Revisiting the Mass Model Assumed by $\mathbf{E = mc^2}$

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Einstein devised a thought experiment where he derived a relationship between the mass and internal energy of an object, and in this derivation he tacitly made a relativistic mass model assumption that is worth revisiting. In this thought experiment, Einstein considered an object at rest that emits energy, in the form of radiation, in two opposite directions, but in equal amounts (so its velocity does not change). Then he considered this same object with the same emission, but viewed from a different inertial reference frame that is moving relative to the object along some arbitrary axis. Figure 1 shows this motion along the direction of emission.

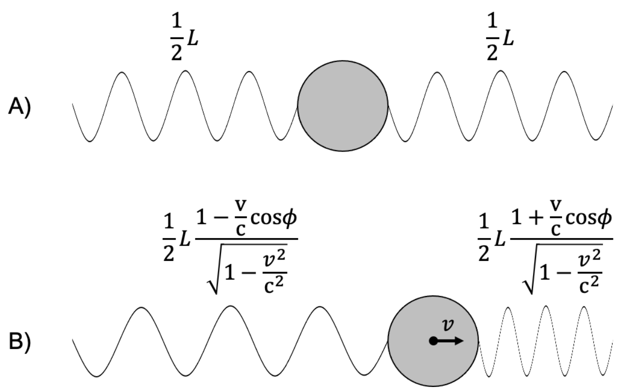

Figure 1. A) Object's inertial reference frame; B) Inertial reference frame with relative motion.

Let E_0 and E_1 be the total energy of the object before and after the radiation emission, respectively, as measured from the object's inertial reference frame. Let H_0 and H_1 be the total energy of the object before an after the radiation emission, respectively, as measured from the inertial reference frame with relative motion. The radiated energy measured from the objects stationary perspective is shown in Equation (1a), while the energy measured from the reference frame that is moving relative to the object is shown in Equation (1b) [1].

$$E_0 - E_1 = L \tag{1a}$$

$$H_0 - H_1 = \frac{L}{\sqrt{1-\frac{v^2}{c^2}}} \tag{1b}$$

Einstein relates the two reference frames, as $H - E$, to find a difference in total energy of the object. In his words, "Thus it is clear that the difference $H - E$ can differ from the kinetic energy K of the body, with respect to the other [reference frame with relative motion], only by an additive constant C, which depends on the choice of additive constants of the energies H and E." The resulting model is shown in in Equation (2) [1].

$$H_0 - E_0 = K_0 + C \tag{2a}$$

$$H_1 - E_1 = K_1 + C \tag{2b}$$

What is the total energy model and kinetic energy model used by Einstein, and upon what do these models depend? The answer to the first part of the question is best expressed by Feynman in his famous lectures. In short, Feynman derives the total energy relation to kinetic energy from relativistic mass as shown in Equation (3) [2].

$$m = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}} = m_0\left(1-\frac{v^2}{c^2}\right)^{-1/2} \tag{3a}$$

$$m = m_0\sum_{i=0}^{\infty}\left((-1)^i\binom{-1/2}{i}\left(\frac{v^2}{c^2}\right)^i\right) \tag{3b}$$

$$m = m_0\left(1+\frac{1}{2}\frac{v^2}{c^2}+\frac{3}{8}\frac{v^4}{c^4}+\ldots\right) \tag{3c}$$

$$m \approx m_0 + \frac{1}{2}m_0\frac{v^2}{c^2} \tag{3d}$$

$$mc^2 \approx m_0c^2 + \frac{1}{2}m_0v^2 \; \blacksquare \tag{3e}$$

How this result relates to Equation (2) is as follows:

- $H = mc^2$
- $E = m_0c^2$
- $H - E = mc^2 - m_0c^2 = \left(\frac{1}{\sqrt{1-\frac{v^2}{c^2}}} - 1\right)m_0c^2$
- $K = \frac{1}{2}m_0v^2$ for small v
- $K = \left(\frac{1}{\sqrt{1-\frac{v^2}{c^2}}} - 1\right)m_0c^2$ is the full kinetic energy model
- $\therefore\ C = 0$ in this case

According to this total energy model, the total energy is the kinetic energy plus the internal energy. Also, this kinetic energy model ultimately depends on rest mass, m_0, being invariant, while relative mass, m, changes as the kinetic energy of an object increases relative to some chosen inertial frame. To see this dependency, one can derive the kinetic energy model from Newtonian first principles relating kinetic energy to force applied over some distance, as shown in some detail in Equation (4).[2]

[2] The full derivation is presented here [3].

$$\text{Let} : \gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{4a}$$

$$\frac{m}{\gamma} = m_0 = \text{invariant} \tag{4b}$$

$$\Delta K = \int F(s) ds \tag{4c}$$

$$\Delta K = \int \frac{dp}{dt} ds \tag{4d}$$

$$\Delta K = \int v d(mv) \tag{4e}$$

$$\Delta K = \int v d(\gamma m_0 v) \tag{4f}$$

$$\Delta K = m_0 \int v d(\gamma v) \tag{4g}$$

$$\Delta K = m_0(\gamma - 1)c^2 = (\gamma - 1)m_0 c^2 \; \blacksquare \tag{4h}$$

If, on the other hand, m turns out to be invariant while m_0 changes, then the relativistic kinetic energy model becomes the familiar Newtonian kinetic energy model: $\Delta K = \frac{1}{2}mv^2$. If this model is correct, then it implies that the total energy model shown in Equation (3e) is incorrect. We would need to derive a new total energy model; however, before diving into a new total energy model derivation, I first want to answer which mass model is correct with the aid of another thought experiment.

For this next thought experiment, consider relativistic effects on gravitational forces. Suppose we managed to craft four Osnium orbs, each having the same shape and size. Assuming each orb is perfectly spherical with a radius of 0.1 $[m]$, then they would each have the same mass of about 92 $[kg]$. Assuming we could set up an experiment in an inertial frame in empty space, where we place a pair of orbs so the distance between their center of masses is 100 $[m]$, then it would take about four months, in proper time, for their gravitational forces to bring the two orbs into contact.

Now, if we send the other pair of orbs away, in the twins paradox fashion, but otherwise the same initial conditions[3], and supposing they returned at the moment the stationary orbs touched, then the traveling orbs would still be some distance apart at the end of four months. This is shown in Figure 2.

These results suggest to me that the rest mass, m_0, of the traveling orbs reduces, but to what? Through testing, it was found that the resulting mass reduction associated with this experiment was $m_0 = \frac{1}{\gamma^2}m$, which makes intuitive sense distance traversed during gravitational acceleration is related to time squared, and the time recorded by the traveler reduced by $\frac{1}{\gamma}$.

Suppose we conducted another similar experiment; however, suppose two new orbs were crafted to serve as the stationary orbs, and their radius were such as to grantee that their mass would be equal to $\frac{1}{\gamma^2}m$. We would find that upon the traveling orbs return, both orb pairs would be the same distance closer.

[3]With the distance between their center of masses being orthogonal to the velocity direction.

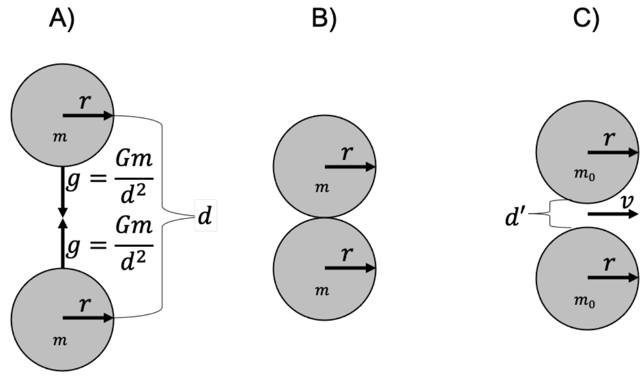

Figure 2. **A) Initial Conditions In Stationary Frame; B) State of stationary orbs at the end of four months; C) State of traveling orbs at the end of four months.**

I actually simulated several trials of such an experimental, numerically solving for the dynamics in each reference frame, and the code is in the Appendix. Each trial had the same relative velocity, $v = 0.6c$, but each trial tested different return times. At the end of each trial, I compared the two pairs of orbs to see if each traveled the same distance. The results are presented in Figure 3.

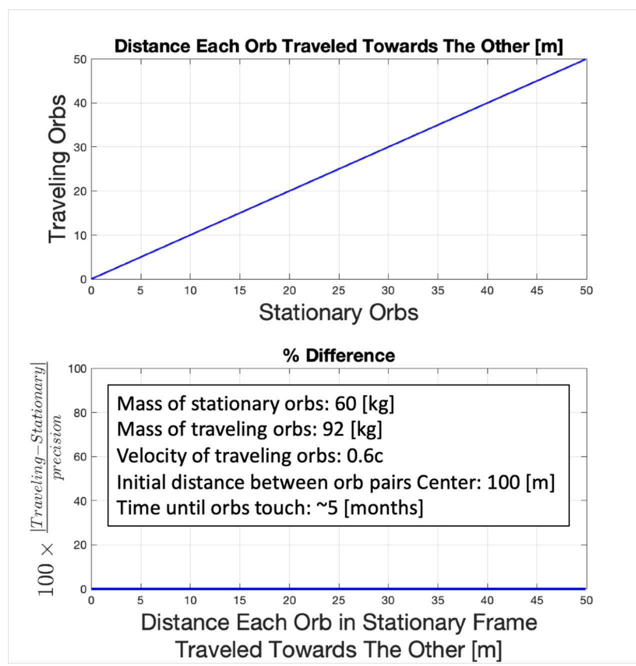

Figure 3. **Results for how far the orbs traveled towards each other.**

The two pairs of orbs have the same gravitational behavior. Therefore, invoking the method of agreement, whereby the same gravitational effect occurred in both cases, proves inductively that the rest mass, m_0, of the traveling orbs matches the mass of the stationary orbs. Thus, the correct mass model is one in which rest mass, m_0, reduces while relative mass, m, remains invariant.

The apparent unchanging rest mass measurements maid by traveling observers, in situations akin to the twins paradox, is actually behavior that is consistent with other measurements they make. These travelers do not measure a change in how fast their clocks tick, but they tick slower; they do not measure a change in the length of their ship, but their ship is shorter; Now, we can add that they do not measure a change in their rest mass, but it is indeed smaller.

$m_0 = \frac{1}{\gamma^2} m$ is not the original relationship that Feynman presented [2]; however, this makes no difference to the kinetic energy model, and the respective total energy model derivation is shown in Equation (5).

$$m = \gamma^2 m_0 = \text{invariant} \tag{5a}$$

$$m \left(1 - \frac{v^2}{c^2}\right) = m_0 \tag{5b}$$

$$m = m_0 + m\frac{v^2}{c^2} \tag{5c}$$

$$mc^2 = m_0c^2 + mv^2 \tag{5d}$$

$$mc^2 = m_0c^2 + 2\frac{1}{2}mv^2 \tag{5e}$$

$$\frac{1}{2}mc^2 = \frac{1}{2}m_0c^2 + \frac{1}{2}mv^2 \blacksquare \tag{5f}$$

According to this total energy model, the total energy is the kinetic energy plus the internal energy, just as it was in Einstein's model. Unlike Einstein's model, the internal energy of an object diminishes as it gains kinetic energy, all the while the total energy of the object and its mass is conserved. Once the object reaches the speed of light, its rest mass becomes zero, and all of its internal energy has been converted into kinetic energy.

REFERENCES

[1] A. Einstein, *Does the inertia of a body depend upon its energy-content?*. [Online]. Available: https://www.fourmilab.ch/etexts/einstein/E_mc2/e_mc2.pdf. [Accessed: 21-Aug-2022].

[2] R. Feynman, *The Feynman Lectures on Physics*, 2012. [Online]. Available: https://www.feynmanlectures.caltech.edu [Accessed: 20-Aug-2022].

[3] *Relativistic kinetic energy: Derivation, formula, definition* Mech Content, 23-Aug-2022. [Online]. Available: https://mechcontent.com/relativistic-kinetic-energy/. [Accessed: 09-Sep-2022].

Appendix
Matlab Code

```
%% constants and functions
G                 = 6.6744e-11; % [m^3/(kg s)] gravitational constant
gamma             = @(v) 1./sqrt(1-v.^2);
seconds2months    = 12/60^2/24/365;

%% Traveling orbs
% initial conditions
rho      = 22000;        % [kg/m^3] density of osmium
r        = 1e-1;         % [m]      radius of each orb
vol      = 4*pi*r^3/3;   % [m^3]    volume of each orb
m        = rho*vol;      % [kg]     mass of each orb
d        = 1e2;          % [m]      initial distance between orbs' surfaces
d_min    = 2*r;          % [m]      minimum distance between center mass of orbs
gd1      = 2*G*m/(d);    % [J/kg]   initial specific potential energy
v        = 0.6;          % [-]      fraction of the speed of light of orbs
gamma_v  = gamma(v);     % [-]      1/sqrt(1-v^2/c^2)

% initialize other variables
dy   = (d-d_min)/1e4;       % increment steps to numerical solution
ds   = d:-dy:d_min;         % all numerical steps
gds  = ones(size(ds))*gd1;  % relative acceleration
vs   = zeros(size(ds));     % relative velocity of orbs
ts   = zeros(size(ds));     % proper time passed

% incremental solution of orb pairs relative velocity and time passed
for id = 2 : length(ds)
    % this relative acceleration for orbs
    gds(id) = 2*G*m/(ds(id));

    % delta relative acceleration for orbs
    delta_gd = gds(id)-gd1;

    % relative velocity between them
    vs(id) = sqrt(2*delta_gd);

    % time for distance to close by mean relative velocity
    ts(id) = ts(id-1) + dy/mean([vs(id),vs(id-1)]);
end

% total passage of proper time until orbs contact in years and months
total_time_months = max(ts)*seconds2months;

%% Stationary orbs
m0        = m/gamma_v^2; % [kg]    mass of stationary orb is traveling orb's rest mass
gd1_m0    = 2*G*m0/(d);  % [J/kg] initial specific potential energy

% time passed, as measured by stationary orbs
ts_gamma = ts*gamma_v;

% total passage of proper time until orbs contact in years and months
total_time_months_m0   = max(ts_gamma)*seconds2months;

% initialize stationary orbs with mass m0 distance steps
dy_m0 = dy;                % increment steps to numerical solution
ds_m0 = d:-dy_m0:d_min;    % all numerical steps

% initialize other variables
vs_m0   = zeros(size(ds_m0));
gds_m0  = ones(size(ds_m0))*gd1_m0;
ts_m0   = zeros(size(ds_m0));

% incremental solution of orb pairs relative velocity and time passed
for id = 2 : length(ds_m0)
    % this relative acceleration
    gs_m0(id) = 2*G*m0/(ds_m0(id));
```

```
    % delta relative acceleration
    delta_gd_m0 = gs_m0(id)-gd1_m0;

    % relative velocity between them
    vs_m0(id) = sqrt(2*delta_gd_m0);

    % time for distance to close by mean relative velocity
    ts_m0(id) = ts_m0(id-1) + dy_m0/mean([vs_m0(id),vs_m0(id-1)]);
end

%% Plot Results
figure(1);
% plot the movement of each orb makes towards its pair
subplot(2,1,1)
hold off
plot((d-ds)/2,(d-interp1(ts_m0,ds_m0,ts_gamma))/2,'-b','LineWidth',1.5)
title({'Distance Each Orb Traveled Towards The Other [m]'},'fontsize',16);
grid on
xlabel('Stationary Orbs','FontSize',20);
ylabel('Traveling Orbs','FontSize',20);

% plot the percent difference in movement between pairs of orbs
percent_difference = 100*abs((d-interp1(ts_m0,ds_m0,ts_gamma))/2 - (d-ds)/2)./(dy);
subplot(2,1,2)
hold off
plot((d-ds)/2,percent_difference,'-b','LineWidth',2)
title({'% Difference'},'fontsize',16);
grid on
xlabel({'Distance Each Orb in Stationary Frame'...
    ,'Traveled Towards The Other [m]'},'FontSize',20);
ylabel({'$100\times\frac{|Traveling-Stationary|}{precision}$'}...
    ,'FontSize',20,'Interpreter','latex');
ylim([0 100]);

% print ellapsed proper (AAK wall) time for each pair or orbs
fprintf('Elapsed Time for Traveling Orbs: %0.1f [months]\n',total_time_months);
fprintf('Elapsed Time for Stationary Orbs: %0.1f [months]\n',total_time_months_m0);
```