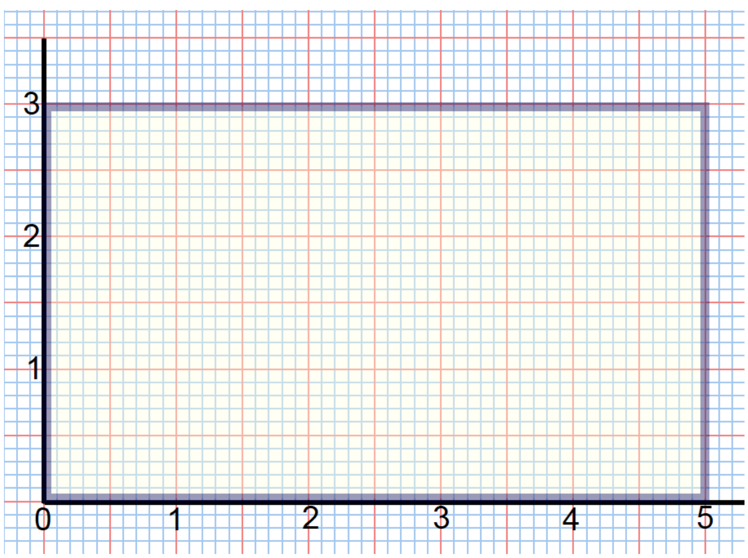
Lesson 1: What Is Area?

Classwork

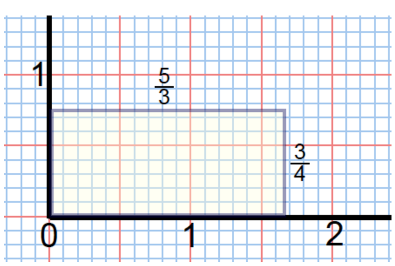
Exploratory Challenge 1

a. What is area?

b. What is the area of the rectangle below whose side lengths measure 3 units by 5 units?

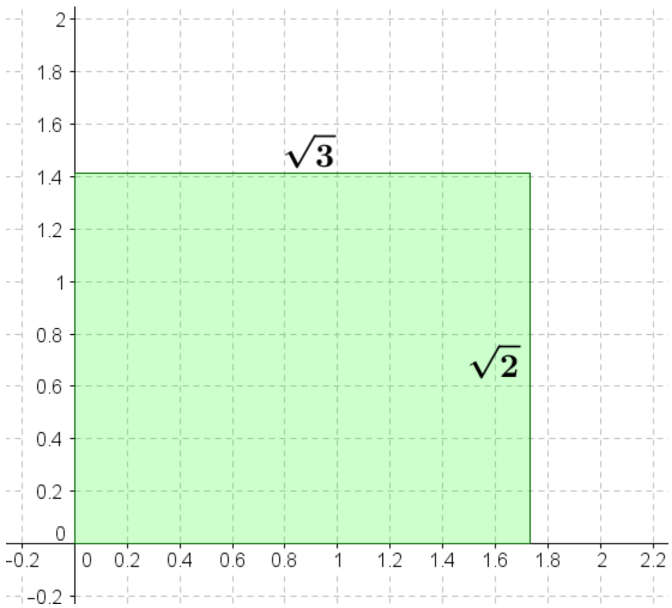


c. What is the area of the $\frac{3}{4} \times \frac{5}{3}$ rectangle below?



Exploratory Challenge 2

a. What is the area of the rectangle below whose side lengths measure $\sqrt{3}$ units by $\sqrt{2}$ units? Use the unit squares on the graph to guide your approximation. Explain how you determined your answer.



b. Is your answer precise?

Discussion

Use Figures 1, 2, and 3 to find upper and lower approximations of the given rectangle.

Figure 1

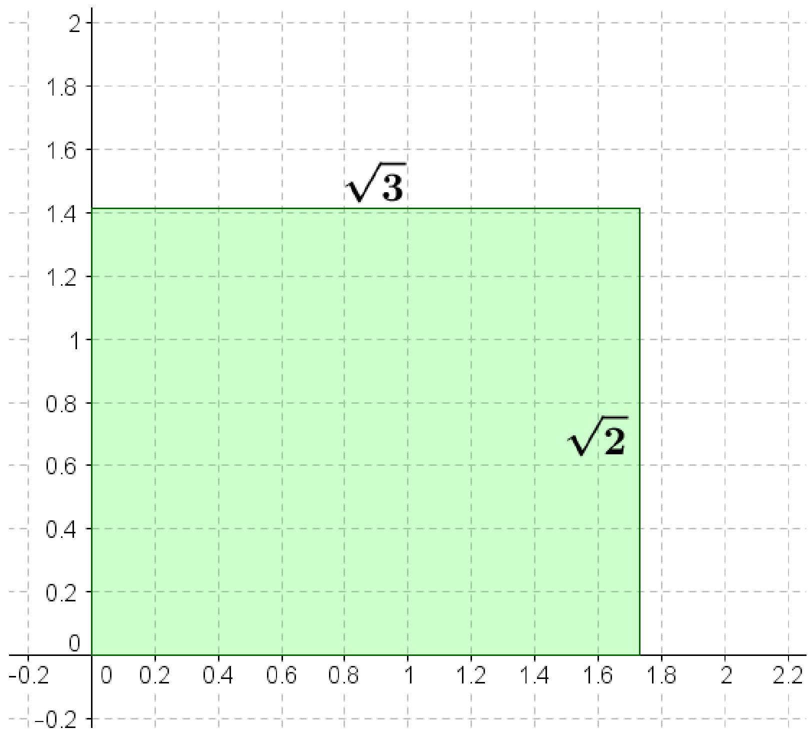


Figure 2

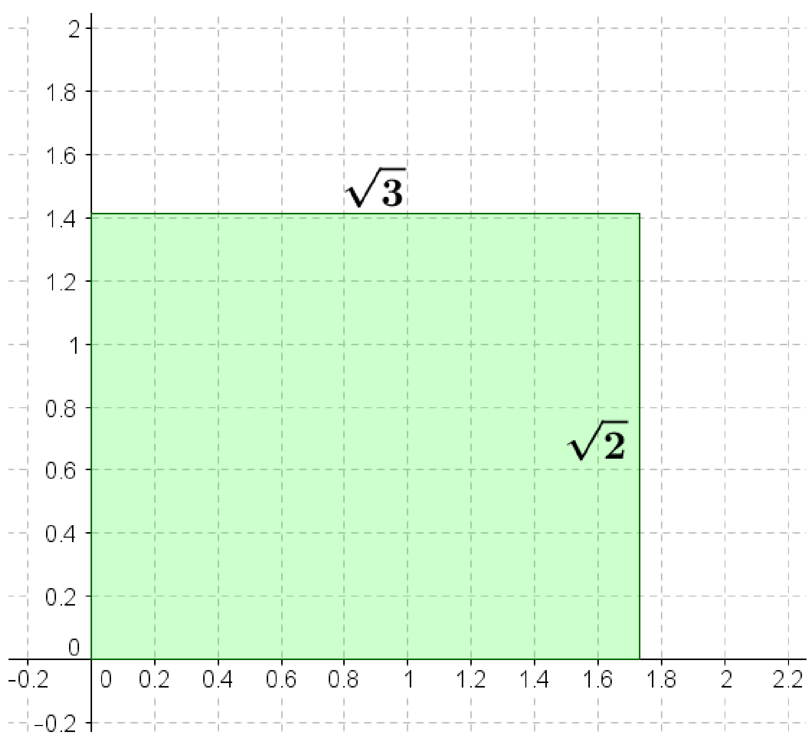
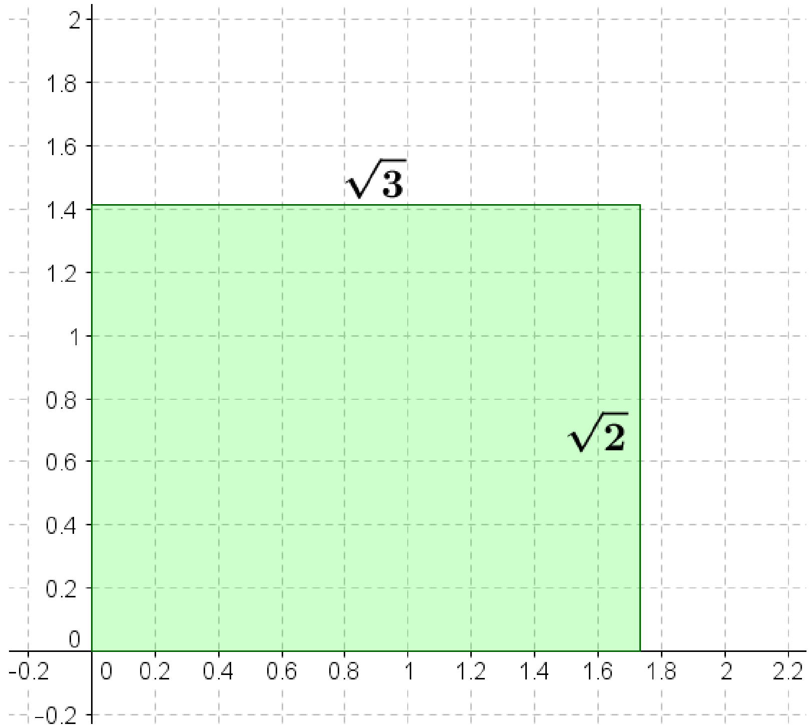


Figure 3

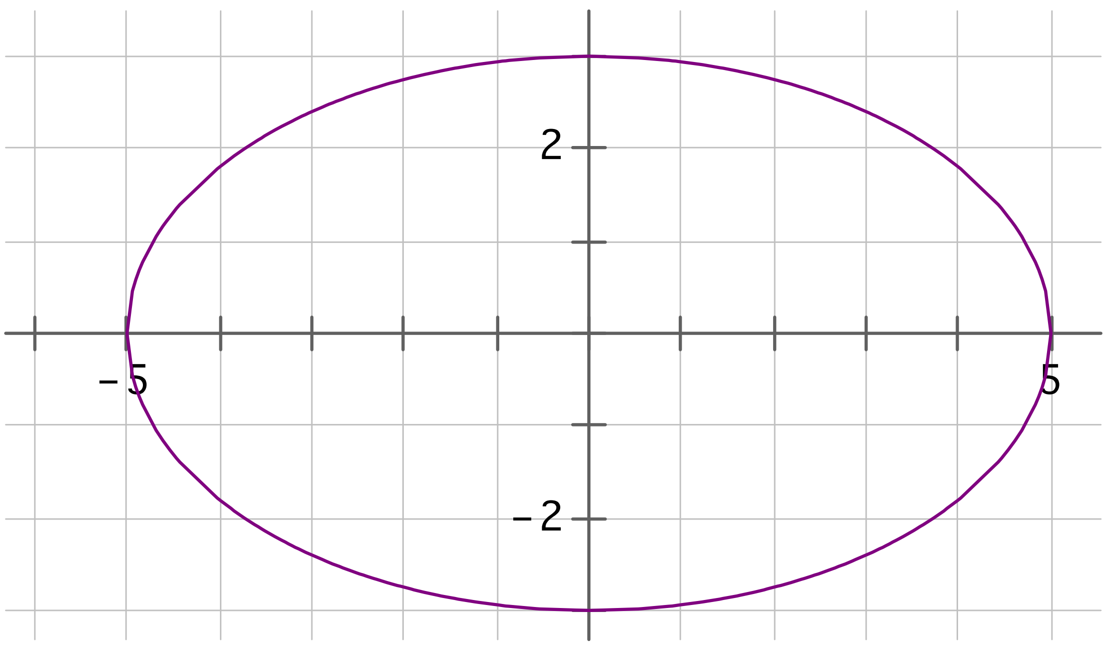


Lower Approximations		
Less than $\sqrt{2}$	Less than $\sqrt{3}$	Less than or equal to A
1	1	$1 \times 1 = 1$
	1.7	$\times 1.7 =$
1.41		$1.41 \times \quad =$
1.414	1.732	$1.414 \times 1.732 =$
1.4142	1.7320	$1.4142 \times 1.7320 = 2.449344$
		$= 2.4494824305$
1.414213	1.732050	$1.414213 \times 1.732050 = 2.44948762665$

Upper Approximations		
Greater than $\sqrt{2}$	Greater than $\sqrt{3}$	Greater than or equal to A
2	2	$2 \times 2 = \ 4$
1.5	1.8	$1.5 \times 1.8 =$
1.42	1.74	$1.42 \times 1.74 = \ 2.4708$
	1.733	$\times 1.733 =$
1.4143	1.7321	$1.4143 \times 1.7321 = \ 2.44970903$
1.41422	1.73206	$1.41422 \times 1.73206 = \ 2.4495138932$
		$= \ 2.449490772914$

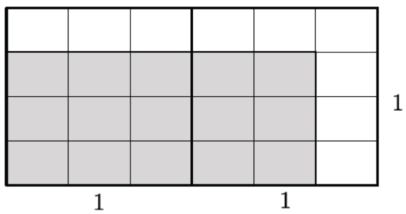
Discussion

If it takes one can of paint to cover a unit square in the coordinate plane, how many cans of paint are needed to paint the region within the curved figure?



Problem Set

1. Use the following picture to explain why $\frac{15}{12}$ is the same as $1\frac{1}{4}$.



2. Figures 1 and 2 below show two polygonal regions used to approximate the area of the region inside an ellipse and above the x-axis.

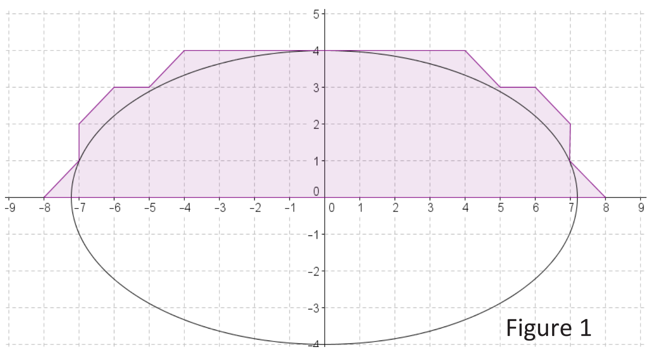


Figure 1

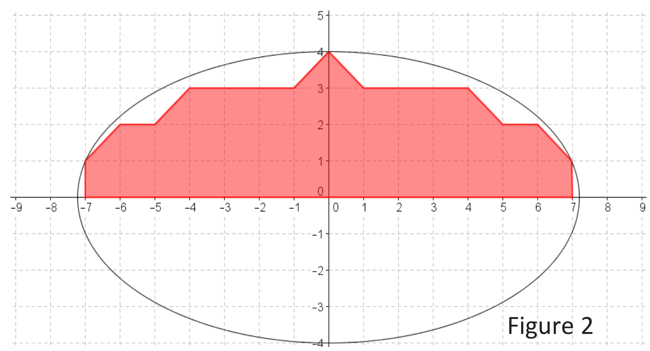


Figure 2

   a. Which polygonal region has a greater area? Explain your reasoning.

   b. Which, if either, of the polygonal regions do you believe is closer in area to the region inside the ellipse and above the x-axis?

3. Figures 1 and 2 below show two polygonal regions used to approximate the area of the region inside a parabola and above the x-axis.

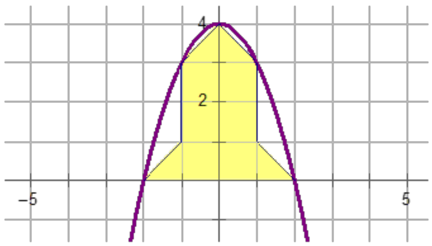


Figure 1

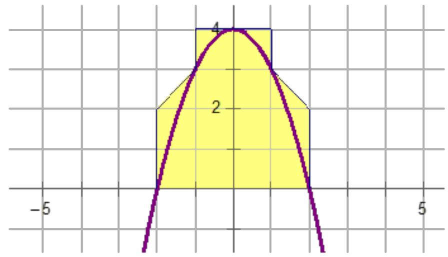


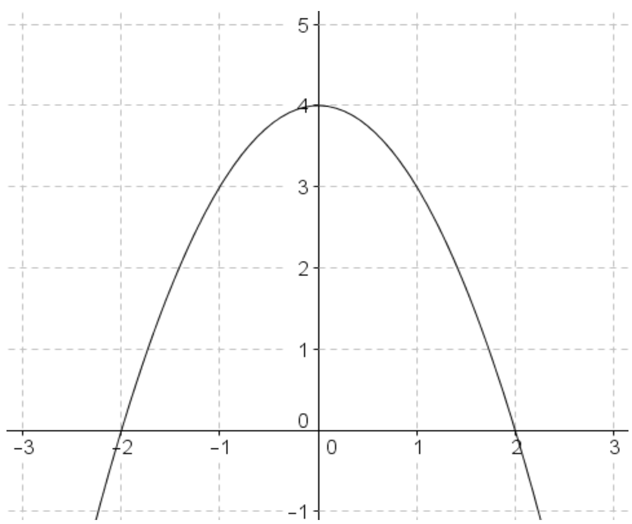
Figure 2

a. Use the shaded polygonal region in Figure 1 to give a lower estimate of the area a under the curve and above the x-axis.

b. Use the shaded polygonal region to give an upper estimate of the area a under the curve and above the x-axis.

c. Use (a) and (b) to give an average estimate of the area a.

4. Problem 4 is an extension of Problem 3. Using the diagram, draw grid lines to represent each $\frac{1}{2}$ unit.



a. What do the new grid lines divide each unit square into?

b. Use the squares described in part (a) to determine a lower estimate of area a in the diagram.

c. Use the squares described in part (a) to determine an upper estimate of area a in the diagram.

d. Calculate an average estimate of the area under the curve and above the x-axis based on your upper and lower estimates in parts (b) and (c).

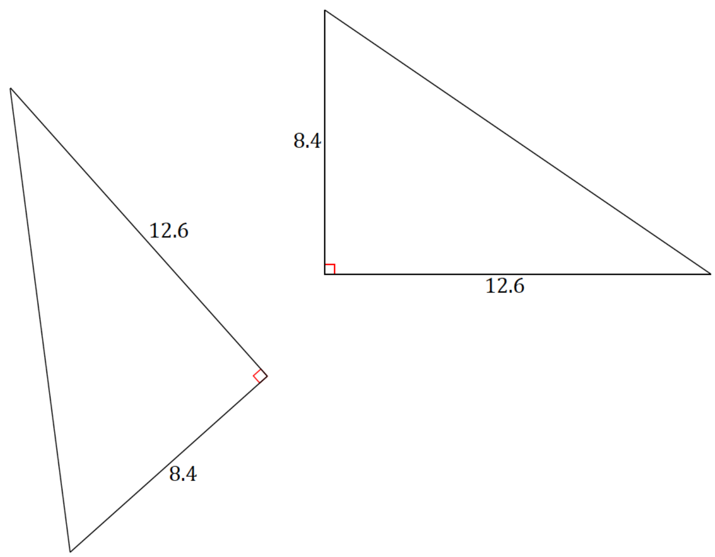
e. Do you think your average estimate in Problem 4 is more or less precise than your estimate from Problem 3? Explain.

Lesson 2: Properties of Area

Classwork

Exploratory Challenge/Exercises 1–4

1. Two congruent triangles are shown below.



   a. Calculate the area of each triangle.

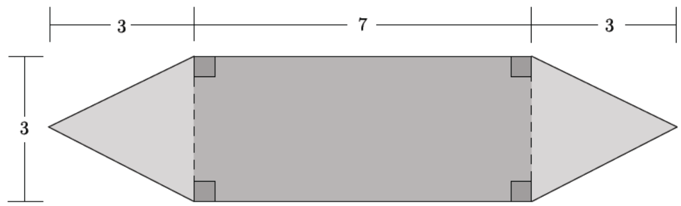
   b. Circle the transformations that, if applied to the first triangle, would always result in a new triangle with the same area:

      Translation   Rotation   Dilation   Reflection

   c. Explain your answer to part (b).

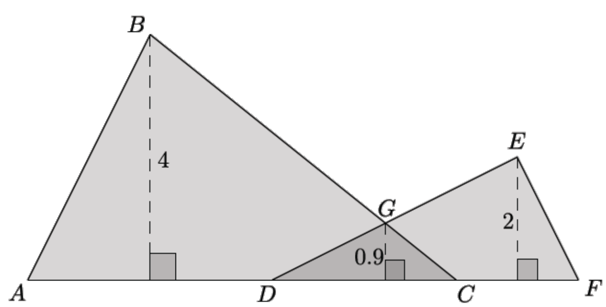
2.

a. Calculate the area of the shaded figure below.



b. Explain how you determined the area of the figure.

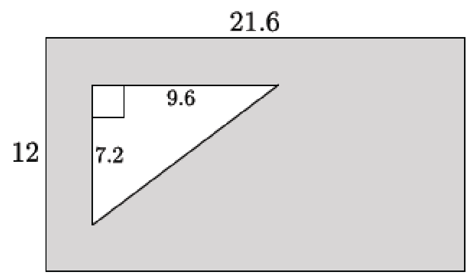
3. Two triangles $\triangle ABC$ and $\triangle DEF$ are shown below. The two triangles overlap forming $\triangle DGC$.



a. The base of figure $ABGEF$ is comprised of segments of the following lengths: $AD = 4$, $DC = 3$, and $CF = 2$. Calculate the area of the figure $ABGEF$.

b. Explain how you determined the area of the figure.

4. A rectangle with dimensions 21.6×12 has a right triangle with a base 9.6 and a height of 7.2 cut out of the rectangle.



a. Find the area of the shaded region.

b. Explain how you determined the area of the shaded region.

Lesson Summary

SET (DESCRIPTION): A *set* is a well-defined collection of objects. These objects are called elements or members of the set.

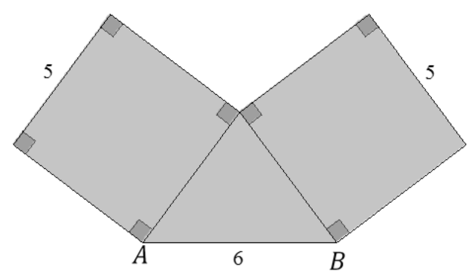
SUBSET: A set A is a *subset* of a set B if every element of A is also an element of B. The notation $A \subseteq B$ indicates that the set A is a subset of set B.

UNION: The *union* of A and B is the set of all objects that are either elements of A or of B, or of both. The union is denoted $A \cup B$.

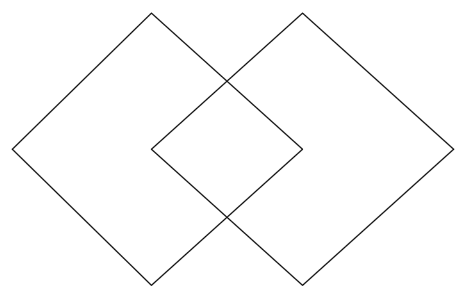
INTERSECTION: The *intersection* of A and B is the set of all objects that are elements of A and also elements of B. The intersection is denoted $A \cap B$.

Problem Set

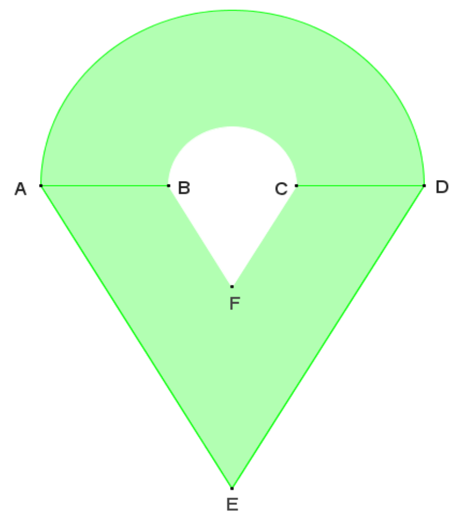
1. Two squares with side length 5 meet at a vertex and together with segment AB form a triangle with base 6 as shown. Find the area of the shaded region.



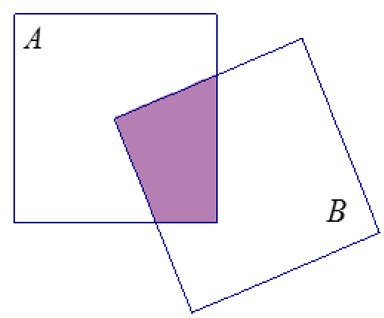
2. If two 2×2 square regions S_1 and S_2 meet at midpoints of sides as shown, find the area of the square region, $S_1 \cup S_2$.



3. The figure shown is composed of a semicircle and a non-overlapping equilateral triangle, and contains a hole that is also composed of a semicircle and a non-overlapping equilateral triangle. If the radius of the larger semicircle is 8, and the radius of the smaller semicircle is $\frac{1}{3}$ that of the larger semicircle, find the area of the figure.



4. Two square regions A and B each have area 8. One vertex of square B is the center point of square A. Can you find the area of $A \cup B$ and $A \cap B$ without any further information? What are the possible areas?



5. Four congruent right triangles with leg lengths a and b and hypotenuse length c are used to enclose the green region in Figure 1 with a square and then are rearranged inside the square leaving the green region in Figure 2.

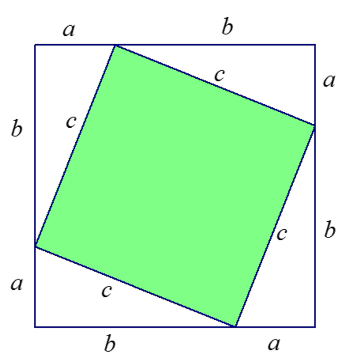


Figure 1

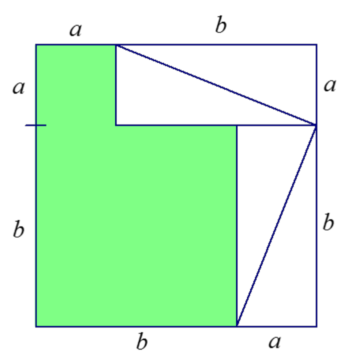


Figure 2

a. Use Property 4 to explain why the green region in Figure 1 has the same area as the green region in Figure 2.

b. Show that the green region in Figure 1 is a square and compute its area.

c. Show that the green region in Figure 2 is the union of two non-overlapping squares and compute its area.

d. How does this prove the Pythagorean theorem?

Lesson 3: The Scaling Principle for Area

Classwork

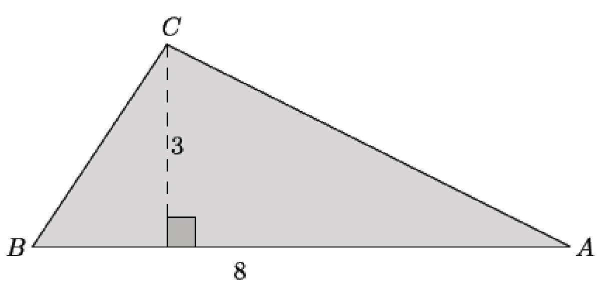
Exploratory Challenge

Complete parts (i)–(iii) of the table for each of the figures in questions (a)–(d): (i) Determine the area of the figure (pre-image), (ii) Determine the scaled dimensions of the figure based on the provided scale factor, (iii) Determine the area of the dilated figure. Then, answer the question that follows.

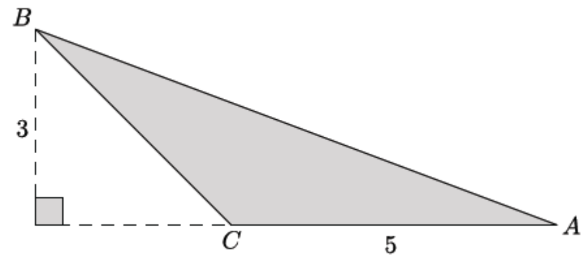
In the final column of the table, find the value of the ratio of the area of the similar figure to the area of the original figure.

(i) Area of Original Figure	Scale Factor	(ii) Dimensions of Similar Figure	(iii) Area of Similar Figure	Ratio of Areas $\text{Area}_{\text{similar}} : \text{Area}_{\text{original}}$
	3			
	2			
	$\frac{1}{2}$			
	$\frac{3}{2}$			

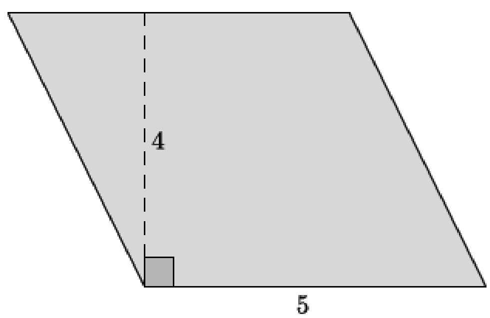
a.



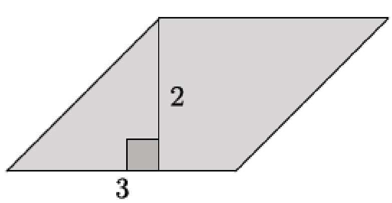
b.



c.



d.



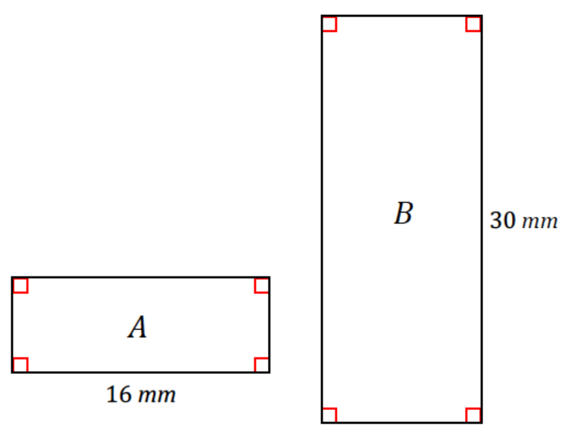
e. Make a conjecture about the relationship between the areas of the original figure and the similar figure with respect to the scale factor between the figures.

THE SCALING PRINCIPLE FOR TRIANGLES:

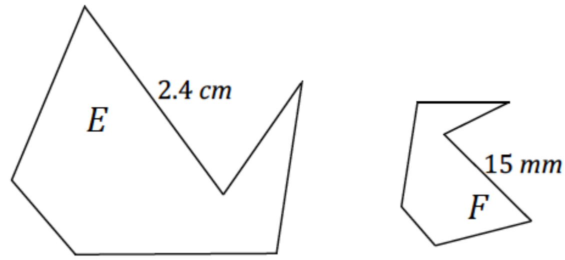
THE SCALING PRINCIPLE FOR POLYGONS:

Exercises 1–2

1. Rectangles A and B are similar and are drawn to scale. If the area of rectangle A is 88 mm^2, what is the area of rectangle B?



2. Figures E and F are similar and are drawn to scale. If the area of figure E is $120\ \mathrm{mm}^2$, what is the area of figure F?



THE SCALING PRINCIPLE FOR AREA:

Lesson Summary

THE SCALING PRINCIPLE FOR TRIANGLES: If similar triangles S and T are related by a scale factor of r, then the respective areas are related by a factor of r^2.

THE SCALING PRINCIPLE FOR POLYGONS: If similar polygons P and Q are related by a scale factor of r, then their respective areas are related by a factor of r^2.

THE SCALING PRINCIPLE FOR AREA: If similar figures A and B are related by a scale factor of r, then their respective areas are related by a factor of r^2.

Problem Set

1. A rectangle has an area of 18. Fill in the table below by answering the questions that follow. Part of the first row has been completed for you.

1	2	3	4	5	6
Original Dimensions	**Original Area**	**Scaled Dimensions**	**Scaled Area**	$\frac{\text{Scaled Area}}{\text{Original Area}}$	**Area ratio in terms of the scale factor**
18×1	18	$9 \times \frac{1}{2}$	$\frac{9}{2}$		

   a. List five unique sets of dimensions of your choice that satisfy the criterion set by the column 1 heading and enter them in column 1.

   b. If the given rectangle is dilated from a vertex with a scale factor of $\frac{1}{2}$, what are the dimensions of the images of each of your rectangles? Enter the scaled dimensions in column 3.

   c. What are the areas of the images of your rectangles? Enter the areas in column 4.

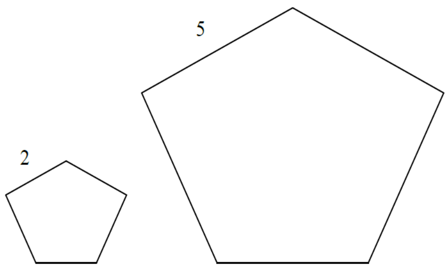
   d. How do the areas of the images of your rectangles compare to the area of the original rectangle? Write the value of each ratio in simplest form in column 5.

   e. Write the values of the ratios of area entered in column 5 in terms of the scale factor $\frac{1}{2}$. Enter these values in column 6.

   f. If the areas of two unique rectangles are the same, x, and both figures are dilated by the same scale factor r, what can we conclude about the areas of the dilated images?

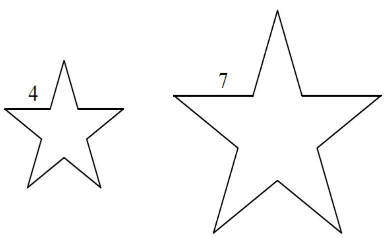
2. Find the ratio of the areas of each pair of similar figures. The lengths of corresponding line segments are shown.

a.

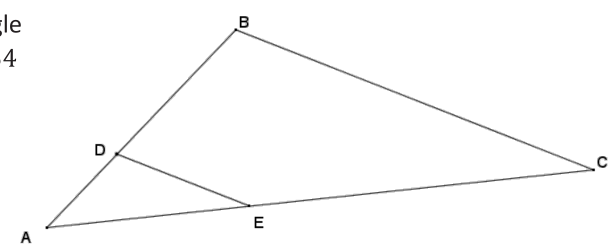


b.

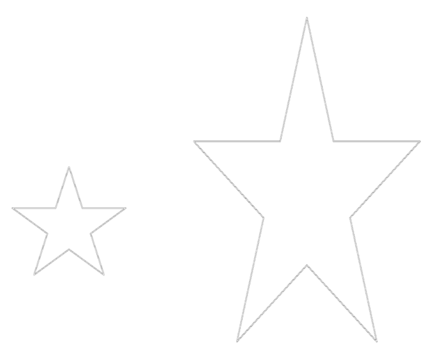
c.



3. In $\triangle ABC$, line segment DE connects two sides of the triangle and is parallel to line segment BC. If the area of $\triangle ABC$ is 54 and $BC = 3DE$, find the area of $\triangle ADE$.

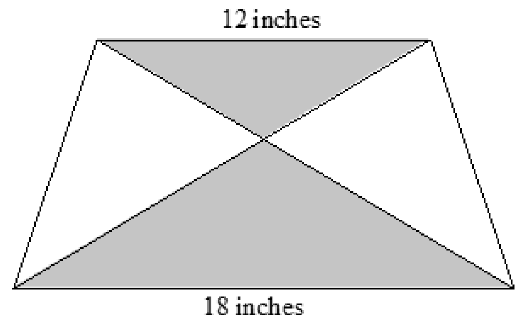


4. The small star has an area of 5. The large star is obtained from the small star by stretching by a factor of 2 in the horizontal direction and by a factor of 3 in the vertical direction. Find the area of the large star.

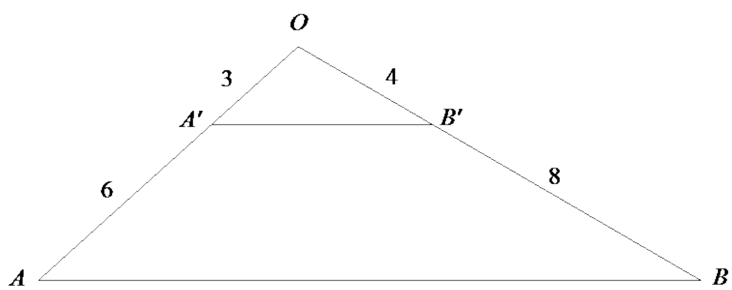


5. A piece of carpet has an area of 50 square yards. How many square inches will this be on a scale drawing that has 1 inch represent 1 yard?

6. An isosceles trapezoid has base lengths of 12 in. and 18 in. If the area of the larger shaded triangle is 72 in^2, find the area of the smaller shaded triangle.



7. Triangle ABO has a line segment $\overline{A'B'}$ connecting two of its sides so that $\overline{A'B'} \parallel \overline{AB}$. The lengths of certain segments are given. Find the ratio of the area of triangle $OA'B'$ to the area of the quadrilateral $ABB'A'$.



8. A square region S is scaled parallel to one side by a scale factor r, $r \neq 0$, and is scaled in a perpendicular direction by a scale factor one-third of r to yield its image S'. What is the ratio of the area of S to the area of S'?

9. Figure T' is the image of figure T that has been scaled horizontally by a scale factor of 4, and vertically by a scale factor of $\frac{1}{3}$. If the area of T' is 24 square units, what is the area of figure T?

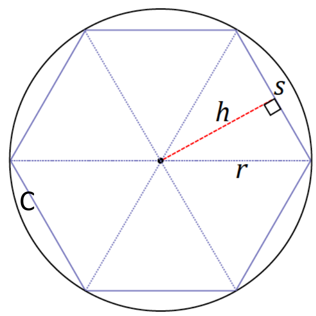
10. What is the effect on the area of a rectangle if …
    a. Its height is doubled and base left unchanged?
    b. If its base and height are both doubled?
    c. If its base were doubled and height cut in half?

Lesson 4: Proving the Area of a Disk

Classwork

Opening Exercise

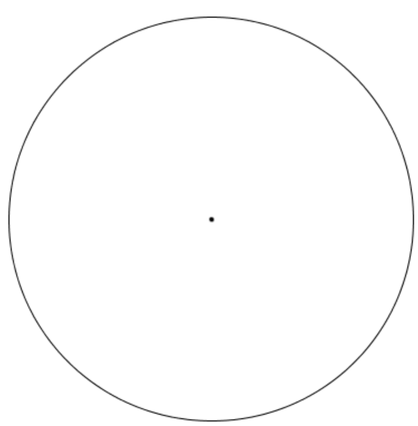
The following image is of a regular hexagon inscribed in circle C with radius r. Find a formula for the area of the hexagon in terms of the length of a side, s, and the distance from the center to a side.



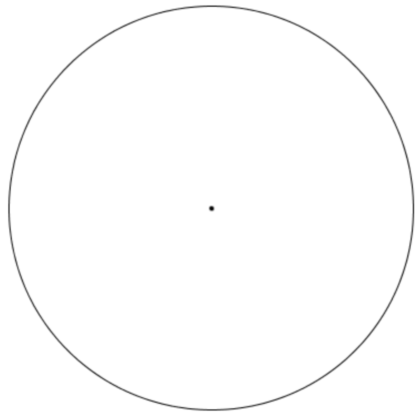
Example

a. Begin to approximate the area of a circle using inscribed polygons.

How well does a square approximate the area of a disk? Create a sketch of P_4 (a regular polygon with 4 sides, a square) in the following circle. Shade in the area of the disk that is not included in P_4.



b. Next, create a sketch of P_8 in the following circle.



c. Indicate which polygon has a greater area.

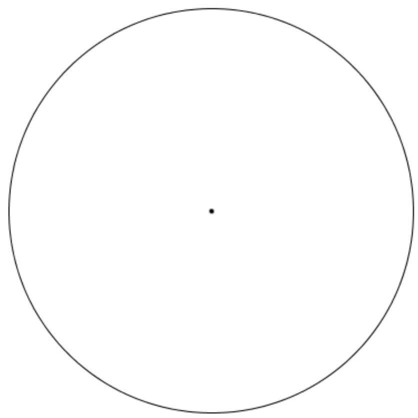
$$\text{Area}(P_4) \ \_\_\_\_\_\_\_\_\_\_ \ \text{Area}(P_8)$$

d. Will the area of inscribed regular polygon P_{16} be greater or less than the area of P_8? Which is a better approximation of the area of the disk?

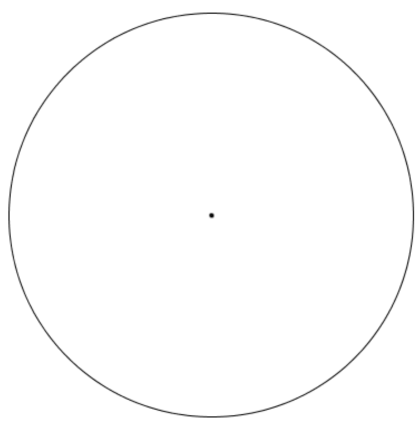
e. We noticed that the area of P_4 was less than the area of P_8 and that the area of P_8 was less than the area of P_{16}. In other words, $\text{Area}(P_n)$ _______ $\text{Area}(P_{2n})$. Why is this true?

f. Now we will approximate the area of a disk using circumscribed (outer) polygons.

Now circumscribe, or draw a square on the outside of, the following circle such that each side of the square intersects the circle at one point. We will denote each of our outer polygons with prime notation; we are sketching P'_4 here.



g. Create a sketch of P'_8.



h. Indicate which polygon has a greater area.

$$\text{Area}(P'_4) \;\_\_\_\_\_\_\_\_\_\_\; \text{Area}(P'_8).$$

i. Which is a better approximation of the area of the circle, P'_4 or P'_8? Explain why.

j. How will $\text{Area}(P'_n)$ compare to $\text{Area}(P'_{2n})$? Explain.

LIMIT (DESCRIPTION): Given an infinite sequence of numbers, $a_1, a_2, a_3, \ldots$, to say that *the limit of the sequence is* A means, roughly speaking, that when the index n is very large, then a_n is very close to A. This is often denoted as, "As $n \to \infty$, $a_n \to A$."

AREA OF A CIRCLE (DESCRIPTION): The *area of a circle* is the limit of the areas of the inscribed regular polygons as the number of sides of the polygons approaches infinity.

Problem Set

1. Describe a method for obtaining closer approximations of the area of a circle. Draw a diagram to aid in your explanation.

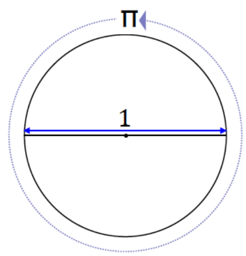
2. What is the radius of a circle whose circumference is π?

3. The side of a square is 20 cm long. What is the circumference of the circle when …
   a. The circle is inscribed within the square?
   b. The square is inscribed within the circle?

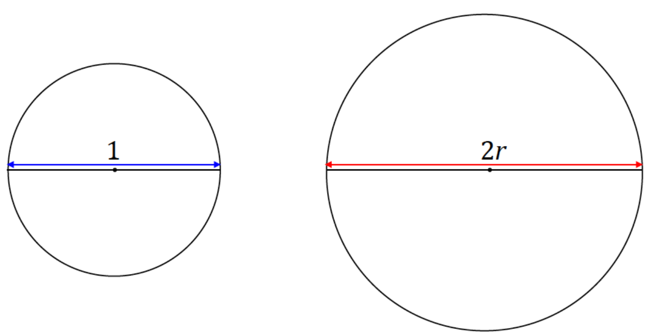
4. The circumference of circle $C_1 = 9$ cm, and the circumference of $C_2 = 2\pi$ cm. What is the value of the ratio of the areas of C_1 to C_2?

5. The circumference of a circle and the perimeter of a square are each 50 cm. Which figure has the greater area?

6. Let us define π to be the circumference of a circle whose diameter is 1.



We are going to show why the circumference of a circle has the formula $2\pi r$. Circle C_1 below has a diameter of $d = 1$, and circle C_2 has a diameter of $d = 2r$.



   a. All circles are similar (proved in Module 2). What scale factor of the similarity transformation takes C_1 to C_2?
   b. Since the circumference of a circle is a one-dimensional measurement, the value of the ratio of two circumferences is equal to the value of the ratio of their respective diameters. Rewrite the following equation by filling in the appropriate values for the diameters of C_1 and C_2:

$$\frac{\text{Circumference}(\mathrm{C_2})}{\text{Circumference}(\mathrm{C_1})} = \frac{\text{diameter}(\mathrm{C_2})}{\text{diameter}(\mathrm{C_1})}.$$

c. Since we have defined π to be the circumference of a circle whose diameter is 1, rewrite the above equation using this definition for C_1.

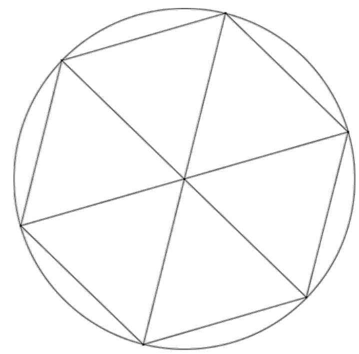
d. Rewrite the equation to show a formula for the circumference of C_2.

e. What can we conclude?

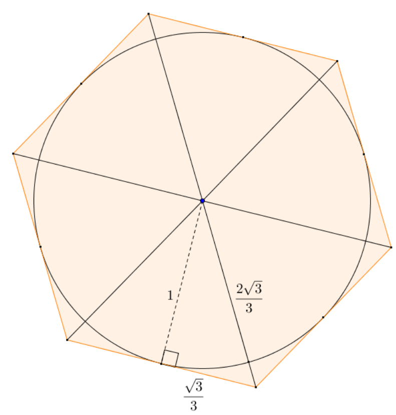
7.

a. Approximate the area of a disk of radius 1 using an inscribed regular hexagon. What is the percent error of the approximation?

(Remember that percent error is the absolute error as a percent of the exact measurement.)



b. Approximate the area of a circle of radius 1 using a circumscribed regular hexagon. What is the percent error of the approximation?



c. Find the average of the approximations for the area of a circle of radius 1 using inscribed and circumscribed regular hexagons. What is the percent error of the average approximation?

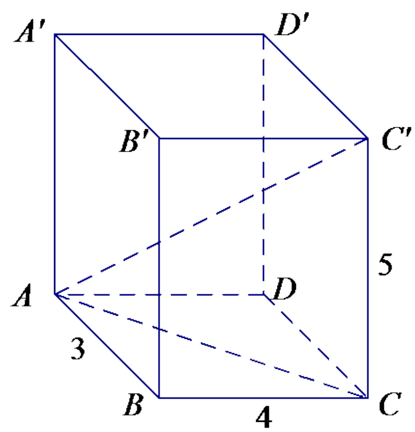
8. A regular polygon with n sides each of length s is inscribed in a circle of radius r. The distance h from the center of the circle to one of the sides of the polygon is over 98% of the radius. If the area of the polygonal region is 10, what can you say about the area of the circumscribed regular polygon with n sides?

Lesson 5: Three-Dimensional Space

Classwork

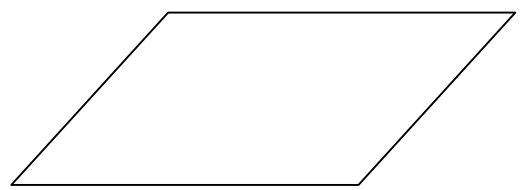
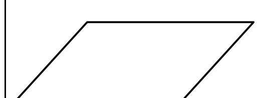
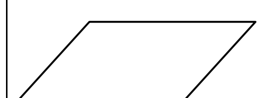
Exercise

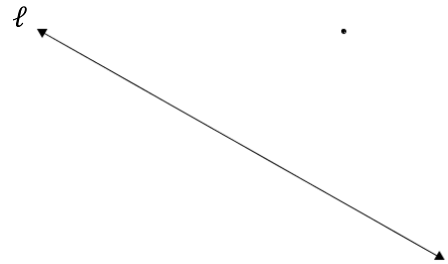
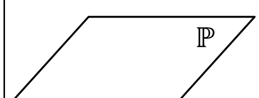
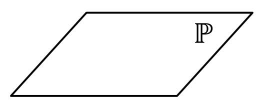
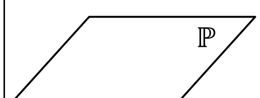
The following three-dimensional right rectangular prism has dimensions $3 \times 4 \times 5$. Determine the length of $\overline{AC'}$. Show a full solution.

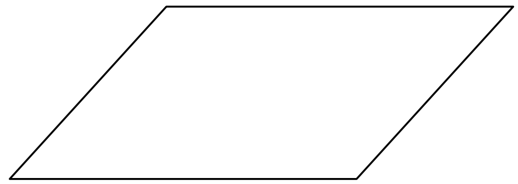
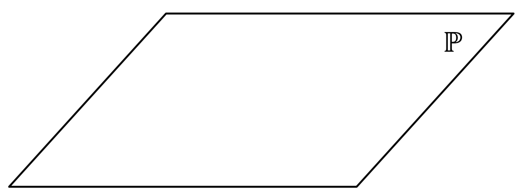
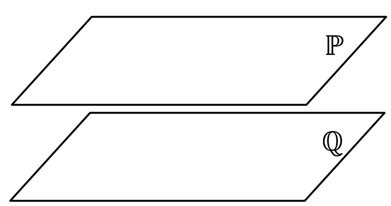


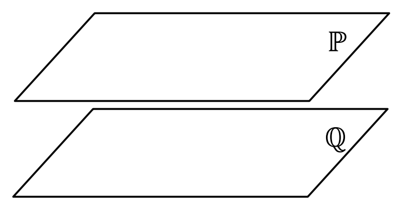
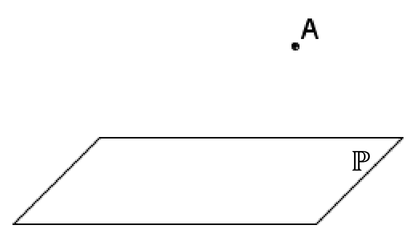
Exploratory Challenge

Table 1: Properties of Points, Lines, and Planes in 3-Dimensional Space

<table>
<tr><th></th><th>Property</th><th colspan="3">Diagram</th></tr>
<tr><td>1</td><td>Two points P and Q determine a distance PQ, a line segment PQ, a ray PQ, a vector PQ, and a line PQ.</td><td colspan="3"></td></tr>
<tr><td>2</td><td>Three non-collinear points A, B, and C determine a plane ABC and, in that plane, determine a triangle ABC.</td><td colspan="3">Given a picture of the plane below, sketch a triangle in that plane.
</td></tr>
<tr><td>3</td><td>Two lines either meet in a single point or they do not meet. Lines that do not meet and lie in a plane are called parallel. Skew lines are lines that do not meet and are not parallel.</td><td>(a) Sketch two lines that meet in a single point.</td><td>(b) Sketch lines that do not meet and lie in the same plane; i.e., sketch parallel lines.
</td><td>(c) Sketch a pair of skew lines.
</td></tr>
</table>

4	Given a line ℓ and a point not on ℓ, there is a unique line through the point that is parallel to ℓ.			
5	Given a line ℓ and a plane P, then ℓ lies in P, ℓ meets P in a single point, or ℓ does not meet P, in which case we say ℓ is *parallel* to P. (Note: This implies that if two points lie in a plane, then the line determined by the two points is also in the plane.)	(a) Sketch a line ℓ that lies in plane P. 	(b) Sketch a line ℓ that meets P in a single point. 	(c) Sketch a line ℓ that does not meet P; i.e., sketch a line ℓ parallel to P. 
6	Two planes either meet in a line or they do not meet, in which case we say the planes are *parallel*.	(a) Sketch two planes that meet in a line.	(b) Sketch two planes that are parallel.	

7	Two rays with the same vertex form an angle. The angle lies in a plane and can be measured by degrees.	Sketch the example in the following plane: 
8	Two lines are *perpendicular* if they meet, and any of the angles formed between the lines is a right angle. Two segments or rays are perpendicular if the lines containing them are perpendicular lines.	
9	A line ℓ is perpendicular to a plane P if they meet in a single point, and the plane contains two lines that are perpendicular to ℓ, in which case every line in P that meets ℓ is perpendicular to ℓ. A segment or ray is perpendicular to a plane if the line determined by the ray or segment is perpendicular to the plane.	Draw an example of a line that is perpendicular to a plane. Draw several lines that lie in the plane that pass through the point where the perpendicular line intersects the plane. 
10	Two planes perpendicular to the same line are parallel.	

11	Two lines perpendicular to the same plane are parallel.	Sketch an example that illustrates this statement using the following plane: 
12	Any two line segments connecting parallel planes have the same length if they are each perpendicular to one (and hence both) of the planes.	Sketch an example that illustrates this statement using parallel planes P and Q. 
13	The *distance between a point and a plane* is the length of the perpendicular segment from the point to the plane. The distance is defined to be zero if the point is on the plane. The *distance between two planes* is the distance from a point in one plane to the other.	Sketch the segment from A that can be used to measure the distance between A and the plane P. 

Lesson Summary

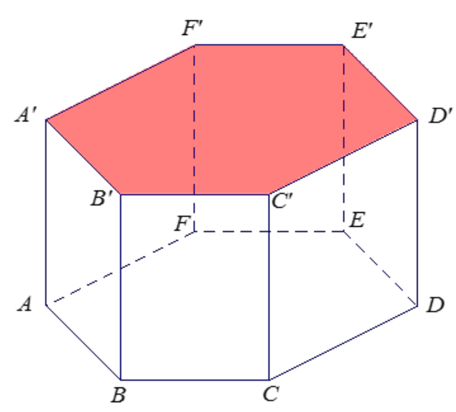
SEGMENT: The *segment between points A and B* is the set consisting of A, B, and all points on the line $\overleftrightarrow{AB}$ between A and B. The segment is denoted by $\overline{AB}$, and the points A and B are called the *endpoints.*

LINE PERPENDICULAR TO A PLANE: A line L intersecting a plane E at a point P is said to be *perpendicular to the plane E* if L is perpendicular to every line that (1) lies in E and (2) passes through the point P. A segment is said to be perpendicular to a plane if the line that contains the segment is perpendicular to the plane.

Problem Set

1. Indicate whether each statement is always true (A), sometimes true (S), or never true (N).
   a. If two lines are perpendicular to the same plane, the lines are parallel.
   b. Two planes can intersect in a point.
   c. Two lines parallel to the same plane are perpendicular to each other.
   d. If a line meets a plane in one point, then it must pass through the plane.
   e. Skew lines can lie in the same plane.
   f. If two lines are parallel to the same plane, the lines are parallel.
   g. If two planes are parallel to the same line, they are parallel to each other.
   h. If two lines do not intersect, they are parallel.

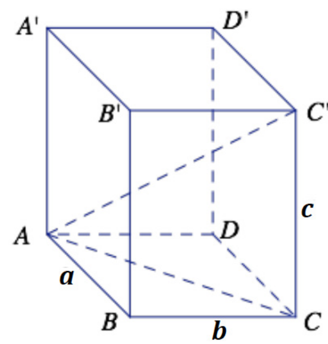
2. Consider the right hexagonal prism whose bases are regular hexagonal regions. The top and the bottom hexagonal regions are called the *base faces*, and the side rectangular regions are called the *lateral faces*.



   a. List a plane that is parallel to plane $C'D'E'$.
   b. List all planes shown that are not parallel to plane CDD'.
   c. Name a line perpendicular to plane ABC.
   d. Explain why $AA' = CC'$.
   e. Is $\overleftrightarrow{AB}$ parallel to $\overleftrightarrow{DE}$? Explain.
   f. Is $\overleftrightarrow{AB}$ parallel to $\overleftrightarrow{C'D'}$? Explain.
   g. Is $\overleftrightarrow{AB}$ parallel to $\overleftrightarrow{D'E'}$? Explain.
   h. If line segments $\overline{BC'}$ and $\overline{C'F'}$ are perpendicular, then is $\overleftrightarrow{BC}$ perpendicular to plane $C'A'F'$? Explain.
   i. One of the following statements is false. Identify which statement is false and explain why.
      (i) $\overleftrightarrow{BB'}$ is perpendicular to $\overleftrightarrow{B'C'}$.
      (ii) $\overleftrightarrow{EE'}$ is perpendicular to $\overleftrightarrow{EF}$.
      (iii) $\overleftrightarrow{CC'}$ is perpendicular to $\overleftrightarrow{E'F'}$.
      (iv) $\overleftrightarrow{BC}$ is parallel to $\overleftrightarrow{F'E'}$.

3. In the following figure, $\triangle ABC$ is in plane P, $\triangle DEF$ is in plane Q, and $BCFE$ is a rectangle. Which of the following statements are true?
   a. $\overline{BE}$ is perpendicular to plane Q.
   b. $BF = CE$.
   c. Plane P is parallel to plane Q.
   d. $\triangle ABC \cong \triangle DEF$.
   e. $AE = AF$.

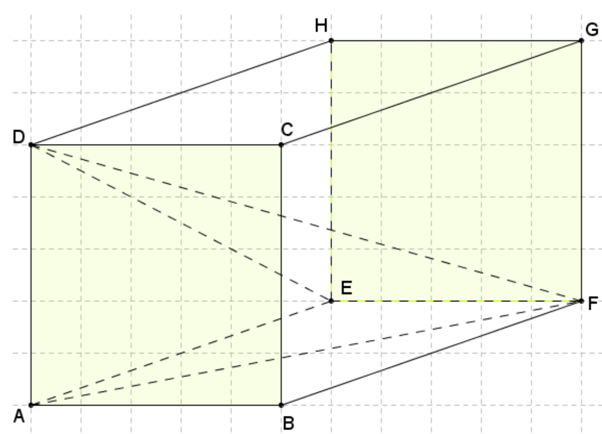
4. Challenge: The following three-dimensional right rectangular prism has dimensions $a \times b \times c$. Determine the length of $\overline{AC'}$.



5. A line ℓ is perpendicular to plane P. The line and plane meet at point C. If A is a point on ℓ different from C, and B is a point on P different from C, show that $AC < AB$.

6. Given two distinct parallel planes P and R, $\overleftrightarrow{EF}$ in P with $EF = 5$, point G in R, $m\angle GEF = 9°$, and $m\angle EFG = 60°$, find the minimum and maximum distances between planes P and R, and explain why the actual distance is unknown.

7. The diagram shows a right rectangular prism determined by vertices A, B, C, D, E, F, G, and H. Square $ABCD$ has sides with length 5, and $AE = 9$. Find DF.



Lesson 6: General Prisms and Cylinders and Their Cross-Sections

Classwork

Opening Exercise

Sketch a right rectangular prism.

RIGHT RECTANGULAR PRISM: Let E and E' be two parallel planes. Let B be a rectangular region[1] in the plane E. At each point P of B, consider the segment $\overline{PP'}$ perpendicular to E, joining P to a point P' of the plane E'. The union of all these segments is called a *right rectangular prism*.

GENERAL CYLINDER: (See Figure 1.) Let E and E' be two parallel planes, let B be a region[2] in the plane E, and let L be a line which intersects E and E' but not B. At each point P of B, consider the segment $\overline{PP'}$ parallel to L, joining P to a point P' of the plane E'. The union of all these segments is called a *general cylinder with base B*.

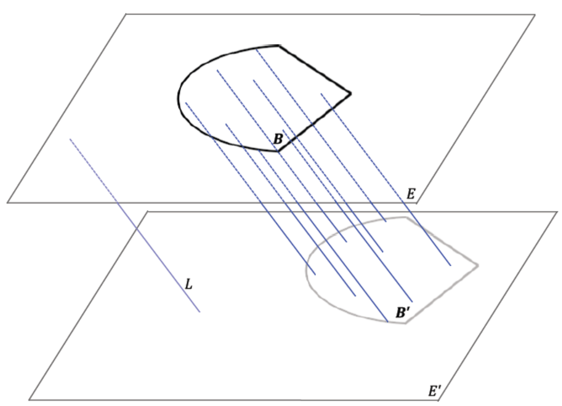


Figure 1

[1] (Fill in the blank.) A rectangular region is the union of a rectangle and ______________________________ .
[2] In Grade 8, a *region* refers to a *polygonal region* (triangle, quadrilateral, pentagon, and hexagon) or a *circular region,* or regions that can be decomposed into such regions.

Discussion

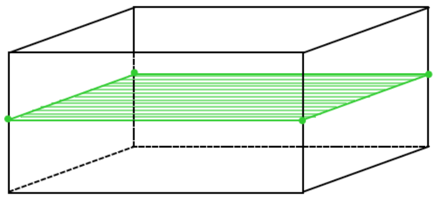


Figure 2

Example of a cross-section of a prism, where the intersection of a plane with the solid is parallel to the base.

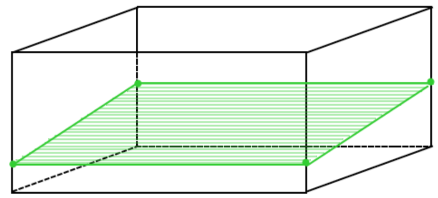
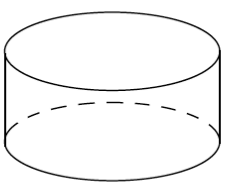
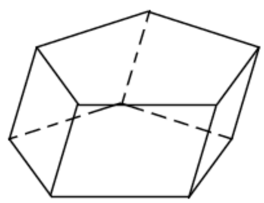
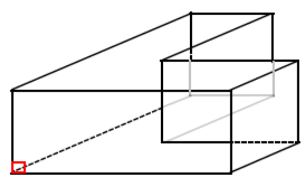
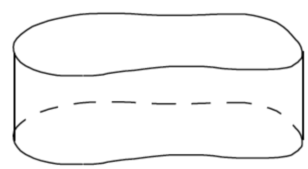


Figure 3

A general intersection of a plane with a prism; sometimes referred to as a slice.

Exercise

Sketch the cross-section for the following figures:

a.	b.	c.	d.
			

Extension

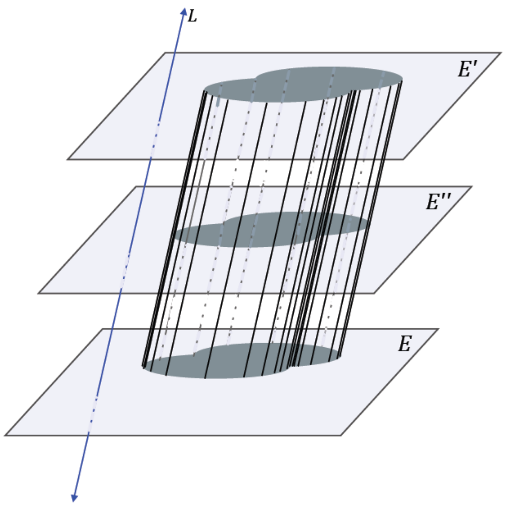


Figure 4

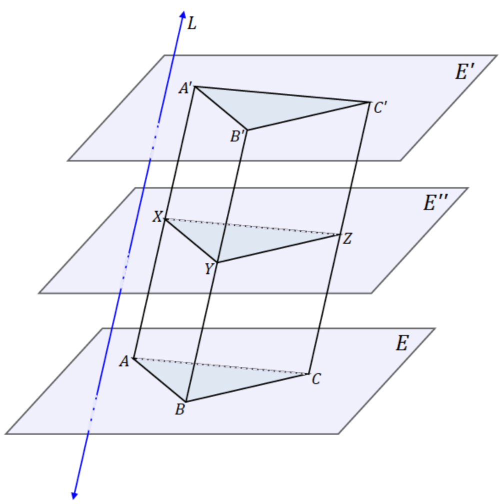


Figure 5

Lesson Summary

Relevant Vocabulary

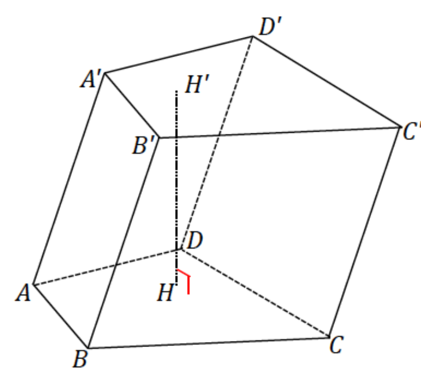
RIGHT RECTANGULAR PRISM: Let E and E' be two parallel planes. Let B be a rectangular region in the plane E. At each point P of B, consider the segment $\overline{PP'}$ perpendicular to E, joining P to a point P' of the plane E'. The union of all these segments is called a *right rectangular prism*.

LATERAL EDGE AND FACE OF A PRISM: Suppose the base B of a prism is a polygonal region and P_i is a vertex of B. Let P'_i be the corresponding point in B' such that $\overline{P_iP'_i}$ is parallel to the line L defining the prism. The segment $\overline{P_iP'_i}$ is called a *lateral edge of the prism*. If $\overline{P_iP_{i+1}}$ is a base edge of the base B (a side of B), and F is the union of all segments $\overline{PP'}$ parallel to L for which P is in $\overline{P_iP_{i+1}}$ and P' is in B', then F is a *lateral face of the prism*. It can be shown that a lateral face of a prism is always a region enclosed by a parallelogram.

GENERAL CYLINDER: Let E and E' be two parallel planes, let B be a region in the plane E, and let L be a line which intersects E and E' but not B. At each point P of B, consider the segment $\overline{PP'}$ parallel to L, joining P to a point P' of the plane E'. The union of all these segments is called a *general cylinder with base B*.

Problem Set

1. Complete each statement below by filling in the missing term(s).



   a. The following prism is called a(n) ____________ prism.

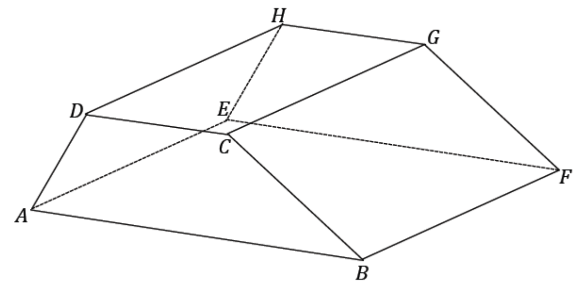
   b. If $\overline{AA'}$ were perpendicular to the plane of the base, then the prism would be called a(n) ____________ prism.

   c. The regions $ABCD$ and $A'B'C'D'$ are called the ____________ of the prism.

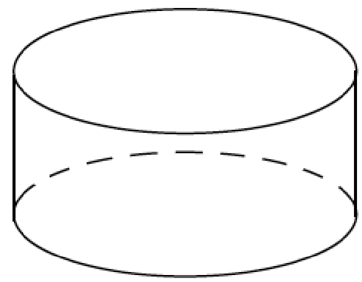
   d. $\overline{AA'}$ is called a(n) ____________.

   e. Parallelogram region $BB'C'C$ is one of four ____________ ____________.

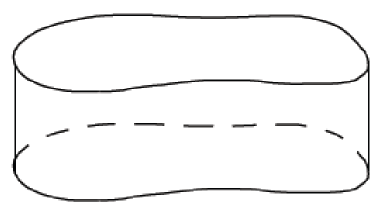
2. The following right prism has trapezoidal base regions; it is a right trapezoidal prism. The lengths of the parallel edges of the base are 5 and 8, and the nonparallel edges are 4 and 6; the height of the trapezoid is 3.7. The lateral edge length DH is 10. Find the surface area of the prism.



3. The base of the following right cylinder has a circumference of 5π and a lateral edge of 8. What is the radius of the base? What is the lateral area of the right cylinder?



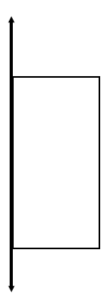
4. The following right general cylinder has a lateral edge of length 8, and the perimeter of its base is 27. What is the lateral area of the right general cylinder?



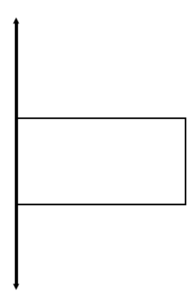
5. A right prism has base area 5 and volume 30. Find the prism's height, h.

6. Sketch the figures formed if the rectangular regions are rotated around the provided axis:

a.



b.



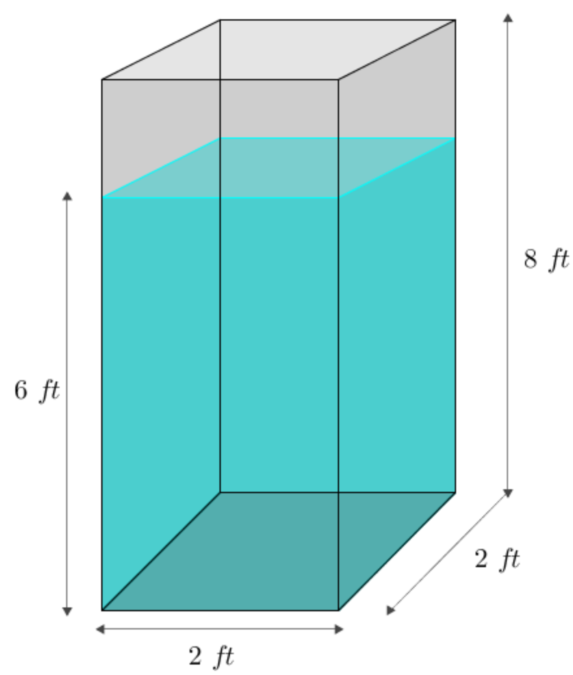
7. A cross-section is taken parallel to the bases of a general cylinder and has an area of 18. If the height of the cylinder is h, what is the volume of the cylinder? Explain your reasoning.

8. A general cylinder has a volume of 144. What is one possible set of dimensions of the base and height of the cylinder if all cross-sections parallel to its bases are …
   a. Rectangles?
   b. Right triangles?
   c. Circles?

9. A general hexagonal prism is given. If P is a plane that is parallel to the planes containing the base faces of the prism, how does P meet the prism?

10. Two right prisms have similar bases. The first prism has height 5 and volume 100. A side on the base of the first prism has length 2, and the corresponding side on the base of the second prism has length 3. If the height of the second prism is 6, what is its volume?

11. A tank is the shape of a right rectangular prism with base 2 ft. $\times$ 2 ft. and height 8 ft. The tank is filled with water to a depth of 6 ft. A person of height 6 ft. jumps in and stands on the bottom. About how many inches will the water be over the person's head? Make reasonable assumptions.

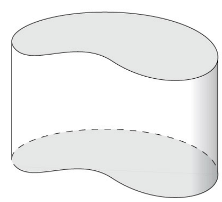


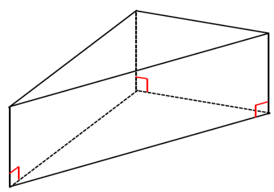
Lesson 7: General Pyramids and Cones and Their Cross-Sections

Classwork

Opening Exercise

Group the following images by shared properties. What defines each of the groups you have made?


1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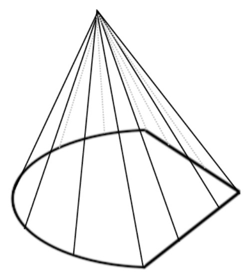
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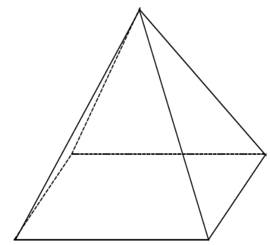
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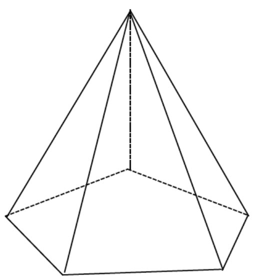
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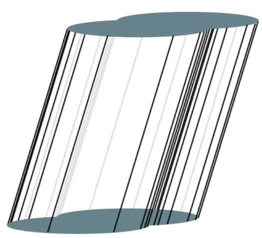
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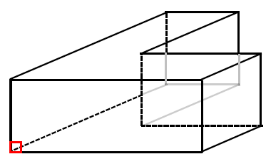
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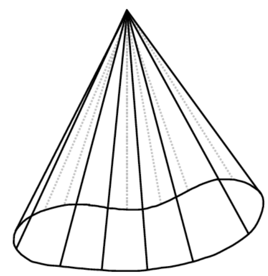
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RECTANGULAR PYRAMID: Given a rectangular region B in a plane E and a point V not in E, the *rectangular pyramid with base* B *and vertex* V is the collection of all segments $\overline{VP}$ for any point P in B.

GENERAL CONE: Let B be a region in a plane E and V be a point not in E. The *cone with base* B *and vertex* V is the union of all segments $\overline{VP}$ for all points P in B (See Figures 1 and 2).

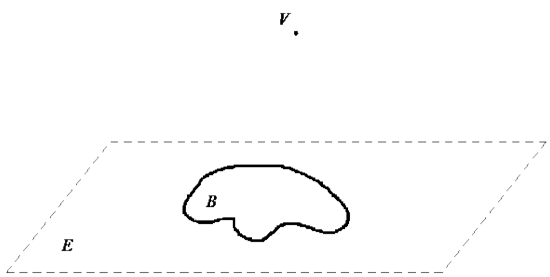


Figure 1

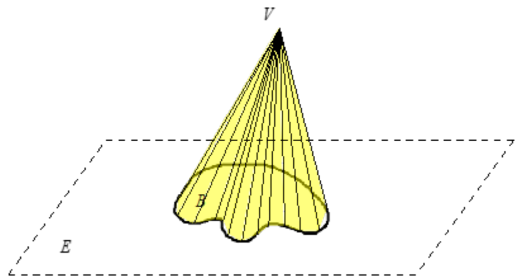


Figure 2

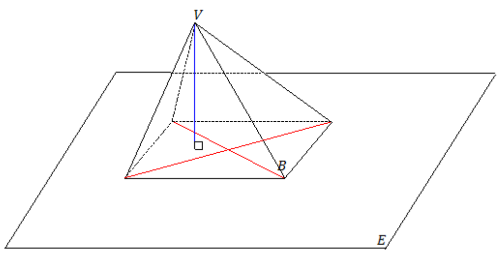


Figure 3

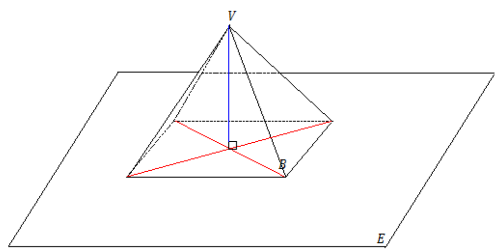


Figure 4

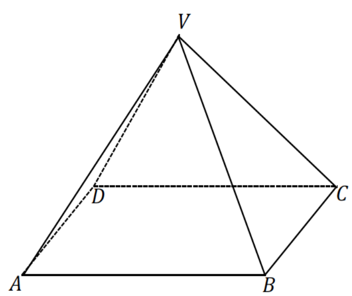


Figure 5

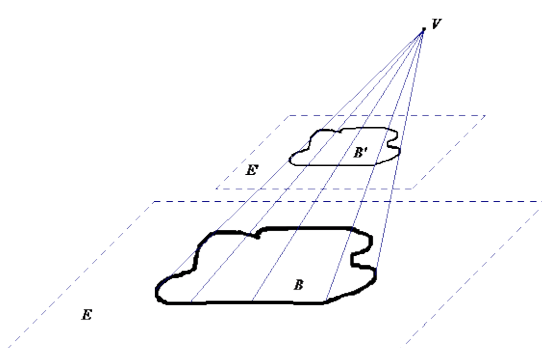
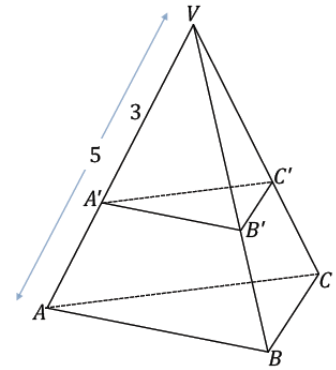


Figure 6

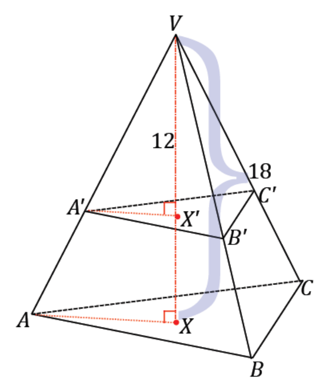
Example 1

In the following triangular pyramid, a plane passes through the pyramid so that it is parallel to the base and results in the cross-section $\triangle A'B'C'$. If the area of $\triangle ABC$ is 25 mm^2, what is the area of $\triangle A'B'C'$?

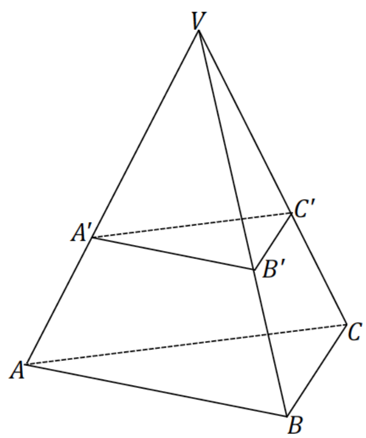


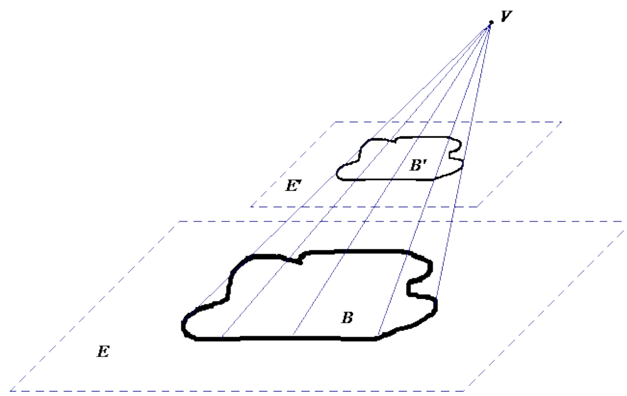
Example 2

In the following triangular pyramid, a plane passes through the pyramid so that it is parallel to the base and results in the cross-section $\triangle A'B'C'$. The altitude from V is drawn; the intersection of the altitude with the base is X, and the intersection of the altitude with the cross-section is X'. If the distance from X to V is 18 mm, the distance from X' to V is 12 mm, and the area of $\triangle A'B'C'$ is 28 mm^2, what is the area of $\triangle ABC$?



Extension





Exercise 1

The area of the base of a cone is 16, and the height is 10. Find the area of a cross-section that is distance 5 from the vertex.

Example 3

GENERAL CONE CROSS-SECTION THEOREM: If two general cones have the same base area and the same height, then cross-sections for the general cones the same distance from the vertex have the same area.

State the theorem in your own words.

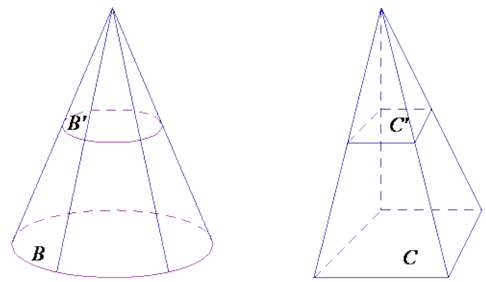
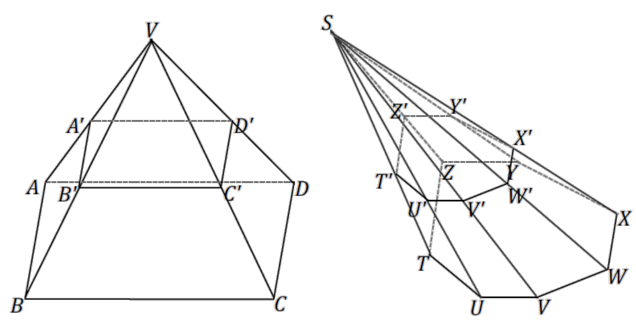


Figure 8

Use the space below to prove the *general cone cross-section theorem*.

Exercise 2

The following pyramids have equal altitudes, and both bases are equal in area and are coplanar. Both pyramids' cross-sections are also coplanar. If $BC = 3\sqrt{2}$ and $B'C' = 2\sqrt{3}$, and the area of $TUVWXYZ$ is 30 units^2, what is the area of cross-section $A'B'C'D'$?



Lesson Summary

CONE: Let B be a region in a plane E and V be a point not in E. The *cone with base* B *and vertex* V is the union of all segments $\overline{VP}$ for all points P in B.

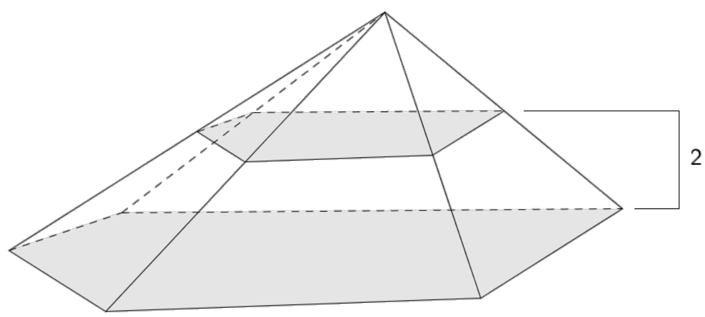
If the base is a polygonal region, then the *cone* is usually called a *pyramid*.

RECTANGULAR PYRAMID: Given a rectangular region B in a plane E and a point V not in E, the *rectangular pyramid with base* B *and vertex* V is the union of all segments $\overline{VP}$ for points P in B.

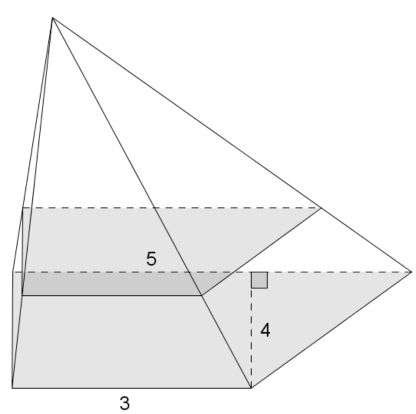
LATERAL EDGE AND FACE OF A PYRAMID: Suppose the base B of a pyramid with vertex V is a polygonal region and P_i is a vertex of B. The segment $\overline{P_iV}$ is called a *lateral edge* of the pyramid. If $\overline{P_iP_{i+1}}$ is a base edge of the base B (a side of B), and F is the union of all segments $\overline{PV}$ for P in $\overline{P_iP_{i+1}}$, then F is called a *lateral face* of the pyramid. It can be shown that the face of a pyramid is always a triangular region.

Problem Set

1. The base of a pyramid has area 4. A cross-section that lies in a parallel plane that is distance of 2 from the base plane, has an area of 1. Find the height, h, of the pyramid.



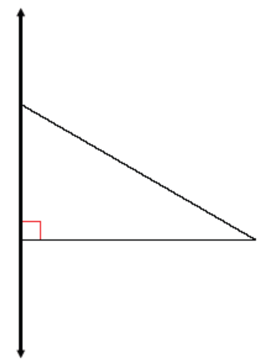
2. The base of a pyramid is a trapezoid. The trapezoidal bases have lengths of 3 and 5, and the trapezoid's height is 4. Find the area of the parallel slice that is three-fourths of the way from the vertex to the base.



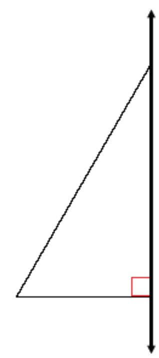
3. A cone has base area $36\ \text{cm}^2$. A parallel slice $5\ \text{cm}$ from the vertex has area $25\ \text{cm}^2$. Find the height of the cone.

4. Sketch the figures formed if the triangular regions are rotated around the provided axis:

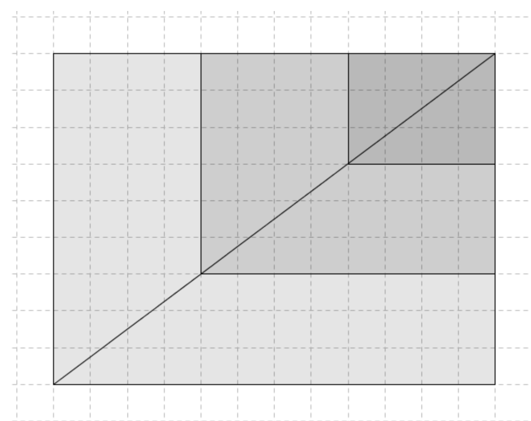
a.



b.



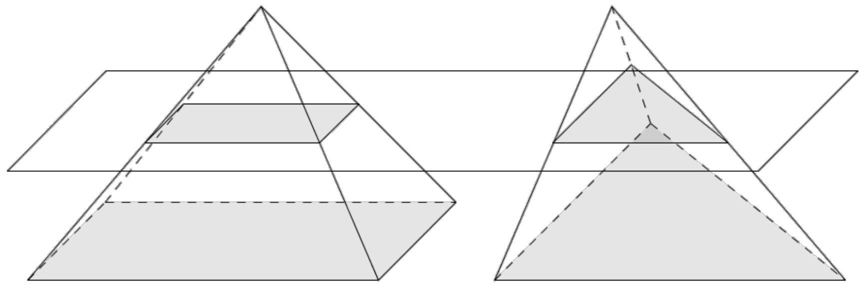
5. Liza drew the top view of a rectangular pyramid with two cross-sections as shown in the diagram and said that her diagram represents one, and only one, rectangular pyramid. Do you agree or disagree with Liza? Explain.



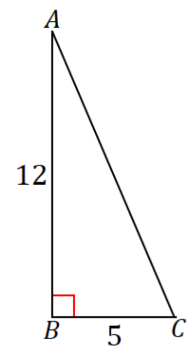
6. A general hexagonal pyramid has height 10 in. A slice 2 in. above the base has area $16\ \text{in}^2$. Find the area of the base.

7. A general cone has base area $3\ \text{units}^2$. Find the area of the slice of the cone that is parallel to the base and $\frac{2}{3}$ of the way from the vertex to the base.

8. A rectangular cone and a triangular cone have bases with the same area. Explain why the cross-sections for the cones halfway between the base and the vertex have the same area.



9. The following right triangle is rotated about side AB. What is the resulting figure, and what are its dimensions?

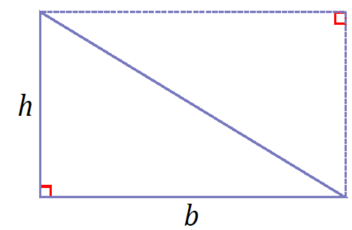


Lesson 8: Definition and Properties of Volume

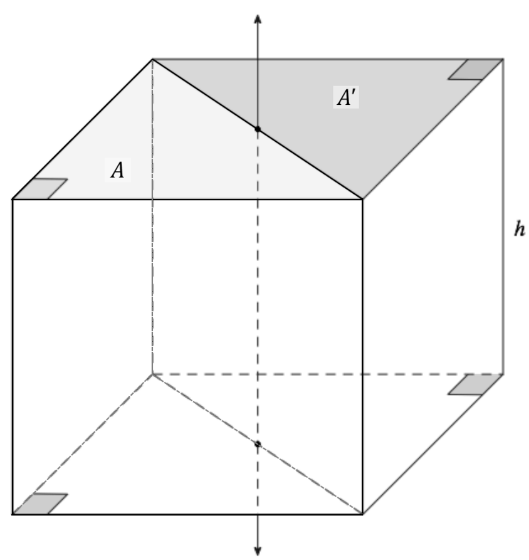
Classwork

Opening Exercise

a. Use the following image to reason why the area of a right triangle is $\frac{1}{2}bh$ (Area Property 2).



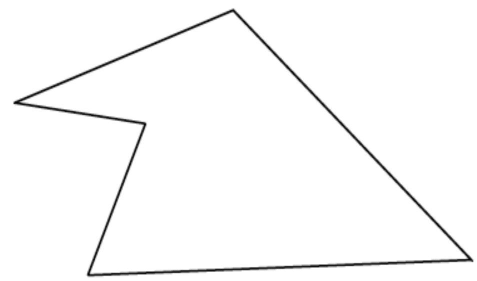
b. Use the following image to reason why the volume of the following triangular prism with base area A and height h is Ah (Volume Property 2).



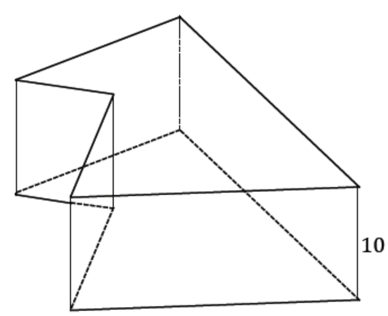
Exercises

Complete Exercises 1–2, and then have a partner check your work.

1. Divide the following polygonal region into triangles. Assign base and height values of your choice to each triangle, and determine the area for the entire polygon.



2. The polygon from Exercise 1 is used here as the base of a general right prism. Use a height of 10 and the appropriate value(s) from Exercise 1 to determine the volume of the prism.



We can use the formula $\text{density} = \frac{\text{mass}}{\text{volume}}$ to find the density of a substance.

3. A square metal plate has a density of $10.2\ \text{g/cm}^3$ and weighs $2.193\ \text{kg}$.
   a. Calculate the volume of the plate.

   b. If the base of this plate has an area of $25\ \text{cm}^2$, determine its thickness.

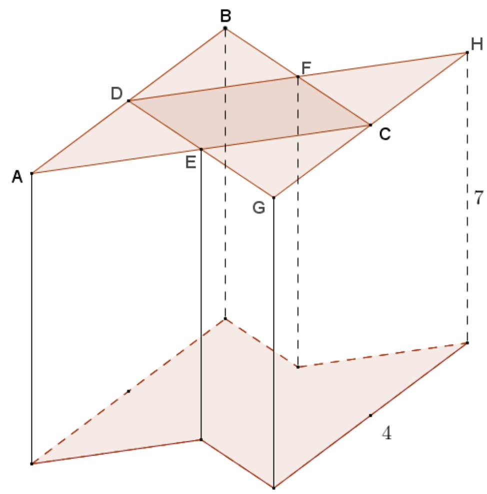
4. A metal cup full of water has a mass of $1{,}000\ \text{g}$. The cup itself has a mass of $214.6\ \text{g}$. If the cup has both a diameter and a height of $10\ \text{cm}$, what is the approximate density of water?

Problem Set

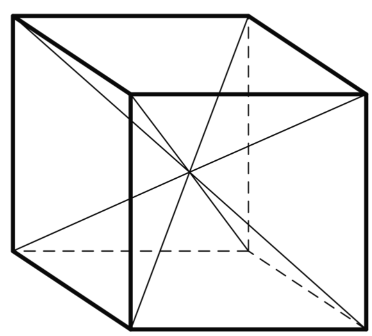
1. Two congruent solids S_1 and S_2 have the property that $S_1 \cap S_2$ is a right triangular prism with height $\sqrt{3}$ and a base that is an equilateral triangle of side length 2. If the volume of $S_1 \cup S_2$ is 25 units^3, find the volume of S_1.

2. Find the volume of a triangle with side lengths 3, 4, and 5.

3. The base of the prism shown in the diagram consists of overlapping congruent equilateral triangles ABC and DGH. Points C, D, E, and F are midpoints of the sides of triangles ABC and DGH. $GH = AB = 4$, and the height of the prism is 7. Find the volume of the prism.



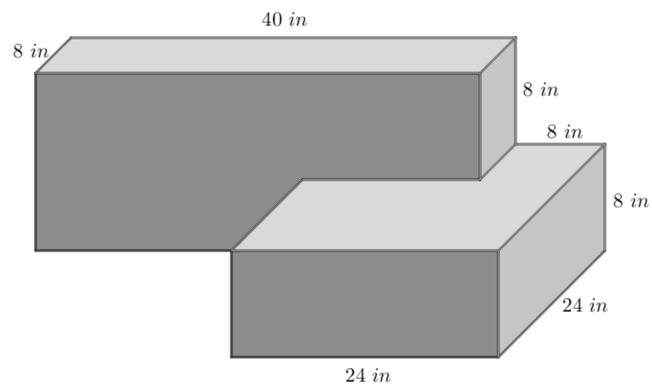
4. Find the volume of a right rectangular pyramid whose base is a square with side length 2 and whose height is 1.
   Hint: Six such pyramids can be fit together to make a cube with side length 2 as shown in the diagram.



5. Draw a rectangular prism with a square base such that the pyramid's vertex lies on a line perpendicular to the base of the prism through one of the four vertices of the square base, and the distance from the vertex to the base plane is equal to the side length of the square base.

6. The pyramid that you drew in Problem 5 can be pieced together with two other identical rectangular pyramids to form a cube. If the side lengths of the square base are 3, find the volume of the pyramid.

7. Paul is designing a mold for a concrete block to be used in a custom landscaping project. The block is shown in the diagram with its corresponding dimensions and consists of two intersecting rectangular prisms. Find the volume of mixed concrete, in cubic feet, needed to make Paul's custom block.



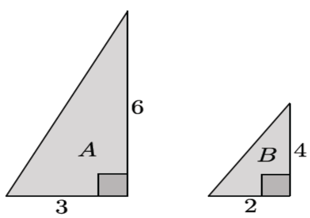
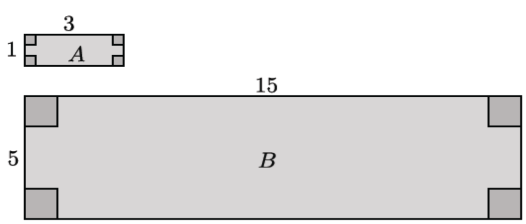
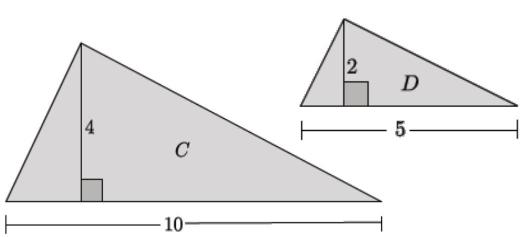
8. Challenge: Use card stock and tape to construct three identical polyhedron nets that together form a cube.

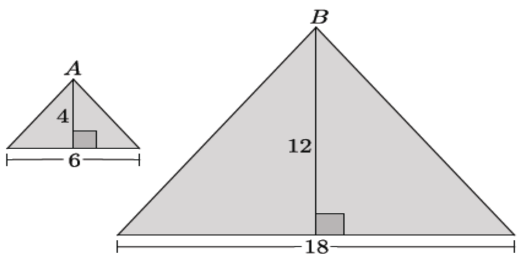
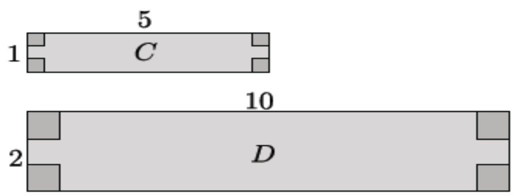
Lesson 9: Scaling Principle for Volumes

Classwork

Opening Exercise

a. For each pair of similar figures, write the ratio of side lengths $a : b$ or $c : d$ that compares one pair of corresponding sides. Then, complete the third column by writing the ratio that compares the areas of the similar figures. Simplify ratios when possible.

Similar Figures	Ratio of Side Lengths $a : b$ or $c : d$	Ratio of Areas $\text{Area}(A) : \text{Area}(B)$ or $\text{Area}(C) : \text{Area}(D)$
$\triangle A \sim \triangle B$ 	$6: 4$ $3: 2$	$9: 4$ $3^2: 2^2$
Rectangle $A \sim$ Rectangle B 		
$\triangle C \sim \triangle D$ 		

$\triangle A \sim \triangle B$ 		
Rectangle $A \sim$ Rectangle B 		

b.

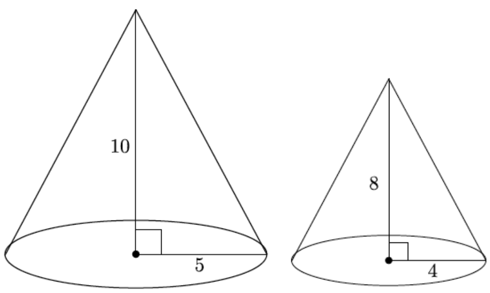
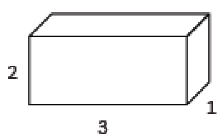
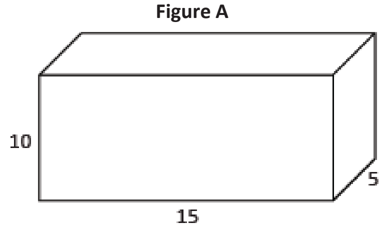
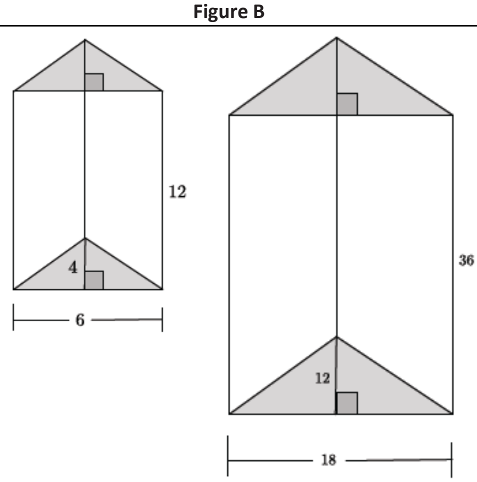
i. State the relationship between the ratio of sides $a : b$ and the ratio of the areas $\text{Area}(A) : \text{Area}(B)$.

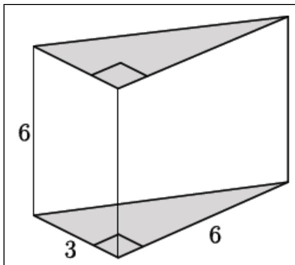
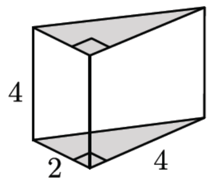
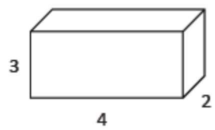
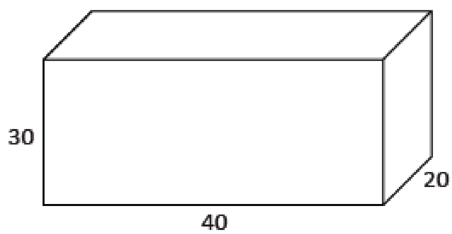
ii. Make a conjecture as to how the ratio of sides $a : b$ will be related to the ratio of volumes $\text{Volume}(S) : \text{Volume}(T)$. Explain.

c. What does is mean for two solids in three-dimensional space to be similar?

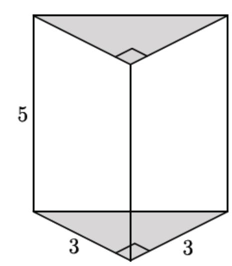
Exercises

1. Each pair of solids shown below is similar. Write the ratio of side lengths $a : b$ comparing one pair of corresponding sides. Then, complete the third column by writing the ratio that compares volumes of the similar figures. Simplify ratios when possible.

Similar Figures	Ratio of Side Lengths $a : b$	Ratio of Volumes $\text{Volume}(A) : \text{Volume}(B)$
 Figure A Figure B		
 Figure A  Figure B		
 Figure A Figure B		

 Figure A  Figure B		
 Figure A  Figure B		

2. Use the triangular prism shown to answer the questions that follow.



   a. Calculate the volume of the triangular prism.

   b. If one side of the triangular base is scaled by a factor of 2, the other side of the triangular base is scaled by a factor of 4, and the height of the prism is scaled by a factor of 3, what are the dimensions of the scaled triangular prism?

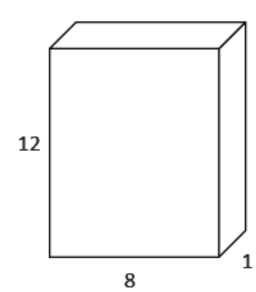
c. Calculate the volume of the scaled triangular prism.

d. Make a conjecture about the relationship between the volume of the original triangular prism and the scaled triangular prism.

e. Do the volumes of the figures have the same relationship as was shown in the figures in Exercise 1? Explain.

3. Use the rectangular prism shown to answer the questions that follow.

   a. Calculate the volume of the rectangular prism.



b. If one side of the rectangular base is scaled by a factor of $\frac{1}{2}$, the other side of the rectangular base is scaled by a factor of 24, and the height of the prism is scaled by a factor of $\frac{1}{3}$, what are the dimensions of the scaled rectangular prism?

c. Calculate the volume of the scaled rectangular prism.

d. Make a conjecture about the relationship between the volume of the original rectangular prism and the scaled rectangular prism.

4. A manufacturing company needs boxes to ship their newest widget, which measures $2 \times 4 \times 5 \text{ in}^3$. Standard size boxes, 5-inch cubes, are inexpensive but require foam packaging so the widget is not damaged in transit. Foam packaging costs $\$0.03$ per cubic inch. Specially designed boxes are more expensive but do not require foam packing. If the standard size box costs $\$0.80$ each and the specially designed box costs $\$3.00$ each, which kind of box should the company choose? Explain your answer.

Problem Set

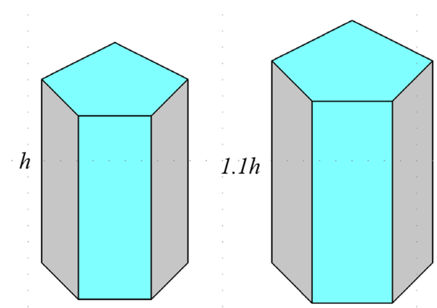
1. Coffees sold at a deli come in similar-shaped cups. A small cup has a height of 4.2" and a large cup has a height of 5". The large coffee holds 12 fluid ounces. How much coffee is in a small cup? Round your answer to the nearest tenth of an ounce.

2. Right circular cylinder A has volume $2{,}700$ and radius 3. Right circular cylinder B is similar to cylinder A and has volume $6{,}400$. Find the radius of cylinder B.

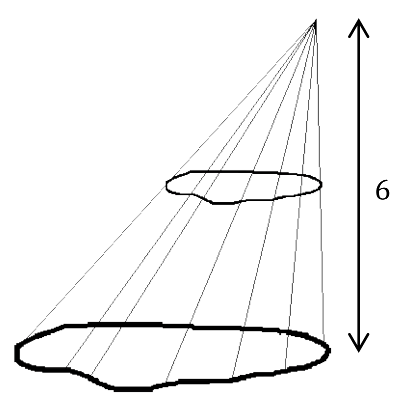
3. The Empire State Building is a 102-story skyscraper. Its height is $1{,}250$ ft. from the ground to the roof. The length and width of the building are approximately 424 ft. and 187 ft., respectively. A manufacturing company plans to make a miniature version of the building and sell cases of them to souvenir shops.
   a. The miniature version is just $\frac{1}{2500}$ of the size of the original. What are the dimensions of the miniature Empire State Building?
   b. Determine the volume of the minature building. Explain how you determined the volume.

4. If a right square pyramid has a 2×2 square base and height 1, then its volume is $\frac{4}{3}$. Use this information to find the volume of a right rectangular prism with base dimensions $a \times b$ and height h.

5. The following solids are similar. The volume of the first solid is 100. Find the volume of the second solid.



6. A general cone has a height of 6. What fraction of the cone's volume is between a plane containing the base and a parallel plane halfway between the vertex of the cone and the base plane?



7. A company uses rectangular boxes to package small electronic components for shipping. The box that is currently used can contain 500 of one type of component. The company wants to package twice as many pieces per box. Michael thinks that the box will hold twice as much if its dimensions are doubled. Shawn disagrees and says that Michael's idea provides a box that is much too large for $1{,}000$ pieces. Explain why you agree or disagree with one or either of the boys. What would you recommend to the company?

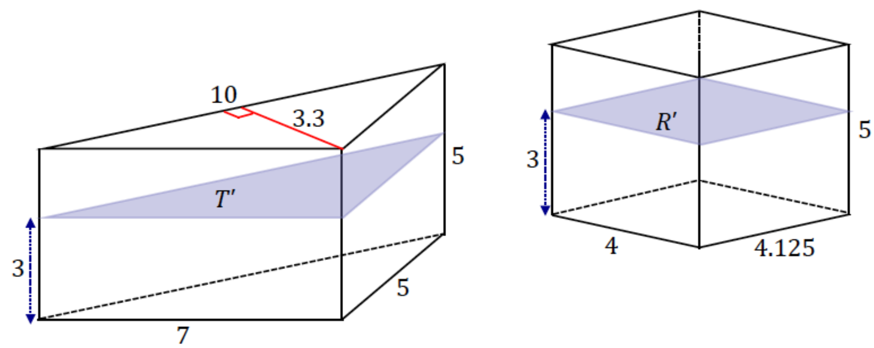
8. A dairy facility has bulk milk tanks that are shaped like right circular cylinders. They have replaced one of their bulk milk tanks with three smaller tanks that have the same height as the original but $\frac{1}{3}$ the radius. Do the new tanks hold the same amount of milk as the original tank? If not, explain how the volumes compare.

Lesson 10: The Volume of Prisms and Cylinders and Cavalieri's Principle

Classwork

Opening Exercise

The bases of the following triangular prism T and rectangular prism R lie in the same plane. A plane that is parallel to the bases and also a distance 3 from the bottom base intersects both solids and creates cross-sections T' and R'.



a. Find $\text{Area}(T')$.

b. Find $\text{Area}(R')$.

c. Find $\text{Vol}(T)$.

d. Find $\text{Vol}(R)$.

e. If a height other than 3 were chosen for the cross-section, would the cross-sectional area of either solid change?

Discussion

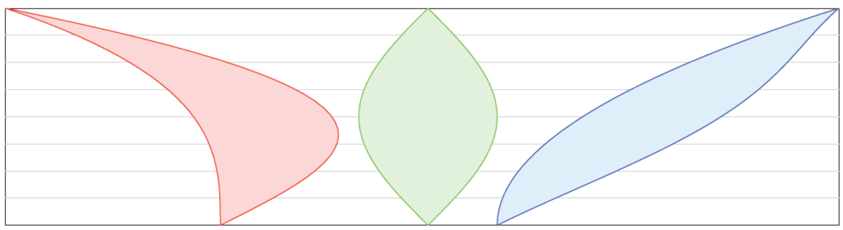
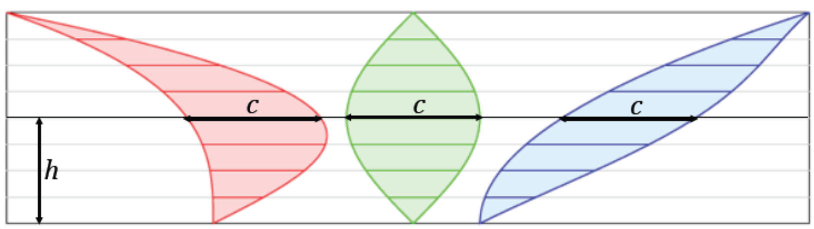
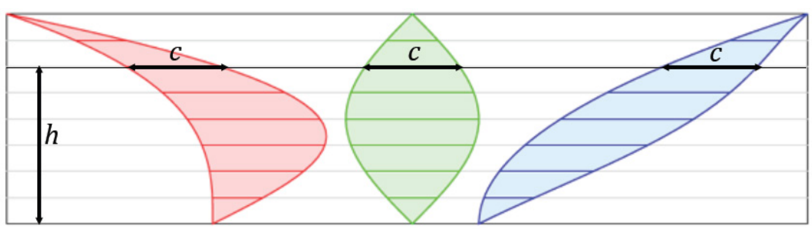


Figure 1

Example 1



Example 2



PRINCIPLE OF PARALLEL SLICES IN THE PLANE: If two planar figures of equal altitude have identical cross-sectional lengths at each height, then the regions of the figures have the same area.

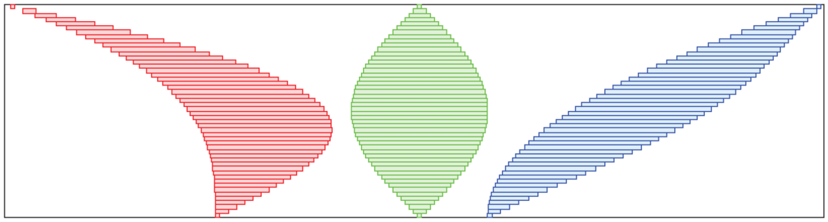
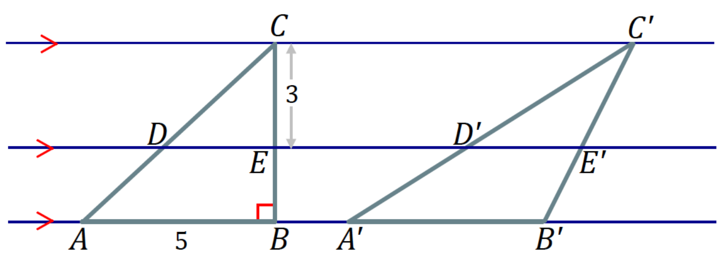


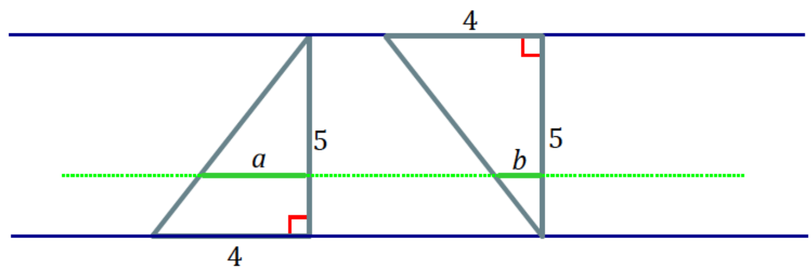
Figure 2

Example

a. The following triangles have equal areas: $\text{Area}(\triangle ABC) = \text{Area}(\triangle A'B'C') = 15 \text{ units}^2$. The distance between $\overleftrightarrow{DE}$ and $\overleftrightarrow{CC'}$ is 3. Find the lengths $\overline{DE}$ and $\overline{D'E'}$.



b. Joey says that if two figures have the same height and the same area, then their cross-sectional lengths at each height will be the same. Give an example to show that Joey's theory is incorrect.



Discussion

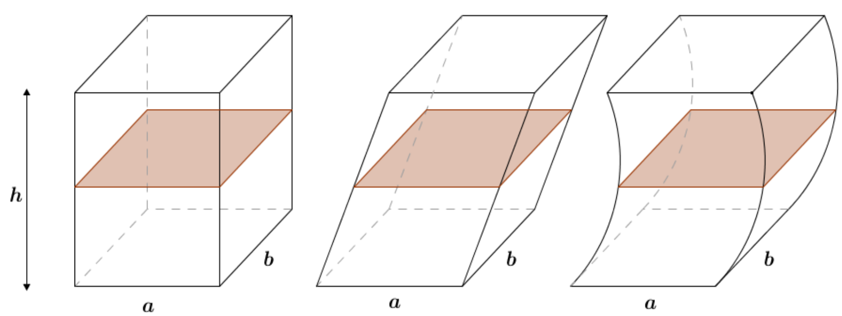


Figure 3

CAVALIERI'S PRINCIPLE: Given two solids that are included between two parallel planes, if every plane parallel to the two planes intersects both solids in cross-sections of equal area, then the volumes of the two solids are equal.

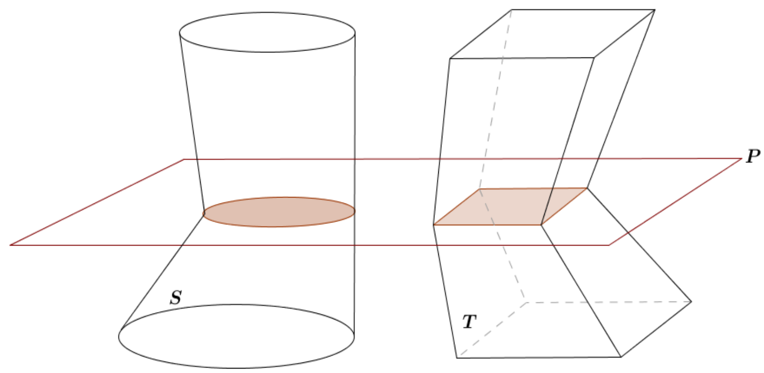


Figure 4

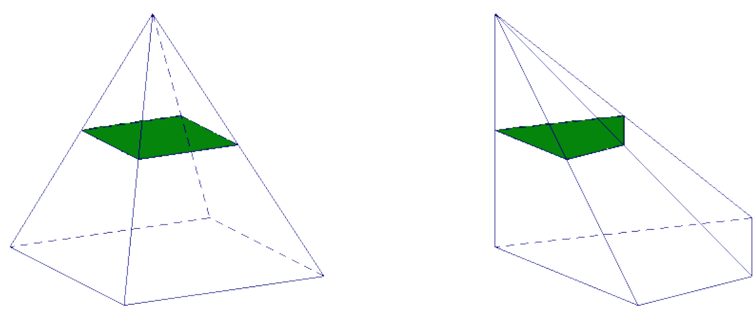


Figure 5

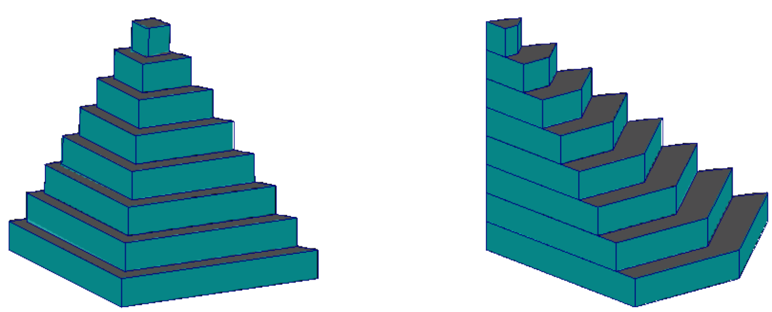


Figure 6

Lesson Summary

PRINCIPLE OF PARALLEL SLICES IN THE PLANE: If two planar figures of equal altitude have identical cross-sectional lengths at each height, then the regions of the figures have the same area.

CAVALIERI'S PRINCIPLE: Given two solids that are included between two parallel planes, if every plane parallel to the two planes intersects both solids in cross-sections of equal area, then the volumes of the two solids are equal.

Problem Set

1. Use the principle of parallel slices to explain the area formula for a parallelogram.

2. Use the principle of parallel slices to show that the three triangles shown below all have the same area.

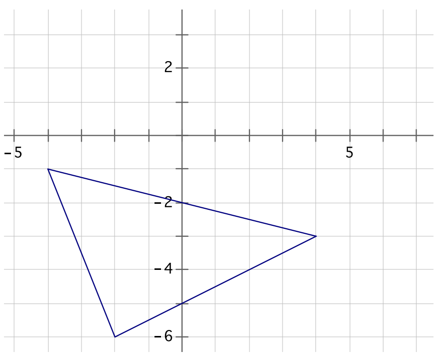


Figure 1

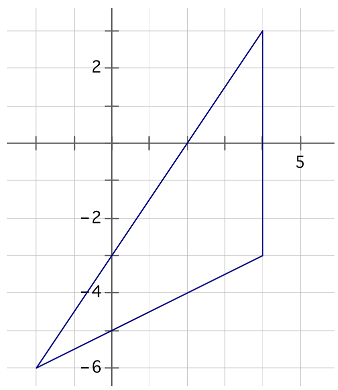


Figure 2

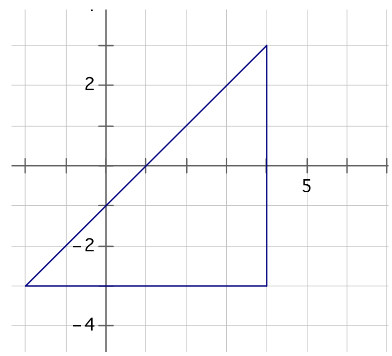
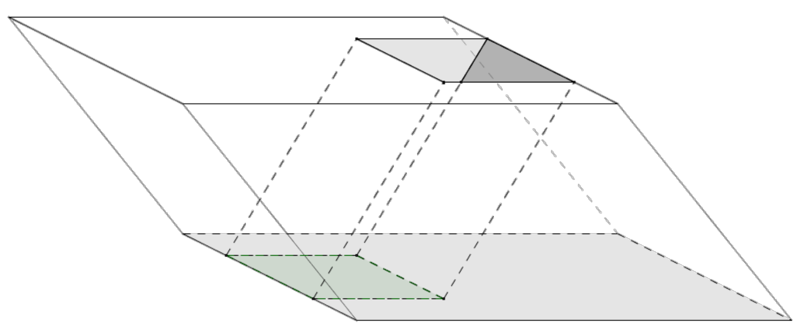
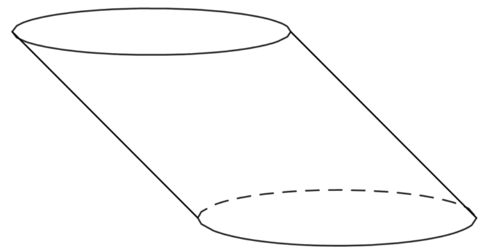


Figure 3

3. An oblique prism has a rectangular base that is $16 \text{ in.} \times 9 \text{ in.}$ A hole in the prism is also the shape of an oblique prism with a rectangular base that is 3 in. wide and 6 in. long, and the prism's height is 9 in. (as shown in the diagram). Find the volume of the remaining solid.



4. An oblique circular cylinder has height 5 and volume 45π. Find the radius of the circular base.



5. A right circular cone and a solid hemisphere share the same base. The vertex of the cone lies on the hemisphere. Removing the cone from the solid hemisphere forms a solid. Draw a picture, and describe the cross-sections of this solid that are parallel to the base.

6. Use Cavalieri's principle to explain why a circular cylinder with a base of radius 5 and a height of 10 has the same volume as a square prism whose base is a square with edge length $5\sqrt{\pi}$ and whose height is also 10.

Lesson 11: The Volume Formula of a Pyramid and Cone

Classwork

Exploratory Challenge

Use the provided manipulatives to aid you in answering the questions below.

a.
   i. What is the formula to find the area of a triangle?

   ii. Explain why the formula works.

b.
   i. What is the formula to find the volume of a triangular prism?

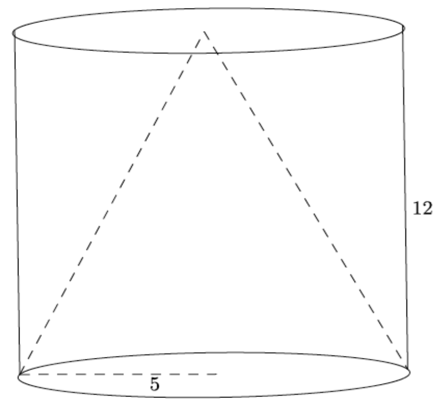
   ii. Explain why the formula works.

c.
   i. What is the formula to find the volume of a cone or pyramid?

   ii. Explain why the formula works.

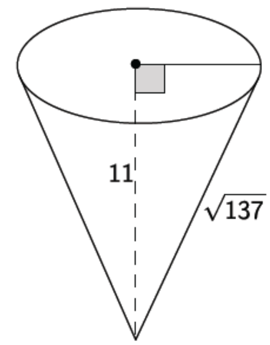
Exercises

1. A cone fits inside a cylinder so that their bases are the same and their heights are the same, as shown in the diagram below. Calculate the volume that is inside the cylinder but outside of the cone. Give an exact answer.



2. A square pyramid has a volume of 245 in^3. The height of the pyramid is 15 in. What is the area of the base of the pyramid? What is the length of one side of the base?

3. Use the diagram below to answer the questions that follow.
   a. Determine the volume of the cone shown below. Give an exact answer.



   b. Find the dimensions of a cone that is similar to the one given above. Explain how you found your answers.

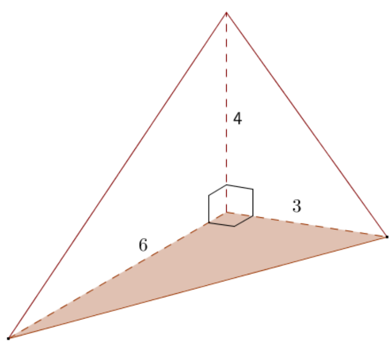
   c. Calculate the volume of the cone that you described in part (b) in two ways. (Hint: Use the volume formula and the scaling principle for volume.)

4. Gold has a density of 19.32 g/cm^3. If a square pyramid has a base edge length of 5 cm, height of 6 cm, and a mass of 942 g, is the pyramid in fact solid gold? If it is not, what reasons could explain why it is not? Recall that density can be calculated with the formula $\text{density} = \frac{\text{mass}}{\text{volume}}$.

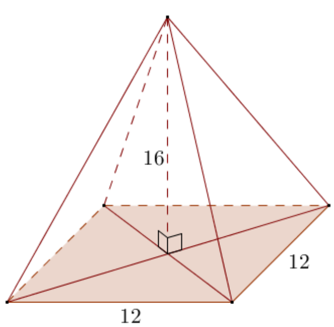
Problem Set

1. What is the volume formula for a right circular cone with radius r and height h?

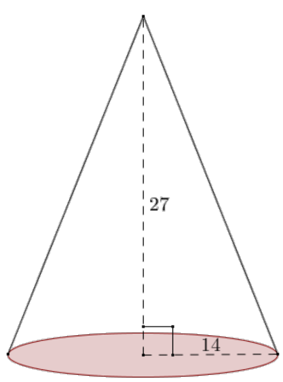
2. Identify the solid shown, and find its volume.



3. Find the volume of the right rectangular pyramid shown.



4. Find the volume of the circular cone in the diagram. (Use $\frac{22}{7}$ as an approximation of pi.)



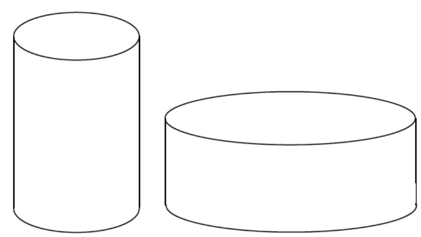
5. Find the volume of a pyramid whose base is a square with edge length 3 and whose height is also 3.

6. Suppose you fill a conical paper cup with a height of $6"$ with water. If all the water is then poured into a cylindrical cup with the same radius and same height as the conical paper cup, to what height will the water reach in the cylindrical cup?

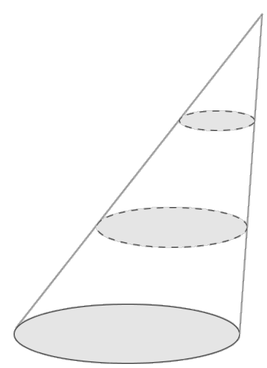
7. Sand falls from a conveyor belt and forms a pile on a flat surface. The diameter of the pile is approximately 10 ft. and the height is approximately 6 ft. Estimate the volume of the pile of sand. State your assumptions used in modeling.

8. A pyramid has volume 24 and height 6. Find the area of its base.

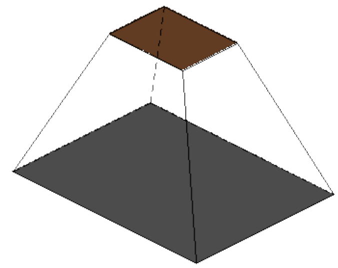
9. Two jars of peanut butter by the same brand are sold in a grocery store. The first jar is twice the height of the second jar, but its diameter is one-half as much as the shorter jar. The taller jar costs $\$1.49$, and the shorter jar costs $\$2.95$. Which jar is the better buy?



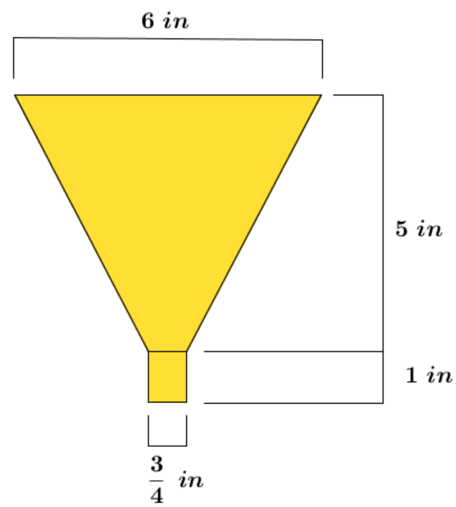
10. A cone with base area A and height h is sliced by planes parallel to its base into three pieces of equal height. Find the volume of each section.



11. The frustum of a pyramid is formed by cutting off the top part by a plane parallel to the base. The base of the pyramid and the cross-section where the cut is made are called the *bases of the frustum*. The distance between the planes containing the bases is called the *height of the frustum*. Find the volume of a frustum if the bases are squares of edge lengths 2 and 3, and the height of the frustum is 4.



12. A bulk tank contains a heavy grade of oil that is to be emptied from a valve into smaller 5.2-quart containers via a funnel. To improve the efficiency of this transfer process, Jason wants to know the greatest rate of oil flow that he can use so that the container and funnel do not overflow. The funnel consists of a cone that empties into a circular cylinder with the dimensions as shown in the diagram. Answer each question below to help Jason determine a solution to his problem.



a. Find the volume of the funnel.

b. If $1\ \text{in}^3$ is equivalent in volume to $\frac{4}{231}$ qt., what is the volume of the funnel in quarts?

c. If this particular grade of oil flows out of the funnel at a rate of 1.4 quarts per minute, how much time in minutes is needed to fill the 5.2-quart container?

d. Will the tank valve be shut off exactly when the container is full? Explain.

e. How long after opening the tank valve should Jason shut the valve off?

f. What is the maximum constant rate of flow from the tank valve that will fill the container without overflowing either the container or the funnel?

Lesson 12: The Volume Formula of a Sphere

Classwork

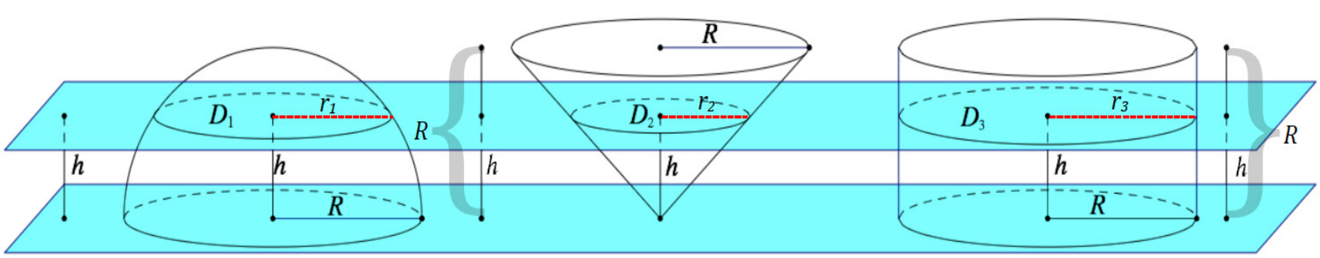
Opening Exercise

Picture a marble and a beach ball. Which one would you describe as a sphere? What differences between the two could possibly impact how we describe what a sphere is?

Definition	Characteristics

Sphere

Examples	Non-Examples



Hemisphere H **Cone T** **Cylinder S**

Example

Use your knowledge about the volumes of cones and cylinders to find a volume for a solid hemisphere of radius R.

Exercises

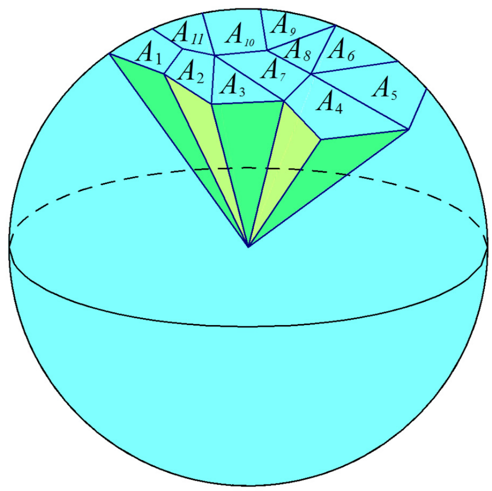
1. Find the volume of a sphere with a diameter of 12 cm to one decimal place.

2. An ice cream cone is 11 cm deep and 5 cm across the opening of the cone. Two hemisphere-shaped scoops of ice cream, which also have diameters of 5 cm, are placed on top of the cone. If the ice cream were to melt into the cone, will it overflow?

3. Bouncy, rubber balls are composed of a hollow, rubber shell 0.4" thick and an outside diameter of 1.2". The price of the rubber needed to produce this toy is $\$0.035/\text{in}^3$.
   a. What is the cost of producing 1 case, which holds 50 such balls? Round to the nearest cent.

   b. If each ball is sold for $\$0.10$, how much profit is earned on each ball sold?

Extension



Lesson Summary

SPHERE: Given a point C in the three-dimensional space and a number $r > 0$, the *sphere with center* C *and radius* r is the set of all points in space that are distance r from the point C.

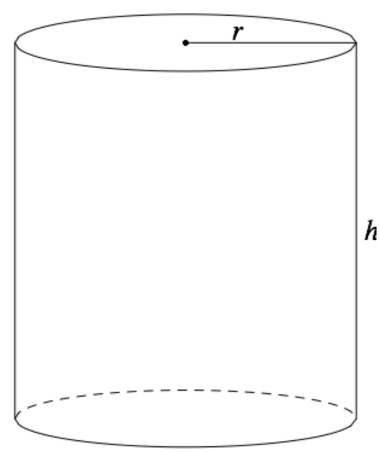
SOLID SPHERE OR BALL: Given a point C in the three-dimensional space and a number $r > 0$, the *solid sphere (or ball) with center* C *and radius* r is the set of all points in space whose distance from the point C is less than or equal to r.

Problem Set

1. A solid sphere has volume 36π. Find the radius of the sphere.

2. A sphere has surface area 16π. Find the radius of the sphere.

3. Consider a right circular cylinder with radius r and height h. The area of each base is πr^2. Think of the lateral surface area as a label on a soup can. If you make a vertical cut along the label and unroll it, the label unrolls to the shape of a rectangle.
   a. Find the dimensions of the rectangle.
   b. What is the lateral (or curved) area of the cylinder?



4. Consider a right circular cone with radius r, height h, and slant height l (see Figure 1). The area of the base is πr^2. Open the lateral area of the cone to form part of a disk (see Figure 2). The surface area is a fraction of the area of this disk.

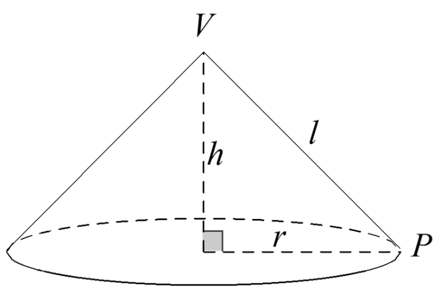


Figure 1

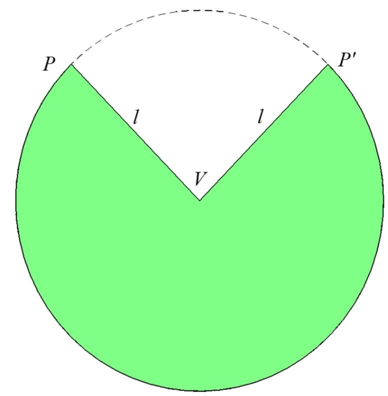


Figure 2

   a. What is the area of the entire disk in Figure 2?
   b. What is the circumference of the disk in Figure 2?

   The length of the arc on this circumference (i.e., the arc that borders the green region) is the circumference of the base of the cone with radius r or $2\pi r$. (Remember, the green region forms the curved portion of the cone and closes around the circle of the base.)

   c. What is the ratio of the area of the disk that is shaded to the area of the whole disk?
   d. What is the lateral (or curved) area of the cone?

5. A right circular cone has radius 3 cm and height 4 cm. Find the lateral surface area.

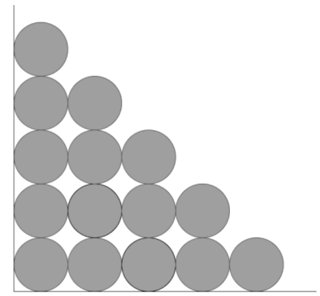
6. A semicircular disk of radius 3 ft. is revolved about its diameter (straight side) one complete revolution. Describe the solid determined by this revolution, and then find the volume of the solid.

7. A sphere and a circular cylinder have the same radius, r, and the height of the cylinder is $2r$.
   a. What is the ratio of the volumes of the solids?
   b. What is the ratio of the surface areas of the solids?

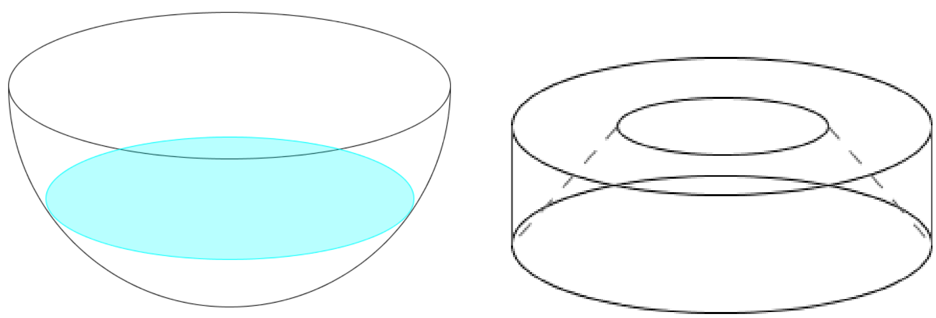
8. The base of a circular cone has a diameter of 10 cm and an altitude of 10 cm. The cone is filled with water. A sphere is lowered into the cone until it just fits. Exactly one-half of the sphere remains out of the water. Once the sphere is removed, how much water remains in the cone?

9. Teri has an aquarium that is a cube with edge lengths of 24 inches. The aquarium is $\frac{2}{3}$ full of water. She has a supply of ball bearings each having a diameter of $\frac{3}{4}$ inch.

   a. What is the maximum number of ball bearings that Teri can drop into the aquarium without the water overflowing?
   b. Would your answer be the same if the aquarium was $\frac{2}{3}$ full of sand? Explain.
   c. If the aquarium is empty, how many ball bearings would fit on the bottom of the aquarium if you arranged them in rows and columns as shown in the picture?
   d. How many of these layers could be stacked inside the aquarium without going over the top of the aquarium? How many bearings would there be altogether?
   e. With the bearings still in the aquarium, how much water can be poured into the aquarium without overflowing?
   f. Approximately how much of the aquarium do the ball bearings occupy?

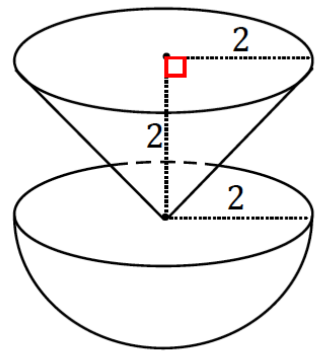


10. Challenge: A hemispherical bowl has a radius of 2 meters. The bowl is filled with water to a depth of 1 meter. What is the volume of water in the bowl? (Hint: Consider a cone with the same base radius and height, and the cross-section of that cone that lies 1 meter from the vertex.)



11. Challenge: A certain device must be created to house a scientific instrument. The housing must be a spherical shell, with an outside diameter of 1 m. It will be made of a material whose density is 14 g/cm^3. It will house a sensor inside that weighs 1.2 kg. The housing, with the sensor inside, must be neutrally buoyant, meaning that its density must be the same as water. Ignoring any air inside the housing, and assuming that water has a density of 1 g/cm^3, how thick should the housing be made so that the device is neutrally buoyant? Round your answer to the nearest tenth of a centimeter.

12. Challenge: An inverted, conical tank has a circular base of radius 2 m and a height of 2 m and is full of water. Some of the water drains into a hemispherical tank, which also has a radius of 2 m. Afterward, the depth of the water in the conical tank is 80 cm. Find the depth of the water in the hemispherical tank.

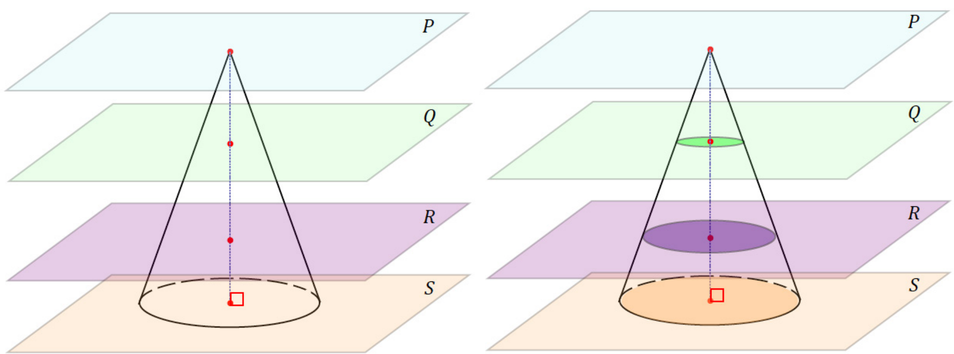


Lesson 13: How Do 3D Printers Work?

Classwork

Opening Exercise

a. Observe the following right circular cone. The base of the cone lies in plane S, and planes P, Q, and R are all parallel to S. Plane P contains the vertex of the cone.

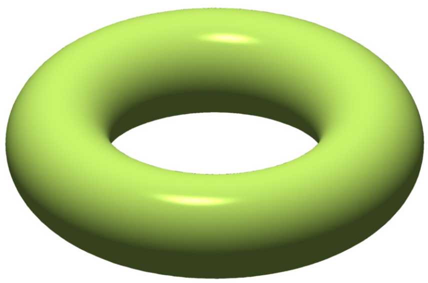


Sketch the cross-section P' of the cone by plane P.	Sketch the cross-section Q' of the cone by plane Q.
Sketch the cross-section R' of the cone by plane R.	Sketch the cross-section S' of the cone by plane S.

b. What happens to the cross-sections as we look at them starting with P' and work toward S'?

Exercise 1

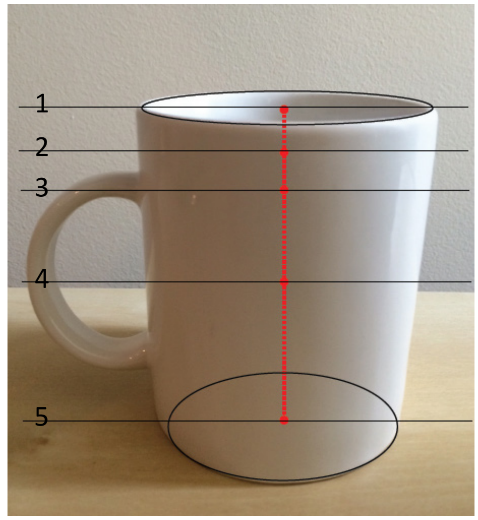
1. Sketch five evenly spaced, horizontal cross-sections made with the following figure.



http://commons.wikimedia.org/wiki/File%3ATorus_illustration.png; By Oleg Alexandrov (self-made, with MATLAB) [Public domain], via Wikimedia Commons. Attribution not legally required.

Example

Let us now try drawing cross-sections of an everyday object, such as a coffee cup.



Sketch the cross-sections at each of the indicated heights.

1	2	3

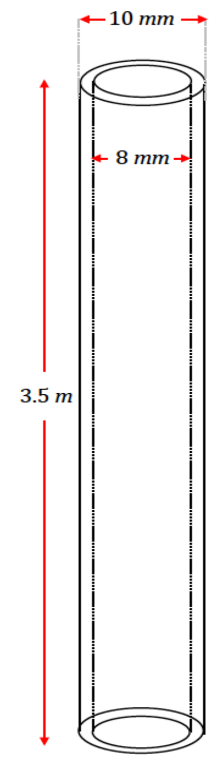
4	5

Exercises 2–4

2. A cone with a radius of 5 cm and height of 8 cm is to be printed from a 3D printer. The medium that the printer will use to print (i.e., the "ink" of this 3D printer) is a type of plastic that comes in coils of tubing which has a radius of $1\frac{1}{3}\text{ cm}$. What length of tubing is needed to complete the printing of this cone?

3. A cylindrical dessert 8 cm in diameter is to be created using a type of 3D printer specially designed for gourmet kitchens. The printer will "pipe" or, in other words, "print out" the delicious filling of the dessert as a solid cylinder. Each dessert requires 300 cm^3 of filling. Approximately how many layers does each dessert have if each layer is 3 mm thick?

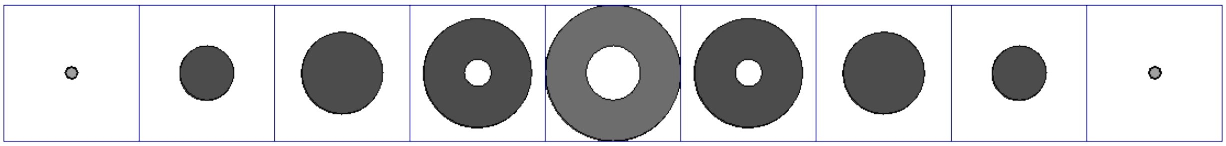
4. The image shown to the right is of a fine tube that is printed from a 3D printer that prints replacement parts. If each layer is 2 mm thick, and the printer prints at a rate of roughly 1 layer in 3 seconds, how many minutes will it take to print the tube?



Note: Figure not drawn to scale.

Problem Set

1. Horizontal slices of a solid are shown at various levels arranged from highest to lowest. What could the solid be?



2. Explain the difference in a 3D printing of the ring pictured in Figure 1 and Figure 2 if the ring is oriented in each of the following ways.

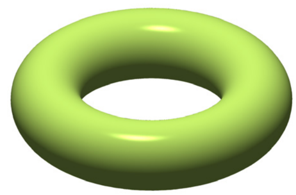


Figure 1

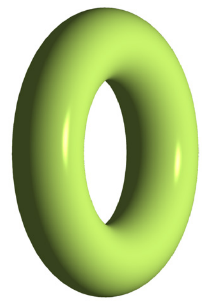


Figure 2

3. Each bangle printed by a 3D printer has a mass of exactly 25 g of metal. If the density of the metal is 14 g/cm^3, what length of a wire 1 mm in radius is needed to produce each bangle? Find your answer to the tenths place.

4. A certain 3D printer uses 100 m of plastic filament that is 1.75 mm in diameter to make a cup. If the filament has a density of 0.32 g/cm^3, find the mass of the cup to the tenths place.

5. When producing a circular cone or a hemisphere with a 3D printer, the radius of each layer of printed material must change in order to form the correct figure. Describe how radius must change in consecutive layers of each figure.

6. Suppose you want to make a 3D printing of a cone. What difference does it make if the vertex is at the top or at the bottom? Assume that the 3D printer places each new layer on top of the previous layer.

7. Filament for 3D printing is sold in spools that contain something shaped like a wire of diameter 3 mm. John wants to make 3D printings of a cone with radius 2 cm and height 3 cm. The length of the filament is 25 meters. About how many cones can John make?

8. John has been printing solid cones but would like to be able to produce more cones per each length of filament than you calculated in Problem 7. Without changing the outside dimensions of his cones, what is one way that he could make a length of filament last longer? Sketch a diagram of your idea and determine how much filament John would save per piece. Then determine how many cones John could produce from a single length of filament based on your design.

9. A 3D printer uses one spool of filament to produce 20 congruent solids. Suppose you want to produce similar solids that are 10% longer in each dimension. How many such figures could one spool of filament produce?

10. A fabrication company 3D-prints parts shaped like a pyramid with base as shown in the following figure. Each pyramid has a height of 3 cm. The printer uses a wire with a density of $12\ \text{g/cm}^3$, at a cost of $\$0.07/\text{g}$.

    It costs $\$500$ to set up for a production run, no matter how many parts they make. If they can only charge $\$15$ per part, how many do they need to make in a production run to turn a profit?