Revisiting the Mass Model Assumed by $\mathbf{E = mc^2}$

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Einstein, in his original proof for his total energy model, $E = mc^2$, made a tacit assumption about the relativistic mass model, which is worth revisiting. The evidence for the genius of Einstein is numerous, and in this particular case he devised a thought experiment to tease out a means to measure the internal energy of stationary objects as a relationship to their mass. Physicists knew at the time that stationary objects were made of particles and these particles had some energy, meaning even stationary objects contained internal energy. However, the means to measure it had remained elusive until Einstein, using previously established principles and relationships from electromagnetism and relativity, found a way [1].

In this thought experiment, Einstein considered an object that emits energy, in the form of radiation, in two opposite directions, but in equal amounts (so its velocity does not change). Then he considered this same object with the same emission, but viewed from a different inertial reference frame that perceives that the object is moving along the axes of emission, as shown in Figure 1.

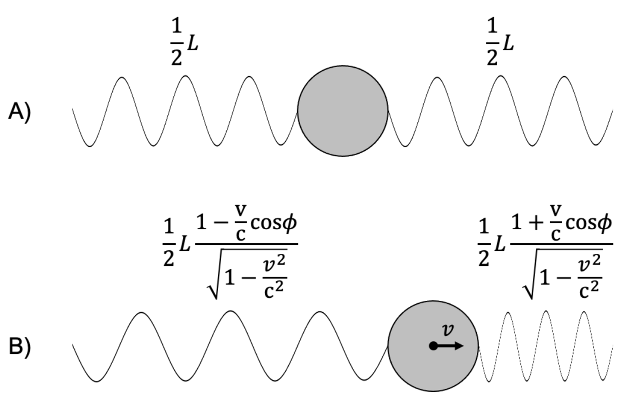

Figure 1. A) Object's inertial reference frame; B) Inertial reference frame with relative motion.

Let E_0 and E_1 be the total energy of the object before and after the radiation emission, respectively, as measured from the object's inertial reference frame. Let H_0 and H_1 be the total energy of the object before an after the radiation emission, respectively, as measured from the inertial reference frame with relative motion. The radiated energy measured from the objects stationary perspective is shown in Equation (1a), while the energy measured from the reference frame that is moving relative to the object is shown in Equation (1b) [1].

$$E_0 - E_1 = L \tag{1a}$$

$$H_0 - H_1 = \frac{L}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{1b}$$

Einstein relates the two reference frames before and after the emission as $H - E$ to find a difference in total energy of the object, as viewed by the two reference frames. Then he finds the relation of total energy to an assumed kinetic energy model. Einstein states, "Thus it is clear that the difference $H - E$ can differ from the kinetic energy K of the body, with respect to the other [reference frame with relative motion], only by an additive constant C." This model is shown in in Equation (2) [1].

$$H_0 - E_0 = K_0 + C \tag{2a}$$

$$H_1 - E_1 = K_1 + C \tag{2b}$$

Where did this kinetic energy model's relation to total energy come from, and upon what does it depend? The answer to the first part of the question is best expressed by Feynman in his famous lectures. In short, Feynman derives the total energy relation to kinetic energy from relativistic mass as shown in Equation (3) [2].

$$m = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}} = m_0 \left(1 - \frac{v^2}{c^2}\right)^{-1/2} \tag{3a}$$

$$m = m_0 \left(\sum_{i=0}^{\infty} (-1)^i \binom{-1/2}{i} \left(\frac{v^2}{c^2}\right)^i \right) \tag{3b}$$

$$m = m_0 \left(1 + \frac{1}{2}\frac{v^2}{c^2} + \frac{3}{8}\frac{v^4}{c^4} + \dots\right) \tag{3c}$$

$$m \approx m_0 + \frac{1}{2} m_0 \frac{v^2}{c^2} \tag{3d}$$

$$mc^2 \approx m_0 c^2 + \frac{1}{2} m_0 v^2 \blacksquare \tag{3e}$$

How this result relates to Equation (2) is as follows:

- $H - E = mc^2$
- $C = m_0 c^2$
- $K = \frac{1}{2} m_0 v^2$ for small v
- $K = \left(\frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} - 1\right) m_0 c^2$ is the full kinetic energy model

According to this total energy model, the internal energy of an object is the constant $C = m_0 c^2$. Also, this is where the kinetic energy model originally comes from, but it depends on rest mass, m_0, being invariant, while relative mass, m, changes as the kinetic energy of an object increases relative to some chosen inertial frame. To see this dependency, one can derive the kinetic energy model from Newtonian first principles relating kinetic energy to force applied over some distance, as shown in Equation (4).

$$\frac{m}{\gamma} = m_0 = \text{invariant} \tag{4a}$$

$$\Delta K = \int F(s)ds \tag{4b}$$

$$\Delta K = \int \frac{dp}{dt}ds \tag{4c}$$

$$\Delta K = \int vd(mv) \tag{4d}$$

$$\Delta K = \int vd(\gamma m_0 v) \tag{4e}$$

$$\Delta K = m_0 \int vd(\gamma v) \tag{4f}$$

$$\Delta K = m_0(\gamma - 1)c^2 = (\gamma - 1)m_0 c^2 \blacksquare \tag{4g}$$

$$\text{Where :}$$

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{4h}$$

If, on the other hand, it turns out the m is invariant while m_0 changes, then the relativistic kinetic energy model changes, which implies a different total energy model. This alternative kinetic energy model is derived in Equation (5).

$$m = \gamma m_0 = \text{invariant} \tag{5a}$$

$$\Delta K = \int F(s)ds \tag{5b}$$

$$\Delta K = \int \frac{dp}{dt}ds \tag{5c}$$

$$\Delta K = \int vd(mv) \tag{5d}$$

$$\Delta K = m \int vd(v) \tag{5e}$$

$$\Delta K = \frac{1}{2}mv^2 = \gamma\frac{1}{2}m_0 v^2 \blacksquare \tag{5f}$$

How this kinetic energy model relates to total energy is shown in Equation (6).

$$m = \gamma m_0 = \text{invariant} \tag{6a}$$

$$m\left(1 - \frac{v^2}{c^2}\right) = \frac{1}{\gamma}m_0 \tag{6b}$$

$$m = \frac{1}{\gamma}m_0 + m\frac{v^2}{c^2} \tag{6c}$$

$$mc^2 = \frac{1}{\gamma}m_0 c^2 + mv^2 \tag{6d}$$

$$mc^2 = \frac{1}{\gamma^2}mc^2 + 2\frac{1}{2}mv^2 \tag{6e}$$

$$\frac{1}{2}mc^2 = \frac{1}{\gamma^2}\frac{1}{2}mc^2 + \frac{1}{2}mv^2 \blacksquare \tag{6f}$$

Therefore, the total energy difference between the two reference frames is $H - E = K + I$, where I is not a constant, and it represents the internal potential energy that can be converted into kinetic energy. Equation (6) relates to this new model in the following way:

- $H - E = \frac{1}{2}mc^2$
- $K = \frac{1}{2}mv^2$
- $I = \frac{1}{\gamma^2}\frac{1}{2}mc^2$

According to this total energy model, when the object is stationary, $\frac{1}{\gamma^2} = 1$; therefore, all the energy of the object is internal potential energy. The internal potential energy of an object diminishes as it gains kinetic energy, all the while the total energy of the object is conserved. Once the object reaches the speed of light, $\frac{1}{\gamma^2} = 0$; therefore, all of its internal potential energy has been converted to kinetic energy, and no more potential remains.

Both total energy models rest on their respective mass model, because the mass model implies a kinetic energy model, which implies a total energy model. The question remains: which mass model is correct? To determine the answer to this question we must conduct another thought experiment.

In this next thought experiment consider relativistic effects on gravitational forces. Suppose we managed to craft four Osnium orbs, each having the same shape and size. Assuming there were perfectly spherical with a radius of about 0.1 $[m]$, each orb would have the same mass of about 92 $[kg]$. Assuming we could set up an experiment in an inertial frame in empty space, where we place a pair of orbs so the distance between their center of masses was 100 $[m]$, it would take about four months, in proper time, for their gravitational forces to bring the two orbs into contact.

Now, if we send the other pair of orbs away with some initial velocity, but otherwise the same initial conditions with their distance apart orthogonal to velocity direction, they would still be some distance apart at the end of four months. This is shown in Figure 2.

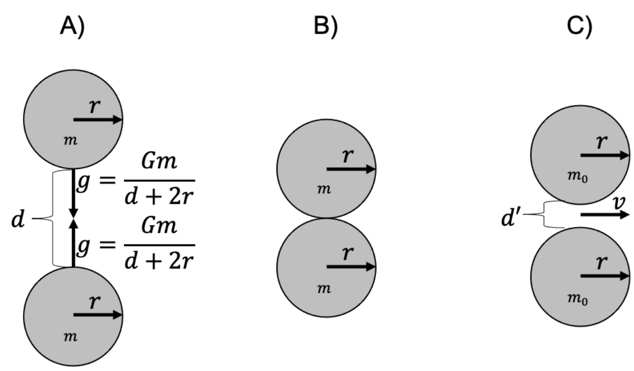

Figure 2. A) Initial Conditions In Stationary Frame; B) State of stationary orbs at the end of four months; C) State of traveling orbs at the end of four months.

These results suggest to me that the rest mass, m_0, of the traveling orbs reduces, but to what? Through testing, it was found that the resulting mass reduction associated was $m_0 = \frac{1}{\gamma^2}m$, which makes intuitive sense distance traversed during acceleration is related to time squared and rest time was reduced by $\frac{1}{\gamma}$.

Suppose we conducted another similar experiment in the twins paradox fashion, where we send a pair of orbs away with the same initial conditions at some velocity. However, suppose two new orbs were crafted to serve as the stationary orbs, and their radius were such as to grantee that their mass would be equal to $\frac{1}{\gamma^2}m$. We would find that upon the traveling orbs return, both orb pairs would be the same distance closer.

I actually simulated several such experimental trials, numerically solving for the dynamics in each reference frame, and the code is in the Appendix. Each trial had the same relative velocity, $v = 0.6c$, but each trial (represented as an iteration in the numerical solution) tested different return times. At the end of each trial, I compared the two pairs of orbs to see if they each traveled the same distance. The results are presented in Figure 3.

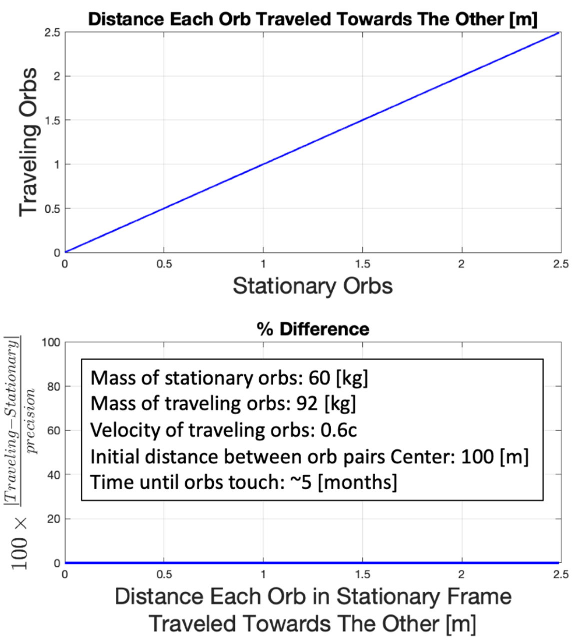

Figure 3. Results for how far the orbs traveled towards each other.

The two pairs of orbs have the same gravitational behavior. Therefore, invoking the method of agreement, whereby the same gravitational effect occurs in both cases, proves inductively that the rest mass, m_0, of the traveling orbs matches the mass of the stationary orbs. Thus, the correct mass model is one in which rest mass, m_0, reduces while relative mass, m, remains invariant.

The apparent unchanging rest mass measurements maid by traveling observers, in a situations akin to the twins paradox, is actually behavior that is consistent with other measurements they make. These travelers do not measure a change in how fast their clocks tick, but they tick slower; they do not measure a change in the length of their ship, but their ship is shorter; Now, we can add that they do not measure a change in their rest mass, but it is indeed smaller.

$m_0 = \frac{1}{\gamma^2}m$ is not the original relationship that Feynman presented [2]; however, this makes little difference to the kinetic energy model, but it has a profound impact on the total energy model, as shown in Equation (7).

$$m = \gamma^2 m_0 = \text{invariant} \tag{7a}$$

$$m\left(1 - \frac{v^2}{c^2}\right) = m_0 \tag{7b}$$

$$m = m_0 + m\frac{v^2}{c^2} \tag{7c}$$

$$mc^2 = m_0c^2 + mv^2 \tag{7d}$$

$$mc^2 = m_0c^2 + 2\frac{1}{2}mv^2 \tag{7e}$$

$$\frac{1}{2}mc^2 = \frac{1}{2}m_0c^2 + \frac{1}{2}mv^2 \tag{7f}$$

$$\frac{1}{2}mc^2 = \frac{1}{\gamma^2}\frac{1}{2}mc^2 + \frac{1}{2}mv^2 \;\blacksquare \tag{7g}$$

The unexpected, but intuitive, discovery is that the internal potential energy of an object is the total energy of its rest mass, or $\frac{1}{2}m_0c^2$.

Appendix

Matlab Code

```
%% constants and functions
G                  = 6.6744e-11; % [m^3/(kg s)] gravitational constant
gamma              = @(v) 1./sqrt(1-v.^2);
seconds2months     = 12/60^2/24/365;

%% Traveling orbs
% initial conditions
rho       = 22000;           % [kg/m^3] density of osmium
r         = 1e-1;            % [m] radius of each orb
vol       = 4*pi*r^3/3;      % [m^3] volume of each orb
m         = rho*vol;         % [kg] mass of each orb
d         = 1e2;             % [m] initial distance between orbs' surfaces
d_end     = 2*r;             % [m] distance between center mass of orbs when they contact
g1        = 2*G*m/(d);   % initial relative acceleration for orbs
v         = 0.6;             % fraction of the speed of light of orbs
gamma_v = gamma(v);          % 1/sqrt(1-v^2/c^2)

% initialize other variables
dy = (d-d_end)/1e4;              % increment steps to numerical solution
ds = d:-dy:d_end;                % all numerical steps
gs = ones(size(ds))*g1; % relative acceleration
vs = zeros(size(ds));    % relative velocity of orbs
```

```matlab
ts = zeros(size(ds));    % proper time passed

% incremental solution of orb pairs relative velocity and time passed
for id = 2 : length(ds)
    % this relative acceleration for orbs
    gs(id) = 2*G*m/(ds(id));

    % delta relative acceleration for orbs
    dg = gs(id)-g1;

    % relative velocity between them
    vs(id) = sqrt(2*dg);

    % time for distance to close by mean relative velocity
    ts(id) = ts(id-1) + dy/mean([vs(id),vs(id-1)]);
end

% total passage of proper time until orbs contact in years and months
total_time_days = max(ts)*seconds2months;

%% Stationary orbs
m0      = m/gamma_v^2;     % mass of stationary orb is predicted rest mass of traveling orbs
g1_m0   = 2*G*m0/(d); % initial relative acceleration for stationary orbs

% time passed, as measured by stationary orbs
ts_gamma = ts*gamma_v;

% total passage of proper time until orbs contact in years and months
total_time_days_m0  = max(ts_gamma)*seconds2months;

% initialize stationary orbs with mass m0 distance steps
dy_m0 = dy;          % increment steps to numerical solution
ds_m0 = d:-dy_m0:d_end; % all numerical steps

% initialize other variables
vs_m0 = zeros(size(ds_m0));
gs_m0 = ones(size(ds_m0))*g1_m0;
ts_m0 = zeros(size(ds_m0));

% incremental solution of orb pairs relative velocity and time passed
for id = 2 : length(ds_m0)
    % this relative acceleration
    gs_m0(id) = 2*G*m0/(ds_m0(id));

    % delta relative acceleration
    dg_m0 = gs_m0(id)-g1_m0;

    % relative velocity between them
    vs_m0(id) = sqrt(2*dg_m0);

    % time for distance to close by mean relative velocity
    ts_m0(id) = ts_m0(id-1) + dy_m0/mean([vs_m0(id),vs_m0(id-1)]);
end

%% Plot Results
% plot the movement of each pair of orbs with respect to each other
figure(1);
subplot(2,1,1)
hold off
plot((d-ds)/2,(d-interp1(ts_m0,ds_m0,ts_gamma))/2,'-b','LineWidth',1.5)
title({'Distance Each Orb Traveled Towards The Other [m]'},'fontsize',16);
grid on
xlabel('Stationary Orbs','FontSize',20);
ylabel('Traveling Orbs','FontSize',20);

% plot the percent difference in movement between pairs of orbs
percent_difference = abs(interp1(ts_m0,ds_m0,ts_gamma) - ds)./(dy);
subplot(2,1,2)
hold off
plot((d-ds)/2,100*percent_difference,'-b','LineWidth',2)
title({'% Difference'},'fontsize',16);
grid on
xlabel({'Distance Each Orb in Stationary Frame','Traveled Towards The Other [m]'},'FontSize',20);
ylabel({'$100\times\frac{|Traveling-Stationary|}{precision}$'},'FontSize',20,'Interpreter','latex');
ylim([0 100]);

disp("Done.");
```

References

[1] A. Einstein, *Does the inertia of a body depend upon its energy-content?*. [Online]. Available: https://www.fourmilab.ch/etexts/einstein/E_mc2/e_mc2.pdf. [Accessed: 21-Aug-2022].

[2] R. Feynman, *The Feynman Lectures on Physics*, 2012. [Online]. Available: https://www.feynmanlectures.caltech.edu [Accessed: 20-Aug-2022].

[3] *Acceleration (special relativity)* Wikipedia, 29-Jan-2022. [Online]. Available: https://en.wikipedia.org/wiki/Acceleration_(special_relativity). [Accessed: 05-Sep-2022].