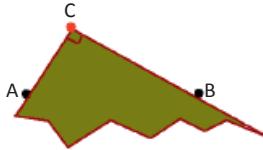


## Lesson 1: Thales' Theorem

### Classwork

#### Opening Exercise

- Mark points  $A$  and  $B$  on the sheet of white paper provided by your teacher.
- Take the colored paper provided, and “push” that paper up between points  $A$  and  $B$  on the white sheet.
- Mark on the white paper the location of the corner of the colored paper, using a different color than black. Mark that point  $C$ . See the example below.



- Do this again, pushing the corner of the colored paper up between the black points but at a different angle. Again, mark the location of the corner. Mark this point  $D$ .
- Do this again and then again, multiple times. Continue to label the points. What curve do the colored points ( $C, D, \dots$ ) seem to trace?

#### Exploratory Challenge

Choose one of the colored points ( $C, D, \dots$ ) that you marked. Draw the right triangle formed by the line segment connecting the original two points  $A$  and  $B$  and that colored point. Draw a rotated copy of the triangle underneath it.

Label the acute angles in the original triangle as  $x$  and  $y$ , and label the corresponding angles in the rotated triangle the same.

Todd says  $ABCC'$  is a rectangle. Maryam says  $ABCC'$  is a quadrilateral, but she's not sure it's a rectangle. Todd is right but doesn't know how to explain himself to Maryam. Can you help him out?

- What composite figure is formed by the two triangles? How would you prove it?

- What is the sum of  $x$  and  $y$ ? Why?

- ii. How do we know that the figure whose vertices are the colored points ( $C, D, \dots$ ) and points  $A$  and  $B$  is a rectangle?
  
- b. Draw the two diagonals of the rectangle. Where is the midpoint of the segment connecting the two original points  $A$  and  $B$ ? Why?
  
- c. Label the intersection of the diagonals as point  $P$ . How does the distance from point  $P$  to a colored point ( $C, D, \dots$ ) compare to the distance from  $P$  to points  $A$  and  $B$ ?
  
- d. Choose another colored point, and construct a rectangle using the same process you followed before. Draw the two diagonals of the new rectangle. How do the diagonals of the new and old rectangle compare? How do you know?
  
- e. How does your drawing demonstrate that all the colored points you marked do indeed lie on a circle?

**Example 1**

In the Exploratory Challenge, you proved the converse of a famous theorem in geometry. Thales' theorem states: *If A, B, and C are three distinct points on a circle and segment  $\overline{AB}$  is a diameter of the circle, then  $\angle ACB$  is right.*

Notice that, in the proof in the Exploratory Challenge, you started with a right angle (the corner of the colored paper) and created a circle. With Thales' theorem, you must start with the circle, and then create a right angle.

Prove Thales' theorem.

- a. Draw circle  $P$  with distinct points  $A$ ,  $B$ , and  $C$  on the circle and diameter  $\overline{AB}$ . Prove that  $\angle ACB$  is a right angle.
  
  
  
  
  
  
- b. Draw a third radius ( $\overline{PC}$ ). What types of triangles are  $\triangle APC$  and  $\triangle BPC$ ? How do you know?
  
  
  
  
  
  
- c. Using the diagram that you just created, develop a strategy to prove Thales' theorem.
  
  
  
  
  
  
- d. Label the base angles of  $\triangle APC$  as  $b^\circ$  and the bases of  $\triangle BPC$  as  $a^\circ$ . Express the measure of  $\angle ACB$  in terms of  $a^\circ$  and  $b^\circ$ .
  
  
  
  
  
  
- e. How can the previous conclusion be used to prove that  $\angle ACB$  is a right angle?

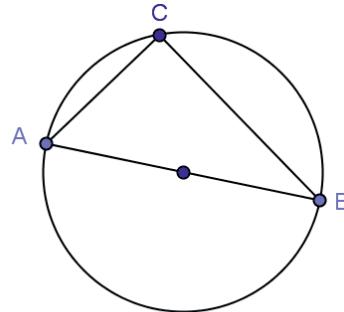
**Exercises 1–2**

1.  $\overline{AB}$  is a diameter of the circle shown. The radius is 12.5 cm, and  $AC = 7$  cm.

a. Find  $m\angle C$ .

b. Find  $AB$ .

c. Find  $BC$ .

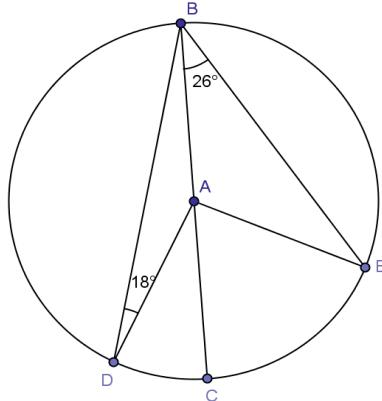


2. In the circle shown,  $\overline{BC}$  is a diameter with center  $A$ .

a. Find  $m\angle DAB$ .

b. Find  $m\angle BAE$ .

c. Find  $m\angle DAE$ .



**Lesson Summary****THEOREMS:**

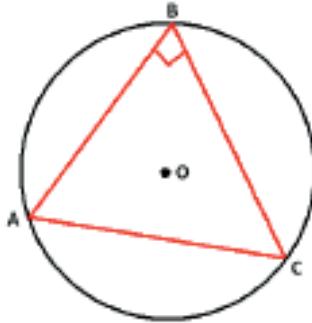
- **THALES' THEOREM:** If  $A, B$ , and  $C$  are three different points on a circle with  $\overline{AB}$  a diameter, then  $\angle ACB$  is a right angle.
- **CONVERSE OF THALES' THEOREM:** If  $\Delta ABC$  is a right triangle with  $\angle C$  the right angle, then  $A, B$ , and  $C$  are three distinct points on a circle with  $\overline{AB}$  a diameter.
- Therefore, given distinct points  $A, B$ , and  $C$  on a circle,  $\Delta ABC$  is a right triangle with  $\angle C$  the right angle if and only if  $\overline{AB}$  is a diameter of the circle.
- Given two points  $A$  and  $B$ , let point  $P$  be the midpoint between them. If  $C$  is a point such that  $\angle ACB$  is right, then  $BP = AP = CP$ .

**Relevant Vocabulary**

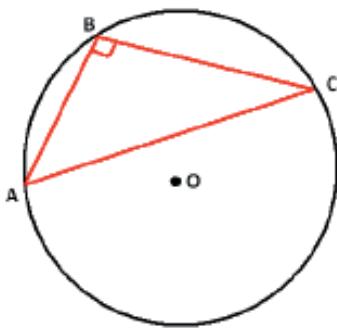
- **CIRCLE:** Given a point  $C$  in the plane and a number  $r > 0$ , the *circle* with center  $C$  and radius  $r$  is the set of all points in the plane that are distance  $r$  from the point  $C$ .
- **RADIUS:** May refer either to the line segment joining the center of a circle with any point on that circle (a *radius*) or to the length of this line segment (the *radius*).
- **DIAMETER:** May refer either to the segment that passes through the center of a circle whose endpoints lie on the circle (a *diameter*) or to the length of this line segment (the *diameter*).
- **CHORD:** Given a circle  $C$ , and let  $P$  and  $Q$  be points on  $C$ . The segment  $\overline{PQ}$  is called a *chord* of  $C$ .
- **CENTRAL ANGLE:** A *central angle* of a circle is an angle whose vertex is the center of a circle.

**Problem Set**

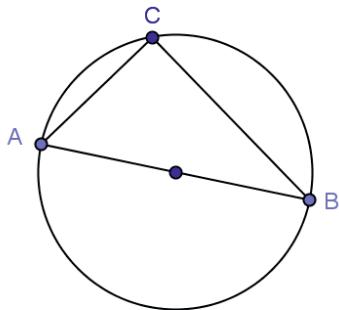
1.  $A, B$ , and  $C$  are three points on a circle, and angle  $ABC$  is a right angle. What's wrong with the picture below? Explain your reasoning.



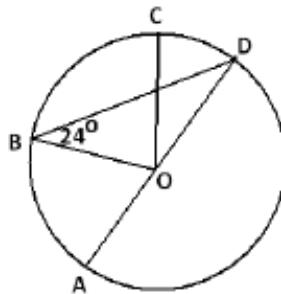
2. Show that there is something mathematically wrong with the picture below.



3. In the figure below,  $\overline{AB}$  is the diameter of a circle of radius 17 miles. If  $BC = 30$  miles, what is  $AC$ ?



4. In the figure below,  $O$  is the center of the circle, and  $\overline{AD}$  is a diameter.



- Find  $m\angle AOB$ .
- If  $m\angle AOB : m\angle COD = 3 : 4$ , what is  $m\angle BOC$ ?

5.  $\overline{PQ}$  is a diameter of a circle, and  $M$  is another point on the circle. The point  $R$  lies on the line  $\overleftrightarrow{MQ}$  such that  $RM = MQ$ . Show that  $m\angle PRM = m\angle PQM$ . (Hint: Draw a picture to help you explain your thinking!)
  
6. Inscribe  $\triangle ABC$  in a circle of diameter 1 such that  $\overline{AC}$  is a diameter. Explain why:
  - a.  $\sin(\angle A) = BC$ .
  - b.  $\cos(\angle A) = AB$ .

## Lesson 2: Circles, Chords, Diameters, and Their Relationships

### Classwork

#### Opening Exercise

Construct the perpendicular bisector of line segment  $\overline{AB}$  below (as you did in Module 1).



Draw another line that bisects  $\overline{AB}$  but is not perpendicular to it.

List one similarity and one difference between the two bisectors.

**Exercises 1–6**

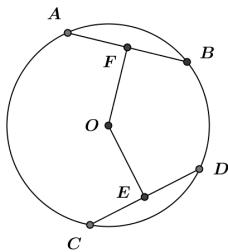
1. Prove the theorem: *If a diameter of a circle bisects a chord, then it must be perpendicular to the chord.*

2. Prove the theorem: *If a diameter of a circle is perpendicular to a chord, then it bisects the chord.*

3. The distance from the center of a circle to a chord is defined as the length of the perpendicular segment from the center to the chord. Note that, since this perpendicular segment may be extended to create a diameter of the circle, therefore, the segment also bisects the chord, as proved in Exercise 2 above.

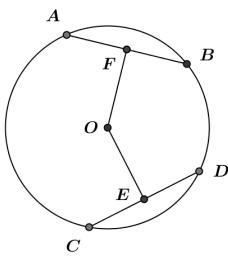
Prove the theorem: *In a circle, if two chords are congruent, then the center is equidistant from the two chords.*

Use the diagram below.



4. Prove the theorem: *In a circle, if the center is equidistant from two chords, then the two chords are congruent.*

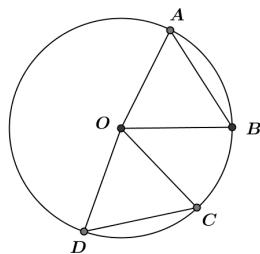
Use the diagram below.



5. A central angle defined by a chord is an angle whose vertex is the center of the circle and whose rays intersect the circle. The points at which the angle's rays intersect the circle form the endpoints of the chord defined by the central angle.

Prove the theorem: *In a circle, congruent chords define central angles equal in measure.*

Use the diagram below.



6. Prove the theorem: *In a circle, if two chords define central angles equal in measure, then they are congruent.*

**Lesson Summary**

**THEOREMS** about chords and diameters in a circle and their converses:

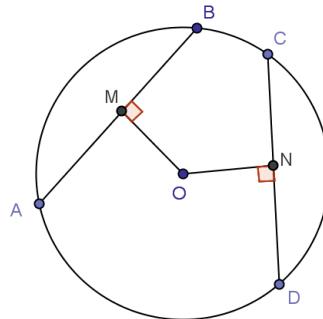
- If a diameter of a circle bisects a chord, then it must be perpendicular to the chord.
- If a diameter of a circle is perpendicular to a chord, then it bisects the chord.
- If two chords are congruent, then the center is equidistant from the two chords.
- If the center is equidistant from two chords, then the two chords are congruent.
- Congruent chords define central angles equal in measure.
- If two chords define central angles equal in measure, then they are congruent.

**Relevant Vocabulary**

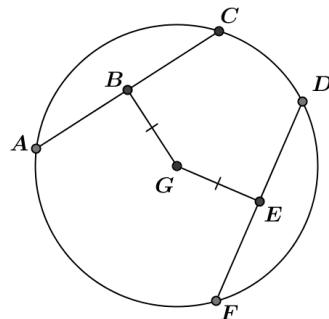
**EQUIDISTANT:** A point  $A$  is said to be *equidistant* from two different points  $B$  and  $C$  if  $AB = AC$ .

**Problem Set**

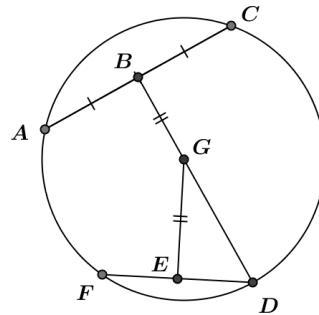
1. In this drawing,  $AB = 30$ ,  $OM = 20$ , and  $ON = 18$ . What is  $CN$ ?



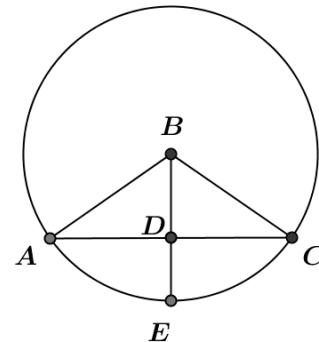
2. In the figure to the right,  $\overline{AC} \perp \overline{BG}$  and  $\overline{DF} \perp \overline{EG}$ ;  $EF = 12$ . Find  $AC$ .



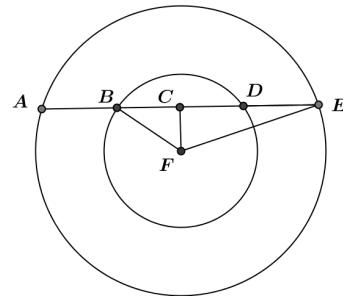
3. In the figure,  $AC = 24$ , and  $DG = 13$ . Find  $EG$ . Explain your work.



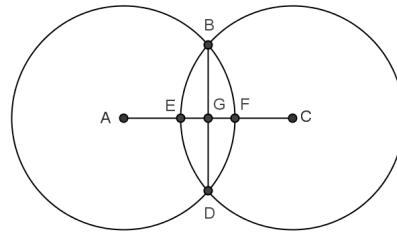
4. In the figure,  $AB = 10$ ,  $AC = 16$ . Find  $DE$ .



5. In the figure,  $CF = 8$ , and the two concentric circles have radii of 10 and 17. Find  $DE$ .

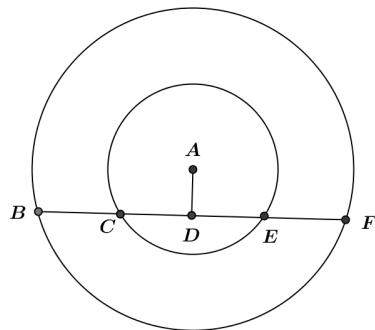


6. In the figure, the two circles have equal radii and intersect at points  $B$  and  $D$ .  $A$  and  $C$  are centers of the circles.  $AC = 8$ , and the radius of each circle is 5.  $\overline{BD} \perp \overline{AC}$ . Find  $BD$ . Explain your work.

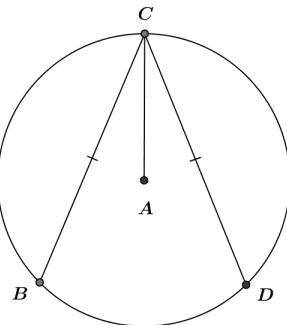


7. In the figure, the two concentric circles have radii of 6 and 14. Chord  $\overline{BF}$  of the larger circle intersects the smaller circle at  $C$  and  $E$ .  $CE = 8$ .  $\overline{AD} \perp \overline{BF}$ .

- Find  $AD$ .
- Find  $BF$ .



8. In the figure,  $A$  is the center of the circle, and  $CB = CD$ . Prove that  $\overline{AC}$  bisects  $\angle BCD$ .



9. In class, we proved: *Congruent chords define central angles equal in measure.*

- Give another proof of this theorem based on the properties of rotations. Use the figure from Exercise 5.
- Give a rotation proof of the converse: *If two chords define central angles of the same measure, then they must be congruent.*

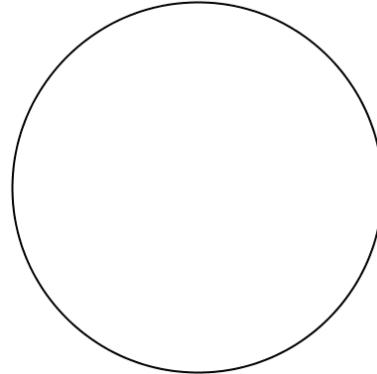
## Lesson 3: Rectangles Inscribed in Circles

### Classwork

#### Opening Exercise

Using only a compass and straightedge, find the location of the center of the circle below. Follow the steps provided.

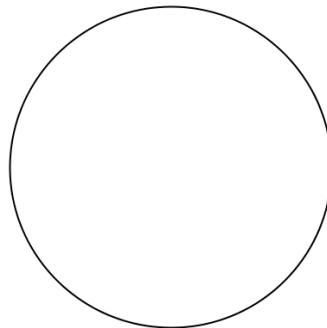
- Draw chord  $\overline{AB}$ .
- Construct a chord perpendicular to  $\overline{AB}$  at endpoint  $B$ .
- Mark the point of intersection of the perpendicular chord and the circle as point  $C$ .
- $\overline{AC}$  is a diameter of the circle. Construct a second diameter in the same way.
- Where the two diameters meet is the center of the circle.



Explain why the steps of this construction work.

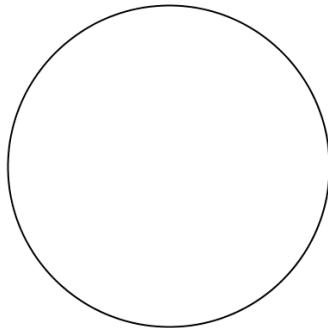
#### Exploratory Challenge

Construct a rectangle such that all four vertices of the rectangle lie on the circle below.



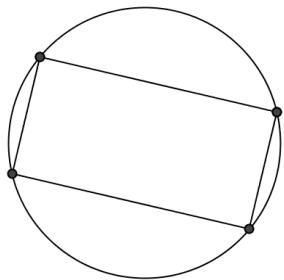
**Exercises 1–5**

1. Construct a kite inscribed in the circle below, and explain the construction using symmetry.



2. Given a circle and a rectangle, what must be true about the rectangle for it to be possible to inscribe a congruent copy of it in the circle?

3. The figure below shows a rectangle inscribed in a circle.



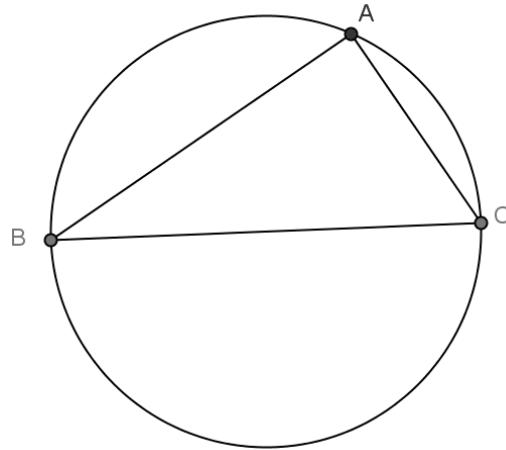
- a. List the properties of a rectangle.
- b. List all the symmetries this diagram possesses.

- c. List the properties of a square.
- d. List all the symmetries of the diagram of a square inscribed in a circle.
4. A rectangle is inscribed into a circle. The rectangle is cut along one of its diagonals and reflected across that diagonal to form a kite. Draw the kite and its diagonals. Find all the angles in this new diagram, given that the acute angle between the diagonals of the rectangle in the original diagram was  $40^\circ$ .

5. **Challenge:** Show that the 3 vertices of a right triangle are equidistant from the midpoint of the hypotenuse by showing that the perpendicular bisectors of the legs pass through the midpoint of the hypotenuse. (This is called the side-splitter theorem.)

- Draw the perpendicular bisectors of  $\overline{AB}$  and  $\overline{AC}$ .
- Label the point where they meet  $P$ . What is point  $P$ ?

- What can be said about the distance from  $P$  to each vertex of the triangle? What is the relationship between the circle and the triangle?



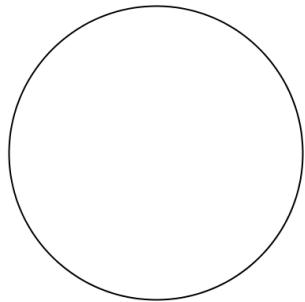
- Repeat this process, this time sliding  $B$  to another place on the circle and call it  $B'$ . What do you notice?
- Using what you have learned about angles, chords, and their relationships, what does the position of point  $P$  depend on? Why?

**Lesson Summary****Relevant Vocabulary**

**INSCRIBED POLYGON:** A polygon is *inscribed* in a circle if all vertices of the polygon lie on the circle.

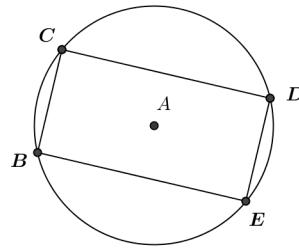
**Problem Set**

1. Using only a piece of  $8.5 \times 11$  inch copy paper and a pencil, find the location of the center of the circle below.

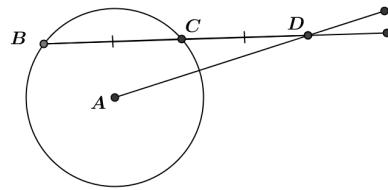


2. Is it possible to inscribe a parallelogram that is not a rectangle in a circle?

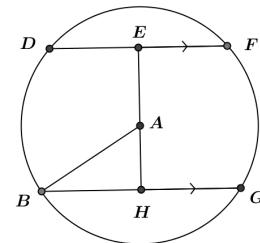
3. In the figure,  $BCDE$  is a rectangle inscribed in circle  $A$ .  $DE = 8$ ;  $BE = 12$ . Find  $AE$ .



4. Given the figure,  $BC = CD = 8$  and  $AD = 13$ . Find the radius of the circle.



5. In the figure,  $\overline{DF}$  and  $\overline{BG}$  are parallel chords 14 cm apart.  $DF = 12$  cm,  $AB = 10$  cm, and  $\overline{EH} \perp \overline{BG}$ . Find  $BG$ .



6. Use perpendicular bisectors of the sides of a triangle to construct a circle that circumscribes the triangle.

## Lesson 4: Experiments with Inscribed Angles

### Classwork

#### Opening Exercise

ARC:

MINOR AND MAJOR ARC:

INSCRIBED ANGLE:

CENTRAL ANGLE:

INTERCEPTED ARC OF AN ANGLE:

**Exploratory Challenge 1**

Your teacher will provide you with a straight edge, a sheet of colored paper in the shape of a trapezoid, and a sheet of plain white paper.

- Draw 2 points no more than 3 inches apart in the middle of the plain white paper, and label them  $A$  and  $B$ .
- Use the acute angle of your colored trapezoid to plot a point on the white sheet by placing the colored cutout so that the points  $A$  and  $B$  are on the edges of the acute angle and then plotting the position of the vertex of the angle. Label that vertex  $C$ .
- Repeat several times. Name the points  $D, E, \dots$

**Exploratory Challenge 2**

- a. Draw several of the angles formed by connecting points  $A$  and  $B$  on your paper with any of the additional points you marked as the acute angle was “pushed” through the points ( $C, D, E, \dots$ ). What do you notice about the measures of these angles?
- b. Draw several of the angles formed by connecting points  $A$  and  $B$  on your paper with any of the additional points you marked as the obtuse angle was “pushed” through the points from above. What do you notice about the measures of these angles?

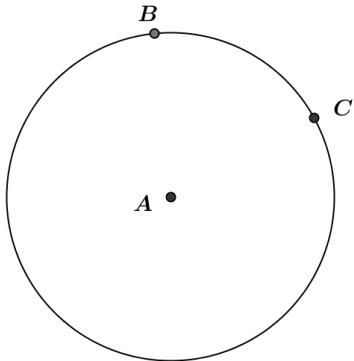
**Exploratory Challenge 3**

- a. Draw a point on the circle, and label it  $D$ . Create angle  $\angle BDC$ .
- b.  $\angle BDC$  is called an inscribed angle. Can you explain why?

- c. Arc  $\widehat{BC}$  is called the intercepted arc. Can you explain why?
- d. Carefully cut out the inscribed angle, and compare it to the angles of several of your neighbors.
- e. What appears to be true about each of the angles you drew?
- f. Draw another point on a second circle, and label it point  $E$ . Create angle  $\angle BEC$ , and cut it out. Compare  $\angle BDC$  and  $\angle BEC$ . What appears to be true about the two angles?
- g. What conclusion may be drawn from this? Will all angles inscribed in the circle from these two points have the same measure?
- h. Explain to your neighbor what you have just discovered.

**Exploratory Challenge 4**

- a. In the circle below, draw the angle formed by connecting points  $B$  and  $C$  to the center of the circle.



- b. Is  $\angle BAC$  an inscribed angle? Explain.
- c. Is it appropriate to call this *the* central angle? Why or why not?
- d. What is the intercepted arc?
- e. Is the measure of  $\angle BAC$  the same as the measure of one of the inscribed angles in Example 2?
- f. Can you make a prediction about the relationship between the inscribed angle and the central angle?

**Lesson Summary**

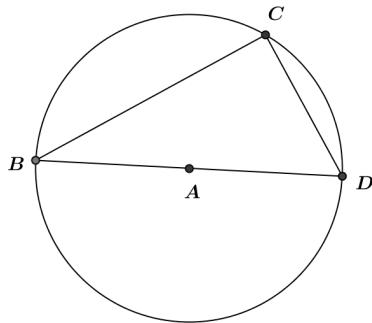
All inscribed angles from the same intercepted arc have the same measure.

**Relevant Vocabulary**

- **ARC:** An *arc* is a portion of the circumference of a circle.
- **MINOR AND MAJOR ARC:** Let  $C$  be a circle with center  $O$ , and let  $A$  and  $B$  be different points that lie on  $C$  but are not the endpoints of the same diameter. The *minor arc* is the set containing  $A$ ,  $B$ , and all points of  $C$  that are in the interior of  $\angle AOB$ . The *major arc* is the set containing  $A$ ,  $B$ , and all points of  $C$  that lie in the exterior of  $\angle AOB$ .
- **INSCRIBED ANGLE:** An *inscribed angle* is an angle whose vertex is on a circle, and each side of the angle intersects the circle in another point.
- **CENTRAL ANGLE:** A *central angle* of a circle is an angle whose vertex is the center of a circle.
- **INTERCEPTED ARC OF AN ANGLE:** An angle *intercepts* an arc if the endpoints of the arc lie on the angle, all other points of the arc are in the interior of the angle, and each side of the angle contains an endpoint of the arc.

**Problem Set**

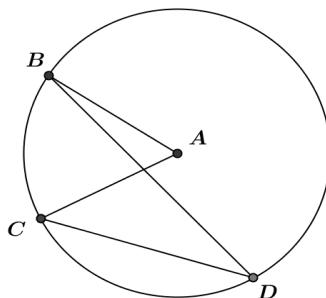
1. Using a protractor, measure both the inscribed angle and the central angle shown on the circle below.



$$m\angle BCD = \underline{\hspace{2cm}}$$

$$m\angle BAD = \underline{\hspace{2cm}}$$

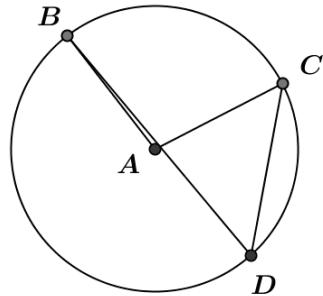
2. Using a protractor, measure both the inscribed angle and the central angle shown on the circle below.



$m\angle BDC = \underline{\hspace{2cm}}$

$m\angle BAC = \underline{\hspace{2cm}}$

3. Using a protractor, measure both the inscribed angle and the central angle shown on the circle below.

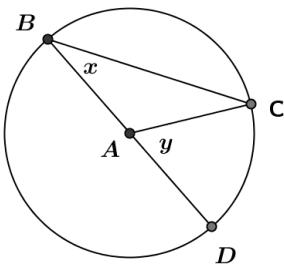


$m\angle BDC = \underline{\hspace{2cm}}$

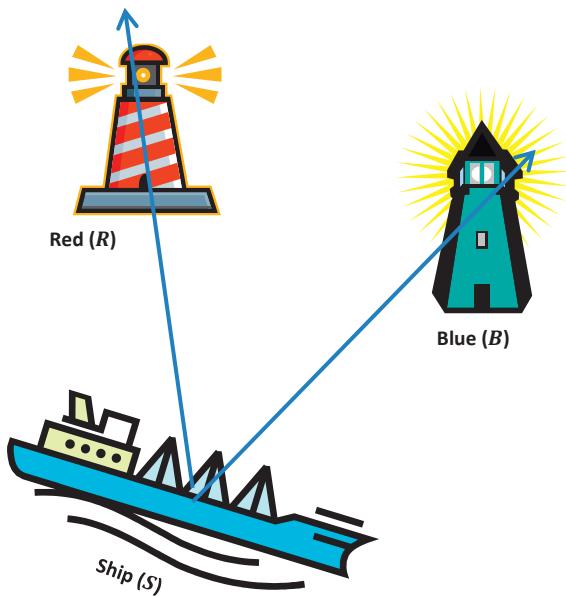
$m\angle BAC = \underline{\hspace{2cm}}$

4. What relationship between the measure of the inscribed angle and the measure of the central angle that intercept the same arc is illustrated by these examples?
5. Is your conjecture at least true for inscribed angles that measure  $90^\circ$ ?

6. Prove that  $y = 2x$  in the diagram below.



7. Red ( $R$ ) and blue ( $B$ ) lighthouses are located on the coast of the ocean. Ships traveling are in safe waters as long as the angle from the ship ( $S$ ) to the two lighthouses ( $\angle RSB$ ) is always less than or equal to some angle  $\theta$  called the “danger angle.” What happens to  $\theta$  as the ship gets closer to shore and moves away from shore? Why do you think a larger angle is dangerous?

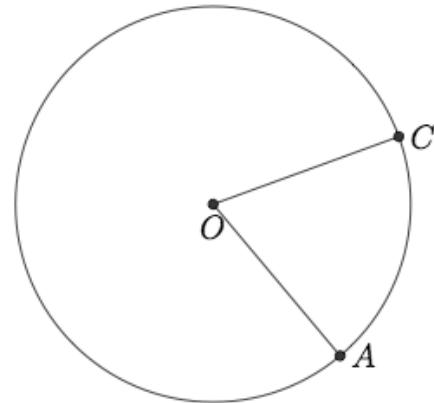


## Lesson 5: Inscribed Angle Theorem and its Applications

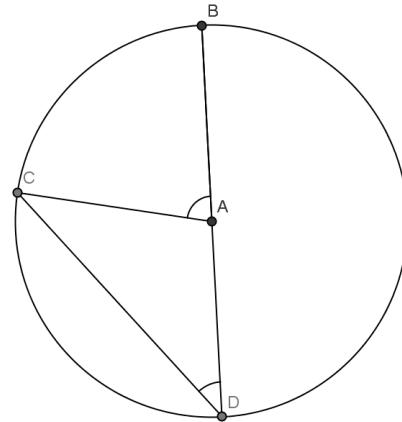
### Classwork

#### Opening Exercise

1.  $A$  and  $C$  are points on a circle with center  $O$ .
  - a. Draw a point  $B$  on the circle so that  $\overline{AB}$  is a diameter. Then draw the angle  $\angle ABC$ .
  - b. What angle in your diagram is an inscribed angle?
  - c. What angle in your diagram is a central angle?
  - d. What is the intercepted arc of angle  $\angle ABC$ ?
  - e. What is the intercepted arc of  $\angle AOC$ ?

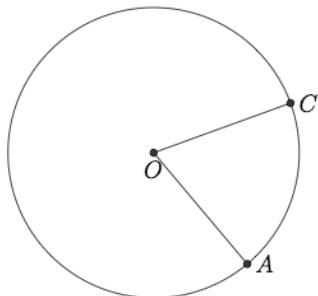


2. The measure of the inscribed angle is  $x$  and the measure of the central angle is  $y$ . Find  $m\angle CAB$  in terms of  $x$ .



**Example 1**

$A$  and  $C$  are points on a circle with center  $O$ .

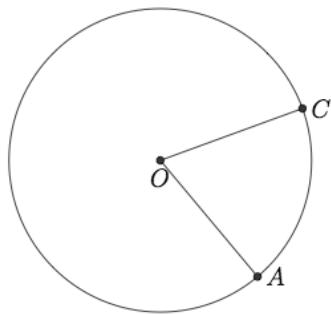


- a. What is the intercepted arc of  $\angle COA$ ? Color it red.
- b. Draw triangle  $AOC$ . What type of triangle is it? Why?
- c. What can you conclude about  $m\angle OCA$  and  $m\angle OAC$ ? Why?
- d. Draw a point  $B$  on the circle so that  $O$  is in the interior of the inscribed angle  $\angle ABC$ .
- e. What is the intercepted arc of angle  $\angle ABC$ ? Color it green.
- f. What do you notice about arc  $\widehat{AC}$ ?

- g. Let the measure of  $\angle ABC$  be  $x$  and the measure of  $\angle AOC$  be  $y$ . Can you prove that  $y = 2x$ ? (Hint: Draw the diameter that contains point  $B$ .)
- h. Does your conclusion support the inscribed angle theorem?
- i. If we combine the opening exercise and this proof, have we finished proving the inscribed angle theorem?

**Example 2**

$A$  and  $C$  are points on a circle with center  $O$ .



- a. Draw a point  $B$  on the circle so that  $O$  is in the exterior of the inscribed angle  $\angle ABC$ .
- b. What is the intercepted arc of angle  $\angle ABC$ ? Color it yellow.
- c. Let the measure of  $\angle ABC$  be  $x$ , and the measure of  $\angle AOC$  be  $y$ . Can you prove that  $y = 2x$ ? (Hint: Draw the diameter that contains point  $B$ .)

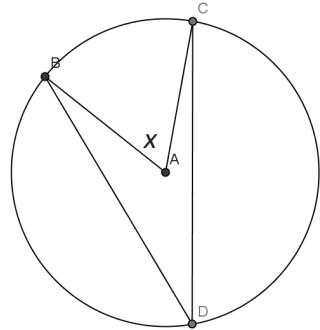
d. Does your conclusion support the inscribed angle theorem?

e. Have we finished proving the inscribed angle theorem?

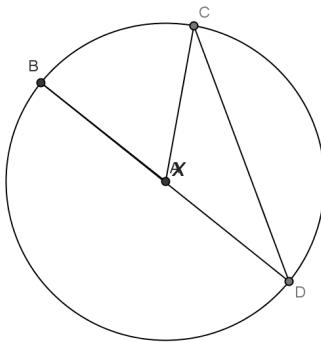
### Exercises 1–5

1. Find the measure of the angle with measure  $x$ .

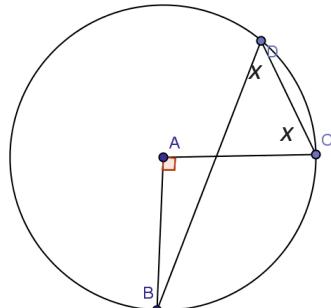
a.  $m\angle D = 25^\circ$



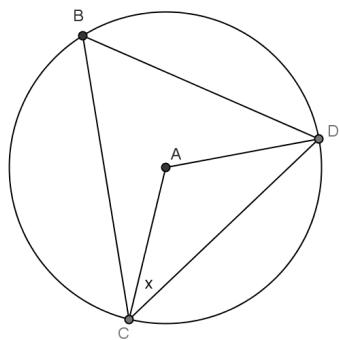
b.  $m\angle D = 15^\circ$



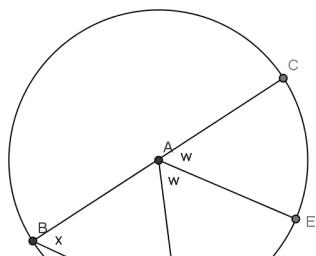
c.  $m\angle BAC = 90^\circ$



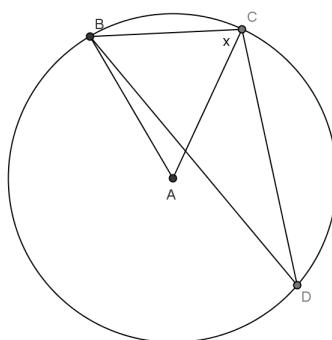
d.  $m\angle B = 32^\circ$



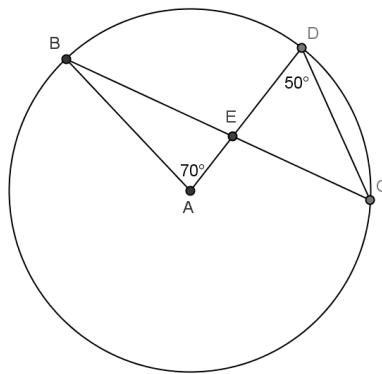
e.



f.  $m\angle D = 19^\circ$

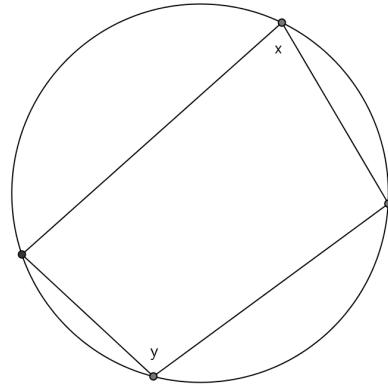


2. Toby says  $\triangle BEA$  is a right triangle because  $m\angle BEA = 90^\circ$ . Is he correct?  
Justify your answer.

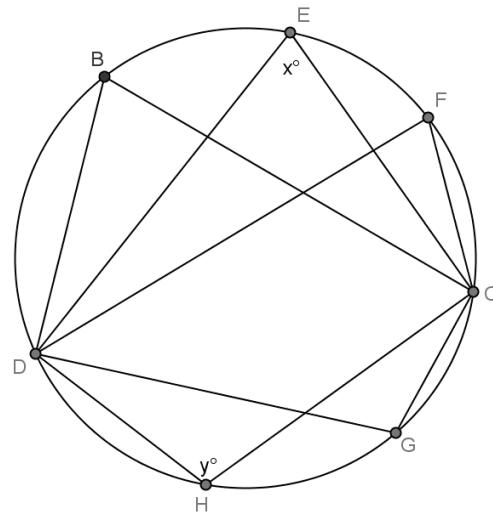


3. Let's look at relationships between inscribed angles.

- a. Examine the inscribed polygon below. Express  $x$  in terms of  $y$  and  $y$  in terms of  $x$ . Are the opposite angles in any quadrilateral inscribed in a circle supplementary? Explain.

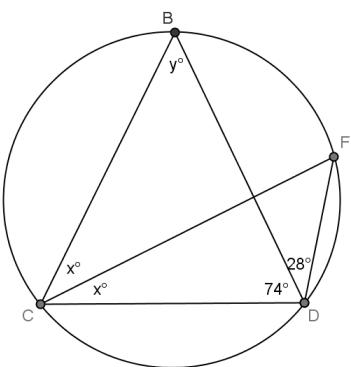


- b. Examine the diagram below. How many angles have the same measure, and what are their measures in terms of  $x$ ?

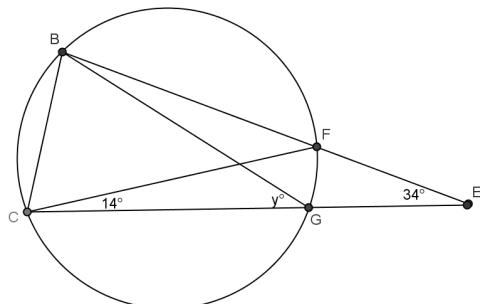


4. Find the measures of the labeled angles.

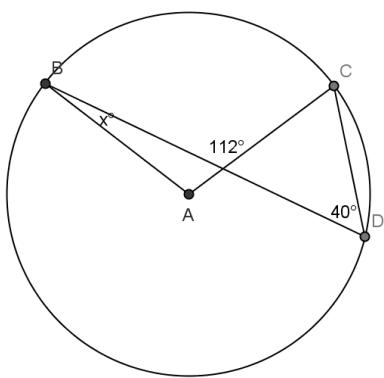
a.



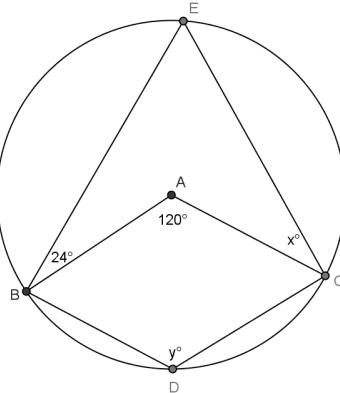
b.



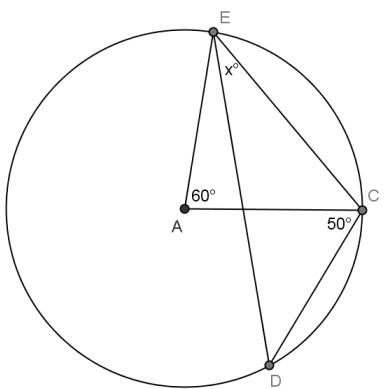
c.



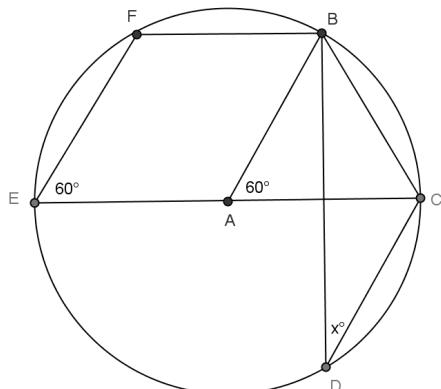
d.



e.



f.



**Lesson Summary****THEOREMS:**

- **THE INSCRIBED ANGLE THEOREM:** The measure of a central angle is twice the measure of any inscribed angle that intercepts the same arc as the central angle.
- **CONSEQUENCE OF INSCRIBED ANGLE THEOREM:** Inscribed angles that intercept the same arc are equal in measure.

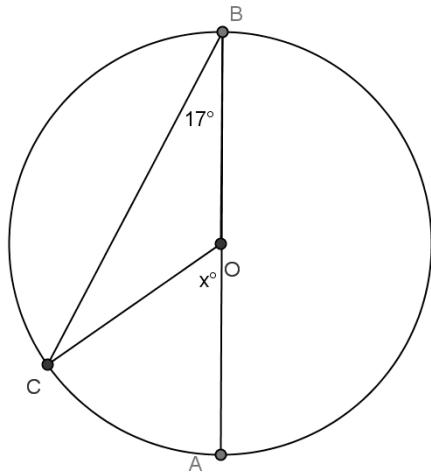
**Relevant Vocabulary**

- **INSCRIBED ANGLE:** An *inscribed angle* is an angle whose vertex is on a circle, and each side of the angle intersects the circle in another point.
- **INTERCEPTED ARC:** An angle *intercepts* an arc if the endpoints of the arc lie on the angle, all other points of the arc are in the interior of the angle, and each side of the angle contains an endpoint of the arc. An angle inscribed in a circle intercepts exactly one arc, in particular, the arc intercepted by a right angle is the semicircle in the interior of the angle.

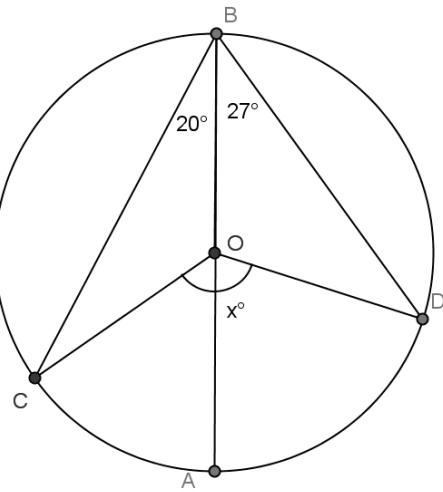
**Problem Set**

Find the value of  $x$  in each exercise.

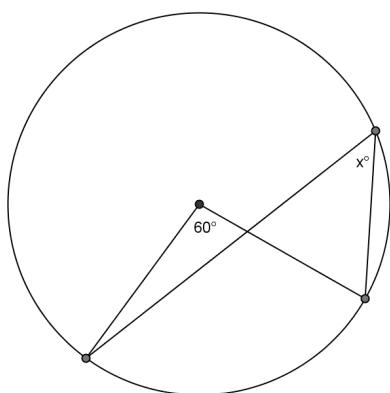
1.



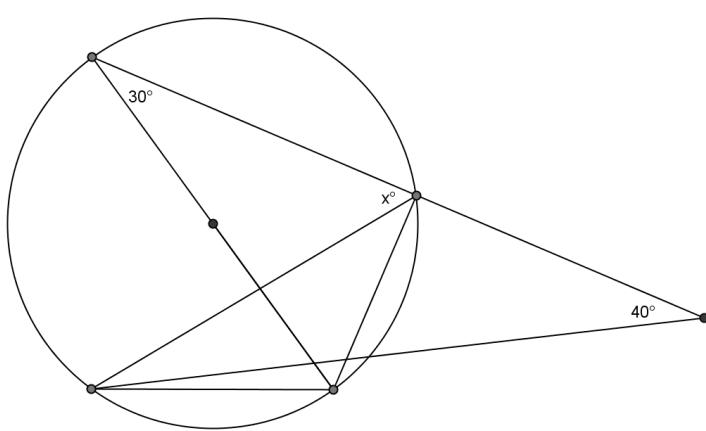
2.



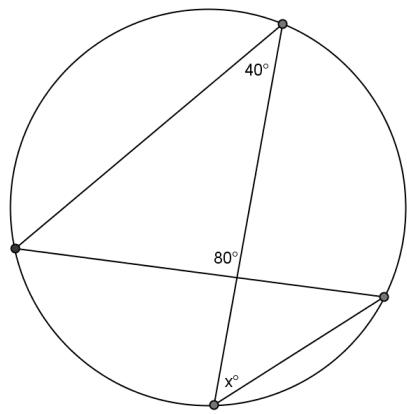
3.



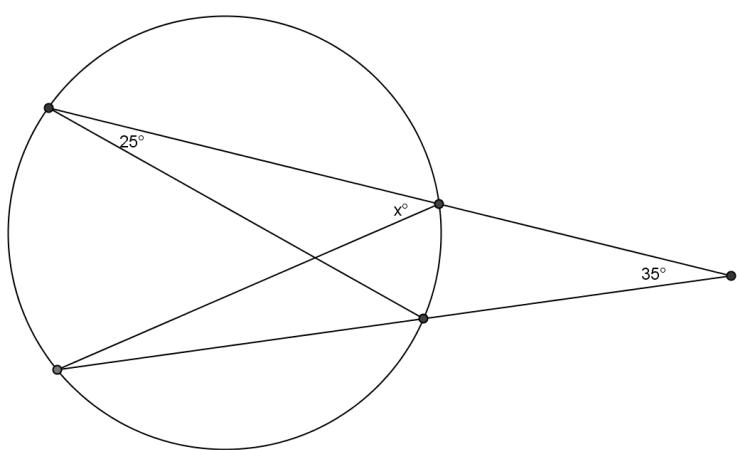
4.



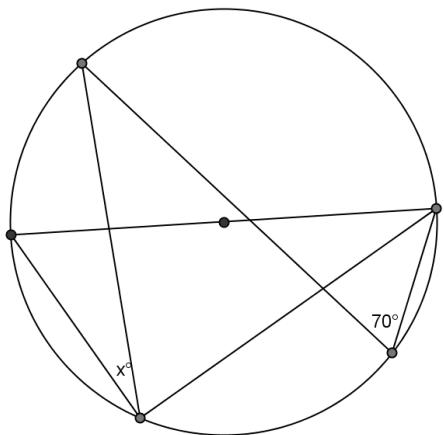
5.



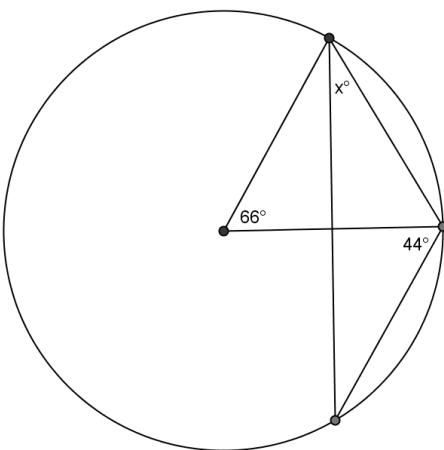
6.



7.

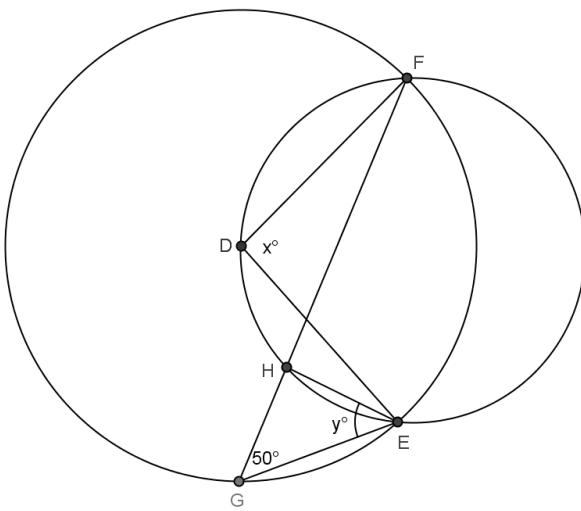


8.



9.

- a. The two circles shown intersect at  $E$  and  $F$ . The center of the larger circle,  $D$ , lies on the circumference of the smaller circle. If a chord of the larger circle,  $\overline{FG}$ , cuts the smaller circle at  $H$ , find  $x$  and  $y$ .



- b. How does this problem confirm the inscribed angle theorem?

10. In the figure below,  $\overline{ED}$  and  $\overline{BC}$  intersect at point E.

Prove:  $m\angle DAB + m\angle EAC = 2(m\angle BFD)$

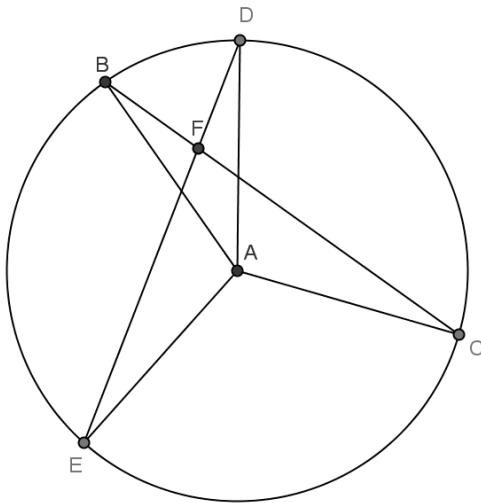
**PROOF:** Join  $\overline{BE}$ .

$$m\angle BED = \frac{1}{2}(m\angle \underline{\hspace{2cm}})$$

$$m\angle EBC = \frac{1}{2}(m\angle \underline{\hspace{2cm}})$$

In  $\triangle BEF$ ,

$$m\angle BEF + m\angle EBF = m\angle \underline{\hspace{2cm}}$$



$$\frac{1}{2}(m\angle \underline{\hspace{2cm}}) + \frac{1}{2}(m\angle \underline{\hspace{2cm}}) = m\angle \underline{\hspace{2cm}}$$

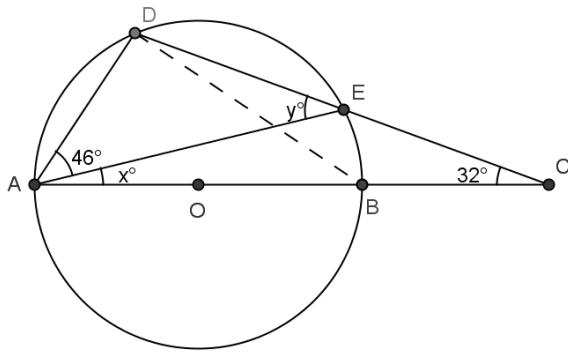
$$\therefore m\angle DAB + m\angle EAC = 2(m\angle BFD)$$

## Lesson 6: Unknown Angle Problems with Inscribed Angles in Circles

### Classwork

#### Opening Exercise

In a circle, a chord  $\overline{DE}$  and a diameter  $\overline{AB}$  are extended outside of the circle to meet at point  $C$ . If  $m\angle DAE = 46^\circ$ , and  $m\angle DCA = 32^\circ$ , find  $m\angle DEA$ .



Let  $m\angle DEA = y$ ,  $m\angle EAE = x$

$$\text{In } \triangle ABD, m\angle DBA = \quad \text{Reason}$$

$$m\angle ADB = \quad \text{Reason}$$

$$\therefore 46 + x + y + 90 = \quad \text{Reason}$$

$$x + y =$$

$$\text{In } \triangle ACE, y = x + 32 \quad \text{Reason}$$

$$x + x + 32 = \quad \text{Reason}$$

$$x =$$

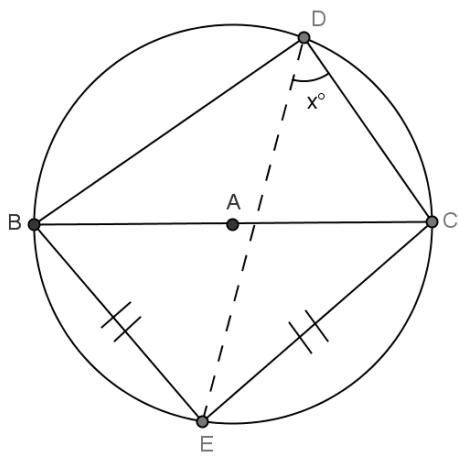
$$y =$$

$$m\angle DEA =$$

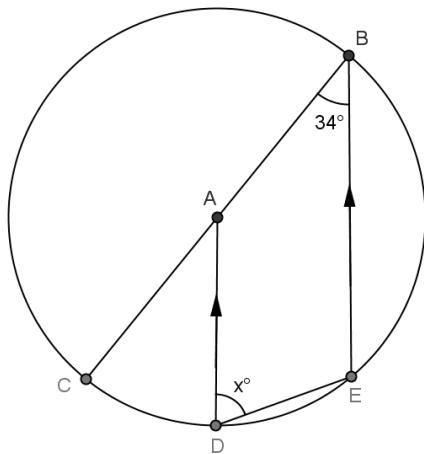
**Exercises 1–4**

Find the value  $x$  in each figure below, and describe how you arrived at the answer.

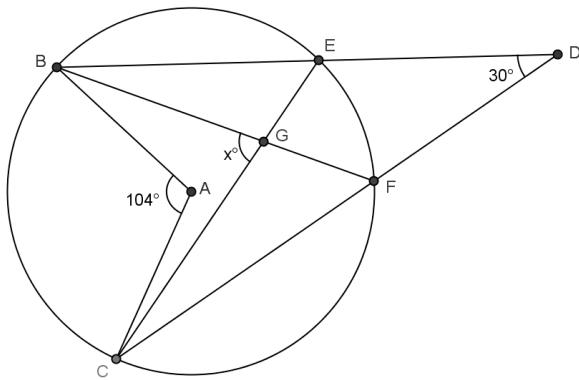
1. Hint: Thales' theorem



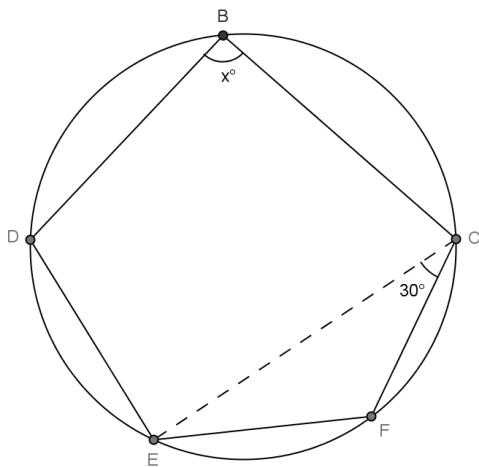
- 2.



- 3.



- 4.



**Lesson Summary:****THEOREMS:**

- **THE INSCRIBED ANGLE THEOREM:** The measure of a central angle is twice the measure of any inscribed angle that intercepts the same arc as the central angle
- **CONSEQUENCE OF INSCRIBED ANGLE THEOREM:** Inscribed angles that intercept the same arc are equal in measure.
- If  $A, B, B'$ , and  $C$  are four points with  $B$  and  $B'$  on the same side of line  $\overline{AC}$ , and angles  $\angle ABC$  and  $\angle AB'C$  are congruent, then  $A, B, B'$ , and  $C$  all lie on the same circle.

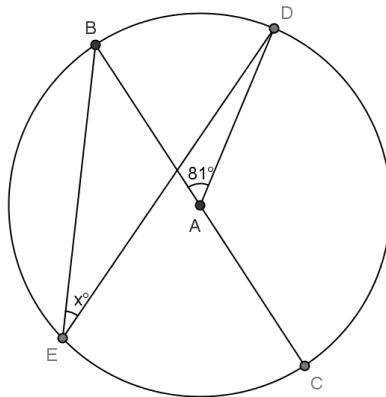
**Relevant Vocabulary**

- **CENTRAL ANGLE:** A *central angle* of a circle is an angle whose vertex is the center of a circle.
- **INSCRIBED ANGLE:** An *inscribed angle* is an angle whose vertex is on a circle, and each side of the angle intersects the circle in another point.
- **INTERCEPTED ARC:** An angle *intercepts* an arc if the endpoints of the arc lie on the angle, all other points of the arc are in the interior of the angle, and each side of the angle contains an endpoint of the arc. An angle inscribed in a circle intercepts exactly one arc, in particular, the arc intercepted by a right angle is the semicircle in the interior of the angle.

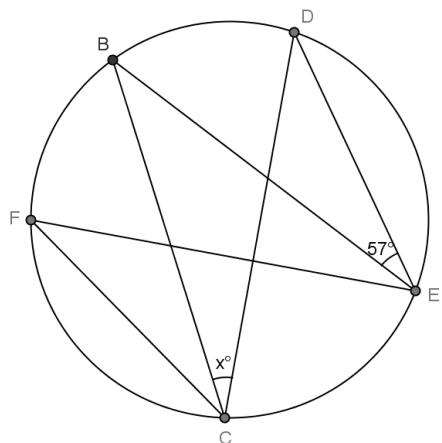
**Problem Set**

In Problems 1–5, find the value  $x$ .

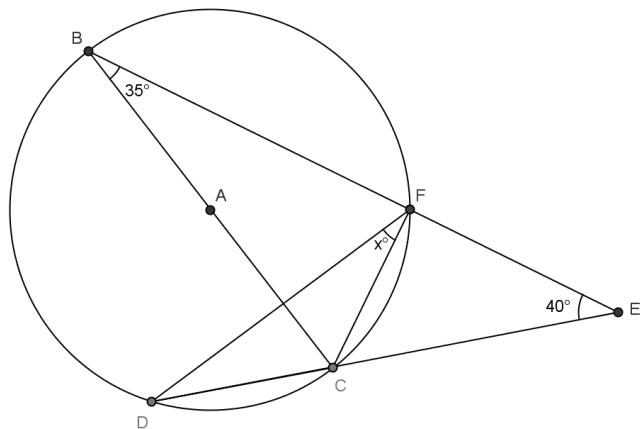
1.



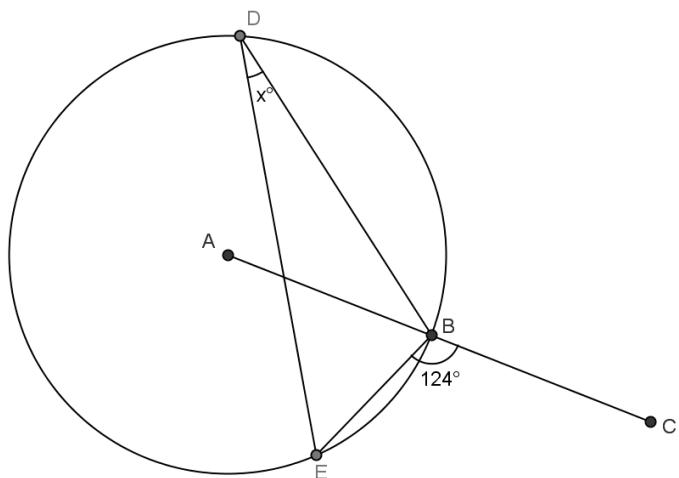
2.



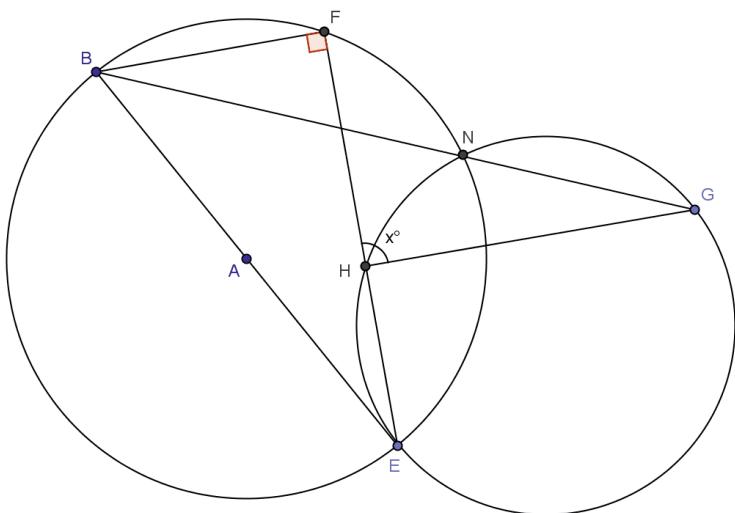
3.



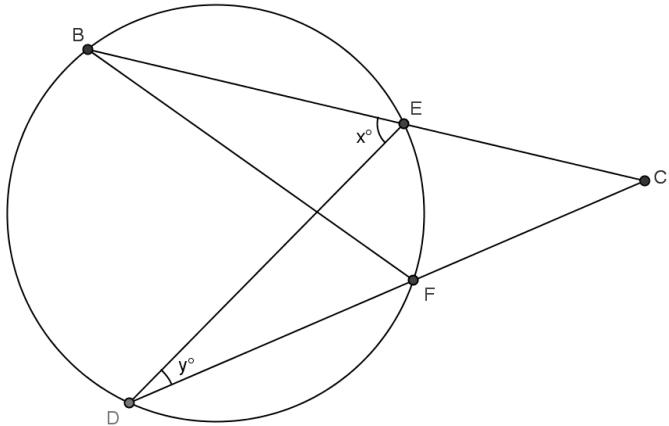
4.



5.

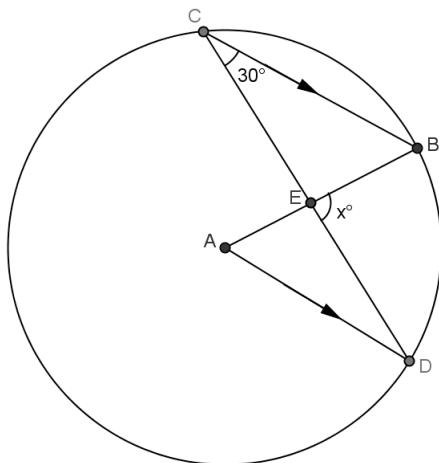


6. If  $BF = FC$ , express  $y$  in terms of  $x$ .



7.

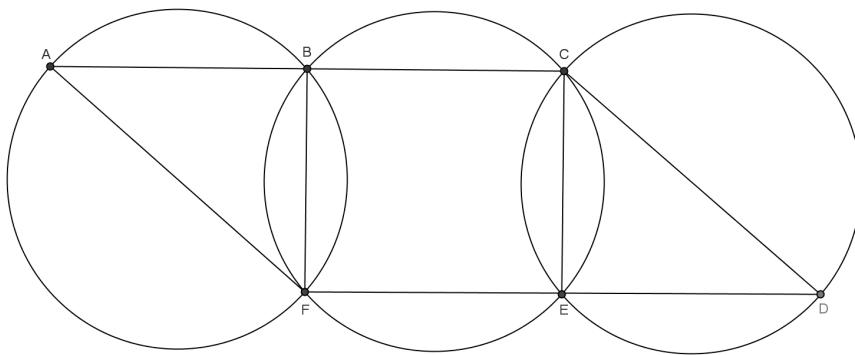
- a. Find the value  $x$ .

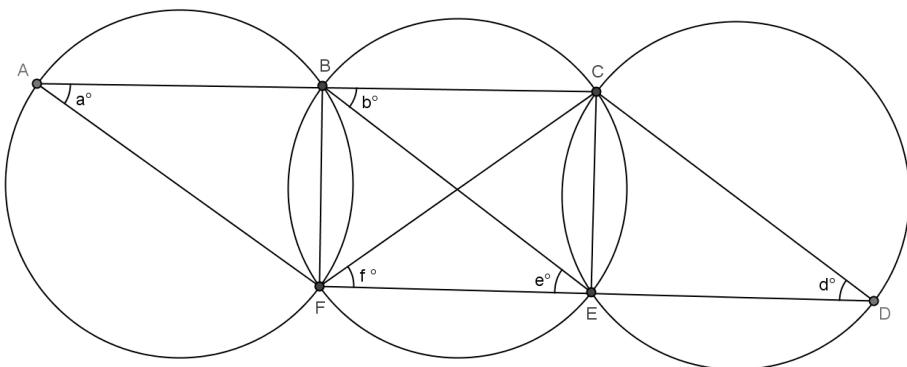


- b. Suppose the  $m\angle C = a^\circ$ . Prove that  $m\angle DEB = 3a^\circ$ .

8. In the figure below, three identical circles meet at  $B, F$  and  $C, E$  respectively.  $BF = CE$ .  $A, B, C$  and  $F, E, D$  lie on straight lines.

Prove  $ACDF$  is a parallelogram.



**PROOF:**Join  $BE$  and  $CF$ .

$$BF = CE$$

Reason: \_\_\_\_\_

$$a = \underline{\hspace{2cm}} = \underline{\hspace{2cm}} = \underline{\hspace{2cm}} = d$$

Reason: \_\_\_\_\_

$$\underline{\hspace{2cm}} = \underline{\hspace{2cm}}$$

 $\overline{AC} \parallel \overline{FD}$  Alternate angles are equal.

$$\underline{\hspace{2cm}} = \underline{\hspace{2cm}}$$

 $\overline{AF} \parallel \overline{BE}$  Corresponding angles are equal.

$$\underline{\hspace{2cm}} = \underline{\hspace{2cm}}$$

 $\overline{BE} \parallel \overline{CD}$ 

Corresponding angles are equal.

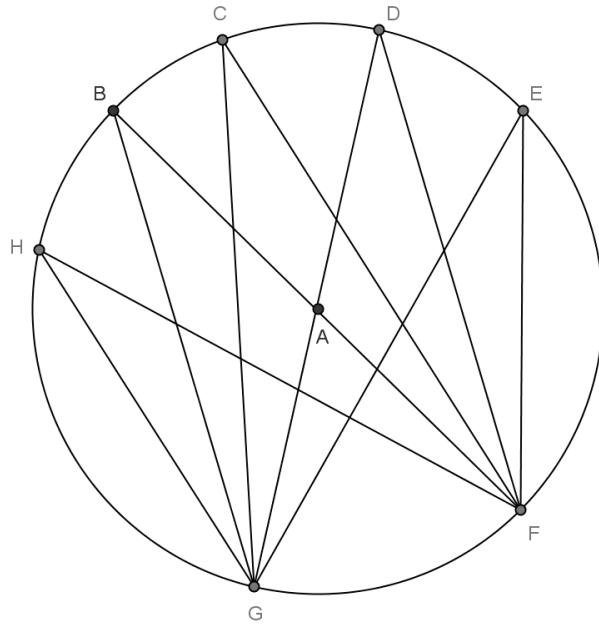
 $\overline{AF} \parallel \overline{BE} \parallel \overline{CD}$ 
 $ACDF$  is a parallelogram.

## Lesson 7: The Angle Measure of an Arc

### Classwork

#### Opening Exercise

If the measure of  $\angle GBF$  is  $17^\circ$ , name 3 other angles that have the same measure and explain why.

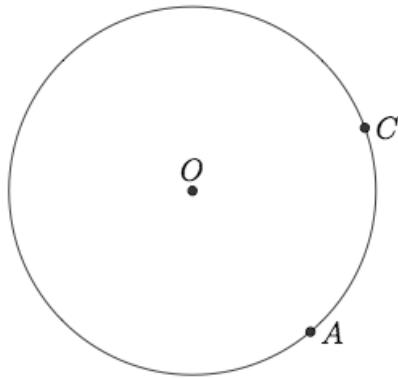


What is the measure of  $\angle GAF$ ? Explain.

Can you find the measure of  $\angle BAD$ ? Explain.

**Example 1**

What if we started with an angle inscribed in the minor arc between  $A$  and  $C$ ?

**Exercises 1–4**

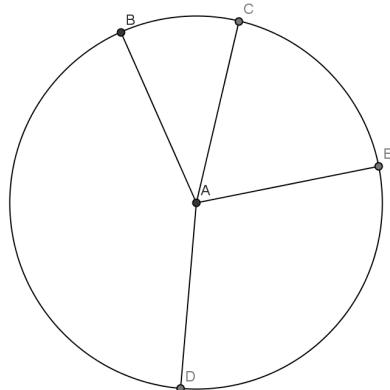
1. In circle  $A$ ,  $\widehat{BC}:\widehat{CE}:\widehat{ED}:\widehat{DB} = 1:2:3:4$ . Find

a.  $m\angle BAC$

b.  $m\angle DAE$

c.  $m\widehat{DB}$

d.  $m\widehat{CED}$

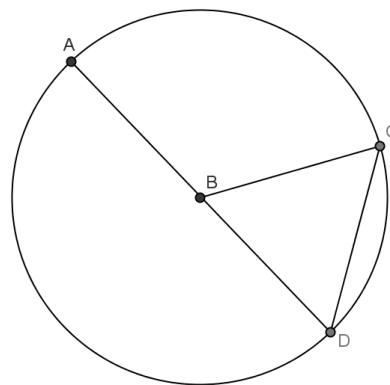


2. In circle  $B$ ,  $AB = CD$ . Find

a.  $m\widehat{CD}$

b.  $m\widehat{CAD}$

c.  $m\widehat{AD}$

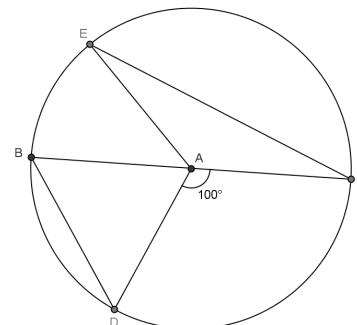


3. In circle  $A$ ,  $\overline{BC}$  is a diameter and  $m\angle DAC = 100^\circ$ . If  $m\widehat{EC} = 2m\widehat{BD}$ , find

a.  $m\angle BAE$

b.  $m\widehat{EC}$

c.  $m\widehat{DEC}$

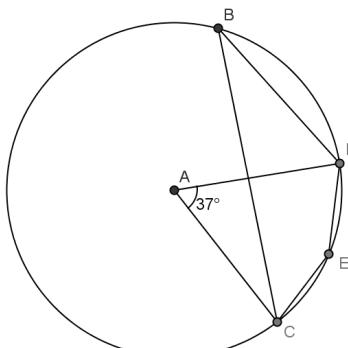


4. Given circle  $A$  with  $m\angle CAD = 37^\circ$ , find

a.  $m\widehat{CBD}$

b.  $m\angle CBD$

c.  $m\angle CED$



**Lesson Summary****THEOREMS:**

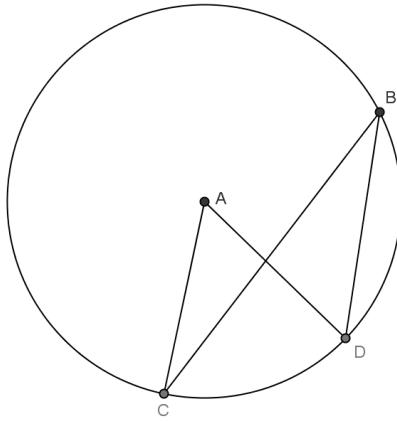
- **INSCRIBED ANGLE THEOREM:** The measure of an inscribed angle is half the measure of its intercepted arc.
- Two arcs (of possibly different circles) are similar if they have the same angle measure. Two arcs in the same or congruent circles are congruent if they have the same angle measure.
- All circles are similar.

**Relevant Vocabulary**

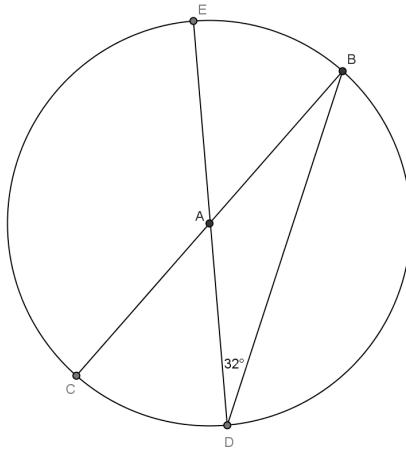
- **ARC:** An *arc* is a portion of the circumference of a circle.
- **MINOR AND MAJOR ARC:** Let  $C$  be a circle with center  $O$ , and let  $A$  and  $B$  be different points that lie on  $C$  but are not the endpoints of the same diameter. The *minor arc* is the set containing  $A, B$ , and all points of  $C$  that are in the interior of  $\angle AOB$ . The *major arc* is the set containing  $A, B$ , and all points of  $C$  that lie in the exterior of  $\angle AOB$ .
- **SEMICIRCLE:** In a circle, let  $A$  and  $B$  be the endpoints of a diameter. A *semicircle* is the set containing  $A, B$ , and all points of the circle that lie in a given half-plane of the line determined by the diameter.
- **INSCRIBED ANGLE:** An *inscribed angle* is an angle whose vertex is on a circle and each side of the angle intersects the circle in another point.
- **CENTRAL ANGLE:** A *central angle* of a circle is an angle whose vertex is the center of a circle.
- **INTERCEPTED ARC OF AN ANGLE:** An angle *intercepts* an arc if the endpoints of the arc lie on the angle, all other points of the arc are in the interior of the angle, and each side of the angle contains an endpoint of the arc.

**Problem Set**

- Given circle  $A$  with  $m\angle CAD = 50^\circ$ ,
  - Name a central angle.
  - Name an inscribed angle.
  - Name a chord.
  - Name a minor arc.
  - Name a major arc.
  - Find  $m\widehat{CD}$ .
  - Find  $m\widehat{CBD}$ .
  - Find  $m\angle CBD$ .

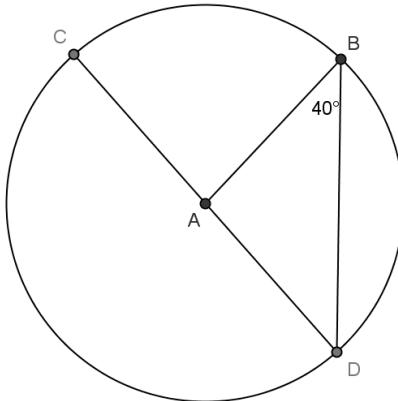


2. Given circle  $A$ , find the measure of each minor arc.

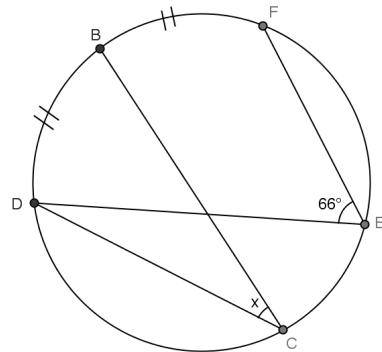


3. Given circle  $A$ , find

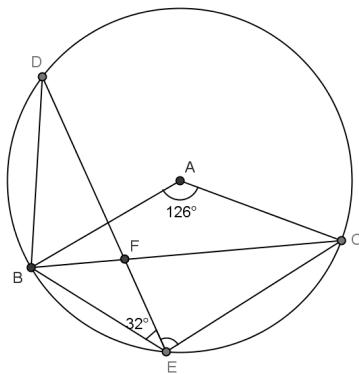
- $m\angle BAD$
- $m\angle CAB$
- $m\widehat{BC}$
- $m\widehat{BD}$
- $m\widehat{BCD}$



4. Find the angle measure of angle  $x$ .



5. In the figure,  $m\angle BAC = 126^\circ$  and  $m\angle BED = 32^\circ$ . Find  $m\angle DEC$ .



6. In the figure  $m\angle BCD = 74^\circ$ , and  $m\angle BDC = 42^\circ$ . K is the midpoint of  $\widehat{CB}$  and J is the midpoint of  $\widehat{BD}$ . Find  $m\angle KBD$  and  $m\angle CKJ$ .

Solution: Join  $BK, KC, KD, KJ, JC$ , and  $JD$ .

$$\widehat{BK} = \widehat{KC}$$

$$m\angle KDC = \frac{42}{2} = 21^\circ$$

$$a = \underline{\hspace{2cm}}$$

$$\text{In } \triangle BCD, \quad b = \underline{\hspace{2cm}}$$

$$c = \underline{\hspace{2cm}}$$

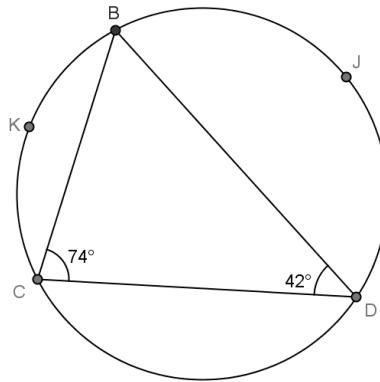
$$\widehat{BJ} = \widehat{JD}$$

$$m\angle JCD = \underline{\hspace{2cm}}$$

$$d = \underline{\hspace{2cm}}$$

$$m\angle KBD = a + b = \underline{\hspace{2cm}}$$

$$m\angle CKJ = c + d = \underline{\hspace{2cm}}$$

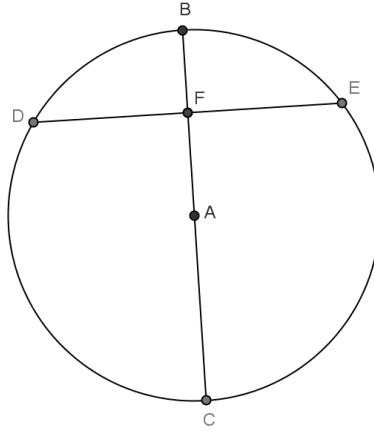


## Lesson 8: Arcs and Chords

### Classwork

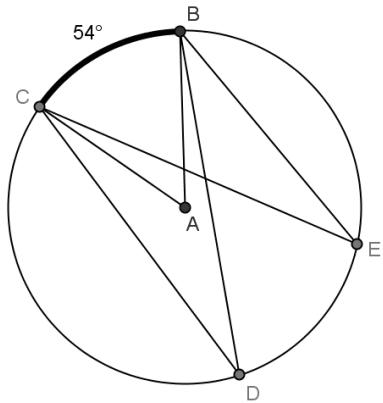
#### Opening Exercise

Given circle  $A$  with  $\overline{BC} \perp \overline{DE}$ ,  $FA = 6$ , and  $AC = 10$ . Find  $BF$  and  $DE$ . Explain your work.



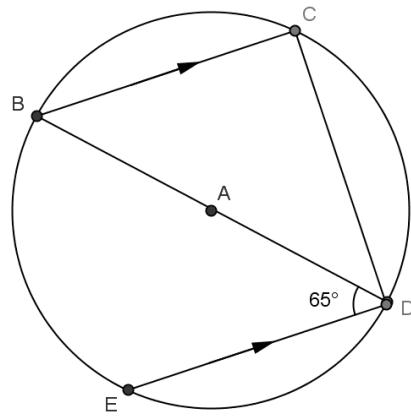
### Exercises

- Given circle  $A$  with  $m\widehat{BC} = 54^\circ$  and  $\angle CDB \cong \angle DBE$ , find  $m\widehat{DE}$ . Explain your work.



2. If two arcs in a circle have the same measure, what can you say about the quadrilateral formed by the four endpoints? Explain.

3. Find the angle measure of  $\widehat{CD}$  and  $\widehat{ED}$ .

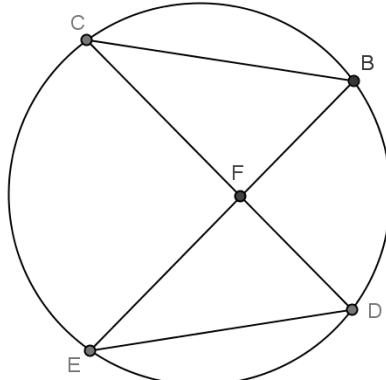


4.  $m\widehat{CB} = m\widehat{ED}$  and  $m\widehat{BC}:m\widehat{BD}:m\widehat{EC} = 1:2:4$ . Find

a.  $m\angle BCF$

b.  $m\angle EDF$

c.  $m\angle CFE$

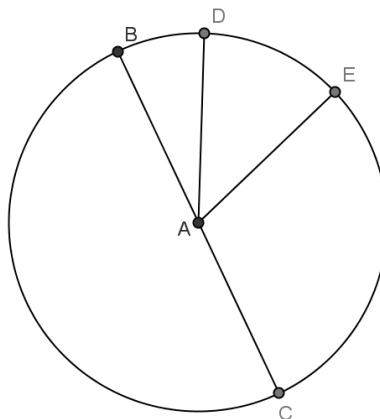


5.  $\overline{BC}$  is a diameter of circle  $A$ .  $m\widehat{BD}:m\widehat{DE}:m\widehat{EC} = 1:3:5$ . Find

a.  $m\widehat{BD}$

b.  $m\widehat{DEC}$

c.  $m\widehat{ECB}$



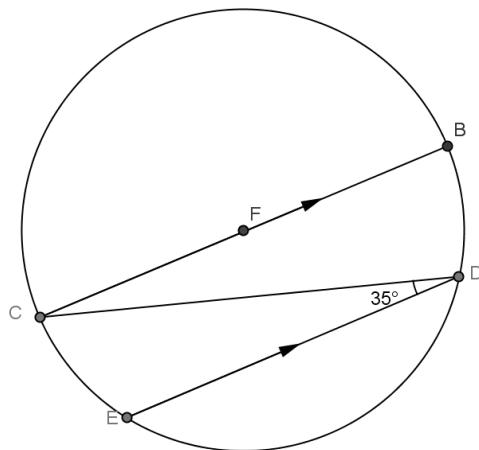
**Lesson Summary****THEOREMS:**

- Congruent chords have congruent arcs.
- Congruent arcs have congruent chords.
- Arcs between parallel chords are congruent.

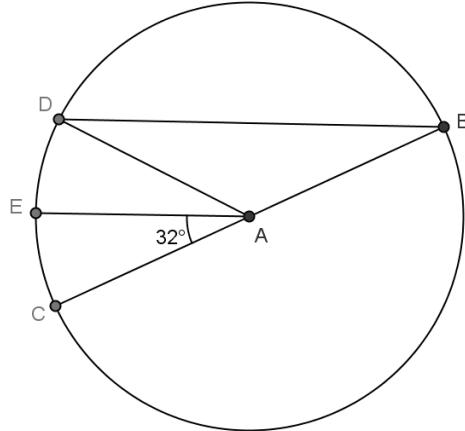
**Problem Set**

1. Find

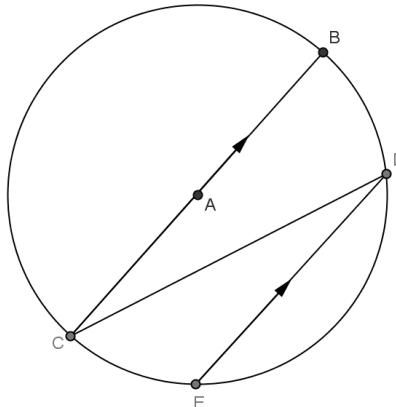
- $m\widehat{CE}$
- $m\widehat{BD}$
- $m\widehat{ED}$

2. In circle A,  $\overline{BC}$  is a diameter,  $m\widehat{CE} = m\widehat{ED}$ , and  $m\angle CAE = 32^\circ$ .

- Find  $m\angle CAD$ .
- Find  $m\angle ADC$ .

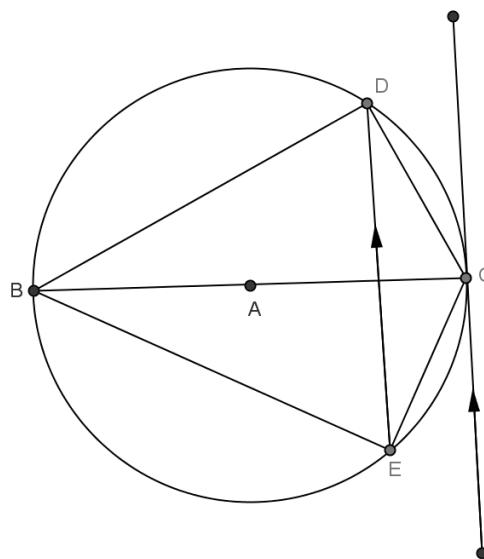


3. In circle A,  $\overline{BC}$  is a diameter,  $2m\widehat{CE} = m\widehat{ED}$ , and  $\overline{BC} \parallel \overline{DE}$ . Find  $m\angle CDE$ .

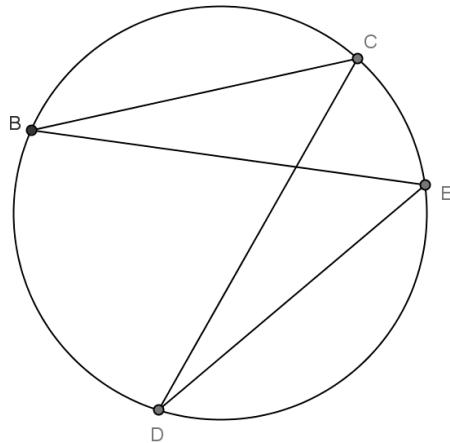


4. In circle A,  $\overline{BC}$  is a diameter and  $\widehat{CE} = 68^\circ$ .

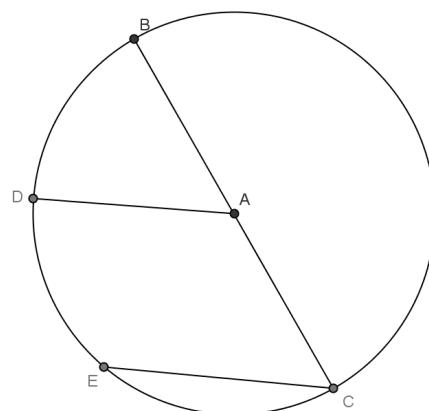
- Find  $m\widehat{CD}$ .
- Find  $m\angle DBE$ .
- Find  $m\angle DCE$



5. In the circle given,  $\widehat{BC} \cong \widehat{ED}$ . Prove  $\overline{BE} \cong \overline{DC}$ .



6. Given circle  $A$  with  $\overline{AD} \parallel \overline{CE}$ , show  $\widehat{BD} \cong \widehat{DE}$ .



7. In circle  $A$ ,  $\overline{AB}$  is a radius and  $\widehat{BC} \cong \widehat{BD}$  and  $m\angle CAD = 54^\circ$ . Find  $m\angle ABC$ . Complete the proof.

$$BC = BD \quad \underline{\hspace{10cm}}$$

$$m\angle \underline{\hspace{1cm}} = m\angle \underline{\hspace{1cm}} \quad \underline{\hspace{10cm}}$$

$$m\angle BAC + m\angle CAD + m\angle BAD = \underline{\hspace{10cm}}$$

$$2m\angle \underline{\hspace{1cm}} + 54^\circ = 360^\circ \quad \underline{\hspace{10cm}}$$

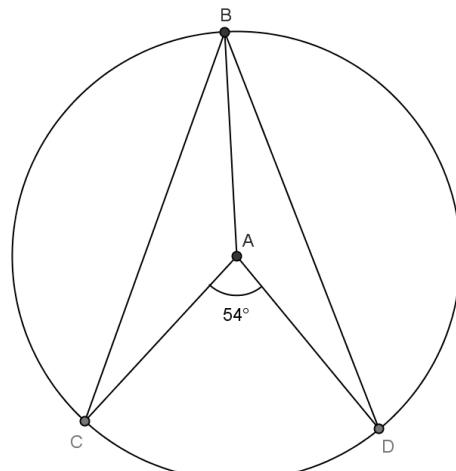
$$m\angle BAC = \underline{\hspace{1cm}}$$

$$AB = AC \quad \underline{\hspace{10cm}}$$

$$m\angle \underline{\hspace{1cm}} = m\angle \underline{\hspace{1cm}} \quad \underline{\hspace{10cm}}$$

$$2m\angle ABC + m\angle BAC = \underline{\hspace{10cm}}$$

$$m\angle ABC = \underline{\hspace{1cm}}$$

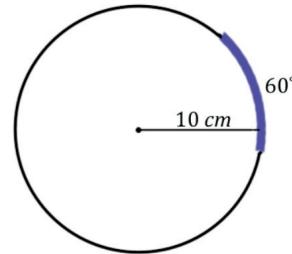


## Lesson 9: Arc Length and Areas of Sectors

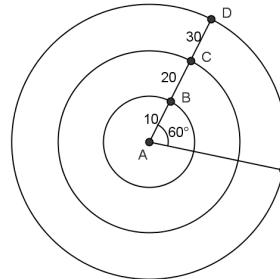
### Classwork

#### Example 1

- a. What is the length of the arc of degree measure  $60^\circ$  in a circle of radius 10 cm?



- b. Given the concentric circles with center  $A$  and with  $m\angle A = 60^\circ$ , calculate the arc length intercepted by  $\angle A$  on each circle. The inner circle has a radius of 10 and each circle has a radius 10 units greater than the previous circle.



- c. An arc, again of degree measure  $60^\circ$ , has an arc length of  $5\pi$  cm. What is the radius of the circle on which the arc sits?

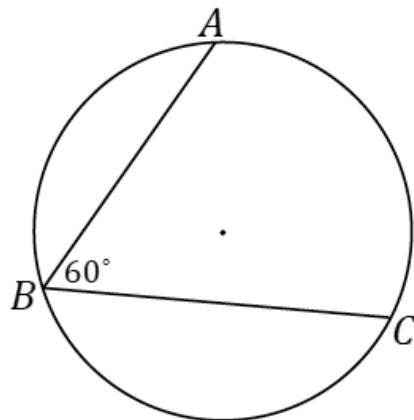
- d. Give a general formula for the length of an arc of degree measure  $x^\circ$  on a circle of radius  $r$ .

- e. Is the length of an arc intercepted by an angle proportional to the radius? Explain.

### Exercise 1

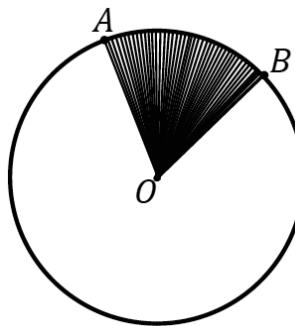
1. The radius of the following circle is 36 cm, and the  $m\angle ABC = 60^\circ$ .

a. What is the arc length of  $\widehat{AC}$ ?



b. What is the radian measure of the central angle?

**SECTOR:** Let  $\widehat{AB}$  be an arc of a circle with center  $O$  and radius  $r$ . The union of all segments  $\overline{OP}$ , where  $P$  is any point of  $\widehat{AB}$ , is called a *sector*.

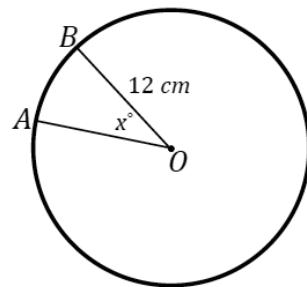


**Example 2**

- a. Circle  $O$  has a radius of 10 cm. What is the area of the circle? Write the formula.
  
  
  
  
  
  
- b. What is the area of half of the circle? Write and explain the formula.
  
  
  
  
  
  
- c. What is the area of a quarter of the circle? Write and explain the formula.
  
  
  
  
  
  
- d. Make a conjecture about how to determine the area of a sector defined by an arc measuring 60 degrees.
  
  
  
  
  
  
- e. Circle  $O$  has a minor arc  $\widehat{AB}$  with an angle measure of  $60^\circ$ . Sector  $AOB$  has an area of  $24\pi$ . What is the radius of circle  $O$ ?
  
  
  
  
  
  
- f. Give a general formula for the area of a sector defined by arc of angle measure  $x^\circ$  on a circle of radius  $r$ ?

**Exercises 2–3**

2. The area of sector  $AOB$  in the following image is  $28\pi$ . Find the measurement of the central angle labeled  $x^\circ$ .

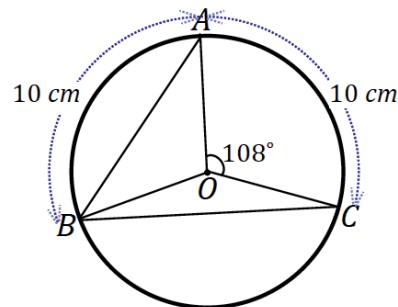


3. In the following figure, circle  $O$  has a radius of 8 cm,  $m\angle AOC = 108^\circ$  and  $\widehat{AB} = \widehat{AC} = 10$  cm. Find:

a.  $\angle OAB$

b.  $\widehat{BC}$

c. Area of sector  $BOC$



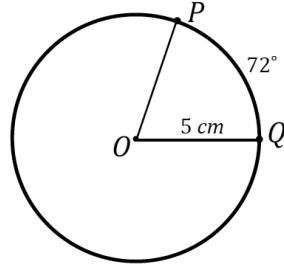
**Lesson Summary****Relevant Vocabulary**

- ARC:** An *arc* is any of the following three figures—a minor arc, a major arc, or a semicircle.
- LENGTH OF AN ARC:** The *length of an arc* is the circular distance around the arc.<sup>1</sup>
- MINOR AND MAJOR ARC:** In a circle with center  $O$ , let  $A$  and  $B$  be different points that lie on the circle but are not the endpoints of a diameter. The *minor arc* between  $A$  and  $B$  is the set containing  $A$ ,  $B$ , and all points of the circle that are in the interior of  $\angle AOB$ . The *major arc* is the set containing  $A$ ,  $B$ , and all points of the circle that lie in the exterior of  $\angle AOB$ .
- RADIAN:** A *radian* is the measure of the central angle of a sector of a circle with arc length of one radius length.
- SECTOR:** Let arc  $\widehat{AB}$  be an arc of a circle with center  $O$  and radius  $r$ . The union of the segments  $\overline{OP}$ , where  $P$  is any point on the arc  $\widehat{AB}$ , is called a *sector*. The arc  $\widehat{AB}$  is called the *arc of the sector*, and  $r$  is called its *radius*.
- SEMICIRCLE:** In a circle, let  $A$  and  $B$  be the endpoints of a diameter. A *semicircle* is the set containing  $A$ ,  $B$ , and all points of the circle that lie in a given half-plane of the line determined by the diameter.

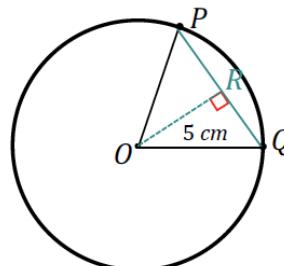
**Problem Set**

1.  $P$  and  $Q$  are points on the circle of radius 5 cm and the measure of arc  $\widehat{PQ}$  is  $72^\circ$ .  
Find, to one decimal place each of the following:

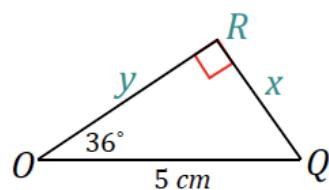
- The length of arc  $\widehat{PQ}$
- Find the ratio of the arc length to the radius of the circle.



- The length of chord  $PQ$



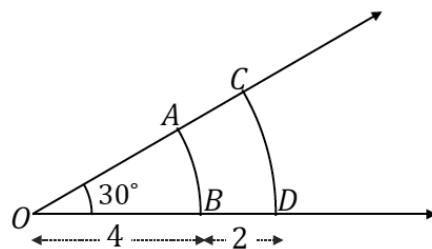
- d. The distance of the chord  $PQ$  from the center of the circle.



- e. The perimeter of sector  $POQ$ .  
f. The area of the wedge between the chord  $PQ$  and the arc  $\widehat{PQ}$   
g. The perimeter of this wedge.

2. What is the radius of a circle if the length of a  $45^\circ$  arc is  $9\pi$ ?

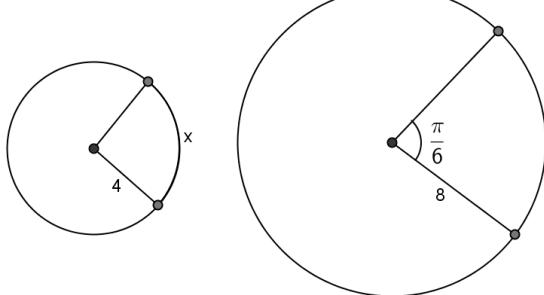
3. Arcs  $\widehat{AB}$  and  $\widehat{CD}$  both have an angle measure of  $30^\circ$ , but their arc lengths are not the same.  $\overline{OB} = 4$  and  $\overline{BD} = 2$ .
- What are the arc lengths of arcs  $\widehat{AB}$  and  $\widehat{CD}$ ?
  - What is the ratio of the arc length to the radius for all of these arcs? Explain.
  - What are the areas of the sectors  $AOB$  and  $COD$ ?



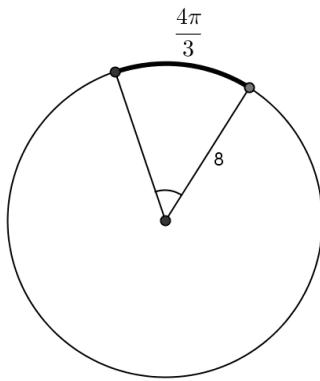
4. In the circles shown, find the value of  $x$ .

The circles shown have central angles that are equal in measure.

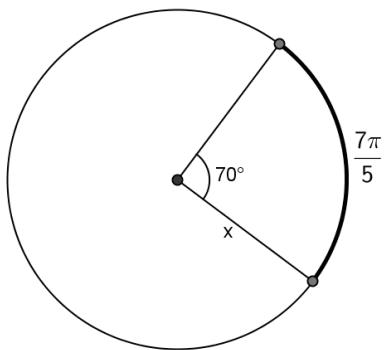
a.



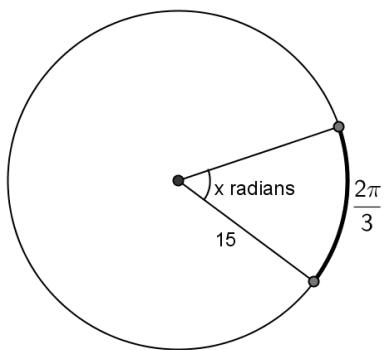
b.



c.

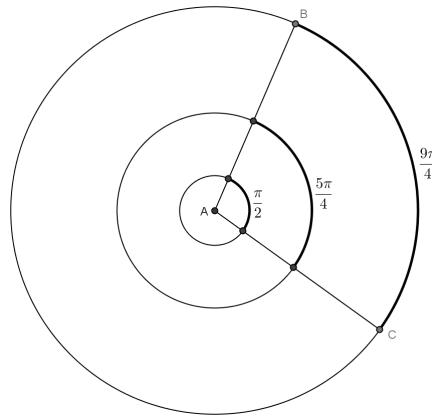


d.

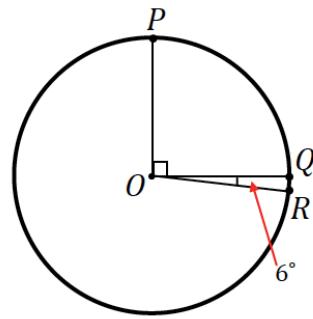


5. The concentric circles all have center  $A$ . The measure of the central angle is  $45^\circ$ . The arc lengths are given.

- Find the radius of each circle.
- Determine the ratio of the arc length to the radius of each circle, and interpret its meaning.

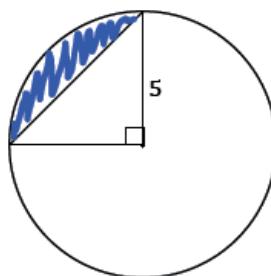


6. In the figure, if  $\widehat{PQ} = 10$  cm, find the length of arc  $\widehat{QR}$ ?

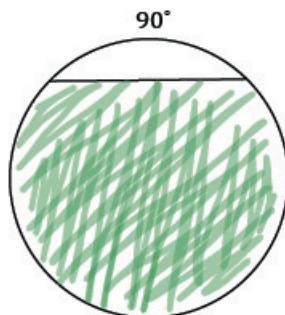


7. Find, to one decimal place, the areas of the shaded regions.

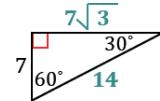
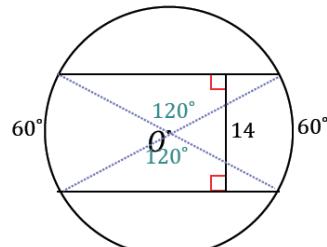
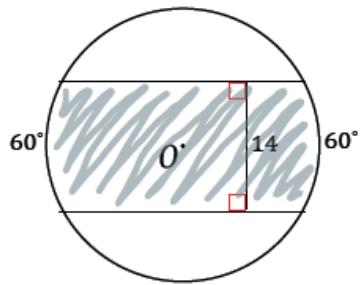
a.



- b. The following circle has a radius of 2.



c.



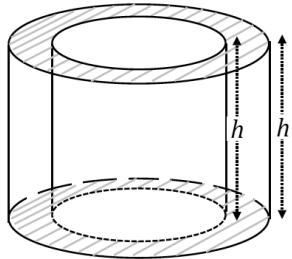
## Lesson 10: Unknown Length and Area Problems

### Classwork

#### Opening Exercise

In the following figure, a cylinder is carved out from within another cylinder of the same height; the bases of both cylinders share the same center.

- Sketch a cross section of the figure parallel to the base.

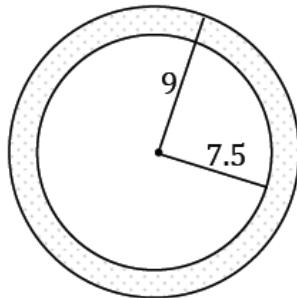


- Mark and label the shorter of the two radii as  $r$  and the longer of the two radii  $s$ .

Show how to calculate the area of the shaded region and explain the parts of the expression.

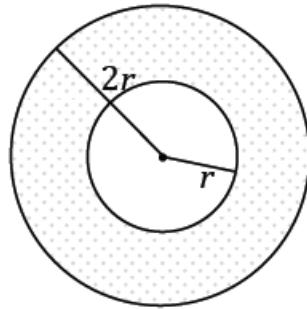
### Exercises 1–13

- Find the area of the following annulus.



2. The larger circle of an annulus has a diameter of 10 cm, and the smaller circle has a diameter of 7.6 cm. What is the area of the annulus?

3. In the following annulus, the radius of the larger circle is twice the radius of the smaller circle. If the area of the following annulus is  $12\pi$  units<sup>2</sup>, what is the radius of the larger circle?



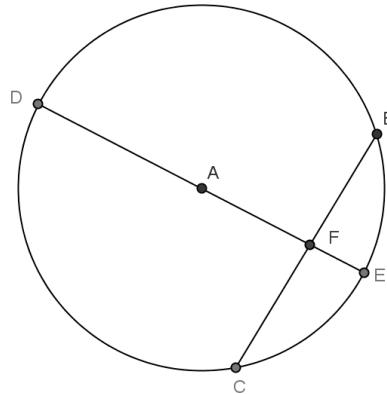
4. An ice cream shop wants to design a super straw to serve with their extra thick milkshakes that is double the width and thickness of a standard straw. A standard straw is 8 mm in diameter and 0.5 mm thick.

a. What is the cross-sectional (parallel to the base) area of the new straw (round to the nearest hundredth)?

b. If the new straw is 23 mm long, what is the maximum volume of milkshake that can be in the straw at one time (round to the nearest hundredth)?

c. A large milkshake is 32 ounces (approximately 950 mL). If Corbin withdraws the full capacity of a straw 10 times a minute, what is the minimum amount of time that it will take him to drink the milkshake (round to the nearest minute)?

5. In the circle given,  $\overline{ED}$  is the diameter and is perpendicular to chord  $\overline{CB}$ .  $DF = 8\text{ cm}$  and  $FE = 2\text{ cm}$ . Find  $AC$ ,  $BC$ ,  $m\angle CAB$ , the arc length of  $\widehat{CEB}$ , and the area of sector  $\widehat{CEB}$  (round to the nearest hundredth, if necessary).



6. Given circle A with  $\angle BAC \cong \angle BAD$ , find the following (round to the nearest hundredth, if necessary):

a.  $m\widehat{CD}$

b.  $m\widehat{CBD}$

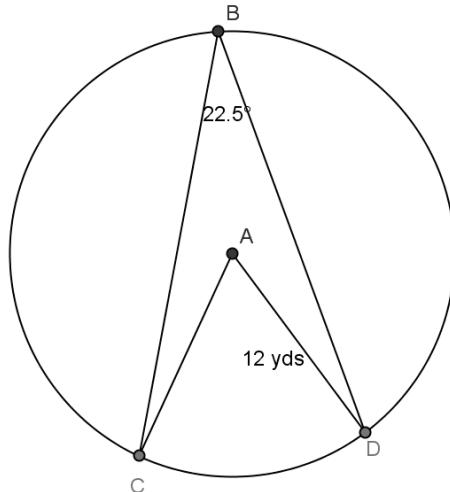
c.  $m\widehat{BCD}$

d. Arc length  $\widehat{CD}$

e. Arc length  $\widehat{CBD}$

f. Arc length  $\widehat{BCD}$

g. Area of sector  $\widehat{CD}$

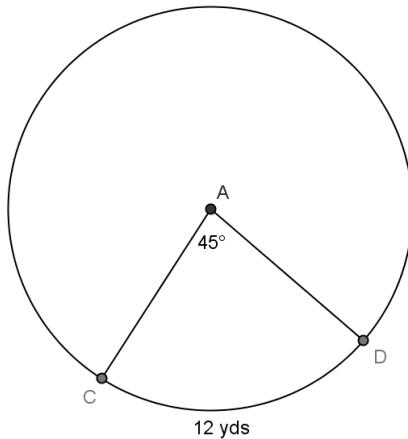


- h. Area of sector  $\widehat{CBD}$
- i. Area of sector  $\widehat{BCD}$
7. Given circle  $A$ , find the following (round to the nearest hundredth, if necessary):

a. Circumference of circle  $A$

b. Radius of circle  $A$

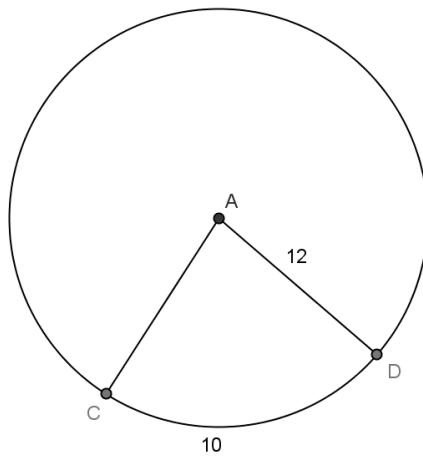
c. Area of sector  $\widehat{CD}$



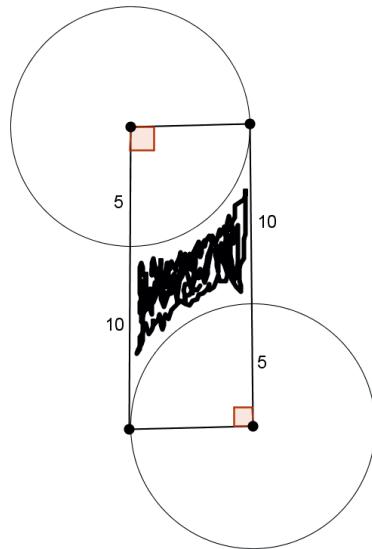
8. Given circle  $A$ , find the following (round to the nearest hundredth, if necessary):

a.  $m\angle CAD$

b. Area of sector  $\widehat{CD}$



9. Find the area of the shaded region (round to the nearest hundredth).



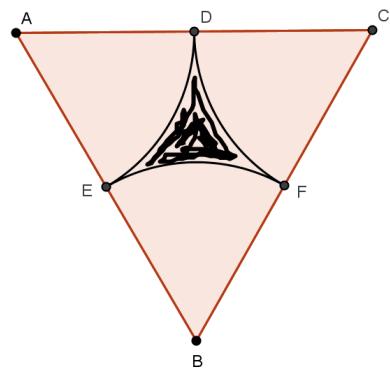
10. Many large cities are building or have built mega Ferris wheels. One is 600 feet in diameter and has 48 cars each seating up to 20 people. Each time the Ferris wheel turns  $\theta$  degrees, a car is in a position to load.

a. How far does a car move with each rotation of  $\theta$  degrees (round to the nearest whole number)?

b. What is the value of  $\theta$  in degrees?

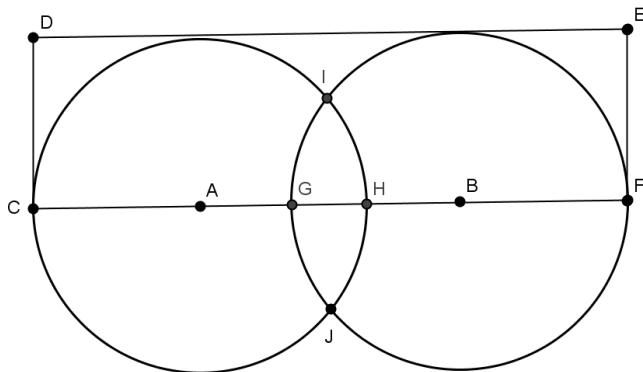
11.  $\triangle ABC$  is an equilateral triangle with edge length 20 cm.  $D$ ,  $E$ , and  $F$  are midpoints of the sides. The vertices of the triangle are the centers of the circles creating the arcs shown. Find the following (round to the nearest hundredth):

a. The area of the sector with center  $A$ .

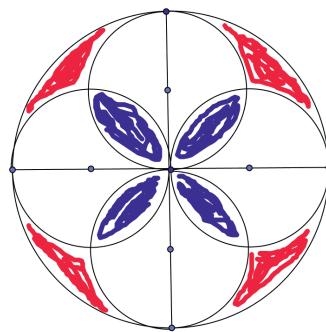


- b. The area of triangle  $ABC$ .
- c. The area of the shaded region.
- d. The perimeter of the shaded region.

12. In the figure shown,  $AC = BF = 5$  cm,  $GH = 2$  cm, and  $m\angle HBI = 30^\circ$ . Find the area in the rectangle, but outside of the circles (round to the nearest hundredth).

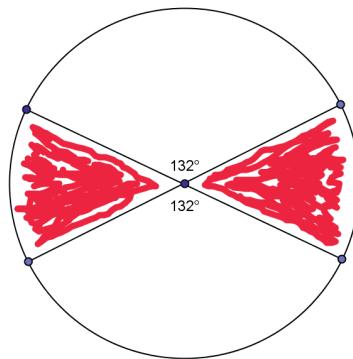


13. This is a picture of a piece of a mosaic tile. If the radius of each smaller circle is 1 inch, find the area of red section, the white section, and the blue section (round to the nearest hundredth).

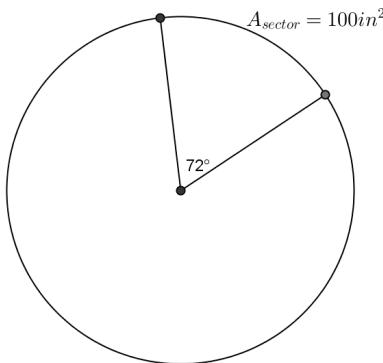


**Problem Set**

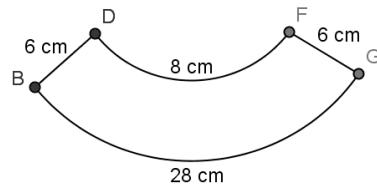
1. Find the area of the shaded region if the diameter is 32 inches (round to the nearest hundredth).



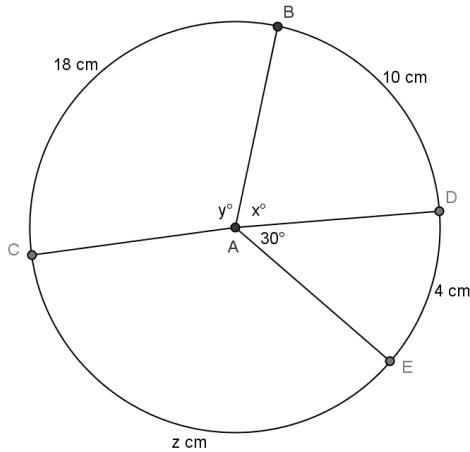
2. Find the area of the entire circle given the area of the sector.



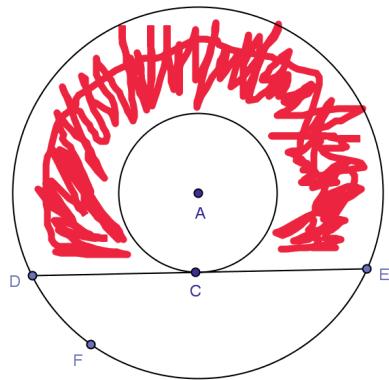
3.  $\widehat{DF}$  and  $\widehat{BG}$  are arcs of concentric circles with  $\overline{BD}$  and  $\overline{FG}$  lying on the radii of the larger circle. Find the area of the region (round to the nearest hundredth).



4. Find the radius of the circle,  $x$ ,  $y$ , and  $z$  (round to the nearest hundredth).



5. In the figure, the radii of two concentric circles are 24 cm and 12 cm.  $m\widehat{DAE} = 120^\circ$ . If a chord  $\overline{DE}$  of the larger circle intersects the smaller circle only at  $C$ , find the area of the shaded region in terms of  $\pi$ .

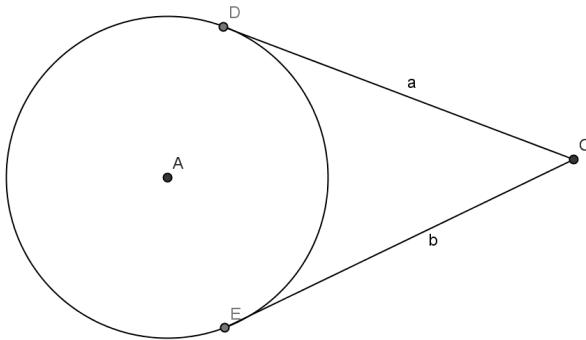


## Lesson 11: Properties of Tangents

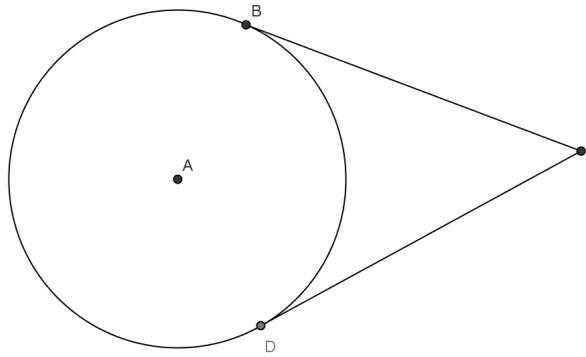
### Classwork

#### Exercises 1–3

1.  $\overline{CD}$  and  $\overline{CE}$  are tangent to circle  $A$  at points  $D$  and  $E$  respectively. Use a two-column proof to prove  $a = b$ .



2. In circle  $A$ , the radius is 9 mm and  $BC = 12$  mm.
- Find  $AC$ .

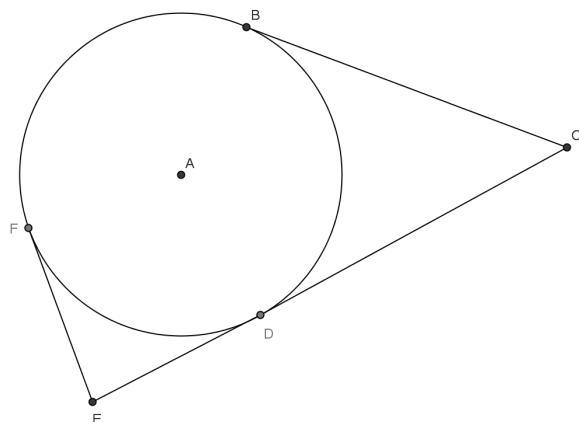


- Find the area of  $\Delta ACD$ .
- Find the perimeter of quadrilateral  $ABCD$ .

3. In circle  $A$ ,  $EF = 12$  and  $AE = 13$ .  $AE:AC = 1:3$ . Find
- The radius of the circle.

- The radius of the circle.
- $BC$  (round to the nearest whole number)

- The radius of the circle.
- $EC$



## Lesson Summary

### THEOREMS:

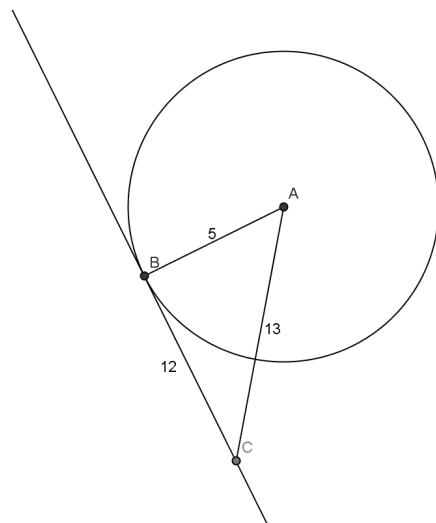
- A tangent line to a circle is perpendicular to the radius of the circle drawn to the point of tangency.
- A line through a point on a circle is tangent at the point if, and only if, it is perpendicular to the radius drawn to the point of tangency.

### Relevant Vocabulary

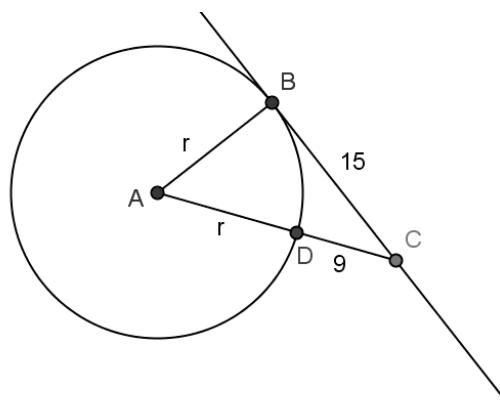
- **INTERIOR OF A CIRCLE:** The *interior of a circle with center  $O$  and radius  $r$*  is the set of all points in the plane whose distance from the point  $O$  is less than  $r$ .  
A point in the interior of a circle is said to be *inside the circle*. A disk is the union of the circle with its interior.
- **EXTERIOR OF A CIRCLE:** The exterior of a circle with center  $O$  and radius  $r$  is the set of all points in the plane whose distance from the point  $O$  is greater than  $r$ .  
A point exterior to a circle is said to be *outside the circle*.
- **TANGENT TO A CIRCLE:** A *tangent line to a circle* is a line in the same plane that intersects the circle in one and only one point. This point is called the *point of tangency*.
- **TANGENT SEGMENT/RAY:** A segment is a *tangent segment to a circle* if the line that contains it is tangent to the circle and one of the end points of the segment is a point of tangency. A ray is called a *tangent ray to a circle* if the line that contains it is tangent to the circle and the vertex of the ray is the point of tangency.
- **SECANT TO A CIRCLE:** A *secant line to a circle* is a line that intersects a circle in exactly two points.
- **POLYGON INSCRIBED IN A CIRCLE:** A polygon is *inscribed in a circle* if all of the vertices of the polygon lie on the circle.
- **CIRCLE INSCRIBED IN A POLYGON:** A circle is *inscribed* in a polygon if each side of the polygon is tangent to the circle.

## Problem Set

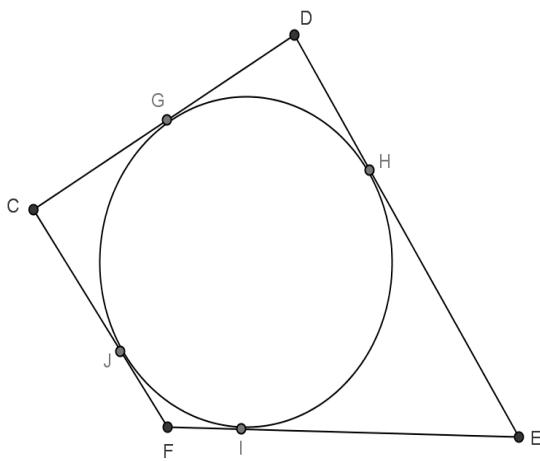
1. If  $AB = 5$ ,  $BC = 12$ , and  $AC = 13$ , is  $\overline{BC}$  tangent to circle  $A$  at point  $B$ ? Explain.



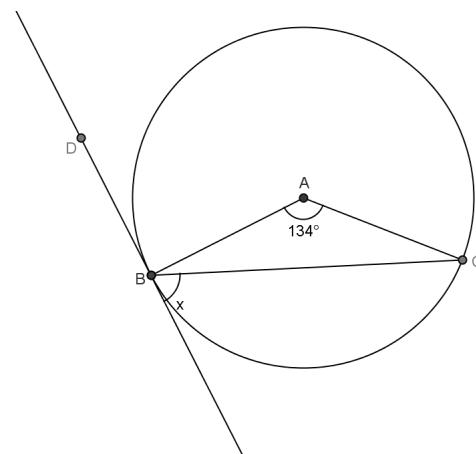
2.  $\overrightarrow{BC}$  is tangent to circle  $A$  at point  $B$ .  $DC = 9$  and  $BC = 15$ .
- Find the radius of the circle.
  - Find  $AC$ .



3. A circular pond is fenced on two opposite sides ( $\overline{CD}, \overline{FE}$ ) with wood and the other two sides with metal fencing. If all four sides of fencing are tangent to the pond, is there more wood or metal fencing used?

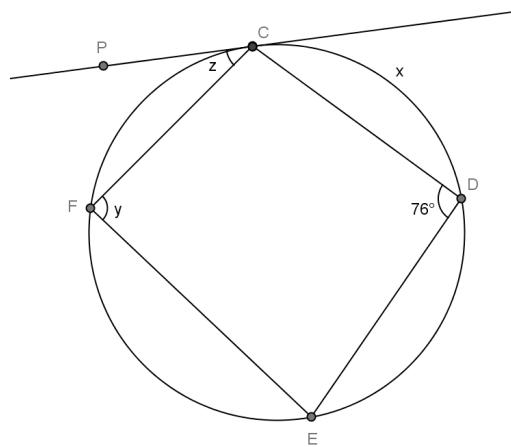


4. Find  $x$  if the line shown is tangent to the circle at point  $B$ .

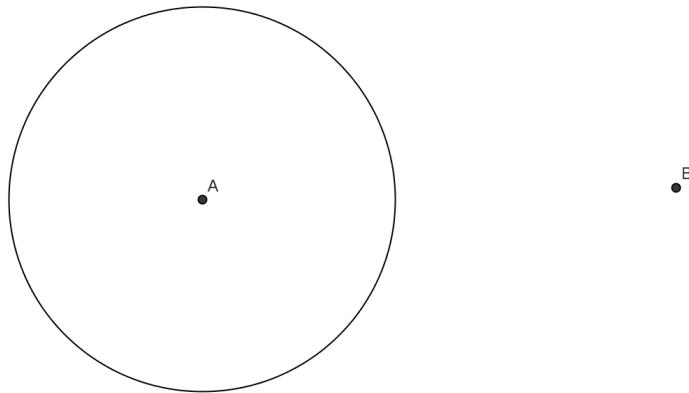


5. Line  $\overleftrightarrow{PC}$  is tangent to the circle at point  $C$ , and  $CD = DE$ . Find

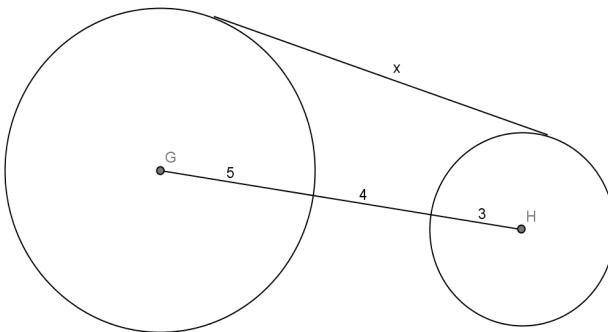
- a.  $x (m\widehat{CD})$
- b.  $y (m\angle CFE)$
- c.  $z(m\angle PCF)$



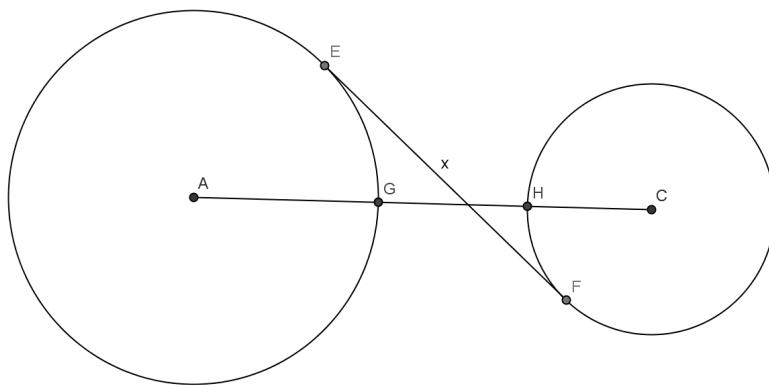
6. Construct two lines tangent to circle  $A$  through point  $B$ .



7. Find  $x$ , the length of the common tangent line between the two circles (round to the nearest hundredth).



8.  $\overline{EF}$  is tangent to both circles  $A$  and  $C$ . The radius of circle  $A$  is 9, and the radius of circle  $C$  is 5. The circles are 2 units apart. Find the length of  $\overline{EF}$ ,  $x$  (round to the nearest hundredth).

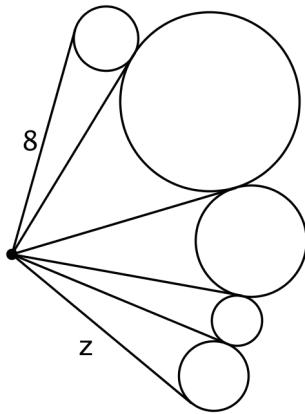


## Lesson 12: Tangent Segments

### Classwork

#### Opening Exercise

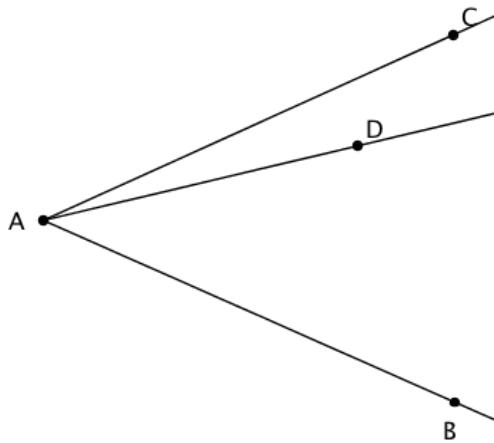
In the diagram to the right, what do you think the length of  $z$  could be? How do you know?



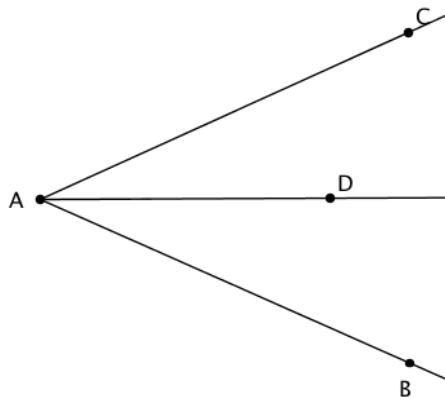
#### Example 1

In each diagram, try to draw a circle with center  $D$  that is tangent to both rays of the angle  $\angle BAC$ .

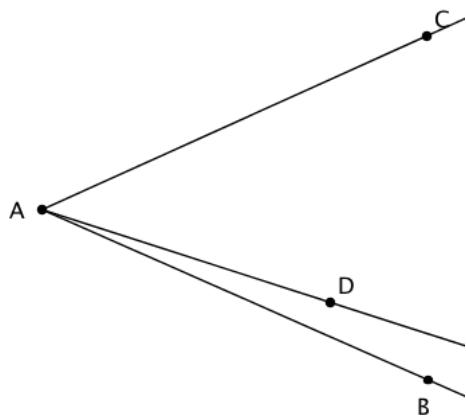
a.



b.



c.



Which diagrams did it seem impossible to draw such a circle? Why did it seem impossible?

What do you conjecture about circles tangent to both rays of an angle? Why do you think that?

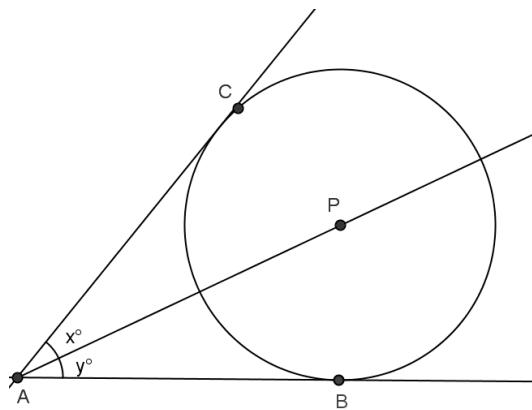
### Exercises 1–5

1. You conjectured that *if a circle is tangent to both rays of a circle, then the center lies on the angle bisector.*

- a. Rewrite this conjecture in terms of the notation suggested by the diagram.

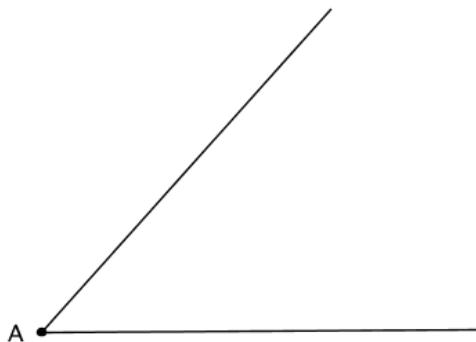
Given:

Need to show:

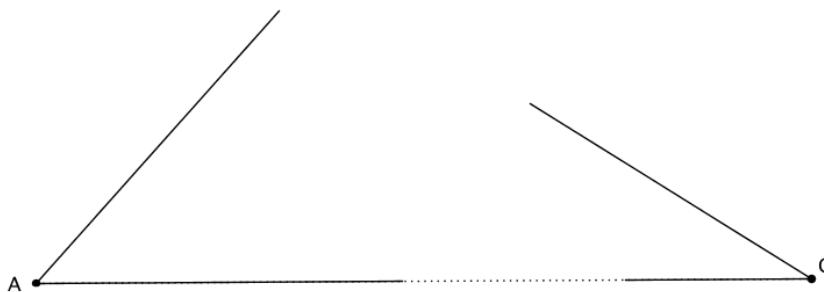


- b. Prove your conjecture using a two-column proof.

2. An angle is shown below.
- Draw at least 3 different circles that are tangent to both rays of the given angle.



- Label the center of one of your circles with  $P$ . How does the distance between  $P$  and the rays of the angle compare to the radius of the circle? How do you know?
3. Construct as many circles as you can that are tangent to both the given angles at the same time. You can extend the rays as needed. These two angles share a side.



Explain how many circles you can draw to meet the above conditions and how you know.

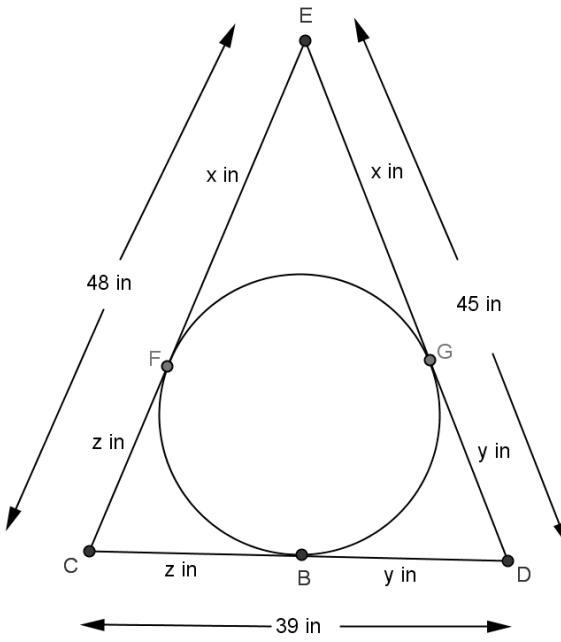
4. In a triangle, let  $P$  be the location where two angle bisectors meet. Must  $P$  be on the third angle bisector as well? Explain your reasoning.
5. Using a straightedge, draw a large triangle  $ABC$ .
  - a. Construct a circle inscribed in the given triangle.
  - b. Explain why your construction works.
- c. Do you know another name for the intersection of the angle bisectors in relation to the triangle?

**Lesson Summary****THEOREMS:**

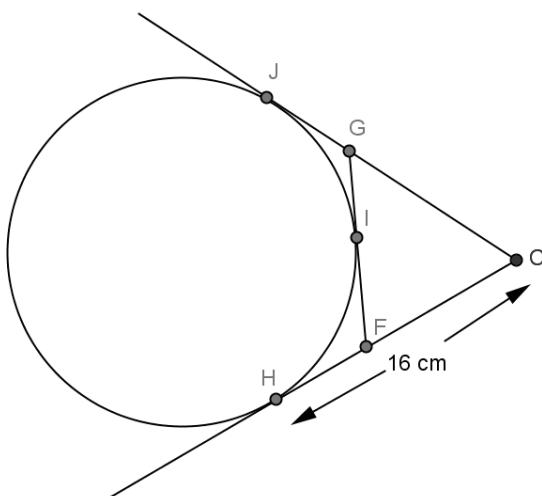
- The two tangent segments to a circle from an exterior point are congruent.
- If a circle is tangent to both rays of an angle, then its center lies on the angle bisector.
- Every triangle contains an inscribed circle whose center is the intersection of the triangle's angle bisectors.

**Problem Set**

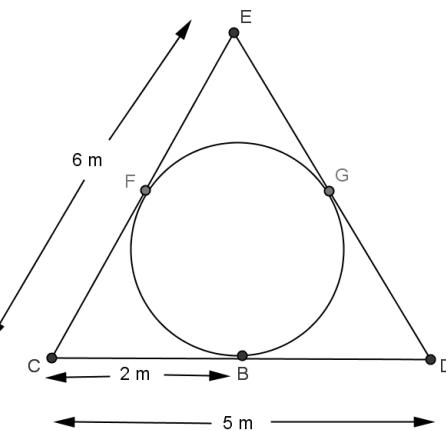
- On a piece of paper, draw a circle with center  $A$  and a point,  $C$ , outside of the circle.
  - How many tangents can you draw from  $C$  to the circle?
  - Draw two tangents from  $C$  to the circle, and label the tangency points  $D$  and  $E$ . Fold your paper along the line  $AC$ . What do you notice about the lengths of  $CD$  and  $CE$ ? About the measures of the angles  $\angle DCA$  and  $\angle ECA$ ?
  - $AC$  is the \_\_\_\_\_ of  $\angle DCE$ .
  - $\overline{CD}$  and  $\overline{CE}$  are tangent to circle  $A$ . Find  $AC$ .
- In the figure at right, the three segments are tangent to the circle at points  $B$ ,  $F$ , and  $G$ . If  $y = \frac{2}{3}x$ , find  $x$ ,  $y$ , and  $z$ .



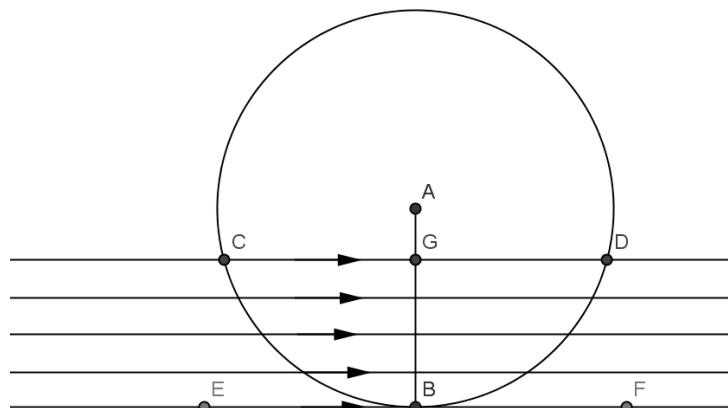
3. In the figure given, the three segments are tangent to the circle at points  $J$ ,  $I$ , and  $H$ .
- Prove  $GF = GJ + HF$
  - Find the perimeter of  $\triangle GCF$ .



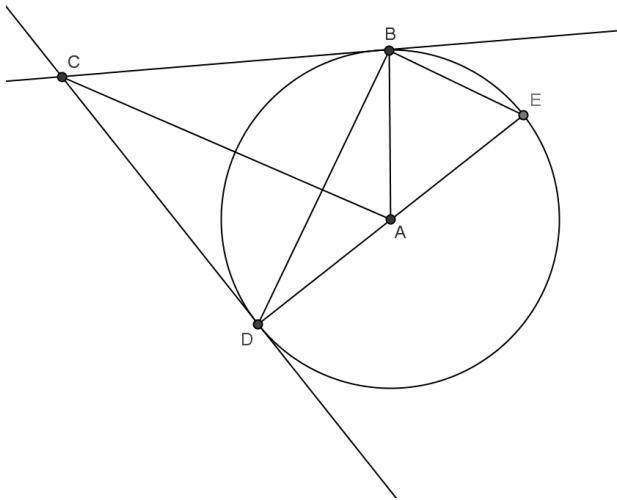
4. In the given figure, the three segments are tangent to the circle at point  $F$ ,  $B$ , and  $G$ . Find  $DE$ .



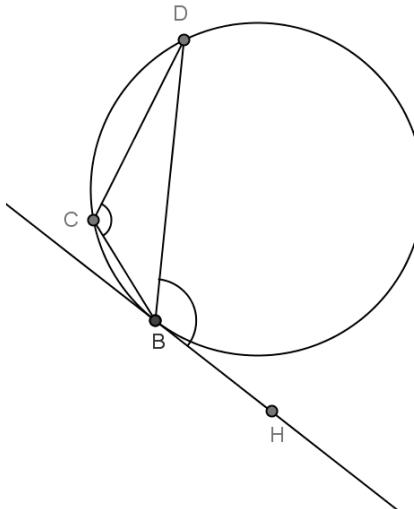
5.  $\overrightarrow{EF}$  is tangent to circle  $A$ . If points  $C$  and  $D$  are the intersection points of circle  $A$  and any line parallel to  $\overrightarrow{EF}$ , answer the following.
- Does  $CG = GD$  for any line parallel to  $\overrightarrow{EF}$ ? Explain.
  - Suppose that  $\overleftrightarrow{CD}$  coincides with  $\overrightarrow{EF}$ . Would  $C$ ,  $G$ , and  $D$  all coincide with  $B$ ?
  - Suppose  $C$ ,  $G$ , and  $D$  have now reached  $B$ , so  $\overleftrightarrow{CD}$  is tangent to the circle. What is the angle between the line  $\overleftrightarrow{CD}$  and  $\overleftrightarrow{AB}$ ?



- d. Draw another line tangent to the circle from some point,  $P$ , in the exterior of the circle. If the point of tangency is point  $T$ , what is the measure of  $\angle PTA$ ?
6. The segments are tangent to circle  $A$  at points  $B$  and  $D$ .  
 $\overline{ED}$  is a diameter of the circle.
- Prove  $\overline{BE} \parallel \overline{CA}$ .
  - Prove quadrilateral  $ABCD$  is a kite.



7. In the diagram shown,  $\overline{BH}$  is tangent to the circle at point  $B$ . What is the relationship between  $\angle DBH$ , the angle between the tangent and a chord, and the arc subtended by that chord and its inscribed angle  $\angle DCB$ ?



## Lesson 13: The Inscribed Angle Alternate a Tangent Angle

### Classwork

#### Opening Exercise

1. In circle A,  $m\widehat{BD} = 56^\circ$ , and  $\overline{BC}$  is a diameter. Find the listed measure, and explain your answer.

a.  $m\angle BDC$

b.  $m\angle BCD$

c.  $m\angle DBC$

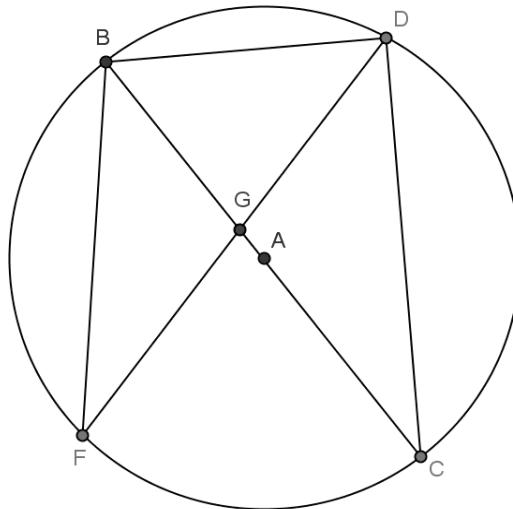
d.  $m\angle BFG$

e.  $m\widehat{BC}$

f.  $m\widehat{DC}$

g. Is the  $m\angle BGD = 56^\circ$ ? Explain.

h. How do you think we could determine the measure of  $\angle BGD$ ?



## Example 1

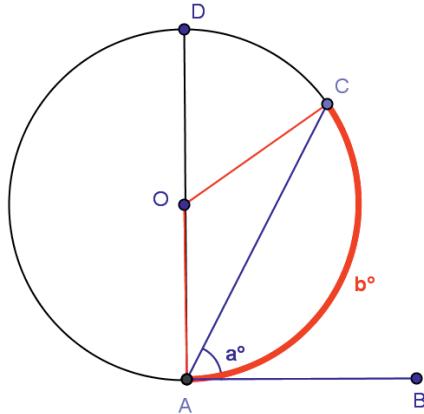


Diagram 1

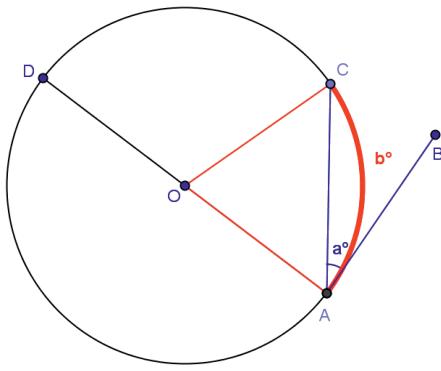


Diagram 2

Examine the diagrams shown. Develop a conjecture about the relationship between  $a$  and  $b$ .

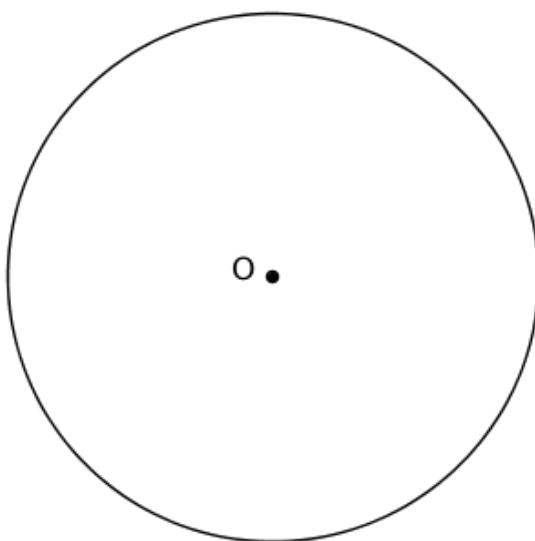
Test your conjecture by using a protractor to measure  $a$  and  $b$ .

	$a$	$b$
Diagram 1		
Diagram 2		

Do your measurements confirm the relationship you found in your homework?

If needed, revise your conjecture about the relationship between  $a$  and  $b$ :

Now test your conjecture further using the circle below.



$a$	$b$

Now, we will prove your conjecture, which is stated below as a theorem.

**THE TANGENT-SECANT THEOREM:** Let  $A$  be a point on a circle, let  $\overrightarrow{AB}$  be a tangent ray to the circle, and let  $C$  be a point on the circle such that  $\overrightarrow{AC}$  is a secant to the circle. If  $a = m\angle BAC$  and  $b$  is the angle measure of the arc intercepted by  $\angle BAC$ , then  $a = \frac{1}{2}b$ .

Given circle  $A$  with tangent  $\overrightarrow{BG}$ , prove what we have just discovered using what you know about the properties of a circle and tangent and secant lines.

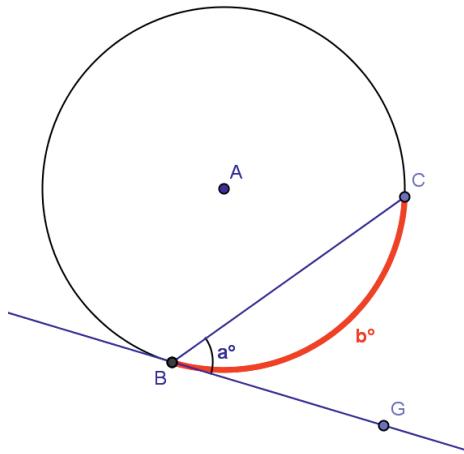
- Draw triangle  $ABC$ . What is the measure of  $\angle BAC$ ? Explain.

- What is the measure of  $\angle ABG$ ? Explain.

- Express the measure of the remaining two angles of triangle  $ABC$  in terms of “ $a$ ” and explain.

- What is the measure of  $\angle BAC$  in terms of “ $a$ ”? Show how you got the answer.

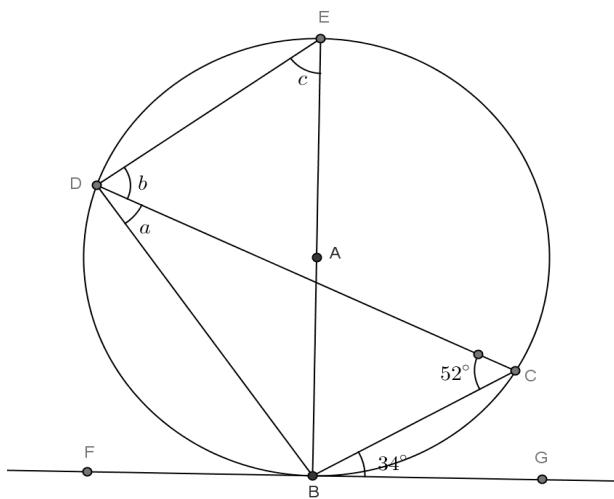
- Explain to your neighbor what we have just proven.



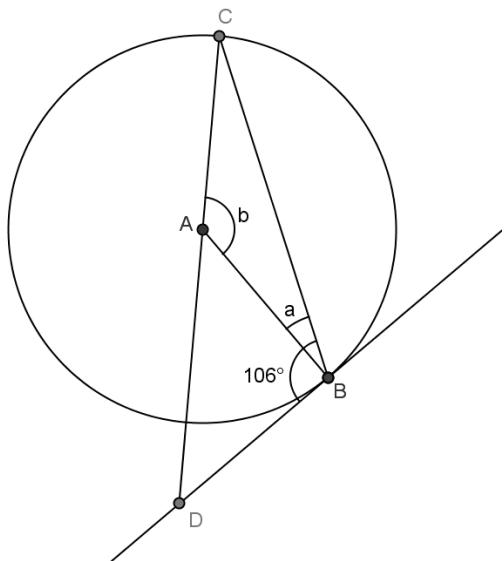
**Exercises**

Find  $x$ ,  $y$ ,  $a$ ,  $b$ , and/or  $c$ .

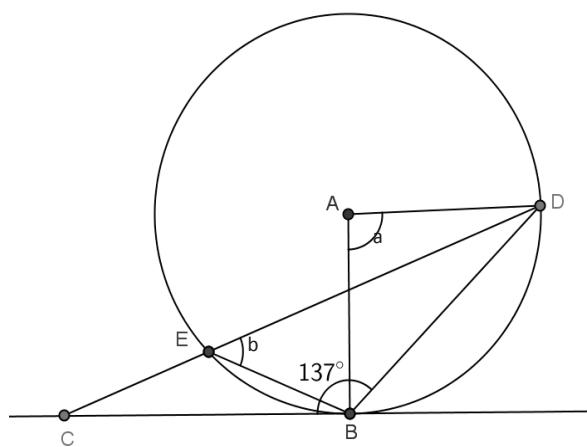
1.



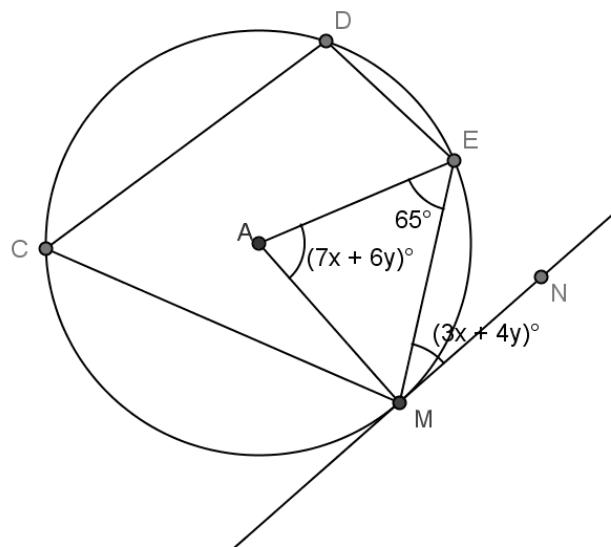
2.



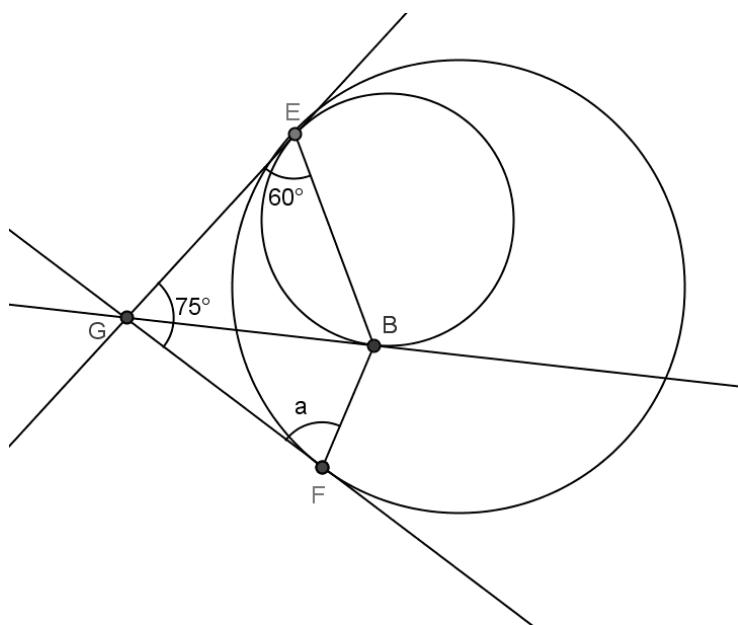
3.



4.



5.



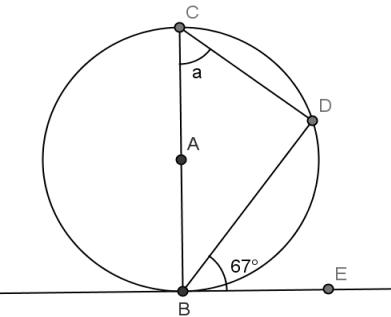
**Lesson Summary****THEOREMS:**

- **CONJECTURE:** Let A be a point on a circle, let  $\overrightarrow{AB}$  be a tangent ray to the circle, and let C be a point on the circle such that  $\overrightarrow{AC}$  is a secant to the circle. If  $a = m\angle BAC$  and  $b$  is the angle measure of the arc intercepted by  $\angle BAC$ , then  $a = \frac{1}{2}b$ .
- **THE TANGENT-SECANT THEOREM:** Let A be a point on a circle, let  $\overrightarrow{AB}$  be a tangent ray to the circle, and let C be a point on the circle such that  $\overrightarrow{AC}$  is a secant to the circle. If  $a = m\angle BAC$  and  $b$  is the angle measure of the arc intercepted by  $\angle BAC$ , then  $a = \frac{1}{2}b$ .
- Suppose  $\overline{AB}$  is a chord of circle C, and  $\overline{AD}$  is a tangent segment to the circle at point A. If E is any point other than A or B in the arc of C on the opposite side of  $\overline{AB}$  from D, then  $m\angle BEA = m\angle BAD$ .

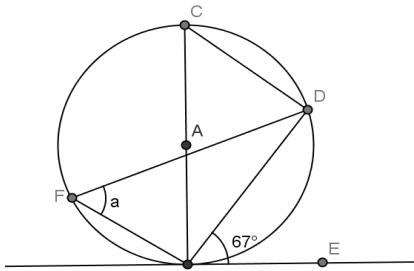
**Problem Set**

In Problems 1–9, solve for  $a$ ,  $b$ , and/or  $c$ .

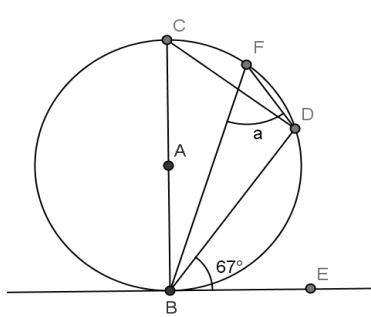
1.



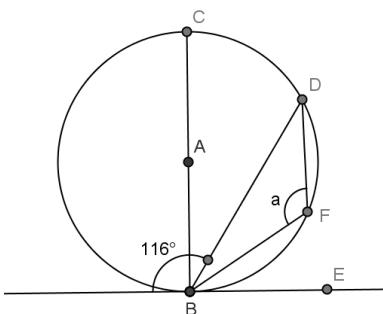
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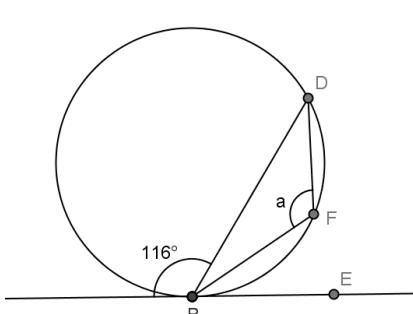
3.



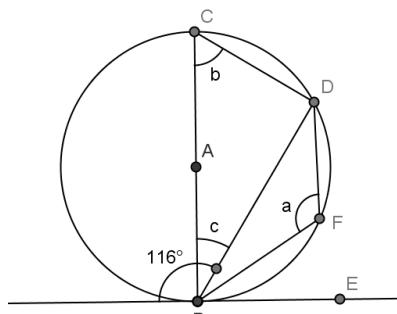
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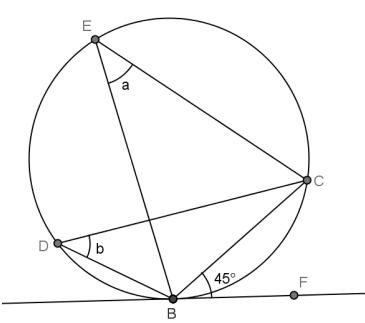
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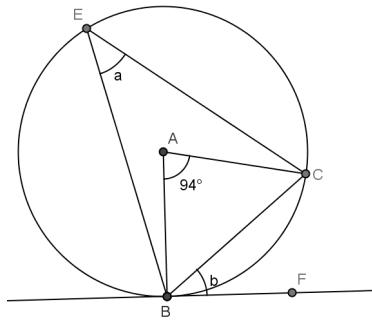
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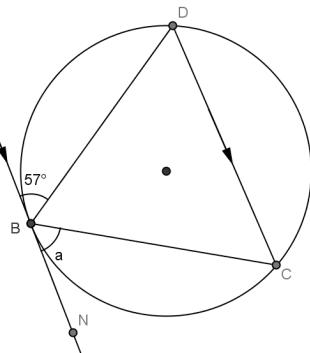
7.



8.

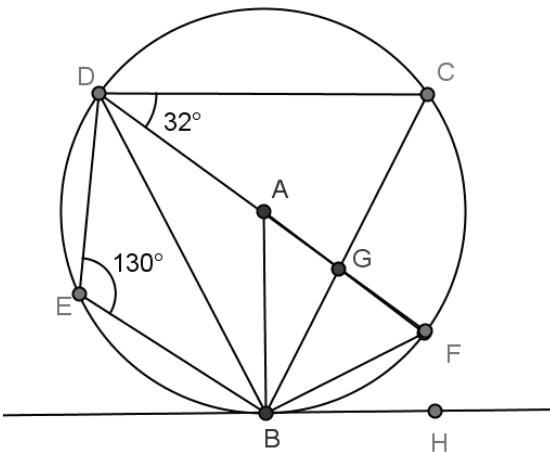


9.

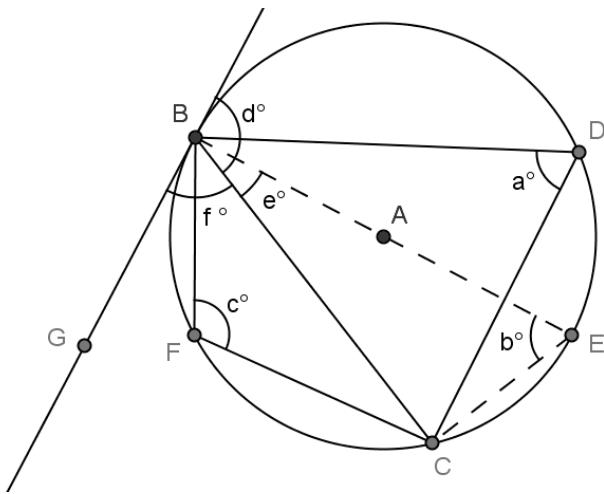


10.  $\overrightarrow{BH}$  is tangent to circle A.  $\overline{DF}$  is a diameter. Find

- $m\angle BCD$
- $m\angle BAF$
- $m\angle BDA$
- $m\angle FBH$
- $m\angle BGF$



11.  $\overline{BG}$  is tangent to circle  $A$ .  $\overline{BE}$  is a diameter. Prove: (i)  $f = a$  and (ii)  $d = c$



## Lesson 14: Secant Lines; Secant Lines That Meet Inside a Circle

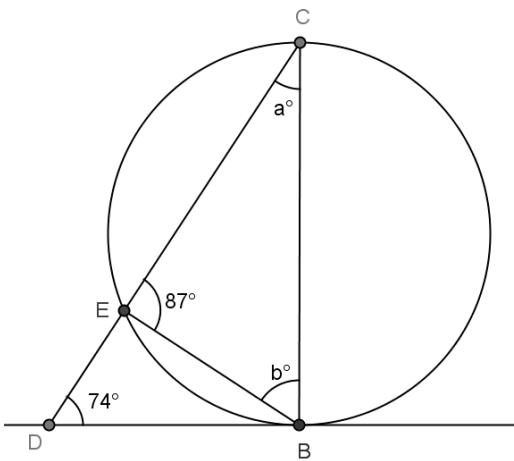
### Classwork

#### Opening Exercise

$\overrightarrow{DB}$  is tangent to the circle as shown.

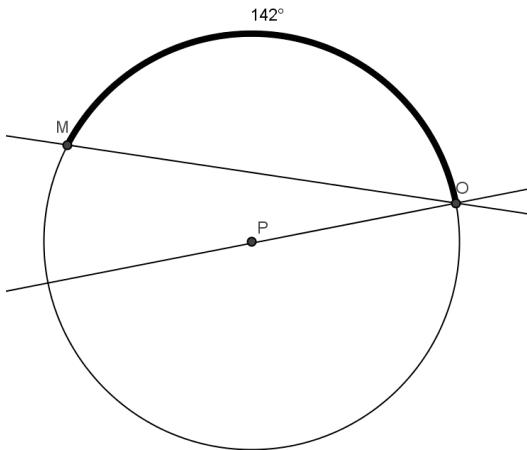
- a. Find the values of  $a$  and  $b$ .

- b. Is  $\overline{CB}$  a diameter of the circle? Explain.

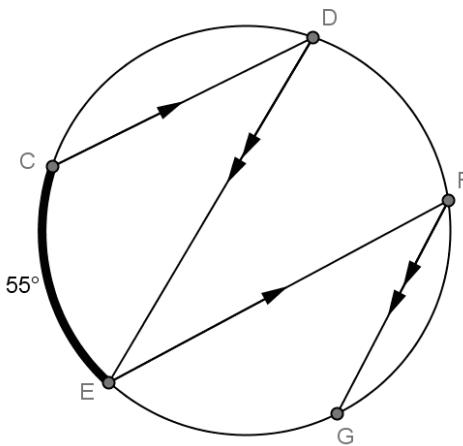


#### Exercises 1–2

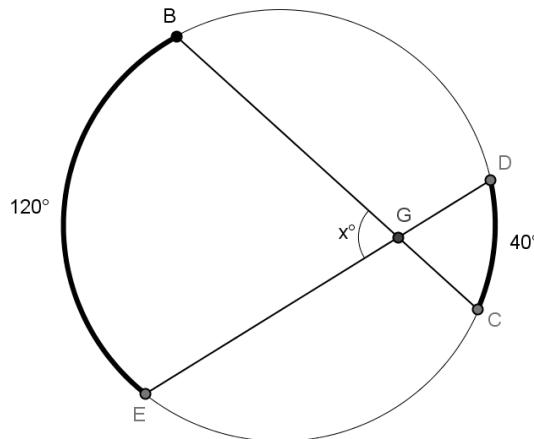
1. In circle  $P$ ,  $\overline{PO}$  is a radius, and  $m\widehat{MO} = 14^\circ$ . Find  $m\angle MOP$ , and explain how you know.



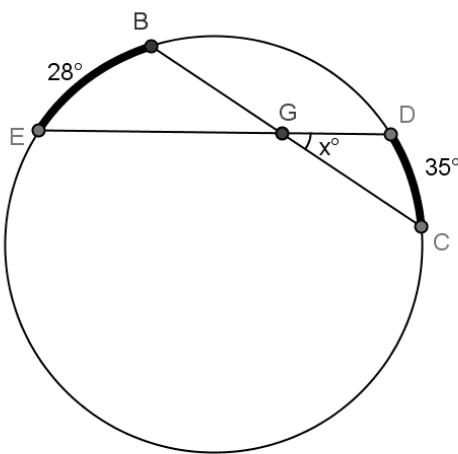
2. In the circle shown,  $m\widehat{CE} = 55^\circ$ . Find  $m\angle DEF$  and  $m\widehat{EG}$ . Explain your answer.

**Example 1**

- a. Find  $x$ . Justify your answer.



- b. Find  $x$ .



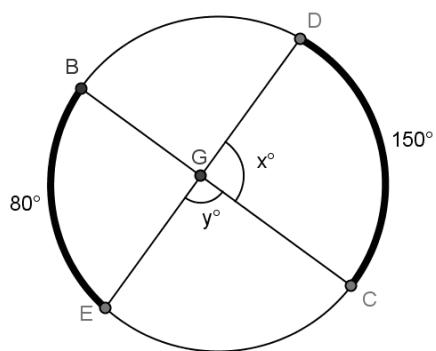
We can state the results of part (b) of this example as the following theorem:

**SECANT ANGLE THEOREM: INTERIOR CASE:** The measure of an angle whose vertex lies in the interior of a circle is equal to half the sum of the angle measures of the arcs intercepted by it and its vertical angle.

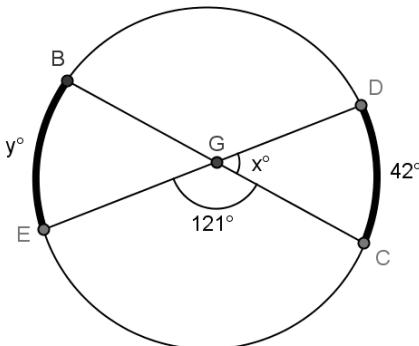
### Exercises 3–7

In Exercises 3–5, find  $x$  and  $y$ .

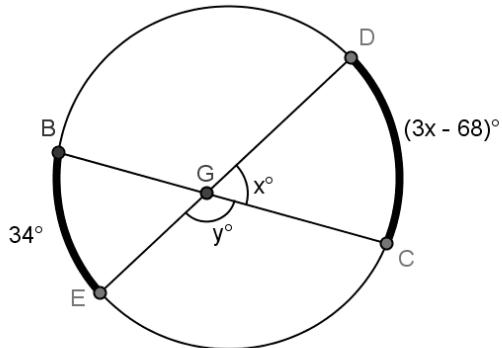
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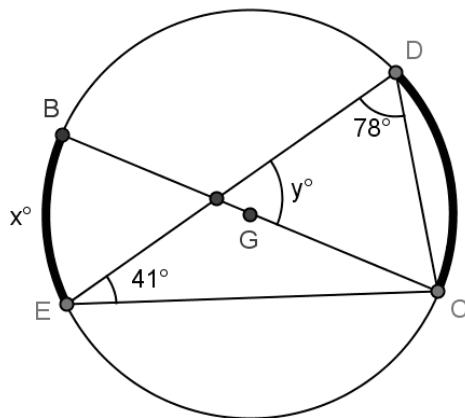
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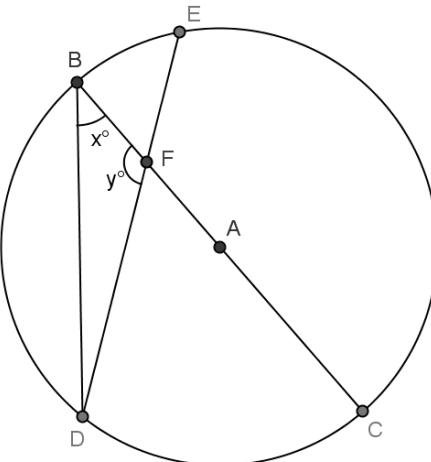
5.



6. In circle,  $\overline{BC}$  is a diameter. Find  $x$  and  $y$ .



7. In the circle shown,  $\overline{BC}$  is a diameter.  $DC:BE = 2:1$ . Prove  $y = 180 - \frac{3}{2}x$  using a two-column proof.



**Lesson Summary****THEOREMS:**

- **SECANT ANGLE THEOREM: INTERIOR CASE.** The measure of an angle whose vertex lies in the interior of a circle is equal to half the sum of the angle measures of the arcs intercepted by it and its vertical angle.

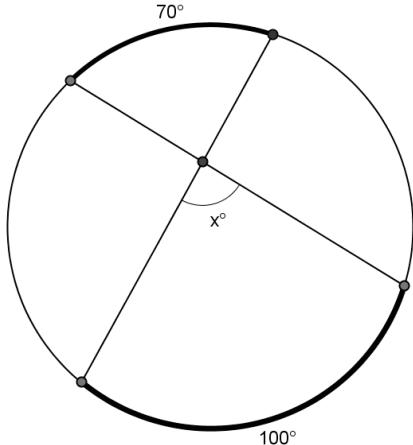
**Relevant Vocabulary**

- **TANGENT TO A CIRCLE:** A *tangent line to a circle* is a line in the same plane that intersects the circle in one and only one point. This point is called the *point of tangency*.
- **TANGENT SEGMENT/RAY:** A segment is a *tangent segment to a circle* if the line that contains it is tangent to the circle and one of the end points of the segment is a point of tangency. A ray is called a *tangent ray to a circle* if the line that contains it is tangent to the circle and the vertex of the ray is the point of tangency.
- **SECANT TO A CIRCLE:** A *secant line to a circle* is a line that intersects a circle in exactly two points.

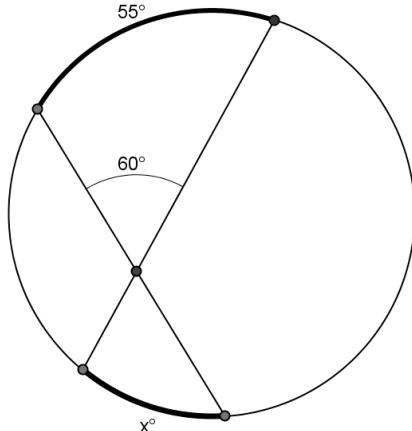
**Problem Set**

In Problems 1–4, find  $x$ .

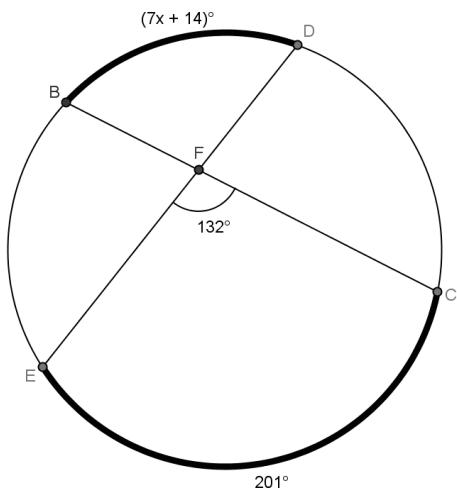
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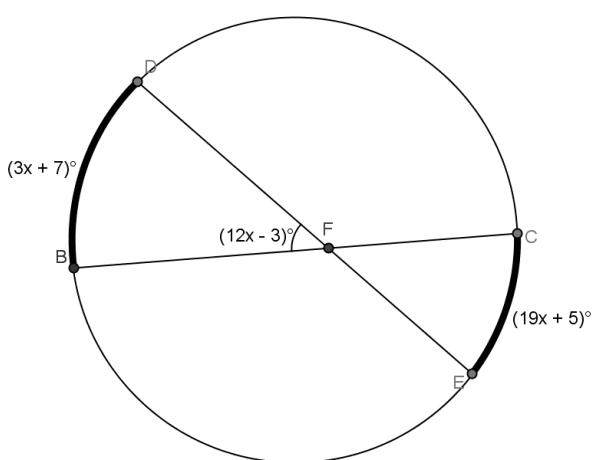
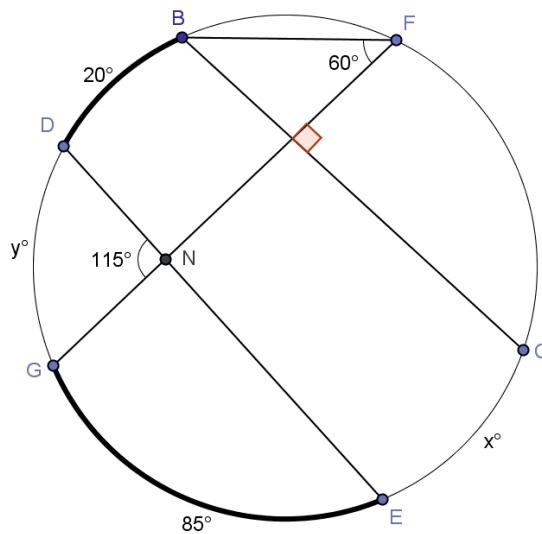
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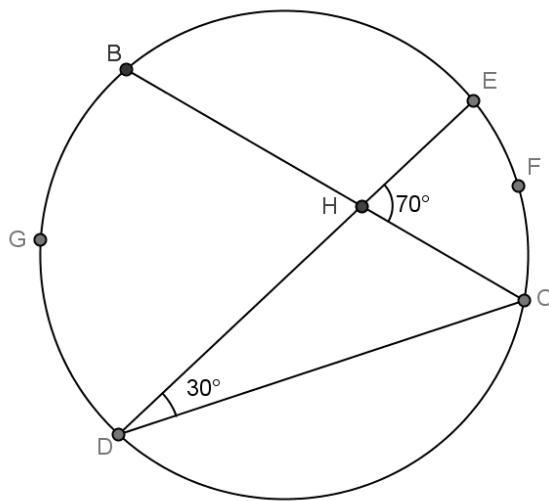
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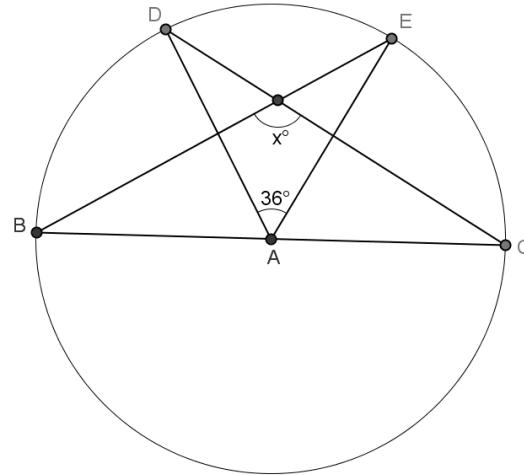
4.

5. Find  $x$  ( $m\widehat{CE}$ ) and  $y$  ( $m\widehat{DG}$ ).

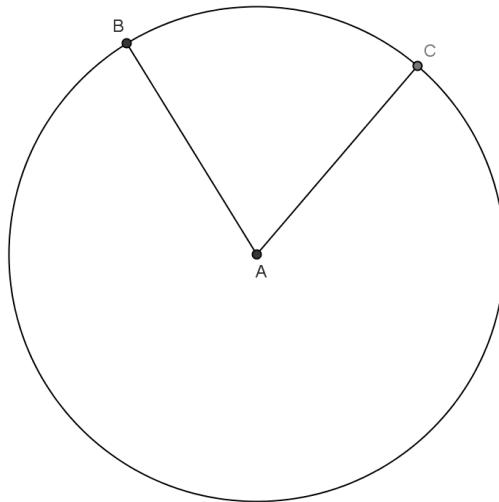
6. Find the ratio of  $m\widehat{EFC}:m\widehat{DGB}$ .



7.  $\overline{BC}$  is a diameter of circle A. Find  $x$ .



8. Show that the general formula we discovered in Example 1 also works for central angles. (Hint: Extend the radii to form 2 diameters, and use relationships between central angles and arc measure.)

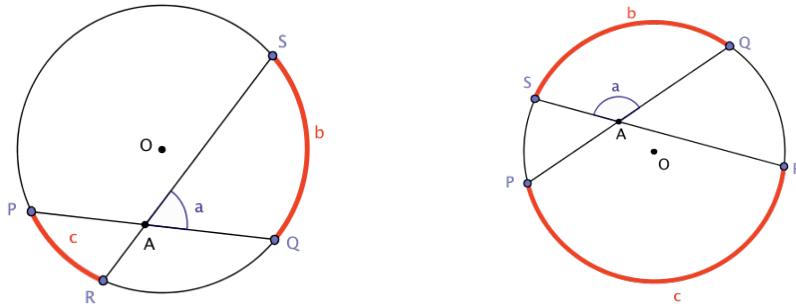


## Lesson 15: Secant Angle Theorem, Exterior Case

### Classwork

#### Opening Exercise

1. Shown below are circles with two intersecting secant chords.



Measure  $a$ ,  $b$ , and  $c$  in the two diagrams. Make a conjecture about the relationship between them.

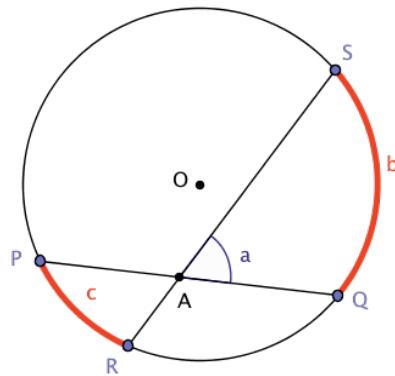
$a$	$b$	$c$

CONJECTURE about the relationship between  $a$ ,  $b$ , and  $c$ :

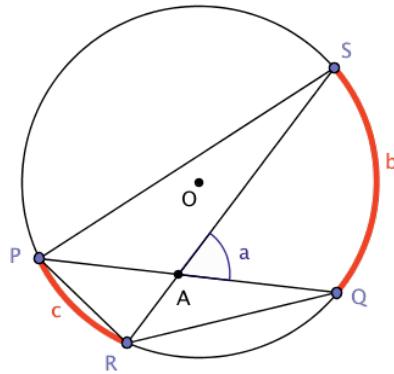
2. We will prove the following.

**SECANT ANGLE THEOREM: INTERIOR CASE.** The measure of an angle whose vertex lies in the interior of a circle is equal to half the sum of the angle measures of the arcs intercepted by it and its vertical angle.

We can interpret this statement in terms of the diagram below. Let  $b$  and  $c$  be the angle measures of the arcs intercepted by the angles  $\angle SAQ$  and  $\angle PAR$ . Then measure  $a$  is the average of  $b$  and  $c$ ; that is,  $a = \frac{b+c}{2}$ .



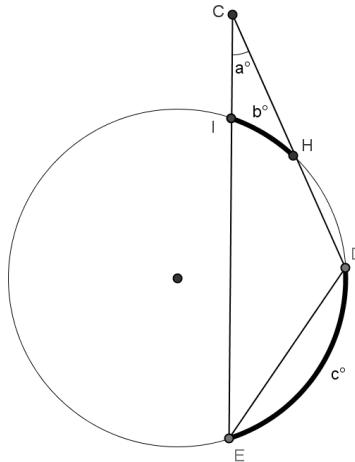
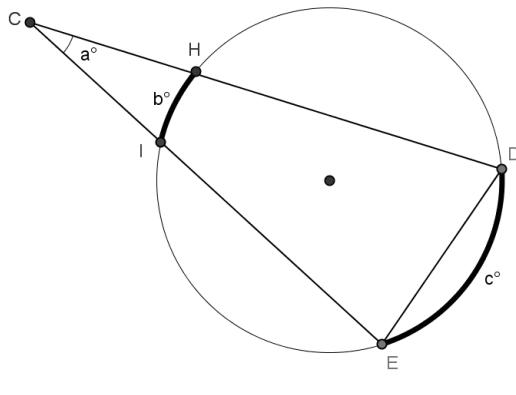
- a. Find as many pairs of congruent angles as you can in the diagram below. Express the measures of the angles in terms of  $b$  and  $c$  whenever possible.



- b. Which triangles in the diagram are similar? Explain how you know.
- c. See if you can use one of the triangles to prove the secant angle theorem: interior case. (Hint: Use the exterior angle theorem.)

**Example 1**

Shown below are two circles with two secant chords intersecting outside the circle.

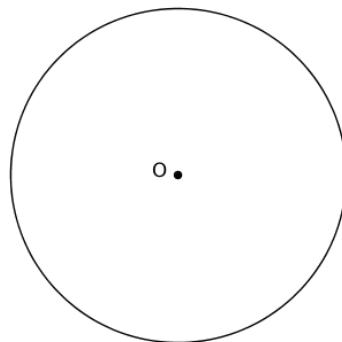


Measure  $a$ ,  $b$ , and  $c$ . Make a conjecture about the relationship between them.

$a$	$b$	$c$

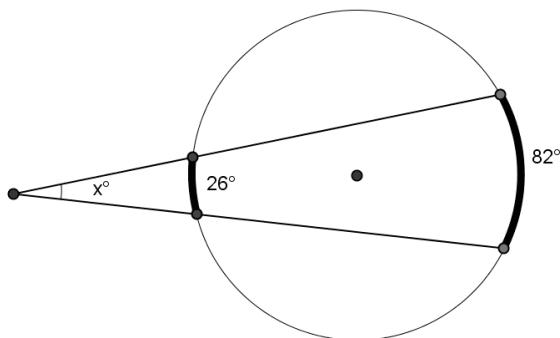
Conjecture about the relationship between  $a$ ,  $b$ , and  $c$ :

Test your conjecture with another diagram.

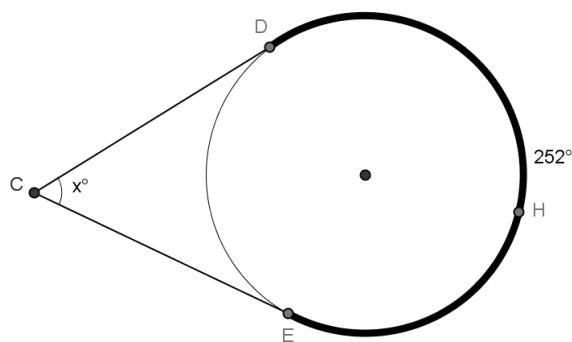


**Exercises**Find  $x$ ,  $y$ , and/or  $z$ .

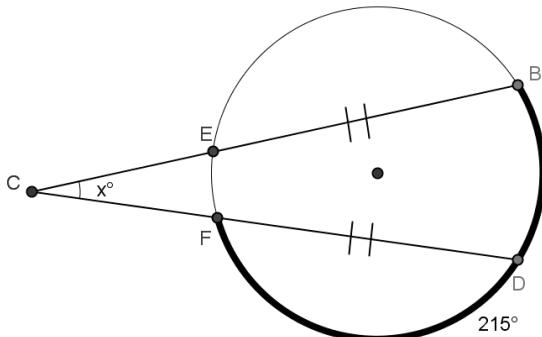
1.



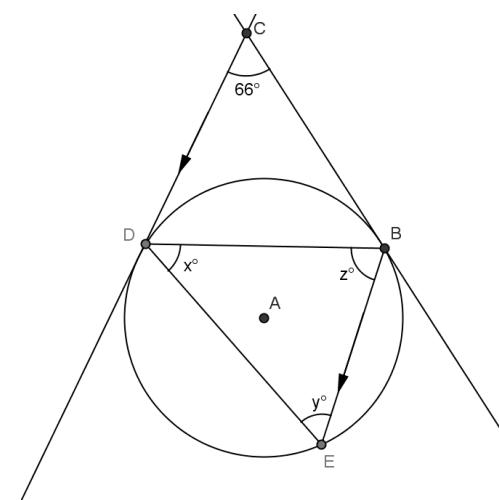
2.



3.



4.

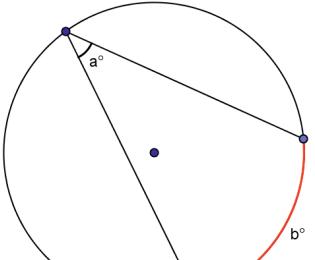
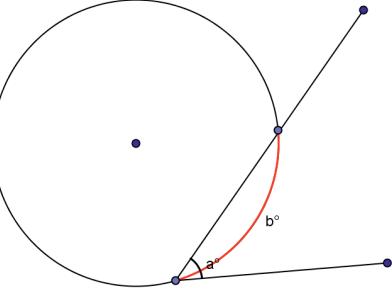


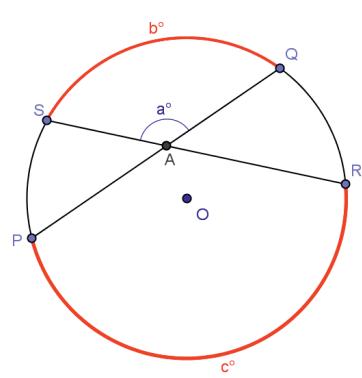
**Lesson Summary:**

We have just developed proofs for an entire family of theorems. Each theorem in this family deals with two shapes and how they overlap. The two shapes are two intersecting lines and a circle.

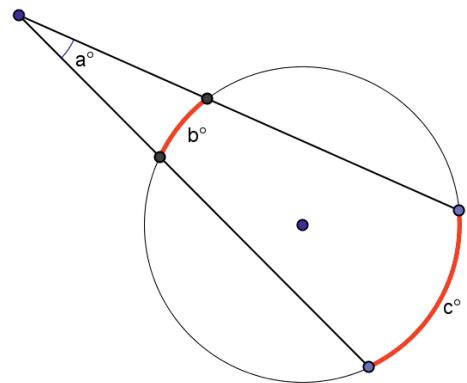
In this exercise, you'll summarize the different cases.

**The Inscribed Angle Theorem and its Family of Theorems**

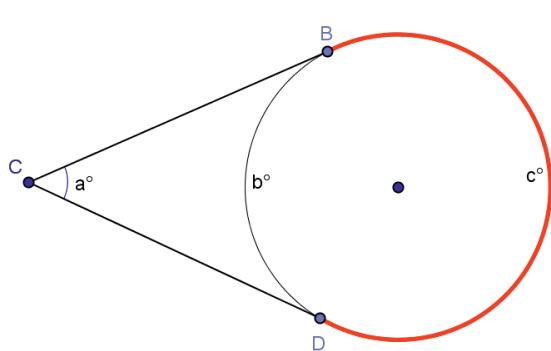
Diagram	How the two shapes overlap	Relationship between $a$ , $b$ , $c$ , and $d$
 (Inscribed Angle Theorem)		
 (Secant – Tangent)		



(Secant Angle Theorem: Interior)



(Secant Angle Theorem: Exterior)



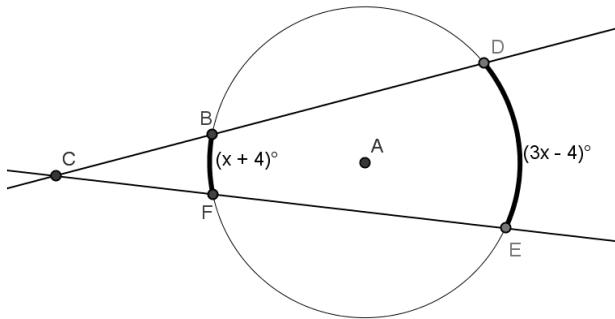
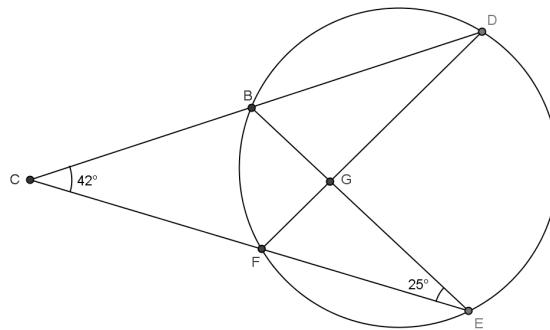
(Two Tangent Lines)

**Lesson Summary****THEOREMS:**

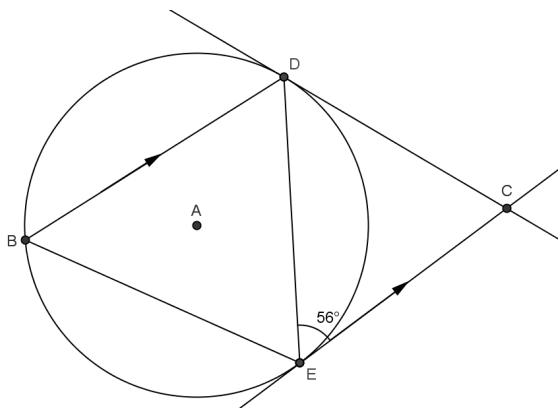
- **SECANT ANGLE THEOREM: INTERIOR CASE.** The measure of an angle whose vertex lies in the interior of a circle is equal to half the sum of the angle measures of the arcs intercepted by it and its vertical angle.
- **SECANT ANGLE THEOREM: EXTERIOR CASE.** The measure of an angle whose vertex lies in the exterior of the circle, and each of whose sides intersect the circle in two points, is equal to half the difference of the angle measures of its larger and smaller intercepted arcs.

**Relevant Vocabulary**

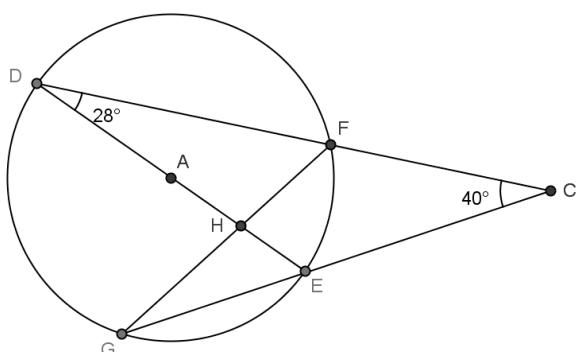
**SECANT TO A CIRCLE:** A *secant line to a circle* is a line that intersects a circle in exactly two points.

**Problem Set**1. Find  $x$ .2. Find  $m\angle DFE$  and  $m\angle DGB$ .

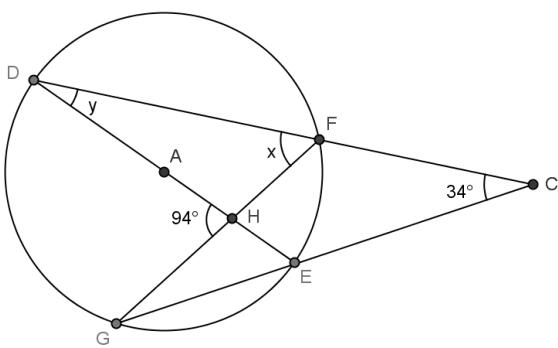
3. Find  $m\angle ECD$ ,  $m\angle DBE$ , and  $m\angle DEB$ .



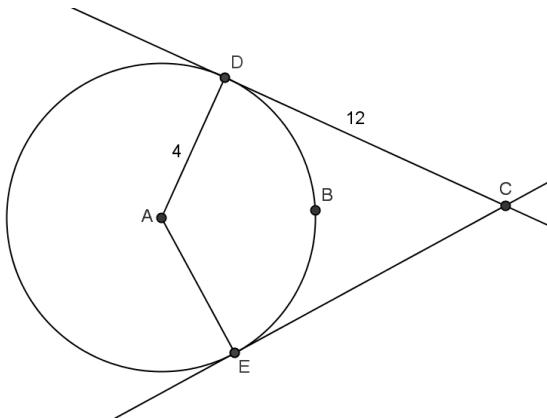
4. Find  $m\angle FGE$  and  $m\angle FHE$ .



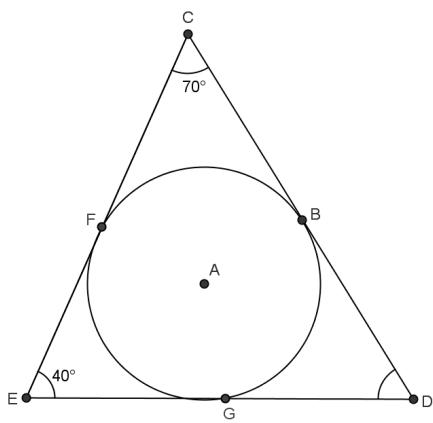
5. Find  $x$  and  $y$ .



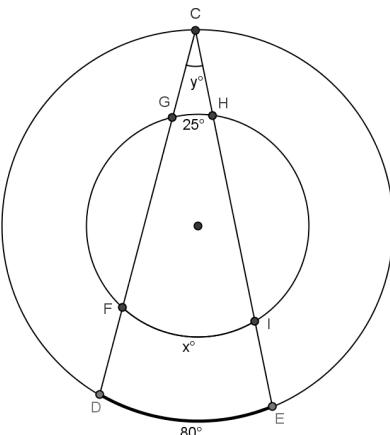
6. The radius of circle A is 4.  $\overline{DC}$  and  $\overline{CE}$  are tangent to the circle with  $DC = 12$ . Find  $m\widehat{EBD}$  and the area of quadrilateral DAEC rounded to the nearest hundredth.



7. Find  $m\widehat{BG}$ ,  $m\widehat{GF}$ , and  $m\widehat{FB}$ .

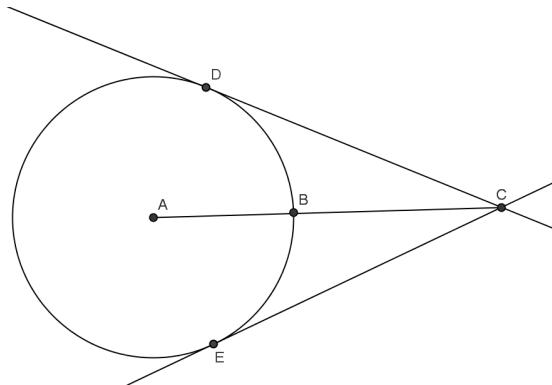


8. Find  $x$  and  $y$ .



9. The radius of a circle is 6.
- If the angle formed between two tangent lines to the circle is  $60^\circ$ , how long are the segments between the point of intersection of the tangent lines and the circle?
  - If the angle formed between the two tangent lines is  $120^\circ$ , how long are the segments between the point of intersection of the tangent lines and the circle? Round to the nearest hundredth.

10.  $\overline{DC}$  and  $\overline{EC}$  are tangent to circle A. Prove  $BD = BE$ .



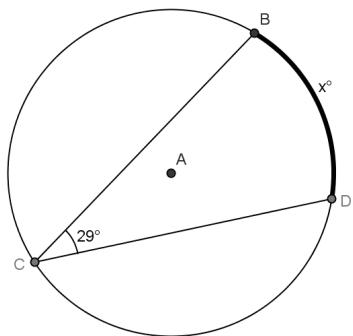
## Lesson 16: Similar Triangles in Circle-Secant (or Circle-Secant-Tangent) Diagrams

### Classwork

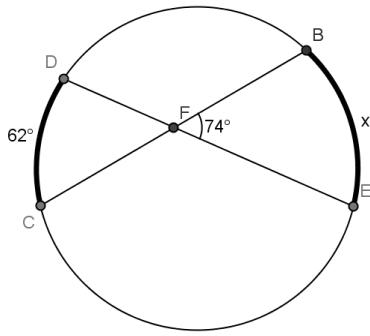
#### Opening Exercise

Identify the type of angle and the angle/arc relationship, and then find the measure of  $x$ .

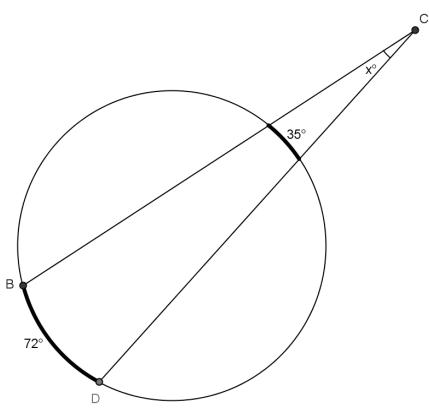
1.



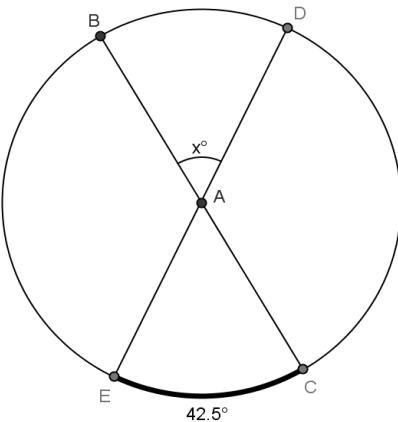
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3.



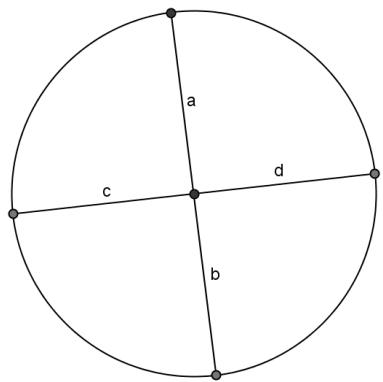
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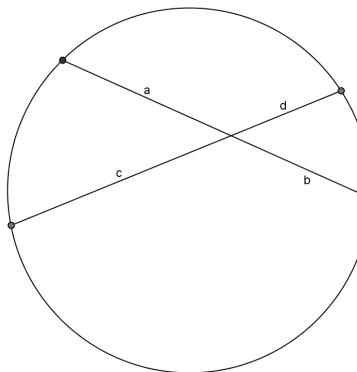
**Example 1**

Measure the lengths of the chords in centimeters and record them in the table.

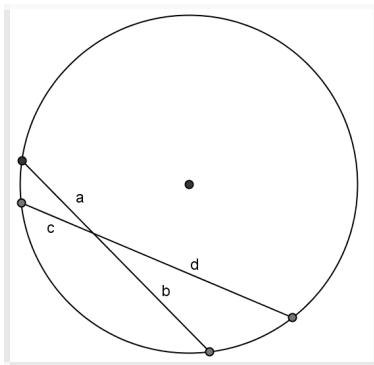
a. s



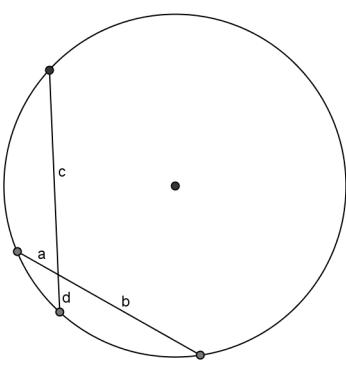
b.



c.



d.

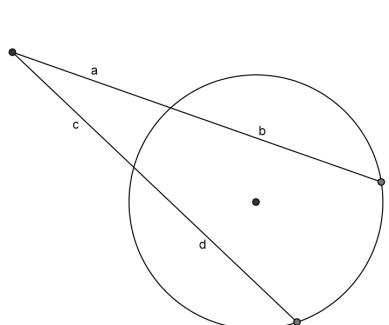


Circle #	$a$ (cm)	$b$ (cm)	$c$ (cm)	$d$ (cm)	Do you notice a relationship?
a					
b					
c					
d					

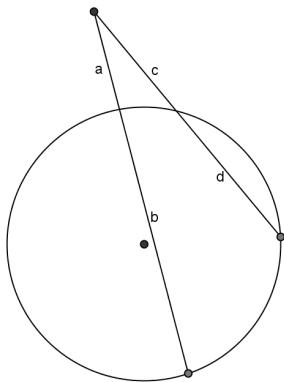
**Example 2**

Measure the lengths of the chords in centimeters and record them in the table.

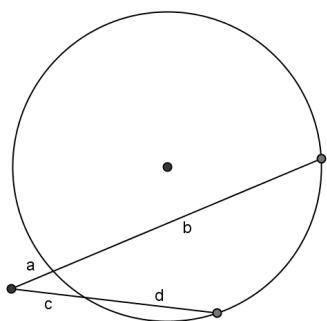
a.



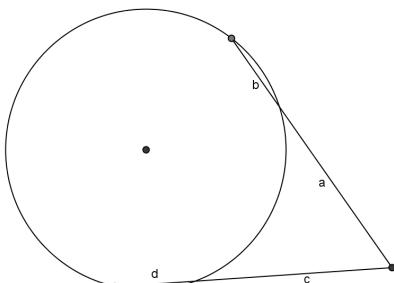
b.



c.

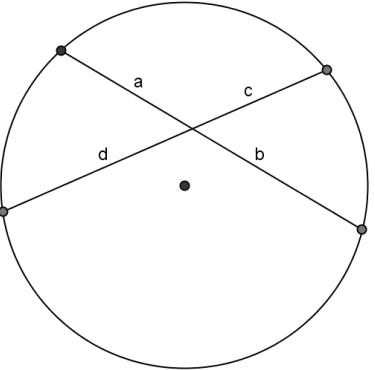
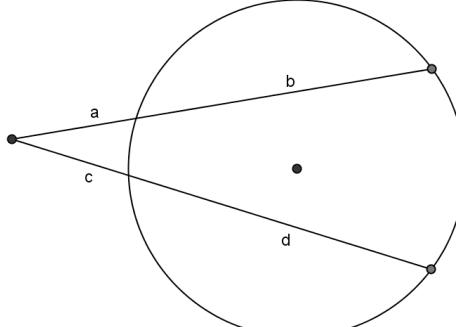
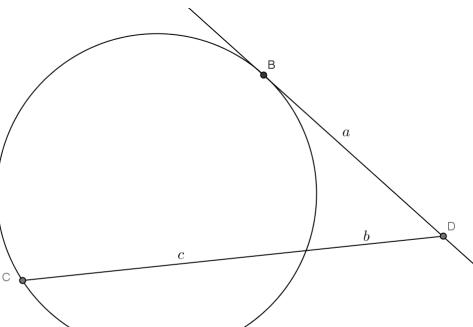


d.



Circle #	$a$ (cm)	$b$ (cm)	$c$ (cm)	$d$ (cm)	Do you notice a relationship?
a					
b					
c					
d					

## The inscribed angle theorem and its family:

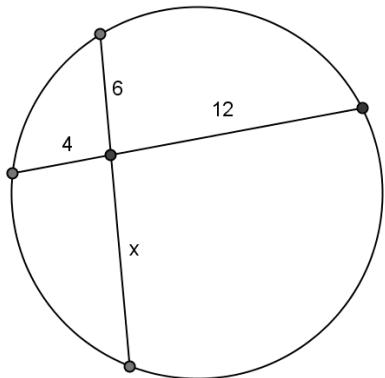
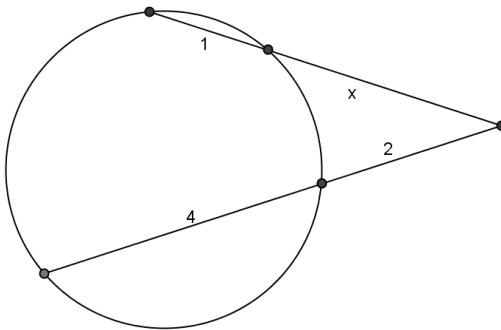
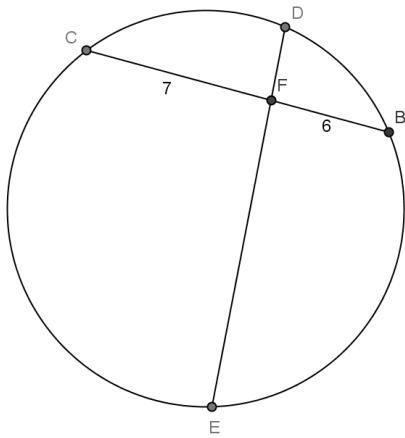
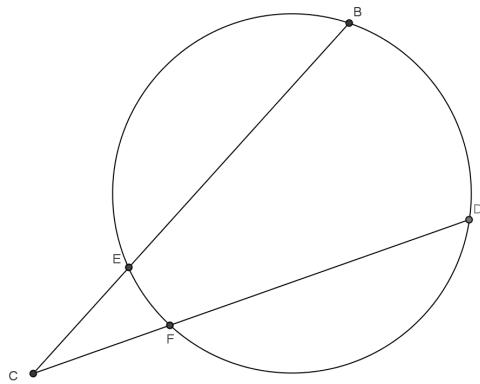
Diagram	How the two shapes overlap	Relationship between $a, b, c$ and $d$
		
		
		

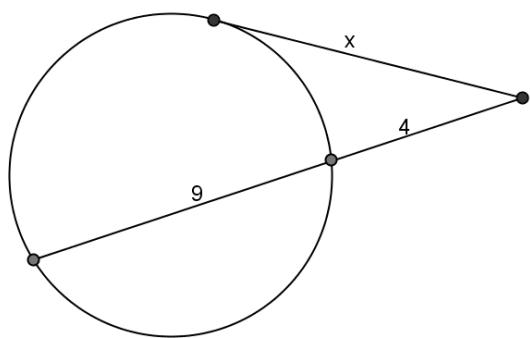
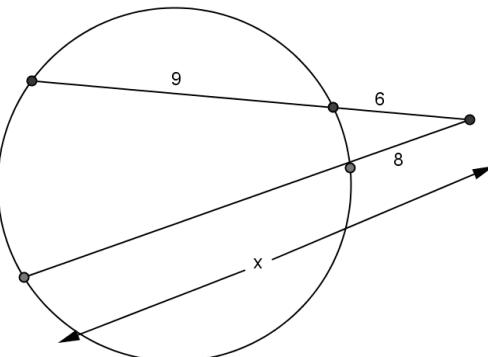
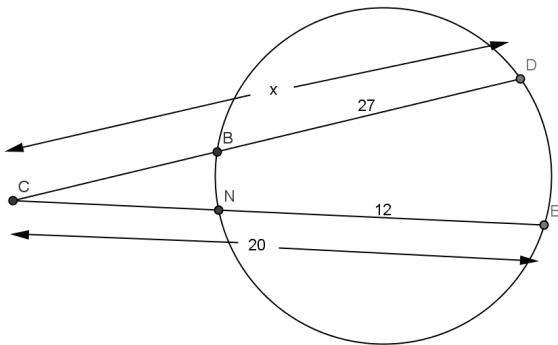
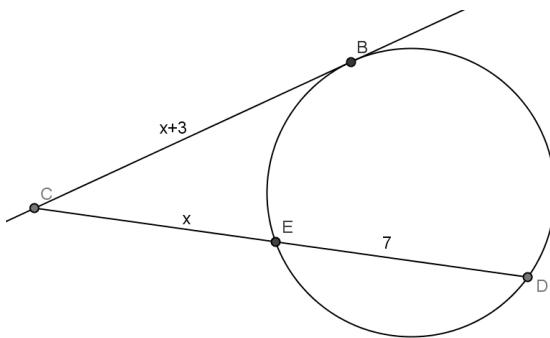
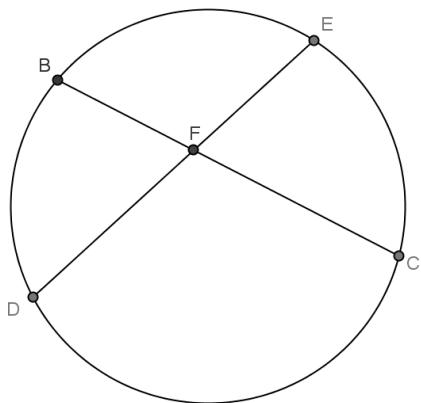
**Lesson Summary****THEOREMS:**

- When secant lines intersect inside a circle, use  $a \cdot b = c \cdot d$ .
- When secant lines intersect outside of a circle, use  $a(a + b) = c(c + d)$ .

**Relevant Vocabulary**

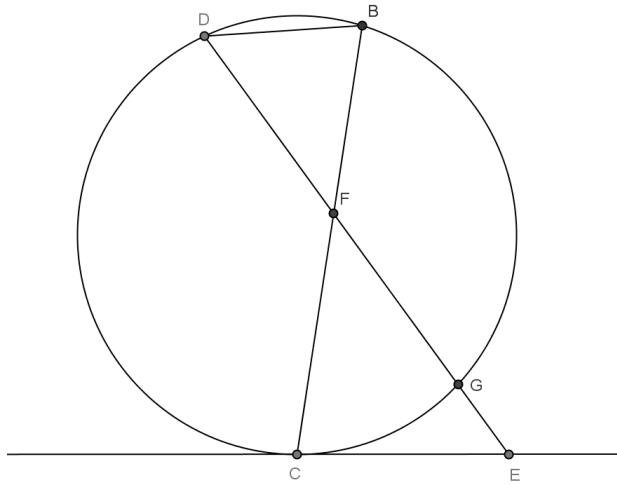
**SECANT TO A CIRCLE:** A *secant line to a circle* is a line that intersects a circle in exactly two points.

**Problem Set**1. Find  $x$ .2. Find  $x$ .3.  $DF < FB, DF \neq 1, DF < FE$ . Prove  $DF = 3$ 4.  $CE = 6, CB = 9, CD = 18$ . Show  $CF = 3$ .

5. Find  $x$ .6. Find  $x$ .7. Find  $x$ .8. Find  $x$ .9. In the circle shown,  $DE = 11$ ,  $BC = 10$ ,  $DF = 8$ . Find  $FE$ ,  $BF$ ,  $FC$ .

10. In the circle shown,  $m\widehat{DBG} = 150^\circ$ ,  $m\widehat{DB} = 30^\circ$ ,  $m\angle CEF = 60^\circ$ ,  $DF = 8$ ,  $DB = 4$ ,  $GF = 12$ .

- Find  $m\angle GDB$ .
- Prove  $\Delta DBF \sim \Delta ECF$ .
- Set up a proportion using sides  $\overline{CE}$  and  $\overline{GE}$ .
- Set up an equation with  $\overline{CE}$  and  $\overline{GE}$  using a theorem for segment lengths from this section.
- Solve for  $CE$  and  $GE$ .

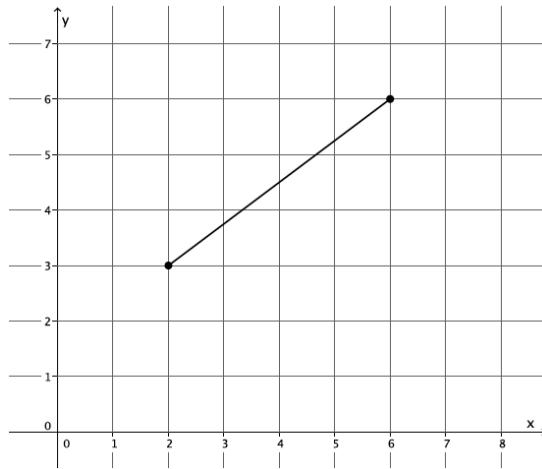


## Lesson 17: Writing the Equation for a Circle

### Classwork

#### Opening Exercise

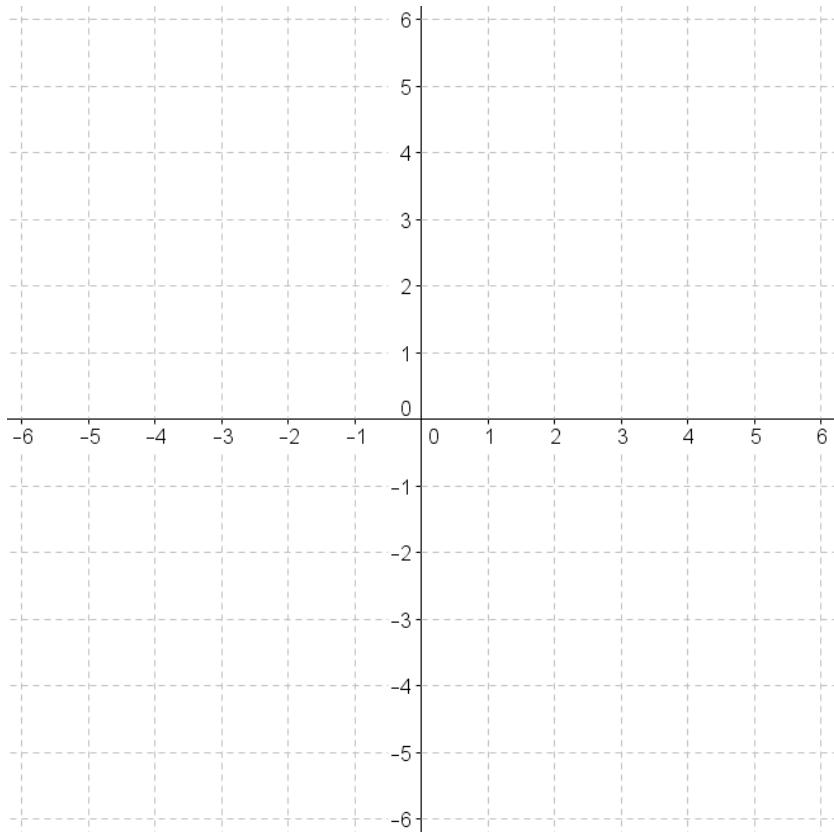
- What is the length of the segment shown on the coordinate plane below?



- Use the distance formula to determine the distance between points  $(9, 15)$  and  $(3, 7)$ .

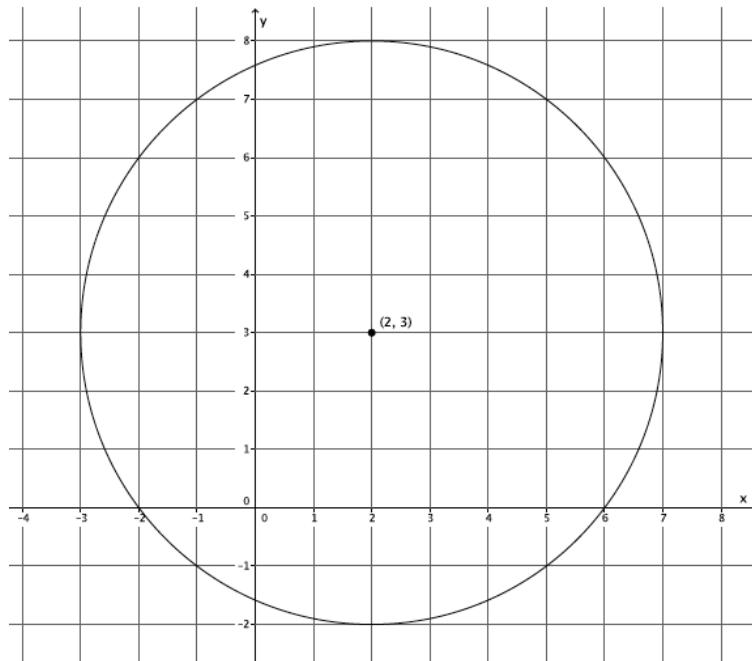
**Example 1**

If we graph all of the points whose distance from the origin is equal to 5, what shape will be formed?



**Example 2**

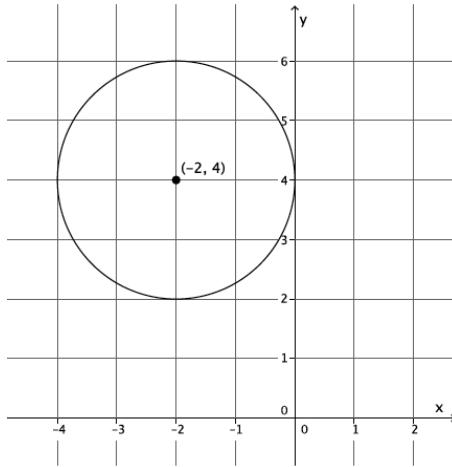
Shown below is a circle with center  $(2, 3)$  with radius 5.



**Exercises**

3. Write an equation for the circle whose center is at  $(9, 0)$  and has radius 7.

4. Write an equation whose graph is the circle below.



5. What is the radius and center of the circle given by the equation  $(x + 12)^2 + (y - 4)^2 = 81$ ?
6. Petra is given the equation  $(x - 15)^2 + (y + 4)^2 = 100$  and identifies its graph as a circle whose center is  $(-15, 4)$  and radius is 10. Has Petra made a mistake? Explain.

- 7.
- What is the radius of the circle with center  $(3, 10)$  that passes through  $(12, 12)$ ?
- 8.
- What is the equation of this circle?
- 9.
- A circle with center  $(2, -5)$  is tangent to the  $x$ -axis.
    - What is the radius of the circle?
  - What is the equation of the circle?
- 9.
- Two points in the plane,  $A = (-3, 8)$  and  $B = (17, 8)$ , represent the endpoints of the diameter of a circle.
    - What is the center of the circle? Explain.
    - What is the radius of the circle? Explain.
    - Write the equation of the circle.

10. Consider the circles with equations:

$$x^2 + y^2 = 25 \text{ and}$$
$$(x - 9)^2 + (y - 12)^2 = 100.$$

a. What are the radii of the circles?

b. What is the distance between the centers of the circles?

c. Make a rough sketch of the two circles to explain why the circles must be tangent to one another.

11. A circle is given by the equation  $(x^2 + 2x + 1) + (y^2 + 4y + 4) = 121$ .

a. What is the center of the circle?

b. What is the radius of the circle?

c. Describe what you had to do in order to determine the center and the radius of the circle.

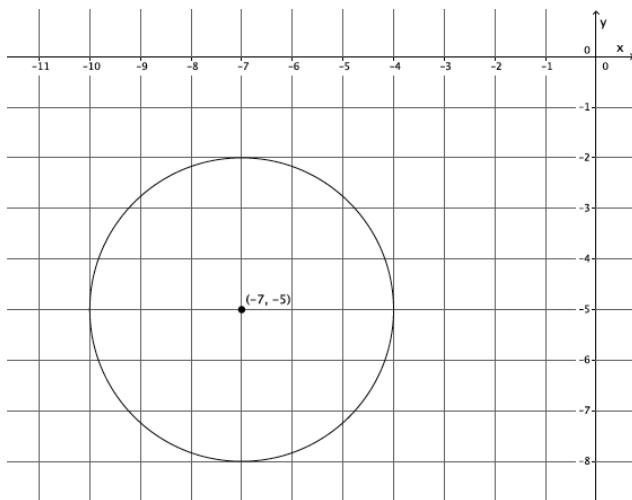
**Lesson Summary**

$(x - a)^2 + (y - b)^2 = r^2$  is the general equation for any circle with radius  $r$  and center  $(a, b)$ .

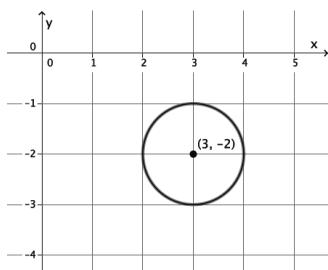
**Problem Set**

1. Write the equation for a circle with center  $\left(\frac{1}{2}, \frac{3}{7}\right)$  and radius  $\sqrt{13}$ .
2. What is the center and radius of the circle given by the equation  $x^2 + (y - 11)^2 = 144$ ?
3. A circle is given by the equation  $x^2 + y^2 = 100$ . Which of the following points are on the circle?
  - a.  $(0, 10)$
  - b.  $(-8, 6)$
  - c.  $(-10, -10)$
  - d.  $(45, 55)$
  - e.  $(-10, 0)$
4. Determine the center and radius of each circle:
  - a.  $3x^2 + 3y^2 = 75$
  - b.  $2(x + 1)^2 + 2(y + 2)^2 = 10$
  - c.  $4(x - 2)^2 + 4(y - 9)^2 - 64 = 0$
5. A circle has center  $(-13, \pi)$  and passes through the point  $(2, \pi)$ .
  - a. What is the radius of the circle?
  - b. Write the equation of the circle.
6. Two points in the plane,  $A = (19, 4)$  and  $B = (19, -6)$ , represent the endpoints of the diameter of a circle.
  - a. What is the center of the circle?
  - b. What is the radius of the circle?
  - c. Write the equation of the circle.

7. Write the equation of the circle shown below.



8. Write the equation of the circle shown below.



9. Consider the circles with equations:

$$x^2 + y^2 = 2 \text{ and}$$

$$(x - 3)^2 + (y - 3)^2 = 32.$$

- What are the radii of the two circles?
- What is the distance between their centers?
- Make a rough sketch of the two circles to explain why the circles must be tangent to one another.

## Lesson 18: Recognizing Equations of Circles

### Classwork

#### Opening Exercise

- a. Express this as a trinomial:  $(x - 5)^2$ .

$x$	$x^2$	$3x$	3
$x$			
3	$3x$	?	

=

40

$x$	$x^2$	$3x$	3
$x$			
3	$3x$	9	

=

49

- b. Express this as a trinomial:  $(x + 4)^2$ .

- c. Factor the trinomial :  $x^2 + 12x + 36$ .

- d. Complete the square to solve the following equation:  $x^2 + 6x = 40$ .

**Example 1**

The following is the equation of a circle with radius 5 and center (1,2). Do you see why?

$$x^2 - 2x + 1 + y^2 - 4y + 4 = 25$$

**Exercise**

1. Rewrite the following equations in the form  $(x - a)^2 + (y - b)^2 = r^2$ .

a.  $x^2 + 4x + 4 + y^2 - 6x + 9 = 36$

b.  $x^2 - 10x + 25 + y^2 + 14y + 49 = 4$

**Example 2**

What is the center and radius of the following circle?

$$x^2 + 4x + y^2 - 12y = 41$$

**Exercises**

2. Identify the center and radius for each of the following circles.

a.  $x^2 - 20x + y^2 + 6y = 35$

b.  $x^2 - 3x + y^2 - 5y = \frac{19}{2}$

3. Could the circle with equation  $x^2 - 6x + y^2 - 7 = 0$  have a radius of 4? Why or why not?
4. Stella says the equation  $x^2 - 8x + y^2 + 2y = 5$  has a center of  $(4, -2)$  and a radius of  $\sqrt{22}$ . Is she correct? Why or why not?

**Example 3**

Could  $x^2 + y^2 + Ax + By + C = 0$  represent a circle?

**Exercise**

5. Identify the graphs of the following equations as a circle, a point, or an empty set.

a.  $x^2 + y^2 + 4x = 0$

b.  $x^2 + y^2 + 6x - 4y + 15 = 0$

c.  $2x^2 + 2y^2 - 5x + y + \frac{13}{4} = 0$

**Problem Set**

1. Identify the center and radii of the following circles.
  - a.  $(x + 25) + y^2 = 1$
  - b.  $x^2 + 2x + y^2 - 8y = 8$
  - c.  $x^2 - 20x + y^2 - 10y + 25 = 0$
  - d.  $x^2 + y^2 = 19$
  - e.  $x^2 + x + y^2 + y = -\frac{1}{4}$
2. Sketch a graph of the following equations.
  - a.  $x^2 + y^2 + 10x - 4y + 33 = 0$
  - b.  $x^2 + y^2 + 14x - 16y + 104 = 0$
  - c.  $x^2 + y^2 + 4x - 10y + 29 = 0$
3. Chante claims that two circles given by  $(x + 2)^2 + (y - 4)^2 = 49$  and  $x^2 + y^2 - 6x + 16y + 37 = 0$  are externally tangent. She is right. Show that she is.
4. Draw a circle. Randomly select a point in the interior of the circle; label the point  $A$ . Construct the greatest radius circle with center  $A$  that lies within the circular region defined by the original circle. Hint: Draw a line through the center, the circle, and point  $A$ .

## Lesson 19: Equations for Tangent Lines to Circles

### Classwork

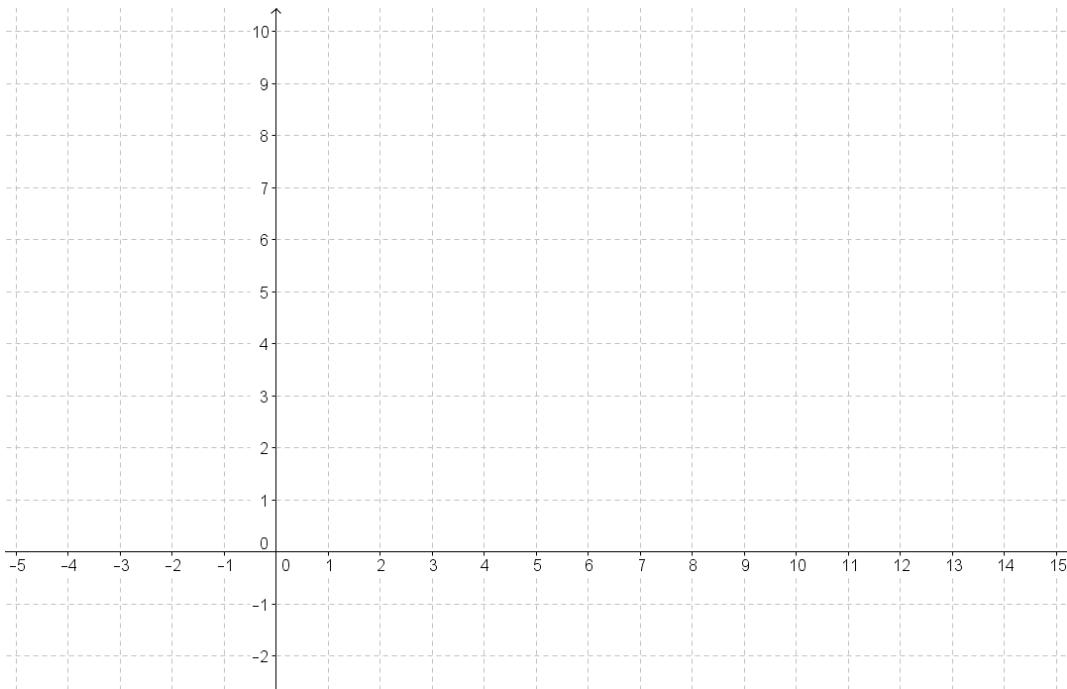
#### Opening Exercise

A circle of radius 5 passes through points  $A(-3,3)$  and  $B(3,1)$ .

- a. What is the special name for segment  $AB$ ?
  
  
  
  
  
  
- b. How many circles can be drawn that meet the given criteria? Explain how you know.
  
  
  
  
  
  
- c. What is the slope of  $\overline{AB}$ ?
  
  
  
  
  
  
- d. Find the midpoint of  $\overline{AB}$ .
  
  
  
  
  
  
- e. Find the equations of the line containing a diameter of the given circle perpendicular to  $\overline{AB}$ .
  
  
  
  
  
  
- f. Is there more than one answer possible for part (e)?

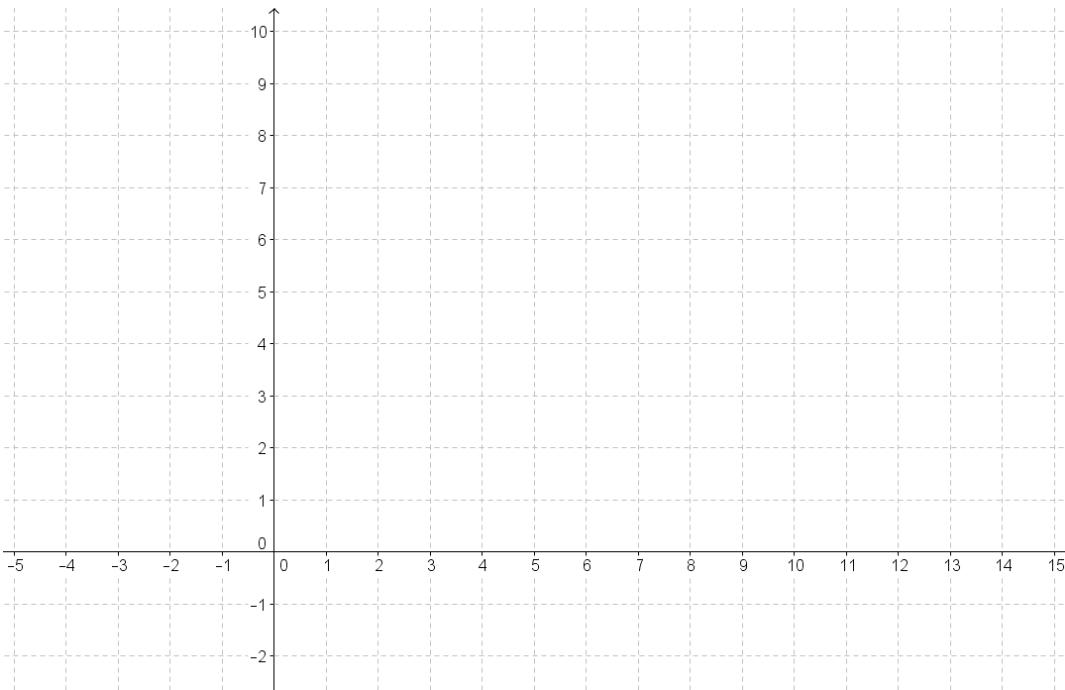
**Example 1**

Consider the circle with equation  $(x - 3)^2 + (y - 5)^2 = 20$ . Find the equations of two tangent lines to the circle that each have slope  $-\frac{1}{2}$ .



**Exercise 1**

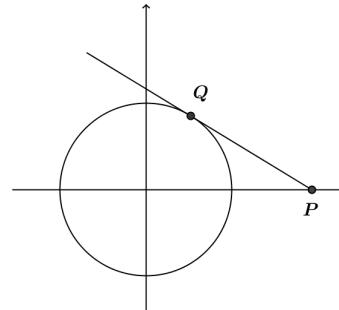
Consider the circle with equation  $(x - 4)^2 + (y - 5)^2 = 20$ . Find the equations of two tangent lines to the circle that each have slope 2.



**Example 2**

Refer to the diagram below.

Let  $p > 1$ . What is the equation of the tangent line to the circle  $x^2 + y^2 = 1$  through the point  $(p, 0)$  on the  $x$ -axis with a point of tangency in the upper half-plane?

**Exercises**

2. Use the same diagram from Example 2 above, but label the point of tangency in the lower half-plane as  $Q'$ .
  - a. What are the coordinates of  $Q'$ ?
  - b. What is the slope of  $\overline{OQ}'$ ?
  - c. What is the slope of  $\overline{Q'P}$ ?
  - d. Find the equation of the second tangent line to the circle through  $(p, 0)$ .

3. Show that a circle with equation  $(x - 2)^2 + (y + 3)^2 = 160$  has two tangent lines with equations

$$y + 15 = \frac{1}{3}(x - 6) \text{ and } y - 9 = \frac{1}{3}(x + 2).$$

4. Could a circle given by the equation  $(x - 5)^2 + (y - 1)^2 = 25$  have tangent lines given by the equations  $y - 4 = \frac{4}{3}(x - 1)$  and  $y - 5 = \frac{3}{4}(x - 8)$ ? Explain how you know.

**Lesson Summary****Theorems**

A tangent line to a circle is perpendicular to the radius of the circle drawn to the point of tangency.

**Relevant Vocabulary**

**TANGENT TO A CIRCLE.** A *tangent line to a circle* is a line in the same plane that intersects the circle in one and only one point. This point is called the *point of tangency*.

**Problem Set**

1. Consider the circle  $(x - 1)^2 + (y - 2)^2 = 16$ . There are two lines tangent to this circle having a slope of 0.
  - a. Find the coordinates of the points of tangency.
  - b. Find the equations of the two tangent lines.
2. Consider the circle  $x^2 - 4x + y^2 + 10y + 13 = 0$ . There are two lines tangent to this circle having a slope of  $\frac{2}{3}$ .
  - a. Find the coordinates of the two points of tangency.
  - b. Find the equations of the two tangent lines.
3. What are the coordinates of the points of tangency of the two tangent lines through the point  $(1,1)$  each tangent to the circle  $x^2 + y^2 = 1$ ?
4. What are the coordinates of the points of tangency of the two tangent lines through the point  $(-1, -1)$  each tangent to the circle  $x^2 + y^2 = 1$ ?
5. What is the equation of the tangent line to the circle  $x^2 + y^2 = 1$  through the point  $(6,0)$ ?
6. D'Andre said that a circle with equation  $(x - 2)^2 + (y - 7)^2 = 13$  has a tangent line represented by the equation  $y - 5 = -\frac{3}{2}(x + 1)$ . Is he correct? Explain.

7. Kamal gives the following proof that  $y - 1 = \frac{8}{9}(x + 10)$  is the equation of a line that is tangent to a circle given by  $(x + 1)^2 + (y - 9)^2 = 145$ .

The circle has center  $(-1, 9)$  and radius 12. The point  $(-10, 1)$  is on the circle because

$$(-10 + 1)^2 + (1 - 9)^2 = (-9)^2 + (-8)^2 = 145.$$

The slope of the radius is  $\frac{9-1}{-1-10} = \frac{8}{-11}$ ; therefore, the equation of the tangent line is  $y - 1 = \frac{8}{9}(x + 10)$ .

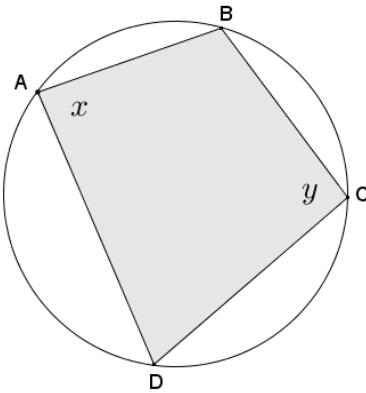
- Kerry said that Kamal has made an error. What was Kamal's error? Explain what he did wrong.
  - What should the equation for the tangent line be?
8. Describe a similarity transformation that maps a circle given by  $x^2 + 6x + y^2 - 2y = 71$  to a circle of radius 3 that is tangent to both axes in the first quadrant.

## Lesson 20: Cyclic Quadrilaterals

### Classwork

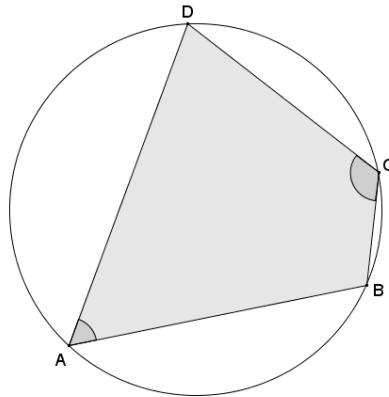
#### Opening Exercise

Given cyclic quadrilateral  $ABCD$  shown in the diagram, prove that  $x + y = 180^\circ$ .



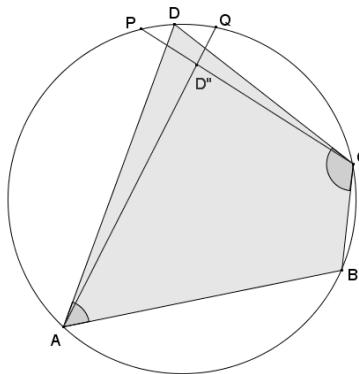
#### Example 1:

Given quadrilateral  $ABCD$  with  $m\angle A + m\angle C = 180^\circ$ , prove that quadrilateral  $ABCD$  is cyclic; in other words, prove that points  $A, B, C$ , and  $D$  lie on the same circle.

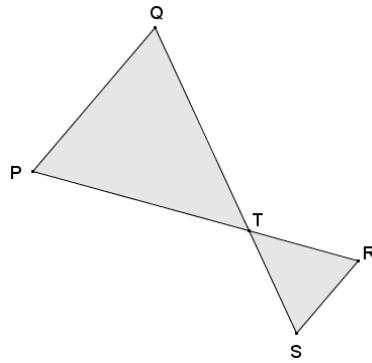


**Exercises**

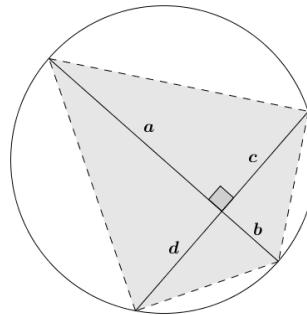
1. Assume that vertex  $D''$  lies inside the circle as shown in the diagram. Use a similar argument to Example 1 to show that vertex  $D''$  cannot lie inside the circle.



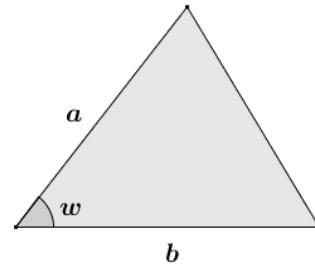
2. Quadrilateral  $PQRS$  is a cyclic quadrilateral. Explain why  $\triangle PQT \sim \triangle SRT$ .



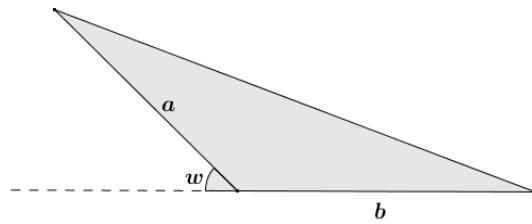
3. A cyclic quadrilateral has perpendicular diagonals. What is the area of the quadrilateral in terms of  $a$ ,  $b$ ,  $c$ , and  $d$  as shown?



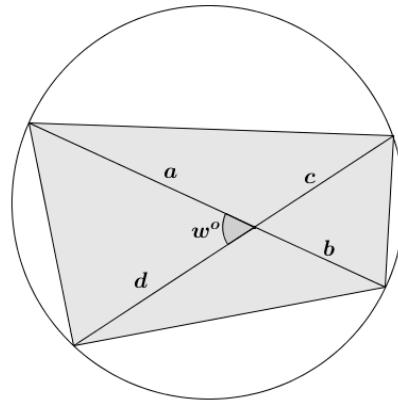
4. Show that the triangle in the diagram has area  $\frac{1}{2}ab \sin(w)$ .



5. Show that the triangle with obtuse angle  $(180 - w)^\circ$  has area  $\frac{1}{2}ab \sin(w)$ .



6. Show that the area of the cyclic quadrilateral shown in the diagram is  $Area = \frac{1}{2}(a + b)(c + d) \sin(w)$ .



**Lesson Summary****THEOREMS:**

Given a convex quadrilateral, the quadrilateral is cyclic if and only if one pair of opposite angles is supplementary.

The area of a triangle with side lengths  $a$  and  $b$  and acute included angle with degree measure  $w$ :

$$\text{Area} = \frac{1}{2}ab \cdot \sin(w).$$

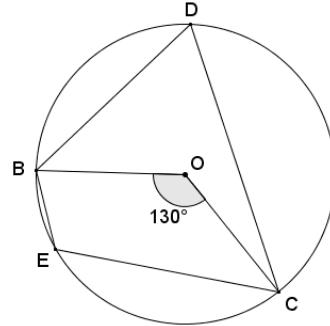
The area of a cyclic quadrilateral  $ABCD$  whose diagonals  $\overline{AC}$  and  $\overline{BD}$  intersect to form an acute or right angle with degree measure  $w$ :  $\text{Area}(ABCD) = \frac{1}{2} \cdot AC \cdot BD \cdot \sin(w)$ .

**Relevant Vocabulary**

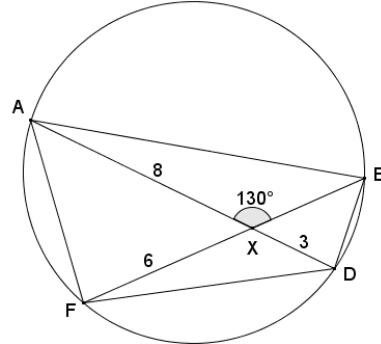
**CYCLIC QUADRILATERAL:** A quadrilateral inscribed in a circle is called a *cyclic quadrilateral*.

**Problem Set**

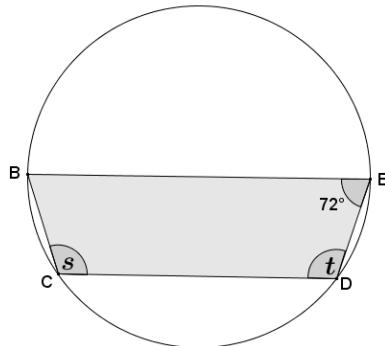
1. Quadrilateral  $BDEC$  is cyclic,  $O$  is the center of the circle, and  $m\angle BOC = 130^\circ$ . Find  $m\angle BEC$ .



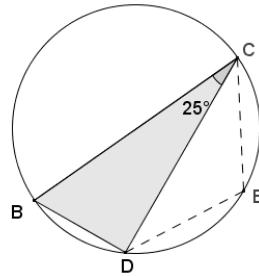
2. Quadrilateral  $FAED$  is cyclic,  $AX = 8$ ,  $FX = 6$ ,  $XD = 3$ , and  $m\angle AXE = 130^\circ$ . Find the area of quadrilateral  $FAED$ .



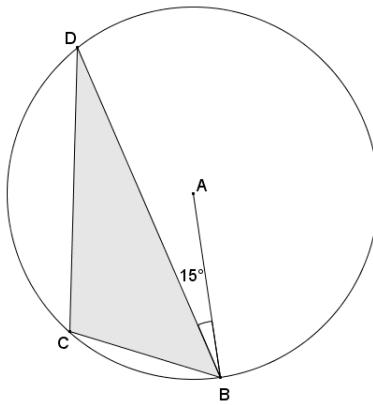
3. In the diagram below,  $\overline{BE} \parallel \overline{CD}$ , and  $m\angle BED = 72^\circ$ . Find the value of  $s$  and  $t$ .



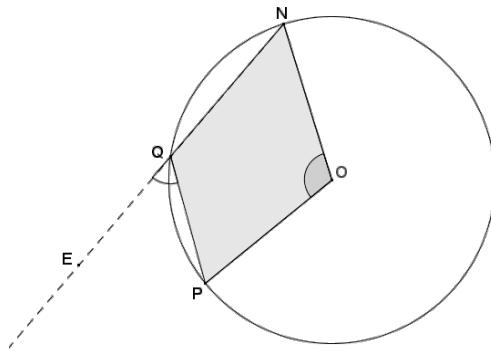
4. In the diagram below,  $\overline{BC}$  is the diameter,  $m\angle BCD = 25^\circ$ , and  $\overline{CE} \cong \overline{DE}$ . Find  $m\angle CED$ .



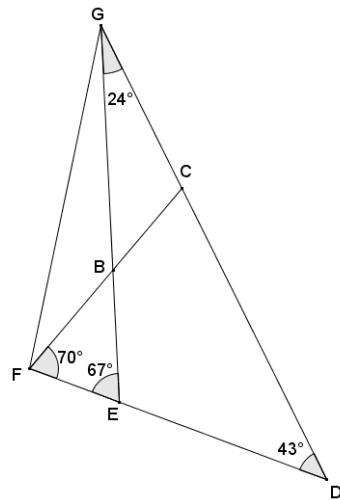
5. In circle A,  $m\angle ABD = 15^\circ$ . Find  $m\angle BCD$ .



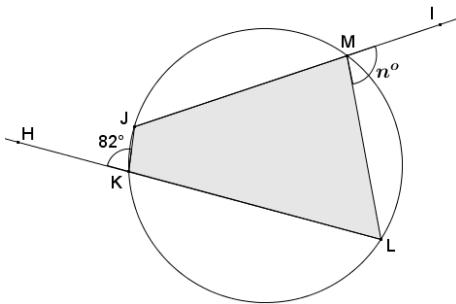
6. Given the diagram below,  $O$  is the center of the circle. If  $m\angle NOP = 112^\circ$ , find  $m\angle PQE$ .



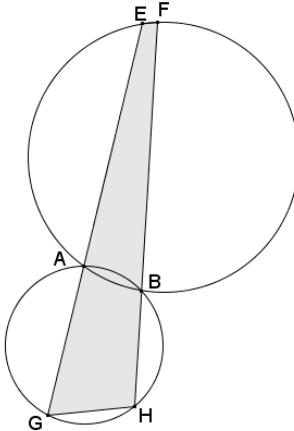
7. Given the angle measures as indicated in the diagram below, prove that vertices  $C, B, E$ , and  $D$  lie on a circle.



8. In the diagram below, quadrilateral  $JKLM$  is cyclic. Find the value of  $n$ .

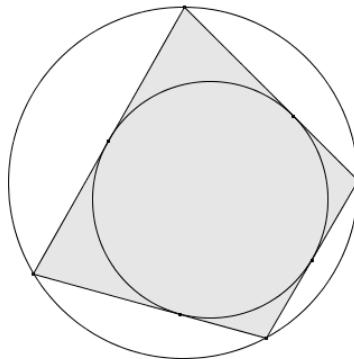


9. Do all four perpendicular bisectors of the sides of a cyclic quadrilateral pass through a common point? Explain.
10. The circles in the diagram below intersect at points  $A$  and  $B$ . If  $m\angle FHG = 100^\circ$  and  $m\angle HGE = 70^\circ$ , find  $m\angle GEF$  and  $m\angle EFH$ .

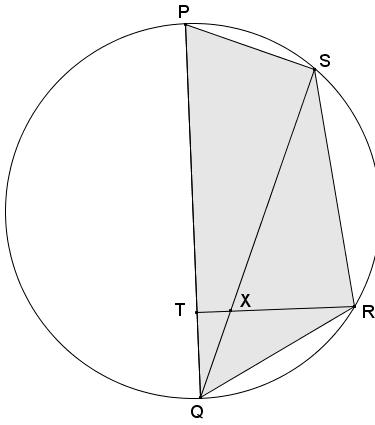


11. A quadrilateral is called *bicentric* if it is both cyclic and possesses an inscribed circle. (See diagram to the right.)

- What can be concluded about the opposite angles of a bicentric quadrilateral? Explain.
- Each side of the quadrilateral is tangent to the inscribed circle. What does this tell us about the segments contained in the sides of the quadrilateral?
- Based on the relationships highlighted in part (b), there are four pairs of congruent segments in the diagram. Label segments of equal length with  $a$ ,  $b$ ,  $c$ , and  $d$ .
- What do you notice about the opposite sides of the bicentric quadrilateral?



12. Quadrilateral  $PSRQ$  is cyclic such that  $\overline{PQ}$  is the diameter of the circle. If  $\angle QRT \cong \angle QSR$ , prove that  $\angle PTR$  is a right angle, and show that  $S$ ,  $X$ ,  $T$ , and  $P$  lie on a circle.



## Lesson 21: Ptolemy's Theorem

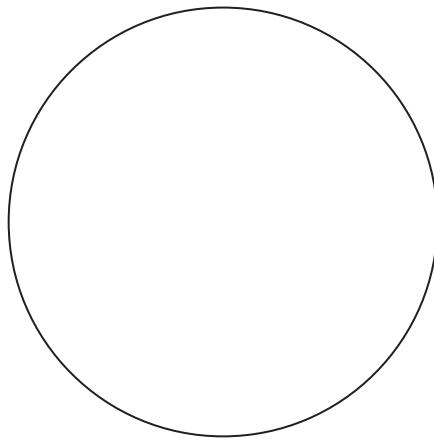
### Classwork

#### Opening Exercise

Ptolemy's theorem says that for a cyclic quadrilateral  $ABCD$ ,  $AC \cdot BD = AB \cdot CD + BC \cdot AD$ .

With ruler and a compass, draw an example of a cyclic quadrilateral. Label its vertices  $A$ ,  $B$ ,  $C$ , and  $D$ .

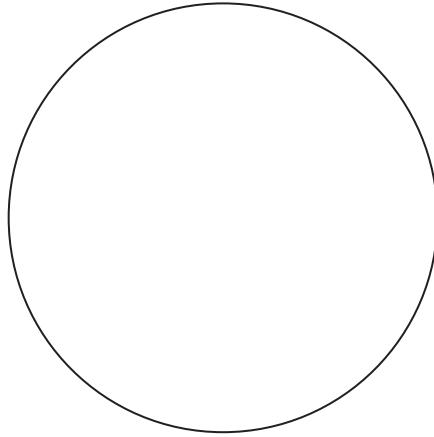
Draw the two diagonals  $\overline{AC}$  and  $\overline{BD}$ .



With a ruler, test whether or not the claim that  $AC \cdot BD = AB \cdot CD + BC \cdot AD$  seems to hold true.

Repeat for a second example of a cyclic quadrilateral.

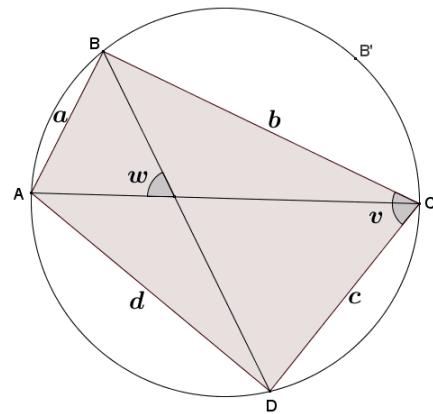
**Challenge:** Draw a cyclic quadrilateral with one side of length zero. What shape is this cyclic quadrilateral? Does Ptolemy's claim hold true for it?



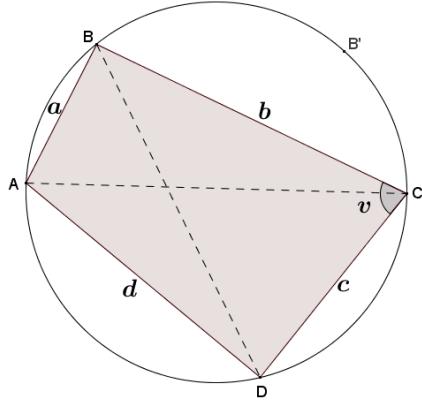
### Exploratory Challenge: A Journey to Ptolemy's Theorem

The diagram shows cyclic quadrilateral  $ABCD$  with diagonals  $\overline{AC}$  and  $\overline{BD}$  intersecting to form an acute angle with degree measure  $w$ .  $AB = a$ ,  $BC = b$ ,  $CD = c$ , and  $DA = d$ .

- From last lesson, what is the area of quadrilateral  $ABCD$  in terms of the lengths of its diagonals and the angle  $w$ ? Remember this formula for later on!

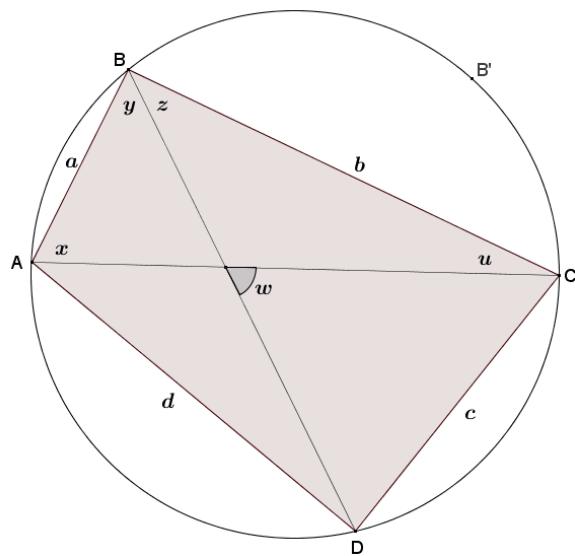


- Explain why one of the angles,  $\angle BCD$  or  $\angle BAD$ , has a measure less than or equal to  $90^\circ$ .
- Let's assume that  $\angle BCD$  in our diagram is the angle with a measure less than or equal to  $90^\circ$ . Call its measure  $v$  degrees. What is the area of triangle  $BCD$  in terms of  $b$ ,  $c$ , and  $v$ ? What is the area of triangle  $BAD$  in terms of  $a$ ,  $d$ , and  $v$ ? What is the area of quadrilateral  $ABCD$  in terms of  $a$ ,  $b$ ,  $c$ ,  $d$ , and  $v$ ?



- We now have two different expressions representing the area of the same cyclic quadrilateral  $ABCD$ . Does it seem to you that we are close to a proof of Ptolemy's claim?

- e. Trace the circle and points  $A$ ,  $B$ ,  $C$ , and  $D$  onto a sheet of patty paper. Reflect triangle  $ABC$  about the perpendicular bisector of diagonal  $\overline{AC}$ . Let  $A'$ ,  $B'$ , and  $C'$  be the images of the points  $A$ ,  $B$ , and  $C$ , respectively.
- What does the reflection do with points  $A$  and  $C$ ?
  - Is it correct to draw  $B'$  as on the circle? Explain why or why not.
  - Explain why quadrilateral  $AB'CD$  has the same area as quadrilateral  $ABCD$ .
- f. The diagram shows angles having degree measures  $u$ ,  $w$ ,  $x$ ,  $y$ , and  $z$ . Find and label any other angles having degree measures  $u$ ,  $w$ ,  $x$ ,  $y$ , or  $z$ , and justify your answers.



- g. Explain why  $w = u + z$  in your diagram from part (f).
- h. Identify angles of measures  $u, x, y, z$ , and  $w$  in your diagram of the cyclic quadrilateral  $AB'CD$  from part (e).
- i. Write a formula for the area of triangle  $B'AD$  in terms of  $b, d$ , and  $w$ . Write a formula for the area of triangle  $B'CD$  in terms of  $a, c$ , and  $w$ .
- j. Based on the results of part (i), write a formula for the area of cyclic quadrilateral  $ABCD$  in terms of  $a, b, c, d$ , and  $w$ .
- k. Going back to part (a), now establish Ptolemy's theorem.

**Lesson Summary****Theorems**

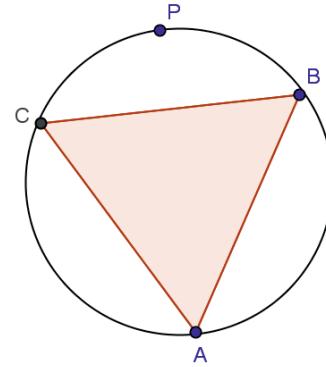
**PTOLEMY'S THEOREM:** For a cyclic quadrilateral  $ABCD$ ,  $AC \cdot BD = AB \cdot CD + BC \cdot AD$ .

**Relevant Vocabulary**

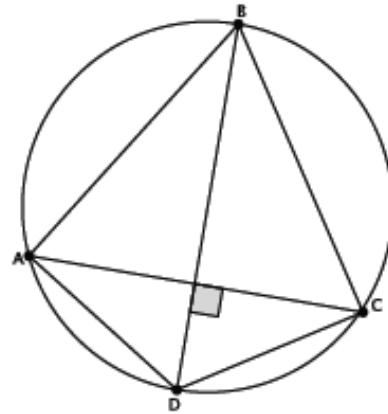
**CYCLIC QUADRILATERAL:** A quadrilateral with all vertices lying on a circle is known as a cyclic quadrilateral.

**Problem Set**

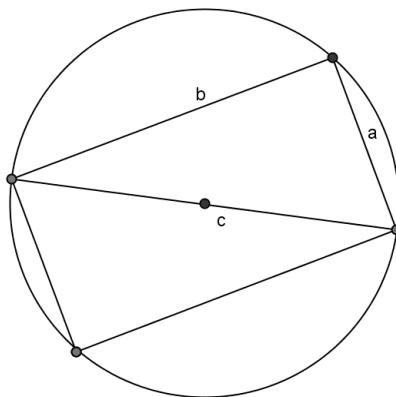
1. An equilateral triangle is inscribed in a circle. If  $P$  is a point on the circle, what does Ptolemy's theorem have to say about the distances from this point to the three vertices of the triangle?



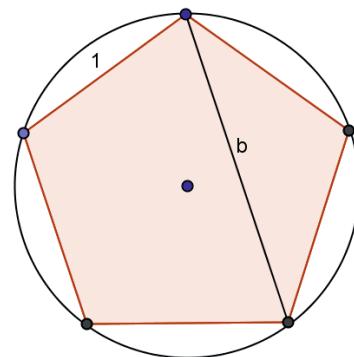
2. Kite  $ABCD$  is inscribed in a circle. The kite has an area of 108 sq. in., and the ratio of the lengths of the non-congruent adjacent sides is 3 : 1. What is the perimeter of the kite?



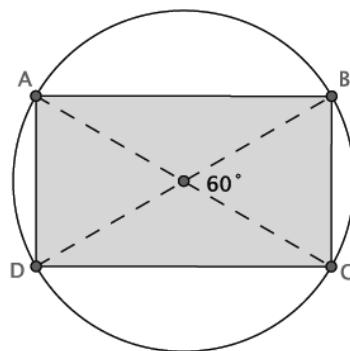
3. Draw a right triangle with leg lengths  $a$  and  $b$ , and hypotenuse length  $c$ .  
 Draw a rotated copy of the triangle such that the figures form a rectangle.  
 What does Ptolemy have to say about this rectangle?



4. Draw a regular pentagon of side length 1 in a circle. Let  $b$  be the length of its diagonals. What does Ptolemy's theorem say about the quadrilateral formed by four of the vertices of the pentagon?



5. The area of the inscribed quadrilateral is  $\sqrt{300}$  mm<sup>2</sup>. Determine the circumference of the circle.



6. Extension: Suppose  $x$  and  $y$  are two acute angles, and the circle has a diameter of 1 unit. Find  $a$ ,  $b$ ,  $c$ , and  $d$  in terms of  $x$  and  $y$ . Apply Ptolemy's theorem, and determine the exact value of  $\sin(75^\circ)$ .

- Explain why  $\frac{a}{\sin(x)}$  equals the diameter of the circle.
- If the circle has a diameter of 1, what is  $a$ ?
- Use Thales' theorem to write the side lengths in the original diagram in terms of  $x$  and  $y$ .
- If one diagonal of the cyclic quadrilateral is 1, what is the other?
- What does Ptolemy's theorem give?
- Using the result from part (e), determine the exact value of  $\sin(75^\circ)$ .

