SIGNALS AND SYSTEMS

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SOLUTION OF EXERCISE 2

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۱-سیگنال‌های زیر پاسخ ضربه‌های سیستم‌های LTI هستند. آیا این سیستم‌ها پایدار و علی هستند؟ (با راه حل)

A) $h[n] = \left(\frac{1}{5}\right)^n u[n]$

سیستم پایدار است زیرا مطلقا جمع پذیر نمی‌باشد:

$$\sum_{k=-\infty}^{\infty} |h[k]| = \sum_{k=-\infty}^{\infty} \left(\frac{1}{5}\right)^k u[k] = \sum_{k=0}^{\infty} \left(\frac{1}{5}\right)^k = \frac{1}{1-\frac{1}{5}} = \frac{5}{4} < \infty$$

سیستم علی است زیرا:

$\forall n < 0 \ \rightarrow h[n] = 0$

B) h[n] = $(0.8)^n\, u[n+2]$

سیستم پایدار است:

$$\sum_{k=-\infty}^{\infty} |h[k]| = \sum_{k=-\infty}^{\infty} \left|0.8^k\, u[k+2]\right| = \sum_{k=-2}^{\infty} 0.8^k = \sum_{k=0}^{\infty} 0.8^{k-2} = (0.8)^{-2} \sum_{k=0}^{\infty} 0.8^k$$

$$= \frac{(0.8)^{-2}}{1-0.8} \ < \ \infty$$

سیستم علی نیست زیرا به عنوان مثال:

$n = -2 < 0 \ \rightarrow h[-2] = (0.8)^{-2} \neq 0$

C) $h[n] = \left(-\frac{1}{2}\right)^n u[n] + (1.01)^n u[n-1]$

$$\sum_{k=-\infty}^{\infty} |h[k]| = \sum_{k=-\infty}^{\infty} \left|\left(-\frac{1}{2}\right)^k u[k] \ + \ (1.01)^k\, u[k-1]\right| = \frac{1}{2} + \sum_{k=1}^{\infty} \left|\left(-\frac{1}{2}\right)^k \ + \ (1.01)^k\right|$$

$$\frac{1}{2} + \sum_{k=0}^{\infty} \left| (1.01)^{2k+1} \ - \left(\frac{1}{2}\right)^{2k+1} \right| + \sum_{k=1}^{\infty} \left| (1.01)^{2k} \ + \left(\frac{1}{2}\right)^{2k} \right|$$

$$= \frac{1}{2} + \sum_{k=0}^{\infty} \left((1.01)^{2k+1} \ - \left(\frac{1}{2}\right)^{2k+1} \right) + \sum_{k=1}^{\infty} \left(\frac{1}{2}\right)^{2k} + \sum_{k=1}^{\infty} (1.01)^{2k} \ \rightarrow \infty$$

پس سیستم ناپایدار است

سیستم علی است زیرا $\forall n < 0 \rightarrow h[n] = 0$

D) $h(t) = e^{-6|t|}$

$$\int_{-\infty}^{+\infty} |h(\tau)| d\tau = \int_{-\infty}^{+\infty} \left|e^{-6|\tau|}\right| d\tau = \int_{-\infty}^{+\infty} e^{-6|\tau|}\, d\tau = \int_{-\infty}^{0} e^{-6|\tau|}\, d\tau \; + \int_{0}^{+\infty} e^{-6|\tau|}\, d\tau$$

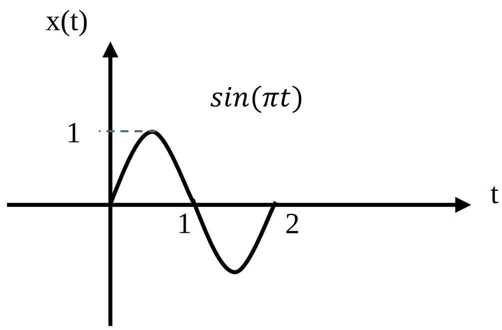
$$= \int_{0}^{+\infty} e^{-6\tau}\, d\tau \; + \int_{0}^{+\infty} e^{-6\tau}\, d\tau = 2\int_{0}^{+\infty} e^{-6\tau}\, d\tau = \frac{1}{3} < \infty$$

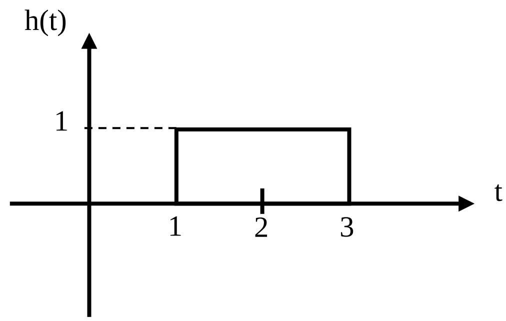
پس سیستم پایدار است.

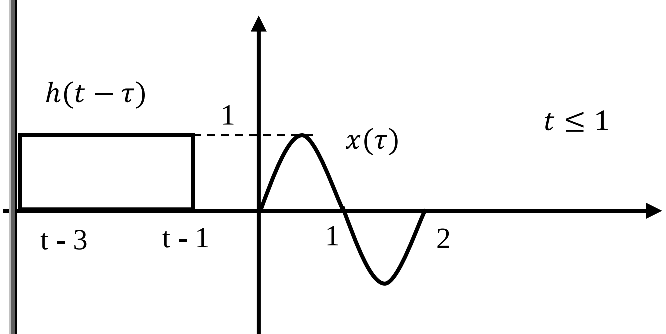
سیستم علی نیست زیرا: $\exists t < 0 \rightarrow h(t) \neq 0$

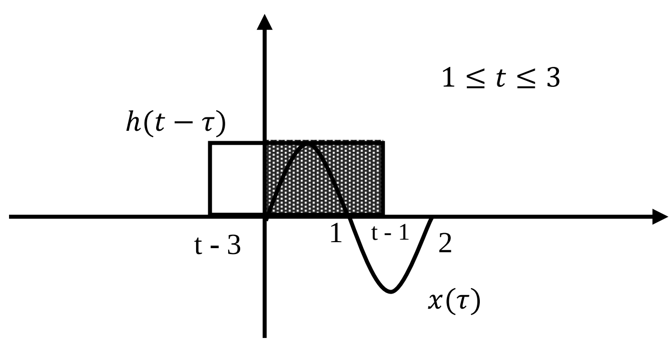
۲- حاصل کانولوشن‌های زیر را بدست آورید.

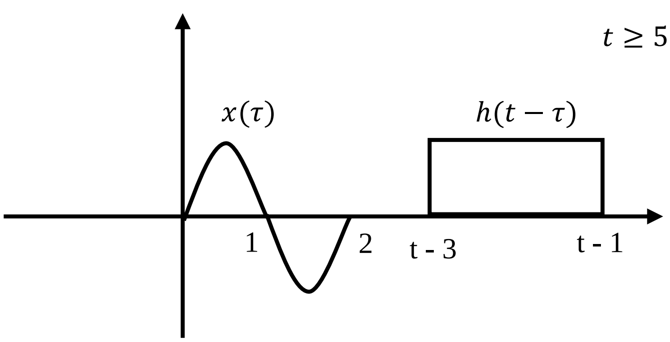
A)





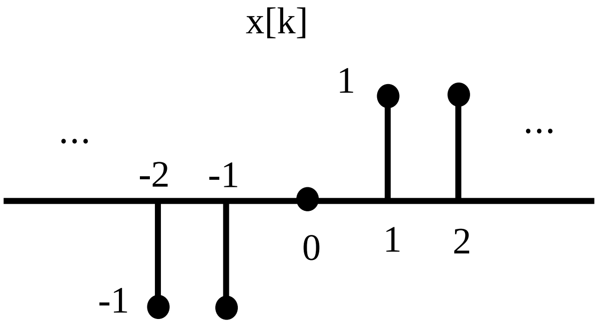


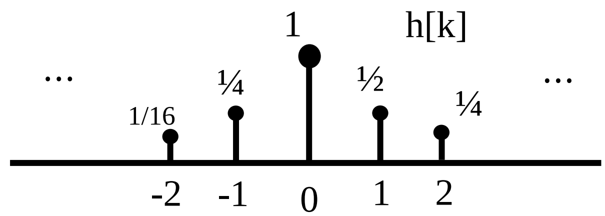


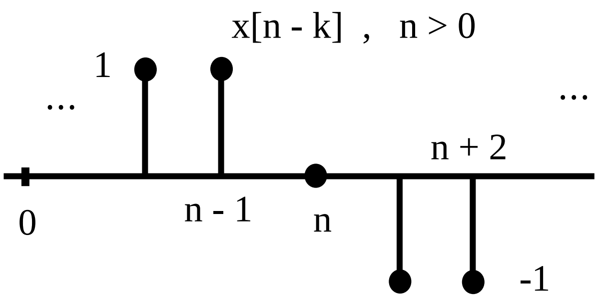


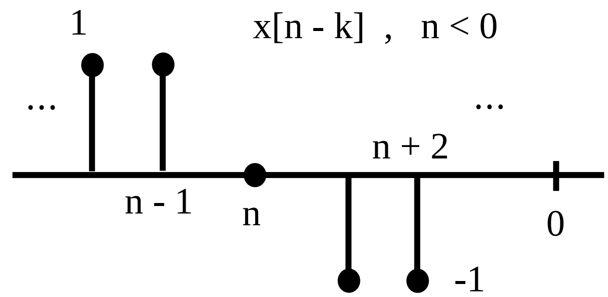
$$y(t) = \begin{cases} \int_{0}^{t-1} x(\tau)d\tau = \int_{0}^{t-1} sin(\pi\tau)\, d\tau = \dfrac{1}{\pi}(1 - cos(\pi(t-1))) \ ; \ 1 \le t \le 3 \\ \int_{t-3}^{2} x(\tau)d\tau = \int_{t-3}^{2} sin(\pi\tau)\, d\tau = \dfrac{1}{\pi}(cos(\pi(t-3)) - 1) \ ; \quad 3 \le t \le 5 \\ 0 \qquad\qquad\qquad\qquad\qquad\qquad ; \ elsewhere \end{cases}$$

B) x[n] = u[n] – u[-n] , $h[n] = \begin{cases} \left(\frac{1}{2}\right)^n & n \ge 0 \\ 4^n & n < 0 \end{cases}$









For n ≥ 0 :

$$y[n] = \sum_{k=-\infty}^{+\infty} x[n-k]\, h[k] = \sum_{k=-\infty}^{-1} 4^k + \sum_{k=0}^{n-1} \left(\frac{1}{2}\right)^k - \sum_{k=n+1}^{\infty} \left(\frac{1}{2}\right)^k$$

for n < 0:

$$y[n] = \sum_{k=-\infty}^{+\infty} x[n-k]\, h[k] = \sum_{k=-\infty}^{n-1} 4^k - \sum_{k=n+1}^{-1} (4)^k - \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k$$

$$y[n] = \begin{cases} \sum_{k=1}^{\infty}\left(\frac{1}{4}\right)^k + \sum_{k=0}^{n-1}\left(\frac{1}{2}\right)^k - \sum_{k=n+1}^{\infty}\left(\frac{1}{2}\right)^k & n \geq 0 \\ -\sum_{k=0}^{\infty}\left(\frac{1}{2}\right)^k - \sum_{k=n+1}^{-1} 4^k + \sum_{k=-\infty}^{n-1} 4^k & n < 0 \end{cases}$$

$$\sum_{k=0}^{n-1}\left(\frac{1}{2}\right)^k = \frac{1-\left(\frac{1}{2}\right)^n}{1-\frac{1}{2}} = 2\left[1-\left(\frac{1}{2}\right)^n\right]$$

$$\sum_{k=-\infty}^{-1} 4^k = \sum_{k=1}^{\infty}\left(\frac{1}{4}\right)^k = -1 + \sum_{k=0}^{\infty}\left(\frac{1}{4}\right)^k = -1 + \frac{1}{1-\frac{1}{4}} = -1 + \frac{1}{\frac{3}{4}} = \frac{1}{3}$$

$$\sum_{k=n+1}^{\infty}\left(\frac{1}{2}\right)^k = \sum_{k=0}^{\infty}\left(\frac{1}{2}\right)^k - \sum_{k=0}^{n}\left(\frac{1}{2}\right)^k = \frac{1}{1-\frac{1}{2}} - \frac{1-\left(\frac{1}{2}\right)^{n+1}}{1-\frac{1}{2}}$$

$$= 2 - 2\left[1-\left(\frac{1}{2}\right)^{n+1}\right] = 2\left(\frac{1}{2}\right)^{n+1} = 2^{-n}$$

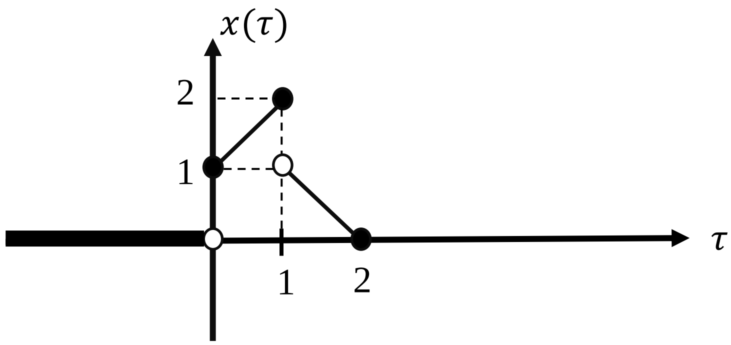
$$\sum_{k=0}^{\infty}\left(\frac{1}{2}\right)^k = \frac{1}{1-\frac{1}{2}} = 2$$

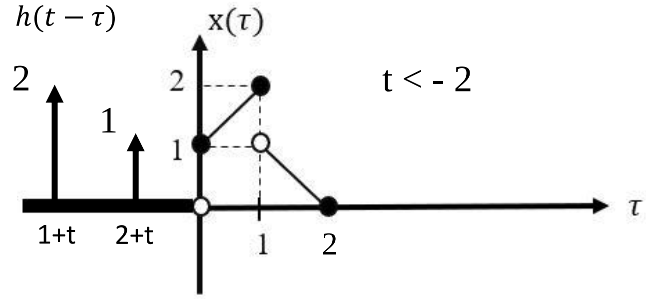
$$\sum_{k=-1}^{n+1} 4^k = \sum_{k=1}^{-n-1}\left(\frac{1}{4}\right)^k = \left(\sum_{k=0}^{-n-1}\left(\frac{1}{4}\right)^k\right) - 1 = -1 + \frac{1-\left(\frac{1}{4}\right)^{-n}}{1-\frac{1}{4}} = -1 + \frac{4}{3}[1-4^n]$$

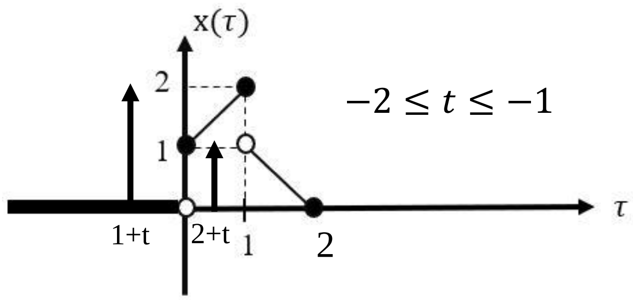
$$\sum_{k=-\infty}^{n-1} 4^k = \sum_{k=-n+1}^{\infty}\left(\frac{1}{4}\right)^k = \sum_{k=0}^{\infty}\left(\frac{1}{4}\right)^k - \sum_{k=0}^{-n}\left(\frac{1}{4}\right)^k = \frac{1}{1-\frac{1}{4}} - \frac{1-\left(\frac{1}{4}\right)^{-n+1}}{1-\frac{1}{4}}$$

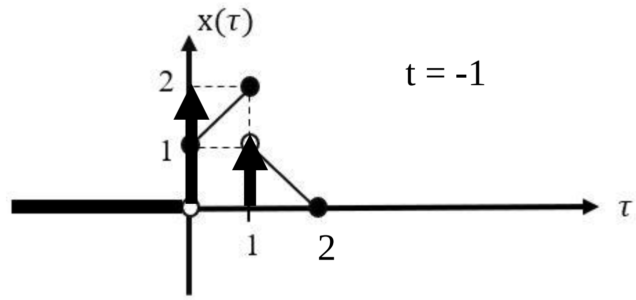
$$= \frac{4}{3}\left(\frac{1}{4}\right)^{-n+1} = \frac{1}{4}4^n$$

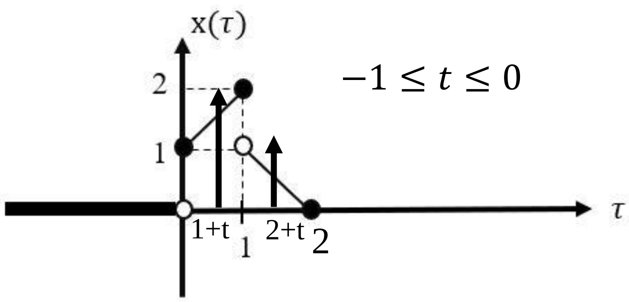
C) $x(t) = \begin{cases} t+1 & ; \quad 0 \le t \le 1 \\ 2-t & ; \quad 1 \le t \le 2 \\ 0 & ; \quad O.w \end{cases}$, $\quad h(t) = \delta(t+2) + 2\delta(t+1)$

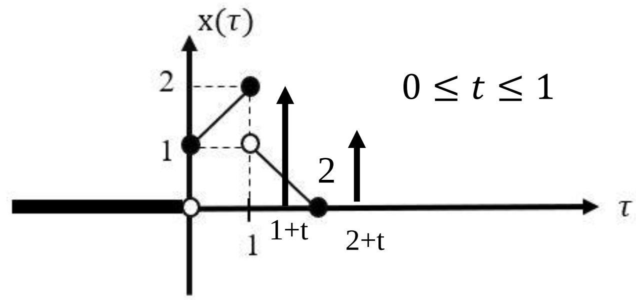












$$t < -2 \rightarrow x(\tau)h(t-\tau) = 0 \rightarrow y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\, d\tau = 0 \quad , \quad t < -2$$

$$-2 \le t \le -1 \rightarrow y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\, d\tau = \int_{-\infty}^{\infty} x(\tau)\left(\delta\big((2+t)-\tau\big)\right) d\tau$$

$$= \int_{-\infty}^{\infty} x(2+t)\left(\delta\big((2+t)-\tau\big)\right) d\tau = x(t+2)\int_{-\infty}^{\infty}\left(\delta\big((2+t)-\tau\big)\right) d\tau = x(t+2)$$

$$\rightarrow y(t) = x(t+2) = (2+t)+1 = t+3 \qquad , \qquad -2 \le t \le -1$$

$$t = -1 \rightarrow x(\tau)h(t-\tau) = x(\tau)\{2\delta(-\tau) + \delta(1-\tau)\} = 2x(0)\delta(-\tau) + x(1)\delta(1-\tau)$$

$$= 2\delta(-\tau) + 2\delta(1-\tau)$$

$$\rightarrow y(-1) = \int_{-\infty}^{\infty} \{2\delta(-\tau) + 2\delta(1-\tau)\} d\tau = 2 + 2 = 4 \quad , \quad t = -1$$

$$-1 \le t \le 0 \rightarrow x(\tau)h(t-\tau) = x(\tau)\{\delta\big((2+t)-\tau\big) + 2\delta\big((1+t)-\tau\big)\}$$

$$= (2-(2+t))\delta\big((2+t)-\tau\big) + 2(1+(1+t))\delta\big((1+t)-\tau\big)$$

$$\rightarrow y(t) = -t\int_{-\infty}^{\infty}\delta\big((2+t)-\tau\big)d\tau + 2(2+t)\int_{-\infty}^{\infty}\delta\big((1+t)-\tau\big)d\tau$$

$$= -t + (4+2t) = 4+t \quad , \quad -1 \le t \le 0$$

$$0 \le t \le 1 \rightarrow y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau = \int_{-\infty}^{\infty}(2-\tau)\Big(2\delta\big((1+t)-\tau\big)\Big)\,d\tau$$

$$= 2\int_{-\infty}^{\infty}(2-(1+t))\delta\big((1+t)-\tau\big)d\tau = 2(1-t)\int_{-\infty}^{\infty}\delta\big((1+t)-\tau\big)d\tau$$

$$= 2(1-t) = 2-2t \quad , \quad 0 \le t \le 1$$

$$for\ t > 1 \ \rightarrow y(t) \ = \ 0$$

$$\rightarrow y(t) = \begin{cases} t+3 & ; \quad -2 \le t \le -1 \\ 4 & ; \quad t = -1 \\ t+4 & ; \quad -1 \le t \le 0 \\ 2-2t & ; \quad 0 \le t \le 1 \\ 0 & ; \quad elsewhere \end{cases}$$

D) $x[n] = 3^n u[-n-1] + \left(\frac{1}{3}\right)^n u[n]$, $\mathrm{h[n]} = \left(\frac{1}{4}\right)^n u[n+3]$ **(امتیازی)**

$$x[n] = \begin{cases} 3^n & ;\ n \le -1 \\ \left(\dfrac{1}{3}\right)^n & ;\ \ n \ge 0 \end{cases}$$

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = \sum_{k=-\infty}^{-1} 3^k h[n-k] + \sum_{k=0}^{\infty}\left(\frac{1}{3}\right)^k h[n-k]$$

$$= \sum_{k=-\infty}^{-1} 3^k \left(\frac{1}{4}\right)^{n-k} u[n-k+3] + \sum_{k=0}^{+\infty}\left(\frac{1}{3}\right)^k \left(\frac{1}{4}\right)^{n-k} u[n-k+3]$$

$$= \left(\frac{1}{4}\right)^n \sum_{k=-\infty}^{-1} 12^k\, u[-k+n+3] + \left(\frac{1}{4}\right)^n \sum_{k=0}^{+\infty}\left(\frac{4}{3}\right)^k u[-k+n+3]$$

دو حالت را در نظر میگیریم:

حالت اول: $n \ge -3$

$$y[n]=\left(\frac{1}{4}\right)^n\sum_{k=-\infty}^{-1}12^k-\left(\frac{1}{4}\right)^n\sum_{k=0}^{n+3}\left(\frac{4}{3}\right)^k=\left(\frac{1}{4}\right)^n\left[\sum_{k=-\infty}^{0}12^{k-1}+\sum_{k=0}^{n+3}\left(\frac{4}{3}\right)^k\right]$$

$$=\left(\frac{1}{4}\right)^n\left[12^{-1}\sum_{k=0}^{\infty}12^{-k}+\sum_{k=0}^{n+3}\left(\frac{4}{3}\right)^k\right]=\left(\frac{1}{4}\right)^n\left[\frac{\frac{1}{12}}{1-\frac{1}{12}}+\frac{1-\left(\frac{4}{3}\right)^{n+3+1}}{1-\frac{4}{3}}\right]$$

$$=\left(\frac{1}{4}\right)^n\left[\frac{1}{11}+3\left[\left(\frac{4}{3}\right)^{n+4}-1\right]\right]\quad ,\ n\geq -3$$

حالت دوم: $n\leq -4$

$$y[n]=\left(\frac{1}{4}\right)^n\sum_{k=-\infty}^{n+3}12^k+0=\left(\frac{1}{4}\right)^n\sum_{k=-\infty}^{0}12^{k+n+3}=12^3\times 3^n\sum_{k=0}^{+\infty}\left(\frac{1}{12}\right)^k$$

$$=12^3\times 3^n\frac{1}{1-\frac{1}{12}}=\frac{12^4\times 3^n}{11}\quad ,\quad n\leq -4$$

۳-یک سیستم LTI علی S با ورودی x(t) و خروجی y(t) را در نظر بگیرید:

$$a_2\frac{d^2y(t)}{d^2t}+a_1\frac{dy(t)}{dt}+a_0y(t)=b_0x(t)+b_1\frac{dx(t)}{dt}+b_2\frac{d^2x(t)}{d^2t}$$

نشان دهید که

$$y(t)=A\int_{-\infty}^{t}y(\tau)d\tau+B\int_{-\infty}^{t}\left(\int_{-\infty}^{\tau}y(\sigma)d\sigma\right)d\tau+Cx(t)+D\int_{-\infty}^{t}x(\tau)d\tau+E\int_{-\infty}^{t}\left(\int_{-\infty}^{\tau}x(\sigma)d\sigma\right)d\tau$$

و ثابت‌های A, B, C, D, E را بر حسب $a_0, a_1, a_2, b_0, b_1, b_2$ بیان کنید. در ادامه در نظر داشته باشید که S را می‌توان حاصل اتصال سری دو سیستم LTI مقابل دانست:

$$S_1: y_1(t)=Cx_1(t)+D\int_{-\infty}^{t}x_1(\tau)d\tau+E\int_{-\infty}^{t}\left(\int_{-\infty}^{\tau}x_1(\sigma)d\sigma\right)d\tau$$

$$S_2: y_2(t)=A\int_{-\infty}^{t}y_2(\tau)d\tau+B\int_{-\infty}^{t}\left(\int_{-\infty}^{\tau}y_2(\sigma)d\sigma\right)d\tau+x_2(t)$$

که در آن به دلیل اتصال سری می‌توان گفت $x_2(t)=y_1(t)$ است. نمودار جعبه‌ای S را به صورت اتصال سری نمودار جعبه‌ای S_1 و S_2 به صورت Direct Form I, II رسم کنید

$$a_2\frac{d^2y(t)}{d^2t}+a_1\frac{dy(t)}{dt}+a_0y(t)=b_0x(t)+b_1\frac{dx(t)}{dt}+b_2\frac{d^2x(t)}{d^2t}$$

$$\Longrightarrow\int_{-\infty}^{t}\left[a_2\frac{d^2y(\tau)}{d^2\tau}+a_1\frac{dy(\tau)}{d\tau}+a_0y(\tau)\right]d\tau=\int_{-\infty}^{t}\left[b_0x(\tau)+b_1\frac{dx(\tau)}{d\tau}+b_2\frac{d^2x(\tau)}{d^2\tau}\right]d\tau$$

$$\to a_2\left[\frac{dy(t)}{dt}-\frac{dy(-\infty)}{dt}\right]+a_1[y(t)-y(-\infty)]+a_0\int_{-\infty}^{t}y(\tau)d\tau$$

$$=b_0\int_{-\infty}^{t}x(\tau)d\tau+b_1[x(t)-x(-\infty)]+b_2\left[\frac{dx(t)}{dt}-\frac{dx(-\infty)}{dt}\right]$$

سیستم علی است بنابراین:

$$y(-\infty)=\frac{dy(-\infty)}{dt}=\cdots=0\quad,\quad x(-\infty)=\frac{dx(-\infty)}{dt}=\cdots=0$$

$$\Longrightarrow a_2\frac{dy(t)}{dt}+a_1y(t)+a_0\int_{-\infty}^{t}y(\tau)d\tau=b_0\int_{-\infty}^{t}x(\tau)d\tau+b_1x(t)+b_2\frac{dx(t)}{dt}$$

$$\xrightarrow{\int .}\int_{-\infty}^{t}\left[a_2\frac{dy(t)}{dt}+a_1y(t)+a_0\int_{-\infty}^{t}y(\tau)d\tau\right]d\tau=\int_{-\infty}^{t}\left[b_0\int_{-\infty}^{t}x(\tau)d\tau+b_1x(t)+b_2\frac{dx(t)}{dt}\right]d\tau$$

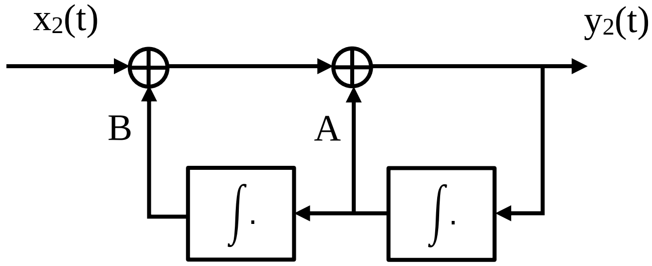
$$\Longrightarrow a_2[y(t)-y(-\infty)]+a_1\int_{-\infty}^{t}y(\tau)d\tau+a_0\int_{-\infty}^{t}\left(\int_{-\infty}^{t}y(\tau)d\tau\right)d\xi$$

$$=b_0\int\left(\int_{-\infty}^{t}x(\tau)d\tau\right)d\xi+b_1\int_{-\infty}^{t}x(\tau)d\tau+b_2[x(t)-x(-\infty)]$$

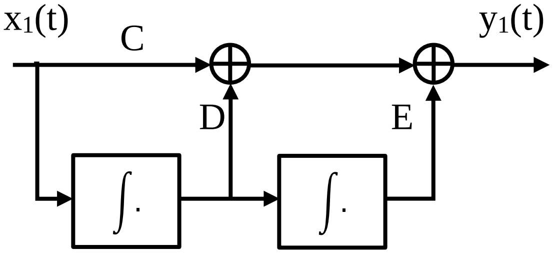
همچنان به دلیل علی بودن داریم:

$$y(t)=-\frac{a_1}{a_2}\int_{-\infty}^{t}y(\tau)d\tau-\frac{a_0}{a_2}\int_{-\infty}^{t}\left(\int_{-\infty}^{t}y(\tau)d\tau\right)d\xi+\frac{b_2}{a_2}x(t)+\frac{b_1}{a_2}\int_{-\infty}^{t}x(\tau)d\tau+\frac{b_0}{a_2}\int\left(\int_{-\infty}^{t}x(\tau)d\tau\right)d\xi$$

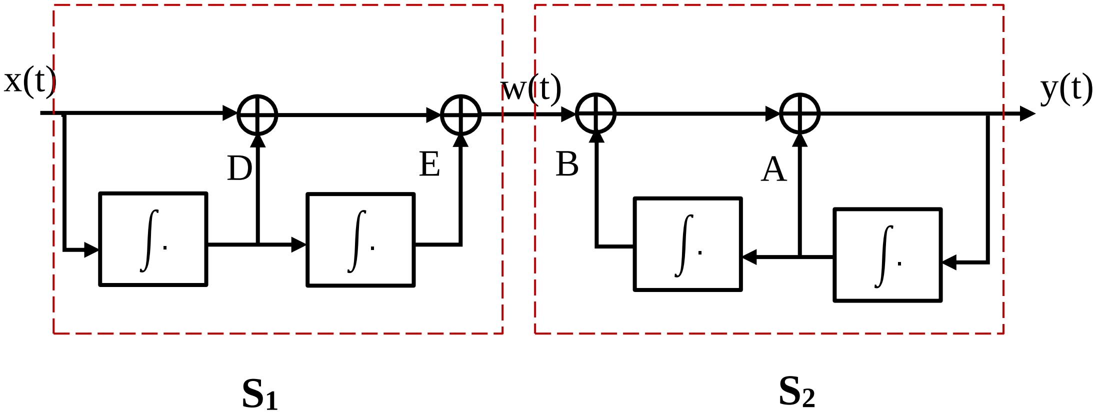
$$\Longrightarrow A=-\frac{a_1}{a_2}\quad,\quad B=-\frac{a_0}{a_2}\quad,\quad C=\frac{b_2}{a_2}\quad,\quad D=\frac{b_1}{a_2}\quad,\quad E=\frac{b_0}{a_2}$$

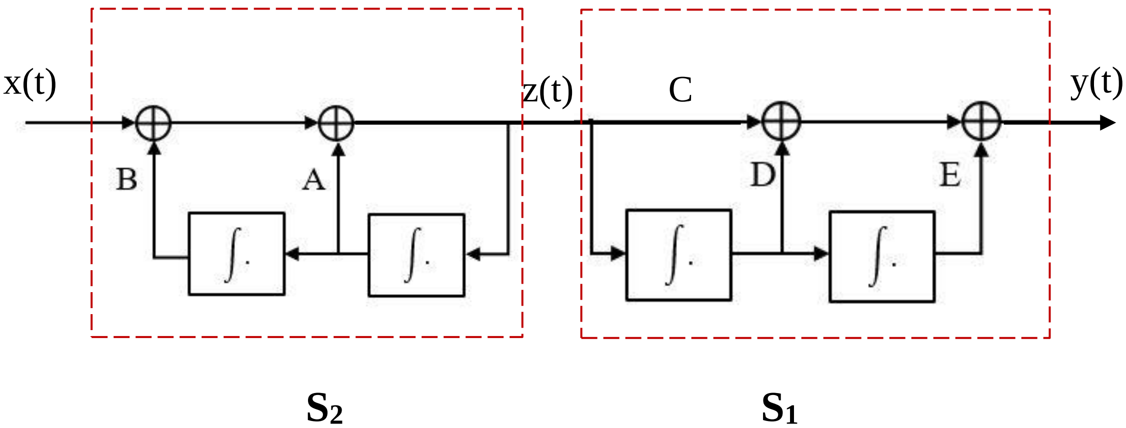


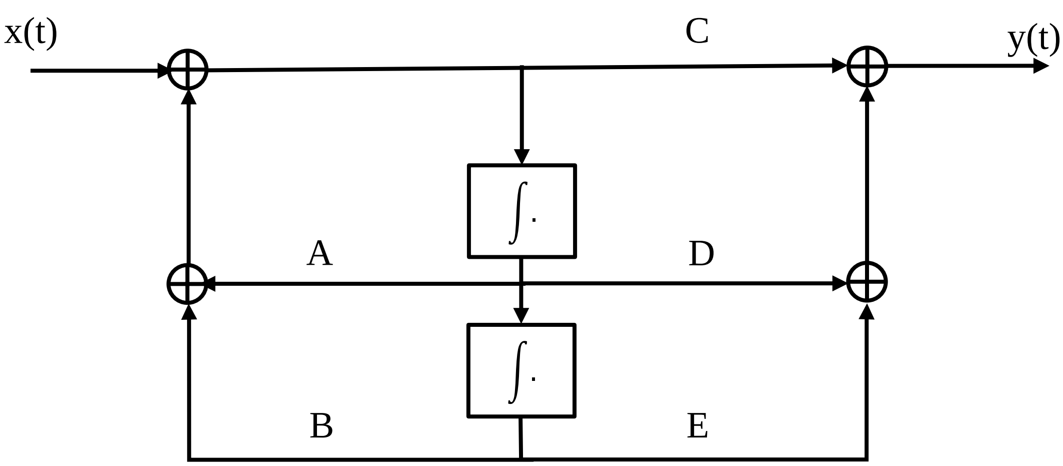
S2



S1







۴- معادله تفاضلی زیر را حل کنید. (با فرض شرایط استراحت اولیه)

$$y[n]-\frac{1}{2}y[n-1]=x[n] \quad , \quad x[n]=k\cos(\Omega_0 n)\,u[n]$$

برای یافتن پاسخ همگن داریم:

$$y_h[n] - \frac{1}{2} y_h[n-1] = 0$$

در این معادله پاسخی به صورت $y_h[n] = A\alpha^n$ صدق می‌کند بنابراین:

$$A\alpha^n - \frac{1}{2} A\alpha^{n-1} = 0 \rightarrow A\alpha^{n-1}\left(\alpha - \frac{1}{2}\right) = 0 \rightarrow \alpha = \frac{1}{2} \Longrightarrow y_h[n] = A\left(\frac{1}{2}\right)^n$$

با توجه به نوع ورودی می‌توان شکل پاسخ خصوصی را حدس زد و سپس آن را در معادله تفاضلی قرار داد.

$$y_p[n] = \mathrm{B}\cos(\Omega_0 n + \theta) \quad , \quad y_p[n-1] = \mathrm{B}\cos(\Omega_0(n-1) + \theta) = B\, cos(\Omega_0 n - \Omega_0 + \theta)$$

$$\Rightarrow B\{cos\,(\Omega_0 n)\, cos\,\theta\; - sin(\Omega_0 n)\, sin\,\theta\} - \frac{B}{2}\{cos\,(\Omega_0 n)\, cos\,(-\Omega_0 + \theta)\; - sin(\Omega_0 n)\, sin\,(-\Omega_0 + \theta)\} =$$
$$k\cos(\Omega_0 n)$$

$$\Rightarrow cos\,(\Omega_0 n)\,\left\{B\, cos(\theta) - \frac{1}{2}B\;\; cos\,(-\Omega_0 + \theta)\right\} + sin(\Omega_0 n)\left\{-B\, sin(\theta) + \frac{B}{2} sin\,(-\Omega_0 + \theta)\right\} = k\cos(\Omega_0 n)$$

$$\Rightarrow \begin{cases} B\, cos(\theta) - \dfrac{B}{2}\, cos(-\Omega_0 + \theta) = k & (I) \\ -B\, sin(\theta) + \dfrac{B}{2}\, sin(-\Omega_0 + \theta) = 0 & (II) \end{cases}$$

$$\xrightarrow{(II)} sin(-\Omega_0 + \theta) = \; 2\, sin(\theta)$$

$$\rightarrow sin(\Omega_0)\, cos(\theta) + \; cos(\Omega_0) sin(\theta) = 2\, sin(\theta)$$

$$\rightarrow sin(\Omega_0)\, cos(\theta) = \; sin(\theta)(cos(\Omega_0) - 2) \Longrightarrow \theta = \tan^{-1}\left(\frac{sin(\Omega_0)}{cos(\Omega_0) - 2}\right)$$

$$\Longrightarrow B = \frac{k}{cos(\theta) - \dfrac{1}{2}\, cos(-\Omega_0 + \theta)}$$

$$\Rightarrow y[n] = \left[A\left(\frac{1}{2}\right)^n + B\cos(\Omega_0 n + \theta)\right] u[n]$$

شرط استراحت اولیه (initial rest) : خروجی سیستم و مشتقات آن دقیقا قبل از اعمال ورودی برابر صفر است. به عنوان مثال اگر ورودی در لحظه t_0^+ به سیستم اعمال شده باشد، خروجی سیستم و کلیه مشتقات آن در لحظه t_0^- برابر صفر در نظر گرفته می‌شود.

$$y[0] = k \Rightarrow A + B\, cos(\theta) = k \Rightarrow A = k - B\, cos(\theta)$$