NCERT 12.9.6.4

Solving a Differential Equation

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January 9, 2025

Problem Statement

Solve the following differential equation:

$$\frac{dy}{dx} + y \sec x = \tan x \tag{1}$$

with the initial conditions:

$$x = 0, \quad y = 1. \tag{2}$$

Step 1: Recognize the Linear Form

The given equation is a first-order linear differential equation of the form:

$$\frac{dy}{dx} + P(x)y = Q(x),$$

where

$$P(x) = \sec x,$$
$$Q(x) = \tan x.$$

Step 2: Find the Integrating Factor

Step Find the Integrating Factor: Find the Integrating Factor

The integrating factor (IF) is given by:

$$\mu(x) = e^{\int P(x)dx} = e^{\int \sec x \, dx}. \tag{3}$$

The integral of $\sec x$ is:

$$\int \sec x \, dx = \ln|\sec x + \tan x|. \tag{4}$$

Thus, the integrating factor becomes:

$$\mu(x) = \sec x + \tan x. \tag{5}$$

Step 3: Multiply Through by the Integrating Factor

Multiply the entire differential equation by $\mu(x)$:

$$(\sec x + \tan x)\frac{dy}{dx} + y(\sec x + \tan x)\sec x = (\sec x + \tan x)\tan x. \tag{6}$$

This simplifies to:

$$\frac{d}{dx}\left(y(\sec x + \tan x)\right) = (\sec x + \tan x)\tan x. \tag{7}$$

Step 4: Integrate Both Sides

Integrate both sides:

$$\int \frac{d}{dx}\left(y(\sec x + \tan x)\right) dx = \int (\sec x + \tan x) \tan x \, dx. \tag{8}$$

The solution to the integral gives:

$$y(\sec x + \tan x) = \sec x + \tan x - x + C, \tag{9}$$

where C is the constant of integration.

Step 5: Solve for y

Solve for y:

$$y = \frac{\sec x + \tan x - x + C}{\sec x + \tan x}. \tag{10}$$

Step 6: Determine C

Using the initial conditions $x = 0$, $y = 1$:

$$1 = \frac{\sec 0 + \tan 0 - 0 + C}{\sec 0 + \tan 0}. \tag{11}$$

This simplifies to:

$$1 = 1 + C \implies C = 0. \tag{12}$$

Thus, the solution is:

$$y = \frac{\sec x + \tan x - x}{\sec x + \tan x}. \tag{13}$$

Step 7: Numerical Solution

For a numerical solution, we use the iterative logic:

$$x_0 = 0, \quad y_0 = 1, \quad h = 0.001, \tag{14}$$

$$y_{n+1} = y_n + h \cdot (\tan x_n - y_n \sec x_n), \tag{15}$$

$$x_{n+1} = x_n + h. \tag{16}$$

Verification

Below is a graph verifying the solution:

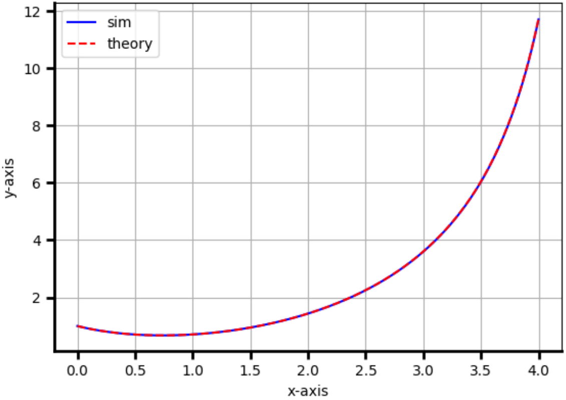

Figure: Graphical Verification of the Solution