NCERT 9.1.4

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Question: Solve the ODE $(\frac{d^2y}{dx^2})^2 + \cos\left(\frac{dy}{dx}\right) = 0$
Solution: The given equation is:

$$\left(\frac{d^2y}{dx^2}\right)^2 + \cos\left(\frac{dy}{dx}\right) = 0.$$

1) **Rewrite the equation as a system of first-order ODEs.** Let $v = \frac{dy}{dx}$ and $u = \frac{d^2y}{dx^2}$. The equation becomes:

$$u^2 + \cos(v) = 0.$$

From this, we solve for u:

$$u = \pm\sqrt{-\cos(v)}, \quad \text{valid only when } \cos(v) < 0.$$

Thus, the system of equations is:

$$\frac{dy}{dx} = v,$$
$$\frac{dv}{dx} = u = \pm\sqrt{-\cos(v)}.$$

2) **Solve numerically using the RK4 method.** Using the RK4 formula, we compute successive values of y, v, and u iteratively:

$$k_1^y = h \cdot v,$$
$$k_1^v = h \cdot \left(\pm\sqrt{-\cos(v)}\right),$$
$$k_2^y = h \cdot \left(v + \frac{k_1^v}{2}\right),$$
$$k_2^v = h \cdot \left(\pm\sqrt{-\cos\left(v + \frac{k_1^v}{2}\right)}\right),$$
$$k_3^y = h \cdot \left(v + \frac{k_2^v}{2}\right),$$
$$k_3^v = h \cdot \left(\pm\sqrt{-\cos\left(v + \frac{k_2^v}{2}\right)}\right),$$
$$k_4^y = h \cdot \left(v + k_3^v\right),$$
$$k_4^v = h \cdot \left(\pm\sqrt{-\cos\left(v + k_3^v\right)}\right).$$

The updated values are:

$$
\begin{aligned}
y_{n+1} &= y_n + \frac{1}{6}\left(k_1^y + 2k_2^y + 2k_3^y + k_4^y\right), \\
v_{n+1} &= v_n + \frac{1}{6}\left(k_1^v + 2k_2^v + 2k_3^v + k_4^v\right).
\end{aligned}
$$

3) **Numerical Implementation.** For a numerical solution, choose initial values $y(0) = y_0$ and $v(0) = v_0$, and iterate using the formulas above.

Explanation of the Runge-Kutta Method:

The Runge-Kutta method is a powerful numerical technique for solving ordinary differential equations (ODEs). It is particularly effective for approximating solutions to initial value problems of the form:

$$
\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0.
$$

In this document, we focus on the 4th-order Runge-Kutta method (RK4), which is given by the iterative formula:

$$
y_{n+1} = y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4),
$$

where

$$
\begin{aligned}
k_1 &= h \cdot f(x_n, y_n), \\
k_2 &= h \cdot f\left(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right), \\
k_3 &= h \cdot f\left(x_n + \frac{h}{2}, y_n + \frac{k_2}{2}\right), \\
k_4 &= h \cdot f(x_n + h, y_n + k_3).
\end{aligned}
$$

Here, h is the step size, and x_n and y_n are the values of x and y at the n-th step.

1) **Procedure for the RK4 Method:**
   a) Choose the step size h.
   b) Compute k_1, k_2, k_3, and k_4 as defined above.
   c) Update y_{n+1} using the RK4 formula.
   d) Repeat for the desired number of steps or until the solution reaches a specified x-value.
2) **Conclusion:** The Runge-Kutta method is a robust numerical tool for solving systems of ODEs, including complex equations like $(\frac{d^2y}{dx^2})^2 + \cos\left(\frac{dy}{dx}\right) = 0$. By converting the second-order ODE into a system of first-order ODEs, the method can be applied systematically to obtain an approximate solution.

Below is the plot Fig. 2

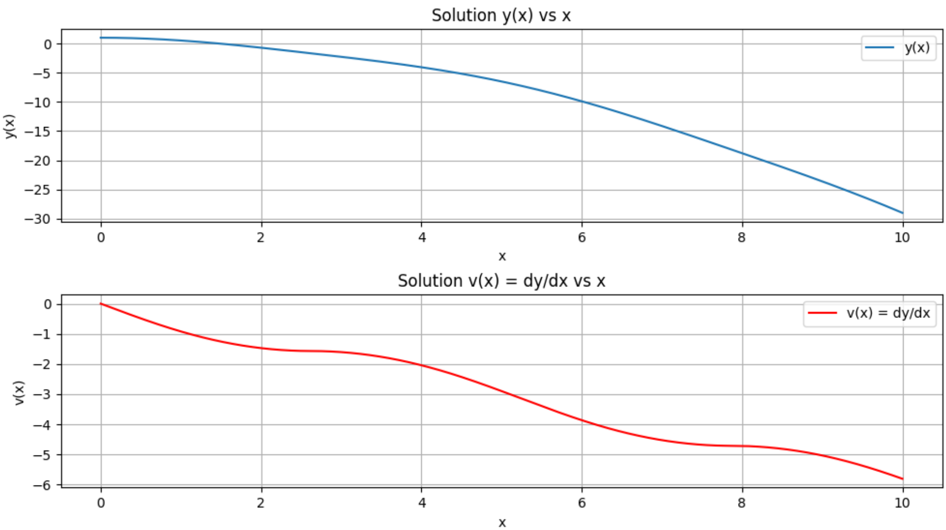

Fig. 2: Verification