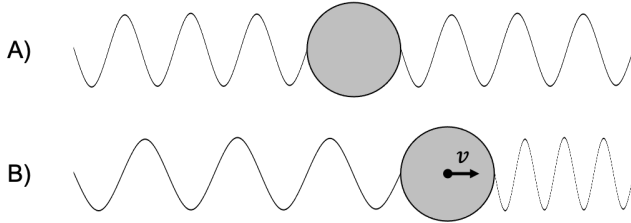


Revisiting the Mass Model Assumed by $E = mc^2$

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Einstein devised a thought experiment where he derived a relationship between the mass and internal energy of an object, and in this derivation he tacitly made a relativistic mass model assumption that is worth revisiting. In this thought experiment, Einstein considered an object at rest that emits energy, in the form of radiation, in two opposite directions, but in equal amounts (so its velocity does not change). Then he considered this same object with the same emission, but viewed from a different inertial reference frame that is moving relative to the object along some arbitrary axis. Figure 1 shows this motion along the direction of emission for ease of presentation.



**Figure 1. A) Object's inertial reference frame;
B) Inertial reference frame with relative motion.**

Before relating energy lost to mass lost, he first compared the total energies between the two systems, where E represents the total energy of the object as measured from the object's inertial reference frame, and H represents the total energy of the object as measured from the reference frame with relative motion. Einstein said, "[t]hus it is clear that the difference $H - E$ can differ from the kinetic energy K of the body, with respect to the other [reference frame with relative motion], only by an additive constant C , which depends on the choice of arbitrary additive constants of the energies H and E ." The resulting model is: $H - E = K + C$ [1].

What is the total energy model and kinetic energy model used by Einstein, and upon what do these models depend? The answer to the first part of the question is best expressed by Feynman in his famous lectures. In short, Feynman derives the total energy relation to kinetic energy from relativistic mass as shown in Equation (1) [2].

$$m = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}} = m_0 \left(1 - \frac{v^2}{c^2}\right)^{-1/2} \quad (1a)$$

$$m = m_0 \left(1 + \frac{1}{2} \frac{v^2}{c^2} + \frac{3}{8} \frac{v^4}{c^4} + \dots\right) \quad (1b)$$

$$m \approx m_0 + \frac{1}{2} m_0 \frac{v^2}{c^2} \quad (1c)$$

$$mc^2 \approx m_0 c^2 + \frac{1}{2} m_0 v^2 \blacksquare \quad (1d)$$

This result relates to $H - E = K + C$ as follows:

- $H = mc^2$
- $E = m_0 c^2$
- $K = H - E = mc^2 - m_0 c^2 \approx \frac{1}{2} m_0 v^2$ for small v
- $\therefore C = 0$ in this case

According to this total energy model, the total energy is the kinetic energy plus the internal energy. Also, this kinetic energy model ultimately depends on rest mass, m_0 , being invariant, while relative mass, m , changes as the kinetic energy of an object increases relative to some chosen inertial frame. To see this dependency, one can derive this kinetic energy model from Newtonian first principles relating kinetic energy to force applied over some distance, as shown in some detail in Equation (2).²

$$\text{Let : } \gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \quad (2a)$$

$$\frac{m}{\gamma} = m_0 = \text{invariant} \quad (2b)$$

$$\Delta K = \int F(s) ds \quad (2c)$$

$$\Delta K = \int \frac{dp}{dt} ds \quad (2d)$$

$$\Delta K = \int v d(mv) \quad (2e)$$

$$\Delta K = \int v d(\gamma m_0 v) \quad (2f)$$

$$\Delta K = m_0 \int v d(\gamma v) \quad (2g)$$

$$\Delta K = m_0 (\gamma - 1) c^2 = mc^2 - m_0 c^2 \blacksquare \quad (2h)$$

If, on the other hand, m turns out to be invariant while m_0 changes, then the relativistic kinetic energy model becomes the familiar Newtonian kinetic energy model: $\Delta K = \frac{1}{2} m v^2$. If this model is correct, then it implies that the total energy model shown in Equation (1d) is incorrect. We would need to derive a new total energy model; however, before diving into this derivation, I first want to answer which mass model is correct with the aid of another thought experiment.

For this next thought experiment, consider relativistic effects on gravitational forces. Suppose we managed to craft four Osnium orbs, each having the same shape and size. Assuming each orb is perfectly spherical with a radius of 0.1 [m], then the mass of each would be about 92 [kg]. The first pair of orbs are setup in an inertial frame in empty space so that the

²The full derivation is presented here [3].

distance between their center of masses is 100 [m]. It would take about four months, in the orb's proper time, for their gravitational forces to bring them into contact.

Now, if we send the other pair of orbs away, in the twins paradox fashion, but otherwise the same initial conditions³, and supposing they returned at the moment the stationary orbs touched, then the traveling orbs would still be some distance apart at the end of four months. This is shown in Figure 2.

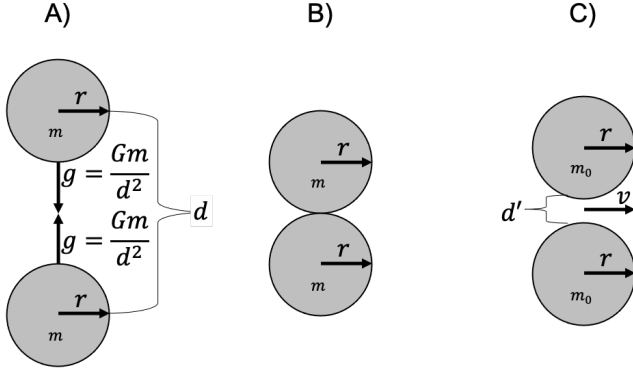


Figure 2. A) Initial Conditions In Stationary Frame; B) State of stationary orbs at the end of four months; C) State of traveling orbs at the end of four months.

These results suggest to me that the rest mass, m_0 , of the traveling orbs reduces, but to what? Through testing, it was found that the resulting mass reduction was $m_0 = \frac{1}{\gamma^2}m$, which makes intuitive sense, because distance traversed during gravitational acceleration is related to time squared, and the time recorded for the traveling orbs is smaller than the time recorded for the stationary orbs by a factor of $\frac{1}{\gamma}$.

Suppose we conducted another similar experiment; however, for this experiment two new orbs are crafted just like the others, except their radii were reduced to grantee that their mass would be equal to $\frac{1}{\gamma^2}m$. These new smaller orbs will replace the stationary orbs. Given this new setup, we would find that upon the traveling orbs return, both pairs of orbs would be the same distance closer.

I actually simulated several trials of such an experiment, numerically solving for the dynamics in each reference frame, and the code is in the Appendix. Each trial had the same relative velocities, $v = 0.6c$, but each trial tested different return times. At the end of each trial, I compared the two pairs of orbs to see if each traveled the same distance. The results are presented in Figure 3.

The two pairs of orbs had the same gravitational behavior. Therefore, invoking the method of agreement, whereby the same gravitational effect occurred in both cases, proves inductively that the rest mass, m_0 , of the traveling orbs is equivalent to the mass of the stationary orbs. Thus, the correct mass model is one in which rest mass, m_0 , reduces while relative mass, m , remains invariant.

$m_0 = \frac{1}{\gamma^2}m$ is not the original relationship that Feynman presented [2]; however, this makes no difference to the

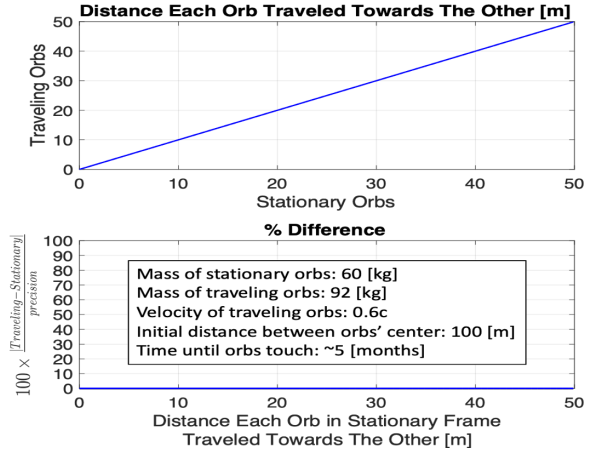


Figure 3. Results for how far the orbs traveled towards each other.

kinetic energy model, and the respective total energy model derivation is shown in Equation (3).

$$m = \gamma^2 m_0 = \text{invariant} \quad (3a)$$

$$m \left(1 - \frac{v^2}{c^2} \right) = m_0 \quad (3b)$$

$$m = m_0 + m \frac{v^2}{c^2} \quad (3c)$$

$$mc^2 = m_0 c^2 + mv^2 \quad (3d)$$

$$mc^2 = m_0 c^2 + 2 \frac{1}{2} mv^2 \quad (3e)$$

$$\frac{1}{2} mc^2 = \frac{1}{2} m_0 c^2 + \frac{1}{2} mv^2 \blacksquare \quad (3f)$$

According to this total energy model, the total energy is the kinetic energy plus the internal energy, just as it was in Einstein's model. Unlike Einstein's model, the internal energy of an object diminishes as it gains kinetic energy, all the while the total energy of the object and its mass is conserved.

REFERENCES

- [1] A. Einstein, *Does the inertia of a body depend upon its energy-content?*. [Online]. Available: https://www.fourmilab.ch/etexts/einstein/E_mc2/e_mc2.pdf. [Accessed: 21-Aug-2022].
- [2] R. Feynman, *The Feynman Lectures on Physics*, 2012. [Online]. Available: <https://www.feynmanlectures.caltech.edu> [Accessed: 20-Aug-2022].
- [3] *Relativistic kinetic energy: Derivation, formula, definition* Mech Content, 23-Aug-2022. [Online]. Available: <https://mechcontent.com/relativistic-kinetic-energy/>. [Accessed: 09-Sep-2022].

³With the distance between their center of masses being orthogonal to the velocity direction.

APPENDIX

MATLAB CODE

```

1 %% constants and functions
2 G = 6.6744e-11; % [m^3/(kg s)] gravitational constant
3 gamma = @(v) 1./sqrt(1-v.^2);
4 seconds2months = 12/60^2/24/365;
5
6 %% Traveling orbs
7 % initial conditions
8 rho = 22000; % [kg/m^3] density of osmium
9 r = 1e-1; % [m] radius of each orb
10 vol = 4*pi*r^3/3; % [m^3] volume of each orb
11 m = rho*vol; % [kg] mass of each orb
12 d = 1e2; % [m] initial distance between orbs' surfaces
13 d_min = 2*r; % [m] minimum distance between center mass of orbs
14 gd1 = 2*G*m/(d); % [J/kg] initial specific potential energy
15 v = 0.6; % [-] fraction of the speed of light of orbs
16 gamma_v = gamma(v); % [-] 1/sqrt(1-v^2/c^2)
17
18 % initialize other variables
19 dy = (d-d_min)/1e4; % increment steps to numerical solution
20 ds = d:-dy:d_min; % all numerical steps
21 gds = ones(size(ds))*gd1; % relative acceleration
22 vs = zeros(size(ds)); % relative velocity of orbs
23 ts = zeros(size(ds)); % proper time passed
24
25 % incremental solution of orb pairs relative velocity and time passed
26 for id = 2 : length(ds)
27     % this relative acceleration for orbs
28     gds(id) = 2*G*m/(ds(id));
29
30     % delta relative acceleration for orbs
31     delta_gd = gds(id)-gd1;
32
33     % relative velocity between them
34     vs(id) = sqrt(2*delta_gd);
35
36     % time for distance to close by mean relative velocity
37     ts(id) = ts(id-1) + dy/mean([vs(id),vs(id-1)]);
38 end
39
40 % total passage of proper time until orbs contact in years and months
41 total_time_months = max(ts)*seconds2months;
42
43 %% Stationary orbs
44 m0 = m/gamma_v^2; % [kg] mass of stationary orb is traveling orb's rest mass
45 gd1_m0 = 2*G*m0/(d); % [J/kg] initial specific potential energy
46
47 % time passed, as measured by stationary orbs
48 ts_gamma = ts*gamma_v;
49
50 % total passage of proper time until orbs contact in years and months
51 total_time_months_m0 = max(ts_gamma)*seconds2months;
52
53 % initialize stationary orbs with mass m0 distance steps
54 dy_m0 = dy; % increment steps to numerical solution
55 ds_m0 = d:-dy_m0:d_min; % all numerical steps
56
57 % initialize other variables
58 vs_m0 = zeros(size(ds_m0));
59 gds_m0 = ones(size(ds_m0))*gd1_m0;
60 ts_m0 = zeros(size(ds_m0));
61
62 % incremental solution of orb pairs relative velocity and time passed
63 for id = 2 : length(ds_m0)
64     % this relative acceleration
65     gs_m0(id) = 2*G*m0/(ds_m0(id));

```

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66
67     % delta relative acceleration
68     delta_gd_m0 = gs_m0(id)-gd1_m0;
69
70     % relative velocity between them
71     vs_m0(id) = sqrt(2*delta_gd_m0);
72
73     % time for distance to close by mean relative velocity
74     ts_m0(id) = ts_m0(id-1) + dy_m0/mean([ vs_m0(id),vs_m0(id-1)]);
75 end
76
77 %% Plot Results
78 figure(1);
79 % plot the movement of each orb makes towards its pair
80 subplot(2,1,1)
81 hold off
82 plot((d-ds)/2,(d-interp1(ts_m0,ds_m0,ts_gamma))/2,'-b','LineWidth',1.5)
83 title({'Distance Each Orb Traveled Towards The Other [m]'},'fontsize',16);
84 grid on
85 xlabel('Stationary Orbs','FontSize',20);
86 ylabel('Traveling Orbs','FontSize',20);
87
88 % plot the percent difference in movement between pairs of orbs
89 percent_difference = 100*abs((d-interp1(ts_m0,ds_m0,ts_gamma))/2 - (d-ds)/2)./(dy);
90 subplot(2,1,2)
91 hold off
92 plot((d-ds)/2,percent_difference,'-b','LineWidth',2)
93 title({'% Difference'},'fontsize',16);
94 grid on
95 xlabel({'Distance Each Orb in Stationary Frame'...
96         'Traveled Towards The Other [m]'},'FontSize',20);
97 ylabel({'$100\times\frac{|Traveling-Stationary|}{precision}$'}...
98         'FontSize',20,'Interpreter','latex');
99 ylim([0 100]);
100
101 % print ellapsed proper (AAK wall) time for each pair or orbs
102 fprintf('Elapsed Time for Traveling Orbs: %0.1f [months]\n',total_time_months);
103 fprintf('Elapsed Time for Stationary Orbs: %0.1f [months]\n',total_time_months_m0);

```