Các pp Runge – Kutta hiện giải bài toán Cauchy cho phương trình vi phân thường

Bài toán Cauchy

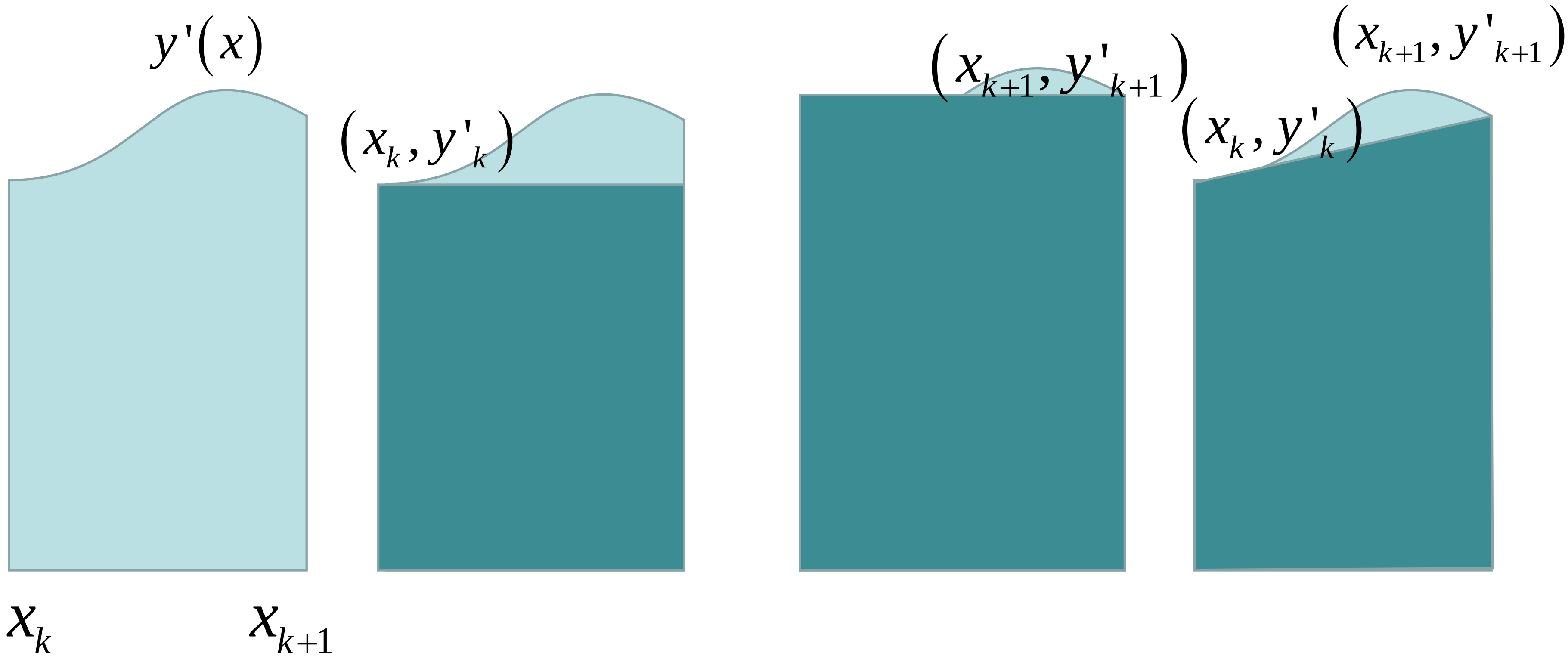
$$\begin{cases} y' = f(x, y), x \in I = \left[x_0, X\right], \\ y \in C^1\left(I, R^k\right) \\ y(x_0) = y_0 \end{cases}$$

Phương trình tích phân

$$y(x) = y(x_0) + \int_{x_0}^{x} f(t, y(t))\,dt$$

$$y(x_{k+1}) = y(x_k) + \int_{x_k}^{x_{k+1}} f(t, y(t))\,dt$$

Ý nghĩa hình học của các CT



Euler hiện

Euler ẩn

Hình thang

- Euler forward (hiện)

$$y_{n+1} = y_n + hf(x_n, y_n)$$

- Euler backward (ẩn)

$$y_{n+1} = y_n + hf\left(x_{n+1}, y_{n+1}\right)$$

- Công thức hình thang

$$y_{n+1} = y_n + \frac{h}{2}\left(f\left(x_n, y_n\right) + f\left(x_{n+1}, y_{n+1}\right)\right)$$

R-K làm gì?

- Tính tích phân trong phương trình tích phân qua s nấc trung gian

- Đảm bảo việc tính thông qua các nấc trung gian có hiệu quả giống như khai triển Taylor hàm y(x) đến bậc cao

Công thức R-K tổng quát

$$y_{n+1} = y_n + r_1 k_1^{(n)} + r_2 k_2^{(n)} + \ldots + r_s k_s^{(n)}$$

$$k_i^{(n)} = hf\left(x_n + \alpha_i h, y_n + \beta_{i-1,1} k_1^{(n)} + \ldots + \beta_{i-1,i-1} k_{i-1}^{(n)}\right)$$

$$\alpha_1 = 0, \alpha_i \in [0,1]$$

R-K 1 nấc

$$s = 1$$

$$y_{n+1} = y_n + r_1 k_1^{(n)}$$

$$k_1^{(n)} = hf\left(x_n, y_n\right)$$

$$y\left(x_{n+1}\right) = y\left(x_n\right) + hy'\left(x_n\right) + O\left(h^2\right)$$

$$\Rightarrow r_1 = 1$$

R-K 2 nấc

$$s = 2$$

$$y_{n+1} = y_n + r_1 k_1^{(n)} + r_2 k_2^{(n)}$$

$$k_1^{(n)} = hf\left(x_n, y_n\right) = hf_n$$

$$k_2^{(n)} = hf\left(x_n + \alpha_2 h, y_n + \beta_{11} k_1^{(n)}\right)$$

$$\Rightarrow k_2^{(n)} = h\left[f_n + \alpha_2 h f_{x,n}^{'} + \beta_{11} k_1^{(n)} f_{y,n}^{'} + O(h^2) \right]$$

$$y\left(x_{n+1}\right) = y\left(x_n\right) + hf_n + \frac{h^2}{2}\left[f_{x,n}^{'} + f_{y,n}^{'}.f_n \right] + O\left(h^3\right)$$

R-K 2 nấc

$$r_1 + r_2 = 1; r_2\alpha_2 = \frac{1}{2}; r_2\beta_{11} = \frac{1}{2}$$

$$r_1 = 0; r_2 = 1; \alpha_2 = \frac{1}{2}; \beta_{11} = \frac{1}{2}$$

$$r_1 = r_2 = \frac{1}{2}; \alpha_2 = \beta_{11} = 1$$

$$r_1 = \frac{1}{3}; r_2 = \frac{2}{3}; \alpha_2 = \beta_{11} = \frac{3}{4}$$

.....

R-K 3 nấc

$$y_{n+1} = y_n + r_1 k_1^{(n)} + r_2 k_2^{(n)} + r_3 k_3^{(n)}$$

$$k_1^{(n)} = hf\left(x_n, y_n\right)$$

$$k_2^{(n)} = hf\left(x_n + \alpha_2 h, y_n + \beta_{11} k_1^{(n)}\right)$$

$$k_3^{(n)} = hf\left(x_n + \alpha_3 h, y_n + \beta_{21} k_1^{(n)} + \beta_{22} k_2^{(n)}\right)$$

R-K 3 nấc

$$k_2^{(n)} = h\begin{bmatrix} f_n + \alpha_2 h f'_{x,n} + \beta_{11} h f_n f'_{y,n} + \\ + \frac{h^2}{2}\alpha_2^2 f''_{x,n} + h^2 \alpha_2 \beta_{11} f_n f''_{xy,n} + \frac{h^2}{2}\beta_{11}^2 f_n^2 f''_{y,n} + O\left(h^3\right) \end{bmatrix}$$

$$k_3^{(n)} = h\begin{bmatrix} f_n + \alpha_3 h f'_{x,n} + \left(\beta_{21} k_1^{(n)} + \beta_{22} k_2^{(n)}\right) f'_{y,n} + \frac{h^2}{2}\alpha_3^2 f''_{x,n} + \\ + \alpha_3 h\left(\beta_{21} k_1^{(n)} + \beta_{22} k_2^{(n)}\right) f''_{xy,n} + \frac{1}{2}\left(\beta_{21} k_1^{(n)} + \beta_{22} k_2^{(n)}\right)^2 f''_{y,n} + O\left(h^3\right) \end{bmatrix}$$

$$y\left(x_{n+1}\right) = y\left(x_n\right) + h f_n + \frac{h^2}{2}\left[f'_{x,n} + f'_{y,n}.f_n \right]$$

$$+ \frac{h^3}{6}\left[f''_{xx} + 2 f''_{xy} f_n + f''_{yy} f_n^2 + f'_y f'_x + f_y'^2 f_n \right] + O\left(h^4\right)$$

$$r_1 + r_2 + r_3 = 1$$

$$r_2\alpha_2 + r_3\alpha_3 = \frac{1}{2}$$

$$r_2\beta_{11} + r_3\left(\beta_{21} + \beta_{22}\right) = \frac{1}{2}$$

$$\frac{1}{2}r_2\alpha_2^2 + \frac{1}{2}r_3\alpha_3^2 = \frac{1}{6}$$

$$r_2\alpha_2\beta_{11} + r_3\alpha_3\left(\beta_{21} + \beta_{22}\right) = \frac{1}{3}$$

$$r_2\beta_{11}^2 + r_3\left(\beta_{21} + \beta_{22}\right)^2 = \frac{1}{3}$$

$$r_3\beta_{22}\alpha_2 = \frac{1}{6}$$

$$r_3\beta_{11}\beta_{22} = \frac{1}{6}$$

R-K3 thường dùng

$$r_1 = \frac{1}{6}; r_2 = \frac{2}{3}; r_3 = \frac{1}{6}; \alpha_2 = \frac{1}{2}; \alpha_3 = 1; \beta_{11} = \frac{1}{2}; \beta_{21} = -1; \beta_{22} = 2$$

$$y_{n+1} = y_n + \frac{1}{6}\left(k_1^{(n)} + 4k_2^{(n)} + k_3^{(n)}\right)$$

$$k_1^{(n)} = hf\left(x_n, y_n\right)$$

$$k_2^{(n)} = hf\left(x_n + \frac{1}{2}h, y_n + \frac{1}{2}k_1^{(n)}\right)$$

$$k_3^{(n)} = hf\left(x_n + h, y_n - k_1^{(n)} + 2k_2^{(n)}\right)$$

R-K3 thường dùng (Heun)

$$r_1 = \frac{1}{4}; r_2 = 0; r_3 = \frac{3}{4}; \alpha_2 = \beta_{11} = \frac{1}{3}; \alpha_3 = \beta_{22} = \frac{2}{3}; \beta_{21} = 0$$

$$y_{n+1} = y_n + \frac{1}{4}\left(k_1^{(n)} + 3k_3^{(n)}\right)$$

$$k_1^{(n)} = hf\left(x_n, y_n\right)$$

$$k_2^{(n)} = hf\left(x_n + \frac{1}{3}h, y_n + \frac{1}{3}k_1^{(n)}\right)$$

$$k_3^{(n)} = hf\left(x_n + \frac{2}{3}h, y_n + \frac{2}{3}k_2^{(n)}\right)$$

R-K 4 thường dùng

$$y_{n+1} = y_n + \frac{1}{6}\left(k_1^{(n)} + 2k_2^{(n)} + 2k_3^{(n)} + k_4^{(n)}\right)$$

$$k_1^{(n)} = hf\left(x_n, y_n\right)$$

$$k_2^{(n)} = hf\left(x_n + \frac{1}{2}h, y_n + \frac{1}{2}k_1^{(n)}\right)$$

$$k_3^{(n)} = hf\left(x_n + \frac{1}{2}h, y_n + \frac{1}{2}k_2^{(n)}\right)$$

$$k_4^{(n)} = hf\left(x_n + h, y_n + k_3^{(n)}\right)$$

$$
\begin{aligned}
k_1 &= hf(t_k, y_k),\\
k_2 &= hf\left(t_k + \frac{h}{3}, y_k + \frac{k_1}{3}\right),\\
k_3 &= hf\left(t_k + \frac{2h}{3}, y_k - \frac{k_1}{3} + k_2\right),\\
k_4 &= hf\left(t_k + h, y_k + k_1 - k_2 + k_3\right),\\
y_{k+1} &= y_k + \frac{1}{8}(k_1 + 3k_2 + 3k_3 + k_4).
\end{aligned}
$$

Bậc cao nhất của các công thức R_K s nấc

s	1	2	3	4	5	6	7	8	9
p	1	2	3	4	4	5	6	6	7

Ví dụ mô hình hệ thú mồi

$$\begin{cases} N' = rN\left(1 - \dfrac{N}{K}\right) - aNP \\ P' = -\mu P + aNP \end{cases}$$