Revisiting the Mass Model Assumed by $\mathbf{E = mc^2}$

Daniel Harris
Northrop Grumman
Huntsville, USA
daniel.harris2@ngc.com

Einstein, in his original proof for his total energy model, $E = mc^2$, made a tacit assumption about the relativistic mass model when he assumed a kinetic energy model. The evidence for the genius of Einstein is numerous, and in this particular case he devised a thought experiment to tease out a means to measure the internal energy of stationary objects as a relationship to their mass. Physicists knew at the time stationary objects were made of particles and these particles had some energy, meaning even stationary objects contained energy. However, the means to measure it had remained elusive until Einstein, using previously established principles and relationships from electromagnetism and relativity, found a way [1].

In this thought experiment, Einstein considered an object that emits energy, in the form of radiation in two equal but opposite directions (so its velocity does not change). Then he considered this same object with the same emission, but viewed from a different inertial reference frame that perceives that the object is moving relative along the axes of emission, as shown in Figure 1.

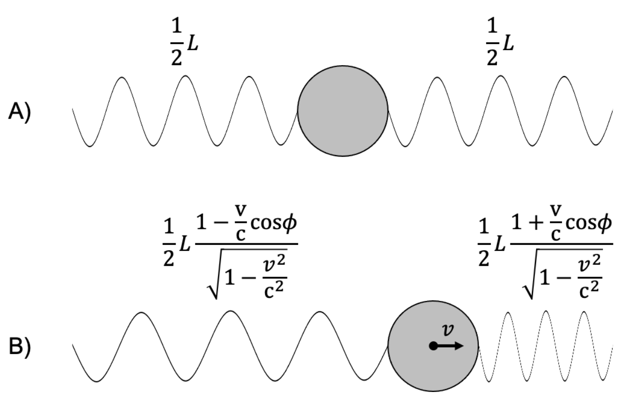

Figure 1. A) Object's inertial reference frame; B) Inertial reference frame with relative motion.

Let E_0 and E_1 be the total energy of the object before and after radiation emission, respectively, as measured from the object's inertial reference frame. Let H_0 and H_1 be the total energy of the object before an after radiation emission, respectively, as measured from the inertial reference frame with relative motion. The radiated energy measured from the objects stationary perspective is shown in Equation (1a), while the energy measured from the reference frame that is moving relative to the object is shown in Equation (1b) [1].

$$E_0 - E_1 = L \tag{1a}$$

$$H_0 - H_1 = \frac{L}{\sqrt{1-\frac{v^2}{c^2}}} \tag{1b}$$

Einstein relates the two reference frames before and after the emission as $H - E$ to find a difference in total energy of the object, as seen by the two reference frames. Then he finds the relation of total energy to an assumed kinetic energy model. Einstein states, "Thus it is clear that the difference $H - E$ can differ from the kinetic energy K of the body, with respect to the other [reference frame with relative motion], only by an additive constant C." This model is shown in in Equation (2) [1].

$$H_0 - E_0 = K_0 + C \tag{2a}$$
$$H_1 - E_1 = K_1 + C \tag{2b}$$

Where did this kinetic energy model's relation to total energy come from, and upon what does it depend? The answer to the first part of the question is best expressed by Feynman in his famous lectures. In short, Feynman derives the total energy relation to kinetic energy from relativistic mass as show in Equation (3) [2].

$$m = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}} = m_0\left(1-\frac{v^2}{c^2}\right)^{-1/2} \tag{3a}$$

$$m = m_0\left(\sum_{i=0}^{\infty}(-1)^i\binom{-1/2}{i}\left(\frac{v^2}{c^2}\right)^i\right) \tag{3b}$$

$$m = m_0\left(1+\frac{1}{2}\frac{v^2}{c^2}+\frac{3}{8}\frac{v^4}{c^4}+\ldots\right) \tag{3c}$$

$$m \approx m_0 + \frac{1}{2}m_0\frac{v^2}{c^2} \tag{3d}$$

$$mc^2 \approx m_0c^2 + \frac{1}{2}m_0v^2 \blacksquare \tag{3e}$$

How this result relates to Equation (2) is as follows:

- $H - E = mc^2$
- $C = m_0c^2$
- $K = \frac{1}{2}mc^2$ for small v
- $K = \left(\frac{1}{\sqrt{1-\frac{v^2}{c^2}}} - 1\right)m_0c^2$ is the full kinetic energy model

That is where the kinetic energy model comes from, but it depends on rest mass, m_0, being invariant, while relative mass, m, changes as the kinetic energy of an object increases relative to some chosen inertial frame. To see this dependency, one can derive the kinetic energy model from Newtonian first principles relating kinetic energy to force applied over some distance, as shown in Equation (4).

$$\frac{m}{\gamma} = m_0 = \text{invariant} \tag{4a}$$
$$\Delta K = \int F(s)ds \tag{4b}$$
$$\Delta K = \int \frac{dp}{dt}ds \tag{4c}$$
$$\Delta K = \int vd(mv) \tag{4d}$$
$$\Delta K = \int vd(\gamma m_0 v) \tag{4e}$$
$$\Delta K = m_0 \int vd(\gamma v) \tag{4f}$$
$$\Delta K = m_0(\gamma - 1)c^2 = (\gamma - 1)m_0c^2 \blacksquare \tag{4g}$$
Where :
$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{4h}$$

If, on the other hand, it turns out the m is invariant while m_0 changes, then the relativistic kinetic energy model changes, and so does its relation to total energy. This alternative kinetic energy model is derived in Equation (5).

$$m = \gamma m_0 = \text{invariant} \tag{5a}$$
$$\Delta K = \int F(s)ds \tag{5b}$$
$$\Delta K = \int \frac{dp}{dt}ds \tag{5c}$$
$$\Delta K = \int vd(mv) \tag{5d}$$
$$\Delta K = m \int vd(v) \tag{5e}$$
$$\Delta K = \frac{1}{2}mv^2 = \gamma\frac{1}{2}m_0v^2 \blacksquare \tag{5f}$$

How this kinetic energy model relates to total energy is shown in Equation (6).

$$m = \gamma m_0 = \text{invariant} \tag{6a}$$
$$m\left(1 - \frac{v^2}{c^2}\right) = \frac{1}{\gamma}m_0 \tag{6b}$$
$$m = \frac{1}{\gamma}m_0 + m\frac{v^2}{c^2} \tag{6c}$$
$$mc^2 = \frac{1}{\gamma}m_0c^2 + mv^2 \tag{6d}$$
$$mc^2 = \frac{1}{\gamma^2}mc^2 + 2\frac{1}{2}mv^2 \tag{6e}$$
$$\frac{1}{2}mc^2 = \frac{1}{\gamma^2}\frac{1}{2}mc^2 + \frac{1}{2}mv^2 \blacksquare \tag{6f}$$

Therefore, the total energy difference between the two reference frames is $H - E = K + I$, where I is the internal potential energy that can be converted into kinetic energy. Equation (6) relates to this new model in the following way:

- $H - E = \frac{1}{2}mc^2$
- $K = \frac{1}{2}mv^2$
- $I = \frac{1}{\gamma^2}\frac{1}{2}mc^2$

According to this total energy model, when the object is stationary, $\frac{1}{\gamma^2} = 1$; therefore, all the energy of the object is internal potential energy. The internal potential energy of an object diminishes as it gains kinetic energy, all the while the total energy of the object is conserved. Once the object reaches the speed of light, $\frac{1}{\gamma^2} = 0$; therefore, all of its internal potential energy has been converted to kinetic energy, and no more potential remains to gain kinetic energy.

The total energy model rests ultimately on the mass model, because the mass model implies a kinetic energy model, and the kinetic energy model implies a total energy model. The question remains: which mass model is correct? To determine the answer to this question we must conduct another thought experiment.

In this next thought experiment lets consider relativistic effects on gravitational forces. Suppose we managed craft four Osnium orbs, each having the same shape and size. Assuming there were perfectly spherical with a radius of about 0.1 $[m]$, each orb would have the same mass of about 92 $[kg]$. Assuming we could set up an experiment in an inertial frame in empty space, where we place two of the orbs at a distance of 100 $[m]$ apart, it would take about four months for their gravitational forces to bring the two orbs into contact.

Now, if we send the other pair of orbs away with some initial velocity, but otherwise the same initial conditions with their distance apart orthoginal to velocity direction, they would still be some distance apart at the end of four months. This is shown in Figure 2.

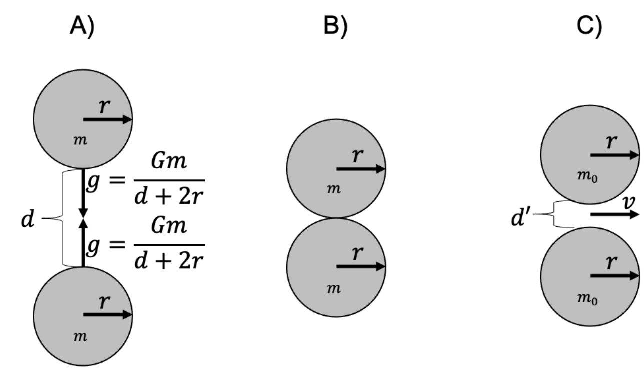

Figure 2. A) Initial Conditions In Stationary Frame; B) State of stationary orbs at the end of four months; C) State of traveling orbs at the end of four months.

These results suggest to me that the mass of the traveling orbs reduces, but to what? To gain insight into this question we turn to the effects of relativistic acceleration, which is well understood in the direction orthogonal to velocity, assuming no acceleration along the velocity direction, which is $a_0 = \frac{1}{\gamma^2}a$ [3]. The resulting mass reduction associated with this acceleration would then be $m_0 = \frac{1}{\gamma^2} m$.

Suppose we conducted another similar experiment in the twins paradox fashion, where we send a pair of orbs away with the same initial conditions at some velocity. However, suppose two new orbs were crafted to serve as the stationary orbs, and their radius were such as to grantee that their mass would be equal to $\frac{1}{\gamma^2} m$. We would find that upon the traveling orbs return, both pairs would have traveled the same distance.

I actually simulated several such experimental trials, numerically solving for the dynamics in each reference frame, and the code is in the Appendix. Each trial had the same relative velocity, $v = 0.6c$, but each trial (represented as an iteration in the numerical solution) tested different return times. At the end of each trial, I compared the two pairs of orbs to see if they each traveled the same distance. The results are presented in Figure 3.

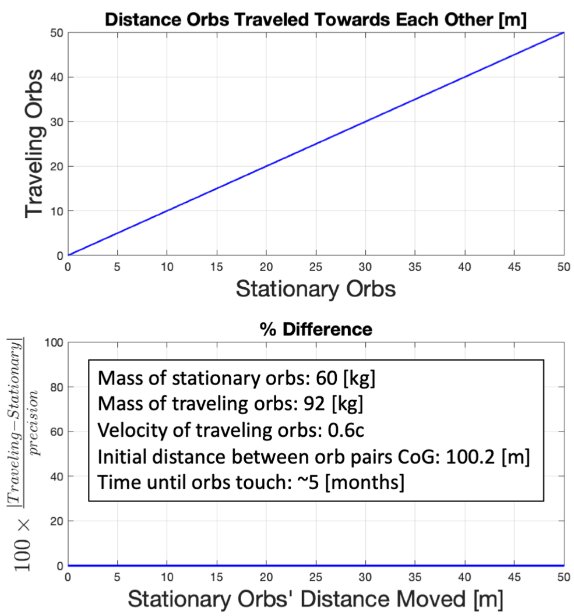

Figure 3. Results for how far the orbs traveled towards each other.

The two pairs of orbs have the same gravitational behavior. Invoking the method of agreement, whereby the same gravitational effect occurs in both cases, proves the mass of the traveling orbs matches the reduced mass of the stationary orbs. Thus, the correct mass model is one in which rest mass, m_0, reduces and relative mass, m, is invariant.

$m_0 = \frac{1}{\gamma^2} m$ is not the original relationship that Feynman presented [2]; however, this makes little difference to the kinetic energy model, and to the total energy model, as shown in Equation (7).

$$m = \gamma^n m_0 \tag{7a}$$

$$m\left(1 - \frac{v^2}{c^2}\right) = \gamma^{n-2} m_0 \tag{7b}$$

$$m = \gamma^{n-2} m_0 + m\frac{v^2}{c^2} \tag{7c}$$

$$mc^2 = \gamma^{n-2} m_0 c^2 + mv^2 \tag{7d}$$

$$mc^2 = \frac{1}{\gamma^2} mc^2 + 2\frac{1}{2} mv^2 \tag{7e}$$

$$\frac{1}{2} mc^2 = \frac{1}{\gamma^2}\frac{1}{2} mc^2 + \frac{1}{2} mv^2 \blacksquare \tag{7f}$$

The apparent unchanging mass measurements maid by traveling observers, in the twins paradox situation as an example, is actually behavior that is consistent with other measurements they make. These travelers do not measure a change in how fast their clocks tick, but they tick slower. These observers do not measure a change in the length of their ship, but their ship is shorter. Now, we can add they do not measure a change in their mass, but it is indeed smaller.

The last point worth mentioning is that Feynman (and others) have mentioned that in labs it is observed that an increased magnetic force is required to accelerate an electron that approaches the speed of light (compared to when it is stationary), and that this observation matches predictions for assuming m_0 is invariant. It is true that these observations do match those predictions. Feynman suggests the reason that more magnetic force is required is because m is not invariant, and m increases towards infinity as objects approach the speed of light. This is also the explanation for why it is impossible for any massed object to reach the speed of light, as it would require an infinite amount of energy to accelerate an infinitely massive object [2].

Mass, however, is not the only factor at play. The force of the magnetic field only effects charged objects, and only to the degree of their charge. To integrate all the presented evidence without contradiction, would require one to investigate the charge of the electron as its velocity approaches c to determine if it actually remains constant, or if it reduces, like many of its other properties (e.g., rate of time, length, rest mass). A reduced charge would reduce the effectiveness of a constant magnetic force requiring an infinitely strong magnetic field to continue to accelerate a particle whose charge approaches zero.

This suggests that the reason objects cannot exceed the speed of light is that their properties become closer to that of light. Meaning, in a kinetic context, using rockets to accelerate particles backward to accelerate a ship forward, would essentially be rocketing objects whose rest mass was approaching zero yielding a negligible change in kinetic energy of the ship. Also, in a electromagnetic context, any magnetic field's effectiveness reduces to zero as the charge of fast moving particles approaches zero.

APPENDIX

MATLAB CODE

```matlab
%% constants and functions
G              = 6.6744e-11; % [m^3/(kg s)] gravitational constant
gamma          = @(v) 1./sqrt(1-v.^2);
seconds2years  = 1/60^2/24/365;
years2months   = 12;

%% Traveling orbs
% initial conditions
rho      = 22000;          % [kg/m^3] density of osmium
r        = 1e-1;           % [m] radius of each orb
vol      = 4*pi*r^3/3;     % [m^3] volume of each orb
m        = rho*vol;        % [kg] mass of each orb
d        = 1e2;            % [m] initial distance between orbs' surfaces
g1       = 2*G*m/(d+2*r);  % initial relative acceleration for orbs
v        = 0.6;            % fraction of the speed of light of orbs
gamma_v  = gamma(v);       % 1/sqrt(1-v^2/c^2)

% initialize other variables
dy = d/1e4;                % increment steps to numerical solution
ds = d:-dy:0;              % all numerical steps
gs = ones(size(ds))*g1;    % relative acceleration
vs = zeros(size(ds));      % relative velocity of orbs
ts = zeros(size(ds));      % proper time passed

% incremental solution of orb pairs relative velocity and time passed
for id = 2 : length(ds)
    % this relative acceleration for orbs
    gs(id) = 2*G*m/(ds(id)+2*r);

    % delta relative acceleration for orbs
    dg = gs(id)-g1;

    % relative velocity between them
    vs(id) = sqrt(2*dg);

    % time for distance to close by mean relative velocity
    ts(id) = ts(id-1) + dy/mean([vs(id),vs(id-1)]);
end

% total passage of proper time until orbs contact in years and months
total_time_years = max(ts)*seconds2years;
total_time_months = total_time_years*years2months;

%% Stationary orbs
m0       = m/gamma_v^2;     % mass of stationary orb is predicted rest mass of traveling orbs
g1_m0    = 2*G*m0/(d+2*r); % initial relative acceleration for stationary orbs

% time passed, as measured by stationary orbs
ts_gamma = ts*gamma_v;

% total passage of proper time until orbs contact in years and months
total_time_years_m0  = max(ts_gamma)*seconds2years;
total_time_months_m0 = total_time_years_m0*years2months;

% initialize stationary orbs with mass m0 distance steps
dy_m0 = dy;             % increment steps to numerical solution
ds_m0 = d:-dy_m0:0; % all numerical steps

% initialize other variables
vs_m0 = zeros(size(ds_m0));
gs_m0 = ones(size(ds_m0))*g1_m0;
ts_m0 = zeros(size(ds_m0));

% incremental solution of orb pairs relative velocity and time passed
for id = 2 : length(ds_m0)
    % this relative acceleration
    gs_m0(id) = 2*G*m0/(ds_m0(id)+2*r);

    % delta relative acceleration
    dg_m0 = gs_m0(id)-g1_m0;

    % relative velocity between them
    vs_m0(id) = sqrt(2*dg_m0);

    % time for distance to close by mean relative velocity
    ts_m0(id) = ts_m0(id-1) + dy_m0/mean([vs_m0(id),vs_m0(id-1)]);
end

%% Plot Results
% plot the movement of each pair of orbs with respect to each other
figure(1);
subplot(2,1,1)
hold off
plot(ds/2,interp1(ts_m0,ds_m0,ts_gamma)/2,'-b','LineWidth',1.5)
title({'Distance Orbs Traveled Towards Each Other [m]'},'fontsize',16);
grid on
xlabel('Stationary Orbs','FontSize',20);
ylabel('Traveling Orbs','FontSize',20);

% plot the percent difference in movement between pairs of orbs
percent_difference = abs(interp1(ts_m0,ds_m0,ts_gamma) - ds)./(dy);
```

```
subplot(2,1,2)
hold off
plot(ds/2,100*percent_difference,'-b','LineWidth',2)
title({'% Difference'},'fontsize',16);
grid on
xlabel('Stationary Orbs'' Distance Moved [m]','FontSize',20);
ylabel({'$100\times\frac{|Traveling-Stationary|}{precision}$'},'FontSize',20,'Interpreter','latex');
ylim([0 100]);

disp("Done.");
```

REFERENCES

[1] A. Einstein, *Does the inertia of a body depend upon its energy-content?*. [Online]. Available: https://www.fourmilab.ch/etexts/einstein/E_mc2/e_mc2.pdf. [Accessed: 21-Aug-2022].

[2] R. Feynman, *The Feynman Lectures on Physics*, 2012. [Online]. Available: https://www.feynmanlectures.caltech.edu [Accessed: 20-Aug-2022].

[3] *Acceleration (special relativity)* Wikipedia, 29-Jan-2022. [Online]. Available: https://en.wikipedia.org/wiki/Acceleration_(special_relativity). [Accessed: 05-Sep-2022].