AOS 575 Problem Set 3 **Due October 13, 2022**

1. Compute solutions to the constant wind speed advection equation $\frac{\partial\psi}{\partial t}+c\frac{\partial\psi}{\partial x}=0$ on the periodic domain $0\le x\le 1$ with the initial condition $\psi(x,0)=(\sin(2\pi x))^6$. Set c = 0.1 and examine the solutions at time t = 50. Use 4th order centered differences in space and the 3rd order Adams Bashforth time stepping scheme. (Be sure to start this scheme off with time steps that retain 3rd order accuracy in time.) Include the Python, Matlab, or other code that you use to compute these solutions in your answers.

   (a) Use a Courant number $\frac{c\Delta t}{\Delta x}=0.1$ and compare and explain the results using $\Delta x=1/20$, $\Delta x=1/40$, $\Delta x=1/80$, and $\Delta x=1/160$. Plot the solutions (these 4 curves should be overlaid on the same plot) and then plot the RMS errors of each of these solutions with respect to the exact solution of the continuous equation against Δx on a log-log plot and explain what you observed. (Hint: If this has been coded correctly, the RMS errors with $\Delta x=1/160$ should be of order 0.0005. If not, make sure you have taken the right number of time-steps.)
   (b) Use $\Delta x=1/20$ and compare and explain the results using Courant numbers $\frac{c\Delta t}{\Delta x}=0.1$, $\frac{c\Delta t}{\Delta x}=0.2$, $\frac{c\Delta t}{\Delta x}=0.4$, and $\frac{c\Delta t}{\Delta x}=0.8$. Again plot the solutions (being sure to use a physically sensible range for the y-axis) and plot the RMS errors against Courant number on a log-log plot.
   (c) Repeat part (b) but using $\Delta x=1/80$.

2. Repeat the solution of the same test case as in problem 1, but this time using Lele's 4th-order compact difference scheme (See Durran's book for details) in space and leapfrog time stepping with an Asselin filter with coefficient $\gamma=0.1$ in time. That is, compute solutions to the constant wind speed advection equation $\frac{\partial\psi}{\partial t}+c\frac{\partial\psi}{\partial x}=0$ on the periodic domain $0\le x\le 1$ with the initial condition $\psi(x,0)=(\sin(2\pi x))^6$. Set c = 0.1 and examine the solutions at time t = 50.
   Lele's 4th-order compact difference scheme is given by solving
   $$\frac{1}{24}\left[5\left.\frac{\partial\psi}{\partial x}\right|_{j+1}+14\left.\frac{\partial\psi}{\partial x}\right|_{j}+5\left.\frac{\partial\psi}{\partial x}\right|_{j-1}\right]=\frac{1}{12}\left(11\frac{\psi_{j+1}-\psi_{j-1}}{2\Delta x}+\frac{\psi_{j+2}-\psi_{j-2}}{4\Delta x}\right)$$
   for $\left.\frac{\partial\psi}{\partial x}\right|_{j}$. A cyclic tridiagonal solver will be needed. Examples of one can be found in the appendix of Durran, or in sections 2.4 and 2.7 of Numerical Recipes in C, 2nd edition, or it should be possible to derive a solver directly; ask if you need help! Include the Python, Matlab, or other code that you use to compute these solutions in your answers.

(a) Use a Courant number $\frac{c\Delta t}{\Delta x} = 0.1$ and compare and explain the results using $\Delta x = 1/20$, $\Delta x = 1/40$, $\Delta x = 1/80$, and $\Delta x = 1/160$. Plot the solutions themselves, and then plot the RMS errors of each of these solutions with respect to the exact solution of the continuous equation against Δx on a log-log plot and explain what you observed. (Hint: If this has been coded correctly, the RMS errors with $\Delta x = 1/160$ should be of order 0.017.)

(b) Use $\Delta x = 1/20$ and compare and explain the results using Courant numbers $\frac{c\Delta t}{\Delta x} = 0.1$, $\frac{c\Delta t}{\Delta x} = 0.2$, $\frac{c\Delta t}{\Delta x} = 0.4$, and $\frac{c\Delta t}{\Delta x} = 0.8$. Again plot the solutions and plot the RMS errors against Courant number on a log-log plot.

(c) Repeat part (b) but using $\Delta x = 1/80$.

3.(a) Derive a third order, direct space time method (with third order accuracy in both space and time) finite difference approximation to the linear advection problem $\frac{\partial \psi}{\partial t} + c\frac{\partial \psi}{\partial x} = 0$ where $c > 0$ is a positive constant flow. The resulting scheme should take the form

$$\frac{1}{\Delta t}(\psi_j^{n+1} - \psi_j^n) = -\frac{c}{\Delta x}(\alpha\psi_{j-2}^n + \beta\psi_{j-1}^n + \gamma\psi_j^n + \delta\psi_{j+1}^n)$$

where α, β, γ, and δ are factors that you will determine. Assume a regular grid with index j, such that $x_j = j\Delta x$ and $\psi_j = \psi(x_j)$. Hint: you will need higher time derivatives of the original equation to eliminate the two leading temporal truncation error terms.

(b) Derive the discrete flux F that when used in the difference equation

$$\frac{1}{\Delta t}(\psi_j^{n+1} - \psi_j^n) = -\frac{1}{\Delta x}(F_{j+1/2} - F_{j-1/2})$$

gives the same result as the finite difference equation derived in part (a). Hint: F will take the form

$$F_{j+1/2} = c\lfloor \psi_j + d_1(\psi_j - \psi_{j-1}) + d_0(\psi_{j+1} - \psi_j)\rfloor$$

where d_1 and d_0 are functions of the Courant number $C = \frac{c\Delta t}{\Delta x}$.

(c) Consider this flux in the limit that the Courant number goes to 0. What finite difference spatial discretization does this correspond to? Consult the lecture notes on spatial discretization if necessary.