Formule di elettro- e magnetostatica

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1 Equazioni di Maxwell

$$\mathrm{div}\vec{E} = 4\pi\rho \qquad \mathrm{rot}\vec{E} = 0 \qquad \vec{E} = -\nabla V$$
$$\mathrm{div}\vec{B} = 0 \qquad \mathrm{rot}\vec{B} = \tfrac{4\pi}{c}\vec{J} \qquad \vec{B} = \mathrm{rot}\vec{A}$$

$$\vec{F} = q\left(\vec{E} + \frac{\vec{v}}{c} \times \vec{B}\right)$$

2 Risolvere Poisson o Laplace

2.1 Funzione di Green

$$V\left(\vec{r}\right) = \int dr'^3 \rho\left(\vec{r}'\right) G\left(\vec{r}, \vec{r}'\right) - \frac{1}{4\pi}\oint_S \left(V\frac{\partial G}{\partial u} - G\frac{\partial V}{\partial u}\right)$$

Per il piano:

$$G\left(\vec{r}, \vec{r}'\right) = \frac{1}{\left|\vec{r} - \vec{r}'\right|} - \frac{1}{\left|\vec{r} + \vec{r}'\right|}$$

Per la sfera (raggio a):

$$G\left(\vec{r}, \vec{r}'\right) = \frac{1}{\left|\vec{r} - \vec{r}'\right|} - \frac{a}{r'}\frac{1}{\left|\vec{r} - \frac{a^2}{r'^2}\vec{r}'\right|}$$

2.2 Polinomi di Legendre

$$V\left(r, \theta\right) = \sum_{l=0}^{\infty}\left(A_l r^l + \frac{B_l}{r^{l+1}}\right) P_l\left(\cos\theta\right)$$

$$P_0 = 1 \quad P_1 = \cos\theta \quad P_2 = \frac{1}{2}\left(3\cos^2\theta - 1\right) \quad P_l\left(-x\right) = \left(-1\right)^l P_l\left(x\right)$$

Normalizzazione: $\int_{-1}^{1} P_{l'}\left(x\right) P_l\left(x\right) = \frac{2}{2l+1}\delta_{l,l'}$

$$\frac{1}{\left|\vec{r} - \vec{r}'\right|} = \sum_{l=0}^{\infty}\frac{r_<^l}{r_>^{l+1}} P_l\left(\cos\theta\right)$$

2.3 Armoniche sferiche

$$V(r,\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left(A_{l,m}r^l + B_{l,m}\frac{1}{r^{l+1}}\right)Y_{l,m}(\theta,\phi)$$

Lo sviluppo in multipoli di una distribuzione $\rho(\vec{r}\,')$ limitata a grande distanza dall'origine è:

$$V(\vec{r}) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} C_{l,m}\frac{Y_{l,m}(\theta,\phi)}{r^{l+1}}$$

con $C_{l,m} = \int dr'^3 r'^l \rho(\vec{r}\,')\, Y*_{l,m}(\theta,\phi)$. Similmente vicino all'origine. Le prime armoniche sferiche sono:

	0	1	2	3
0	$\frac{1}{2}\sqrt{\frac{1}{\pi}}$	$\frac{1}{2}\sqrt{\frac{3}{\pi}}\cos\theta$	$\frac{1}{4}\sqrt{\frac{5}{\pi}}\left(3\cos^2\theta-1\right)$	$\frac{1}{4}\sqrt{\frac{7}{\pi}}\left(5\cos^3\theta-3\cos\theta\right)$
1		$-\frac{1}{2}\sqrt{\frac{3}{2\pi}}e^{\mathrm{i}\phi}\sin\theta$	$-\frac{1}{2}\sqrt{\frac{15}{2\pi}}e^{\mathrm{i}\phi}\cos\theta\sin\theta$	$-\frac{1}{8}\sqrt{\frac{21}{2\pi}}e^{\mathrm{i}\phi}\sin\theta\left(5\cos^2\theta-1\right)$
2			$\frac{1}{4}\sqrt{\frac{15}{2\pi}}e^{2\mathrm{i}\phi}\sin^2\theta$	$\frac{1}{4}\sqrt{\frac{105}{2\pi}}e^{2\mathrm{i}\phi}\sin^2\theta\cos\theta$
3				$-\frac{1}{8}\sqrt{\frac{35}{\pi}}e^{3\mathrm{i}\phi}\sin^3\theta$

3 Biot-Savart e altro magnetismo

Se la distribuzione di corrente è localizzata

$$\vec{B} = \frac{1}{c}\int dr'^3\frac{\vec{J}\times(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3} \quad \vec{A} = \frac{1}{c}\int dr'^3\frac{\vec{J}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}$$

3.1 Multipoli magnetici

$$\vec{A} = \frac{1}{rc}\int dr'^3\vec{J}(\vec{r}\,')\left(1+\frac{\vec{r}\cdot\vec{r}\,'}{r^2}\right) + \frac{1}{r^3c}\int dr'^3\vec{J}(\vec{r}\,')\left(xx'+yy'+zz'\right)$$

4 Dipoli e altri cazzilli

Si indica con $\vec{p}$ il dipolo elettrico, con $\vec{m}$ quello magnetico e con $\vec{s}$ uno qualsiasi dei due. Inoltre si indica con $\vec{G}$ un campo qualsiasi (elettrico o magnetico).

$$\vec{G} = \frac{3\vec{r}(\vec{s}\cdot\vec{r}) - r^2\vec{s}}{r^5}$$

$$\vec{F} = \left(\vec{p} \cdot \vec{\nabla}\right) \vec{E} \quad U = -\vec{p} \cdot \vec{E} \quad U = \frac{1}{8\pi} \int |\vec{E}|^2 dr^3$$

$$\vec{m} = \frac{1}{2c} \int dr^3 \vec{r} \times \vec{J} \quad \vec{A} = \frac{\vec{m} \times \vec{r}}{r^3} \quad \vec{\tau} = \vec{m} \times \vec{B} \quad \vec{F} = \vec{\nabla}\left(\vec{m} \cdot \vec{B}\right)$$

5 Operatori differenziali

5.1 Gradiente

$$\left(\frac{\partial f}{\partial \rho}, \frac{1}{\rho}\frac{\partial f}{\partial \phi}, \frac{\partial f}{\partial z}\right)$$

$$\left(\frac{\partial f}{\partial \rho}, \frac{1}{\rho}\frac{\partial f}{\partial \theta}, \frac{1}{\rho \sin\theta}\frac{\partial f}{\partial \phi}\right)$$

5.2 Divergenza

$$\frac{1}{\rho}\frac{\partial\left(\rho F_\rho\right)}{\partial \rho} + \frac{1}{\rho}\frac{\partial F_\phi}{\partial \phi} + \frac{\partial F_z}{\partial z}$$

$$\frac{1}{\rho^2}\frac{\partial\left(\rho^2 F_\rho\right)}{\partial r} + \frac{1}{\rho \sin\theta}\frac{\partial\left(\sin\theta F_\theta\right)}{\partial \theta} + \frac{1}{\rho \sin\theta}\frac{\partial F_\phi}{\partial \phi}$$

5.3 Rotore

$$\left(\frac{1}{\rho}\frac{\partial F_z}{\partial \phi} - \frac{\partial F_\phi}{\partial z}, \frac{\partial F_\rho}{\partial z} - \frac{\partial F_z}{\partial \rho}, \frac{1}{\rho}\left(\frac{\partial\left(\rho F_\phi\right)}{\partial \rho} - \frac{\partial F_\rho}{\partial \phi}\right)\right)$$

$$\left(\frac{1}{\rho\sin\theta}\left(\frac{\partial\left(\sin\theta F_\phi\right)}{\partial\theta} - \frac{\partial F_\theta}{\partial\phi}\right), \frac{1}{\rho}\left(\frac{1}{\sin\theta}\frac{\partial F_\rho}{\partial\phi} - \frac{\partial\left(\rho F_\phi\right)}{\partial\rho}\right), \frac{1}{\rho}\left(\frac{\partial\left(\rho F_\theta\right)}{\partial\rho} - \frac{\partial F_\rho}{\partial\theta}\right)\right)$$

5.4 Laplaciano

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial f}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial\phi^2} + \frac{\partial^2 f}{\partial z^2}$$

$$\frac{1}{\rho^2}\frac{\partial}{\partial\rho}\left(\rho^2\frac{\partial f}{\partial\rho}\right) + \frac{1}{\rho^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{\rho^2\sin^2\theta}\frac{\partial^f}{\partial\phi^2}$$