Lesson 1: Ferris Wheels—Tracking the Height of a Passenger Car

Classwork

Exploratory Challenge 1: The Height of a Ferris Wheel Car

George Ferris built the first Ferris wheel in 1893 for the World's Columbian Exhibition in Chicago. It had 30 passenger cars, was 264 feet tall, and rotated once every 9 minutes when all the cars were loaded. The ride cost $\$0.50$.

Image:The New York Times/Redux

a. Create a sketch of the height of a passenger car on the original Ferris wheel as that car rotates around the wheel 4 times. List any assumptions that you are making as you create your model.

b. What type of function would best model this situation?

Exercises 1–5

1. Suppose a Ferris wheel has a diameter of 150 feet. From your viewpoint, the Ferris wheel is rotating counterclockwise. We will refer to a rotation through a full $360°$ as a "turn".

   a. Create a sketch of the height of a car that starts at the bottom of the wheel for two turns.

   b. Explain how the features of your graph relate to this situation.

2. Suppose a Ferris wheel has a diameter of 150 feet. From your viewpoint, the Ferris wheel is rotating counterclockwise.

   a. Your friends board the Ferris wheel, and the ride continues boarding passengers. Their car is in the three o'clock position when the ride begins. Create a sketch of the height of your friends' car for two turns.

   b. Explain how the features of your graph relate to this situation.

3. How would your sketch change if the diameter of the wheel changed?

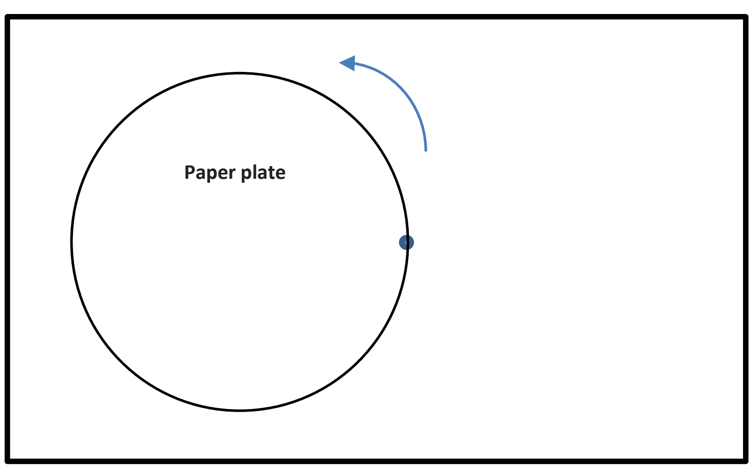
4. If you translated the sketch of your graph down by the radius of the wheel, what would the x-axis represent in this situation?

5. How could we create a more precise sketch?

Exploratory Challenge 2: The Paper Plate Model

Use a paper plate mounted on a sheet of paper to model a Ferris wheel, where the lower edge of the paper represents the ground. Use a ruler and protractor to measure the height of a Ferris wheel car above the ground for various amounts of rotation. Suppose that your friends board the Ferris wheel near the end of the boarding period and the ride begins when their car is in the three o'clock position as shown.

a. Mark the diagram below to estimate the location of the Ferris wheel passenger car every $15°$. The point on the circle below represents the passenger car in the 3 o'clock position. Since this is the beginning of the ride, consider this position to be the result of rotating by $0°$.



b. Using the physical model you created with your group, record your measurements in the table, and then graph the ordered pairs (rotation, height) on the coordinate grid shown below. Provide appropriate labels on the axes.

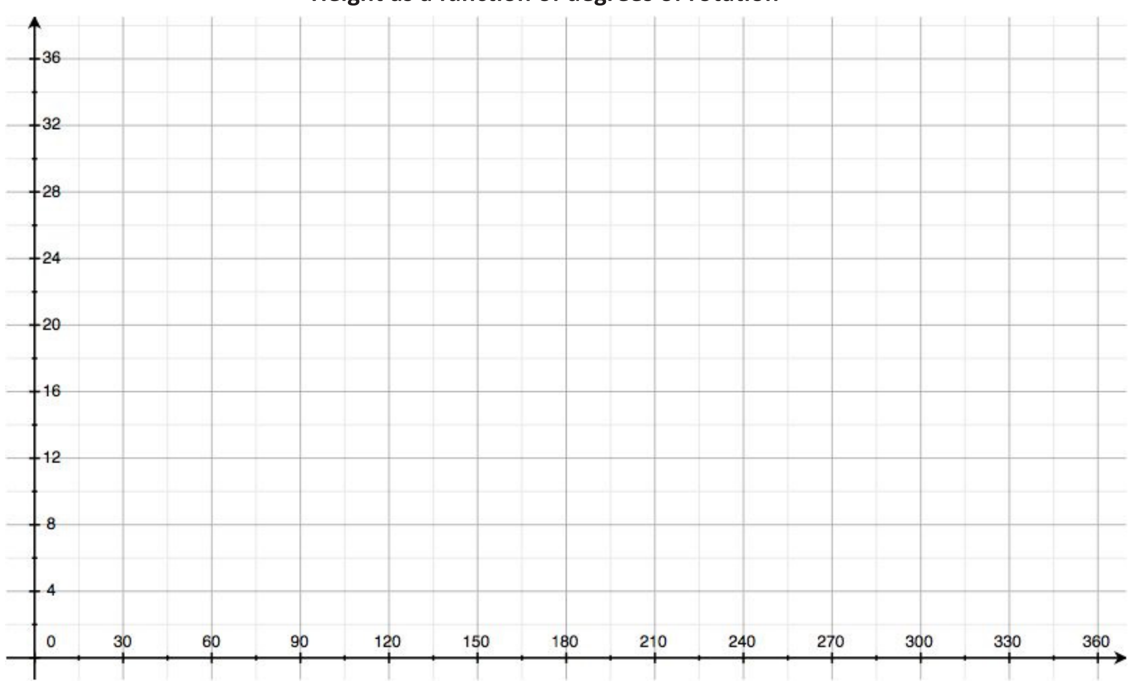
Rotation (degrees)	Height (cm)
0	
15	
30	
45	
60	
75	
90	

Rotation (degrees)	Height (cm)
105	
120	
135	
150	
180	
195	
210	

Rotation (degrees)	Height (cm)
225	
240	
255	
270	
285	
300	
315	

Rotation (degrees)	Height (cm)
330	
345	
360	

Height as a function of degrees of rotation



c. Explain how the features of your graph relate to the paper plate model you created.

Closing

- How does a function like the one that represents the height of a passenger car on a Ferris wheel differ from other types of functions you have studied such as linear, polynomial, and exponential functions?

- What is the domain of your Ferris wheel height function? What is the range?

- Provide a definition of periodic function in your own words. Why is the Ferris wheel height function an example of a periodic function?

- What other situations might be modeled by a periodic function?

Problem Set

1. Suppose that a Ferris wheel is 40 feet in diameter, rotates counterclockwise, and when a passenger car is at the bottom of the wheel it is located 2 feet above the ground.
   a. Sketch a graph of a function that represents the height of a passenger car that starts at the 3 o'clock position on the wheel for one turn.
   b. Sketch a graph of a function that represents the height of a passenger car that starts at the top of the wheel for one turn.
   c. The sketch you created in part (a) represents a graph of a function. What is the domain of the function? What is the range?
   d. The sketch you created in part (b) represents a graph of a function. What is the domain of the function? What is the range?
   e. Describe how the graph of the function in part (a) would change if you sketched the graph for two turns.
   f. Describe how the function in part (a) and its graph would change if the Ferris wheel had a diameter of 60 feet.

2. A small pebble is lodged in the tread of a tire with radius 25 cm. Sketch the height of the pebble above the ground as the tire rotates counterclockwise through 5 turns. Start your graph when the pebble is at the 9 o'clock position.

3. The graph you created in Exercise 2 represents a function.
   a. Describe how the function and its graph would change if the tire's radius was 24 inches instead of 25 cm.
   b. Describe how the function and its graph would change if the wheel was turning in the opposite direction.
   c. Describe how the function and its graph would change if we started the graph when the pebble was at ground level.

4. Justice believes that the height of a Ferris wheel passenger car is best modeled with a piecewise linear function. Make a convincing argument why a piecewise linear function IS NOT a good model for the height of a car on a rotating Ferris wheel.

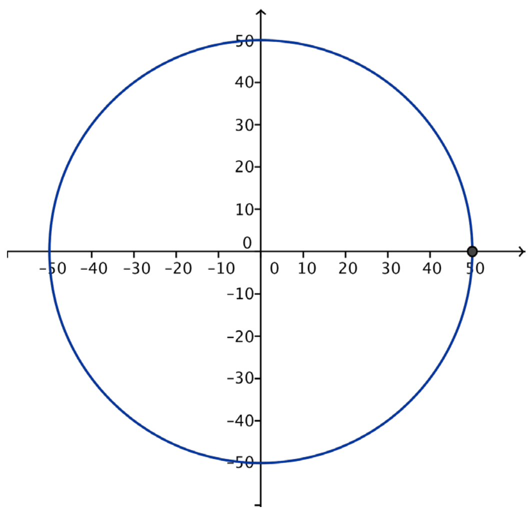
Lesson 2: The Height and Co-Height Functions of a Ferris Wheel

Classwork

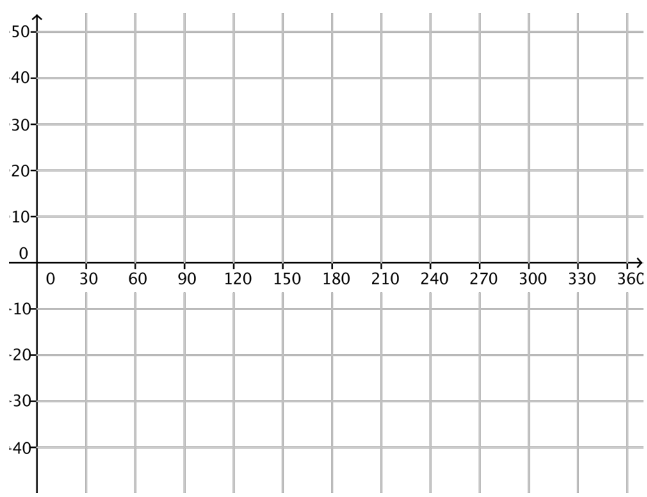
Opening Exercise

Suppose a Ferris wheel has a radius of 50 feet. We will measure the height of a passenger car that starts in the 3 o'clock position with respect to the horizontal line through the center of the wheel. That is, we consider the height of the passenger car at the outset of the problem (that is, after a $0°$ rotation) to be 0 feet.

a. Mark the diagram to show the position of a passenger car at 30 degree intervals as it rotates counterclockwise around the Ferris wheel.



b. Sketch the graph of the height function of the passenger car for one turn of the wheel. Provide appropriate labels on the axes.

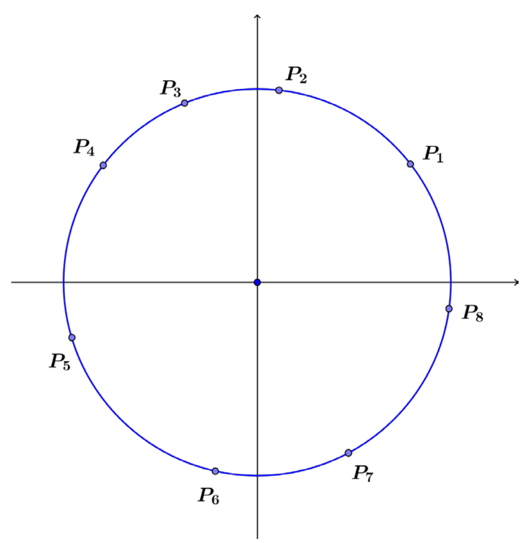


c. Explain how you can identify the radius of the wheel from the graph in part (b).

d. If the center of the wheel is 55 feet above the ground, how high is the passenger car above the ground when it is at the top of the wheel?

Exercises 1–3

1. Each point P_1, P_2, … P_8 on the circle in the diagram at right represents a passenger car on a Ferris wheel.



a. Draw segments that represent the co-height of each car. Which cars have a positive co-height? Which cars have a negative co-height?

b. List the points in order of increasing co-height; that is, list the point with the smallest co-height first and the point with the largest co-height last.

2. Suppose that the radius of a Ferris wheel is 100 feet and the wheel rotates counterclockwise through one turn. Define a function that measures the co-height of a passenger car as a function of the degrees of rotation from the initial 3 o'clock position.

   a. What is the domain of the co-height function?

   b. What is the range of the co-height function?

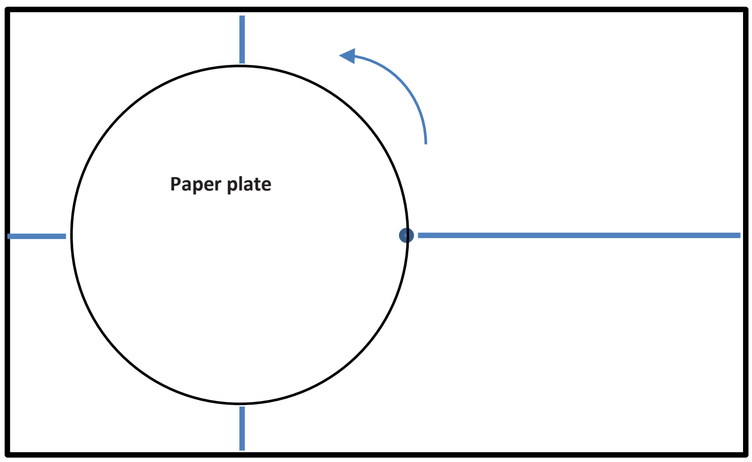
   c. How does changing the wheel's radius affect the domain and range of the co-height function?

3. For a Ferris wheel of radius 100 feet going through one turn, how do the domain and range of the height function compare to the domain and range of the co-height function? Is this true for any Ferris wheel?

Exploratory Challenge: The Paper Plate Model Again

Use a paper plate mounted on a sheet of paper to model a Ferris wheel, where the lower edge of the paper represents the ground. Use a ruler and protractor to measure the height and co-height of a Ferris wheel car at various amounts of rotation, measured with respect to the horizontal and vertical lines through the center of the wheel. Suppose that your friends board the Ferris wheel near the end of the boarding period and the ride begins when their car is in the three o'clock position as shown.

a. Mark horizontal and vertical lines through the center of the wheel on the card stock behind the plate as shown. We will measure the height and co-height as the displacement from the horizontal and vertical lines through the center of the plate.



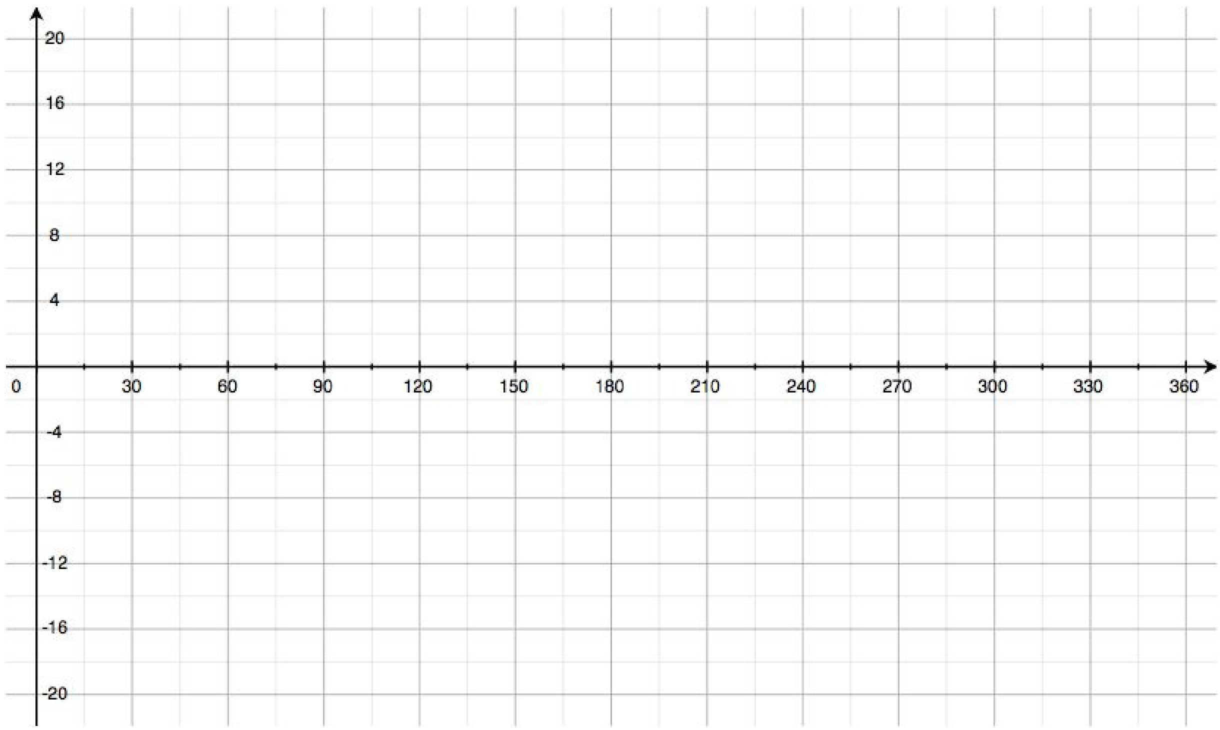
b. Using the physical model you created with your group, record your measurements in the table, and then graph each of the two sets of ordered pairs (rotation angle, height) and (rotation angle, co-height) on separate coordinate grids below. Provide appropriate labels on the axes.

Rotation (degrees)	Height (cm)	Co-height (cm)
0		
15		
30		
45		
60		
75		
90		
105		
120		

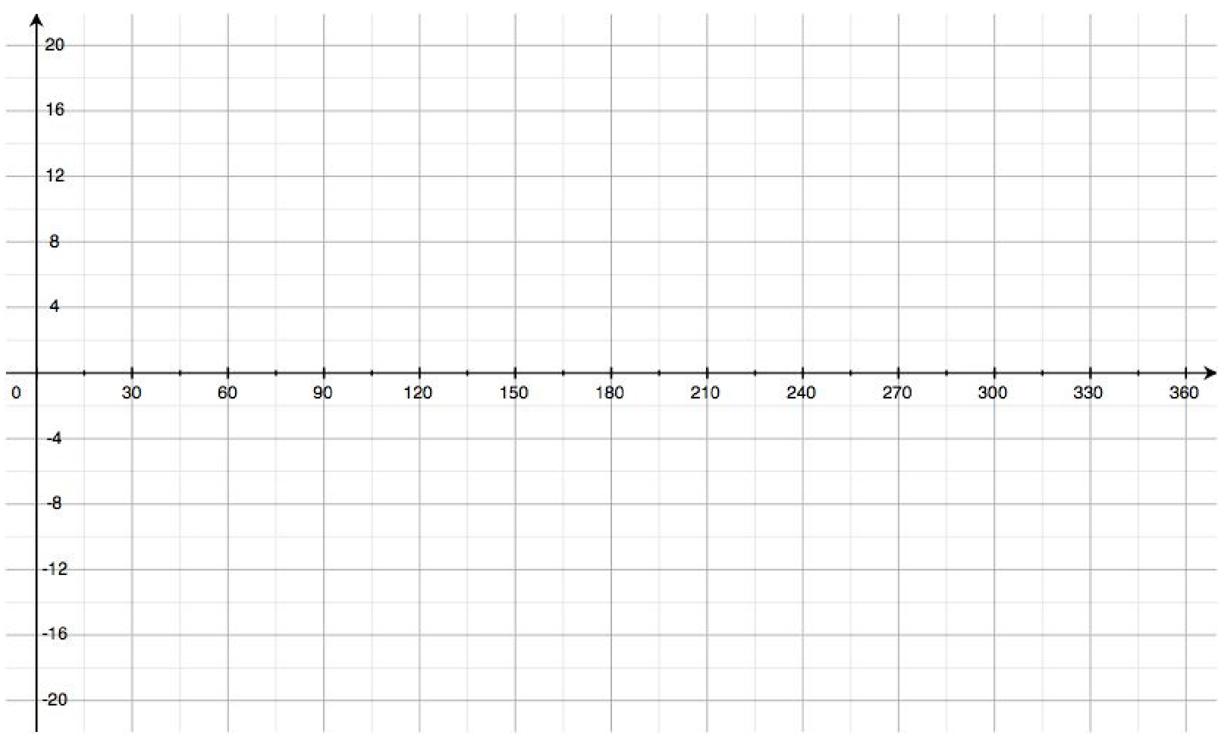
Rotation (degrees)	Height (cm)	Co-height (cm)
135		
150		
165		
180		
195		
210		
225		
240		

Rotation (degrees)	Height (cm)	Co-height (cm)
255		
270		
285		
300		
315		
330		
345		
360		

Height as a Function of Degrees of Rotation



Co-height as a Function of Degrees of Rotation



Closing

- Why do you think we named the new function the co-height?

- How are the graphs of these two functions alike? How are they different?

- What does a negative value of the height function tell us about the location of the passenger car at various positions around a Ferris wheel? What about a negative value of the co-height function?

Problem Set

1. The Seattle Great Wheel, with an overall height of 175 feet, was the tallest Ferris wheel on the west coast at the time of its construction in 2012. For this exercise, assume that the diameter of the wheel is 175 feet.
   a. Create a diagram that shows the position of a passenger car on the Great Wheel as it rotates counterclockwise at 45 degree intervals.
   b. On the same set of axes, sketch graphs of the height and co-height functions for a passenger car starting at the 3 o'clock position on the Great Wheel and completing one turn.
   c. Discuss the similarities and differences between the graph of the height function and the graph of the co-height function.
   d. Explain how you can identify the radius of the wheel from either graph.

2. In 2014, the "High Roller" Ferris wheel opened in Las Vegas, dwarfing the Seattle Great Wheel with a diameter of 520 feet. Sketch graphs of the height and co-height functions for one complete turn of the High Roller.

3. Consider a Ferris wheel with a 50-foot radius. We will track the height and co-height of passenger cars that begin at the 3 o'clock position. Sketch graphs of the height and co-height functions for the following scenarios.
   a. A passenger car on the Ferris wheel completes one turn, traveling counterclockwise.
   b. A passenger car on the Ferris wheel completes two full turns, traveling counterclockwise.
   c. The Ferris wheel is stuck in reverse, and a passenger car on the Ferris wheel completes two full *clockwise* turns.

4. Consider a Ferris wheel with radius 40 feet that is rotating counterclockwise. At which amounts of rotation are the values of the height and co-height functions equal? Does this result hold for a Ferris wheel with a different radius?

5. Yuki is on a passenger car of a Ferris wheel at the 3 o'clock position. The wheel then rotates $135°$ counterclockwise and gets stuck. Lee argues that she can compute the value of the co-height of Yuki's car if she is given one of the following two pieces of information:
   i. The value of the height function of Yuki's car, or
   ii. The diameter of the Ferris wheel itself.

   Is Lee correct? Explain how you know.

Lesson 3: The Motion of the Moon, Sun, and Stars—Motivating Mathematics

Classwork

Opening

- Why does it look like the sun moves across the sky?

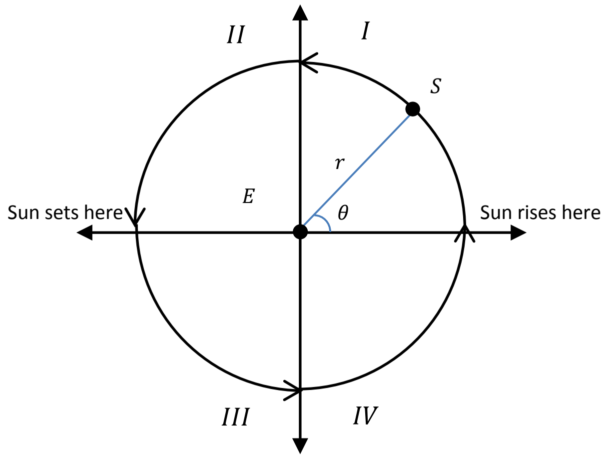
- Is the sun moving, or are you moving?

- In ancient Greek mythology, the god Helios was the personification of the sun. He rode across the sky every day in his chariot led by four horses. Why do your answers make it believable that in ancient times, people imagined the sun was pulled across the sky each day?

Discussion

In mathematics, counterclockwise rotation is considered to be the positive direction of rotation, which runs counter to our experience with a very common example of rotation: the rotation of the hands on a clock.

- Is there a connection between counterclockwise motion being considered to be positive and the naming of the quadrants on a standard coordinate system?



- What does the circle's radius, r, represent?

- How has the motion of the sun influenced the development of mathematics?

- How is measuring the "height" of the sun like measuring the Ferris wheel passenger car height in the previous lessons?

Exercises 1–4

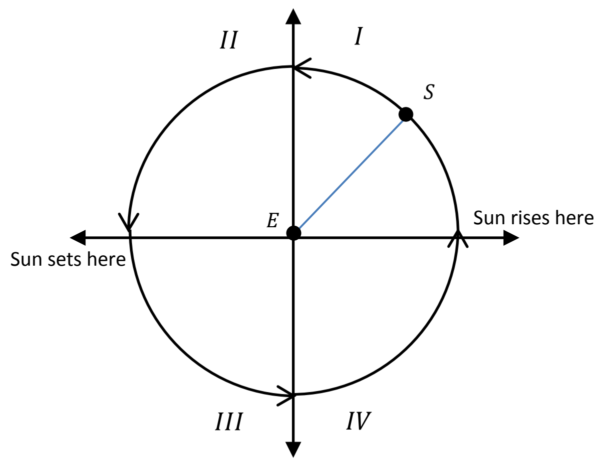
1. Calculate $\text{jya}(7\frac{1}{2}°)$, $\text{jya}(11\frac{1}{4}°)$, $\text{jya}(15°)$, and $\text{jya}(18\frac{3}{4}°)$ using Aryabhata's formula[1], round to the nearest integer, and add your results to the table below. Leave the rightmost column blank for now.

n	$\theta°$	$\text{jya}(\theta)$	$3438\sin(\theta)$
1	$3\frac{3}{4}°$	225	
2	$7\frac{1}{2}°$		
3	$11\frac{1}{4}°$		
4	$15°$		
5	$18\frac{3}{4}°$		
6	$22\frac{1}{2}°$	1315	
7	$26\frac{1}{4}°$	1520	
8	$30°$	1719	
9	$33\frac{3}{4}°$	1910	
10	$37\frac{1}{2}°$	2093	
11	$41\frac{1}{4}°$	2267	
12	$45°$	2431	

n	$\theta°$	$\text{jya}(\theta)$	$3438\sin(\theta)$
13	$48\frac{3}{4}°$	2585	
14	$52\frac{1}{2}°$	2728	
15	$56\frac{1}{4}°$	2859	
16	$60°$	2978	
17	$63\frac{3}{4}°$	3084	
18	$67\frac{1}{2}°$	3177	
19	$71\frac{1}{4}°$	3256	
20	$75°$	3321	
21	$78\frac{3}{4}°$	3372	
22	$82\frac{1}{2}°$	3409	
23	$86\frac{1}{4}°$	3431	
24	$90°$	3438	

[1] In constructing the table, Aryabhata made adjustments to the values of his approximation to the jya to match his observational data. The first adjustment occurs in the calculation of $\text{jya}(30°)$. Thus, the entire table cannot be accurately constructed using this formula.

2. Label the angle θ, $\text{jya}(\theta)$, $\text{kojya}(\theta)$, and r in the diagram shown below.



a. How does this relate to something you have done before?

b. How does $\text{jya}(\theta)$ relate to a length *we* already know?

3. Use your calculator to compute $r\sin(\theta)$ for each value of θ in the table from Exercise 1, where $r = 3438$. Record this in the blank column on the right in Exercise 1, rounding to the nearest integer. How do Aryabhata's approximated values from around the year A.D. 500 compare to the value we can calculate with our modern technology?

4. We will assume that the sun rises at 6:00 am, is directly overhead at 12:00 noon, and sets at 6:00 pm. We measure the "height" of the sun by finding its vertical distance from the horizon line; the horizontal line that connects the eastern-most point, where the sun rises, to the western-most point, where the sun sets.

   a. Using $r = 3438$, as Aryabhata did, find the "height" of the sun at the times listed in the following table:

Time of day	Height
6:00 a.m.	
7:00 a.m.	
8:00 a.m.	
9:00 a.m.	
10:00 a.m.	
11:00 a.m.	
12:00 p.m.	

   b. Now, find the height of the sun at the times listed in the following table using the actual distance from the earth to the sun, $r = 93$ million miles.

Time of day	Height
6:00 a.m.	
7:00 a.m.	
8:00 a.m.	
9:00 a.m.	
10:00 a.m.	
11:00 a.m.	
12:00 p.m.	

Closing

- Our exploration of the historical development of the sine table is based on observations of the motion of the planets and stars in Babylon and India. Is it based on a geocentric or heliocentric model? What does that term mean?

- How is Aryabhata's function jya related to the sine of an angle of a triangle?

- How does the apparent motion of the sun in the sky relate to the motion of a passenger car of a Ferris wheel?

Lesson Summary

Ancient scholars in Babylon and India conjectured that celestial motion was circular; the sun and other stars orbited the earth in circular fashion. The earth was presumed the center of the sun's orbit.

The quadrant numbering in a coordinate system is consistent with the counterclockwise motion of the sun, which rises from the east and sets in the west.

The 6th century Indian scholar Aryabhata created the first sine table, using a measurement he called *jya*. The purpose of his table was to calculate the position of the sun, the stars, and the planets.

Problem Set

1. An Indian astronomer noted that the angle of his line of sight to Venus measured $52\frac{1}{2}^\circ$. We now know that the average distance from the Earth to Venus is 162 million miles. Use Aryabhata's table to estimate the apparent height of Venus. Round your answer to the nearest million miles.

2. Later, the Indian astronomer saw that the angle of his line of sight to Mars measured $82\frac{1}{2}^\circ$. We now know that the average distance from the Earth to Mars is 140 million miles. Use Aryabhata's table to estimate the apparent height of Mars. Round your answer to the nearest million miles.

3. The moon orbits the earth in an elongated orbit, with an average distance of the moon from the earth of roughly $239{,}000$ miles. It takes the moon 27.32 days to travel around the earth, so the moon moves with respect to the stars roughly $0.5°$ every hour. Suppose that angle of inclination of the moon with respect to the observer measures $45°$ at midnight. As in Example 1, an observer is standing still and facing north. Use Aryabhata's jya table to find the apparent height of the moon above the observerat the times listed in the table below, to the nearest thousand miles.

Time (hour:min)	Angle of elevation θ	Height
12:00 a.m.		
7:30 a.m.		
3:00 p.m.		
10:30 p.m.		
6:00 a.m.		
1:30 p.m.		
9:00 p.m.		

4. George wants to apply Aryabhata's method to estimate the height of the International Space Station, which orbits earth at a speed of about $17{,}500$ miles per hour. This means that the space station makes one full rotation around the earth roughly every 90 minutes. The space station maintains a low earth orbit, with an average distance from earth of 238 miles.

   a. George supposes that the space station is just visible on the eastern horizon at 12:00 midnight, so its apparent height at that time would be 0 miles above the horizon. Use Aryabhata's jya table to find the apparent height of the space station above the observer at the times listed in the table below.

Time (hour:min:sec)	Angle of elevation θ	Height
12:00:00 a.m.		
12:03:45 a.m.		
12:07:30 a.m.		
12:11:15 a.m.		
12:15:00 a.m.		
12:18:45 a.m.		
12:22:30 a.m.		

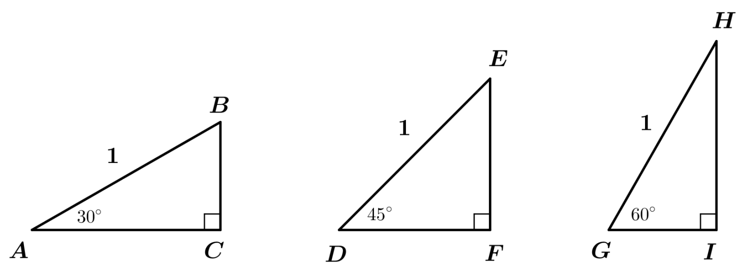
   b. When George presents his solution to his classmate Jane, she tells him that his model isn't appropriate for this situation. Is she correct? Explain how you know. (Hint: As we set up our model in the first discussion, we treated our observer as if he was the center of the orbit of the sun around the earth. In part (a) of this problem, we treated our observer as if she was the center of the orbit of the International Space Station around the earth. The radius of the earth is approximately 3963 miles, the space station orbits about 238 miles above the earth's surface, and the distance from the earth to the sun is roughly $93{,}000{,}000$ miles. Draw a picture of the earth and the path of the space station, then compare that to the points with heights and rotation angles from part (a).)

Lesson 4: From Circle-ometry to Trigonometry

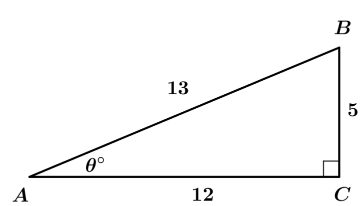
Classwork

Opening Exercises

1. Find the lengths of the sides of the right triangles below, each of which has hypotenuse of length 1.

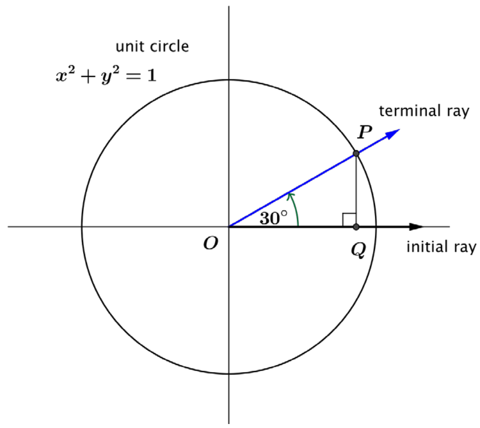


2. Given the following right triangle ΔABC with $m(\angle A) = \theta$, find $\sin(\theta°)$ and $\cos(\theta°)$.



Example 1

Suppose that point P is the point on the unit circle obtained by rotating the initial ray through $30°$. Find $\sin(30°)$ and $\cos(30°)$.



What is the length OQ of the horizontal leg of our triangle?

What is the length QP of the vertical leg of our triangle?

What is $\sin(30°)$?

What is $\cos(30°)$?

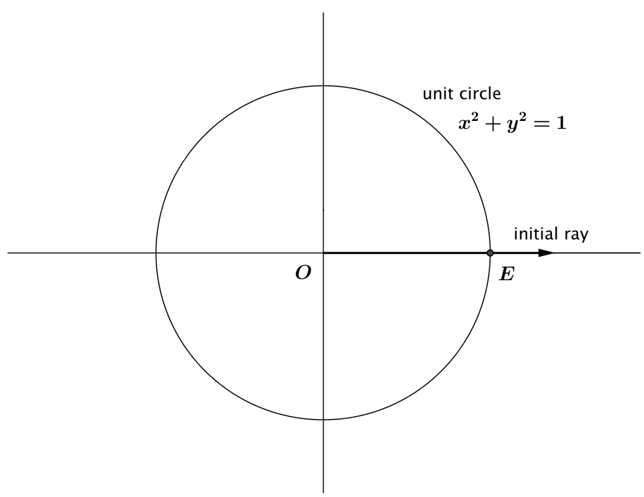
Exercises 1–2

1. Suppose that P is the point on the unit circle obtained by rotating the initial ray through $45°$. Find $\sin(45°)$ and $\cos(45°)$.

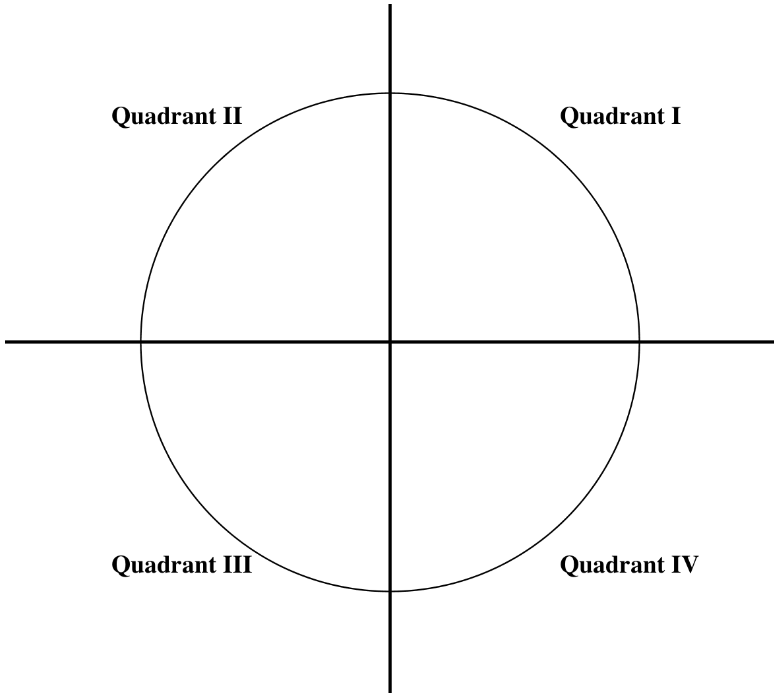
2. Suppose that P is the point on the unit circle obtained by rotating the initial ray through $60°$. Find $\sin(60°)$ and $\cos(60°)$.

Example 2

Suppose that P is the point on the unit circle obtained by rotating the initial ray through $150°$. Find $\sin(150°)$ and $\cos(150°)$.



Discussion



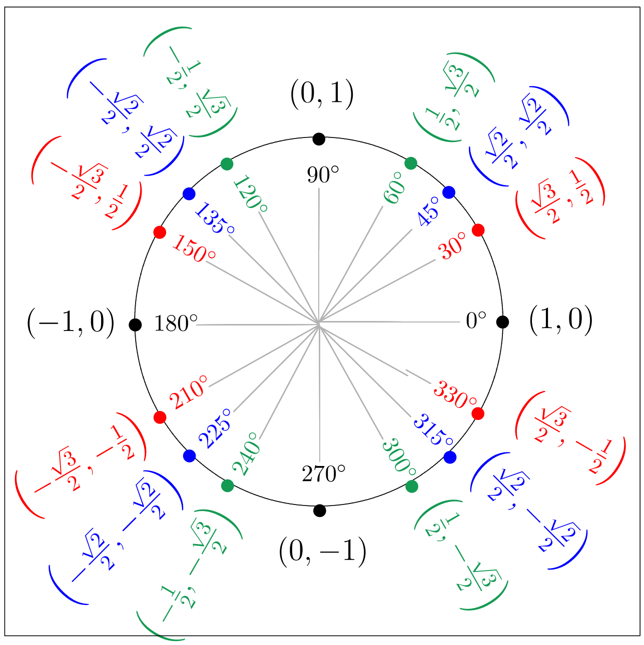
Exercises 3–5

3. Suppose that P is the point on the unit circle obtained by rotating the initial ray through 120 degrees. Find the measure of the reference angle for $120°$, then find $\sin(120°)$ and $\cos(120°)$.

4. Suppose that P is the point on the unit circle obtained by rotating the initial ray through $240°$, . Find the measure of the reference angle for $240°$, then find $\sin(240°)$ and $\cos(240°)$.

5. Suppose that P is the point on the unit circle obtained by rotating the initial ray through 330 degrees. Find the measure of the reference angle for $330°$, then find $\sin(330°)$ and $\cos(330°)$.

Discussion



Lesson Summary

In this lesson we formalized the idea of the height and co-height of a Ferris wheel and defined the sine and cosine functions that give the x and y coordinates of the intersection of the unit circle and the initial ray rotated through θ degrees, for most values of θ with $0 < \theta < 360$.

- The value of $\cos(\theta)$ is the x-coordinate of the intersection point of the terminal ray and the unit circle.
- The value of $\sin(\theta)$ is the y-coordinate of the intersection point of the terminal ray and the unit circle.
- The sine and cosine functions have domain of all real numbers and range $[-1,1]$.

Problem Set

1. Fill in the chart, and write in the reference angles and the values of the sine and cosine for the indicated rotation numbers.

Amount of rotation, θ, in degrees	Measure of Reference Angle	$\cos\theta$	$\sin\theta$
$330°$			
$90°$			
$120°$			
$150°$			
$135°$			
$270°$			
$225°$			

2. Using geometry, Jennifer correctly calculated that $\sin(15°) = \frac{1}{2}\sqrt{2-\sqrt{3}}$. Based on this information, fill in the chart:

Amount of rotation, θ, in degrees	Measure of Reference Angle	$\cos(\theta)$	$\sin(\theta)$
$15°$			
$165°$			
$195°$			
$345°$			

3. Suppose $0 < \theta < 90$ and $\sin(\theta) = \frac{1}{\sqrt{3}}$. What is the value of $\cos(\theta)$?

4. Suppose $90° < \theta < 180°$ and $\sin(\theta) = \frac{1}{\sqrt{3}}$. What is the value of $\cos(\theta)$?

5. If $\cos(\theta) = -\frac{1}{\sqrt{5}}$, what are two possible values of $\sin(\theta)$?

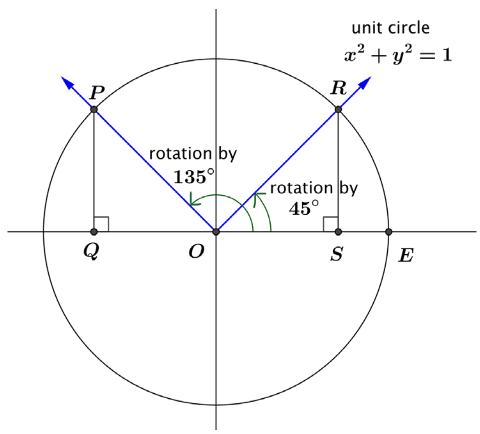
6. Johnny rotated the initial ray through θ degrees, found the intersection of the terminal ray with the unit circle, and calculated that $\sin(\theta) = \sqrt{2}$. Ernesto insists that Johnny made a mistake in his calculation. Explain why Ernesto is correct.

7. If $\sin(\theta) = 0.5$, and we know that $\cos(\theta) < 0$, then what is the smallest possible positive value of θ?

8. The vertices of triangle ΔABC have coordinates $A = (0,0)$, $B = (12,5)$, and $C = (12,0)$.
   a. Argue that ΔABC is a right triangle.
   b. What are the coordinates where the hypotenuse of ΔABC intersects the unit circle $x^2 + y^2 = 1$?
   c. Let θ denote the degrees of rotation from $\overrightarrow{AC}$ to $\overrightarrow{AB}$. Calculate $\sin(\theta)$ and $\cos(\theta)$.

9. The vertices of triangle ΔABC have coordinates $A = (0,0)$, $B = (4,3)$, and $C = (4,0)$. The vertices of triangle ΔADE are at the points $A = (0,0)$, $D = (3,4)$, and $E = (3,0)$.
   a. Argue that ΔABC is a right triangle.
   b. What are the coordinates where the hypotenuse of ΔABC intersects the unit circle $x^2 + y^2 = 1$?
   c. Let θ denote the degrees of rotation from $\overrightarrow{AC}$ to $\overrightarrow{AB}$. Calculate $\sin(\theta)$ and $\cos(\theta)$.
   d. Argue that ΔADE is a right triangle.
   e. What are the coordinates where the hypotenuse of ΔADE intersects the unit circle $x^2 + y^2 = 1$?
   f. Let ϕ denote the degrees of rotation from $\overrightarrow{AE}$ to $\overrightarrow{AD}$. Calculate $\sin \phi$ and $\cos \phi$.
   g. What is the relation between the sine and cosine of θ and the sine and cosine of ϕ?

10. Use a diagram to explain why $\sin(135°) = \sin(45°)$, but $\cos(135°) \neq \cos(45°)$.



Lesson 5: Extending the Domain of Sine and Cosine to All Real Numbers

Classwork

Opening Exercises

a. Suppose that a group of 360 coworkers pool their money, buying a single lottery ticket every day with the understanding that if any ticket was a winning ticket, the group would split the winnings evenly, and they would donate any left over money to the local high school. Using this strategy, the group won $\$1,000$. How much money was donated to the school?

b. What if the winning ticket was worth $\$250,000$? Using the same plan as in part (a), how much money would be donated to the school?

c. What if the winning ticket was worth $\$540,000$? Using the same plan as in part (a), how much money would be donated to the school?

Exercises 1–5

1. Find $\cos(405°)$ and $\sin(405°)$. Identify the measure of the reference angle.

2. Find $\cos(840°)$ and $\sin(840°)$. Identify the measure of the reference angle.

3. Find $\cos(1680°)$ and $\sin(1680°)$. Identify the measure of the reference angle.

4. Find $\cos(2115°)$ and $\sin(2115°)$. Identify the measure of the reference angle.

5. Find $\cos(720030°)$ and $\sin(720030°)$. Identify the measure of the reference angle.

Exercises 6–10

6. Find $\cos(-30°)$ and $\sin(-30°)$. Identify the measure of the reference angle.

7. Find $\cos(-135°)$ and $\sin(-135°)$. Identify the measure of the reference angle.

8. Find $\cos(-1320°)$ and $\sin(-1320°)$. Identify the measure of the reference angle.

9. Find $\cos(-2205°)$ and $\sin(-2205°)$. Identify the measure of the reference angle.

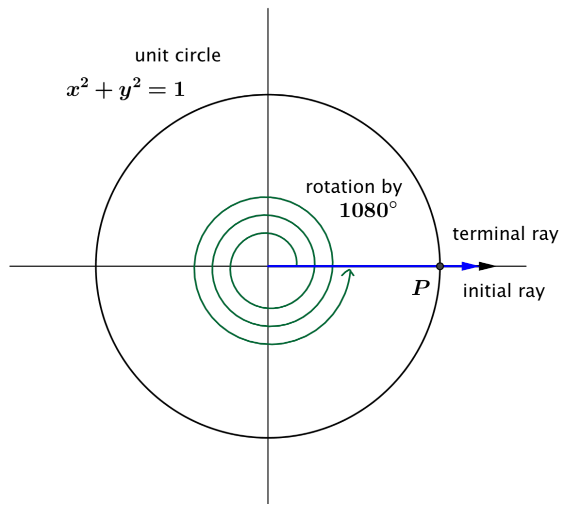
10. Find $\cos(-2835°)$ and $\sin(-2835°)$. Identify the measure of the reference angle.

Discussion

Case 1: What about the values of the sine and cosine function of other amounts of rotation that produce a terminal ray along the positive x-axis, such as $1080°$?

Our definition of a reference angle is the angle formed by the terminal ray and the x-axis, but our terminal ray lies along the x-axis so the terminal ray and the x-axis form a zero angle.

How would we assign values to $\cos(1080°)$ and $\sin(1080°)$?



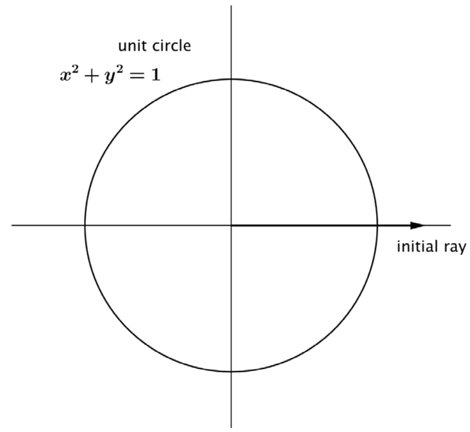
What if we rotated around $24000°$, which is 400 turns? What are $\cos(24000°)$ and $\sin(24000°)$?

State a generalization of these results:

If $\theta = n \cdot 360°$, for some integer n, then $\cos(\theta) =$ ____, and $\sin(\theta) =$ _____.

Case 2: What about the values of the sine and cosine function of other amounts of rotation that produce a terminal ray along the negative x-axis, such as $540°$?

How would we assign values to $\cos(540°)$ and $\sin(540°)$?

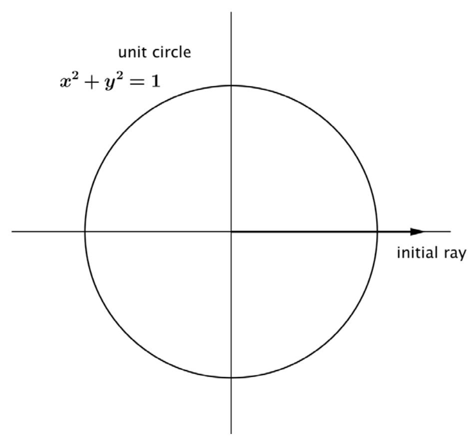


What are the values of $\cos(900°)$ and $\sin(900°)$? How do you know?

State a generalization of these results:

If $\theta = n \cdot 360° + 180°$, for some integer n, then $\cos(\theta) =$ ____, and $\sin(\theta) =$ _____.

Case 3: What about the values of the sine and cosine function for rotations that are $90°$ more than a number of full turns, such as $-630°$?
How would we assign values to $\cos(-630°)$, and $\sin(-630°)$?



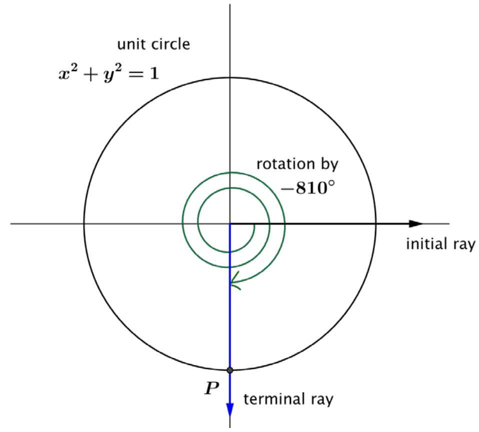
Can we generalize to any rotation that produces a terminal ray along the positive y-axis?

State a generalization of these results:

If $\theta = n \cdot 360° + 90°$*, for some integer* n*, then* $\cos(\theta) =$ ____*, and* $\sin(\theta) =$ _____.

Case 4: What about the values of the sine and cosine function for rotations whose terminal ray lies along the negative y-axis, such as $-810°$?

How would we assign values to $\cos(-810°)$ and $\sin(-810°)$?



Can we generalize to any rotation that produces a terminal ray along the negative y-axis?

State a generalization of these results:

If $\theta = n \cdot 360° + 270°$*, for some integer* n*, then* $\cos(\theta) =$ ____*, and* $\sin(\theta) =$ _____.

Discussion

Let θ be any real number. In the Cartesian plane, rotate the initial ray by θ degrees about the origin. Intersect the resulting terminal ray with the unit circle to get a point (x_θ, y_θ) in the coordinate plane. The value of $\sin(\theta)$ is y_θ, and the value of $\cos(\theta)$ is x_θ.

Lesson Summary

In this lesson we formalized the definition of the sine and cosine functions of a number of degrees of rotation, θ. We rotate the initial ray made from the positive x-axis through θ degrees, going counterclockwise if $\theta > 0$ and clockwise if $\theta < 0$. The point P is defined by the intersection of the terminal ray and the unit circle.

- The value of $\cos(\theta)$ is the x-coordinate of P.
- The value of $\sin(\theta)$ is the y-coordinate of P.
- The sine and cosine functions have domain of all real numbers and range $[-1,1]$.

Problem Set

1. Fill in the chart; write the quadrant where the terminal ray is located after rotation by θ, the measures of the reference angles, and the values of the sine and cosine functions for the indicated rotation numbers.

Number of degrees of rotation, θ	Quadrant	Measure of Reference Angle	$\cos(\theta)$	$\sin(\theta)$
690				
810				
1560				
1440				
855				
-330				
-4500				
-510				
-135				
-1170				

2. Using geometry, Jennifer correctly calculated that $\sin 15° = \frac{1}{2}\sqrt{2-\sqrt{3}}$. Based on this information, fill in the chart:

Number of degrees of rotation, θ	Quadrant	Measure of Reference Angle	$\cos(\theta)$	$\sin(\theta)$
525				
705				
915				
-15				
-165				
-705				

3. Suppose θ represents a quantity in degrees, and that $\sin(\theta) = 0.5$. List the first six possible positive values that θ can take.

4. Suppose θ represents a quantity in degrees, and that $\sin(\theta°) = -0.5$. List six possible negative values that θ can take.

5. Suppose θ represents a quantity in degrees. Is it possible that $\cos(\theta°) = \frac{1}{2}$ and $\sin(\theta°) = \frac{1}{2}$?

6. Jane says that since the reference angle for a rotation through $-765°$ has measure $45°$, then $\cos(-765°) = \cos(45°)$, and $\sin(-765°) = \sin(45°)$. Explain why she is or is not correct.

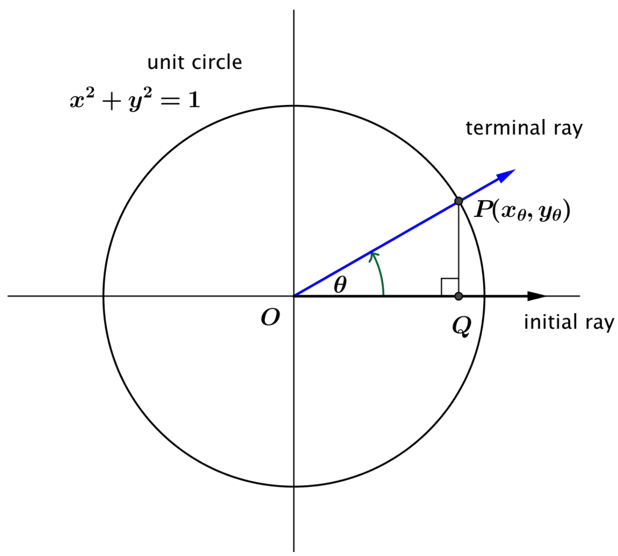
7. Doug says that since the reference angle for a rotation through $765°$ has measure $45°$, then $\cos(765°) = \cos(45°)$, and $\sin(765°) = \sin(45°)$. Explain why he is or is not correct.

Lesson 6: Why Call It Tangent?

Classwork

Opening Exercise

Let $P(x_\theta, y_\theta)$ be the point where the terminal ray intersects the unit circle after rotation by θ degrees, as shown in the diagram below.



a. Using triangle trigonometry, what are the values of x_θ and y_θ in terms of θ?

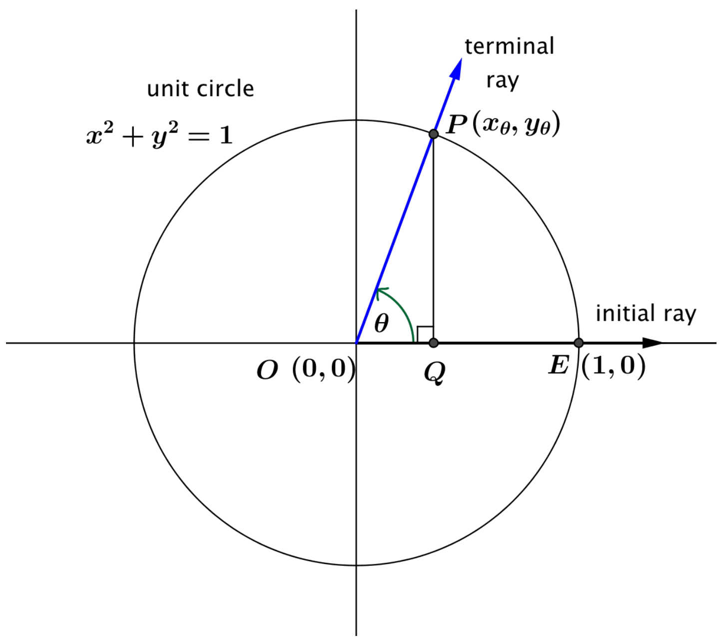
b. Using triangle trigonometry, what is the value of $\tan(\theta)$ in terms of x_θ and y_θ?

c. What is the value of $\tan(\theta)$ in terms of θ?

Discussion

A description of the tangent function is provided below. Be prepared to answer questions based on your understanding of this function and to discuss your responses with others in your class.

> Let θ be any real number. In the Cartesian plane, rotate the non-negative x-axis by θ degrees about the origin. Intersect the resulting terminal ray with the unit circle to get a point (x_θ, y_θ). If $x_\theta \neq 0$, then the value of $\tan(\theta)$ is $\frac{y_\theta}{x_\theta}$. In terms of the sine and cosine functions, $\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$ for $\cos(\theta) \neq 0$.



Exercise 1

1. For each value of θ in the table below, use the given values of $\sin(\theta)$ and $\cos(\theta)$ to approximate $\tan(\theta)$ to two decimal places.

θ (degrees)	$\sin(\theta)$	$\cos(\theta)$	$\tan(\theta)$
-89.9	-0.999998	0.00175	
-89	-0.9998	0.0175	
-85	-0.996	0.087	
-80	-0.98	0.17	
-60	-0.87	0.50	
-40	-0.64	0.77	
-20	-0.34	0.94	
0	0	1.00	
20	0.34	0.94	
40	0.64	0.77	
60	0.87	0.50	
80	0.98	0.17	
85	0.996	0.087	
89	0.9998	0.0175	
89.9	0.999998	0.00175	

a. As $\theta \rightarrow -90$ and $\theta > -90$, what value does $\sin(\theta)$ approach?

b. As $\theta \rightarrow -90$ and $\theta > -90$, what value does $\cos(\theta)$ approach?

c. As $\theta \rightarrow -90$ and $\theta > -90$, how would you describe the value of $\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$?

d. As $\theta \rightarrow 90$ and $\theta < 90$, what value does $\sin(\theta)$ approach?

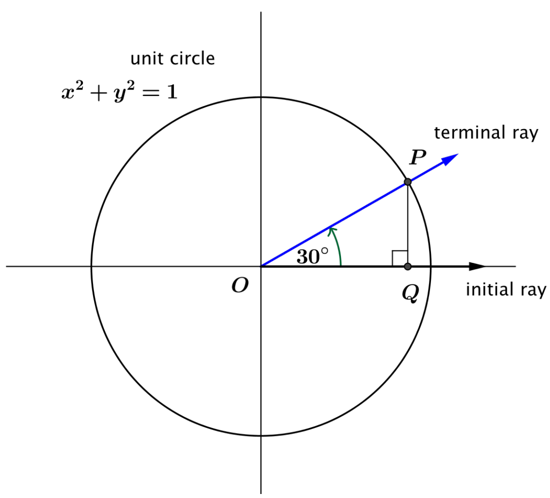
e. As $\theta \rightarrow 90$ and $\theta < 90$, what value does $\cos(\theta)$ approach?

f. As $\theta \rightarrow 90$ and $\theta < 90$, how would you describe the behavior of $\tan(\theta) = \sin(\theta)/\cos(\theta)$?

g. How can we describe the range of the tangent function?

Example 1

Suppose that point P is the point on the unit circle obtained by rotating the initial ray through $30°$. Find $\tan(30°)$.



Exercises 2–6: Why Do We Call it Tangent?

2. Let P be the point on the unit circle with center O that is the intersection of the terminal ray after rotation by θ degrees as shown in the diagram at right. Let Q be the foot of the perpendicular line from P to the x-axis, and let the line ℓ be the line perpendicular to the x-axis at $S(1,0)$. Let R be the point where the secant line $\overleftrightarrow{OP}$ intersects the line ℓ. Let m be the length of the segment $\overline{RS}$.

   a. Show that $m = \tan(\theta)$.

   b. Using a segment in the figure, make a conjecture why mathematicians named the function $f(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$ the tangent function.

   c. Why can you use either triangle, ΔPOQ or ΔROS, to calculate the length m?

   d. Imagine that YOU are the mathematician who gets to name the function (how cool would that be?). Based upon what you know about the equations of lines, what might you have named the function instead?

3. Draw four pictures similar to the diagram above to illustrate what happens to the value of $\tan(\theta)$ as the rotation of the terminal ray contained on a secant line through the origin increases towards $90°$. How does your diagram relate to the work done in Exercise 1?

4. When the terminal ray is vertical, what is the relationship between the secant line and the tangent line? Explain why you cannot determine the measure of m in this instance. What is the value of $\tan(90°)$?

5. When the terminal ray is horizontal, what is the relationship between this secant line and the x-axis? Explain what happens to the value of m in this instance. What is the value of $\tan(0°)$?

6. When the terminal ray is rotated counterclockwise about the origin by $45°$, what is relationship between the value of m and length of $\overline{OS}$? What is the value of $\tan(45°)$?

Exercises 7–8

7. Rotate the initial ray about the origin the stated number of degrees. Draw a sketch and label the coordinates of point P where the terminal ray intersects the unit circle. What is the slope of the line containing this ray?

   a. 30

   b. 45

   c. 60

d. Use the definition of tangent to find $\tan(30°)$, $\tan(45°)$, and $\tan(60°)$. How do your answers compare your work in parts (a)–(c)?

e. If the initial ray is rotated θ degrees about the origin, show that the slope of the line containing the terminal ray is equal to $\tan(\theta)$. Explain your reasoning.

f. Now that you have shown that the tangent function is equal to the slope of the terminal ray, would you prefer using the name "tangent function" or "slope function"? Why do you think we use "tangent" instead of "slope" as the name of the tangent function?

8. Rotate the initial ray about the origin the stated number of degrees. Draw a sketch and label the coordinates of point P where the terminal ray intersects the unit circle. How does your diagram in this Exercise relate to the diagram in the corresponding part of Exercise 7? What is $\tan(\theta)$ for these values of θ?

a. 210

b. 225

c. 240

d. What do the results of parts (a)–(c) suggest about the value of the tangent function after rotating an additional 180 degrees?

e. What is the period of the tangent function? Discuss with a classmate and write your conclusions.

f. Use the results of Exercise 7(d) to explain why $\tan(0°) = 0$.

g. Use the results of Exercise 7(d) to explain why $\tan(90°)$ is undefined.

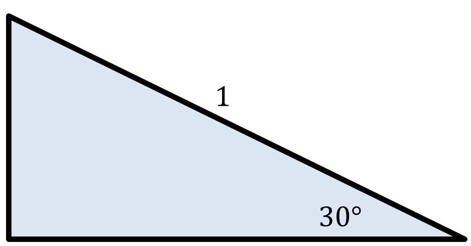
Lesson Summary

- $\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$, where $\cos(\theta) \neq 0$.
- The value of $\tan(\theta)$ is the length of the line segment on the tangent line to the unit circle centered at the origin from the intersection with the unit circle and the intersection with the secant line created by the x-axis rotated $\theta°$ (this is why we call it tangent).
- The value of $\tan(\theta)$ is the slope of the line obtained by rotating the x-axis θ degrees about the origin.
- The domain of the tangent function is $\{\theta \in \mathbb{R} | \theta \neq 90 + 180k, \text{for all integers } k\}$ which is equivalent to $\{\theta \in \mathbb{R} | \cos(\theta) \neq 0\}$.
- The range of the tangent function is all real numbers.
- The period of the tangent function is 180.

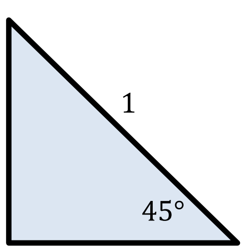
$\tan(0°)$	$\tan(30°)$	$\tan(45°)$	$\tan(60°)$	$\tan(90°)$
0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	undefined

Problem Set

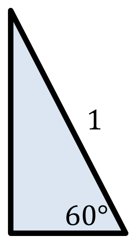
1. Label the missing side lengths, and find the value of $\tan(\theta)$ in the following right triangles.
   a. $\theta = 30°$



   b. $\theta = 45°$



c. $\theta = 60°$



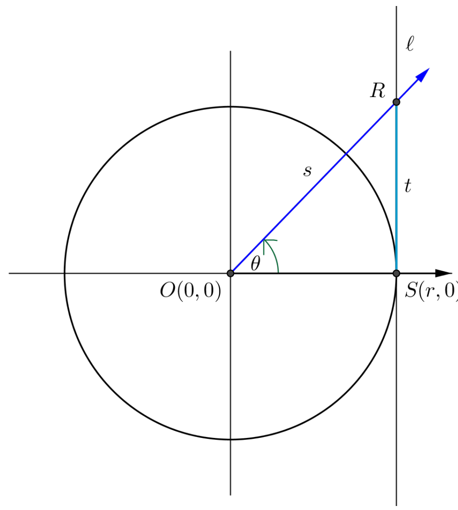
2. Let θ be any real number. In the Cartesian plane, rotate the initial ray by θ degrees about the origin. Intersect the resulting terminal ray with the unit circle to get point $P(x_\theta, y_\theta)$.

a. Complete the table by finding the slope of the line through the origin and the point P.

θ	Slope	θ	Slope
0		180	
30		210	
45		225	
60		240	
90		270	
120		300	
135		315	
150		330	

b. Explain how these slopes are related to the tangent function.

3. Consider the following diagram of a circle of radius r centered at the origin. The line ℓ is tangent to the circle at $S(r,0)$, so ℓ is perpendicular to the x-axis.



a. If $r = 1$, then state the value of t in terms of one of the trigonometric functions.

b. If r is any positive value, then state the value of t in terms of one of the trigonometric functions.

For the given values of r and θ, find t.

c. $\theta = 30, r = 2$

d. $\theta = 45, r = 2$

e. $\theta = 60, r = 2$

f. $\theta = 45, r = 4$

g. $\theta = 30, r = 3.5$

h. $\theta = 0, r = 9$

i. $\theta = 90, r = 5$

j. $\theta = 60, r = \sqrt{3}$

k. $\theta = 30, r = 2.1$

l. $\theta = A, r = 3$

m. $\theta = 30, r = b$

n. Knowing that $\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$, for $r = 1$, find the value of s in terms of one of the trigonometric functions.

4. Using what you know of the tangent function, show that $-\tan(\theta) = \tan(-\theta)$ for $\theta \neq 90 + 180k$, for all integers k.

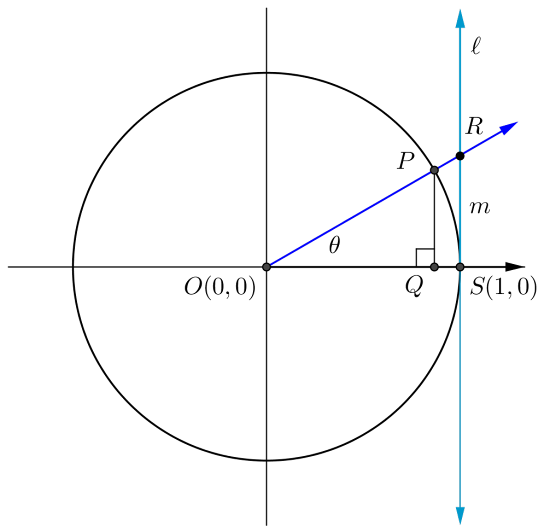
Lesson 7: Secant and the Co-Functions

Classwork

Opening Exercise

Give the measure of each segment below in terms of a trigonometric function.

$OQ =$ ___________ $PQ =$ ____________ $RS =$ ____________



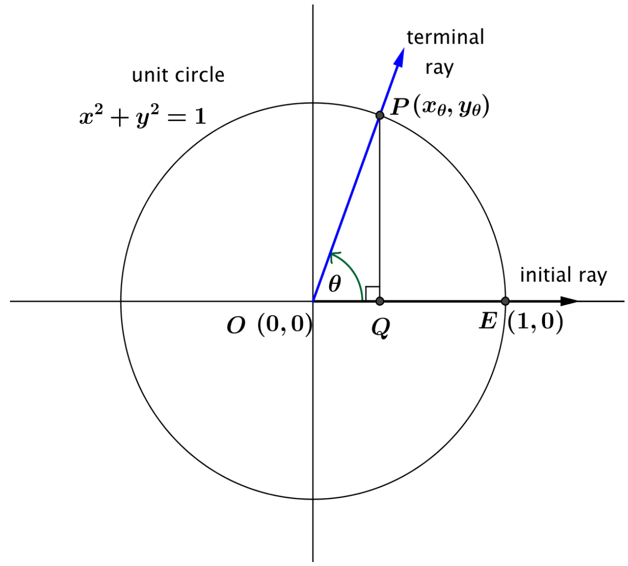
Example 1

Use similar triangles to find the value of $\sec(\theta)$ in terms of one other trigonometric function.

Exercise 1

A definition of the secant function is offered below. Answer the questions to better understand this definition and the domain and range of this function. Be prepared to discuss your responses with others in your class.

> Let θ be any real number.
>
> In the Cartesian plane, rotate the non-negative x-axis by θ degrees about the origin. Intersect this new ray with the unit circle to get a point (x_θ, y_θ).
>
> If $x_\theta \neq 0$, then the value of $\sec(\theta)$ is $\frac{1}{x_\theta}$.
>
> Otherwise, $\sec(\theta)$ is undefined.
>
> In terms of the cosine function, $\sec(\theta) = \frac{1}{\cos(\theta)}$ for $\cos(\theta) \neq 0$.



a. What is the domain of the secant function?

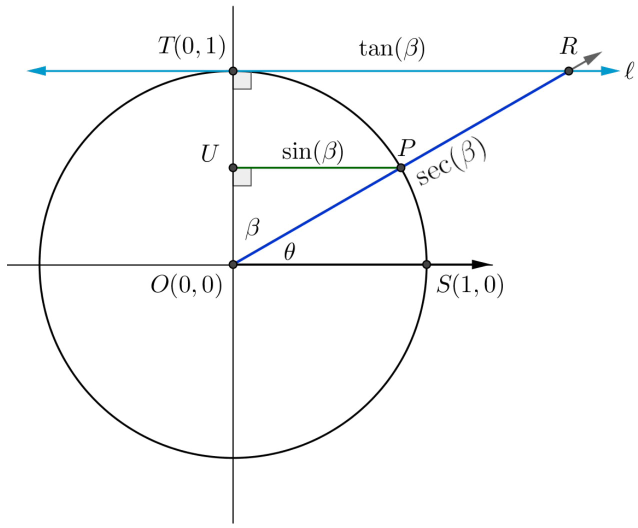
b. The domains of the secant and tangent functions are the same. Why?

c. What is the range of the secant function? How is this range related to the range of the cosine function?

d. Is the secant function a periodic function? If so, what is its period?

Exercise 2

In the diagram below, the blue line is tangent to the unit circle at $(0,1)$.



a. How does this diagram compare to the one given in the Opening Exercise?

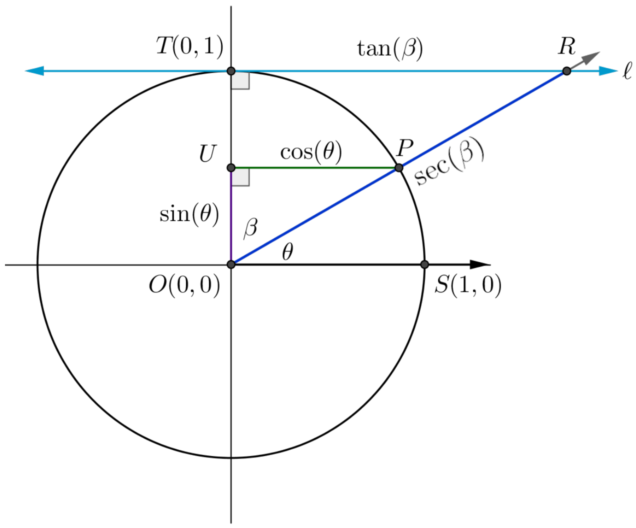
b. What is the relationship between β and θ?

c. Which segment in the figure has length $\sin(\theta)$? Which segment has length $\cos(\theta)$?

d. Which segment in the figure has length $\sin(\beta)$? Which segment has length $\cos(\beta)$?

e. How can you write $\sin(\theta)$ and $\cos(\theta)$ in terms of the trigonometric functions of β?

Example 2



The blue line is tangent to the circle at $(0,1)$.

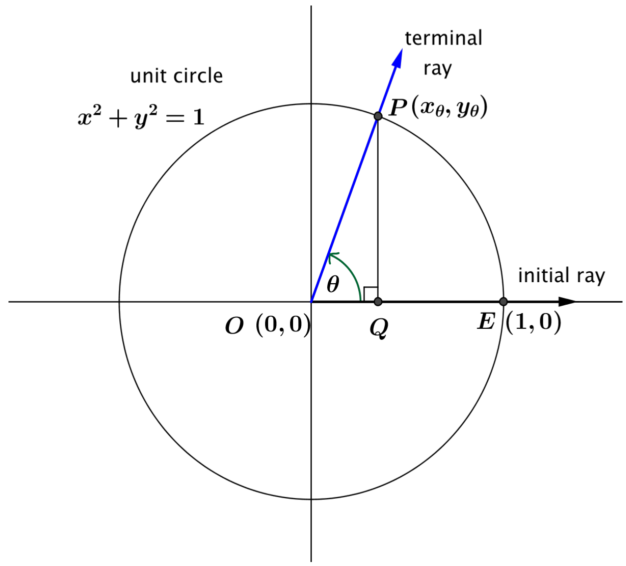
a. If two angles are complements with measures β and θ as shown in the diagram at right, use similar triangles to show that $\sec(\beta) = \frac{1}{\sin(\theta)}$.

b. If two angles are complements with measures β and θ as shown in the diagram above, use similar triangles to show that $\tan(\beta) = \frac{1}{\tan(\theta)}$.

Discussion

Descriptions of the cosecant and cotangent functions are offered below. Answer the questions to better understand the definitions and the domains and ranges of these functions. Be prepared to discuss your responses with others in your class.

> Let θ be any real number such that $\theta \neq 180k$ for all integers k.
>
> In the Cartesian plane, rotate the initial ray by θ degrees about the origin. Intersect the resulting terminal ray with the unit circle to get a point (x_θ, y_θ).
>
> The value of $\csc(\theta)$ is $\frac{1}{y_\theta}$.
>
> The value of $\cot(\theta)$ is $\frac{x_\theta}{y_\theta}$.



The secant, cosecant, and cotangent functions are often referred to as reciprocal functions. Why do you think these functions are so named?

Why is the domain of these functions restricted?

The domains of the cosecant and cotangent functions are the same. Why?

What is the range of the cosecant function? How is this range related to the range of the sine function?

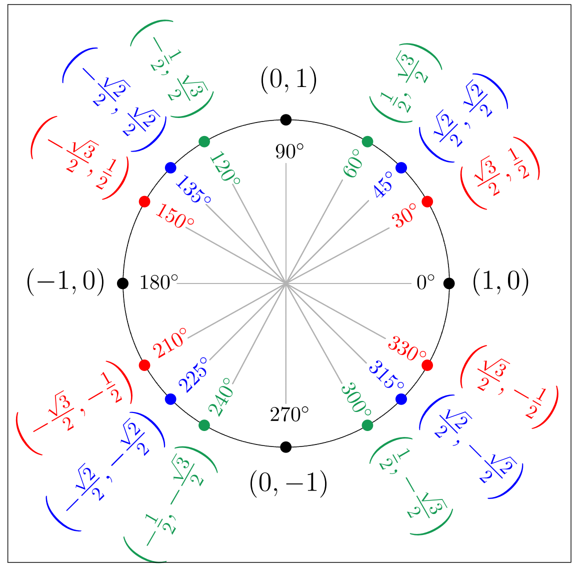
What is the range of the cotangent function? How is this range related to the range of the tangent function?

Let θ be any real number. In the Cartesian plane, rotate the initial ray by θ degrees about the origin. Intersect the resulting terminal ray with the unit circle to get a point (x_θ, y_θ). Then:

Function	Value	For any θ such that…	Formula
Sine	y_θ	θ is a real number	
Cosine	x_θ	θ is a real number	
Tangent	$\frac{y_\theta}{x_\theta}$	$\theta \neq 90 + 180k$, for all integers k	$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$
Secant	$\frac{1}{x_\theta}$	$\theta \neq 90 + 180k$, for all integers k	$\sec(\theta) = \frac{1}{\cos(\theta)}$
Cosecant	$\frac{1}{y_\theta}$	$\theta \neq 180k$, for all integers k	$\csc(\theta) = \frac{1}{\sin(\theta)}$
Cotangent	$\frac{x_\theta}{y_\theta}$	$\theta \neq 180k$, for all integers k	$\cot(\theta) = \frac{\cos(\theta)}{\sin(\theta)}$

Problem Set

1. Use the reciprocal interpretations of $\sec(\theta)$, $\csc(\theta)$, and $\cot(\theta)$ and the unit circle provided to complete the table.



θ	$\sec(\theta)$	$\csc(\theta)$	$\cot(\theta)$
0			
30			
45			
60			
90			
120			
180			
225			
240			
270			
315			
330			

2. Find the following values from the information given.
   a. $\sec(\theta)$; $\cos(\theta) = 0.3$
   b. $\csc(\theta)$; $\sin(\theta) = -0.05$
   c. $\cot(\theta)$; $\tan(\theta) = 1000$
   d. $\sec(\theta)$; $\cos(\theta) = -0.9$
   e. $\csc(\theta)$; $\sin(\theta) = 0$
   f. $\cot(\theta)$; $\tan(\theta) = -0.0005$

3. Choose three θ values from the table in Problem 1 for which $\sec(\theta)$, $\csc(\theta)$, and $\tan(\theta)$ are defined and not zero. Show that for these values of θ, $\frac{\sec(\theta)}{\csc(\theta)} = \tan(\theta)$.

4. Find the value of $\sec(\theta)\cos(\theta)$ for the following values of θ.
   a. $\theta = 120$
   b. $\theta = 225$
   c. $\theta = 330$
   d. Explain the reasons for the pattern you see in your responses to parts (a)–(c).

5. Draw a diagram representing the two values of θ between 0 and 360 so that $\sin(\theta) = -\frac{\sqrt{3}}{2}$. Find the values of $\tan(\theta)$, $\sec(\theta)$, and $\csc(\theta)$ for each value of θ.

6. Find the value of $\left(\sec(\theta)\right)^2 - \left(\tan(\theta)\right)^2$ when $\theta = 225$.

7. Find the value of $\left(\csc(\theta)\right)^2 - \left(\cot(\theta)\right)^2$ when $\theta = 330$.

Extension Problems:

8. Using the descriptions $\sec(\theta) = \frac{1}{\cos(\theta)}$, $\csc(\theta) = \frac{1}{\sin(\theta)}$, and $\cot(\theta) = \frac{1}{\tan(\theta)}$, show that $\sec(\theta)/\csc(\theta) = \tan(\theta)$. where these functions are defined and not zero.

9. Tara showed that $\frac{\sec(\theta)}{\csc(\theta)} = \tan(\theta)$, for values of θ for which the functions are defined and $csc(\theta) \neq 0$, and then concluded that $\sec(\theta) = \sin(\theta)$ and $\csc(\theta) = \cos(\theta)$. Explain what is wrong with her reasoning.

10. From Lesson 6, Ren remembered that the tangent function is odd, meaning that $-\tan(\theta) = \tan(-\theta)$ for all θ in the domain of the tangent function. He concluded because of the relationship between the secant function, cosecant function, and tangent function developed in Problem 9, it is impossible for both the secant and the cosecant functions to be odd. Explain why he is correct.

Lesson 8: Graphing the Sine and Cosine Functions

Classwork

Exploratory Challenge 1

Your group will be graphing: $f(\theta) = \sin(\theta)$ $g(\theta) = \cos(\theta)$

The circle on the next page is a unit circle, meaning that the length of the radius is one unit.

1. Mark axes on the poster board, with a horizontal axis in the middle of the board and a vertical axis near the left edge, as shown.

2. Measure the radius of the circle using a ruler. Use the length of the radius to mark 1 and -1 on the vertical axis.
3. Wrap the yarn around the circumference of the circle starting at 0. Mark each $15°$ increment on the yarn with the marker. Unwind the yarn and lay it on the horizontal axis. Transfer the marks on the yarn to corresponding increments on the horizontal axis. Label these marks as $0, 15, 30, \ldots, 360$.
4. We will record the number of degrees of rotation θ on the horizontal axis of the graph, and we will record the value of either $\sin(\theta)$ or $\cos(\theta)$ on the vertical axis. Notice that the scale is wildly different on the vertical and horizontal axes.
5. If you are graphing $g(\theta) = \cos(\theta)$: For each θ marked on your horizontal axis, beginning at 0, use the spaghetti to measure the *horizontal* displacement from the vertical axis to the relevant point on the unit circle. The horizontal displacement is the value of the cosine function. Break the spaghetti to mark the correct length, and place it vertically at the appropriate tick mark on the horizontal axis.
6. If you are graphing $f(\theta) = \sin(\theta)$: For each θ marked on your horizontal axis, beginning at 0, use the spaghetti to measure the *vertical* displacement from the horizontal to the relevant point on the unit circle. The vertical displacement is the value of the sine function. Break the spaghetti to mark the correct length, and place it vertically at the appropriate tick mark on the horizontal axis.
7. Remember to place the spaghetti below the horizontal axis when the value of the sine function or the cosine function is negative. Glue each piece of spaghetti in place.
8. Draw a smooth curve that connects the points at the end of each piece of spaghetti.

Exploratory Challenge 2

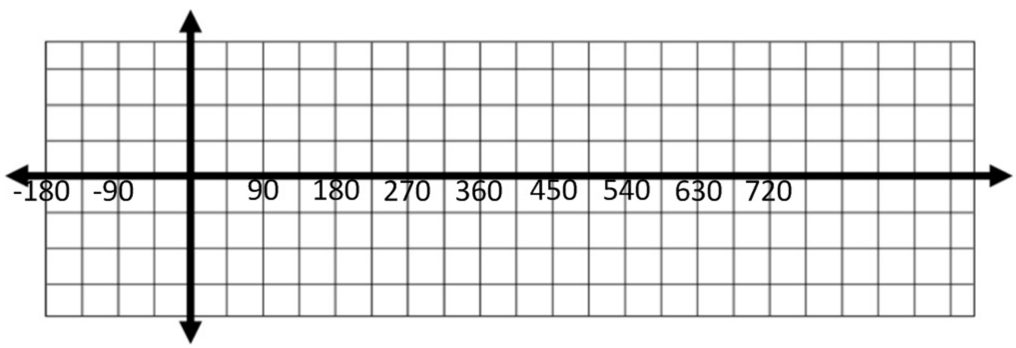
Part I: Consider the function $f(\theta) = \sin(\theta)$.

a. Complete the following table by using the special values learned in Lesson 4. Give values as an approximation to one decimal place.

θ	0	30	45	60	90	120	135	150	180
$\sin(\theta)$									

θ	210	225	240	270	300	315	330	360
$\sin(\theta)$								

b. Using the values in the table, sketch the graph of the sine function on the interval $[0,360]$.



c. Extend the graph of the sine function above so that it is graphed on the interval from $[-180,720]$.

d. For the interval $[-180,720]$, describe the values of θ at which the sine function has relative maxima and minima.

e. For the interval $[-180,720]$, describe the values of θ for which the sine function is increasing and decreasing.

f. For the interval $[-180,720]$, list the values of θ at which the graph of the sine function crosses the horizontal axis.

g. Describe the end behavior of the sine function.

h. Based on the graph, is sine an odd function, even function, or neither? How do you know?

i. Describe how the sine function repeats.

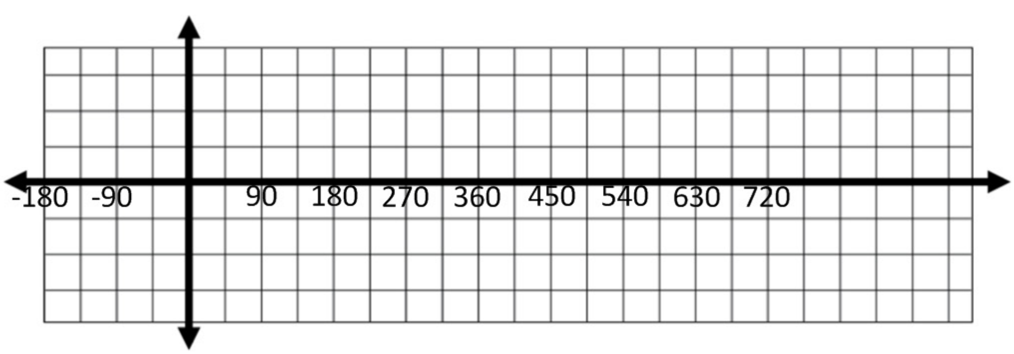
Part II: Consider the function $g(\theta) = \cos(\theta)$.

a. Complete the following table giving answers as approximations to one decimal place.

θ	0	30	45	60	90	120	135	150	180
$\cos(\theta)$									

θ	210	225	240	270	300	315	330	360
$\cos(\theta)$								

b. Using the values in the table, sketch the graph of the cosine function on the interval $[0,360]$.



c. Extend the graph of the cosine function above so that it is graphed on the interval from $[-180, 720]$.

d. For the interval $[-180, 270]$, describe the values of θ at which the cosine function has relative maxima and minima.

e. For the interval $[-180, 720]$, describe the values of θ for which the cosine function is increasing and decreasing.

f. For the interval $[-180, 720]$, list the values of θ at which the graph of the cosine function crosses the horizontal axis.

g. Describe the end behavior of the graph of the cosine function.

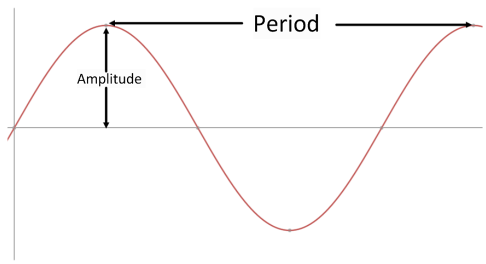
h. Based on the graph, is cosine an odd function, even function, or neither? How do you know?

i. Describe how the cosine function repeats.

j. How are the sine function and the cosine function related to each other?

Lesson Summary

- A function f whose domain is a subset of the real numbers is said to be *periodic with period* $P > 0$ if the domain of f contains $x + P$ whenever it contains x, and if $f(x + P) = f(x)$ for all real numbers x in its domain.
- If a least positive number P exists that satisfies this equation, it is called the *fundamental period* or, if the context is clear, just the *period* of the function.
- The *amplitude* of the sine or cosine function is the average of the maximum value and the minimum value of the function.



Problem Set

1. Graph the sine function on the interval $[-360, 360]$ showing all key points of the graph (horizontal and vertical intercepts and maximum and minimum points). Then, use the graph to answer each of the following questions.

   a. On the interval $[-360,360]$, what are the relative minima of the sine function? Why?

   b. On the interval $[-360,360]$, what are the relative maxima of the sine function? Why?

   c. On the interval $[-360, 360]$, for what values of θ is $\sin(\theta) = 0$? Why?

   d. If we continued to extend the graph in either direction, what would it look like? Why?

   e. Arrange the following values in order from smallest to largest by using their location on the graph.

   $\sin(170°)$ $\sin(85°)$ $\sin(-85°)$ $\sin(200°)$

   f. On the interval $(90, 270)$, is the graph of the sine function increasing or decreasing? Based on that, name another interval not included in $(90, 270)$ where the sine function must have the same behavior.

2. Graph the cosine function on the interval $[-360, 360]$ showing all key points of the graph (horizontal and vertical intercepts and maximum and minimum points). Then, use the graph to answer each of the following questions.

   a. On the interval $[-360,360]$, what are the relative minima of the cosine function? Why?

   b. On the interval $[-360,360]$, what are the relative maxima of the cosine function? Why?

   c. On the interval $[-360, 360]$, for what values of θ is $\cos(\theta) = 0$? Why?

   d. If we continued to extend the graph in either direction, what would it look like? Why?

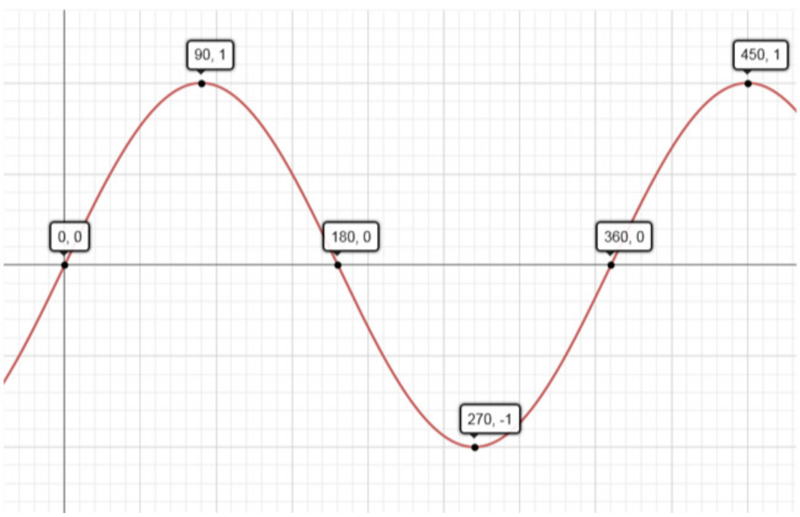
   e. What can be said about the end behavior of the cosine function?

   f. Arrange the following values in order from smallest to largest by using their location on the graph.

   $\cos(135°)$ $\cos(85°)$ $\cos(-15°)$ $\cos(190°)$

3. Write a paragraph comparing and contrasting the sine and cosine functions using their graphs and end behavior.

4. Use the graph of the sine function given below to answer the following questions.



   a. Desmond is trying to determine the value of $\sin(45°)$. He decides that since 45 is halfway between 0 and 90 that $\sin(45°) = \frac{1}{2}$. Use the graph to show him that he is incorrect.

   b. Using the graph, complete each sentence by filling in $>$, $<$, or $=$.

      i. $\sin(250°)$ $\boxed{}$ $\sin(290°)$

      ii. $\sin(25°)$ $\boxed{}$ $\sin(85°)$

      iii. $\sin(140°)$ $\boxed{}$ $\sin(160°)$

   c. On the interval $[0,450]$, list the values of θ such that $\sin(\theta) = \frac{1}{2}$.

   d. Explain why there are no values of θ such that $\sin(\theta) = 2$.

Lesson 9: Awkward! Who Chose the Number 360, Anyway?

Classwork

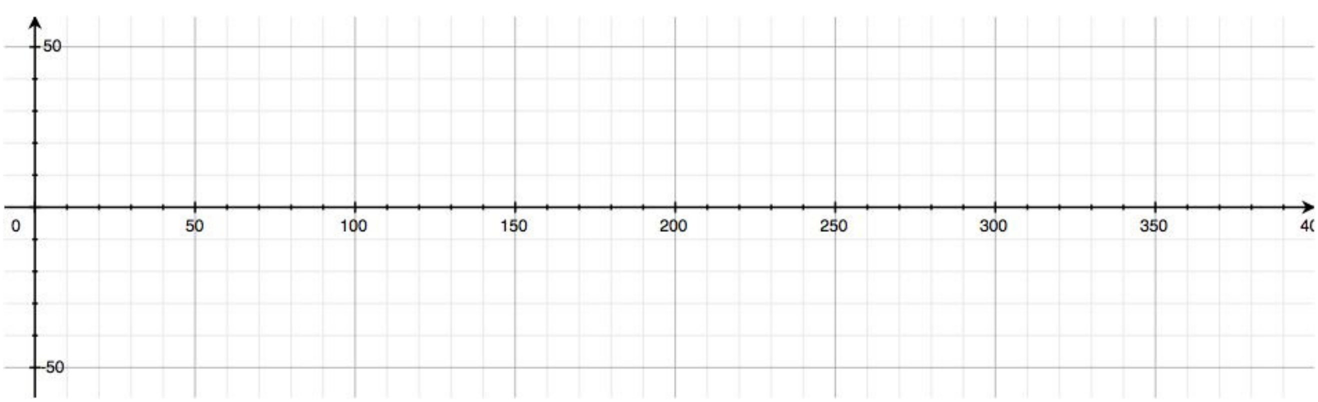
Opening Exercise

Let's construct the graph of the function $y = \sin(x)$, where x is the measure of degrees of rotation. In Lesson 5, we decided that the domain of the sine function is all real numbers and the range is $[-1,1]$. Use your calculator to complete the table below with values rounded to one decimal place, and then graph the function on the axes below. Be sure that your calculator is in degree mode.

x	$y = \sin(x)$
0	
30	
45	
60	
90	
120	

x	$y = \sin(x)$
135	
150	
180	
210	
225	
240	

x	$y = \sin(x)$
270	
300	
315	
330	
360	



Exercises 1–5

Set your calculator's viewing window to $0 \leq x \leq 10$ and $-2.4 \leq y \leq 2.4$, and be sure that your calculator is in degree mode. Plot the following functions in the same window:

$y = \sin(x)$

$y = \sin(2x)$

$y = \sin(10x)$

$y = \sin(50x)$

$y = \sin(100x)$

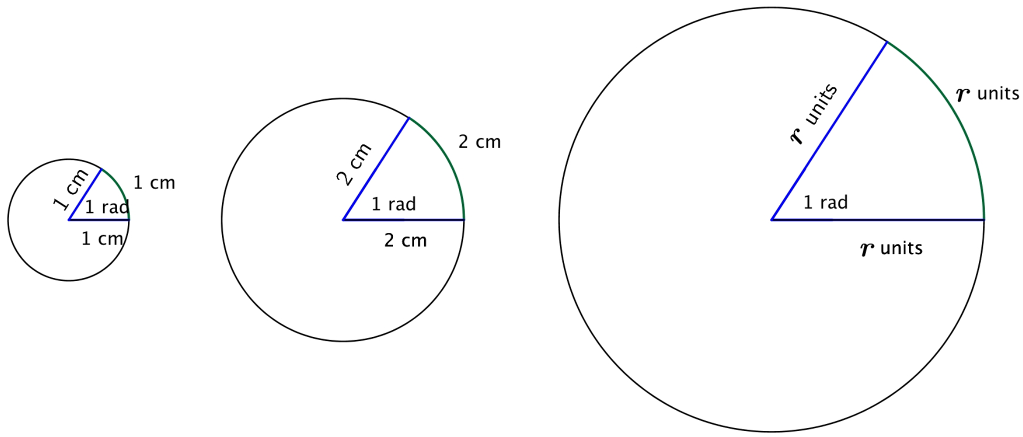
1. This viewing window was chosen because it has close to the same scale in the horizontal and vertical directions. In this viewing window, which of the five transformed sine functions most clearly shows the behavior of the sine function?

2. Describe the relationship between the steepness of the graph $y = \sin(kx)$ near the origin and the value of k.

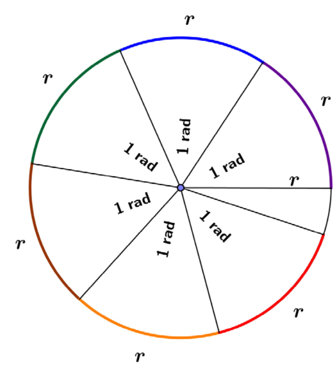
3. Since we can control the steepness of the graph $y = \sin(kx)$ near the origin by changing the value of k, how steep might we want this graph to be? What is your "favorite" positive slope for a line through the origin?

4. In the same viewing window on your calculator, plot $y = x$ and $y = \sin(kx)$ for some value of k. Experiment with your calculator to find a value of k so that the steepness of $y = \sin(kx)$ matches the slope of the line $y = x$ near the origin. You may need to change your viewing window to $0 \leq x \leq 2$ and $0 \leq y \leq 1$ to determine the best value of k.

- A circle is defined by a point and a radius. If we start with a circle of any radius, and look at a sector of that circle with an arc length equal to the length of the radius, then the central angle of that sector is always the same size. We define a *radian* to be the measure of that central angle and denote it by 1 rad.



- Thus, a radian measures how far one radius will "wrap around" the circle. For any circle, it takes $2\pi \approx 6.3$ radius lengths to wrap around the circumference. In the figure at right, 6 radius lengths are shown around the circle, with roughly 0.3 radius lengths left over.



5. Use a protractor that measures angles in degrees to find an approximate degree measure for an angle with measure 1 rad. Use one of the figures above.

Examples 1–4

1. Convert from degrees to radians: $45°$

2. Convert from degrees to radians: $33°$

3. Convert from radians to degrees: $-\frac{\pi}{3}$ rad

4. Convert from radians to degrees: $\frac{19\pi}{17}$ rad

Exercises 6–7

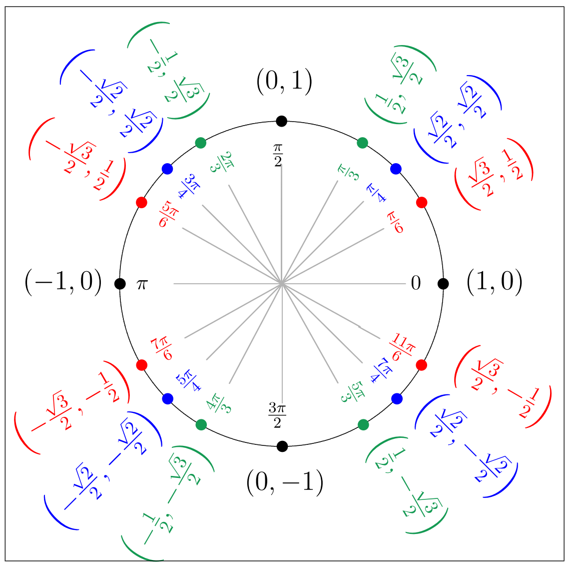
6. Complete the table below, converting from degrees to radians or from radians to degrees as necessary. Leave your answers in exact form, involving π.

Degrees	Radians
$45°$	$\frac{\pi}{4}$
$120°$	
	$-\frac{5\pi}{6}$
	$\frac{3\pi}{2}$
$450°$	
$x°$	
	x

7. On your calculator, graph the functions $y = x$ and $y = \sin\left(\frac{180}{\pi}x\right)$. What do you notice near the origin? What is the decimal approximation to the constant $\frac{180}{\pi}$ to one decimal place? Explain how this relates to what we've done in Exercise 4.

Lesson Summary

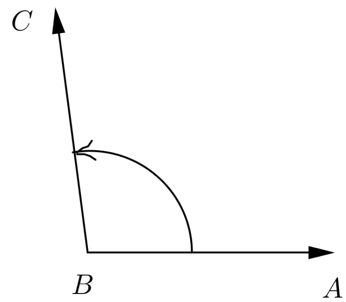
- A *radian* is the measure of the central angle of a sector of a circle with arc length of one radius length.
- There are 2π radians in a $360°$ rotation, also known as a *turn*, so we convert degrees to radians and radians to degrees by:
$$2\pi \text{ rad} = 1 \text{ turn} = 360\,°.$$
- **SINE FUNCTION (DESCRIPTION).** The *sine function*, $\sin: \mathbb{R} \to \mathbb{R}$, can be defined as follows: Let θ be any real number. In the Cartesian plane, rotate the initial ray by θ radians about the origin. Intersect the resulting terminal ray with the unit circle to get a point (x_θ, y_θ). The value of $\sin(\theta)$ is y_θ.
- **COSINE FUNCTION (DESCRIPTION).** The *cosine function*, $\cos: \mathbb{R} \to \mathbb{R}$, can be defined as follows: Let θ be any real number. In the Cartesian plane, rotate the initial ray by θ radians about the origin. Intersect the resulting terminal ray with the unit circle to get a point (x_θ, y_θ). The value of $\cos(\theta)$ is x_θ.



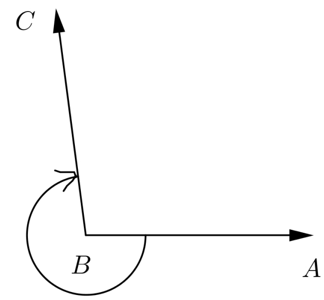
Problem Set

1. Use a radian protractor to measure the amount of rotation in radians of ray $\overrightarrow{BA}$ to $\overrightarrow{BC}$ in the indicated direction. Measure to the nearest 0.1 radian. Use negative measures to indicate clockwise rotation.

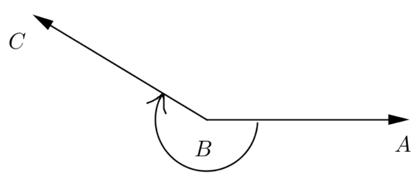
a.



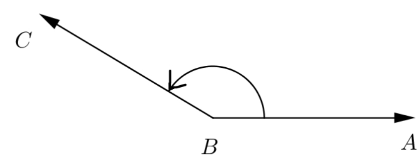
b.



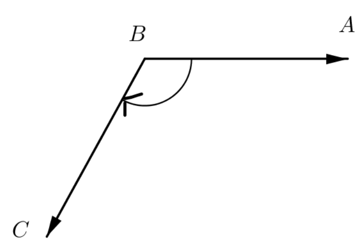
c.



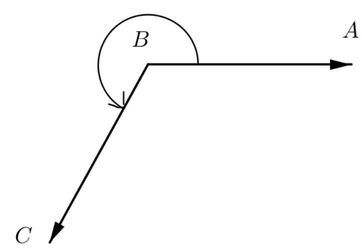
d.



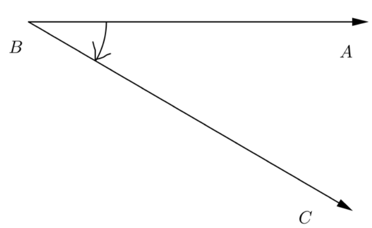
e.



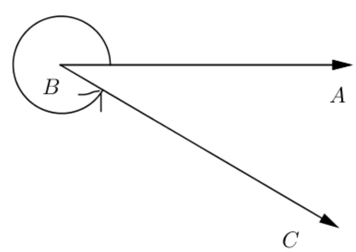
f.



g.



h.



2. Complete the table below, converting from degrees to radians. Where appropriate, give your answers in the form of a fraction of π.

Degrees	Radians
$90°$	
$300°$	
$-45°$	
$-315°$	
$-690°$	
$3\frac{3}{4}°$	
$90\pi°$	
$-\frac{45°}{\pi}$	

3. Complete the table below, converting from radians to degrees.

Radians	Degrees
$\frac{\pi}{4}$	
$\frac{\pi}{6}$	
$\frac{5\pi}{12}$	
$\frac{11\pi}{36}$	
$-\frac{7\pi}{24}$	
$-\frac{11\pi}{12}$	
49π	
$\frac{49\pi}{3}$	

4. Use the unit circle diagram from the end of the lesson and your knowledge of the six trigonometric functions to complete the table below. Give your answers in exact form, as either rational numbers or radical expressions.

θ	$\cos(\theta)$	$\sin(\theta)$	$\tan(\theta)$	$\cot(\theta)$	$\sec(\theta)$	$\csc(\theta)$
$\frac{\pi}{3}$						
$\frac{3\pi}{4}$						
$\frac{5\pi}{6}$						
0						
$-\frac{3\pi}{4}$						
$-\frac{7\pi}{6}$						
$-\frac{11\pi}{3}$						

5. Use the unit circle diagram from the end of the lesson and your knowledge of the sine, cosine, and tangent functions to complete the table below. Select values of θ so that $0 \leq \theta < 2\pi$.

θ	$\cos(\theta)$	$\sin(\theta)$	$\tan(\theta)$
	$\frac{1}{2}$		$-\sqrt{3}$
		$-\frac{\sqrt{2}}{2}$	1
	$-\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	
	-1		0
	0	-1	
		$-\frac{1}{2}$	$\frac{\sqrt{3}}{3}$

6. How many radians does the minute hand of a clock rotate through over 10 minutes? How many degrees?

7. How many radians does the minute hand of a clock rotate through over half an hour? How many degrees?

8. How many radians is an angle subtended by an arc of a circle with radius 4 cm if the intercepted arc has length 14 cm? How many degrees?

9. How many radians is the angle formed by the minute and hour hands of a clock when the clock reads $1:30$? How many degrees? (Hint: you must take into account that the hour hand is not directly on the '1'.)

10. How many radians is the angle formed by the minute and hour hands of a clock when the clock reads $5:45$? How many degrees?

11. How many degrees does the earth revolve on its axis each hour? How many radians?

12. The distance from the Equator to the North Pole is almost exactly $10{,}000$ km.
    a. Roughly how many kilometers is 1 degree of latitude?
    b. Roughly how many kilometers is 1 radian of latitude?

Lesson 10: Basic Trigonometric Identities from Graphs

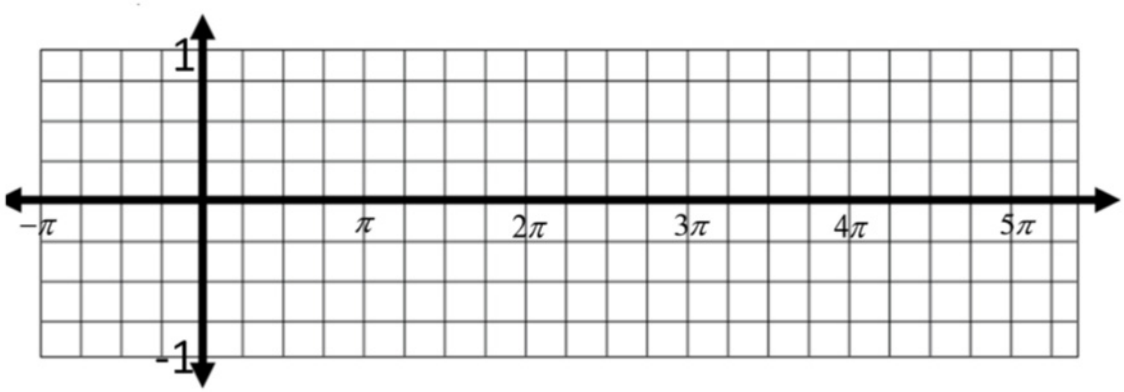
Classwork

Exploratory Challenge 1

Consider the function $f(x) = \sin(x)$ where x is measured in radians.

Graph $f(x) = \sin(x)$ on the interval $[-\pi, 5\pi]$ by constructing a table of values. Include all intercepts, relative maximum points, and relative minimum points of the graph. Then, use the graph to answer the questions that follow.

x	
$f(x)$	



a. Using one of your colored pencils, mark the point on the graph at each of the following pairs of x-values.

$$x = -\frac{\pi}{2} \text{ and } x = -\frac{\pi}{2} + 2\pi$$

$$x = \pi \text{ and } x = \pi + 2\pi$$

$$x = \frac{7\pi}{4} \text{ and } x = \frac{7\pi}{4} + 2\pi$$

b. What can be said about the y-values for each pair of x-values marked on the graph?

c. Will this relationship hold for any two x-values that differ by 2π? Explain how you know.

d. Based on these results, make a conjecture by filling in the blank below.

For any real number x, $\sin(x + 2\pi) =$ ____________________.

e. Test your conjecture by selecting another x-value from the graph and demonstrating that the equation from part (d) holds for that value of x.

f. How does the conjecture in part (d) support the claim that the sine function is a periodic function?

g. Use this identity to evaluate $\sin\left(\frac{13\pi}{6}\right)$.

h. Using one of your colored pencils, mark the point on the graph at each of the following pairs of x-values.

$x = -\frac{\pi}{4}$ and $x = -\frac{\pi}{4} + \pi$

$x = 2\pi$ and $x = 2\pi + \pi$

$x = \frac{5\pi}{2}$ and $x = \frac{5\pi}{2} + \pi$

i. What can be said about the y-values for each pair of x-values marked on the graph?

j. Will this relationship hold for any two x-values that differ by π? Explain how you know.

k. Based on these results, make a conjecture by filling in the blank below.

For any real number x, $\sin(x + \pi) =$ ________________.

l. Test your conjecture by selecting another x-value from the graph and demonstrating that the equation from part (k) holds for that value of x.

m. Is the following statement true or false? Use the conjecture from (k) to explain your answer.

$$\sin\left(\frac{4\pi}{3}\right) = -\sin\left(\frac{\pi}{3}\right)$$

n. Using one of your colored pencils, mark the point on the graph at each of the following pairs of x-values.

$x = -\frac{3\pi}{4}$ and $x = \frac{3\pi}{4}$
$x = -\frac{\pi}{2}$ and $x = \frac{\pi}{2}$

o. What can be said about the y-values for each pair of x-values marked on the graph?

p. Will this relationship hold for any two x-values with the same magnitude but opposite sign? Explain how you know.

q. Based on these results, make a conjecture by filling in the blank below.

For any real number x, $\sin(-x) =$ ________________.

r. Test your conjecture by selecting another x-value from the graph and demonstrating that the equation from part (q) holds for that value of x.

s. Is the sine function an odd function, even function, or neither? Use the identity from part (q) to explain.

t. Describe the x-intercepts of the graph of the sine function.

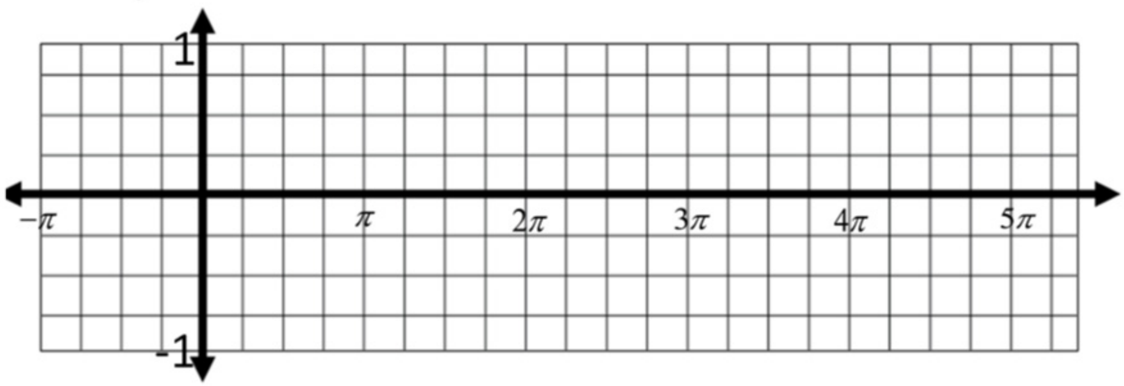
u. Describe the end behavior of the sine function.

Exploratory Challenge 2

Consider the function $g(x) = \cos(x)$ where x is measured in radians.

Graph $g(x) = \cos(x)$ on the interval $[-\pi, 5\pi]$ by constructing a table of values. Include all intercepts, relative maximum points, and relative minimum points. Then, use the graph to answer the questions that follow.

x	
$g(x)$	



a. Using one of your colored pencils, mark the point on the graph at each of the following pairs of $\boldsymbol{x}$-values.

$x = -\frac{\pi}{2}$ and $x = -\frac{\pi}{2} + 2\pi$

$x = \pi$ and $x = \pi + 2\pi$

$x = \frac{7\pi}{4}$ and $x = \frac{7\pi}{4} + 2\pi$

b. What can be said about the y-values for each pair of x-values marked on the graph?

c. Will this relationship hold for any two x-values that differ by 2π? Explain how you know.

d. Based on these results, make a conjecture by filling in the blank below.

For any real number x, $\cos(x + 2\pi) =$ ___________.

e. Test your conjecture by selecting another x-value from the graph and demonstrating that the equation from part (d) holds for that value of x.

f. How does the conjecture from part (d) support the claim that the cosine function is a periodic function?

g. Use this identity to evaluate $\cos\left(\frac{9\pi}{4}\right)$.

h. Using one of your colored pencils, mark the point on the graph at each of the following pairs of x-values.

$x = -\frac{\pi}{4}$ and $x = -\frac{\pi}{4} + \pi$

$x = 2\pi$ and $x = 2\pi + \pi$

$x = \frac{5\pi}{2}$ and $x = \frac{5\pi}{2} + \pi$

i. What can be said about the y-values for each pair of x-values marked on the graph?

j. Will this relationship hold for any two x-values that differ by π? Explain how you know.

k. Based on these results, make a conjecture by filling in the blank below.

For any real number x, $\cos(x + \pi) =$ ___________.

l. Test your conjecture by selecting another x-value from the graph and demonstrating that the equation from part (k) holds for that value of x.

m. Is the following statement true or false? Use the identity from part (k) to explain your answer.

$$\cos\left(\frac{5\pi}{3}\right) = -\cos\left(\frac{2\pi}{3}\right)$$

n. Using one of your colored pencils, mark the point on the graph at each of the following pairs of x-values.

$x = -\frac{3\pi}{4}$ and $x = \frac{3\pi}{4}$

$x = -\pi$ and $x = \pi$

o. What can be said about the y-values for each pair of x-values marked on the graph?

p. Will this relationship hold for any two x-values with the same magnitude and same sign? Explain how you know.

q. Based on these results, make a conjecture by filling in the blank below.

For any real number , $\cos(-x) =$ ____________.

r. Test your conjecture by selecting another x-value from the graph and demonstrating that the identity is true for that value of x.

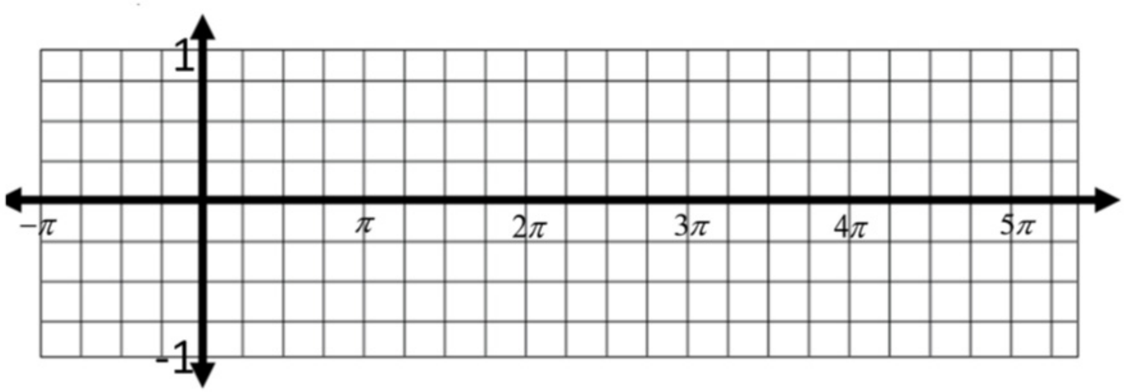
s. Is the cosine function an odd function, even function, or neither? Use the identity from part (n) to explain.

t. Describe the x-intercepts of the graph of the cosine function.

u. Describe the end behavior of $g(x) = \cos(x)$.

Exploratory Challenge 3

Graph both $f(x) = \sin(x)$ and $g(x) = \cos(x)$ on the graph below. Then, use the graphs to answer the questions that follow.



a. List ways in which the graphs of the sine and cosine functions are alike.

b. List ways in which the graphs of the sine and cosine functions are different.

c. What type of transformation would be required to make the graph of the sine function coincide with the graph of the cosine function?

d. What is the smallest possible horizontal translation required to make the graph of $f(x) = \sin(x)$ coincide with the graph of $g(x) = \cos(x)$?

e. What is the smallest possible horizontal translation required to make the graph of $g(x) = \cos(x)$ coincide with the graph of $f(x) = \sin(x)$?

f. Use your answers from parts (d) and (e) to fill in the blank below.

For any real number x, ____________ $= \cos\left(x - \frac{\pi}{2}\right)$.

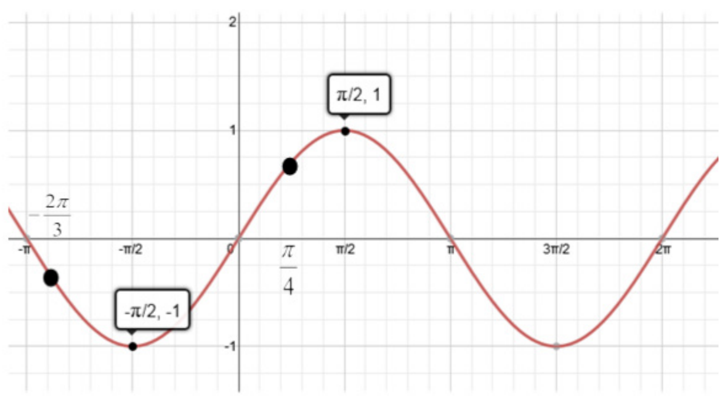
For any real number x, ____________ $\sin\left(x + \frac{\pi}{2}\right)$.

Problem Set

1. Describe the values of x for which each of the following is true.
   a. The cosine function has a relative maximum.
   b. The sine function has a relative maximum.

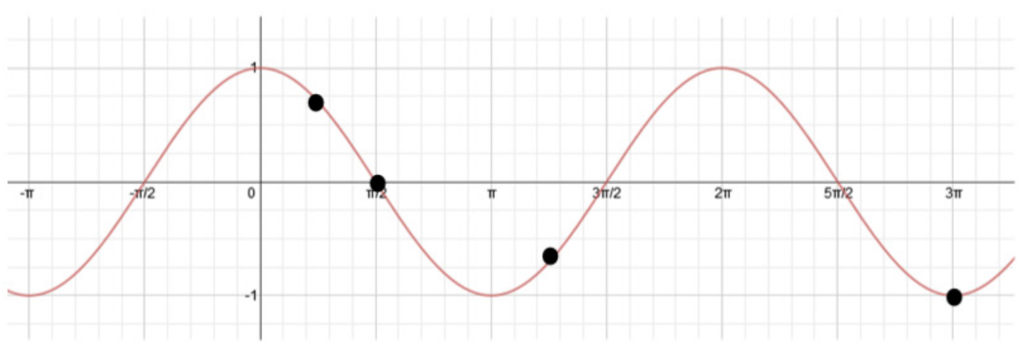
2. Without using a calculator, rewrite each of the following in order from least to greatest. Use the graph to explain your reasoning.

$$\sin\left(\frac{\pi}{2}\right) \qquad \sin\left(-\frac{2\pi}{3}\right) \qquad \sin\left(\frac{\pi}{4}\right) \qquad \sin\left(-\frac{\pi}{2}\right)$$



3. Without using a calculator, rewrite each of the following in order from least to greatest. Use the graph to explain your reasoning.

$$\cos\left(\frac{\pi}{2}\right) \qquad \cos\left(\frac{5\pi}{4}\right) \qquad \cos\left(\frac{\pi}{4}\right) \qquad \cos(5\pi)$$



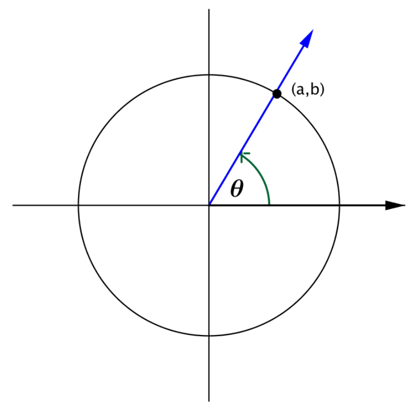
4. Evaluate each of the following without a calculator using a trigonometric identity when needed.

$$\cos\left(\frac{\pi}{6}\right) \qquad \cos\left(-\frac{\pi}{6}\right) \qquad \cos\left(\frac{7\pi}{6}\right) \qquad \cos\left(\frac{13\pi}{6}\right)$$

5. Evaluate each of the following without a calculator using a trigonometric identity when needed.

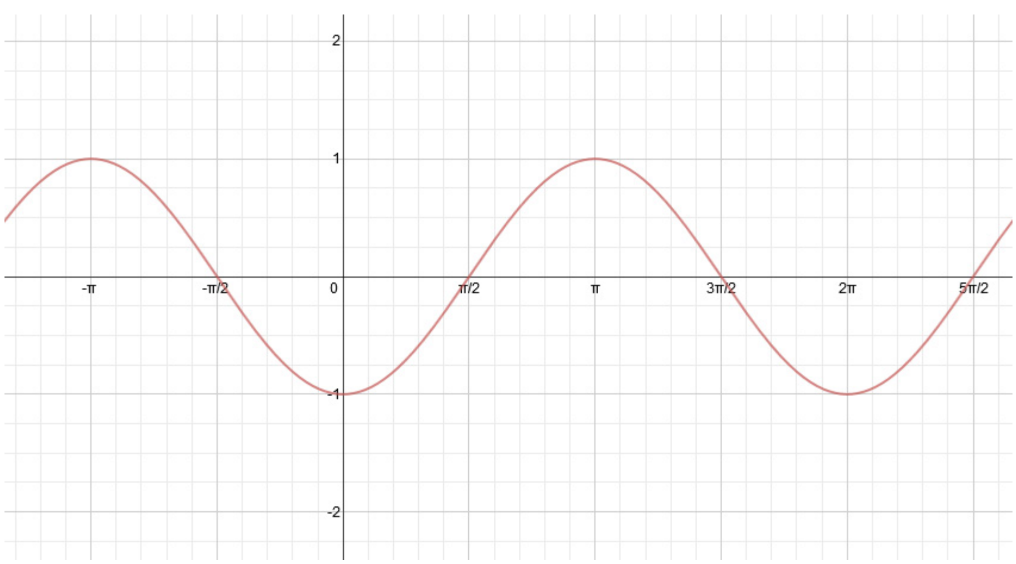
$$\sin\left(\frac{3\pi}{4}\right) \qquad \sin\left(\frac{11\pi}{4}\right) \qquad \sin\left(\frac{7\pi}{4}\right) \qquad \sin\left(\frac{-5\pi}{4}\right)$$

6. Use the rotation through θ radians shown to answer each of the following questions.
   a. Explain why $\sin(-\theta) = -\sin(\theta)$ for all real numbers θ.
   b. What symmetry does this identity demonstrate about the graph of $y = \sin(x)$?

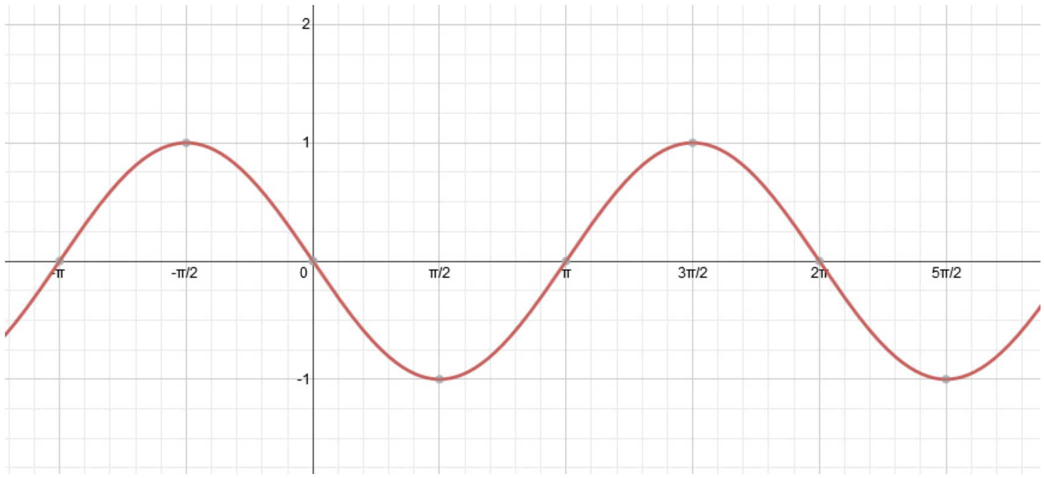


7. Use the same rotation shown in Problem 6 to answer each of the following questions.
   a. Explain why $\cos(-\theta) = \cos(\theta)$.
   b. What symmetry does this identity demonstrate about the graph of $y = \cos(x)$?

8. Find equations of two different functions that can be represented by the graph shown below—one sine and one cosine—using different horizontal transformations.



9. Find equations of two different functions that can be represented by the graph shown below—one sine and one cosine—using different horizontal translations.



Lesson 11: Transforming the Graph of the Sine Function

Classwork

Opening Exercise

Explore your assigned parameter in the sinusoidal function $f(x) = A\sin\big(\omega(x-h)\big) + k$. Select several different values for your assigned parameter and explore the effects of changing the parameter's value on the graph of the function compared to the graph of $f(x) = \sin(x)$. Record your observations in the table below. Include written descriptions and sketches of graphs.

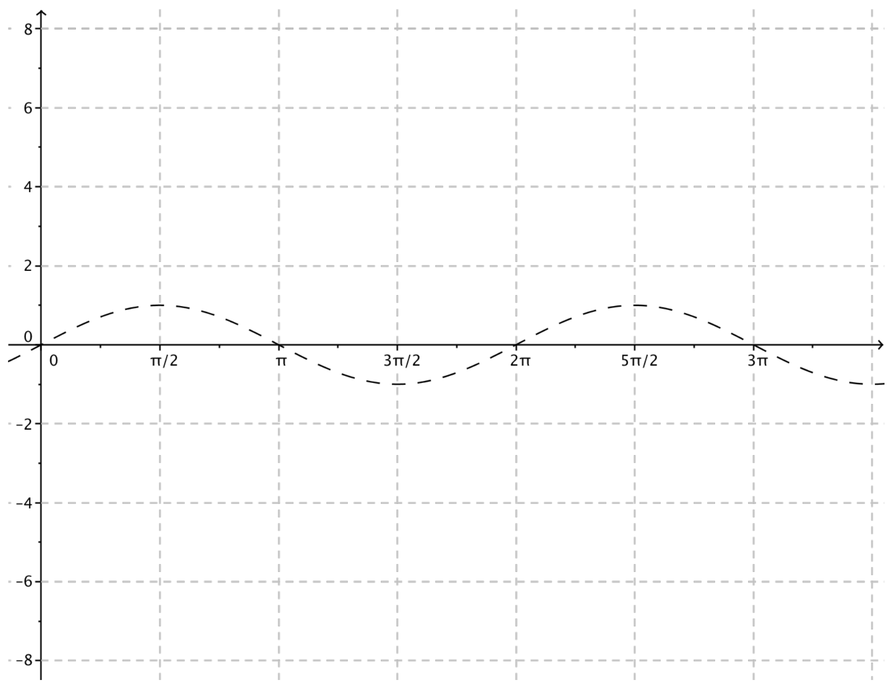
***A*-Team**	***ω*-Team**
$f(x) = A\sin(x)$ Suggested A values: $2, 3, 10, 0, -1, -2, \frac{1}{2}, \frac{1}{5}, -\frac{1}{3}$	$f(x) = \sin(\omega x)$ Suggested ω values: $2,3,5, \frac{1}{2}, \frac{1}{4}, 0, -1, -2, \pi, 2\pi, 3\pi, \frac{\pi}{2}, \frac{\pi}{4}$

***k*-Team**	***h*-Team**
$f(x) = \sin(x) + k$ Suggested k values: $2, 3, 10, 0, -1, -2, \frac{1}{2}, \frac{1}{5}, -\frac{1}{3}$	$f(x) = \sin(x - h)$ Suggested h values: $\pi, -\pi, \frac{\pi}{2}, -\frac{\pi}{4}, 2\pi, 2, 0, -1, -2, 5, -5$

Example

Graph the following function:

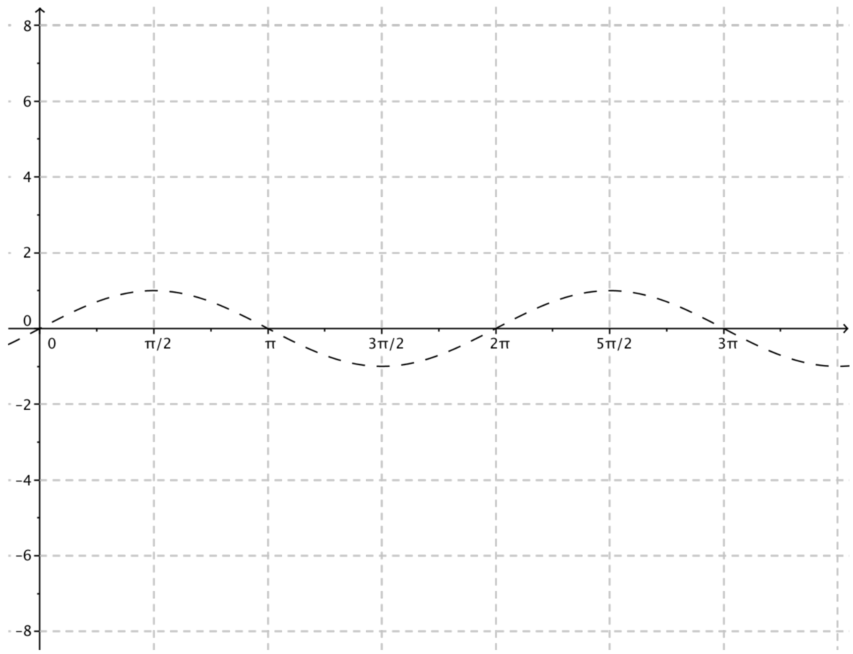
$$f(x) = 3\sin\left(4\left(x - \frac{\pi}{6}\right)\right) + 2.$$



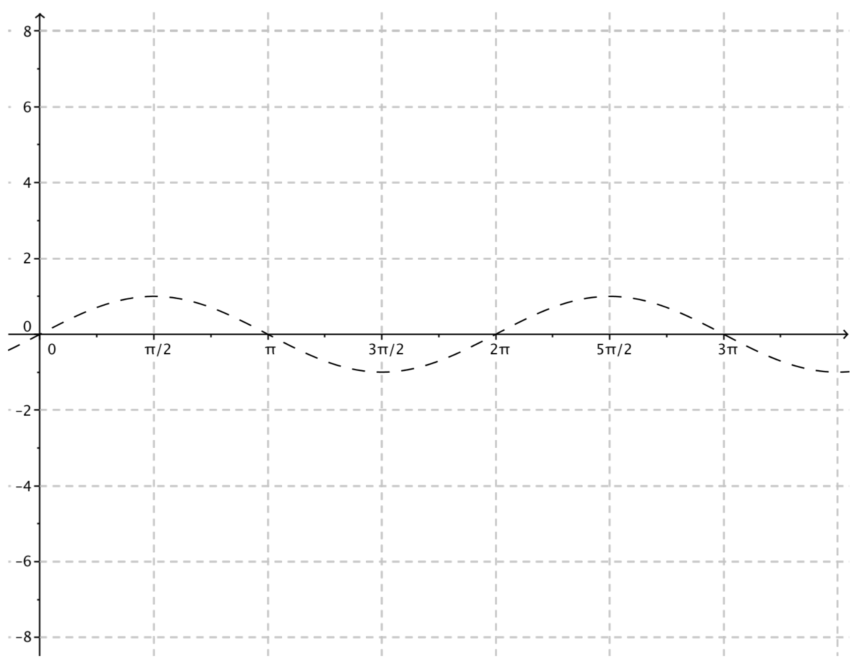
Exercise

For each function, indicate the amplitude, frequency, period, phase shift, vertical translation, and equation of the midline. Graph the function together with a graph of the sine function $f(x) = \sin(x)$ on the same axes. Graph at least one full period of each function.

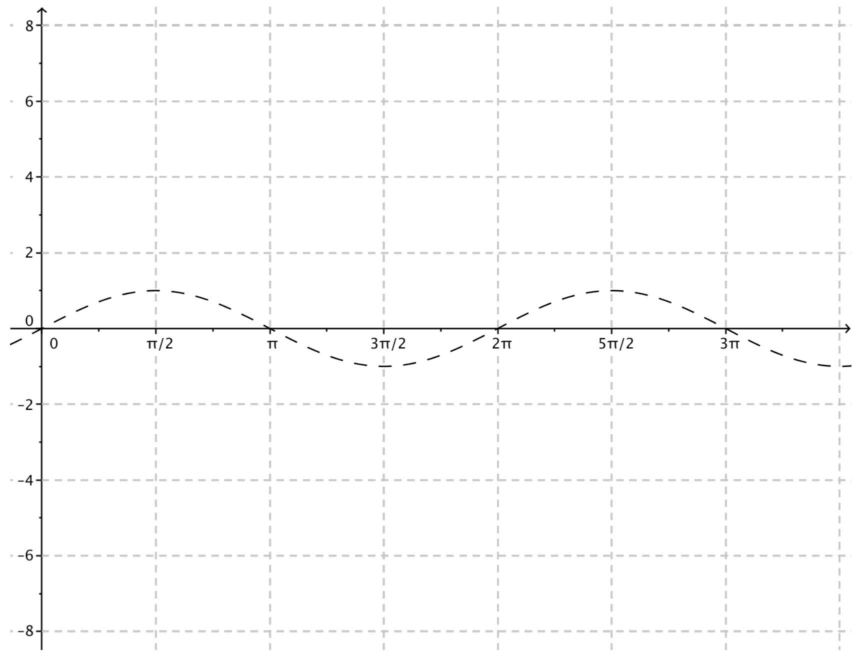
a. $g(x) = 3\sin(2x) - 1$



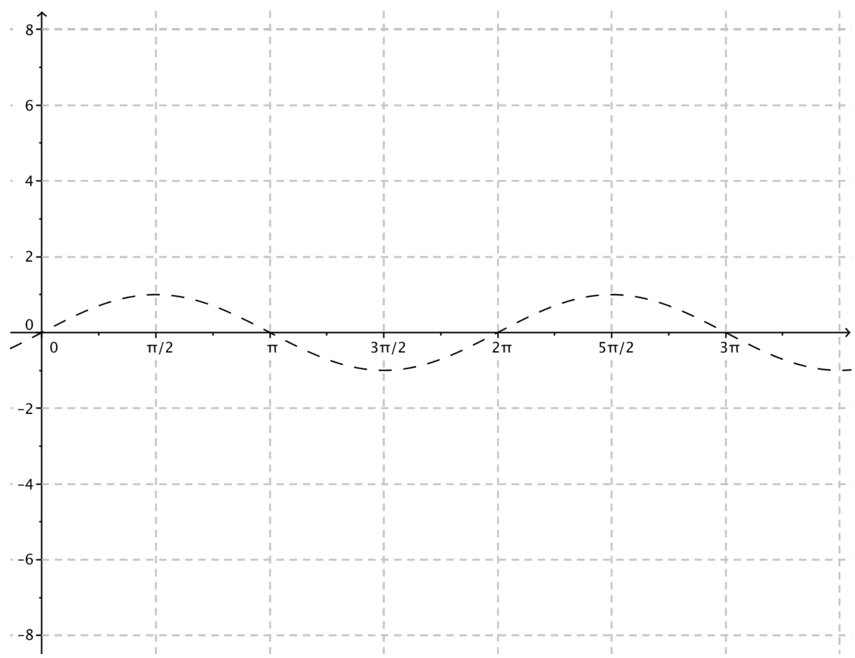
b. $g(x) = \frac{1}{2}\sin\left(\frac{1}{4}(x + \pi)\right)$



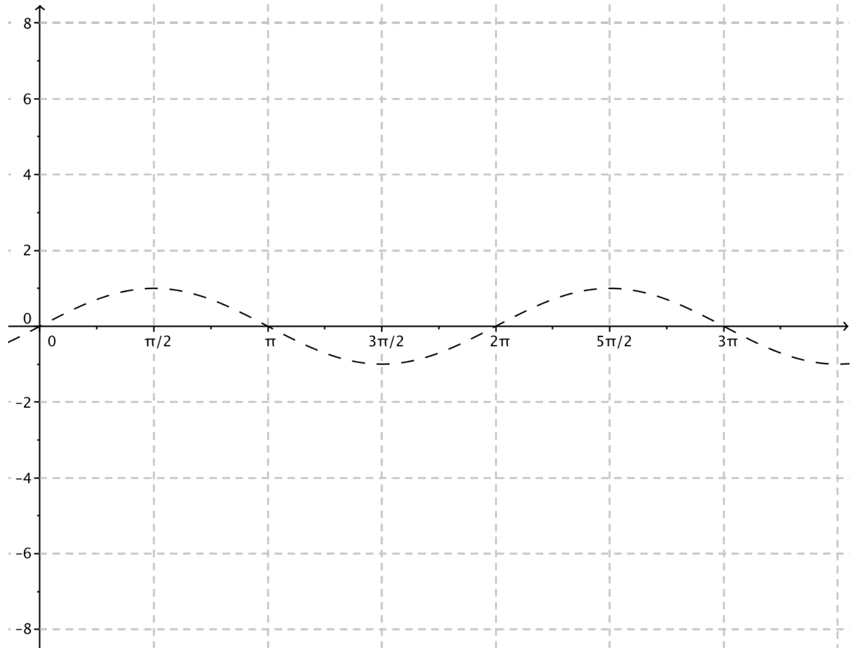
c. $g(x) = 5\sin(-2x) + 2$



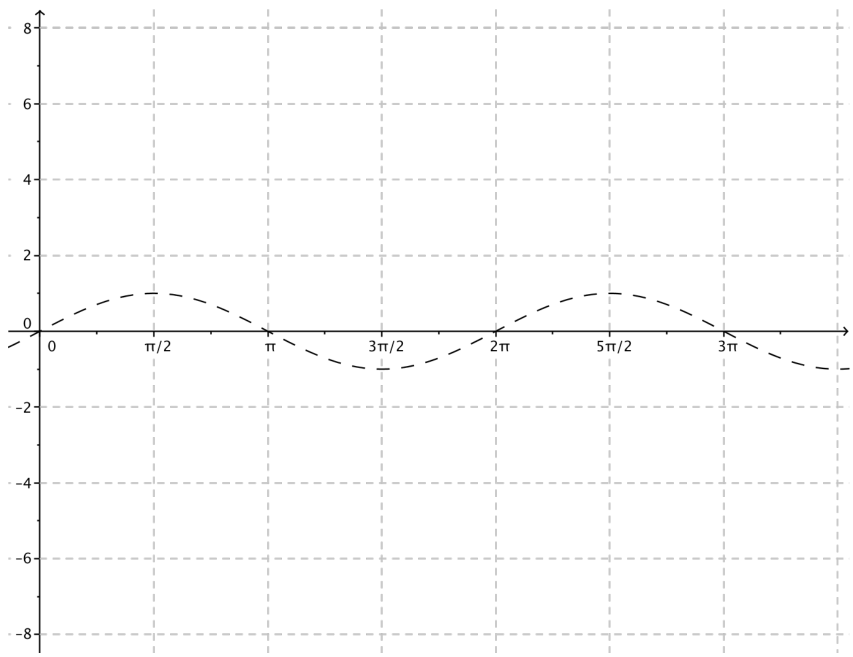
d. $g(x) = -2\sin\left(3\left(x + \frac{\pi}{6}\right)\right)$



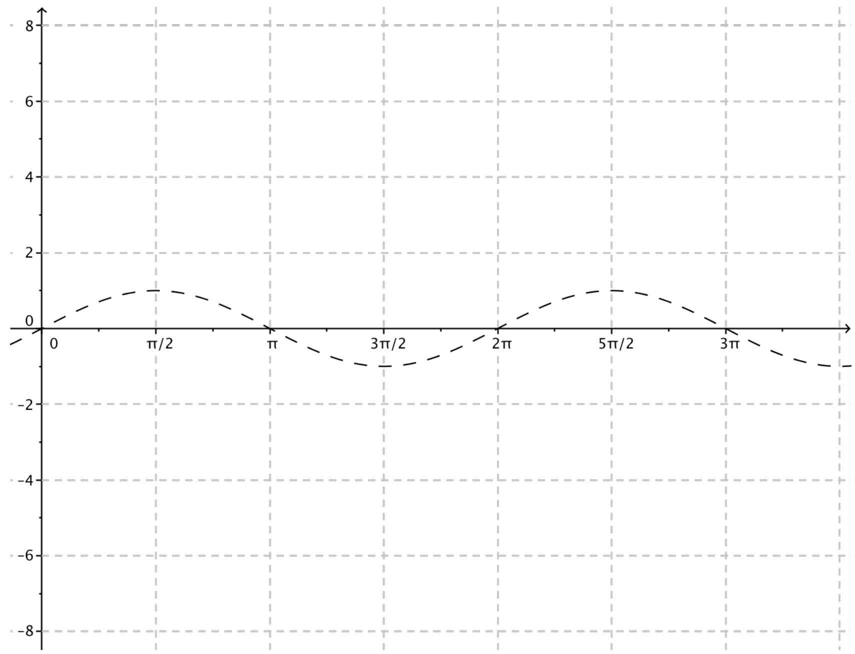
e. $g(x) = 3\sin(x + \pi) + 3$



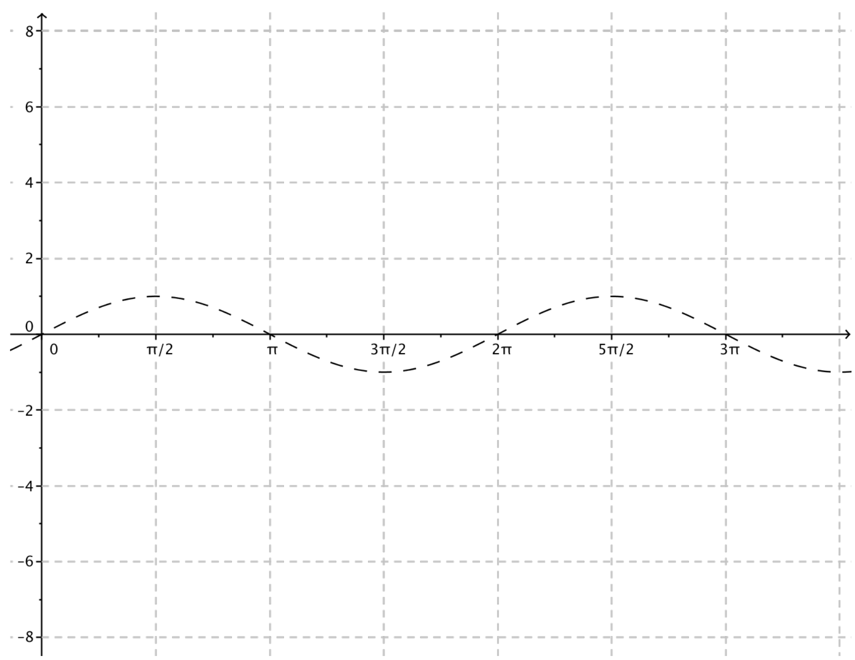
f. $g(x) = -\frac{2}{3}\sin(4x) - 3$



g. $g(x) = \pi \sin\left(\frac{x}{2}\right) + \pi$



h. $g(x) = 4\sin\left(\frac{1}{2}(x - 5\pi)\right)$



Lesson Summary

In this lesson, we investigated the effects of the parameters A, ω, h, and k on the graph of the function $f(x) = A\sin\big(\omega(x-h)\big) + k$.

- The graph of $y = k$ is the **midline**. The value of k determines the vertical translation of the graph compared to the graph of the sine function. If $k > 0$, then the graph shifts k units upwards. If $k < 0$, then the graph shifts k units downward.
- The **amplitude** of the function is $|A|$; the vertical distance from a maximum point to the midline of the graph is $|A|$.
- The **phase shift** is h. The value of h determines the horizontal translation of the graph from the graph of the sine function. If $h > 0$, the graph is translated h units to the right, and if $h < 0$, the graph is translated h units to the left.
- The **frequency** of the function is $f = \frac{|\omega|}{2\pi}$, and the period is $P = \frac{2\pi}{|\omega|}$. The **period** is the vertical distance between two consecutive maximal points on the graph of the function.

These parameters affect the graph of $f(x) = A\cos\big(\omega(x-h)\big) + k$ similarly.

Problem Set

1. For each function, indicate the amplitude, frequency, period, phase shift, horizontal, and vertical translations, and equation of the midline. Graph the function together with a graph of the sine function $f(x) = \sin(x)$ on the same axes. Graph at least one full period of each function. No calculators allowed.

   a. $g(x) = 3\sin\left(x - \frac{\pi}{4}\right)$

   b. $g(x) = 5\sin(4x)$

   c. $g(x) = 4\sin\left(3\left(x + \frac{\pi}{2}\right)\right)$

   d. $g(x) = 6\sin(2x + 3\pi)$ (Hint: First, rewrite the function in the form $g(x) = A\sin\big(\omega(x-h)\big)$.)

2. For each function, indicate the amplitude, frequency, period, phase shift, horizontal, and vertical translations, and equation of the midline. Graph the function together with a graph of the sine function $f(x) = \cos(x)$ on the same axes. Graph at least one full period of each function. No calculators allowed.

   a. $g(x) = \cos(3x)$

   b. $g(x) = \cos\left(x - \frac{3\pi}{4}\right)$

   c. $g(x) = 3\cos\left(\frac{x}{4}\right)$

   d. $g(x) = 3\cos(2x) - 4$

   e. $g(x) = 4\cos\left(\frac{\pi}{4} - 2x\right)$ (Hint: First, rewrite the function in the form $g(x) = A\cos\big(\omega(x-h)\big)$.)

3. For each problem, sketch the graph of the pairs of indicated functions on the same set of axes without using a calculator or other graphing technology.

   a. $f(x) = \sin(4x)$, $g(x) = \sin(4x) + 2$
   b. $f(x) = \sin\left(\frac{1}{2}x\right)$, $g(x) = 3\sin\left(\frac{1}{2}x\right)$
   c. $f(x) = \sin(-2x)$, $g(x) = \sin(-2x) - 3$
   d. $f(x) = 3\sin(x)$, $g(x) = 3\sin\left(x - \frac{\pi}{2}\right)$
   e. $f(x) = -4\sin(x)$, $g(x) = -4\sin\left(\frac{1}{3}x\right)$
   f. $f(x) = \frac{3}{4}\sin(x)$, $g(x) = \frac{3}{4}\sin(x - 1)$
   g. $f(x) = \sin(2x)$, $g(x) = \sin\left(2\left(x - \frac{\pi}{6}\right)\right)$
   h. $f(x) = 4\sin(x) - 3$, $g(x) = 4\sin\left(x - \frac{\pi}{4}\right) - 3$

Extension Problems

4. Show that if the graphs of the functions $f(x) = A\sin\big(\omega(x - h_1)\big) + k$ and $g(x) = A\sin\big(\omega(x - h_2)\big) + k$ are the same, then h_1 and h_2 differ by an integer multiple of the period.

5. Show that if h_1 and h_2 differ by an integer multiple of the period, then the graphs of $f(x) = A\sin\big(\omega(x - h_1)\big) + k$ and $g(x) = A\sin\big(\omega(x - h_2)\big) + k$ are the same graph.

6. Find the x-intercepts of the graph of the function $f(x) = A\sin\big(\omega(x - h)\big)$ in terms of the period P, where $\omega > 0$.

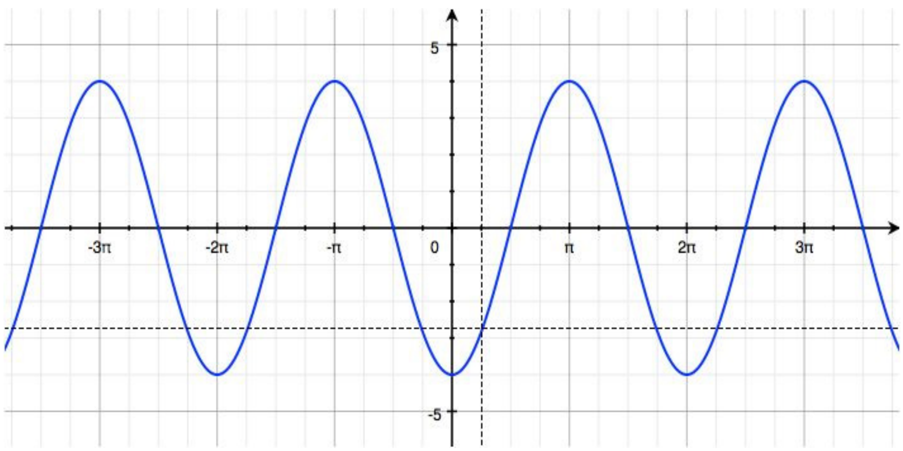
Lesson 12: Ferris Wheels—Using Trigonometric Functions to Model Cyclical Behavior

Classwork

Opening Exercise

Ernesto thinks that this is the graph of $f(x) = 4\sin\left(x - \frac{\pi}{2}\right)$. Danielle thinks it is the graph of $f(x) = 4\cos(x)$.

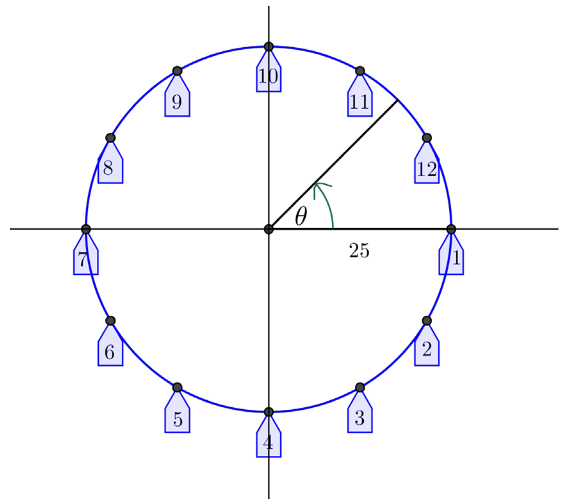
Who is correct, and why?



Exploratory Challenge

Exercises 1–5

A carnival has a Ferris wheel that is 50 feet in diameter with 12 passenger cars. When viewed from the side where passengers board, the Ferris wheel rotates counterclockwise and makes two full turns each minute. Riders board the Ferris wheel from a platform that is 15 feet above the ground. We will use what we have learned about periodic functions to model the position of the passenger cars from different mathematical perspectives. We will use the points on the circle in the diagram at right to represent the position of the cars on the wheel.



1. For this exercise, we will consider the height of a passenger car to be the vertical displacement from the horizontal line through the center of the wheel, and the co-height of a passenger car to be the horizontal displacement from the vertical line through the center of the wheel.
   a. Let $\theta = 0$ represent the position of car 1 in the diagram at right. Sketch the graphs of the co-height and the height of car 1 as functions of θ, the number of radians through which the car has rotated.

   b. What is the amplitude, $|A|$, of the height and co-height functions for this Ferris wheel?

c. Let $X(\theta)$ represent the co-height function after rotation by θ radians, and let $Y(\theta)$ represent the height function after rotation by θ radians. Write functions X for the co-height and Y for the height in terms of θ.

d. Graph the functions X and Y from part (c) on a graphing calculator set to parametric mode. Use a viewing window $[-48,48] \times [-30,30]$. Sketch the graph below.

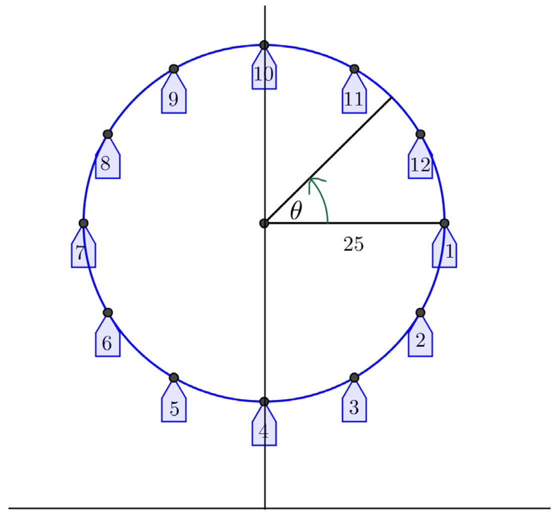
e. Why did we choose the symbols X and Y to represent the co-height and height functions?

f. Evaluate $X(0)$ and $Y(0)$ and explain their meaning in the context of the Ferris wheel.

g. Evaluate $X\left(\frac{\pi}{2}\right)$ and $Y\left(\frac{\pi}{2}\right)$ and explain their meaning in the context of the Ferris wheel.

2. The model we created in Exercise 1 measures the height of car 1 above the horizontal line through the center of the wheel. We now want to alter this model so that it measures the height of car 1 above the ground.

   a. If we measure the height of car 1 above the ground instead of above the horizontal line through the center of the wheel, how will the functions X and Y need to change?



   b. Let $\theta = 0$ represent the position of car 1 in the diagram at right. Sketch the graphs of the co-height and the height of car 1 as functions of the number of radians through which the car has rotated, θ.

   c. How are the graphs from Exercise 2(b) related to the graphs from Exercise 1(a)?

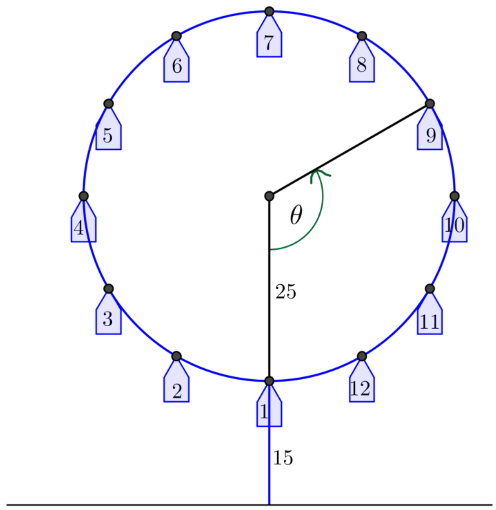
d. From this perspective, find the equations for the functions X and Y that model the position of car 1 with respect to the number of radians the car has rotated.

e. Change the viewing window on your calculator to $[-60,60] \times [-5,70]$, and graph the functions X and Y together. Sketch the graph.

f. Evaluate $X(0)$ and $Y(0)$ and explain their meaning in the context of the Ferris wheel.

g. Evaluate $X\left(\frac{\pi}{2}\right)$ and $Y\left(\frac{\pi}{2}\right)$ and explain their meaning in the context of the Ferris wheel.

3. In reality, no one boards a Ferris wheel halfway up; passengers board at the bottom of the wheel. To truly model the motion of a Ferris wheel, we need to start with passengers on the bottom of the wheel. Refer to the diagram below.



   a. Let $\theta = 0$ represent the position of car 1 at the bottom of the wheel in the diagram at right. Sketch the graphs of the height and the co-height of car 1 as functions of θ, the number of radians through which the car has rotated from the position at the bottom of the wheel.

   b. How are the graphs from Exercise 3(a) related to the graphs from Exercise 2(b)?

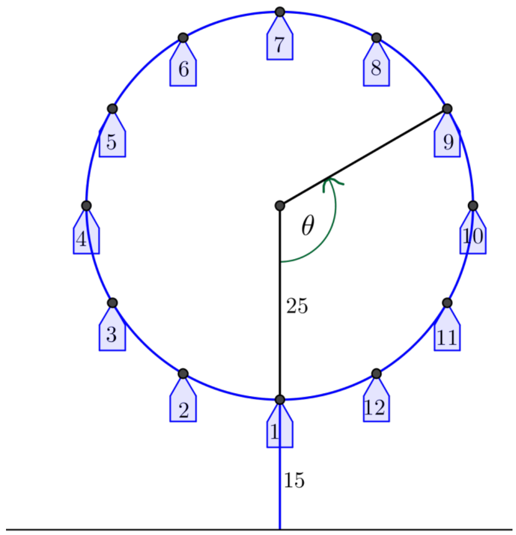
c. From this perspective, find the equations for the functions X and Y that model the position of car 1 with respect to the number of radians the car has rotated.

d. Graph the functions X and Y from part (c) together on the graphing calculator. Sketch the graph. How is this graph different from the one from Exercise 2(d)?

e. Evaluate $X(0)$ and $Y(0)$ and explain their meaning in the context of the Ferris wheel.

f. Evaluate $X\left(\frac{\pi}{2}\right)$ and $Y\left(\frac{\pi}{2}\right)$ and explain their meaning in the context of the Ferris wheel.

4. Finally, it is not very useful to track the position of a Ferris wheel as a function of how much it has rotated. It would make more sense to keep track of the Ferris wheel as a function of time. Recall that the Ferris wheel completes two full turns per minute.



a. Let $\theta = 0$ represent the position of car 1 at the bottom of the wheel. Sketch the graphs of the co-height and the height of car 1 as functions of time.

b. The co-height and height functions from part (a) can be written in the form $X(t) = A\cos\big(\omega(t-h)\big) + k$ and $Y(t) = A\sin\big(\omega(t-h)\big) + k$. From the graphs in part (a), identify the values of A, ω, h, and k for each function X and Y.

c. Write the equations $X(t)$ and $Y(t)$ using the values you identified in part (b).

d. In function mode, graph your functions from part (c) on a graphing calculator for $0 < t < 2$ and compare against your sketches in part (a) to confirm your equations.

e. Explain the meaning of the parameters in your equation for X in terms of the Ferris wheel scenario.

f. Explain the meaning of the parameters in your equation for Y in terms of the Ferris wheel scenario.

5. In parametric mode, graph the functions X and Y from Exercise 3(c) on a graphing calculator for $0 \leq t \leq \frac{1}{2}$.

   a. Sketch the graph. How is this graph different from the graph in Exercise 3(d)?

   b. What would the graph look like if you graphed the functions X and Y from Exercise 3(c) for $0 \leq t \leq \frac{1}{4}$? Why?

   c. Evaluate $X(0)$ and $Y(0)$ and explain their meaning in the context of the Ferris wheel.

   d. Evaluate $X\left(\frac{1}{8}\right)$ and $Y\left(\frac{1}{8}\right)$ and explain their meaning in the context of the Ferris wheel.

Exercise 6–9

6. You are in in car 1, and you see your best friend in car 3. How much higher than your friend are you when you reach the top?

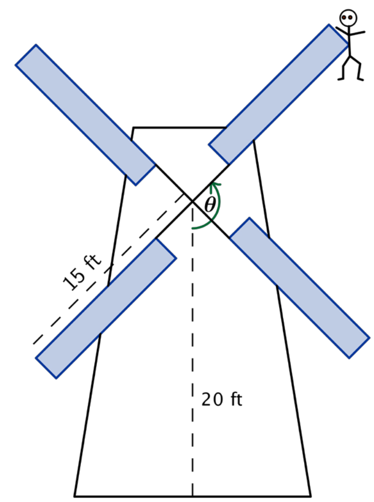
7. Find an equation of the function H that represents the difference in height between you in car 1 and your friend in car 3 as the wheel rotates through θ radians, beginning with $\theta = 0$ at the bottom of the wheel.

8. Find an equation of the function that represents the difference in height between car 1 and car 3 with respect to time, t, in minutes. Let $t = 0$ minutes correspond to a time when car 1 is located at the bottom of the Ferris wheel. Assume the wheel is moving at a constant speed starting at $t = 0$.

9. Use a calculator to graph the function H in Exercise 8 for $0 \leq t \leq 2$. What type of function does this appear to be? Does that make sense?

Problem Set

1. In the classic novel, *Don Quixote*, the title character famously battles a windmill. In this problem, you will model what happens when Don Quixote battles a windmill, and the windmill wins. Suppose the center of the windmill is 20 feet off the ground, and the sails are 15 feet long. Don Quixote is caught on a tip of one of the sails. The sails are turning at a rate of one counterclockwise rotation every 60 seconds.



   a. Explain why a sinusoidal function could be used to model Don Quixote's height above the ground as a function of time.
   b. Sketch a graph of Don Quixote's height above the ground as a function of time. Assume $t = 0$ corresponds to a time when he was closest to the ground. What are the amplitude, period, and midline of the graph?
   c. Model Don Quixote's height H above the ground as a function of time t since he was closest to the ground.
   d. After 1 minute and 40 seconds, Don Quixote fell off the sail and straight down to the ground. How far did he fall?

2. The High Roller, a Ferris wheel in Las Vegas, Nevada, opened in March 2014. The 550 ft. tall wheel has a diameter of 520 feet. A ride on one of its 28 passenger cars lasts 30 minutes, the time it takes the wheel to complete one full rotation. Riders board the passenger cars at the bottom of the wheel. Assume that once the wheel is in motion, it maintains a constant speed for the 30 minute ride and is rotating in a counterclockwise direction.
   a. Sketch a graph of the height of a passenger car on the High Roller as a function of the time the ride began.
   b. Write a sinusoidal function H that represents the height of a passenger car t minutes after the ride begins.
   c. Explain how the parameters of your sinusoidal function relate to the situation.
   d. If you were on this ride, how high would you be above the ground after 20 minutes?
   e. Suppose the ride costs $25. How much does 1 minute of riding time cost? How much does 1 foot of riding distance cost? How much does 1 foot of height above the ground cost?
   f. What are some of the limitations of this model based on the assumptions that we made?

3. Once in motion, a pendulum's distance varies sinusoidally from 12 feet to 2 feet away from a wall every 12 seconds.
   a. Sketch the pendulum's distance D from the wall over a 1 minute interval as a function of time t. Assume $t = 0$ corresponds to a time when the pendulum was furthest from the wall.
   b. Write a sinusoidal function for D, the pendulum's distance from the wall, as a function of the time since it was furthest from the wall.
   c. Identify two different times when the pendulum was 10 feet away from the wall. (Hint: Write an equation and solve it graphically.)

4. The height in meters relative to the starting platform height of a car on a portion of a roller coaster track is modeled by the function $H(t) = 3\sin\left(\frac{\pi}{4}(t - 10)\right) - 7$. It takes a car 24 seconds to travel on this portion of the track, which starts 10 seconds into the ride.
   a. Graph the height relative to the starting platform as a function of time over this time interval.
   b. Explain the meaning of each parameter in the height function in terms of the situation.

5. Given the following function, use the parameters to formulate a real world situation involving one dimension of circular motion that could be modeled using this function. Explain how each parameter of the function relates to your situation.

$$f(x) = 10\sin\left(\frac{\pi}{8}(x - 3)\right) + 15$$

Lesson 13: Tides, Sound Waves, and Stock Markets

Classwork

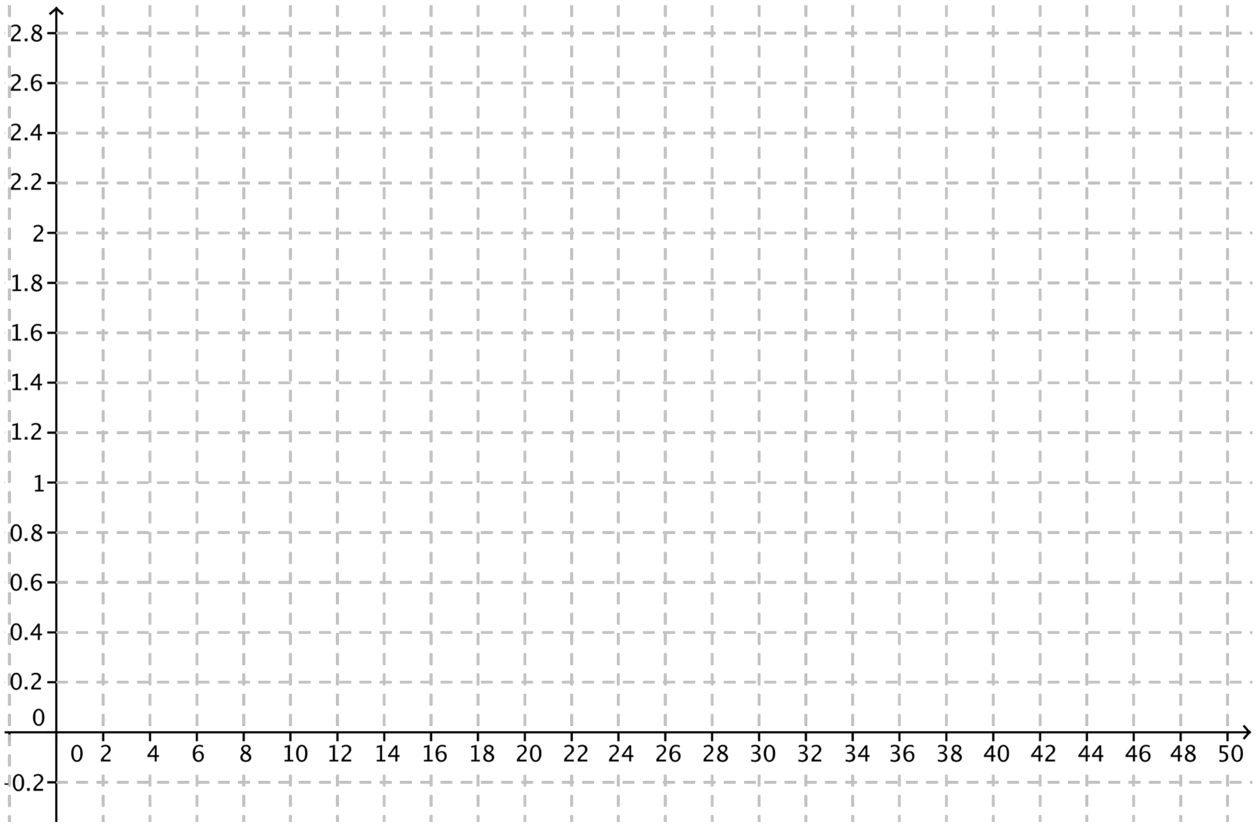
Opening Exercise

Anyone who works on or around the ocean needs to have information about the changing tides to conduct their business safely and effectively. People who go to the beach or out onto the ocean for recreational purposes also need information about tides when planning their trip. The table below shows tide data for Montauk, NY, for May 21–22, 2014. The heights reported in the table are relative to the Mean Lower Low Water (MLLW). The MLLW is the average height of the lowest tide recorded at a tide station each day during the recording period. This reference point is used by the National Oceanic and Atmospheric Administration (NOAA) for the purposes of reporting tidal data throughout the United States. Each different tide station throughout the United States has its own MLLW. High and low tide levels are reported relative to this number. Since it is an average, some low tide values can be negative. The MLLW values are reset approximately every 20 years by the NOAA.

MONTAUK, NY TIDE CHART

Date	Day	Time	Height in Feet	High/Low
2014/05/21	Wed.	$02:47$ a.m.	2.48	H
2014/05/21	Wed.	$09:46$ a.m.	-0.02	L
2014/05/21	Wed.	$03:31$ p.m.	2.39	H
2014/05/21	Wed.	$10:20$ p.m.	0.27	L
2014/05/22	Thurs.	$03:50$ a.m.	2.30	H
2014/05/22	Thurs.	$10:41$ a.m.	0.02	L
2014/05/22	Thurs.	$04:35$ p.m.	2.51	H
2014/05/22	Thurs.	$11:23$ p.m.	0.21	L

a. Create a scatter plot of the data with the horizontal axis representing time since midnight on May 21 and the vertical axis representing the height in feet relative to the MLLW.



b. What type of function would best model this set of data? Explain your choice.

Example 1: Write a Sinusoidal Function to Model a Set of Data

a. On the scatter plot you created in the Opening Exercise, sketch the graph of a sinusoidal function that could be used to model the data.

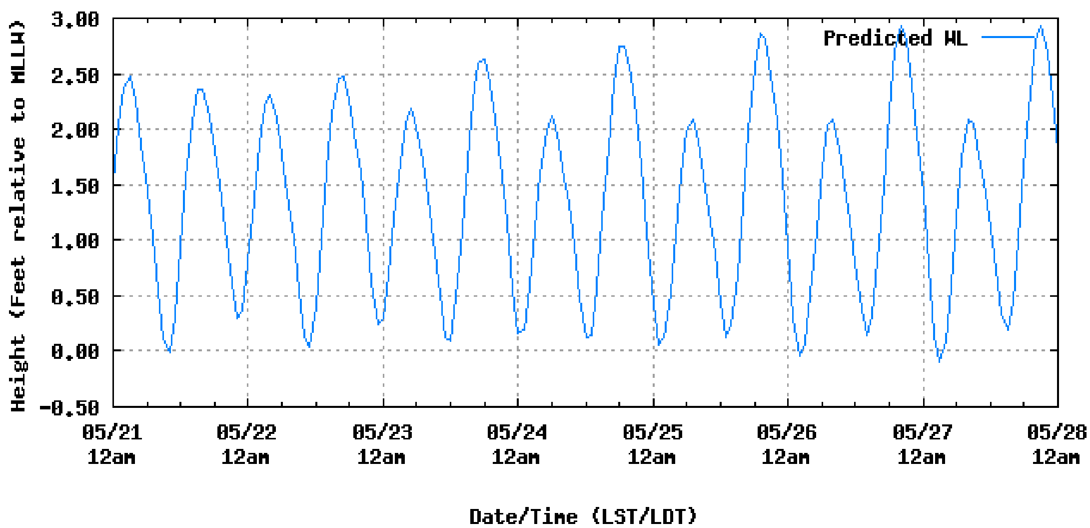
b. What are the midline, amplitude, and period of the graph?

c. Estimate the horizontal distance between a point on the graph that crosses the midline and the vertical axis.

d. Write a function of the form $f(x) = A\sin\big(\omega(x-h)\big) + k$ to model these data where x is the hours since midnight on May 21 and $f(x)$ is the height in feet relative to the MLLW.

Exercise 1

1. The graph of the tides at Montauk for the week of May 21–28 is shown below. How accurately does your model predict the time and height of the high tide on May 28?



Source: http://tidesandcurrents.noaa.gov/

Example 2: Digital Sampling of Sound

When sound is recorded or transmitted electronically, the continuous waveform is sampled to convert it to a discrete digital sequence. If you increase the sampling rate (represented by the horizontal scaling) or the resolution (represented by the vertical scaling), the sound quality of the recording or transmission improves.

The data graphed below represent two different samples of a pure tone. One is sampled 16 times per unit of time at a 4-bit (16 equal intervals in the vertical direction) resolution, and the other is sampled 8 times per unit of time at a 2-bit (4 equal intervals in the vertical direction) resolution. The graph of the actual sound wave is also shown.

Which sample points would produce a better model of the actual sound wave? Explain your reasoning.

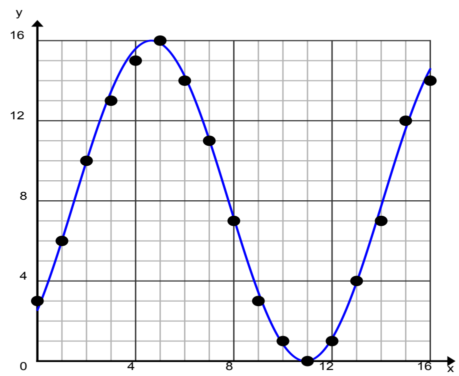


Figure 2: Sample rate = 16, 4-bit resolution

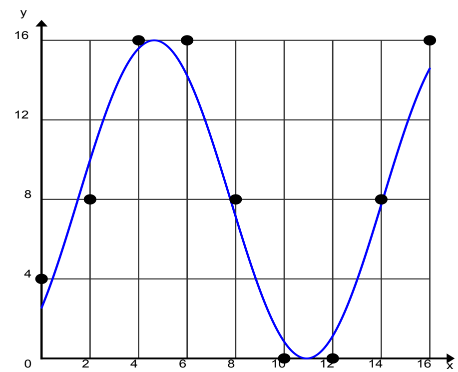


Figure 2: Sample rate = 8, 2-bit resolution

Exercises 2–6

Stock prices have historically increased over time, but they also vary in a cyclical fashion. Create a scatter plot of the data for the monthly stock price for a 15-month time period since January 1, 2003.

Months Since Jan. 1, 2003	Price at Close in dollars
0	20.24
1	19.42
2	18.25
3	19.13
4	19.20
5	20.91
6	20.86
7	20.04
8	20.30
9	20.54
10	21.94
11	21.87
12	21.51
13	20.65
14	19.84

2. Would a sinusoidal function be an appropriate model for this data? Explain your reasoning.

We can model the slight upward trend in this data with the linear function $f(x) = 19.5 + 0.13x$.

If we add a sinusoidal function to this linear function, we can model this data with an algebraic function that displays an upward trend but also varies in a cyclical fashion.

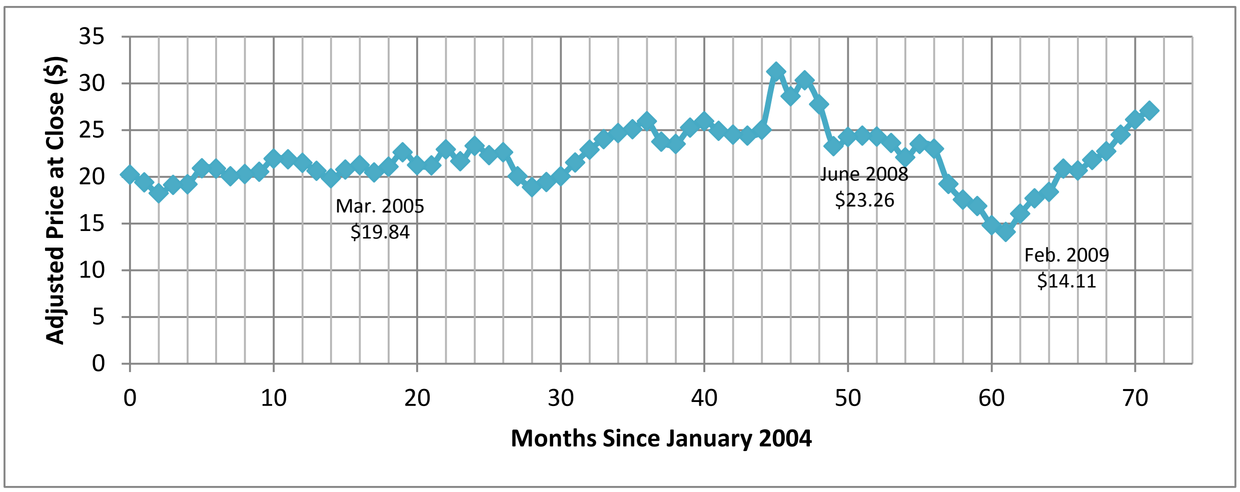
3. Find a sinusoidal function, g, that when added to the linear function f will model this data.

4. Let S be the stock price function that is the sum of the linear function listed above and the sinusoidal function in Exercise 3.

$$S(x) = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_.$$

5. Add the graph of this function to your scatter plot. How well does it appear to fit the data?

6. Here is a graph of the same stock through December 2009.



a. Will your model do a good job of predicting stock values past the year 2005?

b. What event occurred in 2008 to account for the sharp decline in the value of stocks?

c. What are the limitations of using any function to make predictions regarding the value of a stock at some point in the future?

Lesson Summary

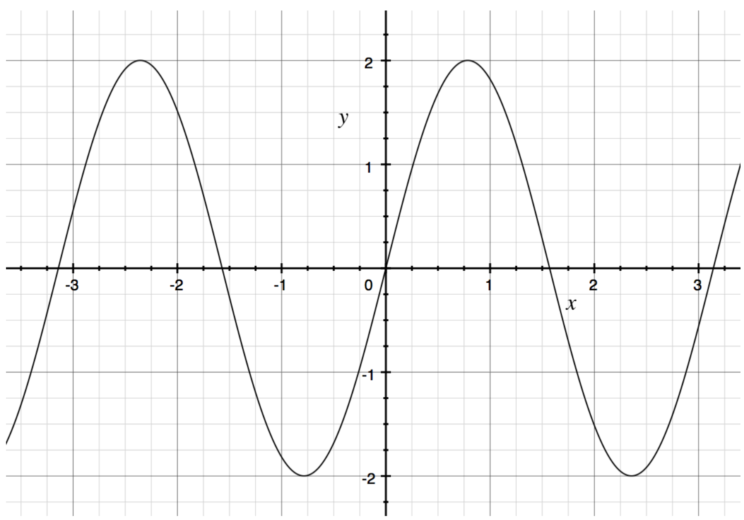
We can model periodic data with either a sine or a cosine function by extrapolating values of the parameters A, ω, h, and k from the data and defining a function $f(t) = A\sin\big(\omega(t-h)\big) + k$ or $g(t) = A\cos\big(\omega(t-h)\big) + k$, as appropriate.

Sine or cosine functions may not perfectly fit most data sets from actual measurements; therefore, there are often multiple functions used to model a data set.

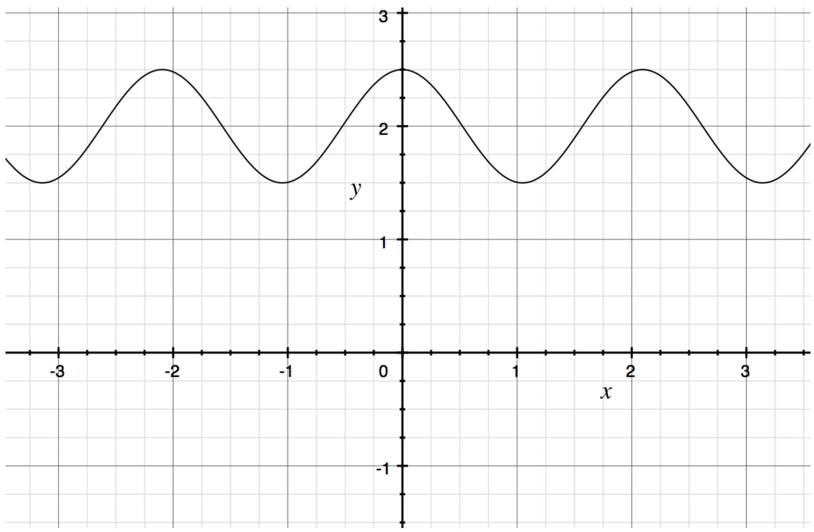
If possible, plot the data together with the function that appears to fit the graph. If it is not a good fit, adjust the model and try again.

Problem Set

1. Find equations of both a sine function and a cosine function that could each represent the graph given below.



2. Find equations of both a sine function and a cosine function that could each represent the graph given below.



3. Rapidly vibrating objects send pressure waves through the air which are detected by our ears and then interpreted by our brains as sound. Our brains analyze the amplitude and frequency of these pressure waves.

   A speaker usually consists of a paper cone attached to an electromagnet. By sending an oscillating electric current through the electromagnet, the paper cone can be made to vibrate. By adjusting the current, the amplitude and frequency of vibrations can be controlled.

   The following graph shows the pressure intensity (I) as a function of time (x), in seconds, of the pressure waves emitted by a speaker set to produce a single pure tone.

   a. Does it seem more natural to use a sine or a cosine function to fit to this graph?
   b. Find the equation of a trigonometric function that fits this graph.

4. Suppose that the following table represents the average monthly ambient air temperature, in degrees Fahrenheit, in some subterranean caverns in southeast Australia for each of the twelve months in a year. We wish to model this data with a trigonometric function. (Notice that the seasons are reversed in the Southern Hemisphere, so January is in summer and July is in winter.)

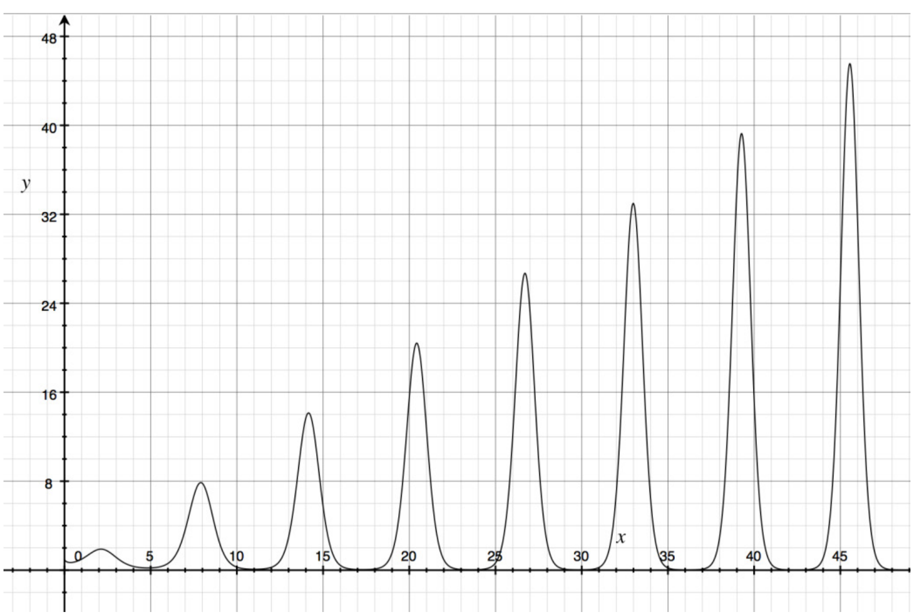
Month	Jan.	Feb.	Mar.	Apr.	May	June	July	Aug.	Sept.	Oct.	Nov.	Dec.
°F	64.04	64.22	61.88	57.92	53.60	50.36	49.10	49.82	52.34	55.22	58.10	61.52

a. Does it seem reasonable to assume that this data, if extended beyond the one year, should be roughly periodic?
b. What seems to be the amplitude of the data?
c. What seems to be the midline of the data (equation of the line through the middle of the graph)?
d. Would it be easier to use sine or cosine to model this data?
e. What is a reasonable approximation for the horizontal shift?
f. Write an equation for a function that could fit this data.

5. The table below provides data for the number of daylight hours as a function of day of the year, where day 1 represents January 1st. Graph the data and determine if it could be represented by a sinusoidal function. If it can, determine the period, amplitude, and midline of the function, and find an equation for a function that models the data.

Day of Year	0	50	100	150	175	200	250	300	350
Hours	4.0	7.9	14.9	19.9	20.5	19.5	14.0	7.1	3.6

6. The function graphed below is $y = x^{\sin(x)}$. Blake says, "The function repeats on a fixed interval, so it must be a sinusoidal function." Respond to his argument.



Lesson 14: Graphing the Tangent Function

Classwork

Exploratory Challenge/Exercises 1–5

1. Use your calculator to calculate each value of $\tan(x)$ to two decimal places in the table for your group.

Group 1 $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$		Group 2 $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$		Group 3 $\left(-\frac{3\pi}{2}, -\frac{\pi}{2}\right)$		Group 4 $\left(\frac{3\pi}{2}, \frac{5\pi}{2}\right)$	
x	$\tan(x)$	x	$\tan(x)$	x	$\tan(x)$	x	$\tan(x)$
$-\frac{11\pi}{24}$		$\frac{13\pi}{24}$		$-\frac{35\pi}{24}$		$\frac{37\pi}{24}$	
$-\frac{5\pi}{12}$		$\frac{7\pi}{12}$		$-\frac{17\pi}{12}$		$\frac{19\pi}{12}$	
$-\frac{4\pi}{12}$		$\frac{8\pi}{12}$		$-\frac{16\pi}{12}$		$\frac{20\pi}{12}$	
$-\frac{3\pi}{12}$		$\frac{9\pi}{12}$		$-\frac{15\pi}{12}$		$\frac{21\pi}{12}$	
$-\frac{2\pi}{12}$		$\frac{10\pi}{12}$		$-\frac{14\pi}{12}$		$\frac{22\pi}{12}$	
$-\frac{\pi}{12}$		$\frac{11\pi}{12}$		$-\frac{13\pi}{12}$		$\frac{23\pi}{12}$	
0		π		$-\pi$		2π	
$\frac{\pi}{12}$		$\frac{13\pi}{12}$		$-\frac{11\pi}{12}$		$\frac{25\pi}{12}$	
$\frac{2\pi}{12}$		$\frac{14\pi}{12}$		$-\frac{10\pi}{12}$		$\frac{26\pi}{12}$	
$\frac{3\pi}{12}$		$\frac{15\pi}{12}$		$-\frac{9\pi}{12}$		$\frac{27\pi}{12}$	
$\frac{4\pi}{12}$		$\frac{16\pi}{12}$		$-\frac{8\pi}{12}$		$\frac{28\pi}{12}$	
$\frac{5\pi}{12}$		$\frac{17\pi}{12}$		$-\frac{7\pi}{12}$		$\frac{29\pi}{12}$	
$\frac{11\pi}{24}$		$\frac{35\pi}{24}$		$-\frac{13\pi}{24}$		$\frac{59\pi}{24}$	

Group 5 $\left(-\frac{5\pi}{2}, -\frac{3\pi}{2}\right)$		Group 6 $\left(\frac{5\pi}{2}, \frac{7\pi}{2}\right)$		Group 7 $\left(-\frac{7\pi}{2}, -\frac{5\pi}{2}\right)$		Group 8 $\left(\frac{7\pi}{2}, \frac{9\pi}{2}\right)$	
x	$\tan(x)$	x	$\tan(x)$	x	$\tan(x)$	x	$\tan(x)$
$-\frac{59\pi}{24}$		$\frac{61\pi}{24}$		$-\frac{83\pi}{24}$		$\frac{37\pi}{24}$	
$-\frac{29\pi}{12}$		$\frac{31\pi}{12}$		$-\frac{41\pi}{12}$		$\frac{43\pi}{12}$	
$-\frac{28\pi}{12}$		$\frac{32\pi}{12}$		$-\frac{40\pi}{12}$		$\frac{44\pi}{12}$	
$-\frac{27\pi}{12}$		$\frac{33\pi}{12}$		$-\frac{39\pi}{12}$		$\frac{45\pi}{12}$	
$-\frac{26\pi}{12}$		$\frac{34\pi}{12}$		$-\frac{38\pi}{12}$		$\frac{46\pi}{12}$	
$-\frac{25\pi}{12}$		$\frac{35\pi}{12}$		$-\frac{37\pi}{12}$		$\frac{47\pi}{12}$	
-2π		3π		-3π		4π	
$-\frac{23\pi}{12}$		$\frac{37\pi}{12}$		$-\frac{35\pi}{12}$		$\frac{49\pi}{12}$	
$-\frac{22\pi}{12}$		$\frac{38\pi}{12}$		$-\frac{34\pi}{12}$		$\frac{50\pi}{12}$	
$-\frac{21\pi}{12}$		$\frac{39\pi}{12}$		$-\frac{33\pi}{12}$		$\frac{51\pi}{12}$	
$-\frac{20\pi}{12}$		$\frac{40\pi}{12}$		$-\frac{32\pi}{12}$		$\frac{52\pi}{12}$	
$-\frac{19\pi}{12}$		$\frac{41\pi}{12}$		$-\frac{31\pi}{12}$		$\frac{53\pi}{12}$	
$-\frac{37\pi}{24}$		$\frac{83\pi}{24}$		$-\frac{61\pi}{24}$		$\frac{107\pi}{24}$	

2. The tick marks on the axes provided are spaced in increments of $\frac{\pi}{12}$. Mark the horizontal axis by writing the number of the left endpoint of your interval at the left-most tick mark, the multiple of π that is in the middle of your interval at the point where the axes cross, and the number at the right endpoint of your interval at the right-most tick mark. Fill in the remaining values at increments of $\frac{\pi}{12}$.

3. On your plot, sketch the graph of $y = \tan(x)$ on your specified interval by plotting the points in the table and connecting the points with a smooth curve. Draw the graph with a bold marker.

4. What happens to the graph near the edges of your interval? Why does this happen?

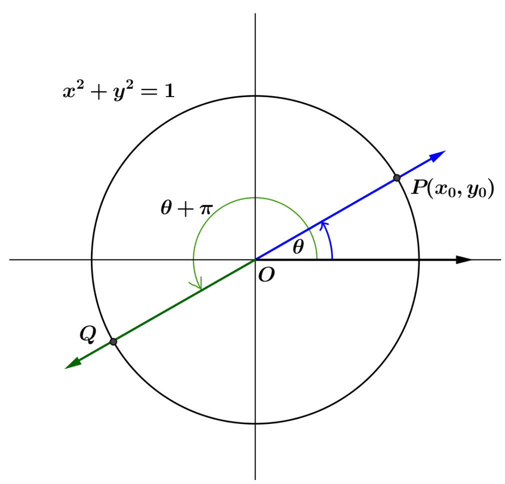
5. When you are finished, affix your graph to the board in the appropriate place, matching endpoints of intervals.

Exploratory Challenge 2/Exercises 6–16

For each exercise below, let $m = \tan(\theta)$ be the slope of the terminal ray in the definition of the tangent function, and let $P = (x_0, y_0)$ be the intersection of the terminal ray with the unit circle after being rotated by θ radians for $0 < \theta < \frac{\pi}{2}$. We know that the tangent of θ is the slope m of $\overleftrightarrow{OP}$.

6. Let Q be the intersection of the terminal ray with the unit circle after being rotated by $\theta + \pi$ radians.

   a. What is the slope of $\overleftrightarrow{OQ}$?



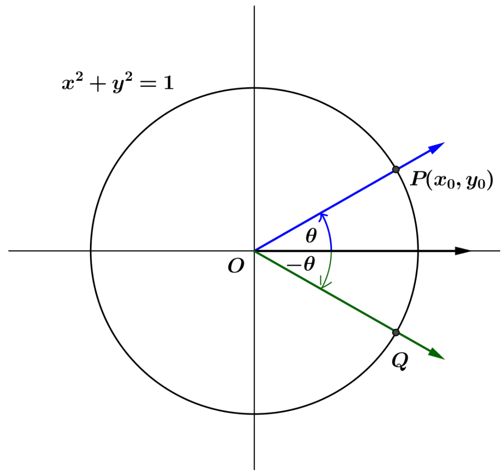
   b. Find an expression for $\tan(\theta + \pi)$ in terms of m.

   c. Find an expression for $\tan(\theta + \pi)$ in terms of $\tan(\theta)$.

   d. How can the expression in part (c) be seen in the graph of the tangent function?

7. Let Q be the intersection of the terminal ray with the unit circle after being rotated by $-\theta$ radians.

   a. What is the slope of $\overleftrightarrow{OQ}$?



   b. Find an expression for $\tan(-\theta)$ in terms of m.

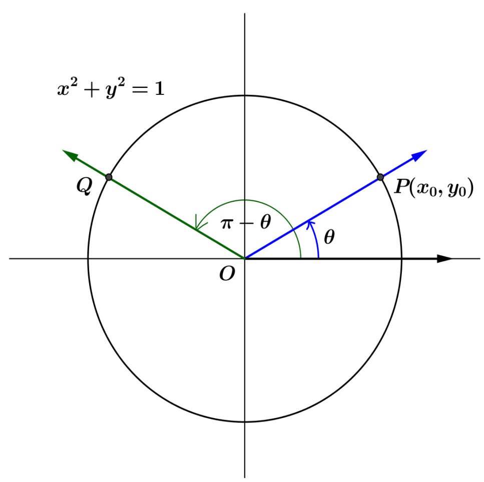
   c. Find an expression for $\tan(-\theta)$ in terms of $\tan(\theta)$.

   d. How can the expression in part (c) be seen in the graph of the tangent function?

8. Is the tangent function an even function, an odd function, or neither? How can you tell your answer is correct from the graph of the tangent function?

9. Let Q be the intersection of the terminal ray with the unit circle after being rotated by $\pi - \theta$ radians.

   a. What is the slope of $\overleftrightarrow{OQ}$?

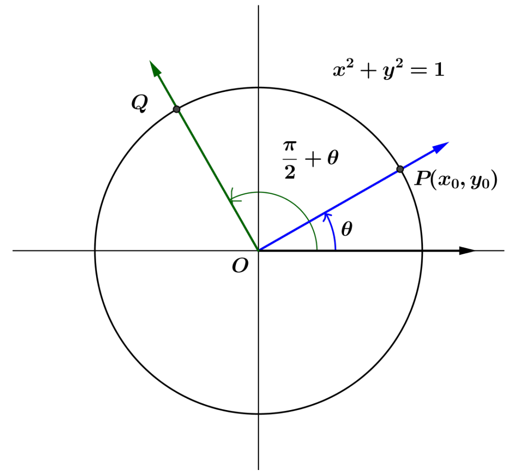


   b. Find an expression for $\tan(\pi - \theta)$ in terms of m.

   c. Find an expression for $\tan(\pi - \theta)$ in terms of $\tan(\theta)$.

10. Let Q be the intersection of the terminal ray with the unit circle after being rotated by $\frac{\pi}{2} + \theta$ radians.

    a. What is the slope of $\overleftrightarrow{OQ}$?

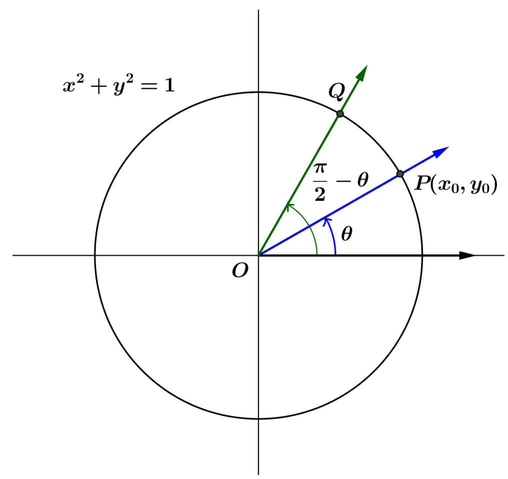


    b. Find an expression for $\tan\left(\frac{\pi}{2} + \theta\right)$ in terms of m.

    c. Find an expression for $\tan\left(\frac{\pi}{2} + \theta\right)$ first in terms of $\tan(\theta)$ and then in terms of $\cot(\theta)$.

11. Let Q be the intersection of the terminal ray with the unit circle after being rotated by $\frac{\pi}{2} - \theta$ radians.

    a. What is the slope of $\overleftrightarrow{OQ}$?



    b. Find an expression for $\tan\left(\frac{\pi}{2} - \theta\right)$ in terms of m.

    c. Find an expression for $\tan\left(\frac{\pi}{2} - \theta\right)$ in terms of $\tan(\theta)$ or other trigonometric functions.

12. Summarize your results from Exercises 6, 7, 9, 10, and 11.

13. We have only demonstrated that the identities in Exercise 12 are valid for $0 < \theta < \frac{\pi}{2}$ because we only used rotations that left point P in the first quadrant. Argue that $\tan\left(-\frac{2\pi}{3}\right) = -\tan\left(\frac{2\pi}{3}\right)$. Then, using similar logic, we could argue that all of the above identities extend to any value of θ for which the tangent (and cotangent for the last two) are defined.

14. For which values of θ are the identities in Exercise 7 valid?

15. Derive an identity for $\tan(2\pi + \theta)$ from the graph.

16. Use the identities you summarized in Exercise 7 to show $\tan(2\pi - \theta) = -\tan(\theta)$ where $\theta \neq \frac{\pi}{2} + k\pi$, for all integers k.

Lesson Summary

The tangent function $\tan(x) = \frac{\sin(x)}{\cos(x)}$ is periodic with period π. We have established the following identities:

- $\tan(x + \pi) = \tan(x)$ for all $\theta \neq \frac{\pi}{2} + k\pi$, for all integers k.
- $\tan(-x) = -\tan(x)$ for all $\theta \neq \frac{\pi}{2} + k\pi$, for all integers k.
- $\tan(\pi - x) = -\tan(x)$ for all $\theta \neq \frac{\pi}{2} + k\pi$, for all integers k.
- $\tan\left(\frac{\pi}{2} + x\right) = -\cot(x)$ for all $\theta \neq k\pi$, for all integers k.
- $\tan\left(\frac{\pi}{2} - x\right) = \cot(x)$ for all $\theta \neq k\pi$, for all integers k.
- $\tan(2\pi + x) = \tan(x)$ for all $\theta \neq \frac{\pi}{2} + k\pi$, for all integers k.
- $\tan(2\pi - x) = -\tan(x)$ for all $\theta \neq \frac{\pi}{2} + k\pi$, for all integers k.

Problem Set

1. Recall that the cotangent function is defined by $\cot(x) = \frac{\cos(x)}{\sin(x)} = \frac{1}{\tan(x)}$, where $\sin(x) \neq 0$.

   a. What is the domain of the cotangent function? Explain how you know.

   b. What is the period of the cotangent function? Explain how you know.

   c. Use a calculator to complete the table of values of the cotangent function on the interval $(0, \pi)$ to two decimal places.

x	$\cot(x)$
$\frac{\pi}{24}$	
$\frac{\pi}{12}$	
$\frac{2\pi}{12}$	
$\frac{3\pi}{12}$	

x	$\cot(x)$
$\frac{4\pi}{12}$	
$\frac{5\pi}{12}$	
$\frac{\pi}{2}$	

x	$\cot(x)$
$\frac{7\pi}{12}$	
$\frac{8\pi}{12}$	
$\frac{9\pi}{12}$	

x	$\cot(x)$
$\frac{10\pi}{12}$	
$\frac{11\pi}{12}$	
$\frac{23\pi}{24}$	

   d. Plot your data from part (c) and sketch a graph of $y = \cot(x)$ on $(0, \pi)$.

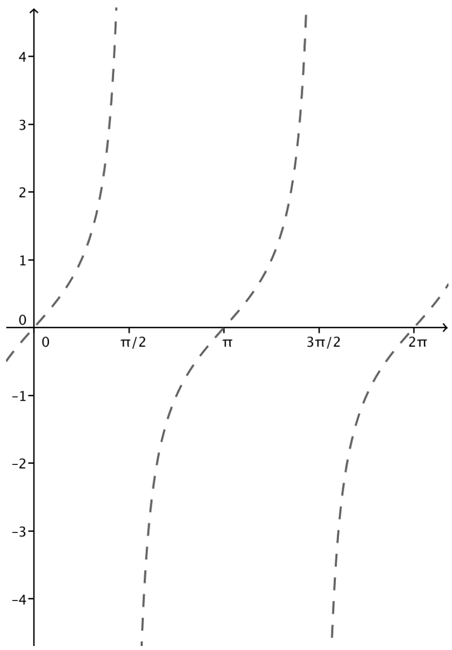
   e. Sketch a graph of $y = \cot(x)$ on $(-2\pi, 2\pi)$ without plotting points.

   f. Discuss the similarities and differences between the graphs of the tangent and cotangent functions.

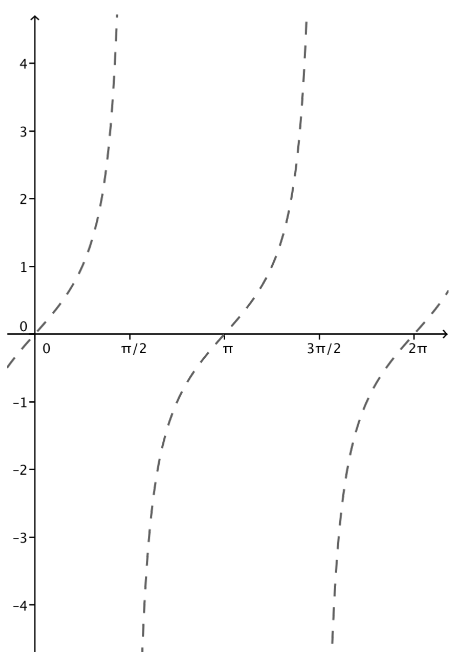
   g. Find all x-values where $\tan(x) = \cot(x)$ on the interval $(0, 2\pi)$.

2. Each set of axes below shows the graph of $f(x) = \tan(x)$. Use what you know about function transformations to sketch a graph of $y = g(x)$ for each function g on the interval $(0, 2\pi)$.

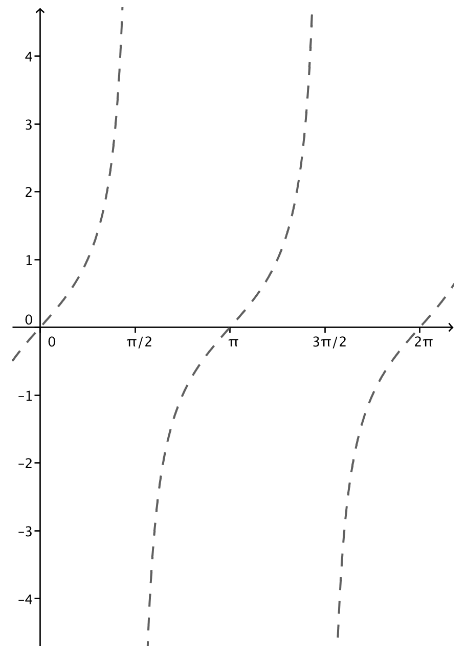
a. $g(x) = 2\tan(x)$



b. $g(x) = \frac{1}{3}\tan(x)$



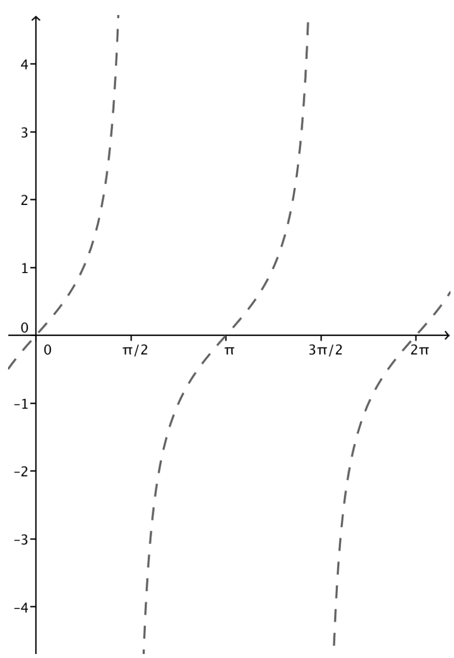
c. $g(x) = -2\tan(x)$



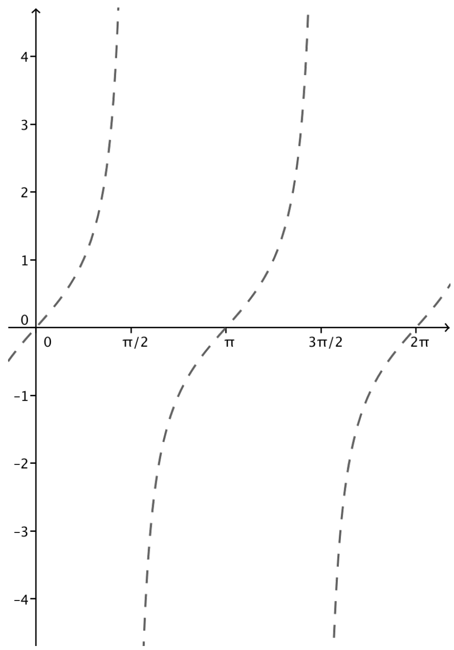
d. How does changing the parameter A affect the graph of $g(x) = A\tan(x)$?

3. Each set of axes below shows the graph of $f(x) = \tan(x)$. Use what you know about function transformations to sketch a graph of $y = g(x)$ for each function g on the interval $(0, 2\pi)$.

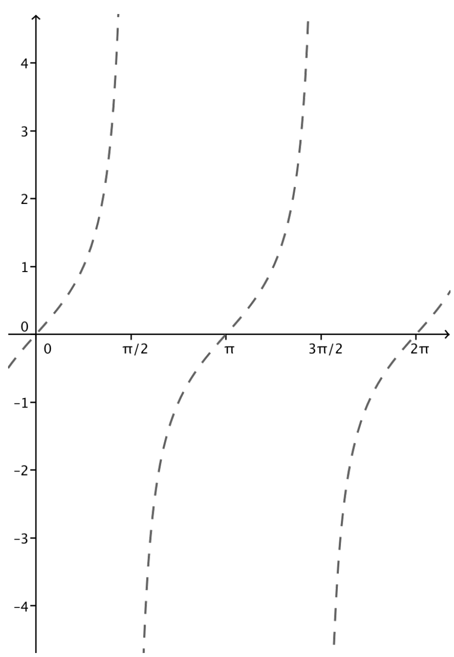
a. $g(x) = \tan\left(x - \frac{\pi}{2}\right)$



b. $g(x) = \tan\left(x - \frac{\pi}{6}\right)$



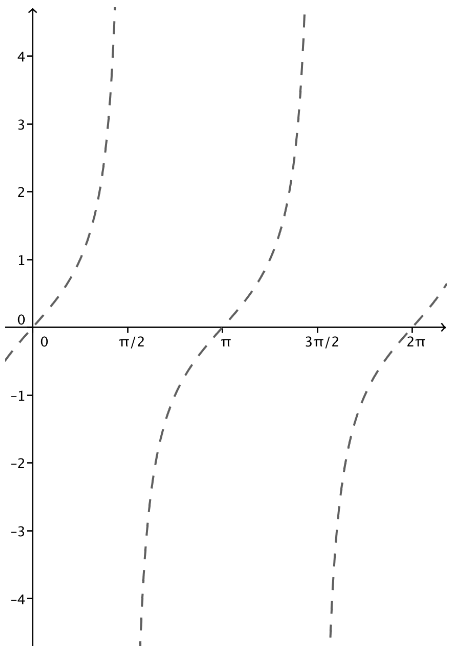
c. $g(x) = \tan\left(x + \frac{\pi}{4}\right)$



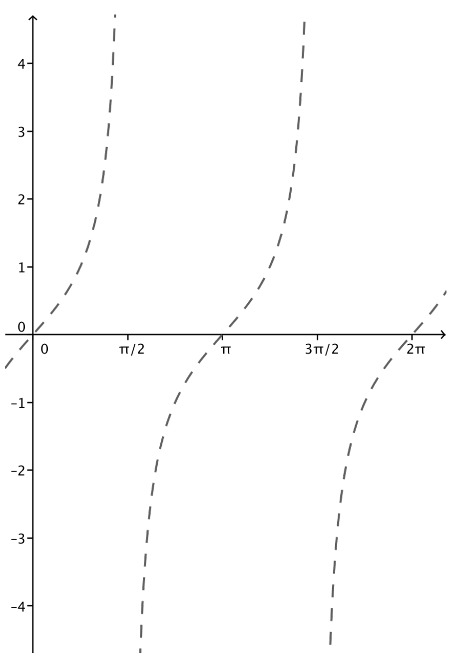
d. How does changing the parameter h affect the graph of $g(x) = \tan(x - h)$?

4. Each set of axes below shows the graph of $f(x) = \tan(x)$. Use what you know about function transformations to sketch a graph of $y = g(x)$ for each function g on the interval $(0, 2\pi)$.

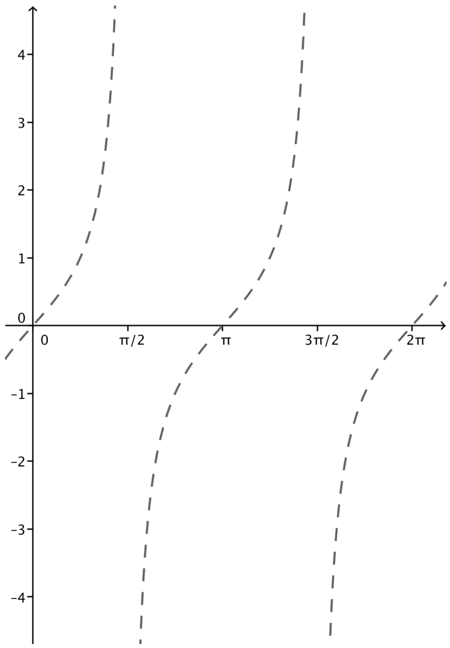
   a. $g(x) = \tan(x) + 1$



   b. $g(x) = \tan(x) + 3$



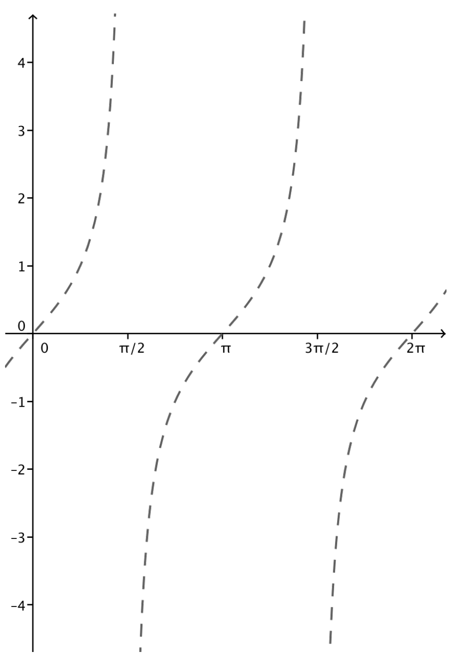
c. $g(x) = \tan(x) - 2$



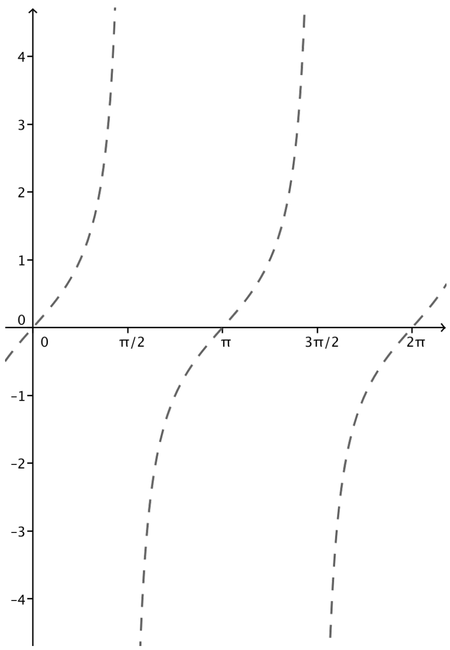
d. How does changing the parameter k affect the graph of $g(x) = \tan(x) + k$?

5. Each set of axes below shows the graph of $f(x) = \tan(x)$. Use what you know about function transformations to sketch a graph of $y = g(x)$ for each function g on the interval $(0, 2\pi)$.

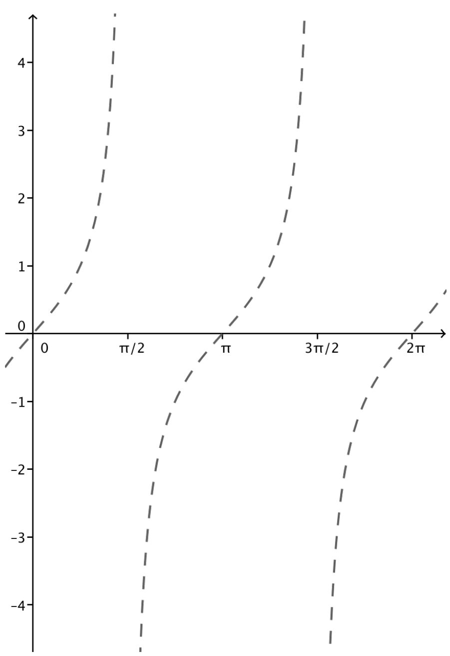
a. $g(x) = \tan(3x)$



b. $g(x) = \tan\left(\frac{x}{2}\right)$



c. $g(x) = \tan(-3x)$



d. How does changing the parameter ω affect the graph of $g(x) = \tan(\omega x)$?

6. Use your knowledge of function transformation and the graph of $y = \tan(x)$ to sketch graphs of the following transformations of the tangent function.

   a. $y = \tan(2x)$

   b. $y = \tan\left(2\left(x - \frac{\pi}{4}\right)\right)$

   c. $y = \tan\left(2\left(x - \frac{\pi}{4}\right)\right) + 1.5$

7. Find parameters A, ω, h, and k so that the graphs of $f(x) = A\tan\left(\omega(x - h)\right) + k$ and $g(x) = \cot(x)$ are the same.

Lesson 15: What Is a Trigonometric Identity?

Classwork

Exercises 1–3

1. Recall the Pythagorean identity $\sin^2(\theta) + \cos^2(\theta) = 1$, where θ is any real number.

   a. Find $\sin(x)$, given $\cos(x) = \frac{3}{5}$, for $-\frac{\pi}{2} < x < 0$.

   b. Find $\tan(y)$, given $\cos(y) = -\frac{5}{13}$, for $\frac{\pi}{2} < y < \pi$.

   c. Write $\tan(z)$ in terms of $\cos(z)$, for $\pi < z < \frac{3\pi}{2}$.

2. Use the Pythagorean identity to do the following:
   a. Rewrite the expression $\cos(\theta)\sin^2(\theta) - \cos(\theta)$ in terms of a single trigonometric function.

   b. Rewrite the expression $(1 - \cos^2(\theta))\csc(\theta)$ in terms of a single trigonometric function.

   c. Find all the solutions of the equation $2\sin^2(\theta) = 2 + \cos(\theta)$ in the interval $[0,2\pi)$. Draw a unit circle that shows the solutions.

3. Which of the following equations are identities? For those equations that are identities, which ones are defined for all real numbers and which are not? For the latter, for which values of x are they not defined?

   a. $\sin(x + 2\pi) = \sin(x)$ where the functions on both sides are defined.

   b. $\sec(x) = 1$ where the functions on both sides are defined.

   c. $\sin(-x) = \sin(x)$ where the functions on both sides are defined.

   d. $1 + \tan^2(x) = \sec^2(x)$ where the functions on both sides are defined.

   e. $\sin\left(\frac{\pi}{2} - x\right) = \cos(x)$ where the functions on both sides are defined.

   f. $\sin^2(x) = \tan^2(x)$ for all real x.

Lesson Summary

The Pythagorean identity: $\sin^2(\theta) + \cos^2(\theta) = 1$ for all real numbers θ.

Problem Set

1. Which of the following are trigonometric identities? Graph the functions on each side of the equation.
   a. $\tan(x) = \frac{\sin(x)}{\cos(x)}$ where the functions on both sides are defined.
   b. $\cos^2(x) = 1 + \sin(x)$ where the functions on both sides are defined.
   c. $\cos\left(\frac{\pi}{2} - x\right) = \sin(x)$ where the functions on both sides are defined.

2. Determine the domain of the following trigonometric identities:
   a. $\cot(x) = \frac{\cos(x)}{\sin(x)}$ where the functions on both sides are defined.
   b. $\cos(-u) = \cos(u)$ where the functions on both sides are defined.
   c. $\sec(y) = \frac{1}{\cos(y)}$ where the functions on both sides are defined.

3. Rewrite $\sin(x)\cos^2(x) - \sin(x)$ as an expression containing a single term.

4. Suppose $0 < \theta < \frac{\pi}{2}$, and $\sin(\theta) = \frac{1}{\sqrt{3}}$. What is the value of $\cos(\theta)$?

5. If $\cos(\theta) = -\frac{1}{\sqrt{5}}$, what are possible values of $\sin(\theta)$?

6. Use the Pythagorean identity $\sin^2(\theta) + \cos^2(\theta) = 1$, where θ is any real number, to find the following:
   a. $\cos(\theta)$, given $\sin(\theta) = \frac{5}{13}$, for $\frac{\pi}{2} < \theta < \pi$.
   b. $\tan(x)$, given $\cos(x) = -\frac{1}{\sqrt{2}}$, for $\pi < x < \frac{3\pi}{2}$.

7. The three identities below are all called Pythagorean identities. The second and third follow from the first, as you saw in Example 1 and the Exit Ticket.

   a. For which values of θ are each of these identities defined?

      i. $\sin^2(\theta) + \cos^2(\theta) = 1$, where the functions on both sides are defined.

      ii. $\tan^2(\theta) + 1 = \sec^2(\theta)$, where the functions on both sides are defined.

      iii. $1 + \cot^2(\theta) = \csc^2(\theta)$, where the functions on both sides are defined.

   b. For which of the three identities is 0 in the domain of validity?

   c. For which of the three identities is $\frac{\pi}{2}$ in the domain of validity?

   d. For which of the three identities is $-\frac{\pi}{4}$ in the domain of validity?

Lesson 16: Proving Trigonometric Identities

Classwork

Opening Exercise

Which of these statements is a trigonometric identity? Provide evidence to support your claim.

Statement 1: $\sin^2(\theta) = 1 - \cos^2(\theta)$ for θ any real number.

Statement 2: $1 - \cos(\theta) = 1 - \cos(\theta)$ for θ any real number.

Statement 3: $1 - \cos(\theta) = 1 + \cos(\theta)$ for θ any real number.

Using Statements 1 and 2, create a third identity, Statement 4, whose left side is $\frac{\sin^2(\theta)}{1-\cos(\theta)}$.

For which values of θ is this statement valid?

Discuss in pairs what it might mean to "prove" an identity. What might it take to prove, for example, that the following statement is an identity?

$$\frac{\sin^2(\theta)}{1-\cos(\theta)} = 1 + \cos(\theta) \text{ where } \theta \neq 2\pi k, \text{ for all integers } k.$$

To prove an identity, you have to use logical steps to show that one side of the equation in the identity can be transformed into the other side of the equation using already established identities such as the Pythagorean identity or the properties of operation (commutative, associative, and distributive properties). It is not correct to start with what you want to prove and work on both sides of the equation at the same time, as the following exercise shows.

Exercise

Take out your calculators and quickly graph the equations $y = \sin(x) + \cos(x)$ and $y = -\sqrt{1 + 2\sin(x)\cos(x)}$ to determine whether $\sin(\theta) + \cos(\theta) = -\sqrt{1 + 2\sin(\theta)\cos(\theta)}$ for all θ for which both functions are defined is a valid identity. You should see from the graphs that the functions are not equivalent.

Suppose that Charles did not think to graph the equations to see if the given statement was a valid identity, so he set about proving the identity using algebra and a previous identity. His argument is shown below.

First, [1] $\sin(\theta) + \cos(\theta) = -\sqrt{1 + 2\sin(\theta)\cos(\theta)}$ for θ any real number.

Now, using the multiplication property of equality, square both sides, which gives

[2] $\sin^2(\theta) + 2\sin(\theta)\cos(\theta) + \cos^2(\theta) = 1 + 2\sin(\theta)\cos(\theta)$ for θ any real number.

Using the subtraction property of equality, subtract $2\sin(\theta)\cos(\theta)$ from each side, which gives

[3] $\sin^2(\theta) + \cos^2(\theta) = 1$ for θ any real number.

Statement [3] is the Pythagorean identity. So, replace $\sin^2(\theta) + \cos^2(\theta)$ by 1 to get

[4] $1 = 1$, which is definitely true.

Therefore, the original statement must be true.

Does this mean that Charles has proven that Statement [1] is an identity? Discuss with your group whether it is a valid proof. If you decide it is not a valid proof, then discuss with your group how and where his argument went wrong.

Example 1: Two Proofs of our New Identity

Work through these two different ways to approach proving the identity $\frac{\sin^2(\theta)}{1-\cos(\theta)} = 1 + \cos(\theta)$ where $\theta \neq 2\pi k$, for integers k. The proofs make use of some of the following properties of equality and real numbers. Here *a*, *b,* and *c* stand for arbitrary real numbers.

Reflexive property of equality	$a = a$
Symmetric property of equality	If $a = b$, then $b = a$.
Transitive property of equality	If $a = b$ and $b = c$, then $a = c$.
Addition property of equality	If $a = b$, then $a + c = b + c$.
Subtraction property of equality	If $a = b$, then $a - c = b - c$.
Multiplication property of equality	If $a = b$, then $a \times c = b \times c$.
Division property of equality	If $a = b$ and $c \neq 0$, then $a \div c = b \div c$.
Substitution property of equality	If $a = b$, then b may be substituted for a in any expression containing a.
Associative properties	$(a + b) + c = a + (b + c)$ and $a(bc) = (ab)c$.
Commutative properties	$a + b = b + a$ and $ab = ba$.
Distributive property	$a(b + c) = ab + ac$ and $(a + b)c = ac + bc$.

Fill in the missing parts of the proofs below.

A. We start with Statement 1 from the opening activity and divide both sides by the same expression, $1 - \cos(\theta)$. This step will introduce division by zero when $1 - \cos(\theta) = 0$ and will change the set of values of θ for which the identity is valid.

PROOF:

Step	Left Side of Equation		Equivalent Right Side	Domain	Reason
1	$\sin^2(\theta) + \cos^2(\theta)$	$=$	1	θ any real number	Pythagorean identity
2	$\sin^2(\theta)$	$=$	$1 - \cos^2(\theta)$	θ any real number	
3		$=$	$(1 - \cos(\theta))(1 + \cos(\theta))$	θ any real number	
4		$=$	$\frac{(1 - \cos(\theta))(1 + \cos(\theta))}{1 - \cos(\theta)}$		
5	$\frac{\sin^2(\theta)}{1 - \cos(\theta)}$	$=$		$\theta \neq 2\pi k$ for all integers k	Substitution property of equality using $\frac{1-\cos(\theta)}{1-\cos(\theta)} = 1$

B. Or, we can start with the more complicated side of the identity we want to prove and use algebra and prior trigonometric definitions and identities to transform it to the other side. In this case, the more complicated expression is $\frac{\sin^2(\theta)}{1-\cos(\theta)}$.

PROOF:

Step	Left Side of Equation		Equivalent Right Side	Domain	Reason
1	$\frac{\sin^2(\theta)}{1-\cos(\theta)}$	$=$	$\frac{1-\cos^2(\theta)}{1-\cos(\theta)}$	$\theta \neq 2\pi k$ for all integers k	Substitution property of equality using $\sin^2(\theta) = 1-\cos^2(\theta)$
2		$=$	$\frac{(1-\cos(\theta))(1+\cos(\theta))}{1-\cos(\theta)}$		Distributive property
3	$\frac{\sin^2(\theta)}{1-\cos(\theta)}$	$=$	$1+\cos(\theta)$		

Exercises 1–2

Prove that the following are trigonometric identities, beginning with the side of the equation that seems to be more complicated and starting the proof by restricting x to values where the identity is valid. Make sure that the complete identity statement is included at the end of the proof.

1. $\tan(x) = \frac{\sec(x)}{\csc(x)}$ for real numbers $x \neq \frac{\pi}{2} + \pi k$, for all integers k.

2. $\cot(x) + \tan(x) = \sec(x)\csc(x)$ for all real $x \neq \frac{\pi}{2}n$ for integer n.

Problem Set

1. Does $\sin(x+y)$ equal $\sin(x)+\sin(y)$ for all real numbers x and y?
   a. Find each of the following: $\sin\left(\frac{\pi}{2}\right), \sin\left(\frac{\pi}{4}\right), \sin\left(\frac{3\pi}{4}\right)$.
   b. Are $\sin\left(\frac{\pi}{2}+\frac{\pi}{4}\right)$ and $\sin\left(\frac{\pi}{2}\right)+\sin\left(\frac{\pi}{4}\right)$ equal?
   c. Are there any values of x and y for which $\sin(x+y)=\sin(x)+\sin(y)$?

2. Use $\tan(x)=\frac{\sin(x)}{\cos(x)}$ and identities involving the sine and cosine functions to establish the following identities for the tangent function. Identify the values of x where the equation is an identity.
   a. $\tan(\pi-x)=\tan(x)$
   b. $\tan(x+\pi)=\tan(x)$
   c. $\tan(2\pi-x)=-\tan(x)$
   d. $\tan(-x)=-\tan(x)$

3. Rewrite each of the following expressions as a single term. Identify the values of theta for which the original expression and your expression are equal:
   a. $\cot(\theta)\sec(\theta)\sin(\theta)$
   b. $\left(\frac{1}{1-\sin(x)}\right)\left(\frac{1}{1+\sin(x)}\right)$
   c. $\frac{1}{\cos^2(x)}-\frac{1}{\cot^2(x)}$
   d. $\frac{(\tan(x)-\sin(x))(1+\cos(x))}{\sin^3(x)}$

4. Prove that for any two real numbers a and b,
$$\sin^2(a)-\sin^2(b)+\cos^2(a)\sin^2(b)-\sin^2(a)\cos^2(b)=0.$$

5. Prove that the following statements are identities for all values of θ for which both sides are defined, and describe that set.
   a. $\cot(\theta)\sec(\theta)=\csc(\theta)$
   b. $\big(\csc(\theta)+\cot(\theta)\big)\big(1-\cos(\theta)\big)=\sin(\theta)$
   c. $\tan^2(\theta)-\sin^2(\theta)=\tan^2(\theta)\sin^2(\theta)$
   d. $\frac{4+\tan^2(x)-\sec^2(x)}{\csc^2(x)}=3\sin^2(x)$
   e. $\frac{(1+\sin(\theta))^2+\cos^2(\theta)}{1+\sin(\theta)}=2$

6. Prove that the value of the following expression does not depend on the value of y:
$$\cot(y)\frac{\tan(x)+\tan(y)}{\cot(x)+\cot(y)}.$$

Lesson 17: Trigonometric Identity Proofs

Classwork

Opening Exercise

We have seen that $\sin(\alpha+\beta) \neq \sin(\alpha)+\sin(\beta)$. So, what is $\sin(\alpha+\beta)$? Begin by completing the following table:

$\boldsymbol{\alpha}$	$\boldsymbol{\beta}$	$\mathbf{sin}(\boldsymbol{\alpha})$	$\mathbf{sin}(\boldsymbol{\beta})$	$\mathbf{sin}(\boldsymbol{\alpha}+\boldsymbol{\beta})$	$\mathbf{sin}(\boldsymbol{\alpha})\ \mathbf{cos}(\boldsymbol{\beta})$	$\mathbf{sin}(\boldsymbol{\alpha})\ \mathbf{sin}(\boldsymbol{\beta})$	$\mathbf{cos}(\boldsymbol{\alpha})\ \mathbf{cos}(\boldsymbol{\beta})$	$\mathbf{cos}(\boldsymbol{\alpha})\ \mathbf{sin}(\boldsymbol{\beta})$
$\frac{\pi}{6}$	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{4}$	$\frac{1}{2}$		
$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	1			$\frac{\sqrt{3}}{4}$	
$\frac{\pi}{4}$	$\frac{\pi}{6}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	$\frac{1+\sqrt{3}}{2\sqrt{2}}$	$\frac{\sqrt{3}}{2\sqrt{2}}$			
$\frac{\pi}{4}$	$\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1		$\frac{1}{2}$	$\frac{1}{2}$	
$\frac{\pi}{3}$	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}$				$\frac{\sqrt{3}}{4}$
$\frac{\pi}{3}$	$\frac{\pi}{4}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1+\sqrt{3}}{2\sqrt{2}}$		$\frac{\sqrt{3}}{2\sqrt{2}}$	$\frac{1}{2\sqrt{2}}$	

Use the following table to formulate a conjecture for $\cos(\alpha + \beta)$:

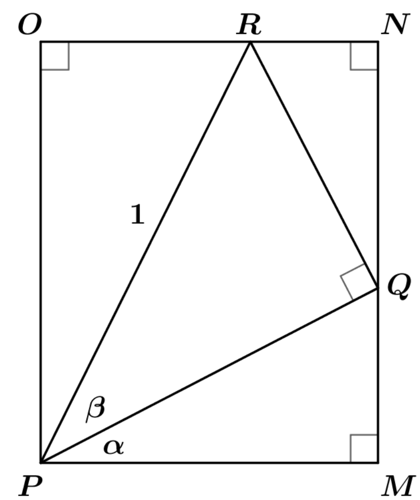
$\boldsymbol{\alpha}$	$\boldsymbol{\beta}$	$\mathbf{cos}(\boldsymbol{\alpha})$	$\mathbf{cos}(\boldsymbol{\beta})$	$\mathbf{cos}(\boldsymbol{\alpha}+\boldsymbol{\beta})$	$\mathbf{sin}(\boldsymbol{\alpha})\,\mathbf{cos}(\boldsymbol{\beta})$	$\mathbf{sin}(\boldsymbol{\alpha})\,\mathbf{sin}(\boldsymbol{\beta})$	$\mathbf{cos}(\boldsymbol{\alpha})\,\mathbf{cos}(\boldsymbol{\beta})$	$\mathbf{cos}(\boldsymbol{\alpha})\,\mathbf{sin}(\boldsymbol{\beta})$
$\frac{\pi}{6}$	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{1}{2}$	$\mathbf{\frac{1}{2}}$	$\frac{\sqrt{3}}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	$\frac{\sqrt{3}}{4}$
$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\mathbf{0}$	$\frac{1}{4}$	$\frac{\sqrt{3}}{4}$	$\frac{\sqrt{3}}{4}$	$\frac{3}{4}$
$\frac{\pi}{4}$	$\frac{\pi}{6}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	$\mathbf{\frac{\sqrt{3}-1}{2\sqrt{2}}}$	$\frac{\sqrt{3}}{2\sqrt{2}}$	$\frac{1}{2\sqrt{2}}$	$\frac{\sqrt{3}}{2\sqrt{2}}$	$\frac{1}{2\sqrt{2}}$
$\frac{\pi}{4}$	$\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	$\mathbf{0}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
$\frac{\pi}{3}$	$\frac{\pi}{3}$	$\frac{1}{2}$	$\frac{1}{2}$	$\mathbf{-\frac{1}{2}}$	$\frac{\sqrt{3}}{4}$	$\frac{3}{4}$	$\frac{1}{4}$	$\frac{\sqrt{3}}{4}$
$\frac{\pi}{3}$	$\frac{\pi}{4}$	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\mathbf{\frac{1-\sqrt{3}}{2\sqrt{2}}}$	$\frac{\sqrt{3}}{2\sqrt{2}}$	$\frac{\sqrt{3}}{2\sqrt{2}}$	$\frac{1}{2\sqrt{2}}$	$\frac{1}{2\sqrt{2}}$

Examples 1–2: Formulas for $\sin(\alpha + \beta)$ and $\cos(\alpha + \beta)$

1. One conjecture is that the formula for the sine of the sum of two numbers is $\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$. The proof can be a little long, but it is fairly straightforward. We will prove only the case when the two numbers are positive, and their sum is less than $\frac{\pi}{2}$.

   a. Let α and β be positive real numbers such that $0 < \alpha + \beta < \frac{\pi}{2}$.

   b. Construct rectangle $MNOP$ such that $PR = 1, m\angle PQR = 90°$, $m\angle RPQ = \beta$, and $m\angle QPM = \alpha$. See the figure at the right.



   c. Fill in the blanks in terms of α and β:

      i. $m\angle RPO =$ _____________.

      ii. $m\angle PRO =$ _______________.

      iii. Therefore, $\sin(\alpha+\beta) = PO$.

      iv. $RQ = \sin($_____$)$.

      v. $PQ = \cos($_____$)$.

   d. Let's label the angle and length measurements as shown.

   e. Use this new figure to fill in the blanks in terms of α and β:

      i. Why does $\sin(\alpha) = \frac{MQ}{\cos(\beta)}$?

      ii. Therefore, $MQ =$ _________.

      iii. $m\angle RQN =$ ____________.

   f. Now consider ΔRQN. Since $cos(\alpha) = \frac{QN}{sin(\beta)}$,

      $QN =$ _________.

   g. Label these lengths and angle measurements in the figure.

h. Since $MNOP$ is a rectangle, $OP = MQ + QN$.

i. Thus, $\sin(\alpha + \beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$.

Note that we have only proven the formula for the sine of the sum of two real numbers α and β in the case where $0 < \alpha + \beta < \frac{\pi}{2}$. A proof for all real numbers α and β breaks down into cases that are proven similarly to the case we have just seen. Although we are omitting the full proof, this formula holds for all real numbers α and β.

Thus, for any real numbers $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$,

$$\mathbf{sin}(\boldsymbol{\alpha} + \boldsymbol{\beta}) = \mathbf{sin}(\boldsymbol{\alpha})\mathbf{cos}(\boldsymbol{\beta}) + \mathbf{cos}(\boldsymbol{\alpha})\mathbf{sin}(\boldsymbol{\beta}).$$

2. Now let's prove our other conjecture, which is that the formula for the cosine of the sum of two numbers is

$$\cos(\alpha + \beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta).$$

Again, we will prove only the case when the two numbers are positive, and their sum is less than $\frac{\pi}{2}$. This time, we will use the sine addition formula and identities from previous lessons instead of working through a geometric proof.

Fill in the blanks in terms of α and β:

Let α and β be any real numbers. Then,

$$\begin{aligned}
\cos(\alpha + \beta) &= \sin\left(\frac{\pi}{2} - (\underline{\qquad\qquad})\right) \\
&= \sin\big((\underline{\qquad\qquad}) - \beta\big) \\
&= \sin\big((\underline{\qquad\qquad}) + (-\beta)\big) \\
&= \sin(\underline{\qquad\qquad})\cos(-\beta) + \cos(\underline{\qquad\qquad})\sin(-\beta) \\
&= \cos(\alpha)\cos(-\beta) + \sin(\alpha)\sin(-\beta) \\
&= \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta).
\end{aligned}$$

Thus, for all real numbers $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$,

$$\mathbf{cos}(\boldsymbol{\alpha} + \boldsymbol{\beta}) = \mathbf{cos}(\boldsymbol{\alpha})\mathbf{cos}(\boldsymbol{\beta}) - \mathbf{sin}(\boldsymbol{\alpha})\mathbf{sin}(\boldsymbol{\beta}).$$

Exercises 1–2: Formulas for $\sin(\alpha - \beta)$ and $\cos(\alpha - \beta)$

1. Rewrite the expression $\sin(\alpha - \beta)$ as follows: $\sin(\alpha + (-\beta))$. Use the rewritten form to find a formula for the sine of the difference of two angles, recalling that the sine is an odd function.

2. Now use the same idea to find a formula for the cosine of the difference of two angles. Recall that the cosine is an even function.

Thus, for all real numbers α and β,

$$\sin(\alpha - \beta) = \sin(\alpha)\cos(\beta) - \cos(\alpha)\sin(\beta), \text{ and}$$

$$\cos(\alpha - \beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta).$$

Exercises 3–5

3. Derive a formula for $\tan(\alpha + \beta)$ in terms of $\tan(\alpha)$ and $\tan(\beta)$ for $\frac{2n+1}{2}\pi < \theta < \frac{2n+3}{2}\pi$, for any integer n.

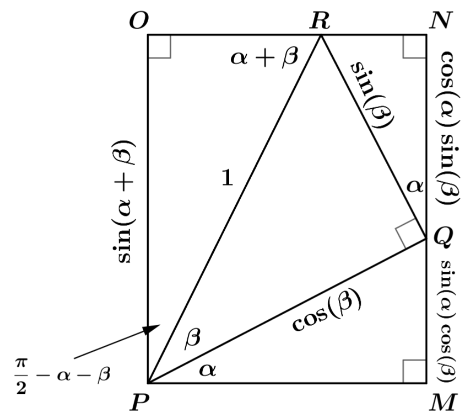
   Hint: Use the addition formulas for sine and cosine.

4. Derive a formula for $\sin(2u)$ in terms of $\sin(u)$ and $\cos(u)$ for all real numbers u.

5. Derive a formula for $\cos(2u)$ in terms of $\sin(u)$ and $\cos(u)$ for all real numbers u.

Problem Set

1. Prove the formula
   $\cos(\alpha + \beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$ for $0 < \alpha + \beta < \frac{\pi}{2}$
   using the rectangle $MNOP$ in the figure at the right and calculating PM, RN, and RO in terms of α and β.



2. Derive a formula for $\tan(2u)$ for $u \neq \frac{\pi}{4} + \frac{k\pi}{2}$ and $u \neq \frac{\pi}{2} + k\pi$, for all integers k.

3. Prove that $\cos(2u) = 2\cos^2(u) - 1$ is true for any real number u.

4. Prove that $\frac{1}{\cos(x)} - \cos(x) = \sin(x) \cdot \tan(x)$ is true for $x \neq \frac{\pi}{2} + k\pi$, for all integers k.

5. Write as a single term: $\cos\left(\frac{\pi}{4} + \theta\right) + \cos\left(\frac{\pi}{4} - \theta\right)$.

6. Write as a single term: $\sin(25°)\cos(10°) - \cos(25°)\sin(10°)$.

7. Write as a single term: $\cos(2x)\cos(x) + \sin(2x)\sin(x)$.

8. Write as a single term: $\frac{\sin(\alpha+\beta)+\sin(\alpha-\beta)}{\cos(\alpha)\cos(\beta)}$, where $\cos(\alpha) \neq 0$ and $\cos(\beta) \neq 0$.

9. Prove that for all values of θ, $\cos\left(\frac{3\pi}{2} + \theta\right) = \sin(\theta)$.

10. Prove that for all values of θ, $\cos(\pi - \theta) = -\cos(\theta)$.