70| = kt + c_1$$

$$T - 70 = c_2 e^{kt}$$

$$T = 70 + c_2 e^{kt}.$$

When $t = 0$, $T = 300$ so that $300 = 70 + c_2$ gives $c_2 = 230$ and, therefore, $T = 70 + 230e^{kt}$.

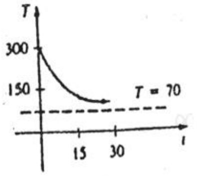
From $T(3) = 200$ we find

$$e^{3k} = \frac{13}{23} \quad \text{or} \quad k = \frac{1}{3}\ln\frac{13}{23} = -0.19018.$$

Thus

$$T(t) = 70 + 230e^{-0.19018t}. \tag{5}$$

We note that (5) furnishes no finite solution to $T(t) = 70$ since $\lim_{t\to\infty} T(t) = 70$. Yet intuitively we expect the cake to reach the room temperature after a reasonably long period of time. How long is long? Of course, we should not be disturbed by the fact that the model (4) does not quite live up to our physical intuition. Parts (a) and (b) of Figure 3.9 clearly show that the cake will be approximately at room temperature in about one-half hour.



(a)

$T(t)$	t (minutes)
75°	20.1
74°	21.3
73°	22.8
72°	24.9
71°	28.6
70.5°	32.3

(b)

Figure 3.9

Series Circuits

In a series circuit containing only a resistor and an inductor, Kirchhoff's second law states that the sum of the voltage drop across the inductor $(L(di/dt))$ and the voltage drop across the resistor (iR) is the same as the impressed voltage $(E(t))$ on the circuit. See Figure 3.10.

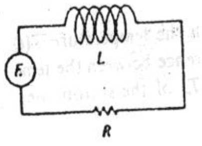
Thus we obtain the linear differential equation for the current $i(t)$,

$$L\frac{di}{dt} + Ri = E(t), \tag{6}$$

where L and R are constants known as the inductance and the resistance, respectively. The current $i(t)$ is sometimes called the response of the system.

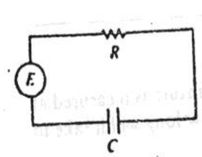
The voltage drop across a capacitor with capacitance C is given by $q(t)/C$, where q is the charge on the capacitor. Hence, for the series circuit shown in Figure 3.11, Kirchhoff's second law gives

$$Ri + \frac{1}{C}q = E(t). \tag{7}$$



L-R Series Circuit

Figure 3.10



R-C Series Circuit

Figure 3.11

But current i and charge q are related by $i = dq/dt$, so (7) becomes the linear differential equation

$$R\frac{dq}{dt} + \frac{1}{C}q = E(t). \tag{8}$$

EXAMPLE 5

A 12-volt battery is connected to a series circuit in which the inductance is 1/2 henry and the resistance is 10 ohms. Determine the current i if the initial current is zero.

Solution From (6) we see that we must solve

$$\frac{1}{2}\frac{di}{dt} + 10i = 12$$

subject to $i(0) = 0$. First, we multiply the differential equation by 2 and read off the integrating factor e^{20t}. We then obtain

$$\frac{d}{dt}[e^{20t}i] = 24e^{20t}$$

$$e^{20t}i = \frac{24}{20}e^{20t} + c$$

$$i = \frac{6}{5} + ce^{-20t}.$$

Now $i(0) = 0$ implies $0 = 6/5 + c$, or $c = -6/5$. Therefore the response is

$$i(t) = \frac{6}{5} - \frac{6}{5}e^{-20t}.$$

■

From (7) of Section 2.5 we can write a general solution of (6):

$$i(t) = \frac{e^{-(R/L)t}}{L}\int e^{(R/L)t}E(t)\,dt + ce^{-(R/L)t}. \tag{9}$$

In particular, when $E(t) = E_0$ is a constant, (9) becomes

$$i(t) = \frac{E_0}{R} + ce^{-(R/L)t}. \tag{10}$$

Note that as $t \to \infty$, the second term in equation (10) approaches zero. Such a term is usually called a **transient term**; any remaining terms are called the **steady-state** part of the solution. In this case E_0/R is also called the **steady-state current**; for large values of time it then appears that the current in the circuit is simply governed by Ohm's law ($E = iR$).

Mixture Problem

The mixing of two fluids sometimes gives rise to a linear first-order differential equation. In the next example we consider the mixture of two salt solutions with different concentrations.

EXAMPLE 6

Initially 50 pounds of salt is dissolved in a large tank holding 300 gallons of water. A brine solution is pumped into the tank at a rate of 3 gallons per minute, and the well-stirred solution is then pumped out at the same rate. See Figure 3.12. If the concentration of the solution entering is 2 pounds per gallon, determine the amount of salt in the tank at any time. How much salt is present after 50 minutes? after a long time?

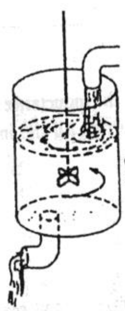


Figure 3.12

Solution Let $A(t)$ be the amount of salt (in pounds) in the tank at any time. For problems of this sort, the net rate at which $A(t)$ changes is given by

$$\frac{dA}{dt} = \left(\begin{matrix}\textit{rate of}\\ \textit{substance entering}\end{matrix}\right) - \left(\begin{matrix}\textit{rate of}\\ \textit{substance leaving}\end{matrix}\right) = R_1 - R_2. \tag{11}$$

Now the rate at which the salt enters the tank is, in pounds per minute,

$$R_1 = (3 \text{ gal/min}) \cdot (2 \text{ lb/gal}) = 6 \text{ lb/min},$$

whereas the rate at which salt is leaving is

$$R_2 = (3 \text{ gal/min}) \cdot \left(\frac{A}{300} \text{ lb/gal}\right) = \frac{A}{100} \text{ lb/min}.$$

Thus equation (11) becomes

$$\frac{dA}{dt} = 6 - \frac{A}{100}, \tag{12}$$

which we solve subject to the initial condition $A(0) = 50$.

Since the integrating factor is $e^{t/100}$, we can write (12) as

$$\frac{d}{dt}[e^{t/100}A] = 6e^{t/100}$$

and therefore

$$e^{t/100}A = 600e^{t/100} + c$$

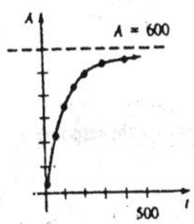
$$A = 600 + ce^{-t/100}. \tag{13}$$

When $t = 0$, $A = 50$, so we find that $c = -550$. Finally, we obtain

$$A(t) = 600 - 550e^{-t/100}. \tag{14}$$

At $t = 50$ we find $A(50) = 266.41$ pounds. Also, as $t \to \infty$ it is seen from (14) and Figure 3.13 that $A \to 600$. Of course this is what we would expect; over a long period of time the number of pounds of salt in the solution must be

$$(300 \text{ gal})(2 \text{ lb/gal}) = 600 \text{ lb}.$$ ■



(a)

t (minutes)	A(lbs)
50	266.41
100	397.67
150	477.27
200	525.57
300	572.62
400	589.93

(b)

Figure 3.13

In Example 6 we assumed that the rate at which the solution was pumped in was the same as the rate at which the solution was pumped out. However, this need not be the case; the mixed brine solution could be pumped out at a rate faster or slower than the rate at which the other solution is pumped in. The resulting differential equation in this latter situation is linear with a variable coefficient.

EXAMPLE 7

If the well-stirred solution in Example 6 is pumped out at a slower rate of 2 gallons per minute, then the solution is *accumulating* at a rate of

$$(3 - 2)\text{ gal/min} = 1\text{ gal/min}.$$

After t minutes there are $300 + t$ gallons of brine in the tank. The rate at which the salt is leaving is then

$$R_2 = (2\text{ gal/min}) \cdot \left(\frac{A}{300 + t}\text{ lb/gal}\right) = \frac{2A}{300 + t}\text{ lb/min}.$$

Hence equation (11) becomes

$$\frac{dA}{dt} = 6 - \frac{2A}{300 + t} \quad \text{or} \quad \frac{dA}{dt} + \frac{2A}{300 + t} = 6.$$

Finding the integrating factor and solving the last equation, we get

$$A(t) = 2(300 + t) + c(300 + t)^{-2}.$$

The initial condition $A(0) = 50$ yields $c = -4.95 \times 10^7$ and so

$$A(t) = 2(300 + t) - (4.95 \times 10^7)(300 + t)^{-2}.$$ ■

Remark Consider the differential equation in Example 1 that describes the growth of bacteria. The solution $N(t) = N_0 e^{0.4055t}$ of the initial-value problem $dN/dt = kN$, $N(t_0) = N_0$, is of course a continuous function. But in the example we are talking about a population of bacteria and so common sense dictates that N take on only positive integer values. Moreover, the population does not necessarily grow continuously, that is, every second, every microsecond, and so on, as predicted by the function $N(t) = N_0 e^{0.4055t}$; there may be time intervals $[t_1, t_2]$ over which there is no growth at all. Perhaps, then, the graph shown in Figure 3.14(a) is a more realistic description of N than that given by the graph of an exponential function. The point is, in many instances a mathematical model describes a system in only approximate terms. It is often more convenient than accurate to use a continuous function to describe a discrete phenomenon. However, for some purposes we may be satisfied if

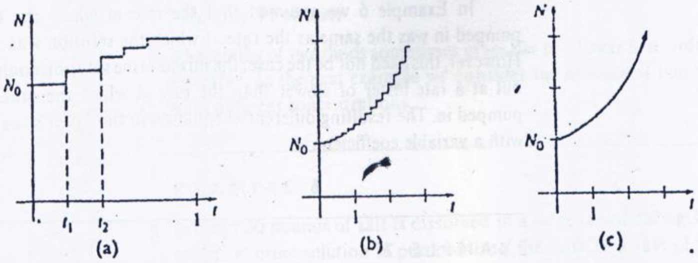


Figure 3.14

our model describes the system fairly accurately when viewed macroscopically in time as in Figures 3.14(b) and (c) rather than microscopically.

3.2 EXERCISES

Answers to odd-numbered problems begin on page A-6.

1. The population of a certain community is known to increase at a rate proportional to the number of people present at any time. If the population has doubled in 5 years, how long will it take to triple? to quadruple?
2. Suppose it is known that the population of the community in Problem 1 is 10,000 after 3 years. What was the initial population? What will be the population in 10 years?
3. The population of a town grows at a rate proportional to the population at any time. Its initial population of 500 increases by 15% in 10 years. What will be the population in 30 years?
4. The population of bacteria in a culture grows at a rate proportional to the number of bacteria present at any time. After 3 hours it is observed that there are 400 bacteria present. After 10 hours there are 2000 bacteria present. What is the initial number of bacteria?
5. The radioactive isotope of lead, Pb-209, decays at a rate proportional to the amount present at any time and has a half-life of 3.3 hours. If 1 gram of lead is present initially, how long will it take for 90% of the lead to decay?
6. Initially there were 100 milligrams of a radioactive substance present. After 6 hours the mass decreased by 3%. If the rate of decay is proportional to the amount of the substance present at any time, find the amount remaining after 24 hours.
7. Determine the half-life of the radioactive substance described in Problem 6.

8. Show that the half-life of a radioactive substance is, in general,

$$t = \frac{(t_2 - t_1)\ln 2}{\ln(A_1/A_2)},$$

where $A_1 = A(t_1)$ and $A_2 = A(t_2)$, $t_1 < t_2$.

9. When a vertical beam of light passes through a transparent substance, the rate at which its intensity I decreases is proportional to $I(t)$, where t represents the thickness of the medium (in feet). In clear seawater the intensity 3 feet below the surface is 25% of the initial intensity I_0 of the incident beam. What is the intensity of the beam 15 feet below the surface?

10. When interest is compounded continuously, the amount of money S increases at a rate proportional to the amount present at any time: $dS/dt = rS$, where r is the annual rate of interest (see (26) of Section 1.2).
    (a) Find the amount of money accrued at the end of 5 years when $5000 is deposited in a savings account drawing $5\frac{3}{4}$% annual interest compounded continuously.
    (b) In how many years will the initial sum deposited be doubled?
    (c) Use a hand calculator to compare the number obtained in part (a) with the value

$$S = 5000\left(1 + \frac{0.0575}{4}\right)^{5(4)}.$$

This value represents the amount accrued when interest is compounded quarterly.

11. In a piece of burned wood, or charcoal, it was found that 85.5% of the C-14 had decayed. Use the information in Example 3 to determine the approximate age of the wood. (It is precisely these data that archaeologists used to date prehistoric paintings in a cave in Lascaux, France.)

12. A thermometer is taken from an inside room to the outside where the air temperature is 5°F. After 1 minute the thermometer reads 55°F, and after 5 minutes the reading is 30°F. What is the initial temperature of the room?

13. A thermometer is removed from a room where the air temperature is 70°F to the outside where the temperature is 10°F. After $\frac{1}{2}$ minute the thermometer reads 50°F. What is the reading at $t = 1$ minute? How long will it take for the thermometer to reach 15°F?

14. Formula (3) also holds when an object absorbs heat from the surrounding medium. If a small metal bar whose initial temperature is 20°C is dropped into a container of boiling water, how long will it take for the bar to reach 90°C if it is known that its temperature increased 2° in 1 second? How long will it take the bar to reach 98°C?

15. A 30-volt electromotive force is applied to an L-R series circuit in which

the inductance is 0.1 henry and the resistance is 50 ohms. Find the current $i(t)$ if $i(0) = 0$. Determine the current as $t \to \infty$.

16. Solve equation (6) under the assumption that $E(t) = E_0 \sin \omega t$ and $i(0) = i_0$.

17. A 100-volt electromotive force is applied to an R-C series circuit in which the resistance is 200 ohms and the capacitance is 10^{-4} farad. Find the charge $q(t)$ on the capacitor if $q(0) = 0$. Find the current $i(t)$.

18. A 200-volt electromotive force is applied to an R-C series circuit in which the resistance is 1000 ohms and the capacitance is 5×10^{-6} farad. Find the charge $q(t)$ on the capacitor if $i(0) = 0.4$. Determine the charge and current at $t = 0.005$ second. Determine the charge as $t \to \infty$.

19. An electromotive force

$$E(t) = \begin{cases} 120, & 0 \le t \le 20 \\ 0, & t > 20 \end{cases}$$

is applied to an L-R series circuit in which the inductance is 20 henry and the resistance is 2 ohms. Find the current $i(t)$ if $i(0) = 0$.

20. Suppose an R-C series circuit has a variable resistor. If the resistance at any time t is given by $R = k_1 + k_2 t$, where $k_1 > 0$ and $k_2 > 0$ are known constants, then (8) becomes

$$(k_1 + k_2 t)\frac{dq}{dt} + \frac{1}{C} q = E(t).$$

Show that if $E(t) = E_0$ and $q(0) = q_0$, then

$$q(t) = E_0 C + (q_0 - E_0 C)\left(\frac{k_1}{k_1 + k_2 t}\right)^{1/Ck_2}$$

21. A tank contains 200 liters of fluid in which 30 g of salt is dissolved. Brine containing 1 g of salt per liter is then pumped into the tank at a rate of 4 liters per minute; the well-mixed solution is pumped out at the same rate. Find the number of grams of salt $A(t)$ in the tank at any time.

22. Solve Problem 21 assuming pure water is pumped into the tank.

23. A large tank is filled with 500 gallons of pure water. Brine containing 2 lb of salt per gallon is pumped into the tank at a rate of 5 gallons per minute. The well-mixed solution is pumped out at the same rate. Find the number of pounds of salt $A(t)$ in the tank at any time.

24. Solve Problem 23 under the assumption that the solution is pumped out at a faster rate of 10 gallons per minute. When is the tank empty?

25. A large tank is partially filled with 100 gallons of fluid in which 10 lb of salt is dissolved. Brine containing $\frac{1}{2}$ lb of salt per gallon is pumped into the tank at a rate of 6 gallons per minute. The well-mixed solution is then

pumped out at a slower rate of 4 gallons per minute. Find the number of pounds of salt in the tank after 30 minutes.

26. Beer containing 6% alcohol per gallon is pumped into a vat that initially contains 400 gallons of beer at 3% alcohol. The rate at which the beer is pumped in is 3 gallons per minute, whereas the mixed liquid is pumped out at a rate of 4 gallons per minute. Find the number of gallons of alcohol $A(t)$ in the tank at any time. What is the percentage of alcohol in the tank after 60 minutes? When is the tank empty?

Miscellaneous Applications

27. The differential equation governing the velocity v of a falling mass m subjected to air resistance proportional to the instantaneous velocity is

$$m\frac{dv}{dt} = mg - kv,$$

where k is a positive constant of proportionality.

(a) Solve the equation subject to the initial condition $v(0) = v_0$.

(b) Determine the limiting, or terminal, velocity of the weight.

(c) If distance s is related to velocity $ds/dt = v$, find an explicit expression for s if it is further known that $s(0) = s_0$.

28. The rate at which a drug disseminates into the bloodstream is governed by the differential equation

$$\frac{dX}{dt} = A - BX,$$

where A and B are positive constants. The function $X(t)$ describes the concentration of the drug in the bloodstream at any time t. Find the limiting value of X as $t \to \infty$. At what time is the concentration one-half this limiting value? Assume that $X(0) = 0$.

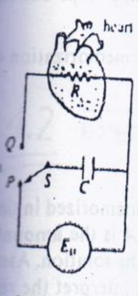


Figure 3.15

29. A heart pacemaker, as shown in Figure 3.15, consists of a battery, a capacitor, and the heart as a resistor. When the switch S is at P the capacitor charges; when S is at Q the capacitor discharges, sending an electrical stimulus to the heart. During this time the voltage E applied to the heart is given by

$$\frac{dE}{dt} = -\frac{1}{RC}E, \qquad t_1 < t < t_2,$$

where R and C are constants. Determine $E(t)$ if $E(t_1) = E_0$. (Of course, the opening and closing of the switch are periodic in time, to simulate the natural heartbeat.)

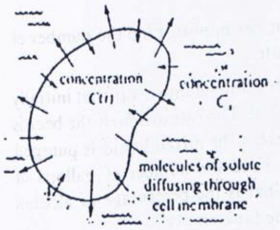


Figure 3.16

30. Suppose a cell is suspended in a solution containing a solute of constant concentration C_s. Suppose further that the cell has constant volume V and that the area of its permeable membrane is the constant A. By Fick's law the rate of change of its mass m is directly proportional to the area A and the difference $C_s - C(t)$, where $C(t)$ is the concentration of the solute inside the cell at any time t. Find $C(t)$ if $m = VC(t)$ and $C(0) = C_0$. See Figure 3.16.

31. In one model of the changing population $P(t)$ of a community, it is assumed that

$$\frac{dP}{dt} = \frac{dB}{dt} - \frac{dD}{dt},$$

where dB/dt and dD/dt are the birth and death rates, respectively.

(a) Solve for $P(t)$ if

$$\frac{dB}{dt} = k_1 P \quad \text{and} \quad \frac{dD}{dt} = k_2 P.$$

(b) Analyze the cases $k_1 > k_2$, $k_1 = k_2$, and $k_1 < k_2$.

32. The differential equation

$$\frac{dP}{dt} = (k \cos t)P,$$

where k is a positive constant, is often used as a model of a population that undergoes yearly seasonal fluctuations. Solve for $P(t)$ and graph the solution. Assume $P(0) = P_0$.

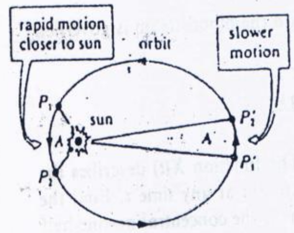


Figure 3.17

33. In polar coordinates the angular momentum of a moving body of mass m is defined to be $L = mr^2(d\theta/dt)$. Assume that the coordinates of the body are (r_1, θ_1) and (r_2, θ_2) at times $t = a$ and $t = b$, $a < b$, respectively. If L is constant, show that the area A swept out by r is $A = L(b - a)/2m$. When the sun is taken to be at the origin, this proves Kepler's second law of planetary motion: The radius vector joining the sun sweeps out equal areas in equal intervals of time. See Figure 3.17.

34. When forgetfulness is taken into account, the rate of memorization of a subject is given by

$$\frac{dA}{dt} = k_1(M - A) - k_2 A,$$

where $k_1 > 0$, $k_2 > 0$, $A(t)$ is the amount of material memorized in time t, M is the total amount to be memorized, and $M - A$ is the amount remaining to be memorized. Solve for $A(t)$ and graph the solution. Assume $A(0) = 0$. Find the limiting value of A as $t \to \infty$ and interpret the result.

EXAMPLE 3

It can be verified that $y_1 = \dfrac{\sin x}{\sqrt{x}}$ is a solution of $x^2y'' + xy' + (x^2 - \frac{1}{4})y = 0$ on $(0, \pi)$. Find a second solution.

Solution First put the equation into the form

$$y'' + \frac{1}{x}y' + \left(1 - \frac{1}{4x^2}\right)y = 0.$$

Then from (4) we have

$$y_2 = \frac{\sin x}{\sqrt{x}} \int \frac{e^{-\int dx/x}}{\left(\dfrac{\sin x}{\sqrt{x}}\right)^2} dx \qquad \boxed{e^{-\int dx/x} = e^{\ln x^{-1}} = x^{-1}}$$

$$= \frac{\sin x}{\sqrt{x}} \int \csc^2 x \, dx$$

$$= \frac{\sin x}{\sqrt{x}}(-\cot x) = -\frac{\cos x}{\sqrt{x}}.$$

Since the differential equation is homogeneous, we can disregard the negative sign and take the second solution to be $y_2 = (\cos x)/\sqrt{x}$. ■

Observe that $y_1(x)$ and $y_2(x)$ of Example 3 are linearly independent solutions of the given differential equation on the larger interval $(0, \infty)$.

Remark We have derived and illustrated how to use (4) because you will see this formula again in the next section and in Section 6.1. We use (4) simply to save time in obtaining a desired result. Your instructor will tell you whether you should memorize (4) or whether you should know the first principles of reduction of order.

4.2 EXERCISES

Answers to odd-numbered problems begin on page A-9.

In Problems 1–30 find a second solution of each differential equation. Use reduction of order or formula (4) as instructed. Assume an appropriate interval of validity.

1. $y'' + 5y' = 0; \quad y_1 = 1$
2. $y'' - y' = 0; \quad y_1 = 1$
3. $y'' - 4y' + 4y = 0; \quad y_1 = e^{2x}$
4. $y'' + 2y' + y = 0; \quad y_1 = xe^{-x}$
5. $y'' + 16y = 0; \quad y_1 = \cos 4x$
6. $y'' + 9y = 0; \quad y_1 = \sin 3x$
7. $y'' - y = 0; \quad y_1 = \cosh x$
8. $y'' - 25y = 0; \quad y_1 = e^{5x}$
9. $9y'' - 12y' + 4y = 0; \quad y_1 = e^{2x/3}$
10. $6y'' + y' - y = 0; \quad y_1 = e^{x/3}$

11. $x^2y'' - 7xy' + 16y = 0; \quad y_1 = x^4$
12. $x^2y'' + 2xy' - 6y = 0; \quad y_1 = x^2$
13. $xy'' + y' = 0; \quad y_1 = \ln x$
14. $4x^2y'' + y = 0; \quad y_1 = x^{1/2} \ln x$
15. $(1 - 2x - x^2)y'' + 2(1 + x)y' - 2y = 0; \quad y_1 = x + 1$
16. $(1 - x^2)y'' - 2xy' = 0; \quad y_1 = 1$
17. $x^2y'' - xy' + 2y = 0; \quad y_1 = x \sin(\ln x)$
18. $x^2y'' - 3xy' + 5y = 0; \quad y_1 = x^2 \cos(\ln x)$
19. $(1 + 2x)y'' + 4xy' - 4y = 0; \quad y_1 = e^{-2x}$
20. $(1 + x)y'' + xy' - y = 0; \quad y_1 = x$
21. $x^2y'' - xy' + y = 0; \quad y_1 = x$
22. $x^2y'' - 20y = 0; \quad y_1 = x^{-4}$
23. $x^2y'' - 5xy' + 9y = 0; \quad y_1 = x^3 \ln x$
24. $x^2y'' + xy' + y = 0; \quad y_1 = \cos(\ln x)$
25. $x^2y'' - 4xy' + 6y = 0; \quad y_1 = x^2 + x^3$
26. $x^2y'' - 7xy' - 20y = 0; \quad y_1 = x^{10}$
27. $(3x + 1)y'' - (9x + 6)y' + 9y = 0; \quad y_1 = e^{3x}$
28. $xy'' - (x + 1)y' + y = 0; \quad y_1 = e^x$
29. $y'' - 3(\tan x)y' = 0; \quad y_1 = 1$
30. $xy'' - (2 + x)y' = 0; \quad y_1 = 1$

In Problems 31–34 use the method of reduction of order to find a solution of the given nonhomogeneous equation. The indicated function $y_1(x)$ is a solution of the associated homogeneous equation. Determine a second solution of the homogeneous equation and a particular solution of the nonhomogeneous equation.

31. $y'' - 4y = 2; \quad y_1 = e^{-2x}$
32. $y'' + y' = 1; \quad y_1 = 1$
33. $y'' - 3y' + 2y = 5e^{3x}; \quad y_1 = e^x$
34. $y'' - 4y' + 3y = x; \quad y_1 = e^x$
35. Verify by direct substitution that formula (4) satisfies equation (2).

4.3 HOMOGENEOUS LINEAR EQUATIONS WITH CONSTANT COEFFICIENTS

We have seen that the linear first-order equation $dy/dx + ay = 0$, where a is a constant, has the exponential solution $y = c_1e^{-ax}$ on $(-\infty, \infty)$. Therefore, it is natural to seek to determine whether exponential solutions exist on $(-\infty, \infty)$ for higher-order equations such as

$$a_ny^{(n)} + a_{n-1}y^{(n-1)} + \dots + a_2y'' + a_1y' + a_0y = 0, \tag{1}$$

where the a_i, $i = 0, 1, \ldots, n$, are constants. The surprising fact is that *all* solutions of (1) are exponential functions or constructed out of exponential functions. We begin by considering the special case of the second-order equation

$$ay'' + by' + cy = 0. \tag{2}$$

Auxiliary Equation

If we try a solution of the form $y = e^{mx}$, then $y' = me^{mx}$ and $y'' = m^2e^{mx}$ so that equation (2) becomes

$$am^2e^{mx} + bme^{mx} + ce^{mx} = 0 \quad \text{or} \quad e^{mx}(am^2 + bm + c) = 0.$$

Because e^{mx} is never zero for real values of x, it is apparent that the only way that this exponential function can satisfy the differential equation is to choose m so that it is a root of the quadratic equation

$$am^2 + bm + c = 0. \tag{3}$$

This latter equation is called the **auxiliary equation**, or **characteristic equation**, of the differential equation (2). We consider three cases, namely, the solutions of the auxiliary equation corresponding to distinct real roots, real but equal roots, and a conjugate pair of complex roots.

CASE I DISTINCT REAL ROOTS Under the assumption that the auxiliary equation (3) has two unequal real roots m_1 and m_2, we find two solutions

$$y_1 = e^{m_1x} \quad \text{and} \quad y_2 = e^{m_2x}.$$

We have seen that these functions are linearly independent on $(-\infty, \infty)$ (see Example 13, Section 4.1) and hence form a fundamental set. It follows that the general solution of (2) on this interval is

$$y = c_1e^{m_1x} + c_2e^{m_2x}. \tag{4}$$

CASE II REPEATED REAL ROOTS When $m_1 = m_2$ we necessarily obtain only one exponential solution $y_1 = e^{m_1x}$. However, it follows immediately from the discussion of Section 4.2 that a second solution is

$$y_2 = e^{m_1x}\int \frac{e^{-(b/a)x}}{e^{2m_1x}}\,dx. \tag{5}$$

But from the quadratic formula, we have $m_1 = -b/2a$ since the only way to have $m_1 = m_2$ is to have $b^2 - 4ac = 0$. In view of the fact that $2m_1 = -b/a$, (5) becomes

$$y_2 = e^{m_1x}\int \frac{e^{2m_1x}}{e^{2m_1x}}\,dx = e^{m_1x}\int dx = xe^{m_1x}.$$

The general solution of (2) is then

$$y = c_1 e^{m_1 x} + c_2 x e^{m_1 x}. \tag{6}$$

CASE III CONJUGATE COMPLEX ROOTS If m_1 and m_2 are complex, then we can write

$$m_1 = \alpha + i\beta \quad \text{and} \quad m_2 = \alpha - i\beta,$$

where α and $\beta > 0$ are real and $i^2 = -1$. Formally there is no difference between this case and Case I, and hence

$$y = C_1 e^{(\alpha + i\beta)x} + C_2 e^{(\alpha - i\beta)x}.$$

However, in practice we prefer to work with real functions instead of complex exponentials. To this end we use Euler's formula:*

$$e^{i\theta} = \cos\theta + i\sin\theta,$$

where θ is any real number. It follows from this formula that

$$e^{i\beta x} = \cos\beta x + i\sin\beta x \quad \text{and} \quad e^{-i\beta x} = \cos\beta x - i\sin\beta x, \tag{7}$$

where we have used $\cos(-\beta x) = \cos\beta x$ and $\sin(-\beta x) = -\sin\beta x$. Note that by first adding and then subtracting the two equations in (7), we obtain, respectively,

$$e^{i\beta x} + e^{-i\beta x} = 2\cos\beta x \quad \text{and} \quad e^{i\beta x} - e^{-i\beta x} = 2i\sin\beta x.$$

Since $y = C_1 e^{(\alpha + i\beta)x} + C_2 e^{(\alpha - i\beta)x}$ is a solution of (2) for any choice of the constants C_1 and C_2, the choices $C_1 = C_2 = 1$ and $C_1 = 1, C_2 = -1$ give, in turn, two solutions:

$$y_1 = e^{(\alpha + i\beta)x} + e^{(\alpha - i\beta)x} \quad \text{and} \quad y_2 = e^{(\alpha + i\beta)x} - e^{(\alpha - i\beta)x}.$$

Leonhard Euler (1707–1783) A man with a prodigious memory and phenomenal powers of concentration, Euler had almost universal interests: he was a theologian, physicist, astronomer, linguist, physiologist, classical scholar, and, primarily, mathematician. Euler is considered to be a true genius of the era. In mathematics, he made lasting contributions to algebra, trigonometry, analytic geometry, calculus, calculus of variations, differential equations, complex variables, number theory, and topology. The volume of his mathematical output did not seem to be affected by the distractions of thirteen children or the fact that he was totally blind for the last seventeen years of his life. Euler wrote over 700 papers and 32 books on mathematics and was responsible for introducing many of the symbols (such as e, π, and $i = \sqrt{-1}$) and notations that are still used (such as $f(x)$, Σ, $\sin x$, and $\cos x$). Euler was born in Basel, Switzerland, on April 15, 1707, and died of a stroke in St. Petersburg on September 18, 1783, while serving in the court of the Russian empress Catherine the Great.

See Appendix IV for a review of complex numbers and a derivation of Euler's formula.

But $$y_1 = e^{\alpha x}(e^{i\beta x} + e^{-i\beta x}) = 2e^{\alpha x}\cos\beta x$$

and $$y_2 = e^{\alpha x}(e^{i\beta x} - e^{-i\beta x}) = 2ie^{\alpha x}\sin\beta x.$$

Hence, from Corollary (A) of Theorem 4.3 the last two results show that the *real* functions $e^{\alpha x}\cos\beta x$ and $e^{\alpha x}\sin\beta x$ are solutions of (2). Moreover, from Example 14 of Section 4.1 we have $W(e^{\alpha x}\cos\beta x, e^{\alpha x}\sin\beta x) = \beta e^{2\alpha x} \neq 0, \beta > 0$, and so we can conclude that $e^{\alpha x}\cos\beta x$ and $e^{\alpha x}\sin\beta x$ themselves form a fundamental set of solutions of the differential equation on $(-\infty, \infty)$. By the superposition principle, the general solution is

$$\begin{aligned} y &= c_1e^{\alpha x}\cos\beta x + c_2e^{\alpha x}\sin\beta x \\ &= e^{\alpha x}(c_1\cos\beta x + c_2\sin\beta x). \end{aligned} \tag{8}$$

EXAMPLE 1

Solve the following differential equations:

(a) $2y'' - 5y' - 3y = 0$ (b) $y'' - 10y' + 25y = 0$ (c) $y'' + y' + y = 0$

Solution (a) $2m^2 - 5m - 3 = (2m+1)(m-3) = 0$

$$m_1 = -\tfrac{1}{2}, \quad m_2 = 3$$

$$y = c_1e^{-x/2} + c_2e^{3x}$$

(b) $m^2 - 10m + 25 = (m-5)^2 = 0$

$$m_1 = m_2 = 5$$

$$y = c_1e^{5x} + c_2xe^{5x}$$

(c) $m^2 + m + 1 = 0$

$$m_1 = -\frac{1}{2} + \frac{\sqrt{3}}{2}i, \quad m_2 = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$$

$$y = e^{-x/2}\left(c_1\cos\frac{\sqrt{3}}{2}x + c_2\sin\frac{\sqrt{3}}{2}x\right)$$ ■

EXAMPLE 2

Solve the initial-value problem

$$y'' - 4y' + 13y = 0, \quad y(0) = -1, \quad y'(0) = 2.$$

Solution The roots of the auxiliary equation $m^2 - 4m + 13 = 0$ are $m_1 = 2 + 3i$ and $m_2 = 2 - 3i$ so that

$$y = e^{2x}(c_1 \cos 3x + c_2 \sin 3x).$$

The condition $y(0) = -1$ implies

$$-1 = e^0(c_1 \cos 0 + c_2 \sin 0) = c_1,$$

from which we can write

$$y = e^{2x}(-\cos 3x + c_2 \sin 3x).$$

Differentiating this latter expression and using $y'(0) = 2$ give

$$y' = e^{2x}(3 \sin 3x + 3c_2 \cos 3x) + 2e^{2x}(-\cos 3x + c_2 \sin 3x)$$

$$2 = 3c_2 - 2$$

and so $c_2 = 4/3$. Hence

$$y = e^{2x}\left(-\cos 3x + \frac{4}{3} \sin 3x\right).$$ ■

EXAMPLE 3

The two equations

$$y'' + k^2 y = 0 \tag{9}$$

$$y'' - k^2 y = 0 \tag{10}$$

are frequently encountered in the study of applied mathematics. For the former differential equation, the auxiliary equation $m^2 + k^2 = 0$ has the roots $m_1 = ki$ and $m_2 = -ki$. It follows from (8) that the general solution of (9) is

$$y = c_1 \cos kx + c_2 \sin kx. \tag{11}$$

The differential equation (10) has the auxiliary equation $m^2 - k^2 = 0$, with real roots $m_1 = k$ and $m_2 = -k$, so that its general solution is

$$y = c_1 e^{kx} + c_2 e^{-kx}. \tag{12}$$

Notice that if we choose $c_1 = c_2 = 1/2$ in (12), then

$$y = \frac{e^{kx} + e^{-kx}}{2} = \cosh kx$$

is also a solution of (10). Furthermore, if $c_1 = 1/2$, $c_2 = -1/2$, then (12) becomes

$$y = \frac{e^{kx} - e^{-kx}}{2} = \sinh kx.$$

Since cosh kx and sinh kx are linearly independent on any interval of the x-axis, they form a fundamental set. Thus an alternative form for the general solution of (10) is

$$y = c_1 \cosh kx + c_2 \sinh kx. \tag{13}$$

■

Higher-Order Equations

In general, to solve an nth-order differential equation

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_2 y'' + a_1 y' + a_0 y = 0, \tag{14}$$

where the a_i, $i = 0, 1, \ldots, n$, are real constants, we must solve an nth-degree polynominal equation

$$a_n m^n + a_{n-1} m^{n-1} + \cdots + a_2 m^2 + a_1 m + a_0 = 0 \tag{15}$$

If all the roots of (15) are real and distinct, then the general solution of (14) is

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \cdots + c_n e^{m_n x}. \tag{16}$$

It is somewhat harder to summarize the analogues of Cases II and III because the roots of an auxiliary equation of degree greater than two can occur in many combinations. For example, a fifth-degree equation could have five distinct real roots, or three distinct real and two complex roots, or one real and four complex roots, or five real but equal roots, or five real roots but two of them equal, and so on. When m_1 is a root of multiplicity k of an nth-degree auxiliary equation (that is, k roots are equal to m_1), it can be shown that the linearly independent solutions are

$$e^{m_1 x}, xe^{m_1 x}, x^2 e^{m_1 x}, \ldots, x^{k-1} e^{m_1 x}$$

and the general solution must contain the linear combination

$$c_1 e^{m_1 x} + c_2 x e^{m_1 x} + c_3 x^2 e^{m_1 x} + \cdots + c_k x^{k-1} e^{m_1 x}.$$

Lastly, it should be remembered that when the coefficients are real, complex roots of an auxiliary equation always appear in conjugate pairs. Thus, for example, a cubic polynomial equation can have at most two complex roots.

EXAMPLE 4

Solve $$y''' + 3y'' - 4y = 0.$$

Solution It should be apparent from inspection of

$$m^3 + 3m^2 - 4 = 0$$

that one root is $m_1 = 1$. Now if we divide $m^3 + 3m^2 - 4$ by $m - 1$, we find

$$m^3 + 3m^2 - 4 = (m-1)(m^2 + 4m + 4) = (m-1)(m+2)^2,$$

and so the other roots are $m_2 = m_3 = -2$. Thus the general solution is

$$y = c_1e^x + c_2e^{-2x} + c_3xe^{-2x}.$$ ■

Of course, the most difficult aspect of solving constant-coefficient equations is finding the roots of auxiliary equations of degree greater than two. As illustrated in Example 4, one way to solve an equation is to guess a root m_1. If we have found one root m_1, we then know from the **factor theorem** that $m - m_1$ is a factor of the polynomial. By dividing the polynomial by $m - m_1$, we obtain the factorization $(m - m_1)Q(m)$. We then try to find the roots of the quotient $Q(m)$. The algebraic technique of **synthetic division** is also very helpful in finding **rational** roots of polynomial equations. Specifically, if $m_1 = p/q$ is a rational real root (p and q integers, p/q in lowest terms) of an auxiliary equation

$$a_nm^n + \cdots + a_1m + a_0 = 0$$

with integer coefficients, then p is a factor of a_0 and q is a factor of a_n. Thus to determine whether a polynomial equation has rational roots we need only examine all ratios of each factor of a_0 to each factor of a_n. In this manner we construct a list of all the possible rational roots of the equation. We test each of these numbers by synthetic division. If the remainder is zero, the number m_1 being tested is a root of the equation and thus $m - m_1$ is a factor of the polynomial.

The next example illustrates this method.

EXAMPLE 5

Solve $3y''' + 5y'' + 10y' - 4y = 0.$

Solution The auxiliary equation is

$$3m^3 + 5m^2 + 10m - 4 = 0.$$

All the factors of $a_0 = -4$ and $a_n = 3$ are

$$p: \pm 1, \pm 2, \pm 4 \quad \text{and} \quad q: \pm 1, \pm 3,$$

respectively. Therefore the possible rational roots of the auxiliary equation are

$$\frac{p}{q}: -1, 1, -2, 2, -4, 4, -\frac{1}{3}, \frac{1}{3}, -\frac{2}{3}, \frac{2}{3}, -\frac{4}{3}, \frac{4}{3}.$$

Testing each of these numbers in turn by synthetic division, we eventually find

coefficients of auxiliary equation

$$\begin{array}{r|rrrr} \frac{1}{3} & 3 & 5 & 10 & -4 \\ & & 1 & 2 & 4 \\ \hline & 3 & 6 & 12 & \boxed{0} \end{array} \quad \text{remainder}$$

Consequently $m_1 = 1/3$ is a root. Furthermore, you should verify that it is the *only* rational root. The numbers highlighted in the preceding division are the coefficients of the quotient. Thus the auxiliary equation can be written as

$$\left(m - \frac{1}{3}\right)(3m^2 + 6m + 12) = 0 \quad \text{or} \quad (3m - 1)(m^2 + 2m + 4) = 0.$$

Solving $m^2 + 2m + 4 = 0$ by the quadratic formula leads to the complex roots $m_2 = -1 + \sqrt{3}i$ and $m_3 = -1 - \sqrt{3}i$. Thus the general solution of the differential equation is

$$y = c_1 e^{x/3} + e^{-x}(c_2 \cos \sqrt{3}x + c_3 \sin \sqrt{3}x). \quad ■$$

EXAMPLE 6

Solve $$\frac{d^4y}{dx^4} + 2\frac{d^2y}{dx^2} + y = 0.$$

Solution The auxiliary equation

$$m^4 + 2m^2 + 1 = (m^2 + 1)^2 = 0$$

has roots $m_1 = m_3 = i$ and $m_2 = m_4 = -i$. Thus from Case II the solution is

$$y = C_1 e^{ix} + C_2 e^{-ix} + C_3 x e^{ix} + C_4 x e^{-ix}.$$

By Euler's formula the grouping $C_1 e^{ix} + C_2 e^{-ix}$ can be rewritten as

$$c_1 \cos x + c_2 \sin x$$

after a relabeling of constants. Similarly, $x(C_3 e^{ix} + C_4 e^{-ix})$ can be expressed as $x(c_3 \cos x + c_4 \sin x)$. Hence the general solution is

$$y = c_1 \cos x + c_2 \sin x + c_3 x \cos x + c_4 x \sin x. \quad ■$$

Example 6 illustrates a special case when the auxiliary equation has repeated complex roots. In general, if $m_1 = \alpha + i\beta$ is a complex root of multiplicity k of an auxiliary equation with real coefficients, then its conjugate $m_2 = \alpha - i\beta$ is also a root of multiplicity k. From the $2k$ complex-valued solutions

$$e^{(\alpha+i\beta)x}, xe^{(\alpha+i\beta)x}, x^2e^{(\alpha+i\beta)x}, \ldots, x^{k-1}e^{(\alpha+i\beta)x}$$

$$e^{(\alpha-i\beta)x}, xe^{(\alpha-i\beta)x}, x^2e^{(\alpha-i\beta)x}, \ldots, x^{k-1}e^{(\alpha-i\beta)x}$$

we conclude, with the aid of Euler's formula, that the general solution of the corresponding differential equation must then contain a linear combination of the $2k$ real linearly independent solutions

$$e^{\alpha x} \cos \beta x, xe^{\alpha x} \cos \beta x, x^2e^{\alpha x} \cos \beta x, \ldots, x^{k-1}e^{\alpha x} \cos \beta x$$

$$e^{\alpha x} \sin \beta x, xe^{\alpha x} \sin \beta x, x^2e^{\alpha x} \sin \beta x, \ldots, x^{k-1}e^{\alpha x} \sin \beta x.$$

In Example 6 we identify $k = 2$, $\alpha = 0$, and $\beta = 1$.

4.3 EXERCISES

Answers to odd-numbered problems begin on page A-9.

In Problems 1–36 find the general solution of the given differential equation.

1. $4y'' + y' = 0$
2. $2y'' - 5y' = 0$
3. $y'' - 36y = 0$
4. $y'' - 8y = 0$
5. $y'' + 9y = 0$
6. $3y'' + y = 0$
7. $y'' - y' - 6y = 0$
8. $y'' - 3y' + 2y = 0$
9. $\dfrac{d^2y}{dx^2} + 8\dfrac{dy}{dx} + 16y = 0$
10. $\dfrac{d^2y}{dx^2} - 10\dfrac{dy}{dx} + 25y = 0$
11. $y'' + 3y' - 5y = 0$
12. $y'' + 4y' - y = 0$
13. $12y'' - 5y' - 2y = 0$
14. $8y'' + 2y' - y = 0$
15. $y'' - 4y' + 5y = 0$
16. $2y'' - 3y' + 4y = 0$
17. $3y'' + 2y' + y = 0$
18. $2y'' + 2y' + y = 0$
19. $y''' - 4y'' - 5y' = 0$
20. $4y''' + 4y'' + y' = 0$
21. $y''' - y = 0$
22. $y''' + 5y'' = 0$
23. $y''' - 5y'' + 3y' + 9y = 0$
24. $y''' + 3y'' - 4y' - 12y = 0$
25. $y''' + y'' - 2y = 0$
26. $y''' - y'' - 4y = 0$
27. $y''' + 3y'' + 3y' + y = 0$
28. $y''' - 6y'' + 12y' - 8y = 0$
29. $\dfrac{d^4y}{dx^4} + \dfrac{d^3y}{dx^3} + \dfrac{d^2y}{dx^2} = 0$
30. $\dfrac{d^4y}{dx^4} - 2\dfrac{d^2y}{dx^2} + y = 0$
31. $16\dfrac{d^4y}{dx^4} + 24\dfrac{d^2y}{dx^2} + 9y = 0$
32. $\dfrac{d^4y}{dx^4} - 7\dfrac{d^2y}{dx^2} - 18y = 0$
33. $\dfrac{d^5y}{dx^5} - 16\dfrac{dy}{dx} = 0$
34. $\dfrac{d^5y}{dx^5} - 2\dfrac{d^4y}{dx^4} + 17\dfrac{d^3y}{dx^3} = 0$
35. $\dfrac{d^5y}{dx^5} + 5\dfrac{d^4y}{dx^4} - 2\dfrac{d^3y}{dx^3} - 10\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} + 5y = 0$
36. $2\dfrac{d^5y}{dx^5} - 7\dfrac{d^4y}{dx^4} + 12\dfrac{d^3y}{dx^3} + 8\dfrac{d^2y}{dx^2} = 0$

In Problems 37–52 solve the given differential equation subject to the indicated initial conditions.

37. $y'' + 16y = 0$, $y(0) = 2$, $y'(0) = -2$
38. $y'' - y = 0$, $y(0) = y'(0) = 1$
39. $y'' + 6y' + 5y = 0$, $y(0) = 0$, $y'(0) = 3$
40. $y'' - 8y' + 17y = 0$, $y(0) = 4$, $y'(0) = -1$
41. $2y'' - 2y' + y = 0$, $y(0) = -1$, $y'(0) = 0$
42. $y'' - 2y' + y = 0$, $y(0) = 5$, $y'(0) = 10$
43. $y'' + y' + 2y = 0$, $y(0) = y'(0) = 0$

44. $4y'' - 4y' - 3y = 0, \quad y(0) = 1, y'(0) = 5$

45. $y'' - 3y' + 2y = 0, \quad y(1) = 0, y'(1) = 1$

46. $y'' + y = 0, \quad y(\pi/3) = 0, y'(\pi/3) = 2$

47. $y''' + 12y'' + 36y' = 0, \quad y(0) = 0, y'(0) = 1, y''(0) = -7$

48. $y''' + 2y'' - 5y' - 6y = 0, \quad y(0) = y'(0) = 0, y''(0) = 1$

49. $y''' - 8y = 0, \quad y(0) = 0, y'(0) = -1, y''(0) = 0$

50. $\dfrac{d^4y}{dx^4} = 0, \quad y(0) = 2, y'(0) = 3, y''(0) = 4, y'''(0) = 5$

51. $\dfrac{d^4y}{dx^4} - 3\dfrac{d^3y}{dx^3} + 3\dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} = 0, \quad y(0) = y'(0) = 0, y''(0) = y'''(0) = 1$

52. $\dfrac{d^4y}{dx^4} - y = 0, \quad y(0) = y'(0) = y''(0) = 0, y'''(0) = 1$

In Problems 53–56 solve the given differential equation subject to the indicated boundary conditions.

53. $y'' - 10y' + 25y = 0, \quad y(0) = 1, y(1) = 0$

54. $y'' + 4y = 0, \quad y(0) = 0, y(\pi) = 0$

55. $y'' + y = 0, \quad y'(0) = 0, y'\left(\dfrac{\pi}{2}\right) = 2$

56. $y'' - y = 0, \quad y(0) = 1, y'(1) = 0$

57. The roots of an auxiliary equation are $m_1 = 4, m_2 = m_3 = -5$. What is the corresponding differential equation?

58. The roots of an auxiliary equation are $m_1 = -\frac{1}{2}, m_2 = 3 + i, m_3 = 3 - i$. What is the corresponding differential equation?

In Problems 59 and 60 find the general solution of the given equation if it is known that y_1 is a solution.

59. $y''' - 9y'' + 25y' - 17y = 0; \quad y_1 = e^x$

60. $y''' + 6y'' + y' - 34y = 0; \quad y_1 = e^{-4x}\cos x$

In Problems 61–64 determine a homogeneous linear differential equation with constant coefficients having the given solutions.

61. $4e^{6x}, 3e^{-3x}$

62. $10\cos 4x, -5\sin 4x$

63. $3, 2x, -e^{7x}$

64. $8\sinh 3x, 12\cosh 3x$

65. Use the facts

$$i = \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right)^2 \quad \text{and} \quad -i = \left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i\right)^2$$

to solve the differential equation

$$\frac{d^4y}{dx^4} + y = 0.$$

[*Hint:* Write the auxiliary equation $m^4 + 1 = 0$ as $(m^2 + 1)^2 - 2m^2 = 0$. See what happens when you factor.]

4.4 UNDETERMINED COEFFICIENTS—SUPERPOSITION APPROACH

To the instructor In this section the method of undetermined coefficients is developed from the viewpoint of the superposition principle for nonhomogeneous differential equations (Theorem 4.9). In Section 4.6 an entirely different approach to this method will be presented, one utilizing the concept of differential annihilator operators. Take your pick.

To obtain the general solution of a nonhomogeneous linear differential equation we must do two things:

(i) Find the complementary function y_c.

(ii) Find *any* particular solution y_p of the nonhomogeneous equation.

Recall from the discussion of Section 4.1 that a particular solution is any function, free of arbitrary constants, that satisfies the differential equation identically. The general solution of a nonhomogeneous equation on an interval is then $y = y_c + y_p$.

As in the Section 4.3 we begin with second-order equations, but in this case nonhomogeneous equations of the form

$$ay'' + by' + cy = g(x), \tag{1}$$

where a, b, and c are constants. Although the method of undetermined coefficients presented in this section is *not* limited to second-order equations, it is limited to nonhomogeneous linear equations

- that have constant coefficients, and
- where $g(x)$ is a constant k, a polynomial function, an exponential function $e^{\alpha x}$, $\sin \beta x$, $\cos \beta x$, or finite sums and products of these functions.

Note Strictly speaking $g(x) = k$ (a constant) is a polynomial function. Since a constant function is probably not the first thing that comes to mind when you think of polynomial functions, for emphasis we continue to use the redundancy "constant functions, polynomials, ..."

The following are some examples of the types of input functions $g(x)$ that are appropriate for this discussion:

$$g(x) = 10$$

$$g(x) = x^2 - 5x$$

$$g(x) = 15x - 6 + 8e^{-4x}$$

$$g(x) = \sin 3x - 5x \cos 2x$$

$$g(x) = e^x \cos x - (3x^2 - 1)e^{-x}$$

and so on. That is, $g(x)$ is a linear combination of functions of the type

$$k \text{ (constant)}, \quad x^n, \quad x^n e^{\alpha x}, \quad x^n e^{\alpha x} \cos \beta x, \quad \text{and} \quad x^n e^{\alpha x} \sin \beta x,$$

where n is a nonnegative integer and α and β are real numbers. The method of undetermined coefficients is not applicable to equations of form (1) when

$$g(x) = \ln x, \quad g(x) = \frac{1}{x}, \quad g(x) = \tan x, \quad g(x) = \sin^{-1} x,$$

and so on. Differential equations with this latter kind of input function will be considered in Section 4.7.

The set of functions that consists of constants, polynomials, exponentials $e^{\alpha x}$, sines, and cosines has the remarkable property that derivatives of their sums and products are again sums and products of constants, polynomials, exponentials $e^{\alpha x}$, sines, and cosines. Since the linear combination of derivatives $ay_p'' + by_p' + cy_p$ must be identically equal to $g(x)$, it seems reasonable to assume then that y_p *has the same form as* $g(x)$. This assumption could be better characterized as an educated conjecture or guess. The next two examples illustrate the basic method.

EXAMPLE 1

Solve $$y'' + 4y' - 2y = 2x^2 - 3x + 6. \tag{2}$$

Solution **Step 1.** We first solve the associated homogeneous equation $y'' + 4y' - 2y = 0$. From the quadratic formula we find that the roots of the auxiliary equation $m^2 + 4m - 2 = 0$ are $m_1 = -2 - \sqrt{6}$ and $m_2 = -2 + \sqrt{6}$. Hence the complementary function is

$$y_c = c_1 e^{-(2+\sqrt{6})x} + c_2 e^{(-2+\sqrt{6})x}.$$

Step 2. Now since the input function $g(x)$ is a quadratic polynomial, let us assume a particular solution that is also in the form of a quadratic polynomial:

$$y_p = Ax^2 + Bx + C.$$

We seek to determine *specific* coefficients A, B, and C for which y_p is a solution of (2). Substituting y_p and the derivatives

$$y_p' = 2Ax + B \quad \text{and} \quad y_p'' = 2A$$

into the given differential equation (2), we get

$$y_p'' + 4y_p' - 2y_p = 2A + 8Ax + 4B - 2Ax^2 - 2Bx - 2C = 2x^2 - 3x + 6.$$

Since the last equation is supposed to be an identity, the coefficients of like powers of x must be equal:

equal

$$\boxed{-2A}x^2 + \boxed{8A - 2B}x + \boxed{2A + 4B - 2C} = 2x^2 - 3x + 6.$$

That is,

$$-2A = 2$$
$$8A - 2B = -3$$
$$2A + 4B - 2C = 6.$$

Solving this system of equations leads to the values $A = -1$, $B = -5/2$, and $C = -9$. Thus a particular solution is

$$y_p = -x^2 - \frac{5}{2}x - 9.$$

Step 3. The general solution of the given equation is

$$y = y_c + y_p = c_1e^{-(2+\sqrt{6})x} + c_2e^{(-2+\sqrt{6})x} - x^2 - \frac{5}{2}x - 9.$$ ■

EXAMPLE 2

Find a particular solution of $y'' - y' + y = 2\sin 3x$.

Solution A natural first guess for a particular solution would be $A\sin 3x$. But since successive differentiations of $\sin 3x$ produce $\sin 3x$ *and* $\cos 3x$, we are prompted instead to assume a particular solution that includes both of these terms:

$$y_p = A\cos 3x + B\sin 3x.$$

Differentiating y_p and substituting the results into the differential equation give, after regrouping,

$$y_p'' - y_p' + y_p = (-8A - 3B)\cos 3x + (3A - 8B)\sin 3x = 2\sin 3x$$

or

equal

$$\boxed{-8A-3B}\cos 3x + \boxed{3A-8B}\sin 3x = 0\cos 3x + 2\sin 3x.$$

From the resulting system of equations

$$-8A - 3B = 0$$

$$3A - 8B = 2$$

we get $A = 6/73$ and $B = -16/73$. A particular solution of the equation is

$$y_p = \frac{6}{73}\cos 3x - \frac{16}{73}\sin 3x.$$

As we mentioned, the form that we assume for the particular solution y_p is an educated guess; it is not a blind guess. This educated guess must take into consideration not only the types of functions that make up $g(x)$, but also, as we shall see in Example 4, the functions that make up the complementary function y_c. ■

EXAMPLE 3

Solve

$$y'' - 2y' - 3y = 4x - 5 + 6xe^{2x}. \tag{3}$$

Solution Step 1. First, the solution of the associated homogeneous equation $y'' - 2y' - 3y = 0$ is found to be $y_c = c_1e^{-x} + c_2e^{3x}$.

Step 2. Next, the presence of $4x - 5$ in $g(x)$ suggests that the particular solution includes a linear polynomial. Furthermore, since the derivative of the product xe^{2x} produces $2xe^{2x}$ and e^{2x}, we also assume that the particular solution includes both xe^{2x} and e^{2x}. In other words, g is the sum of two basic kinds of functions:

$$g(x) = g_1(x) + g_2(x) = \textit{polynomial} + \textit{exponentials}.$$

Correspondingly, the superposition principle for nonhomogeneous equations (Theorem 4.9) suggests that we seek a particular solution

$$y_p = y_{p_1} + y_{p_2},$$

where $y_{p_1} = Ax + B$ and $y_{p_2} = Cxe^{2x} + De^{2x}$. Substituting

$$y_p = Ax + B + Cxe^{2x} + De^{2x}$$

into the given equation (3) and grouping like terms give

$$y_p'' - 2y_p' - 3y_p = -3Ax - 2A - 3B - 3Cxe^{2x} + (2C - 3D)e^{2x} = 4x - 5 + 6xe^{2x}. \tag{4}$$

From this identity we obtain the system of four equations in four unknowns:

$$-3A = 4$$
$$-2A - 3B = -5$$
$$-3C = 6$$
$$2C - 3D = 0.$$

The last equation in this system results from the interpretation that the coefficient of e^{2x} in the right member of (4) is zero. Solving, we find $A = -4/3$, $B = 23/9$, $C = -2$, and $D = -4/3$. Consequently,

$$y_p = -\frac{4}{3}x + \frac{23}{9} - 2xe^{2x} - \frac{4}{3}e^{2x}.$$

Step 3. The general solution of the equation is

$$y = c_1e^{-x} + c_2e^{3x} - \frac{4}{3}x + \frac{23}{9} - \left(2x + \frac{4}{3}\right)e^{2x}. \quad \blacksquare$$

In light of the superposition principle (Theorem 4.9) we can also approach Example 3 from the viewpoint of solving two simpler problems. You should verify that substituting

$$y_{p_1} = Ax + B \quad \text{into} \quad y'' - 2y' - 3y = 4x - 5$$

and

$$y_{p_2} = Cxe^{2x} + De^{2x} \quad \text{into} \quad y'' - 2y' - 3y = 6xe^{2x}$$

yields in turn $y_{p_1} = -(4/3)x + 23/9$ and $y_{p_2} = -(2x + 4/3)e^{2x}$. A particular solution of (3) is then $y_p = y_{p_1} + y_{p_2}$.

The next example illustrates that sometimes the "obvious" assumption for the form of y_p is not a correct assumption.

EXAMPLE 4

Find a particular solution of $y'' - 5y' + 4y = 8e^x$.

Solution Differentiation of e^x produces no new functions. Thus, proceeding as we did in the earlier examples, we can reasonably assume a particular solution of the form

$$y_p = Ae^x.$$

But in this case substitution of this expression into the differential equation yields the contradictory statement

$$0 = 8e^x,$$

and so we have clearly made the wrong guess for y_p.

The difficulty here is apparent upon examining the complementary function $y_c = c_1e^x + c_2e^{4x}$. Observe that our assumption Ae^x is already present in y_c. This means that e^x is a solution of the associated homogeneous differential equation, and a constant multiple Ae^x when substituted into the differential equation necessarily produces zero.

What then should be the form of y_p? Inspired by Case II of Section 4.3, let's see whether we can find a particular solution of the form

$$y_p = Axe^x.$$

Using $y_p' = Axe^x + Ae^x$ and $y_p'' = Axe^x + 2Ae^x$, we obtain

$$y_p'' - 5y_p' + 4y_p = Axe^x + 2Ae^x - 5Axe^x - 5Ae^x + 4Axe^x = 8e^x$$

or

$$-3Ae^x = 8e^x.$$

From this last equation we see that the value of A is now determined as $A = -8/3$. Therefore

$$y_p = -\frac{8}{3}xe^x$$

must be a particular solution of the given equation. ■

The difference in the procedures used in Examples 1–3 and in Example 4 suggests that we consider two cases. The first case reflects the situation in Examples 1–3.

CASE I *No function in the assumed particular solution is a solution of the associated homogeneous differential equation.*

In the following table we illustrate some specific examples of $g(x)$ in (1) along with the corresponding form of the particular solution. We are, of course, taking for granted that no function in the assumed particular solution y_p is duplicated by a function in the complementary function y_c.

Trial Particular Solutions

$g(x)$	Form of y_p
1. 1 (any constant)	A
2. $5x + 7$	$Ax + B$
3. $3x^2 - 2$	$Ax^2 + Bx + C$
4. $x^3 - x + 1$	$Ax^3 + Bx^2 + Cx + D$
5. $\sin 4x$	$A\cos 4x + B\sin 4x$
6. $\cos 4x$	$A\cos 4x + B\sin 4x$
7. e^{5x}	Ae^{5x}
8. $(9x - 2)e^{5x}$	$(Ax + B)e^{5x}$
9. x^2e^{5x}	$(Ax^2 + Bx + C)e^{5x}$
10. $e^{3x}\sin 4x$	$Ae^{3x}\cos 4x + Be^{3x}\sin 4x$
11. $5x^2\sin 4x$	$(Ax^2 + Bx + C)\cos 4x + (Dx^2 + Ex + F)\sin 4x$
12. $xe^{3x}\cos 4x$	$(Ax + B)e^{3x}\cos 4x + (Cx + D)e^{3x}\sin 4x$

EXAMPLE 5

Determine the form of a particular solution of

(a) $y'' - 8y' + 25y = 5x^3e^{-x} - 7e^{-x}$ and **(b)** $y'' + 4y = x \cos x$.

Solution (a) We can write

$$g(x) = (5x^3 - 7)e^{-x}.$$

Using entry 9 in the table as a model, we assume a particular solution of the form

$$y_p = (Ax^3 + Bx^2 + Cx + D)e^{-x}.$$

Note that there is no duplication between the terms in y_p and the terms in the complementary function $y_c = e^{4x}(c_1 \cos 3x + c_2 \sin 3x)$.

(b) The function $g(x) = x \cos x$ is similar to entry 11 in the table, except of course we use a linear rather than a quadratic polynomial and $\cos x$ and $\sin x$ instead of $\cos 4x$ and $\sin 4x$ in the form of y_p:

$$y_p = (Ax + B) \cos x + (Cx + D) \sin x.$$

Again observe that there is no duplication of terms between y_p and $y_c = c_1 \cos 2x + c_2 \sin 2x$. ■

If $g(x)$ consists of a sum of, say, m terms of the kind listed in the table, then (as in Example 3) the assumption for a particular solution y_p consists of the sum of the trial forms $y_{p_1}, y_{p_2}, \ldots, y_{p_m}$ corresponding to these terms:

$$y_p = y_{p_1} + y_{p_2} + \cdots + y_{p_m}.$$

Put another way:

The form of y_p is a linear combination of all linearly independent functions that are generated by repeated differentiations of $g(x)$.

EXAMPLE 6

Determine the form of a particular solution of

$$y'' - 9y' + 14y = 3x^2 - 5 \sin 2x + 7xe^{6x}.$$

Solution

Corresponding to $3x^2$ we assume: $y_{p_1} = Ax^2 + Bx + C$

Corresponding to $-5 \sin 2x$ we assume: $y_{p_2} = D \cos 2x + E \sin 2x$

Corresponding to $7xe^{6x}$ we assume: $y_{p_3} = (Fx + G)e^{6x}$

The assumption for the particular solution is then

$$y_p = y_{p_1} + y_{p_2} + y_{p_3} = Ax^2 + Bx + C + D \cos 2x + E \sin 2x + (Fx + G)e^{6x}.$$

No term in this assumption duplicates a term in $y_c = c_1e^{2x} + c_2e^{7x}$. ■

CASE II *A function in the assumed particular solution is also a solution of the associated homogeneous differential equation.*

The next example is similar to Example 4.

EXAMPLE 7

Find a particular solution of $y'' - 2y' + y = e^x$.

Solution The complementary function is $y_c = c_1e^x + c_2xe^x$. As in Example 4, the assumption $y_p = Ae^x$ will fail since it is apparent from y_c that e^x is a solution of the associated homogeneous equation $y'' - 2y' + y = 0$. Moreover, we will not be able to find a particular solution of the form $y_p = Axe^x$ since the term xe^x is also duplicated in y_c. We next try

$$y_p = Ax^2e^x.$$

Substituting into the given differential equation yields

$$2Ae^x = e^x \quad \text{and so} \quad A = \frac{1}{2}.$$

Thus a particular solution is

$$y_p = \frac{1}{2}x^2e^x.$$

■

Suppose again that $g(x)$ consists of m terms of the kind given in the table and suppose further that the usual assumption for a particular solution is

$$y_p = y_{p_1} + y_{p_2} + \cdots + y_{p_m},$$

where the y_{p_i}, $i = 1, 2, \ldots, m$, are the trial particular solution forms corresponding to these terms. Under the circumstance described in Case II, we can make up the following general rule:

If any y_{p_i} contains terms that duplicate terms in y_c, then that y_{p_i} must be multiplied by x^n, where n is the smallest positive integer that eliminates that duplication.

EXAMPLE 8

Solve the initial-value problem

$$y'' + y = 4x + 10\sin x, \qquad y(\pi) = 0, \quad y'(\pi) = 2.$$

Solution The solution of the associated homogeneous equation $y'' + y = 0$ is

$$y_c = c_1\cos x + c_2\sin x.$$

Now since $g(x)$ is the sum of a linear polynomial and a sine function, our normal assumption for y_p from entries 2 and 5 of the trial solutions table would be the sum of $y_{p_1} = Ax + B$ and $y_{p_2} = C\cos x + D\sin x$:

$$y_p = Ax + B + C\cos x + D\sin x. \tag{5}$$

But there is an obvious duplication of the terms $\cos x$ and $\sin x$ in this assumed form and two terms in the complementary function. This duplication can be eliminated by simply multiplying y_{p_2} by x. Instead of (5) we now use

$$y_p = Ax + B + Cx\cos x + Dx\sin x. \tag{6}$$

Differentiating this expression and substituting the results into the differential equation give

$$y_p'' + y_p = Ax + B - 2C\sin x + 2D\cos x = 4x + 10\sin x$$

and so

$$A = 4$$
$$B = 0$$
$$-2C = 10$$
$$2D = 0.$$

The solutions of the system are immediate: $A = 4$, $B = 0$, $C = -5$, and $D = 0$. Therefore from (6) we obtain

$$y_p = 4x - 5x\cos x.$$

The general solution of the given equation is

$$y = y_c + y_p = c_1\cos x + c_2\sin x + 4x - 5x\cos x.$$

We now apply the prescribed initial conditions to the general solution of the equation. First, $y(\pi) = c_1\cos\pi + c_2\sin\pi + 4\pi - 5\pi\cos\pi = 0$ yields $c_1 = 9\pi$ since $\cos\pi = -1$ and $\sin\pi = 0$. Next, from the derivative

$$y' = -9\pi\sin x + c_2\cos x + 4 + 5x\sin x - 5\cos x$$

and $\quad y'(\pi) = -9\pi\sin\pi + c_2\cos\pi + 4 + 5\pi\sin\pi - 5\cos\pi = 2$

we find $c_2 = 7$. The solution of the initial value is then

$$y = 9\pi\cos x + 7\sin x + 4x - 5x\cos x.$$

EXAMPLE 9

Solve $\quad y'' - 6y' + 9y = 6x^2 + 2 - 12e^{3x}.$

Solution The complementary function is

$$y_c = c_1e^{3x} + c_2xe^{3x}$$

and based on entries 3 and 7 of the table, the usual assumption for a particular solution would be

$$y_p = \underbrace{Ax^2 + Bx + C}_{\boxed{y_{p_1}}} + \underbrace{De^{3x}}_{\boxed{y_{p_2}}}.$$

Inspection of these functions shows that the one term in y_{p_2} is duplicated in y_c. If we multiply y_{p_2} by x, we note that the term xe^{3x} is still part of y_c. But multiplying y_{p_2} by x^2 eliminates all duplications. Thus the operative form of a particular solution is

$$y_p = Ax^2 + Bx + C + Dx^2e^{3x}.$$

Differentiating this last form, substituting into the differential equation, and collecting like terms give

$$y_p'' - 6y_p' + 9y_p = 9Ax^2 + (-12A + 9B)x + 2A - 6B + 9C + 2De^{3x} = 6x^2 + 2 - 12e^{3x}.$$

It follows from this identity that $A = 2/3$, $B = 8/9$, $C = 2/3$, and $D = -6$. Hence the general solution $y = y_c + y_p$ is

$$y = c_1e^{3x} + c_2xe^{3x} + \frac{2}{3}x^2 + \frac{8}{9}x + \frac{2}{3} - 6x^2e^{3x}.$$ ■

Higher-Order Equations

The method of undetermined coefficients given here is not restricted to second-order equations but can be used as well with higher-order equations

$$a_ny^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1y' + a_0y = g(x)$$

with constant coefficients. It is only necessary that $g(x)$ consist of the proper kinds of functions discussed above.

EXAMPLE 10

Solve $$y''' + y'' = e^x \cos x.$$

Solution From the characteristic equation $m^3 + m^2 = 0$ we find $m_1 = m_2 = 0$ and $m_3 = -1$. Hence the complementary solution of the equation is $y_c = c_1 + c_2x + c_3e^{-x}$. With $g(x) = e^x \cos x$ we see from entry 10 of the table of trial particular solutions that we should assume

$$y_p = Ae^x \cos x + Be^x \sin x.$$

Since there are no functions in y_p that duplicate functions in the complementary solution, we proceed in the usual manner. From

$$y_p''' + y_p'' = (-2A + 4B)e^x \cos x + (-4A - 2B)e^x \sin x = e^x \cos x$$

we get

$$-2A + 4B = 1$$
$$-4A - 2B = 0.$$

This system gives $A = -1/10$ and $B = 1/5$, so that a particular solution is

$$y_p = -\frac{1}{10}e^x \cos x + \frac{1}{5}e^x \sin x.$$

The general solution of the equation is

$$y = y_c + y_p = c_1 + c_2x + c_3e^{-x} - \frac{1}{10}e^x \cos x + \frac{1}{5}e^x \sin x.$$

EXAMPLE 11

Determine the form of a particular solution of

$$y^{(4)} + y''' = 1 - e^{-x}.$$

Solution Comparing the complementary function

$$y_c = c_1 + c_2x + c_3x^2 + c_4e^{-x}$$

with our normal assumption for a particular solution

$$y_p = \underbrace{A}_{\boxed{y_{p_1}}} + \underbrace{Be^{-x}}_{\boxed{y_{p_2}}},$$

we see that the duplications between y_c and y_p are eliminated when y_{p_1} is multiplied by x^3 and y_{p_2} is multiplied by x. Thus the correct assumption for a particular solution is

$$y_p = Ax^3 + Bxe^{-x}.$$

4.4 EXERCISES

Answers to odd-numbered problems begin on page A-10.

In Problems 1–26 solve the given differential equation by undetermined coefficients.

1. $y'' + 3y' + 2y = 6$
2. $4y'' + 9y = 15$
3. $y'' - 10y' + 25y = 30x + 3$
4. $y'' + y' - 6y = 2x$
5. $\frac{1}{4}y'' + y' + y = x^2 - 2x$
6. $y'' - 8y' + 20y = 100x^2 - 26xe^x$
7. $y'' + 3y = -48x^2e^{3x}$
8. $4y'' - 4y' - 3y = \cos 2x$
9. $y'' - y' = -3$
10. $y'' + 2y' = 2x + 5 - e^{-2x}$

11. $y'' - y' + \frac{1}{4}y = 3 + e^{x/2}$

12. $y'' - 16y = 2e^{4x}$

13. $y'' + 4y = 3\sin 2x$

14. $y'' + 4y = (x^2 - 3)\sin 2x$

15. $y'' + y = 2x\sin x$

16. $y'' - 5y' = 2x^3 - 4x^2 - x + 6$

17. $y'' - 2y' + 5y = e^x \cos 2x$

18. $y'' - 2y' + 2y = e^{2x}(\cos x - 3\sin x)$

19. $y'' + 2y' + y = \sin x + 3\cos 2x$

20. $y'' + 2y' - 24y = 16 - (x + 2)e^{4x}$

21. $y''' - 6y'' = 3 - \cos x$

22. $y''' - 2y'' - 4y' + 8y = 6xe^{2x}$

23. $y''' - 3y'' + 3y' - y = x - 4e^x$

24. $y''' - y'' - 4y' + 4y = 5 - e^x + e^{2x}$

25. $y^{(4)} + 2y'' + y = (x - 1)^2$

26. $y^{(4)} - y'' = 4x + 2xe^{-x}$

In Problems 27 and 28 use a trigonometric identity as an aid in finding a particular solution of the given differential equation.

27. $y'' + 4y = 8\sin^2 x$

28. $y'' + y = \sin x \cos 2x$

In Problems 29–40 solve the given differential equation subject to the indicated initial conditions.

29. $y'' + y = -2, \quad y\left(\frac{\pi}{8}\right) = \frac{1}{2}, y'\left(\frac{\pi}{8}\right) = 2$

30. $2y'' + 3y' - 2y = 14x^2 - 4x - 11, \quad y(0) = 0, y'(0) = 0$

31. $5y'' + y' = -6x, \quad y(0) = 0, y'(0) = -10$

32. $y'' + 4y' + 4y = (3 + x)e^{-2x}, \quad y(0) = 2, y'(0) = 5$

33. $y'' + 4y' + 5y = 35e^{-4x}, \quad y(0) = -3, y'(0) = 1$

34. $y'' - y = \cosh x, \quad y(0) = 2, y'(0) = 12$

35. $\frac{d^2x}{dt^2} + \omega^2 x = F_0 \sin \omega t, \quad x(0) = 0, x'(0) = 0$

36. $\frac{d^2x}{dt^2} + \omega^2 x = F_0 \cos \gamma t, \quad x(0) = 0, x'(0) = 0$

37. $y'' + y = \cos x - \sin 2x, \quad y\left(\frac{\pi}{2}\right) = 0, y'\left(\frac{\pi}{2}\right) = 0$

38. $y'' - 2y' - 3y = 2\cos^2 x, \quad y(0) = -\frac{1}{3}, y'(0) = 0$

39. $y''' - 2y'' + y' = 2 - 24e^x + 40e^{5x}, \quad y(0) = \frac{1}{2}, y'(0) = \frac{5}{2}, y''(0) = -\frac{9}{2}$

40. $y''' + 8y = 2x - 5 + 8e^{-2x}, \quad y(0) = -5, y'(0) = 3, y''(0) = -4$

2.5 LINEAR EQUATIONS

In Chapter 1 we defined the general form of a linear differential equation of order n to be

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x).$$

We remind you that linearity means that all coefficients are functions of x only, and that y and all its derivatives are raised to the first power. Now when $n = 1$, we obtain a linear first-order equation.

DEFINITION 2.5 Linear Equation

A differential equation of the form

$$a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

is said to be a linear equation.

Dividing by the lead coefficient $a_1(x)$ gives a more useful form of a linear equation:

$$\frac{dy}{dx} + P(x)y = f(x). \tag{1}$$

We seek a solution of (1) on an interval I for which the functions $P(x)$ and $f(x)$ are continuous. In the discussion that follows, we tacitly assume that (1) has a solution.

Integrating Factor

Using differentials, we can rewrite equation (1) as

$$dy + [P(x)y - f(x)]\,dx = 0. \tag{2}$$

Linear equations possess the pleasant property that a function $\mu(x)$ can always be found such that the multiple of (2),

$$\mu(x)\,dy + \mu(x)[P(x)y - f(x)]\,dx = 0, \tag{3}$$

is an exact differential equation. By Theorem 2.2 we know that the left side of equation (3) is an exact differential if

$$\frac{\partial}{\partial x}\mu(x) = \frac{\partial}{\partial y}\mu(x)[P(x)y - f(x)] \tag{4}$$