9.ex.5

EE24BTECH11036 - Krishna Patil

Question

Question: Verify that the function $y = a\cos x + b\sin x$, where $a, b \in \mathbb{R}$, is a solution of the differential equation

$$\frac{d^2y}{dx^2} + y = 0$$

if $y(0) = a$ and $y'(0) = b$.

Theoretical Solution

The given differential equation is a second-order linear ordinary differential equation.
Let $y(0) = a$ and $y'(0) = b$.
By the definition of Laplace transform:

$$\mathcal{L}\left(f(t)\right) = \int_0^\infty e^{-st} f(t)\, dt$$

Some important properties of Laplace transform:

$$\mathcal{L}\left(y''\right) = s^2\mathcal{L}\left(y\right) - sy(0) - y'(0) = s^2\mathcal{L}\left(y\right) - sa - b$$

$$\mathcal{L}\left(\cos t\right) = \frac{s}{s^2+1}$$

$$\mathcal{L}\left(\sin t\right) = \frac{1}{s^2+1}$$

Laplace Transform of the Differential Equation

Applying the Laplace transform on the given differential equation:

$$y'' + y = 0$$

we get:

$$\mathcal{L}\left(y''\right) + \mathcal{L}\left(y\right) = 0$$

$$s^2\mathcal{L}\left(y\right) - sa - b + \mathcal{L}\left(y\right) = 0$$

$$\mathcal{L}\left(y\right) = \frac{sa + b}{s^2 + 1} = a\frac{s}{s^2 + 1} + b\frac{1}{s^2 + 1}$$

Taking the inverse Laplace transform:

$$y = a\mathcal{L}^{-1}\left(\frac{s}{s^2 + 1}\right) + b\mathcal{L}^{-1}\left(\frac{1}{s^2 + 1}\right)$$

$$y = a\cos x + b\sin x$$

Thus, the function $y = a\cos x + b\sin x$ is indeed a solution to the differential equation.

Computational Solution: Trapezoidal Method

The given differential equation can be represented as:

$$y'' + y = 0$$

Let $y = y_1$ and $y' = y_2$, then:

$$\frac{dy_2}{dx} = -y_1 \quad \text{and} \quad \frac{dy_1}{dx} = y_2$$

Discretizing the steps using the Trapezoid rule:

$$y_{2,n+1} - y_{2,n} = -\frac{h}{2}(y_{1,n} + y_{1,n+1})$$

$$y_{1,n+1} - y_{1,n} = \frac{h}{2}(y_{2,n} + y_{2,n+1})$$

Solving for $y_{1,n+1}$ and $y_{2,n+1}$:

$$y_{1,n+1} = \frac{(4 - h^2)y_{1,n} + 4hy_{2,n}}{(4 + h^2)}$$

$$y_{2,n+1} = \frac{(4 - h^2)y_{2,n} - 4hy_{1,n}}{(4 + h^2)}$$

Computational Solution: Bilinear Transform

We apply the Laplace transform:

$$Y(s) = \frac{sa + b}{s^2 + 1}$$

Using the Bilinear transform:

$$s = \frac{2}{h} \frac{1 - z^{-1}}{1 + z^{-1}}$$

$$Y(z) = \frac{2ha(z^2 - 1) + bh^2(z+1)^2}{(h^2+4)z^2 + 2(h^2-4)z + (h^2+4)}$$

Rewriting this, we get the difference equation:

$$\boxed{z^2 Y(z) + 2\frac{h^2-4}{h^2+4} zY(z) + Y(z) = \frac{(2ha + bh^2)z^2 + (2h^2 b)z + (h^2 b - 2}{h^2+4}}$$

By taking the inverse z-transform, we obtain the difference equation:

$$\boxed{y_{n+2} + 2\frac{h^2-4}{h^2+4} y_{n+1} + y_n = \frac{(h^2 b - 2ha - y_0)}{h^2+4}}$$

Region of Convergence (ROC)

Laplace Transform ROC:

$$\boxed{\mathrm{Re}(s) > 0}$$

The ROC is the right half of the complex plane.

Bilinear Transform ROC:

$$\boxed{|z| < 1}$$

The ROC is the inside of the unit circle in the Z-plane.

Results: Plot Comparison

Iteratively solving the difference equations and plotting the results using the Trapezoidal method and Bilinear transform, we get the following plot:

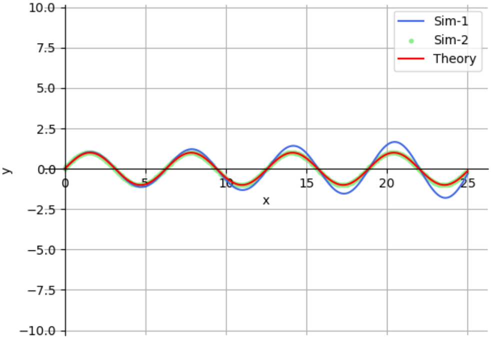

Figure: Sim-1 is from the Trapezoidal Method and Sim-2 is from the Bilinear Transform, showing the accuracy of the Bilinear method.