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Complex Numbers and Transformations

Draft Copy

[1] Each lesson is ONE day, and ONE day is considered a 45-minute period.

Precalculus and Advanced Topics • Module 1

Complex Numbers and Transformations

OVERVIEW

Module 1 sets the stage for expanding students understanding of transformations by first exploring the notion of linearity in an algebraic context ("Which familiar algebraic functions are linear?"). This quickly leads to a return to the study of complex numbers and a study of linear transformations in the complex plane. Thus Module 1 builds on standards **N-CN.A.1** and **N-CN.A.2** introduced in Grade 11 and standards **G-CO.A.2**, **G-CO.A.4**, and **G-CO.A.5** introduced in Grade 10.

Topic A opens with a study of common misconceptions by asking questions such as "For which numbers a and b does $(a+b)^2 = a^2 + b^2$ happen to hold?"; "Are there numbers a and b for which $\frac{1}{a+b} = \frac{1}{a} + \frac{1}{b}$?", and so on. The second exercise has only complex solutions which launch a study of quotients of complex numbers and the use of conjugates to find moduli and quotients (**N-CN.A.3**). The topic ends by classifying real and complex functions that satisfy linearity conditions. (A function L is linear if, and only if, there is a real or complex value w such that $L(z) = wz$ for all real or complex z.) Complex number multiplication is emphasized in the last lesson.

In Topic B, students develop an understanding that when complex numbers are considered points in the Cartesian plane, complex number multiplication has the geometric effect of a rotation followed by a dilation in the complex plan. This is a concept that has been developed since Grade 11 and builds upon standards **N-CN.A.1** and **N-CN.A.2**, which when introduced, were accompanied with the observation that multiplication by i has the geometric effect of rotating a given complex number $90°$ about the origin in a counter-clockwise direction. The algebraic inverse of a complex number (its reciprocal) provides the inverse geometric operation. Analysis of the angle of rotation and the scale of the dilation brings a return to topics in trigonometry first introduced in Grade 10 (**G-SRT.C.6, G-SRT.C.7, G-SRT.C.8**) and expanded on in Grade 11 (**F-TF.A.1, F-TF.A.2, F-TF.C.8**). It also reinforces the geometric interpretation of the modulus of a complex number and introduces the notion of the argument of a complex number.

The theme of Topic C is to highlight the effectiveness of changing notations and the power provided by certain notations such as matrices. By exploiting the connection to trigonometry, students see how much of complex arithmetic is simplified. By regarding complex numbers as points in the Cartesian plane, we can begin to write analytic formulas for translations, rotations, and dilations in the plane and revisit the ideas of Grade 10 geometry (**G-CO.A.2, G-CO.A.4** and **G-CO.A.5**) in this light. By taking this work one step further we develop the 2×2 matrix notation for planar transformations represented by complex number arithmetic. This work sheds light on how geometry software and video games efficiently perform rigid motion calculations. Finally, taking another step further, the flexibility implied by 2×2 matrix notation allows us to study additional matrix transformations (shears, for example) that do not necessarily arise from our original complex-number thinking context.

The Mid-Module Assessment follows Topic B. The End-of-Module Assessment follows Topic C.

Focus Standards

Perform arithmetic operations with complex numbers.

N-CN.A.3 (+) Find the conjugate of a complex number; use conjugates to find moduli and quotients of complex numbers.

Represent complex numbers and their operations on the complex plane.

N-CN.B.4 (+) Represent complex numbers on the complex plane in rectangular and polar form (including real and imaginary numbers), and explain why the rectangular and polar forms of a given complex number represent the same number.

N-CN.B.5 (+) Represent addition, subtraction, multiplication, and conjugation of complex numbers geometrically on the complex plane; use properties of this representation for computation. *For example,* $(-1 + \sqrt{3}\,i)^3 = 8$ *because* $(-1 + \sqrt{3}\,i)$ *has modulus 2 and argument 120°.*

N-CN.B.6 (+) Calculate the distance between numbers in the complex plane as the modulus of the difference and the midpoint of a segment as the average of the numbers at its endpoints.

Perform operations on matrices and use matrices in applications.

N-VM.C.10 (+) Understand that the zero and identity matrices play a role in matrix addition and multiplication similar to the role of 0 and 1 in the real numbers. The determinant of a square matrix is nonzero if and only if the matrix has a multiplicative inverse.

N-VM.C.11 (+) Multiply a vector (regarded as a matrix with one column) by a matrix of suitable dimensions to produce another vector. Work with matrices as transformations of vectors.

N-VM.C.12 (+) Work with 2 x 2 matrices as transformations of the plane, and interpret the absolute value of the determinant in terms of area.

Experiment with transformations in the plane.

G-CO.A.2 Represent transformations in the plane using, e.g., transparencies and geometry software; describe transformations as functions that take points in the plane as inputs and give other points as outputs. Compare transformations that preserve distance and angle to those that do not (e.g., translation versus horizontal stretch).

G-CO.A.4 Develop definitions of rotations, reflections, and translations in terms of angles, circles, perpendicular lines, parallel lines, and line segments.

G-CO.A.5 Given a geometric figure and a rotation, reflection, or translation, draw the transformed figure using, e.g., graph paper, tracing paper, or geometry software. Specify a sequence of transformations that will carry a given figure onto another.

Foundational Standards

Reason quantitatively, and use units to solve problems.

N-Q.A.2 Define appropriate quantities for the purpose of descriptive modeling.★

Perform arithmetic operations with complex numbers.

N-CN.A.1 Know there is a complex number i such that $i^2 = -1$, and every complex number has the form $a + bi$ with a and b real.

N-CN.A.2 Use the relation $i^2 = -1$ and the commutative, associative, and distributive properties to add, subtract, and multiply complex numbers.

Use complex numbers in polynomial identities and equations.

N-CN.C.7 Solve quadratic equations with real coefficients that have complex solutions.

N-CN.C.8 (+) Extend polynomial identities to the complex numbers. *For example, rewrite* $x^2 + 4$ *as* $(x + 2i)(x - 2i)$.

Interpret the structure of expressions.

A-SSE.A.1 Interpret expressions that represent a quantity in terms of its context.★

a. Interpret parts of an expression, such as terms, factors, and coefficients.

b. Interpret complicated expressions by viewing one or more of their parts as a single entity. *For example, interpret P(1+r)n as the product of P and a factor not depending on P.*

Write expressions in equivalent forms to solve problems.

A-SSE.B.3 Choose and produce an equivalent form of an expression to reveal, and explain properties of the quantity represented by the expression.★

a. Factor a quadratic expression to reveal the zeros of the function it defines.

b. Complete the square in a quadratic expression to reveal the maximum or minimum value of the function it defines.

c. Use the properties of exponents to transform expressions for exponential functions. *For example, the expression* 1.15^t *can be rewritten as* $(1.15^{1/12})^{12t} \approx 1.012^{12t}$ *to reveal the approximate equivalent monthly interest rate if the annual rate is 15%.*

Create equations that describe numbers or relationships.

A-CED.A.1 Create equations and inequalities in one variable and use them to solve problems. *Include equations arising from linear and quadratic functions, and simple rational and exponential functions.*

A-CED.A.2 Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales.

A-CED.A.3 Represent constraints by equations or inequalities, and by systems of equations and/or inequalities, and interpret solutions as viable or nonviable options in a modeling context. *For example, represent inequalities describing nutritional and cost constraints on combinations of different foods.*

A-CED.A.4 Rearrange formulas to highlight a quantity of interest, using the same reasoning as in solving equations. *For example, rearrange Ohm's law $V = IR$ to highlight resistance R.*

Understand solving equations as a process of reasoning and explain the reasoning.

A-REI.A.1 Explain each step in solving a simple equation as following from the equality of numbers asserted at the previous step, starting from the assumption that the original equation has a solution. Construct a viable argument to justify a solution method.

Solve equations and inequalities in one variable.

A-REI.B.3 Solve linear equations and inequalities in one variable, including equations with coefficients represented by letters.

A-REI.C.6 Solve systems of linear equations exactly and approximately (e.g., with graphs), focusing on pairs of linear equations in two variables.

Extend the domain of trigonometric functions using the unit circle.

F-TF.A.1 Understand radian measure of an angle as the length of the arc on the unit circle subtended by the angle.

F-TF.A.2 Explain how the unit circle in the coordinate plane enables the extension of trigonometric functions to all real numbers, interpreted as radian measures of angles traversed counterclockwise around the unit circle.

F-TF.A.3 (+) Use special triangles to determine geometrically the values of sine, cosine, tangent for $\pi/3$, $\pi/4$ and $\pi/6$, and use the unit circle to express the values of sine, cosine, and tangent for $\pi-x$, $\pi+x$, and $2\pi-x$ in terms of their values for x, where x is any real number.

Prove and apply trigonometric identities.

F-TF.C.8 Prove the Pythagorean identity $\sin^2(\theta) + \cos^2(\theta) = 1$ and use it to find $\sin(\theta)$, $\cos(\theta)$, or $\tan(\theta)$ given $\sin(\theta)$, $\cos(\theta)$, or $\tan(\theta)$ and the quadrant of the angle.

Focus Standards for Mathematical Practice

MP.2 **Reason abstractly and quantitatively.** Students come to recognize that a multiplication by a complex number corresponds to the geometric action of a rotation and dilation from the origin in the complex plane. Students apply this knowledge to understand that multiplication by the reciprocal provides the inverse geometric operation to a rotation and dilation. Much of the module is dedicated to helping students quantify the rotations and dilations in increasingly abstract ways so they do not depend on the ability to visualize the transformation. That is, they reach a point where they do not need a specific geometric model in mind to think about a rotation or dilation. Instead, they can make generalizations about the rotation or dilation based on the problems they have previously solved.

MP.3 **Construct viable arguments and critique the reasoning of others.** Throughout the module, students study examples of work by algebra students. This work includes a number of common mistakes that algebra students make, but it is up to the student to decide about the validity of the argument. Deciding on the validity of the argument focuses the students on justification and argumentation as they work decide when purported algebraic identities do or do not hold. In cases where they decide that the given student work is incorrect, the students work to develop the correct general algebraic results and justify them by reflecting on what they perceived as incorrect about the original student solution.

MP.4 **Model with mathematics.** As students work through the module, they become attuned to the geometric effect that occurs in the context of complex multiplication. However, initially it is unclear to them why multiplication by complex numbers entails specific geometric effects. Thus, the students initially have a model that includes two seemingly associated actions (geometric transformation and multiplication with complex numbers) but no formal way to explain why they should be connected. In the module, the students create a model of computer animation in the plane. The focus of the mathematics in the computer animation is such that the students come to rotating and translating as dependent on matrix operations and the addition of 2×1 vectors. Thus, their initial geometric model becomes more formal with the notion of complex numbers.

Terminology

New or Recently Introduced Terms

- Linear Transformation
- Conjugate
- Modulus
- Argument
- Polar Form
- Matrix
- Determinant

Familiar Terms and Symbols[2]

- Complex Number
- Rotation
- Dilation
- Translation
- Rectangular Coordinates
- Trigonometry
- Rectangular Form

Suggested Tools and Representations

- Graphing Calculator
- Wolfram Alpha Software
- Geometer's Sketchpad Software

Assessment Summary

Assessment Type	Administered	Format	Standards Addressed
Mid-Module Assessment Task	After Topic B	Constructed response with rubric	N-CN.A.3, N-CN.B.4, N-CN.B.5, N-CN.B.6, G-CO.A.2, G-CO.A.4, G-CO.A.5
End-of-Module Assessment Task	After Topic C	Constructed response with rubric	N-CN.B.4, N-CN.B.5, N-VM.C.10, N-VM.C.11, N-VM.C.12, G-CO.A.2, G-CO.A.4, G-CO.A.5

[2] These are terms and symbols students have seen previously.

Name ____ Date ____________________

1. Let $z_1 = 2 - 2i$, and $z_2 = (1 - i) + \sqrt{3}(1 + i)$.

   a. What is the modulus and argument of z_1?

   b. Find a complex number w, written in the form $w = a + ib$, such that $wz_1 = z_2$.

   c. What is the modulus and argument of w?

d. When the points z_1 and z_2 are plotted in the complex plane, what is the angle between the line segment connecting the origin to z_1 and the line segment connecting z_2 to z_1? What type of triangle is formed by the origin and the two points represented by the complex numbers z_1 and z_2?

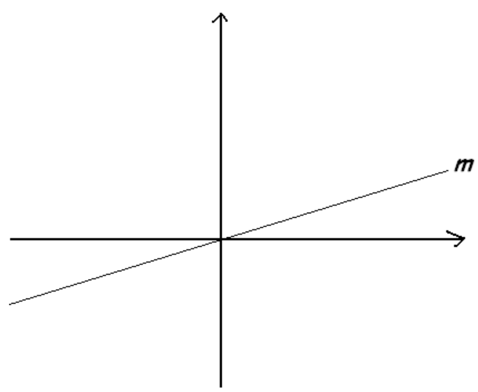
e. Find the complex number, v, closest to the origin that lies on the line segment connecting z_1 and z_2. Write v in rectangular form.

2. Let z be the complex number $2 + 3i$ lying in the complex plane.

   a. Write down the complex number that is the reflection of z across the horizontal axis.

   b. Write down the complex number that is the reflection of z across the vertical axis.

Let m be the line through the origin of slope $\frac{1}{2}$ in the complex plane.



   c. Write down a complex number, w, of modulus 1 that lies on m in the first quadrant.

d. What is the modulus of wz and how does it compare to the modulus of z? First use properties of modulus to answer this question, and then give a geometric explanation for what you observe.

e. When asked,
"What is the argument of $\frac{1}{w}z$ *?"*
Paul gave the answer: $\arctan\left(\frac{3}{2}\right) - \arctan\left(\frac{1}{2}\right)$, which he then computed to two decimal places.
Provide a geometric explanation that yields Paul's answer.

f. When asked,

"What is the argument of $\frac{1}{w}z$*?"*

Mable did the complex number arithmetic and computed $z \div w$. She then gave an answer in the form $\arctan\left(\frac{a}{b}\right)$ for some fraction $\frac{a}{b}$. What fraction did Mable find? Up to two decimal places, is Mable's final answer the same as Paul's?

g. Write down the complex number (in rectangular form $a + ib$) that is the reflection of z across line m.

3.

a. A non-zero complex number $a + ib$, with a and b real numbers, happens to satisfy $(a + ib)^2 = a^2 + ib^2$. What is the argument of that complex number? Explain.

b. Is there a complex number $a + ib$, with a and b non-zero real numbers, which happens to satisfy: $\frac{1}{a+ib} = \frac{1}{a} + \frac{1}{ib}$?

If so, give an example of such a complex number. If none exists, explain why there can be no such complex number.

c. Find three non-zero complex numbers which each satisfy the equation $z^2 = \bar{z}$.

A Progression Toward Mastery					
Assessment Task Item		**STEP 1** Missing or incorrect answer and little evidence of reasoning or application of mathematics to solve the problem.	**STEP 2** Missing or incorrect answer but evidence of some reasoning or application of mathematics to solve the problem.	**STEP 3** A correct answer with some evidence of reasoning or application of mathematics to solve the problem, or an incorrect answer with substantial evidence of solid reasoning or application of mathematics to solve the problem.	**STEP 4** A correct answer supported by substantial evidence of solid reasoning or application of mathematics to solve the problem.
1	**a** N-CN.B.5 N-CN.B.6	Students response is incorrect, and there is no evidence to support that the student understands how to compute the modulus and argument.	Student uses the correct method and answer for either the modulus or the argument, OR student uses the correct method for both but makesminor errors.	Student uses the correct methods for both, but either the modulus or argument is incorrect due to a minor error, OR student gives correct answers for both with no supporting work shown.	Student computes modulus AND argument correctly. Work is shown to support the answer.
	b N-CN.A.3	Student does not attempt to divide z_2 by z_1 to find w.	Student attempts to divide z_2 by z_1 without applying the correct algorithm involving multiplication 1 in the form of the conjugate z_1 divided by the conjugate of z_1.	Student applies the division algorithm correctly AND shows work, but has minor mathematical errors leading to an incorrect final answer.	Student applies the division algorithm correctly and gives a correct answer with sufficient work shown to demonstrate understanding of the process.
	c N-CN.B.5 N-CN.B.6	Student does not compute either answer correctly nor is there evidence to support that the student understands how to compute the modulus and argument.	Student computes modulus or argument correctly OR uses correct method for both, but arrives at incorrect answers due to minor errors.	Student uses correct methods for both, but either the modulus or the argument is incorrect due to a minor error OR correct answers for both are given with no supporting work shown.	Student computes modulus AND argument correctly for the answer to part (b) and work is shown to support answer. NOTE: Student can earn full points for this even if the answer to (b) is incorrect.

	d N-CN.B.4 N-CN.B.5 G-CO.A.2 G-CO.A.4 G-CO.A.5	Student makes little or no attempt to identify the triangle and angle measure.	Student may identify the triangle as isosceles but cannot determine the angle between the two complex numbers whose vertex is at the origin. Student's explanation fails to connect multiplication with transformations. Explanation may include a comparison of the moduli of z_1 and z_2. Answer may include a sketch to support the answer.	Student identifies the triangle as isosceles or equilateral but may not indicate the requested angle measurement or gives an incorrect angle measurement. Student's explanation does not fully address the connection between multiplication and transformations but may include a comparison of the moduli of z_1 and z_2. Answer may be supported with a sketch.	Student identifies the triangle as equilateral and gives the measure of the angle as *60°*. Student's explanation clearly connects multiplication with the correct transformations by explaining that z_2 is the image of z_1 achieved by rotating z_1 by the $\arg(w)$ and no dilation since $\lvert w\rvert = 1$. Answer may be supported with a sketch.
	e N-CN.B.4	Student makes little or no attempt to find v.	Student may attempt to sketch the situation, but more than one misconception or mathematical error leads to an incorrect or incomplete solution.	Student finds v correctly, but there is little or no explanation or work explaining why v is the midpoint of the line segment, <u>OR</u> student identifies that v would be at the midpoint, but fails to compute it correctly.	Student averages z_1 and z_2 to find v <u>AND</u> clearly explains why using a geometric argument regarding v's location on the perpendicular bisector of the triangle.
2	**a** N-CN.A.3 G-CO.A.2	Student incorrectly answers both the real and the imaginary part.	Student incorrectly answers either the real or the imaginary part.	Student gives the correct answer but does not give an explanation.	Student gives the correct answer and identifies it as the conjugate.
	b N-CN.B.5 G-CO.A.2	Student incorrectly answers both the real and the imaginary part.	Student incorrectly answers either the real or the imaginary part.	Student gives the correct answer but does not give an explanation.	Student gives the correct answer and explains that the real part is the opposite, but the imaginary part stays the same.
	c N-CN.B.4 G-CO.A.2	Student answer is not a complex number <u>OR</u> is a complex number with both an incorrect modulus and argument. There is little or no supporting work shown.	Student gives an incorrect answer with little evidence of correct reasoning (solution may fail to address modulus of 1 but be a complex number on the line m <u>OR</u> may have a modulus of 1 but not be a complex number on the line m).	Student gives correct answer with limited reasoning or work to support the answer, <u>OR</u> student uses correct reasoning but minor mathematical errors lead to an incorrect solution.	Student gives a correct answer with work shown to support approach. Student's reasoning could use the polar form of a complex number <u>OR</u> apply proportional reasoning to find w with the correct argument and modulus.
	d	Student gives an incorrect answer with	Student gives a correct answer but does not	Student gives a correct answer, but one of the	Student gives a correct answer with work shown

	N-CN.B.4 G-CO.A.5	little or no work shown.	include a geometric or modulus operation explanation.	two explanations is limited or missing, <u>OR</u> student gives an incorrect answer due to a minor error with sufficient evidence to support understanding of both the modulus and geometric arguments.	to clearly support both explanations (properties of modulus and geometric).
	e N-CN.B.5 N-CN.B.6	Student shows little or no work. Explanation fails to address transformations in a meaningful way.	Student explanation may include references to rotations and dilations but does not clearly address the fact that multiplication by $1/w$ represents a clockwise rotation of $\arg(w)$.	Student identifies $\arg(z)$ and $\arg(w)$, explains that multiplication by $1/w$ would create a clockwise rotation, but the explanation lacks a clear reason why the two should be subtracted or contains other minor errors.	Student explains both the rotation and lack of dilation correctly since $\lvert w \rvert = 1$. In the explanation, student identifies $\arg(z)$ and $\arg(w)$ <u>AND</u> supports why their difference in the solution. A sketch may be included.
	f N-CN.A.3 N-CN.B.5	Students answer both z/w and $\arg(z/w)$ incorrectly due to major misconceptions or calculation errors. Student may argue incorrectly that the arguments in e and f should be different.	Student answers either z/w or $\arg(z/w)$ incorrectly due to mathematical errors. Student s indicates that both arguments should be the same.	Student correctly computes z/w and $\arg(z/w)$ but fails to compare the arguments in part d and e or explain why they should be the same.	Student correctly computes both arguments to two decimal places. Work shown uses appropriate notation <u>AND</u> sufficient steps to follow the solution.
	g N-CN.B.4 G-CO.A.4	Student makes little or no attempt to solve the problem.	Student explains an approach that would partially accomplish a correct solution and supports it with correct work, <u>OR</u> student explains a correct approach but makes limited progress on a correct solution due to major errors.	Student explains an approach but may not be able to execute an approach that would yield a correct solution. Work is limited and may contain minor errors.	Student explains <u>AND</u> executes an approach that yields a correct solution. Student work is supported both verbally <u>AND</u> symbolically with proper terminology and notation.
3	**a** N-CN.A.3	Student makes no attempt to solve for the argument.	Student gives an incorrect explanation as to how the numbers a and b were reached and an incorrect answer for the argument.	Student gives the correct answer for the argument but is not thorough, <u>OR</u> student does not give an explanation of how the answer was reached.	Student gives the correct answer for the argument, <u>AND</u> has a complete explanation as to how the numbers a and b were found to calculate the argument.

	b **N-CN.A.3**	Student does not reach the conclusion that no complex numbers exist, and there is little to no attempt to solve for a complex number.	Student does not reach the conclusion that no complex numbers exist is not reached but there is some correct reasoning and progress made on solving for the terms a and b.	Student correctly states the answer but has flaws in the process used to reach the answer OR has correct algebraic reasoning and verbal explanations only made errors that prevented a correct final answer.	Student correctly states the answer that no complex numbers exist AND provides correct step-by-step equations used to reach the answer, as well as verbal explanations.
	c **N-CN.A.3**	Student makes little to no attempt to solve for any of the three non-zero complex numbers.	Student correctly uses algebraic reasoning to solve for the three solutions, but calculations contain errors that prevent any correct final solutions.	Student correctly uses algebraic reasoning to solve for the three solutions, but only correctly solves one of the “cases” correctly due to errors in calculations, OR is able to find three solutions but one of them isn’t a non-zero complex number.	Student correctly solves for three non-zero complex numbers that satisfy the given equation and correctly realizes that a potential solution would yield a zero complex number, thus not including it as an answer.

Name ____ Date ____________________

1. Let $z_1 = 2 - 2i$, and $z_2 = (1 - i) + \sqrt{3}(1 + i)$.

   a. What is the modulus and argument of z_1?

$$|z_1| = \sqrt{(2)^2 + (-2)^2} = \sqrt{8} = 2\sqrt{2}$$

$$arg(z_1) = \tan^{-1}\left(\frac{-2}{2}\right) = \frac{-\pi}{4}$$

   b. Find a complex number w, written in the form $w = a + ib$, such that $wz_1 = z_2$.

$$z_2 = 1 - i + \sqrt{3} + i\sqrt{3} = \left(\sqrt{3} + 1\right) + \left(\sqrt{3} - 1\right)i$$

$wz_1 = z_2$ *implies that:*

$$w = \frac{z_2}{z_1} = \frac{\left[\left(\sqrt{3} + 1\right) + \left(\sqrt{3} - 1\right)i\right]}{(2 - 2i)} \times \frac{(2 + 2i)}{(2 + 2i)} = \frac{2\sqrt{3} + 2 + 2i\sqrt{3} + 2i + 2i\sqrt{3} - 2i - 2\sqrt{3} + 2}{4 + 4}$$

$$= \frac{4 + 4i\sqrt{3}}{8}$$

$$= \frac{1}{2} + \frac{\sqrt{3}}{2}i$$

   c. What is the modulus and argument of w?

$$|w| = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = \sqrt{\frac{1}{4} + \frac{3}{4}} = 1$$

$$arg(w) = \tan^{-1}\left(\frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}}\right) = \frac{\pi}{3}$$

d. When the points z_1 and z_2 are plotted in the complex plane, what is the angle between the line segment connecting the origin to z_1 and the line segment connecting z_2 to z_1? What type of triangle is formed by the origin and the two points represented by the complex numbers z_1 and z_2?

Since $z_2 = wz_1, z_2$ is the transformation of z_1 rotated counterclockwise by $arg(w)$, which is $\frac{\pi}{3}$ and dilated by the modulus of w, which is 1.

Since $|w| = 1$ and $arg(w) = \frac{\pi}{3}$, the triangle formed by the origin and the points representing z_1 and z_2 will be equilateral. All of the angles are 60° in this triangle.

e. Find the complex number, v, closest to the origin that lies on the line segment connecting z_1 and z_2. Write v in rectangular form.

The point that represents v is the midpoint of the segment connecting z_1 and z_2 since it must be on the perpendicular bisector of the triangle with vertex at the origin.

To find the midpoint, average z_1 and z_2.

$$v = \frac{\sqrt{3}+1+2}{2} + \frac{\sqrt{3}-1-2}{2}i = \frac{\sqrt{3}+3}{2} + \frac{\sqrt{3}-3}{2}i$$

2. Let z be the complex number $2 + 3i$ lying in the complex plane.

   a. Write down the complex number that is the reflection of z across the horizontal axis.

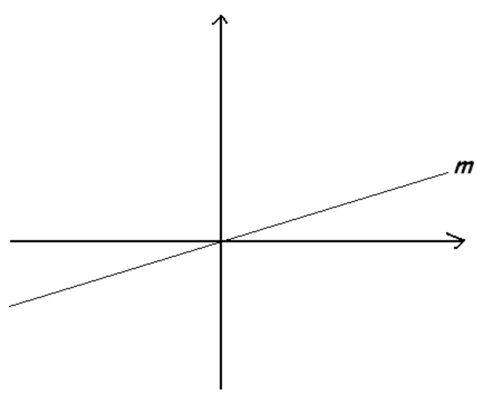
   This number is the conjugate of z.

   $$\bar{z} = 2 - 3i$$

   b. Write down the complex number that is the reflection of z across the vertical axis.

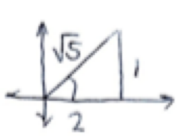
   This number is $-2+3i$

Let m be the line through the origin of slope $\frac{1}{2}$ in the complex plane.



   c. Write down a complex number, w, of modulus 1 that lies on m in the first quadrant.

   Because the slope of m is $\frac{1}{2}$, the argument of w is $tan^{-1}\left(\frac{1}{2}\right)$.
   Using the polar form of w,
   $w = 1\left[cos\left(tan^{-1}\frac{1}{2}\right) + i\,sin\left(tan^{-1}\frac{1}{2}\right)\right]$. From the triangle shown below, $w = \frac{2}{\sqrt{5}} + \frac{1}{\sqrt{5}}i$



d. What is the modulus of wz, and how does it compare to the modulus of z? First use properties of modulus to answer this question, and then give a geometric explanation for what you observe.

The modulus of wz is $\sqrt{13}$, and is the same as $|z|$.

Using the properties of modulus,

$$|wz| = |w| \times |z| = 1 \times |z| = 1 \times \sqrt{2^2 + 3^2} = \sqrt{13}$$

Geometrically, multiplying by w will rotate z by the $arg(w)$ and dilate z by $|w|$. Since $|w| = 1$, the transformation is a rotation only, so both w and z are the same distance from the origin.

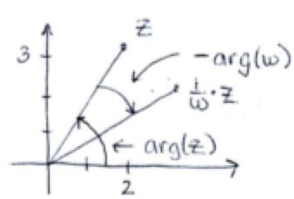
e. When asked,

"What is the argument of $\frac{1}{w}z$ *?"*

Paul gave the answer: $\arctan\left(\frac{3}{2}\right) - \arctan\left(\frac{1}{2}\right)$, which he then computed to two decimal places. Provide a geometric explanation that yields Paul's answer.

The product $\frac{1}{w}z$ would result in a clockwise rotation of z by the $arg(w)$. There would be no dilation since $|w| = 1$.

$$arg(z) = \tan^{-1}\left(\frac{3}{2}\right)$$
$$arg(w) = \tan^{-1}\left(\frac{1}{2}\right)$$
$$arg\left(\frac{1}{w}z\right) = \tan^{-1}\left(\frac{3}{2}\right) - \tan^{-1}\left(\frac{1}{2}\right)$$



f. When asked,

"What is the argument of $\frac{1}{w}z$?"

Mable did the complex number arithmetic and computed $z \div w$. She then gave an answer in the form arctan $\left(\frac{a}{b}\right)$ for some fraction $\frac{a}{b}$. What fraction did Mable find? Up to two decimal places, is Mable's final answer the same as Paul's?

$$\frac{z}{w} = \frac{z\overline{w}}{w\overline{w}} = \frac{z\overline{w}}{|w|} = \frac{(2+3i) \times \left(\frac{2}{\sqrt{5}} - \frac{1}{\sqrt{5}}i\right)}{1}$$

$$= \frac{4}{\sqrt{5}} + \frac{6i}{\sqrt{5}} - \frac{2i}{\sqrt{5}} + \frac{3}{\sqrt{5}}$$

$$= \frac{7}{\sqrt{5}} + \frac{4}{\sqrt{5}}i$$

Comparing these angles shows they are the same.

$$\tan^{-1}\left(\frac{4}{7}\right) \approx 0.52$$

$$\tan^{-1}\left(\frac{3}{2}\right) - \tan^{-1}\left(\frac{1}{2}\right) \approx 0.52$$

g. Write down the complex number (in rectangular form $a + ib$) that is the reflection of z across line m.

Using transformations, rotate z clockwise by $\arg(w)$ so the line m corresponds to the x-axis. Then, reflect the new point horizontally. Finally, rotate this point counterclockwise by $\arg(w)$.

Symbolically,

$$\frac{1}{w}z = \frac{7}{\sqrt{5}} + \frac{4}{\sqrt{5}}i \qquad \textit{clockwise rotation}$$

$$\frac{1}{w}z = \frac{7}{\sqrt{5}} - \frac{4}{\sqrt{5}}i \qquad \textit{the conjugate, horizontal reflection}$$

$$w\frac{1}{w}z = \left(\frac{2}{\sqrt{5}} + \frac{1}{\sqrt{5}}i\right) \times \left(\frac{7}{\sqrt{5}} - \frac{4}{\sqrt{5}}i\right) = \frac{14}{5} + \frac{4}{5} - \frac{8}{5}i + \frac{7}{5}i$$

$$= \frac{18}{5} - \frac{1}{5}i \qquad \textit{counter clockwise rotation}$$

3.

a. A non-zero complex number $a + ib$, with a and b real numbers, happens to satisfy $(a + ib)^2 = a^2 + ib^2$. What is the argument of that complex number? Explain.

We have $a^2 - b^2 + 2abi = a^2 + ib^2$ from which it follows that $a^2 - b^2 = a^2$ and so $b = 0$. As the complex number is non-zero, $a \neq 0$. The complex number is a non-zero real number and so has argument zero.

b. Is there a complex number $a + ib$, with a and b non-zero real numbers, which happens to satisfy $\frac{1}{a+ib} = \frac{1}{a} + \frac{1}{ib}$?

If so, give an example of such a complex number. If none exists, explain why there can be no such complex number.

If so, give an example of such a complex number. If none exists, explain why there can be no such complex number.

We have:

$$\frac{a - ib}{a^2 + b^2} = \frac{1}{a} - i\frac{1}{b}$$

from which it follows that $\frac{a}{a^2+b^2} = \frac{1}{a}$ and $\frac{b}{a^2+b^2} = \frac{1}{b}$. Consequently $\frac{a^2}{a^2+b^2} = \frac{b^2}{a^2+b^2}$, giving, $a^2 = b^2$, and so $a = \pm b$.

From $1 = \frac{a^2}{a^2+b^2} = \frac{a^2}{2a^2}$, we see that no value of a, and hence no value for b, exists to make this equation hold. There are no complex numbers that satisfy this equation.

c. Find three non-zero complex numbers which each satisfy the equation $z^2 = \bar{z}$.

Write $z = a + ib$ for some real numbers a and b. We wish to solve the following:

$$a^2 - b^2 + 2abi = a = ib.$$

From $2ab = -b$ we obtain have either that $b = 0$ or $a = -\frac{1}{2}$.

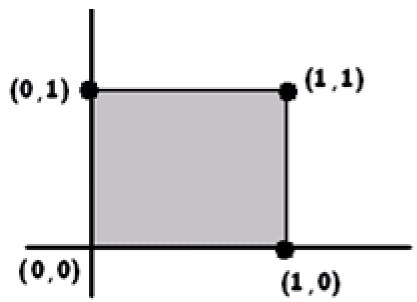
In the first case $a^2 - 0^2 = a$ gives $a = 1$ (we can't have $a = 0$ as well), and $z = 1$ is one complex number satisfying $z^2 = \bar{z}$.

In the second case, $\frac{1}{4} - b^2 = -\frac{1}{2}$ gives $b = \pm\frac{\sqrt{3}}{2}$, and $z = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$ are two more solutions to $z^2 = \bar{z}$.

Name __________________________________ Date ____________________

1. Consider the transformation on the plane given by the 2×2 matrix $\begin{pmatrix} 1 & k \\ 0 & k \end{pmatrix}$ for a fixed positive number k.

   a. Draw a sketch of the image of the unit square under this transformation (the unit square has vertices$(0,0)$, $(1,0)$, $(0,1)$, $(1,1)$). Be sure to label all four vertices of the image figure.



The Unit Square

   b. What is the area of the image parallelogram?

c. Find the coordinates of a point $\begin{pmatrix} x \\ y \end{pmatrix}$ whose image under the transformation is $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$.

d. The transformation $\begin{pmatrix} 1 & k \\ 0 & k \end{pmatrix}$ is applied once to the point $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, then once to the image point, then once to the image of the image point, and then once to the image of the image of the image point, and so on. What are the coordinates of a ten-fold image of the point $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, that is, the image of the point after the transformation has been applied ten times?

2. Consider the transformation given by $\begin{pmatrix} \cos(1) & -\sin(1) \\ \sin(1) & \cos(1) \end{pmatrix}$.

   a. Describe the geometric effect of applying this transformation to a point $\begin{pmatrix} x \\ y \end{pmatrix}$ in the plane.

   b. Describe the geometric effect of applying this transformation to a point $\begin{pmatrix} x \\ y \end{pmatrix}$ in the plane twice: once to the point and then once to its image.

   c. Use part (b) to prove $\cos(2) = \cos^2(1) - \sin^2(1)$, and $\sin(2) = 2\sin(1)\cos(1)$.

3.

a. Write $(1+i)^{10}$ as a complex number of the form $a+ib$ for real numbers a and b.

b. Find a complex number $a+ib$, with a and b positive real numbers, such that $(a+ib)^3=i$.

c. If z is a complex number, is there sure to exist, for any positive integer n, a complex number w such that $w^n = z$? Explain your answer.

d. If z is a complex number, is there sure to exist, for any negative integer n, a complex number w such that $w^n = z$? Explain your answer.

4. In programming a computer video game, Mavis coded the changing location of a space-rocket as follows:

At a time t seconds between $t = 0$ seconds and $t = 2$ seconds, the location $\begin{pmatrix} x \\ y \end{pmatrix}$ of the rocket is given by

$$\begin{pmatrix} \cos\left(\frac{\pi}{2}t\right) & -\sin\left(\frac{\pi}{2}t\right) \\ \sin\left(\frac{\pi}{2}t\right) & \cos\left(\frac{\pi}{2}t\right) \end{pmatrix} \begin{pmatrix} -1 \\ -1 \end{pmatrix}.$$

At a time of t seconds between $t = 2$ seconds and $t = 4$ seconds, the location of the rocket is given by: $\begin{pmatrix} 3 - t \\ 3 - t \end{pmatrix}$.

a. What is the location of the rocket at time $t = 0$? What is its location at time $t = 4$?

b. Petrich is worried that Mavis may have made a mistake and the location of the rocket is unclear at time $t = 2$ seconds. Explain why there is no inconsistency in the location of the rocket at this time.

c. What is the area of the region enclosed by the path of the rocket from time $t = 0$ to time $t = 4$?

d. Mavis later decided that the moving rocket should be shifted five places further to the right. How should she adjust her formulations above to accomplish this translation?

5. Let $P = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ and $O = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

a. Give an example of a 2×2 matrix A, not with all entries equal to zero, such that $PA = O$.

b. Give an example of a 2×2 matrix B with $PB \neq O$.

c. Give an example of a 2×2 matrix C such that $CR = R$ for all 2×2 matrices R.

d. If a 2×2 matrix D has the property that $D + R = R$ for all 2×2 matrices R, must D be the zero matrix O? Explain.

e. Let $E = \begin{pmatrix} 2 & 4 \\ 3 & 6 \end{pmatrix}$. Is there 2×2 matrix F so that $EF = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $FE = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$? If so, find one. If not, explain why no such matrix F can exist.

A Progression Toward Mastery					
Assessment Task Item		**STEP 1** **Missing or incorrect answer and little evidence of reasoning or application of mathematics to solve the problem.**	**STEP 2** **Missing or incorrect answer but evidence of some reasoning or application of mathematics to solve the problem.**	**STEP 3** **A correct answer with some evidence of reasoning or application of mathematics to solve the problem, <u>or</u> an incorrect answer with substantial evidence of solid reasoning or application of mathematics to solve the problem.**	**STEP 4** **A correct answer supported by substantial evidence of solid reasoning or application of mathematics to solve the problem.**
1	**a** N-VM.C.11 N-VM.C.12 G-CO.A.2 G-CO.A.5	Student's solution doesn't apply matrix multiplication or transformations to determine the coordinates of the resulting image. Sketch is missing.	Student computes twoor more coordinates of the image incorrectly and the sketch of the image is incomplete or poorly labeled, <u>OR</u> the image is a parallelogram with no work shown and no vertices labeled.	Student computes coordinates of the image correctly, but the sketch of the image may be slightly inaccurate. Work to support the calculation of the image coordinates is limited, <u>OR</u> student computes three out of four coordinates correctly and sketch accurately reflects the student's coordinates.	Student applies matrix multiplication to each coordinate of the unit square to get the image coordinates and draws a fairly accurate sketch of a parallelogram with vertices correctly labeled. Values for k will vary, but the resulting image should look like a parallelogram and the distance k in the vertical and horizontal direction should appear equal.
	b N-VM.C.12 G-CO.A.2	Student does not compute the area of a parallelogram or his sketched figure correctly.	Student computes the area of his sketched figure correctly but does not use determinant of the 2 x 2 matrix in their calculation.	Student computes the area of his figure using the determinant of the 2 x 2 matrix, but the solution may contain minor errors.	Student computes area of the parallelogram correctly using a determinant. Work shows understanding that the area of the image is the product of the area of the original figure and the absolute value of the determinant of the transformation matrix.
	c N-VM.C.10 N-VM.C.11	Student does not provide a solution ,<u>OR</u> work is unrelated to the standards addressed in this	Student computes an incorrect solution or set up of the original matrix equation. Limited evidence is evident that	Student creates a correct matrix equation to solve for the point and translates the equation to a system of linear	Student creates a correct matrix equation to solve for the point. Student translates the equation to a system of linear

	G-CO.A.2	problem.	the student understands that the solution to the matrix equation will find the point in question OR correct matrix equation and no additional work is given OR correct system of linear equations and no additional work Is given.	equations. Work shown may be incomplete and final answer may contain minor errors, OR student has the correct solution, but the matrix equation or the system of equations is missing from their solution. Very little work shown to provide evidence of student thinking.	equations AND solves the system correctly. Work shown is organized in a manner that is easy to follow and uses proper mathematical notation.
	d N-VM.C.11	Student solution does not correctly apply the transformation one time. Student does not attempt a generalization for the 10-fold image.	Student solution does not correctly apply the transformation more than one time. Student may attempt to generalize to the 10-fold image, but their answer contains major conceptual errors.	Student solution provides evidence that the student understood the problem and observed patterns, but minor errors prevent a correct solution for the 10-fold image OR student solution shows correct repeated application of the transformation at least three times, but the student is unable to extend the pattern to the 10-fold image.	Student gives correct solution for the 10-fold image. Student solution provides enough evidence AND explanation to clearly illustrate how they observed and extended the pattern.
2	**a** N-VM.C.11 N-VM.C.12 N-CN.B.4 N-CN.B.5 G-CO.A.2 G-CO.A.4	Student does not recognize the transformation as a rotation of the point about the origin.	Student identifies the transformation as a rotation but cannot correctly state the direction or the angle measure.	Student correctly identifies the transformation as a rotation about the origin, but the answer contains an error, such as the wrong direction or the wrong angle measurement.	Student correctly identifies the transformation as a counterclockwise rotation about the origin through an angle of 1 radian.
	b N-VM.C.11 N-VM.C.12 N-CN.B.4 N-CN.B.5 G-CO.A.2 G-CO.A.4	Student does not identify the repeated transformation as a rotation.	Student identifies the transformation as a rotation, but the solution does not make it clear that the 2nd rotation applies to the image of the original point OR identifies the transformation as an additional rotation, but the answer contains two or more errors.	Student correctly identifies the repeated transformation as an additional rotation, but the answer contains no more than one error.	Student correctly identifies the repeated transformation as a rotation of the image of the point another 1 radian clockwise about the origin for a total of 2 radians.

	c **N-CN.B.4** **N-CN.B.5** **G-CO.A.2**	Student makes little or no attempt at multiplying the point (*x*,*y*) by either of the rotation matrices.	Student sets up and attempts the necessary matrix multiplications, but solution has too many major errors, <u>OR</u> there is too little work for the student to make significant progress on the proof.	Student's solution includes multiplication of (*x*,*y*) by the original rotation matrix twice <u>AND</u> multiplication of (*x*,*y*) by the 2-radian rotation matrix. Student fails to equate the two answers to finish the proof. Solution may contain minor computation errors.	Student's solution details multiplication by the original rotation matrix twice, compares that result to multiplication by the 2-radian rotation matrix, <u>AND</u> equates the two answers to verify the identities. Student uses correct notation <u>AND</u> the solution illustrates her thinking clearly. The solution is free from minor errors.
3	**a** **N-CN.A.3** **N-CN.B.4** **N-CN.B.5**	Student makes little or no attempt to find the modulus and argument.	Student attempts to find the modulus and argument, but solution has major errors that lead to an incorrect answer.	Student has the correct answer but may not be in proper form <u>OR</u> makes minor computational errors in finding the modulus and argument.	Student writes the correct answer in the proper form <u>AND</u> correctly solves for the modulus and argument of the expression, showing all steps.
	b **N-CN.A.3** **N-CN.B.4** **N-CN.B.5**	Student makes little or no attempt to solve for a complex number.	Student attempts to find a complex number, but lacks the proper steps in order to do so, resulting in an incorrect answer.	Student may find a correct answer,but doesn't show any steps taken to solve the problem <u>OR</u> has an answer that does not have a and b as positive real numbers.	Student correctly finds a complex number in the form a+ib, where a and b are positive real numbers, that satisfies the given equation and shows all steps such as finding the modulus and argument of i.
	c **N-CN.A.3** **N-CN.B.4** **N-CN.B.5**	Student does not give any explanation as to whether a complex number, w, exists for the given equation and conditions, and answers incorrectly.	Student answers incorrectly but gives an explanation that has somewhat valid points but is lacking proper information.	Student answers correctly but doesn't give an accurate written and algebraic explanation such as stating the modulus and argument of z and w for both zero and non-zero cases.	Student answers correctly <u>AND</u> provides correct reasoning as to why w is sure to exist, including stating the modulus and argument of z and w if they are non-zero.
	d **N-CN.A.3** **N-CN.B.4** **N-CN.B.5**	Student does not give any explanation as to whether a complex number, w, exists for the given equation and conditions, and answers incorrectly.	Student answers incorrectly but gives an explanation that has somewhat valid points but is lacking proper information.	Student answers correctly but lacks proper reasoning to support the answer.	Student answers correctly <u>AND</u> provides correct reasoning as to why w is sure to exist, including an algebraic solution.

4	a N-VM.C.10 N-VM.C.11 N-VM.C.12 G-CO.A.2	Student makes little to no attempt to solve for the location of the rocket at either time given.	Student sets up a matrix equation, but does not use the correct matrices in order to solve the problem.	Student correctly sets up the matrix equation but due to errors in calculations, fails to reach a correct final answer for the location of the rocket at both times.	Student correctly solves for the location of the rocket at both times given, using the correct matrix equation.
	b N-VM.C.10 N-VM.C.11 N-VM.C.12 G-CO.A.2	Student makes little to no attempt to find the location of the rocket at the given time for either set of instructions and gives no explanation.	Student sets up matrix equations to solve for the location of the rocket but fails to properly solve the equations and produce an accurate explanation.	Student correctly finds the location of the rocket for one set of instructions but fails to verify the location of the rocket for the other set of instructions is consistent with the first.	Student correctly gives the location of the rocket for the given time for both sets of instructions <u>AND</u> correctly makes the correlations between the two.
	c N-VM.C.10 N-VM.C.11 N-VM.C.12 G-CO.A.2	Student makes little to no attempt to solve for the area.	Student attempts to find the area of the region enclosed by the path of the rocket but does not make the correct conclusion that it travels in a semicircle.	Student correctly finds that the path traversed is a semicircle but has minor errors in calculations that prevent the correct area from being found.	Student correctly finds the area of the enclosed path of the rocket including finding the radius of the traversed path.
	d N-VM.C.10 N-VM.C.11 N-VM.C.12 G-CO.A.2	Student makes little to no attempt to adjust the matrix five places further right.	Student sets up matrix/matrices for one or both sets of instructions but incorrectly translates the points 5 units to the right.	Student orrectly sets up the shifted matrix for one set of instructions but fails to correctly set up the shifted matrices for both sets of instructions.	Student correctly sets up the matrices for both sets of instructions that results in a shift of the rocket five places to the right.
5	a N-VM.C.10	Student makes little to no attempt to find matrix.	Student sets up a matrix equation, but does not use the correct matrices in order to solve the problem.	Student correctly sets up the matrix equation but due to errors in calculations, fails to find the correct matrix.	Student correctly sets up and solves the matrix equation leading to the correct matrix.
	b N-VM.C.10	Student makes little to no attempt to find matrix.	Student sets up a matrix equation, but does not use the correct matrices in order to solve the problem.	Student correctly sets up the matrix equation but due to errors in calculations, fails to find the correct matrix.	Student correctly sets up and solves the matrix equation leading to the correct matrix.

	c **N-VM.C.10**	Student makes little to no attempt to find matrix.	Student sets up a matrix equation, but does not use the identity matrix in order to solve the problem.	Student identifies the identity matrix as the answer, but writes the matrix incorrectly.	Student identifies the identity matrix as the answer and writes it correctly.
	d **N-VM.C.10**	Student makes little to no attempt to find matrix.	Student sets up a matrix equation, but does not use the correct matrices in order to solve the problem.	Student correctly sets up the matrix equation but due to errors in calculations, fails to find the correct matrix.	Student correctly sets up and solves the matrix equation leading to the correct matrix.
	a **N-VM.C.10**	Student makes little to no attempt to find matrix.	Student sets up a matrix equation, but does not use the correct matrices in order to answer the question.	Student correctly sets up one or both matrix equations but due to errors in calculations, fails to arrive at the correct answer.	Student correctly sets up and solves both matrix equations leading to the correct answer.

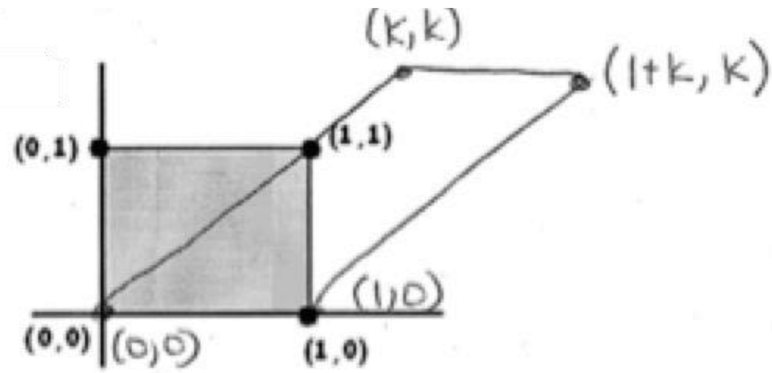
Name ____ Date ____________________

1. Consider the transformation on the plane given by the 2×2 matrix $\begin{pmatrix} 1 & k \\ 0 & k \end{pmatrix}$ for a fixed positive number k.

   a. Draw a sketch of the image of the unit square under this transformation (the unit square has vertices $(0,0)$, $(1,0)$, $(0,1)$, and $(1,1)$). Be sure to label all four vertices of the image figure.

   To find the coordinates of the image, multiply the vertices of the unit square by the matrix.

   $$\begin{pmatrix} 1 & k \\ 0 & k \end{pmatrix}\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
   $$\begin{pmatrix} 1 & k \\ 0 & k \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} k \\ k \end{pmatrix}$$
   $$\begin{pmatrix} 1 & k \\ 0 & k \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1+k \\ k \end{pmatrix}$$
   $$\begin{pmatrix} 1 & k \\ 0 & k \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$



The Unit Square

   The image is a parallelogram with base = 1 and height = k.

   b. What is the area of the image parallelogram?

   To find the area of the image figure, multiply the area of the unit square by the absolute value of $\begin{vmatrix} 1 & k \\ 0 & k \end{vmatrix}$.

   $$\begin{vmatrix} 1 & k \\ 0 & k \end{vmatrix} = (1 \times k) - (0 \times k) = k$$

   $$Area = 1x|k| = k \text{ since } k > 0$$

c. Find the coordinates of a point $\begin{pmatrix} x \\ y \end{pmatrix}$ whose image under the transformation is $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$.

Solve the equation to find the coordinates of $\begin{pmatrix} x \\ y \end{pmatrix}$.

$$\begin{pmatrix} 1 & k \\ 0 & k \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

Converting the matrix equation to a system of linear equations gives us

$$x + ky = 2$$
$$ky = 3$$

Solve this system.

$$y = \frac{3}{k}$$
$$x + k\left(\frac{3}{k}\right) = 2$$
$$x + 3 = 2$$
$$x = -1$$

The point is $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ \frac{3}{k} \end{pmatrix}$

d. The transformation $\begin{pmatrix} 1 & k \\ 0 & k \end{pmatrix}$ is applied once to the point $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, then once to the image point, then once to the image of the image point, and then once to the image of the image of the image point, and so on. What are the coordinates of ten-fold image of the point $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, that is, the image of the point after the transformation has been applied ten times?

Multiply to apply the transformation once: $\begin{pmatrix} 1 & k \\ 0 & k \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1+k \\ k \end{pmatrix}$

Multiply again by the 2×2 matrix: $\begin{pmatrix} 1 & k \\ 0 & k \end{pmatrix}\begin{pmatrix} 1+k \\ k \end{pmatrix} = \begin{pmatrix} 1+k+k^2 \\ k^2 \end{pmatrix}$

Multiply again by the 2×2 matrix: $\begin{pmatrix} 1 & k \\ 0 & k \end{pmatrix}\begin{pmatrix} 1+k+k^2 \\ k^2 \end{pmatrix} = \begin{pmatrix} 1+k+k^2+k^3 \\ k^3 \end{pmatrix}$

By observing the patterns, we can see that the result of n multiplications is a 2×1 matrix whose top row is the previous row plus k^n and whose bottom row is k^n.

The 10-fold image would be $\begin{pmatrix} 1+k+k^2+k^3+\cdots k^{10} \\ k^{10} \end{pmatrix}$

2. Consider the transformation given by $\begin{pmatrix}\cos(1) & -\sin(1)\\ \sin(1) & \cos(1)\end{pmatrix}$.

a. Describe the geometric effect of applying this transformation to a point $\begin{pmatrix}x\\ y\end{pmatrix}$ in the plane.

This transformation will rotate the point $\binom{x}{y}$ counterclockwise about the origin through an angle of 1 radian.

b. Describe the geometric effect of applying this transformation to a point $\begin{pmatrix}x\\ y\end{pmatrix}$ in the plane twice: once to the point, and then once to its image.

This transformation will rotate the point $\binom{x}{y}$ counterclockwise about the origin an additional 1 radian for a total rotation of 2 radians.

c. Use part (b) to prove $\cos(2)=\cos^2(1)-\sin^2(1)$ and $\sin(2)=2\sin(1)\cos(1)$.

To prove this, multiply $\binom{x}{y}$ by the transformation matrix.

$$\begin{pmatrix}\cos(1) & -\sin(1)\\ \sin(1) & \cos(1)\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}x\cos(1)-y\sin(1)\\ x\sin(1)+y\cos(1)\end{pmatrix}$$

Then multiply this answer by the transformation matrix

$$\begin{pmatrix}\cos(1) & -\sin(1)\\ \sin(1) & \cos(1)\end{pmatrix}\begin{pmatrix}x\cos(1)-y\sin(1)\\ x\sin(1)+y\cos(1)\end{pmatrix}$$

Apply matrix multiplication: $\begin{pmatrix}\cos(1)(x\cos(1)-y\sin(1))-\sin(1)(x\sin(1)+y\cos(1))\\ \sin(1)(x\cos(1)-y\sin(1))+\cos(1)(x\sin(1)+y\cos(1))\end{pmatrix}$

Distribute: $\begin{pmatrix}x(\cos(1))^2-y\cos(1)\sin(1)-x\sin(1)^2-y\sin(1)\cos(1)\\ x\sin(1)\cos(1)-y\sin(1)^2+x\cos(1)\sin(1)+y(\cos(1))^2\end{pmatrix}$

Rearrange and factor: $\begin{pmatrix}x((\cos(1))^2-\sin(1)^2)-y(2\sin(1)\cos(1))\\ x(2\sin(1)\cos(1))+y(\cos(1)^2-\sin(1)^2)\end{pmatrix}$

This matrix is equal to the matrix resulting from the 2 radian rotation.

$$\begin{pmatrix}\cos(2) & -\sin(2)\\ \sin(2) & \cos(2)\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}x\cos(2)-y\sin(2)\\ x\sin(2)+y\cos(2)\end{pmatrix}$$

When you equate the answers and compare the coefficients of x and y, you can see that $\cos(2)=\cos(1)^2-\sin(1)^2$ and $\sin(2)=2\sin(1)\cos(1)$

The matrices are equal because they represent the same transformation.

$$\begin{pmatrix}x((\cos(1))^2-\sin(1)^2)-y(2\sin(1)\cos(1))\\ x(2\sin(1)\cos(1))+y(\cos(1)^2-\sin(1)^2)\end{pmatrix}=\begin{pmatrix}x\cos(2)-y\sin(2)\\ x\sin(2)+y\cos(2)\end{pmatrix}$$

3.

a. Write $(1+i)^{10}$ as a complex number of the form $a+ib$ for real numbers a and b.

$1+i$ has argument $\frac{\pi}{4}$ and modulus $\sqrt{2}$, and so $(1+i)^{10}$ has argument $10\times\frac{\pi}{4}=\frac{\pi}{2}+2\pi$ and modulus $\left(\sqrt{2}\right)^{10}=2^5=32$. Thus, $(1+i)^{10}=32i$.

b. Find a complex number $a+ib$, with a and b positive real numbers, such that $(a+ib)^3=i$.

i has argument $\frac{\pi}{2}$ and modulus 1. Thus a complex number $a+ib$ of argument $\frac{\pi}{6}$ and modulus 1 will satisfy $(a+ib)^3=i$. We have $a+ib=\frac{\sqrt{3}}{2}+i\frac{1}{2}$.

c. If z is a complex number, is there sure to exist, for any positive integer n, a complex number w such that $w^n = z$? Explain your answer.

Yes. If $z = 0$, then $w = 0$ works. If, on the other hand, z is not zero and has argument θ and modulus m, then let w, be the complex number with argument $\frac{\theta}{n}$ and modulus $m^{\frac{1}{n}}$:

$$w = m^{\frac{1}{n}}\left(\cos\left(\frac{\theta}{n}\right) + i\sin\left(\frac{\theta}{n}\right)\right).$$

d. If z is a complex number, is there sure to exist, for any negative integer n, a complex number w such that $w^n = z$? Explain your answer.

If $z = 0$, then there is no such complex number w,. if $z \neq 0$, then $\frac{1}{w}$, with w as given in part c satisfies $\left(\frac{1}{w}\right)^{-n} = z$, showing that the answer to the question is yes in this case.

4. In programming a computer video game, Mavis coded the changing location of a space-rocket as follows:

At a time t seconds between $t = 0$ seconds and $t = 2$ seconds, the location $\begin{pmatrix} x \\ y \end{pmatrix}$ of the rocket is given by:

$$\begin{pmatrix} \cos\left(\frac{\pi}{2}t\right) & -\sin\left(\frac{\pi}{2}t\right) \\ \sin\left(\frac{\pi}{2}t\right) & \cos\left(\frac{\pi}{2}t\right) \end{pmatrix}\begin{pmatrix} -1 \\ -1 \end{pmatrix}.$$

At a time of t seconds between $t = 2$ seconds and $t = 4$ seconds, the location of the rocket is given by:

$$\begin{pmatrix} 3-t \\ 3-t \end{pmatrix}.$$

a. What is the location of the rocket at time $t = 0$? What is its location at time $t = 4$?

At time $t = 0$ the location of the rocket is

$$\begin{pmatrix} \cos(0) & -\sin(0) \\ \sin(0) & \cos(0) \end{pmatrix}\begin{pmatrix} -1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -1 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$$

At time $t = 4$ the location of the rocket is

$$\begin{pmatrix} 3-4 \\ 3-4 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$$

the same as start.

b. Petrich is worried that Mavis may have made a mistake and the location of the rocket is unclear at time $t = 2$ seconds. Explain why there is no inconsistency in the location of the rocket at this time.

According to the first set of instructions, the location of the rocket at time $t = 2$ is

$$\begin{pmatrix} \cos(\pi) & -\sin(\pi) \\ \sin(\pi) & \cos(\pi) \end{pmatrix}\begin{pmatrix} -1 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} -1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

According to the second set of instructions, its location at this time is

$$\begin{pmatrix} 3-2 \\ 3-2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

These are consistent.

c. What is the area of the region enclosed by the path of the rocket from time $t = 0$ to time $t = 4$?

The path traversed is a semicircle with a radius of $\sqrt{2}$. The area enclosed is $\frac{1}{2}x2\pi = \pi$ squared units.

d. Mavis later decided that the moving rocket should be shifted five places further to the right. How should she adjust her formulations above to accomplish this translation?

Notice that:

$$\begin{pmatrix} \cos\left(\frac{\pi}{2}t\right) & -\sin\left(\frac{\pi}{2}t\right) \\ \sin\left(\frac{\pi}{2}t\right) & \cos\left(\frac{\pi}{2}t\right) \end{pmatrix} \begin{pmatrix} -1 \\ -1 \end{pmatrix} = \begin{pmatrix} -\cos\left(\frac{\pi}{2}t\right) + \sin\left(\frac{\pi}{2}t\right) \\ -\sin\left(\frac{\pi}{2}t\right) - \cos\left(\frac{\pi}{2}t\right) \end{pmatrix}$$

To translate these points 5 units to the right, use

$\begin{pmatrix} -\cos\left(\frac{\pi}{2}t\right) + \sin\left(\frac{\pi}{2}t\right) + 5 \\ -\sin\left(\frac{\pi}{2}t\right) - \cos\left(\frac{\pi}{2}t\right) \end{pmatrix}$ *for* $0 \le t \le 2$

Also use

$\begin{pmatrix} 3 - t + 5 \\ 3 - t \end{pmatrix} = \begin{pmatrix} 8 - t \\ 3 - t \end{pmatrix}$ *for* $2 \le t \le 4$

5. Let $P = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ and $O = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

   a. Give an example of a 2×2 matrix A, not with all entries equal to zero, such that $PA = O$.

      Notice that for any matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, *we have* $PA = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ a & b \end{pmatrix}$.

      If we choose $A = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$, *for example, then* $PA = O$.

   b. Give an example of a 2×2 matrix B with $PB \neq O$.

      Following the discussion in part a), we see that choosing $A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ *gives* $PA = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$, *which is different from* O.

   c. Give an example of a 2×2 matrix C such that $CR = R$ for all 2×2 matrices R.

      Choose $C = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. *The identity matrix has this property.*

d. If a 2×2 matrix D has the property that $D+R=R$ for all 2×2 matrices R, must D be the zero matrix O? Explain.

Write $D=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *and* $R=\begin{pmatrix} x & y \\ z & w \end{pmatrix}$. *Then for* $D+R=\begin{pmatrix} a+x & b+y \\ c+z & d+w \end{pmatrix}$ *to equal* $\begin{pmatrix} x & y \\ z & w \end{pmatrix}$ *no matter the values of* x, y, z, *and* w, *we need:*

$$a+x=x$$
$$b+y=y$$
$$c+z=z$$
$$d+w=w$$

to hold for all values x, y, z, and w. *Thus we need* $a=0$, $b=0$, $c=0$, *and* $d=0$. *That is,* D *must indeed be the zero matrix.*

e. Let $E=\begin{pmatrix} 2 & 4 \\ 3 & 6 \end{pmatrix}$. Is there 2×2 matrix F so that $EF=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $FE=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$? If so, find one. If not, explain why no such matrix F can exist.

The determinant of E *is* $|2\cdot 6-3\cdot 4|=0$ *and so no inverse matrix like* F *can exist.*

[Alternatively: Write $F=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. *Then* $EF=\begin{pmatrix} 2a+4c & 2b+4d \\ 3a+6c & 3b+3d \end{pmatrix}$. *For this t equal* $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ *we need, at the very least:*

$$2a+4c=1$$
$$3a+6c=0$$

The first of these equations gives $a+2c=\frac{1}{2}$ *and the second* $a+2c=0$. *There is no solution to this system of equations and so there can be no matrix* F *with the desired property.]*

GRADE 12 MODULE 1
OUTLINE OF LESSONS
(To be fleshed out and added to, of course!)

TOPIC A: A QUESTION OF LINEARITY

N-CN.3 (+) Find the conjugate of a complex number; use conjugates to find moduli and quotients of complex numbers.

N-CN.4 (+) Represent complex numbers on the complex plane in rectangular and polar form (including real and imaginary numbers), and explain why the rectangular and polar forms of a given complex number represent the same number.

Comment: This topic bounces off of N-CN.1, N-CN.2, N-CN.7, and the extensions N-CN.8 and N-CN.9 done in G.11.M1.

LESSONS 1–2: Wishful Thinking—Does Linearity Hold?

Wouldn't it be lovely if functions were "nice" and just did what we expected them to do?

EXERCISE: A common student mistake is to write $(a+b)^2 = a^2 + b^2$.

a) Substitute in some values for a and b to show this statement is not true in general.
b) Find some values for a and b for which the statement, by coincidence, happens to work.
c) Find *all* values for a and b for which the statement is true.

EXERCISE: A common student mistake is to write $\sqrt{a+b} = \sqrt{a} + \sqrt{b}$. (Here assume that neither a nor b is negative.)

a) Substitute in some non-negative values for a and b to show this statement is not true in general.
b) Find some non-negative values for a and b for which the statement, by coincidence, happens to work.
c) Find *all* values for a and b for which the statement is true.

Comment: The video http://www.jamestanton.com/?p=677 could be of interest here.

HOMEWORK: Examine $(a+b)^3$ and $(a+b)^{\frac{1}{3}}$.

EXERCISE: Trigonometry would be so much easier if a statement like the following were true:

$$\sin(x+y) = \sin x + \sin y$$

a) Show that this statement is not true in general.
b) Are there any values for x and y for which such a statement does hold?
c) Repeat these two questions for the statement: $\sin(2x) = 2\sin(x)$.

HOMEWORK: Examine $\cos(x+y)$, $\cos(2x)$, $\tan(x+y)$, and $\tan(2x)$.

EXERCISE:

a) Find a value for a such that $\log(2a) = 2\log a$. Is there one?
b) Find values for a and b so that $10^{a+b} = 10^a + 10^b$. Are there any?

[Offer the hint, if needed, of looking for values with $a = b$ in the second case, if needed.]

[Answer: $a = 2$ works in the first, and $a = b = \log 2$ in the second.]

HOMEWORK: Examine $\log(3a) = 3\log a$ and $\log(ka) = k\log a$.

EXERCISE: Are there any real numbers a and b so that $\dfrac{1}{a+b} = \dfrac{1}{a} + \dfrac{1}{b}$?

[Answer: No. If the relation holds, then neither a nor b can be zero. Multiplying through we get: $ab = a(a+b) + b(a+b)$ giving $ab = (a+b)^2$. Since the right side is positive, so is ab.

Expanding gives: $ab = a^2 + 2ab + b^2$, and so $a^2 + b^2 + ab = 0$. A sum of three positive numbers can't be zero!]

(HOMEWORK?) EXERCISE: Let $f(x) = ax^2 + bx + c$. Find some values for a, b, and c so that $f(x+y) = f(x) + f(y)$. Describe the set of *all* values for a, b, and c that make $f(x+y) = f(x) + f(y)$ valid for all real numbers x and y.

The point is that it is very unusual for a function L to satisfy the dream conditions:

1) $L(x+y) = L(x) + L(y)$
2) $L(kx) = kL(x)$

for all real values x, y, and k.

LESSON 3: Which Real-Number Functions Are Linear?

So … Are there any real-number functions L that satisfy the dream conditions:

1) $L(x+y)=L(x)+L(y)$
2) $L(kx)=kL(x)$

for all real values x, y, and k?

Suppose we have one. What do the two conditions above imply about the function?

Since these conditions are meant to hold for *all* real values x, y, and k, we can choose any particular values of x, y, and k we like and see what we can learn! For example, in 2) let's choose $x=1$ and $k=0$. Then 2) says:

$$L(0\times 1)=0\times L(1)$$

We don't know the value of $L(1)$, but $0\times L(1)$ is for sure zero. (Also, $0\times 1=0$.)
So condition 2) shows us that

$$L(0)=0.$$

Question: What does this say about the graph of $y=L(x)$?

EXERCISE: We can also prove that $L(0)=0$ using condition 1) instead. How?

[Answer: Choose $x=0$ and $y=0$.]

What more can we deduce? Well,

$$L(2x)=2L(x)$$
$$L(3x)=3L(x)$$

and so on

Enlightening? Doesn't seem to be.

Choose $k = -1$ (!) and we learn:

$$L(-x) = -L(x)$$

Question: What does this say about the graph of $y = L(x)$? (Recall the meaning of an "odd function" and the anti-symmetry it has.)

Anything more?

SOMETHING SNEAKY … (An epiphany!)

From $L(kx) = kL(x)$ we must have that $L(x) = L(x \cdot 1) = xL(1)$.

Again, we don't know what the value of $L(1)$ is. But let's call this value m. (So $m = L(1)$.) Then we have just shown:

$$L(x) = mx.$$

(And one should check that a function of this form really does satisfy $L(x+y) = L(x) + L(y)$ and $L(kx) = kL(x)$.)

Question: What is the graph of $y = L(x)$? Is it indeed an odd function through the origin?

[Answer: We have $y = mx$, which is a straight-line graph through the origin. It is indeed an odd function.]

We have just established:

If a real-number function L satisfies the conditions

1) $L(x+y) = L(x) + L(y)$
2) $L(kx) = kL(x)$

for all real values x, y, and k, then that function is of the form $L(x) = mx$, and its graph is a straight line through the origin.

EXERCISE: Explain why a general linear function $L(x) = mx + b$ with $b \neq 0$ fails to satisfy the two conditions. (Do both conditions break down or just one of them?)

Confusing Jargon: Mathematicians call any function that satisfies 1) and 2) a *linear transformation*. As we have just seen, for real-number functions, the graph of a linear transformation must be a straight line through the origin.

Functions of the form $f(x) = mx + b$ are called *linear functions* because their graphs are straight lines in the coordinate plane (not necessarily through the origin). As we have seen, a linear function need not be a linear transformation!

Moral of Lessons 1, 2, and 3:

Only a very special class of functions have the nice properties $L(x+y) = L(x) + L(y)$ *and* $L(kx) = kL(x)$. *The algebra dream of students is usually never true! Sad!*

EXERCISE: Study integer-valued functions L defined only on the integers.

Suppose an integer-valued function satisfies:

1) $L(a+b) = L(a) + L(b)$
2) $L(ka) = kL(a)$

for all integers a, b, and k.

Must there be an integer m so that $L(a) = ma$ for all integers a?

[Yes: Follow the reasoning in the lesson, or follow the reasoning in the next exercise.]

EXERCISE: Suppose an integer-valued function L satisfies just:

$$L(a+b) = L(a) + L(b)$$

for all integers a and b.

Must there be an integer m so that $L(a) = ma$ for all integers a?

[Yes: Let $m = L(1)$.]

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Now

$$L(2) = L(1+1) = 2L(1) = 2m$$
$$L(3) = L(1) + L(2) = 3L(1) = 3m$$

etc.

We see in this way that $L(a) = ma$ for all positive integers a.

$L(0) = L(0+0) = L(0) + L(0)$ gives $L(0) = 0$, and so. $L(0) = m \cdot 0$

$0 = L(0) = L(a + (-a)) = L(a) + L(-a)$ shows $L(-a) = -L(a) = -ma = m(-a)$.

So $L(a) = ma$ for a a positive integer, a negative integer, and for a equal to zero.]

EXERCISE: Give an example of a real-number function L that satisfies $L(xy) = L(x) + L(y)$ for all real numbers x and y with $L(x) \neq 0$ for at least one real number x.

[Answer: Any logarithm function will do.]

EXERCISE: Give an example of a real-number function L that satisfies $L(x+y) = L(x) \cdot L(y)$ for all real numbers x and with $L(x) \neq 0$ for at least one real number x.

[Answer: Any exponential function will do.]

EXERCISE: Give an example of a real-number function L that satisfies $L(xy) = L(x) \cdot L(y)$ for all real numbers x and y with $L(x) \neq 0$ for at least one real number x.

[Answer: $L(x) = x^{33}$, for example, works.]

LESSONS 4–5: An Appearance of Complex Numbers

A few lessons ago we asked about the dream relation:

$$\frac{1}{a+b}=\frac{1}{a}+\frac{1}{b}$$

We proved that there are no real numbers that make this a true statement.

In doing so, we multiplied through by a and by b and by $(a+b)$ to get:

$$a^2+ab+b^2=0$$

We showed that there are no real-number solutions to this equation.

EXERCISE: But show that there are complex-number solutions to $a^2+ab+b^2=0$!

[Answer: Viewing this as a quadratic in a we get:

$$a^2+ab+\frac{1}{4}b^2=-\frac{3}{4}b^2$$

$$\left(a+\frac{1}{2}b\right)^2=-\frac{3}{4}b^2$$

$$a+\frac{1}{2}b=\pm i\frac{\sqrt{3}}{2}b$$

$$a=\frac{-1\pm i\sqrt{3}}{2}b$$

Choose $b=2$ and $a=-1+i\sqrt{3}$, for example, and verify that these do indeed satisfy $a^2+ab+b^2=0$. In fact, there are infinitely many pairs of complex numbers satisfying $a^2+ab+b^2=0$.]

So is it true that $\dfrac{1}{\left(-1+\sqrt{3}i\right)+2}=\dfrac{1}{-1+\sqrt{3}i}+\dfrac{1}{2}$?

We need to review our understanding of complex numbers from G.11.M1.

<Go through the relevant material:

Definition of i, as motivated geometrically by rotating the number line 90° counterclockwise and thus giving a "number" i with the property $i^2 = -1$. (See http://www.jamestanton.com/?p=625 and then http://www.jamestanton.com/?p=564.)
Definition of a general complex number $a+ib$.
Representing complex numbers as points in the complex plane.
Basic arithmetic: addition, scalar multiplication, and multiplication.
The appearance of complex solutions to some quadratics.

Do some practice exercises.>

We see that we know how to add, subtract, and multiply complex numbers.

If $a = -1+\sqrt{3}i$ and $b = 2$, then $a+b = 1+\sqrt{3}i$.

Is it possible to divide them?

Do $\dfrac{1}{a+b} = \dfrac{1}{1+\sqrt{3}i}$ and $\dfrac{1}{a} = \dfrac{1}{-1+\sqrt{3}i}$ make sense?
Does $\dfrac{1}{-1+i\sqrt{3}} + \dfrac{1}{2}$ equal $\dfrac{1}{1+i\sqrt{3}}$?

This is a mystery to be picked up next lesson.

LESSONS 6–7: Complex-Number Division

UNDERSTANDING RECIPROCALS:

In real-number arithmetic, $\frac{1}{5}$ is multiplicative inverse of 5 ; that is, $\frac{1}{5}$ is the number which, when multiplied by 5, gives 1 :

$$\frac{1}{5}\times 5=1$$

EXERCISE: Is there a multiplicative inverse of i ? Is there a (complex) number z such that $z\times i=1$?

[Answer: A little mulling gives that $z=-i$ does the trick.]

We have $\frac{1}{i}=-i$.

EXERCISE: What is the multiplicative inverse of $2i$?

EXERCISE: Does $3+4i$ have a multiplicative inverse? Is there a complex number $p+iq$ such that $(3+4i)(p+iq)=1$?

[Answer: Have students start with $(3+4i)(p+iq)=1$ and expand to see possible candidates for a and b . Get:

$$3p-4q=1$$
$$4p+3q=0$$

Solving gives $p=\frac{3}{5}$ and $q=-\frac{4}{5}$. This suggests: $\frac{1}{3+4i}=\frac{3-4i}{5}$. Now check that it works!]

EXERCISE: Quickly—what must be the multiplicative inverse of $3-4i$?

QUESTION: Make a guess—what does the number 5 that appeared in the multiplicative inverse of $3+4i$ have to do with the numbers 3 and 4 ?

EXERCISE: IN THE ABSRACT Suppose $z = a + ib$. Find a general formula for the multiplicative inverse of z. (Hint: Solve the system $(a+ib)(p+iq) = 1$ for p and q and check that your candidate solution actually works.)

[Answer: Get $\frac{1}{a+ib} = \frac{a-ib}{\sqrt{a^2+b^2}}$.]

EXERCISE: Show that $a = -1+\sqrt{3}i$ and $b = 2$ really do satisfy $\frac{1}{a+b} = \frac{1}{a} + \frac{1}{b}$.

Let's pause and look at the features of the formula for the multiplicative inverse of a complex number:

$$\frac{1}{a+ib} = \frac{a-ib}{\sqrt{a^2+b^2}}$$

Question: What does this formula give for the reciprocal of i ? For the reciprocal of 2 ?

Features of this formula often reappear in complex-number arithmetic, and so mathematicians have given these features names.

JARGON: The *conjugate* of a complex number $a+ib$ is $a-ib$.

For example: The conjugate of $3+4i$ is $3-4i$.
The conjugate of $-2-i$ is $-2+i$.
The conjugate of 7 is 7.
The conjugate of i is $-i$.

If $z = a+ib$, then the conjugate of z is denoted $\bar{z}$. We have $\bar{z} = a-ib$.

JARGON: The *modulus* of a complex number $a+ib$ is the real number $\sqrt{a^2+b^2}$.

For example: The modulus of $3+4i$ is 5.
The modulus of $3-4i$ is 5.
The modulus of 7 is 7.
The modulus of i is 1.

If $z = a+ib$, then the modulus of z is denoted $|z|$. We have $|z| = \sqrt{a^2+b^2}$.

SOME PROPERTIES TO EXPLORE:

CONJUGATES:

a) What is the geometric effect of taking the conjugate of a complex number? If z is a point in the complex plane, where does $\bar{z}$ sit in relation to z ?
b) What can you say about the conjugate of the conjugate of a complex number?
c) Is the following relation always true: $\overline{z+w}=\bar{z}+\bar{w}$?
d) Is the following relation always true: $\overline{z\cdot w}=\bar{z}\cdot\bar{w}$?

MODULUS:

a) If $z=a+ib$ is a point in the complex plane, what is the geometric interpretation of $|z|$?
b) The notation for the modulus of a complex number matches the notation for the absolute value of a real number. Surely this is not coincidental. If a complex number happens to be real, what can you say about its modulus?
c) Show that $|iz|=|z|$ for all complex numbers.
d) (OPTIONAL): More generally, prove that $|zw|=|z|\cdot|w|$ for all complex numbers z and w .

CONJUGATES AND MODULUS:

a) Show that for all complex numbers z we have: $z\cdot\bar{z}=|z|^2$
b) What does the difference of two squares formula say about a^2+b^2 ?
<Answer: $a^2+b^2=a^2-(ib)^2=(a-ib)(a+ib)$.>

HOMEWORK:

a) By substituting it in, show that $z=\dfrac{i\sqrt{3}-1}{2}$ satisfies the equation $z^2+z+1=0$.
b) Without doing any work, explain why the complex conjugate $\bar{z}=\dfrac{-i\sqrt{3}-1}{2}$ must also be a solution to this equation.

HOMEWORK:
Let $z=a+ib$ and $w=c+id$. Show that:

a) $|\bar{z}|=|z|$
b) $\left|\dfrac{1}{z}\right|=\dfrac{1}{|z|}$
c) If $|z|=0$, must it be that $z=0$?
d) Give a specific example to show that $|z+w|$ usually does not equal $|z|+|w|$.

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UPSHOT OF THIS LESSON:

EXERCISE: Explain the following.

Every non-zero complex number z *has a multiplicative inverse. It is given by* $\dfrac{1}{z} = \dfrac{\bar{z}}{|z|}$.

<Now do exercises about computing things like $\dfrac{2-6i}{2+5i}$, and so on. Lots of practice.>

Introduce multiplying by the conjugate in both numerator and denominator:

$$\frac{2-6i}{2+5i} = \frac{2-6i}{2+5i} \cdot \frac{2-5i}{2-5i} = \frac{-26-22i}{29}$$
$$\frac{1}{1-i} = \frac{1+i}{(1-i)(1+i)} = \frac{1+i}{2}$$

Complete this topic with lots of review practice, having students really see complex numbers as points in the plane. In particular, understand the geometric placement of $z+w$ and $z-w$ when given z and w.

Have students try to explain why $|z+w| \le |z| + |w|$ and $|z-w| \le |z| + |w|$ geometrically (and then prove it analytically too?).

HOMEWORK or IN LESSON: We started off this topic looking for real-number functions L for which the student dreams:

1) $L(x+y) = L(x) + L(y)$
2) $L(kx) = kL(x)$

are true. We found that they rarely are! Only for functions of the form $L(x) = mx$ for some fixed number m.
But we did see that sometimes these conditions can hold if we permit complex-number solutions. For example, $\dfrac{1}{a+b} = \dfrac{1}{a} + \dfrac{1}{b}$ does not have real-number solutions, but it does have complex-number solutions.
This makes me think that maybe there are other types of functions that satisfy the student dream relations if we consider complex-number functions. SO ....

Suppose L is a complex-number function satisfying:

1) $L(z+w) = L(z) + L(w)$
2) $L(kz) = kL(z)$

for all complex numbers z ¸ w , and k .

Is $L(z) = mz$ for a fixed complex number m the only type of complex-number function that satisfies these conditions?

TOPIC B: COMPLEX-NUMBER OPERATIONS AS TRANSFORMATIONS

N-CN.3 (+) Find the conjugate of a complex number; use conjugates to find moduli and quotients of complex numbers.

N-CN.4 (+) Represent complex numbers on the complex plane in rectangular and polar form (including real and imaginary numbers), and explain why the rectangular and polar forms of a given complex number represent the same number.

N-CN.5 (+) Represent addition, subtraction, multiplication, and conjugation of complex numbers geometrically on the complex plane; use properties of this representation for computation. *For example, $(–1 + \sqrt{3}\, i)^3 = 8$ because $(–1 + \sqrt{3}\, i)$ has modulus 2 and argument 120°.*

N-CN.6 (+) Calculate the distance between numbers in the complex plane as the modulus of the difference, and the midpoint of a segment as the average of the numbers at its endpoints.

G-CO.2 Represent transformations in the plane using, e.g., transparencies and geometry software; describe transformations as functions that take points in the plane as inputs and give other points as outputs. Compare transformations that preserve distance and angle to those that do not (e.g., translation versus horizontal stretch).

G-CO.4 Develop definitions of rotations, reflections, and translations in terms of angles, circles, perpendicular lines, parallel lines, and line segments.

G-CO.5 Given a geometric figure and a rotation, reflection, or translation, draw the transformed figure using, e.g., graph paper, tracing paper, or geometry software. Specify a sequence of transformations that will carry a given figure onto another.

LESSONS 8–9: The Geometric Effect of Some Complex Arithmetic

We saw last topic that the only complex-number functions L satisfying $L(z+w) = L(z) + L(w)$ and $L(kz) = kL(z)$ for all complex numbers z ¸ w , and k , are those of the form $L(z) = mz$ for some complex number m . (Even in the realm of complex numbers do the student dream formulas rarely hold true!)

But we also saw in the last topic that the complex numbers have associated with them geometric interpretations. After all, points in the complex plane seem very much akin to points in the coordinate plane. Every complex operation must have some geometric interpretation.

EXERCISE: *Taking the conjugate of a complex number corresponds to reflecting a complex number about the real axis. What operation on complex numbers induces a reflection about the imaginary axis?*

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The Geometry of Complex-Number Addition and Subtraction:

EXERCISE: <Develop an exercise to help students see that complex addition/subtraction has the geometric effect of performing a translation to points in the complex plane:

$T(z) = z + 2$ shifts all points two units to the right.

$T(z) = z + i$ shifts all points one unit upward.

$T(z) = z + w$ shifts all points $\sqrt{2}$ units in a southwest direction. What is the value of w?

The Geometry of Complex-Number Multiplication:

EXERCISE: Describe the effect of $L(z) = 2z$ on points in the complex plane.

<Develop an appropriate exercise to have students see this is a dilation from the origin with scale factor 2.>

EXERCISE: Describe the effect of $L(z) = iz$ on points in the complex plane.

<Lead students to see, first with concrete numbers, then abstractly, that $i(a+ib) = -b+ia$ is the point $a+ib$ rotated 90° counterclockwise about the origin. (Actually have students plot points on graph paper.) Mention that this is consistent with how we defined i in the first place: as some object that has the geometric effect of inducing a 90° counterclockwise rotation about the origin! >

HOMEWORK: What is the geometric effect of $L(z) = -z$?

Class Struggle: Can we describe the geometric effect of $L(z) = (3+4i)z$?

Mess around with this. Flail. Come to the conclusion that it is hard to think one's way through this. There has got to be an easier way to get to and understand all this!

Easier Class Struggle: Can we describe the geometric effect of $L(z) = (1+i)z$?

Do some experimentation with plotting actual points, $z = 1$, $z = i$, $z = 1+i$, $z = 4+6i$, and their images. Perhaps try to get to points where students can conjecture: dilation with factor $\sqrt{2}$ and rotation of 45°.

We can see that the geometric interpretation of complex multiplication is tricky to figure out. Let's put this on hold while we step back and collect our thoughts.

HOMEWORK:
Consider a point z in the complex plane, the point iz, and the origin O.

We showed that *iz* is a 90° rotation of z (about the origin). Let's check that our math is hanging together.

Look at the three distances $|z|$ and $|iz|$ and $|z-iz|$. What does the converse of Pythagoras's theorem tell you about the triangle formed by the points O, z, and iz?

LESSONS 10–11: Distance and Complex Numbers

BE SURE TO MIX DEGREE AND RADIAN MEASURE FOR ANGLES THROUGHOUT THIS MODULE. (I've mostly written degrees in these notes.)

Last lesson we got stuck on understanding the geometric effect of multiplication by a complex number. What is the geometric action of multiplication by a complex number w on all the points in the complex plane?

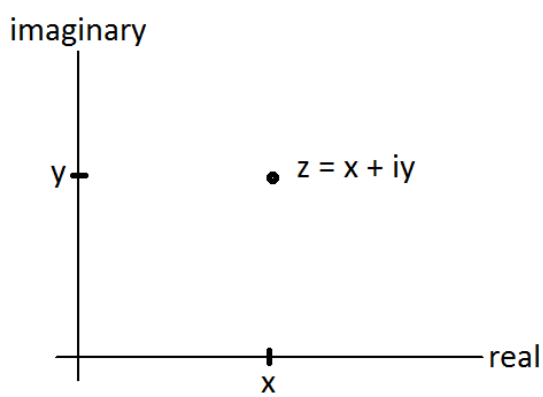
To understand this, let's be very clear what we mean by the geometry here. After all, the geometry we studied in Grade 10 was about points in the coordinate plane, not complex numbers in the complex plane!

BEING EXPLICIT ABOUT THE GEOMETRY OF COMPLEX NUMBERS:

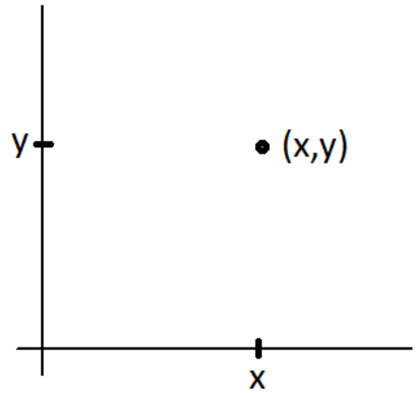
We say that "complex numbers are points in the complex plane."

What do we mean by this, really? In what way are complex numbers "points"?

Here's a complex number $z = x + iy$ drawn in the complex plane:



Oh! It looks like the point (x, y) drawn in the coordinate plane:



So it looks like we can interchangeably think of a complex number $x+iy$ in the complex plane as a point (x, y) in the coordinate plane, and vice versa.

For example: The complex number $2+3i$ matches the point $(2,3)$.
The point $(-1,8)$ matches the complex number $-1+8i$.

Question: What point matches the complex number i ? What complex number matches the point $(-90,0)$?

All the work we did in geometry class thus translates into the language of complex numbers and, vice versa, any work we do with complex numbers should translate back to results about geometry.

But here's the kicker:

IT MAKES NO SENSE TO ADD POINTS IN GEOMETRY.
(If A and B are points, what could $A+B$ possibly mean?)

IT MAKES PERFECT SENSE TO ADD COMPLEX NUMBERS.
If $A=x+iy$ and $B=a+ib$, then $A+B=(x+a)+i(y+b)$.

IF WE VIEW POINTS IN THE PLANE AS COMPLEX NUMBERS, THEN WE CAN ADD POINTS IN GEOMETRY!

This is powerful!

EXAMPLE: The Midpoint Formula

In ordinary geometry thinking: *If* $A=(a_1,a_2)$ *and* $B=(b_1,b_2)$ *, then their midpoint is*

$$M=\left(\frac{a_1+b_1}{2},\frac{a_2+b_2}{2}\right).$$

Now view these points as complex numbers:

$$A=a_1+ia_2$$
$$B=b_1+ib_2$$

and

$$M=\frac{a_1+b_1}{2}+i\frac{a_2+b_2}{2}.$$

We can rewrite M as $\dfrac{a_1+ia_2+b_1+ib_2}{2}=\dfrac{A+B}{2}$.

In complex-number thinking:

The midpoint of points A *and* B *is just* $\dfrac{A+B}{2}$*, the arithmetic average of the two numbers!*

EXAMPLE: The Distance Formula

In ordinary geometry thinking: *If* $A=(a_1,a_2)$ *and* $B=(b_1,b_2)$ *, then*

$$AB=\sqrt{(a_1-b_1)^2+(a_2-b_2)^2}.$$

Show that in complex-number thinking, the distance between $A=a_1+ia_2$ and $B=b_1+ib_2$ is $|A-B|$.

EXERCISE: Suppose $z=2+7i$ and $w=-3+i$. Find the midpoint m of z and w, and verify that $|z-m|=|w-m|$.

EXERCISE: Let $z=-100+100i$ and $w=1000-1000i$. Find a point one quarter of the way along the line segment connecting segment connecting z and w, closer to z than to w. Write this point in the form $\alpha z+\beta w$ for some real numbers α and β.

Describe the location of the point $\frac{2}{5}z+\frac{3}{5}w$ on this line segment.

EXERCISE:

a) Let $A=2+3i$ and $B=-4-8i$. Find a point C so that B is the midpoint of A and C.

b) Give two points A and B, and find a formula for a point C in terms of A and B so that B is the midpoint of A and C. Verify that your formula is correct with the specific examples from part a).

<Answer: $C=2B-A$ >

ACTIVITY: LEAPFROG PUZZLE

Draw three points in the plane A, B, and C. Start at any position  *and leapfrog over A to a new position P_1. (So A is the midpoint of $\overline{P_0P_1}$.) Then leapfrog over B, then C, then A again, then B again, then C again, then A again, and so on. What magic thing happens?*

<Actually do the activity. (Warning: It is easy to go off the page if your three points A, B, and C aren't close together.) You'll find that you end up back at start after six jumps.

To prove this:

By the previous exercise we have:

$$P_1=2A-P_0$$
$$P_2=2B-P_1=2B-2A+P_0$$
$$P_3=2C-P_2=2C-2B+2A-P_0$$
$$P_4=2A-P_3=-2C+2B+P_0$$
$$P_5=2B-P_4=2C-P_0$$
$$P_6=2C-P_5=P_0$$

FURTHER EXPLORATIONS:

Repeatedly jump over just one point A. (Easy to see what happens.)

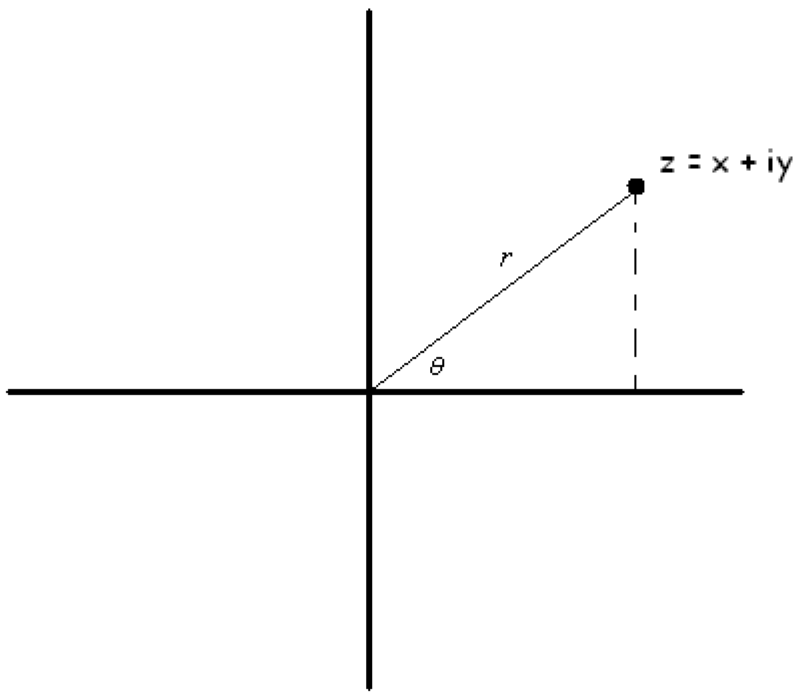
Repeatedly jump over two points A and B.

Four points? Five points?

Why stick in two dimensional? What if the three points A, B, and C are positioned in three-dimensional space?

LESSON 12: Trigonometry and Complex Numbers

Every complex number $z = x + iy$ appears as a point on the complex plane with coordinates (x, y) as a point in the coordinate plane. But one can't help but notice a right triangle in such a picture. We're now thinking trigonometry!



Each complex number has an angle of elevation θ from the positive real axis and a distance r from the origin.

JARGON: In the theory of complex numbers, the angle θ is called the *argument* of the complex number and the distance r the *modulus* of the complex number.

Comment: Is "modulus" indeed the right word? Does $r = |z|$?

Comment: The argument is defined only up to multiples of 360° (or if you are thinking in terms of radians, up to multiples of 2π).

We see from the picture:

$$x = r\cos\theta$$
$$y = r\sin\theta$$

which means that every complex number can be written in the form:

$$z = x + iy = r\cos\theta + ir\sin\theta = r\left(\cos\theta + i\sin\theta\right)$$

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JARGON: If a complex number z has argument θ and modulus r, then $z = r(\cos\theta + i\sin\theta)$ is called the *polar form* of the complex number.

EXERCISES: Lots of practice converting between polar form and rectangular forms of complex numbers:

From a picture it is clear that i has modulus 1 and argument 90°, so we should have $i = \cos(90^\circ) + i\sin(90^\circ)$. Is this right?

Practice using arctangent to find arguments.

Be sure to know that $a + ib = r(\cos\theta + i\sin\theta)$ with $r = \sqrt{a^2 + b^2}$ and $\theta = \arctan\left(\frac{b}{a}\right)$, for the appropriate value of arctan. (This takes some care to develop!) The angle for $\arctan\left(\frac{b}{a}\right)$ is the one in the same quadrant as $a + ib$.)

LESSONS 13–15: MODELING MATHEMATICS ITSELF—Discovering the Geometric Effect of Complex Multiplication

BACK TO $L(z) = wz$ **.**

We're stuck on understanding what multiplication by complex numbers means geometrically.

We've seen that $L(z) = 2z$ is a dilation from the origin with scale factor 2.

We've seen that $L(z) = iz$ is a 90° counterclockwise rotation about the origin.

EXERCISE: What is the geometric effect of $L(z) = 2iz$?

We got stuck on $L(z) = (3+4i)z$.

Our work on distance gives us the answer!

<The following is done in full generality. We need to lead students through this next math gently and appropriately.>

Suppose we are given a complex number z. We are going to multiply it by $w = c + id$.
For now, assume $c > 0$.

Now $wz = cz + idz$.

We know cz is on the same ray from the origin through z. (I've drawn a picture here assuming $c > 1$. But $0 < c \le 1$ is analogous: Have students draw the picture for this case while the teacher draws the picture for the case $c > 1$ on the board.)

Who knows where the point $cz + idz$ is? I've just drawn it somewhere randomly for now.

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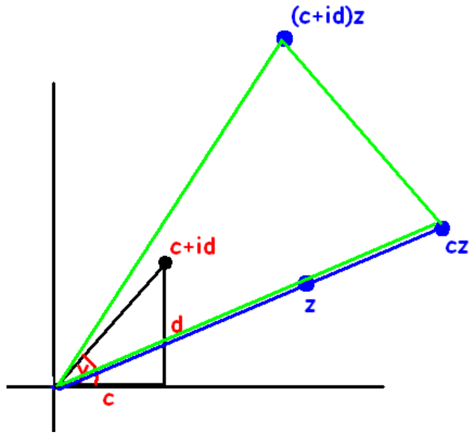

Let's assume the argument of $w = c + id$ is y (that is, the angle of elevation of this complex number from the real axis is y) .

The green triangle has sides of lengths:

$$|cz| = c|z|$$
$$|cz + idz - cz| = |idz| = d|z|$$
$$|(c+id)z| = |c+id| \cdot |z|$$

and is similar, by side-side-side (SSS), to the black triangle, and so also contains angle y .

We see $(c+id)z$ has angle to the horizontal axis, the argument of z changed by angle y ; that is, z has been rotated y degrees. Also, the point $(c+id)z$ is a different distance from the origin than z . In fact $|(c+id)z| = |c+id| \cdot |z| = \sqrt{c^2+d^2}\,|z|$ shows we have a dilation by factor $\sqrt{c^2+d^2}$. This is the modulus of w .

AN ISSUE:

Has z been rotated through an angle y in a counterclockwise direction as we happened to have drawn, or in a clockwise direction? (What would the picture look like in this case?) How can we tell?

Things are a bit clearer if we write our complex numbers in polar form.

Write $z = r\left(\cos\theta + i\sin\theta\right)$ and $w = c + id = m\left(\cos y + i\sin y\right)$.
Then

$$\begin{aligned} wz &= rm\left(\cos\theta\cos y - \sin\theta \sin y\right) + rmi\left(\sin\theta\cos y + \sin y\cos\theta\right) \\ &= rm\cos\left(\theta + y\right) + rmi\sin\left(\theta + y\right) \\ &= rm\left(\cos\left(\theta + y\right) + i\sin\left(\theta + y\right)\right) \end{aligned}$$

using the angle-sum formulas from trigonometry.
So wz has modulus rm and argument $\theta + y$; that is, since the argument of a complex number is measured in a counterclockwise direction from the real axis, the argument of z, which was θ, had increased to $\theta + y$. We have a counterclockwise rotation.

<u>UPSHOT</u>: MULTIPLICATION BY COMPLEX NUMBER w CORRESONDS TO A COUNTERCLOCKWISE ROTATION ABOUT THE ORIGIN AND DILATION FROM THE ORIGIN. The angle of rotation is the argument of w, and the scale factor of the dilation is the modulus of w.

Another way of saying this: *Every complex-number linear transformation L is a counterclockwise rotation and dilation about the origin.*

EXERCISE: We proved our result for $w = c + id$ with $c > 0$.

a) Case $c = 0$: Is multiplication by $w = id$ a counterclockwise rotation about the origin through an angle equal to the argument of w followed by a dilation with scale factor the modulus of w? (Check cases: $d > 0$, $d < 0$, and $d = 0$!)

b) Case $c < 0$: Prove that multiplication by $w = c + id$ with $c < 0$ corresponds to a counterclockwise rotation about the origin through an angle equal to the argument of w followed by a dilation with scale factor the modulus of w. (HINT: Is it easier to multiply first by $-w$ and then by -1 ?)

<The above exercise should be teased apart and nicely scaffolded for kids. Maybe have students explore the geometry of multiplication by $-3+4i$ by comparing it with multiplication by $3-4i$ and then by -1. This sort of work will lead to understanding b).>

EXERCISE: *What is the reflection of the point* $5+8i$ *about the diagonal line from the origin in the first quadrant?*

<This needs a lot of scaffolding for kids. Need to lead them to:

Rotate 45° clockwise (that is, -45° counterclockwise) (multiply by $\cos\left(-45^\circ\right)+i\sin\left(-45^\circ\right)=\dfrac{1-i}{\sqrt{2}}$);

reflect about the real axis; rotate back 45° counterclockwise (multiply by $\cos\left(45^\circ\right)+i\sin\left(45^\circ\right)=\dfrac{1+i}{\sqrt{2}}$. We get:

$$8+5i \quad\rightarrow\quad \left(\frac{1-i}{\sqrt{2}}\right)(8+5i)=\frac{13-3i}{\sqrt{2}} \quad\rightarrow\quad \overline{\frac{13-3i}{\sqrt{2}}}=\frac{13+3i}{\sqrt{2}} \quad\rightarrow\left(\frac{1+i}{\sqrt{2}}\right)\left(\frac{13+3i}{\sqrt{2}}\right)=5+8i$$

Ooh! We should have guessed that!>

Now do a reflection across at diagonal ray at an angle of 60° ! (Try guessing that one!)

LESSON 16: The Geometric Effect of Multiplying by a Reciprocal

The transformation $L(z)=(1+i)z$ has the geometric effect of rotating a complex number about the origin though 45° and performing a dilation with scale factor $\sqrt{2}$.

Suppose I wish to "undo" that effect.

Then, geometrically, we need to rotate -45° and dilate by a factor of $\frac{1}{\sqrt{2}}$. So we need to multiply by the complex number $\frac{1}{\sqrt{2}}\left(\cos\left(-45^\circ\right)+i\sin\left(-45^\circ\right)\right)=\frac{1-i}{\sqrt{2}}$.

But hang on! The way to undo multiplication by $1+i$ is to multiply by $\frac{1}{1+i}$!

But note:

$$\frac{1}{1+i}=\frac{1-i}{(1+i)(1-i)}=\frac{1-i}{\sqrt{2}}.$$

Of course!

We have:

Multiplication by $\frac{1}{c+id}$ *has the reverse geometric effect of multiplication by* $c+id$.

CONSEQUENCE: *If* z *has modulus* r *and argument* θ, *then* $\frac{1}{z}$ *has modulus* $\frac{1}{r}$ *and argument* $-\theta$.

Since $1+i$ has modulus $\sqrt{2}$ and argument $\frac{\pi}{4}$, we must have

$$\frac{1}{1+i}=\frac{1}{\sqrt{2}}\left(\cos\left(-\frac{\pi}{4}\right)+i\sin\left(\frac{-\pi}{4}\right)\right)=\frac{1}{\sqrt{2}}\left(\frac{1-i}{\sqrt{2}}\right)=\frac{1-i}{2}.$$

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SUMMARY OF TRANSFORMATIONS

We have seen translations as $L(z) = z + a$.

We've seen $L(z) = \overline{z}$ is reflection about the real axis.

> **EXERCISE**: Does $L(z) = \overline{z}$ satisfy the dream conditions $L(z+w) = L(z) + L(w)$ and $L(mz) = mL(z)$? If not, in what way does it fail?

We have that $L(z) = wz$ is a rotation about the origin followed by a dilation from the origin.

OTHER FORMULAS:

EXERCISE: Describe a reflection across the line that passes through the origin and a given complex number α.

> <Scaffold this quite a bit more: The answer is $L(z) = \alpha\left(\overline{\frac{1}{\alpha}z}\right)$. (Rotate the diagonal line to the real axis by multiplying by $\frac{1}{\alpha}$, reflect, and rotate back .) >

EXERCISE: What is the geometric effect of this function $L(z) = \alpha\overline{\left(\frac{z - a_0}{\alpha}\right)} + a_0$?

TOPIC C: THE POWER OF THE RIGHT NOTATION

N-CN.B.4 (+) Represent complex numbers on the complex plane in rectangular and polar form (including real and imaginary numbers), and explain why the rectangular and polar forms of a given complex number represent the same number.

N-CN.B.5 (+) Represent addition, subtraction, multiplication, and conjugation of complex numbers geometrically on the complex plane; use properties of this representation for computation. *For example, $(-1 + \sqrt{3}\,i)^3 = 8$ because $(-1 + \sqrt{3}\,i)$ has modulus 2 and argument 120°.*

N-VM.C.10 (+) Understand that the zero and identity matrices play a role in matrix addition and multiplication similar to the role of 0 and 1 in the real numbers. The determinant of a square matrix is nonzero if and only if the matrix has a multiplicative inverse.

N-VM.C.11 (+) Multiply a vector (regarded as a matrix with one column) by a matrix of suitable dimensions to produce another vector. Work with matrices as transformations of vectors.

N-VM.C.12 (+) Work with 2 x 2 matrices as transformations of the plane, and interpret the absolute value of the determinant in terms of area.

G-CO.A.2 Represent transformations in the plane using, e.g., transparencies and geometry software; describe transformations as functions that take points in the plane as inputs and give other points as outputs. Compare transformations that preserve distance and angle to those that do not (e.g., translation versus horizontal stretch).

G-CO.A.4 Develop definitions of rotations, reflections, and translations in terms of angles, circles, perpendicular lines, parallel lines, and line segments.

G-CO.A.5 Given a geometric figure and a rotation, reflection, or translation, draw the transformed figure using, e.g., graph paper, tracing paper, or geometry software. Specify a sequence of transformations that will carry a given figure onto another.

LESSONS 17–19: Exploiting the Connection to Trigonometry

<Probably not three lessons worth!>

We have seen that every complex number z can be written in the form $z = r(\cos\theta + i\sin\theta)$ where r is its modulus (distance from the origin) and θ is its argument (angle of elevation from the real axis).

We have also seen that:

If z has modulus r, argument θ
w has modulus m and argument α

Then

zw has modulus rm and argument $\theta+\alpha$.

Let's use this to our advantage!

EXAMPLE: $1+i$ has modulus $\sqrt{2}$ and argument $\theta=\frac{\pi}{4}$. (Draw a sketch to see this!)

Then $(1+i)^2$ must have modulus $\sqrt{2}\cdot\sqrt{2}=2$ and argument $\frac{\pi}{4}+\frac{\pi}{4}=\frac{\pi}{2}$. It must be the number $2\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)=2(0+i)=2i$. (Is it?)

Also, $(1+i)^3=(1+i)(1+i)^2$ has modulus $\sqrt{2}\cdot 2=2\sqrt{2}$ and argument $\frac{\pi}{4}+\frac{\pi}{2}=\frac{3\pi}{4}$. It must be the number $2\sqrt{2}\left(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right)=2\sqrt{2}\left(-\frac{1}{\sqrt{2}}+i\frac{1}{\sqrt{2}}\right)=-2+2i$. (Is it?)

EXERCISE: What is the modulus of $(1+i)^{20}$? What is its argument? Explain why $(1+i)^{20}$ is a real number.

EXERCISE: If z has modulus r and argument θ , what is the modulus and argument of z^2 ? Of z^N ? Of $\frac{1}{z}$?

EXERCISE: Write $\left(\frac{1-i}{\sqrt{2}}\right)^7$ as a complex number in the form $a+ib$ for real values a and b .

EXERCISE: Write $\left(1+\sqrt{3}i\right)^9$ as a complex number in the form $a+ib$ for real values a and b .

EXERCISE: Find all integers N such that $\left(\sqrt{3}+i\right)^N$ is a real number.

EXERCISE: We say that w is a square root of a complex number z if $w^2 = z$.

a) Verify that $2+i$ is a square root of $3+4i$. Find a second square root of $3+4i$.
b) If z has modulus r and argument θ, explain why $w=\sqrt{r}\left(\cos\dfrac{\theta}{2}+i\sin\dfrac{\theta}{2}\right)$ is a square root of z.
c) Find two square roots of i.
d) Find two square roots of -1.
e) There are three complex numbers that are cube roots of -1. Find all three! (Write each in the form $a+ib$ with a and b real numbers.)

Do lots of practice like the above, suitably scaffolded for students.

LESSON 20: Exploiting the Connection to Cartesian Coordinates

We have seen that complex multiplication has the effect of performing a dilation and a rotation in the complex plane:

> If w has modulus m and argument α, then $L(z) = wz$ corresponds to a rotation about the origin (through the angle α) and a dilation from the origin (with scale factor r).

And if we like, we can think of this as an action in the Coordinate plane of ordinary geometry—complex numbers correspond to points:

$$x + iy \leftrightarrow (x, y)$$

EXERCISE: a) Find a complex number w so that multiplication by w induces a one degree rotation about the origin in the *clockwise* direction with *no* dilation.
b) Find a complex number w such that multiplication by w induces a dilation from the origin with scale factor 0.1 with *no* rotation.

Video game makers are very interested in the mathematics of rotations and dilations. In a first-player game (is that the term?), you are centered at the origin and if you move forward in the game, then the images on the screen need to undergo a translation (to mimic what you would see as you walk past them), possibly with dilations involved (as you walk closer to objects, they look larger), and if you turn, the images on the screen need to undergo a rotation.

ONE PROBLEM: We have set up the mathematics of rotations and dilations for two-dimensional geometry, no worries, but in this day and age video games are set in three-dimensional space. We need to translate our work from two dimensions to three dimensions.

Complex numbers are inherently two dimensional:

$$x + iy \leftrightarrow (x, y)$$

Three-dimensional geometry is about points in, well, three-dimensional space:

$$(x, y, z)$$

So, in order to understand the mathematics of three-dimensional video game design, say, we need to do two things:

1. Rewrite all our work about rotations and dilations for complex numbers $x+iy$ in terms of points (x,y), and see what rotations and dilations look like from that perspective.
2. Then see if we can generalize the mathematics of rotations on two-dimensional points (x,y) to three-dimensional points (x,y,z).

We can certainly do the first of these tasks in this module. We'll get to the second one as best we can in the next module.

TRANSLATING COMPLEX MULTIPLICATION INTO THE LANGUAGE OF COORDINATES

In complex-number notation:

If w has modulus r and argument α, then wz is a rotation of z through an angle α about the origin followed by a dilation from the origin with scale factor r.

Being more explicit:

If $w=a+ib$, then $r=\sqrt{a^2+b^2}$ and $\alpha=\arctan\left(\frac{b}{a}\right)$, for the appropriate value of the inverse tangent.

Being even more explicit about the complex numbers:

Multiplying $x+iy$ by $a+ib$ rotates $x+iy$ about the origin through an angle $\arctan\left(\frac{b}{a}\right)$ and then performs a dilation on that point from the origin with scale factor $\sqrt{a^2+b^2}$.

Saying this another way:

The action $x+iy\to(a+ib)(x+iy)$ corresponds to a rotation and a dilation of the complex number $x+iy$ through an angle $\arctan\left(\frac{b}{a}\right)$ and scale factor $\sqrt{a^2+b^2}$.

Working it out:

The action $x+iy \to (ax-by)+i(ay+bx)$ corresponds to a rotation and a dilation of the point $x+iy$ through an angle $\arctan\left(\frac{b}{a}\right)$ and scale factor $\sqrt{a^2+b^2}$.

Now writing this in terms of points in Coordinate geometry, rather than complex numbers:

The action $(x,y) \to (\mathrm{ax-by}, ay+bx)$ corresponds to a rotation and a dilation of the point (x,y) through an angle $\arctan\left(\frac{b}{a}\right)$ and scale factor $\sqrt{a^2+b^2}$.

That's it! Here is what a rotation followed by a dilation looks like in two-dimensional geometry:

For real numbers a and b , $L(x,y)=(ax-by,\ bx+ay)$ corresponds to a rotation and dilation about the origin. The angle of rotation is $\arctan\left(\frac{b}{a}\right)$ (chosen appropriately), and the scale factor of the dilation is $\sqrt{a^2+b^2}$.

ICK!

This looks ugly and very difficult to work with.

Is there a better way to express this? (I certainly don't see any obvious way to generalize this to three-dimensional geometry.)

Hang on to next lesson!

EXERCISES: <This is going to be tricky! Some possible ideas—to be scaffolded and worded properly:

a) Which values of a and b give a pure dilation with scale factor 2 ?

<Answer: In complex-number notation, $w=2$ does the trick (argument here is $\alpha=0$). So $a=2, b=0$ should do the trick. Check: $L(x,y)=(ax-by,bx+ay)=(2x,2y)$. Yep. That's a dilation.>

b) What values for a and b give a pure 90° rotation? A 120° rotation?

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c) What values for a and b give a pure 45° rotation? Check your answer by computing $L\left(L(x,y)\right)$. (What map should this correspond to?)

There has to be some way of getting at the following through some exercises:

The issue of the "appropriate value of $\arctan\left(\frac{b}{a}\right)$" is tricky: If you choose the wrong one, then you are off by 180°. How do you know which one to choose? In complex-number thinking, you want the one that corresponds to the quadrant $w = a + ib$ is in. Now $w = w \cdot 1$, so $a + ib$ is the image of the complex number 1. In Coordinate speak, (a,b) is the image of $(1,0)$. So we want the value of $\arctan\left(\frac{b}{a}\right)$ that matches the quadrant the point (a,b) is in.

d) $L(x,y) = (2x + 3y, -3x + 2y)$ is a rotation and dilation about the origin. What angle? What scale factor?

(Just do a couple of exercises like this. We don't want to dwell on this at all because the notation is horrid, and we want to let it all go!)

LESSON 21: The Hunt for Better Notation

In the mid-1800s and all through the early 1900s, various situations arose where formulas like $L(x, y) = (ax - by,\ bx + ay)$ and worse kept appearing, and people were hunting and struggling to find a friendlier way to express these expressions. (After all, who wants to be bogged down by difficult notation?)

We don't want to go through 70 years of struggle here but will take advantage of what they—eventually—found to be a wonderful way to express these things, even though it may be a little strange to us at first. (In G.12.M3 we'll encounter another, more general, situation in which the notation we develop here naturally appears. Seeing this notation in two different contexts highlights the power of good notation!)

> **<PERSONAL INAPPROPRIATE COMMENT:** I personally would not be introducing matrix notation. I feel the following is terribly forced. If there is a better student-friendly and natural way to get to this—and swiftly—then go for it. Matrix notation does reappear in G.12.M3 in terms of networks, and all makes much better sense there. (It follows http://www.jamestanton.com/wp-content/uploads/2013/12/Pamphlet_-Vectors-and-Matrices3.pdf.) >

For starters… Mathematicians found it is actually more convenient to write points (x, y) in columns: $\begin{pmatrix} x \\ y \end{pmatrix}$.

This is the first piece of weirdness.

Secondly, using different letter names for coordinates can be confusing, especially if you want to work in three—or more(!)—dimensions. Instead of using x and y, we could use x_1 and x_2. So a point now appears as: $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$. (And a point in three dimensions is $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$, and so on.)

Now the statement $L(x, y) = (ax - by,\ bx + ay)$ looks like:

$$L\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} ax_1 - bx_2 \\ bx_1 + ax_2 \end{pmatrix}.$$

The a s and b s are confusing, and they appear in four places: first in the first slot with x_1, and then in the first slot with x_2, and then in the second slot with x_1, and then in the second slot with x_2. To develop a notation that is as general as possible, let's use the same one symbol a, and index by which slot it goes in with subscripts: a_{11}, a_{12}, a_{21} and a_{22}.

So we'll now write:

$$L\begin{pmatrix}x_1\\x_2\end{pmatrix}=\begin{pmatrix}a_{11}x_1+a_{12}x_2\\a_{21}x_1+a_{22}x_2\end{pmatrix}$$

where $a_{11}=a$, $a_{12}=-b$, $a_{21}=a$, and $a_{22}=a$.

The key feature of this formula is the four values a_{11}, a_{12}, a_{21} and a_{22}, and it seems natural to display them in a table:

$$\begin{pmatrix}a_{11} & a_{12}\\a_{21} & a_{22}\end{pmatrix}$$

A table of values like this is called a *matrix.* (In geology, the matrix is the background material that holds a gemstone or a fossil in place. Here we have a table holding the information about a function in place.)

In fact, since all the information about L is encoded in the matrix, mathematicians will write:

$$L\begin{pmatrix}x_1\\x_2\end{pmatrix}=\begin{pmatrix}a_{11} & a_{12}\\a_{21} & a_{22}\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}$$

where this notation—and it is just notation—is interpreted to mean:

$$\begin{pmatrix}a_{11} & a_{12}\\a_{21} & a_{22}\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}=\begin{pmatrix}a_{11}x_1+a_{12}x_2\\a_{21}x_1+a_{22}x_2\end{pmatrix}$$

<Teach row-by-column multiplication for this very basic 2×2 and 2×1 case.>

SO … back to our rotations and dilations in the plane, with the variables x, y, a, b we have:

$$L\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}a & -b\\b & a\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}$$

This is a rotation of angle $\arctan\left(\frac{b}{a}\right)$ (with angle to match the quadrant of (a,b)) and dilation with scale factor $\sqrt{a^2+b^2}$.

EXERCISES: Practice matrix multiplication.

Practice writing rotations and dilations in matrix notation.

A pure rotation is:

$$\begin{pmatrix} \cos\theta & -\cos\theta \\ \sin\theta & \sin\theta \end{pmatrix}.$$

A pure dilation is:

$$\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}.$$

LESSONS 22–23: Modeling Video Game Motion with Matrices

In the new notation:

$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

takes a point (x, y) and rotates it through an angle θ counterclockwise about the origin.

$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

"pushes" the point (x, y) away from the origin (along the ray connecting the origin to the point) with scale factor k,

and

$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} a & -b \\ b & a \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

is a combination of both.

EXERCISE: How do we perform a translation in this notation?

Find a matrix $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$ such that $\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$ corresponds to a translation two units to the right and six units vertically.

<Answer: This is a trick question! There is no matrix that represents a translation.

Notice that any map that is given by a matrix must take the point $(0,0)$ to $(0,0)$:

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} \to \begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} a0 + c0 \\ b0 + d0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Since a translation does not keep the origin fixed, there is no matrix representation for it!>

But we can represent the desired translation as:

$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} x+2 \\ y+6 \end{pmatrix}.$$

Now do a series of exercises that get students to the point that they can answer the following question that appears in the end-of-module assessment:

In programming a computer video game, Mavis coded the changing location of a space rocket as follows:

At a time t seconds between $t = 0$ seconds and $t = 2$ seconds, the location $\begin{pmatrix} x \\ y \end{pmatrix}$ of the rocket is given by:

$$\begin{pmatrix} \cos\left(\frac{\pi}{2}t\right) & -\sin\left(\frac{\pi}{2}t\right) \\ \sin\left(\frac{\pi}{2}t\right) & \cos\left(\frac{\pi}{2}t\right) \end{pmatrix} \begin{pmatrix} -1 \\ -1 \end{pmatrix}.$$

At a time of t seconds between $t = 2$ seconds and $t = 4$ seconds, the location of the rocket is given by:

$$\begin{pmatrix} 3-t \\ 3-t \end{pmatrix}.$$

a) What is the location of the rocket at time $t = 0$? What is its location at time $t = 4$?

At time $t = 0$ the location of the rocket is

$$\begin{pmatrix} \cos(0) & -\sin(0) \\ \sin(0) & \cos(0) \end{pmatrix} \begin{pmatrix} -1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$$

At time $t = 4$ the location of the rocket is

$$\begin{pmatrix} 3-4 \\ 3-4 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$$

the same as start.

b) Petrich is worried that Mavis may have made a mistake and the location of the rocket is unclear at time $t = 2$ seconds. Explain why there is no inconsistency in the location of the rocket at this time.

According to the first set of instructions, the location of the rocket at time $t = 2$ is

$$\begin{pmatrix} \cos(\pi) & -\sin(\pi) \\ \sin(\pi) & \cos(\pi) \end{pmatrix} \begin{pmatrix} -1 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} -1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

According to the second set of instructions, its location at this time is

$$\begin{pmatrix} 3-2 \\ 3-2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

These are consistent.

c) What is the area of the region enclosed by the path of the rocket from time $t = 0$ to time $t = 4$?

The path traversed is a semicircle with a radius of $\sqrt{2}$. The area enclosed is $\frac{1}{2}x2\pi = \pi$ squared units.

d) Mavis later decided that the moving rocket should be shifted five places further to the right. How should she adjust her formulations above to accomplish this translation?

Notice that:

$$\begin{pmatrix} \cos\left(\frac{\pi}{2}t\right) & -\sin\left(\frac{\pi}{2}t\right) \\ \sin\left(\frac{\pi}{2}t\right) & \cos\left(\frac{\pi}{2}t\right) \end{pmatrix}\begin{pmatrix} -1 \\ -1 \end{pmatrix} = \begin{pmatrix} -\cos\left(\frac{\pi}{2}t\right) + \sin\left(\frac{\pi}{2}t\right) \\ -\sin\left(\frac{\pi}{2}t\right) - \cos\left(\frac{\pi}{2}t\right) \end{pmatrix}$$

To translate these points 5 units to the right, use

$$\begin{pmatrix} -\cos\left(\frac{\pi}{2}t\right) + \sin\left(\frac{\pi}{2}t\right) + 5 \\ -\sin\left(\frac{\pi}{2}t\right) - \cos\left(\frac{\pi}{2}t\right) \end{pmatrix} \text{ for } 0 \le t \le 2$$

Also use

$$\begin{pmatrix} 3-t+5 \\ 3-t \end{pmatrix} = \begin{pmatrix} 8-t \\ 3-t \end{pmatrix} \text{ for } 2 \le t \le 4$$

TEASER COMMENT: The notation suggests how we can get transformations in three dimensions! Points are $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$, and we can have transformations of the form: $\begin{pmatrix} m & q & t \\ n & r & u \\ p & s & v \end{pmatrix}$.

(And four dimensions! And five!)

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LESSONS 24–25: Matrix Notation Encompasses New Transformations

We've seen how to represent rotations and dilations in matrix notation.

We've seen that not all transformations can be represented this way (translations, for example).

But we still haven't fully explored the power of this new notation for finding new transformations, some that we may not have even conceived of!

EXAMPLE: Can the reflection about the real axis $L(z)=\bar{z}$ be expressed in matrix notation?

Yes, it's $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$!

EXERCISE: Express a reflection about the vertical axis in matrix notation.

Notice that our reflection $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is *not* of the form $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$. In fact, any matrix $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ must correspond to a transformation of some kind, and it is likely not to be a combination of a rotation and a dilation.

EXERCISE: Explore the transformation given by $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. What does it seem to do to points in the plane?

Do some more exercises like the one above.

EXERCISE: Discover that the transformation $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ doesn't do much. Call it the *identity* transformation.

EXERCISE: Discover that $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ does even less! (Takes all points to the origin.) (Do you see that it is a dilation with scale factor zero?)

COMBINING MATRICES:

Discuss the role of doing one transformation followed by another and discover:

If L is given by $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$ and M by $\begin{pmatrix} p & r \\ q & s \end{pmatrix}$, show that $ML\begin{pmatrix} x \\ y \end{pmatrix}$ is the same as applying the matrix $\begin{pmatrix} pa+rb & pc+rd \\ qa+sb & qc+sd \end{pmatrix}$ to $\begin{pmatrix} x \\ y \end{pmatrix}$.

This motivates our definition of matrix multiplication. (This is really ick, but we have to do it!)

Notice that our definition matches our early definition of 2×2 by 2×1, but columnwise.

Notice that $I=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ acts as a multiplicative identity. (And this makes sense, since I corresponds to a transformation that leaves points alone: so geometrically we must have $IL=L$ and $LI=L$.)

NOW … if there is matrix multiplication, might there be matrix addition?

Seems natural to define:

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix}+\begin{pmatrix} p & r \\ q & s \end{pmatrix}=\begin{pmatrix} a+p & c+r \\ b+q & d+s \end{pmatrix}.$$

(Hard to know what this means geometrically in terms of transformations, but we'll see a natural interpretation of matrix addition in G.12.M3.)

Notice that $O=\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ acts as an additive identity.

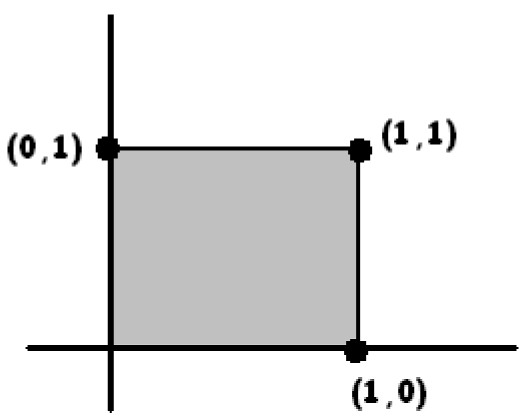
LESSONS 26–27: Getting a Handle on New Transformations

So we've seem that every matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ corresponds to some kind of transformation of the plane. But it can be hard to see what the transformation actually does!

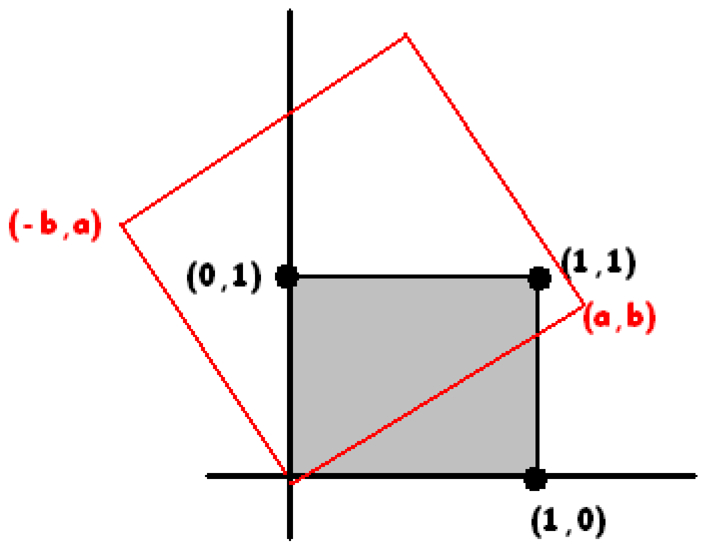
For example, what does the matrix $\begin{pmatrix} 109 & 3 \\ 1 & -2 \end{pmatrix}$ do to points and shapes and lines in the plane? It is wild and new!

GETTING A FEEL FOR NEW TRANSFORMATIONS:

One way to get a feel for what a transformation does is to get a sense of what it does on a very basic geometric object. People often refer to the unit square. (It contains/represents the unit of length and a unit of area, two key ideas one usually wants to think about in two-dimensional geometry.)

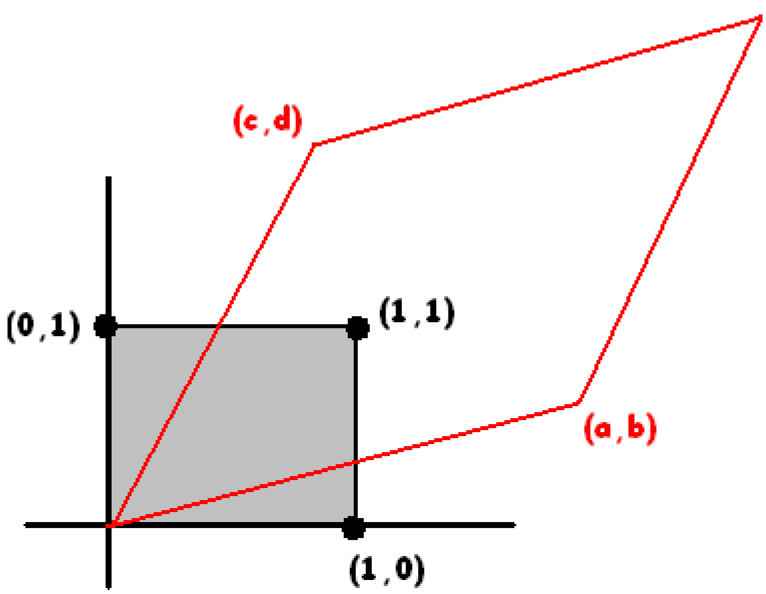


Any matrix of the form $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ is a rotation and a dilation. Its effect on this basic object is a rotation of this square followed by a dilation.



A picture like this does indeed give a sense of what the transformation does to points in the plane.

A transformation given by a general matrix $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$ will do something that looks like this:



Check that the point $(1,0)$ goes to (a,b), the point $(0,1)$ to (c,d), and $(1,1)$ to $(a+c,b+d)$ (and, as we know, the origin is mapped to the origin).

Comment: One can prove that a transformation given by a matrix is sure to take straight-line segments to straight-line segments. (See HOMEWORK EXERCISES). So the image of the unit square is going to be a parallelogram with vertices the images of the four vertices of the unit square.

Now we see in the picture that a general transformation given by a matrix is likely to do more than just rotating and dilating. Is there some skew stretching as well?

To get a sense of the transformation, we can ask:

Basic question: *How has this transformation changed our basic unit of area? What is the area of the image parallelogram?*

<Enclose the parallelogram in a rectangle. Compute its area by subtracting off areas of right triangles. Get area of parallelogram is $ad - bc$.

Warning: We've drawn a picture which has kept the orientation of the vertices. The picture could switch the order of the vertices. Redraw the picture and redo the work and show get area is $bc - ad$.

In all cases, the area of the parallelogram is $|ad - bc|$.

Definition: The *determinant* of a 2×2 matrix $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$ is $|ad - bc|$. Geometrically this is the area of the image of the unit square.

EXERCISES:

a) The identity transformation does nothing to the unit square. Its image area is still 1. Is the determinant of the identity matrix indeed 1?

b) A dilation with scale factor k changes all areas by k^2. Does this gel well with our determinant thinking?

c) Rotations don't change area. Is the determinant of a rotation matrix indeed one?

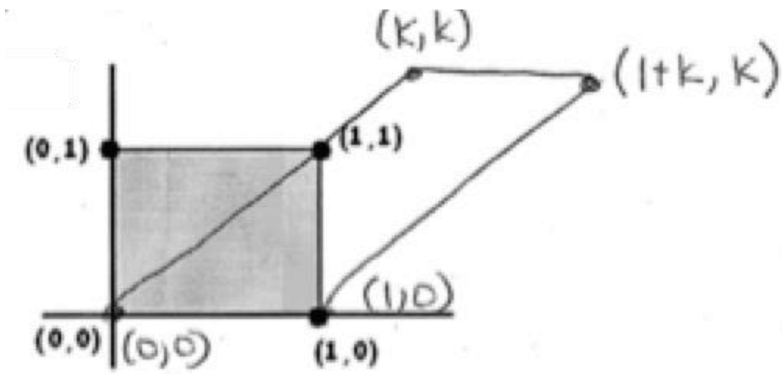
Here's the end-of-module assessment question on this stuff. Make sure we're at the level to be able to handle it:

1. Consider the transformation on the plane given by the 2×2 matrix $\begin{pmatrix} 1 & k \\ 0 & k \end{pmatrix}$ for a fixed positive number k.

   a) Draw a sketch of the image of the unit square under this transformation (the unit square has vertices $(0,0)$, $(1,0)$, $(0,1)$, and $(1,1)$). Be sure to label all four vertices of the image figure.

To find the coordinates of the image, multiply the vertices of the unit square by the matrix.

$$\begin{pmatrix} 1 & k \\ 0 & k \end{pmatrix}\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\begin{pmatrix} 1 & k \\ 0 & k \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} k \\ k \end{pmatrix}$$
$$\begin{pmatrix} 1 & k \\ 0 & k \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1+k \\ k \end{pmatrix}$$
$$\begin{pmatrix} 1 & k \\ 0 & k \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$



The Unit Square

The image is a parallelogram with base = 1 and height = k.

   b) What is the area of the image parallelogram?

To find the area of the image figure, multiply the area of the unit square by the absolute value of $\begin{vmatrix} 1 & k \\ 0 & k \end{vmatrix}$.

$$\begin{vmatrix} 1 & k \\ 0 & k \end{vmatrix} = (1 \times k) - (0 \times k) = k$$

$$Area = 1x|k| = k \text{ since } k > 0$$

   c) Find the coordinates of a point $\begin{pmatrix} x \\ y \end{pmatrix}$ whose image under the transformation is $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$.

Solve the equation to find the coordinates of $\begin{pmatrix} x \\ y \end{pmatrix}$.

$$\begin{pmatrix} 1 & k \\ 0 & k \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

Converting the matrix equation to a system of linear equations gives us

$$x + ky = 2$$
$$ky = 3$$

Solve this system.

$$y = \frac{3}{k}$$
$$x + k\left(\frac{3}{k}\right) = 2$$
$$x + 3 = 2$$
$$x = -1$$

The point is $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ \frac{3}{k} \end{pmatrix}$

d) The transformation $\begin{pmatrix} 1 & k \\ 0 & k \end{pmatrix}$ is applied once to the point $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, then once to the image point, then once to the image of the image point, and then once to the image of the image of the image point, and so on. What are the coordinates of the tenfold image of the point $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, that is, the image of the point after the transformation has been applied ten times?

Multiply to apply the transformation once: $\begin{pmatrix} 1 & k \\ 0 & k \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1+k \\ k \end{pmatrix}$

Multiply again by the 2 × 2 matrix: $\begin{pmatrix} 1 & k \\ 0 & k \end{pmatrix}\begin{pmatrix} 1+k \\ k \end{pmatrix} = \begin{pmatrix} 1+k+k^2 \\ k^2 \end{pmatrix}$

Multiply again by the 2 × 2 matrix: $\begin{pmatrix} 1 & k \\ 0 & k \end{pmatrix}\begin{pmatrix} 1+k+k^2 \\ k^2 \end{pmatrix} = \begin{pmatrix} 1+k+k^2+k^3 \\ k^3 \end{pmatrix}$

By observing the patterns, we can see that the result of n multiplications is a 2 × 1 matrix whose top row is the previous row plus k^n and whose bottom row is k^n. The tenfold image would be $\begin{pmatrix} 1+k+k^2+k^3+\cdots k^{10} \\ k^{10} \end{pmatrix}$

HOMEWORK EXERCISE:

<The sticky point of this lesson is the claim that a matrix transformation takes straight lines to straight lines.

Perhaps create an (optional?) homework exercise of the type:

We made the claim in the lesson that a transformation given by a matrix takes straight-line segments to straight-line segments. Let's show this is true with one example. (How to prove it is true, in general, should be clear from this example.)

Consider the matrix $L = \begin{pmatrix} 2 & 5 \\ -1 & 3 \end{pmatrix}$.

For each real number $0 \le t \le 1$ consider the point $(3+t, 10+2t)$. When $t = 0$, this is the point $A = (3,10)$. When $t = 1$, this is the point $B = (4,12)$.

a) Show that for $t = \frac{1}{2}$, $(3+t, 10+2t)$ is the midpoint of $\overline{AB}$.

b) Show that for each value of t , $(3+t, 10+2t)$ is a point on the line through A and B .

<Answer: The line through A and B has equation $y-10=2(x-3)$. We see that $x=3+t$ and $y=10+2t$ fit this equation.)

c) What is the equation of the line through LA and LB ? Show that $L\begin{pmatrix} 3+t \\ 10+2t \end{pmatrix}$ is sure to lie on this line.

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LESSONS 28–30: When Can We Reverse a Transformation?

The transformation represented by the matrix:

$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

is a counterclockwise rotation though an angle θ about the origin.

We can "undo" by then applying a rotation through an angle $-\theta$. This is represented by the matrix

$$R_{-\theta} = \begin{pmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

Thus applying R_θ and then $R_{-\theta}$ should have the same final effect as applying no transformation at all.

EXERCISE: Verify this through matrix algebra. What matrix do you expect to see when you compute the product $R_{-\theta}R_\theta$? Do you?

What about $R_\theta R_{-\theta}$?

EXERCISE: What transformation "undoes" a dilation from the origin with scale factor k ? <Translate into statement about a product of two matrices being the identity.>

Definition: A matrix A is called an *inverse matrix* to a matrix B if $AB = I$ and $BA = I$.

The transformation represented by A is the one that "undoes" the transformation represented by B .

EXERCISE: Find the matrix inverse to $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$.

<Have students work through the algebra, solving a system of equations:

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Verify that both $AB = I$ and $BA = I$ for the candidate matrix.>

EXERCISE: Does every matrix have a matrix inverse? Try to find a matrix inverse for $\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$.

<Students should get a system of equations with no solution.>

Let's look at the previous example in more detail.

EXERCISE: a) For $L = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$ draw the image parallelogram of the unit square.

<Students should see that it collapses to a line.>

b) Find two distinct points (a,b) and (c,d) with $L(a,b) = L(c,d)$; that is, find two distinct points with the same image under L.

<Since the unit square "collapses" to a straight line under L, there must be points landing in the same location. Some muddling gives $(1,0)$ and $\left(0,\frac{1}{2}\right)$ both have image $(1,2)$, for example.>

c) Can there be a transformation that undoes L?

<No! If there were one that undoes L, then there must be a clear place where each image point came from. But we've just seen that $(1,2)$ came from two different places. There is no clear way to "undo" L.>

d) Is there an inverse matrix for L?

<No. There can't be one. If there were, we'd have a transformation that undoes L, which we don't!>

This exercise shows that if the image of the unit square under L "collapses" to a figure of zero area, then we have distinct points being mapped to the same location under L. There can be no transformation that "undoes" L, which means that the matrix representing L has no inverse matrix.

Recall that the area of the image of the unit square is the determinant of L. So we have:

If the determinant of a matrix L is zero, then there can be no inverse matrix for L.

One can prove that if the determinant is not zero (that is, the image of the unit square is a parallelogram of positive area), then an inverse matrix exists. (See EXERCISE.)

So we actually have:

> *The inverse of a 2×2 matrix exists precisely when the determinant of the matrix is non-zero* (that is, when the image of the unit square is a parallelogram of positive area).

EXERCISE: Suppose $A=\begin{pmatrix} a & c \\ b & d \end{pmatrix}$ is a matrix with non-zero determinant: $ad-bc\neq 0$. Let's prove that an inverse matrix for A exists.

<Now, your choice: Either do this abstractly and let $B=\begin{pmatrix} p & r \\ q & s \end{pmatrix}$ and solve the system of equations $BA=I$ and get $p=\dfrac{d}{ad-bc}$ etc., or just give students

$$B=\begin{pmatrix} \dfrac{d}{ad-bc} & \dfrac{-c}{ad-bc} \\ \dfrac{-b}{ad-bc} & \dfrac{a}{ad-bc} \end{pmatrix}$$

and ask them to show that $AB=I$ and $BA=I$.>

EXERCISES:

<Make sure students have a clear geometric understanding of matrices and their inverses:

The inverse matrix of $\begin{pmatrix} \dfrac{1}{2} & -\dfrac{\sqrt{3}}{2} \\ \dfrac{\sqrt{3}}{2} & \dfrac{1}{2} \end{pmatrix}$ (can you see this is a rotation of 60° ?) must be

$\begin{pmatrix} \dfrac{1}{2} & \dfrac{\sqrt{3}}{2} \\ -\dfrac{\sqrt{3}}{2} & \dfrac{1}{2} \end{pmatrix}$ (a rotation of -60°).

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The inverse of $\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$ (a dilation scale factor of one half) has to be $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$.

Play with rotations of 90° and 180°.

Play with the interplay of composition of rotations and matrix multiplication. (See end-of-module assessment questions for the level of work needed on this.)

Ask questions like:

For which values of a does $\begin{pmatrix} 3 & -100 \\ 900 & a \end{pmatrix}$ have a matrix inverse?