

9.ex.5

EE24BTECH11036 - Krishna Patil

Question : Verify that the function $y = a \cos x + b \sin x$, where, $a, b \in \mathbf{R}$ is a solution of the differential equation $\frac{d^2y}{dx^2} + y = 0$ if $f(0) = a$ and $f'(0) = b$.

Solution:

Theoretical solution:

The given differential equation is a second-order linear ordinary differential equation. Let $y(0) = a$ and $y'(0) = b$. By definition of Laplace transform,

$$\mathcal{L}(f(t)) = \int_0^{\infty} e^{-st} f(t) dt \quad (1)$$

Some used properties of Laplace transform include,

$$\mathcal{L}(y'') = s^2 \mathcal{L}(y) - sy(0) - y'(0) = s^2 \mathcal{L}(y) - sa - b \quad (2)$$

$$\mathcal{L}(\cos t) = \frac{s}{s^2 + 1} \quad (3)$$

$$\mathcal{L}(\sin t) = \frac{1}{s^2 + 1} \quad (4)$$

$$\mathcal{L}(cf(t)) = c\mathcal{L}(f(t)) \quad (5)$$

$$\mathcal{L}(f(t)) = F(s) \implies \mathcal{L}(e^{at}f(t)) = F(s-a) \quad (6)$$

Applying Laplace transform on the given differential equation, we get,

$$y'' + y = 0 \quad (7)$$

$$\mathcal{L}(y'') + \mathcal{L}(y) = 0 \quad (8)$$

$$s^2 \mathcal{L}(y) - sa - b + \mathcal{L}(y) = 0 \quad (9)$$

$$\mathcal{L}(y) = \frac{sa + b}{s^2 + 1} = a \frac{s}{s^2 + 1} + b \frac{1}{s^2 + 1} \quad (10)$$

$$(11)$$

Taking laplace inverse on both sides, we get,

$$y = a\mathcal{L}^{-1}\left(\frac{s}{s^2 + 1}\right) + b\mathcal{L}^{-1}\left(\frac{1}{s^2 + 1}\right) \quad (12)$$

$$y = a \cos x + b \sin x \quad (13)$$

$$(14)$$

Computational Solution: Trapezoid Method

The given differential equation can be represented as

$$y'' + y = 0 \quad (15)$$

Let $y = y_1$ and $y' = y_2$, then,

$$\frac{dy_2}{dx} = -y_1 \text{ and } \frac{dy_1}{dx} = y_2 \quad (16)$$

$$\int_{y_{2,n}}^{y_{2,n+1}} dy_2 = \int_{x_n}^{x_{n+1}} -y_1 dx \quad (17)$$

$$\int_{y_{1,n}}^{y_{1,n+1}} dy_1 = \int_{x_n}^{x_{n+1}} y_2 dx \quad (18)$$

$$(19)$$

Discretizing the steps (Trapezoid rule),

$$y_{2,n+1} - y_{2,n} = -\frac{h}{2} (y_{1,n} + y_{1,n+1}) \quad (20)$$

$$y_{1,n+1} - y_{1,n} = \frac{h}{2} (y_{2,n} + y_{2,n+1}) \quad (21)$$

Solving for $y_{1,n+1}$ and $y_{2,n+1}$, we get,

$$y_{1,n+1} = y_{1,n} + \frac{h}{2} \left(2y_{2,n} - \frac{h}{2} (y_{1,n} + y_{1,n+1}) \right) \quad (22)$$

$$(23)$$

The difference equations can be written as,

$$y_{1,n+1} = \frac{(4 - h^2)y_{1,n} + 4hy_{2,n}}{(4 + h^2)} \quad (24)$$

$$y_{2,n+1} = \frac{(4 - h^2)y_{2,n} - 4hy_{1,n}}{(4 + h^2)} \quad (25)$$

$$(26)$$

Iteratively plotting the above system taking initial conditions as

$$x_0 = 0, y_{1,0} = 0, y_{2,0} = 1 \quad (27)$$

we get the plot of the given differential equation.

Alternative Computational Solution: Bilinear transform

We have to apply laplace transformation on the given differential equation. From (10), we get,

$$Y(s) = \frac{sa + b}{s^2 + 1} \quad (28)$$

$$Y(s) = \frac{sa + b}{s^2 + 1} \quad (29)$$

Applying Bilinear transform, with $T = h$, we get,

$$s = \frac{2}{T} \frac{1 - z^{-1}}{1 + z^{-1}} = \frac{2}{h} \frac{1 - z^{-1}}{1 + z^{-1}} \quad (30)$$

$$\Rightarrow Y(z) = \frac{2ha(z^2 - 1) + bh^2(z + 1)^2}{(h^2 + 4)z^2 + 2(h^2 - 4)z + (h^2 + 4)} \quad (31)$$

$$\Rightarrow \left(z^2 + 2\frac{h^2 - 4}{h^2 + 4}z + 1 \right) Y(z) = \frac{2ha(z^2 - 1) + bh^2(z^2 + 2z + 1)}{h^2 + 4} \quad (32)$$

$$\Rightarrow z^2 Y(z) + 2\frac{h^2 - 4}{h^2 + 4}zY(z) + Y(z) = \frac{(2ha + bh^2)z^2 + (2h^2b)z + (h^2b - 2ha)}{h^2 + 4} \quad (33)$$

Some properties of one sided z transform,

$$\mathcal{Z}(y[n + 2]) = z^2 Y(z) - y[1]z - y[0] \quad (34)$$

$$\mathcal{Z}(y[n + 1]) = zY(z) - zy[0] \quad (35)$$

$$\mathcal{Z}(\delta[n]) = 1, z \neq 0 \quad (36)$$

$$\mathcal{Z}(y[n]) = Y(z) \Rightarrow \mathcal{Z}(y[n - n_0]) = z^{-n_0} Y(z) \quad (37)$$

By the time shift property (37),

$$\mathcal{Z}(\delta[n + 2]) = z^2, z \neq 0 \quad (38)$$

$$\mathcal{Z}(\delta[n + 1]) = z, z \neq 0 \quad (39)$$

Rewriting equation (33), we get,

$$\begin{aligned} & z^2 Y(z) + 2\frac{h^2 - 4}{h^2 + 4}zY(z) + Y(z) + (-y[1]z - y[0]) + 2\left(\frac{h^2 - 4}{h^2 + 4}\right)(-zy[0]) \\ &= \frac{(2ha + bh^2)z^2 + (2h^2b - y[1] - 2\left(\frac{h^2 - 4}{h^2 + 4}\right)y[0])z + (h^2b - 2ha - y[0])}{h^2 + 4} \end{aligned} \quad (40)$$

$$\text{where } z \neq 0 \quad (41)$$

Region of convergence (**ROC**) is given by $z \neq 0$.

Taking z inverse transform on both sides of equation (40), we get the **difference equation** which is given by,

$$\begin{aligned} & y[n + 2] + 2\left(\frac{h^2 - 4}{h^2 + 4}\right)y[n + 1] + y[n] \\ &= \frac{(2ha + bh^2)\delta[n + 2] + (2h^2b - y[1] - 2\left(\frac{h^2 - 4}{h^2 + 4}\right)y[0])\delta[n + 1] + (h^2b - 2ha - y[0])\delta[n]}{h^2 + 4} \end{aligned} \quad (42)$$

Here, δ is given by,

$$\delta[n - n_0] = \begin{cases} 1 & n = n_0 \\ 0 & n \neq n_0 \end{cases} \quad (43)$$

As $n > 0$,

$$\delta[n+2] = \delta[n+1] = 0 \quad (44)$$

The equation (42) is now given by,

$$y[n+2] + 2\left(\frac{h^2 - 4}{h^2 + 4}\right)y[n+1] + y[n] = \frac{(h^2b - 2ha - y[0])\delta[n]}{h^2 + 4} \quad (45)$$

At this point we drop the notation $y[n]$ and replace it with y_n , and we replace $a = y(0)$ and $b = y'(0)$,

$$y_{n+2} + 2\left(\frac{h^2 - 4}{h^2 + 4}\right)y_{n+1} + y_n = \frac{(h^2y'(0) - 2hy(0) - y_0)\delta[n]}{h^2 + 4} \quad (46)$$

Note that for computationally plotting the above difference equation, we need $y_0 = y(0)$ as well as y_1 . To find $y_1 = y(0 + h) = y(h)$ we employ first principle of derivative,

$$y'(x) = \lim_{h \rightarrow 0} \frac{y(x+h) - y(x)}{h} \quad (47)$$

$$y(x+h) = y(x) + hy'(x), h \rightarrow 0 \quad (48)$$

$$y_1 = y(h) = y(0) + hy'(0) \quad (49)$$

Iteratively plotting the above system taking initial conditions as

$$x_0 = 0, y_0 = y(0) = 0, y'(0) = 1 \quad (50)$$

we get the plot of the given differential equation.

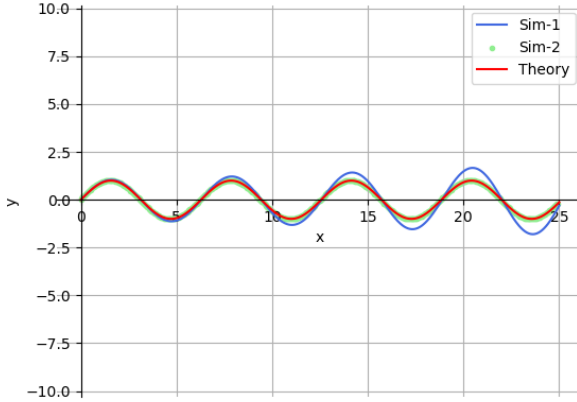


Fig. 0: Here Sim-1 plot represents the plot given by Trapezoid Method, and Sim-2 which is given by Bilinear transform using the same value of h . This plot clearly shows the accuracy of the Bilinear transform method.