

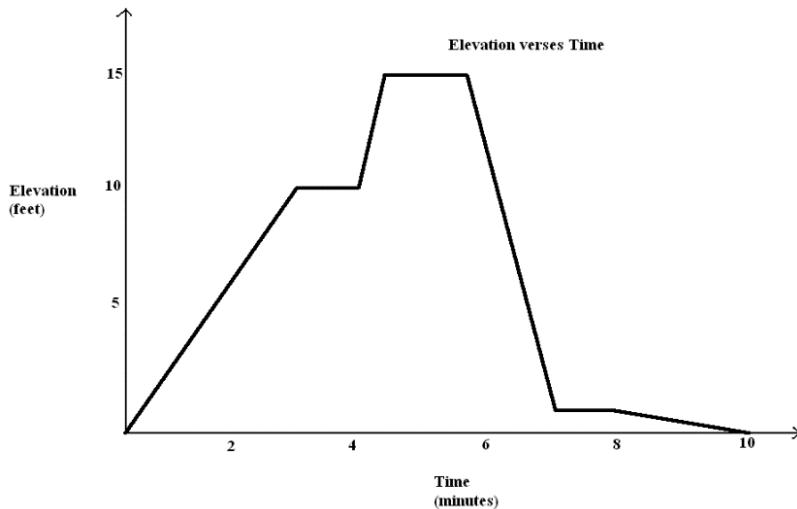
# Lesson 1: Graphs of Piecewise Linear Functions

## Classwork

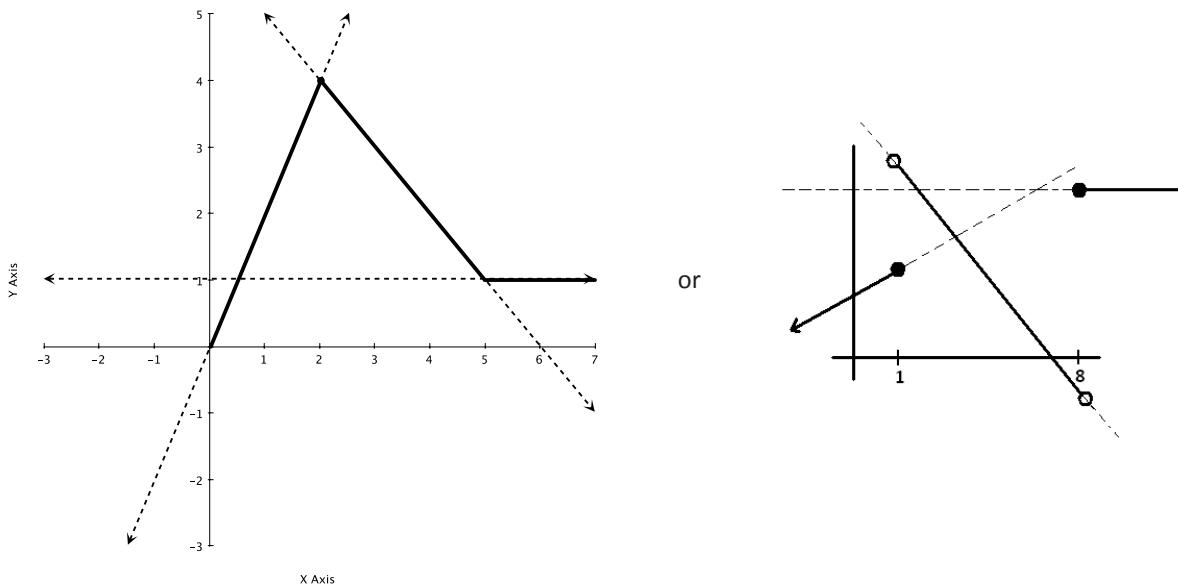
### Exploratory Challenge

**Example 1**

Here is an elevation-versus-time graph of a person's motion. Can we describe what the person might have been doing?



**Piecewise-Defined Linear Function:** Given non-overlapping intervals on the real number line, a (*real*) *piecewise linear function* is a function from the union of the intervals on the real number line that is defined by (possibly different) linear functions on each interval.



**Problem Set**

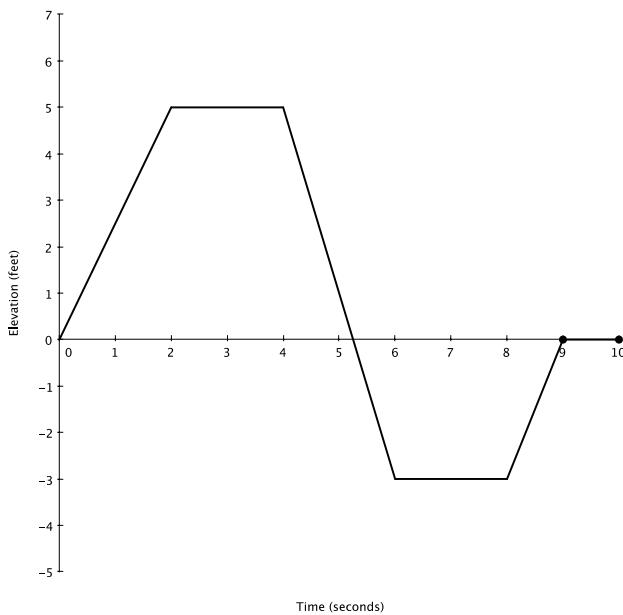
1. Watch the video, “Elevation vs. Time #3” (below)

<http://www.mrmeyer.com/graphingstories1/graphingstories3.mov>. (This is the third video under “Download Options” at the site <http://blog.mrmeyer.com/?p=213> called “Elevation vs. Time #3.”)

It shows a man climbing down a ladder that is 10 feet high. At time 0 seconds, his shoes are at 10 feet above the floor, and at time 6 seconds, his shoes are at 3 feet. From time 6 seconds to the 8.5 second mark, he drinks some water on the step 3 feet off the ground. After drinking the water, he takes 1.5 seconds to descend to the ground and then he walks into the kitchen. The video ends at the 15 second mark.

- a. Draw your own graph for this graphing story. Use straight line segments in your graph to model the elevation of the man over different time intervals. Label your  $x$ -axis and  $y$ -axis appropriately, and give a title for your graph.
- b. Your picture is an example of a graph of a piecewise linear function. Each linear function is defined over an interval of time, represented on the horizontal axis. List those time intervals.
- c. In your graph in part (a), what does a horizontal line segment represent in the graphing story?
- d. If you measured from the top of the man’s head instead (he is 6.2 feet tall), how would your graph change?
- e. Suppose the ladder descends into the basement of the apartment. The top of the ladder is at ground level (0 feet) and the base of the ladder is 10 feet below ground level. How would your graph change in observing the man following the same motion descending the ladder?
- f. What is his average rate of descent between time 0 seconds and time 6 seconds? What was his average rate of descent between time 8.5 seconds and time 10 seconds? Over which interval does he descend faster? Describe how your graph in part (a) can also be used to find the interval during which he is descending fastest.

2. Create an elevation-versus-time graphing story for the following graph:



3. Draw an elevation-versus-time graphing story of your own, and then create a story for it.

## Lesson 2: Graphs of Quadratic Functions

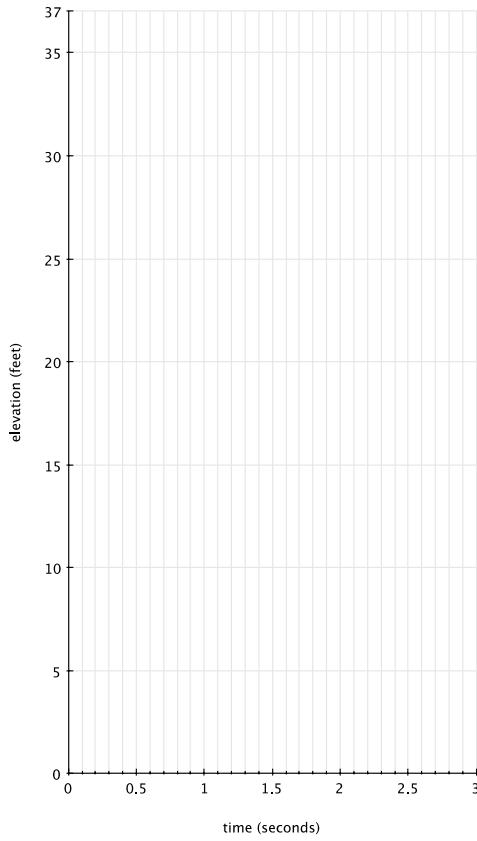
### Classwork

#### Exploratory Challenge

Plot a graphical representation of change in elevation over time for the following “graphing story.” It is a video of a man jumping from 36 feet above ground into 1 foot of water.

<http://www.youtube.com/watch?v=ZCFBC8aXz-g> or

<http://youtu.be/ZCFBC8aXz-g> (If neither link works, search for “OFFICIAL Professor Splash World Record Video!”)



**Example 2**

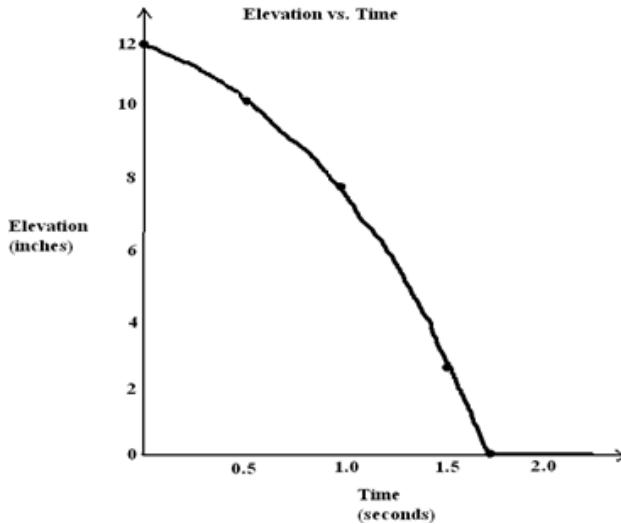
The table below gives the area of a square with sides of whole number lengths. Have students plot the points in the table on a graph and draw the curve that goes through the points.

<i>Side (cm)</i>	0	1	2	3	4
<i>Area (cm<sup>2</sup>)</i>	0	1	4	9	16

On the same graph, reflect the curve across the  $y$ -axis. This graph is an example of a “graph of a quadratic function.”

**Problem Set**

1. Here is an elevation versus time graph of a ball rolling down a ramp. The first section of the graph is slightly curved.



- From the time of about 1.7 seconds onwards, the graph is a flat horizontal line. If Ken puts his foot on the ball at time 2 seconds to stop the ball from rolling, how will this graph of elevation versus time change?
- Estimate the number of inches of change in elevation of the ball from 0 seconds to 0.5 seconds. Also estimate the change in elevation of the ball between 1.0 seconds and 1.5 seconds.
- At what point is the speed of the ball the fastest, near the top of the ramp at the beginning of its journey or near the bottom of the ramp? How does your answer to part (b) support what you say?

2. Watch the following graphing story:

Elevation vs. Time #4 [<http://www.mrmeyer.com/graphingstories1/graphingstories4.mov>. This is the second video under “Download Options” at the site <http://blog.mrmeyer.com/?p=213> called “Elevation vs. Time #4.”]

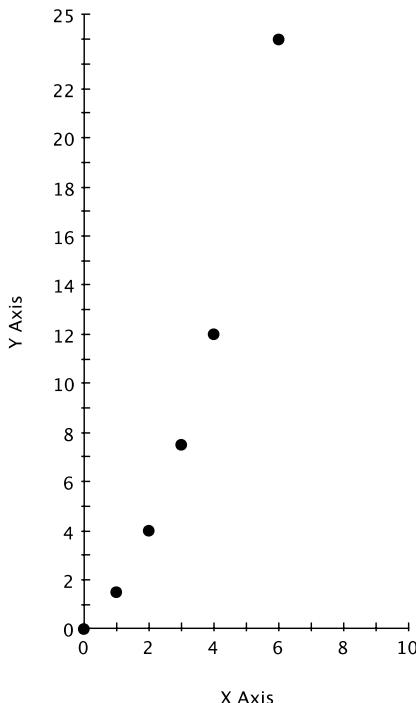
The video is of a man hopping up and down several times at three different heights (first, five medium-sized jumps immediately followed by three large jumps, a slight pause, and then 11 very quick small jumps).

- What object in the video can be used to estimate the height of the man’s jump? What is your estimate of the object’s height?
- Draw your own graph for this graphing story. Use parts of graphs of quadratic functions to model each of the man’s hops. Label your  $x$ -axis and  $y$ -axis appropriately and give a title for your graph.

3. Use the table below to answer the following questions.

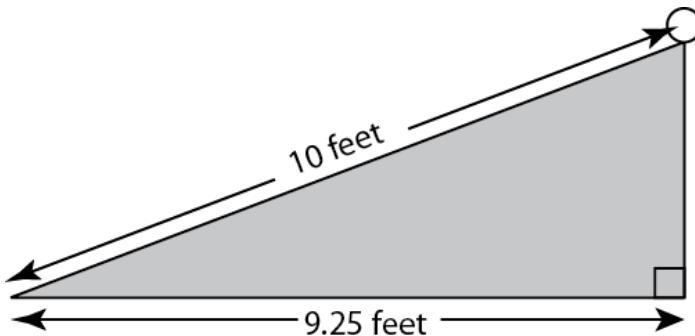
$x$	0	1	2	3	4	5	6
$y$	0	$3/2$	4	$15/2$	12		24

- a. Plot the points  $(x, y)$  in this table on a graph (except when  $x$  is 5).



- b. The  $y$ -values in the table follow a regular pattern that can be discovered by computing the differences of consecutive  $y$ -values. Find the pattern and use it to find the  $y$ -value when  $x$  is 5.  
 c. Plot the point you found in part (b). Draw a curve through the points in your graph. Does the graph go through the point you plotted?  
 d. How is this graph similar to the graphs you drew in Examples 1 and 2 and the Exploratory Challenge? Different?

4. A ramp is made in the shape of a right triangle using the dimensions described in the picture below. The ramp length is 10 feet from the top of the ramp to the bottom, and the horizontal width of the ramp is 9.25 feet.



A ball is released at the top of the ramp and takes 1.6 seconds to roll from the top of the ramp to the bottom. Find each answer below to the nearest 0.1 feet/sec.

- Find the average speed of the ball over the 1.6 seconds.
- Find the average rate of horizontal change of the ball over the 1.6 seconds.
- Find the average rate of vertical change of the ball over the 1.6 seconds.
- What relationship do you think holds for the values of the three average speeds you found in parts (a), (b), and (c)? (Hint: Use the Pythagorean Theorem.)

## Lesson 3: Graphs of Exponential Functions

### Classwork

#### Example 1

Consider the story:

Darryl lives on the third floor of his apartment building. His bike is locked up outside on the ground floor. At 3:00 p.m., he leaves to go run errands, but as he is walking down the stairs, he realizes he forgot his wallet. He goes back up the stairs to get it and then leaves again. As he tries to unlock his bike, he realizes that he forgot his keys. One last time, he goes back up the stairs to get his keys. He then unlocks his bike, and he is on his way at 3:10 p.m.

Sketch a graph that depicts Darryl's change in elevation over time.

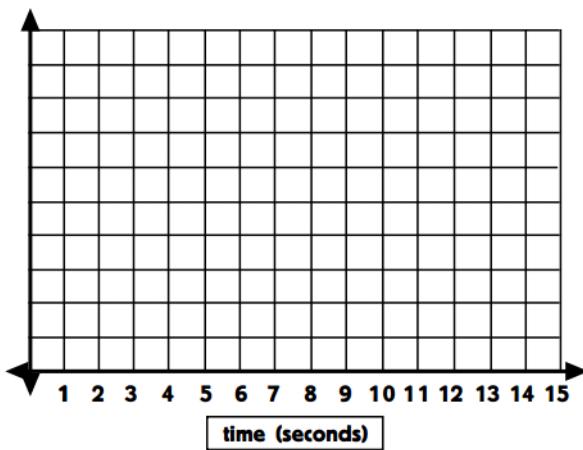
**Exploratory Challenge**

Watch the following graphing story:

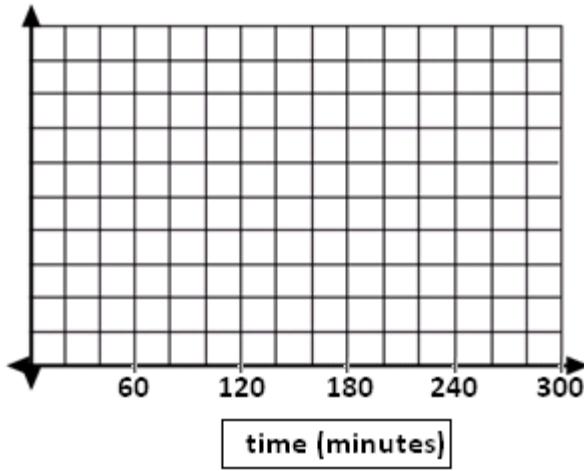
<https://www.youtube.com/watch?v=gEwzDydcIWc>

The video shows bacteria doubling every second.

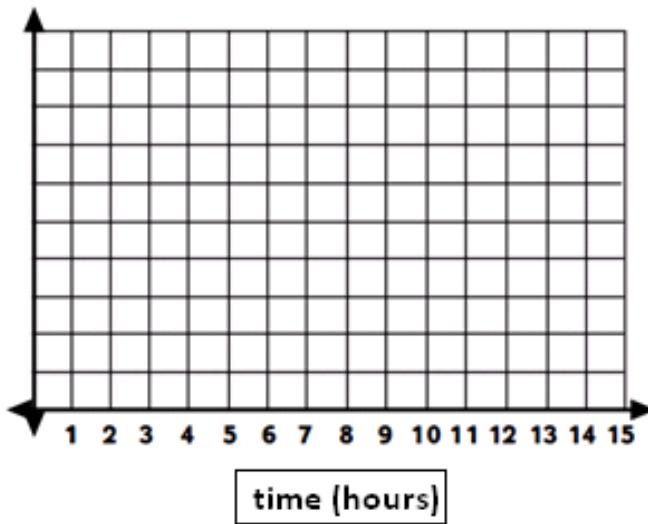
- Graph the number of bacteria versus time in seconds. Begin by counting the number of bacteria present at each second and plotting the appropriate points on the set of axes below. Consider how you might handle estimating these counts as the population of the bacteria grows.



- Graph the number of bacteria versus time in minutes.



3. Graph the number of bacteria versus time in hours (for the first five hours).



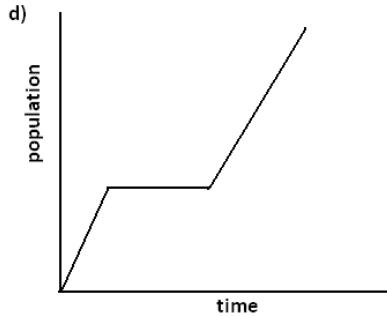
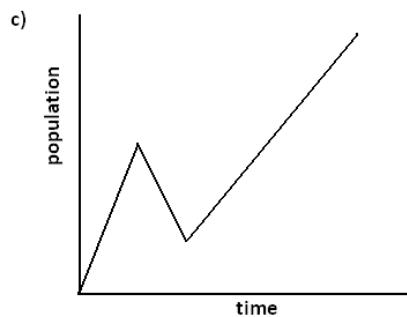
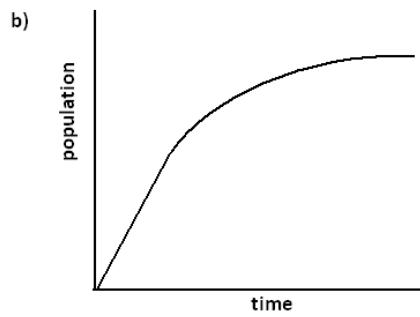
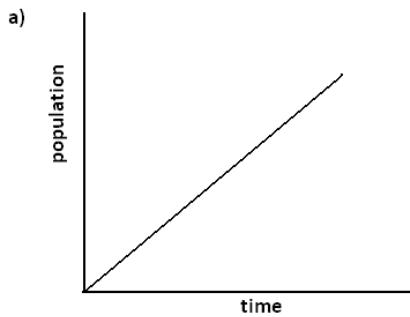
## Problem Set

1. Below are three stories about the population of a city over a period of time and four population-versus-time graphs. Two of the stories each correspond to a graph. Match the two graphs and the two stories. Write stories for the other two graphs, and draw a graph that matches the third story.

Story 1: The population size grows at a constant rate for some time, then doesn't change for a while, and then grows at a constant rate once again.

Story 2: The population size grows somewhat fast at first, and then the rate of growth slows.

Story 3: The population size declines to zero.



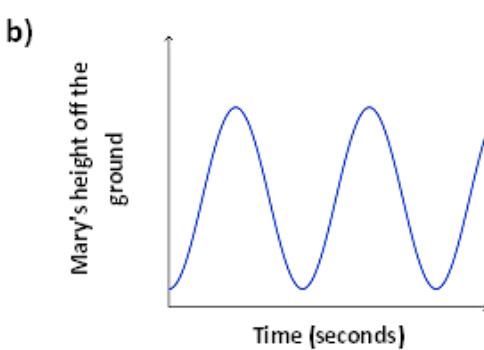
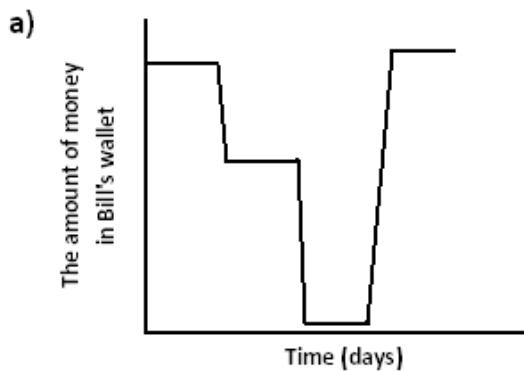
2. In the video, the narrator says:

"Just one bacterium, dividing every 20 minutes, could produce nearly 5,000 billion bacteria in one day. That is 5,000,000,000,000,000,000,000 bacteria."

This seems WAY too big. Could this be correct, or did she make a mistake? (Feel free to experiment with numbers using a calculator.)

3. *Bacillus cereus* is a soil-dwelling bacterium that sometimes causes food poisoning. Each cell divides to form two new cells every 30 minutes. If a culture starts out with exactly 100 bacterial cells, how many bacteria will be present after 3 hours?

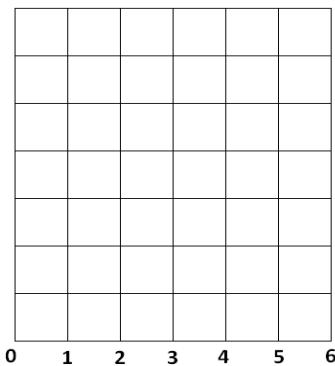
4. Create a story to match each graph below:



5. Consider the following story about skydiving:

Julie gets into an airplane and waits on the tarmac for 2 minutes before it takes off. The airplane climbs to 10,000 feet over the next 15 minutes. After 2 minutes at that constant elevation, Julie jumps from the plane and free falls for 45 seconds until she reaches a height of 5,000 feet. Deploying her chute, she slowly glides back to Earth over the next 7 minutes where she lands gently on the ground.

- Draw an elevation versus time graph to represent Julie's elevation with respect to time.
  - According to your graph, describe the manner in which the plane climbed to its elevation of 10,000 feet.
  - What assumption(s) did you make about falling after she opened the parachute?
6. Draw a graph of the number of bacteria versus time for the following story: Dave is doing an experiment with a type of bacteria that he assumes divides in half exactly every 30 minutes. He begins at 8:00 a.m. with 10 bacteria in a Petri dish and waits for 3 hours. At 11:00 a.m., he decides this is too large of a sample and adds Chemical A to the dish, which kills half of the bacteria almost immediately. The remaining bacteria continue to grow in the same way. At noon, he adds Chemical B to observe its effects. After observing the bacteria for two more hours, he observes that Chemical B seems to have cut the growth rate in half.
7. Decide how to label the vertical axis so that you can graph the data set on the axes below. Graph the data set and draw a curve through the data points.

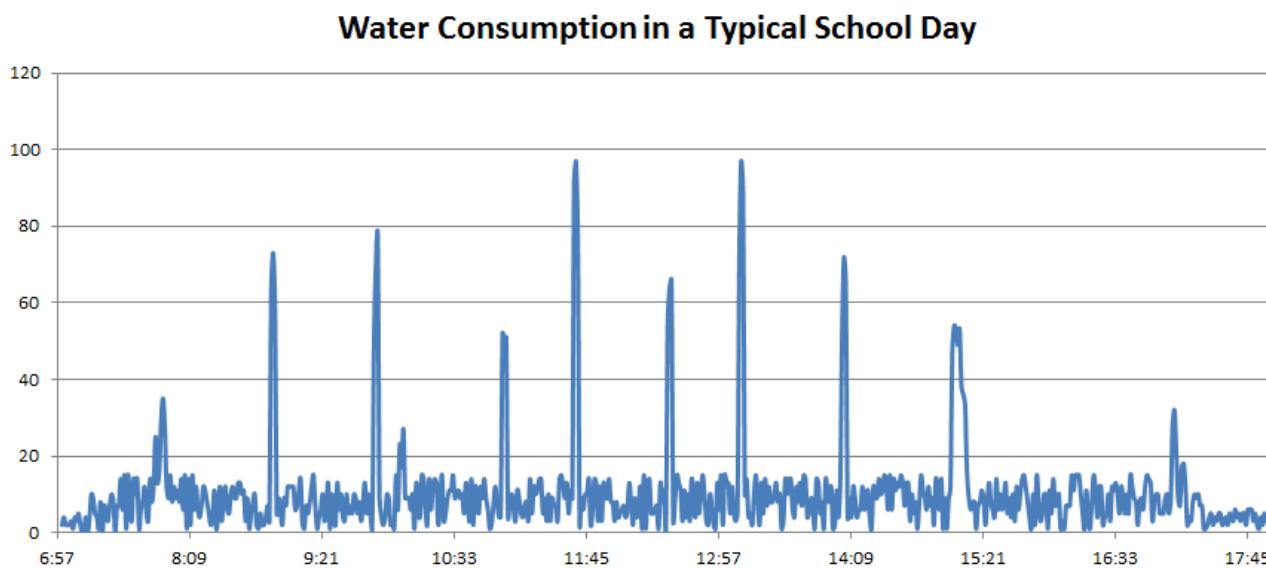


$x$	$y$
0	-1
1	-2
2	-4
3	-8
4	-16
5	-32
6	-64

## Lesson 4: Analyzing Graphs—Water Usage during a Typical Day at School

### Classwork

#### Example 1



### Exercises

1. The bulk of water usage is due to the flushing of toilets. Each flush uses 2.5 gallons of water. Samson estimates that 2% of the school population uses the bathroom between 10:00 a.m. and 10:01 a.m. right before homeroom. What is a good estimate of the population of the school?

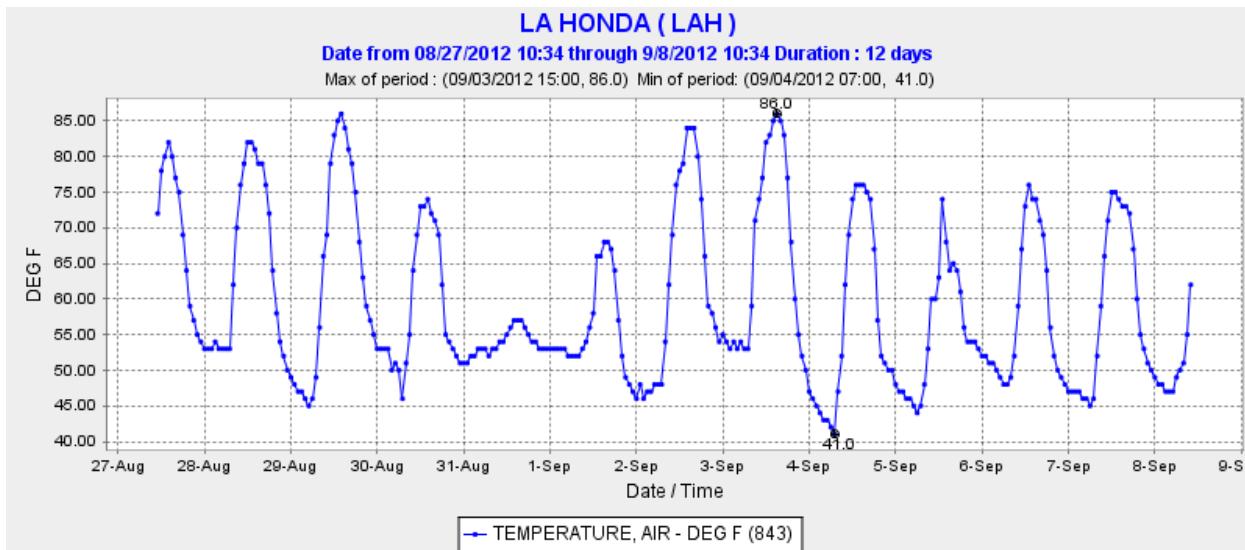
2. Samson then wonders this: If everyone at the school flushed a toilet at the same time, how much water would go down the drain (if the water-pressure of the system allowed)? Are we able to find an answer for Samson?

3. Estimation Exercise

- Make a guess as to how many toilets are at the school.
- Make a guess as to how many students are in the school, and what percentage of students might be using the bathroom at break times between classes, just before the start of school, and just after the end of school. Are there enough toilets for the count of students wishing to use them?
- Using the previous two considerations, estimate the number of students using the bathroom during the peak minute of each break.
- Assuming each flush uses 2.5 gallons of water, estimate the amount of water being used during the peak minute of each break.
- What time of day do these breaks occur? (If the school schedule varies, consider today's schedule.)
- Draw a graph that could represent the water consumption in a typical school day of your school.

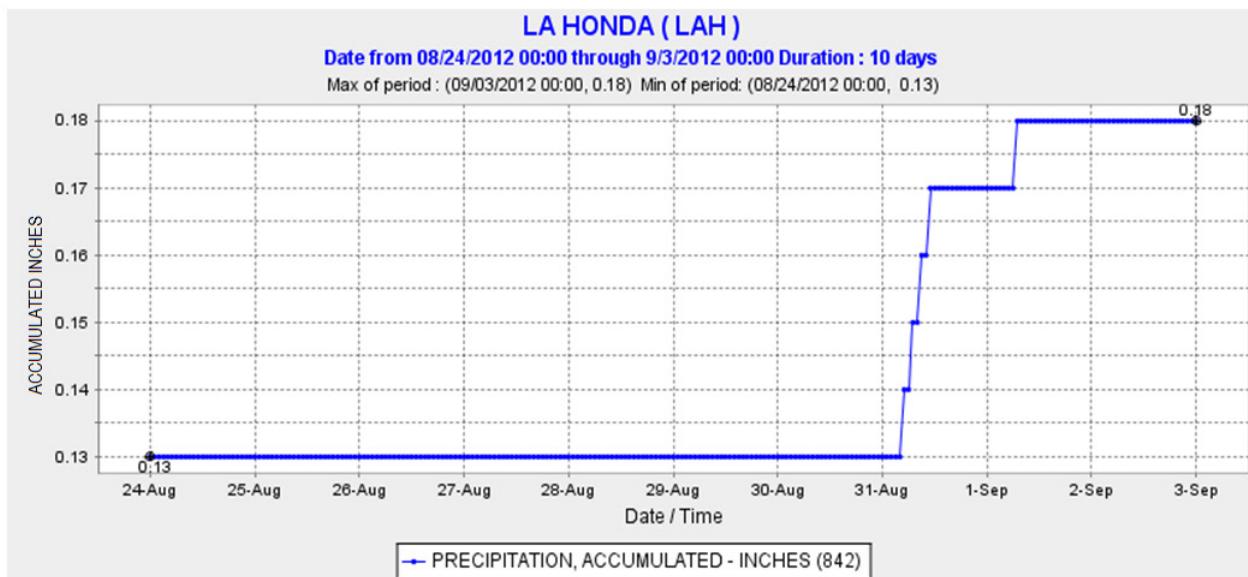
## Problem Set

1. The following graph shows the temperature (in degrees Fahrenheit) of La Honda, CA in the months of August and September of 2012. Answer the questions following the graph.

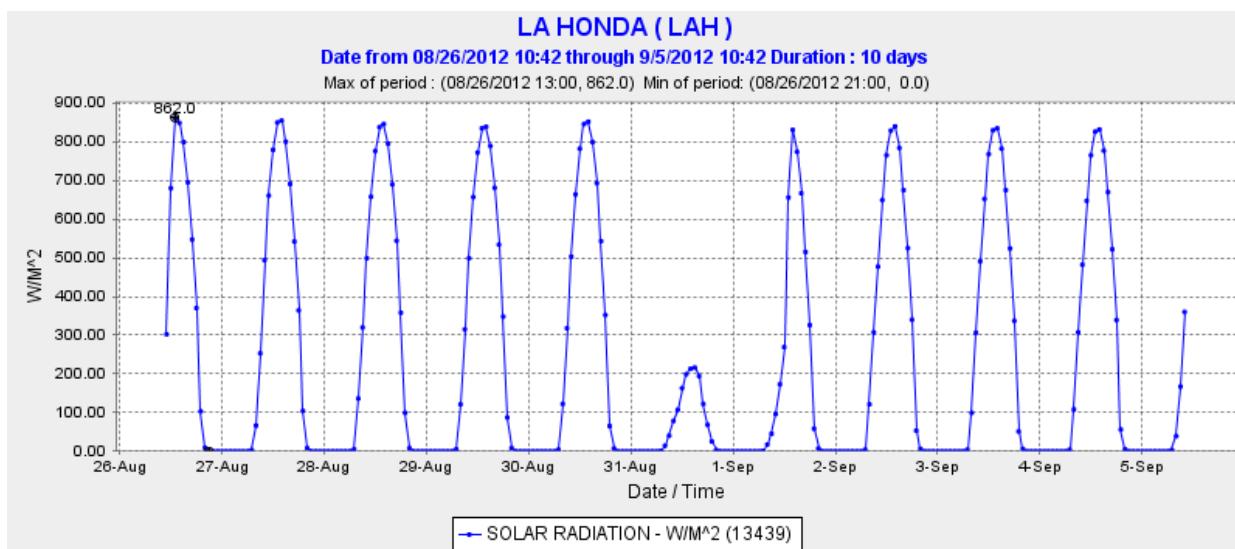


- The graph seems to alternate between peak, valley, peak. Explain why.
- When do you think it should be the warmest during each day? Circle the peak of each day to determine if the graph matches your guess.
- When do you think it should be the coldest during each day? Draw a dot at the lowest point of each day to determine if the graph matches your guess.
- Does the graph do anything unexpected such as not following a pattern? What do you notice? Can you explain why it's happening?

2. The following graph shows the amount of precipitation (rain, snow, or hail) that accumulated over a period of time in La Honda, CA.

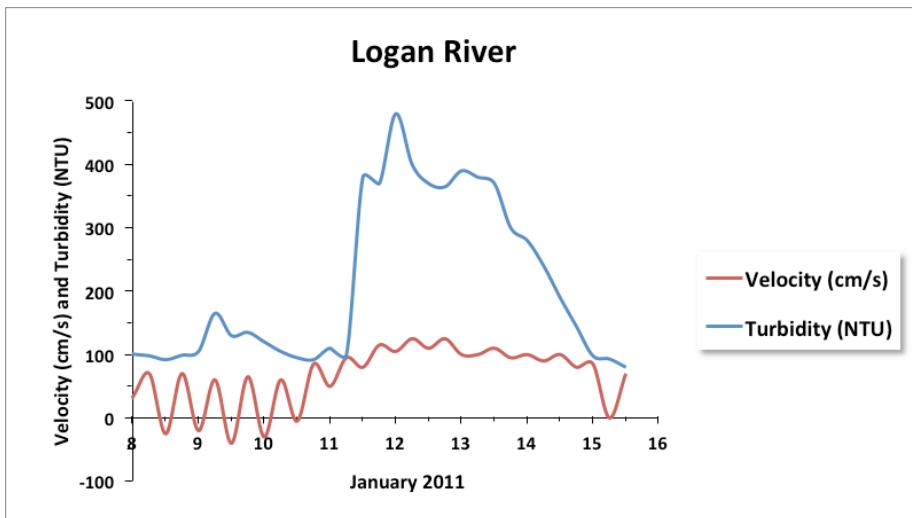


- Tell the complete story of this graph.
  - The term “accumulate” in context of the graph means to add up the amounts of precipitation over time. The graph starts on August 24<sup>th</sup>. Why didn’t the graph start at 0 instead of starting at almost 10 inches?
3. The following graph shows the solar radiation over a period of time in La Honda, CA. Solar radiation is the amount of the sun’s rays that reach the Earth’s surface.



- What happens in La Honda when the graph is flat?
- What do you think is happening when the peaks are very low?
- Looking at all three graphs above, what do you conclude happened on August 31<sup>st</sup>, 2012 in La Honda, CA?

4. The following graph shows the velocity (in centimeters per second) and turbidity of the Logan River in Queensland, Australia during a flood. Turbidity refers to the clarity of the water (higher turbidity means murkier water) and is related to the total amount of suspended solids, such as clay, silt, sand, and phytoplankton, present in the water.



- For recreation, Jill visited the river during the month of January and saw clean and beautiful water. On which day do you think she visited?
- What do the negative velocities (below the grey line) that appear periodically at the beginning represent?
- The behavior of the river seems to follow a normal pattern at the beginning and at the very end of the time period shown. Approximately when does the flood start? Describe its effects on velocity and turbidity.

## Lesson 5: Two Graphing Stories

### Classwork

#### Example 1

Consider the story:

*Maya and Earl live at opposite ends of the hallway in their apartment building. Their doors are 50 feet apart. Each starts at his or her own door and walks at a steady pace toward each other and stop when they meet.*

What would their graphing stories look like if we put them on the *same* graph? When the two people meet in the hallway, what would be happening on the graph? Sketch a graph that shows their distance from Maya's door.

### Exploratory Challenge

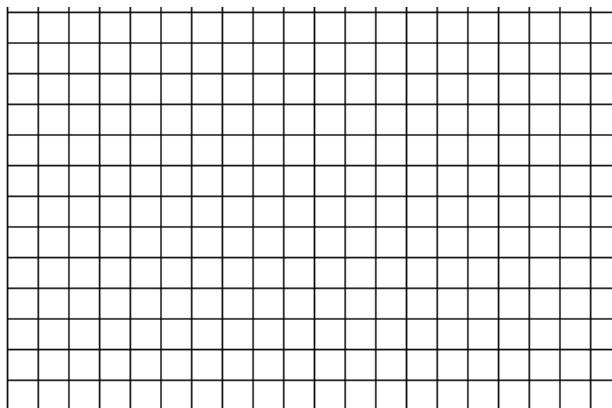
Watch the following graphing story.

<http://youtu.be/X956EvmCevI>

The video shows man and a girl walking on the same stairway.

### Exercises 1–4

- Graph the man's elevation on the stairway versus time in seconds.
- Add the girl's elevation to the same graph. How did you account for the fact that the two people did not start at the same time?



3. Suppose the two graphs intersect at the point  $P(24, 4)$ . What is the meaning of this point in this situation?
  
  
  
  
4. Is it possible for two people, walking in stairwells, to produce the same graphs you have been using and NOT pass each other at time 12 seconds? Explain your reasoning.

### Example 2

Consider the story:

*Duke starts at the base of a ramp and walks up it at a constant rate. His elevation increases by three feet every second. Just as Duke starts walking up the ramp, Shirley starts at the top of the same 25 foot high ramp and begins walking down the ramp at a constant rate. Her elevation decreases two feet every second.*

### Exercises 5–7

5. Sketch two graphs on the same set of elevation-versus-time axes to represent Duke's and Shirley's motions.
  
  
  
  
6. What are the coordinates of the point of intersection of the two graphs? At what time do Duke and Shirley pass each other?
  
  
  
  
7. Write down the equation of the line that represents Duke's motion as he moves up the ramp and the equation of the line that represents Shirley's motion as she moves down the ramp. Show that the coordinates of the point you found in the question above satisfy both equations.

**Lesson Summary**

The **intersection point** of the graphs of two equations is an ordered pair that is a solution to *BOTH* equations. In the context of a distance (or elevation) story, this point represents the fact that both distances (or elevations) are equal at the given time.

Graphing stories with quantities that change at a constant rate can be represented using piecewise linear equations.

**Problem Set**

1. Reread the story about Maya and Earl from Example 1. Suppose that Maya walks at a constant rate of 3 feet every second and Earl walks at a constant rate of 4 feet every second starting from 50 feet away. Create equations for each person's distance from Maya's door and determine exactly when they meet in the hallway. How far are they from Maya's door at this time?
  
2. Consider the story:

*May, June, and July were running at the track. May started first and ran at a steady pace of 1 mile every 11 minutes. June started 5 minutes later than May and ran at a steady pace of 1 mile every 9 minutes. July started 2 minutes after June and ran at a steady pace, running the first lap ( $\frac{1}{4}$  mile) in 1.5 minutes. She maintained this steady pace for 3 more laps and then slowed down to 1 lap every 3 minutes.*

- a. Sketch May, June, and July's distance versus time graphs on a coordinate plane.
- b. Create linear equations that represent each girl's mileage in terms of time in minutes. You will need two equations for July since her pace changes after 4 laps (1 mile).
- c. Who was the first person to run 3 miles?
- d. Did June and July pass May on the track? If they did, when and at what mileage?
- e. Did July pass June on the track? If she did, when and at what mileage?

3. Suppose two cars are travelling north along a road. Car 1 travels at a constant speed of 50 mph for two hours, then speeds up and drives at a constant speed of 100 mph for the next hour. The car breaks down and the driver has to stop and work on it for two hours. When he gets it running again, he continues driving recklessly at a constant speed of 100 mph. Car 2 starts at the same time that Car 1 starts, but Car 2 starts 100 miles farther north than Car 1 and travels at a constant speed of 25 mph throughout the trip.
  - a. Sketch the distance versus time graphs for Car 1 and Car 2 on a coordinate plane. Start with time 0 and measure time in hours.
  - b. Approximately when do the cars pass each other?
  - c. Tell the entire story of the graph from the point of view of Car 2. (What does the driver of Car 2 see along the way and when?)
  - d. Create linear equations representing each car's distance in terms of time (in hours). Note that you will need four equations for Car 1 and only one for Car 2. Use these equations to find the exact coordinates of when the cars meet.
4. Suppose that in Problem 3 above, Car 1 travels at the constant speed of 25 mph the entire time. Sketch the distance versus time graphs for the two cars on the graph below. Do the cars ever pass each other? What is the linear equation for Car 1 in this case?
5. Generate six distinct random whole numbers between 2 and 9 inclusive and fill in the blanks below with the numbers in the order in which they were generated.

$A(0, \underline{\hspace{1cm}})$ ,  $B(\underline{\hspace{1cm}}, \underline{\hspace{1cm}})$ ,  $C(10, \underline{\hspace{1cm}})$

$D(0, \underline{\hspace{1cm}})$ ,  $E(10, \underline{\hspace{1cm}})$ .

(Link to a random number generator [http://www.mathgoodies.com/calculators/random\\_no\\_custom.html](http://www.mathgoodies.com/calculators/random_no_custom.html))

- a. On a coordinate plane, plot points  $A$ ,  $B$ , and  $C$ . Draw line segments from point  $A$  to point  $B$ , and from point  $B$  to point  $C$ .
- b. On the same coordinate plane, plot points  $D$  and  $E$  and draw a line segment from point  $D$  to point  $E$ .
- c. Write a graphing story that describes what is happening in this graph. Include a title,  $x$ - and  $y$ -axis labels, and scales on your graph that correspond to your story.

6. The following graph shows the revenue (or income) a company makes from designer coffee mugs and the total cost (including overhead, maintenance of machines, etc.) that the company spends to make the coffee mugs.



- How are revenue and total cost related to the number of units of coffee mugs produced?
- What is the meaning of the point  $(0, 4000)$  on the total cost line?
- What are the coordinates of the intersection point? What is the meaning of this point in this situation?
- Create linear equations for revenue and total cost in terms of units produced and sold. Verify the coordinates of the intersection point.
- Profit for selling 1,000 units is equal to revenue generated by selling 1,000 units minus the total cost of making 1,000 units. What is the company's profit if 1,000 units are produced and sold?

## Lesson 6: Algebraic Expressions—The Distributive Property

### Classwork

#### Exercise 2

Using the numbers 1, 2, 3, 4 only once and the operations + or  $\times$  as many times as you like, write an expression that evaluates to 16. Use this expression and any combination of those symbols as many times as you like to write an expression that evaluates to 816.

#### Exercise 3

Define the rules of a game as follows:

- Begin by choosing an initial set of symbols, variable or numeric, as a starting set of expressions.
- Generate more expressions by placing any previously created expressions into the blanks of the addition operator: \_\_\_\_ + \_\_\_\_.

#### Exercise 4

Roma says that collecting like terms can be seen as an application of the distributive property. Is writing  $x + x = 2x$  an application of the distributive property?

**Exercise 5**

Leela is convinced that  $(a + b)^2 = a^2 + b^2$ . Do you think she is right? Use a picture to illustrate your reasoning.

**Exercise 6**

Draw a picture to represent the expression  $(a + b + 1) \times (b + 1)$ .

**Exercise 7**

Draw a picture to represent the expression  $(a + b) \times (c + d) \times (e + f + g)$ .

**A Key Belief of Arithmetic**

The Distributive Property: If  $a$ ,  $b$ , and  $c$  are real numbers, then  $a(b + c) = ab + ac$ .

**Lesson Summary**

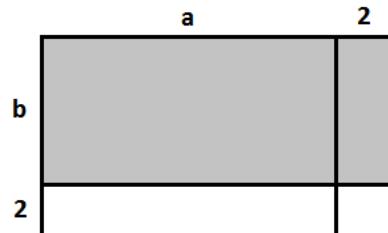
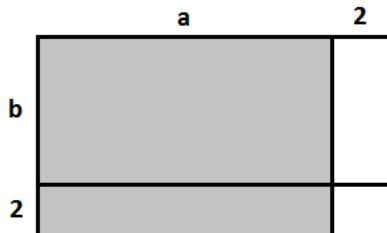
The distributive property represents a key belief about the arithmetic of real numbers. This property can be applied to algebraic expressions using variables that represent real numbers.

**Problem Set**

1. Insert parentheses to make each statement true.
  - a.  $2 + 3 \times 4^2 + 1 = 81$
  - b.  $2 + 3 \times 4^2 + 1 = 85$
  - c.  $2 + 3 \times 4^2 + 1 = 51$
  - d.  $2 + 3 \times 4^2 + 1 = 53$
2. Using starting symbols of  $w$ ,  $q$ , 2, and  $-2$ , which of the following expressions will NOT appear when following the rules of the game played in Exercise 3?
  - a.  $7w + 3q + (-2)$
  - b.  $q - 2$
  - c.  $w - q$
  - d.  $2w + 6$
  - e.  $-2w + 2$
3. Luke wants to play the 4-number game with the numbers 1, 2, 3, and 4 and the operations of addition, multiplication, AND subtraction.  
Leoni responds, “Or we just could play the 4-number game with just the operations of addition and multiplication, but now with the numbers  $-1$ ,  $-2$ ,  $-3$ ,  $-4$ , 1, 2, 3, and 4 instead.”  
What observation is Leoni trying to point out to Luke?
4. Consider the expression:  $(x + 3) \cdot (y + 1) \cdot (x + 2)$ .
  - a. Draw a picture to represent the expression.
  - b. Write an equivalent expression by applying the distributive property.

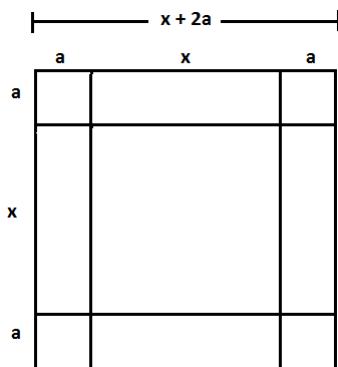
5.

- a. Given that  $a > b$ , which of the shaded regions is larger and why?



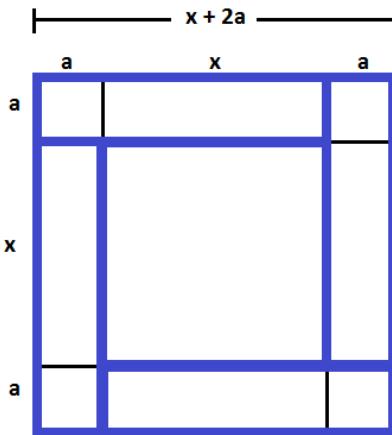
- b. Consider the expressions  $851 \times 29$  and  $849 \times 31$ . Which would result in a larger product? Use a diagram to demonstrate your result

6. Consider the following diagram.



Edna looked at the diagram and then highlighted the four small rectangles shown and concluded:

$$(x + 2a)^2 = x^2 + 4a(x + a).$$



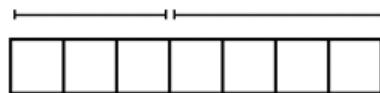
- a. Michael, when he saw the picture, highlighted four rectangles and concluded:  
 $(x + 2a)^2 = x^2 + 2ax + 2a(x + 2a)$ .  
Which four rectangles and one square did he highlight?
- b. Jill, when she saw the picture, highlighted eight rectangles and squares (not including the square in the middle) to conclude:  
 $(x + 2a)^2 = x^2 + 4ax + 4a^2$ .  
Which eight rectangles and squares did she highlight?
- c. When Fatima saw the picture, she exclaimed:  $(x + 2a)^2 = x^2 + 4a(x + 2a) - 4a^2$ . She claims she highlighted just four rectangles to conclude this. Identify the four rectangles she highlighted and explain how using them she arrived at the expression  $x^2 + 4a(x + 2a) - 4a^2$ .
- d. Is each student's technique correct? Explain why or why not.

## Lesson 7: Algebraic Expressions—The Commutative and Associative Properties

### Classwork

#### Exercise 1

Suzy draws the following picture to represent the sum  $3 + 4$ :



Ben looks at this picture from the opposite side of the table and says, "You drew  $4 + 3$ ."

Explain why Ben might interpret the picture this way.

#### Exercise 2

Suzy adds more to her picture and says, "The picture now represents  $(3 + 4) + 2$ ."



How might Ben interpret this picture? Explain your reasoning.

**Exercise 3**

Suzy then draws another picture of squares to represent the product  $3 \times 4$ . Ben moves to the end of the table and says, "From my new seat, your picture looks like the product  $4 \times 3$ ."

What picture might Suzy have drawn? Why would Ben see it differently from his viewpoint?

**Exercise 4**

Draw a picture to represent the quantity  $(3 \times 4) \times 5$  that also could represent the quantity  $(4 \times 5) \times 3$  when seen from a different viewpoint.

**Four Properties of Arithmetic:**

The Commutative Property of Addition: If  $a$  and  $b$  are real numbers, then  $a + b = b + a$ .

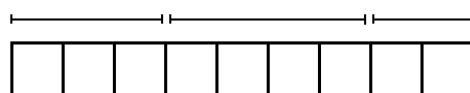
The Associative Property of Addition: If  $a$ ,  $b$ , and  $c$  are real numbers, then  $(a + b) + c = a + (b + c)$ .

The Commutative Property of Multiplication: If  $a$  and  $b$  are real numbers, then  $a \times b = b \times a$ .

The Associative Property of Multiplication: If  $a$ ,  $b$ , and  $c$  are real numbers, then  $(ab)c = a(bc)$ .

**Exercise 5**

Viewing the diagram below from two different perspectives illustrates that  $(3 + 4) + 2$  equals  $2 + (4 + 3)$ .



Is it true for all real numbers  $x$ ,  $y$ , and  $z$  that  $(x + y) + z$  should equal  $(z + y) + x$ ?

(Note: The direct application of the associative property of addition only gives  $(x + y) + z = x + (y + z)$ .)

**Exercise 6**

Draw a flow diagram and use it to prove that  $(xy)z = (zy)x$  for all real numbers  $x$ ,  $y$ , and  $z$ .

**Exercise 7**

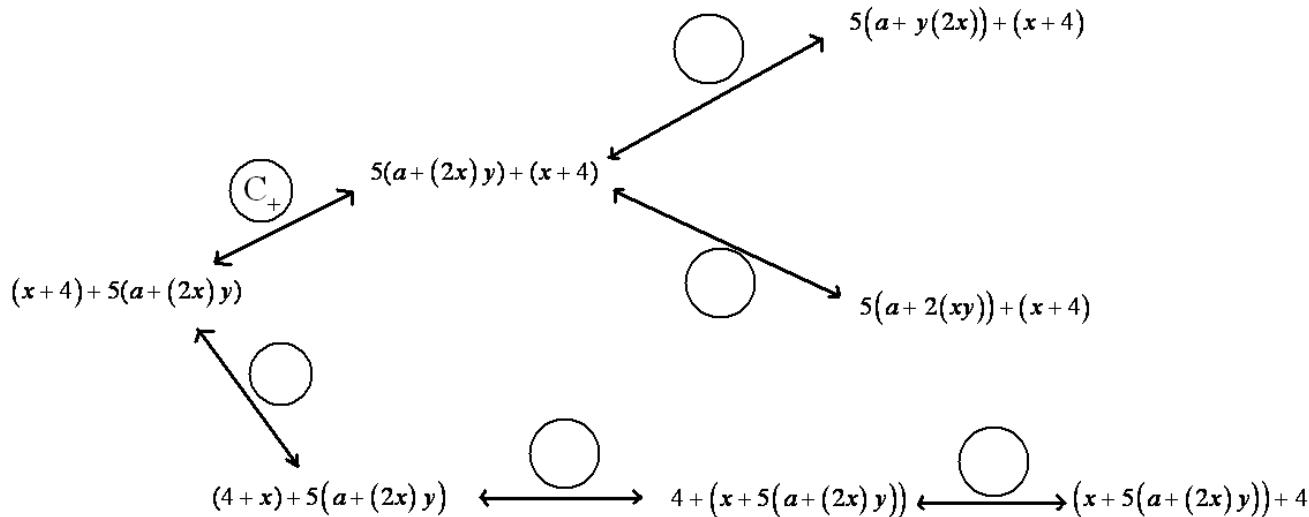
Use these abbreviations for the properties of real numbers and complete the flow diagram.

$C_+$  for the commutative property of addition

$C_x$  for the commutative property of multiplication

$A_+$  for the associative property of addition

$A_x$  for the associative property of multiplication



**Exercise 8**

Let  $a$ ,  $b$ ,  $c$ , and  $d$  be real numbers. Fill in the missing term of the following diagram to show that  $((a + b) + c) + d$  is sure to equal  $a + (b + (c + d))$ .

$$((a + b) + c) + d \xleftarrow{\textcircled{A}} (a + (b + c)) + d \xleftarrow{\textcircled{A}} \boxed{\phantom{000}} \xleftarrow{\textcircled{A}} a + (b + (c + d))$$

**Numerical Symbol:** A *numerical symbol* is a symbol that represents a specific number.

For example,  $0, 1, 2, 3, \frac{2}{3}, -3, -124.122, \pi, e$  are numerical symbols used to represent specific points on the real number line.

**Variable Symbol:** A *variable symbol* is a symbol that is a placeholder for a number.

It is possible that a question may restrict the type of number that a placeholder might permit; e.g., integers only or positive real numbers.

**Algebraic Expression:** An *algebraic expression* is either

1. A numerical symbol or a variable symbol, or
2. The result of placing previously generated algebraic expressions into the two blanks of one of the four operators  $((\underline{\quad}) + (\underline{\quad}), (\underline{\quad}) - (\underline{\quad}), (\underline{\quad}) \times (\underline{\quad}), (\underline{\quad}) \div (\underline{\quad}))$  or into the base blank of an exponentiation with an exponent that is a rational number.

Two algebraic expressions are *equivalent* if we can convert one expression into the other by repeatedly applying the commutative, associative, and distributive properties and the properties of rational exponents to components of the first expression.

**Numerical Expression:** A *numerical expression* is an algebraic expression that contains only numerical symbols (no variable symbols), which evaluate to a single number.

The expression,  $3 \div 0$ , is not a numerical expression.

**Equivalent Numerical Expressions:** Two numerical expressions are *equivalent* if they evaluate to the same number.

Note that  $1 + 2 + 3$  and  $1 \times 2 \times 3$ , for example, are equivalent numerical expressions (they are both 6) but  $a + b + c$  and  $a \times b \times c$  are not equivalent expressions.

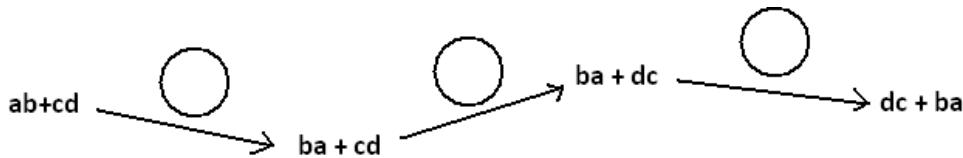
**Lesson Summary**

The commutative and associative properties represent key beliefs about the arithmetic of real numbers. These properties can be applied to algebraic expressions using variables that represent real numbers.

Two algebraic expressions are **equivalent** if we can convert one expression into the other by repeatedly applying the commutative, associative, and distributive properties and the properties of rational exponents to components of the first expression.

**Problem Set**

1. The following portion of a flow diagram shows that the expression  $ab + cd$  is equivalent to the expression  $dc + ba$ .

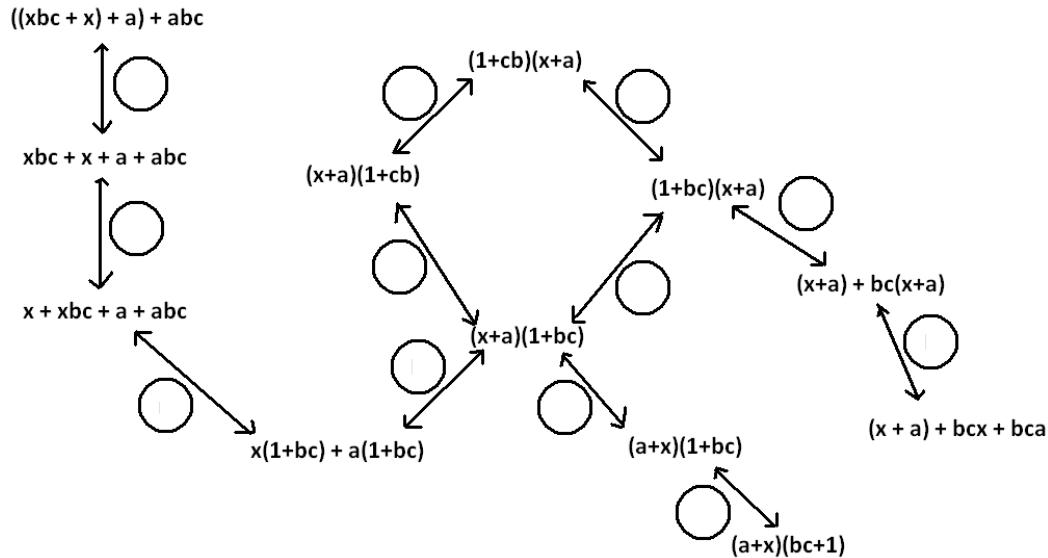


Fill in each circle with the appropriate symbol: Either  $C_+$  (for the “commutative property of addition”) or  $C_\times$  (for the “commutative property of multiplication”).

2. Fill in the blanks of this proof showing that  $(w + 5)(w + 2)$  is equivalent  $w^2 + 7w + 10$ . Write either “commutative property,” “associative property,” or “distributive property” in each blank.

$$\begin{aligned}
 (w + 5)(w + 2) &= (w + 5)w + (w + 5) \times 2 && \text{_____} \\
 &= w(w + 5) + (w + 5) \times 2 && \text{_____} \\
 &= w(w + 5) + 2(w + 5) && \text{_____} \\
 &= w^2 + w \times 5 + 2(w + 5) && \text{_____} \\
 &= w^2 + 5w + 2(w + 5) && \text{_____} \\
 &= w^2 + 5w + 2w + 10 && \text{_____} \\
 &= w^2 + (5w + 2w) + 10 && \text{_____} \\
 &= w^2 + 7w + 10 && \text{_____}
 \end{aligned}$$

3. Fill in each circle of the following flow diagram with one of the letters: C for commutative property (for either addition or multiplication), A for associative property (for either addition or multiplication), or D for distributive property.



4. What is a quick way to see that the value of the sum  $53 + 18 + 47 + 82$  is 200?
- 5.
- If  $ab = 37$  and  $xy = \frac{1}{37}$ , what is the value of the product  $x \times b \times y \times a$ ?
  - Give some indication as to how you used the commutative and associative properties of multiplication to evaluate  $x \times b \times y \times a$  in part (a).
  - Did you use the associative and commutative properties of addition to answer Question 4?
6. The following is a proof of the algebraic equivalency of  $(2x)^3$  and  $8x^3$ . Fill in each of the blanks with either the statement “commutative property” or “associative property.”

$$\begin{aligned}
 (2x)^3 &= 2x \cdot 2x \cdot 2x \\
 &= 2(x \times 2)(x \times 2)x && \text{_____} \\
 &= 2(2x)(2x)x && \text{_____} \\
 &= 2 \cdot 2(x \times 2)x \cdot x && \text{_____} \\
 &= 2 \cdot 2(2x)x \cdot x && \text{_____} \\
 &= (2 \cdot 2 \cdot 2)(x \cdot x \cdot x) && \text{_____} \\
 &= 8x^3
 \end{aligned}$$

7. Write a mathematical proof of the algebraic equivalency of  $(ab)^2$  and  $a^2b^2$ .
- 8.
- Suppose we are to play the 4-number game with the symbols  $a$ ,  $b$ ,  $c$ , and  $d$  to represent numbers, each used at most once, combined by the operation of addition ONLY. If we acknowledge that parentheses are unneeded, show there are essentially only 15 expressions one can write.
  - How many answers are there for the multiplication ONLY version of this game?
9. Write a mathematical proof to show that  $(x + a)(x + b)$  is equivalent to  $x^2 + ax + bx + ab$ .
10. Recall the following rules of exponents:

$$\begin{aligned} x^a \cdot x^b &= x^{a+b} & \frac{x^a}{x^b} &= x^{a-b} & (x^a)^b &= x^{ab} \\ (xy)^a &= x^a y^a & \left(\frac{x}{y}\right)^a &= \frac{x^a}{y^a} \end{aligned}$$

Here  $x$ ,  $y$ ,  $a$ , and  $b$  are real numbers with  $x$  and  $y$  non-zero.

Replace each of the following expressions with an equivalent expression in which the variable of the expression appears only once with a positive number for its exponent. (For example,  $\frac{7}{b^2} \cdot b^{-4}$  is equivalent to  $\frac{7}{b^6}$ .)

- $(16x^2) \div (16x^5)$
- $(2x)^4(2x)^3$
- $(9z^{-2})(3z^{-1})^{-3}$
- $((25w^4) \div (5w^3)) \div (5w^{-7})$
- $(25w^4) \div ((5w^3) \div (5w^{-7}))$

#### Optional Challenge:

11. Grizelda has invented a new operation that she calls the “average operator.” For any two real numbers  $a$  and  $b$ , she declares  $a \oplus b$  to be the average of  $a$  and  $b$ :

$$a \oplus b = \frac{a + b}{2}$$

- Does the average operator satisfy a commutative property? That is, does  $a \oplus b = b \oplus a$  for all real numbers  $a$  and  $b$ ?
- Does the average operator distribute over addition? That is, does  $a \oplus (b + c) = (a \oplus b) + (a \oplus c)$  for all real numbers  $a$ ,  $b$ , and  $c$ ?

## Lesson 8: Adding and Subtracting Polynomials

### Classwork

#### Exercise 1

- a. How many quarters, nickels, and pennies are needed to make \$1.13?

- b. Fill in the blanks:

$$8,943 = \underline{\hspace{2cm}} \times 1000 + \underline{\hspace{2cm}} \times 100 + \underline{\hspace{2cm}} \times 10 + \underline{\hspace{2cm}} \times 1$$

$$= \underline{\hspace{2cm}} \times 10^3 + \underline{\hspace{2cm}} \times 10^2 + \underline{\hspace{2cm}} \times 10 + \underline{\hspace{2cm}} \times 1$$

- c. Fill in the blanks:

$$8,943 = \underline{\hspace{2cm}} \times 20^3 + \underline{\hspace{2cm}} \times 20^2 + \underline{\hspace{2cm}} \times 20 + \underline{\hspace{2cm}} \times 1$$

- d. Fill in the blanks:

$$113 = \underline{\hspace{2cm}} \times 5^2 + \underline{\hspace{2cm}} \times 5 + \underline{\hspace{2cm}} \times 1$$

#### Exercise 2

Now let's be as general as possible by not identifying which base we are in. Just call the base  $x$ .

Consider the expression  $1 \times x^3 + 2 \times x^2 + 7 \times x + 3 \times 1$ , or equivalently  $x^3 + 2x^2 + 7x + 3$ .

- a. What is the value of this expression if  $x = 10$ ?

- b. What is the value of this expression if  $x = 20$ ?

**Exercise 3**

a. When writing numbers in base 10, we only allow coefficients of 0 through 9. Why is that?

b. What is the value of  $22x + 3$  when  $x = 5$ ? How much money is 22 nickels and 3 pennies?

c. What number is represented by  $4x^2 + 17x + 2$  if  $x = 10$ ?

d. What number is represented by  $4x^2 + 17x + 2$  if  $x = -2$  or if  $x = \frac{2}{3}$ ?

e. What number is represented by  $-3x^2 + \sqrt{2}x + \frac{1}{2}$  when  $x = \sqrt{2}$ ?

**Polynomial Expression:** A polynomial expression is either

1. A numerical expression or a variable symbol, or
2. The result of placing two previously generated polynomial expressions into the blanks of the addition operator ( $\underline{\quad} + \underline{\quad}$ ) or the multiplication operator ( $\underline{\quad} \times \underline{\quad}$ ).

**Exercise 4**

Find each sum or difference by combining the parts that are alike.

a.  $417 + 231 = \underline{\hspace{2cm}} \text{ hundreds} + \underline{\hspace{2cm}} \text{ tens} + \underline{\hspace{2cm}} \text{ ones} + \underline{\hspace{2cm}} \text{ hundreds} + \underline{\hspace{2cm}} \text{ tens} + \underline{\hspace{2cm}} \text{ ones}$

$= \underline{\hspace{2cm}} \text{ hundreds} + \underline{\hspace{2cm}} \text{ tens} + \underline{\hspace{2cm}} \text{ ones}$

b.  $(4x^2 + x + 7) + (2x^2 + 3x + 1)$

c.  $(3x^3 - x^2 + 8) - (x^3 + 5x^2 + 4x - 7)$

d.  $(3x^3 + 8x) - 2(x^3 + 12)$

e.  $(5 - t - t^2) + (9t + t^2)$

f.  $(3p + 1) + 6(p - 8) - (p + 2)$

**Lesson Summary**

A **monomial** is a polynomial expression generated using only the multiplication operator ( $\times$ ). Thus, it does not contain + or – operators. Monomials are written with numerical factors multiplied together and variable or other symbols each occurring one time (using exponents to condense multiple instances of the same variable).

A **polynomial** is the sum (or difference) of monomials.

The **degree of a monomial** is the sum of the exponents of the variable symbols that appear in the monomial.

The **degree of a polynomial** is the degree of the monomial term with the highest degree.

**Problem Set**

1. Celina says that each of the following expressions is actually a binomial in disguise:

- i.  $5abc - 2a^2 + 6abc$
- ii.  $5x^3 \cdot 2x^2 - 10x^4 + 3x^5 + 3x \cdot (-2)x^4$
- iii.  $(t+2)^2 - 4t$
- iv.  $5(a-1) - 10(a-1) + 100(a-1)$
- v.  $(2\pi r - \pi r^2)r - (2\pi r - \pi r^2) \cdot 2r$

For example, she sees that the expression in (i) is algebraically equivalent to  $11abc - 2a^2$ , which is indeed a binomial. (She is happy to write this as  $11abc + (-2)a^2$ , if you prefer.)

Is she right about the remaining four expressions?

2. Janie writes a polynomial expression using only one variable,  $x$ , with degree 3. Max writes a polynomial expression using only one variable,  $x$ , with degree 7.
- What can you determine about the degree of the sum of Janie's and Max's polynomials?
  - What can you determine about the degree of the difference of Janie's and Max's polynomials?
3. Suppose Janie writes a polynomial expression using only one variable,  $x$ , with degree of 5, and Max writes a polynomial expression using only one variable,  $x$ , with degree of 5.
- What can you determine about the degree of the sum of Janie's and Max's polynomials?
  - What can you determine about the degree of the difference of Janie's and Max's polynomials?
4. Find each sum or difference by combining the parts that are alike.
- |  |   |
|--|---|
| a. $(2p + 4) + 5(p - 1) - (p + 7)$       | f. $(12x + 1) + 2(x - 4) - (x - 15)$        |
| b. $(7x^4 + 9x) - 2(x^4 + 13)$           | g. $(13x^2 + 5x) - 2(x^2 + 1)$              |
| c. $(6 - t - t^4) + (9t + t^4)$          | h. $(9 - t - t^2) - \frac{3}{2}(8t + 2t^2)$ |
| d. $(5 - t^2) + 6(t^2 - 8) - (t^2 + 12)$ | i. $(4m + 6) - 12(m - 3) + (m + 2)$         |
| e. $(8x^3 + 5x) - 3(x^3 + 2)$            | j. $(15x^4 + 10x) - 12(x^4 + 4x)$           |

## Lesson 9: Multiplying Polynomials

### Classwork

#### Exercise 1

- a. Gisella computed  $342 \times 23$  as follows:

300	40	2	
6000	800	40	20
900	120	6	3
6000	1700	160	6

Can you explain what she is doing? What is her final answer?

Use a geometric diagram to compute the following products:

b.  $(3x^2 + 4x + 2) \times (2x + 3)$

c.  $(2x^2 + 10x + 1)(x^2 + x + 1)$

d.  $(x - 1)(x^3 + 6x^2 - 5)$

**Exercise 2**

Multiply the polynomials using the distributive property:  $(3x^2 + x - 1)(x^4 - 2x + 1)$ .

**Exercise 3**

The expression  $10x^2 + 6x^3$  is the result of applying the distributive property to the expression  $2x^2(5 + 3x)$ . It is also the result of the applying the distributive property to  $2(5x^2 + 3x^3)$  or to  $x(10x + 6x^2)$ , for example, or even to  $1 \cdot (10x^2 + 6x^3)$ !

For (i) to (x) below, write down an expression such that if you applied the distributive property to your expression it will give the result presented. Give interesting answers!

i.  $6a + 14a^2$

ii.  $2x^4 + 2x^5 + 2x^{10}$

iii.  $6z^2 - 15z$

iv.  $42w^3 - 14w + 77w^5$

v.  $z^2(a + b) + z^3(a + b)$

vi.  $\frac{3}{2}s^2 + \frac{1}{2}$

vii.  $15p^3r^4 - 6p^2r^5 + 9p^4r^2 + 3\sqrt{2}p^3r^6$

viii.  $0.4x^9 - 40x^8$

ix.  $(4x + 3)(x^2 + x^3) - (2x + 2)(x^2 + x^3)$

x.  $(2z + 5)(z - 2) - (13z - 26)(z - 3)$

**Exercise 4**

Sammy wrote a polynomial using only one variable,  $x$ , of degree 3. Myisha wrote a polynomial in the same variable of degree 5. What can you say about the degree of the product of Sammy's and Myisha's polynomials?

**Extension**

Find a polynomial that, when multiplied by  $2x^2 + 3x + 1$ , gives the answer  $2x^3 + x^2 - 2x - 1$ .

## Problem Set

- Use the distributive property to write each of the following expressions as the sum of monomials.
  - $3a(4 + a)$
  - $x(x + 2) + 1$
  - $\frac{1}{3}(12z + 18z^2)$
  - $4x(x^3 - 10)$
  - $(x - 4)(x + 5)$
  - $(2z - 1)(3z^2 + 1)$
  - $(10w - 1)(10w + 1)$
  - $(-5w - 3)w^2$
  - $16s^{100} \left(\frac{1}{2}s^{200} + 0.125s\right)$
  - $(2q + 1)(2q^2 + 1)$
  - $(x^2 - x + 1)(x - 1)$
  - $3xz(9xy + z) - 2yz(x + y - z)$
  - $(t - 1)(t + 1)(t^2 + 1)$
  - $(w + 1)(w^4 - w^3 + w^2 - w + 1)$
  - $z(2z + 1)(3z - 2)$
  - $(x + y)(y + z)(z + x)$
  - $\frac{x+y}{3}$
  - $(20f^{10} - 10f^5) \div 5$
  - $-5y(y^2 + y - 2) - 2(2 - y^3)$
  - $\frac{(a+b-c)(a+b+c)}{17}$
  - $(2x \div 9 + (5x) \div 2) \div (-2)$
  - $(-2f^3 - 2f + 1)(f^2 - f + 2)$
- Use the distributive property (and your wits!) to write each of the following expressions as a sum of monomials. If the resulting polynomial is in one variable, write the polynomial in standard form.
  - $(a + b)^2$
  - $(a + 1)^2$
  - $(3 + b)^2$
  - $(3 + 1)^2$
  - $(x + y + z)^2$
  - $(x + 1 + z)^2$
  - $(3 + z)^2$
  - $(p + q)^3$
  - $(p - 1)^3$
  - $(5 + q)^3$
- Use the distributive property (and your wits!) to write each of the following expressions as a polynomial in standard form.
  - $(s^2 + 4)(s - 1)$
  - $3(s^2 + 4)(s - 1)$
  - $s(s^2 + 4)(s - 1)$
  - $(s + 1)(s^2 + 4)(s - 1)$
  - $(u - 1)(u^5 + u^4 + u^3 + u^2 + u + 1)$
  - $\sqrt{5}(u - 1)(u^5 + u^4 + u^3 + u^2 + u + 1)$
  - $(u^7 + u^3 + 1)(u - 1)(u^5 + u^4 + u^3 + u^2 + u + 1)$
- Beatrice writes down every expression that appears in this problem set, one after the other, linking them with “+” signs between them. She is left with one very large expression on her page. Is that expression a polynomial expression? That is, is it algebraically equivalent to a polynomial?

What if she wrote “–” signs between the expressions instead?

What if she wrote “×” signs between the expressions instead?

## Lesson 10: True and False Equations

### Classwork

#### Exercise 1

- a. Consider the statement: "The President of the United States is a United States citizen."  
Is the statement a grammatically correct sentence?  
What is the subject of the sentence? What is the verb in the sentence?  
Is the sentence true?
  
- b. Consider the statement: "The President of France is a United States citizen."  
Is the statement a grammatically correct sentence?  
What is the subject of the sentence? What is the verb in the sentence?  
Is the sentence true?
  
- c. Consider the statement: " $2 + 3 = 1 + 4$ ."  
This is a sentence. What is the verb of the sentence? What is the subject of the sentence?  
Is the sentence true?
  
- d. Consider the statement: " $2 + 3 = 9 + 4$ ."  
Is this statement a sentence? And if so, is the sentence true or false?

A **number sentence** is a statement of equality between two numerical expressions.

A *number sentence* is said to be *true* if both numerical expressions are equivalent (that is, both evaluate to the same number). It is said to be *false* otherwise. True and false are called *truth values*.

**Exercise 2**

Determine whether the following number sentences are TRUE or FALSE.

a.  $4 + 8 = 10 + 5$

b.  $\frac{1}{2} + \frac{5}{8} = 1.2 - 0.075$

c.  $(71 \cdot 603) \cdot 5876 = 603 \cdot (5876 \cdot 71)$

d.  $13 \times 175 = 13 \times 90 + 85 \times 13$

e.  $(7 + 9)^2 = 7^2 + 9^2$

f.  $\pi = 3.141$

g.  $\sqrt{(4 + 9)} = \sqrt{4} + \sqrt{9}$

h.  $\frac{1}{2} + \frac{1}{3} = \frac{2}{5}$

i.  $\frac{1}{2} + \frac{1}{3} = \frac{2}{6}$

j.  $\frac{1}{2} + \frac{1}{3} = \frac{5}{6}$

k.  $3^2 + 4^2 = 7^2$

l.  $3^2 \times 4^2 = 12^2$

m.  $3^2 \times 4^3 = 12^6$

n.  $3^2 \times 3^3 = 3^5$

**Exercise 3**

- a. Could a number sentence be both TRUE and FALSE?
  
- b. Could a number sentence be neither TRUE nor FALSE?

An **algebraic equation** is a statement of equality between two expressions.

*Algebraic equations* can be number sentences (when both expressions are numerical), but often they contain symbols whose values have not been determined.

**Exercise 4**

- a. Which of the following are algebraic equations?
  - i.  $3.1x - 11.2 = 2.5x + 2.3$
  - ii.  $10\pi^4 + 3 = 99\pi^2$
  - iii.  $\pi + \pi = 2\pi$
  - iv.  $\frac{1}{2} + \frac{1}{2} = \frac{2}{4}$
  - v.  $79\pi^3 + 70\pi^2 - 56\pi + 87 = \frac{60\pi + 29,928}{\pi^2}$
  
- b. Which of them are also number sentences?
  
  
- c. For each number sentence, state whether the number sentence is true or false.

**Exercises 5**

When algebraic equations contain a symbol whose value has not yet been determined, we use analysis to determine whether:

1. The equation is true for all the possible values of the variable(s), or
2. The equation is true for a certain set of the possible value(s) of the variable(s), or
3. The equation is never true for any of the possible values of the variable(s).

For each of the three cases, write an algebraic equation that would be correctly described by that case. Use only the variable,  $x$ , where  $x$  represents a real number.

**Example 1**

Consider the following scenario.

Julie is 300 feet away from her friend's front porch and observes, "Someone is sitting on the porch."

Given that she didn't specify otherwise, we would assume that the "someone" Julie thinks she sees is a human. We can't guarantee that Julie's observatory statement is true. It could be that Julie's friend has something on the porch that merely looks like a human from far away. Julie assumes she is correct and moves closer to see if she can figure out who it is. As she nears the porch she declares, "Ah, it is our friend, John Berry."

**Exercise 6**

Name a value of the variable that would make each equation a true number sentence.

Here are several examples of how we can name the value of the variable:

Let  $w = -2$ . Then  $w^2 = 4$  is true.

or

$w^2 = 4$  is true when  $w = -2$

or

$w^2 = 4$  is true if  $w = -2$

or

$w^2 = 4$  is true for  $w = -2$  and  $w = 2$ .

There might be more than one option for what numerical values to write. (And feel free to write more than one possibility.)

Warning: Some of these are tricky. Keep your wits about you!

- a. Let \_\_\_\_\_ . Then  $7 + x = 12$  is true.
  
- b. Let \_\_\_\_\_ . Then  $3r + 0.5 = \frac{37}{2}$  is true.
  
- c.  $m^3 = -125$  is true for \_\_\_\_\_.
  
- d. A number  $x$  and its square,  $x^2$ , have the same value when \_\_\_\_\_.
  
- e. The average of 7 and  $n$  is  $-8$  if \_\_\_\_\_.
  
- f. Let \_\_\_\_\_ . Then  $2a = a + a$  is true.
  
- g.  $q + 67 = q + 68$  is true for \_\_\_\_\_.

**Problem Set**

Determine whether the following number sentences are true or false.

1.  $18 + 7 = \frac{50}{2}$       2.  $3.123 = 9.369 \cdot \frac{1}{3}$       3.  $(123 + 54) \cdot 4 = 123 + (54 \cdot 4)$

4.  $5^2 + 12^2 = 13^2$       5.  $(2 \times 2)^2 = \sqrt{256}$       6.  $\frac{4}{3} = 1.333$

In the following equations, let  $x = -3$  and  $y = \frac{2}{3}$ . Determine whether the following equations are true, false, or neither true nor false.

7.  $xy = -2$       8.  $x + 3y = -1$       9.  $x + z = 4$

10.  $9y = -2x$       11.  $\frac{y}{x} = -2$       12.  $\frac{\frac{2}{x}}{y} = -1$

For each of the following, assign a value to the variable,  $x$ , to make the equation a true statement.

13.  $(x^2 + 5)(3 + x^4)(100x^2 - 10)(100x^2 + 10) = 0$  for \_\_\_\_\_.

14.  $\sqrt{(x+1)(x+2)} = \sqrt{20}$  for \_\_\_\_\_.

15.  $(d+5)^2 = 36$  for \_\_\_\_\_.

16.  $(2z+2)(z^5 - 3) + 6 = 0$  for \_\_\_\_\_.

17.  $\frac{1+x}{1+x^2} = \frac{3}{5}$  for \_\_\_\_\_.

18.  $\frac{1+x}{1+x^2} = \frac{2}{5}$  for \_\_\_\_\_.

19. The diagonal of a square of side length  $L$  is 2 inches long when \_\_\_\_\_.

20.  $(T - \sqrt{3})^2 = T^2 + 3$  for \_\_\_\_\_.

21.  $\frac{1}{x} = \frac{x}{1}$  if \_\_\_\_\_.

22.  $\left(2 + \left(2 - \left(2 + \left(2 - (2 + r)\right)\right)\right)\right) = 1$  for \_\_\_\_\_.

23.  $x + 2 = 9$

24.  $x + 2^2 = -9$

25.  $-12t = 12$

26.  $12t = 24$

27.  $\frac{1}{b-2} = \frac{1}{4}$

28.  $\frac{1}{2b-2} = -\frac{1}{4}$

29.  $\sqrt{x} + \sqrt{5} = \sqrt{x+5}$

30.  $(x-3)^2 = x^2 + (-3)^2$

31.  $x^2 = -49$

32.  $\frac{2}{3} + \frac{1}{5} = \frac{3}{x}$

Fill in the blank with a variable term so that the given value of the variable will make the equation true.

33. \_\_\_\_\_ + 4 = 12;  $x = 8$

34. \_\_\_\_\_ + 4 = 12;  $x = 4$

Fill in the blank with a constant term so that the given value of the variable will make the equation true.

35.  $4y - \underline{\hspace{1cm}} = 100$ ;  $y = 25$

36.  $4y - \underline{\hspace{1cm}} = 0$ ;  $y = 6$

37.  $r + \underline{\hspace{1cm}} = r$ ;  $r$  is any real number

38.  $r \times \underline{\hspace{1cm}} = r$ ;  $r$  is any real number

Generate the following:

39. An equation that is always true

40. An equation that is true when  $x = 0$

41. An equation that is never true

42. An equation that is true when  $t = 1$  or  $t = -1$

43. An equation that is true when  $y = -0.5$

44. An equation that is true when  $z = \pi$

## Lesson 11: Solution Sets For Equations and Inequalities

### Classwork

#### Example 1

Consider the equation,  $x^2 = 3x + 4$ , where  $x$  represents a real number.

- Are the expressions  $x^2$  and  $3x + 4$  algebraically equivalent?
- The following table shows how we might “sift” through various values to assign to the variable symbol  $x$  in the hunt for values that would make the equation true.

$x$ -VALUE	THE EQUATION	TRUTH VALUE
Let $x = 0$	$0^2 = 3(0) + 4$	FALSE
Let $x = 5$	$5^2 = 3(5) + 4$	FALSE
Let $x = 6$	$6^2 = 3(6) + 4$	FALSE
Let $x = -7$	$(-7)^2 = 3(-7) + 4$	FALSE
Let $x = 4$	$4^2 = 3(4) + 4$	TRUE
Let $x = 9$	$9^2 = 3(9) + 4$	FALSE
Let $x = 10$	$10^2 = 3(10) + 4$	FALSE
Let $x = -8$	$(-8)^2 = 3(-8) + 4$	FALSE

#### Example 2

Consider the equation  $7 + p = 12$ .

	THE NUMBER SENTENCE	TRUTH VALUE
Let $p = 0$	$7 + 0 = 12$	FALSE
Let $p = 4$		
Let $p = 1 + \sqrt{2}$		
Let $p = \frac{1}{\pi}$		
Let $p = 5$		

The **solution set** of an equation written with only one variable is the set of all values one can assign to that variable to make the equation a true statement. Any one of those values is said to be a *solution to the equation*.

To *solve an equation* means to *find the solution set* for that equation.

### Example 3

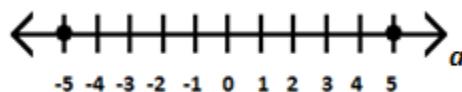
Solve for  $a$ :  $a^2 = 25$ .

One can describe a solution set in any of the following ways:

**IN WORDS:**  $a^2 = 25$  has solutions 5 and  $-5$ . (That is,  $a^2 = 25$  is true when  $a = 5$  or  $a = -5$ .)

**IN SET NOTATION:** The solution set of  $a^2 = 25$  is  $\{-5, 5\}$ .

**IN A GRAPHICAL REPRESENTATION ON A NUMBER LINE:** The solution set of  $a^2 = 25$  is



In this graphical representation, a solid dot is used to indicate a point on the number line that is to be included in the solution set. (WARNING: The dot one physically draws is larger than the point it represents. One hopes that it is clear from the context of the diagram which point each dot refers to.)

How set notation works.

- The curly brackets { } indicate we are denoting a set. A set is essentially a collection of things, e.g., letters, numbers, cars, people. In this case, the things are numbers.
- From this example, the numbers  $-5$  and  $5$  are called elements of the set. No other elements belong in this particular set because no other numbers make the equation  $a^2 = 25$  true.
- When elements are listed, they are listed in increasing order.
- Sometimes, a set is empty; it has no elements. In which case, the set looks like { }. We often denote this with the symbol,  $\emptyset$ . We refer to this as *the empty set* or *the null set*.

**Exercise 1**

Solve for  $a$ :  $a^2 = -25$ . Present the solution set in words, in set notation, and graphically.

**Exercise 2**

Depict the solution set of  $7 + p = 12$  in words, in set notation, and graphically.

**Example 4**

Solve  $\frac{x}{x} = 1$  for  $x$ , over the set of positive real numbers. Depict the solution set in words, in set notation, and graphically.

X-VALUE	THE EQUATION	TRUTH VALUE
Let $x = 2$	$\frac{2}{2} = 1$	TRUE
Let $x = 7$	$\frac{7}{7} = 1$	TRUE
Let $x = 0.01$	$\frac{0.01}{0.01} = 1$	TRUE
Let $x = 562\frac{2}{3}$	$\frac{562\frac{2}{3}}{562\frac{2}{3}} = 1$	TRUE
Let $x = 10^{100}$	$\frac{10^{100}}{10^{100}} = 1$	TRUE
Let $x = \pi$	$\frac{\pi}{\pi} = 1$	TRUE

**Exercise 3**

Solve  $\frac{x}{x} = 1$  for  $x$ , over the set of all non-zero real numbers. Describe the solution set in words, in set notation, and graphically.

**Example 5**

Solve for  $x$ :  $x(3 + x) = 3x + x^2$ .

**Exercise 4**

Solve for  $\alpha$ :  $\alpha + \alpha^2 = \alpha(\alpha + 1)$ . Describe carefully the reasoning that justifies your solution. Describe the solution set in words, in set notation, and graphically.

An **identity** is an equation that is always true.

### Exercise 5

Identify the properties of arithmetic that justify why each of the following equations has a solution set of all real numbers:

a.  $2x^2 + 4x = 2(x^2 + 2x)$

b.  $2x^2 + 4x = 4x + 2x^2$

c.  $2x^2 + 4x = 2x(2 + x)$

### Exercise 6

Create an expression for the right side of each equation such that the solution set for the equation will be all real numbers. (There is more than one possibility for each expression. Feel free to write several answers for each one.)

a.  $2x - 5 = \underline{\hspace{2cm}}$

b.  $x^2 + x = \underline{\hspace{2cm}}$

c.  $4 \cdot x \cdot y \cdot z = \underline{\hspace{2cm}}$

d.  $(x + 2)^2 = \underline{\hspace{2cm}}$

**Example 6**

Solve for  $w$ ,  $w + 2 > 4$ .

**Exercise 7**

- a. Solve for  $B$ :  $B^2 \geq 9$ . Describe the solution set using a number line.
- b. What is the solution set to the statement: “Sticks of lengths 2 yards, 2 yards, and  $L$  yards make an isosceles triangle”? Describe the solution set in words and on a number line.

**Lesson Summary**

The **solution set** of an equation written with only one variable symbol is the set of all values one can assign to that variable to make the equation a true number sentence. Any one of those values is said to be a *solution to the equation*.

To *solve an equation* means to *find the solution set* for that equation.

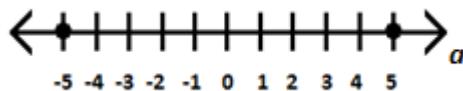
One can describe a solution set in any of the following ways:

**IN WORDS:**  $a^2 = 25$  has solutions 5 and  $-5$ . (That is,  $a^2 = 25$  is true when  $a = 5$  or  $a = -5$ .)

**IN SET NOTATION:** The solution set of  $a^2 = 25$  is  $\{-5, 5\}$ .

It is awkward to express the set of infinitely many numbers in set notation. In these cases we can use the notation: {variable symbol number type | a description}. For example  $\{x \text{ real} \mid x > 0\}$  reads, “ $x$  is a real number where  $x$  is greater than zero.” The symbol  $\mathbb{R}$  can be used to indicate all real numbers.

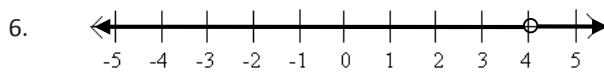
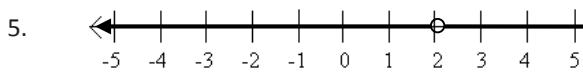
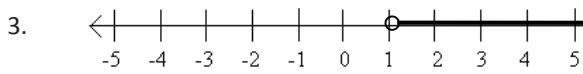
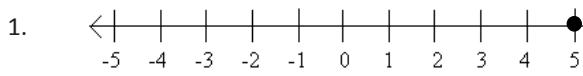
**IN A GRAPHICAL REPRESENTATION ON A NUMBER LINE:** The solution set of  $a^2 = 25$  is as follows:

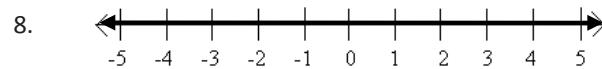
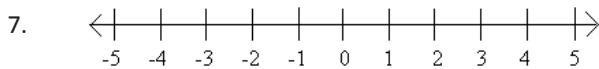


In this graphical representation, a solid dot is used to indicate a point on the number line that is to be included in the solution set. (WARNING: The dot one physically draws is larger than the point it represents! One hopes that it is clear from the context of the diagram which point each dot refers to.)

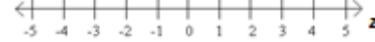
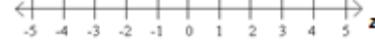
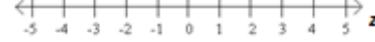
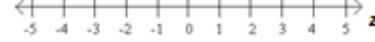
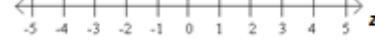
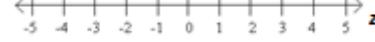
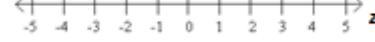
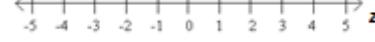
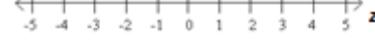
**Problem Set**

For each solution set graphed below, (a) describe the solution set in words, (b) describe the solution set in set notation, and (c) write an equation or an inequality that has the given solution set.





Fill in the chart below.

	SOLUTION SET IN WORDS	SOLUTION SET IN SET NOTATION	GRAPH
9. $z = 2$			
10. $z^2 = 4$			
11. $4z \neq 2$			
12. $z - 3 = 2$			
13. $z^2 + 1 = 2$			
14. $z = 2z$			
15. $z > 2$			
16. $z - 6 = z - 2$			
17. $z - 6 < -2$			
18. $4(z - 1) > 4z - 4$			

For Problems 19–24, answer the following: Are the two expressions algebraically equivalent? If so, state the property (or properties) displayed. If not, state why (the solution set may suffice as a reason) and change the equation, ever so slightly, e.g., touch it up, to create an equation whose solution set is all real numbers.

19.  $x(4 - x^2) = (-x^2 + 4)x$

20.  $\frac{2x}{2x} = 1$

21.  $(x - 1)(x + 2) + (x - 1)(x - 5) = (x - 1)(2x - 3)$

22.  $\frac{x}{5} + \frac{x}{3} = \frac{2x}{8}$

23.  $x^2 + 2x^3 + 3x^4 = 6x^9$

24.  $x^3 + 4x^2 + 4x = x(x + 2)^2$

25. Solve for  $w$ :  $\frac{6w+1}{5} \neq 2$ . Describe the solution set in set notation.
26. Edwina has two sticks, one 2 yards long and the other 2 meters long. She is going to use them, with a third stick of some positive length, to make a triangle. She has decided to measure the length of the third stick in units of feet.
- What is the solution set to the statement: "Sticks of lengths 2 yards, 2 meters, and  $L$  feet make a triangle"? Describe the solution set in words and through a graphical representation.



- What is the solution set to the statement: "Sticks of lengths 2 yards, 2 meters, and  $L$  feet make an isosceles triangle"? Describe the solution set in words and through a graphical representation.
- What is the solution set to the statement: "Sticks of lengths 2 yards, 2 meters, and  $L$  feet make an equilateral triangle"? Describe the solution set in words and through a graphical representation.

## Lesson 12: Solving Equations

### Classwork

#### Opening Exercise

Answer the following questions.

- a. Why should the equations  $(x - 1)(x + 3) = 17 + x$  and  $(x - 1)(x + 3) = x + 17$  have the same solution set?
  
- b. Why should the equations  $(x - 1)(x + 3) = 17 + x$  and  $(x + 3)(x - 1) = 17 + x$  have the same solution set?
  
- c. Do you think the equations  $(x - 1)(x + 3) = 17 + x$  and  $(x - 1)(x + 3) + 500 = 517 + x$  should have the same solution set? Why?
  
- d. Do you think the equations  $(x - 1)(x + 3) = 17 + x$  and  $3(x - 1)(x + 3) = 51 + 3x$  should have the same solution set? Explain why.

#### Exercise 1

- a. Use the commutative property to write an equation that has the same solution set as  
 $x^2 - 3x + 4 = (x + 7)(x - 12)(5)$

- b. Use the associative property to write an equation that has the same solution set as  
 $x^2 - 3x + 4 = (x + 7)(x - 12)(5)$

- c. Does this reasoning apply to the distributive property as well?

**Exercise 2**

Consider the equation  $x^2 + 1 = 7 - x$ .

- a. Verify that this has the solution set  $\{-3, 2\}$ . Draw this solution set as a graph on the number line. *We will later learn how to show that these happen to be the ONLY solutions to this equation.*
- b. Let's add four to both sides of the equation and consider the new equation  $x^2 + 5 = 11 - x$ . Verify 2 and  $-3$  are still solutions.
- c. Let's now add  $x$  to both sides of the equation and consider the new equation  $x^2 + 5 + x = 11$ . Are 2 and  $-3$  still solutions?
- d. Let's add  $-5$  to both sides of the equation and consider the new equation  $x^2 + x = 6$ . Are 2 and  $-3$  still solutions?
- e. Let's multiply both sides by  $\frac{1}{6}$  to get  $\frac{x^2+x}{6} = 1$ . Are 2 and  $-3$  still solutions?
- f. Let's go back to part (d) and add  $3x^3$  to both sides of the equation and consider the new equation  $x^2 + x + 3x^3 = 6 + 3x^3$ . Are 2 and  $-3$  still solutions?

**Exercise 3**

- a. Solve for  $r$ :  $\frac{3}{2r} = \frac{1}{4}$
- b. Solve for  $s$ :  $s^2 + 5 = 30$
- c. Solve for  $y$ :  $4y - 3 = 5y - 8$

**Exercise 4**

Consider the equation  $3x + 4 = 8x - 16$ . Solve for  $x$  using the given starting point.

Group 1	Group 2	Group 3	Group 4
<i>Subtract <math>3x</math> from both sides</i>	<i>Subtract 4 from both sides</i>	<i>Subtract <math>8x</math> from both sides</i>	<i>Add 16 to both sides</i>

**Closing Exercise**

Consider the equation  $3x^2 + x = (x - 2)(x + 5)x$ .

- a. Use the commutative property to create an equation with the same solution set.
- b. Using the result from (a), use the associative property to create an equation with the same solution set.
- c. Using the result from (b), use the distributive property to create an equation with the same solution set.
- d. Using the result from (c), add a number to both sides of the equation.
- e. Using the result from (d), subtract a number from both sides of the equation.
- f. Using the result from (e), multiply both sides of the equation by a number.
- g. Using the result from (f), divide both sides of the equation by a number.
- h. What do all seven equations have in common? Justify your answer.

**Lesson Summary**

If  $x$  is a solution to an equation, it will also be a solution to the new equation formed when the same number is added to (or subtracted from) each side of the original equation or when the two sides of the original equation are multiplied by (or divided by) the same non-zero number. These are referred to as the *Properties of Equality*.

If one is faced with the task of solving an equation, that is, finding the solution set of the equation:

Use the *commutative*, *associative*, and *distributive properties*, AND use the *properties of equality* (adding, subtracting, multiplying by non-zeros, dividing by non-zeros) to keep rewriting the equation into one whose solution set you easily recognize. (We believe that the solution set will not change under these operations.)

**Problem Set**

1. Which of the following equations have the same solution set? Give reasons for your answers that do not depend on solving the equations.

I.  $x - 5 = 3x + 7$

II.  $3x - 6 = 7x + 8$

III.  $15x - 9 = 6x + 24$

IV.  $6x - 16 = 14x + 12$

V.  $9x + 21 = 3x - 15$

VI.  $-0.05 + \frac{x}{100} = \frac{3x}{100} + 0.07$

Solve the following equations, check your solutions, and then graph the solution sets.

2.  $-16 - 6v = -2(8v - 7)$

3.  $2(6b + 8) = 4 + 6b$

4.  $x^2 - 4x + 4 = 0$

5.  $7 - 8x = 7(1 + 7x)$

6.  $39 - 8n = -8(3 + 4n) + 3n$

7.  $(x - 1)(x + 5) = x^2 + 4x - 2$

8.  $x^2 - 7 = x^2 - 6x - 7$

9.  $-7 - 6a + 5a = 3a - 5a$

10.  $7 - 2x = 1 - 5x + 2x$

11.  $4(x - 2) = 8(x - 3) - 12$

12.  $-3(1 - n) = -6 - 6n$

13.  $-21 - 8a = -5(a + 6)$

14.  $-11 - 2p = 6p + 5(p + 3)$

15.  $\frac{x}{x+2} = 4$

16.  $2 + \frac{x}{9} = \frac{x}{3} - 3$

17.  $-5(-5x - 6) = -22 - x$

18.  $\frac{x+4}{3} = \frac{x+2}{5}$

19.  $-5(2r - 0.3) + 0.5(4r + 3) = -64$

## Lesson 13: Some Potential Dangers When Solving Equations

In previous lessons we have looked at techniques for solving equations, a common theme throughout algebra. In this lesson, we will examine some potential dangers where our intuition about algebra may need to be examined.

### Classwork

#### Exercise 1

- a. Describe the property used to convert the equation from one line to the next:

$$x(1 - x) + 2x - 4 = 8x - 24 - x^2$$

$$x - x^2 + 2x - 4 = 8x - 24 - x^2$$

---

$$x + 2x - 4 = 8x - 24$$

---

$$3x - 4 = 8x - 24$$

---

$$3x + 20 = 8x$$

---

$$20 = 5x$$

---

In each of the steps above, we applied a property of real numbers and/or equations to create a new equation.

- b. Why are we sure that the initial equation  $x(1 - x) + 2x - 4 = 8x - 24 - x^2$  and the final equation  $20 = 5x$  have the same solution set?

- c. What is the common solution set to all these equations?

**Exercise 2**

Solve the equation for  $x$ . For each step, describe the operation used to convert the equation.

$$3x - [8 - 3(x - 1)] = x + 19$$

**Exercise 3**

Solve each equation for  $x$ . For each step, describe the operation used to convert the equation.

a.  $7x - [4x - 3(x - 1)] = x + 12$

b.  $2[2(3 - 5x) + 4] = 5[2(3 - 3x) + 2]$

c.  $\frac{1}{2}(18 - 5x) = \frac{1}{3}(6 - 4x)$

**Exercise 4**

Consider the equations  $x + 1 = 4$  and  $(x + 1)^2 = 16$ .

- a. Verify that  $x = 3$  is a solution to both equations.
  
  
  
  
  
  
- b. Find a second solution to the second equation.
  
  
  
  
  
  
- c. Based on your results, what effect does squaring both sides of an equation appear to have on the solution set?

**Exercise 5**

Consider the equations  $x - 2 = 6 - x$  and  $(x - 2)^2 = (6 - x)^2$ .

- a. Did squaring both sides of the equation affect the solution sets?
  
  
  
  
  
  
- b. Based on your results, does your answer to part (c) of the previous question need to be modified?

**Exercise 6**

Consider the equation  $x^3 + 2 = 2x^2 + x$ .

- a. Verify that  $x = 1$ ,  $x = -1$ , and  $x = 2$  are each solutions to this equation.
- b. Bonzo decides to apply the action “Ignore the exponents” on each side of the equation. He gets  $x + 2 = 2x + x$ . In solving this equation, what does he obtain? What seems to be the problem with his technique?
- c. What would Bonzo obtain if he applied his “method” to the equation  $x^2 + 4x + 2 = x^4$ ? Is it a solution to the original equation?

**Exercise 7**

Consider the equation  $x - 3 = 5$ .

- a. Multiply both sides of the equation by a constant, and show that the solution set did not change.

Now, multiply both sides by  $x$ .

- b. Show that  $x = 8$  is still a solution to the new equation.

- c. Show that  $x = 0$  is also a solution to the new equation.

Now, multiply both sides by the factor  $x - 1$ .

- d. Show that  $x = 8$  is still a solution to the new equation.

- e. Show that  $x = 1$  is also a solution to the new equation.

- f. Based on your results, what effect does multiplying both sides of an equation by a constant have on the solution set of the new equation?

- g. Based on your results, what effect does multiplying both sides of an equation by a variable factor have on the solution set of the new equation?

**Lesson Summary**

Assuming that there is a solution to an equation, applying the distributive, commutative, and associative properties and the properties of equality to equations will not change the solution set.

Feel free to try doing other operations to both sides of an equation, but be aware – the new solution set you get contains possible candidates for solutions. You have to plug each one into the original equation to see if it really is a solution to your original equation.

**Problem Set**

1. Solve each equation for  $x$ . For each step, describe the operation used to convert the equation. How do you know that the initial equation and the final equation have the same solution set?
  - a.  $\frac{1}{5}[10 - 5(x - 2)] = \frac{1}{10}(x + 1)$
  - b.  $x(5 + x) = x^2 + 3x + 1$
  - c.  $2x(x^2 - 2) + 7x = 9x + 2x^3$
2. Consider the equation  $x + 1 = 2$ .
  - a. Find the solution set.
  - b. Multiply both sides by  $x + 1$  and find the solution set of the new equation.
  - c. Multiply both sides of the original equation by  $x$  and find the solution set of the new equation.
3. Solve the equation  $x + 1 = 2x$  for  $x$ . Square both sides of the equation and verify that your solution satisfies this new equation. Show that  $-\frac{1}{3}$  satisfies the new equation but not the original equation.
4. Consider the equation  $x^3 = 27$ .
  - a. What is the solution set?
  - b. Does multiplying both sides by  $x$  change the solution set?
  - c. Does multiplying both sides by  $x^2$  change the solution set?
5. Consider the equation  $x^4 = 16$ .
  - a. What is the solution set?
  - b. Does multiplying both sides by  $x$  change the solution set?
  - c. Does multiplying both sides by  $x^2$  change the solution set?

## Lesson 14: Solving Inequalities

### Classwork

#### Exercise 1

Consider the inequality  $x^2 + 4x \geq 5$ .

- a. Sift through some possible values to assign to  $x$  that make this inequality a true statement. Find at least two positive values that work and at least two negative values that work.
  
  
  
  
  
  
- b. Should your four values also be solutions to the inequality  $x(x + 4) \geq 5$ ? Explain why or why not. Are they?
  
  
  
  
  
  
- c. Should your four values also be solutions to the inequality  $4x + x^2 \geq 5$ ? Explain why or why not. Are they?
  
  
  
  
  
  
- d. Should your four values also be solutions to the inequality  $4x + x^2 - 6 \geq -1$ ? Explain why or why not. Are they?
  
  
  
  
  
  
- e. Should your four values also be solutions to the inequality  $12x + 3x^2 \geq 15$ ? Explain why or why not. Are they?

**Example 1**

What is the solution set to the inequality  $5q + 10 > 20$ ? Express the solution set in words, in set notation, and graphically on the number line.

**Exercise 2**

Find the solution set to each inequality. Express the solution in set notation and graphically on the number line.

a.  $x + 4 \leq 7$

b.  $\frac{m}{3} + 8 \neq 9$

c.  $8y + 4 < 7y - 2$

d.  $6(x - 5) \geq 30$

e.  $4(x - 3) > 2(x - 2)$

**Exercise 3**

Recall the discussion on all the strange ideas for what could be done to both sides of an equation. Let's explore some of the same issues here but with inequalities. Recall, in this lesson we have established that adding (or subtracting) and multiplying through by positive quantities does not change the solution set of an inequality. We've made no comment about other operations.

- Squaring:* Do  $B \leq 6$  and  $B^2 \leq 36$  have the same solution set? If not, give an example of a number that is in one solution set but not the other.
- Multiplying through by a negative number:* Do  $5 - C > 2$  and  $-5 + C > -2$  have the same solution set? If not, give an example of a number that is in one solution set but not the other.
- Bonzo's ignoring exponents: Do  $y^2 < 5^2$  and  $y < 5$  have the same solution set?

**Example 2**

Jojo was asked to solve  $6x + 12 < 3x + 6$ , for  $x$ . She answered as follows:

$$6x + 12 < 3x + 6$$

$$6(x + 2) < 3(x + 2) \quad \text{Apply the distributive property.}$$

$$6 < 3 \quad \text{Multiply through by } \frac{1}{x+2}.$$

- a. Since the final line is a false statement, she deduced that there is no solution to this inequality (that the solution set is empty).

What is the solution set to  $6x + 12 < 3x + 6$ ?

- b. Explain why Jojo came to an erroneous conclusion.

**Example 3**

Solve  $-q \geq -7$ , for  $q$ .

**Exercise 4**

Find the solution set to each inequality. Express the solution in set notation and graphically on the number line.

a.  $-2f < -16$

b.  $-\frac{x}{12} \leq \frac{1}{4}$

c.  $6 - a \geq 15$

d.  $-3(2x + 4) > 0$

**Recall the Properties of Inequality:**

- Addition Property of Inequality:  
If  $A > B$ , then  $A + c > B + c$  for any real number  $c$ .
- Multiplication Property of Inequality:  
If  $A > B$ , then  $kA > kB$  for any positive real number  $k$ .

**Exercise 5**

Use the properties of inequality to show that each of the following are true for any real numbers  $p$  and  $q$ .

a. If  $p \geq q$ , then  $-p \leq -q$ .

b. If  $p < q$ , then  $-5p > -5q$ .

- c. If  $p \leq q$ , then  $-0.03p \geq -0.03q$ .
- d. Based on the results from (a) through (c), how might we expand the multiplication property of inequality?

**Exercise 6**

Solve  $-4 + 2t - 14 - 18t > -6 - 100t$ , for  $t$  in two different ways: first without ever multiplying through by a negative number and then by first multiplying through by  $-\frac{1}{2}$ .

**Exercise 7**

Solve  $-\frac{x}{4} + 8 < \frac{1}{2}$ , for  $x$  in two different ways: first without ever multiplying through by a negative number and then by first multiplying through by  $-4$ .

**Problem Set**

1. Find the solution set to each inequality. Express the solution in set notation and graphically on the number line.

- $2x < 10$
- $-15x \geq -45$
- $\frac{2}{3}x \neq \frac{1}{2} + 2$
- $-5(x - 1) \geq 10$
- $13x < 9(1 - x)$

2. Find the mistake in the following set of steps in a student's attempt to solve  $5x + 2 \geq x + \frac{2}{5}$ , for  $x$ . What is the correct solution set?

$$\begin{aligned} 5x + 2 &\geq x + \frac{2}{5} \\ 5\left(x + \frac{2}{5}\right) &\geq x + \frac{2}{5} \quad (\text{factoring out } 5 \text{ on the left side}) \\ 5 &\geq 1 \quad (\text{dividing by } \left(x + \frac{2}{5}\right)) \end{aligned}$$

So, the solution set is the empty set.

3. Solve  $-\frac{x}{16} + 1 \geq -\frac{5x}{2}$ , for  $x$  without multiplying by a negative number. Then solve by multiplying through by  $-16$ .
4. Lisa brought half of her savings to the bakery and bought 12 croissants for \$14.20. The amount of money she brings home with her is more than \$2. Use an inequality to find how much money she had in her savings before going to the bakery. (Write the inequalities that represents the situation and solve it.)

## Lesson 15: Solution Sets of Two or More Equations (or Inequalities) Joined by “And” or “Or”

### Classwork

#### Exercise 1

Determine whether each claim given below is true or false.

- a. Right now, I am in math class and English class.
- b. Right now, I am in math class or English class.
- c.  $3 + 5 = 8$  and  $5 < 7 - 1$ .
- d.  $10 + 2 \neq 12$  and  $8 - 3 > 0$ .
- e.  $3 < 5 + 4$  or  $6 + 4 = 9$ .
- f.  $16 - 20 > 1$  or  $5.5 + 4.5 = 11$

These are all examples of declarative compound sentences.

- g. When the two declarations in the sentences above were separated by “and,” what had to be true to make the statement true?
- h. When the two declarations in the sentences above were separated by “or,” what had to be true to make the statement true?

**Example 1**

Solve each system of equations and inequalities.

- a.  $x + 8 = 3$  or  $x - 6 = 2$
- b.  $4x - 9 = 0$  or  $3x + 5 = 2$
- c.  $x - 6 = 1$  and  $x + 2 = 9$
- d.  $2w - 8 = 10$  and  $w > 9$ .

**Exercise 2**

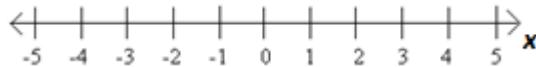
- a. Using a colored pencil, graph the inequality  $x < 3$  on the number line below.



- b. Using a different colored pencil, graph the inequality  $x > -1$  on the number line below.



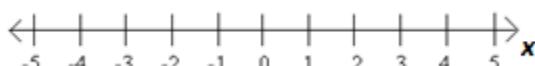
- c. Using a third colored pencil, darken the section of the number line where  $x < 3$  and  $x > -1$ .



- d. Using a colored pencil, graph the inequality  $x < -4$  on the number line below.



- e. Using a different colored pencil, graph the inequality  $x > 0$  on the number line below.



- f. Using a third colored pencil, darken the section of the number line where  $x < -4$  or  $x > 0$ .



- g. Graph the compound sentence  $x > -2$  or  $x = -2$  on the number line below.



- h. How could we abbreviate the sentence  $x > -2$  or  $x = -2$ ?

- i. Rewrite  $x \leq 4$  as a compound sentence and graph the solutions to the sentence on the number line below.



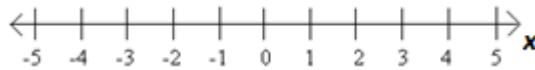
### Example 2

Graph each compound sentence on a number line.

a.  $x = 2$  or  $x > 6$



b.  $x \leq -5$  or  $x \geq 2$



Rewrite as a compound sentence and graph the sentence on a number line.

c.  $1 \leq x \leq 3$



**Exercise 3**

Consider the following two scenarios. For each, specify the variable and say, “ $W$  is the width of the rectangle,” for example, and write a compound inequality that represents the scenario given. Draw its solution set on a number line.

Scenario	Variable	Inequality	Graph
a. Students are to present a persuasive speech in English class. The guidelines state that the speech must be at least 7 minutes but not exceed 12 minutes.			
b. Children and senior citizens receive a discount on tickets at the movie theater. To receive a discount, a person must be between the ages of 2 and 12, including 2 and 12, or 60 years of age or older.			

**Exercise 4**

Determine if each sentence is true or false. Explain your reasoning.

a.  $8 + 6 \leq 14$  and  $\frac{1}{3} < \frac{1}{2}$

b.  $5 - 8 < 0$  or  $10 + 13 \neq 23$

Solve each system and graph the solution on a number line.

c.  $x - 9 = 0$  or  $x + 15 = 0$

d.  $5x - 8 = -23$  or  $x + 1 = -10$

Graph the solution set to each compound inequality on a number line.

e.  $x < -8$  or  $x > -8$

f.  $0 < x \leq 10$

Write a compound inequality for each graph.

g.



h.



- i. A poll shows that a candidate is projected to receive 57% of the votes. If the margin for error is plus or minus 3%, write a compound inequality for the percentage of votes the candidate can expect to get.
- j. Mercury is one of only two elements that is liquid at room temperature. Mercury is non-liquid for temperatures less than  $-38.0^{\circ}\text{F}$  or greater than  $673.8^{\circ}\text{F}$ . Write a compound inequality for the temperatures at which mercury is non-liquid.

**Lesson Summary**

In mathematical sentences, like in English sentences, a compound sentence separated by

AND is true if \_\_\_\_\_.

OR is true if \_\_\_\_\_.

**Problem Set**

1. Consider the inequality  $0 < x < 3$ .
  - a. Rewrite the inequality as a compound sentence.
  - b. Graph the inequality on a number line.
  - c. How many solutions are there to the inequality? Explain.
  - d. What are the largest and smallest possible values for  $x$ ? Explain.
  - e. If the inequality is changed to  $0 \leq x \leq 3$ , then what are the largest and smallest possible values for  $x$ ?



Write a compound inequality for each graph.



Write a single or compound inequality for each scenario.

4. The scores on the last test ranged from 65% to 100%.
5. To ride the roller coaster, one must be at least 4 feet tall.
6. Unsafe body temperatures are those lower than 96°F or above 104°F.

Graph the solution(s) to each of the following on a number line.

7.  $x - 4 = 0$  and  $3x + 6 = 18$



8.  $x < 5$  and  $x \neq 0$



9.  $x \leq -8$  or  $x \geq -1$



10.  $3(x - 6) = 3$  or  $5 - x = 2$



11.  $x < 9$  and  $x > 7$



12.  $x + 5 < 7$  or  $x = 2$



## Lesson 16: Solving and Graphing Inequalities Joined by “And” or “Or”

### Classwork

#### Exercise 1

- a. Solve  $w^2 = 121$ , for  $w$ . Graph the solution on a number line.
  
  
  
  
  
  
- b. Solve  $w^2 < 121$ , for  $w$ . Graph the solution on a number line and write the solution set as a compound inequality.
  
  
  
  
  
  
- c. Solve  $w^2 \geq 121$  for  $w$ . Graph the solution on a number line and write the solution set as a compound inequality.
  
  
  
  
  
  
- d. Quickly solve  $(x + 7)^2 = 121$ , for  $x$ . Graph the solution on a number line.
  
  
  
  
  
  
- e. Use your work from part (d) to quickly graph the solution on a number line to each inequality below.
  - i.  $(x + 7)^2 < 121$
  
  
  
  
  
  
  - ii.  $(x + 7)^2 \geq 121$

**Exercise 2**

Consider the compound inequality  $-5 < x < 4$

- a. Rewrite the inequality as a compound statement of inequality.

- b. Write a sentence describing the possible values of  $x$ .

- c. Graph the solution set on the number line below.

**Exercise 3**

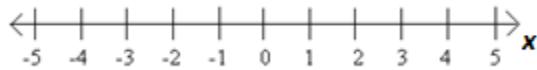
Consider the compound inequality  $-5 < 2x + 1 < 4$ .

- a. Rewrite the inequality as a compound statement of inequality.

- b. Solve each inequality for  $x$ . Then, write the solution to the compound inequality.

- c. Write a sentence describing the possible values of  $x$ .

- d. Graph the solution set on the number line below.



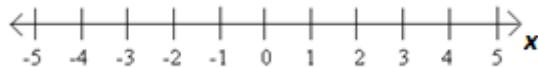
**Exercise 4**

Given  $x < -3$  or  $x > -1$

- a. What must be true in order for the compound inequality to be a true statement?

- b. Write a sentence describing the possible values of  $x$ .

- c. Graph the solution set on the number line below.

**Exercise 5**

Given  $x + 4 < 6$  or  $x - 1 > 3$

- a. Solve each inequality for  $x$ . Then, write the solution to the compound inequality.

- b. Write a sentence describing the possible values of  $x$ .

- c. Graph the solution set on the number line below.



**Exercise 6**

Solve each compound inequality for  $x$  and graph the solution on a number line.

a.  $x + 6 < 8$  and  $x - 1 > -1$

b.  $-1 \leq 3 - 2x \leq 10$

c.  $5x + 1 < 0$  or  $8 \leq x - 5$

d.  $10 > 3x - 2$  or  $x = 4$

e.  $x - 2 < 4$  or  $x - 2 > 4$

f.  $x - 2 \leq 4$  and  $x - 2 \geq 4$

**Exercise 7**

Solve each compound inequality for  $x$  and graph the solution on a number line. Pay careful attention to the inequality symbols and the “and” or “or” statements as you work.

a.  $1 + x > -4$  or  $3x - 6 > -12$

b.  $1 + x > -4$  or  $3x - 6 < -12$

c.  $1 + x > 4$  and  $3x - 6 < -12$

**Problem Set**

Solve each inequality for  $x$  and graph the solution on a number line.

1.  $x - 2 < 6$  or  $\frac{x}{3} > 4$

2.  $-6 < \frac{x+1}{4} < 3$

3.  $5x \leq 21 + 2x$  or  $3(x + 1) \geq 24$

4.  $5x + 2 \geq 27$  and  $3x - 1 < 29$

5.  $0 \leq 4x - 3 \leq 11$

6.  $2x > 8$  or  $-2x < 4$

7.  $8 \geq -2(x - 9) \geq -8$

8.  $4x + 8 > 2x - 10$  or  $\frac{1}{3}x - 3 < 2$

9.  $7 - 3x < 16$  and  $x + 12 < -8$

10. If inequalities question 8 were joined by “and” instead of “or,” what would the solution set become?

11. If the inequalities in question 9 were joined by “or” instead of “and,” what would the solution set become?

## Lesson 17: Equations Involving Factored Expressions

### Classwork

#### Exercise 1

1. Solve each equation for  $x$ .

a.  $x - 10 = 0$

b.  $\frac{x}{2} + 20 = 0$

- c. Demanding Dwight insists that you give him two solutions to the following equation:

$$(x - 10) \left( \frac{x}{2} + 20 \right) = 0$$

Can you provide him with two solutions?

- d. Demanding Dwight now wants FIVE solutions to the following equation:

$$(x - 10)(2x + 6)(x^2 - 36)(x^2 + 10) \left( \frac{x}{2} + 20 \right) = 0$$

Can you provide him with five solutions?

Do you think there might be a sixth solution?

Consider the equation  $(x - 4)(x + 3) = 0$ .

e. Rewrite the equation as a compound statement.

f. Find the two solutions to the equation.

### Example 1

Solve  $2x^2 - 10x = 0$ , for  $x$ .

### Example 2

Solve  $x(x - 3) + 5(x - 3) = 0$ , for  $x$ .

### Exercises 2–7

2.  $(x + 1)(x + 2) = 0$

3.  $(3x - 2)(x + 12) = 0$

4.  $(x - 3)(x - 3) = 0$

5.  $(x + 4)(x - 6)(x - 10) = 0$

6.  $x^2 - 6x = 0$

7.  $x(x - 5) + 4(x - 5) = 0$

**Example 3**

Consider the equation  $(x - 2)(2x - 3) = (x - 2)(x + 5)$ . Lulu chooses to multiply through by  $\frac{1}{x-2}$  and gets the answer  $x = 8$ . But Poindexter points out that  $x = 2$  is also an answer, which Lulu missed.

- a. What's the problem with Lulu's approach?
  
  
  
  
  
- b. Use factoring to solve the original equation for  $x$ .

**Exercises 8–11**

8. Use factoring to solve the equation for  $x$ :  $(x - 2)(2x - 3) = (x - 2)(x + 1)$ .
  
  
  
  
  
9. Solve each of the following for  $x$ :
  - a.  $x + 2 = 5$
  - b.  $x^2 + 2x = 5x$
  
  
  
  
  
- c.  $x(5x - 20) + 2(5x - 20) = 5(5x - 20)$

10.

a. Verify:  $(a - 5)(a + 5) = a^2 - 25$ .

b. Verify:  $(x - 88)(x + 88) = x^2 - 88^2$ .

c. Verify:  $A^2 - B^2 = (A - B)(A + B)$ .

d. Solve for  $x$ :  $x^2 - 9 = 5(x - 3)$ .

e. Solve for  $w$ :  $(w + 2)(w - 5) = w^2 - 4$ .

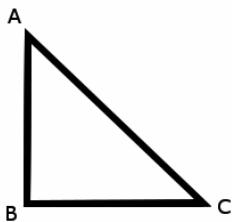
11. A string 60 inches long is to be laid out on a table-top to make a rectangle of perimeter 60 inches. Write the width of the rectangle as  $15 + x$  inches. What is an expression for its length? What is an expression for its area? What value for  $x$  gives an area of largest possible value? Describe the shape of the rectangle for this special value of  $x$ .

**Lesson Summary**

The zero-product property says that if  $ab = 0$ , then either  $a = 0$  or  $b = 0$  or  $a = b = 0$ .

**Problem Set**

1. Find the solution set of each equation:
  - a.  $(x - 1)(x - 2)(x - 3) = 0$
  - b.  $(x - 16.5)(x - 109) = 0$
  - c.  $x(x + 7) + 5(x + 7) = 0$
  - d.  $x^2 + 8x + 15 = 0$
  - e.  $(x - 3)(x + 3) = 8x$
2. Solve  $x^2 - 11x = 0$ , for  $x$ .
3. Solve  $(p + 3)(p - 5) = 2(p + 3)$ , for  $p$ . What solution do you lose if you simply divide by  $p + 3$  to get  $p - 5 = 2$ ?
4. The square of a number plus 3 times the number is equal to 4. What is the number?
5. In the right triangle shown below, the length of side AB is  $x$ , the length of side BC is  $x + 2$ , and the length of the hypotenuse AC is  $x + 4$ . Use this information to find the length of each side. (Use the Pythagorean Theorem to get an equation, and solve for  $x$ .)



6. Using what you learned in this lesson, create an equation that has 53 and 22 as its only solutions.

## Lesson 18: Equations Involving a Variable Expression in the Denominator

### Classwork

#### Opening Exercise

Nolan says that he checks the answer to a division problem by performing multiplication. For example, he says that

$20 \div 4 = 5$  is correct because  $5 \times 4$  is 20, and  $\frac{3}{1/2} = 6$  is correct because  $6 \times \frac{1}{2}$  is 3.

a. Using Nolan's reasoning, explain why there is no real number that is the answer to the division problem  $5 \div 0$ .

b. Quentin says that  $\frac{0}{0} = 17$ . What do you think?

c. Mavis says that the expression  $\frac{5}{x+2}$  has a meaningful value for whatever value one chooses to assign to  $x$ . Do you agree?

d. Bernoit says that the expression  $\frac{3x-6}{x-2}$  always has the value 3 for whichever value one assigns to  $x$ . Do you agree?

**Exercises 1–2**

1. Rewrite  $\frac{10}{x+5}$  as a compound statement.
2. Consider  $\frac{x^2-25}{(x^2-9)(x+4)}$ .
  - a. Is it permissible to let  $x = 5$  in this expression?
  - b. Is it permissible to let  $x = 3$  in this expression?
  - c. Give all the values of  $x$  that are **not** permissible in this expression.

**Example 1**

Consider the equation  $\frac{1}{x} = \frac{3}{x-2}$ .

- a. Rewrite the equation into a system of equations.
- b. Solve the equation for  $x$ , excluding the value(s) of  $x$  that lead to a denominator of zero.

**Example 2**

Consider the equation  $\frac{x+3}{x-2} = \frac{5}{x-2}$ .

- a. Rewrite the equation into a system of equations.

- b. Solve the equation for  $x$ , excluding the value(s) of  $x$  that lead to a denominator of zero.

**Exercises 3–11**

Rewrite each equation into a system of equations excluding the value(s) of  $x$  that lead to a denominator of zero; then, solve the equation for  $x$ .

3.  $\frac{5}{x} = 1$

4.  $\frac{1}{x-5} = 3$

5.  $\frac{x}{x+1} = 4$

6.  $\frac{2}{x} = \frac{3}{x-4}$

7.  $\frac{x}{x+6} = -\frac{6}{x+6}$

8.  $\frac{x-3}{x+2} = 0$

$$9. \frac{x+3}{x+3} = 5$$

$$10. \frac{x+3}{x+3} = 1$$

11. A baseball player's batting average is calculated by dividing the number of times a player got a hit by the total number of times the player was at bat. It is expressed as a decimal rounded to three places. After the first ten games of the season, Samuel had 12 hits off of 33 "at bats."
- What is his batting average after the first ten games?
  - How many hits in a row would he need to get to raise his batting average to above 0.500?
  - How many "at bats" in a row without a hit would result in his batting average dropping below 0.300?

**Problem Set**

1. Consider the equation  $\frac{10(x^2-49)}{3x(x^2-4)(x+1)} = 0$ . Is  $x = 7$  permissible? Which values of  $x$  are excluded? Rewrite as a system of equations.
2. Rewrite each equation as a system of equations excluding the value(s) of  $x$  that lead to a denominator of zero. Then solve the equation for  $x$ .
  - a.  $25x = \frac{1}{x}$
  - b.  $\frac{1}{5x} = 10$
  - c.  $\frac{x}{7-x} = 2x$
  - d.  $\frac{2}{x} = \frac{5}{x+1}$
  - e.  $\frac{3+x}{3-x} = \frac{3+2x}{3-2x}$
3. Ross wants to cut a 40-foot rope into two pieces so that the length of the first piece divided by the length of the second piece is 2.
  - a. Let  $x$  represent the length of the first piece. Write an equation that represents the relationship between the pieces as stated above.
  - b. What values of  $x$  are not permissible in this equation? Describe within the context of the problem, what situation is occurring if  $x$  were to equal this value(s). Rewrite as a system of equations to exclude the value(s).
  - c. Solve the equation to obtain the lengths of the two pieces of rope. (Round to the nearest tenth if necessary.)
4. Write an equation with the restrictions  $x \neq 14$ ,  $x \neq 2$ , and  $x \neq 0$ .
5. Write an equation that has no solution.

## Lesson 19: Rearranging Formulas

### Classwork

#### Exercise 1

Solve each equation for  $x$ . For part (c), remember a variable symbol, like  $a$ ,  $b$ , and  $c$ , represents a number.

a.  $2x - 6 = 10$

b.  $-3x - 3 = -12$

c.  $ax - b = c$

#### Exercise 2

Compare your work in parts (a) through (c) above. Did you have to do anything differently to solve for  $x$  in part (c)?

#### Exercise 3

Solve the equation  $ax - b = c$  for  $a$ . The variable symbols  $x$ ,  $b$ , and  $c$  represent numbers.

**Example 1: Rearranging Familiar Formulas**

The area  $A$  of a rectangle is 25 in $^2$ . The formula for area is  $A = lw$ .

- If the width  $w$  is 10 inches, what is the length  $l$ ?

$$A = 25 \text{ in}^2$$

$l$

- If the width  $w$  is 15 inches, what is the length  $l$ ?

$w$

- Rearrange the area formula to solve for  $l$ .  $A = lw$

$$\frac{A}{w} = \frac{lw}{w}$$

$$\frac{A}{w} = l \text{ or } l = \frac{A}{w}$$

- Verify that the area formula, solved for  $l$ , will give the same results for  $l$  as having solved for  $l$  in the original area formula.

**Exercise 4**

Solve each problem two ways. First, substitute the given values and solve for the given variable. Then, solve for the given variable and substitute the given values.

- a. The perimeter formula for a rectangle is  $p = 2(l + w)$ , where  $p$  represents the perimeter,  $l$  represents the length, and  $w$  represents the width. Calculate  $l$  when  $p = 70$  and  $w = 15$ .
- b. The area formula for a triangle is  $A = \frac{1}{2}bh$ , where  $A$  represents the area,  $b$  represents the length of the base, and  $h$  represents the height. Calculate  $b$  when  $A = 100$  and  $h = 20$ .

**Exercise 5**

Rearrange each formula to solve for the specified variable. Assume no variable is equal to 0.

a. Given  $A = P(1 + rt)$ ,

i. Solve for  $P$ .

ii. Solve for  $t$ .

b. Given  $K = \frac{1}{2}mv^2$ ,

i. Solve for  $m$ .

ii. Solve for  $v$ .

**Example 2: Comparing Equations with One Variable to Those With More Than One Variable**

Equation Containing More Than One Variable	Related Equation
Solve $ax + b = d - cx$ for $x$ .	Solve $3x + 4 = 6 - 5x$ for $x$ .
Solve for $x$ . $\frac{ax}{b} + \frac{cx}{d} = e$	Solve for $x$ . $\frac{2x}{5} + \frac{x}{7} = 3$

**Lesson Summary**

The properties and reasoning used to solve equations apply regardless of how many variables appear in an equation or formula. Rearranging formulas to solve for a specific variable can be useful when solving applied problems.

**Problem Set**

For Problems 1–8, solve for  $x$ .

1.  $ax + 3b = 2f$       2.  $rx + h = sx - k$       3.  $3px = 2q(r - 5x)$       4.  $\frac{x+b}{4} = c$

5.  $\frac{x}{5} - 7 = 2q$       6.  $\frac{x}{6} - \frac{x}{7} = ab$       7.  $\frac{x}{m} - \frac{x}{n} = \frac{1}{p}$       8.  $\frac{3ax+2b}{c} = 4d$

 9. Solve for  $m$ .

$$t = \frac{ms}{m+n}$$

 10. Solve for  $u$ .

$$\frac{1}{u} + \frac{1}{v} = \frac{1}{f}$$

 11. Solve for  $s$ .

$$A = s^2$$

 12. Solve for  $h$ .

$$V = \pi r^2 h$$

 13. Solve for  $m$ .

$$T = 4\sqrt{m}$$

 14. Solve for  $d$ .

$$F = G \frac{mn}{d^2}$$

 15. Solve for  $y$ .

$$ax + by = c$$

 16. Solve for  $b_1$ .

$$A = \frac{1}{2}h(b_1 + b_2)$$

17. The science teacher wrote three equations on a board that relate velocity,  $v$ , distance traveled,  $d$ , and the time to travel the distance,  $t$ , on the board.

$$v = \frac{d}{t}$$

$$t = \frac{d}{v}$$

$$d = vt$$

Would you need to memorize all three equations or could you just memorize one? Explain your reasoning.

## Lesson 20: Solution Sets to Equations with Two Variables

### Classwork

#### Exercise 1

- a. Circle all the ordered pairs  $(x, y)$  that are solutions to the equation  $4x - y = 10$ .

(3,2)      (2, 3)       $(-1, -14)$       (0,0)      (1, -6)  
 (5,10)      (0, -10)      (3,4)      (6,0)      (4, -1)

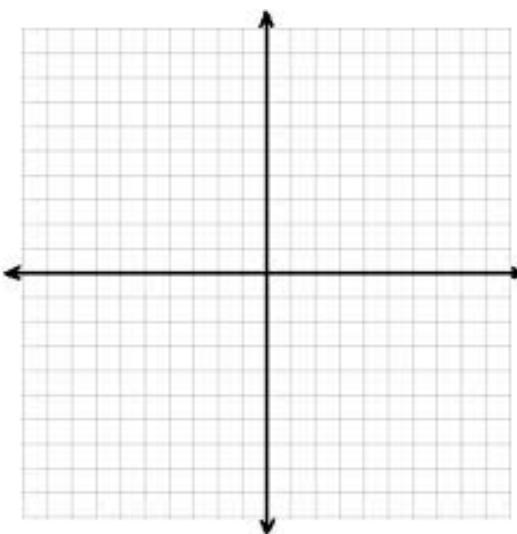
- b. How did you decide whether or not an ordered pair was a solution to the equation?

#### Exercise 2

- a. Discover as many additional solutions to the equation  $4x - y = 10$  as possible. Consider the best way to organize all the solutions you have found. Be prepared to share the strategies you used to find your solutions.
- b. Now, find five more solutions where one or more variables are negative numbers or non-integer values. Be prepared to share the strategies you used to find your solutions.

- c. How many ordered pairs  $(x, y)$  will be in the solution set of the equation  $4x - y = 10$ ?

- d. Create a visual representation of the solution set by plotting each solution as a point  $(x, y)$  in the coordinate plane.



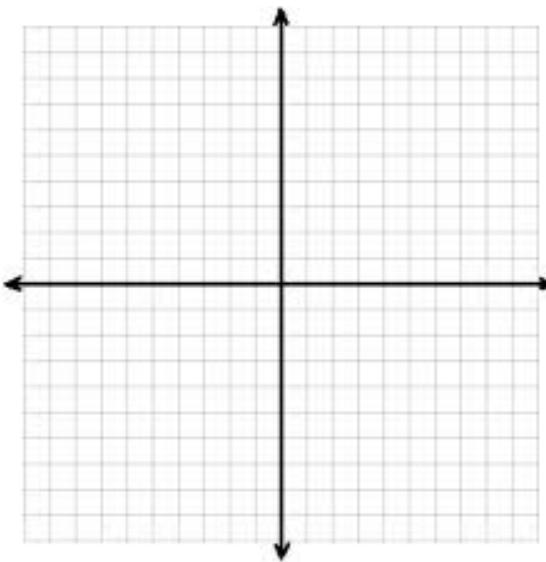
- e. Why does it make sense to represent the solution to the equation  $4x - y = 10$  as a line in the coordinate plane?

**Exercises 3–5**

3. The sum of two numbers is 25. What are the numbers?
- Create an equation using two variables to represent this situation. Be sure to explain the meaning of each variable.

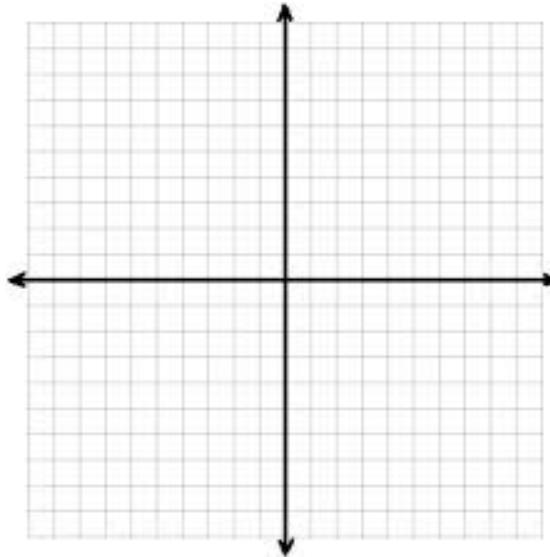
- List at least six solutions to the equation you created in part (a).

- Create a graph that represents the solution set to the equation.



4. Gia had 25 songs in a playlist composed of songs from her two favorite artists, Beyonce and Jennifer Lopez. How many songs did she have by each one in the playlist?
- Create an equation using two variables to represent this situation. Be sure to explain the meaning of each variable.
  - List at least three solutions to the equation you created in part (a).

- Create a graph that represents the solution set to the equation.



5. Compare your solutions to Exercises 3 and 4. How are they alike? How are they different?

**Lesson Summary**

An ordered pair is a **solution** to a two variable equation when each number substituted into its corresponding variable makes the equation a true-number sentence. All of the solutions to a two-variable equation are called the **solution set**.

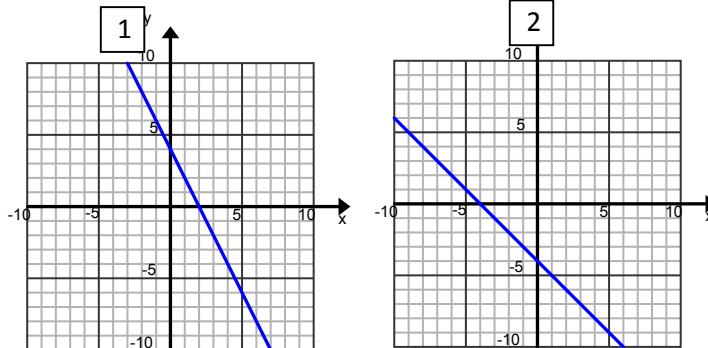
Each ordered pair of numbers in the solution set of the equation corresponds to a point on the coordinate plane. The set of all such points in the coordinate plane is called the **graph of the equation**.

**Problem Set**

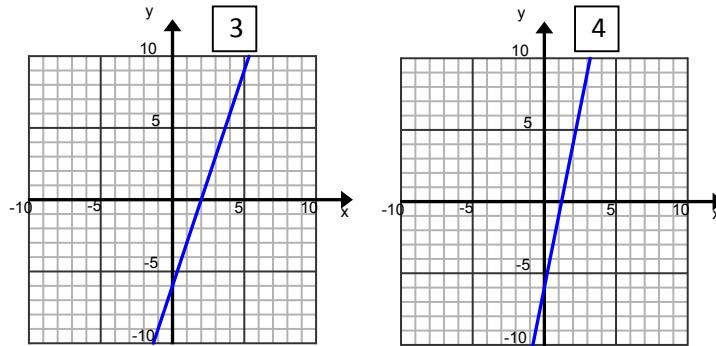
1. Match each equation with its graph.

Explain your reasoning.

a.  $y = 5x - 6$



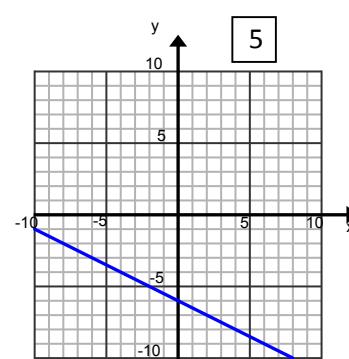
b.  $x + 2y = -12$



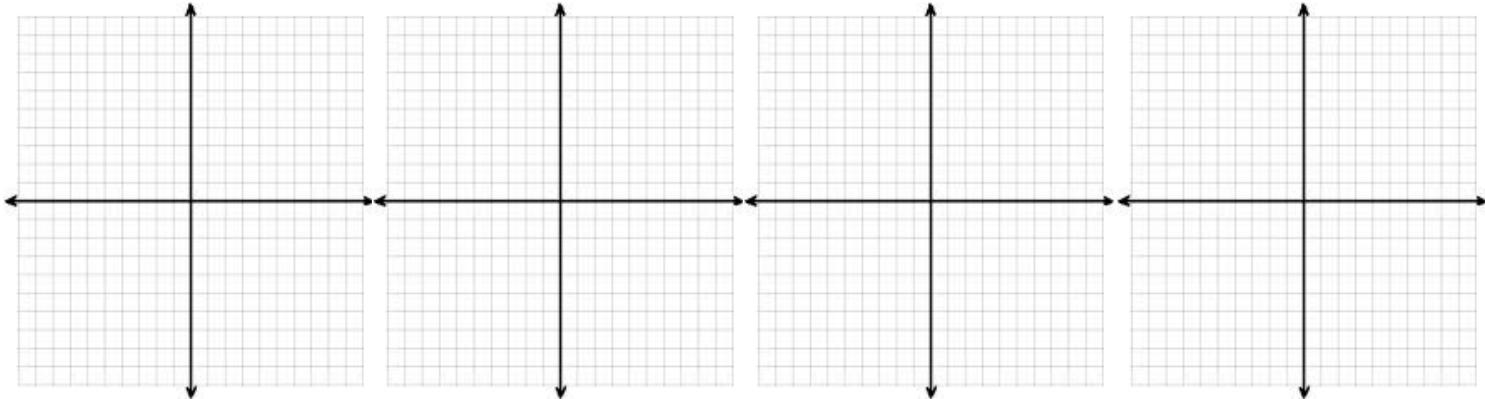
c.  $2x + y = 4$

d.  $y = 3x - 6$

e.  $x = -y - 4$

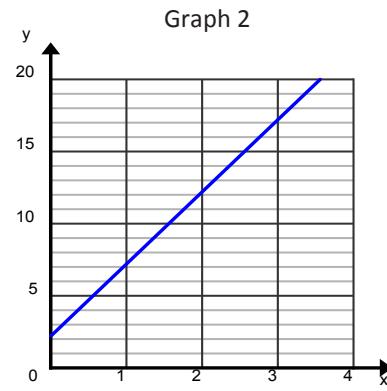
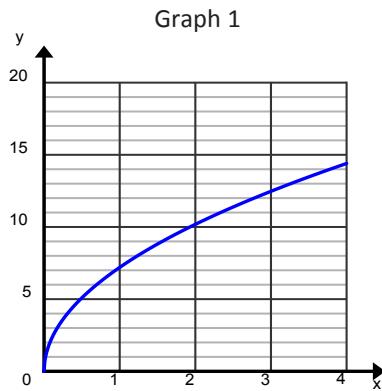


2. Graph the solution set in the coordinate plane. Label at least two ordered pairs that are solutions on your graph.
- $10x + 6y = 100$
  - $y = 9.5x + 20$
  - $7x - 3y = 21$
  - $y = 4(x + 10)$



3. Mari and Lori are starting a business to make gourmet toffee. They gather the following information from another business about prices for different amounts of toffee. Which equation and which graph are most likely to model the price,  $p$ , for  $x$  pounds of toffee? Justify your reasoning.

Pounds $x$	Price, $p$ , for $x$ pounds
0.25	\$3.60
0.81	\$6.48
1	\$7.20
1.44	\$8.64



Equation A:  $p = 5x + 2.2$

Equation B:  $p = 7.2\sqrt{x}$

## Lesson 21: Solution Sets to Inequalities with Two Variables

### Classwork

#### Exercise 1

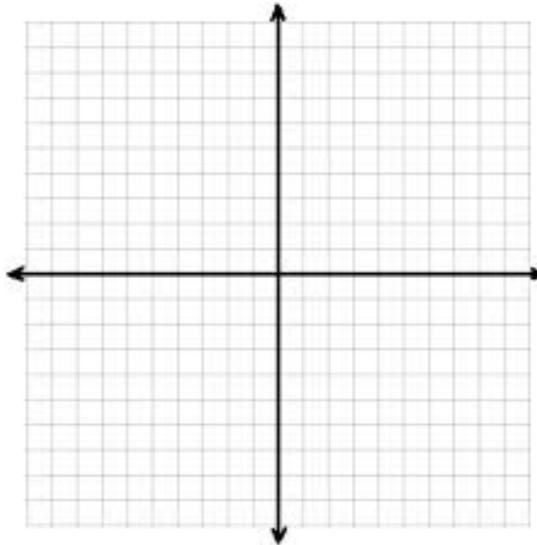
- a. Circle each ordered pair  $(x, y)$  that is a solution to the equation  $4x - y \leq 10$ .

i. (3,2)    (2,3)    (-1,-14)    (0,0)    (1,-6)

ii. (5,10)    (0,-10)    (3,4)    (6,0)    (4,-1)

- b. Plot each solution as a point  $(x, y)$  in the coordinate plane.

- c. How would you describe the location of the solutions in the coordinate plane?



**Exercise 2**

- a. Discover as many additional solutions to the inequality  $4x - y \leq 10$  as possible. Organize your solutions by plotting each solution as a point  $(x, y)$  in the coordinate plane. Be prepared to share the strategies used to find your solutions.

- b. Graph the line  $y = 4x - 10$ . What do you notice about the solutions to the inequality  $4x - y \leq 10$  and the graph of the line  $y = 4x - 10$ ?

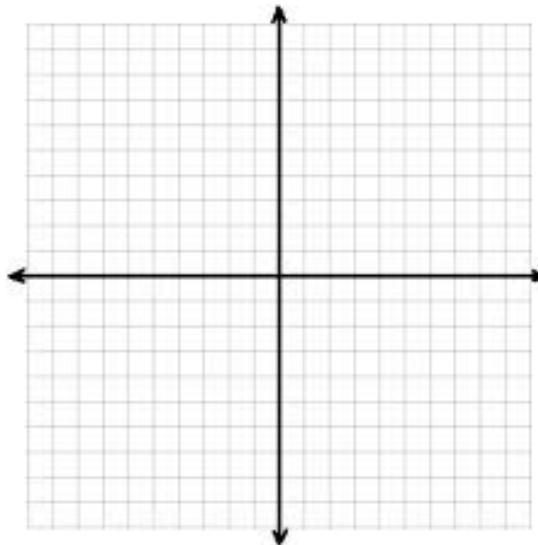
- c. Solve the inequality for  $y$ .

- d. Complete the following sentence:

If an ordered pair is a solution to  $4x - y \leq 10$ , then it will be located \_\_\_\_\_

\_\_\_\_\_ the line  $y = 4x - 10$ .

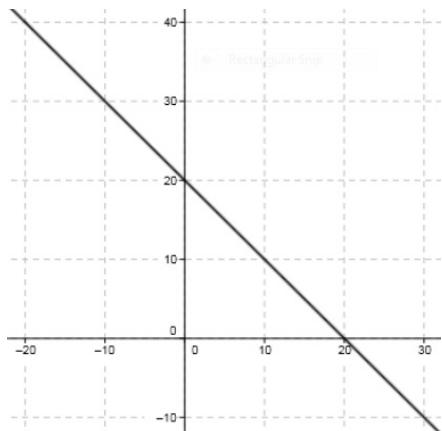
Explain how you arrived at your conclusion.



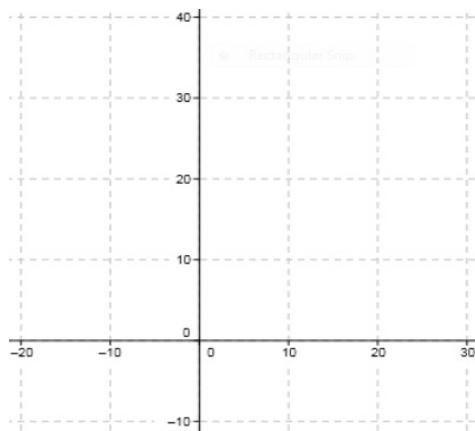
**Example 1**

The solution to  $x + y = 20$  is shown on the graph below.

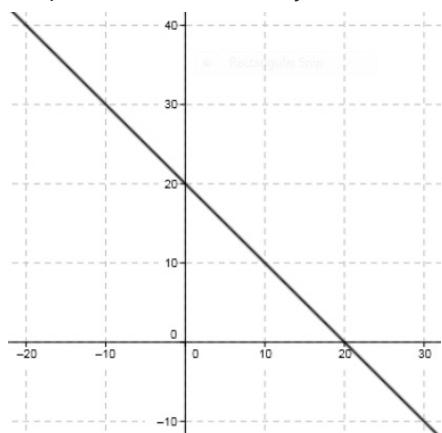
- a. Graph the solution to  $x + y \leq 20$ .



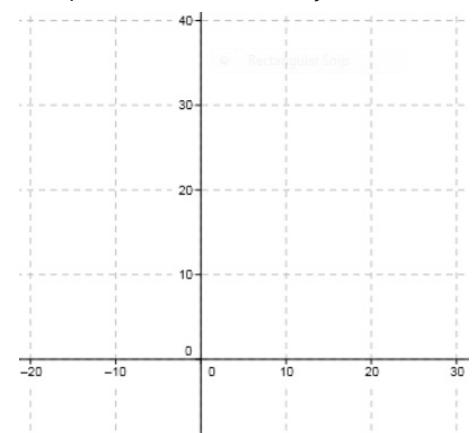
- c. Graph the solution to  $x + y < 20$ .



- b. Graph the solution to  $x + y \geq 20$ .



- d. Graph the solution to  $x + y > 20$ .



**Exercises 3–5**

3. Using a separate sheet of graph paper, plot the solution sets to the following equations and inequalities:

- |                 |               |                    |
|-----------------|---------------|--------------------|
| a. $x - y = 10$ | g. $y = 5$    | m. $x > 0$         |
| b. $x - y < 10$ | h. $y < 5$    | n. $y < 0$         |
| c. $y > x - 10$ | i. $x \geq 5$ | o. $x^2 - y = 0$   |
| d. $y \geq x$   | j. $y \neq 1$ | p. $x^2 + y^2 > 0$ |
| e. $x \geq y$   | k. $x = 0$    | q. $xy \leq 0$     |

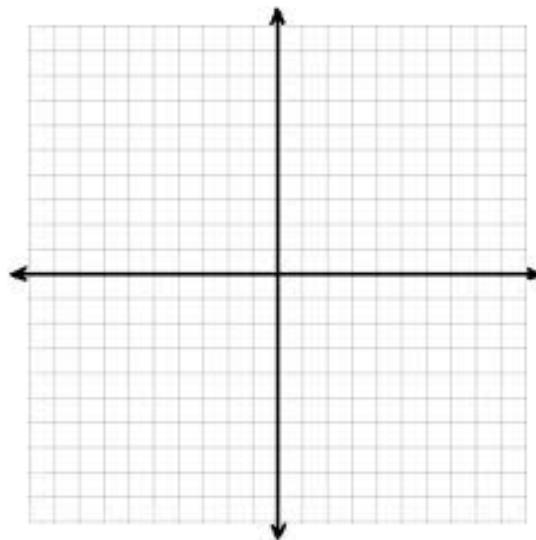
Which of the inequalities in this exercise are *linear* inequalities?

A *half-plane* is the graph of a solution set in the Cartesian coordinate plane of an inequality in two real number variables that is linear and strict.

4. Describe in words the half-plane that is the solution to each inequality.

- a.  $y \geq 0$
- b.  $x < -5$
- c.  $y \geq 2x - 5$
- d.  $y < 2x - 5$

5. Graph the solution set to  $x < -5$ , reading it as an inequality in *one* variable, and describe the solution set in words. Then graph the solution set to  $x < -5$  again, this time reading it as an inequality in *two* variables, and describe the solution set in words.



**Lesson Summary**

An ordered pair is a **solution** to a two-variable inequality if, when each number is substituted into its corresponding variable, it makes the inequality a true number sentence.

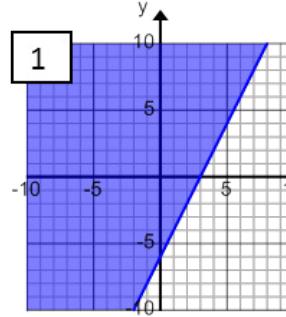
Each ordered pair of numbers in the solution set of the inequality corresponds to a point on the coordinate plane. The set of all such points in the coordinate plane is called the **graph of the inequality**.

The graph of a linear inequality in the coordinate plane is called a **half-plane**.

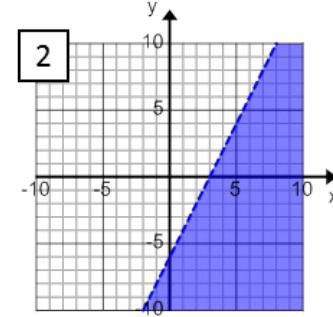
**Problem Set**

1. Match each inequality with its graph. Explain your reasoning.

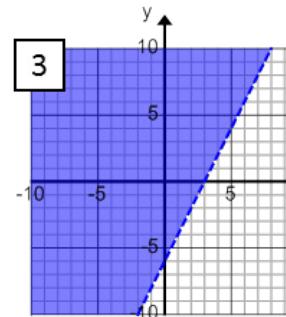
a.  $2x - y > 6$



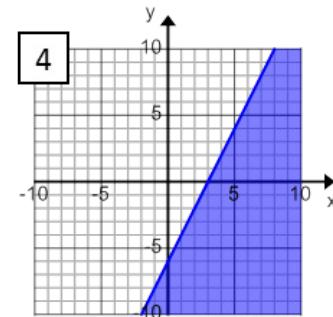
b.  $y \leq 2x - 6$



c.  $2x < y + 6$

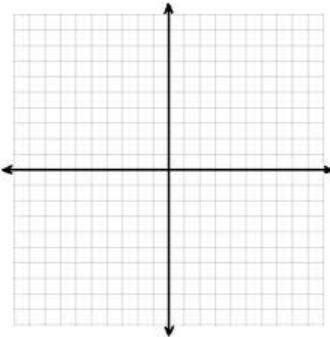


d.  $2x - 6 \leq y$

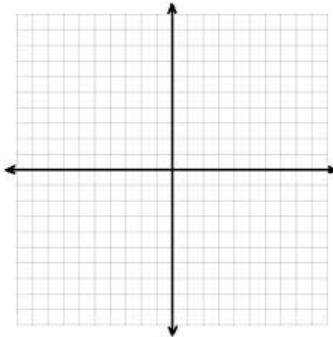


2. Graph the solution set in the coordinate plane. Support your answer by selecting two ordered pairs in the solution set and verifying that they make the inequality true.

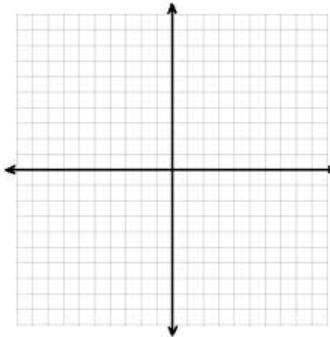
a.  $-10x + y > 25$



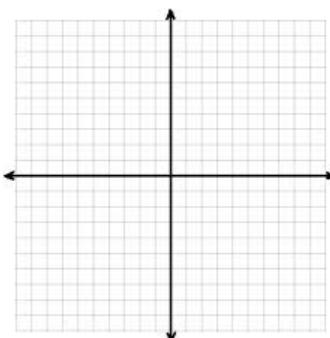
b.  $-6 \leq y$



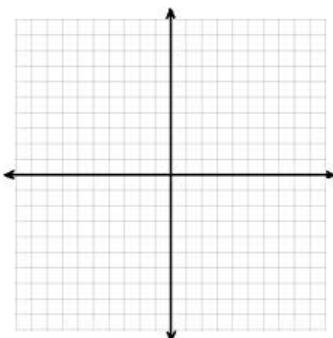
c.  $y \leq -7.5x + 15$



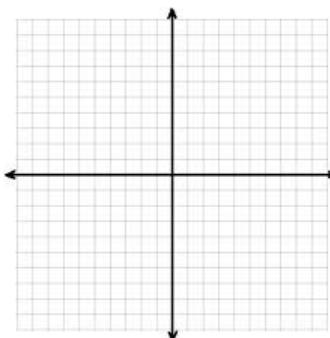
d.  $2x - 8y \leq 24$



e.  $3x < y$



f.  $2x > 0$



3. Marti sells tacos and burritos from a food truck at the farmers market. She sells burritos for \$3.50 each and tacos for \$2.00 each. She hopes to earn at least \$120 at the farmers market this Saturday.
- Identify three combinations of tacos and burritos that will earn Marti more than \$120.
  - Identify three combinations of tacos and burritos that will earn Marti exactly \$120.
  - Identify three combinations of tacos and burritos that will *not* earn Marti at least \$120.
  - Graph your answers to parts (a–c) in the coordinate plane and then shade a half-plane that contains all possible solutions to this problem.
  - Create a linear inequality that represents the solution to this problem. Let  $x$  equal the number of burritos that Marti sells, and let  $y$  equal the number of tacos that Marti sells.
  - Are the points  $(10, 49.5)$  a solution to inequality you created in part (e)? Explain your reasoning.

## Lesson 22: Solution Sets to Simultaneous Equations

### Classwork

#### Opening Exercise

Consider the following compound sentence:  $x + y > 10$  and  $y = 2x + 1$ .

- Circle all the ordered pairs  $(x, y)$  that are solutions to the inequality  $x + y > 10$  (below).
- Underline all the ordered pairs  $(x, y)$  that are solutions to the equation  $y = 2x + 1$ .

(3,7)

(7, 3)

(−1,14)

(0,1)

(12,25)

(5,11)

(0, 12)

(1,8)

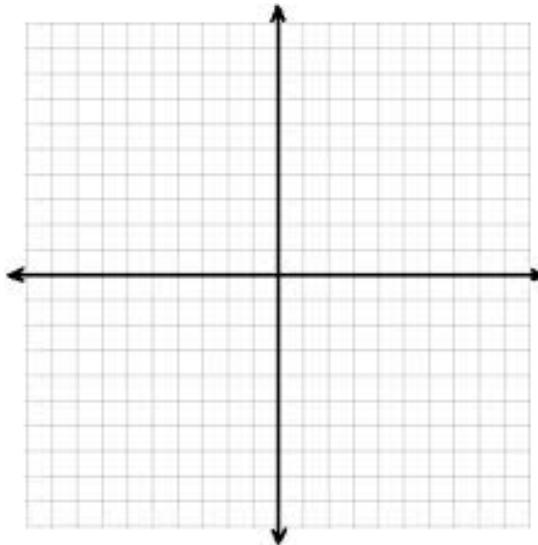
(12,0)

(−1,−1)

- List the ordered pair(s)  $(x, y)$  from above that are solutions to the compound sentence  $x + y > 10$  and  $y = 2x + 1$ .
- List three additional ordered pairs that are solutions to the compound sentence  $x + y > 10$  and  $y = 2x + 1$ .

- Sketch the solution set to the inequality  $x + y > 10$  and the solution set to  $y = 2x + 1$  on the same set of coordinate axes. Highlight the points that lie in BOTH solution sets.

- Describe the solution set to  $x + y > 10$  and  $y = 2x + 1$ .

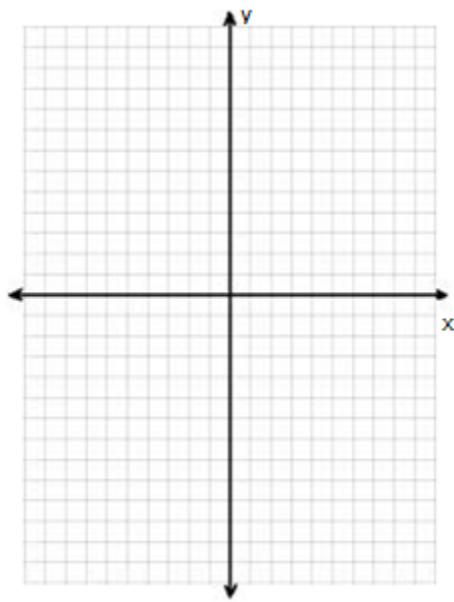


**Example 1**

Solve the following system of equations.

$$\begin{cases} y = 2x + 1 \\ x - y = 7 \end{cases}$$

Graphically:

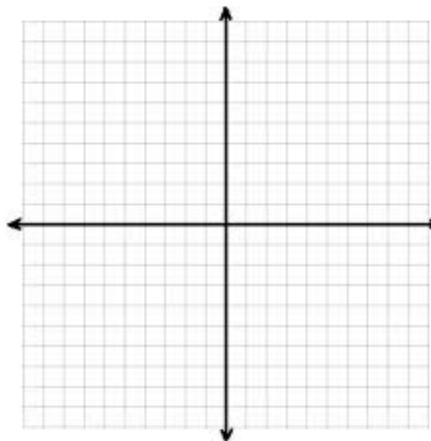


Algebraically:

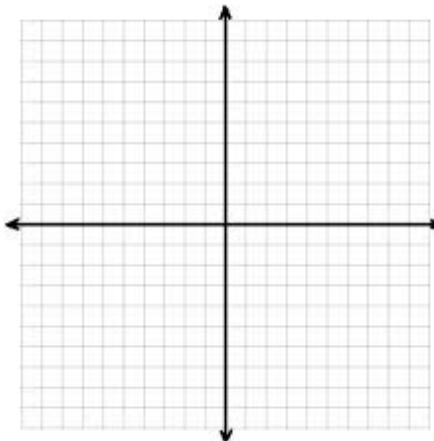
**Exercise 1**

Solve each system first by graphing and then algebraically.

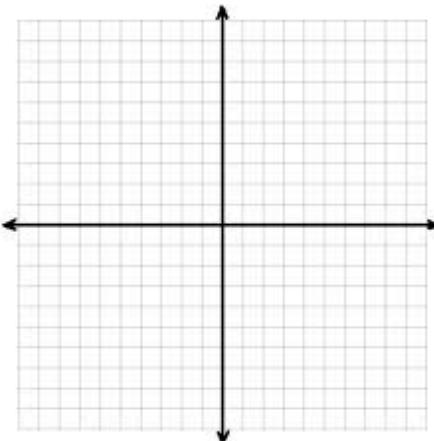
a.  $\begin{cases} y = 4x - 1 \\ y = -\frac{1}{2}x + 8 \end{cases}$



b.  $\begin{cases} 2x + y = 4 \\ 2x + 3y = 9 \end{cases}$



c.  $\begin{cases} 3x + y = 5 \\ 3x + y = 8 \end{cases}$

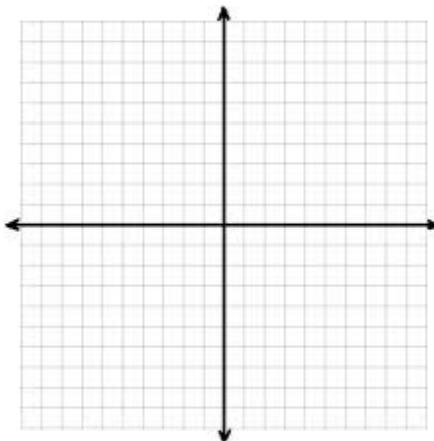


### Example 2

Now suppose the system of equations from Exercise 1(c) was instead a system of inequalities:

$$\begin{cases} 3x + y \geq 5 \\ 3x + y \leq 8 \end{cases}$$

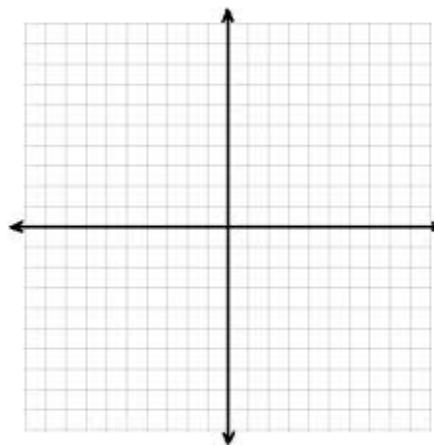
Graph the solution set.



**Example 3**

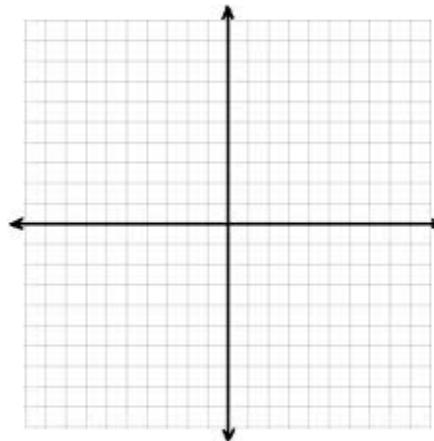
Graph the solution set to the system of inequalities.

$$2x - y < 3 \text{ and } 4x + 3y \geq 0$$

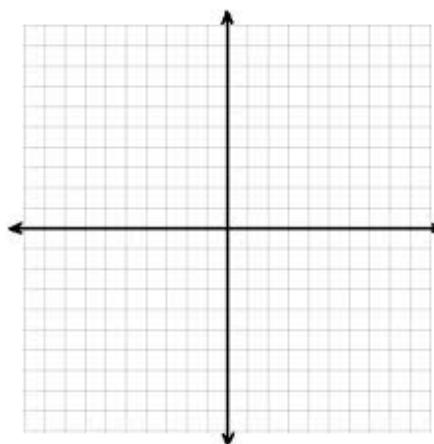
**Exercise 2**

Graph the solution set to each system of inequalities.

a.  $\begin{cases} x - y > 5 \\ x > -1 \end{cases}$



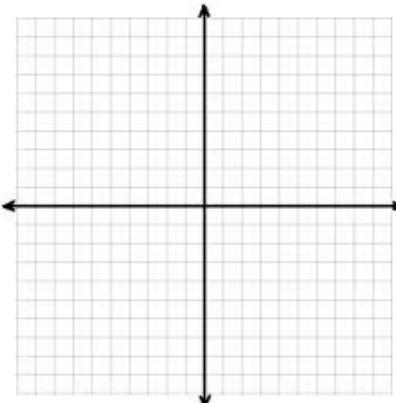
b.  $\begin{cases} y \leq x + 4 \\ y \leq 4 - x \\ y \geq 0 \end{cases}$



## Problem Set

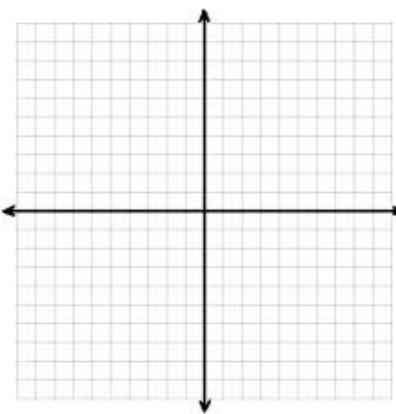
1. Solve the following system of equations first by graphing and then algebraically.

$$\begin{cases} 4x + y = -5 \\ x + 4y = 12 \end{cases}$$



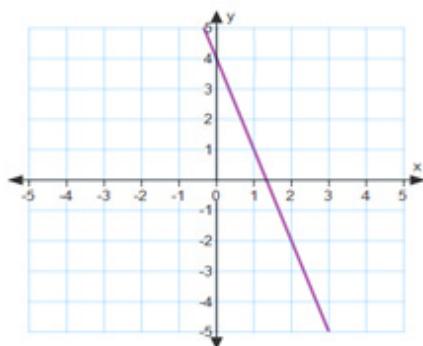
2.

- a. Without graphing, construct a system of two linear equations where  $(0, 5)$  is a solution to the first equation but is not a solution to the second equation, and  $(3, 8)$  is a solution to the system.



- b. Graph the system and label the graph to show that the system you created in part (a) satisfies the given conditions.

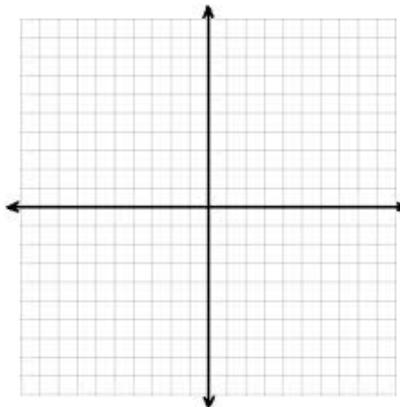
3. Consider two linear equations. The graph of the first equation is shown. A table of values satisfying the second equation is given. What is the solution to the system of the two equations?



$x$	-4	-2	0	2	4
$y$	-26	-18	-10	-2	6

4. Graph the solution to the following system of inequalities:

$$\begin{cases} x \geq 0 \\ y < 2 \\ x + 3y > 0 \end{cases}$$



5. Write a system of inequalities that represents the shaded region of the graph shown.

6. For each question below, provide an explanation or an example to support your claim.
- Is it possible to have a system of equations that has no solution?
  - Is it possible to have a system of equations that has more than one solution?
  - Is it possible to have a system of inequalities that has no solution?

## Lesson 23: Solution Sets to Simultaneous Equations

### Classwork

#### Opening Exercise

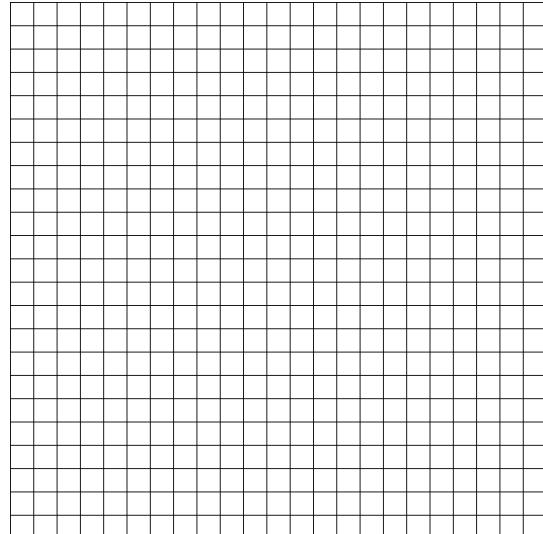
Here is a system of two linear equations. Verify that the solution to this system is (3,4).

Equation A1:  $y = x + 1$

Equation A2:  $y = -2x + 10$

#### Exploratory Challenge

- Write down another system of two linear equations whose solution is (3,4). This time make sure both linear equations have a positive slope.
- Verify that the solution to this system of two linear equations is (3,4).
- Graph equation B1 and B2.
- Are either B1 or B2 equivalent to the original A1 or A2? Explain your reasoning.



- e. Add A1 and A2 to create a new equation C1. Then, multiply A1 by 3 to create a new equation C2. Why is the solution to this system also  $(3, 4)$ ? Explain your reasoning.

The following system of equations was obtained from the original system by adding a multiple of equation A2 to equation A1.

Equation D1:  $y = x + 1$

Equation D2:  $3y = -3x + 21$

- f. What multiple of A2 was added to A1 to create D2?

- g. What is the solution to the system of two linear equations formed by D1 and D2?

- h. Is D2 equivalent to the original A1 or A2? Explain your reasoning.

- i. Start with equation A1. Multiply it by a number of your choice and add the result to equation A2. This creates a new equation E2. Record E2 below to check if the solution is  $(3, 4)$ .

Equation E1:  $y = x + 1$

Equation E2:

**Example 1: Why Does the Elimination Method Work?**

Solve this system of linear equations algebraically.

ORIGINAL SYSTEM

$$2x + y = 6$$

$$x - 3y = -11$$

NEW SYSTEM

SOLUTION

**Exercises 1–2**

1. Explain a way to create a new system of equations with the same solution as the original that eliminates variable  $y$  from one equation. Then determine the solution.

ORIGINAL SYSTEM

$$2x + 3y = 7$$

$$x - y = 1$$

NEW SYSTEM

SOLUTION

2. Explain a way to create a new system of equations with the same solution as the original that eliminates variable  $x$  from one equation. Then determine the solution.

ORIGINAL SYSTEM

$$2x + 3y = 7$$

$$x - y = 1$$

NEW SYSTEM

SOLUTION

**Problem Set**

Try to answer the following without solving for  $x$  and  $y$  first:

1. If  $3x + 2y = 6$  and  $x + y = 4$ , then
  - a.  $2x + y = ?$
  - b.  $4x + 3y = ?$
2. You always get the same solution no matter which two of the four equations you choose from Problem 1 to form a system of two linear equations. Explain why this is true.
3. Solve the system of equations  $\begin{cases} y = \frac{1}{4}x \\ y = -x + 5 \end{cases}$  by graphing. Then, create a new system of equations that has the same solution. Show either algebraically or graphically that the systems have the same solution.
4. Without solving the systems, explain why the following systems must have the same solution.  
System (i):  $4x - 5y = 13$   
 $3x + 6y = 11$       System (ii):  $8x - 10y = 26$   
 $x - 11y = 2$

Solve each system of equations by writing a new system that eliminates one of the variables.

5.  $2x + y = 25$   
 $4x + 3y = 9$
6.  $3x + 2y = 4$   
 $4x + 7y = 1$

## Lesson 24: Applications of Systems of Equations and Inequalities

### Classwork

#### Opening Exercise

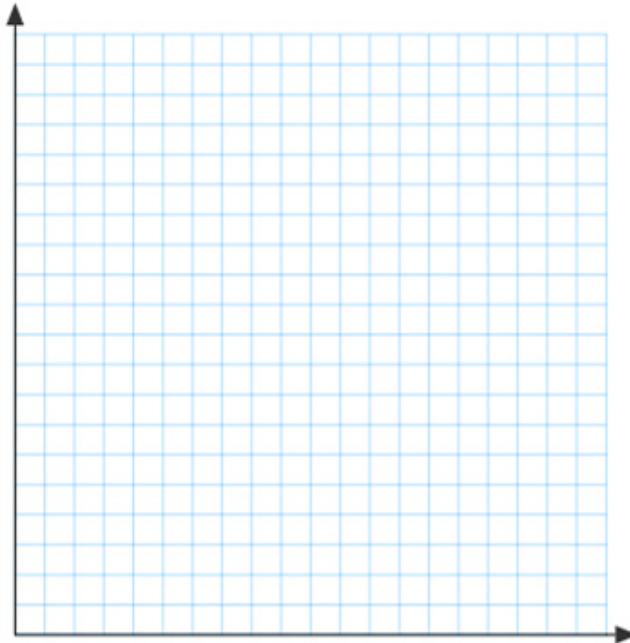
In Lewis Carroll's *Through the Looking Glass*, Tweedledum says, "The sum of your weight and twice mine is 361 pounds." Tweedledee replies, "The sum of your weight and twice mine is 362 pounds." Find both of their weights.

#### Example 1

Lulu tells her little brother, Jack, that she is holding 20 coins all of which are dimes and quarters. They have a value of \$4.10. She says she will give him the coins if he can tell her how many of each she is holding. Solve this problem for Jack.

**Exploratory Challenge**

- a. At a state fair, there is a game where you throw a ball at a pyramid of cans. If you knock over all of the cans, you win a prize. The cost is 3 throws for \$1, but if you have an armband, you get 6 throws for \$1. The armband costs \$10.
- i. Write two cost equations for the game in terms of the number of throws purchased, one without an armband and one with.
- ii. Graph the two cost equations on the same graph. Be sure to label the axes and show an appropriate scale.
- iii. Does it make sense to buy the armband?



- b. A clothing manufacturer has 1,000 yd. of cotton to make shirts and pajamas. A shirt requires 1 yd. of fabric, and a pair of pajamas requires 2 yd. of fabric. It takes 2 hr. to make a shirt and 3 hr. to make the pajamas, and there are 1,600 hr. available to make the clothing.

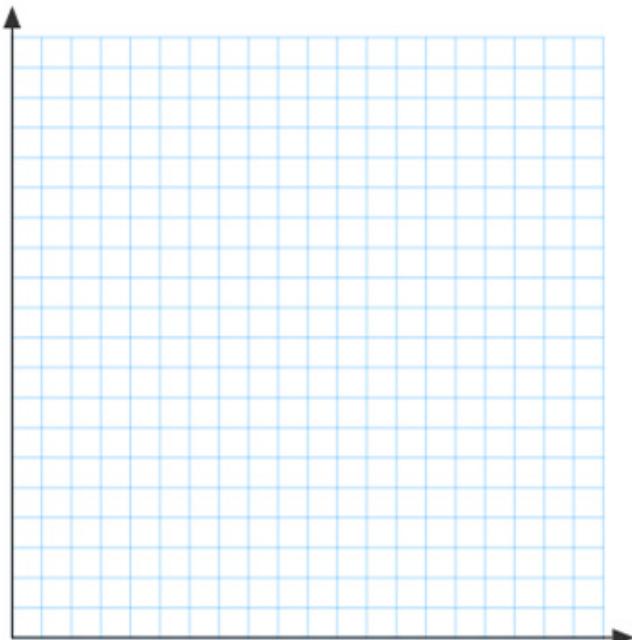
i. What are the variables?

ii. What are the constraints?

iii. Write inequalities for the constraints.

iv. Graph the inequalities and shade the solution set.

v. What does the shaded region represent?



vi. Suppose the manufacturer makes a profit of \$10 on shirts and \$18 on pajamas. How would it decide how many of each to make?

vii. How many of each should the manufacturer make, assuming he will sell all the shirts and pajamas he makes?

**Problem Set**

1. Find two numbers such that the sum of the first and three times the second is 5 and the sum of second and two times the first is 8.
2. A chemist has two solutions: a 50% methane solution and an 80% methane solution. He wants 100 ml of a 70% methane solution. How many ml of each solution does he need to mix?
3. Pam has two part time jobs. At one job, she works as a cashier and makes \$8 per hour. At the second job, she works as a tutor and makes \$12 per hour. One week she worked 30 hours and made \$268. How many hours did she spend at each job?
4. A store sells Brazilian coffee for \$10 per lb. and Columbian coffee for \$14 per lb. If the store decides to make a 150-lb. blend of the two and sell it for \$11 per lb., how much of each type of coffee should be used?
5. A potter is making cups and plates. It takes her 6 min. to make a cup and 3 min. to make a plate. Each cup uses  $\frac{3}{4}$  lb. of clay, and each plate uses 1 lb. of clay. She has 20 hr. available to make the cups and plates and has 250 lb. of clay.
  - a. What are the variables?
  - b. Write inequalities for the constraints.
  - c. Graph and shade the solution set.
  - d. If she makes a profit of \$2 on each cup and \$1.50 on each plate, how many of each should she make in order to maximize her profit?
  - e. What is her maximum profit?

## Lesson 25: Solving Problems in Two Ways—Rates and Algebra

### Classwork

#### Exercise 1

- a. Solve the following problem first using a tape diagram and then using an equation: In a school choir,  $\frac{1}{2}$  of the members were girls. At the end of the year, 3 boys left the choir, and the ratio of boys to girls became 3:4. How many boys remained in the choir?
- b. Which problem solution, the one using a tape diagram or the one using an equation, was easier to set up and solve? Why?

**Exercise 2**

Read the following problem:

All the printing presses at a print shop were scheduled to make copies of a novel and a cookbook. They were to print the same number of copies of each book, but the novel had twice as many pages as the cookbook. All of the printing presses worked for the first day on the larger book, turning out novels. Then, on day two, the presses were split into two equally sized groups. The first group continued printing copies of the novel and finished printing all the copies by the evening of the second day. The second group worked on the cookbook but did not finish by evening. One printing press, working for two additional full days, finished printing the remaining copies of the cookbooks. If all printing presses printed pages (for both the novel and cookbook) at the same constant rate, how many printing presses are there at the print shop?

- a. Solve the problem working with rates to setup a tape diagram or an area model.

- b. Solve the problem by setting up an equation.

**Problem Set**

1. Solve the following problems first using a tape diagram and then by setting up an equation. For each, give your opinion on which solution method was easier. Can you see the connection(s) between the two methods? What does each “unit” in the tape diagram stand for?
  - a. 16 years from now, Pia’s age will be twice her age 12 years ago. Find her present age.
  - b. The total age of a woman and her son is 51 years. Three years ago, the woman was eight times as old as her son. How old is her son now?
  - c. Five years from now, the sum of the ages of a woman and her daughter will be 40 years. The difference in their present age is 24 years. How old is her daughter now?
  - d. Find three consecutive integers such that their sum is 51.
2. Solve the following problems by setting up an equation or inequality.
  - a. If two numbers represented by  $(2m + 1)$  and  $(2m + 5)$  have a sum of 74, find  $m$ .
  - b. Find two consecutive even numbers such that the sum of the smaller number and twice the greater number is 100.
  - c. If 9 is subtracted from a number, and the result is multiplied by 19, the product is 171. Find the number.
  - d. The product of two consecutive whole numbers is less than the sum of the square of the smaller number and 13.
3. The length, 18 meters, is the answer to the following question.  
“The length of a rectangle is three meters longer than its width. The area of the rectangle is 270 square meters. What is the length of the rectangle?”  
Rework this problem: Write an equation using  $L$  as the length (in meters) of the rectangle that would lead to the solution of the problem. Check that the answer above is correct by substituting 18 for  $L$  in your equation.
4. Jim tells you he paid a total of \$23,078.90 for a car, and you would like to know the price of the car before sales tax so that you can compare the price of that model of car at various dealers. Find price of the car before sales tax if Jim bought the car in each of the following states:
  - a. Arizona, where the sales tax is 6.6%.
  - b. New York, where the sales tax is 8.25%.
  - c. A state where the sales tax is  $s\%$ .
5. A checking account is set up with an initial balance of \$9,400, and \$800 is removed from the account at the end of each month for rent (no other user transactions occur on the account).
  - a. Write an inequality whose solutions are the months,  $m$ , in which the account balance is greater than \$3,000. Write the solution set to your equation by identifying all of the solutions.
  - b. Make a graph of the balance in the account after  $m$  months and indicate on the plot the solutions to your inequality in part (a).

6. Axel and his brother like to play tennis. About three months ago they decided to keep track of how many games they have each won. As of today, Axel has won 18 out of the 30 games against his brother.
  - a. How many games would Axel have to win in a row in order to have a 75% winning record?
  - b. How many games would Axel have to win in a row in order to have a 90% winning record?
  - c. Is Axel ever able to reach a 100% winning record? Explain why or why not.
  - d. Suppose that after reaching a winning record of 90% in part (b), Axel had a losing streak. How many games in a row would Axel have to lose in order to drop down to a winning record of 60% again?
7. Omar has \$84 and Calina has \$12. How much money must Omar give to Calina so that Calina will have three times as much as Omar?
  - a. Solve the problem above by setting up an equation.
  - b. In your opinion, is this problem easier to solve using an equation or using a tape diagram? Why?

## Lesson 26: Recursive Challenge Problem—The Double and Add 5

### Game

The *double and add 5* game is *loosely* related to the Collatz conjecture—an *unsolved* conjecture in mathematics named after Lothar Collatz, who first proposed the problem in 1937. The conjecture includes a recurrence relation, *triple and add 1*, as part of the problem statement. A worthwhile activity for you and your class is to read about the conjecture online.

### Classwork

#### Example 1

Fill in the *doubling and adding 5* below:

Number	Double and add 5
1	$1 \cdot 2 + 5 = 7$
7	_____
_____	_____
_____	_____
_____	_____

#### Exercise 1

Complete the tables below for the given starting number.

Number	Double and add 5
2	_____
_____	_____
_____	_____

Number	Double and add 5
3	_____
_____	_____
_____	_____

**Exercise 2**

Given a starting number, double it and add 5 to get the result of round 1. Double the result of Round 1 and add 5, and so on. The goal of the game is to find the smallest starting whole number that produces a result of 100 or greater in three rounds or fewer.

**Exercise 3**

Using a generic initial value,  $a_0$ , and the recurrence relation,  $a_{i+1} = 2a_i + 5$ , for  $i \geq 0$ , find a formula for  $a_1, a_2, a_3, a_4$  in terms of  $a_0$ .

**Vocabulary**

**Sequence:** A *sequence* can be thought of as an ordered list of elements. The elements of the list are called the *terms of the sequence*.

For example,  $(P, O, O, L)$  is a sequence that is different than  $(L, O, O, P)$ . Usually the terms are *indexed* (and therefore ordered) by a subscript starting at either 0 or 1:  $a_1, a_2, a_3, a_4, \dots$ . The “...” symbol indicates that the pattern described is regular, that is, the next term is  $a_5$ , and the next is  $a_6$ , and so on. In the first example,  $a_1 = P$  is the first term,  $a_2 = O$  is the second term, and so on. Both finite and infinite sequences exist everywhere in mathematics. For example, the infinite decimal expansion of  $\frac{1}{3} = 0.\overline{3}$  can be represented as the sequence,  $(0.3, 0.33, 0.333, 0.3333, \dots)$ .

**Recursive Sequence:** An example of a *recursive sequence* is a sequence that is defined by (1) specifying the values of one or more initial terms and (2) having the property that the remaining terms satisfy a recurrence relation that describes the value of a term based upon an algebraic expression in numbers, previous terms, or the index of the term.

The sequence generated by initial term,  $a_1 = 3$ , and recurrence relation,  $a_n = 3a_{n-1}$ , is the sequence  $(3, 9, 27, 81, 243, \dots)$ . Another example, given by the initial terms,  $a_0 = 1, a_1 = 1$ , and recurrence relation,  $a_n = a_{n-1} + a_{n-2}$ , generates the famed *Fibonacci sequence*  $(1, 1, 2, 3, 5, \dots)$ .

**Problem Set**

1. Write down the first 5 terms of the recursive sequences defined by the initial values and recurrence relations below:
  - a.  $a_0 = 0$  and  $a_{i+1} = a_i + 1$ , for  $i \geq 0$ ,
  - b.  $a_1 = 1$  and  $a_{i+1} = a_i + 1$ , for  $i \geq 1$ ,
  - c.  $a_1 = 2$  and  $a_{i+1} = a_i + 2$ , for  $i \geq 1$ ,
  - d.  $a_1 = 3$  and  $a_{i+1} = a_i + 3$ , for  $i \geq 1$ ,
  - e.  $a_1 = 2$  and  $a_{i+1} = 2a_i$ , for  $i \geq 1$ ,
  - f.  $a_1 = 3$  and  $a_{i+1} = 3a_i$ , for  $i \geq 1$ ,
  - g.  $a_1 = 4$  and  $a_{i+1} = 4a_i$ , for  $i \geq 1$ ,
  - h.  $a_1 = 1$  and  $a_{i+1} = (-1)a_i$ , for  $i \geq 1$ ,
  - i.  $a_1 = 64$  and  $a_{i+1} = (-\frac{1}{2})a_i$ , for  $i \geq 1$ ,
2. Look at the sequences you created in Problems 1(b) through 1(d). How would you define a recursive sequence that generates multiples of 31?
3. Look at the sequences you created in problems 1(e) through 1(g). How would you define a recursive sequence that generates powers of 15?
4. The following recursive sequence was generated starting with an initial value of  $a_0$ , and the recurrence relation  $a_{i+1} = 3a_i + 1$ , for  $i \geq 0$ . Fill in the blanks of the sequence  
(\_\_\_\_, \_\_\_, 94, \_\_\_, 850, \_\_\_\_).
5. For the recursive sequence generated by initial value,  $a_0$ , and recurrence relation,  $a_{i+1} = a_i + 2$ , for  $i \geq 0$ , find a formula for  $a_1, a_2, a_3, a_4$  in terms of  $a_0$ . Describe in words what this sequence is generating.
6. For the recursive sequence generated by initial value,  $a_0$ , and recurrence relation,  $a_{i+1} = 3a_i + 1$ , for  $i \geq 0$ , find a formula for  $a_1, a_2, a_3, a_4$  in terms of  $a_0$ .

# Lesson 27: Recursive Challenge Problem—The Double and Add 5 Game

The *double and add 5* game is *loosely* related to the Collatz conjecture—an *unsolved* conjecture in mathematics named after Lothar Collatz, who first proposed the problem in 1937. The conjecture includes a recurrence relation, *triple and add 1*, as part of the problem statement. A worthwhile activity for you and your class is to read about the conjecture online.

## Classwork

### Example 1

Recall Exercise 3 from the previous lesson: Using a generic initial value,  $a_0$ , and the recurrence relation,  $a_{i+1} = 2a_i + 5$ , for  $i \geq 0$ , find a formula for  $a_1, a_2, a_3, a_4$  in terms of  $a_0$ .

### Exercise 1

Using one of the four formulas from Example 1, write an inequality that, if solved for  $a_0$ , will lead to finding the smallest starting whole number for the *double and add 5* game that produces a result of 1,000 or greater in 3 rounds or fewer.

**Exercise 2**

Solve the inequality derived in Exercise 1. Interpret your answer, and validate that it is the solution to the problem. That is, show that the whole number you found results in 1,000 or greater in three rounds, but the previous whole number takes four rounds to reach 1,000.

**Exercise 3**

Find the smallest starting whole number for the *Double and Add 5* game that produces a result of 1,000,000 or greater in four rounds or fewer.

**Lesson Summary**

The formula,  $a_n = 2^n(a_0 + 5) - 5$ , describes the  $n^{\text{th}}$  term of the *double and add 5* game in terms of the starting number  $a_0$  and  $n$ . Use this formula to find the smallest starting whole number for the *double and add 5* game that produces a result of 10,000,000 or greater in 15 rounds or fewer.

**Problem Set**

1. Your older sibling came home from college for the weekend and showed you the following sequences (from her homework) that she claimed were generated from initial values and recurrence relations. For each sequence, find an initial value and recurrence relation that describes the sequence (Your sister showed you an answer to the first problem).
  - (0, 2, 4, 6, 8, 10, 12, 14, 16, ...)
  - (1, 3, 5, 7, 9, 11, 13, 15, 17, ...)
  - (14, 16, 18, 20, 22, 24, 26, ...)
  - (14, 21, 28, 35, 42, 49, ...)
  - (14, 7, 0, -7, -14, -21, -28, -35, ...)
  - (2, 4, 8, 16, 32, 64, 128, ...)
  - (3, 6, 12, 24, 48, 96, ...)
  - (1, 3, 9, 27, 81, 243, ...)
  - (9, 27, 81, 243, ...)
2. Answer the following questions about the recursive sequence generated by initial value,  $a_1 = 4$ , and recurrence relation,  $a_{i+1} = 4a_i$  for  $i \geq 1$ .
  - Find a formula for  $a_1, a_2, a_3, a_4, a_5$  in terms of powers of 4.
  - Your friend, Carl, says that he can describe the  $n^{\text{th}}$  term of the sequence using the formula,  $a_n = 4^n$ . Is Carl correct? Write one or two sentences using the recurrence relation to explain why or why not.
3. The formula,  $a_n = 2^n(a_0 + 5) - 5$ , describes the  $n^{\text{th}}$  term of the *double and add 5* game in terms of the starting number  $a_0$  and  $n$ . Verify that it does describe the  $n^{\text{th}}$  term by filling out the tables below for parts (b) through (e). (The first table is done for you.)
  - Table for  $a_0 = 1$

$n$	$2^n(a_0 + 5) - 5$
1	$2^1 \cdot 6 - 5 = 7$
2	$2^2 \cdot 6 - 5 = 19$
3	$2^3 \cdot 6 - 5 = 43$
4	$2^4 \cdot 6 - 5 = 91$

b. Table for  $a_0 = 8$

$n$	$2^n(a_0 + 5) - 5$
1	_____
2	_____
3	_____
4	_____

c. Table for  $a_0 = 9$

$n$	$2^n(a_0 + 5) - 5$
2	_____
3	_____

d. Table for  $a_0 = 120$

$n$	$2^n(a_0 + 5) - 5$
3	_____
4	_____

e. Table for  $a_0 = 121$

$n$	$2^n(a_0 + 5) - 5$
2	_____
3	_____

4. Bilbo Baggins stated to Samwise Gamgee, “Today, Sam, I will give you \$1. Every day thereafter for the next 14 days, I will take the previous day’s amount, double it and add \$5, and give that new amount to you for that day.”
  - a. How much did Bilbo give Sam on day 15? (Hint: You don’t have to compute each term.)
  - b. Did Bilbo give Sam more than \$350,000 altogether?
  
5. The formula,  $a_n = 2^{n-1}(a_0 + 5) - 5$ , describes the  $n^{\text{th}}$  term of the *Double and Add 5* game in terms of the starting number  $a_0$  and  $n$ . Use this formula to find the smallest starting whole number for the *Double and Add 5* game that produces a result of 10,000,000 or greater in 15 rounds or fewer.

## Lesson 28: Federal Income Tax

### Classwork

#### Important Tax Tables for this Lesson

*Exemption Deductions for Tax Year 2013*

Exemption Class	Exemption Deduction
Single	\$3,900
Married	\$7,800
Married with 1 child	\$11,700
Married with 2 children	\$15,600
Married with 3 children	\$19,500

*Standard Deductions Based Upon Filing Status for Tax Year 2013*

Filing Status	Standard Deduction
Single	\$6,100
Married filing jointly	\$12,200

*Federal Income Tax for Married Filing Jointly for Tax Year 2013*

If taxable income is over--	But not over--	The tax is:	Plus the Marginal Rate	Of the amount over--
\$0	\$17,850	10%		\$0
\$17,850	\$72,500	\$1,785.00	15%	\$17,850
\$72,500	\$146,400	\$9,982.50	25%	\$72,500
\$146,400	\$223,050	\$28,457.50	28%	\$146,400
\$223,050	\$398,350	\$49,919.50	33%	\$223,050
\$398,350	\$450,000	\$107,768.50	35%	\$398,350
\$450,000 +		\$125,846.00	39.6%	\$450,000

**Taxable Income:** The U.S. government considers the *income* of a family (or individual) to include the sum of any money earned from a husband's or wife's jobs, and money made from their personal businesses or investments. The taxes for a household (i.e., an individual or family) are not computed from the income; rather, they are computed from the household's taxable income. For many families, the household's *taxable income* is simply the household's income minus exemption deductions and minus standard deductions:

$$(taxable\ income) = (income) - (exemption\ deduction) - (standard\ deduction)$$

All of the problems we will model in this lesson will use this equation to find a family's taxable income. The only exception is if the family's taxable income is less than zero, in which case we will say that the family's taxable income is just \$0.

Use this formula and the tables above to answer the following questions about taxable income:

### Exercise 1

Find the taxable income of a single person with no kids, who has an income of \$55,000.

### Exercise 2

Find the taxable income of a married couple with two children, who have a combined income of \$55,000.

### Exercise 3

Find the taxable income of a married couple with one child, who has a combined income of \$23,000.

**Federal Income Tax and the Marginal Tax Rate:** Below is an example of how to compute the federal income tax of a household using the Federal Income Tax table above.

### Example 1

Compute the Federal Income Tax for the situation described in Exercise 1 (a single person with no kids making \$55,000).

From the answer in Exercise 1, the taxable income is \$45,000. Looking up \$45,000 in the tax table above, we see that \$45,000 corresponds to the second row because it is between \$17,850 and \$72,500:

If taxable income is over--	But not over--	The tax is:	Plus the Marginal Rate	Of the amount over--
\$17,850	\$72,500	\$1,785.00	15%	\$17,850

To calculate the tax, add \$1,785 plus 15% of the amount of \$45,000 that is over \$17,850. Since  $45,000 - 17,850 = 27,150$ , and 15% of 27,150 is \$4,072.50, the total federal income tax on \$45,000 of taxable income is \$5,857.50.

### Exercise 4

Compute the Federal Income Tax for a married couple with two children making \$127,800.

Taxpayers sometimes misunderstand *marginal tax* to mean: “If my taxable income is \$100,000, and my marginal tax rate is 25%, my federal income taxes are \$25,000.” This statement is not true—they would not owe \$25,000 to the federal government. Instead, a marginal income tax charges a progressively higher tax rate for successively greater levels of income. Therefore, they would really owe:

- 10% on the first \$17,850, or \$1,785 in taxes for the interval from \$0 to \$17,850;
- 15% on the next \$54,650, or \$8,197.50 in taxes for the interval from \$17,850 to \$72,500;
- 25% on the last \$27,500, or \$6,875.00 in taxes for the interval from \$72,500 to \$100,000;

for a total of \$16,857.50 of the \$100,000 of taxable income. Thus, their *effective federal income tax rate* is 16.8575%, not 25% as they claimed. Note that the tax table above incorporates the different intervals so that only one calculation needs to be made (the answer to this problem is the same as the answer in Exercise 5).

**Exercise 5**

Create a table and a graph of federal income tax versus income for a married couple with two children between \$0 of income and \$500,000 of income.

**Exercise 6**

Interpret and validate the graph you created in Exercise 5. Does your graph provide an approximate value for the federal income tax you calculated in Exercise 4?

**Exercise 7**

Use the table you created in Exercise 5 to report on the effective federal income tax rate for a married couple with two children, who makes:

a. \$27,800

b. \$45,650

c. \$500,000

## Problem Set

Use the formula and tax tables provided in this lesson to perform all computations.

1. Find the taxable income of a married couple with two children, who have a combined income of \$75,000.
2. Find the taxable income of a single person with no children, who has an income of \$37,000.
3. Find the taxable income of a married couple with three children, who have a combined income of \$62,000.
4. Find the federal income tax of a married couple with two children, who have a combined income of \$100,000.
5. Find the federal income tax of a married couple with three children, who have a combined income of \$300,000.
6. Find the effective federal income tax rate of a married couple with no children, who have a combined income of \$34,000.
7. Find the effective federal income tax rate of a married couple with one child who have a combined income of \$250,000.
8. The latest report on median household (family) income in the United States is \$50,502 per year. Compute the federal income tax and effective federal income tax rate for a married couple with three children, who have a combined income of \$50,502.
9. Extend the table you created in Exercise 6 by adding a column called, “Effective federal income tax rate.” Compute the effective federal income tax rate to the nearest tenth for each row of the table, and create a graph that shows effective federal income tax rate versus income using the table.