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ME I4600
February 24, 2022

Homework 1

1. **Describe the divergence theorem of Gauss.** (5 pts)

   Gauss' divergence theorem states that the divergence of a field within an enclosed boundary must equal the flux from the surface of the boundary. This theorem is a form of expressing conservation of quantities such as mass, momentum, and energy. In plain English, the theorem roughly states that the expansion or compression of a quantity within a volume is equivalent to the amount of fluid that exits or enters the volume. In mathematical terms,

$$\iiint_V \nabla \mathbf{F} dV = \iint_S \mathbf{F} d\mathbf{S} \tag{1}$$

   where $\mathbf{F}$ is the field, V is the volume, and $\mathbf{S}$ is the surface. This theorem is significant in fluid mechanics as it states that mass must be conserved for some control volume using a relation between different dimensions, although most commonly between 2 (surface) and 3 (volumetric) dimensional flows.

2. **Describe the Taylor series expansion.** (5 pts)

   The Taylor series expansion is a polynomial expansion of a certain function at a given point using derivatives of the function. The Taylor series expansion is useful in computational fluid mechanics for creating numerical approximations for physical processes on a discretized domain (i.e. a grid). The expansion is used to provide an approximation because it allows for a computationally-inexpensive method for modeling processes that are based on functions that may be computationally-expensive, while preserving a high degree of precision. The precision of the numerical method is often controlled by the order of the Taylor expansion, also called *truncation error*, which represents the order of the resultant polynomial chosen for the numerical method. It is worth noting that different schemes exist that allow for the combination of these expansions to increase precision by decreasing the truncation error.

3. **Describe the four different models of flow, and compare the continuity equations derived from these models.** (10 pts)

   (a) **Finite control volume fixed in space**
   Fluid moves through a finite control volume fixed in a flow field. Fluid enters and exits through defined surfaces. Conservation of mass for the volume constrains the mass flow of entering fluid to equal the mass flow of exiting fluid. Therefore, the total mass flow for the volume is the sum of the mass flow in the volume and the mass flow crossing the surfaces. This is presented mathematically:

$$\frac{\partial}{\partial t}\int_V \rho dV + \int_S \rho \mathbf{u}\, d\mathbf{S} = 0 \tag{2}$$

   (b) **Finite control volume moving with the flow**
   A finite control volume moves with the flow. The volume maintains constant mass as it moves. Conservation of mass for the volume allows for variable shape and volume. Therefore, the mass of the volume is equivalent to the summation of the product of fluid density and the differential volumes composing the finite control volume. Because mass is conserved and there is no fluid moving through the volume, the rate change of mass with time is 0. This is presented mathematically:

$$m = \int_V \rho dV \tag{3}$$

$$\frac{\partial m}{\partial t} = \frac{D}{Dt}\int_V \rho dV = 0 \tag{4}$$

(c) **Infinitesimally small element fixed in space**

An infinitesimal fluid element is fixed in a flow field. Fluid enters and exits through defined surfaces. Conservation of mass for the volume constrains entering mass flow to equal exiting mass flow. However, fluid properties are a function of space. Note that spatial dimensions for an infinitesimal (i.e. differential) element can be defined as dx, dy, and dz. To conserve mass, the summation of the mass flow (i.e., product of velocity and density over a spatial dimension) must equal 0 when taken over all surfaces. Additionally, since the field changes with time but the element stays fixed, the change in density with respect to time must be accounted for during mass conservation. This can be expressed mathematically:

$$\frac{\partial \rho}{\partial t} + \left[\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z}\right] = \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0 \tag{5}$$

(d) **Infinitesimally small element moving with the flow**

An infinitesimal (i.e. differential) fluid element moves with the flow. The element maintains constant mass as it moves. Conservation of mass for the volume allows for variable element shape and volume. As the element moves through the fluid with a changing volume, the density must also change to conserve mass. No mass flow exists through the element of its surfaces. This can be expressed mathematically:

$$\frac{D(\delta m)}{Dt} = 0 = \delta V \frac{D\rho}{Dt} + \rho \frac{D\delta V}{Dt} \tag{6}$$

Continuity for the volume or element fixed in space is conservative, whereas the continuity equation for elements moving in space are non-conservative. The difference between conserative and non-conservative is relevant to computational fluid mechanics with regards to modeling phenomena with sharp gradients or discontinuities in the flow field (e.g., shocks, turbulence near surfaces, etc.). This is because the conservative forms of the continuity equation allow for optimal discretization of the field, whereas non-conservative forms require "smooth" properties that may lead to numerical error or artificial fluid properties based on how the discretization expands algebraically.

4. **What is the difference between the integral and differential forms of the governing equations for fluid flow and heat transfer?** (5 pts)

   A significant difference between integral and differential forms of the governing equations is the element analysis approach; with an integral approach, a large region is evaluated while with a differential approach, an infinitesimal fluid element is evaluated. Although these approaches are equivalent, they can be used for different purposes. For larger scales or areas with smoother property gradients, the integral approach is more intuitive whereas for smaller scales or areas with sharper property gradients, the differential approach is more accurate.

5. **What is the difference between the strong and weak conservation forms of the governing equations?** (5 pts)

   The strong conservation form of the governing equations holds at all points for a field with variable quantities (e.g., density, pressure, fluid flow velocity). The weak conservation form holds over a region with some constant properties, such as density and the velocity field. The strong form is valid at all points and no assumptions are required, leading to an accurate but costly set of equations to solve. The weak form is valid for certain regions with an assumption of uniform properties, leading to a less accurate but more cost-effective solution process. It is worth noting that the strong form requires a higher order set of requirements to satisfy for continuity, whereas the weak form is more straightforward.

6. **What are the Euler equations?** (5 pts)

   The Euler equations are the set of equations describing fluid flow assuming inviscid and incompressible flow. The Euler equations are described as:

$$\rho \left[\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u}\right] = -\nabla p + \rho \mathbf{g} \tag{7}$$

In these equations, the acceleration of the fluid (left-hand side) equal the sum of the surface forces and the body forces (right-hand side), which are the sum of the pressure gradient and gravitational forces. The Euler equation is useful in computational fluid mechanics when approximations of fluid flow are needed, especially when viscous effects are negligible (such as in momentum-dominant flows like flow over an airfoil or a wide channel).

7. **What are the complete Navier-Stokes equations?** (5 pts)

   The complete Navier-Stokes equations couple expressions for the conservation of mass, momentum, and energy to describe the total energy of a system and the changes in it with regards to space and time. This set of equations produces 7 equations for 7 unknowns, which are temperature (T), motion in three dimensions (u, v, w), pressure (p), energy (e), and density (ρ). From the complete set of Navier-Stokes equations, simplifications can be made from assumptions to allow for accurate approximations and exact solutions (e.g., isothermal flow, incompressible flow, 1-dimensional flow).

8. **Describe the 3D isothermal incompressible Navier-Stokes equations. Explain the differences between this model and the full system of Navier-Stokes equations. How many unknown variables are in these equations?** (10 pts)

   The equations will be described by their respective groupings: continuity, momentum, and energy. The assumption of incompressibility indicates that $\frac{D\rho}{Dt} = 0$ and the assumption of an isothermal field indicates that $\frac{DT}{Dt} = 0$.

   (a) **Continuity**
   The Navier-Stokes equation for continuity is:

   $$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0 \tag{8}$$

   Assuming incompressibility, ρ will be constant spatially and temporally. Therefore, the equation can be simplified:

   $$\begin{aligned} 0 + \rho \nabla \cdot \mathbf{u} &= 0 \\ \nabla \cdot \mathbf{u} &= 0 \end{aligned}$$

   Since $\mathbf{u}$ is shorthand for (u, v, w), there are 3 unknowns in the continuity equation. The incompressible form of the continuity equation differs from the full form as ρ becomes constant with respect to space and time and does not contribute to the definition for continuity in an incompressible fluid.

   (b) **Momentum**
   Per Anderson (1995), Equation 2.44a, the Navier-Stokes equation for momentum (only x-direction used for brevity) is:

   $$\begin{aligned} &\frac{\partial \rho u}{\partial t} + \frac{\partial (\rho u^2)}{\partial x} + \frac{\partial (\rho u v)}{\partial y} + \frac{\partial (\rho u w)}{\partial z} = \\ &\quad -\frac{\partial p}{\partial x} + \frac{\partial}{\partial x}\left(\lambda \nabla \cdot \mathbf{u} + 2\mu \frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\right] + \frac{\partial}{\partial z}\left[\mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)\right] + \rho f_x \end{aligned} \tag{9}$$

   Incorporating the incompressibility assumption, Equation 9 becomes:

   $$\begin{aligned} &\rho\left[\frac{\partial u}{\partial t} + \frac{\partial u^2}{\partial x} + \frac{\partial u v}{\partial y} + \frac{\partial u w}{\partial z}\right] = \\ &\quad -\frac{\partial p}{\partial x} + \frac{\partial}{\partial x}\left(2\mu \frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\right] + \frac{\partial}{\partial z}\left[\mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)\right] + \rho f_x \end{aligned} \tag{10}$$

   The incompressible form of the momentum equation differs from the complete momentum equation as ρ becomes constant and is factored out on the LHS, whereas the volumetric viscosity ($\lambda \nabla \cdot \mathbf{u}$) can be neglected as $\nabla \cdot \mathbf{u} = 0$.

(c) **Energy**

Per Anderson (1995), Equation 2.64, the Navier-Stokes equation for energy (only x-direction used for brevity) is:

$$\begin{aligned}\frac{\partial}{\partial t}\left[\rho\left(e+\frac{U^2}{2}\right)\right]+\nabla\cdot\left[\rho\left(e+\frac{U^2}{2}\mathbf{u}\right)\right]=\rho\dot{q}+&\\
&\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)+\frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)-\\
&\frac{\partial(up)}{\partial x}-\frac{\partial(vp)}{\partial y}-\frac{\partial(wp)}{\partial z}+\\
&\frac{\partial(u\tau_{xx})}{\partial x}+\frac{\partial(u\tau_{yx})}{\partial y}+\frac{\partial(u\tau_{zx})}{\partial z}+\\
&\frac{\partial(u\tau_{xy})}{\partial x}+\frac{\partial(u\tau_{yy})}{\partial y}+\frac{\partial(u\tau_{zy})}{\partial z}+\\
&\frac{\partial(u\tau_{xz})}{\partial x}+\frac{\partial(u\tau_{yz})}{\partial y}+\frac{\partial(u\tau_{zz})}{\partial z}+\rho\mathbf{f}\cdot\mathbf{u}\end{aligned}\tag{11}$$

Incorporating the incompressibility and isothermal assumptions, Equation 11 becomes:

$$\begin{aligned}\rho\left[\frac{\partial}{\partial t}\left(e+\frac{U^2}{2}\right)\right]=\rho\dot{q}+&\\
&0+0+0-\\
&\frac{\partial(up)}{\partial x}-\frac{\partial(vp)}{\partial y}-\frac{\partial(wp)}{\partial z}+\\
&\frac{\partial}{\partial x}\left[u\left(2\mu\frac{\partial u}{\partial x}\right)\right]+\frac{\partial(u\tau_{yx})}{\partial y}+\frac{\partial(u\tau_{zx})}{\partial z}+\\
&\frac{\partial(u\tau_{xy})}{\partial x}+\frac{\partial}{\partial y}\left[v\left(2\mu\frac{\partial v}{\partial y}\right)\right]+\frac{\partial(u\tau_{zy})}{\partial z}+\\
&\frac{\partial(u\tau_{xz})}{\partial x}+\frac{\partial(u\tau_{yz})}{\partial y}+\frac{\partial}{\partial z}\left[u\left(2\mu\frac{\partial w}{\partial z}\right)\right]+\rho\mathbf{f}\cdot\mathbf{u}\end{aligned}\tag{12}$$

The incompressible and isothermal form of the energy equation differs from the complete energy equation by factoring out ρ on the LHS since it becomes constant. Additionally, because $\nabla\cdot\mathbf{u}=0$, the 2nd energy term simplifies to $\nabla\cdot e$, which is 0. Note that the LHS term remains assuming e and/or U is variable with respect to time. On the RHS, the conduction terms become 0 due to the lack of a temperature gradient. Additionally, the volumetric viscosity terms disappear from the τ_{ij} terms when $i=j$, leading to just surface viscous stresses to be accounted for.

After applying assumptions, the number of unknowns reduces from 7 to 4: u, v, w, and T. It is worth noting that thermal effects can be accounted for if a reference density, ρ_0, is factored into the equations (i.e. Boussinesq approximation).