10.4.4.2.2

EE24BTECH11036 - Krishna Patil

Question: Find the values of k for following quadratic equations, so that they have two equal roots :

$$kx(x-2)+6=0 \tag{1}$$

Solution: For equal roots of a quadratic equation, the discriminant is zero i.e., For the quadratic $ax^2+bx+c=0$ the discriminant $b^2-4ac=0$.

$$kx(x-2)+6=0 \tag{2}$$
$$kx^2-2kx+6=0 \tag{3}$$
$$\therefore a=k, b=-2k, c=6 \tag{4}$$
$$\therefore 4k^2-24k=0 \tag{5}$$
$$k=0 \quad or \quad k=6 \tag{6}$$

but $k=0$ doesn't make sense as no solution is obtained so, $k=6$ is the answer.

So, the equation becomes

$$6x(x-2)+6=0 \tag{7}$$
$$6x^2-12x+6=0 \tag{8}$$
$$x^2-2x+1=0 \tag{9}$$

1) **Thereotical Solution** :

$$x^2-2x+1=0 \tag{10}$$
$$(x-1)^2=0 \implies x=1 \tag{11}$$

2) **Numerical Solution** :

a) **Fixed point iteration Method** :
Say , we have to find roots of

$$f(x)=0 \tag{12}$$

using algebra, we first solve for x i.e., $x=g(x)$ where, $g(x)$ is some other function formed after solving. then we select an initial guess x_0, iterate using the formula

$$x_{n+1}=g(x_n) \tag{13}$$

Repeat this step until the difference between successive approximations $|x_{n+1}-x_n|$ is less than a specified tolerance ϵ.

say for our case , we choose $\epsilon = 10^{-6}$ and $x_0 = 1.5$ and $f(x) = x^2 - 2x + 1$,

$$\because f(x) = 0 \tag{14}$$

$$x^2 - 2x + 1 = 0 \tag{15}$$

$$\sqrt{2x - 1} = x \tag{16}$$

$$\therefore g(x) = \sqrt{2x - 1} \tag{17}$$

So, the iterative equation is ,

$$x_{n+1} = \left(\sqrt{2x_n - 1}\right) \tag{18}$$

after computing , we obtain $x = 1.0014142$ which is pretty near to the theoretical solution.

b) **Newton-Raphson Method**

The Newton-Raphson method is an iterative technique to find the roots of a real-valued function $f(x)$. The update formula is given by:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \tag{19}$$

For the function $f(x) = x^2 - 2x + 1$, the derivative is:

$$f'(x) = 2x - 2 \tag{20}$$

$$f(x) = x^2 - 2x + 1 \tag{21}$$

$$f'(x) = 2x - 2 \tag{22}$$

The Newton-Raphson update formula becomes:

$$x_{n+1} = x_n - \frac{x_n^2 - 2x_n + 1}{2x_n - 2} \tag{23}$$

After computing , we obtain $x = 0.999999$ which is pretty near to the solution.

c) **CODING LOGIC FOR FINDING EIGENVALUES :-**

The quadratic equation

$$x^2 - 2x + 1 = 0 \tag{24}$$

is rewritten in matrix form:

$$\text{Matrix} = \begin{pmatrix} 0 & -\frac{c}{a} \\ 1 & -\frac{b}{a} \end{pmatrix} \tag{25}$$

$$a = 1, \quad b = -2, \quad c = 1. \tag{26}$$

Substituting the values of a, b and c, the matrix becomes:
Let

$$\mathrm{A} = \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix} \tag{27}$$

QR-DECOMPOSITION:-GRAM-SCHMIDT METHOD

i) QR decomposition

$$A = QR \tag{28}$$

A) Q is an $m \times n$ orthogonal matrix
B) R is an $n \times n$ upper triangular matrix.
Given a matrix $A = [a_1, a_2, \ldots, a_n]$, where each a_i is a column vector of size $m \times 1$.

ii) Normalize the first column of A:

$$q_1 = \frac{a_1}{||a_1||} \tag{29}$$

iii) For each subsequent column a_i, subtract the projections of the previously obtained orthonormal vectors from a_i :

$$a_i' = a_i - \sum_{k=1}^{i-1} \langle a_i, q_k \rangle q_k \tag{30}$$

Normalize the result to obtain the next column of Q:

$$q_i = \frac{a_i'}{\left\| a_i' \right\|} \tag{31}$$

Repeat this process for all columns of A.

iv) Finding R:-
After constructing the ortho-normal columns $q_1, q_2, \ldots, q_n$ of Q, we can compute the elements of R by taking the dot product of the original columns of A with the columns of Q:

$$r_{ij} = \langle a_j, q_i \rangle \text{ , for } i \leq j \tag{32}$$

QR-Algorithm

i) Initialization
Let $A_0 = A$, where A is the given matrix.

ii) QR Decomposition
For each iteration $k = 0, 1, 2, \ldots$:
A) Compute the QR decomposition of A_k, such that:

$$A_k = Q_k R_k \tag{33}$$

where:

B) Q_k is an orthogonal matrix ($Q_k^\top Q_k = I$).

C) R_k is an upper triangular matrix.

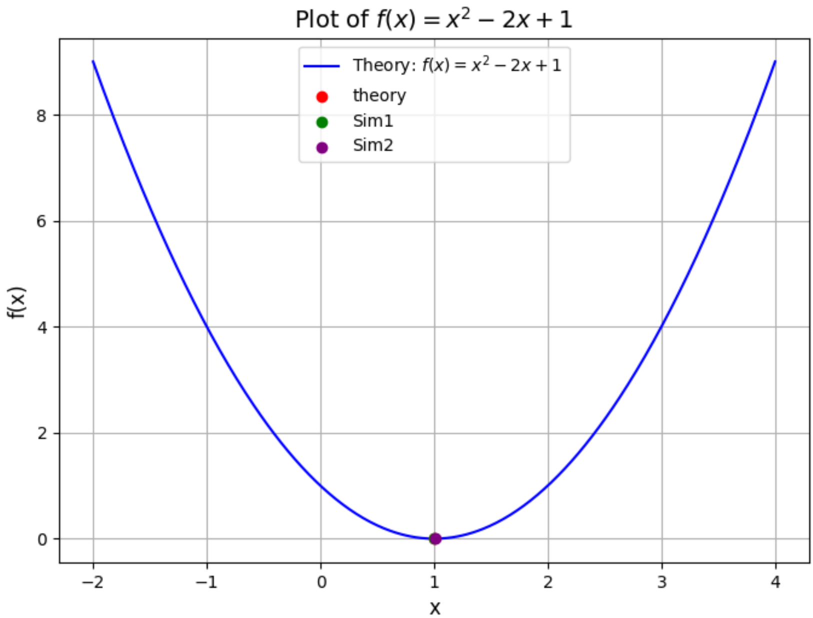
The decomposition ensures $A_k = Q_k R_k$.

D) Form the next matrix A_{k+1} as:

$$A_{k+1} = R_k Q_k \tag{34}$$

iii) Convergence
Repeat Step 2 until A_k converges to an upper triangular matrix T. The diagonal entries of T are the eigenvalues of A.

iv) The eigenvalues of matrix will be the roots of the equation.
how using code we obtain x values as 1.001000 and 0.999000



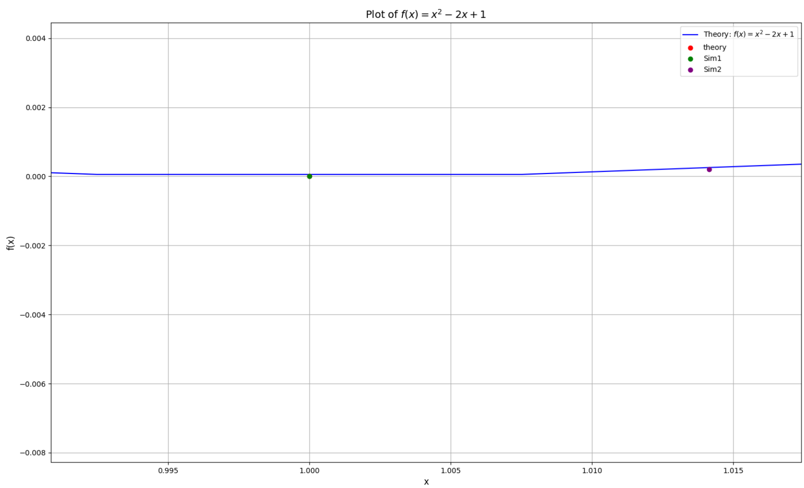


Fig. 2: Zoomed Form