NCERT-8.2.7

EE24BTECH11036 - Krishna Patil

Question : Calculate the area bounded by the curves $y^2 = 4x$ and $y = 2x$.

Solution:
(a) Theoretical Solution :

1) **Find Points of Intersection:**
To find the points of intersection, solve the system of equations:

$$y^2 = 4x, \tag{1}$$
$$y = 2x. \tag{2}$$

Substitute $y = 2x$ into $y^2 = 4x$:

$$(2x)^2 = 4x, \tag{3}$$
$$4x^2 = 4x, \tag{4}$$
$$x(x-1) = 0. \tag{5}$$

Thus, $x = 0$ or $x = 1$. For $x = 0$ $y = 2 \cdot 0 = 0$, so one point is $(0, 0)$. For $x = 1$, $y = 2 \cdot 1 = 2$, so the other point is $(1, 2)$. Therefore, the curves intersect at $(0, 0)$ and $(1, 2)$.

2) **Set Up the Integral:**
The parabola is $y^2 = 4x \implies x = \frac{y^2}{4}$, and the line is $x = \frac{y}{2}$. To calculate the area, we integrate the difference between the parabola and the line in terms of y, from $y = 0$ to $y = 2$:

$$\text{Area} = \int_{y=0}^{y=2} \left(\frac{y}{2} - \frac{y^2}{4}\right) dy. \tag{6}$$

3) **Evaluate the Integral:**
Expand the integral:

$$\text{Area} = \int_0^2 \frac{y}{2}\, dy - \int_0^2 \frac{y^2}{4}\, dy. \tag{7}$$

First term:

$$\int_0^2 \frac{y}{2}\, dy = \frac{1}{2}\int_0^2 y\, dy = \frac{1}{2}\left(\frac{y^2}{2}\right)_0^2 = \frac{1}{2} \cdot \frac{4}{2} = 1. \tag{8}$$

Second term:

$$\int_0^2 \frac{y^2}{4}\,dy = \frac{1}{4}\int_0^2 y^2\,dy = \frac{1}{4}\left(\frac{y^3}{3}\right)_0^2 = \frac{1}{4}\cdot\frac{8}{3} = \frac{2}{3}. \tag{9}$$

Now subtract:

$$\text{Area} = 1 - \frac{2}{3} = \frac{3}{3} - \frac{2}{3} = \frac{1}{3}. \tag{10}$$

(b) Numerical Solution / Simulation :

4) The integral representing the area is:

$$A = \int_{y=0}^{y=2} \left(-x_{\text{parabola}} + x_{\text{line}}\right) dy. \tag{11}$$

From the equations of the curves:

$$x_{\text{parabola}} = \frac{y^2}{4}, \tag{12}$$

$$x_{\text{line}} = \frac{y}{2}. \tag{13}$$

Substituting:

$$A = \int_0^2 \left(-\frac{y^2}{4} + \frac{y}{2}\right) dy. \tag{14}$$

5) Divide the interval $[0,2]$ into n equal subintervals. Each subinterval has a width:

$$h = \frac{2}{n}. \tag{15}$$

Let the points be:

$$y_0 = 0,\ y_1 = h,\ y_2 = 2h,\ \ldots,\ y_n = 2. \tag{16}$$

At each point y_i, the function to evaluate is:

$$f(y_i) = -\frac{y_i^2}{4} + \frac{y_i}{2}. \tag{17}$$

6) Using the trapezoidal rule, the approximation for the area is:

$$A \approx \frac{h}{2}\left(f(y_0) + 2f(y_1) + 2f(y_2) + \cdots + 2f(y_{n-1}) + f(y_n)\right). \tag{18}$$

7) Substitute the expression for $f(y_i)$:

$$f(y_i) = -\frac{(i \cdot h)^2}{4} + \frac{i \cdot h}{2}. \tag{19}$$

Hence:

$$A \approx \frac{h}{2}\left(\left(-\frac{y_0^2}{4} + \frac{y_0}{2}\right) + 2\sum_{i=1}^{n-1}\left(-\frac{(i \cdot h)^2}{4} + \frac{i \cdot h}{2}\right) + \left(-\frac{y_n^2}{4} + \frac{y_n}{2}\right)\right). \tag{20}$$

8) Simplify the terms: - At $y_0 = 0$: $f(y_0) = 0$.
- At $y_n = 2$:

$$f(y_n) = -\frac{2^2}{4} + \frac{2}{2} = 1 - 1 = 0. \tag{21}$$

Thus:

$$A \approx \frac{h}{2} \cdot 2\sum_{i=1}^{n-1}\left(-\frac{(i \cdot h)^2}{4} + \frac{i \cdot h}{2}\right). \tag{22}$$

9) Substitute $h = \frac{2}{n}$:

$$A \approx \frac{\frac{2}{n}}{2} \cdot 2\sum_{i=1}^{n-1}\left(-\frac{(i \cdot \frac{2}{n})^2}{4} + \frac{i \cdot \frac{2}{n}}{2}\right). \tag{23}$$

Simplify:

$$A \approx \frac{1}{n}\sum_{i=1}^{n-1}\left(-\frac{(2i/n)^2}{4} + \frac{2i/n}{2}\right). \tag{24}$$

Further:

$$A \approx \frac{1}{n}\sum_{i=1}^{n-1}\left(-\frac{i^2}{n^2} + \frac{i}{n}\right). \tag{25}$$

10) To derive the difference equation for the area approximation $A_{\text{trap}}(n)$, write:

$$A_{\text{trap}}(n) = \frac{1}{n}\sum_{i=1}^{n-1}\left(\frac{i}{n} - \frac{i^2}{n^2}\right). \tag{26}$$

11) Using the difference equation (26) we can code to simulate the area pretty easily . Choosing n = 1000 we get area as 0.33333 which verifies with the theoretical solution.

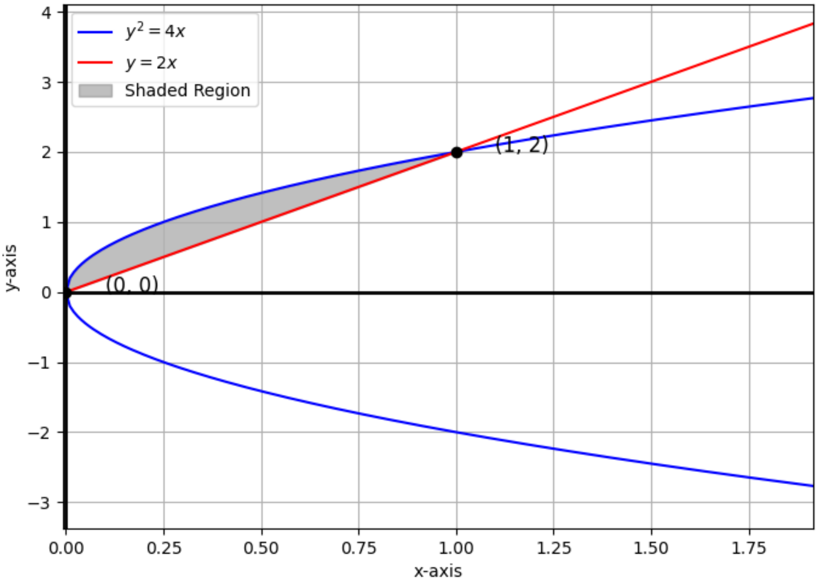
$y^2 = 4x$
$y = 2x$
Shaded Region
(1, 2)
(0, 0)
y-axis
x-axis
4
3
2
1
0
−1
−2
−3
0.00
0.25
0.50
0.75
1.00
1.25
1.50
1.75

Fig. 1: Graph