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# Connecting Algebra and Geometry Through Coordinates

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<sup>1</sup> Each lesson is ONE day, and ONE day is considered a 45-minute period.

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## Geometry • Module 4

# Connecting Algebra and Geometry Through Coordinates

## OVERVIEW

In this module, students explore and experience the utility of analyzing algebra and geometry challenges through the framework of coordinates. The module opens with a modeling challenge (**G-MG.A.1**, **G-MG.A.3**), one that reoccurs throughout the lessons, to use coordinate geometry to program the motion of a robot that is bound within a certain polygonal region of the plane—the room in which it sits. To set the stage for complex work in analytic geometry (computing coordinates of points of intersection of lines and line segments or the coordinates of points that divide given segments in specific length ratios, and so on), students will describe the region via systems of algebraic inequalities (**A-REI.D.12**) and work to constrain the robot motion along line segments within the region (**A-REI.C.6**, **G-GPE.B.7**).

The challenge of programming robot motion along segments parallel or perpendicular to a given segment brings in an analysis of slopes of parallel and perpendicular lines and the need to prove results about these quantities (**G-GPE.B.5**). This work highlights the role of the converse of the Pythagorean theorem in the identification of perpendicular directions of motion (**G-GPE.B.4**).

To fully develop the analysis of perimeter and area of a polygon in terms of the coordinates of its vertices (**G-GPE.B.7**), students will derive the area  $A$  of a triangle with coordinates  $(0,0)$ ,  $(x_1, y_2)$ , and  $(x_2, y_2)$  as  $A = \left(\frac{1}{2}\right) |x_1y_1 - x_2y_2|$  and extend this result to the areas of triangles situated elsewhere in the plane and to simple polygons seen as unions of triangles. Applications to robot motion continue. Students will also find locations on a directed line segment between two given points that partition the segment in given ratios (**G-GPE.B.6**) and connect this work to proving classical results in geometry (**G-GPE.B.4**). For example, proving that the diagonals of a parallelogram bisect one another, and the medians of a triangle meet at the point  $\frac{2}{3}$  of the way from the vertex for each. This study also deepens student understanding of the linear motion of the robot between and beyond two given points.

The module ends with the challenge of locating the point along a line closest to a given point, again given as a robot challenge, and developing the distance formula for a point from a line (**G-GPE.B.4**).

## Focus Standards

### Use coordinates to prove simple geometric theorems algebraically.<sup>2</sup>

- G-GPE.B.4** Use coordinates to prove simple geometric theorems algebraically. *For example, prove or disprove that a figure defined by four given points in the coordinate plane is a rectangle; prove or disprove that the point  $(1, \sqrt{3})$  lies on the circle centered at the origin and containing the point  $(0, 2)$ .*
- G-GPE.B.5** Prove<sup>3</sup> the slope criteria for parallel and perpendicular lines and use them to solve geometric problems (e.g., find the equation of a line parallel or perpendicular to a given line that passes through a given point).
- G-GPE.B.6** Find the point on a directed line segment between two given points that partitions the segment in a given ratio.
- G-GPE.B.7** Use coordinates to compute perimeters of polygons and areas of triangles and rectangles, e.g., using the distance formula.\*

## Foundational Standards

### Solve systems of equations.

- A-REI.C.6** Solve systems of linear equations exactly and approximately (e.g., with graphs), focusing on pairs of linear equations in two variables.

### Represent and solve equations and inequalities graphically.

- A-REI.D.12** Graph the solutions to a linear inequality in two variables as a half-plane (excluding the boundary in the case of a strict inequality), and graph the solution set to a system of linear inequalities in two variables as the intersection of the corresponding half-planes.

### Apply geometric concepts in modeling situations.

- G-MG.A.1** Use geometric shapes, their measures, and their properties to describe objects (e.g., modeling a tree trunk or a human torso as a cylinder).\*
- G-MG.A.3** Apply geometric methods to solve design problems (e.g., designing an object or structure to satisfy physical constraints or minimize cost; working with typographic grid systems based on ratios).\*

<sup>2</sup> In preparation for Regents Exams, a trapezoid is defined as a quadrilateral that has at least one pair of parallel sides.

<sup>3</sup> Prove and apply (in preparation for Regents Exams).

## Focus Standards for Mathematical Practice

- MP.1** **Make sense of problems and persevere in solving them.** Students start the module with the challenge to understand and develop the mathematics for describing of the motion of a robot bound within a certain polygonal region of the plane—the room in which it sits. This a recurring problem throughout the entire module and with it, and through related problems, students discover the slope criteria for perpendicular and parallel lines, the means to find the coordinates of a point dividing a line segment into two lengths in a given ratio, the distance formula of a point from a line, along with a number of geometric results via the tools of coordinate geometry.
- MP.2** **Reason abstractly and quantitatively.** Students rotate line segments about their endpoints and discover the general slope criterion for perpendicular lines and articulate this criterion in an abstract setting. Geometric results (such as “the three medians of a triangle are concurrent”) are examined in concrete settings and students determine that these results hold in general. They also develop a formula for the area of a triangle based solely on the coordinates of its three vertices and generalize this to an area formula for quadrilaterals and other planar polygons.
- MP.4** **Model with mathematics.** Students model the motion of a robot in the plane in two contexts: determining the extent of motion within the bounds of a polygonal region, and determining and moving to the location of the source of beacon signal in the infinite plane.
- MP.7** **Look for and make use of structure.** Students determine slope criteria for perpendicular and parallel lines and use these slope conditions to develop the general equation of a line and the formula for the distance of a point from a line. Students determine the area of polygonal regions using multiple methods including Green’s theorem and decomposition. Definitive geometric properties of special quadrilaterals are explored and properties of special lines in triangles are examined.
- MP.8** **Look for and express regularity in repeated reasoning.** Students use the midpoint to repeatedly separate a segment into proportional parts and derive a formula for calculating the coordinates of a point that will divide a segment into segments of given ratios.

## Terminology

### New or Recently Introduced Terms

- **Normal Segment to a Line** (A line segment with one endpoint on a line and perpendicular to the line is called a *normal segment* to the line.)

## Familiar Terms and Symbols<sup>4</sup>

- Slope
- Parallel
- Perpendicular
- Distance
- Bisect
- Directed Line Segment

## Suggested Tools and Representations

- Graph Paper
- Graphing Calculator
- Wolfram Alpha Software
- Geometer's Sketchpad Software

## Assessment Summary

Assessment Type	Administered	Format	Standards Addressed
Mid-Module Assessment Task	After Topic B	Constructed response with rubric	G-GPE.B.4, G-GPE.B.5, G-GPE.B.7
End-of-Module Assessment Task	After Topic D	Constructed response with rubric	G-GPE.B.4, G-GPE.B.5, G-GPE.B.6, G-GPE.B.7

<sup>4</sup> These are terms and symbols students have seen previously.



## Topic A:

# Rectangular and Triangular Regions Defined by Inequalities

**G-GPE.B.7**

**Focus Standard:** G-GPE.B.7 Use coordinates to compute perimeters of polygons and areas of triangles and rectangles, e.g., using the distance formula.\*

**Instructional Days:** 4

**Lesson 1:** Searching a Region in the Plane (E)<sup>1</sup>

**Lesson 2:** Finding Systems of Inequalities That Describe Triangular and Rectangular Regions (P)

**Lesson 3:** Lines That Pass Through Regions (P)

**Lesson 4:** Designing a Search Robot to Find a Beacon (E)

The module opens with a modeling challenge (**G-MG.A.1**, **G-MG.A.3**) that re-occurs throughout the lessons. Students use coordinate geometry to program the motion of a robot bound in a polygonal region (a room) of the plane. MP.4 is highlighted throughout this module as students transition from the verbal tasks to determining how to use coordinate geometry, algebra, and graphical thinking to complete the task. The modeling task varies in each lesson as students define regions, constrain motion along segments, rotate motion, and move through a real-world task of programming a robot. While this robot moves at a constant speed and its motion is very basic, it allows students to see the usefulness of the concepts taught in this module and put them in context. In Lesson 1 students use the distance formula and previous knowledge of angles to program a robot to search a plane. Students impose a coordinate system and describe the movement of the robot in terms of line segments and points. In Lesson 2, students graph inequalities and discover that a rectangular or triangular region (**G-GPE.B.7**) in the plane can be defined by a system of algebraic inequalities (**A-REI.D.12**). In Lesson 3, students study lines that cut through these previously described regions. Students are given two points in the plane and a region and determine whether a line through those points meets the region. If it does, they describe the intersection as a segment and name the endpoints. Topic A ends with Lesson 4, where students return to programming the robot while constraining motion along line segments within the region (**G-GPE.B.7**, **A-REI.C.6**) and rotating a segment 90° clockwise or

<sup>1</sup> Lesson Structure Key: P-Problem Set Lesson, M-Modeling Cycle Lesson, E-Exploration Lesson, S-Socratic Lesson

counterclockwise about an endpoint (**G-MG.A.1**, **G-MG.A.3**). Revisiting **A-REI.C.6** (solving systems of linear equations in two variables) and **A-REI.D.12** (graphing linear inequalities in two variables and the solution sets of a system of linear inequalities) shows the coherence between algebra and geometry.



## Lesson 1: Searching a Region in the Plane

### Student Outcomes

- Given a physical situation (e.g., a room of a certain shape and dimensions, with objects at certain positions and a robot moving across the room), students impose a coordinate system and describe the given in terms of polygonal regions, line segments, and points **in the coordinate system**.

### Lesson Notes

This lesson will require students to remember concepts such as right triangle trigonometry, graphing on the coordinate plane, and the Pythagorean theorem, and formulas such as the distance formula ( $d = \sqrt{a^2 + b^2}$ ), and the rate formula ( $\text{rate} \times \text{time} = \text{distance}$ ). Some students may need to be reminded how to write the equation of a line. The task at first seems complicated but can be broken down easily if students use graph paper and scale the graph down to show very specific points given.

**MP.4** In this lesson, students transition from verbal, graphical, and algebraic thinking to modeling robot motion in a straight line and using lines of motion and previously learned topics (distance, proportion) to determine the location of impact in a warehouse.

### Classwork

#### Opening (8 minutes)

What type of math (geometry) do you think is involved in programming a robot to vacuum an empty room?

Have students brainstorm and write the mathematical terms they suggest on the board. Do not comment on the ideas; simply write the terms given.

Students will have varying levels of familiarity with robots; show some or all of this video to generate interest and other ideas about the mathematics involved in programming: <http://www.youtube.com/watch?v=gvQKGev56qU>. After showing the video, ask the initial question again. Continue writing terms on the board, but this time, have students explain how the mathematics given would be used.

If students are hesitant to participate, provide scaffolds, such as the following sentence starters:

- To determine how far the robot moves, I could ...
- To identify where the robot starts and ends, I could ...
- To explain the path of the robot, I could ...
- To find the speed of the robot, I could ...

Ask leading questions to elicit the responses below if they are not mentioned by students.

- Distance, angles, slope, lines, equations of lines, graphing on the coordinate plane, scale, rate, time, triangles, and right triangle trigonometry.*

### Exploratory Challenge (27 minutes): Programming a Robot

This example illustrates the types of problems we will be exploring and studying in this module—using mathematics to describe regions in the plane (rooms) and using geometry to fully understand linear motion. This is an exploration activity. Students will work in heterogeneous groups of two or three while the teacher circulates around to answer questions, and then brings the class back together to discuss different strategies for solving and answering each part of the problem.

Present the example and allow five minutes for students to re-read the problem and discuss strategies with their group for solving the problem. Encourage them to brainstorm and do a 30 second quick-write about initial strategies. Circulate and listen to discussions. Bring the class back together and have students share ideas. Use the discussions to analyze where groups are in the problem-solving process. Students then return to their groups and begin working.

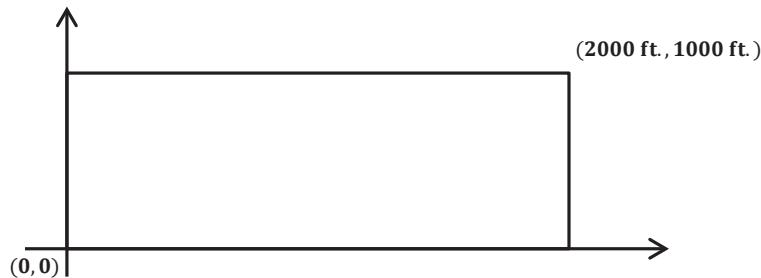
Give students graph paper to start and have them properly scale the axes and plot the known points. Use the exercises as scaffolds to guide students as they work. Groups will need varying levels of scaffolding. Some groups may answer all of the questions and some just a few. Continue walking around the room and answering questions or asking guiding questions where appropriate such as, “If a robot could travel 10 feet in 1 second, what is its rate of speed?” or “If a robot could travel 400 feet in 20 seconds, what is its rate of speed?” Let students really think through the solutions. As you monitor student progress, some strategies that could be used to help groups or the entire class are the following: pause the groups after every five minutes of work time and ask for group updates on what is working and what is not working; call team leaders to huddle and give hints; station yourself near a struggling group to guide them; and pair groups to share ideas and generate new ideas. Remind students that the robot’s motion is constant.

#### Scaffolding:

- If students are struggling, refer to Grade 5, Module 6, Lessons 2, 3, and 4. For a fun review activity on graphing, set up your room as a coordinate plane and have students plot themselves and objects in the room.
- Provide smaller sized, labeled graph paper.
- As an extension, have students determine paths of motion to move from given points to other points using translations.

#### Exploratory Challenge

Students in a robotics class must program a robot to move about an empty rectangular warehouse. The program specifies location at a given time,  $t$ , seconds. The room is twice as long as it is wide. Locations are represented as points in a coordinate plane with the southwest corner of the room deemed the origin,  $(0, 0)$ , and the northeast corner deemed the point  $(2000 \text{ ft.}, 1000 \text{ ft.})$ , as shown in the diagram below.



The first program written has the robot moving at a constant speed in a straight line. At time  $t = 1$  second, the robot is at position  $(30, 45)$ , and at  $t = 3$  seconds, it is at position  $(50, 75)$ . Complete the exercises and answer the questions below to program the robot’s motion.

MP.4

- a. Where is the location of impact?

*It is at position  $(666\frac{2}{3}, 1000)$ .*

- b. At what speed will the robot hit the wall?

*It will hit at approximately 18 ft/sec.*

- c. At what time will the robot hit the wall?

*It will hit at about  $t = 64.8$  seconds.*

#### Exercises 1–8

1. Plot the points on a coordinate plane.

2. Draw the line connecting the segments.

3. How much did the  $x$ -coordinate change in 2 seconds?

*It changed 20 ft.*

4. How much did the  $y$ -coordinate change in 2 seconds?

*It changed 30 ft.*

5. What is the ratio of change in  $y$  to change in  $x$ ?

*The ratio of change is  $\frac{3}{2}$ .*

6. What is the equation of the line of motion?

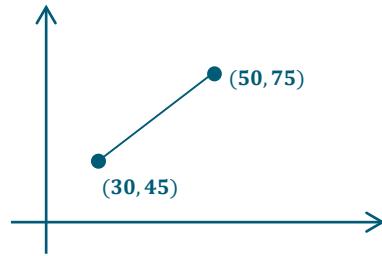
*The equation is  $y = \frac{3}{2}x$ .*

7. What theorem could be used to find the distance between the points?

*The Pythagorean theorem could be used.*

8. How far did the robot travel in 2 seconds?

*It traveled  $\approx 36.06$  ft.*



Students intuitively know that if they let the robot continue on with this motion, it will hit the wall. The questions below will give them information so that they can program the robot. This discussion should begin after groups have completed the exercises and as they are starting to begin answering the example questions. The questions can be answered in a class discussion or in groups, depending on the needs of your students. Have students predict the location of impact and then verify their answer algebraically. Use the questions listed as scaffolds to the discussion.

- Predict the location of impact based on the diagram and then verify your answer algebraically.
- Which wall will it hit? Explain.
  - *It will hit the top wall because if the line is extended, it intersects the top wall first.*
- Where will it hit? Explain.
  - *Answers will vary. Some students may give estimates of (600, 1000), some may say x is between 600 and 700, but y is always 1000, and some may do the calculation immediately.*
- Where will the robot be at  $t = 23$  seconds? Explain.
  - *Rate times time gives us distance, so  $18 \frac{\text{ft}}{\text{sec}} \times 23 \text{ seconds} = 414 \text{ feet}$*
- What can we use to find the location of impact? Explain.
  - *We can use the equation of the line of motion with  $y = 1000$ .*
- What is needed to calculate speed? Explain.
  - *Distance and time are needed. Speed is equal to the quotient of distance and time.*
- What are the units of speed in this problem?
  - *The units of speed are feet/seconds.*
- How far did the robot travel between the two points given? How did you calculate that?
  - *It traveled  $\approx 36.06 \text{ ft}$ . Using the distance formula, you have*
$$d = \sqrt{(50 - 30)^2 + (75 - 45)^2}.$$
- How long did it take the robot to move this distance?
  - $36.06 \text{ ft} \div 18 \frac{\text{ft}}{\text{sec}} \approx 2 \text{ seconds.}$
- If we know distance and time, how can we find rate?
  - *Use distance  $\div$  time.*
- What do we need to calculate time?
  - *We need rate and distance.*
- Where did the robot start its motion? At what time did it start?
  - *Using the graph, students should see that the x-coordinate is changing 10 feet each second and the y-coordinate is changing 15 feet each second. If the robot is at (30,45) after 1 second, then it started at (20,30) at time  $t = 0$ .*
- What is the distance from (20,30) to the wall?
  - *Students must use the distance formula to find the distance between (20,30) and  $(66\frac{2}{3}, 1000)$   $\approx 1111.71 \text{ ft.}$*
- What is the constant rate?
  - *The constant rate is  $18 \frac{\text{ft}}{\text{sec}}$ .*
- Knowing distance and rate, how can you find time?
  - *Use distance  $\div$  time.*

If any group finishes early, have the students program a robot to travel around the classroom and pick up certain objects. Have students measure the classroom and locate objects on a coordinate plane, then program their own robot. They should record their information on a poster to share with the class. This activity could be done in groups of four, pairing the groups in the order that they finish the above example.



### Closing (5 minutes)

Ask students to respond to these questions in writing, to a partner, or as a class.

- When programming a robot, what needs to be known to calculate the speed? What theorem helps you find necessary information?
  - *Since the units of speed are ft/sec in this example, we would need to find distance in feet and know the time the robot needed to travel that distance to calculate speed. The Pythagorean theorem and distance formula allow us to calculate distance. Students may also realize that speed is the slope of this line of motion.*
- What are some methods that can be used to determine where the robot will hit the wall? Explain.
  - *If the equation of motion is known or can be calculated, substitute in the boundary coordinate of the side of the room the robot will hit. For our example, it hit the top wall that had a y-coordinate of 1000. If it had hit the right wall, we would have substituted  $x = 2000$ .*

### Exit Ticket (5 minutes)



Name \_\_\_\_\_

Date \_\_\_\_\_

## Lesson 1: Searching a Region in the Plane

### Exit Ticket

You are moving the robot to your classroom, which measures 30 feet by 40 feet.

1. Draw the classroom set up on a coordinate plane using  $(30, 40)$  as the northeast vertex.

2. The robot was initially placed at position  $(6, 9)$ , and at  $t = 2$  seconds, its position is  $(10, 15)$ .

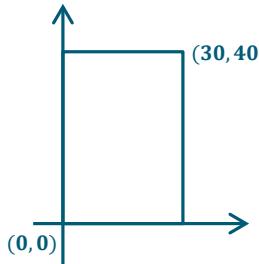
a. How far did the robot travel in 2 seconds?

b. What is the speed of the robot?

## Exit Ticket Sample Solutions

You are moving the robot to your classroom, which measures 30 feet by 40 feet.

1. Draw the classroom set up on the coordinate plane using  $(30, 40)$  as the northeast vertex.



## Scaffolding:

- Have the robot start at  $(0, 0)$  and move to  $(2, 2)$  in a  $10 \text{ ft.} \times 10 \text{ ft.}$  room.
- For students with spatial or eye-hand coordination difficulties, just assess with Problem 2, and provide them the completed grid for Problem 1.

2. The robot was initially placed at position  $(6, 9)$ , and at  $t = 2$  seconds, its position is  $(10, 15)$ .

- a. How far did the robot travel in 2 seconds?

*It traveled approximately 7.2 ft.*

- b. What is the speed of the robot?

*It traveled approximately 3.6 ft/sec.*

## Problem Set Sample Solutions

Encourage students to use graph paper to help them get started on these problems.

1. The robot in the video is moving around an empty  $100 \text{ ft.} \times 100 \text{ ft.}$  storage room at a constant speed. If the robot crosses  $(10, 10)$  at 1 second and  $(30, 30)$  at 6 seconds:

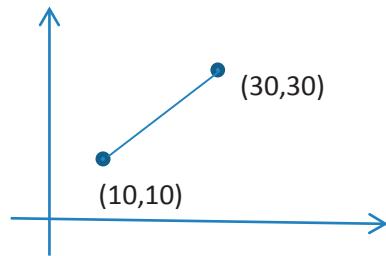
- a. Plot the points and draw the segment connecting the points.

- b. What was the change in the  $x$ -coordinate?

*It changed 20 ft.*

- c. What was the change in the  $y$ -coordinate?

*It changed 20 ft.*



- d. What is the ratio of the change in  $y$  to the change in  $x$ ?

*The ratio of change is 1:1.*

- e. How far did the robot travel between the two points?

*It traveled  $\approx 28.28 \text{ ft.}$*

- f. What was the speed of the robot?

*The speed was  $\approx 5.66$  ft./sec.*

- g. Where did the robot start?

*The robot started at position (6, 6).*

2. Your mother received a robot vacuum cleaner as a gift and wants you to help her program it to clean a vacant 30 ft. by 30 ft. room. If the vacuum is at position (12, 9) at time  $t = 2$  seconds and at position (24, 18) at  $t = 5$  seconds, answer the following:

- a. How far did the robot travel over 3 seconds?

*It traveled 15 ft.*

- b. What is the speed?

*The speed is 5 ft/sec.*

- c. What is the ratio of the change in the  $x$ -coordinate to the change in the  $y$ -coordinate?

*The ratio of change is  $\frac{4}{3}$ , or  $y = \frac{3}{4}x$ .*

- d. Where did the robot start?

*It started at position (4, 3).*

- e. Where will the robot be at  $t = 3$  seconds? Explain how you know.

*It will be at position (16, 12). I know that each second the  $x$ -value changed 4 units and the  $y$ -value changed 3 units. The  $x$ -value started at 4; therefore, in 3 seconds, it increased  $3 \times 4$  units, which means the  $x$ -value at 3 seconds is  $4 + 12$ , or 16. The  $y$ -value started at 3 and increased  $3 \times 3$  units, giving a  $y$ -value of  $3 + 9$ , or 12.*

- f. What is the location of impact?

*The location of impact is  $(30, 22\frac{1}{2})$ .*

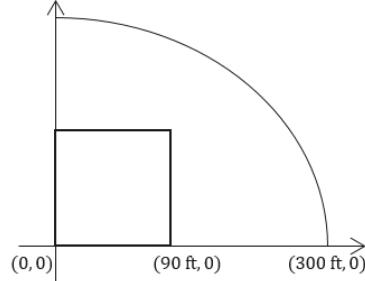
- g. At what time will the robot hit the wall?

*It will hit the wall at about  $t = 6.5$  seconds.*

3. A baseball player hits a ball at home plate at position (0, 0). It travels at a constant speed across first base at position (90, 0) in 2 seconds.

- a. What was the speed of the ball?

*The speed was 45 ft/sec.*





- b. When will it cross the fence at position  $(300, 0)$ ? Explain how you know.

*It will cross the fence in  $\approx 6.67$  seconds. I know because the distance is 300 feet at a constant speed of 45 ft/sec.  $300 \div 45 \approx 6.67$  seconds.*

4. The tennis team has a robot that picks up tennis balls. The tennis court is 36 feet wide and 78 feet long. The robot starts at position  $(8, 10)$  and is at position  $(16, 20)$  at  $t = 4$  seconds. When will it pick up the ball located at position  $(28, 35)$ ?

*It will pick up that ball in approximately 6 seconds.*



## Lesson 2: Finding Systems of Inequalities That Describe Triangular and Rectangular Regions

### Student Outcomes

- Students describe rectangles (with edges parallel to the axes) and triangles in the coordinate plane by means of inequalities. For example, the rectangle in the coordinate plane with lower left vertex  $(1, 2)$  and upper right vertex  $(10, 15)$  is  $\{(x, y) \mid 1 \leq x \leq 10 \text{ & } 2 \leq y \leq 15\}$ ; the triangle with vertices at  $(0, 0)$ ,  $(1, 3)$ , and  $(2, 1)$  is  $\{(x, y) \mid \frac{x}{2} \leq y \leq 3x \text{ & } y \leq -2x + 5\}$ .

### Lesson Notes

In Algebra I, students graphed the solutions of linear inequalities in two variables as half-planes and the solution sets of a system of linear inequalities as the intersection of corresponding half-planes (**A-REI.D.12**). In this lesson, students are given the region in the plane and write systems to describe the regions, reversing the process studied in Algebra I.

### Classwork

#### Opening Exercises (5 minutes)

The purpose of the Opening Exercises is to review the basics of graphing the solution sets of systems of inequalities in two variables. Teachers can model these exercises or use them to supplement instruction as necessary.

##### Opening Exercises

Graph each system of inequalities.

1.  $\begin{cases} y \geq 1 \\ x \leq 5 \end{cases}$

- a. Is  $(1, 2)$  a solution? Explain.

*Yes,  $(1, 2)$  is inside the region.*

- b. Is  $(1, 1)$  a solution? Explain.

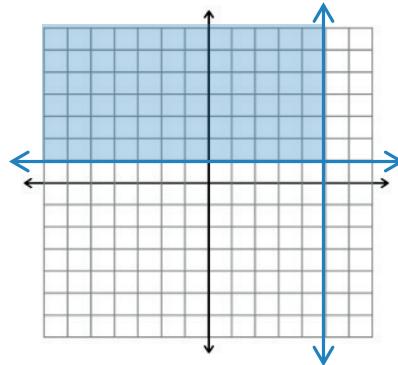
*Yes,  $(1, 1)$  is on the border of the region and included in the region.*

- c. The region is the intersection of how many half-planes? Explain how you know.

*It is the intersection of 2 half planes,  $x \leq 5$  and  $y \geq 1$ .  $x \leq 5$  splits the plane in half at the vertical line  $x = 5$  and to the left.  $y \geq 1$  cuts the plane in half horizontally from the line  $y = 1$  and above.*

##### Scaffolding:

- If students need more practice graphing, have students graph the following:  
 $y = 2x + 1$ ,  
 $y = -3x - 1$ ,  
 $y < 2x + 1$ , and  
 $y \geq 2x + 1$ .
- Advanced learners should be asked to write a system of equations where  $(-1, 2)$  is a solution, but  $(5, -3)$  is not.



MP.7

2.  $\begin{cases} y < 2x + 1 \\ y \geq -3x - 2 \end{cases}$

- a. Is  $(-2, 4)$  in the solution set?

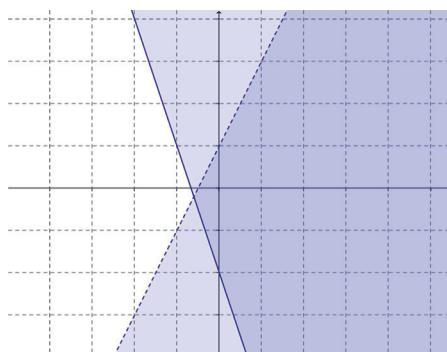
*No, it is out of the shaded region.*

- b. Is  $(1, 3)$  in the solution set?

*No,  $(1, 3)$  is not in the solution set because it is on the boundary, but the boundary is not included.*

- c. The region is the intersection of how many half-planes? Explain how you know.

*It is the intersection of 2 half planes.  $y < 2x + 1$  cuts the plane in half diagonally at the line  $y = 2x + 1$  and below.  $y \geq -3x - 2$  also cuts the plane in half vertically, but at  $y = -3x - 2$  and above.*



MP.1

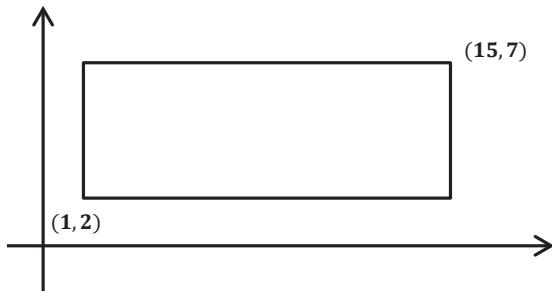
Teachers may consider the open-ended question in the example and let students struggle and problem-solve before asking the leading questions.

### Example 1 (8 minutes)

- Let's explore how we might describe regions in the coordinate plane mathematically. Here is a rectangular region in the plane. (Project the image below onto the screen or draw it on the board.)

This image is in the student materials, but the questions are not.

#### Example 1



#### Scaffolding:

If students are struggling, have them name the point in the top left corner,  $(1, 7)$ , and in the bottom right corner,  $(15, 2)$ . They could also list points between corners, but on boundaries. This should help them see that all points on the boundaries have the same  $x$ - or the same  $y$ -coordinates.

- Post a visual of the distance formula:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Suppose a point  $(x, y)$  is within this rectangular region, or possibly on its boundary.

- Name three points inside the rectangular region.
  - Answers will vary. Possible answers include  $(2, 2)$ ,  $(5, 4)$ , and  $(14, 6)$ .*
- Name three points on the boundary of the rectangular region.
  - Answers will vary. Possible answers include  $(3, 2)$ ,  $(15, 6)$ , and  $(7, 7)$ .*

- What if we want to know all of the points in the region and on the boundary? How can I describe those points?
  - *Let students brainstorm and do a 30-second Quick Write; then, share ideas with the class before asking the following.*
- Can you say anything about the possible value of  $x$ ?
  - *The x-values are to the right of 1 but to the left of 15; the x-values are greater than or equal to 1; the x-values are less than or equal to 15;  $x \geq 1$  and  $x \leq 15$ ;  $1 \leq x \leq 15$ .*
- Can you say anything about the possible value of  $y$ ?
  - *The y-values are above 2 but below 7; the y-values are greater than or equal to 2; the y-values are less than or equal to 7;  $y \geq 2$  and  $y \leq 7$ ;  $2 \leq y \leq 7$ .*
- What can you say about the coordinates of points that lie on the left side of this rectangular region?
  - *The points on the left border of this rectangle all have x-coordinates of 1.*
- On the top side?
  - *The points on the top border of this rectangle all have y-coordinates of 7.*
- The region is a rectangle. Let's review the properties of a rectangle. (Have students share ideas on the board.)
  - *All rectangles have opposite sides parallel, opposite sides congruent, 4 right angles, and diagonals congruent.*
- What is the length of a diagonal of the rectangular region?
  - *The diagonal is approximately 14.9.*
- Does it matter which of the two diagonals you work with?
  - *No, the diagonals of a rectangle are the same length.*
- Can you give the coordinates of a point within the rectangular region that lies on the diagonal that connects  $(1, 2)$  to  $(15, 7)$ ?
  - *There are an infinite number of points on the diagonal. Students may find the point lying in the middle by averaging the values and get  $(8, 4.5)$ , or they could use the rise over run triangle and find other points such as  $(3.8, 3)$ .*

Students can do the exercises below in pairs while the teacher walks around to provide support. The exercises can be scaffolded; give students problems of the appropriate difficulty for their level of knowledge. Another option is to assign different problems to different groups and bring the class together to share after.

## Exercises 1–3 (10 minutes)

## Exercises 1–3

1. Given the region shown to the right:

- a. Name three points in the region.

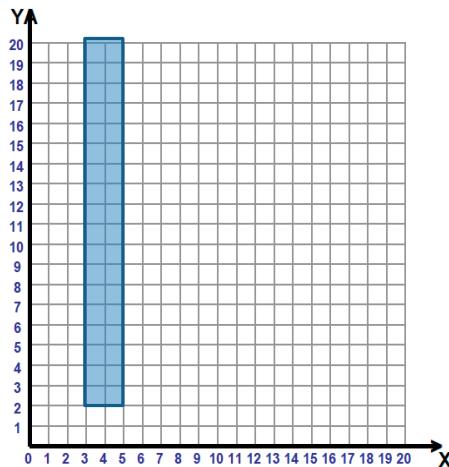
*Answers will vary but could include (4, 3), (3.5, 7), and (4.5, 19).*

- b. Name three points on the boundary.

*Answers will vary but could include (4, 2), (3, 15), and (5, 18).*

- c. Explain in words the points in the region.

*All x-coordinates are greater than or equal to 3, and less than or equal to 5, and all y-coordinates are greater than or equal to 2, and less than or equal to 20.*



- d. Write the inequality describing the x-values.

$$x \geq 3, x \leq 5$$

- e. Write the inequality describing the y-values.

$$y \geq 2, y \leq 20$$

- f. Write this as a system of equations.

$$\{(x, y) | 3 \leq x \leq 5, 2 \leq y \leq 20\}$$

- g. Will the lines  $x = 4$  and  $y = 1$  pass through the region? Draw them.

*$x = 4$  vertically cuts the region in half.  $y = 1$  is below the region and horizontal.*

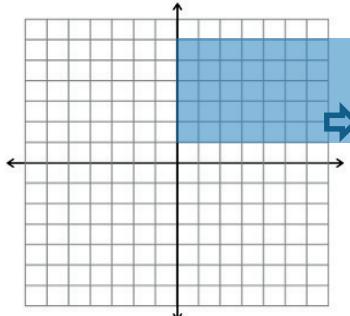
2. Given the region that continues unbound to the right as shown to the right:

- a. Name three points in the region.

*Answers will vary but could include (2, 2), (4, 3), and (5, 4).*

- b. Describe in words the points in the region.

*The region is above  $y = 1$  and below  $y = 5$ . It starts at  $x = 0$  and continues to the right without bound.*



- c. Write the system of inequalities that describe the region.

$$\{(x, y) | x \geq 0, 1 \leq y \leq 5\}$$

- d. Name a horizontal line that passes through the region.

*Answers will vary but may include  $y = 4$ .*

3. Given the region that continues down without bound as shown to the right:

- a. Describe the region in words.

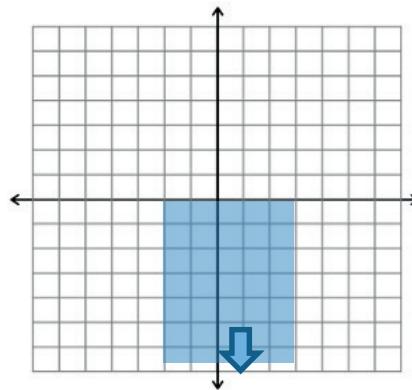
*The region is to the right of  $x = -2$  and to the left of  $x = 3$ . It starts at  $y = 0$  and continues down without bound.*

- b. Write the system of inequalities that describe the region.

$$\{(x, y) \mid -2 \leq x \leq 3, y \leq 0\}$$

- c. Name a vertical line that passes through the region.

*Answers will vary but may include  $x = -1$ .*



### Example 2 (10 minutes)

MP.1

This example can be done as a whole class or in groups. Allow students time to study the triangular region and problem-solve before asking the leading questions.

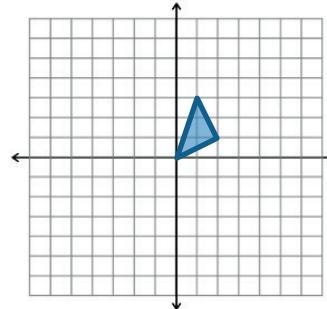
#### Example 2

Draw the triangular region in the plane given by the triangle with vertices  $(0, 0)$ ,  $(1, 3)$ , and  $(2, 1)$ . Can we write a set of inequalities that describes this region?

$$y \leq 3x$$

$$y \geq \frac{1}{2}x$$

$$y \leq -2x + 5$$



- This region is formed by the intersection of how many half-planes?
  - 3
  - Name the endpoints of the left most segment.
  - $(0, 0), (1, 3)$
- Find the slope of the line containing this segment.
  - 3
- Write the equation of the line containing this segment.
  - $y = 3x$

#### Scaffolding:

- If students need support finding slope, model the process of drawing “slope” triangles.
- Model substituting the value of the slope for  $m$  in  $y = mx + b$  and determining the value of  $b$ .
- For students in need of a challenge, use a triangular region with vertices located at non-whole number coordinates.

- Write the equations of the lines that containing the other two segments that are the sides of the triangle.
  - The line containing  $(0,0)$  and  $(2,1)$  is  $y = \frac{1}{2}x$ .
  - The line containing  $(1,3)$  and  $(2,1)$  is  $y = -2x + 5$ .
- What would we do to these equations to indicate shading?
  - Make them inequalities.

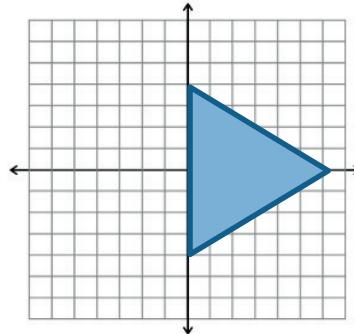
Have students turn to a partner and summarize what they have learned. Use this as an opportunity to informally assess understanding.

### Exercises 4–5 (5 minutes)

#### Exercises 4–5

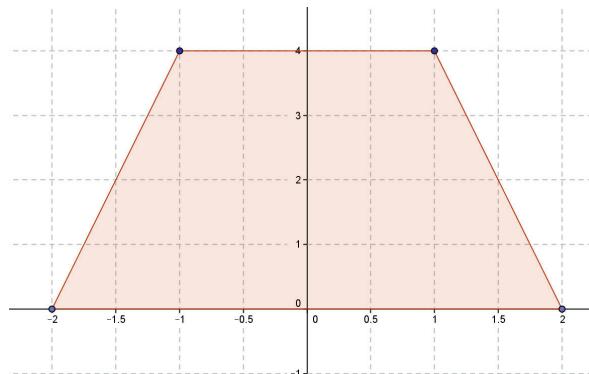
4. Given the triangular region shown, describe this region with a system of inequalities.

$$\begin{aligned}x &\geq 0 \\y &\geq \frac{2}{3}x - 4 \\y &\leq -\frac{2}{3}x + 4\end{aligned}$$



5. Given the trapezoid with vertices  $(-2, 0)$ ,  $(-1, 4)$ ,  $(1, 4)$ , and  $(2, 0)$ , describe this region with a system of inequalities.

$$\begin{aligned}y &\geq 0 \\y &\leq 4 \\y &\geq 4x + 8 \\y &\leq -4x + 8\end{aligned}$$





### Closing (2 minutes)

Discuss the following as a class.

- Describe the process of writing the system of inequalities to describe a triangular region in the plane.
  - Identify the endpoints of each segment and find the slope. Write the equation of the line containing each segment. Identify the segment of the line by restricting the domain. Make the equations inequalities that include points inside the region.
- What if I needed to describe a vertical segment? What is the general equation of a vertical line, and which variable would we restrict?
  - $x = c$ , where  $c$  is a constant. The  $y$ -values would have to be restricted.

### Exit Ticket (5 minutes)

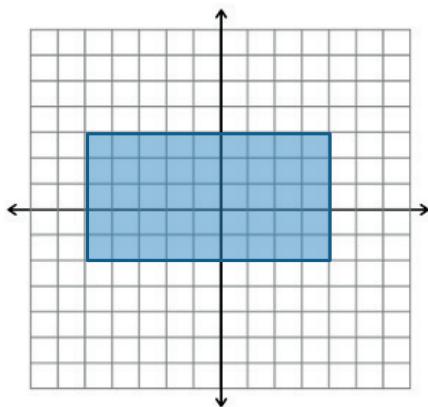
Name \_\_\_\_\_

Date \_\_\_\_\_

## Lesson 2: Finding Systems of Inequalities That Describe Triangular and Rectangular Regions

### Exit Ticket

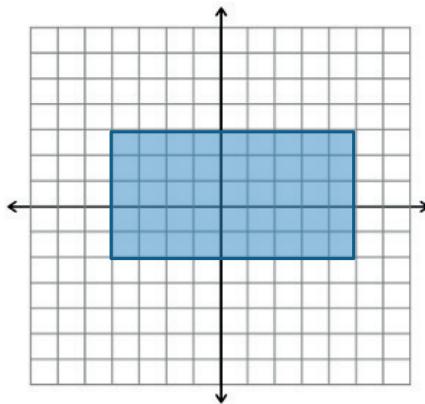
Given the region shown:



- Name three points in the region.
- Identify the coordinates of the four vertices.
- Write the system of inequalities describing this region.

## Exit Ticket Sample Solutions

Given the region shown:



- Name three points in the region.  
*Answers will vary but may include (0, 0), (-2, 3), (4, -1)*
- Identify the coordinates of the four vertices.  
 $(-4, -2), (-4, 3), (5, 3), \text{ and } (5, -2)$
- Write the system of inequalities describing the region  
 $\{(x, y) | -4 \leq x \leq 5, -2 \leq y \leq 3\}$

## Problem Set Sample Solutions

This Problem Set contains two challenge problems. You do not have to assign all problems to all students.

- Given the region shown:

- How many half-planes intersect to form this region?

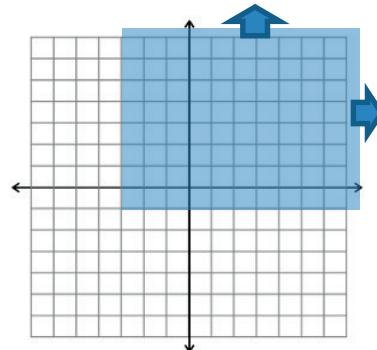
**2**

- Name three points on the boundary of the region.

$(-1, -1), (-3, 4), \text{ and } (5, -1)$

- Describe the region in words.

*The region is above and includes  $y = -1$ , is to the right of and includes  $x = -3$ , and extends without bound to the top and right.*



2. Region  $T$  is shown to the right.

- a. Write the coordinates of the vertices.

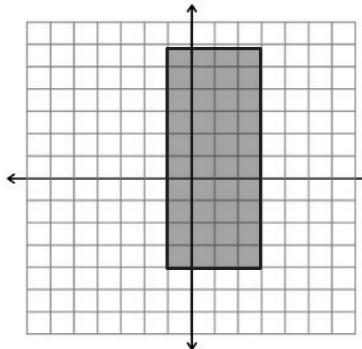
$(-1, -4), (-1, 6), (3, 6)$ , and  $(3, -4)$

- b. Write an inequality that describes the region.

$\{(x, y) | -1 \leq x \leq 3, -4 \leq y \leq 6\}$

- c. What is the length of the diagonals?

The diagonal is approximately 10.8 units long.



- d. Give the coordinates of a point that is both in the region and on one of the diagonals.

Answers will vary.  $(1, 1)$  lies on the diagonal and in the interior of region  $T$ .

3. Jack wants to plant a garden in his back yard. His yard is 120 feet wide and 80 feet deep. He wants to plant a garden that is 20 feet by 30 feet.

- a. Set up a grid for the backyard and place the garden on the grid. Explain why you placed your garden in its place on the grid.

Answers will vary, but the backyard should be on the grid with length 120 feet in the  $x$ -direction and 80 feet in the  $y$ -direction, or this can be set up in the other direction with 80 feet in the  $x$ -direction and 120 feet in the  $y$ -direction. The garden should form a rectangle somewhere on the grid.

- b. Write a system of inequalities to describe the garden.

Answers will vary.

- c. Write the equation of three lines that would go through the region that he could plant on, and explain your choices.

Answers will vary.

4. Given the trapezoidal region shown to the right:

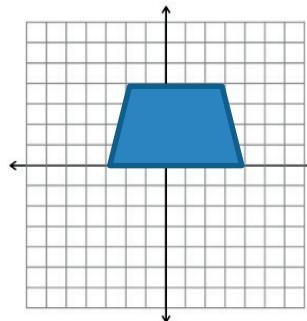
- a. Write the system of inequalities describing the region.

$$y \leq 4x + 12$$

$$y \leq 4$$

$$y \leq -4x + 16$$

$$y \geq 0$$



- b. Translate the region to the right 3 units and down 2 units. Write the system of inequalities describing the translated region.

$$y \leq 4x - 2$$

$$y \leq 2$$

$$y \leq -4x + 26$$

$$y \geq -2$$

## Challenge Problems:

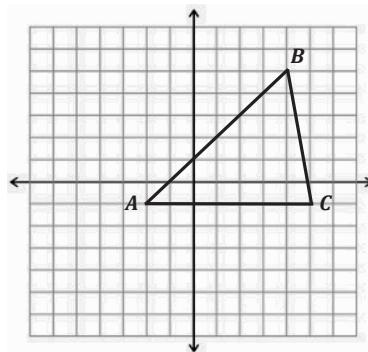
5. Given the triangular region shown with vertices  $A(-2, -1)$ ,  $B(4, 5)$ , and  $C(5, -1)$ :

- a. Describe the systems of inequalities that describe the region enclosed by the triangle.

$$y \leq x + 1$$

$$y \leq -6x + 29$$

$$y \geq -1$$



- b. Rotate the region  $90^\circ$  counterclockwise about Point A. How will this change the coordinates of the vertices?

$$A(-2, -1), B(-8, 5), \text{ and } C(-2, 6)$$

- c. Write the system of inequalities that describe the region enclosed in the rotated triangle.

$$y \geq -x - 3$$

$$y \leq \frac{1}{6}x + \frac{19}{3}$$

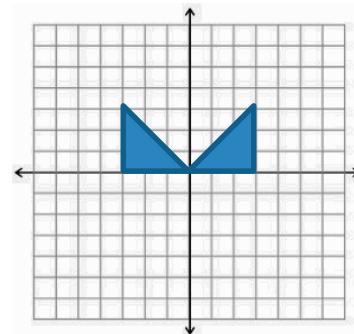
$$x \leq -1$$

6. Write a system of inequalities for the region shown.

$$y \leq |x|$$

$$y \geq 0$$

$$|x| \leq 3$$





## Lesson 3: Lines That Pass Through Regions

### Student Outcomes

- Given two points in the coordinate plane and a rectangular or triangular region, students determine whether the line through those points meets the region, and if it does, they describe the intersections as a segment and name the coordinates of the endpoints.

### Lesson Notes

Give students graph paper and rulers to begin this lesson. Remind students to graph points as carefully and accurately as possible. Playing with the scale on the axes will also make intersection points more obvious and easier to read.

### Classwork

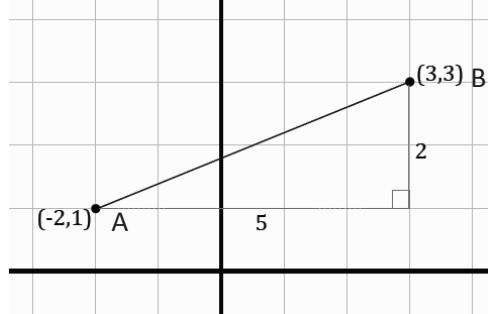
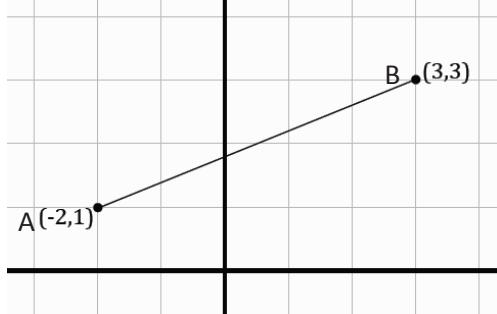
#### Opening Exercise (7 minutes)

The objective in the Opening Exercise is to reactivate student knowledge on how to find the distance between two points in the coordinate plane (refer to Grade 8, Module 7, Topic C, Lesson 17) using the Pythagorean Theorem.

##### Opening Exercise

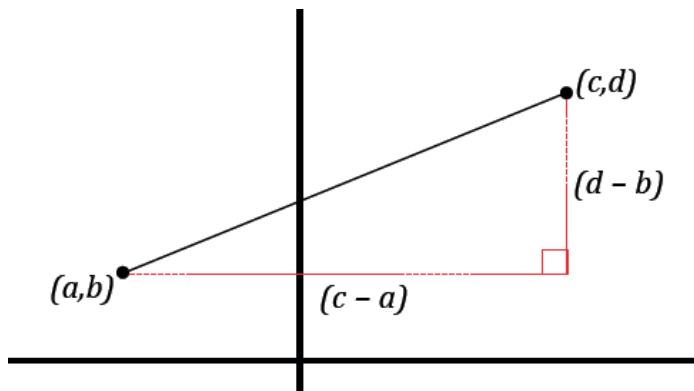
How can we use the Pythagorean theorem to find the length of  $\overline{AB}$ , or in other words, the distance between  $A(-2, 1)$  and  $B(3, 3)$ ? Find the distance between  $A$  and  $B$ .

*The distance between  $A$  and  $B$  is  $\sqrt{29}$ . By treating the length between the provided points as the hypotenuse of a right triangle, we can build a right triangle and determine the leg lengths in order to apply the Pythagorean Theorem and calculate the length of the hypotenuse, which is the length we were trying to find.*



Prompt students with the following hint if needed: Treat  $AB$  as the hypotenuse of a right triangle and build the appropriate right triangle around the segment.

- Notice that in using the Pythagorean theorem, we are taking both the horizontal and vertical distances between the provided points, squaring each distance, and taking the square root of the sum.
- Using the Pythagorean theorem to find the distance between two points can be summarized in the following formula:

**Scaffolding:**

- Consider providing struggling students with the second image of the graph with the grid of the coordinate plane and have them label the axes.

$$d[(a, b), (c, d)] = \sqrt{(c - a)^2 + (d - b)^2}.$$

- This formula is referred to as the distance formula.

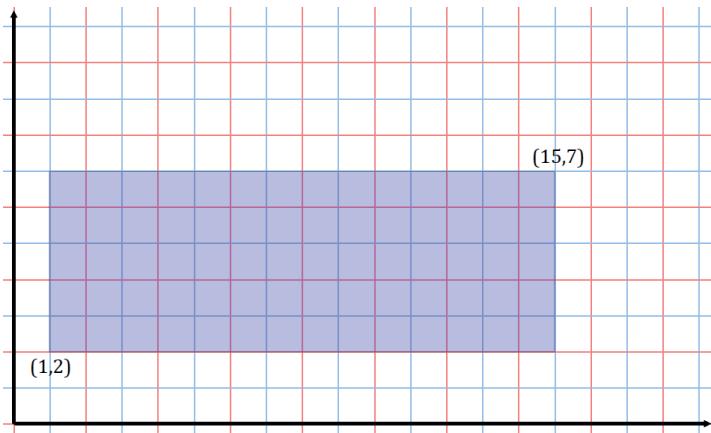
Have students summarize what they learned in the Opening Exercise and use this to informally assess student progress before moving on with the lesson.

**Example 1 (10 minutes)**

For each part below, give students two minutes to graph and find the points of intersection on their own. After two minutes, graph (or have a student come up to graph) the line on the board and discuss the answer as a whole class.

**Example 1**

Consider the rectangular region:





MP.3

- a. Does a line of slope 2 passing through the origin intersect this rectangular region? If so, which boundary points of the rectangle does it intersect? Explain how you know.

*Yes, it intersects twice, at (1, 2) and (3.5, 7). Student justification may vary. One response may include the use of the algebraic equation  $y = 2x$ , or students may create a table of values using their understanding of slope and compare the y-values of their table against the y-values of the boundary points of the rectangle.*

- b. Does a line of slope  $\frac{1}{2}$  passing through the origin intersect this rectangular region? If so, which boundary points of the rectangle does it intersect?

*Yes, it intersects twice, at (4, 2) and (14, 7).*

- c. Does a line of slope  $\frac{1}{3}$  passing through the origin intersect this rectangular region? If so, which boundary points of the rectangle does it intersect?

*Yes, it intersects twice, at (6, 2) and (15, 5).*

Part (d) is meant to encourage a discussion about the interior versus boundary of the rectangular region; have students talk in pairs before sharing out in a whole class debate. Once opinions are discussed, confirm that points that lie on the segments that join the vertices of the rectangle make up the boundary of the rectangle, whereas the interior of the region consists of all other points *within* the rectangle.

- d. A line passes through the lower right vertex of the rectangle? Does the line pass through the interior of the rectangular region or the boundary of the rectangular region? Does the line pass through both?

*The line passes through the boundary of the rectangular region.*

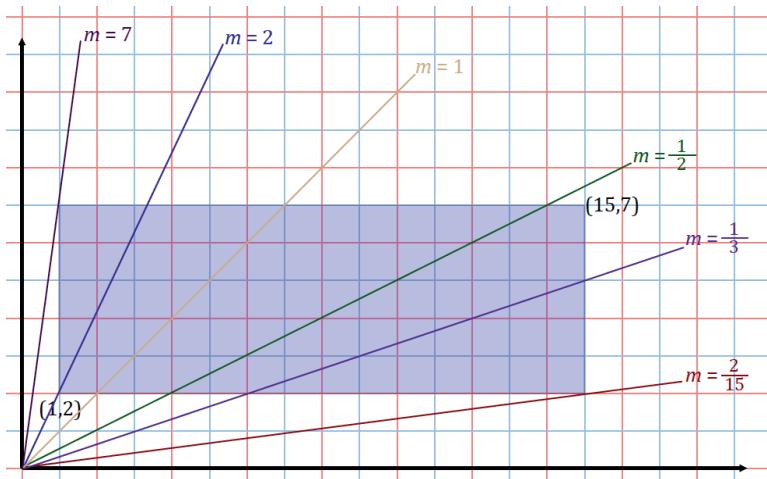
- e. For which values of  $m$  would a line of slope  $m$  through the origin intersect this region?

$$\frac{2}{15} \leq m \leq 7$$

Take time to explore the slopes of the lines above and their intersection points. The more you explore, the more students will start to see what happens as slopes get bigger and smaller to the points of intersection of the region. Project the figure on the board and color code each line with the slope listed on it so students have a visual to see the change in relation to other slopes. This activity can be started with the blank rectangle; as you proceed, ask the following questions:

- Let's draw a line through the origin with slope 1. What other point will it pass through? How many times does it intersect the boundary of the rectangle?
  - Answers will vary, but possible answers include (1, 1) and (2, 2). It intersects twice.
- Let's draw a line through the origin with slope 2. Will it go through the same point as the previous line? What point will it pass through? How many times does it intersect the boundary of the rectangle?
  - It will not pass through the same points as the previous line. Answers will vary, but possible answers include (1, 2) and (2, 4). It intersects twice.

Continue this line of questioning and ask students about the differences between the lines.



MP.7

- Describe how the point or points of intersection change as the slope decreases from  $m = 7$  to  $m = \frac{2}{15}$ .
  - As the slope decreases, the points of intersection occur further and further to the right of the rectangular region.
  - As the slope decreases, the  $x$ -coordinate of each intersection point is increasing for all the points that have a  $y$ -coordinate of 7, and then the  $y$ -coordinate of each intersection point is decreasing for all the points that have an  $x$ -coordinate of 15.
- What would be the slope of a line passing through the origin that only intersected one point of the region and did not pass through the region?
  - If the intersection point was the top left corner,  $(1, 7)$ , the slope of the line through the origin would be 7. If the point of intersection was the bottom right,  $(15, 2)$ , the slope would be  $\frac{2}{15}$ .
- What can you say about the slopes of lines through the origin that don't intersect the region?
  - They are either greater than 7 or less than  $\frac{2}{15}$ .

f. For which values of  $m$  would a line of slope  $m$  through the point  $(0, 1)$  intersect this region?

$$\frac{1}{15} \leq m \leq 6$$

- What changes in the previous problem if we shift the  $y$ -intercept from the origin to  $(0, 1)$ ?
  - The graph moves up 1 unit.
- How does that affect the slope?
  - The rise of the slope decreases by 1; so, instead of going up 7 and over 1 from the origin, you would go up 6 and over 1 to get to the point  $(1, 7)$  if the  $y$ -intercept is  $(0, 1)$ , changing the slope from 7 to 6. The same is true for the slope to the point  $(15, 2)$ . Instead of moving up 2 and right 15, we move up 1 and right 15, changing the slope.

**Example 2 (15 minutes)**

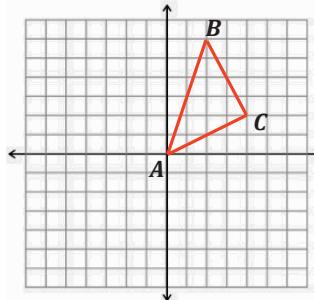
Depending on student ability, consider working on Example 2 as a whole class. Allow students one minute after reading each question to process and begin considering a strategy before continuing with guided prompts.

**Example 2**

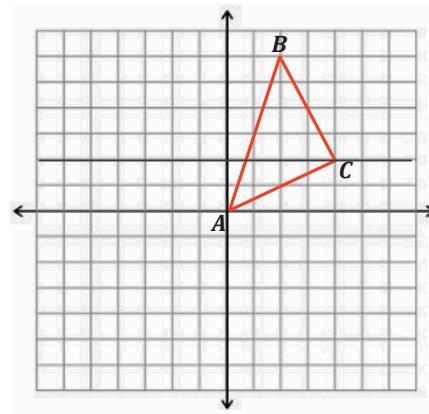
Consider the triangular region in the plane given by the triangle with vertices  $A(0, 0)$ ,  $B(2, 6)$ , and  $C(4, 2)$ .

- a. The horizontal line  $y = 2$  intersects this region. What are the coordinates of the two boundary points it intersects? What is the length of the horizontal segment within the region along this line?

*The coordinates are  $(\frac{2}{3}, 2)$  and  $(4, 2)$ ; the length of the segment is  $3\frac{1}{3}$ .*



- How many times does the line  $y = 2$  intersect the boundary of the triangle?
  - It intersects 2 times.
- Is it easy to find these points from the graph? What can you do to find these points algebraically?
  - No, it is not easy because one of the intersection points does not have whole number coordinates.
  - We can find the equation of the line that passes through  $(0, 0)$  and  $(2, 6)$ , and set the equation equal to 2 and solve for  $x$ . This gives us the  $x$ -coordinate  $\frac{2}{3}$  of the point  $(\frac{2}{3}, 2)$ .



- b. Graph the line  $3x - 2y = 5$ . Find the points of intersection with the triangular region and label them as  $X$  and  $Y$ .

*The points of intersection are  $(1.25, 0.625)$  and  $(\frac{25}{7}, \frac{20}{7})$ .*

**Scaffolding:**

- Depending on student ability, consider using the equation  $-3x + y = 5$  for part (b) in order to ensure whole-number coordinate intersections.
- In this case, the intersections are  $(2, 1)$  and  $(3, 4)$ , and the distance between them is approximately 3.2.

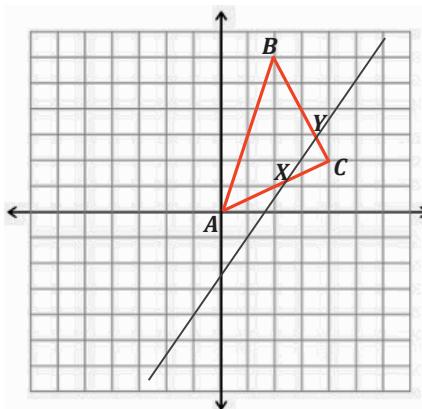
- How will you find the points of intersection of the line with the triangular region?

From graphing the line, we know that the given line intersects  $\overrightarrow{AC}$  and  $\overrightarrow{BC}$ . We need to find the equations of  $\overrightarrow{AC}$  and  $\overrightarrow{BC}$  and put them, as well as the provided equation of the line  $3x - 2y = 5$ , into slope intercept form.

$$\overrightarrow{AC} \Rightarrow y = \frac{1}{2}x.$$

$$\overrightarrow{BC} \Rightarrow y = 10 - 2x.$$

$$3x - 2y = 5 \Rightarrow y = 1.5x - 2.5.$$



The x-coordinates of the points of intersection can be found by setting each of  $\overrightarrow{AC}$  and  $\overrightarrow{BC}$  with the provided equation of the line:

$$(1) \frac{1}{2}x = 1.5x - 2.5$$

$$x = 1.25, y = 0.625.$$

$$(2) 10 - 2x = 1.5x - 2.5$$

$$x = \frac{25}{7}, y = \frac{20}{7}.$$

- c. What is the length of the segment  $\overline{XY}$ ?

$$d\left[(1.25, 0.625), \left(\frac{25}{7}, \frac{20}{7}\right)\right] = \sqrt{\left(\frac{25}{7} - 1.25\right)^2 + \left(\frac{20}{7} - 0.625\right)^2} \approx 3.2$$

- d. A robot starts at position  $(1, 3)$  and moves vertically downward towards the  $x$ -axis at a constant speed of 0.2 units per second. When will it hit the lower boundary of the triangular region that falls in its vertical path?

$$\frac{2.5 \text{ units}}{0.2 \frac{\text{units}}{\text{s}}} = 12.5 \text{ s}$$

- What do you have to calculate before you find the time it hits the boundary?

Find the point of intersection between the vertical path and the lower boundary of the triangular region:

-  $x$ -coordinate of intersection must be 1 and equation of  $\overrightarrow{AC}$ :

$$y = \frac{1}{2}x.$$

- By substituting for  $x$ , the point of intersection must happen at  $(1, 0.5)$ .

Find the distance between  $(1, 3)$  and the point of intersection.

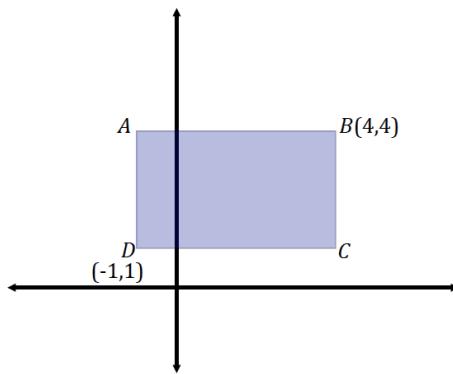
- The distance between  $(1, 3)$  and  $(1, 0.5)$  is 2.5.

#### Scaffolding:

- Depending on student ability, consider using vertex  $B$  for (d) to facilitate the idea behind the question.
- In this case, the robot would hit the lower boundary (at  $(2, 1)$ ) in 30 seconds.

**Exercise (6 minutes)****Exercise**

Consider the given rectangular region:



- a. Draw lines that pass through the origin and through each of the vertices of the rectangular region. Do each of the four lines cross multiple points in the region? Explain.

*Out of the four lines that pass through the origin and each of the vertices, only the lines that pass through A and B pass through multiple points in the region. The lines that pass through C and D do not intersect any other points belonging to the region.*

- b. Write the equation of a line that does not intersect the rectangular region at all.

*Responses will vary; one example is  $x = -2$ .*

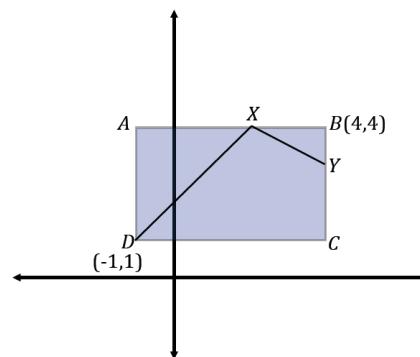
- c. A robot is positioned at  $D$  and begins to move in a straight line with slope  $m = 1$ . When it intersects with a boundary, it then reorients itself and begins to move in a straight line with a slope of  $m = -\frac{1}{2}$ . What is the location of the next intersection the robot makes with the boundary of the rectangular region?

(4, 3)

- d. What is the approximate distance of the robot's path in part (c)?

$$\text{distance}(\overline{DX}) + \text{distance}(\overline{XY}) = \text{distance}(\text{path})$$

$$\sqrt{(2-4)^2 + (4-3)^2} + \sqrt{(-1-2)^2 + (1-4)^2} \approx 6.5$$



**Closing (2 minutes)**

Gather the class and discuss one or both of the following. Use this as an informal assessment of student understanding.

- What is the difference between a rectangle and a rectangular region (or a triangle and a triangular region)?
  - *A rectangle (triangle) only includes the segments forming the sides. A rectangular (triangular) region includes the segments forming the sides of the figure and the area inside.*
- We have found points of intersection in this lesson. Some situations involved the intersection of two non-vertical/non-horizontal lines, and some situations involved the intersection of one non-vertical/non-horizontal line with a vertical or horizontal line. Does one situation require more work than the other? Why?
  - *Finding the intersection of two non-vertical/non-horizontal lines takes more work because neither variable value is obvious and, therefore, a system of equations must be solved.*

**Exit Ticket (5 minutes)**

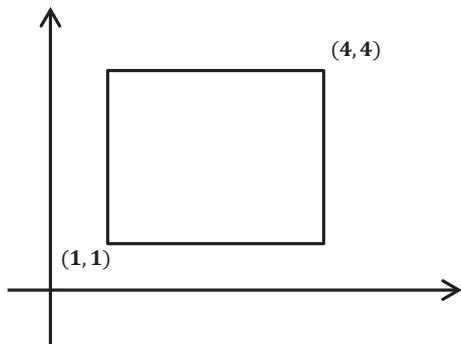
Name \_\_\_\_\_

Date \_\_\_\_\_

## Lesson 3: Lines That Pass Through Regions

### Exit Ticket

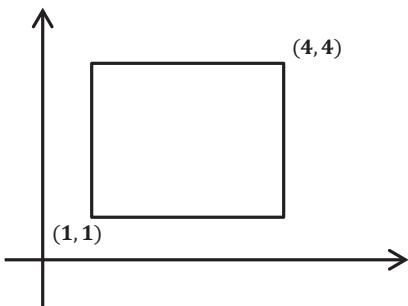
Consider the rectangular region:



- a. Does a line with slope  $\frac{1}{2}$  passing through the origin intersect this region? If so, what are the boundary points it intersects? What is the length of the segment within the region?
  
- b. Does a line with slope 3 passing through the origin intersect this region? If so, what are the boundary points it intersects?

## Exit Ticket Sample Solutions

Consider the rectangular region:



- a. Does a line with slope  $\frac{1}{2}$  passing through the origin intersect this region? If so, what are the boundary points it intersects? What is the length of the segment within the region?

*Yes, it intersects at (2, 1) and (4, 2); the length is approximately 2.24.*

- b. Does a line with slope 3 passing through the origin intersect this region? If so, what are the boundary points it intersects?

*Yes, it intersects at (1, 3) and  $(1\frac{1}{3}, 4)$ .*

## Problem Set Sample Solutions

Problems should be completed with graph paper and a ruler.

1. A line intersects a triangle at least once, but not at any of its vertices. What is the maximum number of sides that a line can intersect a triangle? Similarly, a square? A convex quadrilateral? A quadrilateral, in general?

*For any convex polygon, the maximum number of sides a line can intersect the polygon is 2. For a general quadrilateral, the line could intersect up to all 4 sides.*

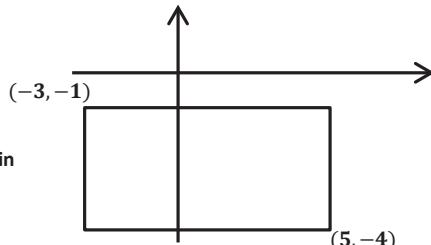
2. Consider the rectangular region:

- a. What boundary points does a line through the origin with a slope of  $-2$  intersect? What is the length of the segment within this region along this line?

*It intersects at  $(\frac{1}{2}, -1)$  and  $(2, -4)$ ; the length is approximately 3.35.*

- b. What boundary points does a line through the origin with a slope of  $3$  intersect? What is the length of the segment within this region along this line?

*It intersects at  $(-\frac{1}{3}, -1)$  and  $(-1\frac{1}{3}, -4)$ ; the length is approximately 3.2.*



- c. What boundary points does a line through the origin with a slope of  $-\frac{1}{5}$  intersect?

*It intersects at  $(5, -1)$  only.*

- d. What boundary points does a line through the origin with a slope of  $\frac{1}{4}$  intersect?

*The line does not intersect the rectangular region.*

3. Consider the triangular region in the plane given by the triangle  $(-1, 3)$ ,  $(1, -2)$ , and  $(-3, -3)$ .

- a. The horizontal line  $y = 1$  intersects this region. What are the coordinates of the two boundary points it intersects? What is the length of the horizontal segment within the region along this line?

*It intersects at  $(-1\frac{2}{3}, 1)$  and  $(-\frac{1}{5}, 1)$ ; the length is approximately 1.47*

- b. What is the length of the section of the line  $2x + 3y = -4$  that lies within this region?

*The length is approximately 3.6.*

- c. If a robot starts at  $(1, -2)$  and moves vertically downward at a constant speed of 0.75 units per second, when will it hit the lower boundary of the triangular region?

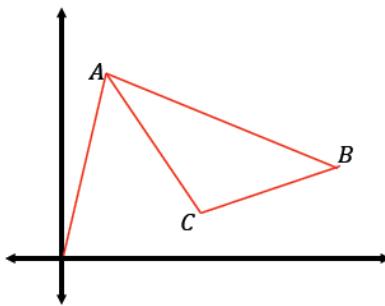
*It will hit in approximately 7.3 seconds.*

- d. If the robot starts at  $(1, -2)$  and moves horizontally left at a constant speed of 0.6 units per second, when will it hit the left boundary of the triangular region?

*It will hit in approximately 6.1 seconds.*

4. A computer software exists so that the cursor of the program begins and ends at the origin of the plane. A program is written to draw a triangle with vertices  $A(1, 4)$ ,  $B(6, 2)$ , and  $C(3, 1)$  so that the cursor only moves in straight lines and travels from the origin to  $A$ , then to  $B$ , then to  $C$ , then to  $A$ , and then back "home" to the origin.

- a. Sketch the cursor's path, i.e., sketch the entire path from when it begins until when it returns "home."



- b. What is the approximate total distance traveled by the cursor?

$\sqrt{17} + \sqrt{29} + \sqrt{10} + \sqrt{13} + \sqrt{17} \approx 20.4$

- c. Assume the cursor is positioned at  $B$  and is moving horizontally towards the  $y$ -axis at  $\frac{2}{3}$  units per second. How long will it take to reach the boundary of the triangle?

$$\text{Equation of } \overrightarrow{AC}: \quad y = -1.5x + 5.5$$

$$x\text{-coordinate on } \overrightarrow{AC} \text{ at } y = 2: \quad \frac{7}{3}$$

$$\text{Distance from } B(6, 2) \text{ to } \left(\frac{7}{3}, 2\right): \quad \frac{11}{3}$$

$$\text{Time needed to reach } \left(\frac{7}{3}, 2\right): \quad \frac{11}{3} \text{ units} \div \frac{2}{3} \text{ units/s} = 5.5 \text{ s}$$

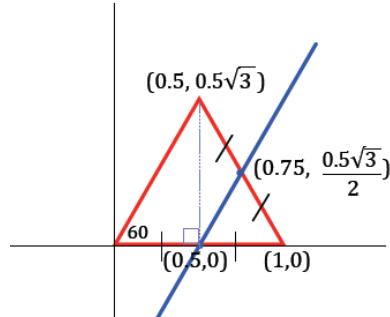
5. An equilateral triangle with side length 1 is placed in the first quadrant so that one of its vertices is at the origin and another vertex is on the  $x$ -axis. A line passes through the point half the distance between the endpoints on one side and half the distance between the endpoints on the other side.

- a. Draw a picture that satisfies these conditions.

*Students can use trigonometry to find the vertex at  $(0.5, 0.5\sqrt{3})$ . Then, average the coordinates of the endpoints, and  $\left(0.75, \frac{0.5\sqrt{3}}{2}\right)$  can be established.*

- b. Find the equation of the line that you drew.

*The line through  $(0.5, 0)$  and  $\left(0.75, \frac{0.5\sqrt{3}}{2}\right)$  is  $y = \sqrt{3}x - 0.5\sqrt{3}$ .*





## Lesson 4: Designing a Search Robot to Find a Beacon

### Student Outcomes

- Given a segment in the coordinate plane, students find the segments obtained by rotating the given segment by  $90^\circ$  counterclockwise and clockwise about one endpoint.

### Lesson Notes

In Lesson 4, students are introduced to the idea of finding the equation of a line perpendicular to a given line through a given point. Students are led to the notion of the slopes of perpendicular lines being negative reciprocals by showing the  $90^\circ$  rotation of a slope triangle. This idea is first demonstrated by having students sketch a perpendicular line to make sense of how the slope triangle of the original line is related to the slope triangle of the perpendicular line. Once students understand the idea of how the slope of a line is related to the slope of its perpendicular, it is applied to the rotation of a segment in order to predict where one endpoint moves to after a  $90^\circ$  rotation clockwise or counterclockwise about the other endpoint. This is generalized to  $90^\circ$  rotations of a point clockwise or counterclockwise about the origin.

### Classwork

#### Opening Exercise (5 minutes)

The goal of these exercises is to reactivate students' knowledge of lines, slope, and  $y$ -intercepts.

##### Opening Exercise

Write the equation of the line that satisfies the following conditions:

- a. Has a slope of  $m = -\frac{1}{4}$  and passes through the point  $(0, -5)$ .

$$y = -\frac{1}{4}x - 5$$

- b. Passes through the points  $(1, 3)$  and  $(-2, -1)$ .

$$y - 3 = \frac{4}{3}(x - 1) \text{ or } y + 2 = \frac{4}{3}(x + 2)$$

#### Exploratory Challenge (15 minutes)

The following prompt is meant to introduce students to the idea of determining the equation of a line perpendicular to a given line, through a given point. The context of the prompt mimics the coordinate plane, and students should find entry to the problem by sketching the known information on graph paper. Read the prompt as a class, and allow students a minute to process their ideas before asking them to proceed with the initial sketch of the situation. Then, proceed to the Discussion, where students will fine tune the approximate southeast direction ping to be the shortest distance from the line, i.e., along the line perpendicular to the line the robot is traveling.

Expect to spend 5–6 minutes on the prompt and the portion of the Discussion that leads to students' understanding that we are trying to establish the equation of the perpendicular line through  $(400, 600)$ .

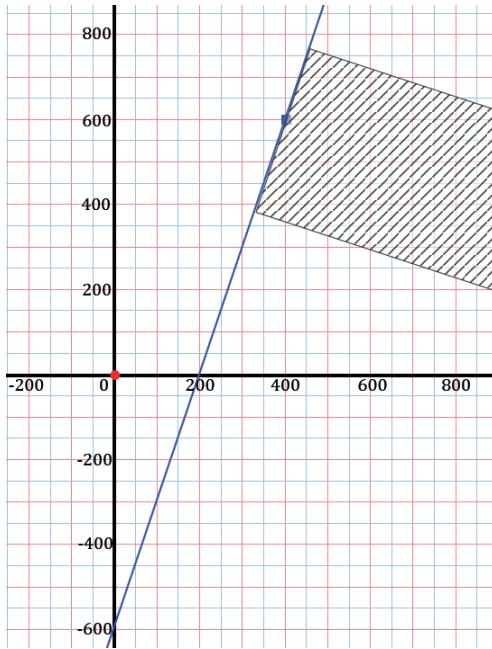
#### Exploratory Challenge

A search robot is sweeping through a flat plane in search of the homing beacon that is admitting a signal. (A homing beacon is a tracking device that sends out signals to identify the location). Programmers have set up a coordinate system so that their location is the origin, the positive  $x$ -axis is in the direction of east, and the positive  $y$ -axis is in the direction of north. The robot is currently 600 units south of the programmers' location and is moving in an approximate northeast direction along the line  $y = 3x - 600$ .

Along this line, the robot hears the loudest “ping” at the point  $(400, 600)$ . It detects this ping coming from approximately a southeast direction. The programmers have the robot return to the point  $(400, 600)$ . What is the equation of the path the robot should take from here to reach the beacon?

Begin by sketching the location of the programmers and the path traveled by the robot on graph paper; then, shade the general direction the ping is coming from.

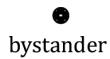
*A student response may look something like the following:*



Once students have formulated a rough idea of where the ping is coming from, they need to be convinced that the loudest sound will come from the shortest distance to the line the robot is on. You can compare the situation to the following using visuals as described below.

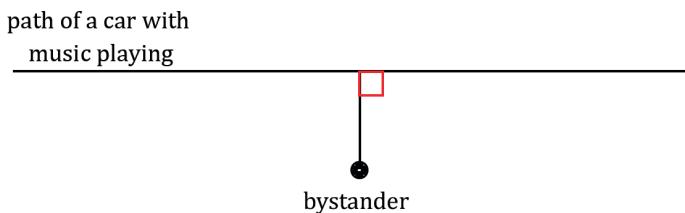
- Imagine a bystander on a sidewalk. A car with music playing passes by. Sketch a model of this.

path of a car with  
music playing



- At what point do you think the music will sound the loudest to the bystander on the sidewalk?
  - *It will sound loudest when the car is directly in front of the bystander.*

Confirm that this is true and that the shortest distance away is when the car is on the line perpendicular from the bystander.



Here is the crux of what students need to know about the question being posed:

- Similarly, if the loudest ping is heard at  $(400, 600)$  on the line  $y = 3x - 600$ , we need to find the line perpendicular to the path of the robot that passes through  $(400, 600)$ . Now, the challenge is to find the equation of this line.

Students should be able to express the objective of the problem:

- *We need to find the equation of the line perpendicular to the robot's path through  $(400, 600)$ .*
- To find the equation of this line, we need to establish the slope of the perpendicular line.
- With the slope and the point  $(400, 600)$ , we can find the equation of the line.
- How can we find the slope of the perpendicular line?

In the remaining 9–10 minutes, consider student ability and either let students develop their own informal methods for doing this, or use the prompts below (provided after the following paragraph) to guide the conversation as a whole class.

Students who are able to construct an argument as to how to determine the perpendicular may use the following line of reasoning. By looking at the “rise-over-run triangle” for the line  $y = 3x - 600$ , they may say, “one step east gives three steps north.” Students might also argue, for example, that the perpendicular line requires “three steps east gives one step south” and, therefore, has a slope of  $-\frac{1}{3}$ . The equation of the perpendicular line is  $y - 600 = -\frac{1}{3}(x - 400)$ .

Use the following prompts to do either whole class discussion or frame student discussion for small-groups.

- What is the slope of the original line?
  - *The slope is 3.*
- What does this mean in terms of direction?
  - *For every three units the robot moves north, it moves one unit east.*

#### Scaffolding:

- Looking at this problem graphically will help students see the location of the robot and the direction of the ping. Show the “rise-over-run” triangles and explain that the robot is originally moving three units north and then one unit east. Draw that triangle. Students can then see the turn will require one unit south for three units east.
- For advanced learners, pose the question: How are the slopes of perpendicular lines related?

Share the following image once the slope of the robot's path has been discussed. The goal is to guide students to the idea of using the slope triangle to help find the value of the perpendicular slope. Consider asking them to sketch a guess of the line by using the fact that it forms a  $90^\circ$  angle with the robot's path.

- Since we know to look for a line perpendicular to the robot's path and, therefore, forms a  $90^\circ$  angle with the path, how can we use the slope triangle to guide us?

Allow time for students to wrestle with the question before presenting the next image. Ask students to describe what they observe in the image and what it implies about the slope of the perpendicular line. Be sure that they note that in marking the right angle formed by the two lines, the edges of the angle coincide with the hypotenuse of each slope triangle.

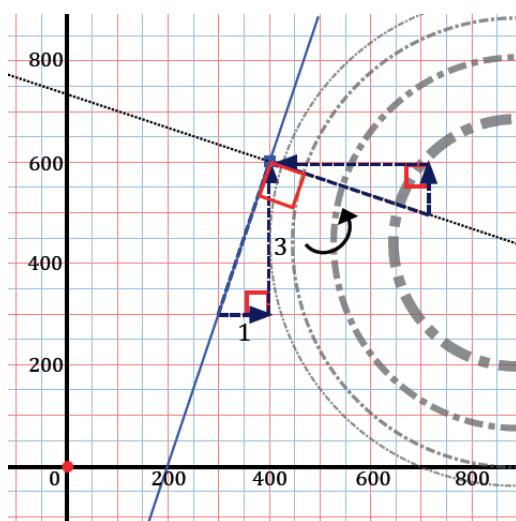
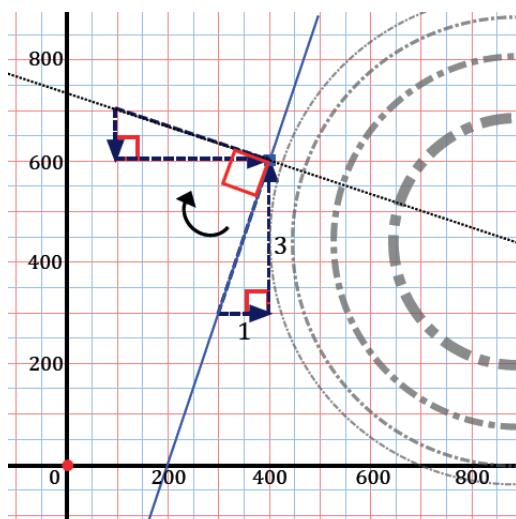
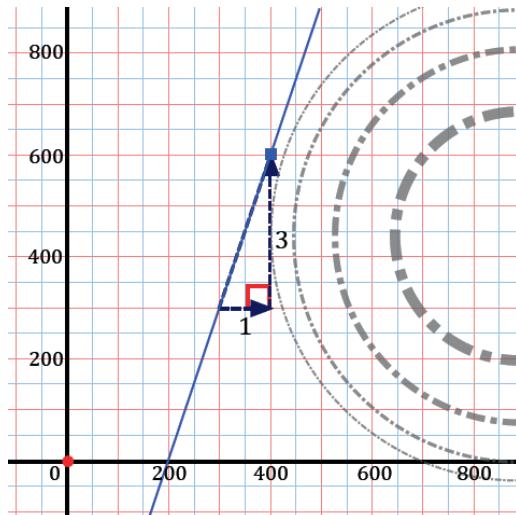
- Has the up and over (slope) triangle been distorted in any way?
  - No, the rotation preserves lengths and angles.*
- Using what you know about the existing slope triangle, what must the slope of the perpendicular line be?
  - The slope must be  $-\frac{1}{3}$ , or down one unit for every three units moved to the right.*
- Is this perpendicular slope algebraically related in any way to the original line's slope?

Consider writing the two slopes side by side on the board for comparison

- It is the negative reciprocal of the original slope.*
- Would it matter if the slope triangle were rotated  $90^\circ$  counterclockwise as opposed to  $90^\circ$  clockwise?

Students may have a difficult time visualizing this; be sure to share the next image and show that the direction of rotation does not impact the discovery of the slope.

- Now that we have established the slope of the perpendicular line, what else is needed to establish the equation of the line?
  - We have the slope, so we can use the known point of  $(400, 600)$  to write the equation of the line.*
- Write the equation of that line.
  - $y - 600 = -\frac{1}{3}(x - 600)$
- We have now achieved our objective of finding the equation of the line perpendicular to the robot's path through the point  $(400, 600)$ .



Students should be asked to summarize to their neighbor what they learned from this exercise. Teachers can use this activity to informally assess understanding so far.

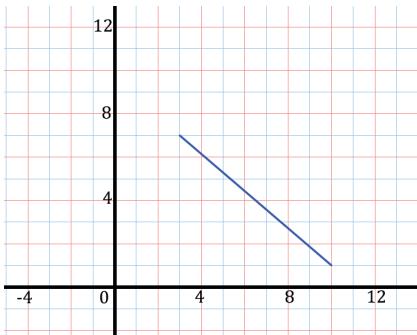
### Example 1 (13 minutes)

In this example, students are rotating line segments about one of the end points 90° counterclockwise and then 90° clockwise.

#### Example 1

The line segment connecting (3, 7) to (10, 1) is rotated counterclockwise 90° about the point (3, 7).

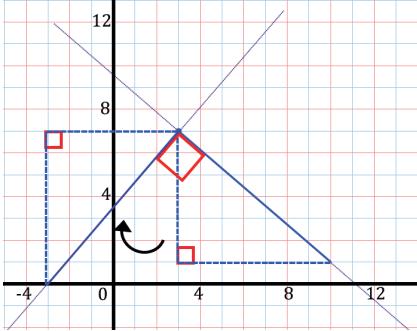
- a. Plot the points.



- Build a right triangle around the segment so that the segment is the hypotenuse of the right triangle.
- What are the lengths of the legs of the triangle?
  - The triangle has leg lengths of 6 and 7.
- What is the slope of the segment (i.e., the hypotenuse)?
  - The slope is  $-\frac{6}{7}$ .

- b. Where will the rotated endpoint land?

( $-3, 0$ )



Just as in the Exploratory Challenge, consider asking students to sketch where they think the rotated segment will fall; for more than just a sketch, provide students with a setsquare and ruler.

- As we saw in the Exploratory Challenge, use this right triangle and the  $90^\circ$  angle of the original segment and rotated segment form to your advantage.

Allow time for students to attempt this independently before reviewing the rotation with them.

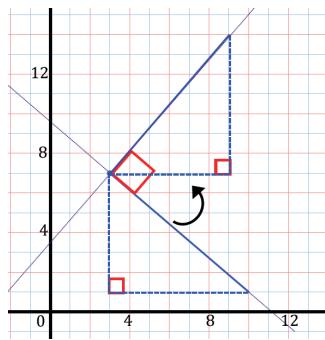
- Where does the endpoint land?
  - $(-3, 0)$
- What is the slope of the rotated segment?
  - $\frac{7}{6}$
- How is this slope related algebraically to the original segment's slope?
  - It is the negative reciprocal of the original slope.*
- If we know that the perpendicular slope is the negative reciprocal of the slope of the original segment, could we have avoided a sketch of the perpendicular line and determined the location of the rotated point? How?
  - We know that the rotated segment has to have a slope of  $\frac{7}{6}$ , so we can move 7 units down and 6 units left, or alternatively, subtract 7 from the y-coordinate and 6 from the x-coordinate of the point  $(3, 7)$ .*

#### Scaffolding:

- If students are having trouble seeing the rotation, use a compass and protractor to show the circle with the center as the point of rotation, and divide the circles into 4 congruent segments. You can also cut out a triangle, attach it to the center with a brad, and show the actual rotation.
- Patty paper can also be used to trace figures, rotate, and fold.

Emphasize this by writing  $(-3, 0)$  next to  $(3, 7)$  on the board and showing the difference between the  $x$ -coordinates and  $y$ -coordinates.

- c. Now rotate the original segment  $90^\circ$  clockwise. Before using a sketch, predict the coordinates of the rotated endpoint using what you know about the perpendicular slope of the rotated segment.



- What do you predict the coordinates of the rotated endpoint to be? Explain how you know.
  - We know that the rotated segment will have a slope of  $\frac{7}{6}$ , so we can add 7 to the y-coordinate and add 6 to the x-coordinate of  $(3, 7)$  to get  $(9, 14)$ .*

Have students verify their prediction by drawing the rotation.

- Let us summarize. Describe how you can use the rise and run values, or the leg lengths of the right triangle built around the segment, to predict the coordinates of a rotated point (clockwise or counterclockwise) relative to the center of rotation.
  - For either a  $90^\circ$  rotation clockwise or counterclockwise, I took the negative reciprocal of the slope of the segment and used what I knew about the general position of the rotated point to accordingly add or subtract the “rise” value from the  $y$ -coordinate and the “run” value from the  $x$ -coordinate of the center.*

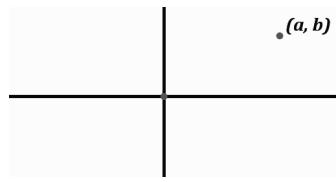
### Exercise 1 (5 minutes)

Students should be given an opportunity to work this in pairs. Have them graph the point and rotate it; students should think of it as a segment between the point and the origin using triangles as in Example 1.

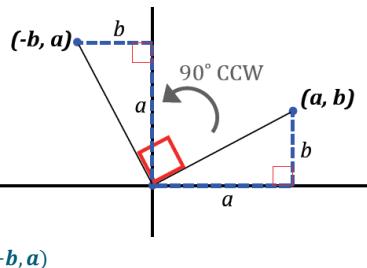
Consider pulling a small group aside for targeted instruction while others are working.

#### Exercise 1

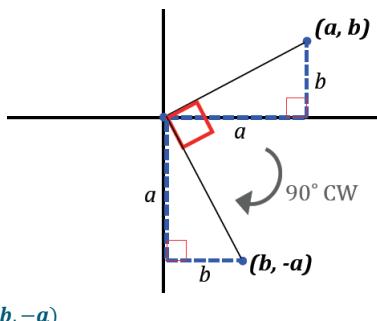
The point  $(a, b)$  is labeled below:



- a. Using  $a$  and  $b$ , describe the location of  $(a, b)$  after a  $90^\circ$  counterclockwise about the origin. Draw a rough sketch to justify your answer.

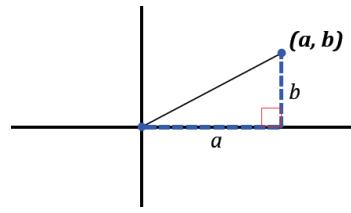


- b. If the rotation was clockwise about the origin, what is the rotated location of  $(a, b)$  in terms of  $a$  and  $b$ ? Draw a rough sketch to justify your answer.

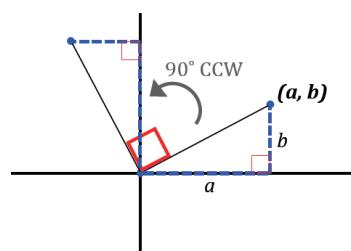


#### Scaffolding:

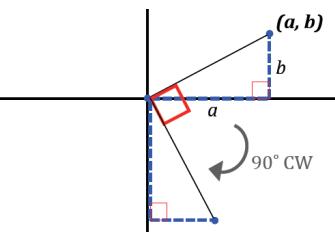
- If scaffolding is needed, provide sketches like the following and have students fill in the missing information.
- Initial drawing:



- Part (a):



- Part (b):



MP.7

- c. What is the slope of the line through the origin and  $(a, b)$ ? What is the slope of the perpendicular line through the origin?

$$\frac{b}{a}; -\frac{a}{b}$$

- d. What do you notice about the relationship between the slope of the line through the origin and  $(a, b)$  and the slope of the perpendicular line?

*The slopes are negative reciprocals of each other. Their product is  $-1$ .*

### Closing (2 minutes)

Ask students to respond to these questions in writing, with a partner, or as a class.

- What is the relationship between the slope of a line and the slope of a line perpendicular to it?
  - *The slopes are negative reciprocals of each other.*
- How do you predict the coordinates of a point to be rotated  $90^\circ$  clockwise or counterclockwise with respect to the center?
  - *For either a  $90^\circ$  rotation clockwise or counterclockwise, I took the negative reciprocal of the slope of the segment and used what I knew about the general position of the rotated point to add or subtract the “rise” value from the  $y$ -coordinate and the “run” value from the  $x$ -coordinate of the center accordingly.*
- How is this special to a rotation about the origin?
  - *Rotating a point  $90^\circ$  counterclockwise about the origin changes the coordinates of  $(a, b)$  to  $(-b, a)$ .*
  - *Rotating a point  $90^\circ$  clockwise about the origin changes the coordinates of  $(a, b)$  to  $(b, -a)$ .*

### Exit Ticket (5 minutes)



Name \_\_\_\_\_

Date \_\_\_\_\_

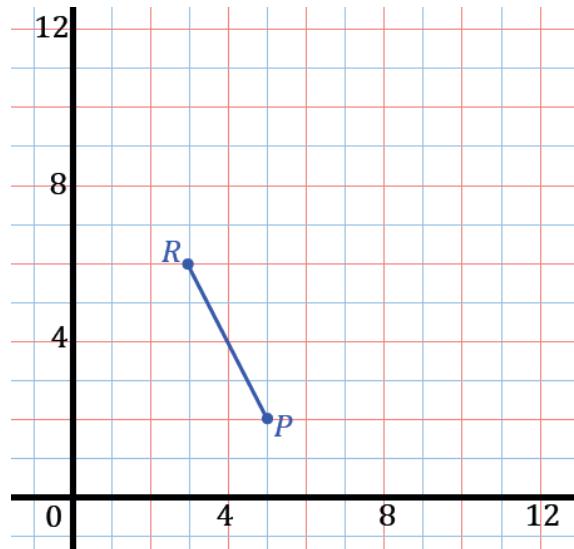
## Lesson 4: Designing a Search Robot to Find a Beacon

### Exit Ticket

If the line segment connecting point  $P(5, 2)$  to  $R(3, 6)$  is rotated  $90^\circ$  counterclockwise about point  $R$ :

- a. Where will point  $P$  land?

- b. What is the slope of the original segment,  $\overline{PR}$ ?



- c. What is the slope of the rotated segment? Explain how you know.

## Exit Ticket Sample Solutions

If the line segment connecting point  $P(5, 2)$  to  $R(3, 6)$  is rotated  $90^\circ$  counterclockwise about point  $R$ :

- a. Where will point  $P$  land?

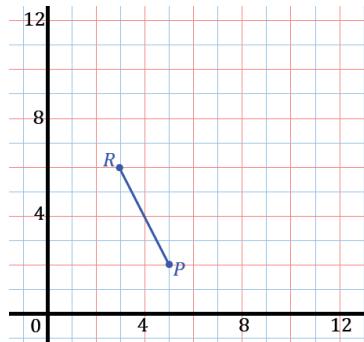
(7, 8)

- b. What is the slope of the original segment,  $\overline{PR}$ ?

-2

- c. What is the slope of the rotated segment? Explain how you know.

The slope is  $\frac{1}{2}$ ; a  $90^\circ$  rotation means the segments are perpendicular, so the slopes are negative reciprocals of each other.



## Problem Set Sample Solutions

1. Find the new coordinates of point  $(0, 4)$  if it rotates:

- a.  $90^\circ$  counterclockwise.

(-4, 0)

- b.  $90^\circ$  clockwise.

(4, 0)

- c.  $180^\circ$  counterclockwise.

(0, -4)

- d.  $270^\circ$  clockwise.

(-4, 0)

2. What are the new coordinates of the point  $(-3, -4)$  if it is rotated about the origin:

- a. Counterclockwise  $90^\circ$ ?

(4, -3)

- b. Clockwise  $90^\circ$ ?

(-4, 3)

3. Line segment  $\overline{ST}$  connects points  $S(7, 1)$  and  $T(2, 4)$ .

- a. Where does point  $T$  land if the segment is rotated  $90^\circ$  counterclockwise about  $S$ ?

(4, -4)



- b. Where does point  $T$  land if the segment is rotated  $90^\circ$  clockwise about  $S$ ?  
 $(10, 6)$
- c. What is the slope of the original segment?  
 $\frac{3}{5}$
- d. What is the slope of the rotated segments?  
 $\frac{5}{3}$
4. Line segment  $\overline{VW}$  connects points  $V(1, 0)$  and  $W(5, -3)$ .
- Where does point  $W$  land if the segment is rotated  $90^\circ$  counterclockwise about  $V$ ?  
 $(4, 4)$
  - Where does point  $W$  land if the segment is rotated  $90^\circ$  clockwise about  $V$ ?  
 $(-2, -4)$
  - Where does point  $V$  land if the segment is rotated  $90^\circ$  counterclockwise about  $W$ ?  
 $(2, -7)$
  - Where does point  $V$  land if the segment is rotated  $90^\circ$  clockwise about  $W$ ?  
 $(8, 1)$
5. If the slope of a line is 0, what is the slope of a line perpendicular to it? If the line has slope 1, what is the slope of a line perpendicular to it?  
*undefined; -1*
6. If a line through the origin has a slope of 2, what is the slope of the line through the origin that is perpendicular to it?  
 $-\frac{1}{2}$
7. A line through the origin has a slope of  $\frac{1}{3}$ . Carlos thinks the slope of a perpendicular line at the origin will be 3. Do you agree? Explain why or why not.  
*I disagree with Carlos. The slope of the perpendicular line would have a slope of -3 because it should be the negative reciprocal of the original slope.*
8. Could a line through the origin perpendicular to a line through the origin with slope  $\frac{1}{2}$  pass through the point  $(-1, 4)$ ? Explain how you know.  
*No, the equation of the line through the origin perpendicular to the line through the origin with slope  $\frac{1}{2}$  has an equation of  $y = -2x$ .  $(-1, 4)$  is not a solution to this equation.*



## Topic B:

## Perpendicular and Parallel Lines in the Cartesian Plane

**G-GPE.B.4, G-GPE.B.5**

<b>Focus Standard:</b>	G-GPE.B.4      Use coordinates to prove simple geometric theorems algebraically. <i>For example, prove or disprove that a figure defined by four given points in the coordinate plane is a rectangle; prove or disprove that the point <math>(1, \sqrt{3})</math> lies on the circle centered at the origin and containing the point <math>(0, 2)</math>.</i>
	G-GPE.B.5      Prove the slope criteria for parallel and perpendicular lines and use them to solve geometric problems (e.g., find the equation of a line parallel or perpendicular to a given line that passes through a given point).
<b>Instructional Days:</b>	4
	<b>Lesson 5:</b> Criterion for Perpendicularity (P) <sup>1</sup>
	<b>Lesson 6:</b> Segments That Meet at Right Angles (P)
	<b>Lesson 7:</b> Equations for Lines Using Normal Segments (S)
	<b>Lesson 8:</b> Parallel and Perpendicular Lines (P)

The challenge of programming robot motion along segments parallel or perpendicular to a given segment leads to an analysis of slopes of parallel and perpendicular lines and the need to prove results about these quantities (**G-GPE.B.5**). MP.3 is highlighted in this topic as students engage in proving criteria and then extending that knowledge to reason about lines and segments. This work highlights the role of the converse of the Pythagorean theorem in the identification of perpendicular directions of motion (**G-GPE.B.4**). In Lesson 5, students explain the connection between the Pythagorean theorem and the criterion for perpendicularity studied in Module 2 (**G-GPE.B.4**). Lesson 6 extends that study by generalizing the criterion for perpendicularity to any two segments and applying this criterion to determine if segments are perpendicular. In Lesson 7, students learn a new format for a line,  $a_1x + a_2y = c$ , and recognize the segment from  $(0, 0)$  to  $(a_1, a_2)$  as normal with a slope of  $-\frac{a_2}{a_1}$ . Lesson 8 concludes Topic B when students recognize parallel and

<sup>1</sup> Lesson Structure Key: P-Problem Set Lesson, M-Modeling Cycle Lesson, E-Exploration Lesson, S-Socratic Lesson

perpendicular lines from their slopes and create equations for parallel and perpendicular lines. The criterion for parallel and perpendicular lines and the work from this topic with the distance formula will be extended in the last two topics of this module as students use these foundations to determine perimeter and area of polygonal regions in the coordinate plane defined by systems of inequalities. Additionally, students will study the proportionality of segments formed by diagonals of polygons.



## Lesson 5: Criterion for Perpendicularity

### Student Outcomes

- Students explain the connection between the Pythagorean theorem and the criterion for perpendicularity.

### Lesson Notes

It is the goal of this lesson to justify and prove the following:

**Theorem:** Given three points on a coordinate plane  $O(0,0)$ ,  $A(a_1, a_2)$ , and  $B(b_1, b_2)$ ,  $\overline{OA} \perp \overline{OB}$  if and only if  $a_1b_1 + a_2b_2 = 0$ .

The proof of this theorem relies heavily on the Pythagorean theorem and its converse. This theorem and its generalization, which will be studied in the next few lessons, will give students an efficient method to prove slope criterion for perpendicular and parallel lines (**G-GPE.B.5**). An explanation using similar triangles was done in Grade 8, Module 4, Lesson 26. The proof we give in geometry has the additional benefit of directly addressing **G-GPE.B.5** in the context of **G-GPE.B.4**. That is, the proof uses coordinates to prove a generic theorem algebraically.

### Classwork

#### Opening Exercise (4 minutes)

##### Opening Exercise

In right triangle  $ABC$ , find the missing side.

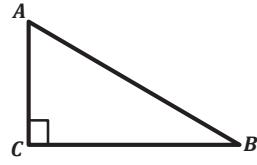
- a. If  $AC = 9$  and  $CB = 12$ , what is  $AB$ ? Explain how you know.

*Because triangle ABC is a right triangle, and we know the length of two of the three sides, we can use the Pythagorean theorem to find the length of the third side, which in this case is the hypotenuse.*

$$\begin{aligned} AB^2 &= AC^2 + CB^2 \\ AB &= \sqrt{AC^2 + CB^2} \\ AB &= \sqrt{9^2 + 12^2} \\ AB &= 15 \end{aligned}$$

- b. If  $AC = 5$  and  $AB = 13$ , what is  $CB$ ?

$$\begin{aligned} AB^2 &= AC^2 + CB^2 \\ 13^2 &= 5^2 + CB^2 \\ CB &= 12 \end{aligned}$$



##### Scaffolding:

- It would be helpful to have posters of the Pythagorean theorem and its converse displayed in the classroom or to allow students to use a graphic organizer containing this information.
- For advanced students, use lengths that are radical expressions.



- c. If  $AC = CB$  and  $AB = 2$ , what is  $AC$  (and  $CB$ )?

$$AC^2 + CB^2 = AB^2$$

$$AC^2 + AC^2 = AB^2$$

$$2AC^2 = 4$$

$$AC^2 = 2$$

$AC = \sqrt{2}$ , and since  $AC = CB$ ,  $CB = \sqrt{2}$  as well.

### Example 1 (9 minutes)

This is a guided example where students are realizing that segments are perpendicular because they form a triangle whose sides satisfy the criterion of the Pythagorean theorem ( $a^2 + b^2 = c^2$ ). The entire focus of this example is that if the sides of a triangle satisfy the Pythagorean theorem, then two of the sides of the triangle meet at right angles (the converse of the Pythagorean theorem).

Pose the following question so that students remember that it is important that “ $c$ ” is the longest side of the right triangle or the hypotenuse.

- Mrs. Stephens asks the class to prove the triangle with sides  $5$ ,  $5\sqrt{2}$ , and  $5$  is a right triangle. Lanya says the triangle is not a right triangle because  $5^2 + (5\sqrt{2})^2 \neq 5^2$ . Is she correct? Explain.
  - *Lanya is not correct.  $5\sqrt{2}$  is greater than  $5$ , so it should be the hypotenuse of the right triangle. The equation should be  $5^2 + 5^2 = (5\sqrt{2})^2$ ; thus, the triangle is right.*

At the beginning of this guided example, give students the following information and ask them if the triangle formed by the three segments is a right triangle. Students may work in pairs and must justify their conclusion.

- $AC = 4$ ,  $AB = 7$ , and  $BC = 5$ ; is triangle  $ABC$  a right triangle? If so, name the right angle. Justify your answer.
  - *Students will need to use the converse of the Pythagorean theorem to answer this question. Does  $4^2 + 5^2 = 7^2$ ?  $41 \neq 49$ ; therefore, triangle  $ABC$  is not a right triangle.*
- Select at least one pair of students to present their answer and rationale. Because we only gave the side lengths and not the coordinates of the vertices, students will have to use the converse of the Pythagorean theorem, not slope, to answer this question.

Plot the following points on a coordinate plane:  $O(0,0)$ ,  $A(6,4)$ , and  $B(-2,3)$ .

- Construct triangle  $ABO$ . Do you think triangle  $ABO$  is a right triangle?
  - *Students will guess either yes or no.*
- How do you think we can determine which of you are correct?
  - *Since we just finished working with the Pythagorean theorem and its converse, most students will probably suggest using the converse of the Pythagorean theorem. Some students may suggest comparing the slopes of the segments. If they do, explain that while this is certainly an appropriate method, the larger goals of the lesson will be supported by the use of the converse of the Pythagorean theorem.*
- We know the coordinates of the vertices of triangle  $ABO$ . What additional information do we need?
  - *We need the lengths of the three sides of the triangle,  $OA$ ,  $OB$ , and  $AB$ .*

MP.3

- Do we have enough information to determine those lengths, and if so, how will we do this?
  - Yes, we will use the distance formula.  $OA = \sqrt{52}$ ,  $OB = \sqrt{13}$ , and  $AB = \sqrt{65}$ .
- If triangle  $ABO$  is a right triangle, which side would be the hypotenuse?
  - The hypotenuse is the longest side of a right triangle. So, if triangle  $ABO$  were a right triangle, its hypotenuse would be side  $AB$ .
- What equation will we use to determine whether triangle  $ABO$  is a right triangle?
  - $OA^2 + OB^2 = AB^2$
- Use the side lengths and the converse of the Pythagorean theorem to determine whether triangle  $ABO$  is a right triangle. Support your findings.
  - Yes, triangle  $ABO$  is a right triangle because  $(\sqrt{52})^2 + (\sqrt{13})^2 = (\sqrt{65})^2$ .
- Which angle is the right angle?
  - $\angle AOB$  is the right angle. It is the angle opposite the hypotenuse  $\overline{AB}$ .
- If  $\angle AOB$  is a right angle, what is the relationship between the legs of the right triangle,  $\overline{OA}$  and  $\overline{OB}$ ?
  - $\overline{OA} \perp \overline{OB}$ .

**Scaffolding:**

- Provide students with a handout with triangle  $OPQ$  already plotted.

Have students summarize to a partner what they learned in this example. In Exercise 1, we will use what we just learned to determine if segments are perpendicular and if triangles are right.

**Exercise 1 (3 minutes)**

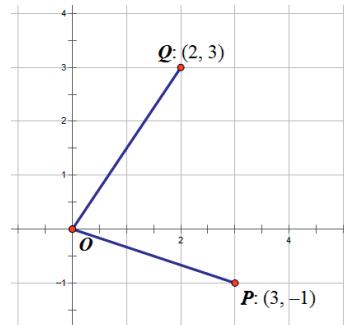
Have students work with a partner, but remind them that each student should record the work on his own student page. Select one pair of students to share their answer and rationale.

**Exercise 1**

1. Use the grid at the right.
  - a. Plot points  $O(0, 0)$ ,  $P(3, -1)$ , and  $Q(2, 3)$  on the coordinate plane.
  - b. Determine whether  $\overline{OP}$  and  $\overline{OQ}$  are perpendicular. Support your findings. No,  $\overline{OA}$  and  $\overline{OB}$  are not perpendicular.

*Using the distance formula, we determined that  $OP = \sqrt{10}$ ,  $OQ = \sqrt{13}$ , and  $PQ = \sqrt{17}$ .*

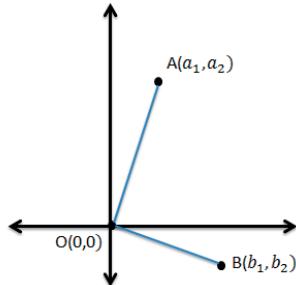
*$(\sqrt{10})^2 + (\sqrt{13})^2 \neq (\sqrt{17})^2$ ; therefore, triangle  $OPQ$  is not a right triangle,  $\angle QOP$  is not a right angle, and  $\overline{OP}$  and  $\overline{OQ}$  are not perpendicular.*



**Example 2 (9 minutes)**

In this example, we will develop the general condition for perpendicularity between two segments  $\overline{OA}$  and  $\overline{OB}$  with endpoints  $O(0,0)$ ,  $A(a_1, a_2)$ , and  $B(b_1, b_2)$ .

Example 2



- Refer to triangle  $ABO$ . What must be true about  $ABO$  if  $\overline{OA}$  is perpendicular to  $\overline{OB}$ ?
  - $ABO$  would be a right triangle, so it must satisfy the Pythagorean theorem.  $OA^2 + OB^2 = AB^2$ .
- Determine the expressions that define  $OA$ ,  $OB$ , and  $AB$ .
  - $OA = \sqrt{a_1^2 + a_2^2}$ ,  $OB = \sqrt{b_1^2 + b_2^2}$ , and  $AB = \sqrt{(b_1 - a_1)^2 + (b_2 - a_2)^2}$ .
- If  $\overline{OA} \perp \overline{OB}$ , the distances will satisfy the Pythagorean theorem.
  - $OA^2 + OB^2 = AB^2$ , and by substituting in the expressions that represent the distances we get:
 
$$\begin{aligned} (\sqrt{a_1^2 + a_2^2})^2 + \left(\sqrt{b_1^2 + b_2^2}\right)^2 &= \left(\sqrt{(b_1 - a_1)^2 + (b_2 - a_2)^2}\right)^2 \\ a_1^2 + a_2^2 + b_1^2 + b_2^2 &= (b_1 - a_1)^2 + (b_2 - a_2)^2 \\ a_1^2 + a_2^2 + b_1^2 + b_2^2 &= b_1^2 - 2a_1b_1 + a_1^2 + b_2^2 - 2a_2b_2 + a_2^2 \\ 0 &= -2a_1b_1 - 2a_2b_2 \\ 0 &= a_1b_1 + a_2b_2. \end{aligned}$$
- We have just demonstrated that if two segments,  $\overline{OA}$  and  $\overline{OB}$  that have a common endpoint at the origin  $O(0,0)$  and other endpoints of  $A(a_1, a_2)$  and  $B(b_1, b_2)$ , are perpendicular, then  $a_1b_1 + a_2b_2 = 0$ .
- Let's revisit Example 1 and verify our formula with a triangle that we have already proven is right. Triangle  $OAB$  has vertices  $O(0,0)$ ,  $A(6,4)$ , and  $B(-2,3)$ . Verify that it is right ( $\overline{OA} \perp \overline{OB}$ ) using the formula.
  - $6 \cdot (-2) + 4 \cdot 3 = 0$ ; the segments are perpendicular.
- In Exercise 1, we found that triangle  $OPQ$  was not right. Let's see if our formula verifies that conclusion. Triangle  $OPQ$  has vertices  $O(0,0)$ ,  $P(3, -1)$ , and  $Q(2, 3)$ .
  - $3 \cdot (2) + (-1) \cdot 3 \neq 0$ ; the segments are not perpendicular.

**Exercises 2–3 (9 minutes)**

In Example 2, we demonstrated that if two segments,  $\overline{OA}$  and  $\overline{OB}$  that have a common endpoint at the origin  $O(0,0)$  and other endpoints of  $A(a_1, a_2)$  and  $B(b_1, b_2)$ , are perpendicular, then  $a_1b_1 + a_2b_2 = 0$ . We did not demonstrate the converse of this statement, so you will need to explain to students that the converse holds as well: If two segments,  $\overline{OA}$  and  $\overline{OB}$  having a common endpoint at the origin  $O(0,0)$  and other endpoints of  $A(a_1, a_2)$  and  $B(b_1, b_2)$  such that  $a_1b_1 + a_2b_2 = 0$ , then  $\overline{OA} \perp \overline{OB}$ . Students will use this theorem frequently during the remainder of this lesson and in upcoming lessons.

In pairs or groups of three, students will determine which pairs of segments are perpendicular. Have students split up the work and then, as they finish, check each other's work.

**Scaffolding:**

- Provide students with a handout that includes these points already plotted.
- Provide students with a list of the pairs of points/segments to explore to make sure they do all of them.
- Provide advanced students with coordinates that are not whole numbers.

**Exercises 2–3**

2. Given points  $A(6, 4)$ ,  $B(24, -6)$ ,  $C(1, 4)$ ,  $P(2, -3)$ ,  $S(-18, -12)$ ,  $T(-3, -12)$ ,  $U(-8, 2)$ , and  $W(-6, 9)$ , find all pairs of segments from the list below that are perpendicular. Support your answer.

$$\overline{OA}, \overline{OB}, \overline{OC}, \overline{OP}, \overline{OS}, \overline{OT}, \overline{OU}, \text{ and } \overline{OW}$$

$$\overline{OA} \perp \overline{OP} \text{ because } 6(2) + 4(-3) = 0.$$

$$\overline{OA} \perp \overline{OW} \text{ because } 6(-6) + 4(9) = 0.$$

$$\overline{OS} \perp \overline{OW} \text{ because } -18(-6) + (-12)(9) = 0.$$

$$\overline{OS} \perp \overline{OP} \text{ because } -18(2) + (-12)(-3) = 0.$$

$$\overline{OB} \perp \overline{OC} \text{ because } 24(1) + (-6)(4) = 0.$$

$$\overline{OB} \perp \overline{OT} \text{ because } 24(-3) + (-6)(-12) = 0.$$

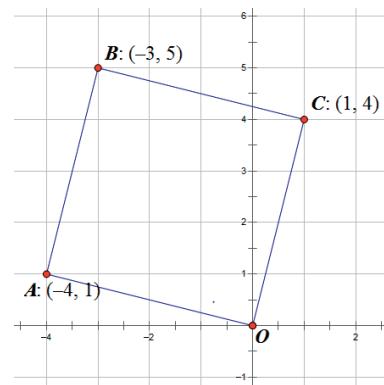
$$\overline{OT} \perp \overline{OU} \text{ because } -3(-8) + (-12)(2) = 0.$$

$$\overline{OC} \perp \overline{OU} \text{ because } 1(-8) + 4(2) = 0.$$

3. The points  $O(0, 0)$ ,  $A(-4, 1)$ ,  $B(-3, 5)$ , and  $C(1, 4)$  are the vertices of parallelogram  $OABC$ . Is this parallelogram a rectangle? Support your answer.

*We are given that the figure is a parallelogram with the properties that the opposite angles are congruent, and the adjacent angles are supplementary. To prove the quadrilateral is a rectangle, we only need to show that one of the four angles is a right angle.*

$\overline{OA} \perp \overline{OC}$  because  $-4(1) + 1(4) = 0$ . Therefore, parallelogram  $OABC$  is a rectangle.



**Example 3 (4 minutes)**

In this example, we will use the theorem students discovered in Example 2 and their current knowledge of slope to define the relationship between the slopes of perpendicular lines (or line segments). The teacher should present the theorem and then the diagram as well as the first two sentences of the proof. Also, consider showing the statements and asking the class to generate the reasons or asking students to try to come up with their own proof of the theorem with a partner.

**Theorem:** Let  $l_1$  and  $l_2$  be two non-vertical lines in the Cartesian plane such that both pass through the origin. The lines  $l_1$  and  $l_2$  are perpendicular if and only if their slopes are negative reciprocals of each other.

**PROOF.** Suppose that  $l_1$  is given by the graph of the equation  $y = m_1x$ , and  $l_2$  by the equation  $y = m_2x$ .

Then  $(0, 0)$  and  $(1, m_1)$  are points on  $l_1$ , and  $(0, 0)$  and  $(1, m_2)$  are points on  $l_2$ .

- By the theorem that we learned in Exercise 2, what can we state?
  - By the theorem of Exercise 2,  $l_1$  and  $l_2$  are perpendicular  $\Leftrightarrow 1 \cdot 1 + m_1 \cdot m_2 = 0$ . ( $\Leftrightarrow$  means "if and only if").
- Let's simplify that.

$$\Leftrightarrow 1 + m_1 \cdot m_2 = 0$$

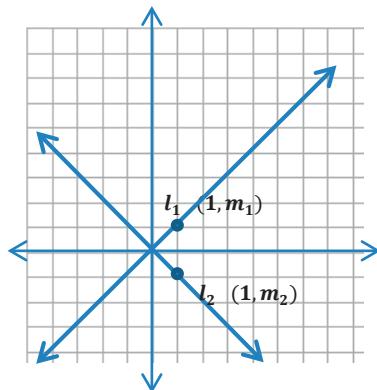
- Now isolate the variables on one side of the equation.

$$\Leftrightarrow m_1 \cdot m_2 = -1$$

- Solve for  $m_1$ .

$$\Leftrightarrow m_1 = -\frac{1}{m_2}$$

- Now, with your partner, restate the theorem and explain it to each other.

**Closing (2 minutes)**

Ask students to respond to these questions in writing, with a partner, or as a class.

- Given  $O(0,0)$ ,  $A(a_1, a_2)$ , and  $B(b_1, b_2)$ , how do we know when  $\overline{OA} \perp \overline{OB}$ ?
  - $a_1b_1 + a_2b_2 = 0$
- How did we derive this formula?
  - First, we had to think about the three points as the vertices of a triangle and assume that the triangle is a right triangle with a right angle at the origin. Next, we used the distance formula to calculate the lengths of the three sides of the triangle. Finally we used the Pythagorean theorem to relate the three lengths. The formula simplified to  $a_1b_1 + a_2b_2 = 0$ .
- What is true about the slopes of perpendicular lines?
  - Their slopes are negative reciprocals of each other.

**Exit Ticket (5 minutes)**

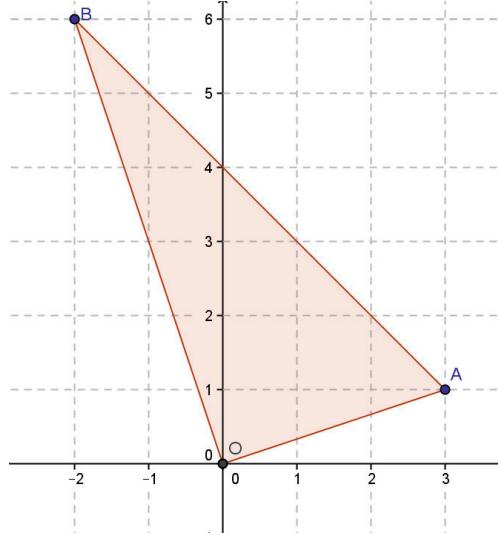
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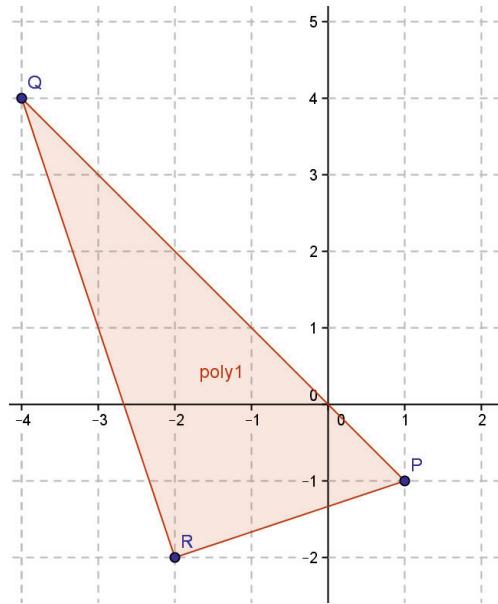
## Lesson 5: Criterion for Perpendicularity

### Exit Ticket

1. Given points  $O(0, 0)$ ,  $A(3, 1)$ , and  $B(-2, 6)$ , prove  $\overline{OA}$  is perpendicular to  $\overline{OB}$ .



2. Given points  $P(1, -1)$ ,  $Q(-4, 4)$ , and  $R(-2, -2)$ , prove  $\overline{PR}$  is perpendicular to  $\overline{QR}$  without the Pythagorean theorem.



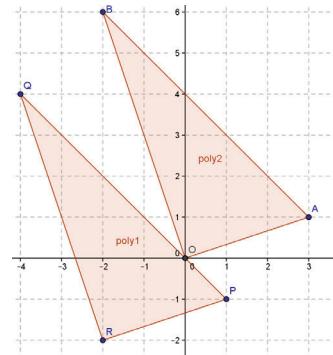
## Exit Ticket Sample Solutions

1. Given points  $O(0, 0)$ ,  $A(3, 1)$ , and  $B(-2, 6)$ , prove  $\overline{OA}$  is perpendicular to  $\overline{OB}$ .

*Since both segments are through the origin,  $(3)(-2) + (1)(6) = 0$ ; therefore, the segments are perpendicular. Students can also prove by the Pythagorean theorem that  $AO^2 + BO^2 = AB^2$ , or  $(\sqrt{10})^2 + (\sqrt{40})^2 = (\sqrt{50})^2$ .*

2. Given points  $P(1, -1)$ ,  $Q(-4, 4)$ , and  $R(-2, -2)$ , prove  $\overline{PR}$  is perpendicular to  $\overline{QR}$  without the Pythagorean theorem.

*The points given are translated points of the point given in Problem 1. Points A, B, and O have all been translated left 2 and down 2. Since  $\overline{AO}$  and  $\overline{BO}$  are perpendicular, translating will not change the angle relationships, so perpendicularity is conserved.*



## Problem Set Sample Solutions

1. Prove using the Pythagorean theorem that  $\overline{AC}$  is perpendicular to  $\overline{AB}$  given  $A(-2, -2)$ ,  $B(5, -2)$ , and  $C(-2, 22)$ .

*$AC = 24$ ,  $BC = 25$ , and  $AB = 7$ . If triangle ABC is right,  $AC^2 + AB^2 = BC^2$ , and  $576 + 49 = 625$ ; therefore, the segments are perpendicular.*

2. Using the general formula for perpendicularity of segments through the origin and  $(90, 0)$ , determine if segments  $\overline{OA}$  and  $\overline{OB}$  are perpendicular.

a.  $A(-3, -4)$ ,  $B(4, 3)$

*$(-3)(4) + (-4)(3) \neq 0$ ; therefore, the segments are not perpendicular.*

b.  $A(8, 9)$ ,  $B(18, -16)$

*$(8)(18) + (9)(-16) = 0$ ; therefore, the segments are perpendicular.*

3. Given points  $O(0, 0)$ ,  $S(2, 7)$ , and  $T(7, -2)$ , where  $\overline{OS}$  is perpendicular to  $\overline{OT}$ , will the images of the segments be perpendicular if the three points  $O$ ,  $S$ , and  $T$  are translated four units to the right and eight units up? Explain your answer.

*$O'(4, 8)$ ,  $S'(6, 15)$ ,  $T'(11, 6)$*

*Yes, the points are all translated 4 units right and 8 units up; since the original segments were perpendicular, the translated segments are perpendicular.*

4. In Example 1, we saw that  $\overline{OA}$  was perpendicular to  $\overline{OB}$  for  $O(0, 0)$ ,  $A(6, 4)$ , and  $B(-2, 3)$ . Suppose  $P(5, 5)$ ,  $Q(11, 9)$ , and  $R(3, 8)$ . Are segments  $\overline{PQ}$  and  $\overline{PR}$  perpendicular? Explain without using triangles or the Pythagorean theorem.

*Yes, the segments are perpendicular. We proved in Example 1 that  $\overline{OA}$  was perpendicular to  $\overline{OB}$ . P, Q, and R are translations of O, A, and B. The original coordinates have been translated 5 right and 5 up.*



5. Challenge: Using what we learned in Exercise 2, if  $C(c_1, c_2)$ ,  $A(a_1, a_2)$ , and  $B(b_1, b_2)$ , what is the general condition of  $a_1, a_2, b_1, b_2, c_1$ , and  $c_2$  that ensures segments  $\overline{CA}$  and  $\overline{CB}$  are perpendicular?

*To translate C to the origin, we move left  $c_1$  and down  $c_2$ . That means  $A(a_1 - c_1, a_2 - c_2)$  and  $B(b_1 - c_1, b_2 - c_2)$ . The condition of perpendicularity is  $a_1b_1 + a_2b_2 = 0$ , meaning that  $(a_1 - c_1)(b_1 - c_1) + (a_2 - c_2)(b_2 - c_2) = 0$ .*

6. A robot that picks up tennis balls is on a straight path from  $(8, 6)$  towards a ball at  $(-10, -5)$ . The robot picks up a ball at  $(-10, -5)$ , then turns  $90^\circ$  right. What are the coordinates of a point that the robot can move towards to pick up the last ball?

*Answer will vary. A possible answer is  $(-21, 13)$ .*

7. Gerry thinks that the points  $(4, 2)$  and  $(-1, 4)$  form a line perpendicular to a line with slope 4. Do you agree? Why or why not?

*I do not agree with Gerry. The segment through the two points has a slope of  $-\frac{2}{5}$ . If it was perpendicular to a line with slope 4, the slope of the perpendicular line would be  $-\frac{1}{4}$ .*

MP.3



## Lesson 6: Segments That Meet at Right Angles

### Student Outcomes

- Students generalize the criterion for perpendicularity of two segments that meet at a point to any two segments in the Cartesian plane.
- Students apply the criterion to determine if two segments are perpendicular.

### Classwork

#### Opening Exercise (2 minutes)

MP.3

As a class, present each problem and have students determine an answer and justify it.

##### Opening Exercise

Carlos thinks that the segment having endpoints  $A(0, 0)$  and  $B(6, 0)$  is perpendicular to the segment with endpoints  $A(0, 0)$  and  $C(-2, 0)$ . Do you agree? Why or why not?

*No, the two segments are not perpendicular. If they were perpendicular, then  $6(-2) + 0(0)$  would be true.*

Working with a partner, given  $A(0, 0)$  and  $B(3, -2)$ , find the coordinates of a point  $C$  so that  $\overline{AC} \perp \overline{AB}$ .

*Let the other endpoint be  $C(c, d)$ . If  $\overline{AC} \perp \overline{AB}$ , then  $3c + -2d = 0$ . This means that as long as we choose values for  $c$  and  $d$  that satisfy the equation  $d = \frac{3}{2}c$ ,  $\overline{AC}$  will be perpendicular to  $\overline{AB}$ .*

*Answers may vary but may include  $(2, 3)$ ,  $(4, 6)$ , and  $(6, 9)$  or any other coordinates that meet the requirement stated above.*

##### Scaffolding:

- Graphing each exercise will help students picture the problem.
- It will be helpful to display posters of the previous day's findings around the room.

**Example 1 (10 minutes)**

In this example, we are trying to get students to see that translating segments may help us determine whether the lines containing the segments are perpendicular.

**Example 1**

Given points  $A(2, 2)$ ,  $B(10, 16)$ ,  $C(-3, 1)$ , and  $D(4, -3)$ , are segments  $\overline{AB}$  and  $\overline{CD}$  perpendicular? Are the lines containing the segments perpendicular? Explain.

*One possible solution would be to translate  $\overline{AB}$  so that  $A'$  is on the origin (using the vector  $\langle -2, -2 \rangle$ , or left 2 down 2) and to translate  $\overline{CD}$  so that  $C'$  is on the origin (using the vector  $\langle 3, -1 \rangle$ , right 3 down 1):*

$$A'(0, 0), B'(10 - 2, 16 - 2), C'(0, 0), \text{ and } D'(4 - (-3), -3 - 1).$$

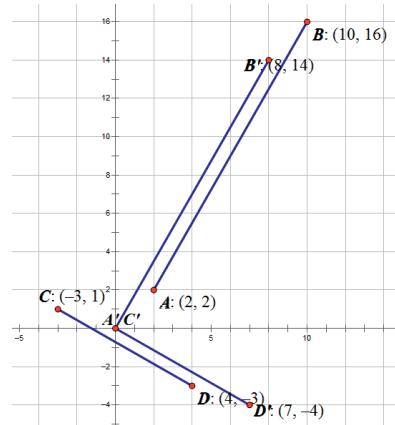
$\overline{AB}$  will be perpendicular to  $\overline{CD}$  if  $\overline{A'B'}$  is perpendicular to  $\overline{C'D'}$ .

$$8(7) + 14(-4) = 0; \text{ therefore, } \overline{A'B'} \text{ is perpendicular to } \overline{C'D'}, \text{ which means } \overline{AB} \perp \overline{CD}.$$

$$A'(0, 0), B'(10 - 2, 16 - 2), C'(0, 0), \text{ and } D'(4 - (-3), -3 - 1).$$

$\overline{AB}$

$$8(7) + 14(-4) = 0.$$



- Working in pairs, each student selects two points to be the endpoint of a segment. One partner identifies the coordinates of points  $A$  and  $B$ , and the other partner chooses coordinates for points  $C$  and  $D$ . Although the likelihood of this happening is low, students may choose one of the same points, but not both. The students then plot the points on the same coordinate plane and construct segments  $\overline{AB}$  and  $\overline{CD}$ .
- The students must now determine whether the segments are perpendicular.
  - *Answers will vary. Some students will say no because they do not intersect. Students should recognize that, unless the segments are parallel, if we extend the segments, the lines containing the segments will intersect.*
- How do your segments differ from those we worked with in Lesson 5?
  - *In that lesson, both segments had one endpoint located at the origin.*
- Would translating your two segments change their orientation?
  - *No. If we translate the segments, thereby translating the lines containing the segments, the angles formed by the two lines will not change. If the lines were perpendicular, the images of the lines under translation will also be perpendicular.*
- With your partner, translate each of your segments so that they have an endpoint at the origin. Then, use the results of yesterday's lesson to determine whether the two segments are perpendicular
  - *Answers will vary.*
- Switch papers with another pair of students and confirm their results.

**Exercise 1 (5 minutes)**

Have students try Exercise 1 using the example just completed as a template. You may pull aside some groups for targeted instruction.

**Exercise 1**

1. Given  $A(a_1, a_2)$ ,  $B(b_1, b_2)$ ,  $C(c_1, c_2)$ , and  $D(d_1, d_2)$ , find a general formula in terms of  $a_1, a_2, b_1, b_2, c_1, c_2, d_1$ , and  $d_2$  that will let us determine whether segments  $\overline{AB}$  and  $\overline{CD}$  are perpendicular.

*After translating the segments so that the image of points A and C lie on the origin, we get  $A'(0, 0)$ ,  $B'(b_1 - a_1, b_2 - a_2)$ ,  $C'(0, 0)$ , and  $D'(d_1 - c_1, d_2 - c_2)$ .*

$$\overline{A'B'} \perp \overline{C'D'} \leftrightarrow (b_1 - a_1)(d_1 - c_1) + (b_2 - a_2)(d_2 - c_2) = 0$$

**Example 2 (7 minutes)**

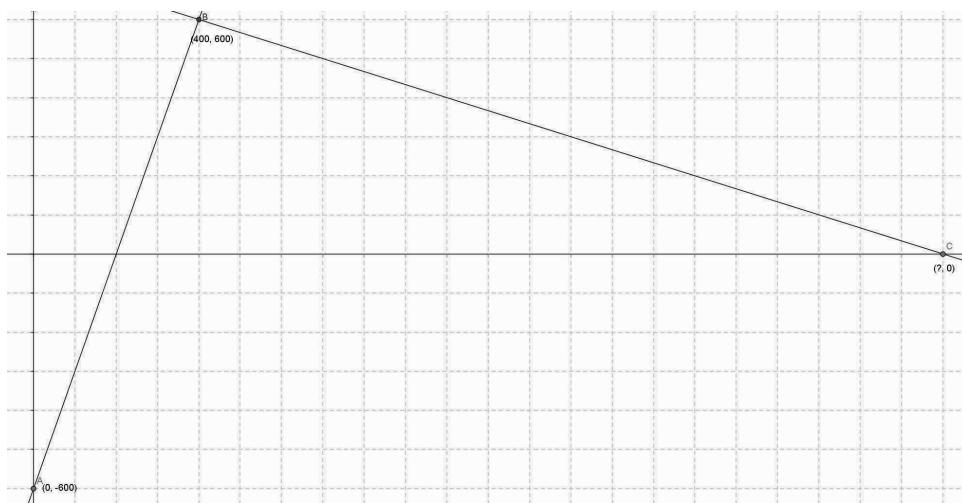
In this example, we are going to have students use the formula they derived in Exercise 1 to find the location of one endpoint of a segment given the other endpoint and the coordinates of the endpoints of a perpendicular segment. Because there are an infinite number of possible endpoints, we will only ask students to determine one unknown coordinate given the other.

- Given  $\overline{AB} \perp \overline{CD}$  and  $A(3, 2)$ ,  $B(7, 10)$ ,  $C(-2, -3)$ , and  $D(4, d_2)$ , find the value of  $d_2$ . While we will be using the formula that we derived in Exercise 1, it will be helpful if you explain each of the steps as you are solving the problem.
    - *Using the formula we derived in the previous exercise, we can write the equation*  
 $(7 - 3)(4 + 2) + (10 - 2)(d_2 + 3) = 0$ .
    - *We translated  $\overline{AB}$  to  $\overline{A'B'}$  so that  $A'$  is located at the origin. We moved three units to the left and down two units giving us  $B'(7 - 3, 10 - 2)$ .*
    - *We translated  $\overline{DC}$  to  $\overline{C'D'}$  so that  $C'$  is located at the origin. We moved two units to the right and up three units giving us  $D'(4 + 2, d_2 + 3)$ .*
- $$4(6) + 8(d_2 + 3) = 0$$
- $$d_2 + 3 = 3$$
- $$d_2 = 0$$

Teachers can assign different exercises to different students or groups and then bring the class back together to share. Some students may be able to complete all exercises while others may need more guidance.

## Exercises 2–4 (14 minutes)

2. Recall the Opening Exercise of Lesson 4 in which a robot is traveling along a linear path given by the equation  $y = 3x - 600$ . The robot hears a ping from a homing beacon when it reaches the point  $B(400, 600)$  and turns to travel along a linear path given by the equation  $y - 600 = -\frac{1}{3}(x - 400)$ . If the homing beacon lies on the  $x$ -axis, what is its exact location? (Use your own graph paper to visualize the scenario.)



- a. If point  $A$  is the  $y$ -intercept of the original equation, what are the coordinates of point  $A$ ?

$A(0, -600)$

- b. What are the endpoints of the original segment of motion?

$A(0, -600)$  and  $B(400, 600)$

- c. If the beacon lies on the  $x$ -axis, what is the  $y$ -value of this point,  $C$ ?

$0$

- d. Translate point  $B$  to the origin. What are the coordinates of  $A'$ ,  $B'$ , and  $C'$ ?

$A'(-400, -1200)$ ,  $B'(0, 0)$ , and  $C'(c_1, -600)$

- e. Use the formula derived in this lesson to determine the coordinates of point  $C$ .

We know that  $\overline{AB} \perp \overline{BC}$ , so  $-400(c_1) + (-1200)(-600) = 0 \leftrightarrow c_1 = 1800$

$C'(1800, -600) \leftrightarrow C(2200, 0)$

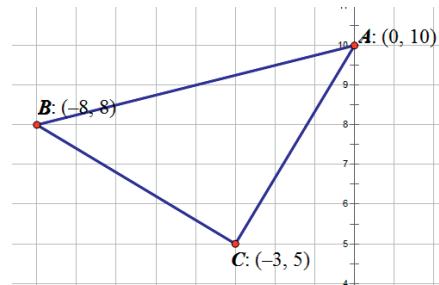
$(2200, 0)$

3. A triangle in the coordinate plane has vertices  $A(0, 10)$ ,  $B(-8, 8)$ , and  $C(-3, 5)$ . Is it a right triangle? If so, at which vertex is the right angle? (Hint: Plot the points and draw the triangle on a coordinate plane to help you determine which vertex is the best candidate for the right angle.)

*The most likely candidate appears to be vertex C. By translating the figure so that C is mapped to the origin, we can use the formula*

$$\begin{aligned} (-8 + 3)(0 + 3) + (8 - 5)(10 - 5) &= 0 \\ -5(3) + 3(5) &= 0 \\ 0 = 0 &\leftrightarrow \overline{BC} \perp \overline{AC}. \end{aligned}$$

*Yes, segments  $\overline{BC}$  and  $\overline{AC}$  are perpendicular. The right angle is C.*



4.  $A(-7, 1)$ ,  $B(-1, 3)$ ,  $C(5, -5)$ , and  $D(-5, -5)$  are vertices of a quadrilateral. If  $\overline{AC}$  bisects  $\overline{BD}$ , but  $\overline{BD}$  does not bisect  $\overline{AC}$ , determine whether  $ABCD$  is a kite.

*We can show  $ABCD$  is a kite if  $\overline{AC} \perp \overline{BD}$ .*

*Translating  $\overline{AC}$  so that the image of point A lies on the origin and translating  $\overline{BD}$  so that the image of point B lies at the origin leads to the equation:*

$$\begin{aligned} (5 + 7)(-5 + 1) + (-5 - 1)(-5 - 3) &= 0 \\ (12)(-4) + (-6)(-8) &= 0, \\ -48 + 48 = 0 &\text{ is a true statement; therefore, } \overline{AC} \perp \overline{BD}, \text{ and } ABCD \text{ is a kite.} \end{aligned}$$

### Closing (2 minutes)

As a class, revisit the two theorems of today's lesson and have students explain the difference between them.

- Given points  $O(0, 0)$ ,  $A(a_1, a_2)$ ,  $B(b_1, b_2)$ ,  $C(c_1, c_2)$ , and  $D(d_1, d_2)$ ,
- $$\overline{OA} \perp \overline{OB} \leftrightarrow (a_1)(b_1) + (a_2)(b_2) = 0;$$
- $$\overline{AB} \perp \overline{CD} \leftrightarrow (b_1 - a_1)(d_1 - c_1) + (b_2 - a_2)(d_2 - c_2) = 0.$$

### Exit Ticket (5 minutes)

Name \_\_\_\_\_

Date \_\_\_\_\_

## Lesson 6: Segments That Meet at Right Angles

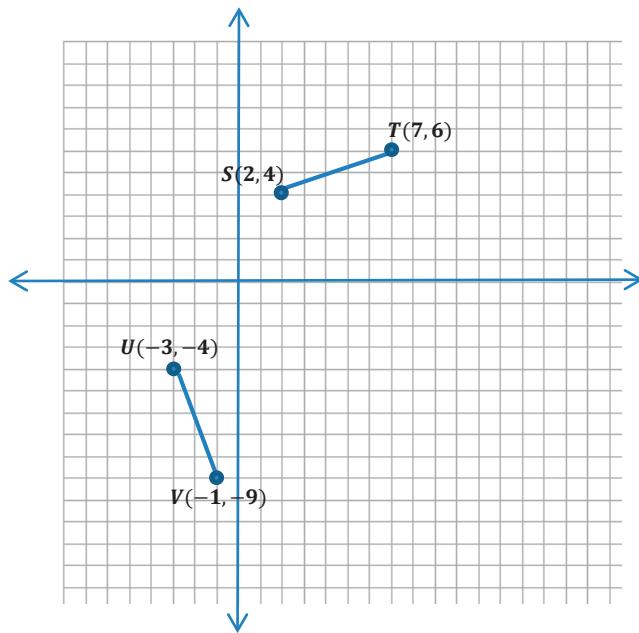
### Exit Ticket

Given points  $S(2, 4)$ ,  $T(7, 6)$ ,  $U(-3, -4)$ , and  $V(-1, -9)$ :

- a. Translate segments  $\overline{ST}$  and  $\overline{UV}$  so that the image of each segment has an endpoint at the origin.

- b. Are the segments perpendicular? Explain.

- c. Are the lines  $\overleftrightarrow{ST}$  and  $\overleftrightarrow{UV}$  perpendicular? Explain.



## Exit Ticket Sample Solutions

Given points  $S(2, 4)$ ,  $T(7, 6)$ ,  $U(-3, -4)$ , and  $V(-1, -9)$ :

- a. Translate segments  $\overline{ST}$  and  $\overline{UV}$  so that the image of each segment has an endpoint at the origin.

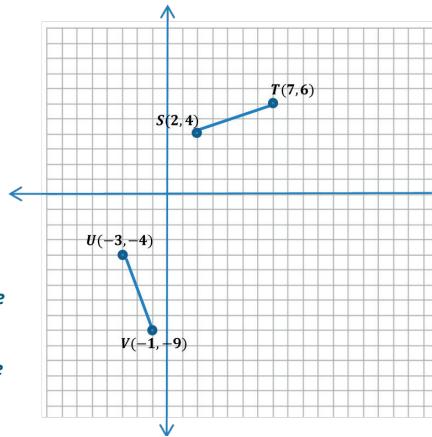
*Answers can vary slightly. Students will have to choose one of the first two and one of the second two.*

*If we translate  $\overline{ST}$  so that the image of  $S$  is at the origin, we get  $S'(0, 0)$ ,  $T'(5, 2)$ .*

*If we translate  $\overline{ST}$  so that the image of  $T$  is at the origin, we get  $S'(-5, -2)$ ,  $T'(0, 0)$ .*

*If we translate  $\overline{UV}$  so that the image of  $U$  is at the origin, we get  $U'(0, 0)$ ,  $V'(2, -5)$ .*

*If we translate  $\overline{UV}$  so that the image of  $V$  is at the origin, we get  $V'(0, 0)$ ,  $U'(-2, 5)$ .*



- b. Are the segments perpendicular? Explain.

*Yes. By choosing any two of the translated segment  $\overline{S'T'}$  and  $\overline{U'V'}$ , we determine whether the equation yields a true statement:  $(b_1 - a_1)(d_1 - c_1) + (b_2 - a_2)(d_2 - c_2) = 0$ .*

*For example, using  $S(-5, -2)$ ,  $T'(0, 0)$ , and  $U'(0, 0)$ ,  $V'(2, -5)$ :*

$$-5(2) + (-2)(5) = 0 \text{ is a true statement; therefore, } \overline{S'T'} \perp \overline{U'V'} \text{ and } \overline{ST} \perp \overline{UV}.$$

- c. Are the lines  $\overleftrightarrow{ST}$  and  $\overleftrightarrow{UV}$  perpendicular? Explain.

*Yes, lines containing perpendicular segments are also perpendicular.*

## Problem Set Sample Solutions

1. Are the segments through the origin and the points listed perpendicular? Explain.

- a.  $A(9, 10)$ ,  $B(10, 9)$

$$\text{No; } 9 \cdot 10 + 10 \cdot 9 \neq 0.$$

- b.  $C(9, 6)$ ,  $D(4, -6)$

$$\text{Yes; } 9 \cdot 4 + 6 \cdot (-6) = 0.$$

2. Given  $M(5, 2)$ ,  $N(1, -4)$ , and  $L$  listed below, are segments  $\overline{LM}$  and  $\overline{MN}$  perpendicular? Translate  $M$  to the origin, write the coordinates of the images of the points, then explain without using slope.

- a.  $L(-1, 6)$

$$M'(0, 0), N'(-4, -6), L(-6, 4)$$

$$\text{Yes; } (-4) \cdot (-6) + (-6) \cdot 4 = 0.$$

## Scaffolding:

Give students steps to follow to make problems more accessible.

1. Plot the points.
2. Find the common endpoint.
3. Translate that to  $(0, 0)$ .
4. Translate the other points.
5. Use the formula  $(a_1)(b_1) + (a_2)(b_2) = 0$  to determine perpendicularity.



b.  $L(11, -2)$

$M'(0, 0), N'(-4, -6), L(6, -4)$

*Yes;  $(-4) \cdot 6 + (-6) \cdot (-4) = 0$ .*

c.  $L(9, 8)$

$M'(0, 0), N'(-4, -6), L(4, 6)$

*No;  $(-4) \cdot 4 + (-6) \cdot 6 \neq 0$ .*

3. Is triangle  $PQR$ , where  $P(-7, 3)$ ,  $Q(-4, 7)$ , and  $R(1, -3)$ , a right triangle? If so, which angle is the right angle? Justify your answer.

*Yes. If the points are translated to  $P'(0, 0)$ ,  $Q'(3, 4)$ , and  $R'(8, -6)$ ,  $3(8) + 4(-6) = 0$ , meaning the segments are perpendicular. The right angle is  $P$ .*

4. A quadrilateral has vertices  $(2 + \sqrt{2}, -1)$ ,  $(8 + \sqrt{2}, 3)$ ,  $(6 + \sqrt{2}, 6)$ , and  $(\sqrt{2}, 2)$ . Prove that the quadrilateral is a rectangle.

*Answers will vary, but it is a rectangle because it has 4 right angles.*

5. Given points  $G(-4, 1)$ ,  $H(3, 2)$ , and  $I(-2, -3)$ , find the  $x$ -coordinate of point  $J$  with  $y$ -coordinate 4 so that the lines  $\overleftrightarrow{GH}$  and  $\overleftrightarrow{IJ}$  are perpendicular.

**-3**

6. A robot begins at position  $(-80, 45)$  and moves on a path to  $(100, -60)$ . It turns  $90^\circ$  counterclockwise.

- a. What point with  $y$ -coordinate 120 is on this path?

**(205, 120)**

- b. Write an equation of the line after the turn.

$$y + 60 = \frac{12}{7}(x - 100)$$

- c. If it stops to charge on the  $x$ -axis, what is the location of the charger?

**(135, 0)**

7. Determine the missing vertex of a right triangle with vertices  $(6, 2)$  and  $(5, 5)$  if the third vertex is on the  $y$ -axis. Verify your answer by graphing.

$$\left(0, \frac{10}{3}\right)$$

8. Determine the missing vertex for a rectangle with vertices  $(3, -2)$ ,  $(5, 2)$ , and  $(-1, 5)$ , and verify by graphing. Then, answer the questions that follow.

**(-3, 1)**

- a. What is the length of the diagonal?

*Approximately 8.06 units*



- b. What is a point on both diagonals in the interior of the figure?

$$\left(1, \frac{3}{2}\right)$$

9. A right triangle has vertices  $(1, 3)$  and  $(6, -1)$  and a third vertex located in Quadrant IV.

- a. Determine the coordinates of the missing vertex.

$$(2, -6)$$

- b. Reflect the triangle across the  $y$ -axis. What are the new vertices?

$$(-1, 3), (-6, -1), \text{ and } (-2, -6)$$

- c. If the original triangle is rotated  $90^\circ$  counterclockwise about the vertex  $(6, -1)$ , what are the coordinates of the other vertices?

$$(2, -6), (11, -5)$$

- d. Now rotate the original triangle  $90^\circ$  clockwise about  $(6, -1)$ . What are the coordinates of the other vertices?

$$(1, 3), (10, 4)$$

- e. What do you notice about both sets of vertices? Explain what you observe.

*One rotated vertex moves onto one of the original vertices, and the third vertex is different. This is because we are rotating the triangle  $90^\circ$  about the right angle.*



## Lesson 7: Equations for Lines Using Normal Segments

### Student Outcomes

- Students state the relationship between previously used formats for equations for lines and the new format  $a_1x + a_2y = c$ , recognizing the segments from  $(0, 0)$  to  $(a_1, a_2)$  as a normal and  $-\frac{a_2}{a_1}$  as a slope.
- Students solve problems that are dependent upon making such interpretations.

### Lesson Notes

This lesson focuses on MP.4 because students work extensively to model robot behavior using coordinates.

### Classwork

#### Opening Exercise (5 minutes)

This exercise can be modeled by the teacher with the whole class, given to groups to present solutions to the class, or used as a supplement to the lesson.

##### Opening Exercise

The equations given are in standard form. Put each equation in slope-intercept form. State the slope and the  $y$ -intercept.

1.  $6x + 3y = 12$

$$y = -2x + 4$$

$$\text{slope} = -2$$

$$y\text{-intercept} = 4$$

2.  $5x + 7y = 14$

$$y = -\frac{5}{7}x + 2$$

$$\text{slope} = -\frac{5}{7}$$

$$y\text{-intercept} = 2$$

3.  $2x - 5y = -7$

$$y = \frac{2}{5}x + \frac{7}{5}$$

$$\text{slope} = \frac{2}{5}$$

$$y\text{-intercept} = \frac{7}{5}$$

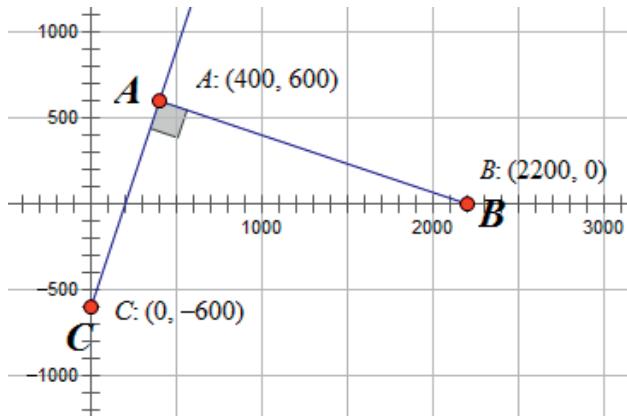
##### Scaffolding:

Provide visuals to reinforce standard and slope-intercept forms:

- Standard form is  $Ax + By = C$  where  $A$ ,  $B$ , and  $C$  are integers.
- Slope-intercept form is  $y = mx + b$  where  $m$  is the slope, and  $b$  is the  $y$ -intercept.

**Discussion (20 minutes)**

Let's revisit the robot from Lesson 4. Recall that it is moving along the line  $y = 3x - 600$ . At the point  $(400, 600)$ , it detected the loudest "ping," and the programmers had the robot change direction at that point and move along a linear path that was perpendicular to his original path.

**Scaffolding:**

Plotting points will help students visualize the problems and understand the translations required.

At the end of Lesson 6, we learned that the beacon happens to lie on the  $x$ -axis and is located at the point  $(2200, 0)$ .

- How did we determine the location of the beacon?
  - We had to find the point  $B$  on the  $x$ -axis so that  $\overline{AB}$  and  $\overline{AC}$  were perpendicular. We knew  $A(400, 600)$  and  $C(0, -600)$  and that the  $y$ -coordinate of point  $B$  was 0 since point  $B$  was on the  $x$ -axis. Using  $B(b, 0)$ , we eventually determined that  $b = 2200$ .
- Let's push this idea a little further. Suppose  $P(x, y)$  is any point on the line containing segment  $\overline{AC}$ . What can you say about  $\overline{AP}$  and  $\overline{AB}$ ?
  - They are also perpendicular.
- Using  $A(400, 600)$ ,  $B(2200, 0)$ , and  $P(x, y)$ , translate the points so that  $A$  is at the origin. What are the new coordinates?
  - If we translate all points in the figure using the translation vector  $\langle -400, -600 \rangle$  that takes  $A$  to the origin  $A' = O(0, 0)$ , then  $B$  becomes  $B'(1800, -600)$ , and  $P$  becomes  $P'(x - 400, y - 600)$ .
- If the condition for perpendicularity is  $ab + cd = 0$ , how could we write an equation involving  $x$  and  $y$ ?
  - The condition that  $\overline{OA'}$  and  $\overline{OP'}$  are perpendicular becomes:
 
$$1800(x - 400) + (-600)(y - 600) = 0$$

$$1800x - 720,000 - 600y + 360,000 = 0$$

$$1800x - 600y = 360,000.$$
- This is the equation in standard form:  $Ax + By = C$ . Look at the values of  $A$  and  $B$  and at the work above. Do you see a relationship between  $A$  and  $B$  and the work above?
  - The coordinates of  $B'$  (the translation of  $B$ ) are  $(1800, -600)$ , which are the same values as  $A$  and  $B$ .

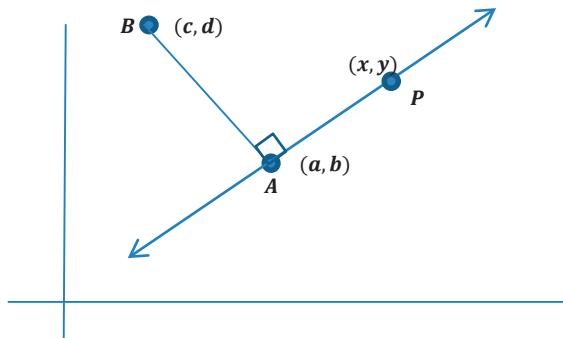
- Let's say that in a more specific way. I will state the relationship, and you repeat it and explain it to your partner.
  - *A is the x-coordinate (abscissa) of the image of the point of the perpendicular segment that does not lie on the line when the point of perpendicularity is at the origin.*
  - *B is the y-coordinate (ordinate) of the image of the point of the perpendicular segment that does not lie on the line when the point of perpendicularity is at the origin.*
- Now, let's put the equation in slope-intercept form.
  - $-600y = -1800x + 360,000$ .
  - $y = 3x - 600$ , which is the equation of  $\overleftrightarrow{AB}$ .
- What does the equation you wrote represent?
  - $\overline{AP}$  will be perpendicular to  $\overline{AB}$  as long as point  $P$  lies on the line containing  $\overline{AC}$ , which is given by the equation  $y = 3x - 600$ .

MP.7

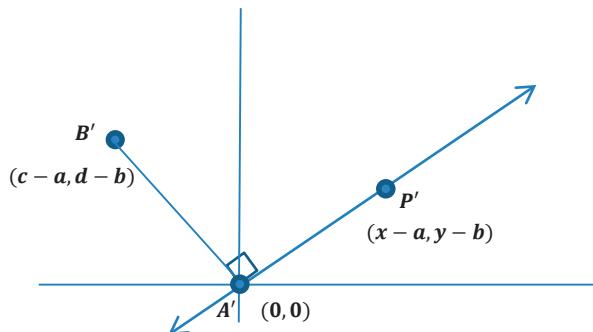
In the next part of this lesson, we generalize what we just discovered. This can be done in several ways.

1. Present the question and explain that we are trying to make this process work for any points with coordinates  $A(a, b)$ ,  $B(c, d)$ , and  $P(x, y)$ . Allow time for students to think and talk to their neighbors for a few minutes; then, show the diagrams and give them more time to talk. Finally, pull everyone together and discuss each step as a class.
2. Assign some groups the task with no leading questions, and let them work independently while other groups are getting different levels of help, some even being directly instructed by the teacher.

- How can we generalize this finding?
- Given point  $A(a, b)$  which lies on line  $l$ , point  $B(c, d)$  not on line  $l$ , and  $\overline{AB}$  perpendicular to line  $l$ , then any point  $P(x, y)$  on line  $l$  will satisfy the relationship  $\overline{AP} \perp \overline{AB}$ . Draw the picture described.



- Translate the points. Which point should be on the origin? What is the translation used? What are the coordinates of the translated points?
  - *A is the common point and should be translated to the origin. The translation is  $(-a, -b)$ , or left a and down b. The translated points are  $A' = O(0, 0)$ ,  $B'(c - a, d - b)$ , and  $P'(x - a, y - b)$ .*



- If the segments are perpendicular, write the equation that must hold true.
  - $(c - a)(x - a) + (d - b)(y - b) = 0$ .
- If  $A = (c - a)$  and  $B = (d - b)$ , write this equation substituting in  $A$  and  $B$ . Which line have we written the equation of?
  - *We end up with the equation of the line that passes through point A that is perpendicular to  $\overline{AB}$ :  
 $A(x - a) + B(y - b) = 0$ .*
- What do  $A$  and  $B$  represent graphically?
  - *A is the abscissa, and B is the ordinate of the image of B.*
- We call segment  $\overline{AB}$  a normal segment to line  $l$  because it has one endpoint on the line and is perpendicular to the line.
  - *Explain to your neighbor what a normal segment is, and write your own definition.*

**Definition:** A line segment with one endpoint on a line and perpendicular to the line is called a normal segment to the line.

#### Scaffolding:

- Have students leave the equations in standard form.
- Provide these steps:  

$$(8 - 5)(x - 5) + (2 + 7)(y + 7) = 0$$

Simplify the parentheses without variables.

$$3(x - 5) + 9(y + 7) = 0$$

Separate the variables—put  $x$  on one side and  $y$  on the other.

$$9(y + 7) = -3(x - 5)$$

Distribute the coefficients.

$$9y + 63 = -3x + 15$$

Bring the constant on the right to the left.

$$9y = -3x - 48$$

Divide by the coefficient of  $y$ .

$$y = -\frac{3}{9}x - \frac{48}{9}$$

Simplify if necessary.

$$y = -\frac{1}{3}x - \frac{16}{3}$$

**Example 1 (5 minutes)****Example 1**Given  $A(5, -7)$  and  $B(8, 2)$ :

- a. Find an equation for the line through  $A$  and perpendicular to  $\overline{AB}$ .

$$A'(0, 0), B'(8 - 5, 2 - (-7)), \text{ and } P'(x - 5, y - (-7))$$

$$(8 - 5)(x - 5) + (2 + 7)(y + 7) = 0$$

$$3(x - 5) = -9(y + 7)$$

$$3x - 15 = -9y - 63$$

$$y = -\frac{1}{3}x - \frac{16}{3}$$

- b. Find an equation for the line through  $B$  and perpendicular to  $\overline{AB}$ .

$$B'(0, 0), A'(5 - 8, -7 - 2), \text{ and } P'(x - 8, y - 2)$$

$$(5 - 8)(x - 8) + (-7 - 2)(y - 2) = 0$$

$$-3(x - 8) = 9(y - 2)$$

$$-3x + 24 = 9y - 18$$

$$y = -\frac{1}{3}x + \frac{14}{3}$$

**Exercises 1–2 (8 minutes)**

Have students plot points to aid in solving problems.

1. Given  $U(-4, -1)$  and  $V(7, 1)$ :

- a. Write an equation for the line through  $U$  and perpendicular to  $\overline{UV}$ .

$$U'(0, 0), V'(7 - (-4), 1 - (-1)), P'(x - (-4), y - (-1))$$

$$(7 + 4)(x + 4) + (1 + 1)(y + 1) = 0$$

$$11x + 44 = -2y - 2$$

$$y = -\frac{11}{2}x - 23$$

- b. Write an equation for the line through  $V$  and perpendicular to  $\overline{UV}$ .

$$V'(0, 0), U'(-4 - 7, -1 - 1), P'(x - 7, y - 1)$$

$$(-7 - 4)(x - 7) + (-1 - 1)(y - 1) = 0$$

$$-11x + 77 = 2y - 2$$

$$y = -\frac{11}{2}x + \frac{79}{2}$$



2. Given  $S(5, -4)$  and  $T(8, 12)$ :
- Write an equation for the line through  $S$  and perpendicular to  $\overline{ST}$ .

$$\begin{aligned} S'(0, 0), T'(-8 - 5, 12 - (-4)), P'(x - 5, y - (-4)) \\ (-8 - 5)(x - 5) + (12 - (-4))(y - (-4)) = 0 \\ -13x + 65 = -16y - 64 \\ y = \frac{13}{16}x - \frac{129}{16} \end{aligned}$$

- Write an equation for the line through  $T$  and perpendicular to  $\overline{ST}$ .

$$\begin{aligned} T'(0, 0), S'(5 - (-8), -4 - 12), P'(x - (-8), y - 12) \\ (5 - (-8))(x - (-8)) + (-4 - 12)(y - 12) = 0 \\ 13x + 104 = 16y - 192 \\ y = \frac{13}{16}x + \frac{37}{2} \end{aligned}$$

### Closing (2 minutes)

Describe the characteristics of a normal segment.

*A line segment with one endpoint on a line and perpendicular to the line is called a normal segment to the line.*

Every equation of a line through a given point  $(a, b)$  has the form  $A(x - a) + B(y - b) = 0$ . Explain how the values of  $A$  and  $B$  are obtained.

*A is the value of the abscissa of the image of the endpoint of the normal segment that does not lie on the line after the figure has been translated so that the image of the endpoint that does lie on the line, the point of perpendicularity, is at the origin.*

*B is the value of the ordinate of the image of the endpoint of the normal segment that does not lie on the line after the figure has been translated so that the image of the endpoint that does lie on the line, the point of perpendicularity, is at the origin.*

### Exit Ticket (5 minutes)

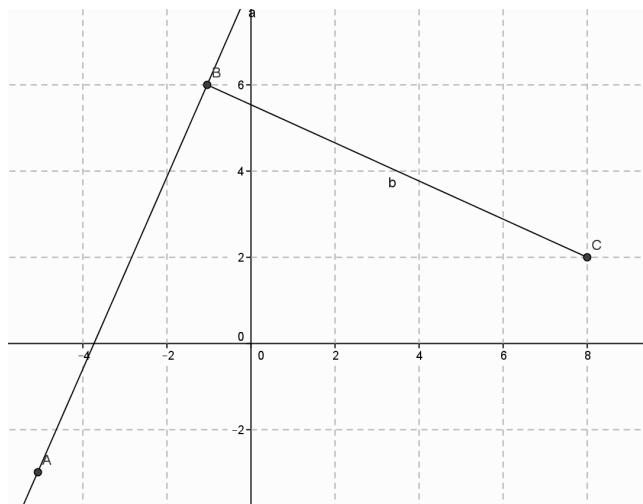
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Date \_\_\_\_\_

## Lesson 7: Equations for Lines Using Normal Segments

### Exit Ticket

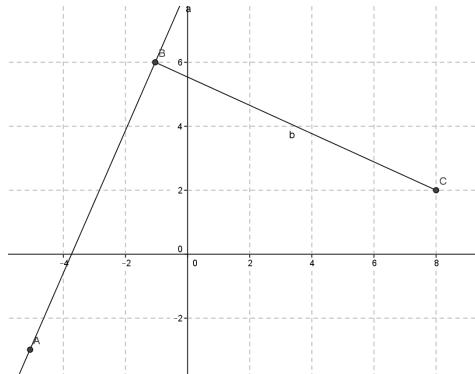
Given  $A(-5, -3)$ ,  $B(-1, 6)$ , and  $C(x, y)$ :



- What are the coordinates of the translated points if  $B$  moves to the origin?
- Write the condition for perpendicularity equation.
- Write the equation for the normal line.

## Exit Ticket Sample Solutions

Given  $A(-5, -3)$ ,  $B(-1, 6)$ , and  $C(x, y)$ :



- a. What are the coordinates of the translated points if  $B$  moves to the origin?

$$A'(-4, -9), B'(0, 0), C'(x + 1, y - 6)$$

- b. Write the condition for perpendicularity equation.

$$-4(x + 1) - 9(y - 6) = 0$$

- c. Write the equation for the normal line in slope-intercept form.

$$y = -\frac{4}{9}x + \frac{50}{9}$$

## Problem Set Sample Solutions

1. Given points  $C(-4, 3)$  and  $D(3, 3)$ :

- a. Write the equation of the line through  $C$  and perpendicular to  $\overline{CD}$ .

$$x = -4$$

- b. Write the equation of the line through  $D$  and perpendicular to  $\overline{CD}$ .

$$x = 3$$

2. Given points  $N(7, 6)$  and  $M(7, -2)$ :

- a. Write the equation of the line through  $M$  and perpendicular to  $\overline{MN}$ .

$$y = -2$$

- b. Write the equation of the line through  $N$  and perpendicular to  $\overline{MN}$ .

$$y = 6$$



3. The equation of a line is given by the equation  $8(x - 4) + 3(y + 2) = 0$ .

- a. What are the coordinates of the image of the endpoint of the normal segment that does not lie on the line? Explain your answer.

*(8, 3) because A(x - 4) + B(y + 2) = 0 is the original formula, and (A, B) are the coordinates of the image of the endpoint of the normal segment not on the line.*

- b. What translation occurred to move the point of perpendicularity to the origin?

*(-4, 2), or left 4 up 2*

- c. What were the coordinates of the original point of perpendicularity? Explain your answer.

*(4, -2) because the translation of (-4, 2) was required to move the point of perpendicularity to the origin.*

- d. What were the endpoints of the original normal segment?

$$8 = c - 4 \leftrightarrow c = 12, \quad \text{and} \quad 3 = d - (-2) \leftrightarrow d = 1.$$

*The endpoints of a normal segment to the given line are A(4, -2) and B(12, 1).*

4. A coach is laying out lanes for a race. The lanes are perpendicular to a segment of the track such that one endpoint of the segment is (2, 50) and the other is (20, 65). What are the equations of the lines through the endpoints?

$$y = -\frac{6}{5}x + \frac{786}{15}; y = -\frac{6}{5}x + 89$$



## Lesson 8: Parallel and Perpendicular Lines

### Student Outcomes

- Students recognize parallel and perpendicular lines from slope.
- Students create equations for lines satisfying criteria of the kind: “Contains a given point and is parallel/perpendicular to a given line.”

### Lesson Notes

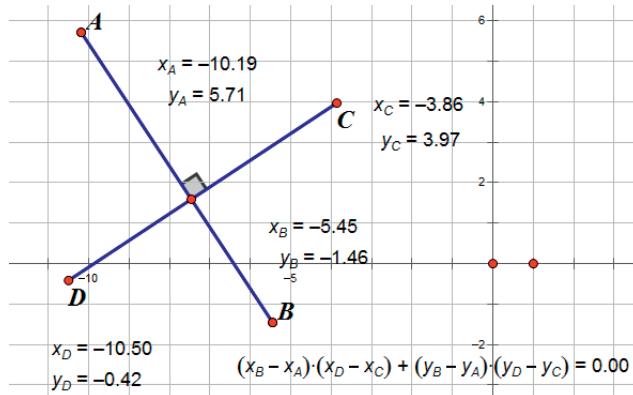
This lesson brings together several of the ideas from the last two lessons. In places where ideas from certain lessons are employed, these are identified with the lesson number underlined (e.g., Lesson 6).

### Classwork

#### Opening (5 minutes)

Students will begin the lesson with the following activity using geometry software to reinforce the theorem studied in Lesson 6, which states that given points  $A(a_1, a_2)$ ,  $B(b_1, b_2)$ ,  $C(c_1, c_2)$ , and  $D(d_1, d_2)$ ,  $\overline{AB} \perp \overline{CD}$  if and only if  $(b_1 - a_1)(d_1 - c_1) + (b_2 - a_2)(d_2 - c_2) = 0$ .

- Construct two perpendicular segments and measure the abscissa (the  $x$ -coordinate) and ordinate (the  $y$ -coordinate) of each of the endpoints of the segments. (You may extend this activity by asking students to determine whether the points used must be the endpoints. This is easily investigated using the dynamic geometry software by creating free moving points on the segment and watching the sum of the products of the differences as the points slide along the segments.)
- Calculate  $(b_1 - a_1)(d_1 - c_1) + (b_2 - a_2)(d_2 - c_2)$ .
- Note the value of the sum and observe what happens to the sum as they manipulate the endpoints of the perpendicular segments.



**Example 1 (10 minutes)**

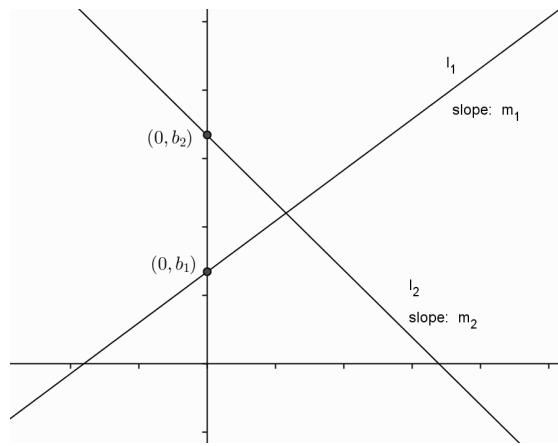
Let  $l_1$  and  $l_2$  be two non-vertical lines in the Cartesian plane.  $l_1$  and  $l_2$  are perpendicular if and only if their slopes are negative reciprocals of each other ([Lesson 7](#)). Explain to students that negative reciprocals also means the product of the slopes is  $-1$ .

PROOF:

Suppose  $l_1$  is given by the equation  $y = m_1x + b_1$ , and  $l_2$  is given by the equation  $y = m_2x + b_2$ . We will start by assuming  $m_1$ ,  $m_2$ ,  $b_1$ , and  $b_2$  are all not zero. We will revisit this proof for cases where one or more of these values is zero in the practice problems.

Let's find two useful points on  $l_1$ :  $A(0, b_1)$  and  $B(1, m_1 + b_1)$ .

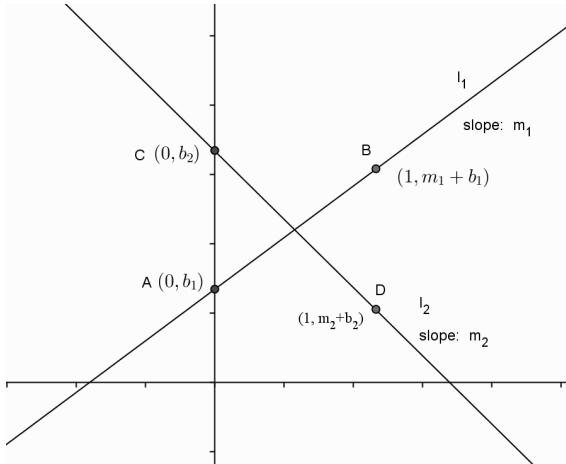
- Why are these points on  $l_1$ ?
  - *A is the y-intercept, and B is the y-intercept plus the slope.*
- Similarly, find two useful points on  $l_2$ . Name them  $C$  and  $D$ .
  - $C(0, b_2)$  and  $D(1, m_2 + b_2)$ .
- Explain how you found them and why they are different from the points on  $l_1$ .
  - *We used the y-intercept of  $l_2$  to find C, then added the slope to the y-intercept to find point D. They are different points because the lines are different, and the lines do not intersect at those points.*
- By the theorem from the Opening Exercise (and [Lesson 6](#)), write the equation that must be satisfied if  $l_1$  and  $l_2$  are perpendicular. Explain your answer



- Take a minute to write your answer, then explain your answer to your neighbor.
- If  $l_1$  is perpendicular to  $l_2$ , then we know from the theorem we studied in Lesson 6 and used in our Opening Exercise that this means
$$(1 - 0)(1 - 0) + (m_1 + b_1 - b_1)(m_2 + b_2 - b_2) = 0$$

$$\Leftrightarrow 1 + m_1 \cdot m_2 = 0$$

$$\Leftrightarrow m_1 m_2 = -1.$$



Summarize this proof and its result to a neighbor ([Lessons 6 and 7](#)).

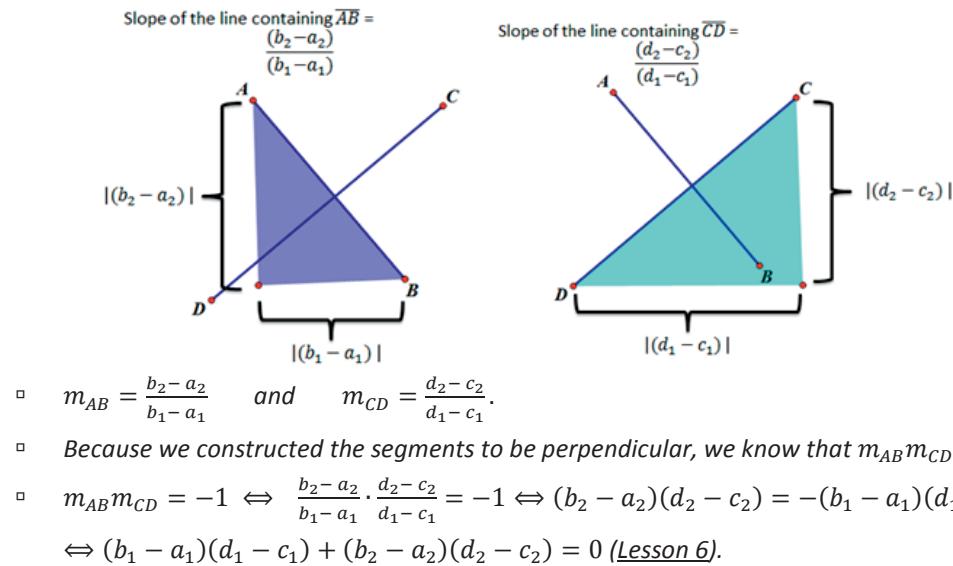
If either  $m_1$  or  $m_2$  are 0, then  $m_1 m_2 = -1$  is false, meaning  $l_1 \perp l_2$  is false, that is they cannot be perpendicular. We will study this case later.

**Discussion (Optional)**

- Working in pairs or groups of three, ask students to use this new theorem to construct an explanation of why  $(b_1 - a_1)(d_1 - c_1) + (b_2 - a_2)(d_2 - c_2) = 0$  when  $\overline{AB} \perp \overline{CD}$  ([Lesson 6](#)).

As the students are constructing their arguments, circulate around the room listening to the progress that is being made with an eye to strategically selecting pairs or groups to share their explanations with the whole group in a manner that will build from basic arguments that may not be fully formed to more sophisticated and complete explanations.

The explanations should include the following understandings:



- $m_{AB} = \frac{b_2 - a_2}{b_1 - a_1}$  and  $m_{CD} = \frac{d_2 - c_2}{d_1 - c_1}$ .
- Because we constructed the segments to be perpendicular, we know that  $m_{AB}m_{CD} = -1$  ([Lesson 7](#)).
- $m_{AB}m_{CD} = -1 \Leftrightarrow \frac{b_2 - a_2}{b_1 - a_1} \cdot \frac{d_2 - c_2}{d_1 - c_1} = -1 \Leftrightarrow (b_2 - a_2)(d_2 - c_2) = -(b_1 - a_1)(d_1 - c_1)$   
 $\Leftrightarrow (b_1 - a_1)(d_1 - c_1) + (b_2 - a_2)(d_2 - c_2) = 0$  ([Lesson 6](#)).

**Exercise 1 (5 minutes)****Exercise 1**

1. a. Write an equation of the line that passes through the origin that intersects the line  $2x + 5y = 7$  to form a right angle.

$$y = \frac{5}{2}x$$

- b. Determine whether the lines given by the equations  $2x + 3y = 6$  and  $y = \frac{3}{2}x + 4$  are perpendicular. Support your answer.

*The slope of the first line is  $-\frac{2}{3}$ , and the slope of the second line is  $\frac{3}{2}$ . The product of these two slopes is  $-1$ ; therefore, the two lines are perpendicular.*

- c. Two lines having the same  $y$ -intercept are perpendicular. If the equation of one of these lines is  $y = -\frac{4}{5}x + 6$ , what is the equation of the second line?

$$y = \frac{5}{4}x + 6$$

**Example 2 (12 minutes)**

In this example, we will study the relationship between a pair of parallel lines and the lines that they are perpendicular to. Students will use the angle congruence axioms developed in Geometry, Module 3 for parallelism.

We are going to investigate two questions in this example.

This example can be done using dynamic geometry software, or students can just sketch the lines and note the angle pair relationships.

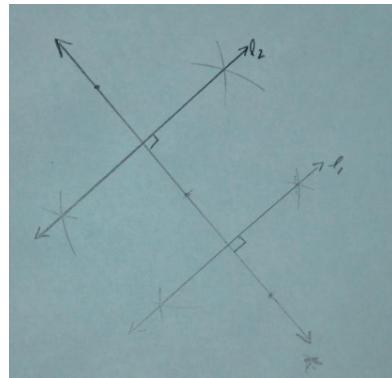
**Example 2**

- a. What is the relationship between two coplanar lines that are perpendicular to the same line?

**Scaffolding:**

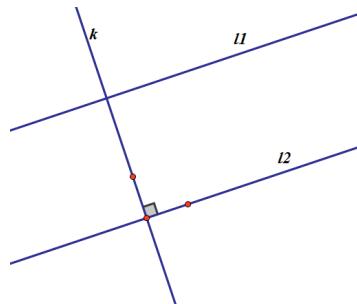
- If students are struggling, change this example to specific lines. Give the coordinates of the two parallel lines and the perpendicular line. Calculate the slopes of each and compare.
- You may also have the students construct two lines that are parallel to a given line using dynamic geometry software and measure and compare their slopes.

- Have students draw a line and label it  $k$ .
- Students will now construct line  $l_1$  perpendicular to line  $k$ .
- Finally, students construct line  $l_2$  not coincident with line  $l_1$ , also perpendicular to line  $k$ .
- What can we say about the relationship between lines  $l_1$  and  $l_2$ ?
  - These lines are parallel because the corresponding angles that are created by the transversal  $k$  are congruent.
- If the slope of line  $k$  is  $m$ , what is the slope of  $l_1$ ? Support your answer. (Lesson 7)
  - The slope of  $l_1$  is  $-\frac{1}{m}$  because the slopes of perpendicular lines are negative reciprocals of each other.
- If the slope of line  $k$  is  $m$ , what is the slope of  $l_2$ ? Support your answer. (Lesson 7)
  - The slope of  $l_2$  is  $-\frac{1}{m}$  because the slopes of perpendicular lines are negative reciprocals of each other.
- Using your answers to the last two questions, what can we say about the slopes of lines  $l_1$  and  $l_2$ ?
  - $l_1$  and  $l_2$  have equal slopes.
- What can be said about  $l_1$  and  $l_2$  if they have equal slopes?
  - $l_1$  and  $l_2$  are parallel.
- What is the relationship between two coplanar lines that are perpendicular to the same line?
  - If two lines are perpendicular to the same line, then the two lines are parallel, and if two lines are parallel, then their slopes are equal.
- Restate this to your partner in your own words and explain it by drawing a picture.

**MP.3**

- b. Given two lines,  $l_1$  and  $l_2$ , with equal slopes and a line  $k$  that is perpendicular to one of these two parallel lines,  $l_1$ :

- What is the relationship between line  $k$  and the other line,  $l_2$ ?
- What is the relationship between  $l_1$  and  $l_2$ ?



Ask students to:

- Construct two lines that have the same slope using construction tools or dynamic geometry software and label the lines  $l_1$  and  $l_2$ .
- Construct a line that is perpendicular to one of these lines and label this line  $k$ .
- If  $l_2$  has a slope of  $m$ , what is the slope of line  $y$ ? Support your answer. ([Lesson 7](#))
  - Line  $k$  must have a slope of  $-\frac{1}{m}$  because line  $k$  is perpendicular to line  $l_2$ , and the slopes of perpendicular lines are negative reciprocals of each other.
- $l_2$  also has a slope of  $m$ , so what is its relationship to line  $k$ ? Support your answer. ([Lesson 7](#))
  - Line  $l_1$  must be perpendicular to line  $k$  because their slopes are negative reciprocals of each other.
- Using your answers to the last two questions, what can we say about two lines that have equal slopes?
  - They must be parallel since they are both perpendicular to the same line.
- Have students combine the results of the two activities to form the following bi-conditional statement:
  - Two non-vertical lines are parallel if and only if their slopes are equal.
- Summarize all you know about the relationships between slope, parallel lines, and perpendicular lines. Share your ideas with your neighbor.

### Exercises 2–7 (8 minutes)

#### Exercises 2–7

2. Given a point  $(-3, 6)$  and a line  $y = 2x - 8$ :
- What is the slope of the line?  
**2**
  - What is the slope of any line parallel to the given line?  
**2**
  - Write an equation of a line through the point and parallel to the line.  
 **$2x - y = -12$**



- d. What is the slope of any line perpendicular to the given line? Explain.

*The slope is  $-\frac{1}{2}$ ; perpendicular lines have slopes that are negative reciprocals of each other.*

3. Find an equation of a line through  $(0, -7)$  and parallel to the line  $y = \frac{1}{2}x + 5$ .

- a. What is the slope of any line parallel to the given line? Explain your answer.

*The slope is  $\frac{1}{2}$ ; lines are parallel if and only if they have equal slopes.*

- b. Write an equation of a line through the point and parallel to the line.

$$x - 2y = 14$$

- c. If a line is perpendicular to  $y = \frac{1}{2}x + 5$ , will it be perpendicular to  $x - 2y = 14$ ? Explain.

*If a line is perpendicular to  $y = \frac{1}{2}x + 5$ , it will also be perpendicular to  $y = \frac{1}{2}x - 7$  because a line perpendicular to one line is perpendicular to all lines parallel to that line.*

4. Find an equation of a line through  $(\sqrt{3}, \frac{1}{2})$  parallel to the line:

a.  $x = -9$

$$x = \sqrt{3}$$

b.  $y = -\sqrt{7}$

$$y = \frac{1}{2}$$

- c. What can you conclude about your answer in parts (a) and (b)?

*They are perpendicular to each other.  $x = \sqrt{3}$  is a vertical line, and  $y = \frac{1}{2}$  is a horizontal line.*

5. Find an equation of a line through  $(-\sqrt{2}, \pi)$  parallel to the line  $x - 7y = \sqrt{5}$ .

$$x - 7y = \sqrt{2} + 7\pi$$

6. Recall that our search robot is moving along the line  $y = 3x - 600$  and wishes to make a right turn at the point  $(400, 600)$ . Find an equation for the perpendicular line on which the robot is to move. Verify that your line intersects the  $x$ -axis at  $(2200, 0)$ .

$$x + 3y = 2200; \text{ when } y = 0, x = 2200$$



7. A robot, always moving at a constant speed of 2 feet per second, starts at position  $(20, 50)$  on the coordinate plane and heads in a south-east direction along the line  $3x + 4y = 260$ . After 15 seconds, it turns left  $90^\circ$  and travels in a straight line in this new direction.

- a. What are the coordinates of the point at which the robot made the turn? What might be a relatively straightforward way of determining this point?

*The coordinates are  $(44, 32)$ . We know the robot moves down 3 units and right 4 units, and the distance moved in 15 seconds is 30 feet. This gives us a right triangle with hypotenuse 5, so we need to move this way 6 times. From  $(20, 50)$ , move a total of down 18 units and right 24 units.*

- b. Find an equation for the second line on which the robot traveled.

$$4x - 3y = 80$$

- c. If, after turning, the robot travels for 20 seconds along this line and then stops, how far will it be from its starting position?

*It will be at the point  $(0, 20)$ , 50 units from its starting position.*

- d. What is the equation of the line the robot needs to travel along in order to now return to its starting position? How long will it take for the robot to get there?

$$x = 20; 10 \text{ seconds}$$

### Closing (2 minutes)

Ask students to respond to these questions in writing, with a partner, or as a class.

- Samantha claims the slopes of perpendicular lines are opposite reciprocals, while Jose says the product of the slopes of perpendicular lines is equal to  $-1$ . Discuss these two claims ([Lesson 7](#)).
  - Both are correct as long as the lines are not horizontal and vertical. If two lines have negative reciprocal slopes,  $m$  and  $-\frac{1}{m}$ , the product  $m \cdot -\frac{1}{m} = -1$ .
  - If one of the lines is horizontal, then the other will be vertical, and the slopes will be zero and undefined, respectively. Therefore, these two claims will not hold.
- How did we use the relationship between perpendicular lines to demonstrate the theorem *two non-vertical lines are parallel if and only if their slopes are equal*, and why did we restrict this statement to non-vertical lines ([Lesson 8](#))?
  - We restricted our discussion to non-vertical lines because the slopes of vertical linear are undefined.
  - We used the fact that the slopes of perpendicular lines are negative reciprocals of each other to show that if two lines are perpendicular to the same line, then they must not only be parallel to each other, but also have equal slopes.

### Exit Ticket (3 minutes)

Teachers can administer the Exit Ticket in several ways, such as the following: assign the entire task; assign some students Problem 1 and others Problem 2; allow student choice.

Name \_\_\_\_\_

Date \_\_\_\_\_

## Lesson 8: Parallel and Perpendicular Lines

### Exit Ticket

1. Are the pairs of lines parallel, perpendicular, or neither? Explain.
  - a.  $3x + 2y = 74$  and  $9x - 6y = 15$
  - b.  $4x - 9y = 8$  and  $18x + 8y = 7$
2. Write the equation of the line passing through  $(-3, 4)$  and perpendicular to  $-2x + 7y = -3$ .



## Exit Ticket Sample Solutions

1. Are the pairs of lines parallel, perpendicular, or neither? Explain.

a.  $3x + 2y = 74$  and  $9x - 6y = 15$

*The lines are neither, and the slopes are  $-\frac{3}{2}$  and  $\frac{3}{2}$ .*

b.  $4x - 9y = 8$  and  $18x + 8y = 7$

*The lines are perpendicular, and the slopes are  $\frac{4}{9}$  and  $-\frac{9}{4}$ .*

2. Write the equation of the line passing through  $(-3, 4)$  and normal to  $-2x + 7y = -3$ .

$7x + 2y = -13$

## Problem Set Sample Solutions

1. Write the equation of the line through  $(-5, 3)$  and:

- a. Parallel to  $x = -1$ .

$x = -5$

- b. Perpendicular to  $x = -1$ .

$y = 3$

- c. Parallel to  $y = \frac{3}{5}x + 2$ .

$3x - 5y = -30$

- d. Perpendicular to  $y = \frac{3}{5}x + 2$ .

$5x + 3y = -16$

2. Write the equation of the line through  $(\sqrt{3}, \frac{5}{4})$  and:

- a. Parallel to  $y = 7$ .

$y = \frac{5}{4}$

- b. Perpendicular to  $y = 7$ .

$x = \sqrt{3}$

- c. Parallel to  $\frac{1}{2}x - \frac{3}{4}y = 10$ .

$8x + 12y = 15 + 8\sqrt{3}$

- d. Perpendicular to  $\frac{1}{2}x - \frac{3}{4}y = 10$ .

$$6x - 4y = -5 + 6\sqrt{3}$$

3. A vacuum robot is in a room and charging at position  $(0, 5)$ . Once charged, it begins moving on a northeast path at a constant speed of  $\frac{1}{2}$  foot per second along the line  $4x - 3y = -15$ . After 60 seconds, it turns right  $90^\circ$  and travels in the new direction.

- a. What are the coordinates of the point at which the robot made the turn?

$$(18, 29)$$

- b. Find an equation for the second line on which the robot traveled.

$$3x + 4y = 194$$

- c. If after turning, the robot travels 80 seconds along this line, how far has it traveled from its starting position?

50 feet

- d. What is the equation of the line the robot needs to travel along in order to return and recharge? How long will it take the robot to get there?

$$y = 50; 100 \text{ seconds}$$

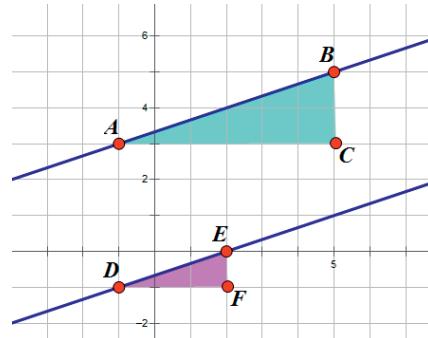
4. Given the statement  $\overleftrightarrow{AB}$  is parallel to  $\overleftrightarrow{DE}$ , construct an argument for or against this statement using the two triangles shown.

*The statement  $\overleftrightarrow{AB}$  is parallel to  $\overleftrightarrow{DE}$  is true.*

*The slope of  $\overleftrightarrow{AB}$  is equal to  $\frac{2}{6}$ , the ratio of the lengths of the legs of the right triangle ABC.*

*The slope of  $\overleftrightarrow{DE}$  is equal to  $\frac{1}{3}$ , the ratio of the lengths of the legs of the right triangle DEF.*

$\frac{2}{6} = \frac{1}{3} \leftrightarrow m_{AB} = m_{DE}$ , and therefore,  $\overleftrightarrow{AB}$  is parallel to  $\overleftrightarrow{DE}$ .





5. Recall the proof we did in Example 1: Let  $l_1$  and  $l_2$  be two non-vertical lines in the Cartesian plane.  $l_1$  and  $l_2$  are perpendicular if and only if their slopes are negative reciprocals of each other. In class, we looked at the case where both  $y$ -intercepts were not zero. In Lesson 5, we looked at the case where both  $y$ -intercepts were equal to zero, when the vertex of the right angle was at the origin. Reconstruct the proof for the case where one line has a  $y$ -intercept of zero, and the other line has a non-zero  $y$ -intercept.

*Suppose  $l_1$  passes through the origin; then, it will be given by the equation  $y = m_1x$ , and  $l_2$  is given by the equation  $y = m_2x + b_2$ . Here, we are assuming  $m_1$ ,  $m_2$ , and  $b_2$  are not zero.*

*We can still use the same two points on  $l_2$  as we used in Example 1,  $(0, b_2)$  and  $\left(-\frac{b_2}{m_2}, 0\right)$ . We will have to find an additional point to use on  $l_2$ , as the  $x$ - and  $y$ -intercepts are the same point because this line passes through the origin,  $(0, 0)$ . Let's let our second point be  $(1, m_2)$ .*

$$\begin{aligned} l_1 \perp l_2 &\leftrightarrow (1 - 0) \left( 0 - \left( -\frac{b_2}{m_2} \right) \right) + (m_2 - 0)(b_2 - 0) = 0 \\ &\leftrightarrow \frac{b_2}{m_2} + m_2 b_2 = 0 \\ &\leftrightarrow \frac{b_2}{m_2} = -m_2 b_2 \\ &\leftrightarrow -\frac{1}{m_1} = m_2 \text{ since } b_2 \neq 0 \end{aligned}$$

6. Challenge: Reconstruct the proof we did in Example 1 if one line has a slope of zero.

*If  $m_1 = 0$ , then  $l_1$  is given by the equation  $y = b_1$ , and  $l_2$  is given by the equation  $y = m_2x + b_2$ . Here we are assuming  $m_2$ ,  $b_1$ , and  $b_2$  are not zero.*

*Choosing the points  $(0, b_1)$  and  $(1, b_1)$  on  $l_1$  and  $(0, b_2)$  and  $(1, m_2 + b_2)$  on  $l_2$ ,*

$$\begin{aligned} l_1 \perp l_2 &\leftrightarrow (1 - 0)(1 - 0) + (b_1 - b_1)(m_2 + b_2 - b_2) = 0 \\ &\leftrightarrow 1 + 0 = 0 \\ &\leftrightarrow 1 = 0. \text{ This is a false statement. If one of the perpendicular lines is horizontal, their slopes cannot be negative reciprocals.} \end{aligned}$$

### Scaffolding:

- Provide students who were absent or who do not take notes well with class notes showing Example 1 with steps clearly explained.

Name \_\_\_\_\_

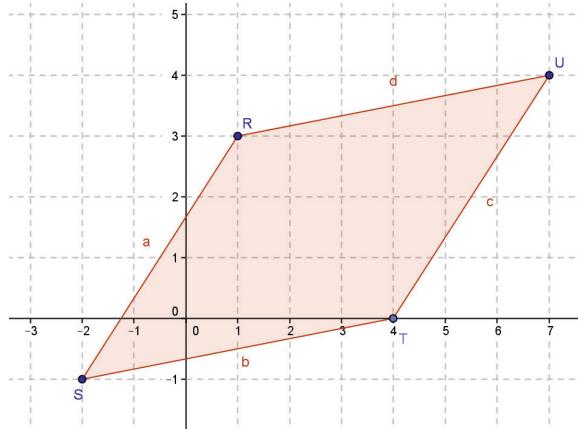
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For problems that require rounding, round answers to the nearest hundredth unless otherwise stated.

1. Given parallelogram  $RSTU$  with vertices  $R(1, 3)$ ,  $S(-2, -1)$ ,  $T(4, 0)$ , and  $U(7, 4)$ :

- a. Find the perimeter of the parallelogram; round to the nearest hundredth.

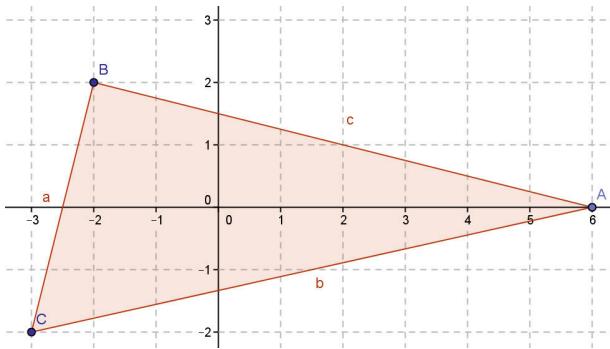
- b. Find the area of the parallelogram.



2. Given triangle  $ABC$  with vertices  $A(6, 0)$ ,  $B(-2, 2)$ , and  $C(-3, -2)$ :

- a. Find the perimeter of the triangle, round to the nearest hundredth.

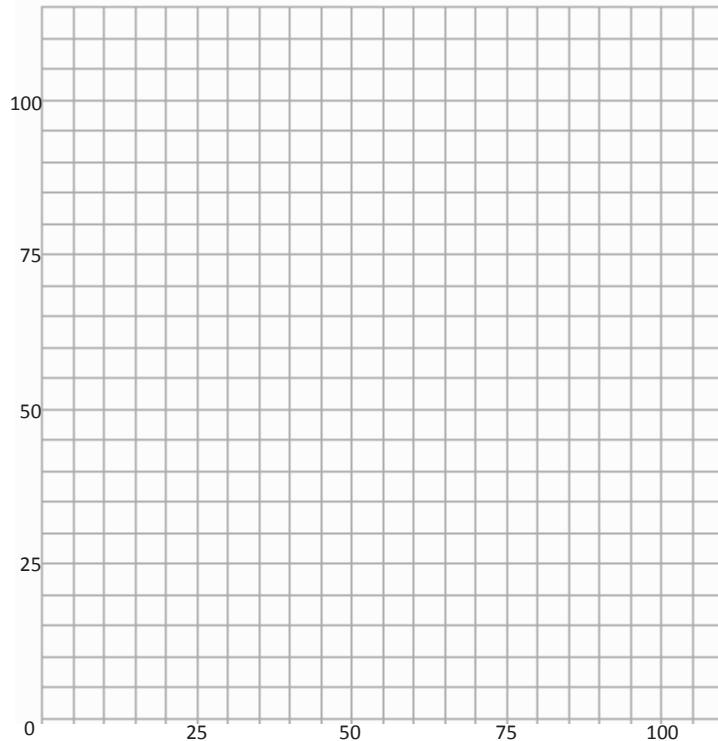
- b. Find the area of the triangle.



3. You are a member of your school's robotics team and are in charge of programming a robot that will pick up ping pong balls. The competition arena is a rectangle with height 90 feet and width 95 feet.

On graph paper, you sketch the arena as a square on the coordinate plane with sides that are parallel to the coordinate axes and with the southwest corner of the arena set at the origin. Each unit width on the paper grid corresponds to 5 feet of length of the arena. You initially set the robot to move along a straight path at a constant speed. In the sketch, the robot's position corresponds to the point  $(10, 30)$  in the coordinate plane at time  $t = 2$  seconds and to the point  $(40, 75)$  at time  $t = 8$  seconds.

- a. Sketch the arena on the graph paper below, and write a system of inequalities that describes the region in the sketch.



- b. Show that at the start, that is, at time  $t = 0$ , the robot was located at a point on the west wall of the arena. How many feet from the southwest corner was it?

- c. What is the speed of the robot? Round to the nearest whole number.

- d. Write down an equation for the line along which the robot moves.

- e. At some time the robot will hit a wall. Which wall will it hit? What are the coordinates of that point of impact?
- f. How far does the robot move between time  $t = 0$  seconds and the time of this impact? What is the time for the impact? Round distance to the nearest hundredth and time to the nearest second.

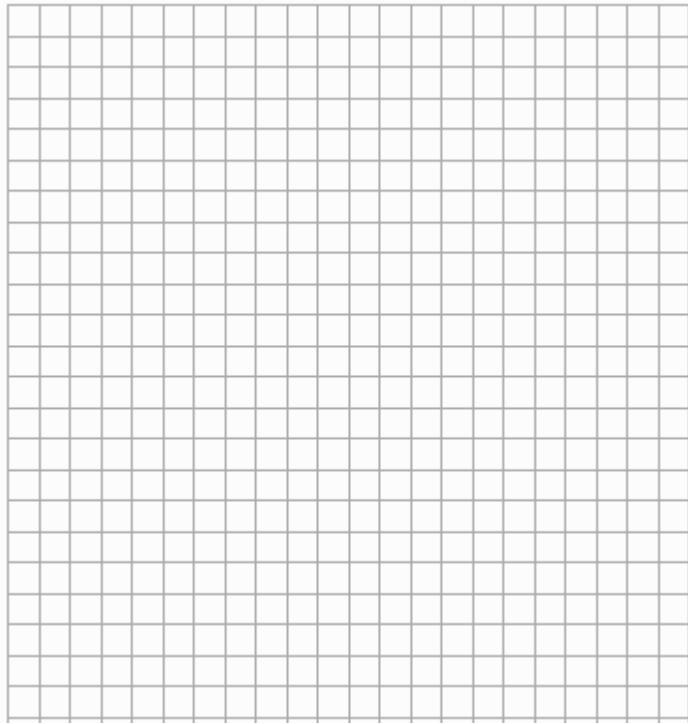
At the time of impact, you have the robot come to a gentle halt, and then turn and head in a direction perpendicular to the wall. Just as the robot reaches the opposite wall, it gently halts, turns, and then returns to start. (We are assuming that the robot does not slow down when it hits a wall.) The robot thus completes of journey composed of three line segments forming a triangle within the arena. Sketch the path of the robot's motion.

- g. What are the coordinates where it hits the east wall?
- h. What is the perimeter of that triangle rounded to the nearest hundredth?
- i. What is the area of the triangle rounded to the nearest tenth?
- j. If the count of ping-pong balls in the arena is large and the balls are spread more or less evenly across the whole arena, what approximate percentage of balls do you expect to lie within the triangle the robot traced? (Assume the robot encountered no balls along any legs of its motion.)

4. Consider the triangular region in the plane given by the triangle  $(1, 6)$ ,  $(6, -1)$ , and  $(1, -4)$ .

a. Sketch the region and write a system of inequalities to describe the region bounded by the triangle.

b. The vertical line  $x = 3$  intersects this region. What are the coordinates of the two boundary points it intersects? What is the length of the vertical segment within the region between these two boundary points?



c. The line  $x = 3$  divides the region into a quadrilateral and a triangle. Find the perimeter of the quadrilateral and the area of the triangle.

5. Is triangle  $RST$ , where  $R(1, 5)$ ,  $S(5, 1)$ ,  $T(-1, -1)$ , a right triangle? If so, which angle is the right angle? Justify your answer.

6. Consider the points  $A(-1, 3)$  and  $B(6, 2)$  in the coordinate plane. Let  $O(0, 0)$  be the origin.
- Find the coordinates of a point,  $C$ , away from the origin on the line  $y = x$  that make triangle  $ABC$  a right triangle with right angle at  $C$ .
  - Find the coordinates of a point,  $D$ , on the line  $y = x$  that make triangle  $OBD$  a right triangle with right angle at  $B$ .
7. Consider the quadrilateral with vertices  $(-2, -1)$ ,  $(2, 2)$ ,  $(5, -2)$ , and  $(1, -5)$ .
- Show that the quadrilateral is a rectangle.
  - Is the quadrilateral a square? Explain.
  - What is the area of the quadrilateral?

- d. What is the area of the region of the quadrilateral that lies to the right of the  $y$ -axis?
- e. What is the equation of the perpendicular bisector of the side of the quadrilateral that lies in the fourth quadrant?
8. Using the general formula for perpendicularity of segments with one endpoint at the origin, determine if the segments from the given points to the origin are perpendicular.
- $(4, 10), (5, -2)$
  - $(-7, 0), (0, -4)$
  - Using the information from part (a), are the segments through the points  $(-3, -2), (1, 8)$ , and  $(2, -4)$  perpendicular? Explain.

9. Write the equation of the line that contains the point  $(-2, 7)$  and is

- a. Parallel to  $x = 3$ .
- b. Perpendicular to  $x = -3$ .
- c. Parallel to  $y = 6x - 13$ .
- d. Perpendicular to  $y = 6x - 13$ .

10. Line  $A$  contains points  $(p - 4, 2)$  and  $(-2, 9)$ . Line  $B$  contains points  $(p, -1)$  and  $(-1, 1)$ .

- a. Find the value of  $p$  if the lines are parallel.
- b. Find the value(s) of  $p$  if the lines are perpendicular.

A Progression Toward Mastery					
Assessment Task Item		STEP 1 Missing or incorrect answer and little evidence of reasoning or application of mathematics to solve the problem.	STEP 2 Missing or incorrect answer but evidence of some reasoning or application of mathematics to solve the problem.	STEP 3 A correct answer with some evidence of reasoning or application of mathematics to solve the problem, or an incorrect answer with substantial evidence of solid reasoning or application of mathematics to solve the problem.	STEP 4 A correct answer supported by substantial evidence of solid reasoning or application of mathematics to solve the problem.
1	<b>a</b> <b>G-GPE.B.7</b>	Student gives no answer, or an incorrect answer is given with no supporting work.	Student gives an incorrect answer but work shows use of the distance formula.	Student gives correct answer with supporting work but does not round to the hundredths.	Student gives correct answer with supporting work and answer is properly rounded.
		<b>b</b> <b>G-GPE.B.7</b>	Student gives no answer, or an incorrect answer is given with no supporting work.	Student attempts to use a formula or decomposition to find area but makes many errors leading to an incorrect answer.	Student shows knowledge of using a formula or decomposition to find error but makes minor errors leading to an incorrect answer.
2	<b>a</b> <b>G-GPE.B.7</b>	Student gives no answer, or an incorrect answer is given with no supporting work.	Student gives an incorrect answer but work shows use of the distance formula.	Student gives a correct answer with supporting work but not rounded to the hundredths.	Student gives correct answer with supporting work, and it is properly rounded.
		<b>b</b> <b>G-GPE.B.7</b>	Student gives no answer, or an incorrect answer is given with no supporting work.	Student attempts to use a formula or decomposition to find area but makes many errors leading to an incorrect answer.	Student shows knowledge of using a formula or decomposition to find error but makes minor errors leading to an incorrect answer.
3	<b>a</b> <b>G-GPE.B.5</b>	Student does not sketch the region nor is the system of inequalities written.	Student sketches the region correctly, but the system of inequalities is either not written or are both incorrect.	Student sketches the region correctly, and one inequality is written correctly.	Student sketches the region correctly, and the system of inequalities is correct.

<b>b</b> <b>G-GPE.B.5</b>	Student gives no answer, or an incorrect answer is given with no supporting work.	Student attempts to use an up and over triangle to determine the starting position, but the position is incorrect.	Student shows correct work to determine starting point but transposes the coordinates (15, 0).	Student has accurate work and the correct starting position.
<b>c</b> <b>G-GPE.B.5</b>	Student does not calculate distance traveled nor the speed, or both are calculated incorrectly.	Student calculates the distance correctly but does not divide by time to arrive at speed.	Student calculates the speed correctly, but units are incorrect or answer is incorrectly rounded.	Student calculates the speed correctly with appropriate units and rounding.
<b>d</b> <b>G-GPE.B.5</b>	Student gives no answer, or an incorrect answer is given with no supporting work.	Student calculates slope correctly, but does not attempt to use a point to write the equation of the line.	Student calculates slope correctly and uses a point to write the equation of the line but makes minor errors.	Student calculates the equation of the line of motion correctly.
<b>e</b> <b>G-GPE.B.5</b>	Student gives no answer, or an incorrect answer with no supporting work is given.	Student realizes the robot will hit the wall at $y = 90$ but finds the $x$ -coordinate incorrectly.	Student uses the equation from part (d) and the $y$ -value of 90 to find the $x$ -coordinate but makes mathematical errors.	Student identifies the correct point of intersection with supporting work.
<b>f</b> <b>G-GPE.B.5</b>	Student has no answer or an incorrect answer with no supporting work.	Student uses the distance formula to find distance but with the wrong set of points.	Student finds the correct distance but does not find time or does not round correctly.	Student finds the correct distance and time of impact, rounding correctly.
<b>g</b> <b>G-GPE.B.5</b>	Student gives no answer, or an incorrect answer with no supporting work is given.	Student realizes the robot will hit the wall at $x = 95$ but has an incorrect $y$ -coordinate with no supporting work or no $y$ -coordinate.	Student uses $x = 95$ and shows proper work to find the $y$ -coordinate but makes a slight error.	Student identifies the correct location of impact with supporting work.
<b>h</b> <b>G-GPE.B.7</b>	Student gives no answer, or an incorrect answer is given with no supporting work.	Student gives an incorrect answer but work shows use of the distance formula using correct points.	Student gives a correct answer with supporting work but not rounded to the hundredths.	Student gives a correct answer with supporting work that is properly rounded.
<b>i</b> <b>G-GPE.B.7</b>	Student gives no answer, or an incorrect answer is given with no supporting work.	Student attempts to use a formula or decomposition to find area with correct points but made many errors leading to an incorrect answer.	Student shows knowledge of using a formula or decomposition to find error but made minor errors leading to an incorrect answer.	Student has correct answer with accurate supporting work.

	<b>j</b> <b>G-GPE.B.7</b>	Student gives no answer, or an incorrect answer given with no supporting work.	Student attempts to set up a ratio of areas but with major errors.	Student set up a ratio of correct areas but did not round correctly and/or did not write as a percent.	Student calculated the percent of balls and rounded correctly with accurate supporting work.
<b>4</b>	<b>a</b> <b>G-GPE.B.7</b>	Student does not sketch the region, nor is the system of inequalities written.	The student sketches the region correctly, but the system of inequalities is either not written or are both incorrect.	The student sketches the region correctly, and one inequality is written correctly.	The student sketches the region correctly, and the system of inequalities is correct.
	<b>b</b> <b>G-GPE.B.7</b>	Student gives no answer, or an incorrect answer is given with no supporting work.	Student shows knowledge of solving a system of equations to find the intersection points but makes errors and does not find segment length.	Student shows knowledge of solving a system of equations to find the intersection points and uses those points to find the segment length, but both are incorrect.	Student finds the correct points of intersection and the length of the segment.
	<b>c</b> <b>G-GPE.B.7</b>	Student gives no answer, or an incorrect answer is given with no supporting work.	Student shows the correct formulas or methods to find the area and perimeter but both are calculated incorrectly.	Student shows the correct formulas or methods to find area and perimeter and only one is calculated correctly.	Student shows the correct formulas or methods to find area and perimeter and both are calculated correctly.
<b>5</b>	<b>G-GPE.B.4</b>	Student gives no answer, or an incorrect answer is given with no supporting work.	Student uses either slopes or the Pythagorean theorem to determine if the triangle is right but makes errors leading to an incorrect answer.	Student uses either slopes or the Pythagorean theorem and determines that the triangle is right but does not or incorrectly identifies the right angle.	Student uses either slopes or the Pythagorean theorem and determines that the triangle is right and correctly identifies the right angle.
<b>6</b>	<b>a</b> <b>G-GPE.B.5</b>	Student gives no answer, or an incorrect answer is given with no supporting work.	Student shows some knowledge of determining perpendicularity but does not consider the line $y = x$ .	Student shows knowledge of perpendicularity and uses the equation of the line $y = x$ but finds the wrong point.	Student shows knowledge of perpendicularity, uses the equation of the line $y = x$ , and identifies the correct point.
	<b>b</b> <b>G-GPE.B.5</b>	Student gives no answer, or an incorrect answer is given with no supporting work.	Student shows some knowledge of determining perpendicularity but does not consider the line $y = x$ .	Student shows knowledge of perpendicularity and uses the equation of the line $y = x$ but finds the wrong point.	Student shows knowledge of perpendicularity, uses the equation of the line $y = x$ , and identifies the correct point.

7	<b>a</b> <b>G-GPE.B.4</b>	Student gives no answer, or an incorrect answer is given with no supporting work.	Student shows knowledge of the properties of a rectangle and attempts to prove both pairs of opposite sides are parallel or the quadrilateral has four right angles but makes errors leading to an incorrect answer.	Student shows knowledge of the properties of a rectangle and proves both pairs of opposite sides are parallel or the quadrilateral has four right angles but does not explain the reasoning.	Student shows knowledge of the properties of a rectangle and proves both pairs of opposite sides are parallel or the quadrilateral has four right angles and writes an explanation of reasoning.
	<b>b</b> <b>G-GPE.B.4</b>	Student gives no answer, or an incorrect answer is given with no supporting work.	Student shows knowledge of the properties of a square and finds the lengths of the sides but makes errors leading to an incorrect answer.	Student shows knowledge of the properties of a square and proves all sides are equal in length but does not explain the reasoning.	Student shows knowledge of the properties of a square, proves all sides are equal in length, and writes an explanation of reasoning.
	<b>c</b> <b>G-GPE.B.7</b>	Student gives no answer, or an incorrect answer is given with no supporting work.	Student attempts to use a formula or decomposition to find the area but makes major errors.	Student uses a formula or decomposition to find the area but makes a small error.	Student finds the correct area with accurate supporting work.
	<b>d</b> <b>G-GPE.B.7</b>	The student gives no answer, or an incorrect answer is given with no supporting work.	Student attempts to use a formula or decomposition to find the area but makes major errors.	Student uses a formula or decomposition to find the area but makes a small error.	Student finds the correct area with accurate supporting work.
	<b>e</b> <b>G-GPE.B.5</b>	The student gives no answer, or an incorrect answer is given with no supporting work.	Student attempts to find the midpoint and slope of the perpendicular bisector, but both are incorrect.	Student finds the correct slope of the perpendicular bisector and the midpoint but does not write an equation.	Student writes the correct equation of the perpendicular bisector with accurate supporting work.
	<b>a</b> <b>G-GPE.B.5</b>	The student gives no answer, or an incorrect answer is given with no supporting work.	Student attempts to use the general formula for perpendicularity but uses it incorrectly.	Student uses the general formula for perpendicularity but makes a slight error leading to an incorrect answer.	Student uses the general formula for perpendicularity correctly and arrives at the correct answer.
8	<b>b</b> <b>G-GPE.B.5</b>	The student gives no answer, or an incorrect answer is given with no supporting work.	Student attempts to use the general formula for perpendicularity but uses it incorrectly.	Student uses the general formula for perpendicularity but makes a slight error leading to an incorrect answer.	Student uses the general formula for perpendicularity correctly and arrives at the correct answer.

	<b>c</b> <b>G-GPE.B.5</b>	The student gives no answer, or an incorrect answer is given with no supporting work.	Student attempts to use the general formula for perpendicularity but uses it incorrectly and does not use part (a) at all.	Student uses the general formula for perpendicularity and gets the correct answer, but does not use part (a).	Student shows that the segments are perpendicular because they are translations of the segments in part (a).
<b>9</b>	<b>a</b> <b>G-GPE.B.5</b>	The student gives no answer, or an incorrect answer is given with no supporting work.	Student writes the equation of the perpendicular line, $y = 7$ .	Student writes a parallel equation but with a wrong constant (i.e., $x = 4$ ).	Student writes the correct equation of the parallel line.
	<b>b</b> <b>G-GPE.B.5</b>	The student gives no answer, or an incorrect answer is given with no supporting work.	Student writes the equation of the parallel line, $x = -2$ .	Student writes a perpendicular equation, but with a wrong constant (i.e., $y = 4$ ).	Student writes the correct equation of the perpendicular line.
	<b>c</b> <b>G-GPE.B.5</b>	The student gives no answer, or an incorrect answer is given with no supporting work.	Student writes the equation of the perpendicular line.	Student uses the correct slope but the wrong point or no point to write the equation of the parallel line.	Student writes the correct equation of the parallel line.
	<b>d</b> <b>G-GPE.B.5</b>	The student gives no answer, or an incorrect answer is given with no supporting work.	Student writes the equation of the parallel line.	Student uses the correct slope but the wrong point or no point to write the equation of the perpendicular line.	Student writes the correct equation of the perpendicular line.
<b>10</b>	<b>a</b> <b>G-GPE.B.5</b>	Student gives no answer, or an incorrect answer is given with no supporting work.	Student attempts to use the general formula for parallel slopes but uses it incorrectly.	Student uses the general formula for parallel slopes but makes a slight error leading to an incorrect answer.	Student uses the general formula for parallel slopes correctly and arrives at the correct answer.
	<b>b</b> <b>G-GPE.B.5</b>	Student gives no answer, or an incorrect answer is given with no supporting work.	Students attempts to use the general formula for perpendicular slopes but uses it incorrectly.	Student uses the general formula for perpendicular slopes but makes a slight error leading to an incorrect error or provides one correct and one incorrect answer.	Students uses the general formula for perpendicular slopes correctly and arrives at both correct answers.

Name \_\_\_\_\_

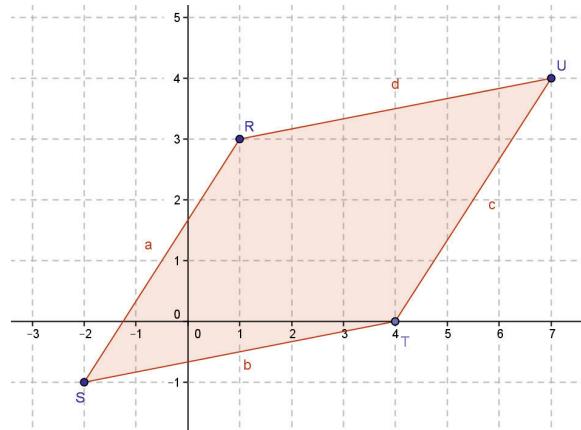
Date \_\_\_\_\_

For problems that require rounding, round answers to the nearest hundredth unless otherwise stated.

1. Given parallelogram  $RSTU$  with vertices  $R(1, 3)$ ,  $S(-2, -1)$ ,  $T(4, 0)$ , and  $U(7, 4)$ :

- a. Find the perimeter of the parallelogram; round to the nearest hundredth.

*22.17 units*



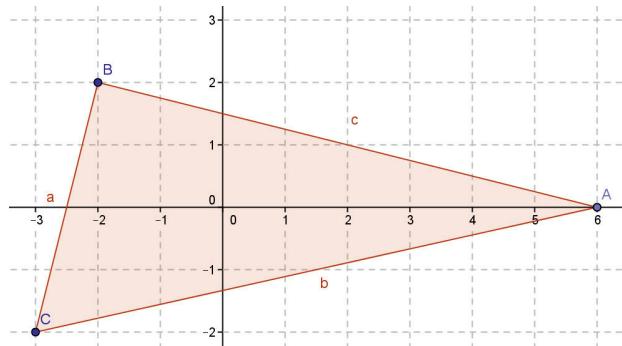
- b. Find the area of the parallelogram.

*18 square units*

2. Given triangle  $ABC$  with vertices  $A(6, 0)$ ,  $B(-2, 2)$ , and  $C(-3, -2)$ :

- a. Find the perimeter of the triangle; round to the nearest hundredth.

*21.60 units*



- b. Find the area of the triangle.

*17 square units*

3. You are a member of your school's robotics team and are in charge of programming a robot that will pick up ping pong balls. The competition arena is a rectangle with height 90 feet and width 95 feet.

On graph paper, you sketch the arena as a square on the coordinate plane with sides that are parallel to the coordinate axes and with southwest corner of the arena set at the origin. Each unit width on the paper grid corresponds to 5 feet of length of the arena. You initially set the robot to move along a straight path at a constant speed. In the sketch, the robot's position corresponds to the point  $(10, 30)$  in the coordinate plane at time  $t = 2$  seconds and to the point  $(40, 75)$  at time  $t = 8$  seconds.

- a. Sketch the arena on the graph paper below, and write a system of inequalities that describes the region in the sketch.

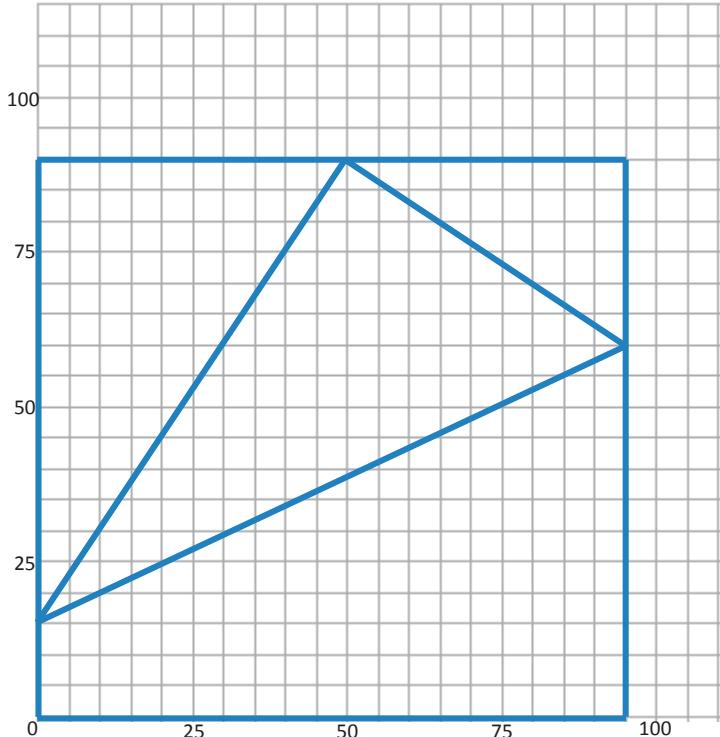
$$0 \leq x \leq 95$$

$$0 \leq y \leq 90$$

- b. Show that at the start, that is, at time  $t = 0$ , the robot was located at a point on the west wall of the arena. How many feet from the southwest corner was it?

$(0, 15)$ , 15 feet North

Each second, the robot travels 5 feet right and 7.5 feet up. If at 2 seconds the robot is at  $(10, 30)$ , so moving left 10 and down 15 brings it to  $(0, 15)$ .



- c. What is the speed of the robot? Round to the nearest whole number.

9 feet/second

- d. Write down an equation for the line along which the robot moves.

$$y = \frac{3}{2}x + 15$$

- e. At some time the robot will hit a wall. Which wall will it hit? What are the coordinates of that point of impact?

North (top) wall at (50, 90)

- f. How far does the robot move between time  $t = 0$  seconds and the time of this impact? What is the time for the impact? Round distance to the nearest hundredth and time to the nearest second.

Distance of 90.14 feet at time 10 seconds.

At the time of impact, you have the robot come to a gentle halt, and then turn and head in a direction perpendicular to the wall. Just as the robot reaches the opposite wall, it gently halts, turns, and then returns to start. (We assume that the robot does not slow down when it hits the wall.) The robot thus completes a journey composed of three line segments forming a triangle within the arena. Sketch the path of the robot's motion.

- g. What are the coordinates where it hits the east wall?

(95, 60)

- h. What is the perimeter of that triangle; round to the nearest hundredth?

249.34 feet

- i. What is the area of the triangle; round to the nearest hundredth?

2437.5 square feet

- j. If the count of ping-pong balls in the arena is large and the balls are spread more or less evenly across the whole arena, what approximate percentage of balls do you expect to lie within the triangle the robot traced? (Assume the robot encountered no balls along any legs of its motion.)

28.57%

4. Consider the triangular region in the plane given by the triangle  $(1, 6)$ ,  $(6, -1)$ , and  $(1, -4)$ .

- a. Sketch the region and write a system of inequalities to describe the region bounded by the triangle.

$$x \geq 1$$

$$7x + 5y \geq 37$$

$$3x - 5y \leq 23$$

- b. The vertical line  $x = 3$  intersects this region. What are the coordinates of the two boundary points it intersects? What is the length of the vertical segment within the region between these two boundary points?

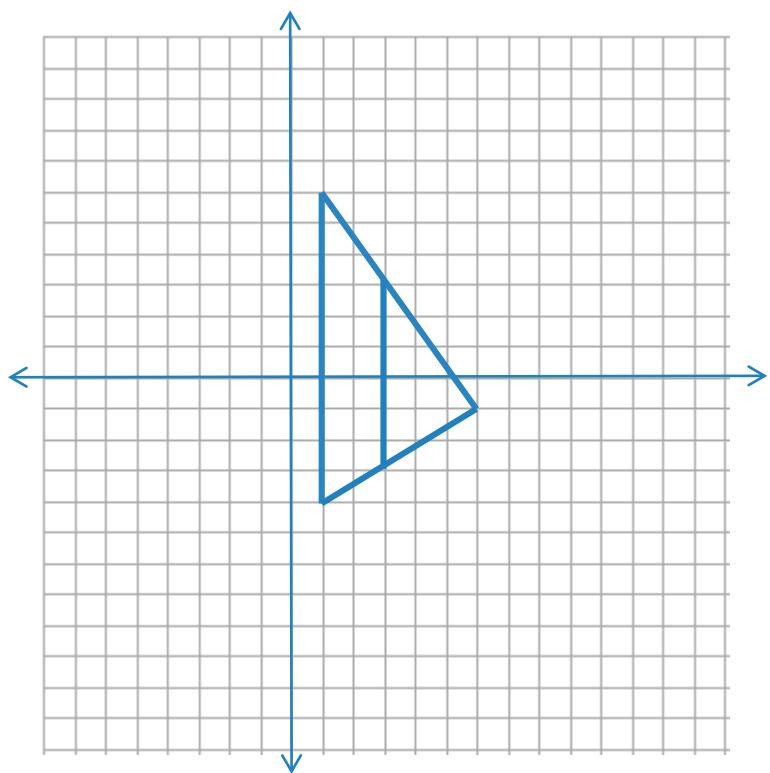
$$\left(3, \frac{16}{5}\right), \left(3, -\frac{14}{5}\right)$$

Length of segment is 6 units.

- c. The line  $x = 3$  divides the region into a quadrilateral and a triangle. Find the perimeter of the quadrilateral and the area of the triangle.

Perimeter of quadrilateral is 21.84 units.

Area of triangle is 9 square units.



5. Is triangle  $RST$ , where  $R(1, 5)$ ,  $S(5, 1)$ ,  $T(-1, -1)$ , a right triangle? If so, which angle is the right angle? Justify your answer.

The triangle is not a right triangle. The slopes of the segments are not negative reciprocals ( $-1, 3$ , and  $\frac{1}{3}$ ), so none of the segments are perpendicular. The converse of the Pythagorean theorem is not true because the side lengths are  $\sqrt{40}$ ,  $\sqrt{40}$ , and  $\sqrt{36}$ .

6. Consider the points  $A(-1, 3)$  and  $B(6, 2)$  in the coordinate plane. Let  $O(0, 0)$  be the origin.
- Find the coordinates of a point,  $C$ , away from the origin on the line  $y = x$  that makes triangle  $ABC$  a right triangle with right angle at  $C$ .

(5, 5)

- Find the coordinates of a point,  $D$ , on the line  $y = x$  that makes triangle  $OB$  $D$  a right triangle with right angle at  $B$ .

(5, 5)

7. Consider the quadrilateral with vertices  $(-2, -1)$ ,  $(2, 2)$ ,  $(5, -2)$ , and  $(1, -5)$ .

- Show that the quadrilateral is a rectangle.

Both pairs of opposite sides of the quadrilateral are parallel (slopes  $\frac{3}{4}$  and  $-\frac{4}{3}$ ) and the slopes are negative reciprocals so the sides are perpendicular to each other, meaning all angles are right angles.

- Is the quadrilateral a square? Explain.

All sides have an equal length of 5 units. A rectangle with all sides of equal lengths is a square.

- What is the area of the quadrilateral?

Area is 25 square units.

- d. What is the area of the region of the quadrilateral that lies to the right of the  $y$ -axis?

Area is  $\frac{14}{3}$  square units.

- e. What is the equation of the perpendicular bisector of the side of the quadrilateral that lies in the fourth quadrant?

$$y = -\frac{4}{3}x + \frac{1}{2}$$

8. Using the general formula for perpendicularity of segments with one endpoint at the origin, determine if the segments from the given points to the origin are perpendicular.

- a.  $(4, 10), (5, -2)$

$4 \cdot 5 + 10 \cdot (-2) = 0$ , the segments are perpendicular.

- b.  $(-7, 0), (0, -4)$

$-7 \cdot 0 + 0 \cdot (-4) = 0$ ; the segments are perpendicular.

- c. Using the information from part (a), are the segments through the points  $(-3, -2), (1, 8)$ , and  $(2, -4)$  perpendicular? Explain.

The segments are perpendicular because they are a translation (right 3, up 2) of the segments in part (a) that are perpendicular.

9. Write the equation of the line that contains the point  $(-2, 7)$  and is:

- a. Parallel to  $x = 3$ .

$$x = -2$$

- b. Perpendicular to  $x = -3$ .

$$y = 7$$

- c. Parallel to  $y = 6x - 13$ .

$$y = 6x + 19$$

- d. Perpendicular to  $y = 6x - 13$ .

$$y = -\frac{1}{6}x + \frac{20}{3}$$

10. Line A contains points  $(p - 4, 2)$  and  $(-2, 9)$ . Line B contains points  $(p, -1)$  and  $(-1, 1)$ .

- a. Find the value of  $p$  if the lines are parallel.

$$p = -\frac{11}{5}$$

- b. Find the value(s) of  $p$  if the lines are perpendicular.

$$p = 4, p = -3$$



## Topic C:

## Perimeters and Areas of Polygonal Regions in the Cartesian Plane

**G-GPE.B.7**

**Focus Standard:** G-GPE.B.7 Use coordinates to compute perimeters of polygons and areas of triangles and rectangles, e.g., using the distance formula.\*

**Instructional Days:** 3

**Lesson 9:** Perimeter and Area of Triangles in the Cartesian Plane (P)<sup>1</sup>

**Lesson 10:** Perimeter and Area of Polygonal Regions in the Cartesian Plane (S)

**Lesson 11:** Perimeters and Areas of Polygonal Regions Defined by Systems of Inequalities (P)

Lesson 9 begins Topic C with students finding the perimeter of triangular regions using the distance formula and deriving the formula for the area of a triangle with vertices  $(0, 0), (x_1, y_1), (x_2, y_2)$  as

$A = \frac{1}{2}|x_1y_2 - x_2y_1|$  (**G-GPE.B.7**). Students are introduced to the “shoelace” formula for area and understand that this formula is useful because only the coordinates of the vertices of a triangle are needed. In Lesson 10, students extend the “shoelace” formula to quadrilaterals, showing that the traditional formulas are verified with general cases of the “shoelace” formula and even extend this work to other polygons (pentagons and hexagons). Students compare the traditional formula for area and area by decomposition of figures and see that the “shoelace” formula is much more efficient in some cases. This work with the “shoelace” formula is the high school Geometry version of Green’s theorem and subtly exposes students to elementary ideas of vector and integral calculus. Lesson 11 concludes this work as the regions are described by a system of inequalities. Students sketch the regions, determine points of intersection (vertices), and use the distance formula to calculate perimeter and the “shoelace” formula to determine area of these regions. Students return to the real-world application of programming a robot and extend this work to robots not just confined to straight line motion but also motion bound by regions described by inequalities and defined areas.

<sup>1</sup> Lesson Structure Key: P-Problem Set Lesson, M-Modeling Cycle Lesson, E-Exploration Lesson, S-Socratic Lesson



## Lesson 9: Perimeter and Area of Triangles in the Cartesian Plane

### Student Outcomes

- Students find the perimeter of a triangle in the coordinate plane using the distance formula.
- Students state and apply the formula for area of a triangle with vertices  $(0, 0)$ ,  $(x_1, y_1)$ , and  $(x_2, y_2)$ .

### Lesson Notes

In this lesson, students find the perimeter and area of triangles in the plane. Students realize that the distance formula is required to find perimeter. Finding the area will be more of a challenge, but the teacher will guide students to understand that surrounding the triangle by a rectangle allows students to compute the area of the rectangle and the three right triangles that surround it and subtract these quantities.

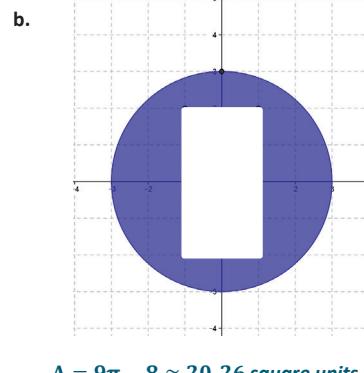
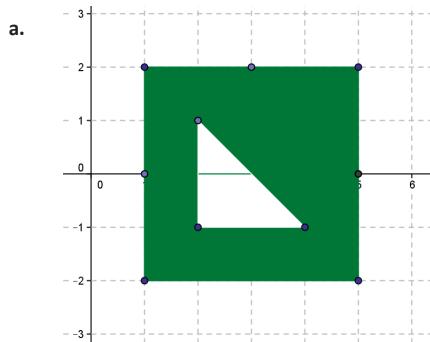
### Classwork

#### Opening Exercises (5 minutes)

The Opening Exercise can be done as a whole-class modeling exercise, small-group work, or independent work. Students found area by decomposing in Grades 6 and 7 (Grade 6, Module 5, Lesson 5; Grade 7, Module 6, Lessons 21 and 22). This Opening Exercise reviews those concepts, allowing students to find dimensions by counting units and can be used as scaffolds for this lesson if needed.

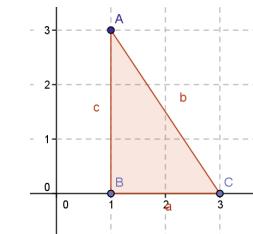
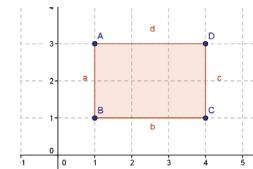
##### Opening Exercises

Find the area of the shaded region.



##### Scaffolding:

- If students are struggling with area, have them calculate the areas of the regions below and review how to find dimensions by counting.



**Example 1 (20 minutes)**

This is a guided example. Have all students find the perimeter of the area using the distance formula. When finding area, assign half the class to find the area of the given triangle using the area formula for a triangle (i.e.,  $\frac{1}{2}bh$ ). Ask the other half of the class to find the area by decomposing; in other words, ask students to find the area of the rectangle shown in the second drawing, and then subtract the shaded triangles to find the area of the unshaded triangle. Ask students to compare the process of each, and then come together for a class discussion. When assigning area, consider assigning the stronger students the area formula and struggling students the decomposition method.

**Example 1**

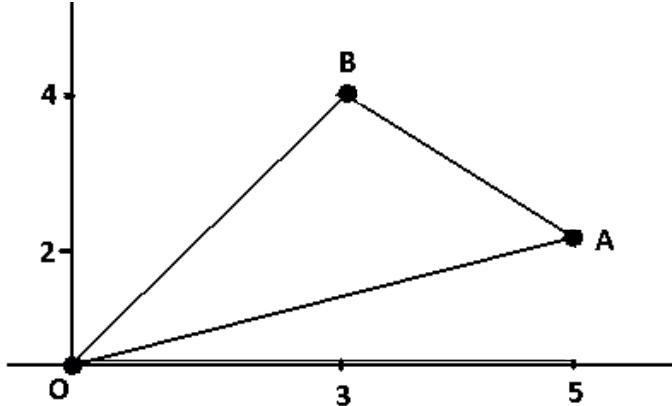
Consider a triangular region in the plane with vertices  $O(0, 0)$ ,  $A(5, 2)$ , and  $B(3, 4)$ . What is the perimeter of the triangular region?

$\approx 13.21$  units

What is the area of the triangular region?

7 square units

- Draw the triangle.
- Can you find the perimeter of  $\triangle OAB$ ?
  - Yes, but we have to find the lengths of the sides.
- Do you have any ideas how to find the side lengths?
  - We can use the distance formula between each set of vertices.
- What is the distance formula?
  - $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$
- Use the distance formula to find the length of  $OA$ ,  $OB$ , and  $AB$ ?
  - $OA = \sqrt{29}$ ,  $OB = 5$ ,  $AB = \sqrt{8}$
- So what is the perimeter of  $\triangle OAB$ ?
  - $\approx 13.21$  units
- Is  $\triangle OAB$  a right triangle? Explain.
  - No,  $(\sqrt{8})^2 + 5^2 \neq (\sqrt{29})^2$ .
- Can you find the area of  $\triangle OAB$ ?
  - If we know the base and height perpendicular to the base, we can.



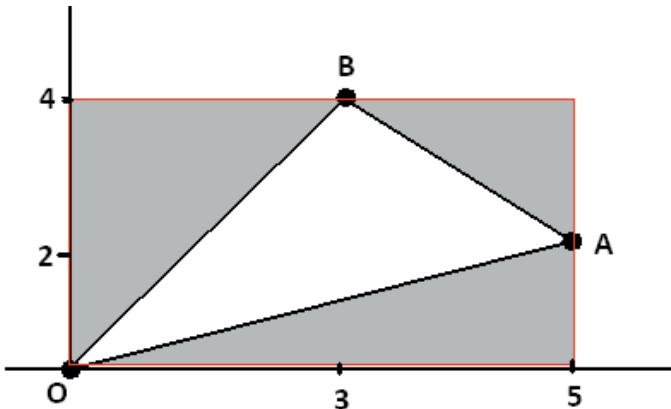
Assign half the class to find the area by using the area formula and constructing the perpendicular height. Assign the other half of the class to think of another way (decomposition). Allow students to work in groups, and then proceed with the questions below as needed to help students using decomposition.

- Can you think of another way to find the area?

Scaffold with the following questions as needed to allow students to see that they can find the area by finding the area of the rectangle that encloses the triangle and then subtracting each of the surrounding right triangles.

- Can you draw a rectangle containing  $\triangle OAB$ ?  
If so, draw it.
  - See shaded rectangle to the right.
- Can you find the area of the rectangle?
  - $4 \times 5 = 20$  square units.
- Can you see a way to find the area of  $\triangle OAB$ ?
  - Subtract the areas of the surrounding triangles.
- Find the areas of each of the triangles.
  - Left  $\triangle$ : 6 square units
  - Right  $\triangle$ : 2 square units
  - Bottom  $\triangle$ : 5 square units
- What is the area of  $\triangle OAB$ ?
  - $20 - 6 - 2 - 5 = 7$  square units

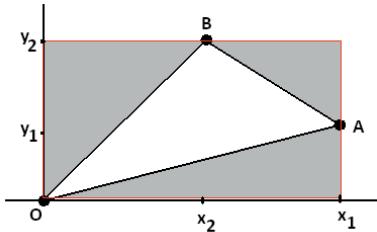
MP.7



Call groups back together, and allow both groups to present their solutions and compare methods. Have them discuss which method is easier to use and when it is easier to use the formula versus decomposition.

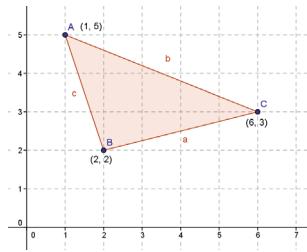
- Did we get the same answer with each method?
  - Yes, both methods gave an area of 7 square units.
- Which method was easier to use and why?
  - Decomposition was easier because constructing a rectangle around the triangle and finding the area of the rectangle was simple. Also, the triangles around the region were all right triangles, so again, the areas of the triangles were easy to find.
- When is it easier to use the area formula, and when is it easier to use decomposition?
  - When the triangle is a right triangle, or when the base and height of the triangle are given, the area formula is easy to use. Otherwise, it is easier to use decomposition.
- Can you help me develop a general formula for the area of any triangle with one vertex at  $(0, 0)$  using decomposition? Let's call  $O(0, 0)$ ,  $A(x_1, y_1)$ , and  $B(x_2, y_2)$ .

Find the general formula for the area of the triangle with vertices  $O(0, 0)$ ,  $A(x_1, y_1)$ , and  $B(x_2, y_2)$  as shown.



#### Scaffolding:

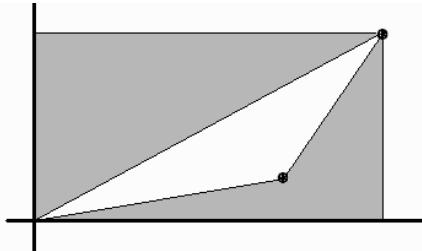
- As students are developing general formulas for the area of triangles, let students struggle, and then ask only questions needed to move students to the next step.
- Students can try this numerical example to solidify this concept.



Let students struggle with this for a while following the steps modeled above. Scaffold as necessary.

- What is the area of the rectangle enclosing the triangle?
  - $x_1y_2$  square units
- What is the area of the surrounding triangles?
  - Left:  $\frac{1}{2}(x_2y_2)$  square units
  - Right:  $\frac{1}{2}(x_2 - x_1)(y_2 - y_1)$  square units
  - Bottom:  $\frac{1}{2}(x_1y_1)$  square units
- Write the general formula for the area of  $\triangle OAB$ ?
  - $x_1y_2 - \frac{1}{2}(x_2y_2) - \frac{1}{2}(x_2 - x_1)(y_2 - y_1) - \frac{1}{2}(x_1y_1)$
- Expand the formula and simplify it.
  - $\frac{1}{2}(x_1y_2 - x_2y_1)$

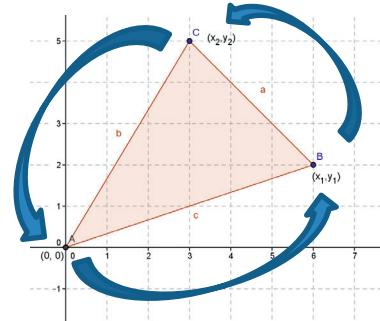
Does the formula work for this triangle?



- Does the formula work for the triangle below?
  - It does, but students may get either of these formulas:  $\frac{1}{2}(x_1y_2 - x_2y_1)$  or  $\frac{1}{2}(x_2y_1 - x_1y_2)$  depending on which point they labeled  $(x_1, y_1)$ .

These formulas differ by a minus sign. Let students compare their answer with their classmates. Make the point that the formulas depend on the choice of labeling the points but differ only by a minus sign. Students will always get a positive area if they order the points counterclockwise. The method that students should follow is (1) pick a starting point; (2) walk around the figure in a counterclockwise direction doing calculations with adjacent points; and (3) when they return to the starting point, reverse directions walking in a clockwise direction and completing the second half of the calculations.

- Do you think this formula works for any triangle with one vertex at  $(0, 0)$ ? Does the quadrant matter?
  - Answers will vary. Yes, it always works no matter the quadrant, and the area will be positive as long as  $(x_1, y_1)$  is the next point in a counterclockwise direction.
- Let's try some problems and see if our formula always works.



**Exercise 1 (5 minutes)**

Students should find the area of the triangles by finding the area of the rectangle enclosing the triangle and subtracting the surrounding triangles. They should compare that area to the area derived by using the above formula.

**Exercise 1**

Find the area of the triangles with vertices listed, first by finding the area of the rectangle enclosing the triangle and subtracting the area of the surrounding triangles, then by using the formula  $\frac{1}{2}(x_1y_2 - x_2y_1)$ .

- a.  $O(0, 0), A(5, 6), B(4, 1)$

**9.5 square units**

- b.  $O(0, 0), A(3, 2), B(-2, 6)$

**11 square units**

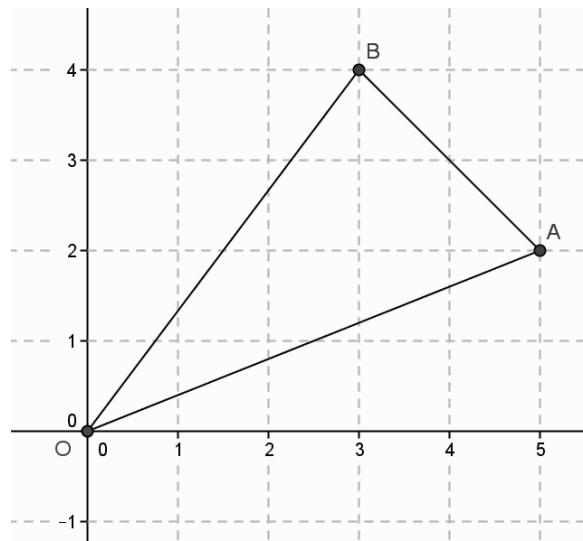
- c.  $O(0, 0), A(5, -3), B(-2, 6)$

**12 square units**

**Example 2 (8 minutes)**

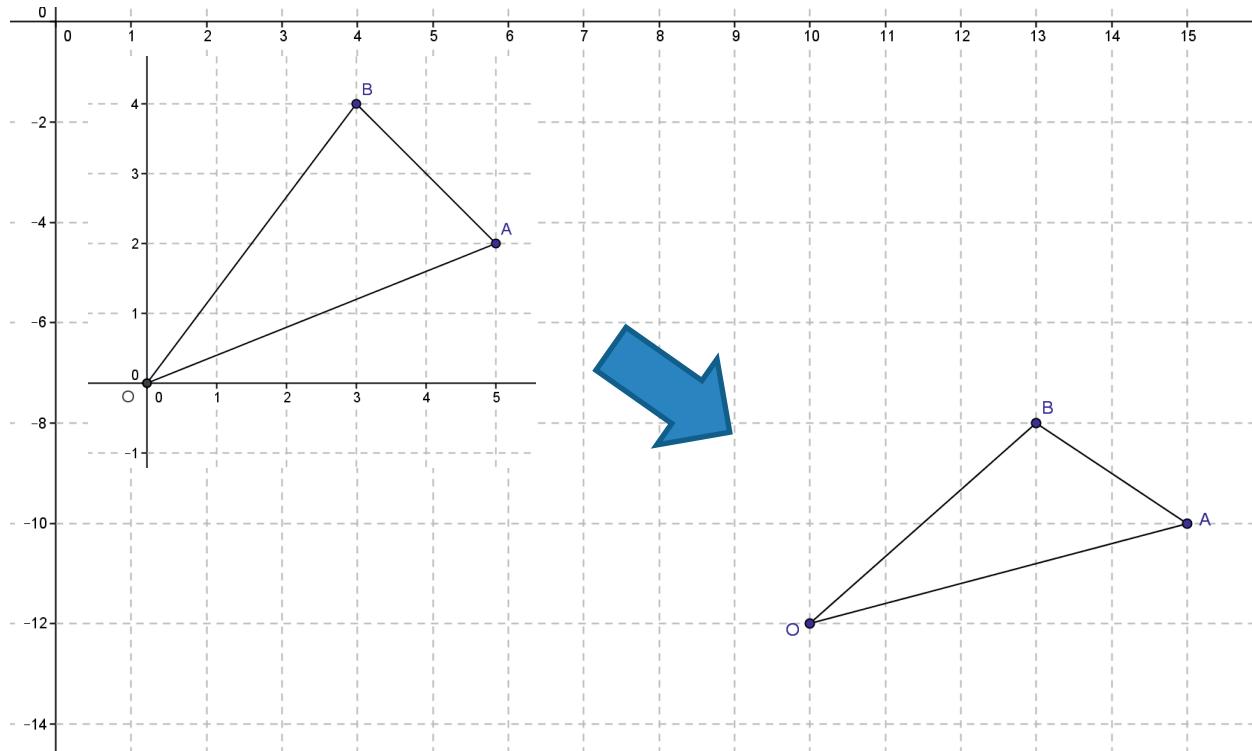
This is a guided example.

- Did the formula work for all of the triangles?
  - Yes, as long as one vertex was at  $(0, 0)$ .
- In the Opening Exercise, we saw that the area of the triangle with vertices  $O(0, 0)$ ,  $A(5, 2)$ , and  $B(3, 4)$  had an area of 7 square units. Can you find the area of a triangle with vertices  $O(10, -12)$ ,  $A(15, -10)$ , and  $B(13, -8)$ ?

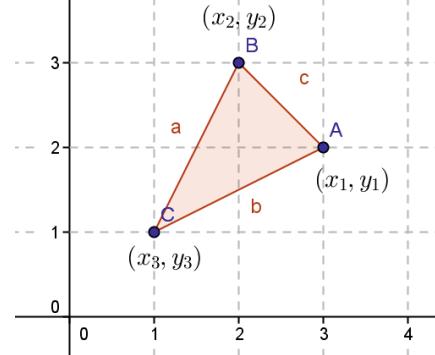


- **7 square units**

- How did you find it so quickly?
  - All of the points were just translated right 10 units and down 12 units, so the area was the same.

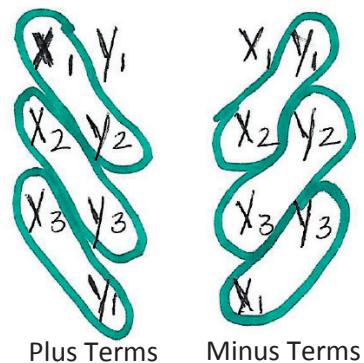


- Can you find a general formula for the area of a triangle with coordinates  $A(x_1, y_1)$ ,  $B(x_2, y_2)$ , and  $C(x_3, y_3)$ ? (Students should see that if we translate one vertex to  $(0, 0)$ , we can use our formula.)
  - We can translate  $C$  to  $(0, 0)$ .
- What are the new coordinates of the vertices?
  - $A(x_1 - x_3, y_1 - y_3)$ ,  $B(x_2 - x_3, y_2 - y_3)$ ,  $C(0, 0)$
- Using our formula, what is the area?
  - $\frac{1}{2}(x_1 - x_3)(y_2 - y_3) - \frac{1}{2}(x_2 - x_3)(y_1 - y_3)$
- Use algebra and simplify this formula.
  - $\frac{1}{2}(x_1y_2 + x_2y_3 + x_3y_1 - y_1x_2 - y_2x_3 - y_3x_1)$
- This is called the “shoelace” area formula. Can you guess why?
  - Answers will vary.
- Let me show you. Do you see the pattern?



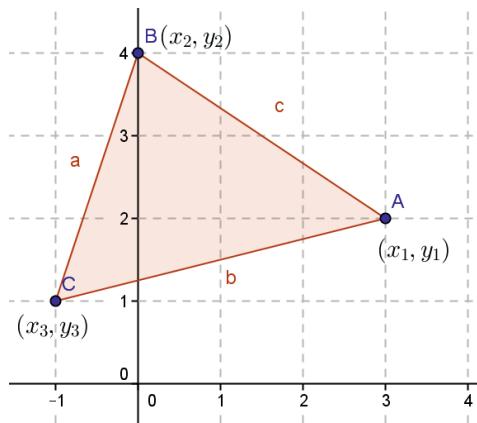
Note to teacher: To remember the formula, order the points going counterclockwise around the figure, then “walk around” the figure counterclockwise summing up the products of each  $x$ -coordinate with the next point’s  $y$ -coordinate,  $x_1y_2 + x_2y_3 + \dots + x_{n-1}y_n + x_ny_1$ . Then “walk around” the figure clockwise doing the same thing,  $x_1y_n + x_ny_{n-1} + \dots + x_3y_2 + x_2y_1$ . Finally, subtract the second expression from the first.

- Now you can see why we call this the “shoelace” area formula!  
Do you think this will work for figures with more than three vertices?



### Closing (2 minutes)

Gather the class together and have a discussion about the following:



- How do you find the perimeter of this triangle?
  - To find the perimeter of this triangle, use the distance formula to find the length of each segment and add the distances together.
- Explain how you would find the area of this triangle.
  - We could draw a rectangle around the triangle and then use the decomposition method to find the area. We could translate one vertex to the origin and use the formula  $\frac{1}{2}(x_1y_2 - x_2y_1)$ , or use the shoelace formula to find the area:

$$\frac{1}{2}(x_1y_2 + x_2y_3 + x_3y_1 - y_1x_2 - y_2x_3 - y_3x_1).$$

### Exit Ticket (5 minutes)

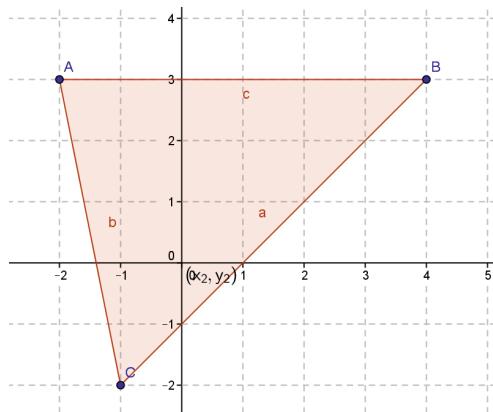
Name \_\_\_\_\_

Date \_\_\_\_\_

## Lesson 9: Perimeter and Area of Triangles in the Cartesian Plane

### Exit Ticket

Given the triangle below with vertices  $A (-2, 3)$ ,  $B (4, 4)$ , and  $C (-1, -2)$ .



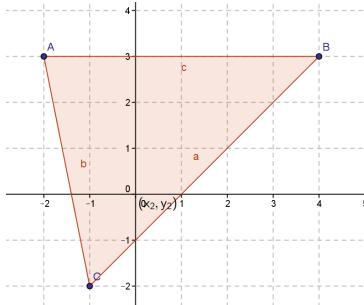
Azha calculated the area using  $5 \cdot 6 - \frac{1}{2}(5 \cdot 1) - \frac{1}{2}(5 \cdot 5)$ ,

while Carson calculated the area using  $\frac{1}{2}((-2) \cdot 4 + 4 \cdot (-2) + (-1) \cdot 3 - 3 \cdot 4 - 4 \cdot (-2) - (-2) \cdot (-2))$ .

Explain the method each one used.

## Exit Ticket Sample Solutions

Given the triangle below with vertices  $A(-2, 3)$ ,  $B(4, 4)$ , and  $C(-1, -2)$ .



Azha calculated the area using  $5 \cdot 6 - \frac{1}{2}(5 \cdot 1) - \frac{1}{2}(5 \cdot 5)$ ,

while Carson calculated the area using  $\frac{1}{2}((-2) \cdot 4 + 4 \cdot (-2) + (-1) \cdot 3 - 3 \cdot 4 - 4 \cdot (-2) - (-2) \cdot (-2))$ .

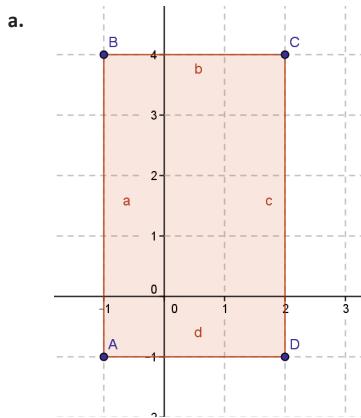
Explain the method each one used.

Azha used the decomposition method. She first determined that the area of the rectangle around the triangle is  $5 \cdot 6$ , and then she subtracted the area of the 3 right triangles surrounding the region. Carson used the “shoelace” method that we learned in this lesson.

## Problem Set Sample Solutions

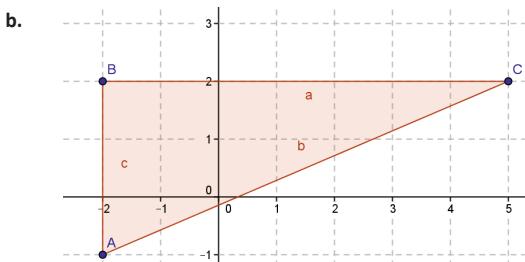
The Problem Set can be assigned as a whole, or problems can be selected to meet student progress.

1. Use coordinates to compute the perimeter and area of each polygon.



$$\text{Perimeter} = 16 \text{ units}$$

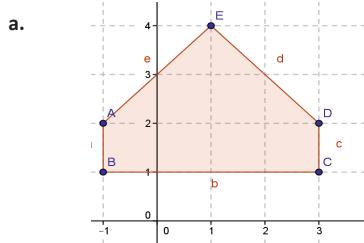
$$\text{Area} = 15 \text{ square units}$$



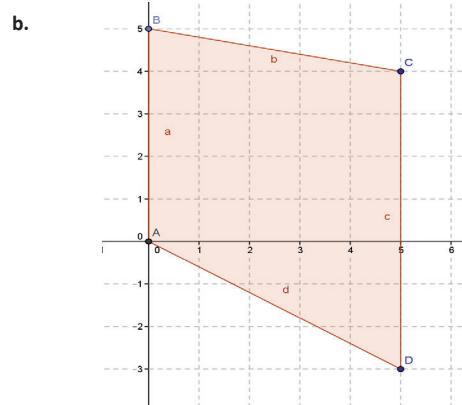
$$\text{Perimeter} \approx 17.62 \text{ units}$$

$$\text{Area} = 10.5 \text{ square units}$$

2. Given the figures below, find the area by decomposing into rectangles and triangles.



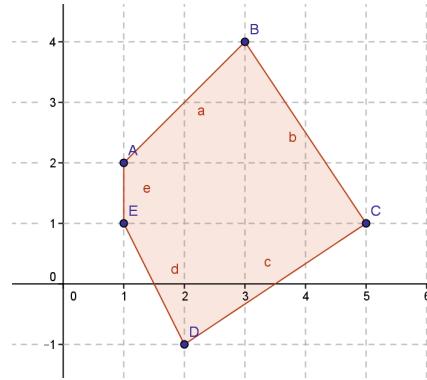
*Area = 8 square units*



*Area = 30 square units*

3. Challenge: Find the area by decomposing the given figure into triangles.

*11 square units*



4. When using the shoelace formula to work out the area of  $\triangle ABC$ , we have some choices to make. For example, we can start at any one of the three vertices  $A$ ,  $B$ , or  $C$ , and we can move either in a clockwise or counterclockwise direction. This gives six options for evaluating the formula.

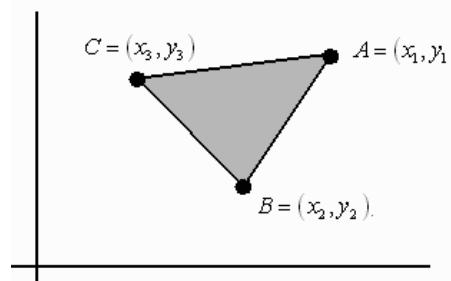
Show that the shoelace formula obtained is identical for the three options that move in a clockwise direction ( $A$  to  $B$  or  $C$  to  $B$  to  $A$  or  $B$  to  $A$  to  $C$ ) and identical for the three options in the reverse direction. Verify that the two distinct formulas obtained differ only by a minus sign.

*Clockwise:*

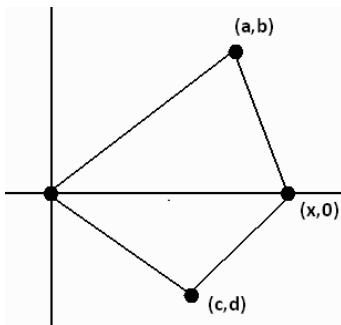
$$\frac{1}{2}(x_1y_2 + x_2y_3 + x_3y_1 - y_1x_2 - y_2x_3 - y_3x_1)$$

*Counterclockwise:*

$$\frac{1}{2}(x_1y_3 + x_3y_2 + x_2y_1 - y_1x_3 - y_3x_2 - y_2x_1)$$



5. Suppose two triangles share a common edge. By translating and rotating the triangles, we can assume that the common edge lies along the  $x$ -axis with one endpoint at the origin.



- a. Show that if we evaluate the shoelace formula for each triangle, both calculated in the same clockwise direction, then the answers are both negative.

$$\text{Top Triangle: } \frac{1}{2}(0 - bx) = -\frac{1}{2}bx, \text{ where } b > 0.$$

$$\text{Bottom Triangle: } \frac{1}{2}(xd - 0) = \frac{1}{2}dx, \text{ where } d < 0.$$

*Both answers are negative when calculated in the clockwise direction.*

- b. Show that if we evaluate them both in a counter-clockwise direction, then both are positive.

$$\text{Top Triangle: } \frac{1}{2}(xb - 0) = \frac{1}{2}bx, \text{ where } b > 0.$$

$$\text{Bottom Triangle: } \frac{1}{2}(0 - xd) = -\frac{1}{2}dx, \text{ where } d < 0.$$

*Both answers are positive when calculated in the counterclockwise direction.*

- c. Explain why evaluating one in one direction and the second in the opposite direction, the two values obtained are opposite in sign.

*If two triangles share a common edge, then evaluating the shoelace formula for each triangle in a consistent direction give answers that are the same sign; they are either each the positive areas of their respective triangles or each the negative versions of those areas.*

6. A textbook has a picture of a triangle with vertices  $(3, 6)$  and  $(5, 2)$ . Something happened in printing the book and the coordinates of the third vertex are listed as  $(-1, \square)$ . The answers in the back of the book give the area of the triangle as 6 square units.

- a. What is the  $y$ -coordinate of the third vertex?

-2

- b. What if both coordinates were missing, but the area was known. Could you use algebra to find the third coordinate? Explain.

*No, you would have an equation with two variables, so you could not solve for both algebraically unless you had a second equation with the same unknowns.*



## Lesson 10: Perimeter and Area of Polygonal Regions in the Cartesian Plane

### Student Outcomes

- Students find the perimeter of a quadrilateral in the coordinate plane given its vertices and edges.
- Students find the area of a quadrilateral in the coordinate plane given its vertices and edges by employing Green's theorem.

### Classwork

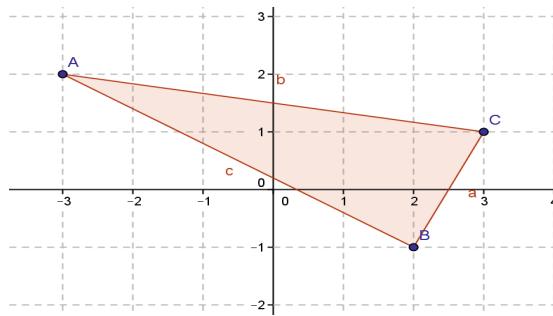
#### Opening Exercise (5 minutes)

The Opening Exercise allows students to practice the shoelace method of finding the area of a triangle in preparation for today's lesson. Have students complete the exercise individually, and then compare their work with a neighbor's. Pull the class back together for a final check and discussion.

If students are struggling with the shoelace method, they can use decomposition.

##### Opening Exercise

Find the area of the triangle given. Compare your answer and method to your neighbor's and discuss differences.



##### Shoelace Formula

Coordinates:  $A(-3, 2), B(2, -1), C(3, 1)$

Area Calculation:

$$\frac{1}{2}(-3 \cdot -1 + 2 \cdot 1 + 3 \cdot 2 - 2 \cdot 2 - (-1) \cdot 3 - 1 \cdot (-3))$$

Area: 6.5 square units

##### Decomposition

Area of Rectangle:  $6 \cdot 3 = 18$  square units

Area of Left Rectangle: 7.5 square units

Area of Bottom Right Triangle: 1 square unit

Area of Top Right Triangle: 3 square units

Area of Shaded Triangle:

$$18 - 7.5 - 1 - 3 = 6.5 \text{ square units}$$

##### Scaffolding:

- Give students time to struggle with these questions. Add more questions as necessary to scaffold for struggling students.
- Consider providing students with pre-graphed quadrilaterals with axes on grid-lines.
- Go from concrete to abstract by starting with finding the area by decomposition, then translating one vertex to the origin, then using the shoelace formula.
- Post shoelace diagram and formula from Lesson 9.

**Discussion (15 minutes)**

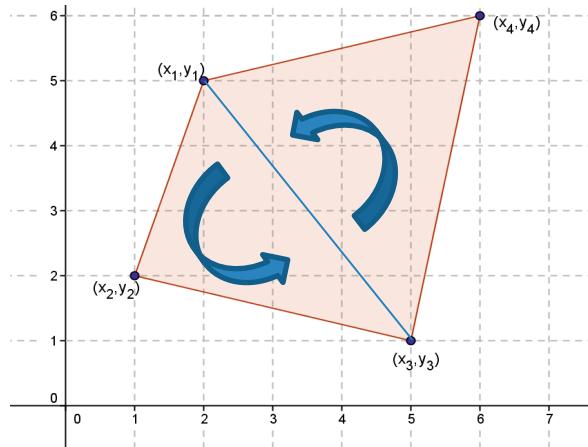
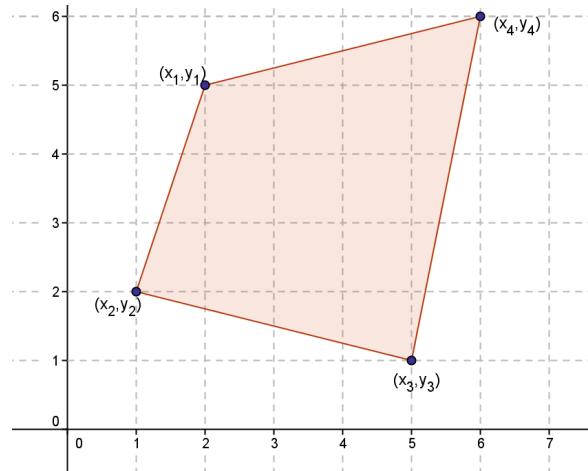
In this lesson, students extend the area formulas for triangles studied in Lesson 9 to quadrilaterals and discover that the formula works for any polygonal region in the coordinate plane. Remind students that they should pick a starting point, and then initially move in a counterclockwise direction as they did in Lesson 9.

- Recall the question that we ended with last lesson:  
Does the shoelace area formula extend to help us find the areas of quadrilaterals in the plane? Look at the quadrilateral given—any thoughts?

Allow students time to come up with ideas, but try to lead them eventually to decomposing the quadrilateral into triangles so you can use the formula from Lesson 9. Have students each try the method they suggest initially and compare and contrast methods.

- *Answers will vary. Students may suggest enclosing the quadrilateral in a rectangle and subtracting triangles as we did in the beginning of Lesson 9. They may suggest measuring and calculating from traditional area formulas. Try to lead them to suggest that we can divide the quadrilateral into two triangles and use the formula from the last lesson to find each area, and then add the two areas.*

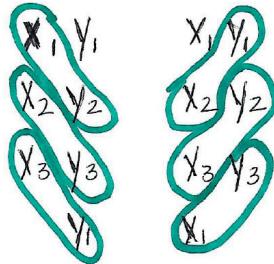
- Let's try it and see if it works. We will assign the numerical values to the coordinates first, find the area of each triangle and add them together, then verify with the shoelace formula for the full quadrilateral. Once we determine that the formula extends, we will write the general formula. For consistency, let's imagine we are walking around the polygon in a counterclockwise direction. This will help us be consistent with the order we list the vertices for the triangle and force us to be consistent in the direction that we apply the shoelace formula within each triangle.
- What are the vertices of the triangle on the left side?
  - $(2, 5), (1, 2), (5, 1)$
- Find the area of that triangle using the shoelace formula.
  - $\frac{1}{2}(2 \cdot 2 + 1 \cdot 1 + 5 \cdot 5 - 5 \cdot 1 - 2 \cdot 5 - 1 \cdot 2) = 6.5 \text{ square units}$
- What are the vertices of the triangle on the right side?
  - $(5, 1), (6, 6), (2, 5)$
- Find the area of that triangle using the shoelace formula.
  - $\frac{1}{2}(5 \cdot 6 + 6 \cdot 5 + 2 \cdot 1 - 1 \cdot 6 - 6 \cdot 2 - 5 \cdot 5) = 9.5 \text{ square units}$



- What is the area of the quadrilateral?
  - $6.5 + 9.5 = 16 \text{ square units}$
- How do you think we could apply the shoelace theorem using four vertices?
  - Pick one vertex, and apply the formula moving in a counterclockwise direction.
- Now let's try the shoelace formula on the quadrilateral without breaking it apart. Let's start with the vertex labeled  $(x_1, y_1)$  and move counterclockwise. Let's start by listing the vertices in order.
  - $(2, 5), (1, 2), (5, 1), (6, 6)$
- Now apply the formula using consecutive vertices.
  - $\frac{1}{2}(2 \cdot 2 + 1 \cdot 1 + 5 \cdot 6 + 6 \cdot 5 - 5 \cdot 1 - 2 \cdot 5 - 1 \cdot 6 - 6 \cdot 2) = 16 \text{ square units}$
- What do you notice?
  - The areas are the same; the formula works for quadrilaterals.

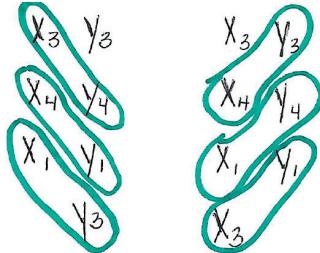
The next part of the lesson will have students developing a general formula for the area of a quadrilateral. Put students in groups and allow them to work according to their understanding. Some groups will be able to follow procedures from today and yesterday and do the work alone. Others may need direct teacher assistance, get guidance only when they signal for it, or use the questions below to guide their thinking. Bring the class back together to summarize.

- Now let's develop a general formula that we can use with coordinates listed as  $(x_n, y_n)$ .
- What is the area of the triangle on the left? Of the triangle on the right?
  - Area left:



$$A_1 = \frac{1}{2}(x_1y_2 + x_2y_3 + x_3y_1 - y_1x_2 - y_2x_3 - y_3x_1)$$

- Area right:



$$A_2 = \frac{1}{2}(x_3y_4 + x_4y_1 + x_1y_3 - y_3x_4 - y_4x_1 - y_1x_3)$$

- Now add the two triangles. Do you see any terms that cancel?
  - Those that involve just the coordinates  $x_1, x_3, y_1, y_3$  cancel; that is, the terms that match the coordinates of the endpoints of the common line to the two triangles we created.
- What is the formula for the area of the quadrilateral?
  - $A = A_1 + A_2 = \frac{1}{2}(x_1y_2 + x_2y_3 + x_3y_1 - y_1x_2 - y_2x_3 - y_3x_1) + \frac{1}{2}(x_3y_4 + x_4y_1 + x_1y_3 - y_1x_3 - y_3x_4 - y_4x_1)$
- Is there a shoelace formula?
  - $\frac{1}{2}(x_1y_2 + x_2y_3 + x_3y_4 + x_4y_1 - y_1x_2 - y_2x_3 - y_3x_4 - y_4x_1)$
- Summarize what you have learned so far in writing and share with your neighbor.
- The shoelace formula can also be called Green's theorem. Green's theorem is a high school Geometry version of the exact same theorem that students learn in Calculus III (along with Stokes' theorem and the divergence theorem). Students only need to know that the shoelace formula can also be referred to as Green's theorem.

The Problem Set in the previous lesson was important. It shows that if one chooses a direction to move around the polygon, and then moves in the same direction for the other triangle, then all areas calculated will have the same sign—they will both be either positive and correct, or they will each be incorrect by a minus sign. This means, when adding all the numbers we obtain, the total will either be the correct positive area or the correct value off by a minus sign. It is important to always move in a counterclockwise direction so that area is always positive.

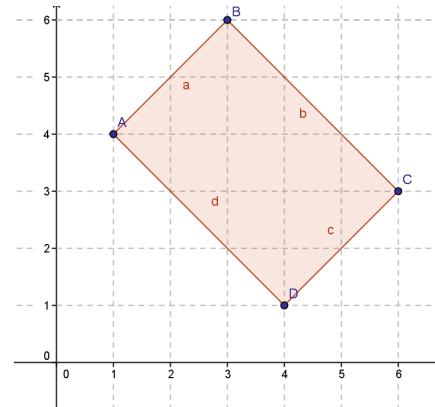
Notice too that in choosing this direction for each triangle, the common interior line is traversed in one direction by one triangle and the opposite direction by the other.

The exercises are designed so that different students can be assigned different problems. All students should do Problem 1 so that they can see the value in the shoelace formula (Green's theorem). After that, problems can be chosen to meet student needs. Some of these exercises can also be included as Problem Set problems.

### Exercises 1–6 (17 minutes)

#### Exercises 1–6

1. Given rectangle  $ABCD$ :
  - a. Identify the vertices.  
 **$A(1, 4), B(3, 6), C(6, 3), D(4, 1)$**
  - b. Find the perimeter using the distance formula.  
 **$\approx 14.14$  units**
  - c. Find the area using the area formula.  
 **$(\sqrt{8})(\sqrt{18}) = 12$  square units**
  - d. List the vertices starting with  $A$  moving counterclockwise.  
 **$A(1, 4), D(4, 1), C(6, 3), B(3, 6)$**



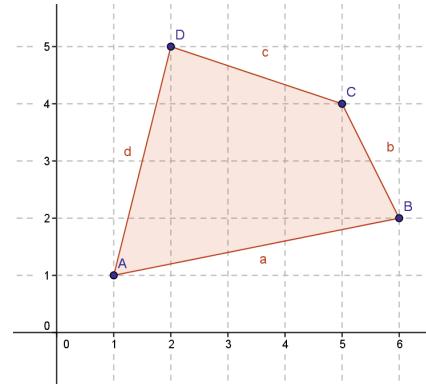
- e. Verify the area using the shoelace formula.

$$\frac{1}{2}(1 \cdot 1 + 4 \cdot 3 + 6 \cdot 6 + 3 \cdot 4 - 4 \cdot 4 - 1 \cdot 6 - 3 \cdot 3 - 6 \cdot 1) = 12 \text{ square units}$$

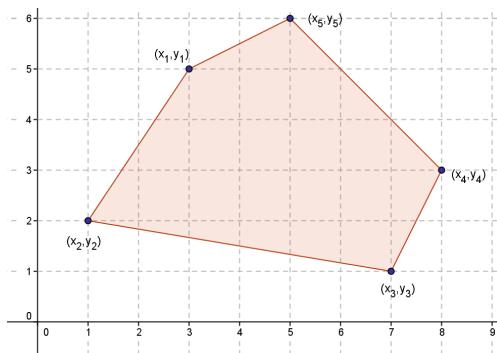
2. Calculate the area and perimeter of the given quadrilateral using the shoelace formula.

$$\text{Area} = \frac{1}{2}(1 \cdot 2 + 6 \cdot 4 + 5 \cdot 5 + 2 \cdot 1 - 1 \cdot 6 - 2 \cdot 5 - 4 \cdot 2 - 5 \cdot 1) = 12 \text{ square units}$$

$\text{Perimeter} \approx 14.62 \text{ units}$



3. Break up the pentagon to find the area using Green's theorem. Compare your method with a partner.



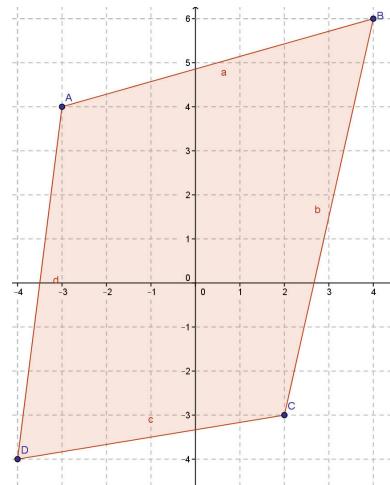
Answers will vary. If broken into a triangle with vertices  $(x_1, y_1)$ ,  $(x_2, y_2)$ , and  $(x_3, y_3)$ , and a quadrilateral with vertices  $(x_1, y_1)$ ,  $(x_3, y_3)$ ,  $(x_4, y_4)$ , and  $(x_5, y_5)$ , the formula would be

$$\frac{1}{2}(x_1y_2 + x_2y_3 + x_3y_1 - y_1x_2 - y_2x_3 - y_3x_1) + \frac{1}{2}(x_3y_4 + x_4y_5 + x_5y_1 + x_1y_3 - y_3x_4 - y_4x_5 + y_5x_1 + y_1x_3)$$

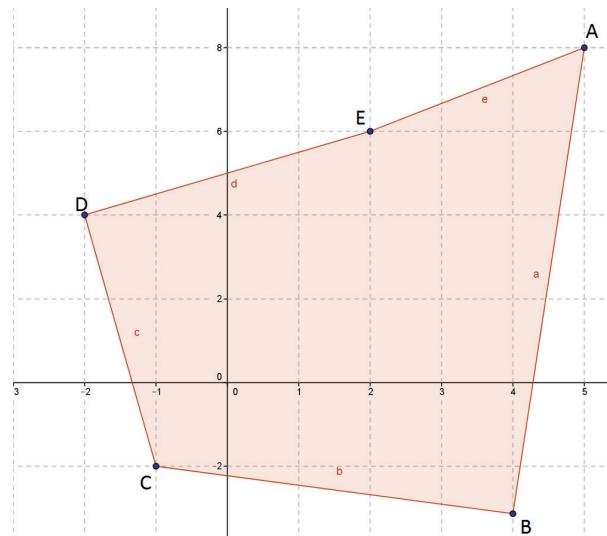
4. Find the perimeter and the area of the quadrilateral with vertices  $A(-3, 4)$ ,  $B(4, 6)$ ,  $C(2, -3)$ , and  $D(-4, -4)$ .

$\text{Perimeter} \approx 30.64 \text{ units}$

$$\text{Area} = \frac{1}{2}(-3 \cdot -4 + (-4) \cdot -3 + 2 \cdot 6 + 4 \cdot 4 - 4 \cdot (-4) - (-4) \cdot 2 - (-3) \cdot 4 - 6 \cdot (-3)) = 53 \text{ square units}$$



5. Find the area of the pentagon with vertices  $A(5, 8)$ ,  $B(4, -3)$ ,  $C(-1, -2)$ ,  $D(-2, 4)$ , and  $E(2, 6)$ .

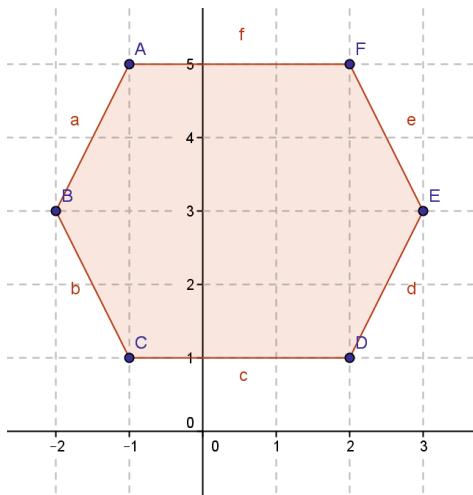


**Area:**

$$\frac{1}{2}(5 \cdot 6 + 2 \cdot 4 + (-2) \cdot (-2) + (-1) \cdot (-3) + 4 \cdot 8 - 8 \cdot 2 - 6 \cdot (-2) - 4 \cdot (-1) - (-2) \cdot 4 - (-3) \cdot 5) =$$

**50 square units**

6. Find the area and perimeter of the hexagon shown.



**Area:**

$$\frac{1}{2}(-1 \cdot 3 + (-2) \cdot 1 + (-1) \cdot 1 + 2 \cdot 3 + 3 \cdot 5 + 5 \cdot 5 - 5 \cdot (-2) - 3 \cdot (-1) - 1 \cdot 2 - 1 \cdot 3 - 3 \cdot 2 - 5 \cdot (-1)) =$$

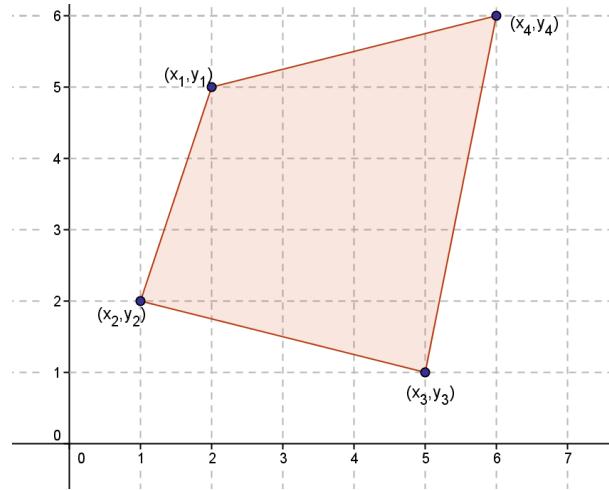
**23.5 square units**

**Perimeter:**  $\approx 14.94$  units

**Closing (3 minutes)**

Gather the class together and present the picture with the following question. Have a class discussion about the value and efficiency of the shoelace formula.

- Describe different ways of calculating the area of this figure. What are the advantages and disadvantages of each?



- You could decompose the quadrilateral into two rectangles and use the area of a triangle formula. You could use the shoelace formula (Green's theorem).
- Decomposing allows you to find the area without memorizing a formula but may be time consuming. Green's theorem may be quicker but more prone to a computational error.

**Exit Ticket (5 minutes)**

Name \_\_\_\_\_

Date \_\_\_\_\_

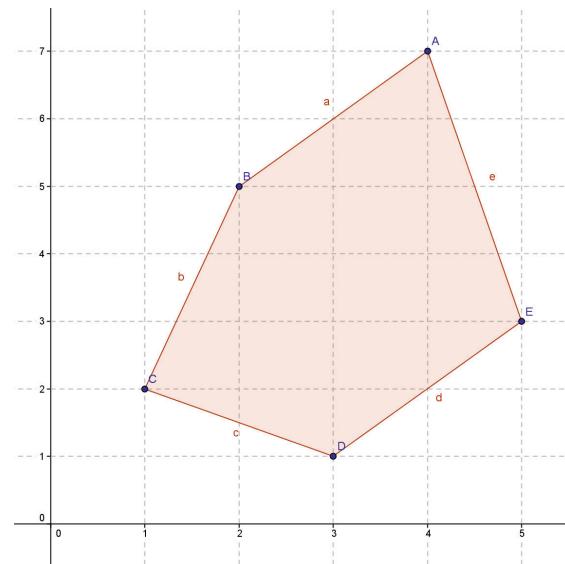
## Lesson 10: Perimeter and Area of Polygonal Regions in the Cartesian Plane

### Exit Ticket

Cory is using the shoelace formula to calculate the area of the pentagon shown. The pentagon has vertices  $A(4, 7)$ ,  $B(2, 5)$ ,  $C(1, 2)$ ,  $D(3, 1)$ , and  $E(5, 3)$ . His calculations are below. Toya says his answer can't be correct because the area in the region is more than 2 square units. Can you identify and explain Cory's error and help him calculate the correct area?

Cory's work:

$$\begin{aligned} & \frac{1}{2}(4 \cdot 5 + 2 \cdot 2 + 1 \cdot 1 + 3 \cdot 3 - 7 \cdot 2 - 5 \cdot 1 - 2 \cdot 3 - 1 \cdot 5) \\ & = 2 \text{ square units} \end{aligned}$$



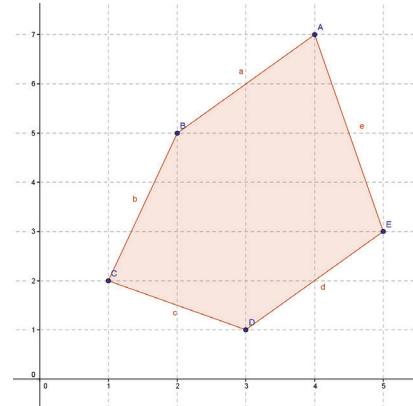
## Exit Ticket Sample Solutions

Cory is using the shoelace formula to calculate the area of the pentagon shown. The pentagon has vertices  $A(4, 7)$ ,  $B(2, 5)$ ,  $C(1, 2)$ ,  $D(3, 1)$ , and  $E(5, 3)$ . His calculations are below. Toya says his answer can't be correct because the area in the region is more than 2 square units. Can you identify and explain Cory's error and help him calculate the correct area?

Cory's work:

$$\frac{1}{2}(4 \cdot 5 + 2 \cdot 2 + 1 \cdot 1 + 3 \cdot 3 - 7 \cdot 2 - 5 \cdot 1 - 2 \cdot 3 - 1 \cdot 5)$$

= 2 square units



*Cory left out the last pair of calculations in both directions. He did not return to his starting point. The calculation should be  $\frac{1}{2}(4 \cdot 5 + 2 \cdot 2 + 1 \cdot 1 + 3 \cdot 3 + 5 \cdot 7 - 7 \cdot 2 - 5 \cdot 1 - 2 \cdot 3 - 1 \cdot 5 - 3 \cdot 4) = 13.5$  square units*

## Problem Set Sample Solutions

1. Given triangle  $ABC$  with vertices  $(7, 4)$ ,  $(1, 1)$ , and  $(9, 0)$ :

- a. Calculate the perimeter using the distance formula.

*Perimeter:  $\approx 19.24$  units*

- b. Calculate the area using the traditional area formula.

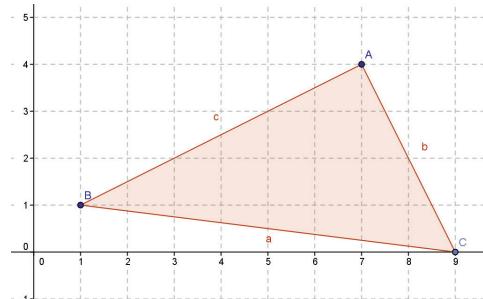
*Area:  $\frac{1}{2}(\sqrt{45})(\sqrt{20}) = 15$  square units*

- c. Calculate the area using the shoelace formula.

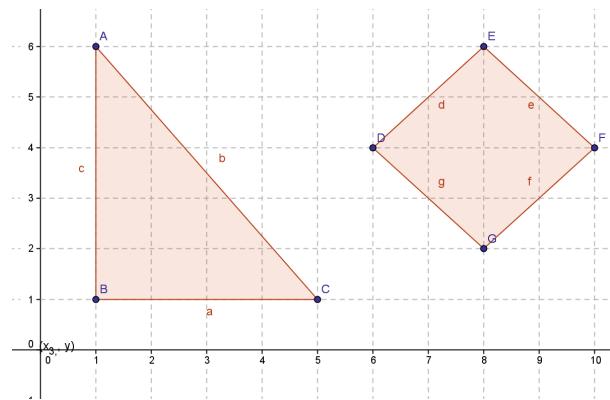
*Area:  $\frac{1}{2}(7 \cdot 1 + 1 \cdot 0 + 9 \cdot 4 - 4 \cdot 1 - 1 \cdot 9 - 0 \cdot 7) = 15$  square units*

- d. Explain why the shoelace formula might be more useful and efficient if you were just asked to find the area.

*To use the shoelace formula, all you need are the coordinates of the vertices; you would not have to use the distance formula.*



2. Given triangle  $ABC$  and quadrilateral  $DEFG$ , describe how you would find the area of each and why you would choose that method, and then find the areas.



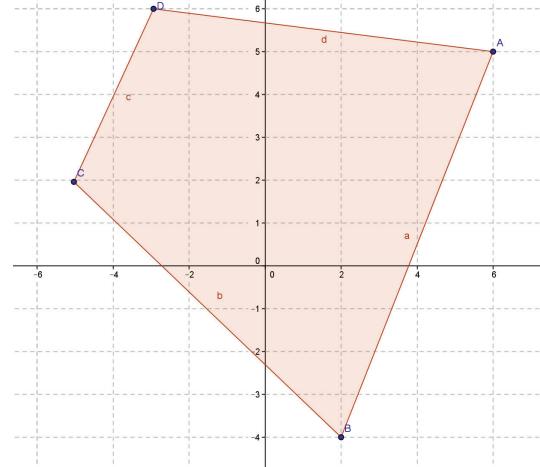
The triangle is a right triangle, and the length and height can be read from the graph, so it would be easiest to find its area by the traditional area formula. Area: 10 square units

To find the area of the quadrilateral, I would use the shoelace formula because the traditional formula would require use of the distance formula first. Area: 8 square units

3. Find the area and perimeter of quadrilateral  $ABCD$  with vertices  $A(6, 5)$ ,  $B(2, -4)$ ,  $C(-5, 2)$ , and  $D(-3, 6)$ .

Area: 62.5 square units

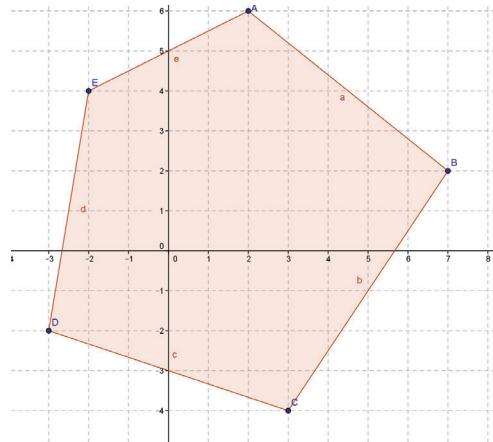
Perimeter:  $\approx 32.60$



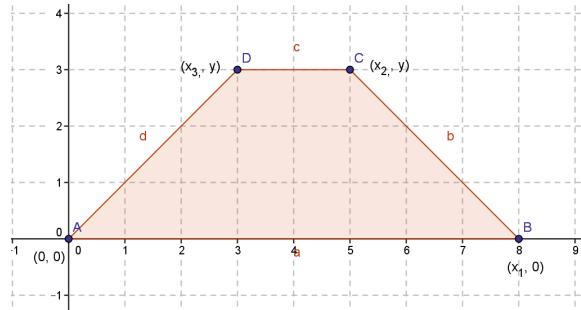
4. Find the area and perimeter of pentagon  $ABCDE$  with vertices  $A(2, 6)$ ,  $B(7, 2)$ ,  $C(3, -4)$ ,  $D(-3, -2)$ , and  $E(-2, 4)$ .

Area: 61 square units

Perimeter:  $\approx 30.49$  units



5. Show that the shoelace formula (Green's theorem) used on the trapezoid shown confirms the traditional formula for the area of a trapezoid  $\frac{1}{2}(b_1 + b_2) \cdot h$ .



*Traditional using coordinates listed:*

$$b_1 = x_1, b_2 = x_3 - x_4, h = y$$

$$\frac{1}{2}(x_1 + x_3 - x_4) \cdot y$$

*Shoelace formula:*

$$\begin{aligned} \frac{1}{2}(0 \cdot 0 + x_1 \cdot y + x_2 \cdot y + x_3 \cdot 0 - 0 \cdot x_1 - 0 \cdot x_2 - y \cdot x_3 - y \cdot 0) &= \frac{1}{2}(x_1 \cdot y + x_2 \cdot y - x_3 \cdot y) \\ &= \frac{1}{2}(x_1 + x_2 - x_3) \cdot y \end{aligned}$$



## Lesson 11: Perimeters and Areas of Polygonal Regions

### Defined by Systems of Inequalities

#### Student Outcomes

- Students find the perimeter of a triangle or quadrilateral in the coordinate plane given a description by inequalities.
- Students find the area of a triangle or quadrilateral in the coordinate plane given a description by inequalities by employing Green's theorem.

#### Lesson Notes

In previous lessons, students found the area of polygons in the plane using the “shoelace” method. In this lesson, we give a name to this method—Green’s theorem. Students will draw polygons described by a system of inequalities, find the perimeter of the polygon, and use Green’s theorem to find the area.

#### Classwork

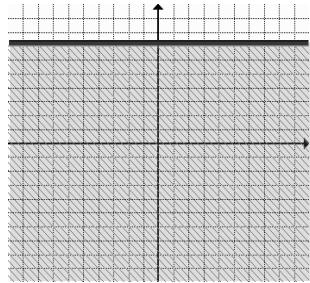
##### Opening Exercises (5 minutes)

The opening exercises are designed to review key concepts of graphing inequalities. The teacher should assign them independently and circulate to assess understanding.

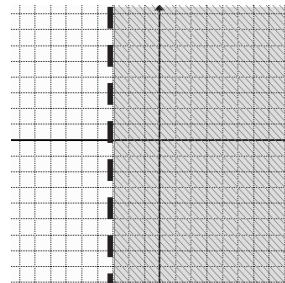
###### Opening Exercises

Graph the following:

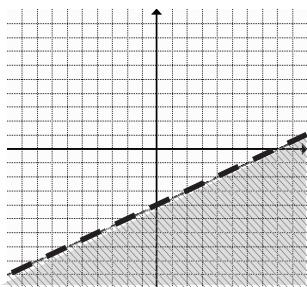
a.  $y \leq 7$



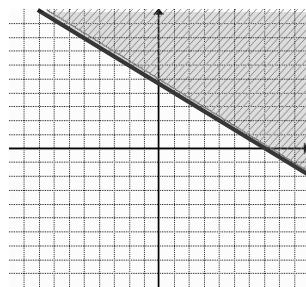
b.  $x > -3$



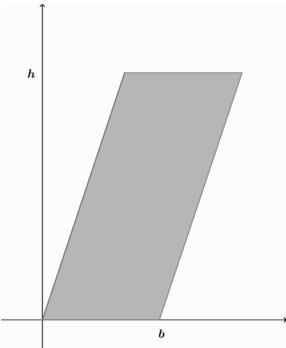
c.  $y < \frac{1}{2}x - 4$



d.  $y \geq -\frac{2}{3}x + 5$

**Example 1 (10 minutes)****Example 1**

A parallelogram with base of length  $b$  and height  $h$  can be situated in the coordinate plane as shown. Verify that the shoelace formula gives the area of the parallelogram as  $bh$ .



- What is the area of a parallelogram?
  - *Base × height*
- The distance from the  $y$ -axis to the top left vertex is some number  $x$ . What are the coordinates of that vertex?
  - $(x, h)$
- Can you determine the coordinates of the top right vertex? What do we know about opposite sides of a parallelogram?
  - *They must be equal.*
- What is the length of the bottom side?
  - $b$  units
- So what is the length of the top side?
  - $b$  units
- The bottom side starts at the origin (where  $x = 0$ ); where does the top side start? Hint: What is the  $x$ -coordinate of the top left vertex?
  - $x$

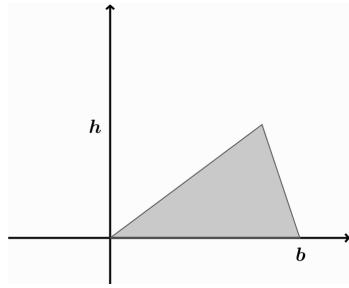
**Scaffolding:**

- If students are confused about the coordinates provide either a graph of a parallelogram with the coordinates showing (more concrete) or a graph with at least tick marks so that students may count the distances for base and height.

- So if the length is  $b$  units, what would the  $x$ -coordinate of the top right vertex be?
  - $x + b$
- So what are the coordinates of the top right vertex?
  - $(x + b, h)$
- List the coordinates of the vertices starting at the origin and moving clockwise.
  - $(0, 0), (x, h), (x + b, h)$ , and  $(b, 0)$
- Use the shoelace formula (Green's theorem) to find the area moving counterclockwise.
  - $\frac{1}{2}(0 \cdot 0 + b \cdot h + (x + b) \cdot h + x \cdot 0 - 0 \cdot b - 0(x + b) - h \cdot x - h \cdot 0) = \frac{1}{2}(b \cdot h + x \cdot h + b \cdot h - h \cdot x) = \frac{1}{2}(2(b \cdot h)) = b \cdot h$

**Example 2 (5 minutes)****Example 2**

A triangle with base  $b$  and height  $h$  can be situated in the coordinate plane as shown. According to Green's theorem, what is the area of the triangle?

**MP.1**

Let students try to do this problem on their own following the steps used above. Scaffold with the following questions as necessary.

- What is the area of a triangle?
  - $\frac{1}{2} \text{base} \times \text{height}$
- Let the distance from the  $y$ -axis to the top vertex be some number  $x$ . What are the coordinates of that vertex?
  - $(x, h)$
- List the coordinates of the vertices starting at the origin and moving clockwise.
  - $(0, 0), (x, h)$ , and  $(b, 0)$
- Use the shoelace formula (Green's theorem) to find the area moving counterclockwise.
  - $\frac{1}{2}(0 \cdot 0 + b \cdot h + x \cdot 0 - 0 \cdot b - 0 \cdot x - h \cdot 0) = \frac{1}{2}(b \cdot h)$
- Summarize what you have learned so far with a partner.
  - *We have verified well known formulas using Green's theorem.*

**Exercises 1–2 (15 minutes)**

In this exercise, students work with a partner to compute the area and perimeter of a quadrilateral region in the plane defined by a set of inequalities. Have each student do one problem, parts (a) and (b), then check in with their partner and check each other's work. Then do parts (c) and (d) and check in again. Students should graph the inequalities, solve pairs of inequalities to find the coordinates of the vertices, use the distance formula to find the perimeter, and apply the shoelace formula (Green's theorem) to find the area.

**Exercises 1–2**

1. A quadrilateral region is defined by the system of inequalities below:

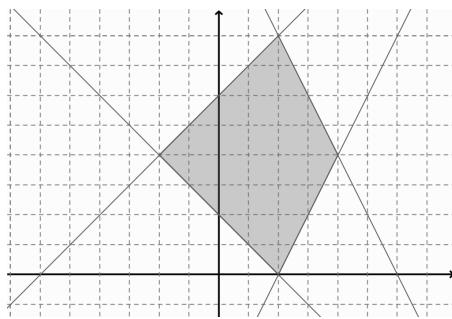
$$y \leq x + 6$$

$$y \leq -2x + 12$$

$$y \geq 2x - 4$$

$$y \geq -x + 2$$

- a. Sketch the region.



- b. Determine the vertices of the quadrilateral.

*(2, 8), (4, 4), (2, 0), (-2, 4). Ask students how they can verify the intersection points. (By showing that each set of coordinates satisfies the equations of both intersecting lines that determine the vertex.)*

- c. Find the perimeter of the quadrilateral region.

*≈ 20.26 units*

- d. Find the area of the quadrilateral region.

*24 square units*

2. A quadrilateral region is defined by the system of inequalities below:

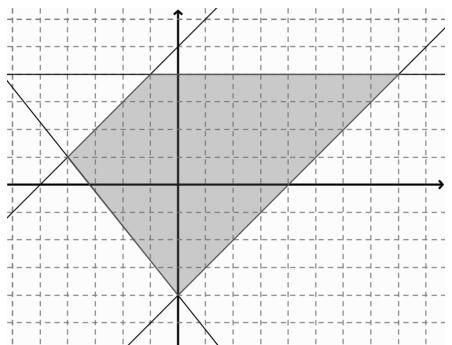
$$y \leq x + 5$$

$$y \geq x - 4$$

$$y \leq 4$$

$$y \geq -\frac{5}{4}x - 4$$

- a. Sketch the region.





- b. Determine the vertices of the quadrilateral.  
 $(-4, 1), (-1, 4), (8, 4), (0, -4)$
- c. Which quadrilateral is defined by these inequalities? How can you prove your conclusion?  
*A trapezoid. We can prove that one pair of opposite sides is parallel.*
- d. Find the perimeter of the quadrilateral region.  
 $\approx 30.95 \text{ units}$
- e. Find the area of the quadrilateral region.  
 $50 \text{ square units}$

### Closing (2 minutes)

Gather the entire class and ask these questions. Have students share answers.

- The shoelace method for finding the area of a polygon is also known as?
  - *Green's theorem*
- How did we verify the formulas for the area of a parallelogram and triangle?
  - *We used Green's theorem with variables as coordinates to verify the known formulas.*

### Exit Ticket (8 minutes)



Name \_\_\_\_\_

Date \_\_\_\_\_

## Lesson 11: Perimeters and Areas of Polygonal Regions Defined by Systems of Inequalities

### Exit Ticket

A quadrilateral region is defined by the system of inequalities below:

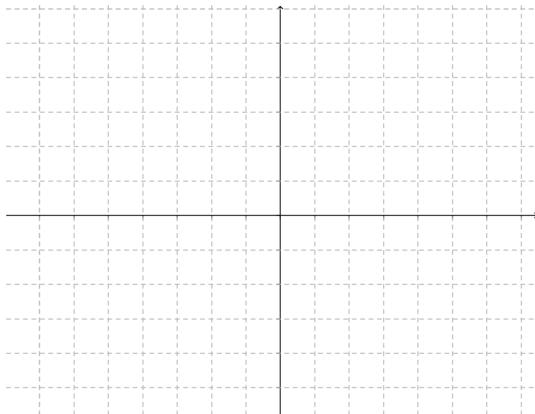
$$y \leq 5$$

$$y \geq -3$$

$$y \leq 2x + 1$$

$$y \geq 2x - 7$$

1. Sketch the region.



2. Determine the coordinates of the vertices.

3. Find the area of the quadrilateral region.

## Exit Ticket Sample Solutions

1. A quadrilateral region is defined by the system of inequalities below:

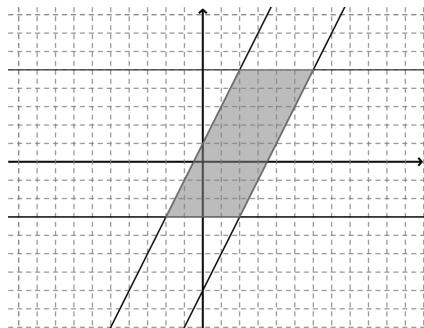
$$y \leq 5$$

$$y \geq -3$$

$$y \leq 2x + 1$$

$$y \geq 2x - 7$$

- a. Sketch the region.



- b. Determine the coordinates of the vertices.

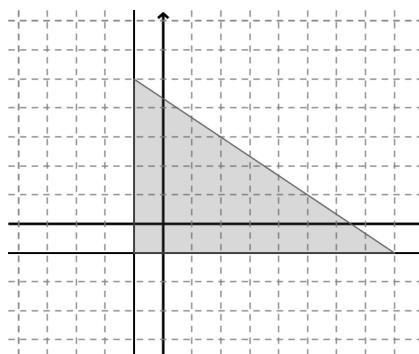
$(2, 5), (6, 5), (2, -3), (-2, -3)$

- c. Find the area of the quadrilateral region.

32 square units

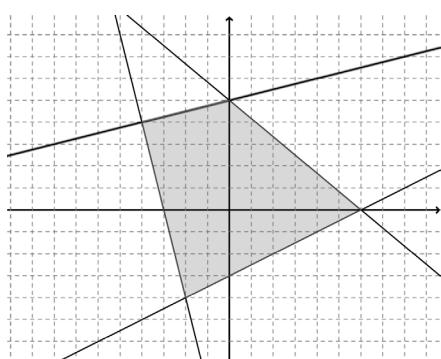
## Problem Set Sample Solutions

For Problems 1–2 below, identify the system of inequalities that defines the region shown.



$$y \geq -1, \quad x \geq -1, \quad y \leq -\frac{2}{3}x + \frac{13}{3}$$

2.



$$y \geq \frac{1}{2}x - 3$$

$$y \leq -\frac{5}{6}x + 5$$

$$y \leq \frac{1}{4}x + 5$$

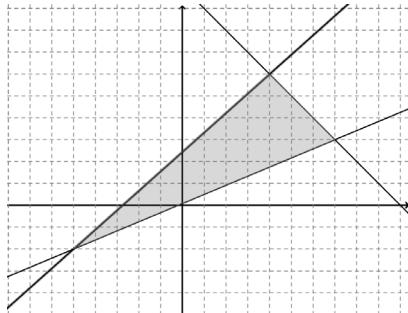
$$y \geq -4x - 12$$

For Problems 3–5 below, a triangular or quadrilateral region is defined by the system of inequalities listed.

- Sketch the region.
- Determine the coordinates of the vertices.
- Find the perimeter of the region rounded to the nearest hundredth if necessary.
- Find the area of the region rounded to the nearest tenth if necessary.

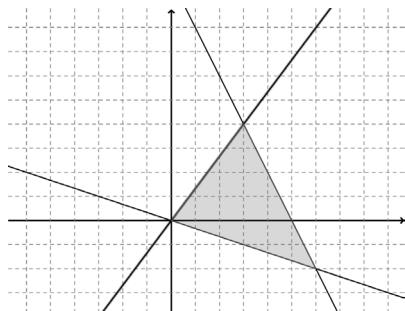
3.  $8x - 9y \geq -22$        $x + y \leq 10$        $5x - 12y \leq -1$

a.



- (4, 6), (7, 3), (-5, -2)
- $\approx 29.28$  units
- 25.5 square units

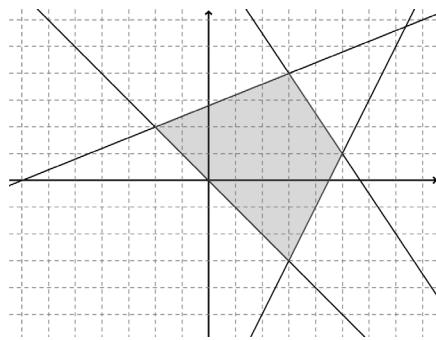
4.  $x + 3y \geq 0$        $4x - 3y \geq 0$        $2x + y \leq 10$

**a.**

**b.**  $(3, 4), (6, -2), (0, 0)$

**c.**  $\approx 18.03$  units**d.** 15 square units

5.  $2x - 5y \geq -14$        $3x + 2y \leq 17$        $2x - y \leq 9$        $x + y \geq 0$

**a.**

**b.**  $(3, 4), (5, 1), (3, -3), (-2, 2)$

**c.**  $\approx 20.53$  units**d.** 24.5 square units



## Topic D:

## Partitioning and Extending Segments and Parameterization of Lines

**G-GPE.B.4, G-GPE.B.6**

<b>Focus Standard:</b>	G-GPE.B.4      Use coordinates to prove simple geometric theorems algebraically. <i>For example, prove or disprove that a figure defined by four given points in the coordinate plane is a rectangle; prove or disprove that the point <math>(1, \sqrt{3})</math> lies on the circle centered at the origin and containing the point <math>(0, 2)</math>.</i>
	G-GPE.B.6      Find the point on a directed line segment between two given points that partitions the segment in a given ratio.
<b>Instructional Days:</b>	4
<b>Lesson 12:</b>	Dividing Segments Proportionately (P) <sup>1</sup>
<b>Lesson 13:</b>	Analytic Proofs of Theorems Previously Proved by Synthetic Means (P)
<b>Lesson 14:</b>	Motion Along a Line—Search Robots Again (Optional) (P)
<b>Lesson 15:</b>	The Distance from a Point to a Line (S)

Topic D concludes the work of Module 4. In Lesson 12, students find midpoints of segments and points that divide segments into 3 or more equal and proportional parts. Students will also find locations on a directed line segment between two given points that partition the segment in given ratios (**G-GPE.B.6**). Lesson 13 requires students to show that if  $B'$  and  $C'$  cut  $\overline{AB}$  and  $\overline{AC}$  proportionately, then the intersection of  $\overline{BC'}$  and  $\overline{C'C}$  lies on the median of  $ABC$  from vertex  $A$  and connects this work to proving classical results in geometry (**G-GPE.B.4**). For instance, the diagonals of a parallelogram bisect one another and the medians of a triangle meet at the point  $\frac{2}{3}$  of the way from the vertex for each.

Lesson 14 is an optional lesson that allows students to explore parametric equations and compare them with more familiar linear equations (**G-GPE.B.6, G-MG.A.1**). Parametric equations make both the  $x$ - and  $y$ -variables in an equation dependent on a third variable, usually time; for example,  $f(t) = (t, 2t - 1)$

<sup>1</sup> Lesson Structure Key: P-Problem Set Lesson, M-Modeling Cycle Lesson, E-Exploration Lesson, S-Socratic Lesson

represents a function,  $f$ , with both  $x$ - and  $y$ -coordinates dependent on the independent variable,  $t$  (time). In this lesson, parametric equations model the robot's horizontal and vertical motion over a period of time,  $P = \left(20 + \frac{t}{2}(100), 10 + \frac{t}{2}(40)\right)$ . Introducing parametric equations in the Geometry course prepares students for higher level courses and also represents an opportunity to show coherence between functions, algebra, and coordinate geometry. Students extend their knowledge of parallel and perpendicular lines to lines given parametrically. Students complete the work of this module in Lesson 15 by deriving and applying the distance formula (**G-GPE.B.4**) and with the challenge of locating the point along a line closest to a given point, again given as a robot challenge.



## Lesson 12: Dividing Segments Proportionately

### Student Outcomes

- Students find midpoints of segments and points that divide segments into 3, 4, or more proportional, equal parts.

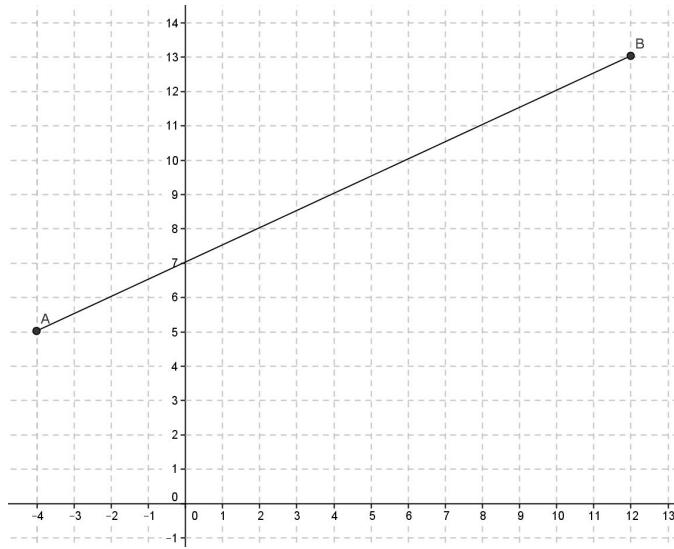
### Classwork

#### Opening Exercise (15 minutes)

The students extend their understanding of the midpoint of a segment to partition segments into ratios other than 1:1.

Give each student a piece of graph paper, and do the following exercise as a class.

- Plot the points  $A(-4, 5)$  and  $B(12, 13)$ .

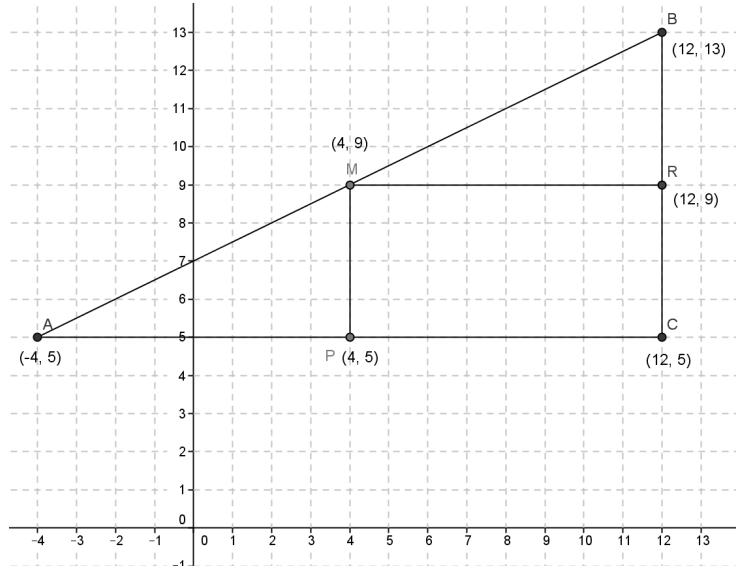


#### Scaffolding:

Suggest students translate the segment so that one endpoint lies at the origin.  $A'(0,0)$  and  $B'(16, 8)$  using the translation vector  $\langle 4, -5 \rangle$  and then translate them back when all points are located using  $\langle -4, 5 \rangle$ . Provide students who are still struggling with an already-graphed translation on a coordinate grid.

- Draw the “up an over” triangle. Label the point at the right angle  $C$ .
- What is the length of  $\overline{AC}$ ?
  - 16 units
- What is the length of  $\overline{BC}$ ?
  - 8 units
- Mark the halfway point on  $\overline{AC}$  and label it point  $P$ . What are the coordinates of point  $P$ ?
  - (4, 5)

- Mark the halfway point on  $\overline{BC}$  and label it point  $R$ . What are the coordinates of point  $R$ ?
  - $(12, 9)$
- Draw a segment from  $P$  to  $\overline{AB}$  perpendicular to  $\overline{AC}$ . Mark the intersection point  $M$ . What are the coordinates of  $M$ ?
  - $(4, 9)$
- Draw a segment from  $R$  to  $\overline{AB}$  perpendicular to  $\overline{BC}$ . What do you notice?
  - *The intersection point is point  $(4, 9)$ .*
- Describe to your neighbor how we found point  $M$ .
  - *We found the halfway point of the  $x$ -distance and the halfway point of the  $y$ -distance and drew perpendicular segments to the segment joining  $A$  and  $B$ .*
- Point  $M$  is called the “midpoint” of  $\overline{AB}$ .
- Ask students to verbally repeat this word and summarize its meaning to a neighbor. Call on students to share their definitions and record them in their notebooks.
- Look at the coordinates of the endpoints and the midpoint. Can you describe how to find the coordinates of the midpoint knowing the endpoints algebraically?
  - *$4$  is the average of the  $x$ -coordinates:  $\frac{-4+12}{2} = 4$ .*
  - *$9$  is the average of the  $y$ -coordinates:  $\frac{5+13}{2} = 9$ .*
- Let’s try to find the midpoint a slightly different way. Starting at point  $A$ , describe how to find the midpoint. Starting at point  $A$  ...
  - *Starting at point  $A$ , find the horizontal distance from  $A$  to  $B$  ( $12 - (-4) = 16$ ), divide by  $2(16 \div 2 = 8)$ , and add that value to the  $x$ -coordinate value of  $A(-4 + 8 = 4)$ . To find the  $y$ -coordinate of the midpoint, find the vertical distance from  $A$  to  $B$  ( $13 - 5 = 8$ ), divide by  $2(8 \div 2 = 4)$ , and add that value to the  $y$ -coordinate of  $B(5 + 4 = 9)$ .*
- Write this process as a formula.
  - $M\left(-4 + \frac{1}{2}(12 - (-4)), 5 + \frac{1}{2}(13 - 5)\right)$
- Explain to your neighbor the two ways that we found the midpoint.
- How would this formula change if we started at endpoint  $B$  instead of  $A$ ?
  - *Instead of adding half the distance between the two endpoints, we would subtract half the distance because we would be moving to the right on our segment.*



- Now write a general formula for the midpoint of a segment with endpoints  $(x_1, y_1)$  and  $(x_2, y_2)$  using the average formula and then the formula starting with endpoint A.

- $\left( \frac{x_1+x_2}{2}, \frac{y_1+y_2}{2} \right)$  or  $\left( \frac{1}{2}(x_1 + x_2), \frac{1}{2}(y_1 + y_2) \right)$
- $M \left( x_1 + \frac{1}{2}(x_2 - x_1), y_1 + \frac{1}{2}(y_2 - y_1) \right)$

- Why do you think all formulas have  $\frac{1}{2}$  in them?
  - The midpoint is half way between the endpoints, so it makes sense that all formulas have  $\frac{1}{2}$  in them.

MP.7

- What if we wanted to find the point that is one-quarter of the way along segment  $\overline{AB}$ , closer to A than to B? How would the formula change? Can you find that point on the graph? Explain how to find that point.

- Instead of  $\frac{1}{2}$  in the second formula, you would use  $\frac{1}{4}$ . Divide the horizontal and vertical distances into four equal segments, start at endpoint A and count  $\frac{1}{4}$  of the way from A toward B. The coordinates of the point are  $(0, 7)$ .

- Can you write a formula using the endpoint to get this point?

- $x: -4 + \frac{(12 - (-4))}{4} = 0$
- $y: 5 + \frac{13 - 5}{4} = 7$

- Another way to ask the question above would be to find the point on the directed segment from  $(-4, 5)$  to  $(12, 13)$  that divides it in the ratio of 1:3. Explain to your neighbor how this is the same question.

- A ratio of 1:3 is the fraction  $\frac{1}{4}$ , so we find the point on the segment with the given endpoints  $\frac{1}{4}$  the distance from  $(-4, 5)$  to  $(12, 13)$ , which is the same question as above.

- Now find the coordinates of the point that sit  $\frac{1}{8}$  of the way along segment  $\overline{AB}$  closer to A than to B, and show how to get that point using a formula.

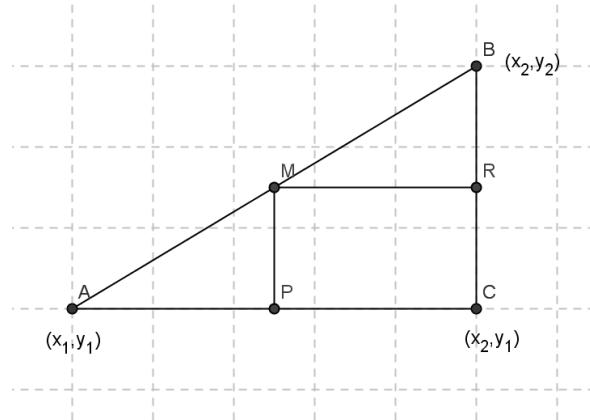
- $x: -4 + \frac{12 - (-4)}{8} = -2$
- $y: 5 + \frac{13 - 5}{8} = 6$
- $(-2, 6)$

- What is another way to state this problem using a ratio and the term *directed segment*?

- Find the point on the directed segment from  $(-4, 5)$  to  $(12, 13)$  that divides it into a ratio of 1:7.

- Find the point on the directed segment from  $(-4, 5)$  to  $(12, 13)$  that divides it into a ratio of 1:15.

- $x: -4 + \frac{12 - (-4)}{16} = -3$
- $y: 5 + \frac{13 - 5}{16} = 5.5$
- $(-3, 5.5)$



As students are working make a note of which students, if any, are calculating the coordinates using the given proportion and which are repeatedly calculating midpoints. When the students have finished finding the coordinates of the points, select two students to share their approaches. The student who used the midpoint approach should present first, followed by the student using the more direct approach.

MP.3

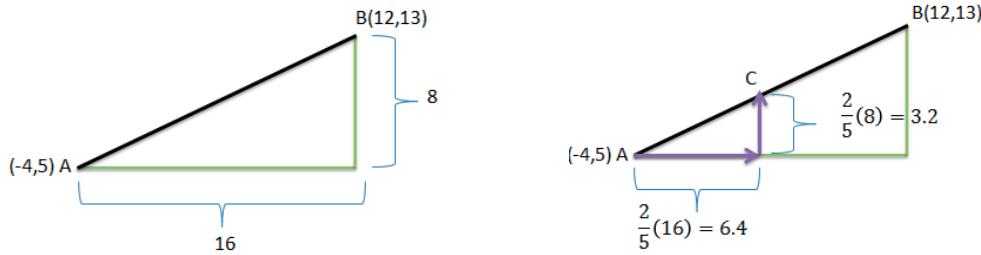
- Approach 1: The student finds the midpoint of each successive segment by calculating the mean of the ordinates and the abscissas.
- Approach 2: The student determines how the vertical and horizontal distances each point lies from point  $A$  based upon the given fraction, calculates these distances and adds them to the ordinate and abscissa of point  $A$ .

### Example 1 (6 minutes)

Students will now extend/apply the understanding of partitioning a segment proportionally to the next problem. Students will not be able to use the method of finding successive midpoints as they may have done in the Opening Exercise. Ask guiding questions as students work through the example.

Given points  $A(-4,5)$  and  $B(12,13)$ , find the coordinates of the point,  $C$ , that sits  $\frac{2}{5}$  of the way along the segment  $\overline{AB}$ , closer to  $A$  than it is to  $B$ .

- Can we find the coordinates of this point by finding the coordinates of midpoints as many of you did in the Opening Exercise?
  - No. In the Opening Exercise we were finding points  $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}$ , and  $\frac{1}{16}$  of the way along a segment. Each of these fractions is a power of  $\frac{1}{2}$ , so we could use successive midpoints to identify the coordinates of these points. This method will not work if the point is  $\frac{2}{5}$  of the way along the segment.
- Can we use proportions to find the coordinates of point  $C$ ?
  - Yes. Use a diagram similar to that shown below. The coordinates of point  $C$  are  $(2.4, 8.2)$ .



Students calculate  $\frac{2}{5}$  of the horizontal and vertical distances (or vectors) between points  $A$  and  $B$  and add these values to the abscissa and ordinate respectively of point  $A$ :  $(-4 + 6.4, 5 + 3.2)$ .

- Can you use what you know about the slope to verify that point  $C$  lies on segment  $AB$ ?
  - Segment  $AB$  lies on a line having a slope of  $\frac{1}{2}$ . We can move from any point on the line to another point on the line by moving to the right a certain distance and up half of that distance, or to the left a certain distance and down half of that distance. To locate point  $C$  we moved to the right 6.4 units and up half of that distance, 3.2 units. Therefore, point  $C$  must lie on the line containing segment  $AB$ .



- How can you relate this idea to our work with similar right triangles?
  - Using the diagrams above, we can see that the perpendiculars to the horizontal base dropped from points C and B divide the original right triangle into two similar triangles whose sides are proportional and whose angles are congruent.
- The problem asked us to find the location of the point that sits  $\frac{2}{5}$  of the way along segment AB, closer to A than to B. How can we use the proportion  $\frac{AC}{AB} = \frac{2}{5}$  to verify that point C meets the original requirement?
  - If point C lies  $\frac{2}{5}$  of the way along segment AB then the distance from A to C will be  $\frac{2}{5}$  distance from A to B:  $AC = \frac{2}{5} AB \rightarrow \frac{AC}{AB} = \frac{\frac{2}{5}AB}{AB} = \frac{2}{5}$ . Using the distance formula we calculate the length of segments AC and AB:
  - $AC = \sqrt{(2.4 - (-4))^2 + (8.2 - 5)^2} = \sqrt{51.2}$  (We could also have just used the distances we calculated in part 1, 6.4, and 3.2.)
  - $AB = \sqrt{(12 - (-4))^2 + (13 - 5)^2} = \sqrt{320}$  (We could also have just used the distances we calculated in part 1, 16, and 8.)
  - $\frac{AC}{AB} = \frac{\sqrt{51.2}}{\sqrt{320}} = \sqrt{\frac{51.2}{320}} = \sqrt{\frac{4}{25}} = \frac{2}{5}$
- Given points A(-4,5) and B(12,13), find the coordinates of the point, D, that sits two-fifths of the way along  $\overline{AB}$ , closer to B than it is to A.
  - $D(5.6, 9.8)$
  - Method 1:  $\left(12 - \frac{2}{5}(16), 13 - \frac{2}{5}(8)\right)$
  - Method 2:  $\left(-4 + \frac{3}{5}(16), 5 + \frac{3}{5}(8)\right)$
- Will point D coincide with point C? Put another way, will the coordinates for point D be the same as the coordinates for point C?
  - No. The only way the two points would occupy the same location is if they were  $\frac{1}{2}$  the distance along the segment (i.e., midpoint).
- Can we use our work from Example 1 to locate point D?
  - Yes. We can do one of two things:
    - (1) We can still calculate  $\frac{2}{5}$  of the vertical and horizontal distances traveled when moving from point A to point B, but we should subtract these values from the abscissa and ordinate of point B.
    - (2) If the point is  $\frac{2}{5}$  of the distance along segment AB but closer to point B than point A, then we will have to move  $\frac{3}{5}$  of the length of the segment from point A to reach the point. Therefore, we can calculate  $\frac{3}{5}$  of the horizontal and vertical distances and add these values to the abscissa and ordinate of point A.

**Example 2 (6 minutes)**

Students further extend their work on partitioning a line segment to determine the location of a point on a segment that divides the segment into a given proportion. In Example 2, students divide the segment based on a *part : part* ratio instead of a *part : whole* ratio.

- Given points  $P(10,10)$  and  $Q(0,4)$ , find point  $R$  on  $\overline{PQ}$  such that  $\frac{PR}{RQ} = \frac{7}{3}$ . Hint: Draw a picture.
- How does this problem differ from the previous examples?
  - In the previous examples, we were comparing the distance from the new point to the entire length of the segment. Now, we are comparing lengths of the two parts of the segment: the length of segment  $PR$  and the length of segment  $RQ$ . Point  $R$  partitions segment  $PQ$  such that  $PR:RQ = 7:3$ . This means  $PR:PQ = 7:10$  and  $RQ:PQ = 3:10$ .
- Just as we used two methods for finding point  $D$  in Example 1 above, we can use either addition or subtraction to find the location of point  $R$  here. Have half the class use addition and half use subtraction; then have volunteers from each group explain their procedure.
  - Since  $R$  is located  $\frac{3}{10}$  of the way along segment  $PQ$ , closer to  $Q$  than to  $P$ , we can calculate the coordinate of  $R$  using one of the following methods:
    - (1) Method 1: Moving  $\frac{3}{10}$  of the length of segment  $PQ$  from point  $Q$ ,  $R\left(0 + \frac{3}{10}(10), 4 + \frac{3}{10}(6)\right)$ .
    - (2) Method 2: Moving  $\frac{7}{10}$  of the length of segment  $PQ$  from point  $P$ ,  $R\left(10 - \frac{7}{10}(10), 10 - \frac{7}{10}(6)\right)$ .
- What are the coordinates of point  $R$ ?
  - $R(3, 5.8)$

**Exercises 1–4 (10 minutes)**

Exercise 4 is intended as an extension for students who complete Exercises 1–3 quickly.

**Exercises 1–5**

1. Find the midpoint of  $\overline{ST}$  given  $S(-2, 8)$  and  $T(10, -4)$ .

$$M\left(\frac{1}{2}(-2+10), \frac{1}{2}(8-4)\right) = M(4, 2)$$

2. Find the point on the directed segment from  $(-2, 0)$  to  $(5, 8)$  that divides it in the ratio of  $1:3$ .

A ratio of  $1:3$  means  $\frac{1}{4}$  of the way from  $(-2, 0)$  to  $(5, 8)$ .

$$\left(-2 + \frac{1}{4}(5 - (-2)), 0 + \frac{1}{4}(8 - 0)\right) = \left(-\frac{1}{4}, 2\right)$$

3. Given  $\overline{PQ}$  and point  $R$  that lies on  $\overline{PQ}$  such that point  $R$  lies  $\frac{7}{9}$  of the length of  $\overline{PQ}$  from point  $P$  along  $\overline{PQ}$ .

- a. Sketch the situation described.





- b. Is point  $R$  closer to  $P$  or closer to  $Q$ , and how do you know?

*R is closer to Q because it lies more than halfway along the segment from point P.*

- c. Use the given information to determine the following ratios:

i.  $PR: PQ$

$$PR: PQ = 7: 9$$

ii.  $RQ: PQ$

$$RQ: PQ = 2: 9$$

iii.  $PR: RQ$

$$PR: RQ = 7: 2$$

iv.  $RQ: PR$

$$RQ: PR = 2: 7$$

- d. If the coordinates of point  $P$  are  $(0, 0)$  and the coordinates of point  $R$  are  $(14, 21)$ , what are the coordinates of point  $Q$ ?

$$(18, 27)$$

4. A robot is at position  $A(40, 50)$  and is heading toward the point  $B(2000, 2000)$  along a straight line at a constant speed. The robot will reach point  $B$  in 10 hours.

- a. What is the location of the robot at the end of the third hour?

$$(628, 635); \text{The robot will be located } \frac{3}{10} \text{ of the length of } \overline{AB} \text{ away from point } A \text{ along } \overline{AB}.$$

$$\left(40 + \frac{3}{10}(2000 - 40), 50 + \frac{3}{10}(2000 - 50)\right)$$

- b. What is the location of the robot five minutes before it reaches point  $B$ ?

$$\left(1983\frac{1}{3}, 1983\frac{3}{4}\right); \text{The robot will be located } \frac{595}{600} \text{ of the length of } \overline{AB} \text{ away from point } A \text{ along } \overline{AB}.$$

$$\left(40 + \frac{595}{600}(2000 - 40), 50 + \frac{595}{600}(2000 - 50)\right)$$

- c. If the robot keeps moving along the straight path at the same constant speed as it passes through point  $B$ , what will be its location at the twelfth hour?

$$(2392, 2390); \text{The robot will be located } \frac{12}{10} \text{ of the length of } \overline{AB} \text{ away from point } A \text{ along } \overline{AB}.$$

$$\left(40 + \frac{12}{10}(2000 - 40), 50 + \frac{12}{10}(2000 - 50)\right)$$



- d. Compare the value of the abscissa ( $x$ -coordinate) to the ordinate ( $y$ -coordinate) before, at, and after the robot passes point  $B$ ?

*Initially, the abscissa was greater than the ordinate. As the robot moved toward point  $B$  these values got closer to being equal. At point  $B$  they were equal and, for all points on the path beyond point  $B$ , the  $y$ -coordinate was greater than the  $x$ -coordinate.*

- e. Could you have predicted the relationship that you noticed in part (d) based on the coordinates of points  $A$  and  $B$ ?

*Yes. If point  $A$  was located at the origin, the path that the robot took would have been described by the equation  $y = x$ ; then at each location the robot occupied, the  $x$ - and the  $y$ -coordinates would have been equal. Point  $A$  actually lies above the origin, making the slope of the line that describes the robot's actual path less than one. Initially, the  $y$ -coordinate of each point (location) is greater than the  $x$ -coordinate because the line has a  $y$ -intercept greater than zero. The slope of the line is less than 1, so as the robot moves to the right, the "gap" closes because for each unit the robot moves to the right, it moves less than one unit up. When the robot reaches point  $B(2000, 2000)$ , the abscissa and the ordinate are equal. Beyond point  $B$  the  $x$ -coordinate will be greater than the  $y$ -coordinate.*

### Closing (3 minutes)

- How did we extend our understanding of midpoint to divide segments proportionally?
  - When we find the location of the midpoint, we are dividing a segment into two congruent segments and, therefore, need to calculate half of the vertical and horizontal distances. We then added or subtracted these values from the coordinates of one of the endpoints of the segment. In this lesson we divided the segment into two segments of different lengths. We had to determine vertical and horizontal distances other than  $\frac{1}{2}$  and then use these values to determine the location of the point. We could use either endpoint to do this, but we had to be careful to add and/or subtract depending on whether we were moving right/left or up/down along the segment and which endpoint the point was closest to.
- What is the midpoint formula? (See Opening Exercise.)
  - $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$
- If you need to partition a segment into fractional parts other than powers of  $\frac{1}{2}$ , you may use a vector approach. Simply multiply the horizontal and vertical vectors between the two points by the requested ratio and either add the results to the first point or subtract from the second point to obtain the coordinates of the partitioning point. (See Example 1.)

### Exit Ticket (5 minutes)

Optional: Allow students to complete Problem 1, Problem 2, or both problems on the Exit Ticket.



Name \_\_\_\_\_

Date \_\_\_\_\_

## Lesson 12: Dividing Segments Proportionately

### Exit Ticket

- Given points  $A(3, -5)$  and  $B(19, -1)$ , find the coordinates of point  $C$  that sit  $\frac{3}{8}$  of the way along  $\overline{AB}$ , closer to  $A$  than to  $B$ .
- Given points  $A(3, -5)$  and  $B(19, -1)$ , find the coordinates of point  $C$  such that  $\frac{CB}{AC} = \frac{1}{7}$ .



## Exit Ticket Sample Solutions

1. Given points  $A(3, -5)$  and  $B(19, -1)$ , find the coordinates of point  $C$  that sit  $\frac{3}{8}$  of the way along  $\overline{AB}$ , closer to  $A$  than to  $B$ .

$$\begin{aligned} \left(3 + \frac{3}{8}(19 - 3), -5 + \frac{3}{8}(-1 - (-5))\right) &= \left(3 + \frac{3}{8}(16), -5 + \frac{3}{8}(4)\right) \\ &= (3 + 6, -5 + 1.5) \\ &= (9, -3.5) \end{aligned}$$

2. Given points  $A(3, -5)$  and  $B(19, -1)$ , find the coordinates of point  $C$  such that  $\frac{CB}{AC} = \frac{1}{7}$ .

$$\begin{aligned} \left(3 + \frac{7}{8}(19 - 3), -5 + \frac{7}{8}(-1 - (-5))\right) &= \left(3 + \frac{7}{8}(16), -5 + \frac{7}{8}(4)\right) \\ &= (3 + 14, -5 + 3.5) \\ &= (17, -1.5) \end{aligned}$$

## Problem Set Sample Solutions

1. Given  $F(0, 2)$  and  $G(2, 6)$ . If point  $S$  lies  $\frac{5}{12}$  of the way along  $\overline{FG}$ , closer to  $F$  than to  $G$ , find the coordinates of  $S$ . Then verify that this point lies on  $\overline{FG}$ .

$$\begin{aligned} S\left(\frac{5}{6}, 3\frac{2}{3}\right) \\ \left(0 + \frac{5}{12}(2 - 0), 2 + \frac{5}{12}(6 - 2)\right) &= \left(0 + \frac{5}{12}(2), 2 + \frac{5}{12}(4)\right) \\ &= \left(\frac{5}{6}, 1\frac{2}{3}\right) \end{aligned}$$

*Verification that  $S$  lies on  $\overline{FG}$ :*

$\overline{FG}$  has a slope of 2 (i.e.,  $\frac{6-2}{2-0}$ ). The slopes of  $\overline{FS}$  and  $\overline{SG}$  must also have a slope of 2 if the point  $S$  lies on the line including segment  $FG$ . The slope of  $\overline{FS}$  is  $\frac{\frac{5}{3}-2}{\frac{5}{6}-0} = \frac{\frac{5}{3}}{\frac{5}{6}} = 2$ , and the slope of  $\overline{SG}$  is  $\frac{6-\frac{5}{3}}{\frac{5}{6}-0} = \frac{\frac{13}{3}}{\frac{5}{6}} = \frac{7}{3} = 2$ . Therefore, point  $S$  does, in fact, lie on  $\overline{FG}$ .

2. Point  $C$  lies  $\frac{5}{6}$  of the way along  $\overline{AB}$ , closer to  $B$  than to  $A$ . If the coordinates of point  $A$  are  $(12, 5)$  and the coordinates of point  $C$  are  $(9.5, -2.5)$ , what are the coordinates of point  $B$ ?

$B(9, -4)$

3. Find the point on the directed segment from  $(-3, -2)$  to  $(4, 8)$  that divides it into a ratio of 3:2.

$\left(\frac{6}{5}, 4\right)$



4. A robot begins its journey at the origin, point  $O$ , and travels along a straight line path at a constant rate. Fifteen minutes into its journey the robot is at  $A(35, 80)$ .

- a. If the robot does not change speed or direction, where will it be 3 hours into its journey (call this point  $B$ )?

$(35 \times 12, 80 \times 12) = (420, 960)$ . Multiply by 12 since there are twelve 15-minute periods in 3 hours.

- b. The robot continues past point  $B$  for a certain period of time until it has traveled an additional  $\frac{3}{4}$  the distance it traveled in the first 3 hours and stops.

- i. How long did the robot's entire journey take?

$$3 \times 1\frac{3}{4} = 5\frac{1}{4} \text{ hours}$$

- ii. What is the robot's final location?

$$(420 \times 1.75, 960 \times 1.75) = (735, 1680)$$

- iii. What was the distance the robot traveled in the last leg of its journey?

$$\sqrt{(735 - 420)^2 + (1680 - 960)^2} \sim 785.9 \text{ units}$$

5. Given  $\overline{LM}$  and point  $R$  that lies on  $\overline{LM}$ , identify the following ratios given that point  $R$  lies  $\frac{a}{b}$  of the way along  $\overline{LM}$ , closer to  $L$  than to  $M$ .

- a.  $LR: LM$

$$LR: LM = a: b$$

- b.  $RM: LM$

$$RM: LM = (b - a): b$$

- c.  $RL: RM$

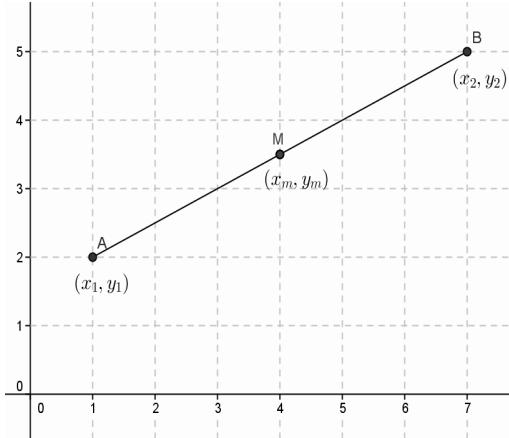
$$RL: RM = a: (b - a)$$

6. Given  $\overline{AB}$  with midpoint  $M$  as shown, prove that the point on the directed segment from  $A$  to  $B$  that divides  $\overline{AB}$  into a ratio of 1:3 is the midpoint of  $\overline{AM}$ .

The point dividing the segment into a ratio of 1:3 is  $\left(x_1 + \frac{1}{4}(x_2 - x_1), y_1 + \frac{1}{4}(y_2 - y_1)\right)$ . The midpoint of  $\overline{AM}$  is  $\left(\frac{1}{2}(x_1 + x_m), \frac{1}{2}(y_1 + y_m)\right)$ . If  $M$  is the midpoint of  $\overline{AB}$ ,  $x_m = \frac{1}{2}(x_1 + x_2)$ , and  $y_m = \frac{1}{2}(y_1 + y_2)$ . Therefore, the x-coordinate of the midpoint of  $\overline{AM}$  can be written as

$$\begin{aligned} \frac{1}{2}\left(x_1 + \frac{1}{2}(x_1 + x_2)\right) &= \frac{1}{2}\left(x_1 + \frac{1}{2}x_1 + \frac{1}{2}x_2\right) = \\ \frac{1}{2}\left(\frac{3}{2}x_1 + \frac{1}{2}x_2\right) &= \frac{3}{4}x_1 + \frac{1}{4}x_2. \end{aligned}$$

If we simplify  $x_1 + \frac{1}{4}(x_2 - x_1) = x_1 + \frac{1}{4}x_2 - \frac{1}{4}x_1 = \frac{3}{4}x_1 + \frac{1}{4}x_2$ . The y-coordinate can be similarly obtained, meaning they are the same point.





## Lesson 13: Analytic Proofs of Theorems Previously Proved by Synthetic Means

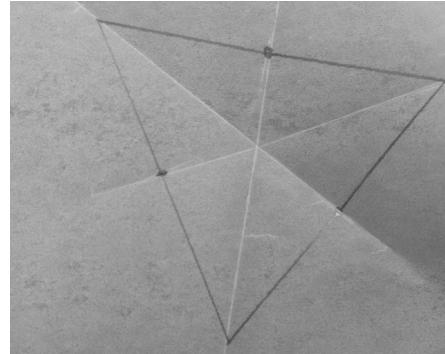
### Student Outcomes

- Using coordinates, students prove that the intersection of the medians of a triangle meet at a point that is two-thirds of the way along each median from the intersected vertex.
- Using coordinates, students prove the diagonals of a parallelogram bisect one another and meet at the intersection of the segments joining the midpoints of opposite sides.

### Lesson Notes

This lesson highlights MP.3 as students develop and justify conjectures. The lesson focuses on proofs and can be extended to a two day lesson if students need additional practice.

In the Opening Exercise, students do a paper folding activity to review the fact that the medians of any triangle intersect at one point. Next, students determine the coordinates of the point of concurrency of the medians of a given triangle. Students then prove that the medians of any triangle are concurrent and that the point of concurrency is located one third of the length of the median from the midpoint of the side of the triangle.



### Classwork

#### Opening (5 minutes)

Have students draw triangles on patty paper then, focusing on one side of the triangle at a time, fold the patty paper so that the two endpoints of the segments coincide and make a small crease marking the midpoint of that segment. To save time, the triangles can be drawn on the patty paper ahead of time, but make sure to draw a variety of triangles: acute, obtuse, right, scalene, isosceles, and equilateral. Repeat the process for all three sides and then mark the midpoints with a pencil. Next, create a crease through the midpoint of one of the sides and the vertex opposite that side. Repeat this for all three sides of the triangle.

- What segments are contained on the creases that you constructed?
- What do you notice about these segments?
- Do you think this will be the case for all triangles? Are all of the triangles you constructed congruent? Did the creases intersect at one point on all of your triangles?
- Given two points  $A(a_1, a_2)$  and  $B(b_1, b_2)$ , what is the midpoint of  $\overline{AB}$ ?

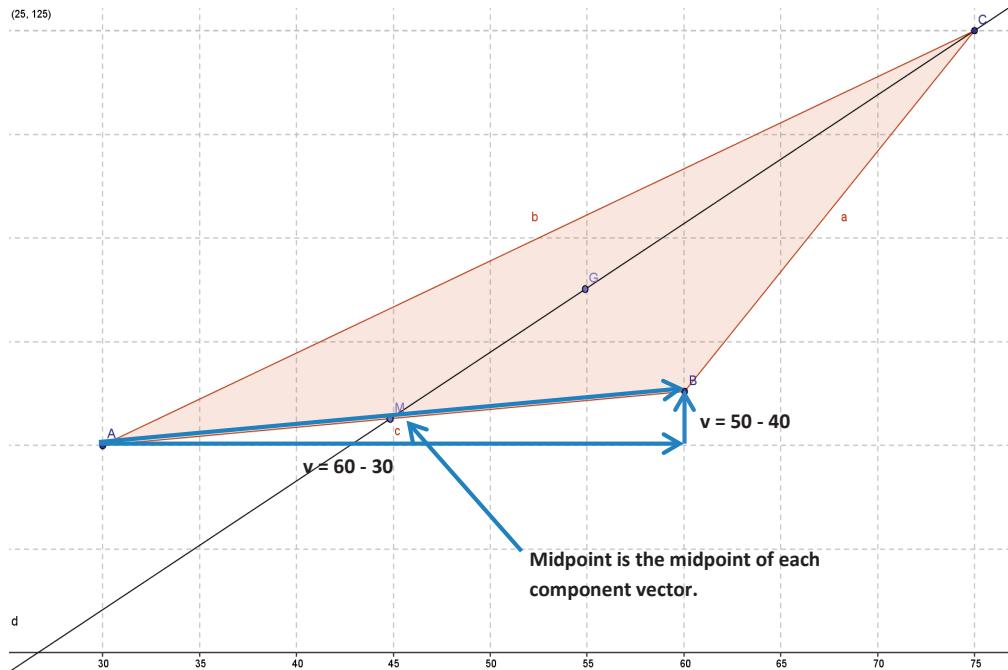
**Opening Exercise (15 minutes)**

In the previous lesson, students learned that given two points  $A(a_1, a_2)$  and  $B(b_1, b_2)$ , the midpoint of  $\overline{AB}$  is  $\left(\frac{1}{2}(a_1 + b_1), \frac{1}{2}(a_2 + b_2)\right)$ . In this exercise, students extend their knowledge of the midpoint of a segment to find the point on each median of a given triangle that is one third of the distance from the side of the triangle to the opposite vertex. The students will discover that this point is the same for all three sides of the triangle hence demonstrating that not only are the medians of a triangle concurrent, they intersect at a point that divides each median into a ratio of 1:2. This opening exercise may be split up in several ways; each group could be given one of each type of triangle and each work with a different vertex, or each group could be given one type of triangle and work with all vertices of that triangle. Regardless, bring the class together in the end to discuss findings.

**Opening Exercise**

Let  $A(30, 40)$ ,  $B(60, 50)$ , and  $C(75, 120)$  be vertices of a triangle.

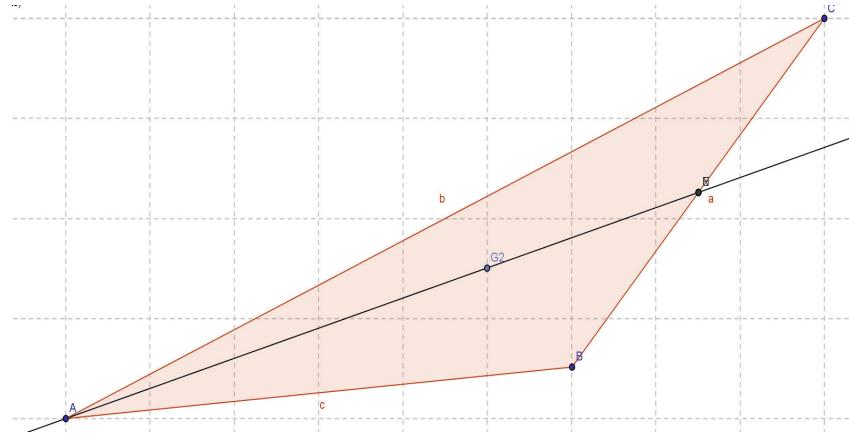
- a. Find the coordinates of the midpoint  $M$  of  $\overline{AB}$  and the point  $G_1$  that is the point one-third of the way along  $\overline{MC}$ , closer to  $M$  than to  $C$ .



$$M = \left(30 + \frac{1}{2}(60 - 30), 40 + \frac{1}{2}(50 - 40)\right) = (45, 45) \text{ or } M = \left(\frac{1}{2}(30 + 60), \frac{1}{2}(40 + 50)\right) = (45, 45)$$

$$G_1: \left(45 + \frac{1}{3}(75 - 45), 45 + \frac{1}{3}(120 - 45)\right) = (55, 70) \text{ or } \left(\frac{1}{3}(45 + 45 + 75), \frac{1}{3}(45 + 45 + 120)\right) = (55, 70)$$

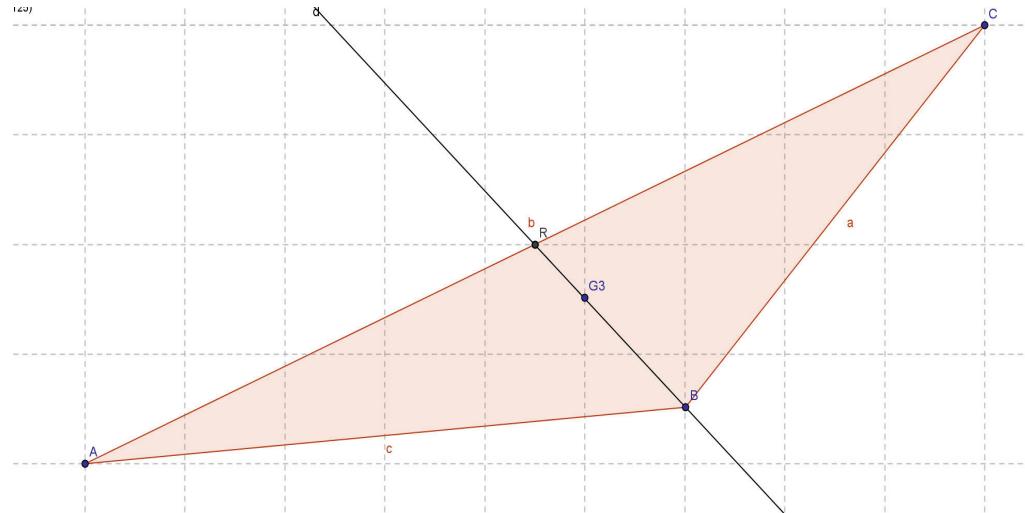
- b. Find the coordinates of the midpoint  $N$  of  $\overline{BC}$  and the point  $G_2$  that is the point one-third of the way along  $\overline{NA}$ , closer to  $N$  than to  $A$ .



$$N = \left( 60 + \frac{1}{2}(75 - 60), 50 + \frac{1}{2}(40 - 50) \right) = (67.5, 85) \text{ or } N = \left( \frac{1}{2}(60 + 75), \frac{1}{2}(50 + 40) \right) = (67.5, 85)$$

$$G_2: \left( 67.5 + \frac{1}{3}(30 - 67.5), 85 + \frac{1}{3}(40 - 85) \right) = (55, 70) \text{ or } \left( \frac{1}{3}(30 + 67.5 + 67.5), \frac{1}{3}(40 + 85 + 85) \right) = (55, 70)$$

- c. Find the coordinates of the midpoint  $R$  of  $\overline{CA}$  and the point  $G_3$  that is the point one-third of the way along  $\overline{RB}$ , closer to  $R$  than to  $B$ .



$$R = \left( 30 + \frac{1}{2}(75 - 30), 40 + \frac{1}{2}(40 - 40) \right) = (52.5, 80) \text{ or } R = \left( \frac{1}{2}(75 + 30), \frac{1}{2}(40 + 40) \right) = (52.5, 80)$$

$$G_3: \left( 52.5 + \frac{1}{3}(60 - 52.5), 80 + \frac{1}{3}(50 - 80) \right) = (55, 70) \text{ or } \left( \frac{1}{3}(52.5 + 60 + 60), \frac{1}{3}(50 + 80 + 80) \right) = (55, 70)$$

**Discussion (2 minutes)**

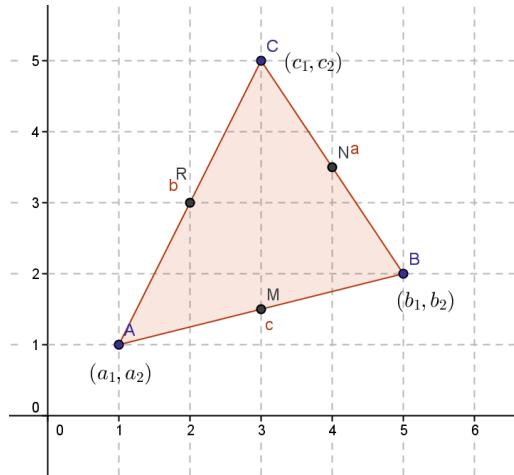
- MP.3**
- What are the coordinates of  $G_1$ ,  $G_2$ , and  $G_3$ ?
    - They all have the same coordinates.
  - Are you surprised that all three points,  $G_1$ ,  $G_2$ , and  $G_3$ , have the same coordinates?
    - Most students will have expected the three medians to intersect at one point, but they may not have known that this point is one-third of the length of each median from each of the midpoints. Some students may have figured this out from the opening activity.
  - What is true about the point of concurrency for the three medians? How do you know?
    - They intersect one-third of the length of the median from the midpoint as proven in the above exercises.
  - Do you think this is true for all triangles?
    - Most students will say yes because it was just shown on the last triangle. If students are not convinced, ask them to take out their patty paper from the Opening Exercise and measure the distances.

**Exercise 1 (10 minutes)**

This exercise asks students to prove that the three medians of any triangle are concurrent. They will also discover that the coordinates of the point of concurrency of the medians, the centroid, can be easily calculated given the coordinates of the three vertices of the triangle.

**Exercise 1**

- a. Given triangle  $ABC$  with vertices  $A(a_1, a_2)$ ,  $B(b_1, b_2)$ , and  $C(c_1, c_2)$ , find the coordinates of the point of concurrency.



$$\text{Midpoint of } \overline{AB}: M = \left( \frac{1}{2}(a_1 + b_1), \frac{1}{2}(a_2 + b_2) \right)$$

**Scaffolding:**

This exercise can be broken down for students with varying levels of ability.

- Give students the steps of each example and have them explain the steps or fill in certain easier parts.
- Talk the group through the first example, and then have them try the other two on their own.
- Assign the exercise as written.



*One third of the length of  $\overline{MC}$  from M:*

$$\left( \frac{1}{2}(a_1 + b_1) + \frac{1}{3}\left(c_1 - \frac{1}{2}(a_1 + b_1)\right), \frac{1}{2}(a_2 + b_2) + \frac{1}{3}\left(c_2 - \frac{1}{2}(a_2 + b_2)\right) \right)$$

$$\left( \frac{1}{2}a_1 + \frac{1}{2}b_1 + \frac{1}{3}c_1 - \frac{1}{6}a_1 - \frac{1}{6}b_1, \frac{1}{2}a_2 + \frac{1}{2}b_2 + \frac{1}{3}c_2 - \frac{1}{6}a_2 + \frac{1}{6}b_2 \right)$$

$$\left( \frac{1}{3}a_1 + \frac{1}{3}b_1 + \frac{1}{3}c_1, \frac{1}{3}a_2 + \frac{1}{3}b_2 + \frac{1}{3}c_2 \right)$$

*Midpoint of  $\overline{BC}$ :  $N = \left( \frac{1}{2}(b_1 + c_1), \frac{1}{2}(b_2 + c_2) \right)$*

*One third of the length of  $\overline{NA}$  from N:*

$$\left( \frac{1}{2}(a_1 + c_1) + \frac{1}{3}\left(b_1 - \frac{1}{2}(a_1 + c_1)\right), \frac{1}{2}(a_2 + c_2) + \frac{1}{3}\left(b_2 - \frac{1}{2}(a_2 + c_2)\right) \right)$$

$$\left( \frac{1}{2}a_1 + \frac{1}{2}c_1 + \frac{1}{3}b_1 - \frac{1}{6}a_1 - \frac{1}{6}c_1, \frac{1}{2}a_2 + \frac{1}{2}c_2 + \frac{1}{3}b_2 - \frac{1}{6}a_2 + \frac{1}{6}c_2 \right)$$

$$\left( \frac{1}{3}a_1 + \frac{1}{3}b_1 + \frac{1}{3}c_1, \frac{1}{3}a_2 + \frac{1}{3}b_2 + \frac{1}{3}c_2 \right)$$

*Midpoint of  $\overline{AC}$ :  $R = \left( \frac{1}{2}(a_1 + c_1), \frac{1}{2}(a_2 + c_2) \right)$*

*One third of the length of  $\overline{RA}$  from R:*

$$\left( \frac{1}{2}(a_1 + c_1) + \frac{1}{3}\left(b_1 - \frac{1}{2}(a_1 + c_1)\right), \frac{1}{2}(a_2 + c_2) + \frac{1}{3}\left(b_2 - \frac{1}{2}(a_2 + c_2)\right) \right)$$

$$\left( \frac{1}{2}a_1 + \frac{1}{2}c_1 + \frac{1}{3}b_1 - \frac{1}{6}a_1 - \frac{1}{6}c_1, \frac{1}{2}a_2 + \frac{1}{2}c_2 + \frac{1}{3}b_2 - \frac{1}{6}a_2 + \frac{1}{6}c_2 \right)$$

$$\left( \frac{1}{3}a_1 + \frac{1}{3}b_1 + \frac{1}{3}c_1, \frac{1}{3}a_2 + \frac{1}{3}b_2 + \frac{1}{3}c_2 \right)$$

After part (a), bring the group back together to discuss the formulas. Name the formula after the person who presents it most clearly—for example: We will call this “Tyler’s formula”.

- b. Let  $A(-23, 12)$ ,  $B(13, 36)$ , and  $C(23, -1)$  be vertices of a triangle. Where will the medians of this triangle intersect? (Use “Tyler’s formula” from part (a) to complete this problem.)

$$\left( \frac{1}{3}(-23) + \frac{1}{3}(13) + \frac{1}{3}(23), \frac{1}{3}(12) + \frac{1}{3}(36) + \frac{1}{3}(-1) \right) \text{ or } \left( \frac{1}{3}(-23 + 13 + 23), \frac{1}{3}(12 + 36 + -1) \right) = \left( \frac{13}{3}, \frac{47}{3} \right)$$

**Exercise 2 (6 minutes)**

In this exercise, students are asked to use this coordinate approach to prove that the diagonals of a rectangle bisect each other.

**Exercise 2**

Prove that the diagonal of a parallelogram bisect each other.

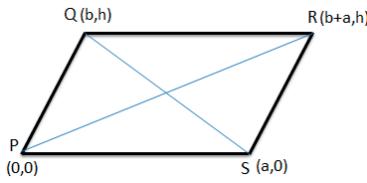
*Students will show that the diagonals are concurrent at their midpoints. Stated another way, both diagonals have the same midpoint.*

$$\text{Midpoint of } \overline{PQ}: \left(\frac{1}{2}(b+a), \frac{1}{2}h\right)$$

$$\text{Midpoint of } \overline{QS}: \left(\frac{1}{2}(b+a), \frac{1}{2}h\right)$$

**Scaffolding:**

You may want to provide students with the following diagram.

**Closing (3 minutes)**

Ask students to respond to this question individually in writing, to a partner, or as a class.

- How did we use coordinates to prove that the medians of any triangle will always meet at a point that is two-thirds of the way along each median from the intersected vertex?
  - We found the point of intersection of the three medians, and then we found the point on each median two-thirds of the way from each vertex and noticed that they were all the same. This was true for any triangle we studied.

**Exit Ticket (4 minutes)**

Name \_\_\_\_\_

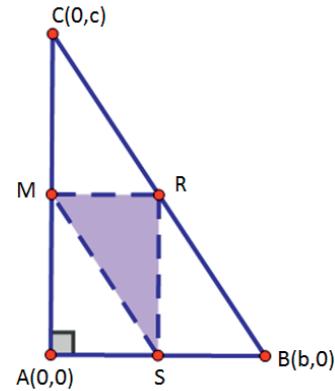
Date \_\_\_\_\_

## Lesson 13: Analytic Proofs of Theorems Previously Proved by Synthetic Means

### Exit Ticket

Prove that the medians of any right triangle form similar right triangles whose area is  $\frac{1}{4}$  the area of the original triangle.

Prove the area of  $\triangle CMR$  is  $\frac{1}{4}$  the area of  $\triangle CAB$ .



## Exit Ticket Sample Solutions

Prove that the medians of any right triangle form a similar right triangles whose area is  $\frac{1}{4}$  the area of the original triangle.

Prove the area of  $\triangle CMR$  is  $\frac{1}{4}$  the area of  $\triangle CAB$ .

Place the triangle on the coordinate plane as shown to the right will allow for the most efficient algebraic solution yielding midpoints

$$M = \left(0, \frac{1}{2}c\right), R = \left(\frac{1}{2}b, \frac{1}{2}c\right), \text{ and } S = \left(\frac{1}{2}b, 0\right).$$

$$\frac{RM}{AB} = \frac{\frac{1}{2}b}{b} = \frac{1}{2}$$

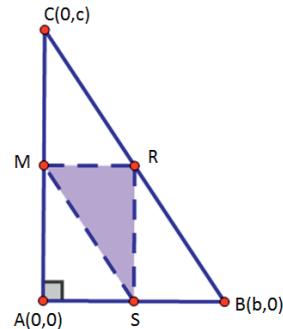
$\overline{AC}$  and  $\overline{RS}$  are both vertical as their slopes are undefined.

$\overline{AB}$  and  $\overline{RM}$  are both horizontal as their slopes are zero.

$\triangle ABC \cong \triangle RMS$  by SAS similarity.

$$\text{The area of } \triangle ABC = \frac{1}{2}b \cdot c.$$

$$\text{The area of } \triangle RMS = \frac{1}{2}\left(\frac{1}{2}b\right) \cdot \left(\frac{1}{2}c\right) = \frac{1}{4}\left(\frac{1}{2}b \cdot c\right) = \frac{1}{8} \triangle ABC.$$



## Problem Set Sample Solutions

- Point  $M$  is the midpoint of segment  $\overline{AC}$ . Find the coordinates of  $M$ :
  - $A(2, 3), C(6, 10)$   
 $(4, 6.5)$
  - $A(-7, 5), C(4, -9)$   
 $(-1.5, -2)$
- $M(-2, 10)$  is the midpoint of segment  $\overline{AB}$ . If  $A$  has coordinates  $(4, -5)$ , what are the coordinates of  $B$ ?  
 $(-8, 25)$
- Line  $A$  is the perpendicular bisector of segment  $\overline{BC}$  with  $B(-2, -1)$  and  $C(4, 1)$ .
  - What is the midpoint of  $\overline{BC}$ ?  
 $(1, 0)$
  - What is the slope of  $\overline{BC}$ ?  
 $\frac{1}{3}$
  - What is the slope of line  $A$ ? (Remember, it is perpendicular to  $\overline{BC}$ ).  
 $-3$



- d. Write the equation of line  $A$ , the perpendicular bisector of  $\overline{BC}$ .

$$y = -3x + 3$$

4. Find the coordinates of the intersection of the medians of  $\triangle ABC$  given  $A(-5, 3)$ ,  $B(6, -4)$ , and  $C(10, 10)$ .

$$\left(\frac{1}{3}(-5+6+10), \frac{1}{3}(3+(-4)+10)\right) = \left(3\frac{2}{3}, 3\right)$$

5. Use coordinates to prove that the diagonals of a parallelogram meet at the intersection of the segments that connect the midpoints of its opposite sides.

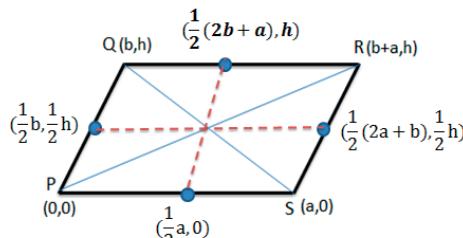
*This problem builds upon the findings of Exercise 2 where students proved that the diagonals of a parallelogram bisected each other by showing that the midpoints of the two diagonals occurred at the same point,  $(\frac{1}{2}(b+a), \frac{1}{2}h)$ .*

A: Midpoint of  $\overline{PS} = \left(\frac{1}{2}a, 0\right)$

B: Midpoint of  $\overline{QR} = \left(\frac{1}{2}(2b+a), h\right)$

C: Midpoint of  $\overline{PQ} = \left(\frac{1}{2}b, \frac{1}{2}h\right)$

D: Midpoint of  $\overline{RS} = \left(\frac{1}{2}(2a+b), \frac{1}{2}h\right)$



Finding the midpoint of the segment connecting the midpoints of  $\overline{PS}$  and  $\overline{QR}$ :

$$\left(\frac{1}{2}\left(\frac{1}{2}a + \frac{1}{2}(2b+a)\right), \frac{1}{2}(0+h)\right) = \left(\frac{1}{2}(a+b), \frac{1}{2}h\right)$$

Finding the midpoint of the segment connecting the midpoints of  $\overline{PQ}$  and  $\overline{RS}$ :

$$\left(\frac{1}{2}\left(\frac{1}{2}b + \frac{1}{2}(2a+b)\right), \frac{1}{2}\left(\frac{1}{2}h + \frac{1}{2}h\right)\right) = \left(\frac{1}{2}(a+b), \frac{1}{2}h\right)$$

The segments connecting the midpoints of the opposite sides of the parallelogram intersect at their midpoints which is located at  $(\frac{1}{2}(a+b), \frac{1}{2}h)$ .

6. Given a quadrilateral with vertices  $E(0, 5)$ ,  $F(6, 5)$ ,  $G(4, 0)$ , and  $H(-2, 0)$ :

- a. Prove quadrilateral  $EFGH$  is a parallelogram.

$\overline{EF}$  and  $\overline{GH}$  are horizontal segments, so they are parallel.

$\overline{HE}$  and  $\overline{GF}$  have slopes of  $\frac{5}{2}$ , so they are parallel.

Both pairs of opposite sides are parallel, so the quadrilateral is a parallelogram.

- b. Prove  $(2, 2.5)$  is a point on both diagonals of the quadrilateral.

Since  $EFGH$  is a parallelogram, the diagonals intersect at their midpoints.  $(2, 2.5)$  is the midpoint of  $\overline{HF}$  and  $\overline{GE}$ , so it is a point on both diagonals.



7. Prove quadrilateral  $WXYZ$  with vertices  $W(1, 3)$ ,  $X(4, 8)$ ,  $Y(10, 11)$ , and  $Z(4, 1)$  is a trapezoid.

$\overline{WX}$  and  $\overline{YZ}$  have slopes of  $\frac{5}{3}$ , so they are parallel.

$\overline{WZ}$  has a slope of  $-\frac{2}{3}$  and  $\overline{XY}$  has a slope of  $\frac{1}{2}$  so they are not parallel.

When one pair of opposite sides is parallel, the quadrilateral is a trapezoid.

8. Given quadrilateral  $JKLM$  with vertices  $J(-4, 2)$ ,  $K(1, 5)$ ,  $L(4, 0)$ , and  $M(-1, -3)$ :

- a. Is it a trapezoid? Explain.

Yes, one pair of opposite sides is parallel.  $\overline{JK}$  and  $\overline{LM}$  both have slopes of  $\frac{3}{5}$ .

When one pair of opposite sides is parallel, the quadrilateral is a trapezoid.

- b. Is it a parallelogram? Explain.

Yes, both pairs of opposite sides are parallel.  $\overline{JM}$  and  $\overline{KL}$  both have slopes of  $-\frac{5}{3}$ .

When both pairs of opposite sides are parallel, the quadrilateral is a parallelogram.

- c. Is it a rectangle? Explain.

$\overline{JK} \perp \overline{KL}$ ,  $\overline{KL} \perp \overline{ML}$ ,  $\overline{ML} \perp \overline{MJ}$ ,  $\overline{MJ} \perp \overline{JK}$  because their slopes are negative reciprocals.

Yes, because a parallelogram with four right angles is a rectangle.

- d. Is it a rhombus? Explain.

$JK = KL = LM = MJ = \sqrt{34}$

Yes, because a parallelogram with four congruent sides is a rhombus.

- e. Is it a square? Explain.

Yes, because a rectangle with four congruent sides is a square.

- f. Name a point on the diagonal of  $JKLM$ . Explain how you know.

(0, 1) is the midpoint of  $\overline{KM}$  and  $\overline{JL}$  and is on both diagonals.



## Lesson 14: Motion Along a Line—Search Robots Again

### Student Outcomes

- Students name several points on a line given by a parametric equation and provide the point-slope equation for a line given by a parametric equation
- Students determine whether lines given parametrically are parallel or perpendicular

### Lesson Notes

This is an optional lesson intended to allow students to explore parametric equations and compare them with more familiar linear equations. Parametric equations make both the  $x$ - and  $y$ -variables in an equation dependent upon a third (independent) variable, usually time. In the search robot scenario revisited here, we use parametric equations to model both the robot's horizontal and vertical motion over a period of time. While this is an optional lesson, it prepares students for the topic of parametric equations in higher-level courses leading toward calculus.

### Classwork

#### Opening Exercises (5 minutes)

We will begin the lesson by having students investigate the image created by a given map as the foundation for parameterizing lines where the values of the  $x$ - and  $y$ -coordinates depend on time ( $t$ ).

During this exercise students work independently.

##### Opening Exercise

- a. If  $f(t) = (t, 2t - 1)$ , find the values of  $f(0)$ ,  $f(1)$ , and  $f(5)$ , and plot them on a coordinate plane.

$$f(0) = (0, -1)$$

$$f(1) = (1, 1)$$

$$f(5) = (5, 9)$$

- b. What is the image of  $f(t)$ ?

A line

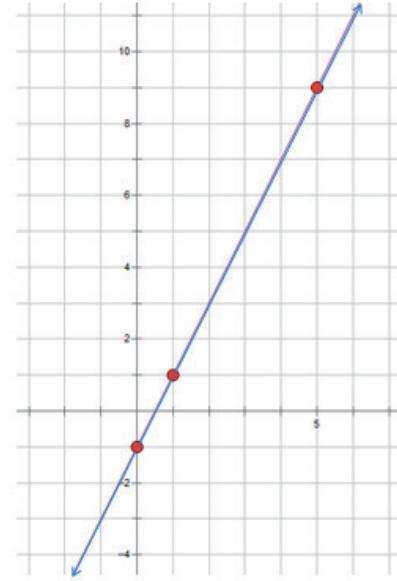
- c. At what time does the graph of the line pass through the  $y$ -axis?

The line passes through the  $y$ -axis when  $t = 0$ .

- d. When does it pass through the  $x$ -axis?

The line passes through the  $x$ -axis when  $2t - 1 = 0 \leftrightarrow$

$$t = \frac{1}{2}$$



- e. Can you write the equation of the line you graphed in slope  $y$ -intercept form?

$$y = 2x - 1$$

- f. How does this equation compare with the definition of  $f(t)$ ?

*The value of the second coordinate is obtained by taking two times the value of the first coordinate and then subtracting 1.*

### Example 1 (8 minutes)

#### Example 1

Programmers want to program a robot so that it moves at a uniform speed along a straight line segment connecting two points  $A$  and  $B$ . If  $A(0, -1)$  and  $B(1, 1)$ , and the robot travels from  $A$  to  $B$  in  $t = 1$  minute,

- a. Where is the robot at  $t = 0$ ?

*The robot will be at  $A(0, -1)$ .*

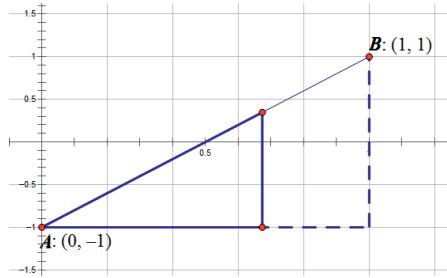
- b. Where is the robot at  $t = 1$ ?

*The robot will be at  $B(1, 1)$ .*

Direct students to choose their own value for  $t$  in part (c).

MP.4

- c. Draw a picture that shows where the robot will be at  $0 < t < 1$ .



*Student pictures will vary, but they should all include similar right triangles. Look around the room at the triangles your fellow students sketched.*

- What do we know about the robots for those students who selected a location closer to  $(1, 1)$  than to  $(0, -1)$ ?
  - These robots have been moving for a longer period of time.
- Is it possible to find the location of the robot for  $t > 1$ ?
  - If the robot is moving along a line that passes through points  $A$  and  $B$  the robot could continue past point  $B$ .

- How far has the robot moved in the horizontal direction after  $\frac{1}{2}$  of a minute?
  - *The robot will move half of the entire horizontal distance between point A and point B because it is moving at a uniform rate:  $\frac{1}{2}(1 - 0) = \frac{1}{2}$ .*
- How far has the robot moved in the vertical direction after  $\frac{1}{2}$  of a minute?
  - *The robot will move half of the entire vertical distance between point A and point B because it is moving at a uniform rate:  $\frac{1}{2}(1 - (-1)) = 1$ .*
- Find the location of the robot for  $t = \frac{1}{2}$ , after the robot has been moving along segment  $\overline{AB}$  for half of a minute.
  - *The robot starts at  $(0, -1)$  when  $t = 0$  and moves  $\frac{1}{2}$  unit horizontally and 1 unit vertically. At  $t = \frac{1}{2}$ , the robot will be at  $(0 + \frac{1}{2}, -1 + 1)$  or  $(\frac{1}{2}, 0)$ .*

### Exercise 1 (4 minutes)

Students extend the work done in Example 1 and discover the equation that will give the coordinate of the robot at any given time  $t$ . This includes positions for  $t > 1$ . Students examine the effect speed has on the location of the robot along the path at any given time. Have students work with a partner on Exercise 1.

#### Exercise 1

A robot is programmed to move along a straight line path through two points  $A$  and  $B$ . It travels at a uniform speed that allows it to make the trip from  $A(0, -1)$  to  $B(1, 1)$  in  $t = 1$  minute. Find the location,  $P$ , when

a.  $t = \frac{1}{4}$

$$\left(0 + \frac{1}{4}(1 - 0), -1 + \frac{1}{4}(1 - (-1))\right)$$

$$\left(\frac{1}{4}, -\frac{1}{2}\right)$$

b.  $t = 0.7$

$$\left(0 + 0.7(1 - 0), -1 + 0.7(1 - (-1))\right)$$

$$(0.7, 0.4)$$

c.  $t = \frac{5}{4}$

$$\left(0 + \frac{5}{4}(1 - 0), -1 + \frac{5}{4}(1 - (-1))\right)$$

$$\left(\frac{5}{4}, \frac{3}{2}\right)$$

d.  $t = 2.2$

$$\left(0 + 2.2(1 - 0), -1 + 2.2(1 - (-1))\right)$$

$$(2.2, 3.4)$$

#### Scaffolding:

- Struggling learners may benefit from using the diagram they created during Example 1.
- While students work in partners, work with a small group that would benefit from targeted instruction.

**Example 2 (6 minutes)**

In this example we investigate the impact the speed of the robot has on its location at any given time. We will use this to create a general equation for the location of the robot at any time  $t$ .

**Example 2**

Our robot has been reprogrammed so that it moves along the same straight line path through two points  $A(0, -1)$  and  $B(1, 1)$  at a uniform rate but makes the trip in 0.6 minutes instead of 1 minute.

How does this change the way we calculate the location of the robot at any time,  $t$ ?

- a. Find the location,  $P$ , of the robot from Example 1 if the robot were traveling at a uniform speed that allowed it to make the trip from  $A$  to  $B$  in  $t = 0.6$  minutes. Is the robot's speed greater or less than the robot's speed in Example 1?

$$P = (a_1 + a_2) + \frac{t}{(0.6)}(b_1 - a_1, b_2 - a_2) \text{ or } P = (a_1, a_2) + \frac{5}{3}t(b_1 - a_1, b_2 - a_2)$$

*The robot is moving faster than it was in Example 1 because it travels the same distance from A and B in a shorter period of time.*

- b. Find the location,  $P$ , of the robot from Example 1 if the robot were traveling at a uniform speed that allowed it to make the trip from  $A$  to  $B$  in  $t = 1.5$  minutes. Is the robot's speed greater or less than the robot's speed in Example 1?

$$P = (a_1 + a_2) + \frac{t}{1.5}(b_1 - a_1, b_2 - a_2) \text{ or } P = (a_1, a_2) + \frac{2}{3}t(b_1 - a_1, b_2 - a_2)$$

*The robot is moving more slowly than it was in Example 1 because it travels the same distance from A and B in a longer period of time.*

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**Exercise 2 (8 minutes)**

This exercise provides students with an opportunity to investigate the effect the straight line path has on the position of the robot.

**Exercise 2**

Two robots are moving along straight line paths in a rectangular room. Robot 1 starts at point  $A(20, 10)$  and travels at a constant speed to point  $B(120, 50)$  in two minutes. Robot 2 starts at point  $C(90, 10)$  and travels at a constant speed to point  $D(60, 70)$  in 90 seconds.

- a. Find the location,  $P$ , of Robot 1 after it has traveled for  $t$  minutes along its path from  $A$  to  $B$ .

$$P = \left( 20 + \frac{t}{2}(100), 10 + \frac{t}{2}(40) \right)$$

$$P = (20, 10) + \frac{t}{2}(100, 40)$$

- b. Find the location,  $Q$ , of Robot 2 after it has traveled for  $t$  minutes along its path from  $A$  to  $B$ .

$$Q = \left( 90 + \frac{t}{1.5}(-30), 10 + \frac{t}{1.5}(60) \right)$$

$$Q = (90, 10) + \frac{t}{1.5}(-30, 60)$$



- c. Are the robots traveling at the same speed? If not, which robot's speed is greater?

$$AB = \sqrt{(120 - 20)^2 + (50 - 10)^2}$$

$$AB = \sqrt{11600}$$

$$AB = 20\sqrt{29}$$

$$AB \approx 107.7$$

$$CD = \sqrt{(60 - 90)^2 + (70 - 10)^2}$$

$$CD = \sqrt{4500}$$

$$CD = 30\sqrt{5}$$

$$CD \approx 67.1$$

*Robot 1 travels approximately 107.7 units in two minutes which is about 53.85 units per minute.*

*Robot 2 travels approximately 67.1 units in 1.5 minutes which is about 44.7 units per minute.*

*Therefore, Robot 1's speed is greater.*

- d. Are the straight line paths that the robots are traveling parallel, perpendicular, or neither? Explain your answer.

$$\text{slope } \overline{AB} = \frac{50-10}{120-20} = \frac{2}{5}$$

$$\text{slope } \overline{CD} = \frac{70-10}{60-90} = -2$$

*The paths are neither parallel nor perpendicular; the slopes of their straight line paths are not equal nor do they have a product equal to -1.*

MP.4

### Example 3 (4 minutes)

#### Example 3

A programmer wants to program a robot so that it moves at a constant speed along a straight line segment connecting the point  $A(30, 60)$  to the point  $B(200, 100)$  over the course of a minute.

At time  $t = 0$ , the robot is at point  $A$ .

At time  $t = 1$ , the robot is at point  $B$ .

- a. Where will the robot be at time  $t = \frac{1}{2}$ ?

*Because the robot is moving at uniform speed, the robot should travel half the distance between points  $A$  and  $B$  in half of the time it takes the robot to travel the entire distance.*

$$\left(30 + \frac{1}{2}(200 - 30), 60 + \frac{1}{2}(100 - 60)\right) = (115, 80)$$

- b. Where will the robot be at time  $t = 0.6$ ?

*The robot will be three-fifths of the way along the segment from  $A$  to  $B$ .*

$$\left(30 + \frac{3}{5}(200 - 30), 60 + \frac{3}{5}(100 - 60)\right) = (132, 84)$$

**Discussion (3 minutes)**

- In the Opening Exercises, what factors were the key pieces of information you used to determine the location of the robot?
  - *That the robot was moving along a line segment.*
  - *The coordinates of the endpoints of the line segment.*
  - *How long the robot had been moving along the segment.*
  - *The time it took the robot to travel the entire length of the segment.*
- Why did we need to know both the time the robot was traveling and the time it took the robot to travel the entire length of the segment?
  - *The robot will cover a fraction of the length of the segment that is equal to the segment which is equal to the ratio of the time the robot was traveling to the total time it takes the robot to travel the entire length.*
- How would changing the speed the robot was traveling along the segment change the Opening Exercises?
  - *If the robot were traveling faster, the time it takes the robot to travel the entire length of the segment would decrease and the robot would be farther along the segment at each of the given times.*
  - *If the robot were traveling more slowly, the time it takes the robot to travel the entire length of the segment would increase, and the robot would not have traveled as far along the segment at each of the given times.*

**Closing (2 minutes)**

Ask students to respond to these questions in writing, with a partner, or as a class.

- What role did time play in determining the locations of the robots during today's lesson?
  - *As time increased, the robot's location moved away from its starting position. The x- and y-coordinates both changed.*
- How did we accommodate this parameter when we wrote the expressions that represented the coordinates of the locations?
  - *The coordinates both were dependent on  $t$ , so each coordinate was written as a function of the variable  $t$ .*

**Exit Ticket (5 minutes)**



Name \_\_\_\_\_

Date \_\_\_\_\_

## Lesson 14: Motion Along a Line—Search Robots Again

### Exit Ticket

Programmers want to program a robot so that it moves along a straight line segment connecting the point  $A(35, 80)$  to the point  $B(150, 15)$  at a uniform speed over the course of five minutes. Find the location of the robot at the following times:

a.  $t = 0$

b.  $t = 2$

c.  $t = 3.5$

d.  $t = 5$



## Exit Ticket Sample Solutions

Programmers want to program a robot so that it moves along a straight line segment connecting the point  $A(35, 80)$  to the point  $B(150, 15)$  at a uniform speed over the course of five minutes. Find the robot's location at the following times:

a.  $t = 0$

*At  $t = 0$ , the robot is at point A(35, 80).*

b.  $t = 2$

*At  $t = 2$ , the robot is at  $(35, 80) + \frac{2}{5}(115, -65)$  or  $(81, 54)$ .*

c.  $t = 3.5$

*At  $t = 3.5$ , the robot is at  $(35, 80) + \frac{3.5}{5}(115, -65)$  or  $(115.5, 34.5)$ .*

d.  $t = 5$

*At  $t = 5$ , the robot is at point B(150, 15).*

## Problem Set Sample Solutions

1. Find the coordinates of the intersection of the medians of  $\triangle ABC$  given  $A(2, 4)$ ,  $B(-4, 0)$ , and  $C(3, -1)$ .

$$\left(\frac{1}{3}(2 + (-4) + 3), \frac{1}{3}(4 + 0 + (-1))\right) = \left(\frac{1}{3}, 1\right)$$

2. Given a quadrilateral with vertices  $A(-1, 3)$ ,  $B(1, 5)$ ,  $C(5, 1)$ , and  $D(3, -1)$ :

- a. Prove that quadrilateral  $ABCD$  is a rectangle.

*$\overline{AB}$  and  $\overline{DC}$  have slopes of 1, so they are parallel.*

*$\overline{BC}$  and  $\overline{AD}$  have slopes of  $-1$ , so they are parallel.*

*Since adjacent sides have slopes that are negative reciprocals, they are perpendicular; therefore, each angle is a right angle.*

*Both pairs of opposite sides are parallel, so the quadrilateral is a parallelogram.*

*All angles are right angles, so the parallelogram is a rectangle.*

- b. Prove that  $(2, 2)$  is a point on both diagonals of the quadrilateral.

*Since  $ABCD$  is a parallelogram, its diagonals intersect at their midpoints.  $(2, 2)$  is the midpoint of both  $\overline{AC}$  and  $\overline{BD}$ , so it is a point on both diagonals.*



3. The robot is programed to travel along a line segment at a constant speed. If  $P$  represents the robot's position at any given time  $t$  in minutes:

$$P = (240, 60) + \frac{t}{10}(100, 100),$$

- a. What was the robot's starting position?

*Starting position:  $(240, 60) + \frac{0}{10}(100, 100)$  or  $(240, 60)$*

- b. Where did the robot stop?

*Stopping position:  $(240, 60) + \frac{10}{10}(100, 100)$  or  $(340, 160)$*

- c. How long did it take the robot to complete the entire journey?

*The robot will travel the entire length of the segment in 10 minutes.*

- d. Did the robot pass through the point  $(310, 130)$  and, if so, how long into its journey did the robot reach this position?

*There are a number of ways to answer part (d). The one most relevant to this lesson might be the following:*

*Find a time such that  $240 + \frac{t}{10}(100) = 310$ , and  $60 + \frac{t}{10}(100) = 130$ . The robot passed through the point  $(310, 130)$  when  $t = 7$ .*

4. Two robots are moving along straight line paths in a rectangular room. Robot 1 starts at point  $A(20, 10)$  and travels at a constant speed to point  $B(120, 50)$  in two minutes. Robot 2 starts at point  $C(90, 10)$  and travels at a constant speed to point  $D(60, 70)$  in 90 seconds. If the robots begin their journeys at the same time, will the robots collide? Why or why not?

*Robot 1's position is modeled by*

$$P = (20, 10) + \frac{t}{2}(100, 40).$$

*Robot 2's position is modeled by*

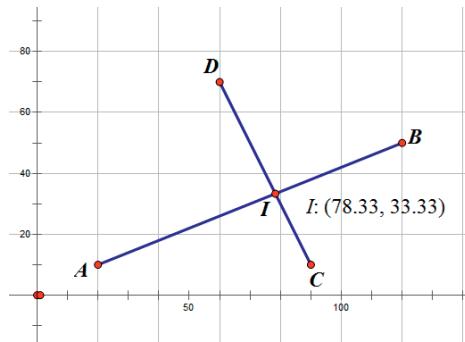
$$Q = (90, 10) + \frac{t}{1.5}(-30, 60).$$

$$(20, 10) + \frac{t}{2}(100, 40) = \left(78\frac{1}{3}, 33\frac{1}{3}\right)$$

*Robot 1 passes through the point  $\left(78\frac{1}{3}, 33\frac{1}{3}\right)$  at  $t = 1\frac{1}{6}$ .*

$$(90, 10) + \frac{t}{1.5}(-30, 60) = \left(78\frac{1}{3}, 33\frac{1}{3}\right)$$

*Robot 2 passes through the point  $\left(78\frac{1}{3}, 33\frac{1}{3}\right)$  at  $t = \frac{7}{12}$ .*



*The paths of the two robots do intersect, but the robots reach this point  $\left(78\frac{1}{3}, 33\frac{1}{3}\right)$  at different times, so the robots will not collide.*



## Lesson 15: The Distance from a Point to a Line

### Student Outcomes

- Students are able to derive a distance formula and apply it.

### Lesson Notes

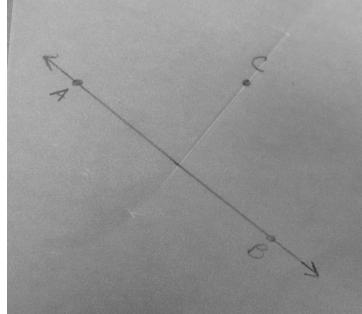
In this lesson, students review the distance formula, the criteria for perpendicularity, and the creation of the equation of a perpendicular line. Students reinforce their understanding that when they are asked to find the distance between a line  $l$  and a point  $P$  not on line  $l$ , they are looking for the shortest distance. This distance is equal to the length of the segment that is perpendicular to  $l$  with one endpoint on line  $l$  and the other endpoint  $P$ . Students derive the general formula for the distance  $d$  of a point  $P(p, q)$  from a line  $y = mx + b$  through a teacher-led exercise, and then practice using the formula.

### Classwork

#### Opening Exercise (5 minutes)

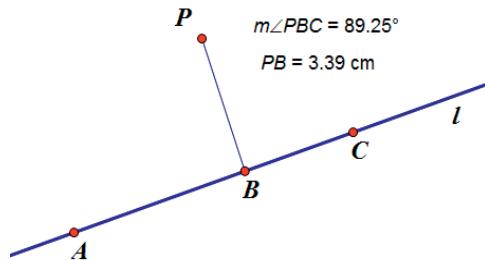
This Opening Exercise has students construct a line that is perpendicular to a given line passing through a point not on the given line. This will lead them to the understanding that the shortest distance from point to a line that does not contain that point is the perpendicular distance. Any other segment from the point to the line would create a right triangle. This new segment would be the hypotenuse of that right triangle and, therefore, would be longer than the segment that is perpendicular to the line.

Students draw a line on a piece of patty paper and a point not on that line. Ask the students to create the shortest segment from the point to the line by folding the patty paper.



Students should fold the patty paper so that the line folds back onto itself and the crease passes through the point not on the line.

A similar exercise can be done using geometry software where the students construct a line  $l$  and a point  $P$  not on line  $l$ . Then construct a segment with one endpoint on line  $l$  and the other endpoint at  $P$ . The students measure the angle formed by the segment and the line and the corresponding length of the segment. The students should recognize that the segment is the shortest when the angle is a right angle.



**Discussion (2 minutes)**

- How do you know the segment you created is the shortest segment from the point to the line?
  - This segment is perpendicular to the line, so it must be the shortest. Any other segment from the point to the line would create a right triangle. This new segment would be the hypotenuse of that right triangle and, therefore, would be longer than the segment that is perpendicular to the line.
- How can we determine the distance point  $C$  is from line  $AB$ ?
  - We could measure the length of the perpendicular segment that we created that has one endpoint on line  $AB$  and the other at point  $P$ .

**Exercise 1 (8 minutes)****Exercise 1**

A robot is moving along the line  $20x + 30y = 600$ . A homing beacon sits at the point  $(35, 40)$ .

- a. Where on this line will the robot hear the loudest ping?

*Students will need to determine the equation of the line passing through the point  $(35, 40)$  that is perpendicular to the line  $20x + 30y = 600$ . The slope of this line is  $-\frac{2}{3}$ .*

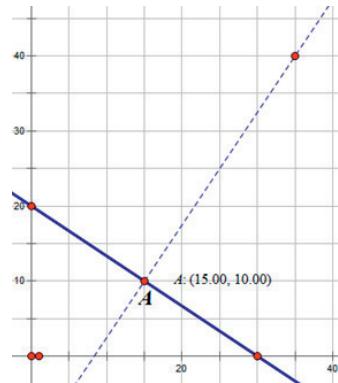
$$y - 40 = \frac{3}{2}(x - 35) \text{ or } y = \frac{3}{2}(x - 35) + 40$$

*There are a variety of methods available to students to use to determine the point where these two lines intersect. Using substitution is one of the more efficient methods.*

$$\begin{aligned} 20x + 30\left(\frac{3}{2}(x - 35) + 40\right) &= 600 \\ 20x + 45x - 1575 + 1200 &= 600 \\ 65x &= 975 \\ x &= 15 \end{aligned}$$

$$\begin{aligned} y &= \frac{3}{2}(15 - 35) + 40 \\ y &= 10 \end{aligned}$$

*The robot will be closest to the beacon when it is on the point  $(15, 10)$ .*



- b. At this point, how far will the robot be from the beacon?

*Students need to calculate the distance between the two points  $B(35, 40)$  and  $A(15, 10)$ . Encourage students to think about the right triangle that is created if one moves from  $A$  to  $B$  in two moves: one horizontal and the other vertical.*

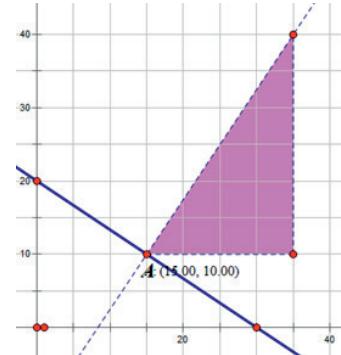
$$AB = \sqrt{(35 - 15)^2 + (40 - 10)^2}$$

$$AB = \sqrt{(20)^2 + (30)^2}$$

$$AB = \sqrt{1300} = 10\sqrt{13}$$

**Scaffolding:**

- For struggling learners the question can be changed to ask, "Where on this line will the robot be closest to the beacon?" Show a picture of the line and the beacon, and ask students to estimate their answer.
- For advanced learners the coordinates of the homing beacon can be changed to  $(40, 50)$ .



**Example 1 (12 minutes)**

In this example students develop a formula for the distance  $d$  that a point  $P(p, q)$  is from the line  $l$  given by the equation  $y = mx + b$ . That is, we only know the point  $P$  and the line  $l$ , we do not know where to drop the perpendicular from  $P$  to  $l$  algebraically. We can do this with a construction, but then we would not know the coordinates of the intersection point  $R$ .

Since  $R$  is some point on the line  $l$ , let's call it  $R(r, s)$ ; it is our goal to first find the values of  $r$  and  $s$  in terms of  $p, q, m$ , and  $b$ . To accomplish this we note that  $\overrightarrow{PR} \perp l$  allows us to use the formula from Lesson 5 (and Lesson 8). To use this formula, we will need a segment on line  $l$ . In this example, we chose to use  $T(r + 1, m(r + 1) + b)$ . By strategically choosing point  $R(r, mr + b)$  and  $T(r + 1, m(r + 1) + b)$  when we apply the formula from Lesson 5, the second set of differences reduces nicely. Recall that we developed this formula by translating the figure so that the image of the vertex of the right angle, in this example, it would be  $R'$ , was at the origin. When we do this, the coordinates of the image of the other endpoint of the segment,  $T'$  in this example, will have coordinates  $(1, m)$ .

**PROOF:**

Let  $l$  be a line given by the graph of the equation  $y = mx + b$  (for some real numbers  $m$  and  $b$ ) and let  $P(p, q)$  be a point not on line  $l$ . To find point  $R(r, s)$  in terms of  $p, q, m$ , and  $b$  such that  $\overrightarrow{RP} \perp l$ , consider points  $R(r, mr + b)$  and  $T(r + 1, m(r + 1) + b)$  on the line  $l$ .

- How do we know that the points  $R(r, mr + b)$  and  $T(r + 1, m(r + 1) + b)$  lie on line  $l$ ?
  - We know both points lie on line  $l$  because their coordinates are of the form  $(x, mx + b)$ .

Direct students to construct a diagram depicting the relationships described above or provide the diagram show to the right.

- If  $PR$  represents the distance from  $P$  to line  $l$ , what do we know about the relationship between  $\overrightarrow{PR}$  and line  $l$ ?
  - $\overrightarrow{PR}$  must be perpendicular line  $l$ .
- What kind of triangle is  $\triangle PTR$ ?
  - $\triangle PTR$  is a right triangle with the right angle at vertex  $R$ . Which means  $\overrightarrow{PR} \perp \overrightarrow{RT}$ .
- This means  $\overrightarrow{PR} \perp \overrightarrow{RT}$ 

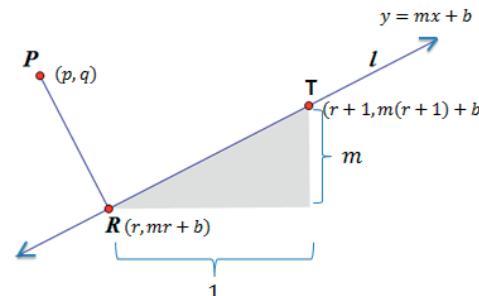
$$\leftrightarrow (p - r)(1) + (q - (mr + b))(m) = 0$$

$$\leftrightarrow p - r + qm - m^2r - bm = 0$$

$$\leftrightarrow -r - m^2r = -p - qm + bm$$

$$\leftrightarrow r(1 + m^2) = p + qm - bm$$

$$\leftrightarrow r = \frac{p + qm - bm}{1 + m^2}$$
- This gives us the coordinates for points  $R$  and  $P$  in terms of  $p, q, m$ , and  $b$ .
  - $R\left(\frac{p+qm-bm}{1+m^2}, m\left(\frac{p+qm-bm}{1+m^2}\right) + b\right)$  and  $P(p, q)$

**Scaffolding:**

- For advanced learners, assign this as a partner or small group exercise instead of a teacher led example.

**Scaffolding:**

- For struggling learners, review Lesson 5, Example 2.

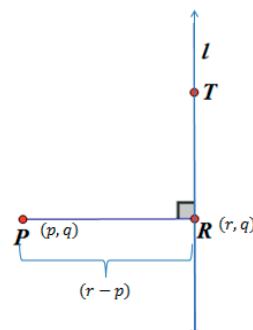
- We can now find the distance  $d$  between points  $P$  and  $R$  using the distance formula.

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$d = \sqrt{\left(\frac{p + qm - bm}{1 + m^2} - p\right)^2 + \left(m\left(\frac{p + qm - bm}{1 + m^2}\right) + b - q\right)^2}$$

Understanding this formula addresses standard **A-SSE.A.1b** (interpret complicated expressions by viewing one or more of their parts as a single entity).

- What are the coordinates of point  $P$  in terms of  $p, q, m$ , and  $b$ ?
  - $(p, q)$
- What are the coordinates of point  $R$  in terms of  $p, q, m$ , and  $b$ ?
  - $\left(\frac{p+qm-bm}{1+m^2}, m\left(\frac{p+qm-bm}{1+m^2}\right) + b\right)$
- Which point represents  $(x_1, y_1)$  in the distance formula?
  - Point P*
- Which point represents  $(x_2, y_2)$  in the distance formula?
  - Point R*
- Does this formula allow us to calculate the distance between the point  $P$  and the line  $l$  knowing only coordinates of the point  $P(p, q)$  and the equation of the line  $y = mx + b$ ?
  - Yes.*
- This formula is quite cumbersome. Do you feel it is realistic to expect us to memorize this formula and use it “by hand” on a regular basis?
  - No.*
- How might this formula be best used?
  - We could create a calculator or computer program that asks us to input the values of  $p, q, m$ , and  $b$  and will return the corresponding distance.*
- Are there limitations to this formula?
  - The formula depends on a real number value for  $m$ ; therefore, the line  $l$  cannot be vertical.*
- How could we calculate the distance from  $P$  to the line  $l$  if  $l$  were vertical?
  - We would only need to look at the differences in the  $x$ -coordinates between points  $P$  and  $R$ .*



**Exercise 2 (10 minutes)**

Students apply the formula for distance found in the previous example to a variety of problems. Students can solve several of these problems using alternate methods to check that the formula does in fact work.

**Exercise 2**

For the following problems, use the formula to calculate the distance between the point  $P$  and the line  $l$ .

$$d = \sqrt{\left(\frac{p + qm - bm}{1 + m^2} - p\right)^2 + \left(m\left(\frac{p + qm - bm}{1 + m^2}\right) + b - q\right)^2}$$

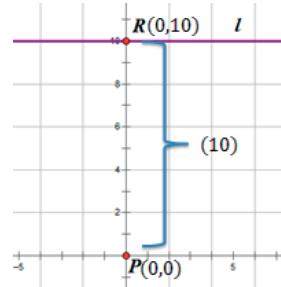
- a.  $P(0, 0)$  and the line  $y = 10$

$$p = 0, q = 0, m = 0, \text{ and } b = 10$$

$$d = \sqrt{\left(\frac{0 + 0 - 10(0)}{1 + 0^2} - 0\right)^2 + \left(0\left(\frac{0 + 0 - 10(0)}{1 + 0^2}\right) + 10 - 0\right)^2}$$

$$d = \sqrt{0 + (10)^2}$$

$$d = 10$$



- b.  $P(0, 0)$  and the line  $y = x + 10$

$$p = 0, q = 0, m = 1, \text{ and } b = 10$$

$$d = \sqrt{\left(\frac{0 + 0(1) - 10(1)}{1 + 1^2} - 0\right)^2 + \left(1\left(\frac{0 + 0(1) - 10(1)}{1 + 1^2}\right) + 10 - 0\right)^2}$$

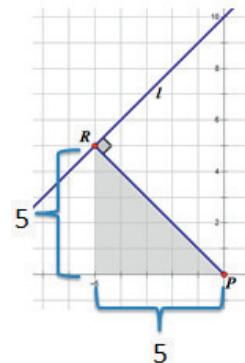
$$d = \sqrt{\left(\frac{10}{2}\right)^2 + \left(\frac{-10}{2} + 10\right)^2}$$

$$d = \sqrt{\left(\frac{10}{2}\right)^2 + \left(\frac{10}{2}\right)^2}$$

$$d = \sqrt{5^2 + 5^2}$$

$$d = \sqrt{2(5^2)}$$

$$d = 5\sqrt{2}$$



c.  $P(0, 0)$  and the line  $y = x - 6$

$$p = 0, q = 0, m = 1, \text{ and } b = -6$$

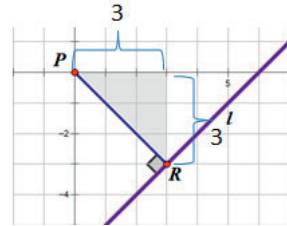
$$d = \sqrt{\left(\frac{0 + 0(1) - (-6)(1)}{1 + 1^2} - 0\right)^2 + \left(1\left(\frac{0 + 0(1) - (-6)(1)}{1 + 1^2}\right) + (-6) - 0\right)^2}$$

$$d = \sqrt{\left(\frac{6}{2}\right)^2 + \left(\frac{6}{2} + (-6)\right)^2}$$

$$d = \sqrt{3^2 + 3^2}$$

$$d = \sqrt{2(3^2)}$$

$$d = 3\sqrt{2}$$



Note: Students can also use the isosceles right triangle ( $45^\circ$ – $45^\circ$ – $90^\circ$ ) that has a hypotenuse of  $\overline{PR}$  to calculate the distance from  $P$  to  $l$  to confirm the results obtained using the formula in parts (b) and (c).

### Closing (3 minutes)

Ask students to respond to these questions individually in writing, with a partner, or as a class.

- Before today's lesson how did you determine the distance from a line to a point not on the line?
  - We had to find the point of intersection of the given line and the line perpendicular to the given line through the point not on the line. Then we used the formula  $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$  to find the distance between the two points.
- What advantage does our newly discovered formula  $d = \sqrt{\left(\frac{p+qm-bm}{1+m^2} - p\right)^2 + \left(m\left(\frac{p+qm-bm}{1+m^2}\right) + b - p\right)^2}$  offer?
  - Now we do not need to find the other endpoint of the shortest segment from the given line to the point not on the line. If we know the equation of the given line and the coordinates of the point not on the line, we have enough information to calculate the distance from the point to the line. The formula is cumbersome to apply in the majority of cases; therefore, it would be most efficient to write a calculator or computer program that uses this formula to calculate distances for given values of  $p$ ,  $q$ ,  $m$ , and  $b$ .
- In each of the exercises we were able to also calculate the distance using an isosceles right triangle. Will this method always apply, or are there certain conditions that make this possible?
  - The segment that represented the distance from the point  $P$  to the line  $l$  was the hypotenuse of an isosceles right triangle because the line had a slope of 1 or  $-1$ .

### Exit Ticket (5 minutes)

Name \_\_\_\_\_

Date \_\_\_\_\_

## Lesson 15: The Distance from a Point to a Line

### Exit Ticket

- Find the distance between the point  $P(0,0)$  and the line  $y = -x + 4$  using the formula from today's lesson.
- Verify using another method.

## Exit Ticket Sample Solutions

1. Find the distance between the point  $P(0, 0)$  and the line  $y = -x + 4$  using the formula from today's lesson.

$$d = \sqrt{\left(\frac{p + qm - bm}{1 + m^2} - p\right)^2 + \left(m\left(\frac{p + qm - bm}{1 + m^2}\right) + b - q\right)^2}$$

$p = 0, q = 0, m = -1, \text{ and } b = 4$

$$d = \sqrt{\left(\frac{0 + 0(-1) - 4(-1)}{1 + (-1)^2} - 0\right)^2 + \left((-1)\left(\frac{0 + 0(-1) - 4(-1)}{1 + 1^2}\right) + 4 - 0\right)^2}$$

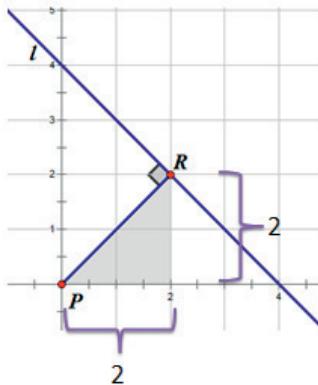
$$d = \sqrt{\left(\frac{4}{2}\right)^2 + \left((-1)\left(\frac{4}{2}\right) + 4\right)^2}$$

$$d = \sqrt{2^2 + 2^2}$$

$$d = \sqrt{2(2^2)}$$

$$d = 2\sqrt{2}$$

2. Verify using another method.



Note: Most students will also use the isosceles right triangle ( $45^\circ$ – $45^\circ$ – $90^\circ$ ) shown below that has a hypotenuse of  $\overline{PR}$  to calculate the distance from  $P$  to  $l$  to confirm the results obtained using the formula.



## Problem Set Sample Solutions

1. Given  $\triangle ABC$  with vertices  $A(3, -1)$ ,  $B(2, 2)$ , and  $C(5, 1)$ .

- a. Find the slope of the angle bisector of  $\angle ABC$ .

$$\text{slope} = -1$$

- b. Prove that the bisector of  $\angle ABC$  is the perpendicular bisector of  $\overline{AC}$ .

Let  $\overline{BD}$  be the bisector of  $\angle ABC$ , where  $D$  is the point of intersection with  $\overline{AC}$ .

$AB = CB = \sqrt{10}$ ; therefore,  $\triangle ABC$  is isosceles, and  $m\angle A = m\angle C$  (base angles of isosceles have equal measures).

$m\angle ABD = m\angle CBD$  by definition of angle bisector.

$\triangle ABD \cong \triangle CBD$  by ASA.

$AD = CD$ , since corresponding sides of congruent triangles have equal lengths; therefore,  $\overline{BD}$  bisects  $\overline{AC}$ .

The slope of  $\overline{AD}$  is  $-1$ ; therefore,  $\overline{AD} \perp \overline{AC}$ .

Therefore,  $\overline{AD}$  is the perpendicular bisector of  $\overline{AC}$ .

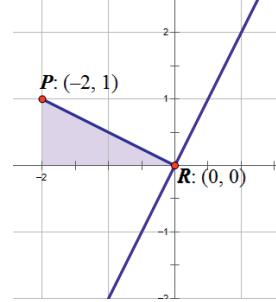
- c. Write the equation of the line containing  $\overline{AD}$ .

$$y = -x + 4$$

2. Use the distance formula from today's lesson to find the distance between the point  $P(-2, 1)$  and the line  $y = 2x$ .

$$p = -2, q = 1, m = 2, \text{ and } b = 0$$

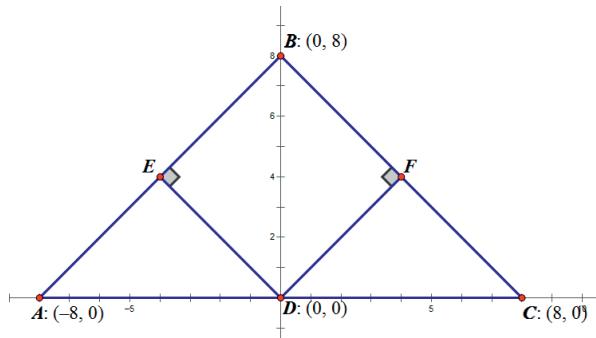
$$\begin{aligned} d &= \sqrt{\left(\frac{p+qm-bm}{1+m^2} - p\right)^2 + \left(m\left(\frac{p+qm-bm}{1+m^2}\right) + b - q\right)^2} \\ d &= \sqrt{\left(\frac{-2+1(2)-0(2)}{1+2^2} - (-2)\right)^2 + \left(2\left(\frac{-2+1(2)-0(2)}{1+2^2}\right) + 0 + 1\right)^2} \\ d &= \sqrt{\left(\frac{0}{5} + 2\right)^2 + \left(2\left(\frac{0}{5}\right) + 1\right)^2} \\ d &= \sqrt{(2)^2 + (1)^2} \\ d &= \sqrt{5} \end{aligned}$$



3. Confirm the results obtained in Problem 1 using another method.

$\overline{PR}$  is the hypotenuse of the right triangle with vertices  $P(-2, 1)$ ,  $R(0, 0)$ , and  $(-2, 0)$ . Using the Pythagorean theorem, we find that  $= \sqrt{2^2 + 1^2} = \sqrt{5}$ .

4. Find the perimeter of quadrilateral  $DEBF$  shown below.



The equation of  $\overline{AB}$  is  $y = x + 8$  (students may also choose to use  $\overline{BC}$  which would make  $m = -1$  instead of 1).

$$p = 0, q = 0, m = 1, \text{ and } b = 8$$

$$\begin{aligned} ED &= \sqrt{\left(\frac{p+qm-bm}{1+m^2}-p\right)^2 + \left(m\left(\frac{p+qm-bm}{1+m^2}\right)+b-q\right)^2} \\ ED &= \sqrt{\left(\frac{0+0(1)-8(1)}{1+1^2}-0\right)^2 + \left(1\left(\frac{0+0(1)-8(1)}{1+1^2}\right)+8-0\right)^2} \\ ED &= \sqrt{\left(\frac{-8}{2}\right)^2 + \left(\left(\frac{-8}{2}\right)+8\right)^2} \\ ED &= \sqrt{\left(\frac{-8}{2}\right)^2 + \left(\left(\frac{-8}{2}\right)+8\right)^2} \\ ED &= \sqrt{4^2 + 4^2} \\ ED &= 4\sqrt{2} \end{aligned}$$

$\triangle AED$  is a right triangle with hypotenuse  $\overline{AD}$  with length 8 and leg  $\overline{ED}$  with length  $4\sqrt{2}$ . This means that  $\triangle AED$  is an isosceles right triangle ( $45^\circ-45^\circ-90^\circ$ ), which means  $AE = 4\sqrt{2}$ . We also know that  $\triangle AED \cong \triangle DEB \cong \triangle BFD \cong \triangle DFC$  because they are all right triangles with hypotenuse of length 8 and leg of length  $4\sqrt{2}$ . Therefore, the perimeter of  $DEBF$  is  $4(4\sqrt{2}) = 16\sqrt{2}$ .

Name \_\_\_\_\_

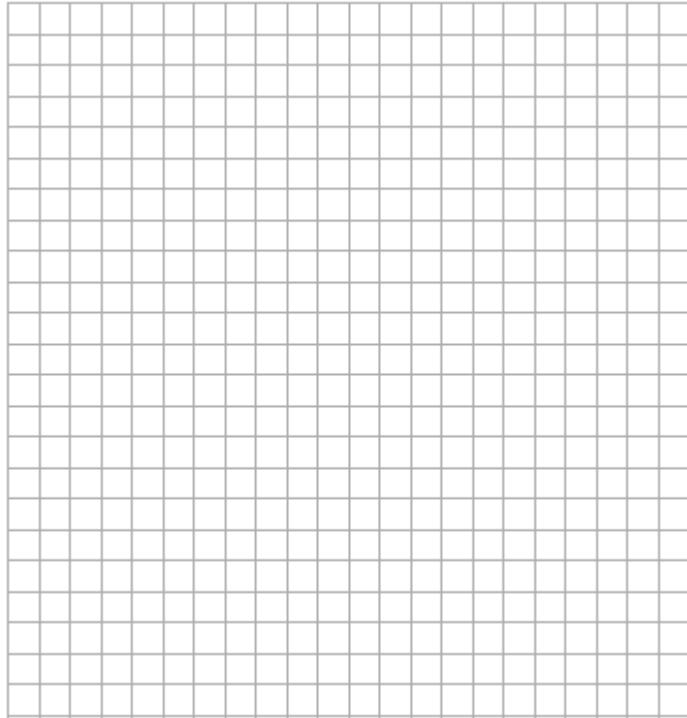
Date \_\_\_\_\_

For problems that require rounding, round answers to the nearest hundredth.

1. A triangular region in the coordinate plane is defined by the system of inequalities

$$y \geq \frac{1}{2}x - 6, \quad y \leq -2x + 9, \quad y \leq 8x + 9.$$

- Determine the coordinates of the vertices of the triangle.
- Sketch the triangular region defined by these inequalities.
- Is the triangle defined by the inequalities a right triangle? Explain your answer.
- Find the perimeter of the triangular region defined by the inequalities, round to the nearest hundredth.



- e. What is the area of this triangular region?
- f. Of the three altitudes of the triangular region defined by the inequalities, what is the length of the shortest of the three? Round to the nearest hundredth.
2. Find the point on the directed line segment from  $(0,3)$  to  $(6,9)$  that divides the segment in the ratio of 2: 1.
3. Consider the points  $A(1,4)$  and  $B(8, -3)$ . Suppose  $C$  and  $D$  are points on the line through  $A$  and  $B$  satisfying  $\frac{AC}{CB} = \frac{1}{3}$  and  $\frac{BD}{DA} = \frac{4}{3}$ , respectively.
- Draw a sketch of the four collinear points  $A, B, C, D$  showing their relative positions to one another.

- b. Find the coordinates of the point  $C$ .
- c. Find the coordinates of the point  $D$ .
4. Two robots are left in a robotics competition. Robot A is programmed to move about the coordinate plane at a constant speed so that, at time  $t$  seconds, its position in the plane is given by

$$(0, 10) + \frac{t}{8}(60, 80).$$

Robot B is also programmed to move about the coordinate plane at a constant speed. Its position in the plane at time  $t$  seconds is given by

$$(70, 0) - \frac{t}{10}(70, -70).$$

a. What was each robot's starting position?

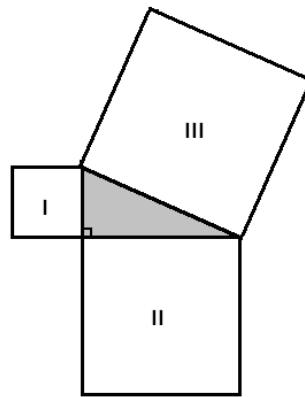
b. Where did each robot stop?

- c. What is the equation of the path of robot A?
- d. What is the equation of the path of robot B?
- e. What is the speed of robot A? (Assume coordinates in the plane are given in units of meters. Give the speed in units of meters per second.)
- f. Do the two robots ever pass through the same point in the plane? Explain. If they do, do they pass through that common point at the same time? Explain.
- g. What is the closest distance robot B will ever be to the origin? Round to the nearest hundredth.

- h. At time  $t = 10$  seconds, robot A will instantaneously turn 90 degrees to the left and travel at the same constant speed it was previously traveling. What will be its coordinates in another 10 seconds' time?
5.  $GDAY$  is a rhombus. If point  $G$  has coordinates  $(2, 6)$  and  $A$  has coordinates  $(8, 10)$ , what is the equation of the line that contains the diagonal  $\overline{DY}$  of the rhombus?
6. a. A triangle has vertices  $A(a_1, a_2)$ ,  $B(b_1, b_2)$ , and  $C(c_1, c_2)$ . Let  $M$  be the midpoint of  $\overline{AC}$  and  $N$  the midpoint of  $\overline{BC}$ . Find a general expression for the slope of  $\overline{MN}$ . What segment of the triangle has the same slope as  $\overline{MN}$ ?

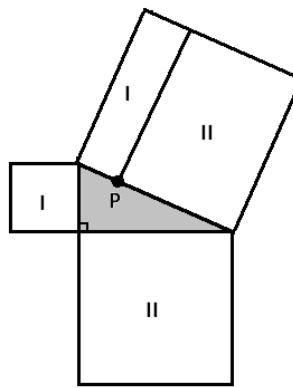
- b. A triangle has vertices  $A(a_1, a_2)$ ,  $B(b_1, b_2)$ , and  $C(c_1, c_2)$ . Let  $P$  be a point on  $\overline{AC}$  with  $AP = \frac{5}{8}AC$ , and let  $Q$  be a point on  $\overline{BC}$  with  $BQ = \frac{5}{8}BC$ . Find a general expression for the slope of  $\overline{PQ}$ . What segment of the triangle has the same slope as  $\overline{PQ}$ ?
- c. A quadrilateral has vertices  $A(a_1, a_2)$ ,  $B(b_1, b_2)$ ,  $C(c_1, c_2)$ , and  $D(d_1, d_2)$ . Let  $R$ ,  $S$ ,  $T$ , and  $U$  be the midpoints of the sides  $\overline{AB}$ ,  $\overline{BC}$ ,  $\overline{CD}$ , and  $\overline{DA}$ , respectively. Demonstrate that  $\overline{RS}$  is parallel to  $\overline{TU}$ . Is  $\overline{ST}$  parallel to  $\overline{UR}$ ? Explain.

7. The Pythagorean theorem states that if three squares are drawn on the sides of a right triangle, then the area of the largest square equals the sum of the areas of the two remaining squares.



$$\text{area III} = \text{area I} + \text{area II}$$

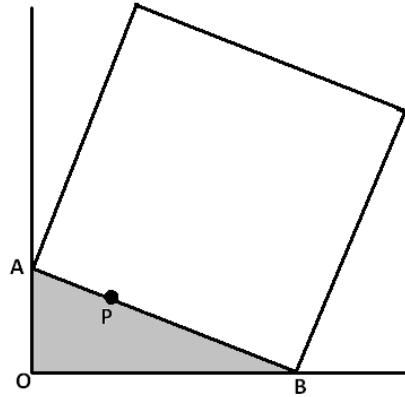
There must be a point  $P$  along the hypotenuse of the right triangle at which the large square is divided into two rectangles as shown, each with area matching the area of one of the smaller squares.



Consider a right triangle  $AOB$  situated on the coordinate plane with vertex  $A$  on the positive  $y$ -axis,  $O$  the origin, and vertex  $B$  on the positive  $x$ -axis.

Suppose  $A$  has coordinates  $(0, a)$ ,  $B$  has coordinates  $(b, 0)$ , and the length of the hypotenuse  $\overline{AB}$  is  $c$ .

- a. Find the coordinates of a point  $P$  on the line segment  $\overline{AB}$  such that  $\overline{OP}$  is perpendicular to  $\overline{AB}$ .



- b. Show that for this point  $P$  we have  $\frac{AP}{PB} = \frac{a^2}{b^2}$ .
- c. Show that if we draw from  $P$  a line perpendicular to  $\overline{AB}$ , then that line divides the square with  $\overline{AB}$  as one of its sides into two rectangles, one of area  $a^2$  and one of area  $b^2$ .

A Progression Toward Mastery					
Assessment Task Item		STEP 1 Missing or incorrect answer and little evidence of reasoning or application of mathematics to solve the problem.	STEP 2 Missing or incorrect answer but evidence of some reasoning or application of mathematics to solve the problem.	STEP 3 A correct answer with some evidence of reasoning or application of mathematics to solve the problem, or an incorrect answer with substantial evidence of solid reasoning or application of mathematics to solve the problem.	STEP 4 A correct answer supported by substantial evidence of solid reasoning or application of mathematics to solve the problem.
1	a  G-GPE.B.5	Student gives no answer or an incorrect answer with no supporting work.	Student equates equations, but no vertices are identified correctly.	Student equated equations and at least one vertex is identified correctly.	Student identifies all vertices correctly.
	b  G-GPE.B.5	Student gives no answer or an incorrect answer with no supporting work.	Student sketches at least one inequality correctly.	Student sketches all inequalities correctly but does not shade region.	Students sketches all inequalities correctly and sketches the region.
	c  G-GPE.B.4	Student gives no answer or an incorrect answer with no supporting work.	Student attempts to find slopes of lines or length of segments, but errors lead to the wrong answer.	Student uses slope or the converse of the Pythagorean theorem to prove the triangle is right but does not explain the answer.	Student uses slope or the converse of the Pythagorean theorem to prove the triangle is right and explains the answer correctly.
	d  G-GPE.B.7	Student gives no answer or an incorrect answer with no supporting work.	Student gives an incorrect answer but work shows use of the distance formula.	Student gives a correct answer with supporting work but not rounded to the hundredths.	Student gives a correct answer with supporting work and properly rounded.
	e  G-GPE.B.7	Student gives no answer or an incorrect answer with no supporting work.	Student attempts to use a formula or decomposition to find area but made many errors leading to an incorrect answer.	Student shows knowledge of using a formula or decomposition to find answer but made minor errors leading to an incorrect answer.	Student has correct answer with accurate supporting work.

	<b>f</b> <b>G-GPE.B.5</b>	Student gives no answer or an incorrect answer with no supporting work.	Student shows some knowledge of altitudes of a triangle, identifying the two legs of the right triangle as two altitudes.	Students shows some knowledge of altitudes of a triangle, identifies the two legs of the right triangle, finds the third altitude of 7.68 units, but does not state the shortest altitude.	Students shows some knowledge of altitudes of a triangle, identifies the two legs of the right triangle, finds the third altitude of 7.68 units, and identifies it as the shortest altitude.
2	<b>G-GPE.B.6</b>	Student gives no answer or an incorrect answer with no supporting work.	Student plots the points and shows some knowledge of using an endpoint and the proportion to find the point but answers incorrectly.	Student finds one of the coordinates correctly.	Student finds both coordinates of the point correctly and writes them as an ordered pair.
3	<b>a</b> <b>G-GPE.B.6</b>	Student gives no answer or an incorrect answer with no supporting work.	Student plots points <i>A</i> and <i>B</i> correctly.	Student plots points <i>A</i> and <i>B</i> as endpoints with point <i>C</i> between but point <i>D</i> is closer to <i>A</i> than <i>C</i> .	Student plots all points correctly: <i>A, C, D, B</i> .
	<b>b</b> <b>G-GPE.B.6</b>	Student gives no answer or an incorrect answer with no supporting work.	Student plots the points and shows some knowledge of using an endpoint and the proportion to find the point but answers incorrectly.	Student finds one of the coordinates correctly.	Student finds both coordinates of the point correctly and writes them as an ordered pair.
	<b>c</b> <b>G-GPE.B.6</b>	Student gives no answer or an incorrect answer with no supporting work.	Student plots the points and shows some knowledge of using an endpoint and the proportion to find the point but answers incorrectly.	Student finds one of the coordinates correctly.	Student finds both coordinates of the point correctly and writes them as an ordered pair.
4	<b>a</b> <b>G-GPE.B.6</b>	No answer is given or an incorrect answer with no supporting work.	Student tries to use formula to calculate starting position but neither is correct.	Student finds one of the starting positions correctly.	Student finds both starting positions correctly.
	<b>b</b> <b>G-GPE.B.6</b>	Student gives no answer or an incorrect answer with no supporting work.	Student tries to use formula to calculate where the robot stopped but neither is correct.	Student finds where one of the robots stops correctly.	Student finds where both of the robots stop correctly.
	<b>c</b> <b>G-GPE.B.6</b>	Student gives no answer or an incorrect answer is given with no supporting work.	Student calculates slope correctly but does not attempt to use a point to write the equation of the line.	Student calculates slope correctly and uses a point to write the equation of the line but makes minor errors.	Student calculates the equation of the line of motion correctly.

	<b>d</b> <b>G-GPE.B.6</b>	Student gives no answer or an incorrect answer is given with no supporting work.	Student calculates slope correctly but does not attempt to use a point to write the equation of the line.	Student calculates slope correctly and uses a point to write the equation of the line but makes minor errors.	Student calculates the equation of the line of motion correctly.
	<b>e</b> <b>G-GPE.B.6</b>	Student does not calculate the distance traveled nor the speed or both are calculated incorrectly.	Student calculates the distance correctly but does not divide by time to arrive at speed.	Student calculates speed correctly but units are incorrect or answer is incorrectly rounded.	Student calculates speed correctly with appropriate units and rounding.
	<b>f</b> <b>G-GPE.B.6</b>	Student gives no answer or an incorrect answer with no supporting work.	Student equates line to find the intersection point but makes mistakes in calculations.	Student finds the correct intersection point but does not find the time that each robot crosses that point.	Student finds the correct intersection point and explains that the robots will not cross at the same time justifying with times.
	<b>g</b> <b>G-GPE.B.5</b>	Student gives no answer or an incorrect answer with no supporting work.	Student uses the equation of motion of robot B but does not use the perpendicular segment to the line through the origin.	Student uses the equation of motion of robot B and the perpendicular line through the origin and find the closest point $(35, 35)$ but not the distance.	Student uses the equation of motion of robot B and the perpendicular line through the origin and finds the distance from $(35, 35)$ to the origin.
	<b>h</b> <b>G-GPE.B.5</b>	Student gives no answer or an incorrect answer with no supporting work.	Student finds the position at 10 seconds $(75, 110)$ but does not move the robot past that point.	Student finds the position at 10 seconds $(75, 100)$ and rotates the motion but calculates the final position incorrectly.	Student finds the position at 10 seconds and rotates and moves the robot to the correct final position $(-35, 185)$ .
<b>5</b>	<b>G-GPE.B.4 G-GPE.B.5</b>	No answer is given or an incorrect answer with no supporting work.	Student shows knowledge of the properties of a rhombus and finds the slope of a line perpendicular to the segment given.	Student shows knowledge of the properties of a rhombus and finds the slope of the line perpendicular to the segment given and the midpoint but does not write the equation of the line.	Student uses the given perpendicular slope and midpoint of segment to write an equation of the other diagonal.
<b>6</b>	<b>a</b> <b>G-GPE.B.6</b>	No answer is given or an incorrect answer with no supporting work.	Student shows knowledge of the midpoint formula, but errors lead to an incorrect answer.	Student finds the midpoints $M$ and $N$ correctly but does not find slope or slope is incorrect.	Student finds the correct slope and sees that it is the same slope as $\overline{AB}$ .

	<b>b</b> <b>G-GPE.B.6</b>	Student gives no answer or an incorrect answer with no supporting work.	Student shows knowledge points on directed segments and finds the coordinates of that point, but errors lead to an incorrect answer.	Student finds the point $P$ and $Q$ correctly but does not find slope or slope is incorrect.	Student finds the correct slope and sees that it is the same slope as $\overline{AB}$ .
	<b>c</b> <b>G-GPE.B.5</b>	Student gives no answer or an incorrect answer with no supporting work.	Student finds the slopes of pairs of segments, but mistakes lead to incorrect answers.	Student finds the slopes of pairs of segments correctly but does not compare slopes and explain answers.	Student finds the slopes of pairs of segments correctly and explains why segments are parallel using slopes.
<b>7</b>	<b>a</b> <b>G-GPE.B.5</b>	Student gives no answer or an incorrect answer with no supporting work.	Student writes the equation of $\overline{AB}$ correctly.	Student writes the equation of $\overline{AB}$ and $\overline{OP}$ but does not find point $P$ or finds it incorrectly.	Student writes the equation of $\overline{AB}$ and $\overline{OP}$ and finds point $P$ correctly.
	<b>b</b> <b>G-GPE.B.4</b>	Student gives no answer or an incorrect answer with no supporting work.	Student finds either $AP$ or $PB$ correctly.	Student finds $AP$ and $PB$ correctly and attempts to simplify the ratio but errors lead to incorrect answers.	Student finds $AP$ and $PB$ , writes them as a ratio, and simplifies the ratio correctly.
	<b>c</b> <b>G-GPE.B.4</b>	Student gives no answer or an incorrect answer with no supporting work.	Student shows knowledge that one dimension of rectangle I and II is $a^2 + b^2$ and tries to use the other dimension but writes the wrong ratio.	Student shows that one dimension of rectangle I and II is $a^2 + b^2$ , writes the other dimension as the correct ratio, and finds the correct areas.	Student shows that one dimension of rectangle I and II is $a^2 + b^2$ , writes the other dimension as the correct ratio, and finds the correct areas.

Name \_\_\_\_\_

Date \_\_\_\_\_

For problems that require rounding, round answers to the nearest hundredth.

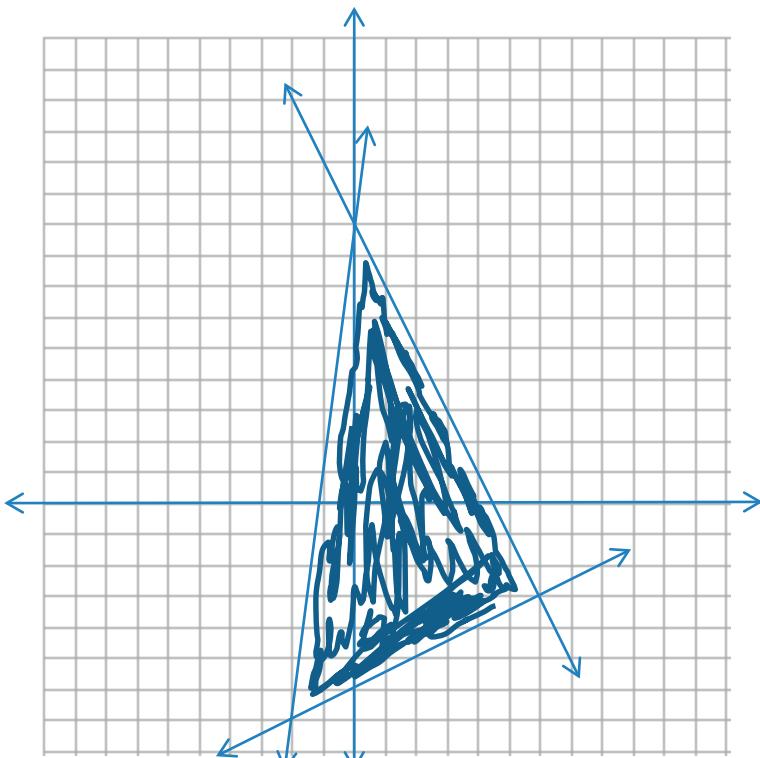
1. A triangular region in the coordinate plane is defined by the system of inequalities

$$y \geq \frac{1}{2}x - 6, \quad y \leq -2x + 9, \quad y \leq 8x + 9.$$

- a. Determine the coordinates of the vertices of the triangle.

$(0, 9), (6, -3), (-2, -7)$

- b. Sketch the triangular region defined by these inequalities.



- c. Is the triangle defined by the inequalities a right triangle? Explain your answer.

Yes, slopes of  $\overline{AB}$  and  $\overline{BC}$  are negative reciprocals, so they are perpendicular.

$AB^2 + BC^2 = AC^2$ ; therefore, the converse of the Pythagorean theorem is true meaning it is a right triangle.

- d. Find the perimeter of the triangular region defined by the inequalities, round to the nearest hundredth.

38.49 units

- e. What is the area of this triangular region?

*60 square units*

- f. Of the three altitudes of the triangular region defined by the inequalities, what is the length of the shortest of the three? Round to the nearest hundredth.

*Altitudes: from  $\angle A = 13.42$ , from  $\angle C = 8.94$ , from  $\angle B = 7.68$  (shortest)*

2. Find the point on the directed line segment from  $(0,3)$  to  $(6,9)$  that divides the segment in the ratio of 2:1.

*(4, 7)*

3. Consider the points  $A(1,4)$  and  $B(8,-3)$ . Suppose  $C$  and  $D$  are points on the line through  $A$  and  $B$  satisfying  $\frac{AC}{CB} = \frac{1}{3}$  and  $\frac{BD}{DA} = \frac{4}{3}$ , respectively.

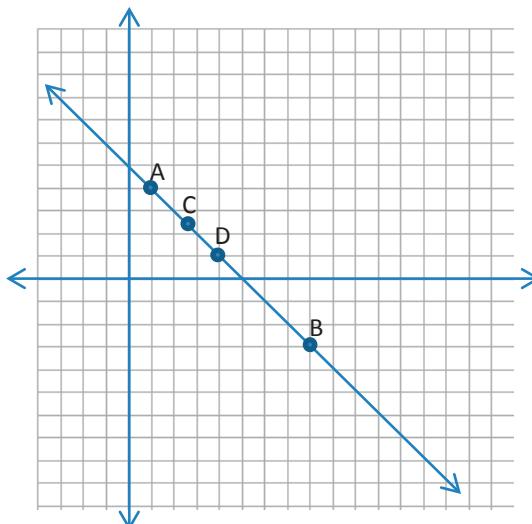
- a. Draw a sketch of the four collinear points  $A, B, C, D$  showing their relative positions to one another.

- b. Find the coordinates of the point  $C$ .

*$\left(\frac{11}{4}, \frac{9}{4}\right)$*

- c. Find the coordinates of the point  $D$ .

*(4, 1)*



4. Two robots are left in a robotics competition. Robot A is programmed to move about the coordinate plane at a constant speed so that, at time  $t$  seconds, its position in the plane is given by

$$(0, 10) + \frac{t}{8}(60, 80).$$

Robot B is also programmed to move about the coordinate plane at a constant speed. Its position in the plane at time  $t$  seconds is given by

$$(70, 0) - \frac{t}{10}(70, -70).$$

- a. What was each robot's starting position?

A (0, 10), B (70, 0)

- b. Where did each robot stop?

A (60, 90), B (0, 70)

- c. What is the equation of the path of robot A?

$y = \frac{4}{3}x + 10$

- d. What is the equation of the path of robot B?

$y = -x + 70$

- e. What is the speed of robot A? (Assume coordinates in the plane are given in units of meters. Give the speed in units of meters per second.)

Distance = 100 m, time = 8 seconds, speed = 12.5 m/sec

- f. Do the two robots ever pass through the same point in the plane? Explain. If they do, do they pass through that common point at the same time? Explain.

The robots both pass through the point  $(28\frac{2}{3}, 48\frac{2}{9})$ . A passes through that point at 3.82 seconds and B at 6.89 seconds. So the robots do not pass through the point at the same time.

- g. What is the closest distance robot B will ever be to the origin? Round to the nearest hundredth.

The closest point that the robot will be to the origin is  $(35, 35)$  and that is a distance of 49.50 meters.

- h. At time  $t = 10$  seconds, robot A will instantaneously turn 90 degrees to the left and travel at the same constant speed it was previously traveling. What will be its coordinates in another 10 seconds' time?

At 10 seconds, robot A will be at  $(75, 110)$ . After turning  $90^\circ$  and continuing for another 10 seconds, the robot will be at  $(-35, 185)$ .

5.  $GDAY$  is a rhombus. If point  $G$  has coordinates  $(2, 6)$  and  $A$  has coordinates  $(8, 10)$ , what is the equation of the line that contains the diagonal  $\overline{DY}$  of the rhombus?

Slope of  $\overline{GA}$  is  $\frac{2}{3}$ , the diagonal will be perpendicular to  $\overline{GA}$  and bisect it, so the slope of  $\overline{DY}$  is  $-\frac{3}{2}$  and it passes through the point  $(5, 8)$ . The equation of the line is  $3x + 2y = 31$ .

6. a. A triangle has vertices  $A(a_1, a_2)$ ,  $B(b_1, b_2)$ , and  $C(c_1, c_2)$ . Let  $M$  be the midpoint of  $\overline{AC}$  and  $N$  the midpoint of  $\overline{BC}$ . Find a general expression for the slope of  $\overline{MN}$ . What segment of the triangle has the same slope as  $\overline{MN}$ ?

$$M\left(\frac{a_1+c_1}{2}, \frac{a_2+c_2}{2}\right), N\left(\frac{b_1+c_1}{2}, \frac{b_2+c_2}{2}\right)$$

$$\text{Slope } \overline{MN} = \frac{\frac{b_2+c_2}{2} - \frac{a_2+c_2}{2}}{\frac{b_1+c_1}{2} - \frac{a_1+c_1}{2}} = \frac{b_2-a_2}{b_1-a_1}$$

Slope  $\overline{MN}$  is the same as slope of  $\overline{AB}$ .

- b. A triangle has vertices  $A(a_1, a_2)$ ,  $B(b_1, b_2)$ , and  $C(c_1, c_2)$ . Let  $P$  be a point on  $\overline{AC}$  with  $AP = \frac{5}{8}AC$ , and let  $Q$  be a point on  $\overline{BC}$  with  $BQ = \frac{5}{8}BC$ . Find a general expression for the slope of  $\overline{PQ}$ . What segment of the triangle has the same slope as  $\overline{PQ}$ ?

$$P\left(a_1 + \frac{5}{8}(c_1 - a_1), a_2 + \frac{5}{8}(c_2 - a_2)\right), Q\left(b_1 + \frac{5}{8}(c_1 - b_1), b_2 + \frac{5}{8}(c_2 - b_2)\right)$$

$$\text{Slope } \overline{PQ} = \frac{\frac{3b_2}{8} - \frac{3a_2}{8}}{\frac{3b_1}{8} - \frac{3a_1}{8}} = \frac{b_2 - a_2}{b_1 - a_1}$$

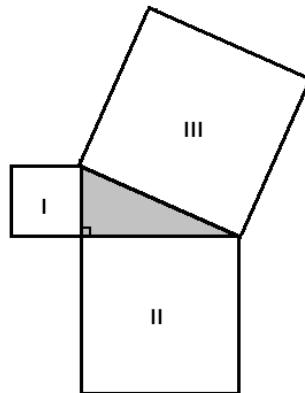
Slope  $\overline{PQ}$  is the same as slope of  $\overline{AB}$ .

- c. A quadrilateral has vertices  $A(a_1, a_2)$ ,  $B(b_1, b_2)$ ,  $C(c_1, c_2)$ , and  $D(d_1, d_2)$ . Let  $R$ ,  $S$ ,  $T$ , and  $U$  be the midpoints of the sides  $\overline{AB}$ ,  $\overline{BC}$ ,  $\overline{CD}$ , and  $\overline{DA}$ , respectively. Demonstrate that  $\overline{RS}$  is parallel to  $\overline{TU}$ . Is  $\overline{ST}$  parallel to  $\overline{UR}$ ? Explain.

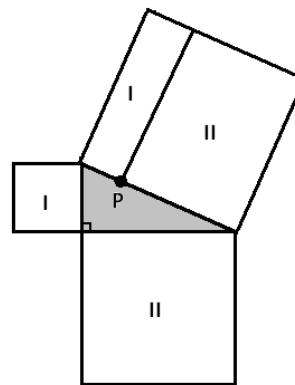
$$\text{Slope } \overline{RS} = \frac{\frac{b_2+c_2}{2} - \frac{a_2+b_2}{2}}{\frac{b_1+c_1}{2} - \frac{a_1+b_1}{2}} = \frac{c_2-a_2}{c_1-a_1} = \text{slope } \overline{TU} \text{ since slopes are equal, segments are parallel.}$$

$$\text{Slope } \overline{ST} = \frac{\frac{b_2+c_2}{2} - \frac{c_2+d_2}{2}}{\frac{b_1+c_1}{2} - \frac{c_1+d_1}{2}} = \frac{b_2-d_2}{b_1-d_1} = \text{slope } \overline{UR} \text{ since slopes are equal, segments are parallel.}$$

7. The Pythagorean theorem states that if three squares are drawn on the sides of a right triangle, then the area of the largest square equals the sum of the areas of the two remaining squares.



There must be a point P along the hypotenuse of the right triangle at which the large square is divided into two rectangles as shown, each with area matching the area of one of the smaller squares.



Consider a right triangle  $AOB$  situated on the coordinate plane with vertex  $A$  on the positive  $y$ -axis,  $O$  the origin, and vertex  $B$  on the positive  $x$ -axis.

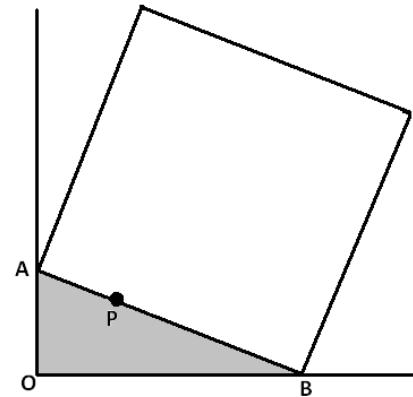
Suppose  $A$  has coordinates  $(0, a)$ ,  $B$  has coordinates  $(b, 0)$ , and the length of the hypotenuse  $\overline{AB}$  is  $c$ .

- a. Find the coordinates of a point  $P$  on the line segment  $\overline{AB}$  such that  $\overline{OP}$  is perpendicular to  $\overline{AB}$ .

$$\text{Equation of } \overline{AB}: y = -\frac{a}{b}x + a$$

$$\text{Equation of } \overline{OP}: y = \frac{b}{a}x$$

$$P \left( \frac{a^2b}{a^2+b^2}, \frac{ab^2}{a^2+b^2} \right)$$



- b. Show that for this point  $P$  we have  $\frac{AP}{PB} = \frac{a^2}{b^2}$ .

$$AP = \sqrt{\left(\frac{a^2b}{a^2+b^2}\right)^2 + \left(\frac{ab^2}{a^2+b^2} - a\right)^2}, PB = \sqrt{\left(b - \frac{a^2b}{a^2+b^2}\right)^2 + \left(\frac{ab^2}{a^2+b^2}\right)^2}$$

$$\frac{AP}{PB} = \sqrt{\frac{a^4b^2 + b^6}{b^6 + a^2b^4}} = \sqrt{\frac{a^4(b^2 + a^2)}{b^4(b^2 + a^2)}} = \sqrt{\frac{a^4}{b^4}} = \frac{a^2}{b^2}$$

- c. Show that if we draw from  $P$  a line perpendicular to  $\overline{AB}$ , then that line divides the square with  $\overline{AB}$  as one of its sides into two rectangles, one of area  $a^2$  and one of area  $b^2$ .

Rectangle I: width =  $a^2+b^2$ , height =  $\frac{a^2}{a^2+b^2}$ , area =  $a^2$

Rectangle II: width =  $a^2+b^2$ , height =  $\frac{b^2}{a^2+b^2}$ , area =  $b^2$