Numerical Solution of $\left(\frac{d^2y}{dx^2}\right)^2 + \cos\left(\frac{dy}{dx}\right) = 0$

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Problem Statement

Given Equation

Solve the ODE:

$$\left(\frac{d^2y}{dx^2}\right)^2 + \cos\left(\frac{dy}{dx}\right) = 0.$$

- This equation models nonlinear systems with oscillatory behavior and is a great candidate for numerical methods.
- We will explore how to solve it using Euler's and Runge-Kutta methods.

Euler's Method: Overview

Euler's Method provides a simple numerical approximation to the ODE $\frac{dy}{dx} = f(x, y)$, given by:

$$y_{n+1} = y_n + h \cdot f(x_n, y_n),$$

where h is the step size.
For the problem at hand, the updates for y and $v = \frac{dy}{dx}$ become:

$$y_{n+1} = y_n + h \cdot v_n,$$
$$v_{n+1} = v_n + h \cdot \pm\sqrt{-\cos(v_n)}.$$

Key Limitations of Euler's Method

- It is prone to numerical instability.
- Large step sizes can lead to significant errors.

Limitations of Euler's Method

Euler's method struggles with this nonlinear equation:

$$\left(\frac{d^2y}{dx^2}\right)^2 + \cos\left(\frac{dy}{dx}\right) = 0.$$

Challenges

- Instability with large step sizes.
- Requires very small step sizes to achieve reasonable accuracy, which increases computation time.

Runge-Kutta Method (RK4)

RK4 Formula: The RK4 method provides a more accurate solution through the following formula:

$$y_{n+1} = y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4),$$

where

$$\begin{aligned}
k_1 &= h \cdot f(x_n, y_n), \\
k_2 &= h \cdot f\left(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right), \\
k_3 &= h \cdot f\left(x_n + \frac{h}{2}, y_n + \frac{k_2}{2}\right), \\
k_4 &= h \cdot f(x_n + h, y_n + k_3).
\end{aligned}$$

Advantages of RK4 over Euler's Method

- Higher-order accuracy ($O(h^4)$).
- Better stability, especially for nonlinear equations.

Numerical Implementation (RK4)

RK4 Algorithm

For $n = 0$ to N:

1. Compute k_1, k_2, k_3, k_4 for y and v.
2. Update y_{n+1} and v_{n+1} using the RK4 formulas.

Comparison of Euler's and RK4 Methods

Why RK4 is Better

Euler's method suffers from:

- Numerical instability.
- Inaccuracy with large step sizes.

On the other hand, RK4:

- Achieves higher accuracy and is more stable.
- Reduces error with smaller computational cost than Euler when considering overall accuracy.

Pseudocode:

for n = 0 to N:

$k_1 = h.f(x_n, y_n)$

$k_2 = h.f(x_n + h/2, y_n + k_1/2)$

$k_3 = h.f(x_n + h/2, y_n + k_2/2)$

$k_4 = h.f(x_n + h, y_n + k_3)$

$y_n + 1 = y_n + (k_1 + 2(k_2) + 2(k_3) + k_4)/6$

Numerical Results

Below are the plots :

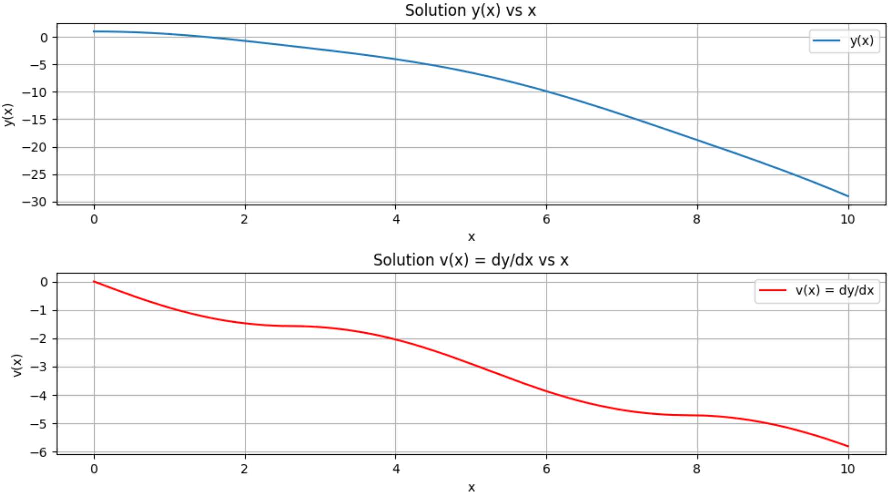

Figure: Numerical result

Conclusion

Summary

- Euler's method is simple but prone to instability and inaccuracy, especially for nonlinear systems.
- The Runge-Kutta method offers higher accuracy and stability, making it the preferred choice for this kind of ODE.
- The RK4 method provides better results with fewer steps due to its higher-order accuracy.