Hàm ghép trơn

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BÀI TOÁN

- Cho bộ điểm $\left(x_i, y_i \right)_{i=\overline{1,n}}$

- Tìm hàm ghép

$$S(x) = S_k(x), x \in \left[x_k, x_{k+1} \right]$$

- thoả mãn $S_k(x)$ là đa thức bậc p, $S(x)$ trơn đến cấp $p - 1$.

Spline tuyến tính

- Cho bộ điểm $\left(x_i, y_i \right)_{i=\overline{1,n}}$

- Tìm hàm ghép

$$S(x) = S_k(x), x \in \left[x_k, x_{k+1} \right]$$

-
$$\deg S_k(x) = 1$$
$$S_k\left(x_{k+1} \right) = y_{k+1} = S_{k+1}\left(x_{k+1} \right)$$
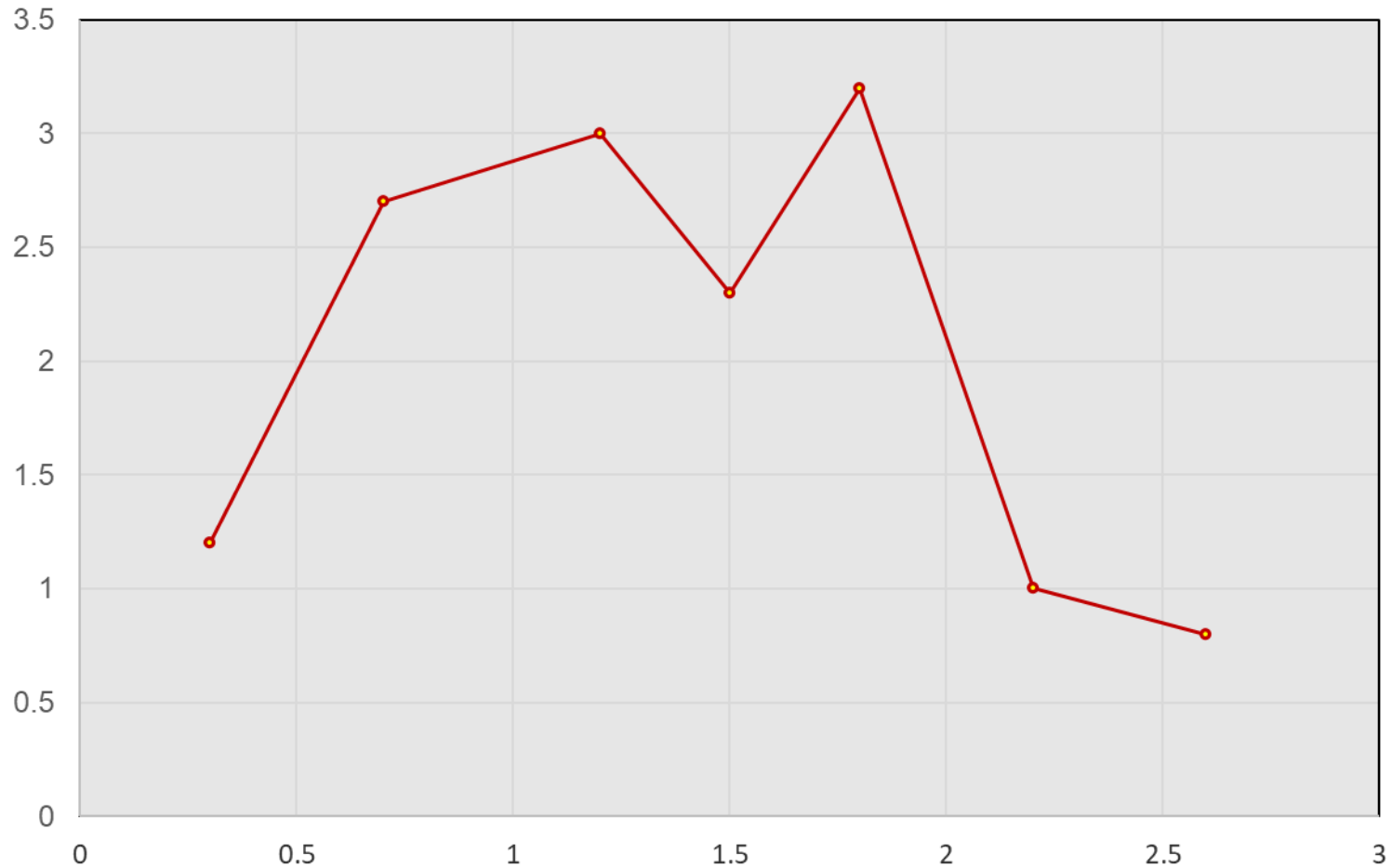
Spline tuyến tính

- $S_k(x), x \in [x_k, x_{k+1}]$ là đa thức nội suy bậc 1 với hai mốc nội suy:

$$(x_k, y_k), \ (x_{k+1}, y_{k+1})$$

$$S_k(x) = \frac{-y_k(x - x_{k+1}) + y_{k+1}(x - x_k)}{h_k} = a_k x + b_k$$

$$a_k = \frac{y_{k+1} - y_k}{h_k}; \qquad b_k = \frac{y_k x_{k+1} - y_{k+1} x_k}{h_k}$$

Spline tuyến tính

Spline cấp 2

- Tìm hàm ghép $\quad S(x) = S_k(x), x \in [x_k, x_{k+1}]$

$$\deg S_k(x) = 2$$

$$S_k(x_{k+1}) = y_{k+1} = S_{k+1}(x_{k+1})$$

$$S_k^{'}(x_{k+1}) = S_{k+1}^{'}(x_{k+1})$$

Spline cấp 2

- Đặt $m_{k+1} = S_k'(x_{k+1}) = S_{k+1}'(x_{k+1})$
- Khi đó

$$S_k'(x) = \frac{-m_k(x - x_{k+1}) + m_{k+1}(x - x_k)}{h_k}$$

$$S_k(x) = \frac{-m_k(x - x_{k+1})^2 + m_{k+1}(x - x_k)^2}{2h_k} + \theta_k$$

$$S_k(x_k) = \frac{-m_k h_k}{2} + \theta_k = y_k \Leftrightarrow \theta_k = y_k + \frac{m_k h_k}{2}$$

$$S_k(x_{k+1}) = y_{k+1} \Leftrightarrow \boxed{m_{k+1} + m_k = 2\frac{y_{k+1} - y_k}{h_k} = \gamma_1}$$

Spline cấp 2

- Ta thu được hệ n-1 phương trình, n ẩn

$$
\begin{bmatrix}
1 & 1 & 0 & & & & & & \\
0 & 1 & 1 & 0 & & & & & \\
& 0 & 1 & 1 & 0 & & & & \\
& & 0 & 1 & 1 & 0 & & & \\
& & & \ddots & \ddots & \ddots & \ddots & & \\
& & & & 0 & 1 & 1 & 0 & \\
& & & & & 0 & 1 & 1 &
\end{bmatrix}
\begin{vmatrix}
\gamma_1 \\
\gamma_2 \\
\gamma_3 \\
\gamma_4 \\
\vdots \\
\gamma_{n-2} \\
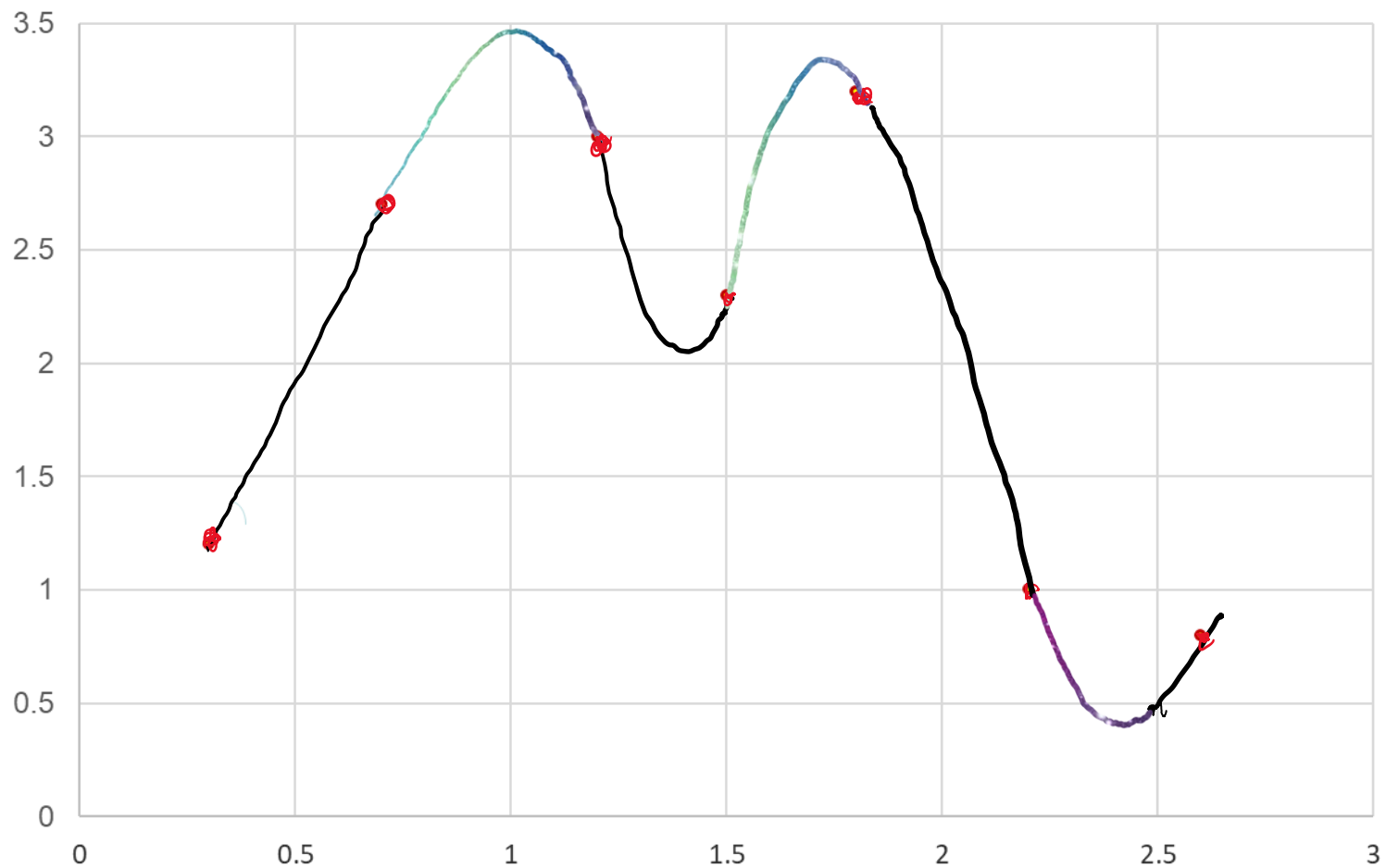\gamma_{n-1}
\end{vmatrix}
$$

Spline cấp 2

- $S_k(x) = a_k x^2 + b_k x + c_k$

$$a_k = \frac{m_{k+1} - m_k}{2h_k};$$

$$b_k = \frac{m_k x_{k+1} - m_{k+1} x_k}{h_k};$$

$$c_k = \frac{-m_k x_{k+1}^2 + m_{k+1} x_k^2}{2h_k} + y_k + \frac{m_k h_k}{2}$$

Spline tuyến tính

Spline cấp 3

- Tìm hàm ghép $\quad S(x) = S_k(x), x \in [x_k, x_{k+1}]$

$$\deg S_k(x) = 3$$

$$S_k(x_{k+1}) = y_{k+1} = S_{k+1}(x_{k+1})$$

$$S_k^{'}(x_{k+1}) = S_{k+1}^{'}(x_{k+1})$$

$$S_k^{''}(x_{k+1}) = S_{k+1}^{''}(x_{k+1})$$

Spline cấp 3

- Đặt $\qquad \alpha_k = S_k^{''}(x_{k+1}) = S_{k+1}^{''}(x_{k+1})$

- Khi đó

$$S_k^{''}(x) = \frac{-\alpha_k(x - x_{k+1}) + \alpha_{k+1}(x - x_k)}{h_k}$$

$$S_k^{'}(x) = \frac{-\alpha_k(x - x_{k+1})^2 + \alpha_{k+1}(x - x_k)^2}{2h_k} + \beta_k$$

$$S_k(x) = \frac{-\alpha_k(x - x_{k+1})^3 + \alpha_{k+1}(x - x_k)^3}{6h_k} + \beta_k x + \theta_k$$

Spline cấp 3

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$$S_k(x) = \frac{-\alpha_k(x - x_{k+1})^3 + \alpha_{k+1}(x - x_k)^3}{6h_k} + \overline{\beta}_k(x - x_k) - \overline{\theta}_k(x - x_{k+1})$$

$$S_k(x_k) = \frac{\alpha_k h_k^2}{6} + \overline{\theta}_k h_k = y_k \Leftrightarrow \overline{\theta}_k = \frac{y_k}{h_k} - \frac{\alpha_k h_k}{6}$$

$$S_k(x_{k+1}) = \frac{\alpha_{k+1} h_k^2}{6} + \overline{\beta}_k h_k = y_{k+1} \Leftrightarrow \overline{\beta}_k = \frac{y_{k+1}}{h_k} - \frac{\alpha_{k+1} h_k}{6}$$

Spline cấp 3

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$$S_k^{'}(x) = \frac{-\alpha_k(x - x_{k+1})^2 + \alpha_{k+1}(x - x_k)^2}{2h_k} + \frac{y_{k+1}}{h_k} - \frac{\alpha_{k+1}h_k}{6} - \left(\frac{y_k}{h_k} - \frac{\alpha_k h_k}{6}\right)$$

$$S_{k+1}^{'}(x_{k+1}) = \frac{-\alpha_{k+1}h_{k+1}}{2} + \frac{y_{k+2}}{h_{k+1}} - \frac{\alpha_{k+2}h_{k+1}}{6} - \left(\frac{y_{k+1}}{h_{k+1}} - \frac{\alpha_{k+1}h_{k+1}}{6}\right)$$

$$S_k^{'}(x_{k+1}) = \frac{\alpha_{k+1}h_k}{2} + \frac{y_{k+1}}{h_k} - \frac{\alpha_{k+1}h_k}{6}_k - \left(\frac{y_k}{h_k} - \frac{\alpha_k h_k}{6}\right)$$

Spline cấp 3

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$$S_k^{'}\left(x_{k+1}\right) - S_{k+1}^{'}\left(x_{k+1}\right) = 0$$

$$\Leftrightarrow \frac{h_{k+1}}{6}\alpha_{k+2} + \left(\frac{h_k + h_{k+1}}{3}\right)\alpha_{k+1} + \frac{h_k}{6}\alpha_k = \gamma_k$$

$$\gamma_k = \frac{y_{k+2} - y_{k+1}}{h_{k+1}} - \frac{y_{k+1} - y_k}{h_k}$$

Spline cấp 3

- $$S_k^{'}\left(x_{k+1}\right) - S_{k+1}^{'}\left(x_{k+1}\right) = 0$$

$$\Leftrightarrow \boxed{\frac{h_{k+1}}{6}\alpha_{k+2} + \left(\frac{h_k + h_{k+1}}{3}\right)\alpha_{k+1} + \frac{h_k}{6}\alpha_k = \gamma_k}$$

$$\gamma_k = \frac{y_{k+2} - y_{k+1}}{h_{k+1}} - \frac{y_{k+1} - y_k}{h_k}$$

Spline cấp 3

Hệ n-2 phương trình, n ẩn

$$
\begin{bmatrix}
\frac{h_1}{6} & \frac{h_1+h_2}{3} & \frac{h_2}{6} & 0 & & & & & & \\
0 & \frac{h_2}{6} & \frac{h_2+h_3}{3} & \frac{h_3}{6} & 0 & & & & & \\
& 0 & \frac{h_3}{6} & \frac{h_3+h_4}{3} & \frac{h_4}{6} & 0 & & & & \\
& & \ddots & \ddots & \ddots & \ddots & \ddots & & & \\
& & & \ddots & \ddots & \ddots & \ddots & \ddots & & \\
& & & 0 & \frac{h_{n-3}}{6} & \frac{h_{n-3}+h_{n-2}}{3} & \frac{h_{n-2}}{6} & 0 & \\
& & & & 0 & \frac{h_{n-2}}{6} & \frac{h_{n-2}+h_{n-1}}{3} & \frac{h_{n-1}}{6} &
\end{bmatrix}
\begin{vmatrix}
\gamma_1 \\
\gamma_2 \\
\gamma_3 \\
\vdots \\
\vdots \\
\gamma_{n-3} \\
\gamma_{n-2}
\end{vmatrix}
$$

Spline cấp 3

$$S_k \bullet (x) = a_k x^3 + b_k x^2 + c_k x + d_k$$

$$a_k = \frac{\alpha_{k+1} - \alpha_k}{6h_k};$$

$$b_k = \frac{3\alpha_k x_{k+1} - 3\alpha_{k+1} x_k}{6h_k};$$

$$c_k = \frac{-3\alpha_k x_{k+1}^2 + 3\alpha_{k+1} x_k^2}{6h_k} + \frac{y_{k+1} - y_k}{h_k} - \frac{(\alpha_{k+1} - \alpha_k)h_k}{6}$$

$$d_k = \frac{\alpha_k x_{k+1}^3 - \alpha_{k+1} x_k^3}{6h_k} + \frac{y_k x_{k+1} - y_{k+1} x_k}{h_k} + \frac{(\alpha_{k+1} x_k - \alpha_k x_{k+1})h_k}{6}$$

Spline tuyến tính