Task 1

$$v = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \rightarrow H = \begin{pmatrix} 1-2|a_1|^2 & -2a_1a_2^* & \cdots & -2a_1a_n^* \\ -2a_2a_1^* & \ddots & & \\ \vdots & & \ddots & \\ -2a_na_1^* & & & 1-2|a_n|^2 \end{pmatrix} = (H^T)^* = H^+$$

очевидно.

$$HH^+ = H^2 = b_{ij}$$

$$b_{11} = (1-2|a_1|^2)^2 + 4|a_1|^2|a_2|^2 + 4|a_1|^2|a_3|^2 \ldots =$$

$$= 1 - 4|a_1|^2 + 4|a_1|^4 + 4|a_1|^2|a_2|^2 + \ldots + 4|a_1|^2|a_n|^2 =$$

$$= 1 + 4|a_1|^2(-1 + \underbrace{(|a_1|^2 + \ldots + |a_n|^2)}_{1}) = 1 + 4|a_1|^2 \cdot 0 = 1 \Rightarrow \underline{b_{ii} = 1}$$

$(j \neq i)$:

$$b_{ij} = (1-2|a_i|^2)2a_ia_j^* - 2a_i^*a_j(1-2|a_j|^2) +$$

$$+ \sum_{k \neq i,j} 4a_ia_k^* \cdot a_ka_j^* =$$

$$= -a_ia_j^*\left(2 - 4|a_i|^2 + 2 - 4|a_j|^2 - 4\sum_{k \neq i,j}|a_k|^2\right) = -a_ia_j^*\left(4 - 4\underbrace{\sum_k |a_k|^2}_{1}\right) = 0$$

$$\Rightarrow HH^+ = 1 \Rightarrow \underline{H \text{ is unitary}}$$

$$H^+ = H^{-1} \Rightarrow \det H \neq 0 \Rightarrow \underline{\mathrm{rk}\, H = \dim v}$$

Task 2

1) $\|x\|_2 \leq \sqrt{m}\,\|x\|_\infty$

$$\lim_{p\to\infty} \sqrt[p]{\sum_i |x_i|^p} = \lim_{p\to\infty} \sqrt[p]{|x_{max}|^p\left(1 + \underbrace{\left(\frac{x_1}{x_{max}}\right)^p}_{0} \ldots + \underbrace{\left(\frac{x_m}{x_{max}}\right)^p}_{0}\right)} = |x_{max}|$$

$$\sqrt{\sum_i |x_i|^2} \;?\; |x_{max}|\sqrt{m}$$

$$x_{max}^2 + \sum |x_i|^2 \leq x_{max}^2\, m$$

2) $\|A\|_\infty \le \sqrt{n}\,\|A\|_2$

$$\|A_2\| = \sqrt{\sum_m \sum_n |a_{mn}|^2} = \sqrt{tr(A^T A)}$$

Task 4

$$\|UA\|_F = \|AU\|_F = \|A\|_F$$

1) Π $\bar{x}$ - вектор.

$$\langle U\bar{x}, U\bar{y} \rangle = x^* U^+ U y = x^* y = \langle x, y \rangle \Rightarrow$$

$$\Rightarrow \|Ux\|_F^2 = \langle x, x \rangle = \|x\|_F^2$$

2) $A = (\bar{a}_1, \dots \bar{a}_n)$

$$\|UA\|_F^2 = \|(Ua_1, \dots Ua_n)\|_F^2 = \sum_{i=1}^{n} \langle Ua_i, Ua_i \rangle = [\text{см } n.1] =$$

$$= \sum_{i=1}^{n} \|a_i\|_F^2 = \sum_{i=1}^{n} \langle a_i, a_i \rangle = \|A\|_F^2$$

3) $\|AU\|_F = \|(AU)^T\|_F = \|U^T A^T\|_F = \|A^T\|_F = \|A\|_F$

Task 3

$A = 1 + uv^*$

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix} (c \;\; d) = \begin{pmatrix} 1+ac & ad \\ bc & 1+bd \end{pmatrix}$$

$$\det A = 1 + \cancel{acbd} + ac + bd - \cancel{bcad} = 0$$

$$ac + bd = -1$$

$-2\ 1 \quad 1\ 1 \quad \Longrightarrow$ A can be singular.

$$AA^{-1} = 1 \longrightarrow (1 + uv^*)(1 + \alpha uv^*) = 1$$

$$\alpha(uv^*)^2 + \alpha uv^* + uv = 0.$$

$$uv^*(\alpha uv^* + \alpha + 1) = 0.$$

$$\cdot A^{-1} \,\Big|\; \alpha(1 + uv^*) = -1 \longrightarrow 1 \cdot \alpha = -A^{-1} \Rightarrow$$

$$\Longrightarrow A^{-1} - \text{diag} \longrightarrow A - \text{diag}.$$

$$uv^* = \frac{\alpha - 1}{\alpha} \longrightarrow uv^* - \text{diag}.$$

не совсем понимаю, что ещё можно сделать тут.