Math 215 - Problem Set 1: Three Dimensional Coordinate Systems, Vectors, Dot Product, Cross Product, Equation of lines and planes, Cylinders, and Quadric Surfaces

Math 215 SI

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1 Review

1.1 Three Dimensional Coordinate Systems

In three-dimensional coordinate systems, points are represented by ordered triples (x, y, z). The three axes (x, y, and z) are mutually perpendicular, and the position of a point is determined by its distances from these axes.

1.2 Vectors

Vectors are mathematical objects that have both magnitude and direction. They are often represented as directed line segments or as ordered triples (x, y, z) in three-dimensional space. Vectors can be added together and multiplied by scalars.

1.3 Dot Product

The dot product (or scalar product) of two vectors is a way of multiplying them to get a scalar. For vectors $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$, the dot product is given by $\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$. It is used to find the angle between vectors and to determine orthogonality. The cosine of the angle θ between two vectors $\mathbf{a}$ and $\mathbf{b}$ can be found using the dot product and the magnitudes of the vectors. The formula is given by:

$$\cos(\theta) = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$$

where $|\mathbf{a}|$ and $|\mathbf{b}|$ are the magnitudes of $\mathbf{a}$ and $\mathbf{b}$, respectively.

1.4 Cross Product

The cross product (or vector product) of two vectors in three-dimensional space results in a third vector that is perpendicular to the plane containing the original vectors. For vectors $\mathbf{a}$ and $\mathbf{b}$, the cross product $\mathbf{a} \times \mathbf{b}$ is given by a determinant involving the unit vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$. The cross product of two vectors $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$ is given by:

$$\mathbf{a} \times \mathbf{b} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix}$$

Expanding the determinant, we get:

$$\mathbf{a} \times \mathbf{b} = (a_2b_3 - a_3b_2)\mathbf{i} - (a_1b_3 - a_3b_1)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}$$

1.5 Equation of lines and planes

The equation of a line in three-dimensional space can be written in parametric form using a point and a direction vector. The equation of a plane can be written in the form $Ax + By + Cz = D$, where A, B, and C are the coefficients that define the normal vector to the plane.

1.6 Cylinders

Cylinders are surfaces generated by moving a line (the generator) parallel to itself along a curve (the directrix). In three-dimensional space, a common type of cylinder is the right circular cylinder, which has a circular base and a fixed height.

1.7 Quadric Surfaces

Quadric surfaces are the graphs of second-degree equations in three variables. Examples include ellipsoids, hyperboloids, paraboloids, and cones. These surfaces can be classified based on the signs and values of the coefficients in their defining equations.

2 Problems

Problem 1

(a) Find the equation of the plane P_1.

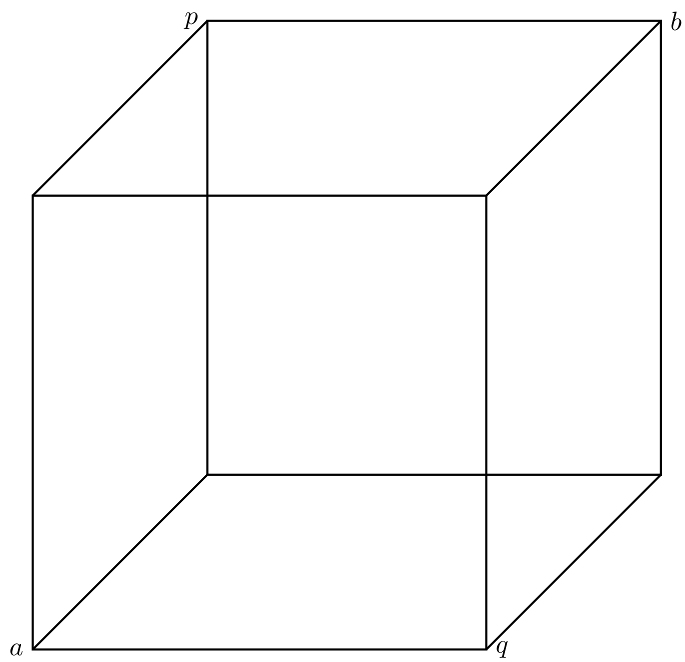
(b) Let P be the plane that contains the point $(0, 2, 1)$ and the line $\mathbf{l}(t) = \langle 2t, t, 1 + 3t \rangle$. Find the angle between P_1 and the plane P_2.

Problem 2

Let $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$, and let θ_1 be the angle $\mathbf{v}$ makes with the x-axis, θ_2 be the angle $\mathbf{v}$ makes with the y-axis, and θ_3 be the angle $\mathbf{v}$ makes with the z-axis. Find $\cos^2(\theta_1) + \cos^2(\theta_2) + \cos^2(\theta_3)$.

Problem 3

The sides of the cube below have a length of six. The line segments ab and pq intersect at the center of the cube, let's call the center c. Let T be the triangle with vertices a, c, and q.



(a) Find the area of the triangle T.

(b) If θ is the angle of T at c, then what is $\cos(\theta)$?