AOS 575 Problem Set 2 **Due September 29, 2022**

1. The chemostat equations can be used as a simple ecosystem model. If N is a nutrient concentration, and P is phytoplankton concentration, the governing equations are:

$$\frac{dN}{dt} = S(N_0 - N) - \frac{\gamma PN}{N_0}$$

$$\frac{dP}{dt} = \frac{\gamma PN}{N_0} - SP$$

Water is introduced into the system at a rate per unit volume S with no phytoplankton and nutrient concentration N_o, and extracted again at the same rate with the average properties. Inside, the phytoplankton grow with a rate γ that is proportional to the nutrient concentration. Suppose that for a given system γ/S = 10.

(1-a1) Devise an *explicit* two-step second-order-accurate time stepping scheme for this set of equations.

(1-a2) Identify the leading order error terms with the equivalent of this scheme for the simple source / damping equation $\frac{d\Psi}{dt} = \gamma\Psi - S\Psi$ (which is related to the P equation).

(1-a3) Use Von Neumann stability analysis to evaluate the range of stable timesteps near the equilibrium values of N and P for this system.

(1-b1) Demonstrate the stability of the discretization of the chemostat equations discussed in the lectures:

$$\frac{N^{n+1} - N^n}{\Delta t} = S\left(N_0 - N^{n+1}\right) - \frac{\gamma P^n N^{n+1}}{N_0}$$

$$\frac{P^{n+1} - P^n}{\Delta t} = \frac{\gamma P^n N^{n+1}}{N_0} - SP^{n+1}$$

by integrating this set of equations numerically, starting from initial conditions $P(0) = 0.5N_0$; $N(0) = 0.5N_0$ out to time 200/γ, making a plot of P and N against time using timesteps of $\Delta t = 20/\gamma$ and $\Delta t = 0.1/\gamma$.

(1-b2) Numerically evaluate the same equations and initial conditions, using the scheme you devised in (a1), using a timestep of $0.1/\gamma$, and compare this solution with the results from (b2).

(1-b3) What happens if you use a timestep of $20/\gamma$ with the scheme from (a1)?

(1-b4) Demonstrate that the discretization in part (b1) is only 1st order accurate, while the scheme derived in part (a1) is 2nd order accurate, by examining the rate of convergence of the solution at time $T = 2/\gamma$ as an increasingly short timestep Δt is taken. Use timesteps that range from $\Delta t = 0.01/\gamma$ up to $\Delta t = 2/\gamma$, comparing the errors with the solution using the scheme from (a1) with timestep of $\Delta t = 0.001/\gamma$.

(1-b5) Discuss the implications of these results for the considerations that might go into selecting a time stepping scheme for ecosystem models.

(Note: Please, use a physically sensible range of values for your y-axis in all plots of your solutions for problem 1.)

(1-c1) (BONUS SECTION): Devise a second (or higher) order accurate discretization of these equations that is well behaved for timesteps up to $\Delta t = 20/\gamma$ and beyond. Evaluate the order of accuracy by examining convergence as in (1-b4), and the stability as in (1-b3). (Hint: there is no simple temporal discretization that satisfies this; timestep dependent weighted averages of different time stepping schemes or multiple semi-implicit steps will likely be required.)

2. Derive the simplest spatially staggered 4th order compact difference approximation to $\frac{\partial \psi}{\partial x}$. That is, find the 4th order accurate approximation to $\left.\frac{\partial \psi}{\partial x}\right|_j$ that uses the values of $\psi_{j-1/2}$, $\psi_{j+1/2}$, $\left.\frac{\partial \psi}{\partial x}\right|_{j-1}$, $\left.\frac{\partial \psi}{\partial x}\right|_j$, and $\left.\frac{\partial \psi}{\partial x}\right|_{j+1}$. Plot the ratio of this approximation to the exact solution over the range of resolvable wavenumbers on a regularly spaced grid.