Question 1: Vector Calculus

Q1.1: Compute the derivative of the function $f(x) = \sin(x)$.

1.1 $f(x) = \sin(x)$

$$\frac{d \sin(x)}{dx} = \cos(x)$$

Q1.2: Use the chain rule to compute the derivative of the function $f(x) = \sin(x^2)$.

1.2 $f(x) = \sin(x^2)$

$$\frac{d \sin(x^2)}{dx} = 2x \cdot \cos(x^2)$$

Q1.3: Compute the partial derivatives (i.e., $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$) of the function $f(x, y) = e^{xy}$.

1.3 $\frac{\partial f}{\partial y} = y e^{xy}$, $\frac{\partial f}{\partial y} = x e^{xy}$

Q1.4: Compute the 2nd-order partial derivatives (i.e., $\frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial y^2}$) of the function $f(x, y) = \sin(xy)$.

1.4

$\frac{\partial f}{\partial x} = y \cos(xy)$ $\frac{\partial f}{\partial y} = x \cos(xy)$

$\frac{\partial^2 f}{\partial x^2} = -y \sin(xy)$ $\frac{\partial^2 f}{\partial y^2} = x \cdot \cos(xy) \cdot xy$

$= x \cdot -\sin(xy) \cdot x$

$= -x^2 \sin^2(xy)$

[2 marks] Q1.5: What is the main difference between a derivative and a gradient?

1. A derivative yields a vector result, while a gradient yields a scalar result.
2. A derivative represents the curvature of a function, while a gradient represents the slope.
3. A derivative operates on scalar-valued functions of multiple variables, while a gradient operates on scalar-valued functions of one variable.
4. A derivative operates on scalar-valued functions of one variable, yielding a scalar, while a gradient operates on scalar-valued functions of multiple variables or vector-valued functions, yielding a vector.

ANSWER : 4 - *A derivative operates on scalar-valued functions of one variable, yielding a scalar, while a gradient operates on scalar-valued functions of multiple variables or vector-valued functions, yielding a vector*

Q1.6: Compute the gradient of the vector-valued function $\vec{F}(x, y) = (x^2, y^2)$.

1.6

$$\vec{F}(x,y) = (x^2, y^2)$$

$$\vec{F}(x,y) = \left(\frac{\partial}{\partial x}(x^2), \frac{\partial}{\partial y}(y^2)\right)$$

$$= (2x, 2y)$$

Q1.7: Compute the gradient of the matrix function $\mathbf{M}(x) = \begin{bmatrix} x & x^2 \\ x^3 & x^4 \end{bmatrix}$.

1.7

$$M(x) = \begin{bmatrix} x & x^2 \\ x^3 & x^4 \end{bmatrix}$$

$$\nabla M(x) = \begin{bmatrix} 1 & 2x \\ 3x^2 & 4x^3 \end{bmatrix}$$

[2 marks] Q1.8: Explain the differences between the Jacobian and the Hessian matrices and their intended use cases. Provide an example to illustrate their roles.

Answer : The Jacobian matrix is used to represent the derivatives of a vector-valued function with respect to its input variables. It is essentially a matrix of first-order partial derivatives.

The Hessian matrix is a square matrix of second-order partial derivatives of a scalar-valued function. It provides information about the curvature of the function's graph.

Both are used to solve optimization problems.

Q1.9: Compute the Jacobian matrix of the vector-valued function $\vec{F}(x, y, z) = (x^2, y^3, z^4)$.

1.9 $\vec{F}(x, y, z) = (x^2, y^3, z^4)$

$$J = \begin{bmatrix} \frac{\partial}{\partial x}(x^2) & \frac{\partial}{\partial y}(x^2) & \frac{\partial}{\partial z}(x^2) \\ \frac{\partial}{\partial x}(y^3) & \frac{\partial}{\partial y}(y^3) & \frac{\partial}{\partial z}(y)^3 \\ \frac{\partial}{\partial x}(z^4) & \frac{\partial}{\partial y}(z^4) & \frac{\partial}{\partial z}(z^4) \end{bmatrix}$$

$$= \begin{bmatrix} 2x & 0 & 0 \\ 0 & 3y^2 & 0 \\ 0 & 0 & 4z^3 \end{bmatrix}$$

Q1.10: Compute the Hessian matrix of the scalar function $f(x, y) = x^2y + y^2x$.

1.10 $\quad f(x, y) = x^2y + y^2x$

$$H = \begin{bmatrix} \frac{\partial^2}{\partial x^2}(x^2y + y^2x) & \frac{\partial^2}{\partial x \partial y}(x^2y + y^2x) \\ \frac{\partial^2}{\partial y \partial x}(x^2y + y^2x) & \frac{\partial^2}{\partial y^2}(x^2y + y^2x) \end{bmatrix}$$

$$= \begin{bmatrix} 2y & 2x+2y \\ 2x+2y & 2x \end{bmatrix}$$

Q1.11: Compute the Hessian matrix of the vector-valued function $\vec{F}(x, y, z) = (x^3, y^2, z^4)$.

1.11

$$\vec{F}(x, y, z) = (x^3, y^2, z^4)$$

$$\text{first order} = (3x^2, 2y, 4z^3)$$

$$\text{Second order derivative} = (6x, 2, 12z^2)$$

$$H = \begin{bmatrix} 6x & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 12z^2 \end{bmatrix}$$

[2 marks] Q1.12: Derivatives and gradients are useful in various domains, e.g., for characterizing local approximations of functions (see, e.g., Taylor Series). Find the linear approximation of the function $f(x, y) = \sqrt{x + y}$ at the point $(1, 1)$.

1.12 $\frac{\partial f}{\partial x} = \frac{1}{2\sqrt{x}}$, $\frac{\partial f}{\partial y} = 1$

At point $(1,1)$:

$f_x(1,1) = \frac{1}{2\sqrt{1}} = \frac{1}{2}$, $f_y(1,1) = 1$

Linear Approximation:

$$L(x,y) = f(1,1) + f_x(1,1)(x-1) + f_y(1,1)(y-1)$$

$$= \sqrt{1} + \frac{1}{2}(x-1) + (y-1)$$

$$= 1 + \frac{1}{2}(x-1) + (y-1)$$

$$\boxed{L(x,y) = \frac{1}{2}x + y - \frac{1}{2}}$$

[3 marks] Q1.13: Compute the partial derivative of the logarithm of a multivariate Gaussian distribution $\mathcal{N}(\mathbf{x}|\mu, \mathbf{\Sigma})$ with respect to the mean vector μ.

$$\mathcal{N}(\mathbf{x}|\mu, \mathbf{\Sigma}) = \frac{1}{\sqrt{(2\pi)^k |\mathbf{\Sigma}|}} \exp\left(-\frac{1}{2}(\mathbf{x} - \mu)^T \mathbf{\Sigma}^{-1} (\mathbf{x} - \mu)\right)$$

Question 2: Analytic Geometry

Q2.1: What is the distance between the points $(1, 2)$ and $(3, 4)$?

2.1

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{4-2}{3-1} = \frac{2}{2} = 1.$$

Q2.2: Find the equation of the line passing through the points $(1, 2)$ and $(3, 4)$.

2.2

$$y = mx + c$$

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{4-2}{3-1} = \frac{2}{2} = 1$$

Using $(3, 4)$:

$$4 = 1 \cdot 3 + c$$
$$4 = 3 + c$$
$$c = 4 - 3 = 1.$$

Eqn of line passing through $(1, 2)$ & $(3, 4)$ is:

$$\boxed{y = x + 1}$$

Q2.3: What are the coordinates of the midpoint of the segment connecting $(1, 2)$ and $(3, 4)$?

2.3 Midpoint : $(x_1 + x_2)/2$, $(y_1 + y_2)/2$

$(1+3)/2$, $(2+4)/2$

$4/2$, $6/2$

$\boxed{\text{Midpoint Coordinates : } (2, 3)}$

Q2.4: Determine the slope of the line perpendicular to the line $3x - 4y = 12$.

2.4

$y = mx + c$

$-4y = -3x + 12$

$y = \frac{-3x}{-4} + \left(\frac{12}{-4}\right)$

$y = \frac{3}{4}x - 3$

gradient $= \frac{3}{4}$

$\boxed{\therefore \text{ Perpendicular} \Rightarrow -\frac{4}{3}}$

Q2.5: Find the equation of a circle with center $(2, -3)$ and radius 4.

2.5 $(x-a)^2 + (y-b)^2 = r^2$

$$(x-2)^2 + (y-(-3))^2 = 4^2$$

$$(x-2)^2 + (y+3)^2 = 16$$

Eqn of circle with center $(2, -3)$ & radius 4 :

$$\boxed{(x-2)^2 + (y+3)^2 = 16}$$

[2 marks] Q2.6: Define the Euclidean norm of a vector in $\mathbb{R}^n$ and show that it satisfies the properties of a norm (list out the three properties).

Answer : The Euclidean norm, also known as the L2 norm or simply the norm, of a vector v is a measure of the vector's length or magnitude. It satisfies the three properties of a norm: **non-negativity**, **homogeneity**, and the **triangle inequality**.

[2 marks] Q2.7: Given the vectors $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix}$, calculate the angle between them using the inner product formula.

2.7

$$\cos\theta = \frac{u \cdot v}{\|u\| \, \|v\|}$$

$$u \cdot v = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix}$$

$$= 1 \cdot (-2) + 2 \cdot 1 + (-3) \cdot 4$$

$$= -2 + 2 - 12$$

$$= -12$$

$$\|u\| \Rightarrow \sqrt{1^2 + 2^2 + (-3)^2}$$

$$= \sqrt{1 + 4 + 9}$$

$$= \sqrt{14}$$

$$\|v\| \Rightarrow \sqrt{(-2)^2 + (1)^2 + (4)^2}$$

$$= \sqrt{4 + 1 + 16}$$

$$= \sqrt{21}$$

$$\cos\theta = \frac{-12}{\sqrt{14} \cdot \sqrt{21}} = \frac{-12}{\sqrt{294}} = \frac{-12}{17.15}$$

$$\theta = \cos^{-1} 0.676$$

$$\boxed{\theta = 132.53^\circ}$$

Q2.8: Define the inner product of two functions $f(x)$ and $g(x)$ over a given interval $[a, b]$.

Answer : The inner product of two functions over a given interval provides a way to quantify their similarity or correlation within that interval, analogous to the dot product of vectors in Euclidean space.

Q2.9: Define what it means for a set of vectors to form an orthonormal basis.

Answer : A set of vectors form an orthonormal basis when their dot product is zero (i.e perpendicular) and if each vector in the set has length (magnitude) is equal to 1.

Q2.10: Describe the rotation of a vector in $\mathbb{R}^2$ about the origin by a given angle.

Answer : the rotation of a vector in R2 about the origin by a given angle involves constructing a rotation matrix and then applying it to the vector. This process effectively rotates the vector by the specified angle.

Question 3: Linear Algebra

Q3.1: Given the matrix $B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, find B^3.

3.1

$$B^2 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \times \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1+6 & 2+8 \\ 3+12 & 6+16 \end{bmatrix} = \begin{bmatrix} 7 & 10 \\ 15 & 22 \end{bmatrix}$$

$$B^3 = \begin{bmatrix} 7 & 10 \\ 15 & 22 \end{bmatrix} \times \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 7+30 & 14+40 \\ 15+66 & 30+88 \end{bmatrix}$$

$$= \begin{bmatrix} 37 & 54 \\ 81 & 118 \end{bmatrix}$$

Q3.2: Solve the system of equations:

$$\begin{cases} 2x + 3y + z = 1 \\ x - 2y + 2z = -2 \\ 3x + y - z = 3 \end{cases}$$

3.2

$$2x + 3y + z = 1 \quad \ldots \text{eqn}(1)$$
$$x - 2y + 2z = -2 \quad \ldots \text{eqn}(2)$$
$$3x + y - z = 3 \quad \ldots \text{eqn}(3)$$

eqn (1) + eqn (3) :

$$\begin{aligned} 2x + 3y + z &= 1 \\ + (3x + 3y - z) &= 3 \\ \hline 5x + 4y &= 4 \end{aligned}$$

$$\vdots$$

$$x = \frac{4}{5} - \frac{4}{5}y \quad \ldots \text{eqn}(6)$$

put x into eqn (1) :

$$2\left(\tfrac{4}{5} - \tfrac{4}{5}y\right) + 3y + z = 1$$

$$\frac{8}{5} - \frac{8}{5}y + 3y + z = 1$$

$$\tfrac{7}{5}y + z = -\frac{3}{5} \quad \ldots \text{(multiply by 3)}$$

$$\frac{21}{5}y + 3z = -\frac{9}{5} \quad \ldots \text{eqn}(4)$$

considering eqn(1) − 2* eqn (2)

$$\begin{aligned} 2x + 3y + z &= 1 \\ - (2x - 4y + 4z &= -4) \\ \hline 7y - 3z &= 6 \quad \ldots \text{eqn}(5) \end{aligned}$$

~~next: 2(eqn(4))/7~~

next : eqn(4) + eqn(5)

$$\left(\frac{21}{5} + 7\right)y = 6 - \frac{9}{5}$$

$$\frac{56}{5}y = \frac{21}{5}$$

$$\boxed{y = \frac{21}{56} = 3/8}$$

plug in y into eqn (6)

$$x = \frac{4}{5} - \frac{4}{5} \cdot \frac{3}{8_2}$$

$$= \frac{4}{5} - \frac{3}{10} = \frac{8-3}{10} = 5/10$$

$$\boxed{x = 1/2}$$

plug x & y into eqn (1)

$$2(1/2) + 3(3/8) + z = 1$$

$$1 + 9/8 + z = 1$$

$$z = 1 - 1 - 9/8$$

$$\boxed{z = -9/8}$$

$$\boxed{\text{Solntn} : (1/2, 3/8, -9/8)}$$

Q3.3: Calculate the eigenvalues and eigenvectors of the matrix:

$$A = \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix}$$

3.3 $\begin{bmatrix} 4-\lambda & -2 \\ 1 & 1-\lambda \end{bmatrix} = 0$

Take determinant:

$(4-\lambda)(1-\lambda) - (-2) = 0$

$4 - 5\lambda + \lambda^2 + 2 = 0$

$\lambda^2 - 5\lambda + 6 = 0$

$\lambda^2 - 3\lambda - 2\lambda + 6 = 0$

$(\lambda - 2)(\lambda - 3) = 0$

$\boxed{\text{Eigen Values: } \lambda = 2, \quad \lambda = 3.}$

Eigen Vector for 2:

$\begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ 1 & -1 \end{bmatrix}_{2R_2 - R_1} \Rightarrow \begin{bmatrix} 2 & -2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$\therefore \; 2x_1 - 2x_2 = 0$

$2x_1 = 2x_2$

$x_1 = x_2$

$$\boxed{\vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}}$$

Eigen Vector for 3:

$\begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 1 & -2 \end{bmatrix}_{R_2 - R_1} \Rightarrow \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$\therefore \; x - 2y = 0$

$x = 2y$

$$\boxed{\vec{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}}$$

Q3.4: Prove that the determinant of a square matrix and its transpose are equal.

3.4

Using an example:

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and its transpose : $A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$

The determinants of each are:

$\det(A) = ad - bc$

and

$\det(A^T) = ad - bc$

Same for a 3x3 matrix:

$B = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ and its transpose : $B^T = \begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix}$

$\det(B) = a(ei - fh) - b(di - fg) + c(dh - eg)$

and

$\det(B^T) = a(ei - fh) - d(bi - ch) + g(bf - ce)$

[3 marks] Q3.5: Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be a linear transformation defined by $T(\mathbf{x}) = A\mathbf{x}$, where A is a 2×3 matrix. Show that the null space of T is a subspace of $\mathbb{R}^3$.

Question 4: Probability and Statistics

Q4.1: What is the probability of rolling a sum of 7 with two six-sided dice?

4.1 To get sum of 7 = 3+4, 4+3, 6+1, 1+6, 5+2, 2+5 ⇒ 6 times.

Total space = 36

∴ probability = $6/36$

$$\boxed{P = 1/6}$$

Q4.2: What is the expected value of the sum of two six-sided dice and why?

4.2

$$E = \frac{\text{Sum of all outcomes}}{\text{no of outcomes}} = \frac{1+2+3+4+5+6}{6}$$

$$= \frac{21}{6} = 3.5$$

$$\boxed{\text{For 2 dice} = 3.5 \times 2 = 7.}$$

WHY: This value represents an average sum ~~the~~ after rolling 2 dice multiple times

[4 marks] Q4.3: Given the dataset $\{10, 15, 20, 25, 30\}$, compute the sample mean, median, mode, and standard deviation.

4.3

$$\text{mean} = \frac{10+15+20+25+30}{5}$$

$$= \frac{100}{5} = 20$$

median = 20 (middle)

mode = All have same frequency

Standard deviatn : der from mean ⇒ squared

$10-20 = -10 \Rightarrow 100$

$15-20 = -5 \Rightarrow 25$

$20-20 = 0 \Rightarrow 0$

$25-20 = 5 \Rightarrow 25$

$30-20 = 10 \Rightarrow 100$

250

Standard deviatn : $\sqrt{\frac{250}{n-1}} = \sqrt{\frac{250}{4}} = \sqrt{62.5}$

$\simeq 7.91$

[2 marks] Q4.4: Explain the difference between population and sample statistics, and provide an example to illustrate each concept.

Answer :

Population statistics refer to measures that describe an entire group or population of interest. **Sample** statistics, on the other hand, are calculated from a subset of the population called a sample. A sample is a smaller, representative group selected from the larger population.

So, for example, if we want to study the income of people in a country, everyone in the country is classified as **Population**, while studying data for adults above 50 years of age would be classified as **Sample**.

[2 marks] Q4.5: Compute the expected value of a binomial distribution with parameters $n = 10$ and $p = 0.3$.

4.5 $E[X] = n \cdot p = 10 \times 0.3 \Rightarrow 3$

Q4.6: What is the variance of a uniform distribution on the interval $[a, b]$, and how is it derived?

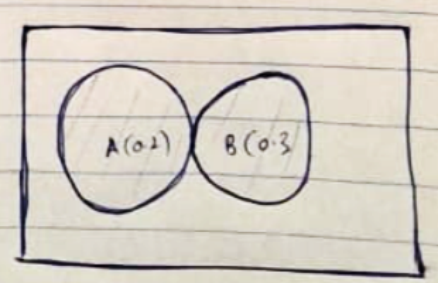
[3 marks] Q4.7: If the probability of an event A is 0.2 and the probability of an event B is 0.3, what is the maximum possible probability that either A or B occurs? Draw a Venn Diagram to complement your derivations.

4.7 $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Max possible for $P(A \cup B)$ is when $P(A \cap B) \simeq 0$

$\therefore P(A \cup B) = 0.2 + 0.3 - 0$

$P(A \cup B) = 0.5$



[3 marks] Q4.8: A company sells computers with two types of hard drives: type A and type B. The probability of a computer having a type A hard drive is 0.6, and the probability of a computer having a type B hard drive is 0.4. If 70% of the computers sold have type A hard drives, what percentage of type A hard drives fail compared to type B hard drives, given that the failure rates for type A and type B hard drives are 10% and 5% respectively?

4.8 Using Bayes Theorem:

$$P(A|F) = \frac{P(F|A) \cdot P(A)}{P(F)}$$

$P(F|A) \cdot P(A) = 0.1 * 0.7 = 0.07$

$P(F|B) \cdot P(B) = 0.05 * 0.3 = 0.015$

$P(F) = 0.07 + 0.015 = 0.085$

$$\therefore P(A|F) = \frac{0.07}{0.085} = \frac{70}{85} \approx 0.8235$$

$$\text{Also: } P(B|F) = \frac{0.015}{0.085} = \frac{15}{85} \approx 0.1765$$

$$\text{From above: } \frac{0.8235}{0.1765} \approx 4.67$$

$\therefore$ Type A hard drives are 4.67 times more likely to fail compared to Type B.

Question 5: Optimization

Q5.1: What is the minimum value of the function $f(x) = x^2 - 4x + 4$?

5.1

$$f'(x) = \frac{d}{dx}(x^2 - 4x + 4)$$

$$= 2x - 4$$

setting $2x - 4 = 0$, min point,

$$\boxed{x = 4/2 = 2.}$$

Q5.2: For the function $f(x, y) = x^2 + y^2$, what are the values of x and y at the minimum?

5.2

$$\frac{\partial f}{\partial x} = 2x \quad , \quad \frac{\partial f}{\partial y} = 2y$$

$$2x = 0 \quad \& \quad 2y = 0$$

$$\boxed{x = 0 \quad , \quad y = 0}$$

[3 marks] Q5.3: Solve the constrained optimization problem: minimize $f(x, y) = x^2 + y^2$ subject to the constraint $x + y = 5$. What is the Lagrange multiplier?

5.3

Lagrange fxn : $L(x, y, \lambda)$

$$L(x, y, \lambda) = f(x, y) - \lambda\,(g(x, y) - C)$$

$$L(x, y, \lambda) = (x^2 + y^2) - \lambda\,(x + y - 5)$$

Taking Partial derivatives :

$$\frac{\partial L}{\partial x} = 2x - \lambda, \quad \frac{\partial L}{\partial y} = 2y - \lambda, \quad \frac{\partial L}{\partial \lambda} = -(x + y - 5)$$

setting each to 0 :

$2x - \lambda = 0 \quad \therefore \quad 2x = \lambda \quad \therefore \quad x = \lambda/2$

$2y - \lambda = 0 \quad \therefore \quad 2y = \lambda \quad \therefore \quad y = \lambda/2$

$-(x + y - 5) = 0$

Substituting $x = \lambda/2$ & $y = \lambda/2$

$\lambda/2 + \lambda/2 = 5 \quad \therefore \quad \boxed{\lambda = 5}$

Q5.4: Define a convex function and explain why convex functions are important in optimization.

Answer : A convex function is a type of mathematical function that has the property that the line segment between any two points on its graph lies above or on the graph.

It is important in optimization because many real-world optimization problems are either convex or can be approximated as convex, and one of its properties i.e. that any local minimum is also a global minimum simplifies optimization, because finding a local minimum guarantees that it is the lowest point in the entire domain.

Q5.5: Define a convex set and provide an example of a convex set in two-dimensional space.

Answer : A convex set is a subset of a vector space such that, for any two points within the set, the line segment connecting them lies entirely within the set.

A well known example of a convex set in 2D space is a **circle.**

Q5.6: Define convex optimization and give an example of a convex optimization problem.

Answer : Convex optimization is a subfield of optimization that deals with problems where the objective function is convex, and the feasible region, as defined by the constraints, is a convex set.

A well known convex optimization problem is Quadratic Programming **(QP)**.

Q5.7: Discuss the tradeoffs between closed-form and iterative optimization techniques.

Precision vs. Approximation: Closed-form solutions provide exact answers, whereas iterative methods offer approximate solutions. In cases where exact precision is crucial, closed-form solutions are preferred. For large-scale or complex problems, iterative methods are often more practical despite their approximate nature.

Speed vs. Scalability: Closed-form solutions are typically faster for small problems but may become impractical for large problems. Iterative methods, although slower per iteration, can efficiently handle larger problems and datasets.

Theory vs. Practice: Closed-form solutions require potentially complex derivations but are simple to use once derived. Iterative methods may be simpler to derive (or already well-established) but can be complex to implement and require careful tuning.

[2 marks] Q5.8: Discuss the concept of global versus local optima in optimization problems and how convexity relates to ease of finding the optimal solution.

A global optimum is the best possible solution among all feasible solutions in the entire search space. It represents the lowest possible value for a minimization problem or the highest possible value for a maximization problem. Finding the global optimum ensures that the solution is optimal regardless of where the optimization process starts from.

A local optimum, on the other hand, is a solution that is the best among a specific neighborhood of feasible solutions but may not be the best overall. In other words, it is a point where the objective function cannot be improved by making small changes to the solution within a limited region of the search space.

Convexity simplifies the optimization process by ensuring that any local minimum is also a global minimum, allowing for the use of efficient algorithms that converge reliably to the optimal solution. This relationship between convexity and ease of optimization makes convex optimization a powerful tool.

Question 6: Python and Algorithms

[2 marks] Q6.1: Write a Python function that checks whether a passed string is a palindrome or not. What is its time complexity?

```python
def isPalindrome(a):
    reduced = ''.join(char.lower() for char in a if char.isalnum())
    return reduced == reduced[::-1]
```

Time Complexity : **O(n)**

Q6.2: Write a Python function to multiply two matrices. What is its time complexity?

```python
def matrix_multiplier(A, B):
    rows_A = len(A)
    rows_B = len(B)

    cols_A = len(A[0])
    cols_B = len(B[0])

    # Check if the matrices can be multiplied
    if cols_A != rows_B:
        raise ValueError("Number of columns in A must be equal to number of rows in B")

    # Initialize the result matrix with zeros
    result = [[0 for _ in range(cols_B)] for _ in range(rows_A)]

    # Matrix multiplication
    for i in range(rows_A):
        for j in range(cols_B):
            for k in range(cols_A):
                result[i][j] += A[i][k] * B[k][j]

    return result
```

Time Complexity : **O(n.m.p) => where:**

n - rows in matrix A,
m - rows in matrix B,
p - columns in matrix B

[2 marks] Q6.3: What is the average-case time complexity of linear search in an unsorted array of n elements?

Answer : Average-case Time Complexity : **O(n)**

[2 marks] Q6.4: What is the worst-case time complexity of inserting an element into a hash table with separate chaining collision resolution?

Answer : Worst-case Time Complexity : **O(n)**

Q6.5: What is the space complexity of the factorial function implemented recursively?

Answer : Space Complexity : **O(n)**