Universal Specificity Investigation 1: Revisiting the Mass Model Assumed by $\mathrm{E = mc^2}$

Daniel Harris
Northrop Grumman
Morrisville, USA
daniel.harris2@ngc.com

Einstein devised a thought experiment by which he derived a relationship between the mass and internal energy of an object, and in this derivation he tacitly made a relativistic mass model assumption that is worth revisiting. In this thought experiment, Einstein considered an object at rest that emits energy, in the form of radiation, in two opposite directions, but in equal amounts. Then he considered this same object with the same emission, but viewed from a different inertial reference frame that is moving relative to the object along some arbitrary axis. Figure 1 depicts this thought experiment [1].

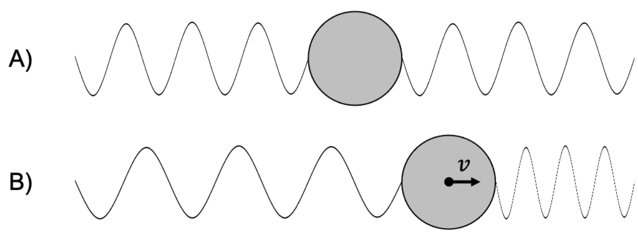

Figure 1. A) Object's inertial reference frame; B) Inertial reference frame with relative motion.

Before relating energy lost to mass lost, Einstein first compared the total energies measured by the two reference frames. He let E represents the total energy of the object as measured from the object's inertial reference frame, and H represents the total energy of the object as measured from the reference frame with relative motion. Einstein said, "[t]hus it is clear that the difference $H - E$ can differ from the kinetic energy K of the body, with respect to the other [reference frame with relative motion], only by an additive constant C..." The resulting model is: $H - E = K + C$ [1].

What is the total energy model and kinetic energy model used by Einstein, and upon what do these models depend? The answer to the first part of the question is best expressed by Feynman in his famous lectures. In short, Feynman derives the total energy relation to kinetic energy from the accepted mass model as shown in Equation (1) [2].

$$m = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}} = m_0 \left(1 - \frac{v^2}{c^2}\right)^{-1/2} \tag{1a}$$

$$m = m_0 \left(1 + \frac{1}{2}\frac{v^2}{c^2} + \frac{3}{8}\frac{v^4}{c^4} + \dots\right) \tag{1b}$$

$$m \approx m_0 + \frac{1}{2} m_0 \frac{v^2}{c^2} \tag{1c}$$

$$mc^2 \approx m_0 c^2 + \frac{1}{2} m_0 v^2 \blacksquare \tag{1d}$$

This result relates to $H - E = K + C$ as follows:

$$H = mc^2 \tag{2a}$$

$$E = m_0 c^2 \tag{2b}$$

$$K = H - E = mc^2 - m_0 c^2 \tag{2c}$$

$$\therefore\ C = 0 \text{ in this case} \tag{2d}$$

According to this total energy model, the total energy is the kinetic energy plus the internal energy.

To now answer the second part of the earlier question, these models ultimately depend on a relativistic mass model where rest mass, m_0, of an object remains invariant, while relative mass, m, increases as the object's kinetic energy increases.

To see this dependency, one can derive this kinetic energy model in Equation (2c) from Newtonian first principles relating kinetic energy to force applied over some distance, as shown in some detail in Equation (3).[2]

$$\text{Let} : \gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{3a}$$

$$m\gamma^{-1} = m_0 = \text{invariant} \tag{3b}$$

$$\Delta K = \int F(s) ds \tag{3c}$$

$$\Delta K = \int \frac{dp}{dt} ds \tag{3d}$$

$$\Delta K = \int v d(mv) \tag{3e}$$

$$\Delta K = \int v d(\gamma m_0 v) = m_0 \int v d(\gamma v) \tag{3f}$$

$$\Delta K = m_0(\gamma - 1)c^2 = mc^2 - m_0 c^2 \blacksquare \tag{3g}$$

If, on the other hand, m turns out to be invariant while m_0 decreases, then the relativistic kinetic energy model becomes the familiar Newtonian kinetic energy model: $\Delta K = \frac{1}{2}mv^2$. If this kinetic energy model is correct, then it implies that the total energy model shown in Equation (1d) is incorrect. A new total energy model would be required; however, before diving into this derivation, I first want to answer which mass model is correct with the aid of another thought experiment.

[2]The full derivation is presented here [3].

For this next thought experiment, consider relativistic effects on gravitational forces. Suppose we managed to craft four Osnium orbs, each having the same shape and size. Assuming each orb has a radius of 0.1 $[m]$, then the mass of each would be about 92 $[kg]$. The first pair of orbs are setup in an inertial frame in empty space with an initial distance of 100 $[m]$ between their center of masses. It would take about four months, in the orb's proper time, for their gravitational forces to bring them into contact.

Now, suppose the other pair of orbs were sent away, in the twins paradox fashion, but otherwise the same initial conditions.[3] Supposing they returned at the moment the stationary orbs touched, then the traveling orbs, being younger, would not be touching, as shown in Figure 2.

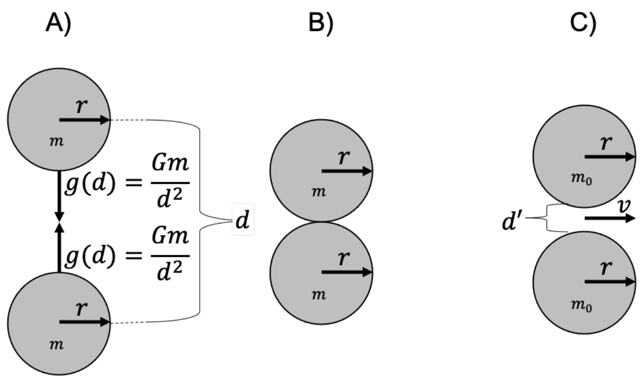

Figure 2. A) Initial Conditions In Stationary Frame; B) State of stationary orbs at the end of four months; C) State of traveling orbs at the end of four months.

These results suggest to me that the rest mass, m_0, of the traveling orbs reduces, but to what? Through testing, it was found that the resulting mass reduction was $m_0 = \gamma^{-2}m$. This result makes intuitive sense, because distance traversed during gravitational acceleration is related to time squared, and the time recorded for the traveling orbs is smaller than the time recorded for the stationary orbs by a factor of γ^{-1}.

Suppose we conducted another similar experiment; however, for this experiment two new orbs are crafted just like the others, except their radii are reduced to grantee that their mass is equal to $\gamma^{-2}m$. Let these new smaller orbs take the place of the stationary orbs. Given this new setup, we would find that upon the traveling orbs return, both pairs of orbs would be the same distance closer.

I actually simulated several trials of such an experiment, numerically solving for the dynamics in each reference frame, and the code is in the Appendix. Each trial had the same relative velocities, but each trial tested different return times. At the end of each trial, I compared the two pairs of orbs to see if each traveled the same distance. The results are presented in Figure 3 to the right.

The two pairs of orbs had the same gravitational behavior. Therefore, invoking the method of agreement, whereby the same gravitational effect occurred in both cases, proves inductively that the rest mass, m_0, of the traveling orbs is equivalent to the mass of the stationary orbs. Thus, the correct mass model is one in which rest mass, m_0, reduces while relative mass, m, remains invariant.

$m_0 = \gamma^{-2}m$ is not the original relativistic mass relationship that Feynman presented [2]; however, this makes no difference to the kinetic energy model, and the respective total energy model derivation is shown in Equation (4).

$$m = \gamma^2 m_0 = \text{invariant} \tag{4a}$$

$$m\left(1 - \frac{v^2}{c^2}\right) = m_0 \tag{4b}$$

$$m = m_0 + m\frac{v^2}{c^2} \tag{4c}$$

$$mc^2 = m_0c^2 + mv^2 = m_0c^2 + 2\frac{1}{2}mv^2 \tag{4d}$$

$$E_T = \frac{1}{2}mc^2 = \frac{1}{2}m_0c^2 + \frac{1}{2}mv^2 \blacksquare \tag{4e}$$

According to this total energy model, the total energy is the kinetic energy plus the internal energy, just as it was in Einstein's model. Unlike Einstein's model, the internal energy of an object diminishes as it gains kinetic energy, all the while the total energy of the object and its mass are conserved.

References

[1] A. Einstein, *Does the inertia of a body depend upon its energy-content?*. [Online]. Available: https://www.fourmilab.ch/etexts/einstein/E_mc2/e_mc2.pdf. [Accessed: 21-Aug-2022].

[2] R. Feynman, *The Feynman Lectures on Physics*, 2012. [Online]. Available: https://www.feynmanlectures.caltech.edu [Accessed: 20-Aug-2022].

[3] *Relativistic kinetic energy: Derivation, formula, definition* Mech Content, 23-Aug-2022. [Online]. Available: https://mechcontent.com/relativistic-kinetic-energy/. [Accessed: 09-Sep-2022].

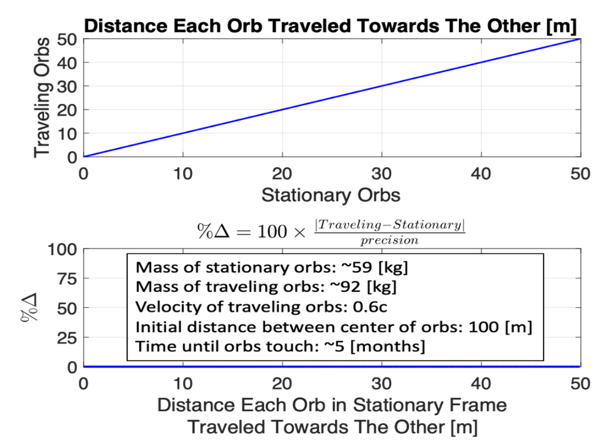

Figure 3. Results for how far the orbs traveled towards each other.

[3] With the distance between their center of masses being orthogonal to the velocity direction.

APPENDIX

MATLAB CODE

```matlab
%% constants and functions
G                 = 6.6744e-11; % [m^3/(kg s)] gravitational constant
gamma             = @(v) 1./sqrt(1-v.^2);
seconds2months    = 12/60^2/24/365;

%% Traveling orbs
% initial conditions
rho      = 22000;       % [kg/m^3] density of osmium
r        = 1e-1;        % [m]      radius of each orb
vol      = 4*pi*r^3/3;  % [m^3]    volume of each orb
m        = rho*vol;     % [kg]     mass of each orb
d        = 1e2;         % [m]      initial distance between orbs' surfaces
d_min    = 2*r;         % [m]      minimum distance between center mass of orbs
gd1      = 2*G*m/(d);   % [J/kg]   initial relative xspecific potential energy
v        = 0.6;         % [-]      fraction of the speed of light of orbs
gamma_v = gamma(v);     % [-]      1/sqrt(1-v^2/c^2)

% initialize other variables
dy  = (d-d_min)/1e4;     % increment steps to numerical solution
ds  = d:-dy:d_min;       % all numerical steps
gds = ones(size(ds))*gd1;% specific potential energy
vs  = zeros(size(ds));   % relative velocity of orbs
ts  = zeros(size(ds));   % proper time passed

% incremental solution of orb pairs relative velocity and time passed
for id = 2 : length(ds)
    % this relative specific potential energy for orbs
    gds(id) = 2*G*m/(ds(id));

    % delta relative specific potential energy for orbs
    delta_gd = gds(id)-gd1;

    % relative velocity between them
    vs(id) = sqrt(2*delta_gd);

    % time for distance to close by mean relative velocity
    ts(id) = ts(id-1) + dy/mean([vs(id),vs(id-1)]);
end

% total passage of proper time until orbs contact in years and months
total_time_months = max(ts)*seconds2months;

%% Stationary orbs
m0       = m/gamma_v^2; % [kg]    mass of stationary orb is traveling orb's rest mass
gd1_m0   = 2*G*m0/(d);  % [J/kg] initial specific potential energy

% time passed, as measured by stationary orbs
ts_gamma = ts*gamma_v;

% total passage of proper time until orbs contact in years and months
total_time_months_m0   = max(ts_gamma)*seconds2months;

% initialize stationary orbs with mass m0 distance steps
dy_m0 = dy;              % increment steps to numerical solution
ds_m0 = d:-dy_m0:d_min; % all numerical steps

% initialize other variables
vs_m0   = zeros(size(ds_m0));
gds_m0  = ones(size(ds_m0))*gd1_m0;
ts_m0   = zeros(size(ds_m0));

% incremental solution of orb pairs relative velocity and time passed
for id = 2 : length(ds_m0)
    % this relative specific potential energy
    gds_m0(id) = 2*G*m0/(ds_m0(id));
```

```matlab

    % delta relative specific potential energy
    delta_gd_m0 = gds_m0(id)-gd1_m0;

    % relative velocity between them
    vs_m0(id) = sqrt(2*delta_gd_m0);

    % time for distance to close by mean relative velocity
    ts_m0(id) = ts_m0(id-1) + dy_m0/mean([vs_m0(id),vs_m0(id-1)]);
end

%% Plot Results
figure(1);
% plot the movement of each orb makes towards its pair
subplot(2,1,1)
plot((d-ds)/2,(d-interp1(ts_m0,ds_m0,ts_gamma))/2,'-b','LineWidth',1.5)
xlim([0 d/2]);
ylim([0 d/2]);
grid on
xlabel('Stationary Orbs','FontSize',20);
ylabel('Traveling Orbs','FontSize',20);
title({'Distance Each Orb Traveled Towards The Other [m]'},'fontsize',16);

% plot the percent difference in movement between pairs of orbs
percent_difference = 100*abs((d-interp1(ts_m0,ds_m0,ts_gamma))/2 - (d-ds)/2)./(dy);
subplot(2,1,2)
plot((d-ds)/2,percent_difference,'-b','LineWidth',2)
xlim([0 d/2]);
ylim([0 100]);
grid on
xlabel({'Distance Each Orb in Stationary Frame'...
    ,'Traveled Towards The Other [m]'},'FontSize',20);
ylabel({'\%$\Delta$'},'FontSize',20,'Interpreter','latex');
title({'\%$\Delta=100\times\frac{|Traveling-Stationary|}{precision}$'}...
    ,'Interpreter','latex','fontsize',16);

% print ellapsed proper (AAK wall) time for each pair or orbs
fprintf('Elapsed Time for Traveling Orbs: %0.1f [months]\n',total_time_months);
fprintf('Elapsed Time for Stationary Orbs: %0.1f [months]\n',total_time_months_m0);
```