PROBABILITY THEORY

Math Tools Lab #7
November 1st, 2019

OUTLINE

(1) Summary statistics: review intuition for norms
(2) Code break 1
(3) Random variables and probability distributions: discrete and continuous
(4) Expected value, variance, and standard deviation
(5) Moments
(6) Code break 2
(7) Covariance and correlation
(8) **Code break 3**

SUMMARY STATISTICS

We've already looked at a few: mean, median, mode, etc.

What is the point of these? We want **a single number** that can represent the data well, which we can measure by looking at **the discrepancies from all data points using a metric**.

What metric should we use? Norms: $L_0, L_1, L_2, \ldots, L_p$

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Generalized L_p norm	$argmin_c[\frac{1}{N}\sum_{n=1}^{N} \vert x_n - c \vert^p]^{1/p}$
L_0 norm	mode
L_1 norm	median
L_2 norm	mean

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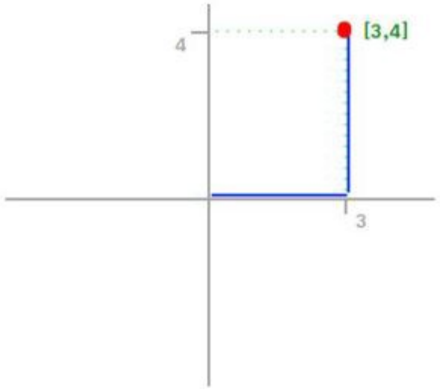

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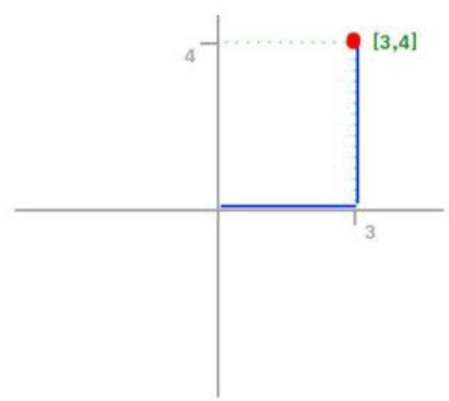

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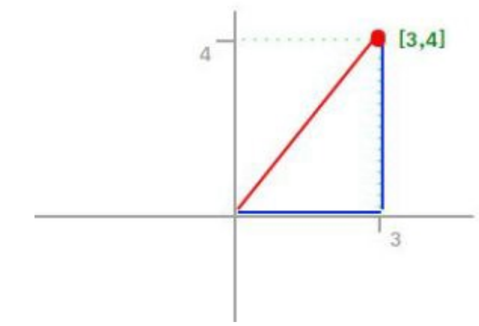
L_2 Norm



Euclidean norm: shortest distance between points

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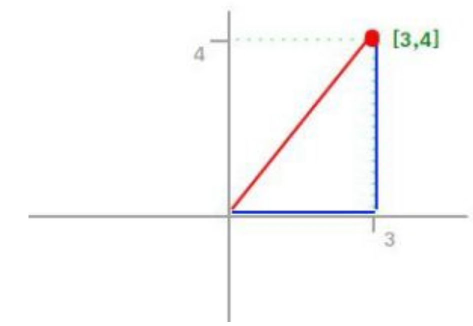

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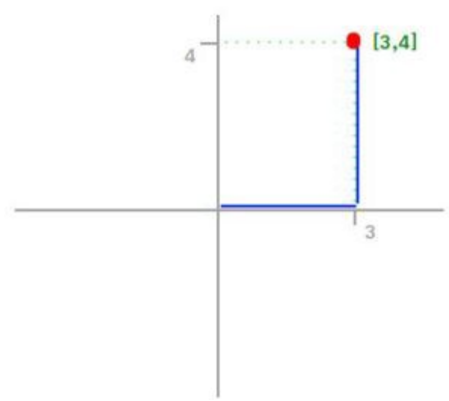

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L_∞ Norm

Largest magnitude element in a vector

$$X = [-6,4,2]$$

$$||X||_\infty = 6$$

CODE BREAK 1

DISCRETE PROBABILITY DISTRIBUTIONS: PMF

A discrete random variable is one that can equal a **finite number of values**. We can describe the distribution of a discrete r.v. using a **probability mass function (pmf)**, which must be non-negative for all inputs and sum to 1. A pmf describes the probability that a discrete r.v. X takes on a particular value, $P(X = x)$.

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Examples of discrete r.v.'s

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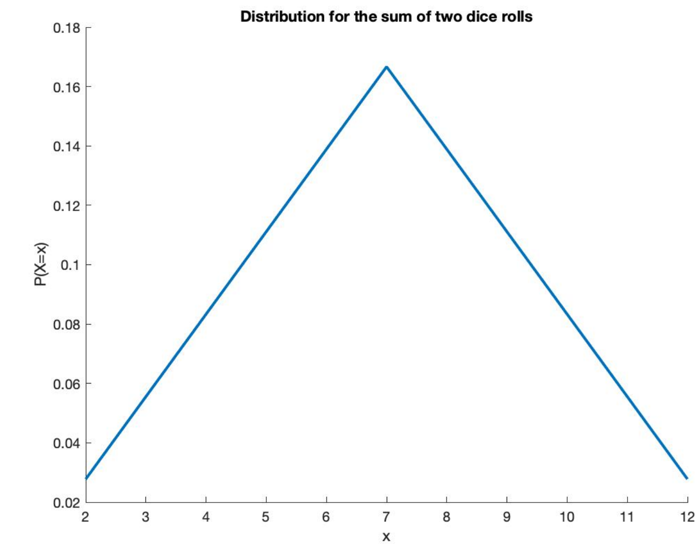

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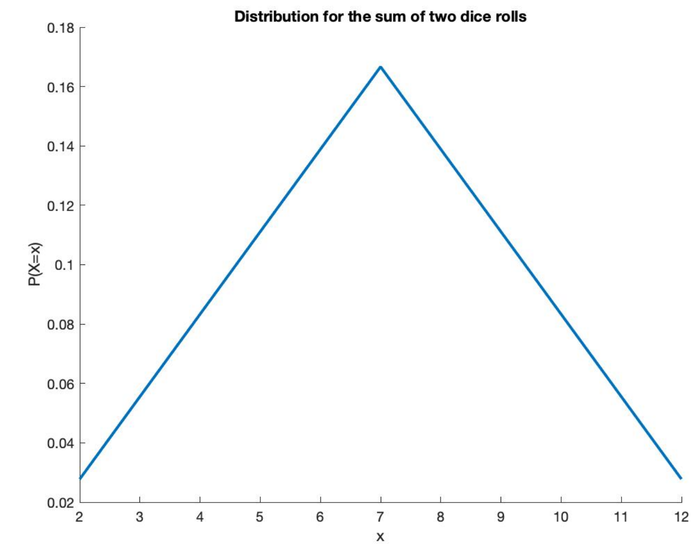

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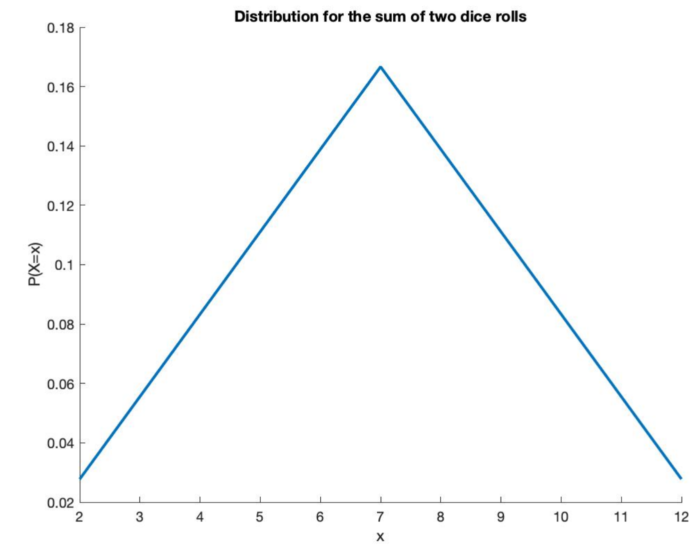

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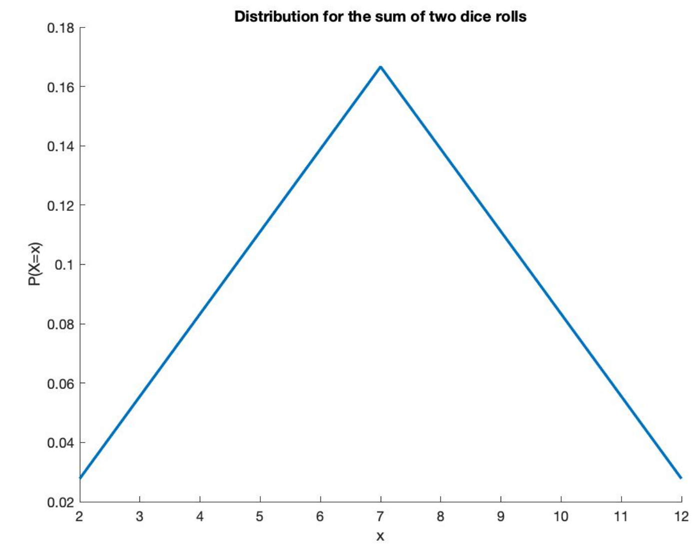

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Another way to describe a r.v.'s distribution is with a **cumulative distribution function (cdf)**, which is a non-decreasing, right continuous step function for discrete r.v.'s. A cdf describes the probability that a discrete r.v. X is less than or equal to a particular value k, $P(X \leq k)$.

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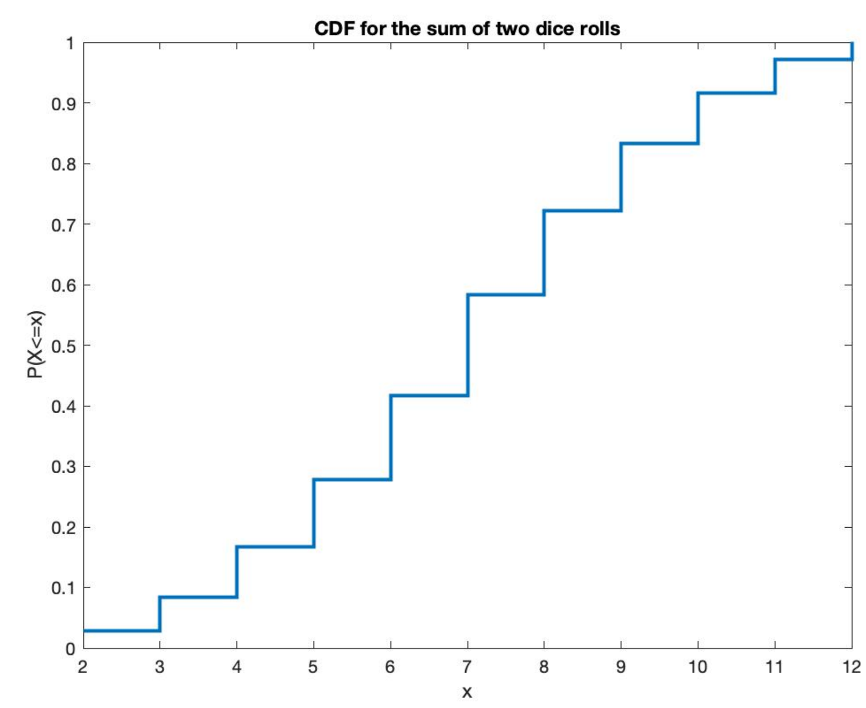

DISCRETE PROBABILITY DISTRIBUTIONS: EXAMPLES

Binomial Distribution

The distribution for the number of successes in a sequence of n independent experiments, each with 2 possible outcomes where p is the probability of success. Example: flipping a coin.

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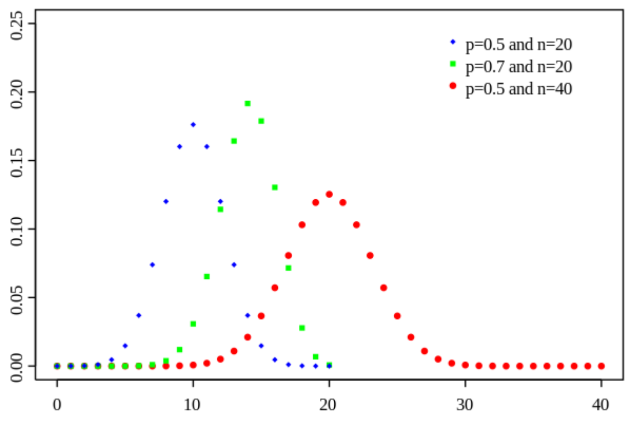

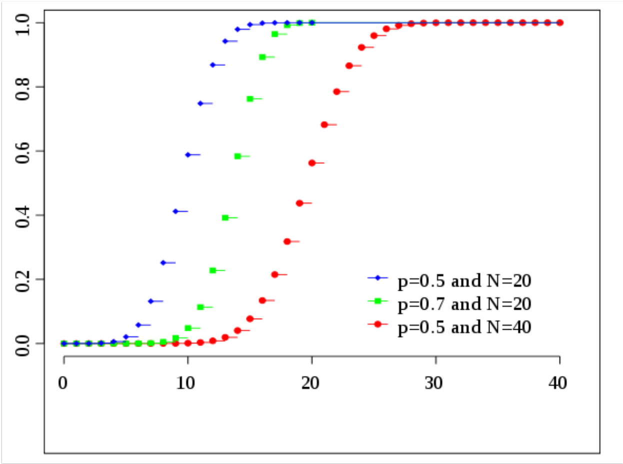

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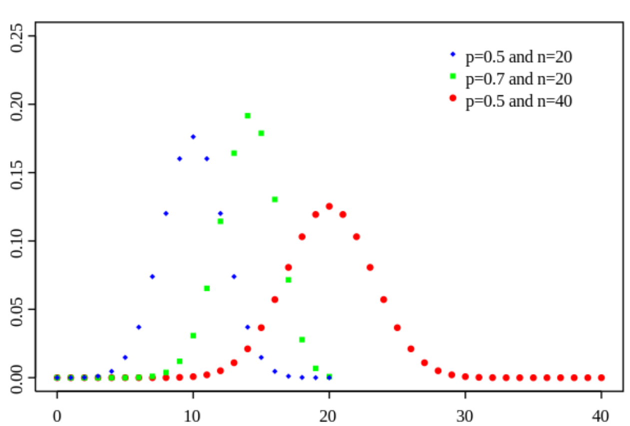

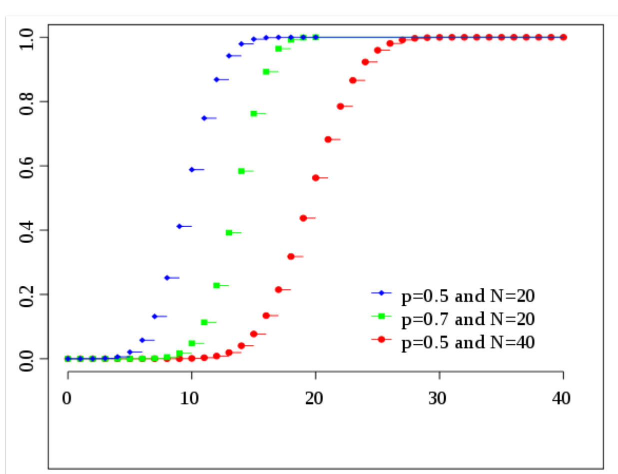

Poisson Distribution

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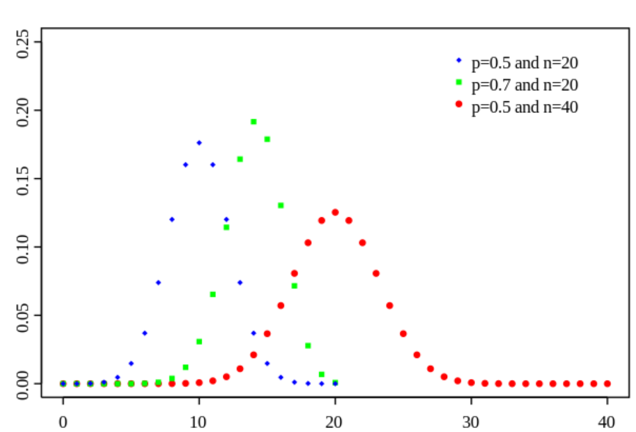

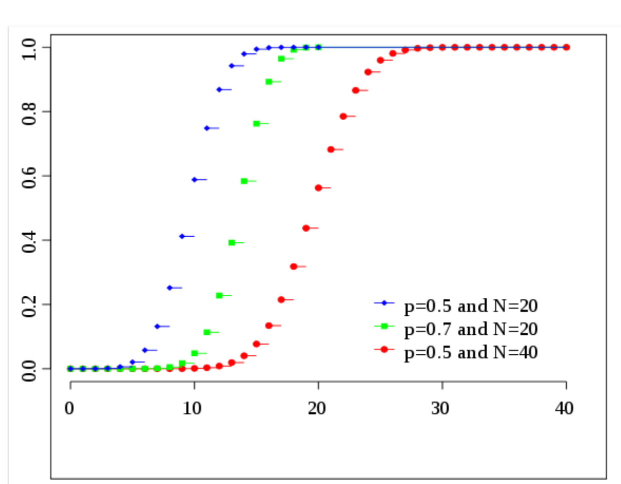

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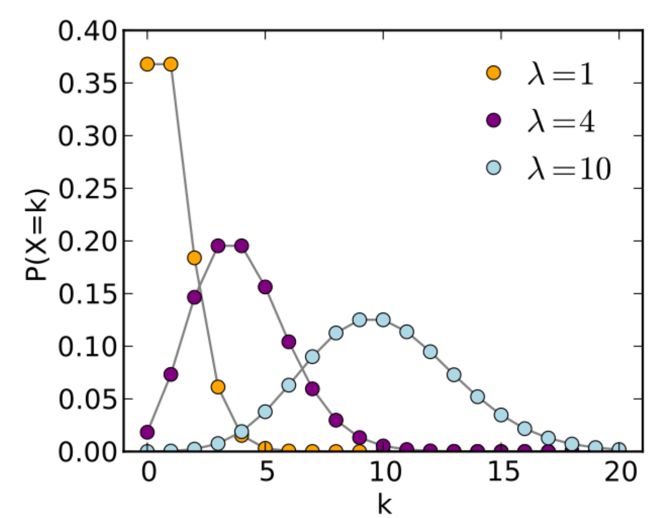

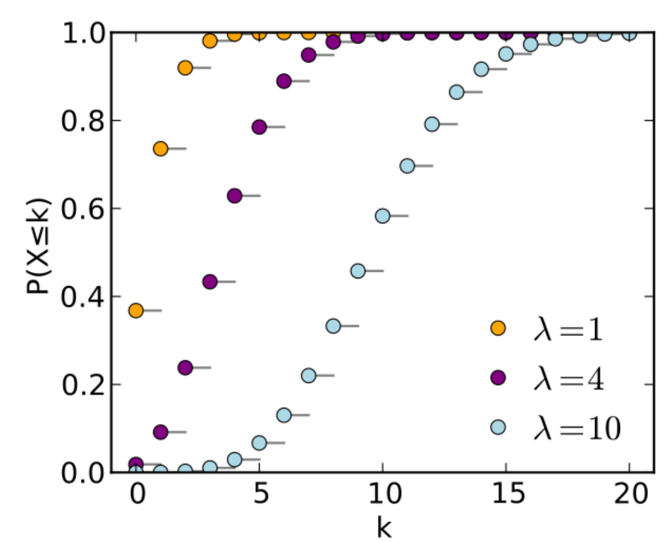

CONTINUOUS PROBABILITY DISTRIBUTIONS: PDF AND CDF

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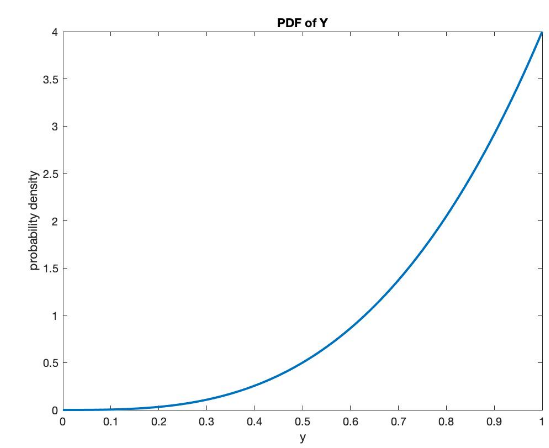
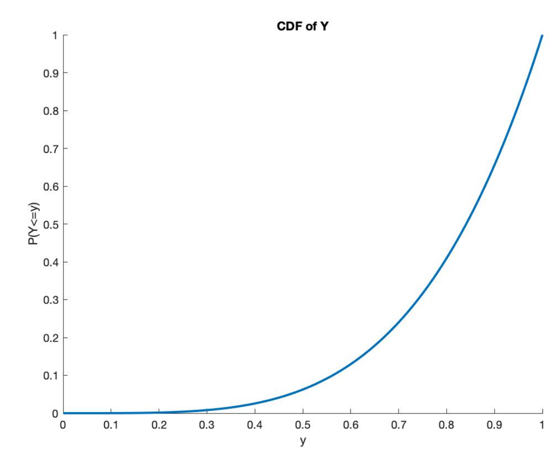
Note: the y axis of a continuous pdf no longer represents probability as in the discrete case, but rather a probability density. Additionally, the probability for a single x value occurring is always 0 and instead we typically look at a range of x values and integrate to find probability.





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Exponential Distribution

The distribution for the time between events that occur continuously and independently at a constant average rate $\lambda = \frac{1}{\beta}$ (this is a specific case of the gamma distribution). Example: time spent waiting at a restaurant before being served.

$$f(x) = \lambda e^{-\lambda x} = \frac{1}{\beta} e^{\frac{-x}{\beta}}$$

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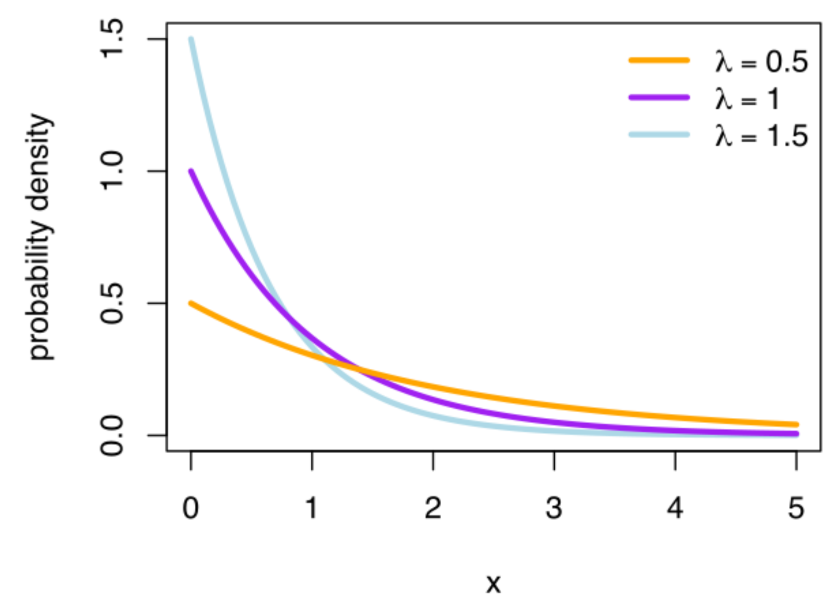

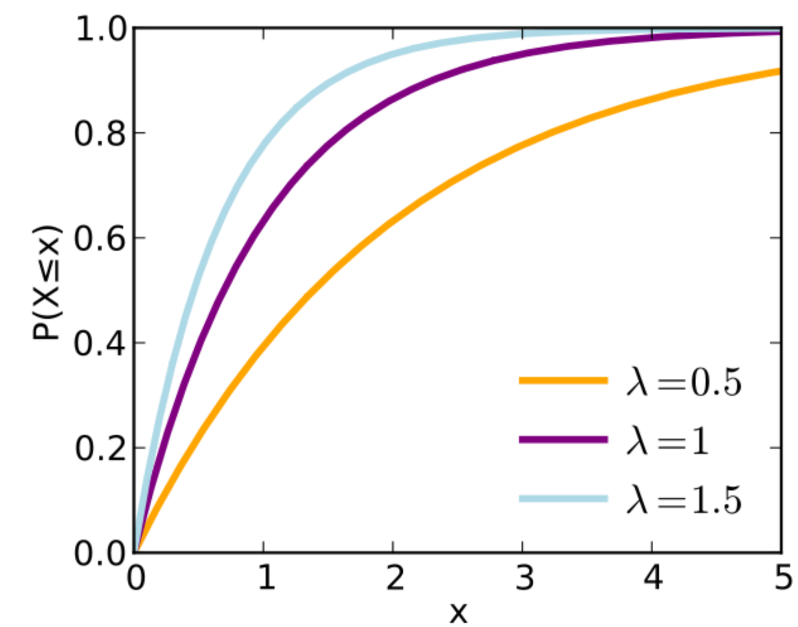

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Amount ($)	1	2	1	2	1	2

$$E(X) = 1(\frac{1}{6}) + 2(\frac{1}{6}) + 1(\frac{1}{6}) + 2(\frac{1}{6}) + 1(\frac{1}{6}) + 2(\frac{1}{6}) = \frac{9}{6} = 1.5$$

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$$E(X) = \sum kP(x = k) = 2(\frac{1}{2}) + 4(\frac{1}{4}) + \ldots = \infty$$

This leads to an ill-advised gambling strategy where you are guaranteed to make \$1 (or an arbitrarily large amount of money)

VARIANCE AND STANDARD DEVIATION

For any r.v., we want to summarize its central tendency but also its spread. The variance of a r.v. X is given by finding the **average squared deviation** of X from its mean, $E(X)$.

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P(X=k)	0.65	0.25	0.06	0.01	0.03

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$$E(X^2) = \sum_{k=0}^{4} k^2 P(X = k) = 1.06$$

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$$E(X) = 0(0.65) + 1(0.25) + 2(0.06) + 3(0.01) + 4(0.03) = 0.52$$

$$E(X^2) = \sum_{k=0}^{4} k^2 P(X = k) = 1.06$$

$$Var(X) = E(X^2) - E(X)^2 = 1.06 - 0.52^2 = 0.7896$$

$$sd(X) = \sqrt{Var(X)} = 0.8886$$

MOMENTS

If X is a r.v., then we can define $E(X^n)$ as the n^{th} **moment of** X. Each moment gives us some insight into the characteristics of the distribution.

In the discrete case, $E(X^n) = \sum_i x_i^n p(x)$ or centered $\sum_i (x_i - \mu)^n p(x_i)$

In the continuous case, $E(X^n) = \int_{-\infty}^{\infty} x^n f(x)dx$ or centered $\int_{-\infty}^{\infty} (x - \mu)^n f(x)dx$

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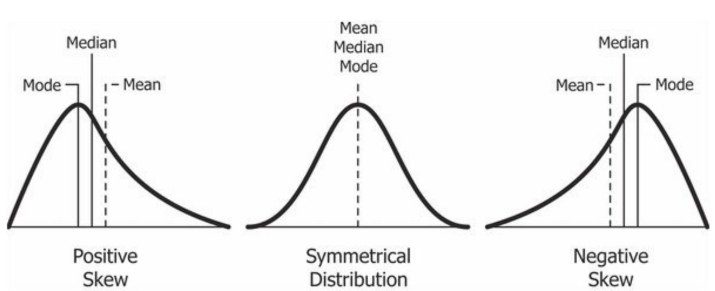
Third moment: skewness

Symmetry of the distribution around the mean



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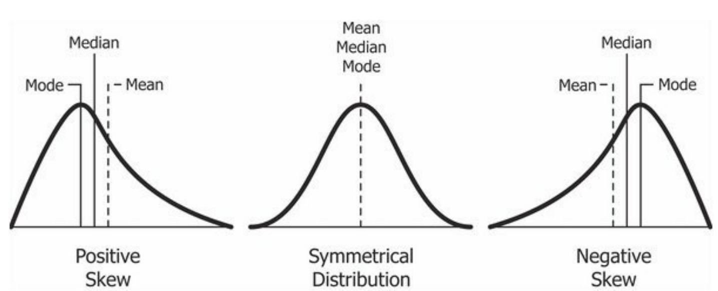
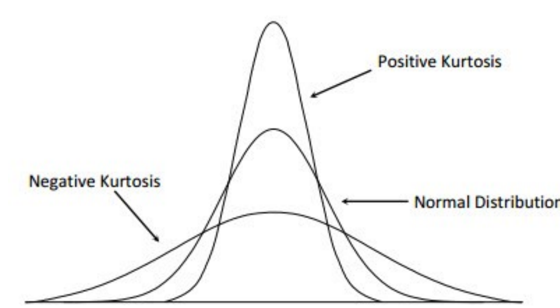


Fourth moment: kurtosis

How heavy the tails of the distribution are



CODE BREAK 2

COVARIANCE AND CORRELATION

Covariance measures the **joint variability of two r.v.'s**. If both X and Y tend to be "big" at the same time, then $Cov(X, Y) > 0$. If one tends to be "big" when the other is "small", then $Cov(X, Y) < 0$.

$$Cov(X, Y) = E[(X - E(X))(Y - E(Y))]$$
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Example: Suppose X and Y are discrete r.v.'s with the joint probability function below. Find $Corr(X, Y)$.

(x,y)	(1,2)	(1,3)	(2,1)	(2,4)
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$$E(Y) = 2(0.5) + 3(0.25) + 1(0.125) + 4(0.125) = 2.375$$
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$$E(X^2) = 1.75, Var(X) = 0.1875$$
$$E(Y^2) = 6.375, Var(Y) = 0.1734$$
$$Cov(X, Y) = E(XY) - E(X)E(Y) = 0.03125$$
$$Corr(X, Y) = \frac{Cov(X, Y)}{\sqrt{Var(X)Var(Y)}} = 0.0842$$

THE COVARIANCE MATRIX

The covariance matrix is a matrix whose element in the i, j position represents **the covariance between the i^{th} and j^{th} element**s of a random vector (a r.v. with multiple dimensions). This generalizes the notion of covariance to multiple dimensions. For example, the variation of a collection of two-dimensional points cannot be fully characterized by a single number or just the variances in the x and y directions: a $2x2$ matrix is needed.

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Assume we have a matrix $X = (X_1, X_2, \ldots, X_n)^T$ where each entry X_i is the entire distribution of a random variable.

We each entry of the covariance matrix, K_{xx}, as:

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So the full matrix for n r.v.'s is equal to the matrix equation $E(XX^T) - \mu_x\mu_x^T$:

$$\mathrm{K}_{\mathbf{XX}} = \begin{bmatrix} \mathrm{E}[(X_1 - \mathrm{E}[X_1])(X_1 - \mathrm{E}[X_1])] & \mathrm{E}[(X_1 - \mathrm{E}[X_1])(X_2 - \mathrm{E}[X_2])] & \cdots & \mathrm{E}[(X_1 - \mathrm{E}[X_1])(X_n - \mathrm{E}[X_n])] \\ \mathrm{E}[(X_2 - \mathrm{E}[X_2])(X_1 - \mathrm{E}[X_1])] & \mathrm{E}[(X_2 - \mathrm{E}[X_2])(X_2 - \mathrm{E}[X_2])] & \cdots & \mathrm{E}[(X_2 - \mathrm{E}[X_2])(X_n - \mathrm{E}[X_n])] \\ \vdots & \vdots & \ddots & \vdots \\ \mathrm{E}[(X_n - \mathrm{E}[X_n])(X_1 - \mathrm{E}[X_1])] & \mathrm{E}[(X_n - \mathrm{E}[X_n])(X_2 - \mathrm{E}[X_2])] & \cdots & \mathrm{E}[(X_n - \mathrm{E}[X_n])(X_n - \mathrm{E}[X_n])] \end{bmatrix}$$

CODE BREAK 3