NCERT-8.2.7

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Question : Calculate the area bounded by the curves $y^2 = 4x$ and $y = 2x$.

Solution:

(a) Theoretical Solution :

1) **Find Points of Intersection:**
To find the points of intersection, solve the system of equations:

$$y^2 = 4x, \tag{1}$$

$$y = 2x. \tag{2}$$

Substitute $y = 2x$ into $y^2 = 4x$:

$$(2x)^2 = 4x, \tag{3}$$

$$4x^2 = 4x, \tag{4}$$

$$x(x-1) = 0. \tag{5}$$

Thus, $x = 0$ or $x = 1$. For $x = 0$ $y = 2 \cdot 0 = 0$, so one point is $(0, 0)$. For $x = 1$, $y = 2 \cdot 1 = 2$, so the other point is $(1, 2)$. Therefore, the curves intersect at $(0, 0)$ and $(1, 2)$.

2) **Set Up the Integral:**
The parabola is $y^2 = 4x \implies x = \frac{y^2}{4}$, and the line is $x = \frac{y}{2}$. To calculate the area, we integrate the difference between the parabola and the line in terms of y, from $y = 0$ to $y = 2$:

$$\text{Area} = \int_{y=0}^{y=2} \left(\frac{y}{2} - \frac{y^2}{4} \right) dy. \tag{6}$$

3) **Evaluate the Integral:**
Expand the integral:

$$\text{Area} = \int_0^2 \frac{y}{2}\, dy - \int_0^2 \frac{y^2}{4}\, dy. \tag{7}$$

First term:

$$\int_0^2 \frac{y}{2}\, dy = \frac{1}{2} \int_0^2 y\, dy = \frac{1}{2} \left(\frac{y^2}{2} \right)_0^2 = \frac{1}{2} \cdot \frac{4}{2} = 1. \tag{8}$$

Second term:

$$\int_0^2 \frac{y^2}{4}\,dy = \frac{1}{4}\int_0^2 y^2\,dy = \frac{1}{4}\left(\frac{y^3}{3}\right)_0^2 = \frac{1}{4}\cdot\frac{8}{3} = \frac{2}{3}. \tag{9}$$

Now subtract:

$$\text{Area} = 1 - \frac{2}{3} = \frac{3}{3} - \frac{2}{3} = \frac{1}{3}. \tag{10}$$

(b) Numerical Solution / Simulation :

We aim to compute the integral:

$$I = \int_0^2 \left(\frac{y}{2} - \frac{y^2}{4}\right) dy$$

using the trapzoidal trule approach.

4) **Discretize the Interval:** Divide the interval $[0,2]$ into $N = 100$ equal subintervals. The step size is:

$$h = \frac{2-0}{N} = \frac{2}{100} = 0.02. \tag{11}$$

The discrete points are:

$$y_i = 0 + i\cdot h, \quad \text{for } i = 0,1,2,\ldots,100. \tag{12}$$

For example:

$$y_0 = 0, \quad y_1 = 0.02, \quad y_2 = 0.04, \ldots, y_{100} = 2. \tag{13}$$

5) **Define the Function:** The function to integrate is:

$$f(y) = \frac{y}{2} - \frac{y^2}{4}. \tag{14}$$

6) **Establish the difference equation:**
Using the trapazoidal rule ,

$$I_n = I_{n-1} + \frac{h}{2}\left(f(y_n) + f(y_{n-1})\right) \tag{15}$$

where:

a) I_n: The approximate integral value up to the n-th point,
b) $h = 0.02$: The step size,
c) $f(y_n) = \frac{y_n}{2} - \frac{y_n^2}{4}$: The function evaluated at y_n.

So, the integral can be approximated as,

$$I_n = I_{n-1} + \frac{h}{2}\left(\left(\frac{y_n}{2} - \frac{y_n^2}{4}\right) + \left(\frac{y_{n-1}}{2} - \frac{y_{n-1}^2}{4}\right)\right) \tag{16}$$

$$y_n = y_{n-1} + h \tag{17}$$

7) **Iterative Computation:** The recurrence relation is applied iteratively starting with the initial condition:

$$I_0 = 0. \tag{18}$$

Each step updates I_n using the values of $f(y_n)$ and $f(y_{n-1})$.

8) **Final Value:** After iterating up to $n = 100$, the value of the integral at the upper bound $y = 2$ is:

$$I[100] \approx 0.3333 \tag{19}$$

Using the difference equation (16) **we can code to simulate the area pretty easily . Choosing n = 100 we get area as 0.3333 which verifies with the theoretical solution.**

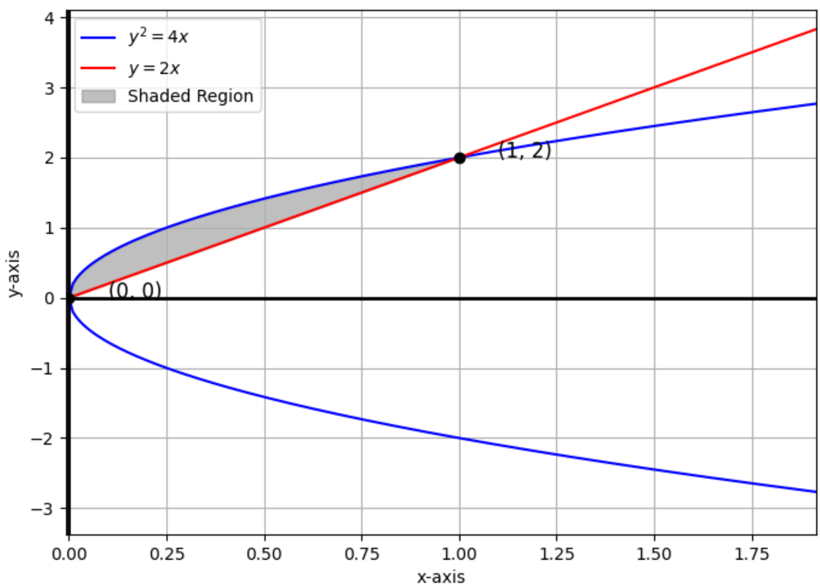

Fig. 8: Graph