# Lecture 4 and 5

# Controllability and Observability: Kalman decompositions

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## I. Observability of DT LTI systems

Consider the DT LTI dynamics again with the observation model. We are interested in deriving the necessary conditions for optimal estimation and thus the additional noise terms can be removed. The system to be considered is the following:

$$\mathbf{x}_{k+1} = A\mathbf{x}_k + B\mathbf{u}_k, \quad (1)$$

$$y_k = C\mathbf{x}_k, \quad (2)$$

as before we consider the worst case of one observation per $k$, i.e., $p = 1$.

**Definition 1** (Observability)**.** *A DT LTI system is said to be observable in $n$ time-steps when the initial condition, $\mathbf{x}_0$, can be recovered from a sequence of observations, $y_0, \ldots, y_{n-1}$, and inputs, $\mathbf{u}_0, \ldots, \mathbf{u}_{n-1}$.*

The observability problem is to find $\mathbf{x}_0$ from a sequence of observations, $y_0, \ldots, y_{k-1}$, and

The lecture material is largely based on: *Fundamentals of Linear State Space Systems*, John S. Bay, McGraw-Hill Series in Electrical Engineering, 1998.

inputs, $\mathbf{u}_0, \ldots, \mathbf{u}_{k-1}$. To this end, note that

$$\begin{aligned}
y_0 &= C\mathbf{x}_0, \\
y_1 &= C\mathbf{x}_1, \\
&= CA\mathbf{x}_0 + CB\mathbf{u}_0, \\
y_2 &= C\mathbf{x}_2, \\
&= CA^2\mathbf{x}_0 + CAB\mathbf{u}_0 + CB\mathbf{u}_1, \\
&\vdots \\
y_{k-1} &= CA^{k-1}\mathbf{x}_0 + CA^{k-2}B\mathbf{u}_0 + \ldots + CB\mathbf{u}_{k-2}.
\end{aligned}$$

In matrix form, the above is given by

$$\begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ \vdots \\ y_{k-1} \end{bmatrix} = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{k-1} \end{bmatrix} \mathbf{x}_0 + \Upsilon \begin{bmatrix} \mathbf{u}_0 \\ \mathbf{u}_1 \\ \mathbf{u}_2 \\ \vdots \\ \mathbf{u}_{k-1} \end{bmatrix}.$$

The above is a linear system of equations compactly written as

$$\Psi_{0:k-1} = \underbrace{\mathcal{O}_{0:k-1}}_{\in\mathbb{R}^{k\times n}} \mathbf{x}_0, \tag{3}$$

and hence, from the *known* inputs and *known* observations, the initial condition, $\mathbf{x}_0$, can be recovered if and only if

$$\text{rank}(\mathcal{O}_{0:k-1}) = n. \tag{4}$$

Using the same arguments from controllability, we note that rank$(\mathcal{O}) < n$ for $k \leq n-2$ as the observability matrix, $\mathcal{O}$, has at most $n-1$ rows. Similarly, adding another observation after the $n$th observation does not help by Cayley-Hamilton arguments as

$$\text{rank}(\mathcal{O}_{0:n-1}) = \text{rank}(\mathcal{O}_{0:n}) = \text{rank}(\mathcal{O}_{0:n+1}) = \ldots. \tag{5}$$

Hence, the observability matrix is defined as

$$\mathcal{O} = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}, \tag{6}$$

without subscripts.

**Remark 1.** *In literature, rank*$(\mathcal{C}) = n$ *is often referred to as the system being* $(A, B)$*–controllable. Similarly, rank*$(\mathcal{O}) = n$ *is often referred to as the system being* $(A, C)$*–observable.*

**Remark 2.** *Show that*

$$\textit{rank}(\mathcal{C}) = n \quad \Leftrightarrow \quad \mathcal{C}\mathcal{C}^T \succ 0 \Leftrightarrow (\mathcal{C}\mathcal{C}^T)^{-1} \textit{ exists}, \tag{7}$$

$$\textit{rank}(\mathcal{O}) = n \quad \Leftrightarrow \quad \mathcal{O}^T\mathcal{O} \succ 0 \Leftrightarrow (\mathcal{O}^T\mathcal{O})^{-1} \textit{ exists}. \tag{8}$$

**Remark 3.** *Show that*

$$(A, C)\textit{–observable} \quad \Leftrightarrow \quad (A^T, C^T)\textit{–controllable}, \tag{9}$$

$$(A, B)\textit{–controllable} \quad \Leftrightarrow \quad (A^T, B^T)\textit{–observable}. \tag{10}$$

## II. Kalman decompositions

Suppose it is known that a SISO (single-input single-output) system has $\text{rank}(\mathcal{C}) = n_c < n$, and $\text{rank}(\mathcal{O}) = n_o < n$. In other words, the SISO system, in question, is neither controllable nor observable. We are interested in a transformation that rearranges the system modes (eigenvalues) into:

- modes that are both controllable and observable;
- modes that are neither controllable nor observable;
- modes that are controllable but not observable; and
- modes that are observable but not controllable.

Such a transformation is called *Kalman decomposition*.

### A. Decomposing controllable modes

Firstly, consider separating only the controllable subspace from the rest of the state-space. To this end, consider a state transformation matrix, $V$, whose first $n_c$ *columns* are a maximal set of linearly independent columns from the controllability matrix, $\mathcal{C}$. The remaining $n - n_c$ columns are chosen to be linearly independent from the first $n_c$ columns (and linearly independent among themselves). The transformation operator (matrix) is $n \times n$, defined as

$$V = \left[ \begin{array}{ccc|ccc} \mathbf{v}_1 & \ldots & \mathbf{v}_{n_c} & \mathbf{v}_{n_c+1} & \ldots & \mathbf{v}_n \end{array} \right] \triangleq \left[ \begin{array}{cc} V_1 & V_2 \end{array} \right]. \tag{11}$$

It is straightforward to show that the transformed system ($\mathbf{x}_k = V\overline{\mathbf{x}}_k$) is

$$\overline{\mathbf{x}}_{k+1} = V^{-1}AV\overline{\mathbf{x}}_k + V^{-1}Bu, \tag{12}$$

$$y_k = CV\overline{\mathbf{x}}_k. \tag{13}$$

Partition $V^{-1}$ as

$$V^{-1} = \left[ \begin{array}{c} \overline{V}_1 \\ \overline{V}_2 \end{array} \right], \tag{14}$$

where $\overline{V}_1$ is $n_c \times n$ and $\overline{V}_2$ is $(n - n_c) \times n$. Now note that

$$I_n = V^{-1}V = \left[ \begin{array}{c} \overline{V}_1 \\ \overline{V}_2 \end{array} \right] \left[ \begin{array}{cc} V_1 & V_2 \end{array} \right] = \left[ \begin{array}{cc} \overline{V}_1 V_1 & \overline{V}_1 V_2 \\ \overline{V}_2 V_1 & \overline{V}_2 V_2 \end{array} \right] = \left[ \begin{array}{cc} I_{n_c} & \\ & I_{n-n_c} \end{array} \right]. \tag{15}$$

The above leads to the following[1] :

$$\overline{V}_2 V_1 = \mathbf{0} \Rightarrow \mathcal{C} \text{ lies in the null space of } \overline{V}_2, \tag{16}$$

further leading to

$$\overline{V}_2 B = \mathbf{0}, \text{ and } \overline{V}_2 A V_1 = \mathbf{0}. \tag{17}$$

The transformed system in Eq. (12) is thus given by

$$\overline{A} \triangleq V^{-1}AV = \begin{bmatrix} \overline{V}_1 \\ \overline{V}_2 \end{bmatrix} A \begin{bmatrix} V_1 & V_2 \end{bmatrix} = \begin{bmatrix} \overline{V}_1 A V_1 & \overline{V}_1 A V_2 \\ \overline{V}_2 A V_1 & \overline{V}_2 A V_2 \end{bmatrix} \triangleq \begin{bmatrix} A_c & A_{c\overline{c}} \\ \mathbf{0} & A_{\overline{c}} \end{bmatrix}, \tag{18}$$

$$\overline{B} \triangleq V^{-1}B = \begin{bmatrix} \overline{V}_1 \\ \overline{V}_2 \end{bmatrix} B = \begin{bmatrix} \overline{V}_1 B \\ \overline{V}_2 B \end{bmatrix} \triangleq \begin{bmatrix} B_c \\ \mathbf{0} \end{bmatrix}. \tag{19}$$

Now recall that a similarity transformation does not change the rank of the controllability matrix, $\overline{\mathcal{C}}$, of the transformed system. Mathematically,

$$\begin{aligned} \text{rank}(\overline{\mathcal{C}}) &= \text{rank}\left(\begin{bmatrix} V^{-1}B & (V^{-1}AV)V^{-1}B & \ldots & (V^{-1}AV)^{n-1}V^{-1}B \end{bmatrix}\right), \\ &= \text{rank}\left(\begin{bmatrix} V^{-1}B & V^{-1}AB & \ldots & V^{-1}A^{n-1}B \end{bmatrix}\right), \\ &= \text{rank}\left(V^{-1}\begin{bmatrix} B & AB & \ldots & A^{n-1}B \end{bmatrix}\right), \\ &= \text{rank}\left(\begin{bmatrix} B & AB & \ldots & A^{n-1}B \end{bmatrix}\right), \\ &= \text{rank}\left(\mathcal{C}\right), \\ &= n_c. \end{aligned}$$

Furthermore,

$$\begin{aligned} \text{rank}(\overline{\mathcal{C}}) &= \text{rank}\left(\left[\begin{array}{cccc|c} \overline{B} & \overline{AB} & \ldots \overline{A}^{n_c-1}\overline{B} & & \ldots \overline{A}^{n-1}\overline{B} \end{array}\right]\right), \\ &= \text{rank}\left(\left[\begin{array}{cccc|c} B_c & A_c B_c & \ldots & A_c^{n_c-1}B_c & \ldots \; A_c^{n-1}B_c \\ \mathbf{0} & \mathbf{0} & \ldots & \mathbf{0} & \mathbf{0} \end{array}\right]\right), \\ &= \text{rank}\left(\begin{bmatrix} B_c & A_c B_c & \ldots & A_c^{n_c-1}B_c \\ \mathbf{0} & \mathbf{0} & \ldots & \mathbf{0} \end{bmatrix}\right), \\ &= n_c, \end{aligned}$$

[1]This is because all of the freedom in the linear independence of $\mathcal{C}$ is already captured in $V_1$.

where to go from second equation to the third, we use Cayley-Hamilton theorem. All of this to show that the subsystem $(A_c, B_c)$ is controllable.

**Conclusions:** Using the transformation matrix, $V$, defined in Eq. (11), we have transformed the original system, $\mathbf{x}_{k+1} = A\mathbf{x}_k + Bu_k$ into the following system:

$$\begin{aligned} \overline{\mathbf{x}}_{k+1} &= \overline{A}\overline{\mathbf{x}}_k + \overline{B}u_k, &(20)\\ \triangleq \begin{bmatrix} \overline{\mathbf{x}}_c(k+1) \\ \overline{\mathbf{x}}_{\overline{c}}(k+1) \end{bmatrix} &= \begin{bmatrix} A_c & A_{c\overline{c}} \\ \mathbf{0} & A_{\overline{c}} \end{bmatrix} \begin{bmatrix} \overline{\mathbf{x}}_c(k) \\ \overline{\mathbf{x}}_{\overline{c}}(k) \end{bmatrix} + \begin{bmatrix} B_c \\ \mathbf{0} \end{bmatrix} u_k, &(21) \end{aligned}$$

such that the transformed state-space is separated into a controllable subspace and an *un*controllable subspace.

Consider the transformed system in Eq. (20) and partition the (transformed) state-vector, $\overline{\mathbf{x}}(k)$, into an $n_c \times 1$ component, $\overline{\mathbf{x}}_c(k)$, and an $n - n_c \times 1$, $\overline{\mathbf{x}}_{\overline{c}}(k)$. We get

$$\begin{aligned} \overline{\mathbf{x}}_c(k+1) &= A_c\overline{\mathbf{x}}_c(k) + A_{c\overline{c}}\overline{\mathbf{x}}_{\overline{c}}(k) + B_c u_k, &(22)\\ \overline{\mathbf{x}}_{\overline{c}}(k+1) &= A_{\overline{c}}\overline{\mathbf{x}}_{\overline{c}}(k). &(23) \end{aligned}$$

Clearly, the lower subsystem, $\overline{\mathbf{x}}_{\overline{c}}(k)$, is not controllable. In the transformed coordinate space, $\overline{\mathbf{x}}_k$, we can only steer a portion, $\overline{\mathbf{x}}_c(k)$, of the transformed states. The subspace of $\mathbb{R}^n$ that can be steered is denoted as $\mathcal{R}(\mathcal{C})$, defined as the row space of $\mathcal{C}$, where $\mathcal{R}(\mathcal{C})$ is called the *controllable subspace*.

Finally, recall that the eigenvalues of similar matrices, $A$ and $V^{-1}AV$, are the same. Recall the structure of $V^{-1}AV$ from Eq. (18); since $V^{-1}AV$ is block upper-triangular, the eigenvalues of $A$ are given by the eigenvalues of $A_c$ and $A_{\overline{c}}$, called the *controllable eigenvalues* (modes) and *uncontrollable eigenvalues* (modes), respectively.

**Example 1.** *Consider the CT LTI system:*

$$\begin{aligned} \dot{\mathbf{x}} &= \begin{bmatrix} 2 & 1 & 1 \\ 5 & 3 & 6 \\ -5 & -1 & -4 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u, \\ \mathcal{C} &= \begin{bmatrix} 1 & 2 & -4 \\ 0 & 5 & -5 \\ 0 & 5 & -5 \end{bmatrix}, \end{aligned}$$

*with rank*$(\mathcal{C}) = 2 < 3$. *Defines* $V$ *as*

$$V = \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 5 & 0 \\ 0 & 5 & 1 \end{array}\right],$$

*and note that the requirement for choosing* $V$ *are satisfied. Finally, verify that the transformed system matrices are*

$$V^{-1}AV = \left[\begin{array}{cc|c} 0 & 6 & 1 \\ 1 & -1 & 0 \\ \hline 0 & 0 & 0 \end{array}\right], \qquad V^{-1}B = \left[\begin{array}{c} 1 \\ 0 \\ \hline 0 \end{array}\right].$$

*Verify that the top-left subsystem is controllable. From this structure, we see that the two-dimensional controllable system is grouped into the first two state variables and that the third state variable has neither an input signal nor a coupling from the other subsystem. It is therefore not controllable.*

### *B. Decomposing observable modes*

Now we can easily follow the same drill and come up with another transformation, $\mathbf{x}_k = W\underline{\mathbf{x}_k}$, where $W$ contains $n_o$ linearly independent *rows* of $\mathcal{O}$ and the remaining rows are chosen to be linearly independent of the first $n_o$ rows (and linearly independent among themselves). Finally, the transformed system,

$$\begin{aligned} \underline{\mathbf{x}}_{k+1} &= \underline{A}\underline{\mathbf{x}}_k + \underline{B}u_k, \\ &= \begin{bmatrix} A_o & \mathbf{0} \\ A_{\overline{o}o} & A_{\overline{o}} \end{bmatrix} \underline{\mathbf{x}}_k + \begin{bmatrix} B_o \\ B_{\overline{o}} \end{bmatrix} u_k, \qquad (24) \\ y_k &= \begin{bmatrix} C_o & \mathbf{0} \end{bmatrix} \underline{\mathbf{x}_k} \qquad (25) \end{aligned}$$

is separated into an observable subspace and an *un*observable subspace. In fact, we can now write this result as a theorem.

**Theorem 1.** *Consider the DT LTI system,*

$$\mathbf{x}_{k+1} = A\mathbf{x}_k,$$

$$y_k = C\mathbf{x}_k,$$

*where* $A \in \mathbb{R}^{n\times n}$ *and* $C \in \mathbb{R}^{p\times n}$. *Let* $\mathcal{O}$ *be the observability matrix as defined in Eq.* (6). *If rank*$(\mathcal{O}) = n_o < n$, *there there exists a non-singular matrix,* $W \in \mathbb{R}^{n\times n}$ *such that*

$$W^{-1}AW \;=\; \begin{bmatrix} A_o & \mathbf{0} \\ A_{\bar{o}o} & A_{\bar{o}} \end{bmatrix}, \qquad CW = \begin{bmatrix} C_o & \mathbf{0} \end{bmatrix},$$

*where* $A_o \in \mathbb{R}^{n_o\times n_o}$, $C_o \in \mathbb{R}^{n_o\times n}$, *and the pair* $(A_o, C_o)$ *is observable.*

By carefully applying a sequence of such controllability and observability transformations, the complete Kalman decomposition results in a system of the following form:

$$\begin{bmatrix} \mathbf{x}_{co}(k+1) \\ \mathbf{x}_{c\bar{o}}(k+1) \\ \mathbf{x}_{\bar{c}o}(k+1) \\ \mathbf{x}_{\overline{co}}(k+1) \end{bmatrix} = \begin{bmatrix} A_{co} & \mathbf{0} & A_{13} & \mathbf{0} \\ A_{21} & A_{c\bar{o}} & A_{23} & A_{24} \\ \mathbf{0} & \mathbf{0} & A_{\bar{c}o} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & A_{43} & A_{\overline{co}} \end{bmatrix} \begin{bmatrix} \mathbf{x}_{co}(k) \\ \mathbf{x}_{c\bar{o}}(k) \\ \mathbf{x}_{\bar{c}o}(k) \\ \mathbf{x}_{\overline{co}}(k) \end{bmatrix} + \begin{bmatrix} B_{co} \\ B_{c\bar{o}} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} u(k), \qquad (26)$$

$$y(k) = \begin{bmatrix} C_{co} & \mathbf{0} & C_{\bar{c}o} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{x}_{co}(k) \\ \mathbf{x}_{c\bar{o}}(k) \\ \mathbf{x}_{\bar{c}o}(k) \\ \mathbf{x}_{\overline{co}}(k) \end{bmatrix}. \qquad (27)$$

### *C. Kalman decomposition theorem*

**Theorem 2.** *Given a DT (or CT) LTI system that is neither controllable nor observable, there exists a non-singular transformation,* $\mathbf{x} = T\widetilde{\mathbf{x}}$*, such that the system dynamics can be transformed into the structure given by Eqs.* (26)–(27).

*Proof: The proof is left as an exercise and constitutes Homework 1.* ■

## APPENDIX A
## SIMILARITY TRANSFORMS

Consider any invertible matrix, $V \in \mathbb{R}^{n\times n}$, and a state transformation, $\mathbf{x}_k = V\mathbf{z}_k$. The transformed DT-LTI system is given by

$$\begin{aligned}
\mathbf{x}_{k+1} &= A\mathbf{x}_k + Bu_k,\\
\Rightarrow V\mathbf{z}_{k+1} &= AV\mathbf{z}_k + Bu_k,\\
\Rightarrow \mathbf{z}_{k+1} &= \underbrace{V^{-1}AV}_{\triangleq\overline{A}}\mathbf{z}_k + \underbrace{V^{-1}B}_{\triangleq\overline{B}}u_k,\\
y_k &= \underbrace{CV}_{\triangleq\overline{C}}\mathbf{z}_k.
\end{aligned}$$

The new (transformed) system is now defined by the system matrices, $\overline{A}$, $\overline{B}$, and $\overline{C}$. The two systems, $\mathbf{x}_k$ and $\mathbf{z}_k$, are called *similar systems.*

**Remark 4.** *Each of the following can be proved.*

- *Given any DT-LTI systems, there are infinite possible transformations.*
- *Similar systems have the same eigenvalues; the eigenvectors of the corresponding system matrices are different.*
- *Similar systems have the same transfer functions; in other words, the transfer function of the DT-LTI dynamics is unique with infinite possible state-space* realizations.
- *The rank of the controllability and observability are invariant to a similar transformation.*

## APPENDIX B
## TRIVIA: NULL SPACES

Consider a set of $m$ vectors, $\mathbf{x}_i \in \mathbb{R}^n, i = 1, \ldots, m$.

**Definition 2** (Span)**.** *The span of $m$ vectors, $\mathbf{x}_i \in \mathbb{R}^n, i = 1, \ldots, m$ is defined as the set of all vectors that can be written as a linear combination of $\{\mathbf{x}_i\}'s$, i.e.,*

$$\textit{span}(\mathbf{x}_1, \ldots, \mathbf{x}_m) = \left\{\mathbf{y} \;\middle|\; \mathbf{y} = \sum_{i=1}^{m} \alpha_i \mathbf{x}_i\right\}, \tag{28}$$

*for* $\alpha_i \in \mathbb{R}$.

**Definition 3.** *The null space, $\mathcal{N}(A) \subseteq \mathbb{R}^n$, of an $n \times n$ matrix, $A$, is defined as the set of all vectors such that*

$$A\mathbf{y} = \mathbf{0}, \tag{29}$$

*for* $\mathbf{y} \in \mathcal{N}(A)$.

Consider a real-valued matrix, $A \in \mathbb{R}^{n \times n}$ and let $\mathbf{x}_1 \neq \mathbf{0}$ and $\mathbf{x}_2 \neq \mathbf{0}$ be the *maximal set of linearly independent vectors*[2] such that

$$A\mathbf{x}_1 = \mathbf{0}, \qquad A\mathbf{x}_2 = \mathbf{0}, \tag{30}$$

which leads to

$$A\sum_{i=1}^{2} \alpha_i \mathbf{x}_i \;=\; \sum_{i=1}^{2} \alpha_i A\mathbf{x}_i = \mathbf{0}. \tag{31}$$

Following the definition of *span* and *null space*, we can say that span$(\mathbf{x}_1, \mathbf{x}_2)$ lies in the null space of $A$, i.e.,

$$\mathcal{N}(A) = \text{span}(\mathbf{x}_1, \mathbf{x}_2). \tag{32}$$

In this particular case, the dimension of $\mathcal{N}(A)$ is 2 and[3] rank$(A) = n - 2$.

[2]In other words, there does not exist any other vector, $\mathbf{x}_3$, that is linearly independent of $\mathbf{x}_1$ and $\mathbf{x}_2$ such that $A\mathbf{x}_3 = \mathbf{0}$.

[3]This further leads to an alternate definition of rank as $n - n_c$, where $n_c \triangleq \dim(\mathcal{N}(A))$.

Lets recall the *un*controllable case with $\text{rank}(\mathcal{C}) = n_c$. Clearly, $\mathcal{C}$ has $n_c$ linearly independent (l.i.) columns (vectors in $\mathbb{R}^n$); let the matrix $V_1 \in \mathbb{R}^{n \times n_c}$ consists of these $n_c$ l.i. columns. On the other hand, let the matrix $\widetilde{V}_1 \in \mathbb{R}^{n \times n - n_c}$ consists of the remaining columns of $\mathcal{C}$ and note that

$$\begin{aligned} & \text{every column of } \widetilde{V}_1 \in \text{span}(\text{columns of } V_1), \\ \text{or,} \quad & \text{span}(\text{columns of } V_1) \text{ contains every column of } \widetilde{V}_1. \end{aligned}$$

The above leads to the following:

**Remark 5.** *If there exists a matrix, $\overline{V}_2$, such that $\overline{V}_2 V_1 = \mathbf{0}$, then*

$$\begin{aligned} & \textit{every column of } V_1 \in \mathcal{N}(\overline{V}_2), \\ \Rightarrow \quad & \textit{span}(\textit{columns of } V_1) \subseteq \mathcal{N}(\overline{V}_2), \\ \Rightarrow \quad & \textit{every column of } \widetilde{V}_1 \in \mathcal{N}(\overline{V}_2). \end{aligned}$$

### A. Kalman decomposition, revisited

Construct a transformation matrix, $V \in \mathbb{R}^{n\times n}$, whose whose first $n \times n_c$ block is $V_1$. Choose the remaining $n - n_c$ columns to be linearly independent from the first $n_c$ columns (and linearly independent among themselves); call the second $n \times (n - n_c)$ block as $V_2$:

$$V \triangleq \begin{bmatrix} V_1 & V_2 \end{bmatrix}, \qquad V^{-1} \triangleq \begin{bmatrix} \overline{V}_1 \\ \overline{V}_2 \end{bmatrix},$$

$$I_n \;=\; V^{-1}V = \begin{bmatrix} \overline{V}_1 \\ \overline{V}_2 \end{bmatrix} \begin{bmatrix} V_1 & V_2 \end{bmatrix} = \begin{bmatrix} \overline{V}_1 V_1 & \overline{V}_1 V_2 \\ \overline{V}_2 V_1 & \overline{V}_2 V_2 \end{bmatrix} = \begin{bmatrix} I_{n_c} & \mathbf{0} \\ \mathbf{0} & I_{n-n_c} \end{bmatrix}.$$

From the above, note that $\overline{V}_2 V_1 = \mathbf{0}$. Now consider the transformed system matrices, $\overline{A}$ and $\overline{B}$:

$$\begin{aligned} \overline{A} \triangleq V^{-1}AV &= \begin{bmatrix} \overline{V}_1 \\ \overline{V}_2 \end{bmatrix} A \begin{bmatrix} V_1 & V_2 \end{bmatrix} = \begin{bmatrix} \overline{V}_1 A V_1 & \overline{V}_1 A V_2 \\ \overline{V}_2 A V_1 & \overline{V}_2 A V_2 \end{bmatrix}, \\ \overline{B} \triangleq V^{-1}B &= \begin{bmatrix} \overline{V}_1 \\ \overline{V}_2 \end{bmatrix} B = \begin{bmatrix} \overline{V}_1 B \\ \overline{V}_2 B \end{bmatrix}. \end{aligned}$$

Note that

$$AV_1 \in \text{span(columns of } V_1) \subseteq \mathcal{N}(\overline{V_2}),$$

$$B \in \mathcal{N}(\overline{V}_2).$$