NCERT-8.2.7

EE24BTECH11036 - Krishna Patil

Question : Calculate the area bounded by the curves $y^2 = 4x$ and $y = 2x$.

Solution:

(a) Theoretical Solution :

1) **Find Points of Intersection:**

2) **Find Points of Intersection:**
   For $y^2 = 4x$:
$$V_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad u_1 = \begin{pmatrix} -2 \\ 0 \end{pmatrix}, \quad f_1 = 0$$
   The quadratic form is:
$$\begin{pmatrix} x \\ y \end{pmatrix}^\top \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + 2 \begin{pmatrix} -2 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + 0 = 0$$
   For $y = 2x$:
$$V_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad u_2 = \begin{pmatrix} -1 \\ 0.5 \end{pmatrix}, \quad f_2 = 0$$
   The quadratic form is:
$$\begin{pmatrix} x \\ y \end{pmatrix}^\top \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + 2 \begin{pmatrix} -1 & 0.5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + 0 = 0$$

3) **Write the two equations in matrix form:**
   - From $Q_1(x, y)$:
$$\begin{pmatrix} x \\ y \end{pmatrix}^\top \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + 2 \begin{pmatrix} -2 \\ 0 \end{pmatrix}^\top \begin{pmatrix} x \\ y \end{pmatrix} = 0$$
     Simplifies to:
$$y^2 - 4x = 0$$
   - From $Q_2(x, y)$:
$$2 \begin{pmatrix} -1 \\ 0.5 \end{pmatrix}^\top \begin{pmatrix} x \\ y \end{pmatrix} = 0$$
     Simplifies to:
$$-2x + y = 0 \quad \text{or equivalently, } y = 2x$$

4) **Substitute $y = 2x$ from $Q_2(x, y)$ into $Q_1(x, y)$:**
$$(2x)^2 - 4x = 0 \tag{1}$$
$$4x^2 - 4x = 0 \tag{2}$$
$$x(x - 1) = 0 \tag{3}$$

5) **Solve for x:**

$$x = 0 \quad \text{or} \quad x = 1$$

6) **Find y for each x:**
   - For $x = 0$, $y = 2(0) = 0$
   - For $x = 1$, $y = 2(1) = 2$

7) **The intersection points are:**

$$(0, 0) \quad \text{and} \quad (1, 2)$$

8) **Set Up the Integral:**
The parabola is $y^2 = 4x \implies x = \frac{y^2}{4}$, and the line is $x = \frac{y}{2}$. To calculate the area, we integrate the difference between the parabola and the line in terms of y, from $y = 0$ to $y = 2$:

$$\text{Area} = \int_{y=0}^{y=2} \left(\frac{y}{2} - \frac{y^2}{4} \right) dy. \tag{4}$$

9) **Evaluate the Integral:**
Expand the integral:

$$\text{Area} = \int_0^2 \frac{y}{2}\, dy - \int_0^2 \frac{y^2}{4}\, dy. \tag{5}$$

First term:

$$\int_0^2 \frac{y}{2}\, dy = \frac{1}{2} \int_0^2 y\, dy = \frac{1}{2} \left(\frac{y^2}{2} \right)_0^2 = \frac{1}{2} \cdot \frac{4}{2} = 1. \tag{6}$$

Second term:

$$\int_0^2 \frac{y^2}{4}\, dy = \frac{1}{4} \int_0^2 y^2\, dy = \frac{1}{4} \left(\frac{y^3}{3} \right)_0^2 = \frac{1}{4} \cdot \frac{8}{3} = \frac{2}{3}. \tag{7}$$

Now subtract:

$$\text{Area} = 1 - \frac{2}{3} = \frac{3}{3} - \frac{2}{3} = \frac{1}{3}. \tag{8}$$

(b) Numerical Solution / Simulation :

We aim to compute the integral:

$$I = \int_0^2 \left(\frac{y}{2} - \frac{y^2}{4} \right) dy$$

using the trapzoidal trule approach.

10) **Discretize the Interval:** Divide the interval $[0, 2]$ into $N = 100$ equal subintervals.

The step size is:

$$h = \frac{2-0}{N} = \frac{2}{100} = 0.02. \tag{9}$$

The discrete points are:

$$y_i = 0 + i \cdot h, \quad \text{for } i = 0, 1, 2, \ldots, 100. \tag{10}$$

For example:

$$y_0 = 0, \quad y_1 = 0.02, \quad y_2 = 0.04, \ldots, y_{100} = 2. \tag{11}$$

11) **Define the Function:** The function to integrate is:

$$f(y) = \frac{y}{2} - \frac{y^2}{4}. \tag{12}$$

12) **Establish the difference equation:**
Using the trapazoidal rule ,

$$I_n = I_{n-1} + \frac{h}{2}\left(f(y_n) + f(y_{n-1})\right) \tag{13}$$

where:

a) I_n: The approximate integral value up to the n-th point,
b) $h = 0.02$: The step size,
c) $f(y_n) = \frac{y_n}{2} - \frac{y_n^2}{4}$: The function evaluated at y_n.

So, the integral can be approximated as,

$$I_n = I_{n-1} + \frac{h}{2}\left(\left(\frac{y_n}{2} - \frac{y_n^2}{4}\right) + \left(\frac{y_{n-1}}{2} - \frac{y_{n-1}^2}{4}\right)\right) \tag{14}$$

$$y_n = y_{n-1} + h \tag{15}$$

13) **Iterative Computation:** The recurrence relation is applied iteratively starting with the initial condition:

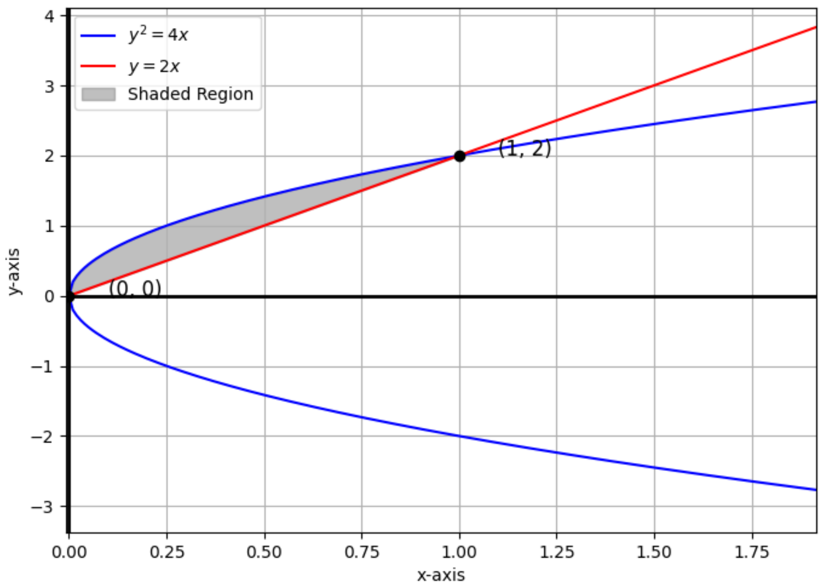
$$I_0 = 0. \tag{16}$$

Each step updates I_n using the values of $f(y_n)$ and $f(y_{n-1})$.

14) **Final Value:** After iterating up to $n = 100$, the value of the integral at the upper bound $y = 2$ is:

$$I[100] \approx 0.3333 \tag{17}$$

Using the difference equation (14) **we can code to simulate the area pretty easily . Choosing n = 100 we get area as 0.3333 which verifies with the theoretical solution.**



$y^2 = 4x$
$y = 2x$
Shaded Region
(1, 2)
(0, 0)
y-axis
x-axis
4
3
2
1
0
−1
−2
−3
0.00
0.25
0.50
0.75
1.00
1.25
1.50
1.75

Fig. 14: Graph