Третьяков А.С. АТ-02
ДЗ-9, 10 задач

№ 10.49)

$$(x^2+xy)y' = x\sqrt{x^2-y^2}+xy+y^2$$

$$u=\frac{y}{x} \quad y=ux \quad y'=u'x+u$$

$$(x^2+x^2u)(u'x+u) = x\sqrt{x^2-x^2u^2}+x^2u+x^2u^2 \quad / : x^2$$

$$(1+u)(u'x+u)=\sqrt{1-u^2}+u+u^2$$

$$u'x=\frac{\sqrt{1-u^2}}{1+u} \qquad \int\frac{1+u}{\sqrt{1-u^2}}=\int\frac{dx}{x}$$

$$\int\frac{du}{\sqrt{1-u^2}} - \frac{1}{2}\int\frac{d(1-u^2)}{\sqrt{1-u^2}} = \arcsin u-\sqrt{1-u^2}$$

$$\boxed{\begin{gathered}1-u^2=0\,?\\ y^2=x^2\\ y=\pm x\end{gathered}}$$

$$\arcsin u-\sqrt{1-u^2}=\ln|x|+C$$

$$\arcsin\frac{y}{x}-\sqrt{1-\frac{y^2}{x^2}}=\ln|x|+C$$

Ответ: $y=\arcsin\frac{y}{x}-\frac{1}{x}\sqrt{x^2-y^2}-\ln x=C,$
$y=\pm x$

№ 10.58) $(x^2+y^2)dy=2xy\,dx$

$$u=\frac{y}{x} \Rightarrow y=ux \Rightarrow dy=u\,dx+x\,du$$

$$\cancel{x^2}(1+u^2)(u\,dx+x\,du)=2\cancel{x^2}u\,dx$$

$$x(1+u^2)du=2u\,dx-u\,dx-u^3dx$$

$$x\,du(1+u^2)=(u-u^3)dx$$

$$\int\frac{1+u^2}{u-u^3}du=\int\frac{dx}{x}$$

$$\int\frac{1+u^2}{u-u^3}du=\int\frac{du}{u}+\int\frac{du}{1-u}-\int\frac{du}{1+u}=$$

$$=\ln|u|-\ln|1-u|-\ln|1+u|$$

$\ln|u| - \ln|1-u| - \ln|1+u| = \ln x + c$

$\ln\left|\frac{u}{1-u^2}\right| = \ln|x| + \ln|c|$

$$\frac{\frac{y}{x}}{1-\frac{y^2}{x^2}} = x \cdot c$$

Ответ:

$y = x^2 c\left(\frac{y^2}{x^2} - 1\right)$

$y = (y^2 c - x^2 c)$

$y = 1x$

№10.60)

$(x+y+1)dx + (2x+2y-1)dy = 0$

$y' = \frac{-x-y-1}{2x+2y-1} = -\frac{1}{2} - \frac{3}{2(2x+2y-1)}$

$u = 2x+2y-1 \Rightarrow y = \frac{u}{2} - x + \frac{1}{2} \Rightarrow y' = \frac{u'}{2} - 1$

$\frac{u'}{2} - 1 = \frac{-1}{2} - \frac{3}{2u} \quad /\cdot 2u$

$u'u - 2u = u - 3$

$\int \frac{u\,du + 3 - 3}{u-3} = \int dx$

$$\boxed{\begin{array}{l} u-3=0 \\ 2x+2y=4 \\ x+y=2 \end{array}}$$

$u + 3\ln|u-3| = x + c$

$2x+2y-1+3\ln|2x+2y-4| = x+c$

Ответ: $x+2y+3\ln|x+y-2| = c, \quad x+y=2$

№10.65) $(\sqrt{xy}-x)\,dy + y\,dx = 0$, $y(1)=1$

$$y' = \frac{-y}{\sqrt{xy}-x} = \frac{-y/x}{\sqrt{\frac{y}{x}}-1}$$

$$u = y/x \Rightarrow y = ux \Rightarrow y' = u'x + u$$

$$u'x + u = \frac{-u}{\sqrt{u}-1} \Rightarrow \int \frac{\sqrt{u}-1}{u\sqrt{u}}\,du = \int \frac{dx}{x}$$

$$-\ln u - \frac{2}{\sqrt{u}} = \ln|x| + C$$

$$-\ln \frac{y}{x} - \frac{2\sqrt{x}}{\sqrt{y}} = \ln|x| + C$$

Ответ: $\ln y + 2\frac{\sqrt{x}}{\sqrt{y}} = 2$

№10.72) $y' + \frac{y}{x} = 2\ln x + 1$

$$y = uv \Rightarrow y' = u'v + v'u$$

$$u'v + v'u + \frac{uv}{x} = 2\ln x + 1$$

$$u'v + u\underbrace{\left(v' + \frac{v}{x}\right)}_{0} = 2\ln x + 1$$

$$v' + \frac{v}{x} = 0 \Rightarrow \int \frac{dv}{v} = -\int \frac{dx}{x} \Rightarrow v = \frac{1}{x}$$

$$u'/x = 2\ln x + 1 \quad / \cdot x$$

$$\int du = \int x(2\ln x + 1)\,dx$$

$$u = x^2 \ln x$$

$$y = \frac{x^2 \ln x + C}{x}$$

Ответ: $y = x\ln x + \frac{C}{x}$

№10.74) $y' = \frac{y}{x+y^3}$

$\frac{1}{x'} = \frac{x+y^3}{y}$

$\frac{1}{x^2} = \frac{x}{y} + y^2$

$x = uv \Rightarrow x^2 = u'v + v'u$

$u'v + v'u = \frac{uv}{y} + y^2$

$u'v + u\underbrace{(v' - \frac{v}{y})}_{0} = y^2$

$v' - \frac{v}{y} = 0$ $\boxed{y=0}$

$\int\frac{dv}{v} = \int\frac{du}{y} \Rightarrow \ln v = \ln y \Rightarrow v = y$

$u'y = y^2 \Rightarrow u' = y \Rightarrow$

$\Rightarrow u = \frac{y^2}{2}$

$x = \frac{y^3}{2} + C$

Ответ: $x = \frac{y^3}{2} + Cy,\ y = 0$

№10.84) $y' = 2y + e^x - x;\ \ y(0) = \frac{1}{4}$

$y = uv \Rightarrow y' = u'v + vu$

$u'v + v'u - 2uv = e^x - x$

$u'v + u\underbrace{(u' - 2v)}_{0} = e^1 - x$

1) $v' - 2v = 0$

$\frac{dv}{dx} = 2v \Rightarrow \int\frac{du}{v} = 2\int dx \Rightarrow \ln v = 2x$

$v = e^{2x}$

2) $u' \cdot e^{2x} = e^x - x \quad | : e^{2x}$

$\frac{du}{dx} = e^{-x} - \frac{x}{e^{2x}}$

$du = \int(e^{-x} - \frac{x}{e^{2x}})dx = -e^{-x} + \frac{x}{2e^{2x}} + \frac{1}{4e^{2x}} + C$

$y = uv = e^{2x}(-e^x + \frac{xe^{-2x}}{2} + \frac{e^{-2x}}{4} + C)$

Ответ: $y = -e^x + \frac{x}{2} + \frac{1}{4} + e^{2x}$

11.10.88) $y' = y(y^3 \cos x + tg\, x)$

$y = uv \Rightarrow y' = u'v + v'u$

$u'v + v'u = u^4 v^4 \cos x + uv\, tg\, x$

$v(\underbrace{u' + u\, tg\, x}_{0}) + v'u = u^4 v^4 \cos x$

$\boxed{\begin{array}{c} u = 0\,? \\ y = 0 \end{array}}$

$\frac{du}{dx} = u\, tg\, x \qquad \int \frac{du}{u} = \int tg\, x\, dx$

$\ln u = -\ln \cos x \qquad u = \frac{1}{\cos x}$

$\frac{dv}{dx} u = u^4 v^4 \cos x \rightarrow \frac{dv}{dx} = \frac{v^4}{\cos^2 x}$

$\int \frac{dv}{v^4} = \int \frac{dx}{\cos^2 x} \rightarrow \frac{v^{-3}}{3} = -tg\, x + C$

$v = \frac{1}{\sqrt[3]{C - 3tg\, x}}$

Ответ: $y = \frac{1}{\cos x \cdot \sqrt[3]{C - 3tg\, x}}$, $y = 0$

11.10.90)

$y' = \frac{2x}{x^2 \cos y + \sin 2y}$

$x' = \frac{x^2 \cos y + \sin 2y}{2x} \quad | \cdot 2x$

$2xx' = x^2 \cos y + \sin 2y$

$z = x^2 \Rightarrow z' = 2xx'$

$z' = z\cos y + \sin 2y; \quad z = uv$

$z' = v'u + u'v$

$u'v + v'u = uv \cos y + \sin 2y$

$v(\underbrace{u' - u\cos y}_{0}) + v'u = \sin 2y$

$u' = u\cos y$

$\int \frac{du}{u} = \int \cos y\, dy \qquad u = e^{\sin y}$

$\frac{dv}{dy} e^{\sin y} = \sin 2y$

$\int dv = \int \frac{2\sin y \cos y}{e^{\sin y}} dy$

$v = \frac{-2\sin y + 2}{e^{\sin y}} + C$

$z = -(2\sin y + 2) + Ce^{2\sin y}$

Ответ: $x^2 = Ce^{\sin y} - 2(\sin y + 1)$

№10.94) $3dy = -(1 + 3y^3) y \sin x\, dx$

$y\left(\frac{\pi}{2}\right) = 1$

$y' = -\frac{y\sin x + 3y^4 \sin x}{3}$

$z = y^{-3} \to z' = -\frac{3y'}{y^4} \Rightarrow y' = -\frac{y^4}{3} z'$

$-\frac{y^4}{3} z' = -\frac{y\sin x + 3y^4 \sin x}{3} \Big| \cdot \left(\frac{-3}{y^4}\right)$

$z' = +\frac{\sin x}{y^3} + 3\sin x$

$z' = z\sin x + 3\sin x = \sin x(z + 3)$

$\int \frac{dz}{z+3} = \int \sin x\, dx \Rightarrow \ln|z+3| = -\cos x \Rightarrow$

$z = Ce^{-\cos x} - 3$

$y^3 = \frac{1}{C \cdot e^{-\cos x} + 3} \qquad 1 = \frac{1}{Ce^{-\cos x} + 3} \Rightarrow C = -2$

Ответ: $y^3 = \frac{e^{\cos x}}{3 - 2e^{\cos x}}$