Math 215 – Problem Set 4

Maximum and Minimum Values, Lagrange Multipliers

Math 215 SI
February 8, 2026

4 Review

4.1 Local and Global Extrema

Local Maximum: A function $f(x, y)$ has a local maximum at (a, b) if

$$f(a, b) \geq f(x, y)$$

for all points (x, y) near (a, b).
Local Minimum: A function $f(x, y)$ has a local minimum at (a, b) if

$$f(a, b) \leq f(x, y)$$

for all points (x, y) near (a, b).
Global (Absolute) Maximum/Minimum: A function has a global maximum (or minimum) at (a, b) if the inequality holds for *all* points in the domain.

Critical Points: A point (a, b) is a critical point if:

$$f_x(a, b) = 0 \quad \text{and} \quad f_y(a, b) = 0$$

or if one of the partial derivatives does not exist.

4.2 Second Derivative Test

Let (a, b) be a critical point of f. Define:

$$D = f_{xx}(a, b)f_{yy}(a, b) - (f_{xy}(a, b))^2$$

- If $D > 0$ and $f_{xx}(a, b) > 0$, then f has a local minimum at (a, b).
- If $D > 0$ and $f_{xx}(a, b) < 0$, then f has a local maximum at (a, b).
- If $D < 0$, then (a, b) is a saddle point.
- If $D = 0$, the test is inconclusive.

4.3 Global Extrema on Closed and Bounded Regions

If f is continuous on a closed and bounded region, then it attains both a global maximum and a global minimum.
Steps to Find Global Extrema:

1. Find all critical points inside the region.
2. Evaluate f at those critical points.
3. Find extrema on the boundary (often by parameterizing or reducing to one variable).
4. Compare all values.

4.4 Lagrange Multipliers

Used to find extrema of a function $f(x, y)$ subject to a constraint $g(x, y) = c$.
At an extreme value under a constraint,

$$\nabla f = \lambda \nabla g$$

This gives the system:

$$f_x = \lambda g_x$$

$$f_y = \lambda g_y$$

$$g(x, y) = c$$

Steps for Lagrange Multipliers:

1. Compute ∇f and ∇g.
2. Set $\nabla f = \lambda \nabla g$.
3. Solve the resulting system.
4. Evaluate f at the candidate points.

Problems

Problem 1

Find and classify the critical points for the function $h(x, y) = x^4 + y^3 - 6y - 2x^2$

Problem 2

Find the maximum and minimum values of the function $f(x, y) = x+y$ on the curve $x^2+y^2-xy = 4$

Problem 3

Consider a hypothetical planet whose surface can be approximated by the sphere $x^2 + y^2 + z^2 = 1$ Suppose the planet's temperature is strange and given by $T(x, y, z) = 2x + 2y + z$. Find the maximum and minimum temperatures on the planet's surface.

Problem 4

You are walking on the graph $f(x, y) = 2y\cos(\pi x) - 2y\cos(\pi y) + 5$ and you are standing at the point $(1, 1, 4)$.
i) Find a direction you should walk in order to stay at a height of 4. Report your answer as a unit vector.
ii) Find a direction you should walk in order to decrease your height the fastest. Report your answer as a unit vector.