3a

(*) $\frac{\partial \psi}{\partial t} + c\frac{\partial \psi}{\partial x} = 0 \rightarrow \frac{\partial \psi}{\partial t} = -c\frac{\partial \psi}{\partial x}$ (1). Rewriting: $\frac{1}{\Delta t}(\psi_j^{n+1} - \psi_j) + \frac{c}{\Delta x}(\alpha\psi_{j-2} + \beta\psi_{j-1} + \gamma\psi_j + \delta\psi_{j+1}) = \frac{\partial \psi}{\partial t} + c\frac{\partial \psi}{\partial x} + O(\Delta x^3, \Delta t^2)$ (2)

→ Build a 3rd order-accurate scheme for $\frac{\partial \psi}{\partial x}$. Equation: $\psi'_j + a_0\psi_{j-2} + a_1\psi_{j-1} + a_2\psi_j + a_3\psi_{j+1} = 0$. (3)

- Building a Taylor table for (3):

	ψ_j	ψ'_j	$\psi_j^{(2)}$	$\psi_j^{(3)}$	$\psi^{(4)}_j$
ψ'_j		1	0	0	0
$a_0\psi_{j-2}$	$-2h$	$2h^2$	$-\frac{4}{3}h^3$	$\frac{2}{3}h^4$	
$a_1\psi_{j-1}$	$-h$	$\frac{1}{2}h^2$	$-\frac{1}{6}h^3$	$\frac{1}{24}h^4$	$-\frac{1}{120}h^5$
$a_2\psi_j$	1	0	0	0	0
$a_3\psi_{j+1}$	h	$\frac{1}{2}h^2$	$\frac{1}{6}h^3$	$\frac{1}{24}h^4$	$\frac{1}{120}h^5$

	ψ_j	ψ'_j	ψ''_j	ψ'''_j	$\psi^{(4)}_j$
ψ'_j	0	1	0	0	0
$a_0\psi_{j-2}$	1	$-2h$	$2h^2$	$-\frac{4}{3}h^3$	$\frac{2}{3}h^4$
$a_1\psi_{j-1}$	1	$-h$	$\frac{1}{2}h^2$	$-\frac{1}{6}h^3$	$\frac{1}{24}h^4$
$a_2\psi_j$	1	0	0	0	0
$a_3\psi_{j+1}$	1	h	$\frac{1}{2}h^2$	$\frac{1}{6}h^3$	$\frac{1}{24}h^4$

→ Using a solver:
$a_0 = -\frac{1}{6h}$
$a_1 = \frac{1}{h}$
$a_2 = -\frac{1}{2h}$
$a_3 = -\frac{1}{3h}$

- Rewriting (3) using a_m values: $\psi'_j \approx -\frac{1}{6\Delta x}\psi_{j-2} + \frac{1}{\Delta x}\psi_{j-1} - \frac{1}{2\Delta x}\psi_j - \frac{1}{3\Delta x}\psi_{j+1} = O(\Delta x^3)$ (4)

- (4) → (2): $\frac{1}{\Delta t}\left[\psi_j^{n+1} - \psi_j\right] + \frac{c}{\Delta x}\left[\overset{\alpha}{}\psi_{j-2} + \beta\psi_{j-1} + \overset{\gamma}{}\psi_j + \overset{\delta}{}\psi_{j+1}\right] = \frac{\partial \psi}{\partial t} + c\frac{\partial \psi}{\partial x} + O(\Delta x^3, \Delta t^2)$. Expand $\frac{\partial \psi}{\partial t}$. (5)
→ (4) represents this

- From (5): RHS: $=$ ~~[illegible]~~ Try again:

$= c\frac{\partial \psi}{\partial x} + \psi'_j + \underbrace{\frac{1}{2}\Delta t\,\psi''_j}_{(6a)} + \underbrace{\frac{1}{6}\Delta t^2\,\psi_j^{(3)}}_{(6b)} + O(\Delta t^4) + O(\Delta t^3)$ (6)

- (6a) and (6b) must be discretized. Writing a Taylor table equation for each:

- (6a): $\psi''_j + b_0\psi_{j-2} + b_1\psi_{j-1} + b_2\psi_j + b_3\psi_{j+1} = O(\Delta t^3)$ (6b): $\psi_j^{(3)} + c_0\psi_{j-2} + c_1\psi_{j-1} + c_2\psi_j + c_3\psi_{j+1} = O(\Delta t^3)$

~~[illegible]~~ However, conversion from time to space must be made using (1).

- (6a): $\frac{\partial^2 \psi_j}{\partial t^2} = \frac{\partial}{\partial t}\left(\frac{\partial \psi_j}{\partial t}\right) = \frac{\partial}{\partial t}\left(-c\frac{\partial \psi_j}{\partial x}\right) = -c\frac{\partial}{\partial x}\left(\frac{\partial \psi_j}{\partial t}\right) = +c^2\frac{\partial^2 \psi_j}{\partial x^2}$ (6a')
- (6b): $\frac{\partial^3 \psi_j}{\partial t^3} = \frac{\partial}{\partial t}\left(\frac{\partial}{\partial t}\left(\frac{\partial \psi_j}{\partial t}\right)\right) = \frac{\partial}{\partial t}\left(\frac{\partial}{\partial t}\left(-c\frac{\partial \psi_j}{\partial x}\right)\right) = \frac{\partial}{\partial t}\left(-c\frac{\partial}{\partial x}\left(\frac{\partial \psi_j}{\partial t}\right)\right) = \frac{\partial}{\partial t}\left(c^2\frac{\partial^2 \psi_j}{\partial x^2}\right) = c^2\frac{\partial}{\partial t}\left(\frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial x}\right)\right) = c^2\left(\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial t}\right)\right)\right) = -c^3\frac{\partial^3 \psi}{\partial x^3}$ (6b')

- Rewrite (6) with (6a') and (6b'): RHS: $= c\frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial t} + \frac{1}{2}c^2\Delta t\frac{\partial^2 \psi_j}{\partial x^2} - \frac{1}{6}c^3\Delta t^2\frac{\partial^3 \psi_j}{\partial x^3} + O(\Delta t^3, \Delta x^3)$ (7)

- Revisiting Taylor tables: (6a): $b_0 = 0$, $b_1 = -\frac{1}{\Delta x^2}$, $b_2 = \frac{2}{\Delta x^2}$, $b_3 = -\frac{1}{\Delta x^2}$ | (6b): $c_0 = \frac{1}{\Delta x^3}$, $c_1 = -\frac{3}{\Delta x^3}$, $c_2 = \frac{3}{\Delta x^3}$, $c_3 = -\frac{1}{\Delta x^3}$

- Applying these to (6a) and (6b) results in:
$\psi''_j = \frac{1}{\Delta x^2}\psi_{j-1} - \frac{2}{\Delta x^2}\psi_j + \frac{1}{\Delta x^2}\psi_{j+1} = \frac{\partial^2 \psi_j}{\partial x^2}$ (8)
$\psi'''_j = -\frac{1}{\Delta x^3}\psi_{j-2} + \frac{3}{\Delta x^3}\psi_{j-1} - \frac{3}{\Delta x^3}\psi_j + \frac{1}{\Delta x^3}\psi_{j+1} = \partial^3\psi_j/\partial x^3$ (9)

- From (7), ~~[illegible]~~ rewrite (7) using (4), (8), (9):

$= \left[\frac{c}{6\Delta x}\psi_{j-2} - \frac{c}{\Delta x}\psi_{j-1} + \frac{c}{2\Delta x}\psi_j + \frac{c}{3\Delta x}\psi_{j+1}\right] + \left[\frac{c^2\Delta t}{2\Delta x^2}\psi_{j-1} - \frac{2c^2\Delta t}{2\Delta x^2}\psi_j + \frac{c^2\Delta t}{2\Delta x^2}\psi_{j+1}\right] + \left[\frac{c^3\Delta t^2}{6\Delta x^3}\psi_{j-2} - \frac{c^3\Delta t^2}{2\Delta x^3}\psi_{j-1} + \frac{c^3\Delta t^2}{2\Delta x^3}\psi_j - \frac{c^3\Delta t^2}{6\Delta x^3}\psi_{j+1}\right] + \frac{\partial \psi}{\partial t} + O(\Delta x^3, \Delta t^3)$ (10)

- Group ψ terms: $\left[\frac{c}{6\Delta x} + \frac{c^3\Delta t^2}{6\Delta x^3}\right]\psi_{j-2} + \left[-\frac{c}{\Delta x} + \frac{c^2\Delta t}{2\Delta x^2} - \frac{c^3\Delta t^2}{2\Delta x^3}\right]\psi_{j-1} + \left[\frac{c}{2\Delta x} - \frac{2c^2\Delta t}{2\Delta x^2} + \frac{c^3\Delta t^2}{2\Delta x^3}\right]\psi_j + \left[\frac{c}{3\Delta x} + \frac{c^2\Delta t}{2\Delta x^2} - \frac{c^3\Delta t^2}{6\Delta x^3}\right]\psi_{j+1} + \frac{\partial \psi}{\partial t} + O(\Delta x^3, \Delta t^3)$ (11)

- Factor out $\frac{c}{\Delta x}$; rewrite $\frac{c\Delta t}{\Delta x} = \mu$: $\frac{c}{\Delta x}\left\{\left[\frac{1}{6}(1+\mu^2)\right]\psi_{j-2} + \left[-1 + \frac{1}{2}\mu - \frac{1}{2}\mu^2\right]\psi_{j-1} + \left[\frac{1}{2} - \mu + \frac{1}{2}\mu^2\right]\psi_j + \left[\frac{1}{3} + \frac{1}{2}\mu - \frac{1}{6}\mu^2\right]\psi_{j+1}\right\} + \frac{\partial \psi}{\partial t} + O(\Delta x^3, \Delta t^2)$ (12)

- By this,

$$\boxed{\begin{aligned} \alpha &= \tfrac{1}{6}(1+\mu^2) \\ \beta &= \tfrac{1}{2}(-2+\mu-\mu^2) \\ \gamma &= \tfrac{1}{2}(1-2\mu+\mu^2) \\ \delta &= \tfrac{1}{6}(2+3\mu-\mu^2) \end{aligned}}$$ ☆

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3b)

- $\frac{1}{\delta t}\left(\psi_j^{n+1} - \psi_j\right) = \underbrace{-\frac{1}{\delta x}\left[F_{j+1/2} - F_{j-1/2}\right]}_{(1a)}$ (1), where $F_{j+1/2} = c\left[\psi_j + d_1(\psi_j - \psi_{j-1}) + d_0(\psi_{j+1} - \psi_j)\right]$ (2a)
  $F_{j-1/2} = c\left[\psi_{j-1} + d_3(\psi_{j-2} - \psi_{j-1}) + d_2(\psi_{j} - \psi_{j-1})\right]$ (2b)
  → assume d_0, d_1 for $F_{j+1/2} \neq d_0, d_1$ for $F_{j-1/2}$

- Note: (1) from 3a) must match $-\frac{c}{\delta x}\left[F_{j+1/2} - F_{j-1/2}\right]$. Write (1a) using (2a) and (2b).

- (1a) $= \cancel{-\frac{c}{\delta x}\left[\psi_j + d_1\psi_j - d_1\psi_{j-1} + d_0\psi_{j+1} - d_0\psi_j - \psi_{j-1} - d_3\psi_{j+1} + d_3\psi_j - d_2\psi_{j+2} + d_2\psi_{j+1}\right]}$ ~~Regrouping by like ψ terms~~ Rewrite: (3)

  $= -\frac{c}{\delta x}\left[\psi_j + d_1\psi_j - d_1\psi_{j-1} + d_0\psi_{j+1} - d_0\psi_j - \psi_{j-1} - d_3\psi_{j-1} + d_3\psi_{j-2} - d_2\psi_j - d_2\psi_{j-1}\right]$ Regrouping by like ψ terms:

  $= -\frac{c}{\delta x}\left[d_3\psi_{j-2} + (-d_1 - 1 - d_3 - d_2)\psi_{j-1} + (1 + d_1 - d_0 - d_2) + d_0\psi_{j+1}\right]$ (4)

- Matching (4) to (11) from 3a):
  $d_3 = \alpha = \frac{1}{6}(1+\mu^2)$ (5a)
  $-d_1 - 1 - d_3 - d_2 = \beta = \frac{1}{2}(-2+\mu-\mu^2)$ (5b)
  $-d_0 + d_1 + 1 - d_2 = \gamma = \frac{1}{2}(1-2\mu+\mu^2)$ (5c)
  $d_0 = \delta = \frac{1}{6}(2+3\mu-\mu^2)$ (5d)

  Try (5b) + (5c) to solve for d_2. ↓ $-d_0 - d_3 - 2d_2 = \beta + \gamma$ (6)

- Expand (6): → $d_2 = -\frac{1}{2}\left[d_0 + d_3 + \frac{1}{2}(-2+\mu-\mu^2) + \frac{1}{2}(1-2\mu+\mu^2)\right]$
  $= -\frac{1}{2}\left[\left(\frac{1}{3} + \frac{1}{2}\mu - \frac{1}{6}\mu^2\right) + \left(\frac{1}{6} + \frac{1}{6}\mu^2\right) + \left(-1 + \frac{1}{2}\mu - \frac{1}{2}\mu^2\right) + \left(\frac{1}{2} - \mu + \frac{1}{2}\mu^2\right)\right]$ (7)

- Simplify (7): $d_2 = -\frac{1}{2}\left[\frac{1}{6}\right] = -\frac{1}{12}$. (8). Use (8) in (5b) to get d_1. → $d_1 = -1 - d_2 - d_3 - \frac{1}{2}(-2+\mu-\mu^2)$ (9)

- Expand (9): $d_1 = -1 - \left(-\frac{1}{12}\right) - \frac{1}{6}(1+\mu^2) - \frac{1}{2}(-2+\mu-\mu^2) = -1 + \frac{1}{12} - \frac{1}{6} - \frac{1}{6}\mu^2 + 1 - \frac{1}{2}\mu + \frac{1}{2}\mu^2$
  $= -\frac{1}{12} - \frac{1}{2}\mu + \frac{1}{3}\mu^2 = \frac{1}{12}\left[-1 - 6\mu + 4\mu^2\right]$ (10)

- Therefore,

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$$\boxed{\begin{aligned} d_0 &= \tfrac{1}{12}\left[4 + 6\mu - 2\mu^2\right] && (11a)\\ d_1 &= \tfrac{1}{12}\left[-1 - 6\mu + 4\mu^2\right] && (11b)\\ d_2 &= -\tfrac{1}{12} && (11c)\\ d_3 &= \tfrac{1}{12}\left[2 + 2\mu^2\right] && (11d)\end{aligned}}$$

☆

- Therefore, $F_{j+1/2} = c\left[\psi_j + \frac{1}{12}(-1-6\mu+4\mu^2)(\psi_j - \psi_{j-1}) + \frac{1}{12}(4+6\mu-2\mu^2)(\psi_{j+1} - \psi_j)\right]$ (12)

  $F_{j-1/2} = c\left[\psi_{j-1} + \frac{1}{12}(2+2\mu^2)(\psi_{j-2} - \psi_{j-1}) + \frac{1}{12}(\psi_j - \psi_{j-1})\right]$ (13)

3c)

- If $\mu \to 0$: $F_{j+1/2} = c\left[\psi_j - \frac{1}{12}(\psi_j - \psi_{j-1}) + \frac{4}{12}(\psi_{j+1} - \psi_j)\right] = c\left[\frac{7}{12}\psi_j + \frac{1}{12}\psi_{j-1} + \frac{4}{12}\psi_{j+1}\right]$ (14)
  (from (13)) $F_{j-1/2} = c\left[\psi_{j-1} + \frac{2}{12}(\psi_{j-2} - \psi_{j-1}) + \frac{1}{12}(\psi_j - \psi_{j-1})\right] = c\left[\frac{10}{12}\psi_{j-1} - \frac{1}{6}\psi_{j-2} + \frac{1}{12}\psi_j\right]$ (15)

- Combining (14) and (15) → (1): $\frac{1}{\delta t}\left[\psi_j^{n+1} - \psi_j\right] = -\frac{c}{\delta x}\left[\left(\frac{7}{12}\psi_j + \frac{1}{12}\psi_{j-1} + \frac{4}{12}\psi_{j+1}\right) - \left(\frac{10}{12}\psi_{j-1} - \frac{1}{6}\psi_{j-2} + \frac{1}{12}\psi_j\right)\right]$
  $= -\frac{c}{\delta x}\left[\frac{1}{2}\psi_j - \frac{6}{6}\psi_{j-1} + \frac{1}{3}\psi_{j+1} + \frac{1}{6}\psi_{j-2}\right)$.

ANS → 3rd order upwind: $= \boxed{-\frac{c}{\delta x}\left[\frac{1}{6}\psi_{j-2} - \frac{6}{6}\psi_{j-1} + \frac{3}{6}\psi_j + \frac{2}{6}\psi_{j+1}\right]}$