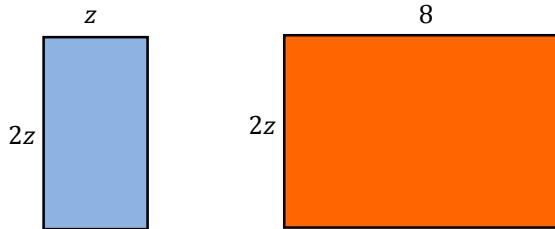


Lesson 1: Multiplying and Factoring Polynomial Expressions

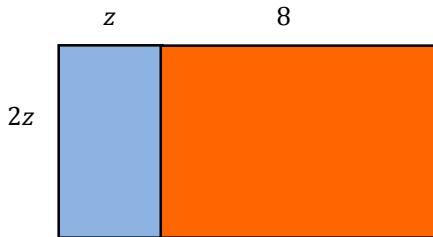
Classwork

Opening Exercise

Write expressions for the areas of the two rectangles in the figures given below.



Now write an expression for the area of this rectangle:



Example 1

Jackson has given his friend a challenge:

The area of a rectangle is represented by $3a^2 + 3a$ for some real number a . Find the dimensions of the length and width. How many possible answers are there for Jackson's challenge to his friend? List the answer(s) you find.

$$3a^2 + 3a \text{ square units}$$

Exercises 1–3

Factor each by factoring out the greatest common factor:

1. $10ab + 5a$

2. $3g^3h - 9g^2h + 12h$

3. $6x^2y^3 + 9xy^4 + 18y^5$

Discussion: Language of Polynomials

PRIME NUMBER: A *prime number* is a positive integer greater than 1 whose only positive integer factors are 1 and itself.

COMPOSITE NUMBER: A *composite number* is a positive integer greater than 1 that is not a prime number.

A composite number can be written as the product of positive integers with at least one factor that is not 1 or itself.

For example, the prime number 7 has only 1 and 7 as its factors. The composite number 6 has factors of 1, 2, 3, and 6; it could be written as the product $2 \cdot 3$.

A nonzero polynomial expression with integer coefficients is said to be *prime or irreducible over the integers* if it satisfies two conditions:

- (1) It is not equivalent to 1 or -1 , and
- (2) If the polynomial is written as a product of two polynomial factors, each with integer coefficients, then one of the two factors must be 1 or -1 .

Given a polynomial in standard form with integer coefficients, let c be the greatest common factor of all of the coefficients. The polynomial is *factored completely over the integers* when it is written as a product of c and one or more prime polynomial factors, each with integer coefficients.

Example 2: Multiply Two Binomials**Using a Table as an Aid**

You have seen the geometric area model used in previous examples to demonstrate the multiplication of polynomial expressions for which each expression was known to represent a measurement of length.

Without a context such as length, we cannot be certain that a polynomial expression represents a positive quantity. Therefore, an area model is not directly applicable to all polynomial multiplication problems. However, a table can be used in a similar fashion to identify each partial product as we multiply polynomial expressions. The table serves to remind us of the area model even though it does not represent area.

For example, fill in the table to identify the partial products of $(x + 2)(x + 5)$. Then, write the product of $(x + 2)(x + 5)$ in standard form.

x	+	5
+		
2		

Without the Aid of a Table

Regardless of whether or not we make use of a table as an aid, the multiplying of two binomials is an application of the distributive property. Both terms of the first binomial distribute over the second binomial. Try it with $(x + y)(x - 5)$. In the example below, the colored arrows match each step of the distribution with the resulting partial product:

Multiply: $(x + y)(x - 5)$

$$\left. \begin{array}{l} x^2 \\ -5x \\ +yx \\ -5y \end{array} \right\} x^2 - 5x + yx - 5y$$

Example 3: The Difference of Squares

Find the product of $(x + 2)(x - 2)$. Use the distributive property to distribute the first binomial over the second.

With the use of a table:

x	+	2
x	x^2	$2x$
+		
-2	$-2x$	-4

$$x^2 - 4$$

Without the use of a table:

$$(x)(x) + (x)(-2) + (2)(x) + (+2)(-2) = x^2 - 2x + 2x - 4 = x^2 - 4$$

Exercise 4

Factor the following examples of the difference of perfect squares.

a. $t^2 - 25$

b. $4x^2 - 9$

c. $16h^2 - 36k^2$

d. $4 - b^2$

e. $x^4 - 4$

f. $x^6 - 25$

Write a General Rule for Finding the Difference of Squares

Write $a^2 - b^2$ in factored form.

Exercises 5–7

Factor each of the following differences of squares completely:

5. $9y^2 - 100z^2$

6. $a^4 - b^6$

7. $r^4 - 16s^4$ (Hint: This one will factor twice.)

Example 4: The Square of a Binomial

To square a binomial, such as $(x + 3)^2$, multiply the binomial by itself.

$$\begin{aligned}(x + 3)(x + 3) &= (x)(x) + (3)(x) + (x)(3) + (3)(3) \\&= x^2 + 3x + 3x + 9 \\&= x^2 + 6x + 9\end{aligned}$$

Square the following general examples to determine the general rule for squaring a binomial:

a. $(a + b)^2$

b. $(a - b)^2$

Exercises 8–9

Square the binomial.

8. $(a + 6)^2$

9. $(5 - w)^2$

Lesson Summary

Factoring is the reverse process of multiplication. When factoring, it is always helpful to look for a GCF that can be pulled out of the polynomial expression. For example, $3ab - 6a$ can be factored as $3a(b - 2)$.

Factor the difference of perfect squares $a^2 - b^2$:

$$(a - b)(a + b).$$

When squaring a binomial $(a + b)$,

$$(a + b)^2 = a^2 + 2ab + b^2.$$

Problem Set

1. For each of the following, factor out the greatest common factor:

- a. $6y^2 + 18$
- b. $27y^2 + 18y$
- c. $21b - 15a$
- d. $14c^2 + 2c$
- e. $3x^2 - 27$

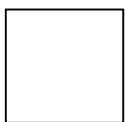
2. Multiply.

- a. $(n - 5)(n + 5)$
- b. $(4 - y)(4 + y)$
- c. $(k + 10)^2$
- d. $(4 + b)^2$

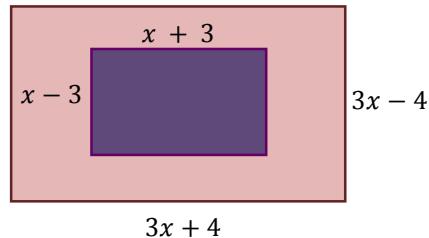
3. The measure of a side of a square is x units. A new square is formed with each side 6 units longer than the original square's side. Write an expression to represent the area of the new square. (Hint: Draw the new square and count the squares and rectangles.)

Original Square

x



4. In the accompanying diagram, the width of the inner rectangle is represented by $x - 3$ and the length by $x + 3$. The width of the outer rectangle is represented by $3x - 4$ and the length by $3x + 4$.



- Write an expression to represent the area of the larger rectangle.
- Write an expression to represent the area of the smaller rectangle.
- Express the area of the region inside the larger rectangle but outside the smaller rectangle as a polynomial in terms of x . (Hint: You will have to add or subtract polynomials to get your final answer.)

Lesson 2: Multiplying and Factoring Polynomial Expressions

Classwork

Example 1: Using a Table as an Aid

Use a table to assist in multiplying $(x + 7)(x + 3)$.

x	+	7
x	x^2	$7x$
+		
3	$3x$	21

Exercise 1

1. Use a table to aid in finding the product of $(2x + 1)(x + 4)$.

POLYNOMIAL EXPRESSION: A *polynomial expression* is either:

- (1) A numerical expression or a variable symbol, or
- (2) The result of placing two previously generated polynomial expressions into the blanks of the addition operator ($\underline{\hspace{1cm}} + \underline{\hspace{1cm}}$) or the multiplication operator ($\underline{\hspace{1cm}} \times \underline{\hspace{1cm}}$).

Exercises 2–6

Multiply the following binomials; note that every binomial given in the problems below is a polynomial in one variable, x , with a degree of one. Write the answers in standard form, which in this case will take the form $ax^2 + bx + c$, where a , b , and c are constants.

2. $(x + 1)(x - 7)$

3. $(x + 9)(x + 2)$

4. $(x - 5)(x - 3)$

5. $\left(x + \frac{15}{2}\right)(x - 1)$

6. $\left(x - \frac{5}{4}\right)\left(x - \frac{3}{4}\right)$

Describe any patterns you noticed as you worked.

Exercises 7–10

Factor the following quadratic expressions.

7. $x^2 + 8x + 7$

8. $m^2 + m - 90$

9. $k^2 - 13k + 40$

10. $v^2 + 99v - 100$

Example 3: Quadratic Expressions

If the leading coefficient for a quadratic expression is not 1, the first step in factoring should be to see if all the terms in the expanded form have a common factor. Then after factoring out the greatest common factor, it may be possible to factor again.

For example, to factor to $2x^3 - 50x$ completely, begin by finding the GCF.

The GCF of the expression is $2x$: $2x(x^2 - 25)$.

Now, factor the difference of squares: $2x(x - 5)(x + 5)$.

Another example: Follow the steps to factor $-16t^2 + 32t + 48$ completely.

- a. First, factor out the GCF. (Remember: When you factor out a negative number all the signs on the resulting factor will change.)

- b. Now look for ways to factor further. (Notice the quadratic expression will factor.)

Lesson Summary

Multiplying binomials is an application of the distributive property; each term in the first binomial is distributed over the terms of the second binomial.

The area model can be modified into a tabular form to model the multiplication of binomials (or other polynomials) that may involve negative terms.

When factoring trinomial expressions (or other polynomial expressions), it is useful to look for a GCF as your first step.

Do not forget to look for these special cases:

- The square of a binomial
- The product of the sum and difference of two expressions.

Problem Set

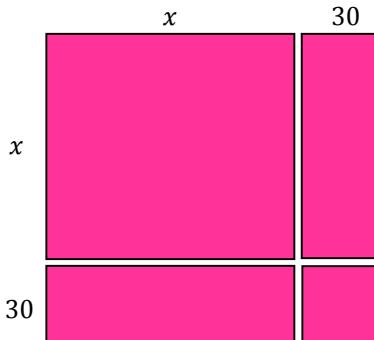
1. Factor these trinomials as the product of two binomials, and check your answer by multiplying.

- $x^2 + 3x + 2$
- $x^2 - 8x + 15$
- $x^2 + 8x + 15$

Factor completely.

- $4m^2 - 4n^2$
- $-2x^3 - 2x^2 + 112x$
- $y^8 - 81x^4$

2. The parking lot at Gene Simon's Donut Palace is going to be enlarged so that there will be an additional 30 ft. of parking space in the front of the lot and an additional 30 ft. of parking space on the side of the lot. Write an expression in terms of x that can be used to represent the area of the new parking lot.



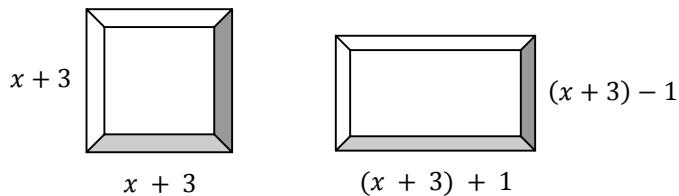
Explain how your solution is demonstrated in the area model.

Lesson 3: Advanced Factoring Strategies for Quadratic Expressions

Classwork

Opening Exercise

Carlos wants to build a sandbox for his little brother. He is deciding between a square sandbox with side lengths that can be represented by $x + 3$ units and a rectangular sandbox with a length 1 unit more than the side of the square and width 1 unit less than the side of the square.



Carlos thinks the areas should be the same because one unit is just moved from one side to the other.

- Do you agree that the two areas should be the same? Why or why not?
- How would you write the expressions that represent the length and width of the rectangular sandbox in terms of the side length of the square?
- If you use the expressions for length and width represented in terms of the side length of the square, can you then write the area of the rectangle in the same terms?

- d. How can this expression be seen as the product of a sum and difference: $(a + b)(a - b)$?
- e. Can you now rewrite the area expression for the rectangle as the difference of squares: $(a + b)(a - b) = a^2 - b^2$?
- f. Look carefully at your answer to the last question. What does it tell you about the areas of the two shapes?
- g. Can you verify that our algebra is correct using a diagram or visual display?

Example 1

In Lesson 2, we saw that factoring is the reverse process of multiplication. We factor a polynomial by reversing the distribution process.

Consider the following example of multiplication:

$$(x + 3)(x + 5) \rightarrow x^2 + 5x + 3x + 15 \rightarrow x^2 + 8x + 15$$

When we compare the numbers in the factored form with the numbers in the expanded form, we see that 15 is the product of the two numbers ($3 \cdot 5$) and 8 is their sum ($3 + 5$). The latter is even more obvious when we look at the expanded form before the like terms are combined.

(Note: When the x -term coefficient is 1, we usually do not write it out algebraically, but it is actually there as a coefficient. Pointing this out here will prepare students for the next example where the x -term coefficients in the factors are not both 1.)

Can you explain why that relationship exists between the numbers in the factors and the numbers in the final expanded form?

Example 2

Now compare the expansion of this binomial product to the one above:

$$(2x + 3)(1x + 5) \rightarrow 2x^2 + 10x + 3x + 15 \rightarrow 2x^2 + 13x + 15$$

In the expression lying between the two arrows (before the like terms are combined), we can see the coefficients of the “split” linear terms ($+10x + 3x$). Also notice that for this example, we have coefficients on both x -terms in the factors and that one of the coefficients is not 1. We have 2 and 1 as the factors of the leading coefficient in the expanded form and 3 and 5 as the factors of the constant term. Get ready for quadratic expressions in factored form where neither of the x -term coefficients are 1.

- a. How is this product different from the first example? How is it similar?

- b. Why are the “split” linear terms different in the two examples?

- c. Now that we have four different numbers (coefficients) in each form of the expression, how can we use the numbers in the expanded form of the quadratic expression on the right to find the numbers in the factors on the left?

- d. Now we need to place those numbers into the parentheses for the factors so that the product matches the expanded form of the quadratic expression. Here is a template for finding the factors using what we call the product-sum method:

$(\underline{x} \pm \underline{})(\underline{x} \pm \underline{})$ [We have four number places to fill in this factor template.]

$(\underline{x} \pm 3)(\underline{x} \pm 5)$ [We know that the 3 and 5 are the correct factors for 15, so we start there.]

$(2x \pm 3)(1x \pm 5)$ [We know that 2 and 1 are the only factors of 2, with the 2 opposite the 5 so that the distribution process gives us $10x$ for one product.]

$(2x + 3)(x + 5)$ [Finally, we know, at least for this example, that all the numbers are positive.]

Example 3

Now try factoring a quadratic expression with some negative coefficients: $3x^2 - x - 4$.

$(\underline{x} \pm \underline{})(\underline{x} \pm \underline{})$ [We have four number places to fill in this factor template.]

$(\underline{x} \pm 1)(\underline{x} \pm 4)$ [We know that ± 1 and ± 4 or ± 2 and ± 2 are the only possible factors for the constant term, -4 , so we start there. Try 1 and 4 to start, and if that does not work, go back and try ± 2 and ± 2 . We know that only one of the numbers can be negative to make the product negative.]

$(1x \pm 1)(3x \pm 4)$ [We know that 3 and 1 are the only factors of 3. We also know that both of these are positive (or both negative). But we do not know which positions they should take, so we will try both ways to see which will give a sum of -1 .]

$(x + 1)(3x - 4)$ [Finally, we determine the two signs needed to make the final product $3x^2 - x - 4$.]

Exercises

Factor the expanded form of these quadratic expressions. Pay particular attention to the negative and positive signs.

1. $3x^2 - 2x - 8$

2. $3x^2 + 10x - 8$

3. $3x^2 + x - 14$ [Notice that there is a 1 as a coefficient in this one.]

4. $2x^2 - 21x - 36$ [This might be a challenge. If it takes too long, try the next one.]

5. $-2x^2 + 3x + 9$ [This one has a negative on the leading coefficient.]

6. $r^2 + \frac{6}{4}r + \frac{9}{16}$ [We need to try one with fractions, too.]

For Exercises 7–10, use the structure of these expressions to factor completely.

7. $100x^2 - 20x - 63$

8. $y^4 + 2y^2 - 3$

9. $9x^2 - 3x - 12$

10. $16a^2b^4 + 20ab^2 - 6$

Lesson Summary

QUADRATIC EXPRESSION: A polynomial expression of degree 2 is often referred to as a *quadratic expression*.

Some quadratics are not easily factored. The following hints will make the job easier:

- In the difference of squares $a^2 - b^2$, either of these terms a or b could be a binomial itself.
- The product-sum method is useful but can be tricky when the leading coefficient is not 1.
- Trial and error is a viable strategy for finding factors.
- Check your answers by multiplying the factors to ensure you get back the original quadratic expression.

Problem Set

Factor the following quadratic expressions.

1. $3x^2 - 2x - 5$
2. $-2x^2 + 5x - 2$
3. $5x^2 + 19x - 4$
4. $4x^2 - 12x + 9$ [This one is tricky, but look for a special pattern.]
5. $3x^2 - 13x + 12$

Lesson 4: Advanced Factoring Strategies for Quadratic Expressions

Classwork

Opening Exercises

Factor the following quadratic expressions.

1. $2x^2 + 10x + 12$

2. $6x^2 + 5x - 6$

Example 1: Splitting the Linear Term

How might we find the factors of $6x^2 + 5x - 6$?

1. Consider the product $(a)(c)$: $(6)(-6) = -36$.
2. Discuss the possibility that a and c are also multiplied when the leading coefficient is 1.
3. List all possible factor pairs of $(a)(c)$: $(1, -36), (-1, 36), (2, -18), (-2, 18), (3, -12), (-3, 12), (4, -9), (-4, 9)$, and $(-6, 6)$.
4. Find the pair that satisfies the requirements of the product-sum method (i.e., a pair of numbers whose product equals ac and whose sum is b): $(-4) + 9 = 5$.
5. Rewrite the expression with the same first and last term but with an expanded b term using that pair of factors as coefficients: $6x^2 - 4x + 9x - 6$.
6. We now have four terms that can be entered into a tabular model or factored by grouping.
7. Factoring by grouping: Take the four terms above and pair the first two and the last two; this makes two groups.

$$[6x^2 - 4x] + [9x - 6] \quad [\text{Form two groups by pairing the first two and the last two.}]$$

$$[2x(3x - 2)] + [3(3x - 2)] \quad [\text{Factor out the GCF from each pair.}]$$

The common binomial factor is now visible as a common factor of each group. Now rewrite by carefully factoring out the common factor, $3x - 2$, from each group: $(3x - 2)(2x + 3)$.

Note: We can factor difficult quadratic expressions, such as $6x^2 + 5x - 6$, using a tabular model or by splitting the linear term algebraically. Try both ways to see which one works best for you.

Exercises 1–4

Factor the following expressions using your method of choice. After factoring each expression completely, check your answers using the distributive property. Remember to always look for a GCF prior to trying any other strategies.

1. $2x^2 - x - 10$

2. $6x^2 + 7x - 20$

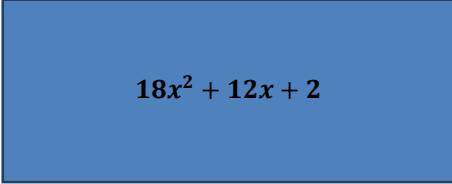
3. $-4x^2 + 4x - 1$

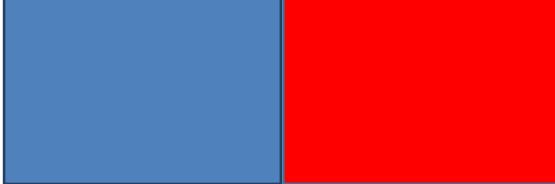
4. The area of a particular triangle can be represented by $x^2 + \frac{3}{2}x - \frac{9}{2}$. What are its base and height in terms of x ?

Lesson Summary

While there are several steps involved in splitting the linear term, it is a relatively more efficient and reliable method for factoring trinomials in comparison to simple guess-and-check.

Problem Set

1. Factor completely.
 - a. $6x^2 + 7x + 2$
 - b. $8x^2 + 20x + 8$
 - c. $3x^2 + 10x + 7$
 - d. $x^2 + \frac{11}{2}x + \frac{5}{2}$
 - e. $6x^3 - 2x^2 - 4x$ [Hint: Look for a GCF first.]
2. The area of the rectangle below is represented by the expression $18x^2 + 12x + 2$ square units. Write two expressions to represent the dimensions, if the length is known to be twice the width.


$18x^2 + 12x + 2$
3. Two mathematicians are neighbors. Each owns a separate rectangular plot of land that shares a boundary and has the same dimensions. They agree that each has an area of $2x^2 + 3x + 1$ square units. One mathematician sells his plot to the other. The other wants to put a fence around the perimeter of his new combined plot of land. How many linear units of fencing will he need? Write your answer as an expression in x .


Note: This question has two correct approaches and two different correct solutions. Can you find them both?

Lesson 5: The Zero Product Property

Classwork

Opening Exercise

Consider the equation $a \cdot b \cdot c \cdot d = 0$. What values of a , b , c , and d would make the equation true?

Exercises 1–4

Find values of c and d that satisfy each of the following equations. (There may be more than one correct answer.)

1. $cd = 0$

2. $(c - 5)d = 2$

3. $(c - 5)d = 0$

4. $(c - 5)(d + 3) = 0$

Example

For each of the related questions below, use what you know about the zero product property to find the answers.

- The area of a rectangle can be represented by the expression $x^2 + 2x - 3$. Write each dimension of this rectangle as a binomial, and then write the area in terms of the product of the two binomials.
- Can we draw and label a diagram that represents the rectangle's area?

c. Suppose the rectangle's area is known to be 21 square units. Can you find the dimensions in terms of x ?

d. Rewrite the equation so that it is equal to zero and solve.

e. What are the actual dimensions of the rectangle?

f. A smaller rectangle can fit inside the first rectangle, and it has an area that can be expressed by the equation $x^2 - 4x - 5$. What are the dimensions of the smaller rectangle in terms of x ?

g. What value for x would make the smaller rectangle have an area of $\frac{1}{3}$ that of the larger?

Exercises 5–8

Solve. Show your work.

5. $x^2 - 11x + 19 = -5$

6. $7x^2 + x = 0$

7. $7r^2 - 14r = -7$

8. $2d^2 + 5d - 12 = 0$

Lesson Summary

Zero Product Property**If $ab = 0$, then $a = 0$ or $b = 0$ or $a = b = 0$.**

When solving for the variable in a quadratic equation, rewrite the equation as a factored quadratic set equal to zero. Using the zero product property, you know that if one factor is equal to zero, then the product of all factors is equal to zero.

Going one step further, when you have set each binomial factor equal to zero and have solved for the variable, all of the possible solutions for the equation have been found. Given the context, some solutions may not be viable, so be sure to determine if each possible solution is appropriate for the problem.

Problem Set

Solve the following equations.

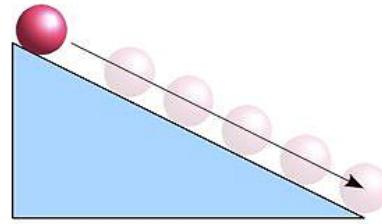
1. $x^2 + 15x + 40 = 4$
2. $7x^2 + 2x = 0$
3. $b^2 + 5b - 35 = 3b$
4. $7r^2 - 14r = -7$

Lesson 6: Solving Basic One-Variable Quadratic Equations

Classwork

Example 1

A physics teacher put a ball at the top of a ramp and let it roll down toward the floor. The class determined that the height of the ball could be represented by the equation $h = -16t^2 + 4$, where the height, h , is measured in feet from the ground and time, t , in seconds.



- What do you notice about the structure of the quadratic expression in this problem? How can this structure help us when we apply this equation?
- In the equation, explain what the 4 represents.
- Explain how you would use the equation to determine the time it takes the ball to reach the floor.
- Now consider the two solutions for t . Which one is reasonable? Does the final answer make sense based on this context? Explain.

Example 2

Lord Byron is designing a set of square garden plots so some peasant families in his kingdom can grow vegetables. The minimum size for a plot recommended for vegetable gardening is at least 2 m on each side. Lord Byron has enough space around the castle to make bigger plots. He decides that each side will be the minimum (2 m) plus an additional x m.

- a. What expression can represent the area of one individual garden based on the undecided additional length x ?
- b. There are 12 families in the kingdom who are interested in growing vegetables in the gardens. What equation can represent the total area, A , of the 12 gardens?
- c. If the total area available for the gardens is 300 sq. m, what are the dimensions of each garden?
- d. Find both values for x that make the equation in part (c) true (the solution set). What value of x will Lord Byron need to add to the 2 m?

Exercises

Solve each equation. Some of them may have radicals in their solutions.

1. $3x^2 - 9 = 0$

2. $(x - 3)^2 = 1$

3. $4(x - 3)^2 = 1$

4. $2(x - 3)^2 = 12$

5. Analyze the solutions for Exercises 2–4. Notice how the questions all had $(x - 3)^2$ as a factor, but each solution was different (radical, mixed number, whole number). Explain how the structure of each expression affected each problem-solution pair.
6. Peter is a painter and he wonders if he would have time to catch a paint bucket dropped from his ladder before it hits the ground. He drops a bucket from the top of his 9-foot ladder. The height, h , of the bucket during its fall can be represented by the equation, $h = -16t^2 + 9$, where the height is measured in feet from the ground, and the time since the bucket was dropped, t , is measured in seconds. After how many seconds does the bucket hit the ground? Do you think he could catch the bucket before it hits the ground?

Lesson Summary

By looking at the structure of a quadratic equation (missing linear terms, perfect squares, factored expressions), you can find clues for the best method to solve it. Some strategies include setting the equation equal to zero, factoring out the GCF or common factors, and using the zero product property.

Be aware of the domain and range for a function presented in context, and consider whether answers make sense in that context.

Problem Set

1. Factor completely: $15x^2 - 40x - 15$.

Solve each equation.

2. $4x^2 = 9$
3. $3y^2 - 8 = 13$
4. $(d + 4)^2 = 5$
5. $4(g - 1)^2 + 6 = 13$
6. $12 = -2(5 - k)^2 + 20$
7. Mischief is a toy poodle that competes with her trainer in the agility course. Within the course, Mischief must leap through a hoop. Mischief's jump can be modeled by the equation $h = -16t^2 + 12t$, where h is the height of the leap in feet and t is the time since the leap, in seconds. At what values of t does Mischief start and end the jump?

Lesson 7: Creating and Solving Quadratic Equations in One Variable

Classwork

Opening Exercise

The length of a rectangle is 5 in. more than twice a number. The width is 4 in. less than the same number. The perimeter of the rectangle is 44 in. Sketch a diagram of this situation, and find the unknown number.

Example 1

The length of a rectangle is 5 in. more than twice a number. The width is 4 in. less than the same number. If the area of the rectangle is 15 in^2 , find the unknown number.

Example 2

A picture has a height that is $\frac{4}{3}$ its width. It is to be enlarged so that the ratio of height to width remains the same, but the area is 192 in^2 . What are the dimensions of the enlargement?

Exercises

Solve the following problems. Be sure to indicate if a solution is to be rejected based on the contextual situation.

1. The length of a rectangle is 4 cm more than 3 times its width. If the area of the rectangle is 15 cm^2 , find the width.

2. The ratio of length to width in a rectangle is $2 : 3$. Find the length of the rectangle when the area is 150 in^2 .

3. One base of a trapezoid is 4 in. more than twice the length of the second base. The height of the trapezoid is 2 in. less than the second base. If the area of the trapezoid is 4 in^2 , find the dimensions of the trapezoid.
(Note: The area of a trapezoid is $A = \frac{1}{2}(b_1 + b_2)h$.)
4. A garden measuring 12 m by 16 m is to have a pedestrian pathway that is w meters wide installed all the way around it, increasing the total area to 285 m^2 . What is the width, w , of the pathway?
5. Karen wants to plant a garden and surround it with decorative stones. She has enough stones to enclose a rectangular garden with a perimeter of 68 ft., and she wants the garden to cover 240 ft^2 . What is the length and width of her garden?

6. Find two consecutive odd integers whose product is 99. (Note: There are two different pairs of consecutive odd integers and only an algebraic solution will be accepted.)
7. Challenge: You have a 500-foot roll of chain link fencing and a large field. You want to fence in a rectangular playground area. What are the dimensions of the largest such playground area you can enclose? What is the area of the playground?

Lesson Summary

When provided with a verbal description of a problem, represent the scenario algebraically. Start by identifying the unknown quantities in the problem and assigning variables. For example, write expressions that represent the length and width of an object.

Solve the equation using techniques previously learned, such as factoring and using the zero product property. The final answer should be clearly stated and should be reasonable in terms of the context of the problem.

Problem Set

Solve the following problems.

1. The length of a rectangle is 2 cm less than its width. If the area of the rectangle is 35 cm^2 , find the width.
2. The ratio of length to width (measured in inches) in a rectangle is $4 : 7$. Find the length of the rectangle if the area is known to be 700 in^2 .
3. One base of a trapezoid is three times the length of the second base. The height of the trapezoid is 2 in. smaller than the second base. If the area of the trapezoid is 30 in^2 , find the lengths of the bases and the height of the trapezoid.
4. A student is painting an accent wall in his room where the length of the room is 3 ft. more than its width. The wall has an area of 130 ft^2 . What are the length and the width, in feet?
5. Find two consecutive even integers whose product is 80. (There are two pairs, and only an algebraic solution will be accepted.)

Lesson 8: Exploring the Symmetry in Graphs of Quadratic Functions

Classwork

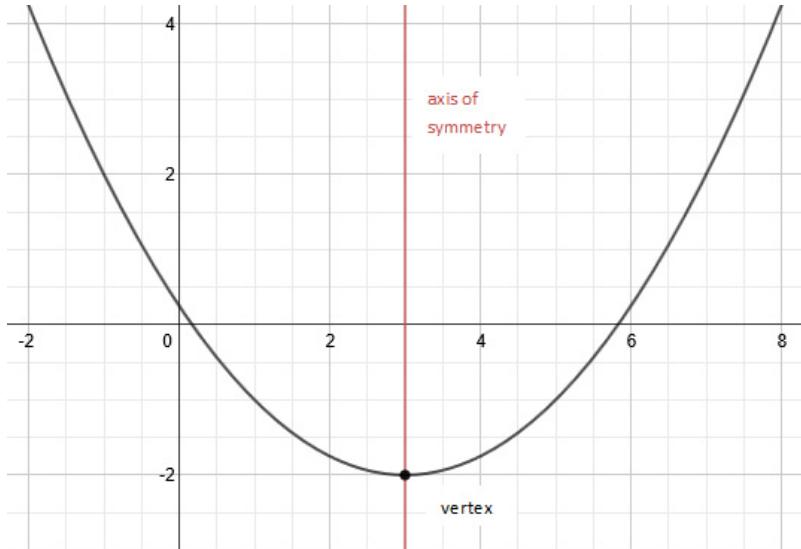
Graph Vocabulary

AXIS OF SYMMETRY: Given a quadratic function in standard form, $f(x) = ax^2 + bx + c$, the vertical line given by the graph of the equation $x = -\frac{b}{2a}$ is called the *axis of symmetry* of the graph of the quadratic function.

VERTEX: The point where the graph of a quadratic function and its axis of symmetry intersect is called the *vertex*.

END BEHAVIOR OF A GRAPH: Given a quadratic function in the form $f(x) = ax^2 + bx + c$ (or $f(x) = a(x - h)^2 + k$), the quadratic function is said to *open up* if $a > 0$ and *open down* if $a < 0$.

- If $a > 0$, then f has a minimum at the x -coordinate of the vertex, i.e., f is decreasing for x -values less than (or to the left of) the vertex, and f is increasing for x -values greater than (or to the right of) the vertex.
- If $a < 0$, then f has a maximum at the x -coordinate of the vertex, i.e., f is increasing for x -values less than (or to the left of) the vertex, and f is decreasing for x -values greater than (or to the right of) the vertex.



End behavior: This quadratic curve opens up. As the values of x approach $+\infty$ and $-\infty$, the values of y approach $+\infty$.

Exploratory Challenge 1

Below are some examples of curves found in architecture around the world. Some of these might be represented by graphs of quadratic functions. What are the key features these curves have in common with a graph of a quadratic function?



St. Louis Arch



Bellos Falls Arch Bridge



Arch of Constantine



Roman Aqueduct

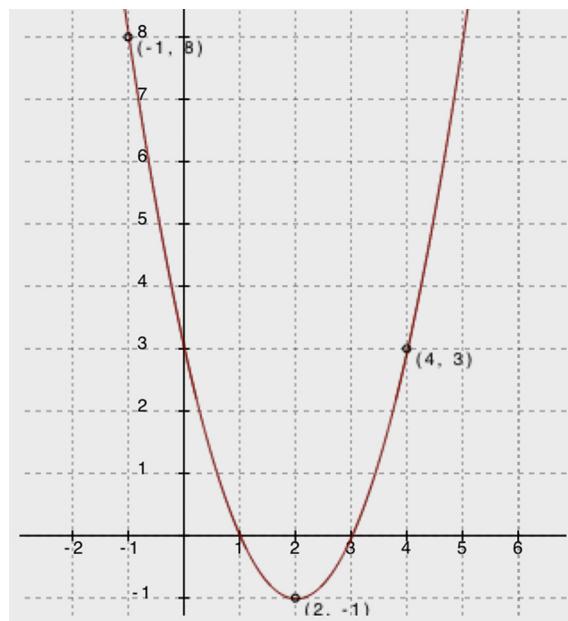
The photographs of architectural features above MIGHT be closely represented by graphs of quadratic functions. Answer the following questions based on the pictures.

- How would you describe the overall shape of a graph of a quadratic function?
- What is similar or different about the overall shape of the above curves?

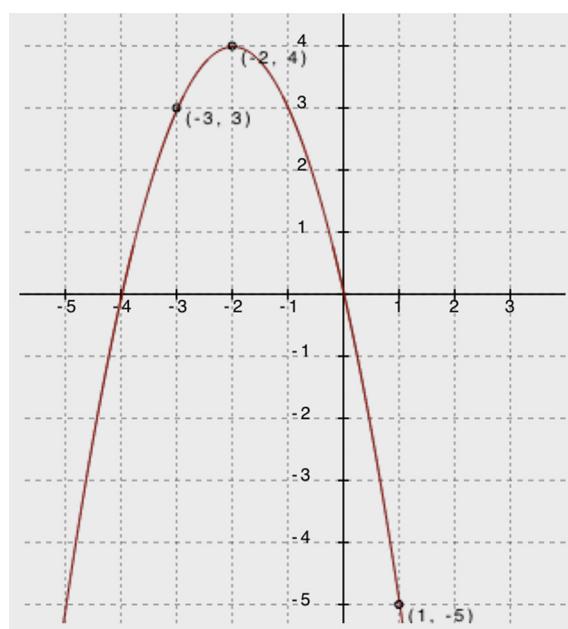
IMPORTANT: Many of the photographs in this activity cannot actually be modeled with a quadratic function but rather are catenary curves. These are “quadratic-like” and can be used for our exploration purposes as they display many of the same features, including the symmetry we are exploring in this lesson.

Exploratory Challenge 2

Use the graphs of quadratic functions (Graph A and Graph B) to fill in the table and answer the questions on the following page.

Graph A

x	$f(x)$
-1	8
2	-1
4	3

Graph B

x	$f(x)$
-3	3
-2	4
1	-5

Use your graphs and tables of values from the previous page to fill in the blanks or answer the questions for each below.

		Graph A	Graph B
1	x -Intercepts		
2	Vertex		
3	Sign of the Leading Coefficient		
4	Vertex Represents a Minimum or Maximum?		
5	Points of Symmetry	<p>Find $f(-1)$ and $f(5)$.</p> <p>Is $f(7)$ greater than or less than 8? Explain</p>	<p>Find $f(-1)$ and $f(-3)$.</p> <p>$f(2) = -12$. Predict the value for $f(-6)$ and explain your answer.</p>
6	Increasing and Decreasing Intervals	<p>On what intervals of the domain is the function depicted by the graph increasing?</p> <p>On what intervals of the domain is the function depicted by the graph decreasing?</p>	<p>On what intervals of the domain is the function depicted by the graph increasing?</p> <p>On what intervals of the domain is the function depicted by the graph decreasing?</p>
7	Average Rate of Change on an Interval	<p>What is the average rate of change for the following intervals?</p> <p>$[-1, 0]$: $[0, 1]$: $[0, 3]$: $[1, 3]$:</p>	<p>What is the average rate of change for the following intervals?</p> <p>$[-5, -4]$: $[-4, -3]$: $[-4, -1]$: $[-3, -1]$:</p>

Understanding the symmetry of quadratic functions and their graphs (Look at row 5 in the chart and the tables.)

- a. What patterns do you see in the tables of values you made next to Graph A and Graph B?

Finding the vertex and axis of symmetry (Look at rows 1 and 2 of the chart.)

- b. How can we know the x -coordinate of the vertex by looking at the x -coordinates of the zeros (or any pair of symmetric points)?

Understanding end behavior (Look at rows 3 and 4 of the chart.)

- c. What happens to the y -values of the functions as the x -values increase to very large numbers? What about as the x -values decrease to very small numbers (in the negative direction)?

- d. How can we know whether a graph of a quadratic function will open up or down?

Identifying intervals on which the function is increasing or decreasing (Look at row 6 in the chart.)

- e. Is it possible to determine the exact intervals that a quadratic function is increasing or decreasing just by looking at a graph of the function?

Computing average rate of change on an interval (Look at row 7 in the chart.)

- f. Explain why the average rate of change over the interval $[1, 3]$ for Graph A was zero.

- g. How are finding the slope of a line and finding the average rate of change on an interval of a quadratic function similar? How are they different?

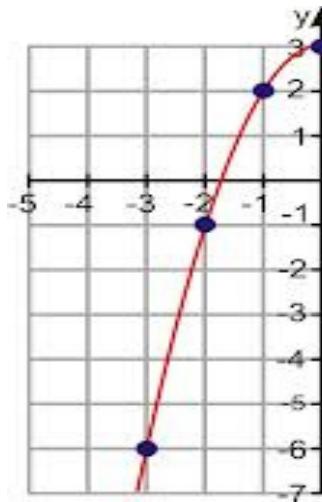
Finding a unique quadratic function

- h. Can you graph a quadratic function if you don't know the vertex? Can you graph a quadratic function if you only know the x -intercepts?

- i. Remember that we need to know at least two points to define a unique line. Can you identify a unique quadratic function with just two points? Explain.
- j. What is the minimum number of points you would need to identify a unique quadratic function? Explain why.

Exploratory Challenge 3

Below you see only one side of the graph of a quadratic function. Complete the graph by plotting three additional points of the quadratic function. Explain how you found these points, and then fill in the table on the right.



x	$f(x)$

- a. What are the coordinates of the x -intercepts?

- b. What are the coordinates of the y -intercept?

- c. What are the coordinates of the vertex? Is it a minimum or a maximum?

- d. If we knew the equation for this curve, what would the sign of the leading coefficient be?

- e. Verify that the average rate of change for the interval $-3 \leq x \leq -2$, or $[-3, -2]$, is 5. Show your steps.

- f. Based on your answer to row 6 in the table for Exploratory Challenge 2, what interval would have an average rate of change of -5 ? Explain.

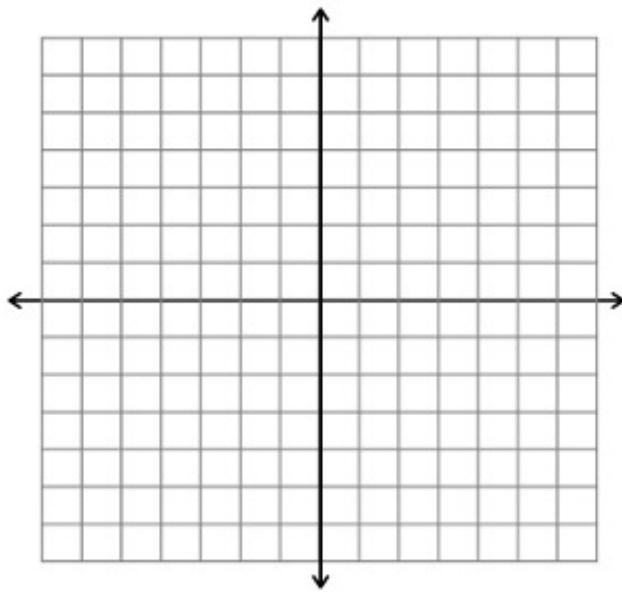
Lesson Summary

Quadratic functions create a symmetrical curve with its highest (maximum) or lowest (minimum) point corresponding to its vertex and an axis of symmetry passing through the vertex when graphed. The x -coordinate of the vertex is the average of the x -coordinates of the zeros or any two symmetric points on the graph.

When the leading coefficient is a negative number, the graph *opens down*, and its end behavior is that both ends move towards negative infinity. If the leading coefficient is positive, the graph *opens up*, and both ends move towards positive infinity.

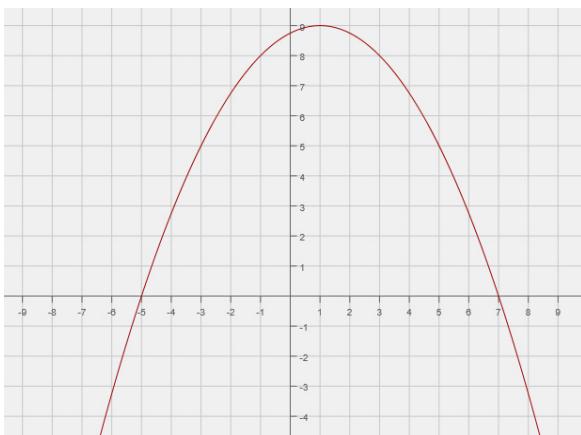
Problem Set

- Khaya stated that every y -value of the graph of a quadratic function has two different x -values. Do you agree or disagree with Khaya? Explain your answer.
- Is it possible for the graphs of two *different* quadratic functions to each have $x = -3$ as its line of symmetry and both have a maximum at $y = 5$? Explain and support your answer with a sketch of the graphs.

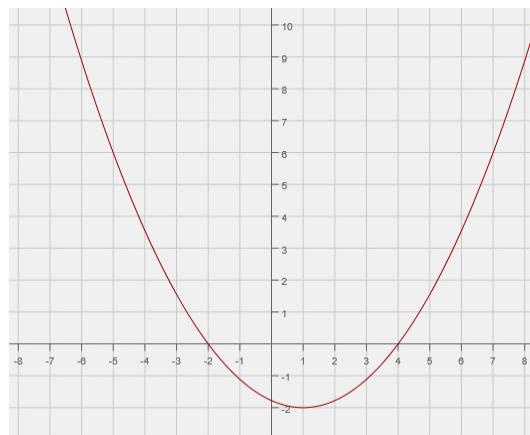


3. Consider the following key features discussed in this lesson for the four graphs of quadratic functions below:
 x -intercepts, y -intercept, line of symmetry, vertex, and end behavior.

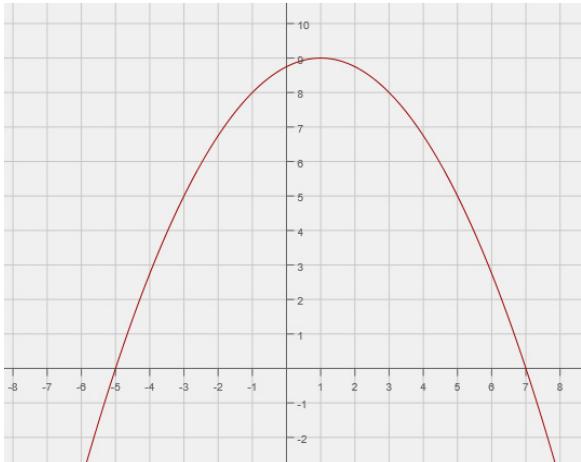
Graph A



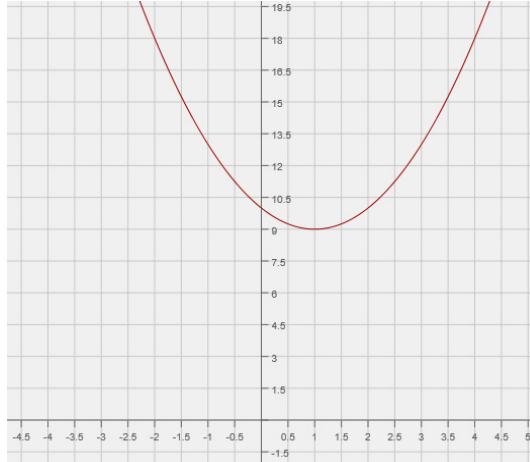
Graph B



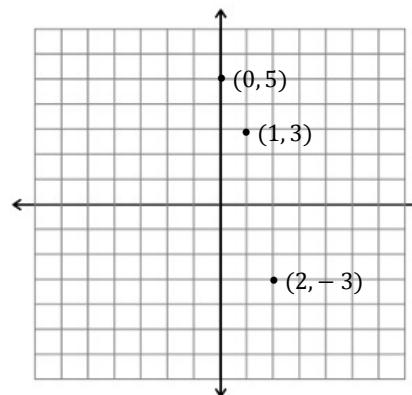
Graph C



Graph D



- a. Which key features of a quadratic function do graphs A and B have in common? Which features are not shared?
- b. Compare graphs A and C and explain the differences and similarities between their key features.
- c. Compare graphs A and D and explain the differences and similarities between their key features.
- d. What do all four of the graphs have in common?
4. Use the symmetric properties of quadratic functions to sketch the graph of the function below, given these points and given that the vertex of the graph is the point $(0, 5)$.



Lesson 9: Graphing Quadratic Functions from Factored Form,

$$f(x) = a(x - m)(x - n)$$

Classwork

Opening Exercise

Solve the following equations.

a. $x^2 + 6x - 40 = 0$

b. $2x^2 + 11x = x^2 - x - 32$

Example 1

Consider the equation $y = x^2 + 6x - 40$.

- Given this quadratic equation, can you find the point(s) where the graph crosses the x -axis?
- How can we write a corresponding quadratic equation if we are given a pair of roots?
- In the last lesson, we learned about the symmetrical nature of the graph of a quadratic function. How can we use that information to find the vertex for the graph?

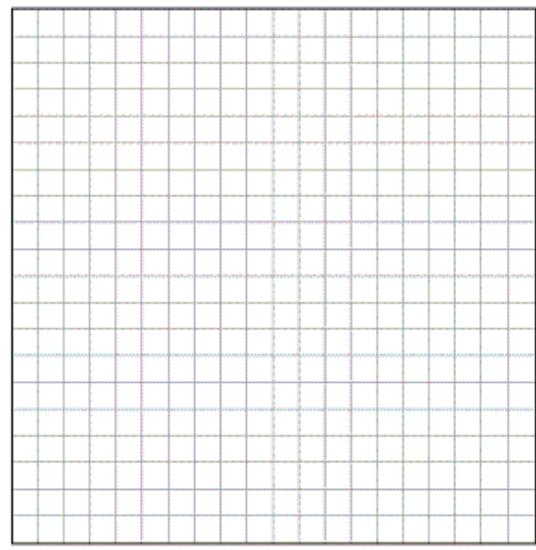
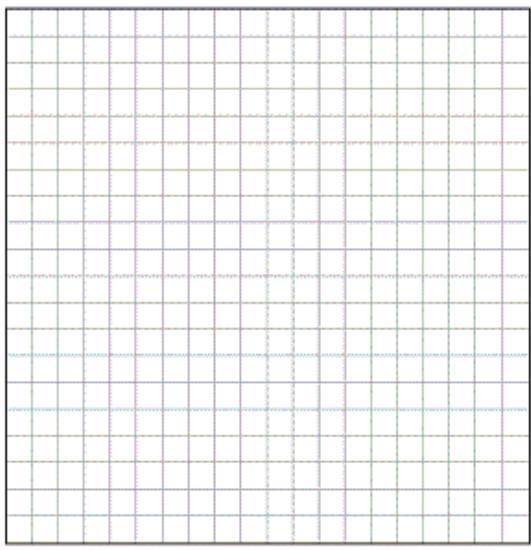
- d. How could we find the y -intercept (where the graph crosses the y -axis and where $x = 0$)?
- e. What else can we say about the graph based on our knowledge of the symmetrical nature of the graph of a quadratic function? Can we determine the coordinates of any other points?
- f. Plot the points you know for this equation on graph paper, and connect them to show the graph of the equation.

Exercise 1

Graph the following functions, and identify key features of the graph.

a. $f(x) = -(x + 2)(x - 5)$

b. $g(x) = x^2 - 5x - 24$



Example 2

A science class designed a ball launcher and tested it by shooting a tennis ball straight up from the top of a 15-story building. They determined that the motion of the ball could be described by the function:

$$h(t) = -16t^2 + 144t + 160,$$

where t represents the time the ball is in the air in seconds and h represents the height, in feet, of the ball above the ground at time t . What is the maximum height of the ball? At what time will the ball hit the ground?

- a. With a graph, we can see the number of seconds it takes for the ball to reach its peak and how long it takes to hit the ground. How can factoring the expression help us graph this function?

- b. Once we have the function in its factored form, what do we need to know in order to graph it? Now graph the function.

- c. Using the graph, at what time does the ball hit the ground?

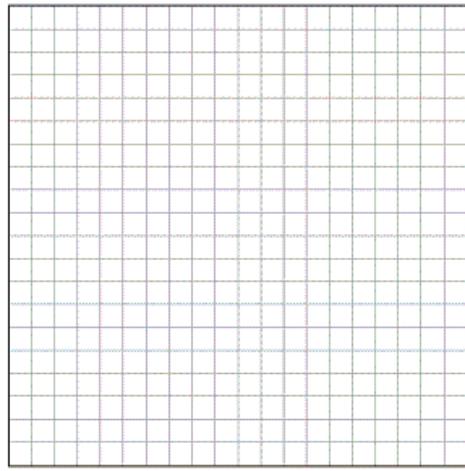
- d. Over what domain is the ball rising? Over what domain is the ball falling?

- e. Using the graph, what is the maximum height the ball reaches?

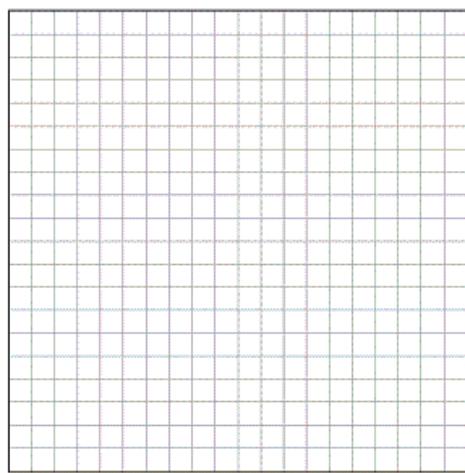
Exercises 2–3

2. Graph the following functions, and identify key features of the graph.

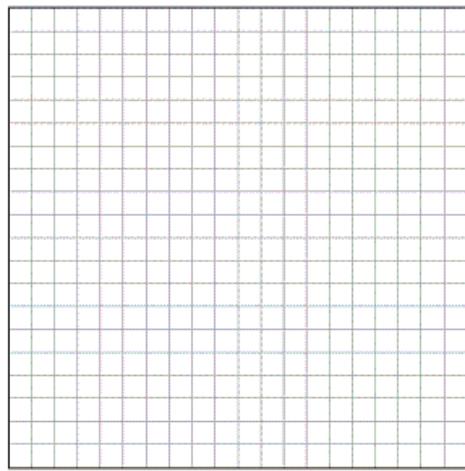
a. $f(x) = 5(x - 2)(x - 3)$



b. $t(x) = x^2 + 8x - 20$

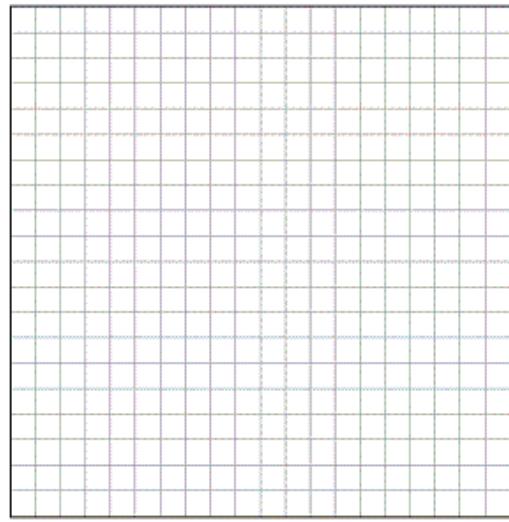


c. $p(x) = -6x^2 + 42x - 60$



3. The science class in Example 3 adjusted their ball launcher so that it could accommodate a heavier ball. They moved the launcher to the roof of a 23-story building and launched an 8.8-pound shot put straight up into the air. (Note: Olympic and high school women use the 8.8-pound shot put in track and field competitions.) The motion is described by the function $h(t) = -16t^2 + 32t + 240$, where h represents the height, in feet, of the shot put above the ground with respect to time in seconds. (Important: No one was harmed during this experiment!)

- a. Graph the function, and identify the key features of the graph.



b. After how many seconds does the shot put hit the ground?

c. What is the maximum height of the shot put?

d. What is the value of $h(0)$, and what does it mean for this problem?



Lesson Summary

- When we have a quadratic function in factored form, we can find its x -intercepts, y -intercept, axis of symmetry, and vertex.
- For any quadratic equation, the roots are the solution(s) where $y = 0$, and these solutions correspond to the points where the graph of the equation crosses the x -axis.
- A quadratic equation can be written in the form $y = a(x - m)(x - n)$, where m and n are the roots of the quadratic. Since the x -value of the vertex is the average of the x -values of the two roots, we can substitute that value back into equation to find the y -value of the vertex. If we set $x = 0$, we can find the y -intercept.
- In order to construct the graph of a unique quadratic function, at least three distinct points of the function must be known.

Problem Set

1. Graph the following on your own graph paper, and identify the key features of the graph.
 - a. $f(x) = (x - 2)(x + 7)$
 - b. $g(x) = -2(x - 2)(x + 7)$
 - c. $h(x) = x^2 - 16$
 - d. $p(x) = x^2 - 2x + 1$
 - e. $q(x) = 4x^2 + 20x + 24$
2. A rocket is launched from a cliff. The relationship between the height, h , in feet, of the rocket and the time, t , in seconds, since its launch can be represented by the following function:

$$h(t) = 16t^2 + 80t + 384.$$

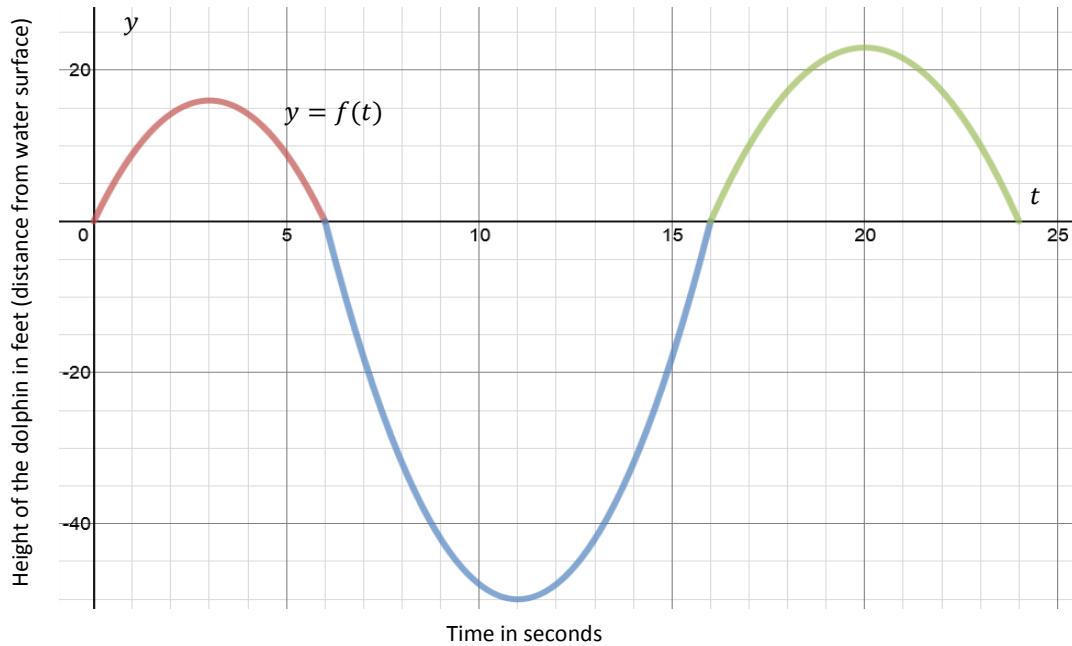
- a. Sketch the graph of the motion of the rocket.
- b. When will the rocket hit the ground?
- c. When will the rocket reach its maximum height?
- d. What is the maximum height the rocket reaches?
- e. At what height was the rocket launched?

Lesson 10: Interpreting Quadratic Functions from Graphs and Tables

Classwork

Example 1

In a study of the activities of dolphins, a marine biologist made a 24-second video of a dolphin swimming and jumping in the ocean with a specially equipped camera that recorded one dolphin's position with respect to time. This graph represents a piecewise function, $f(t)$, that is defined by quadratic functions on each interval. It relates the dolphin's vertical distance from the surface of the water, in feet, to the time from the start of the video, in seconds. Use the graph to answer the questions below.



- Describe what you know for sure about the actions of the dolphin in the time interval from 0–6 sec. Can you determine the horizontal distance the dolphin traveled in that time interval? Explain why or why not.

- b. Where will you find the values for which $f(t) = 0$? Explain what they mean in the context of this problem.
- c. How long was the dolphin swimming under water in the recorded time period? Explain your answer or show your work.
- d. Estimate the maximum height, in feet, that the dolphin jumped in the recorded 24-second time period? Explain how you determined your answer.
- e. Locate the point on the graph where $f(t) = -50$, and explain what information the coordinates of that point give you in the context of this problem.

Example 2

The table below represents the value of Andrew's stock portfolio, where V represents the value of the portfolio in hundreds of dollars and t is the time in months since he started investing. Answer the questions that follow based on the table of values.

t (months)	$V(t)$ (hundreds of dollars)
2	325
4	385
6	405
8	385
10	325
12	225
14	85
16	-95
18	-315

- a. What kind of function could model the data in this table? How can you support your conclusion?

- b. Assuming this data is in fact quadratic, how much did Andrew invest in his stock initially? Explain how you arrived at this answer.

- c. What is the maximum value of his stock, and how long did it take to reach the maximum value?
- d. If the pattern continues to follow the quadratic trend shown above, do you advise Andrew to sell or keep his stock portfolio? Explain why.
- e. How fast is Andrew's stock value decreasing between $[10, 12]$? Find another two-month interval where the average rate of change is faster than $[10, 12]$ and explain why.
- f. Are there other two-month intervals where the rate of change is the same as $[10, 12]$? Explain your answer.

Lesson Summary

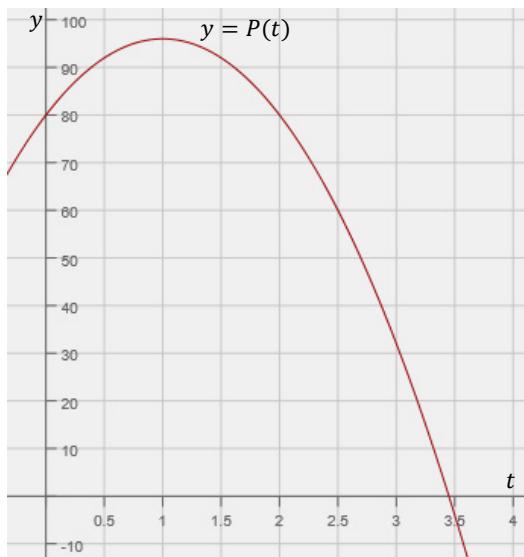
When interpreting quadratic functions and their graphs, it is important to note that the graph does not necessarily depict the path of an object. In the case of free-falling objects, for example, it is height with respect to time.

The y -intercept can represent the initial value of the function, given the context, and the vertex represents the highest (if a maximum) or the lowest (if a minimum) value.

Problem Set

Pettitte and Ryu each threw a baseball into the air.

The vertical height of Pettitte's baseball is represented by the graph $P(t)$ below. P represents the vertical distance of the baseball from the ground in feet, and t represents time in seconds.



The vertical height of Ryu's baseball is represented by the table values $R(t)$ below. $R(t)$ represents the vertical distance of the baseball from the ground in feet, and t represents time in seconds.

t	$R(t)$
0	86
0.5	98
1	102
1.5	98
2	86
2.5	66
3	38
3.52	0

Use the above functions to answer the following questions.

- a. Whose baseball reached the greatest height? Explain your answer.
- b. Whose ball reached the ground fastest? Explain your answer.
- c. Pettitte claims that his ball reached its maximum faster than Ryu's. Is his claim correct or incorrect? Explain your answer.
- d. Find $P(0)$ and $R(0)$ values and explain what they mean in the problem. What conclusion can you make based on these values? Did Ryu and Pettitte throw their baseballs from the same height? Explain your answer.
- e. Ryu claims that he can throw the ball higher than Pettitte. Is his claim correct or incorrect? Explain your answer.

Lesson 11: Completing the Square

Classwork

Opening Exercise

Rewrite the following perfect square quadratic expressions in standard form. Look for patterns in the coefficients, and write two sentences describing what you notice.

FACTORED FORM	WRITE THE FACTORS	DISTRIBUTE	STANDARD FORM
Example: $(x + 1)^2$			
$(x + 2)^2$			
$(x + 3)^2$			
$(x + 4)^2$			
$(x + 5)^2$			
$(x + 20)^2$			

Example

Now try working backwards. Rewrite the following standard form quadratic expressions as perfect squares.

STANDARD FORM	FACTORED FORM
$x^2 + 12x + 36$	
$x^2 - 12x + 36$	
$x^2 + 20x + 100$	
$x^2 - 3x + \frac{9}{4}$	
$x^2 + 100x + 2,500$	
$x^2 + 8x + 3$	

Exploratory Challenge

Find an expression equivalent to $x^2 + 8x + 3$ that includes a perfect square binomial.

Exercises 1–10

Rewrite each expression by completing the square.

1. $a^2 - 4a + 15$

2. $n^2 - 2n - 15$

3. $c^2 + 20c - 40$

4. $x^2 - 1,000x + 60,000$

5. $y^2 - 3y + 10$

6. $k^2 + 7k + 6$

7. $z^2 - 0.2z + 1.5$

8. $p^2 + 0.5p + 0.1$

9. $j^2 - \frac{3}{4}j + \frac{3}{4}$

10. $x^2 - bx + c$

Lesson Summary

Just as factoring a quadratic expression can be useful for solving a quadratic equation, completing the square also provides a form that facilitates solving a quadratic equation.

Problem Set

1. $q^2 + 12q + 32$
2. $m^2 - 4m - 5$
3. $x^2 - 7x + 6.5$
4. $a^2 + 70a + 1,225$
5. $z^2 - 0.3z + 0.1$
6. $y^2 - 6by + 20$
7. Which of these expressions would be most easily rewritten by factoring? Justify your answer.

Lesson 12: Completing the Square

Classwork

Opening Exercise

Rewrite each expression by completing the square.

1. $z^2 - 5z + 8$

2. $x^2 + 0.6x + 1$

Example 1

Now complete the square for $2x^2 + 16x + 3$.

Example 2**Business Application Vocabulary**

UNIT PRICE (PRICE PER UNIT): The price per item a business sets to sell its product, which is sometimes represented as a linear expression.

QUANTITY: The number of items sold, sometimes represented as a linear expression.

REVENUE: The total income based on sales (but without considering the cost of doing business).

UNIT COST (COST PER UNIT) OR PRODUCTION COST: The cost of producing one item, sometimes represented as a linear expression.

PROFIT: The amount of money a business makes on the sale of its product. Profit is determined by taking the total revenue (the quantity sold multiplied by the price per unit) and subtracting the total cost to produce the items (the quantity sold multiplied by the production cost per unit): Profit = Total Revenue – Total Production Costs.

The following business formulas will be used in this and the remaining lessons in the module:

$$\text{Total Production Costs} = (\text{cost per unit})(\text{quantity of items sold})$$

$$\text{Total Revenue} = (\text{price per unit})(\text{quantity of items sold})$$

$$\text{Profit} = \text{Total Revenue} - \text{Total Production Costs}$$

Now solve the following problem:

A certain business is marketing its product and has collected data on sales and prices for the past few years. The company determined that when it raised the selling price of the product, the number of sales went down. The cost of producing a single item is \$10.

- a. Using the data the company collected in this table, determine a linear expression to represent the quantity sold, q .

Selling Price (s)	Quantity Sold (q)
10	1,000
15	900
20	800
25	700

- b. Now find an expression to represent the profit function, P .

Exercises

For Exercises 1–5, rewrite each expression by completing the square.

1. $3x^2 + 12x - 8$

2. $4p^2 - 12p + 13$

3. $\frac{1}{2}y^2 + 3y - 4$

4. $1.2n^2 - 3n + 6.5$

5. $\frac{1}{3}v^2 - 4v + 10$

6. A fast food restaurant has determined that its price function is $3 - \frac{x}{20,000}$, where x represents the number of hamburgers sold.

a. The cost of producing x hamburgers is determined by the expression $5,000 + 0.56x$. Write an expression representing the profit for selling x hamburgers.

b. Complete the square for your expression in part (a) to determine the number of hamburgers that need to be sold to maximize the profit, given this price function.

Lesson Summary

Here is an example of completing the square of a quadratic expression of the form $ax^2 + bx + c$.

$$\begin{aligned}3x^2 - 18x - 2 \\3(x^2 - 6x) - 2 \\3(x^2 - 6x + 9) - 3(9) - 2 \\3(x - 3)^2 - 3(9) - 2 \\3(x - 3)^2 - 29\end{aligned}$$

Problem Set

Rewrite each expression by completing the square.

1. $-2x^2 + 8x + 5$
2. $2.5k^2 - 7.5k + 1.25$
3. $\frac{4}{3}b^2 + 6b - 5$
4. $1,000c^2 - 1,250c + 695$
5. $8n^2 + 2n + 5$

Lesson 13: Solving Quadratic Equations by Completing the Square

Classwork

Opening Exercises

1. Solve the equation for b : $2b^2 - 9b = 3b^2 - 4b - 14$.

2. Rewrite the expression by completing the square: $\frac{1}{2}b^2 - 4b + 13$.

Example 1

Solve for x .

$$12 = x^2 + 6x$$

Rational and Irrational Numbers

The *sum or product of two rational numbers* is always a *rational number*.

The *sum of a rational number and an irrational number* is always an *irrational number*.

The *product of a rational number and an irrational number* is an *irrational number* as long as the rational number is not zero.

Example 2

Now try this one.

$$4x^2 - 40x + 94 = 0$$

Exercises

Solve each equation by completing the square.

1. $x^2 - 2x = 12$

$$2. \quad \frac{1}{2}r^2 - 6r = 2$$

$$3. \quad 2p^2 + 8p = 7$$

$$4. \quad 2y^2 + 3y - 5 = 4$$

Lesson Summary

When a quadratic equation is not conducive to factoring, we can solve by completing the square.

Completing the square can be used to find solutions that are irrational, something very difficult to do by factoring.

Problem Set

Solve each equation by completing the square.

1. $p^2 - 3p = 8$
2. $2q^2 + 8q = 3$
3. $\frac{1}{3}m^2 + 2m + 8 = 5$
4. $-4x^2 = 24x + 11$

Lesson 14: Deriving the Quadratic Formula

Classwork

Opening Exercises

1. Solve for x by completing the square: $x^2 + 2x = 8$.
2. Solve for p by completing the square: $7p^2 - 12p + 4 = 0$.

Discussion

Solve $ax^2 + bx + c = 0$.

Exercises

Use the quadratic formula to solve each equation.

1. $x^2 - 2x = 12 \rightarrow a = 1, b = -2, c = -12$ (Watch the negatives.)

2. $\frac{1}{2}r^2 - 6r = 2 \rightarrow a = \frac{1}{2}, b = -6, c = -2$ (Did you remember the negative?)

3. $2p^2 + 8p = 7 \rightarrow a = 2, b = 8, c = -7$

4. $2y^2 + 3y - 5 = 4 \rightarrow a = 2, b = 3, c = -9$

Exercise 5

Solve these quadratic equations, using a different method for each: solve by factoring, solve by completing the square, and solve using the quadratic formula. Before starting, indicate which method you will use for each.

Method _____

$$2x^2 + 5x - 3 = 0$$

Method _____

$$x^2 + 3x - 5 = 0$$

Method _____

$$\frac{1}{2}x^2 - x - 4 = 0$$

Lesson Summary

The quadratic formula, $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, is derived by completing the square on the general form of a quadratic equation: $ax^2 + bx + c = 0$, where $a \neq 0$. The formula can be used to solve any quadratic equation, and is especially useful for those that are not easily solved using any other method (i.e., factoring or completing the square).

Problem Set

Use the quadratic formula to solve each equation.

1. Solve for z : $z^2 - 3z - 8 = 0$.
2. Solve for q : $2q^2 - 8 = 3q \rightarrow a = 2, b = -3, c = -8$.
3. Solve for m : $\frac{1}{3}m^2 + 2m + 8 = 5$.

Lesson 15: Using the Quadratic Formula

Classwork

Opening Exercises

Solve the following:

1. $4x^2 + 5x + 3 = 2x^2 - 3x$

2. $c^2 - 14 = 5c$

Exercises

Solve Exercises 1–5 using the quadratic formula.

1. $x^2 - 2x + 1 = 0$

$$2. \quad 3b^2 + 4b + 8 = 0$$

$$3. \quad 2t^2 + 7t - 4 = 0$$

$$4. \quad q^2 - 2q - 1 = 0$$

$$5. \quad m^2 - 4 = 3$$

For Exercises 6–9, determine the number of real solutions for each quadratic equation without solving.

6. $p^2 + 7p + 33 = 8 - 3p$

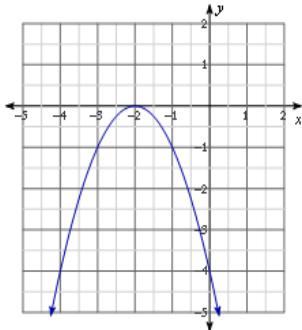
7. $7x^2 + 2x + 5 = 0$

8. $2y^2 + 10y = y^2 + 4y - 3$

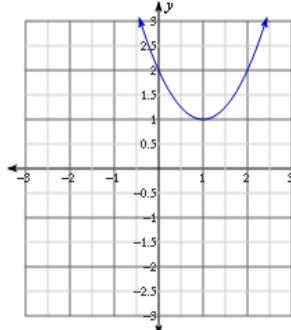
9. $4z^2 + 9 = -4z$

10. On the line below each graph, state whether the discriminant of each quadratic equation is positive, negative, or equal to zero. Then, identify which graph matches the discriminants below.

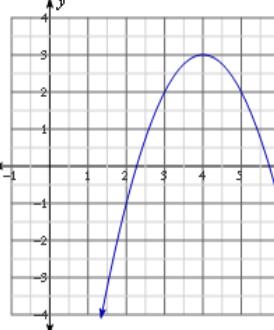
Graph 1



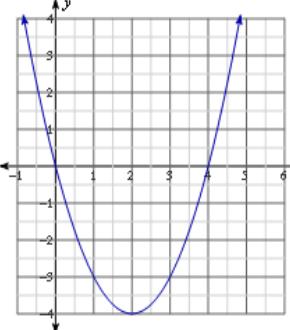
Graph 2



Graph 3



Graph 4



Discriminant A:

$$(-2)^2 - 4(1)(2)$$

Graph: _____

Discriminant B:

$$(-4)^2 - 4(-1)(-4)$$

Graph: _____

Discriminant C:

$$(-4)^2 - 4(1)(0)$$

Graph: _____

Discriminant D:

$$(-8)^2 - 4(-1)(-13)$$

Graph: _____

Lesson Summary

You can use the sign of the discriminant, $b^2 - 4ac$, to determine the number of real solutions to a quadratic equation in the form $ax^2 + bx + c = 0$, where $a \neq 0$. If the equation has a positive discriminant, there are two real solutions. A negative discriminant yields no real solutions, and a discriminant equal to zero yields only one real solution.

Problem Set

Without solving, determine the number of real solutions for each quadratic equation.

1. $b^2 - 4b + 3 = 0$

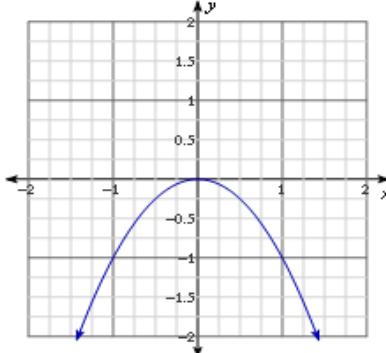
2. $2n^2 + 7 = -4n + 5$

3. $x - 3x^2 = 5 + 2x - x^2$

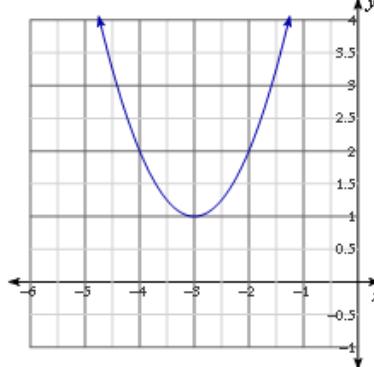
4. $4q + 7 = q^2 - 5q + 1$

Based on the graph of each quadratic function, $y = f(x)$, determine the number of real solutions for each corresponding quadratic equation, $f(x) = 0$.

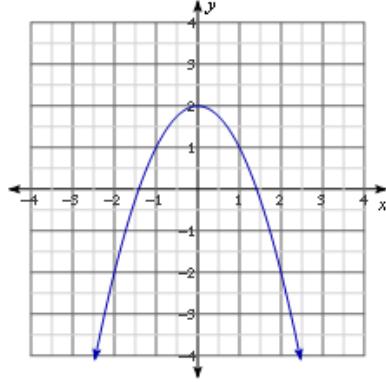
5.



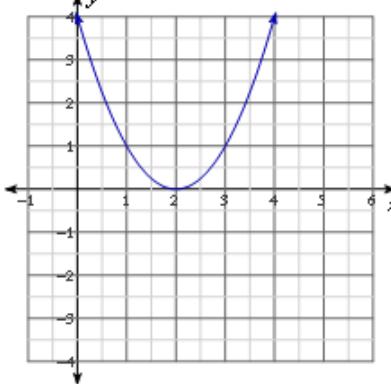
6.



7.



8.



Lesson 16: Graphing Quadratic Equations from the Vertex Form,

$$y = a(x - h)^2 + k$$

Classwork

Opening Exercise

Graph the equations $y = x^2$, $y = (x - 2)^2$, and $y = (x + 2)^2$ on the interval $-3 \leq x \leq 3$.

Exercises 1–2

1. Without graphing, state the vertex for each of the following quadratic equations.

a. $y = (x - 5)^2 + 3$

b. $y = x^2 - 2.5$

c. $y = (x + 4)^2$

2. Write a quadratic equation whose graph will have the given vertex.

a. $(1.9, -4)$

b. $(0, 100)$

c. $(-2, \frac{3}{2})$

Exploratory Challenge

Caitlin has 60 feet of material that can be used to make a fence. Using this material, she wants to create a rectangular pen for her dogs to play in. What dimensions will maximize the area of the pen?

- Let w be the width of the rectangular pen in feet. Write an expression that represents the length when the width is w feet.
- Define a function $A(w)$ that describes the area, A , in terms of the width, w .
- Rewrite $A(w)$ in vertex form.
- What are the coordinates of the vertex? Interpret the vertex in terms of the problem.
- What dimensions maximize the area of the pen? Do you think this is a surprising answer?

Lesson Summary

When graphing a quadratic equation in vertex form, $y = a(x - h)^2 + k$, (h, k) are the coordinates of the vertex.

Problem Set

1. Find the vertex of the graphs of the following quadratic equations.
 - a. $y = 2(x - 5)^2 + 3.5$
 - b. $y = -(x + 1)^2 - 8$
2. Write a quadratic equation to represent a function with the following vertex. Use a leading coefficient other than 1.
 - a. $(100, 200)$
 - b. $\left(-\frac{3}{4}, -6\right)$
3. Use vocabulary from this lesson (i.e., stretch, shrink, opens up, opens down, etc.) to compare and contrast the graphs of the quadratic equations $y = x^2 + 1$ and $y = -2x^2 + 1$.

Lesson 17: Graphing Quadratic Functions from the Standard Form, $f(x) = ax^2 + bx + c$

Classwork

Opening Exercise

A high school baseball player throws a ball straight up into the air for his math class. The math class was able to determine that the relationship between the height of the ball and the time since it was thrown could be modeled by the function $h(t) = -16t^2 + 96t + 6$, where t represents the time (in seconds) since the ball was thrown, and h represents the height (in feet) of the ball above the ground.

- a. What does the domain of the function represent in this context?

- b. What does the range of this function represent?

- c. At what height does the ball get thrown?

- d. After how many seconds does the ball hit the ground?

- e. What is the maximum height that the ball reaches while in the air? How long will the ball take to reach its maximum height?

- f. What feature(s) of this quadratic function are “visible” since it is presented in the standard form,
 $f(x) = ax^2 + bx + c$?

- g. What feature(s) of this quadratic function are “visible” when it is rewritten in vertex form,
 $f(x) = a(x - h)^2 + k$?

A general strategy for graphing a quadratic function from the standard form:

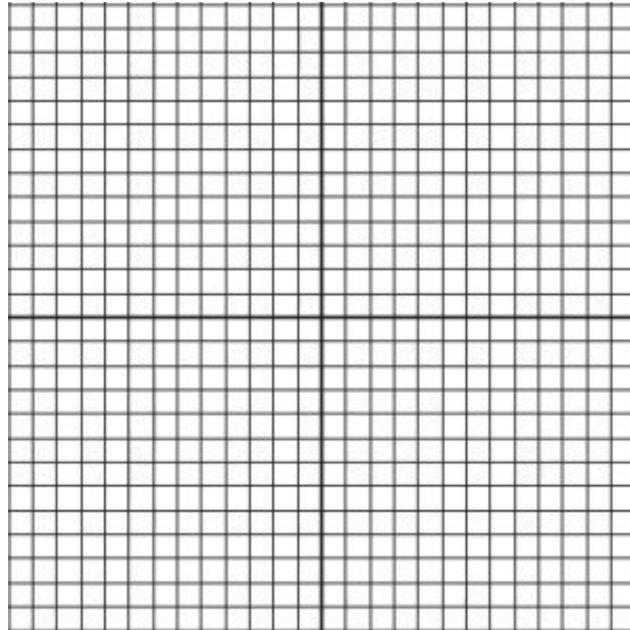
Example 1

A high school baseball player throws a ball straight up into the air for his math class. The math class was able to determine that the relationship between the height of the ball and the time since it was thrown could be modeled by the function $h(t) = -16t^2 + 96t + 6$, where t represents the time (in seconds) since the ball was thrown, and h represents the height (in feet) of the ball above the ground.

- a. What do you notice about the equation, just as it is, that will help us in creating our graph?
- b. Can we factor to find the zeros of the function? If not, solve $h(t) = 0$ by completing the square.

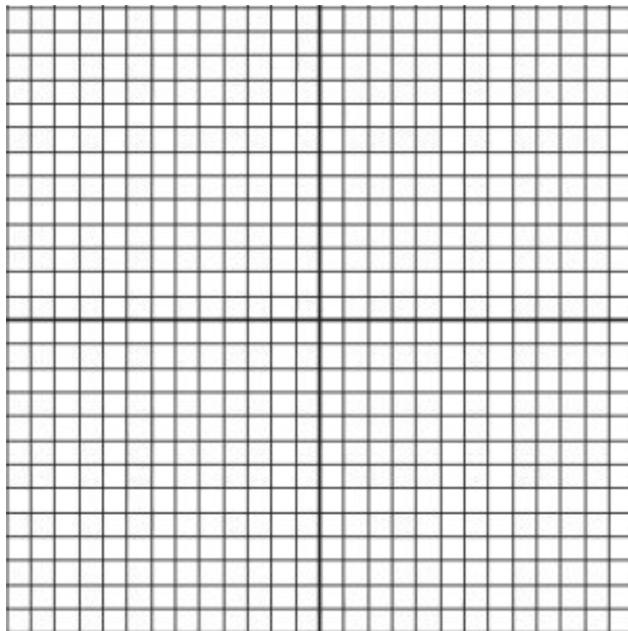
c. Which will you use to find the vertex? Symmetry? Or the completed-square form of the equation?

d. Now we plot the graph of $h(t) = -16t^2 + 96t + 6$ and identify the key features in the graph.

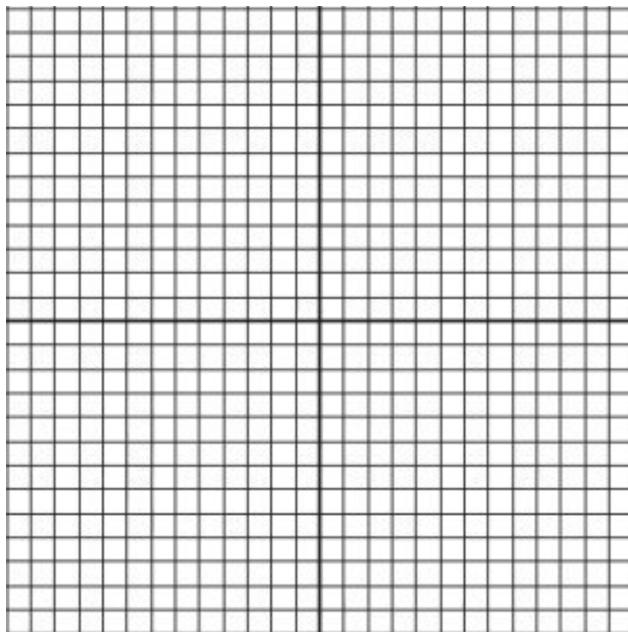


Exercises

1. Graph the equation $n(x) = x^2 - 6x + 5$, and identify the key features.



2. Graph the equation $f(x) = \frac{1}{2}x^2 + 5x + 6$, and identify the key features.

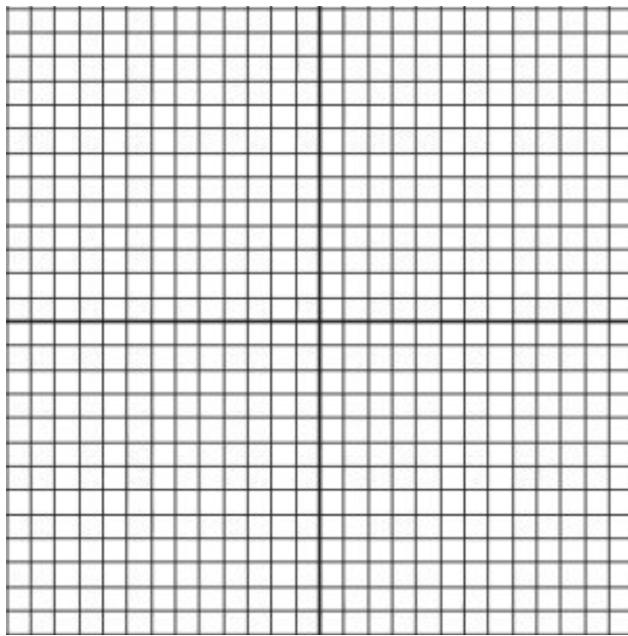


3. Paige wants to start a summer lawn-mowing business. She comes up with the following profit function that relates the total profit to the rate she charges for a lawn-mowing job:

$$P(x) = -x^2 + 40x - 100.$$

Both profit and her rate are measured in dollars. Graph the function in order to answer the following questions.

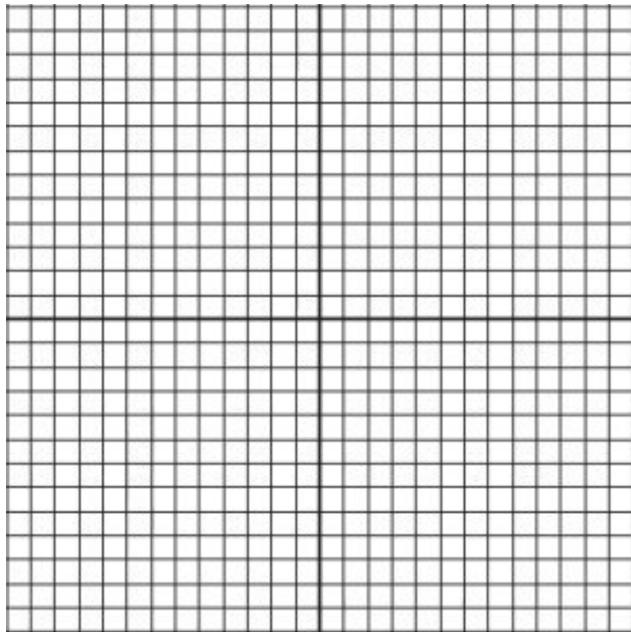
- a. Graph $P(x)$.



- b. According to the function, what is her initial cost (e.g., maintaining the mower, buying gas, advertising)? Explain your answer in the context of this problem.
- c. Between what two prices does she have to charge to make a profit?
- d. If she wants to make \$275 profit this summer, is this the right business choice?

4. A student throws a bag of chips to her friend. Unfortunately, her friend does not catch the chips, and the bag hits the ground. The distance from the ground (height) for the bag of chips is modeled by the function $h(t) = -16t^2 + 32t + 4$, where $h(t)$ is the height (distance from the ground in feet) of the chips, and t is the number of seconds the chips are in the air.

a. Graph $h(t)$.



- b. From what height are the chips being thrown? Tell how you know.
- c. What is the maximum height the bag of chips reaches while airborne? Tell how you know.
- d. How many seconds after the bag was thrown did it hit the ground?

- e. What is the average rate of change of height for the interval from 0 to $\frac{1}{2}$ second? What does that number represent in terms of the context?
- f. Based on your answer to part (e), what is the average rate of change for the interval from 1.5 to 2 sec.?
5. Notice how the profit and height functions both have negative leading coefficients. Explain why this is.

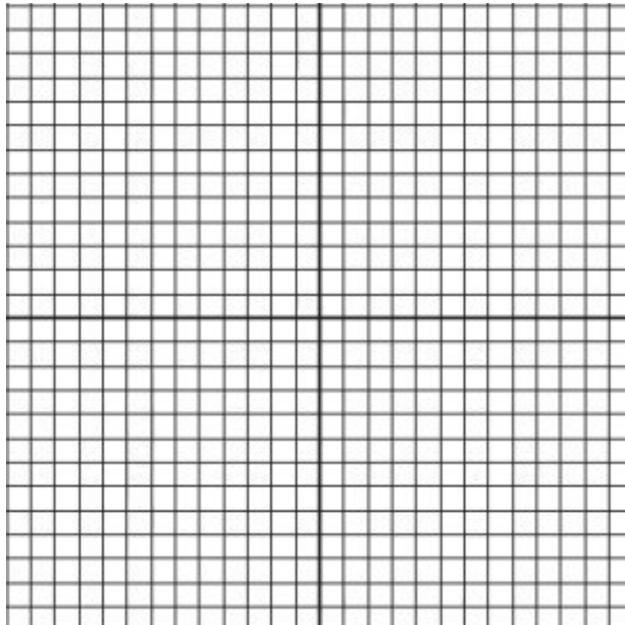
Lesson Summary

The standard form of a quadratic function is $f(x) = ax^2 + bx + c$, where $a \neq 0$. A general strategy to graphing a quadratic function from the standard form:

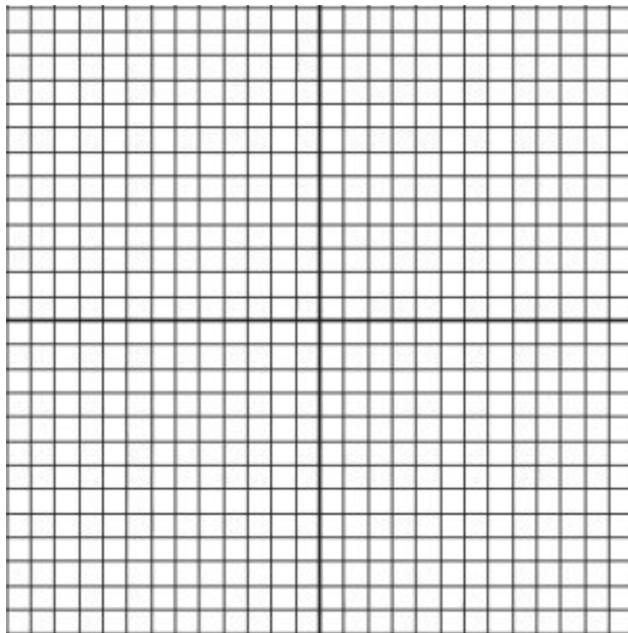
- Look for hints in the function's equation for general shape, direction, and y -intercept.
- Solve $f(x) = 0$ to find the x -intercepts by factoring, completing the square, or using the quadratic formula.
- Find the vertex by completing the square or using symmetry. Find the axis of symmetry and the x -coordinate of the vertex using $\frac{-b}{2a}$ and the y -coordinate of the vertex by finding $f\left(\frac{-b}{2a}\right)$.
- Plot the points that you know (at least three are required for a unique quadratic function), sketch the graph of the curve that connects them, and identify the key features of the graph.

Problem Set

1. Graph $f(x) = x^2 - 2x - 15$, and identify its key features.



2. Graph the following equation $f(x) = -x^2 + 2x + 15$, and identify its key features.



3. Did you recognize the numbers in the first two problems? The equation in the second problem is the product of -1 and the first equation. What effect did multiplying the equation by -1 have on the graph?
4. Giselle wants to run a tutoring program over the summer. She comes up with the following profit function:

$$P(x) = -2x^2 + 100x - 25,$$

where x represents the price of the program. Between what two prices should she charge to make a profit? How much should she charge her students if she wants to make the most profit?

5. Doug wants to start a physical therapy practice. His financial advisor comes up with the following profit function for his business: $P(x) = -\frac{1}{2}x^2 + 150x - 10,000$. How much will it cost for him to start the business? What should he charge his clients to make the most profit?

Lesson 18: Graphing Cubic, Square Root, and Cube Root Functions

Classwork

Opening Exercises

1. Evaluate x^2 when $x = 7$.

2. Evaluate \sqrt{x} when $x = 81$.

3. Evaluate x^3 when $x = 5$.

4. Evaluate $\sqrt[3]{x}$ when $x = 27$.

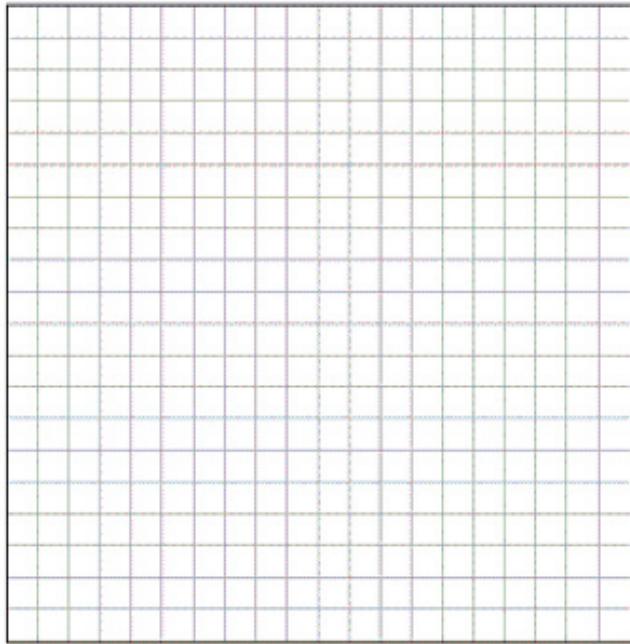
Exploratory Challenge 1

Use your graphing calculator to create a data table for the functions $y = x^2$ and $y = \sqrt{x}$ for a variety of x -values. Use both negative and positive numbers, and round decimal answers to the nearest hundredth.

x	$y = x^2$	$y = \sqrt{x}$

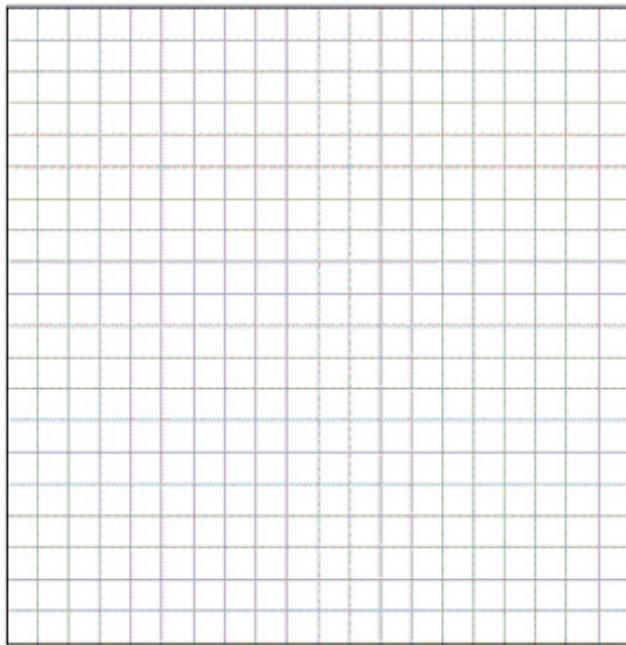
Exploratory Challenge 2

Create the graphs of $y = x^2$ and $y = \sqrt{x}$ on the same set of axes.

**Exploratory Challenge 3**

Create a data table for $y = x^3$ and $y = \sqrt[3]{x}$, and graph both functions on the same set of axes. Round decimal answers to the nearest hundredth.

x	$y = x^3$	$y = \sqrt[3]{x}$
-8		
-2		
-1		
0		
1		
2		
8		



Lesson Summary

- The square root parent function is a reflection of the quadratic parent function across the line $y = x$, when x is non-negative.
- The domain of quadratic, cubic, and cube root parent functions is all real numbers. The domain of square root functions is $x \geq 0$.
- The range of quadratic and square root parent functions is $[0, \infty)$. The range of the cubic and cube root parent functions is all real numbers.
- The cube root and cubic parent functions are symmetrical about the origin and are reflections of each other across the line $y = x$, and the two operations reverse each other.

Problem Set

1. Create the graphs of the functions $f(x) = x^2 + 2$ and $g(x) = \sqrt{x} + 2$ using the given values. Use a calculator to help with decimal approximations.

x	$f(x)$	$g(x)$
-4		
-2		
-1		
0		
1		
2		
4		

2. Why are the first three rows in the table under $g(x)$ crossed out?
3. Describe the relationship between the graphs given by the equations $y = x^2 + 2$ and $y = \sqrt{x} + 2$. How are they alike? How are they different?
4. Refer to your class notes for the graphs of $y = x^2$ and $y = \sqrt{x}$. How are the graphs of $y = x^2$ and $y = \sqrt{x}$ transformed to generate the graphs of $y = x^2 + 2$ and $= \sqrt{x} + 2$?

5. Create the graphs of $p(x) = x^3 - 2$ and $q(x) = \sqrt[3]{x} - 2$ using the given values for x . Use a calculator to help with decimal approximations.

x	$p(x)$	$q(x)$
-8		
-2		
-1		
0		
1		
2		
8		

6. Why aren't there any rows crossed out in the table from Problem 5?
7. Describe the relationship between the domains and ranges of the functions $p(x) = x^3 - 2$ and $q(x) = \sqrt[3]{x} - 2$. Describe the relationship between their graphs.
8. Refer to your class notes for the graphs of $y = x^3$ and $y = \sqrt[3]{x}$. How are the graphs of $y = x^3$ and $y = \sqrt[3]{x}$ transformed to generate the graphs of $y = x^3 - 2$ and $y = \sqrt[3]{x} - 2$?
9. Using your responses to Problems 4 and 8, how do the functions given in Problems 1 and 5 differ from their parent functions? What effect does that difference seem to have on the graphs of those functions?
10. Create your own functions using $r(x) = x^2 - \square$ and $s(x) = \sqrt{x} - \square$ by filling in the box with a positive or negative number. Predict how the graphs of your functions will compare to the graphs of their parent functions based on the number that you put in the blank boxes. Generate a table of solutions for your functions, and graph the solutions.

Lesson 19: Translating Functions

Classwork

Opening Exercise

Graph each set of three functions in the same coordinate plane (on your graphing calculator or a piece of graph paper). Then, explain what similarities and differences you see among the graphs.

a. $f(x) = x$

$g(x) = x + 5$

$h(x) = x - 6$

b. $f(x) = x^2$

$g(x) = x^2 + 3$

$h(x) = x^2 - 7$

c. $f(x) = |x|$

$g(x) = |x + 3|$

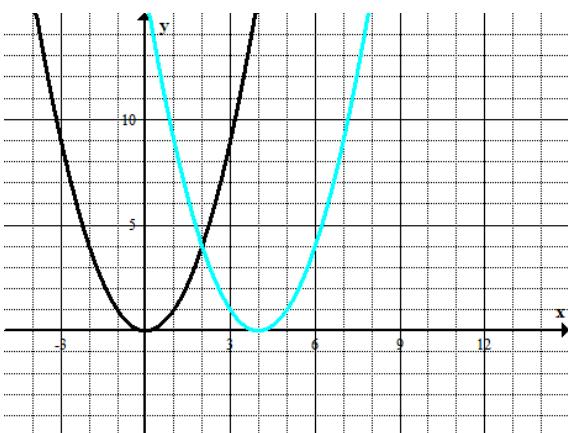
$h(x) = |x - 4|$

Example

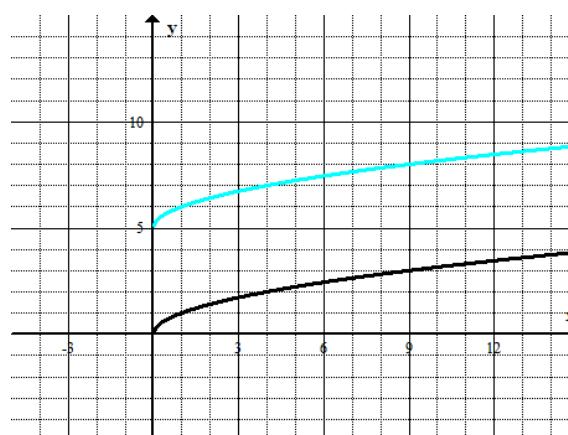
For each graph, answer the following:

- What is the parent function?
- How does the translated graph relate to the graph of the parent function?
- Write the formula for the function depicted by the translated graph.

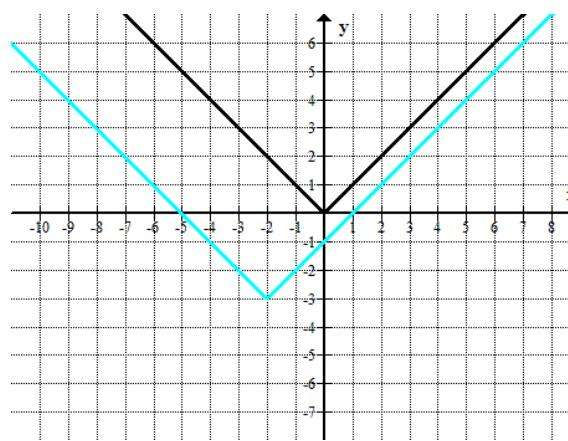
a.



b.

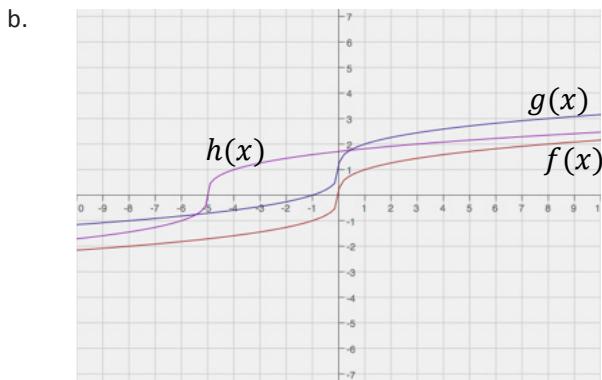
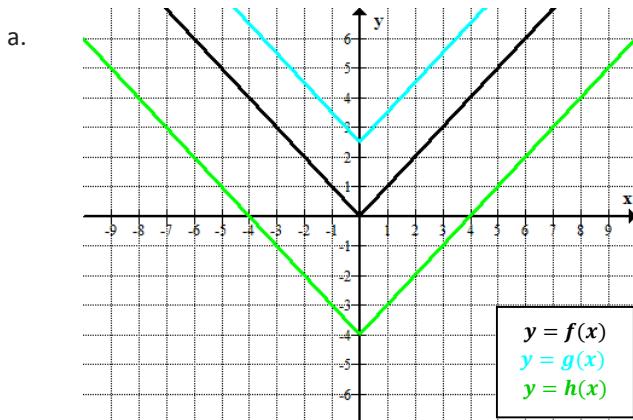


c.



Exercises 1–3

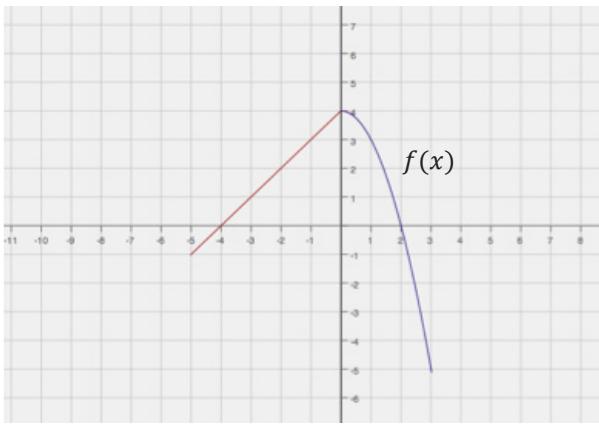
1. For each of the following graphs, use the formula for the parent function f to write the formula of the translated function.



2. Below is a graph of a piecewise function f whose domain is $-5 \leq x \leq 3$. Sketch the graphs of the given functions on the same coordinate plane. Label your graphs correctly.

$$g(x) = f(x) + 3$$

$$h(x) = f(x - 4)$$



3. Match the correct equation and description of the function with the given graphs.

	Graphs	Equation	Description
$f(x)$		E1. $y = (x - 3)^2$ E2. $y = (x + 2)^2 - 3$ E3. $y = -(x - 3)^2 - 2$ E4. $y = (x - 2)^2 - 3$	D1. The graph of the parent function is translated down 3 units and left 2 units. D2. The graph of the function does not have an x -intercept. D3. The coordinate of the y -intercept is $(0, 1)$, and both x -intercepts are positive. D4. The graph of the function has only one x -intercept.
$g(x)$			
$h(x)$			
$p(x)$			

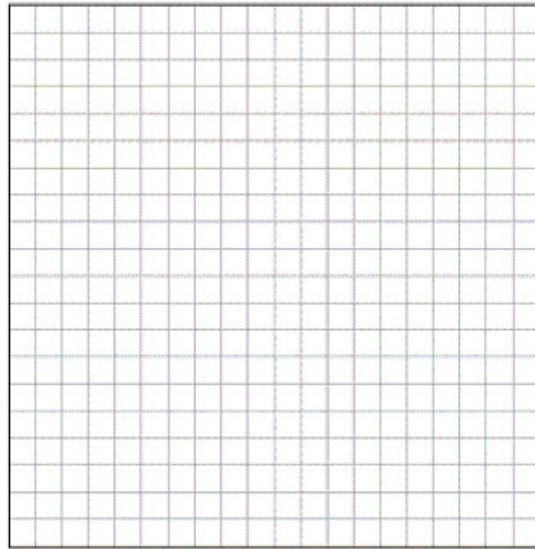
Problem Set

1. Graph the functions in the same coordinate plane. Do not use a graphing calculator.

$$f(x) = \sqrt{x}$$

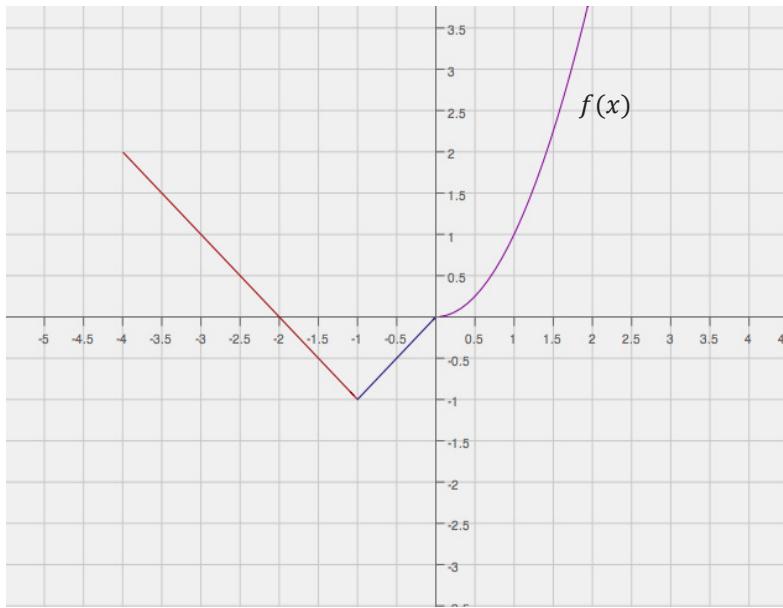
$$p(x) = 10 + \sqrt{x}$$

$$q(x) = \sqrt{x + 8}$$

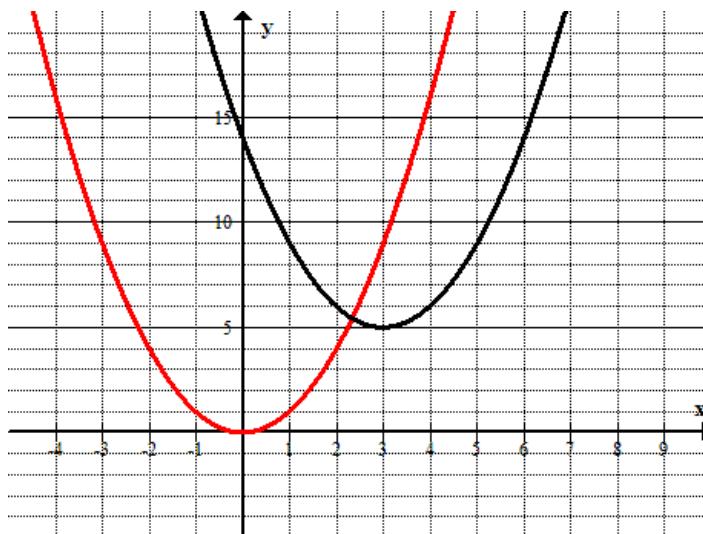


2. Write a function that translates the graph of the parent function $f(x) = x^2$ down 7.5 units and right 2.5 units.
3. How would the graph of $f(x) = |x|$ be affected if the function were transformed to $f(x) = |x + 6| + 10$?
4. Below is a graph of a piecewise function f whose domain is the interval $-4 \leq x \leq 2$. Sketch the graph of the given functions below. Label your graphs correctly.

$$g(x) = f(x) - 1 \quad h(x) = g(x - 2) \quad (\text{Be careful here! This one might be a challenge.})$$



5. Study the graphs below. Identify the parent function and the transformations of that function depicted by the second graph. Then, write the formula for the transformed function.



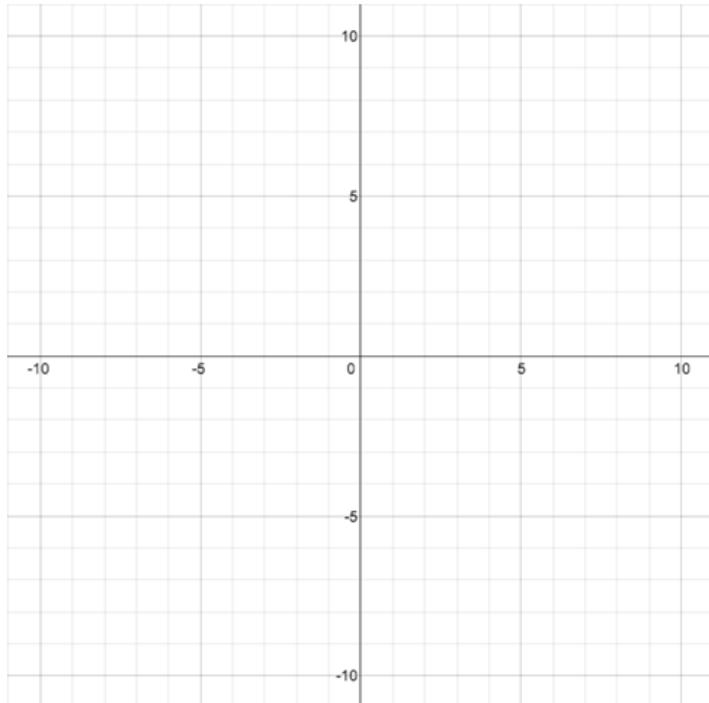
Lesson 20: Stretching and Shrinking Graphs of Functions

Classwork

Opening Exercise

The graph of a quadratic function defined by $f(x) = x^2$ has been translated 5 units to the left and 3 units up. What is the formula for the function, g , depicted by the translated graph?

Sketch the graph of the equation $y = g(x)$.



Example



Exploratory Challenge

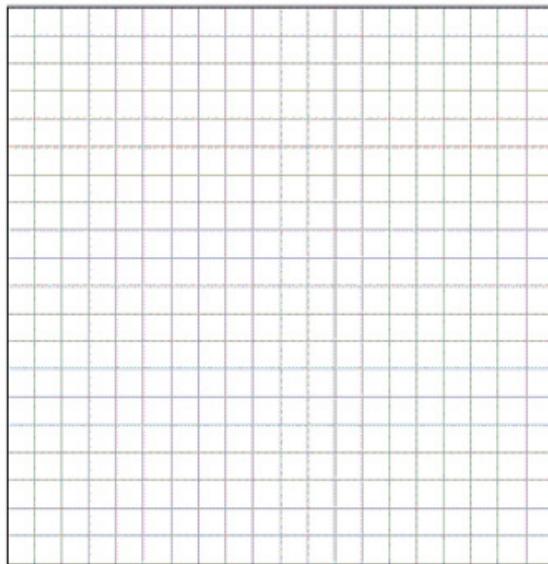
Complete the following to review Module 3 concepts:

- a. Consider the function $f(x) = |x|$. Complete the table of values for $f(x)$. Then, graph the equation $y = f(x)$ on the coordinate plane provided for part (b).

x	$f(x)$
-4	
-2	
0	
2	
4	

- b. Complete the following table of values for each transformation of the function f . Then, graph the equations $y = g(x)$, $y = h(x)$, $y = j(x)$, and $y = k(x)$ on the same coordinate plane as the graph of $y = f(x)$. Label each graph.

x	$f(x)$	$g(x) = 3f(x)$	$h(x) = 2f(x)$	$j(x) = 0.5f(x)$	$k(x) = -2f(x)$
-4					
-2					
0					
2					
4					



- c. Describe how the graph of $y = kf(x)$ relates to the graph of $y = f(x)$ for each case.

i. $k > 1$

ii. $0 < k < 1$

iii. $k = -1$

iv. $-1 < k < 0$

v. $k < -1$

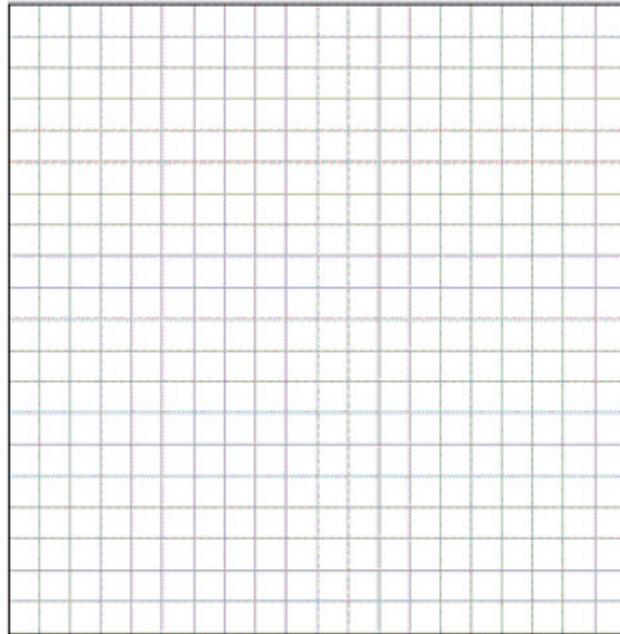
- d. Describe the transformation of the graph of f that results in the graphs of g , h , and k given the following formulas for each function. Then, graph each function and label each graph.

$$f(x) = x^3$$

$$g(x) = 2x^3$$

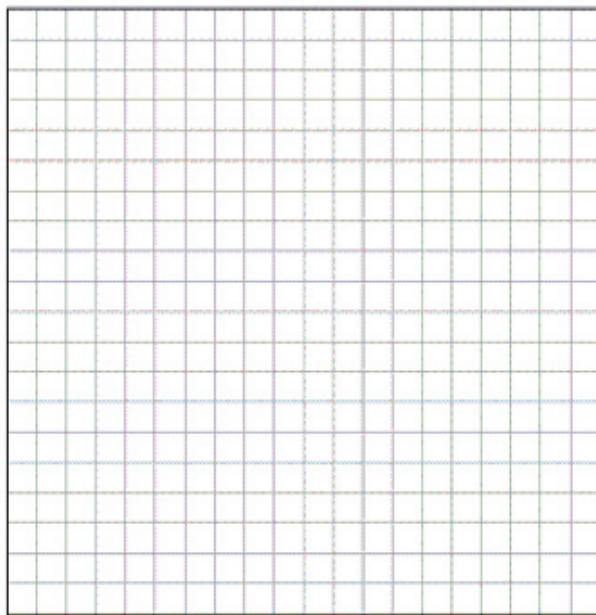
$$h(x) = 0.5x^3$$

$$k(x) = -3x^3$$



- e. Consider the function $f(x) = \sqrt[3]{x}$. Complete the table of values, then graph the equation $y = f(x)$.

x	$f(x)$
-8	
-1	
0	
1	
8	



- f. Complete the following table of values, rounding each value to the nearest hundredth. Graph the equations $y = g(x)$, $y = h(x)$, and $y = j(x)$ on the same coordinate plane as your graph of $y = f(x)$ above. Label each graph.

x	$f(x)$	$g(x) = f(2x)$	$h(x) = f(0.5x)$	$j(x) = f(-2x)$
-8				
-1				
0				
1				
8				

- g. Describe the transformations of the graph of f that result in the graphs of g , h , and j .

- h. Describe how the graph of $y = f\left(\frac{1}{k}x\right)$ relates to the graph of $y = f(x)$ for each case.

i. $k > 1$

ii. $0 < k < 1$

iii. $k = -1$

iv. $-1 < k < 0$

v. $k < -1$

Exercise 1

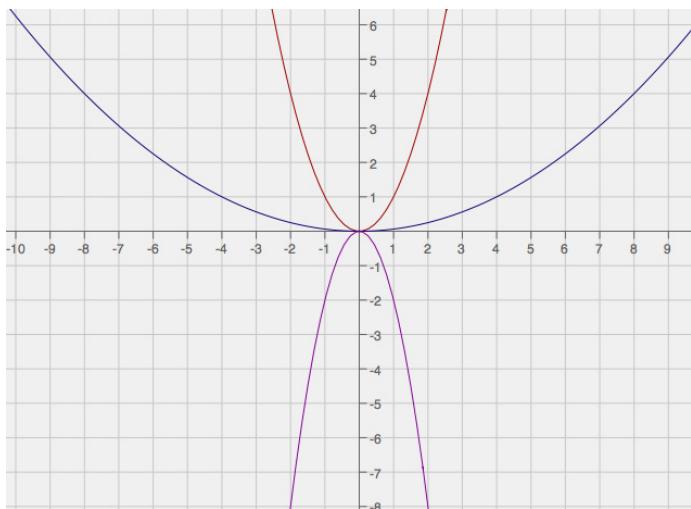
For each of the sets below, answer the following questions:

- What are the parent functions?
- How does the translated graph relate to the graph of the parent function?
- Write the formula for the function depicted by the translated graph.

a.



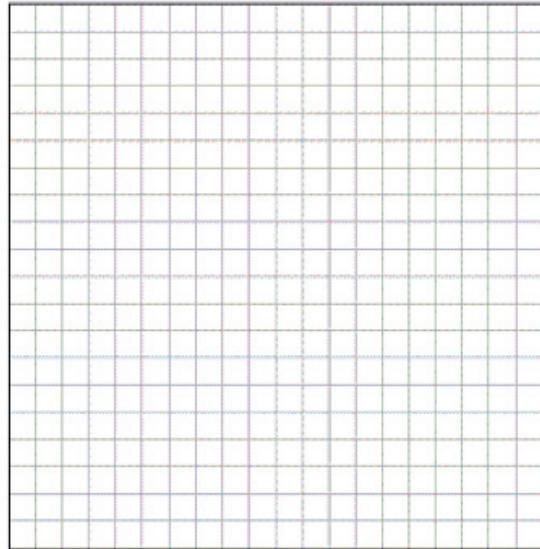
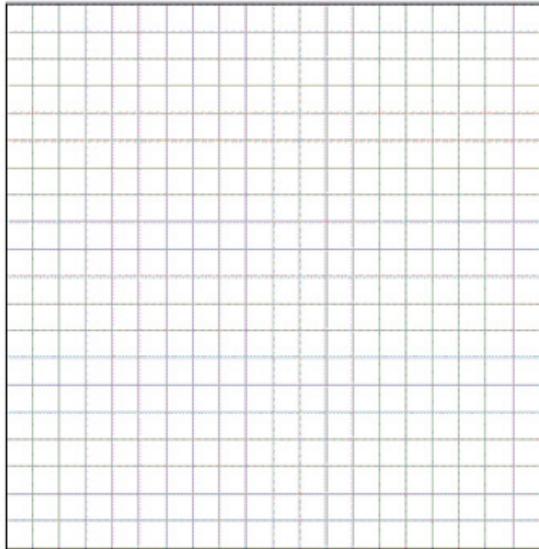
b.

**Exercise 2**

Graph each set of functions in the same coordinate plane. Do not use a graphing calculator.

a. $f(x) = |x|$
 $g(x) = 4|x|$
 $h(x) = |2x|$
 $k(x) = -2|2x|$

b. $g(x) = \sqrt[3]{x}$
 $p(x) = 2\sqrt[3]{x}$
 $q(x) = -2\sqrt[3]{2x}$



Problem Set

1. Graph the functions in the same coordinate plane. Do not use a graphing calculator.

$$f(x) = |x|$$

$$g(x) = 2|x|$$

$$h(x) = |3x|$$

$$k(x) = -3|3x|$$

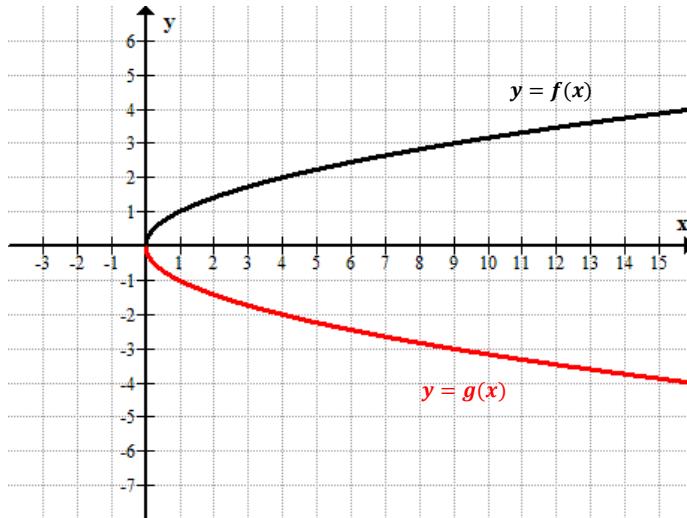
2. Explain how the graphs of functions $g(x) = 3|x|$ and $h(x) = |3x|$ are related.

3. Explain how the graphs of functions $q(x) = -3|x|$ and $r(x) = |-3x|$ are related.

4. Write a function, g in terms of another function f , such that the graph of g is a vertical shrink of the graph f by a factor of 0.75.

5. A teacher wants the students to write a function based on the parent function $f(x) = \sqrt[3]{x}$. The graph of f is stretched vertically by a factor of 4 and shrunk horizontally by a factor of $\frac{1}{3}$. Mike wrote $g(x) = 4\sqrt[3]{3x}$ as the new function, while Lucy wrote $h(x) = 3\sqrt[3]{4x}$. Which one is correct? Justify your answer.

6. Study the graphs of two different functions below. Which is a parent function? What is the constant value(s) multiplied to the parent function to arrive at the transformed graph? Now write the function defined by the transformed graph.



Lesson 21: Transformations of the Quadratic Parent Function,

$$f(x) = x^2$$

Classwork

Example 1: Quadratic Expression Representing a Function

- a. A quadratic function is defined by $g(x) = 2x^2 + 12x + 1$. Write this in the completed-square (vertex) form and show all the steps.

- b. Where is the vertex of the graph of this function located?

- c. Look at the completed-square form of the function. Can you name the parent function? How do you know?

- d. What transformations have been applied to the parent function to arrive at $g(x)$? Be specific.

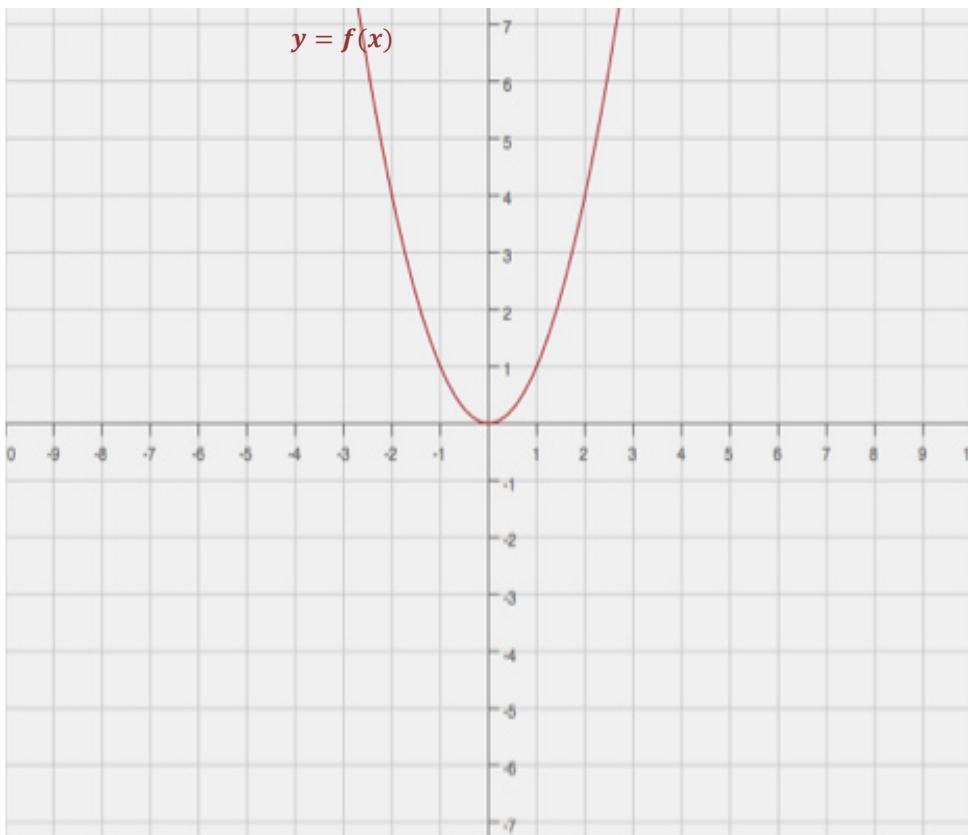
- e. How does the completed-square form relate to the quadratic parent function $f(x) = x^2$?

Example 2

The graph of a quadratic function $f(x) = x^2$ has been translated 3 units to the right, vertically stretched by a factor of 4, and moved 2 units up. Write the formula for the function that defines the transformed graph.

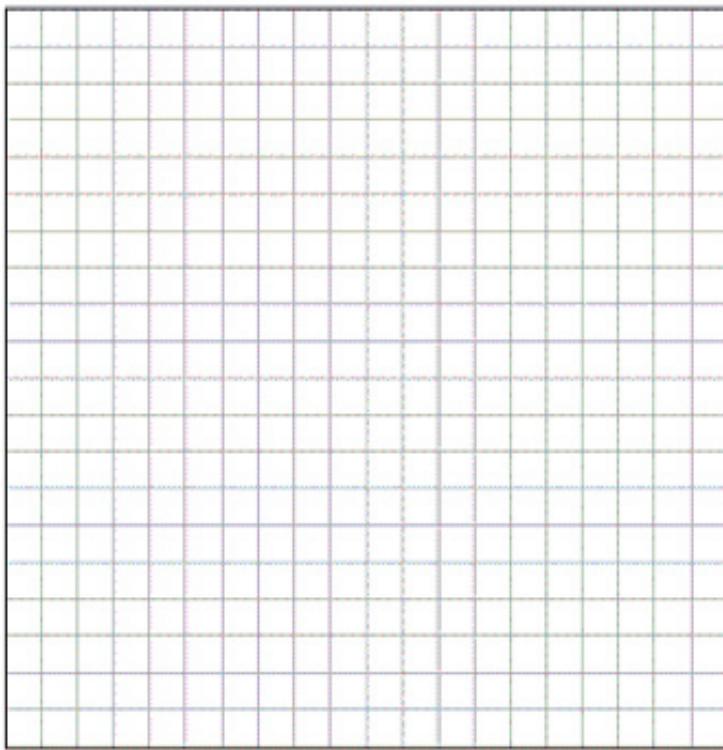
Exercises

1. Without using a graphing calculator, sketch the graph of the following quadratic functions on the same coordinate plane, using transformations of the graph of the parent function $f(x) = x^2$.
 - a. $g(x) = -2(x - 3)^2 + 4$
 - b. $h(x) = -3(x + 5)^2 + 1$
 - c. $k(x) = 2(x + 4)^2 - 3$
 - d. $p(x) = x^2 - 2x$
 - e. $t(x) = x^2 - 2x + 3$



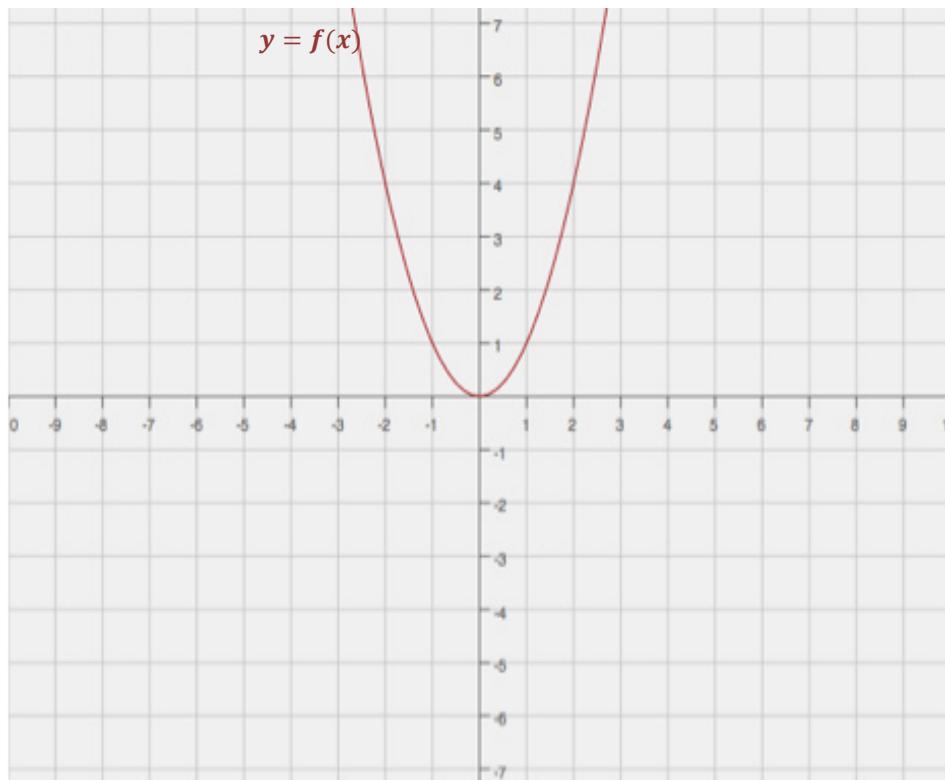
2. Write a formula for the function that defines the described transformation of the graph of the quadratic parent function $f(x) = x^2$.
- 3 units shift to the right
 - Vertical shrink by a factor of 0.5
 - Reflection across the x -axis
 - 4 units shift up

Then, graph both the parent and the transformed functions on the same coordinate plane.



3. Describe the transformation of the quadratic parent function $f(x) = x^2$ that results in the quadratic function $g(x) = 2x^2 + 4x + 1$.

4. Sketch the graphs of the following functions based on the graph of the function $f(x) = x^2$. If necessary, rewrite some of the functions in the vertex (completed-square) form. Label your graphs.
- $g(x) = -(x - 4)^2 + 3$
 - $h(x) = 3(x - 2)^2 - 1$
 - $k(x) = 2x^2 + 8x$
 - $p(x) = x^2 + 6x + 5$



Lesson Summary

Transformations of the quadratic parent function, $f(x) = x^2$, can be rewritten in form $g(x) = a(x - h)^2 + k$, where (h, k) is the vertex of the translated and scaled graph of f , with the scale factor of a , the leading coefficient. We can then quickly and efficiently (without the use of technology) sketch the graph of any quadratic function in the form $f(x) = a(x - h)^2 + k$ using transformations of the graph of the quadratic parent function, $f(x) = x^2$.

Problem Set

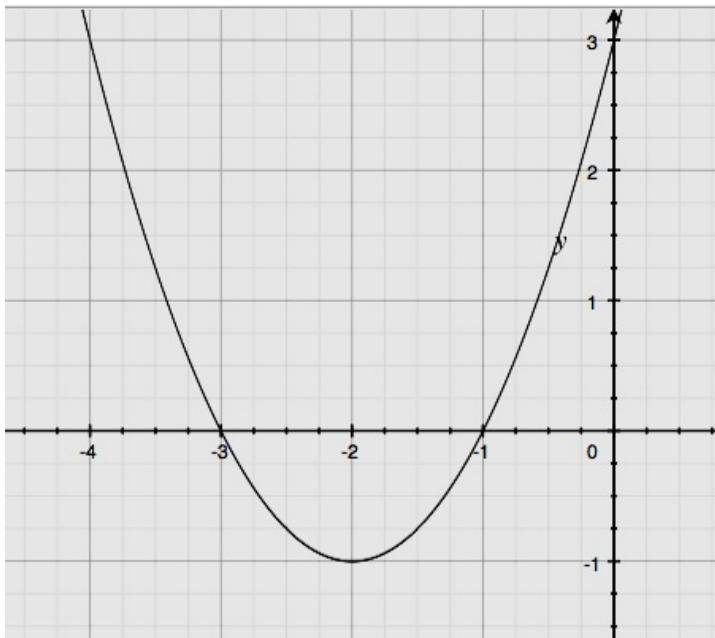
1. Write the function $g(x) = -2x^2 - 20x - 53$ in completed-square form. Describe the transformations of the graph of the parent function $f(x) = x^2$ that result in the graph of g .
2. Write the formula for the function whose graph is the graph of $f(x) = x^2$ translated 6.25 units to the right, vertically stretched by a factor of 8, and translated 2.5 units up.
3. Without using a graphing calculator, sketch the graphs of the functions below based on transformations of the graph of the parent function $f(x) = x^2$. Use your own graph paper and label your graphs.
 - a. $g(x) = (x + 2)^2 - 4$
 - b. $h(x) = -(x - 4)^2 + 2$
 - c. $k(x) = 2x^2 - 12x + 19$
 - d. $p(x) = -2x^2 - 4x - 5$
 - e. $q(x) = 3x^2 + 6x$

Lesson 22: Comparing Quadratic, Square Root, and Cube Root Functions Represented in Different Ways

Classwork

Opening Exercise

Populate the table on the right with values from the graph.



x	y

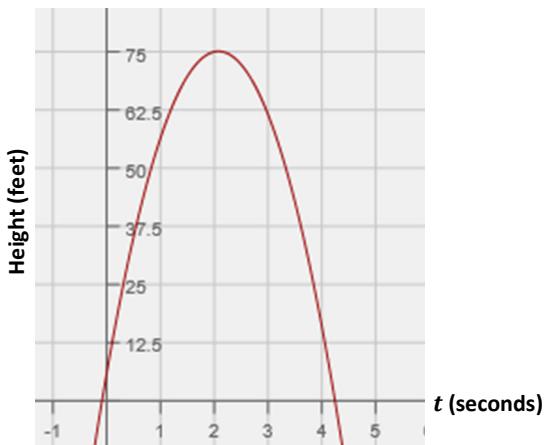
Exploratory Challenges 1–3

Solve each problem, and show or explain how you found your answers.

- Xavier and Sherleese each threw a baseball straight up into the air. The relationship between the height (distance from the ground in feet) of Sherleese's ball with respect to the time since it was thrown, in seconds, is given by the function:

$$S(t) = -16t^2 + 79t + 6.$$

The graph of the height as a function of time of Xavier's ball is represented below.



Xavier claims that his ball went higher than Sherleese's. Sherleese disagrees. Answer the questions below, and support your answers mathematically by comparing the features found in the equation to those in the graph.

- Who is right?
- For how long was each baseball airborne?

- c. Construct a graph of Sherleese's throw as a function of time (t) on the same set of axes as the graph of Xavier's, and use the graph to support your answers to parts (a) and (b).



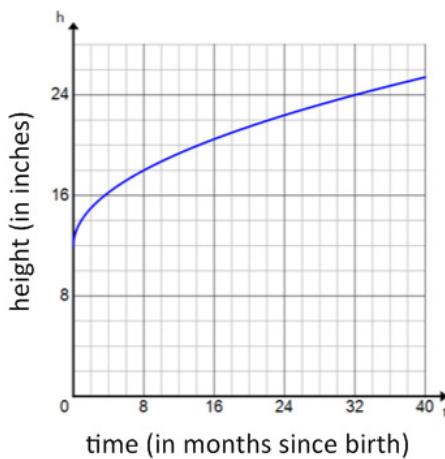
2. At an amusement park, there is a ride called The Centre. The ride is a cylindrical room that spins as the riders stand along the wall. As the ride reaches maximum speed, riders are pinned against the wall and are unable to move. The model that represents the speed necessary to hold the riders against the wall is given by the function $s(r) = 5.05\sqrt{r}$, where s = required speed of the ride (in meters per second) and r = the radius (in meters) of the ride.

In a competing ride called The Spinner, a car spins around a center post. The measurements in the table below show the relationship between the radius (r) of the spin, in meters, and the speed (s) of the car, in meters per second.

r (meters)	s (meters per second)
0	0
1	7.7942
2	11.023
3	13.5
4	15.588
5	17.428

Due to limited space at the carnival, the maximum spin radius of rides is 4 meters. Assume that the spin radius of both rides is exactly 4 meters. If riders prefer a faster spinning experience, which ride should they choose? Show how you arrived at your answer.

3. The growth of a Great Dane puppy can be represented by the graph below, where y represents the shoulder height (in inches) and x represents the puppy's age (in months).



The growth of a lion cub can be modeled by the function represented in the table below.

x (months since birth)	y (height in inches)
0	8
8	18
16	20.599
24	22.422
32	23.874

- Which animal has the greater shoulder height at birth?
- Which animal will have the greater shoulder height at 3 years of age (the age each animal is considered full-grown)?
- If you were told that the domain for these functions is the set of all real numbers, would you agree? Why or why not?

Lesson Summary

The key features of a quadratic function, which are the zeros (roots), the vertex, and the leading coefficient, can be used to interpret the function in a context (e.g., the vertex represents the maximum or minimum value of the function). Graphing calculators and bivariate data tables are useful tools when comparing functions.

Problem Set

1. One type of rectangle has lengths that are always two inches more than their widths. The function f describes the relationship between the width of this rectangle in x inches and its area, $f(x)$, in square inches, and is represented in the table below.

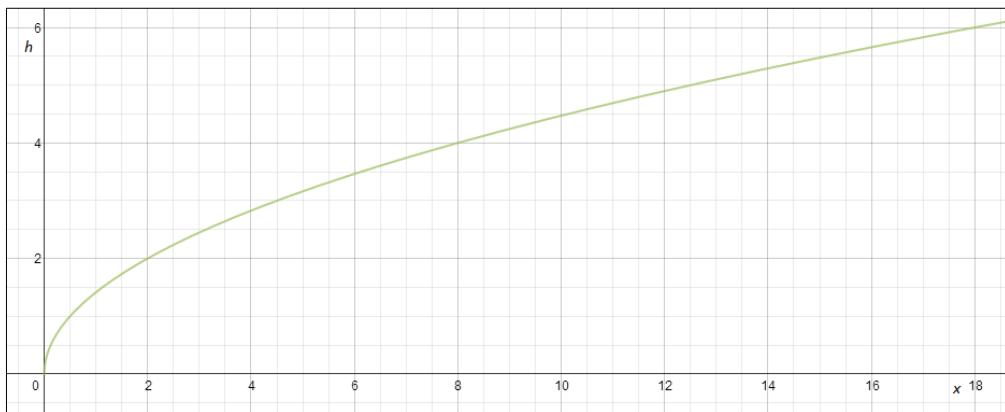
x	$f(x)$
0	0
1	3
2	8
3	15
4	24
5	35

A second type of rectangle has lengths that are always one-half of their widths. The function $g(x) = \frac{1}{2}x^2$ describes the relationship between the width given in x inches and the area, $g(x)$, given in square inches of such a rectangle.

- Use $g(x)$ to determine the area of a rectangle of the second type if the width is 20 inches.
- Why is $(0,0)$ contained in the graphs of both functions? Explain the meaning of $(0,0)$ in terms of the situations that the functions describe.
- Determine which function has a greater average rate of change on the interval $0 \leq x \leq 3$.
- Interpret your answer to part (c) in terms of the situation being described.
- Which type of rectangle has a greater area when the width is 5 inches? By how much?
- Will the first type of rectangle always have a greater area than the second type of rectangle when widths are the same? Explain how you know.

2. The function given by the equation $y = \sqrt{x}$ gives the edge length, y units, of a square with area x square units. Similarly, the graph below describes the length of a leg of an isosceles right triangle whose area is x square units.

Graph of Isosceles Triangle Leg Lengths for Given Areas



- What is the length of a leg of an isosceles right triangle with an area of 12 square units?
- Which function has a greater average rate of change on the interval $0 \leq x \leq 3$?
- Interpret your answer to part (b) in terms of the situation being described.
- Which will have a greater value: the edge length of a square with area 16 square units or the length of a leg of an isosceles right triangle with an area of 16 square units? Approximately by how much?

Lesson 23: Modeling with Quadratic Functions

Classwork

Opening: The Mathematics of Objects in Motion

Read the following explanation of the Mathematics of Objects in Motion:

Any object that is free falling or projected into the air without a power source is under the influence of gravity. All free-falling objects on Earth accelerate toward the center of the Earth (downward) at a constant rate (-32 ft/s^2 , or -9.8 m/s^2) because of the constant force of Earth's gravity (represented by g). That acceleration rate is included in the physics formula used for all objects in a free-falling motion. It represents the relationship of the height of the object (distance from Earth) with respect to the time that has passed since the launch or fall began. That formula is

$$h(t) = \frac{1}{2}gt^2 + v_0t + h_0.$$

For this reason, the leading coefficient for a quadratic function that models the position of a falling, launched, or projected object must either be -16 or -4.9 . Physicists use mathematics to make predictions about the outcome of a falling or projected object.

The mathematical formulas (equations) used in physics commonly use certain variables to indicate quantities that are most often used for motion problems. For example, the following are commonly used variables for an event that includes an object that has been dropped or thrown:

- h is often used to represent the function of height (how high the object is above Earth in feet or meters);
- t is used to represent the time (number of seconds) that have passed in the event;
- v is used to represent *velocity* (the rate at which an object changes position in ft/s or m/s);
- s is used to represent the object's change in position, or *displacement* (how far the object has moved in feet or meters).

We often use subscripts with the variables, partly so that we can use the same variables multiple times in a problem without getting confused, but also to indicate the passage of time. For example:

- v_0 indicates the initial velocity (i.e., the velocity at 0 seconds);
- h_0 tells us the height of the object at 0 seconds, or the initial position.

So putting all of that together, we have a model representing the motion of falling or thrown objects, using U.S. standard units, as a quadratic function:

$$h(t) = -16t^2 + v_0t + h_0,$$

where h represents the height of the object in feet (distance from the Earth), and t is the number of seconds the object has been in motion. Note that the negative sign in front of the 16 (half of $g = 32$) indicates the downward pull of gravity. We are using a convention for quantities with direction here; upward is positive and downward is negative. If units are metric, the following equation is used:

$$h(t) = -4.9t^2 + v_0t + h_0,$$

where everything else is the same, but now the height of the object is measured in meters and the velocity in meters per second.

These physics functions can be used to model many problems presented in the context of free-falling or projected objects (objects in motion without any inhibiting or propelling power source, such as a parachute or an engine).

Mathematical Modeling Exercise 1

Use the information in the Opening to answer the following questions.

Chris stands on the edge of a building at a height of 60 ft. and throws a ball upward with an initial velocity of 68 ft/s. The ball eventually falls all the way to the ground. What is the maximum height reached by the ball? After how many seconds will the ball reach its maximum height? How long will it take the ball to reach the ground?

- a. What units will we be using to solve this problem?

- b. What information from the contextual description do we need to use in the function equation?

- c. What is the maximum point reached by the ball? After how many seconds will it reach that height? Show your reasoning.

- d. How long will it take the ball to land on the ground after being thrown? Show your work.
- e. Graph the function of the height (h) of the ball in feet to the time (t) in seconds. Include and label key features of the graph such as the vertex, axis of symmetry, and t - and y -intercepts.

Mathematical Modeling Exercise 2

Read the following information about Business Applications:

Many business contexts can be modeled with quadratic functions. This is because the expressions representing the price (price per item), the cost (cost per item), and the quantity (number of items sold) are typically linear. The product of any two of those linear expressions will produce a quadratic expression that can be used as a model for the business context. The variables used in business applications are not as traditionally accepted as variables that are used in physics applications, but there are some obvious reasons to use c for cost, p for price, and q for quantity (all lowercase letters). For total production cost, we often use C for the variable, R for total revenue, and P for total profit (all uppercase letters). You have seen these formulas in previous lessons, but we will review them here since we use them in the next two lessons.

Business Application Vocabulary

UNIT PRICE (PRICE PER UNIT): The price per item a business sets to sell its product, sometimes represented as a linear expression.

QUANTITY: The number of items sold, sometimes represented as a linear expression.

REVENUE: The total income based on sales (but without considering the cost of doing business).

UNIT COST (COST PER UNIT) OR PRODUCTION COST: The cost of producing one item, sometimes represented as a linear expression.

PROFIT: The amount of money a business makes on the sale of its product. Profit is determined by taking the total revenue (the quantity sold multiplied by the price per unit) and subtracting the total cost to produce the items (the quantity sold multiplied by the production cost per unit): Profit = Total Revenue – Total Production Cost.

The following business formulas will be used in this lesson:

$$\text{Total Production Costs} = (\text{cost per unit})(\text{quantity of items sold})$$

$$\text{Total Revenue} = (\text{price per unit})(\text{quantity of items sold})$$

$$\text{Profit} = \text{Total Revenue} - \text{Total Production Costs}$$

Now answer the questions related to the following business problem:

A theater decided to sell special event tickets at \$10 per ticket to benefit a local charity. The theater can seat up to 1,000 people, and the manager of the theater expects to be able to sell all 1,000 seats for the event. To maximize the revenue for this event, a research company volunteered to do a survey to find out whether the price of the ticket could be increased without losing revenue. The results showed that for each \$1 increase in ticket price, 20 fewer tickets would be sold.

- a. Let x represent the number of \$1.00 price-per-ticket increases. Write an expression to represent the expected price for each ticket.

- b. Use the survey results to write an expression representing the possible number of tickets sold.
- c. Using x as the number of \$1-ticket price increases and the expression representing price per ticket, write the function, $R(x)$, to represent the total revenue in terms of the number of \$1-ticket price increases.
- d. How many \$1-ticket price increases will produce the maximum revenue? In other words, what value for x produces the maximum R ?
- e. What is the price of the ticket that will provide the maximum revenue?
- f. What is the maximum revenue?

- g. How many tickets will the theater sell to reach the maximum revenue?
- h. How much more will the theater make for the charity by using the results of the survey to price the tickets than they would have if they sold the tickets for their original \$10 price?

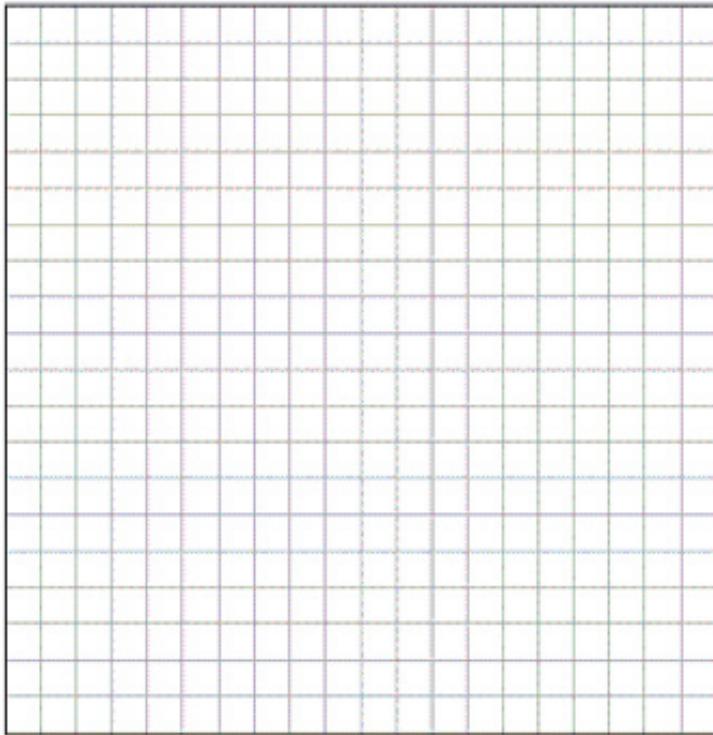
Exercise 1

Two rock climbers try an experiment while scaling a steep rock face. They each carry rocks of similar size and shape up a rock face. One climbs to a point 400 ft. above the ground, and the other climbs to a place below her at 300 ft. above the ground. The higher climber drops her rock, and 1 second later, the lower climber drops his. Note that the climbers are not vertically positioned. No climber is injured in this experiment.

- a. Define the variables in this situation, and write the two functions that can be used to model the relationship between the heights, h_1 and h_2 , of the rocks, in feet, after t seconds.

- b. Assuming the rocks fall to the ground without hitting anything on the way, which of the two rocks will reach the ground last? Show your work, and explain how you know your answer is correct.

- c. Graph the two functions on the same coordinate plane, and identify the key features that show that your answer to part (b) is correct. Explain how the graphs show that the two rocks hit the ground at different times.



- d. Does the graph show how far apart the rocks were when they landed? Explain.

Exercise 2

Amazing Photography Studio takes school pictures and charges \$20 for each class picture. The company sells an average of 12 class pictures in each classroom. They would like to have a special sale that will help them sell more pictures and actually increase their revenue. They hired a business analyst to determine how to do that. The analyst determined that for every reduction of \$2 in the cost of the class picture, there would be an additional 5 pictures sold per classroom.

- a. Write a function to represent the revenue for each classroom for the special sale.

- b. What should the special sale price be?

- c. How much more will the studio make than they would have without the sale?

Lesson Summary

We can write quadratic functions described verbally in a given context. We can also graph, interpret, analyze, or apply key features of quadratic functions to draw conclusions that help us answer questions taken from the problem's context.

- We find quadratic functions commonly applied in physics and business.
- We can substitute known x - and y -values into a quadratic function to create a linear system that, when solved, can identify the parameters of the quadratic equation representing the function.

Problem Set

1. Dave throws a ball upward with an initial velocity of 32 ft/s. The ball initially leaves his hand 5 ft. above the ground and eventually falls back to the ground. In parts (a)–(d), you will answer the following questions: What is the maximum height reached by the ball? After how many seconds will the ball reach its maximum height? How long will it take the ball to reach the ground?
 - a. What units will we be using to solve this problem?
 - b. What information from the contextual description do we need to use to write the formula for the function h of the height of the ball versus time? Write the formula for height of the ball in feet, $h(t)$, where t stands for seconds.
 - c. What is the maximum point reached by the ball? After how many seconds will it reach that height? Show your reasoning.
 - d. How long will it take for the ball to land on the ground after being thrown? Show your work.
 - e. Graph the function of the height of the ball in feet to the time in seconds. Include and label key features of the graph such as the vertex, axis of symmetry, and x - and y -intercepts.
2. Katrina developed an app that she sells for \$5 per download. She has free space on a website that will let her sell 500 downloads. According to some research she did, for each \$1 increase in download price, 10 fewer apps are sold. Determine the price that will maximize her profit.

3. Edward is drawing rectangles such that the sum of the length and width is always six inches.

- Draw one of Edward's rectangles, and label the length and width.
- Fill in the following table with four different possible lengths and widths.

Width (inches)	Length (inches)

- Let x be the width. Write an expression to represent the length of one of Edward's rectangles.
 - Write an equation that gives the area, y , in terms of the width, x .
 - For what width and length will the rectangle have maximum area?
 - Are you surprised by the answer to part (e)? What special name is given for the rectangle in your answer to part (e)?
4. Chase is standing at the base of a 60-foot cliff. He throws a rock in the air hoping to get the rock to the top of the cliff. If the rock leaves his hand 6 ft. above the base at a velocity of 80 ft/s, does the rock get high enough to reach the top of the cliff? How do you know? If so, how long does it take the rock to land on top of the cliff (assuming it lands on the cliff)? Graph the function and label the key features of the graph.

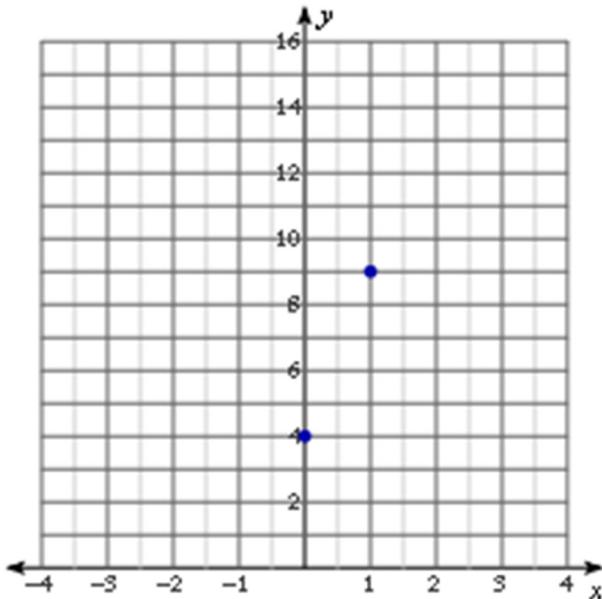
Lesson 24: Modeling with Quadratic Functions

Classwork

Opening Exercise

Draw as many quadratic graphs as possible through the following two points on the graph. Check with your neighbors for ideas. These points are $(0,4)$ and $(1,9)$.

Two Points



Example

Use the example with the blue points $(0, 4)$, $(1, 9)$, and $(-3, 1)$ from above to write the equation for the quadratic containing the three points.

Exercise 1

Write in standard form the quadratic function defined by the points $(0, 5)$, $(5, 0)$, and $(3, -4)$.

Exercise 2

Louis dropped a watermelon from the roof of a tall building. As it was falling, Amanda and Martin were on the ground with a stopwatch. As Amanda called the seconds, Martin recorded the floor the watermelon was passing. They then measured the number of feet per floor and put the collected data into this table. Write a quadratic function to model the following table of data relating the height of the watermelon (distance in feet from the ground) to the number of seconds that had passed.

Height (distance from the ground) for a watermelon that was dropped from a tall building					
Time (t)	0	1	2	3	4
Height $f(t)$	300	284	236	156	44

- a. How do we know this data will be represented by a quadratic function?

- b. Do we need to use all five data points to write the equation?

- c. Are there any points that are particularly useful? Does it matter which we use?
- d. How does this equation for the function match up with what you learned about physics in Lesson 23? Is there a more efficient way to find this equation?
- e. Can you use your quadratic function to predict at what time, t , the watermelon will hit the ground (i.e., $f(t) = 0$)?

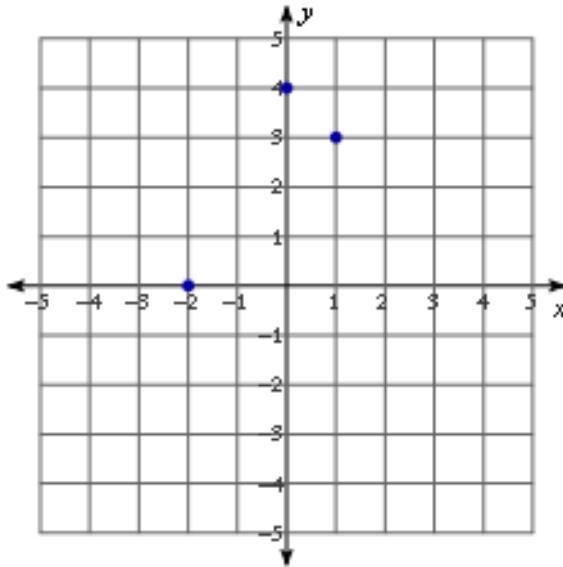
Lesson Summary

We can create a quadratic function from a data set based on a contextual situation, sketch its graph, and interpret both the function and the graph in context. We can then answer questions and make predictions related to the data, the quadratic function, and graph.

To determine a unique quadratic function from a table or graph, we must know at least three distinct points.

Problem Set

1. Write a quadratic function to fit the following points, and state the x -values for both roots. Then, sketch the graph to show that the equation includes the three points.



2. Write a quadratic function to fit the following points: $(0, 0.175)$, $(20, 3.575)$, $(30, 4.675)$.