Planck Time, “Photon On/Off”, and Measurement Context: Rigorous Operational Analysis

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Abstract

Claims like “a photon is on or off” become ambiguous (and sometimes self-contradictory) when classical binary predicates are applied to quantum optical systems without specifying (i) the *mode* defining “the photon” and (ii) the *measurement* defining “on/off.” This paper develops a fully operational account using quantized field modes, number states, coherent states, and explicit detector POVMs (ideal and inefficient on/off detectors with dark counts). We compute detection statistics for several scenarios—vacuum vs. one-photon states, vacuum–one superpositions, coherent and thermal states, Mach–Zehnder interference with and without which-path information (including partial which-path overlap), time-bin superpositions, emitter–field entanglement, and mode-basis dependence—and show that “contradictions” typically arise from implicitly assuming non-contextual, measurement-independent truth values for incompatible observables. We derive Planck units and present a standard heuristic bound suggesting that attempting to operationalize time resolution $\Delta t \ll t_P$ pushes required energies toward the Planck scale, where gravitational backreaction cannot be neglected; we distinguish this from claims that time is discrete in steps of t_P.

1 Notation and assumptions

- We use SI units unless otherwise stated.
- c is the speed of light, G Newton’s constant, $\hbar$ reduced Planck’s constant.
- $\mathcal{H}$ is a Hilbert space; density operators are $\rho \geq 0$ with $\mathrm{Tr}(\rho) = 1$.
- A single bosonic mode has annihilation/creation operators $a, a^\dagger$ with $[a, a^\dagger] = 1$.
- Number operator: $N = a^\dagger a$; Fock states satisfy $N|n\rangle = n|n\rangle$.
- “Photon on/off” will mean a *specific measurement* (typically an on/off threshold detector) applied to a *specific mode* (a chosen wavepacket/spatial-temporal mode). Without that, the question is ill-posed.

2 Introduction: why “photon on/off” can look contradictory

Classical reasoning quietly assumes all of the following:

1. **Definiteness:** a property (e.g., “photon present”) has a definite value at all times.
2. **Non-contextuality:** that value does not depend on how one chooses to measure it.
3. **Measurement revelation:** measurement reveals a pre-existing value rather than creating an outcome via interaction.

Quantum theory rejects this package. A “photon on/off” statement becomes meaningful only after specifying:

- (i) a **mode decomposition** (what counts as “the photon”), and
- (ii) an **observable/POVM** (what counts as “on/off” operationally).

Once those are fixed, the theory makes unambiguous predictions; “contradictions” typically reflect mixing incompatible contexts (e.g., demanding which-path facts *and* interference fringes as if both were simultaneously definite).

2.1 “Square circles” from higher-dimensional projections (explicit construction)

“Square circle” is a useful diagnostic phrase: in a single 2D world, “the same planar figure is both a perfect square and a perfect circle (in the same sense, at the same time)” is a contradiction. But **projection** lets a single higher-dimensional object yield different lower-dimensional appearances.

Example: a right circular cylinder whose shadows are a circle and a square. Let the 3D solid cylinder be

$$C := \{(x, y, z) \in \mathbb{R}^3 : \ x^2 + y^2 \leq R^2, \ 0 \leq z \leq h\}.$$

Define orthographic (drop-coordinate) projections:

$$\pi_{xy}(x, y, z) = (x, y), \qquad \pi_{yz}(x, y, z) = (y, z).$$

Then the image of C under π_{xy} is

$$\pi_{xy}(C) = \{(x, y) \in \mathbb{R}^2 : \exists z \text{ s.t. } (x, y, z) \in C\} = \{(x, y) : x^2 + y^2 \leq R^2\},$$

which is a **filled circle** (disk) of radius R. Similarly,

$$\pi_{yz}(C) = \{(y, z) \in \mathbb{R}^2 : \exists x \text{ s.t. } (x, y, z) \in C\} = \{(y, z) : y^2 \leq R^2, \ 0 \leq z \leq h\} = [-R, R] \times [0, h],$$

which is a **filled rectangle** of width $2R$ and height h. If we choose $h = 2R$, then $\pi_{yz}(C)$ is a **filled square** of side length $2R$. Thus:

- viewing along the cylinder axis gives a **circle**,
- viewing orthogonally gives a **square** (for $h = 2R$).

There is no contradiction because the “circle” and “square” are properties of *different projections* $\pi_{xy}(C)$ and $\pi_{yz}(C)$, not two simultaneous properties of a single 2D object.

Moral. Projections are many-to-one maps: they discard information. Two different projections of the same higher-dimensional object can look mutually incompatible if one incorrectly assumes the projection is “the whole object.”

2.2 Why this maps tightly to quantum measurement (Born rule as projection geometry)

This "projection resolves apparent contradiction" story is not merely metaphorical in quantum theory: **ideal (projective) quantum measurement is literally built from projection operators.**

Let an observable have spectral decomposition

$$A = \sum_a a \, P_a,$$

where P_a are orthogonal projectors ($P_a P_{a'} = \delta_{aa'} P_a$, $\sum_a P_a = I$). For a pure state $|\psi\rangle$, the Born rule can be written as a squared projection norm:

$$p(a) = \langle \psi | P_a | \psi \rangle = \| P_a | \psi \rangle \|^2.$$

The post-measurement state (Lüders rule) is the normalized projection:

$$|\psi\rangle \;\longrightarrow\; \frac{P_a|\psi\rangle}{\sqrt{p(a)}}.$$

So a measurement is, in a precise mathematical sense, a map from a "higher-dimensional" object (state vector or density operator) to a lower-dimensional classical outcome, and it generally discards phase/coherence information relative to other incompatible decompositions. Attempting to demand that a system simultaneously carry measurement-independent truth values for multiple incompatible "views" is analogous to demanding one 2D shadow be both a circle and a square in the same projection.

3 Two-level, self-contained guide (fifth grade and graduate level)

This paper is intentionally written in *two layers*:

- **Layer 1 (fifth grade):** you can read the story and examples without equations.
- **Layer 2 (graduate level):** every key claim is backed by an explicit mathematical model (states, POVMs, Kraus operators) and computed probabilities.

Both layers are self-contained: the same ideas appear in simple language first, then in precise form later.

3.1 Fifth-grade version (no equations required)

What is Planck time? Planck time is a number people get by mixing three "nature constants" (how fast light goes, how strong gravity is, and a quantum constant). It is an *extremely* tiny time: about 0.0000000000000000000000000000000000000000000539 seconds.

Important: This does *not* mean time is made of tiny "ticks" that size. It mainly means: if we try to talk about physics at times that tiny, our usual rules might break because we would need crazy amounts of energy and gravity might matter.

What is a photon? In everyday life, light feels like a smooth beam. In modern physics, we often detect light as little "clicks" in a sensor. A **photon** is a name for the smallest unit of light our detectors can register in the right setup.

But: a photon is not like a tiny marble with a clear path you can track without touching it.

What does "photon on/off" mean? It means we choose a detector and a time window and ask:

- **Off:** no click in that window.
- **On:** a click happens.

Even when there is no light, detectors can sometimes click by accident (noise). So "off" and "on" are really about what the detector does, not a guaranteed label stuck onto the photon.

What is superposition (the big idea)? In normal life, a coin is either heads or tails even if you don't look. In quantum physics, some things are not "secretly one answer" before you measure them. Instead, they can be in a special mix called a **superposition**.

Superposition does *not* mean "we are confused." It means the world can behave in a way where different possibilities can interfere like waves.

A more exact kid definition (and how it differs from guessing). **Definition (kid-level):** A **superposition** is a special kind of "both" where two possibilities are combined in a way that can make *patterns* when they meet again (like waves adding and canceling).

- **Superposition (not just guessing):** The two possibilities can "work together" and make interference patterns.
- **Random guessing (a mix):** Sometimes it is one, sometimes the other, but there is no wave-like pattern from them combining.

The key idea is: superposition is not only about what you know; it is about what *can be observed* when you set up an experiment that lets the possibilities combine.

How can something look like a square and a circle? If you shine a light on a **cylinder**, its shadow can be a **circle** from one direction and a **rectangle** (or even a **square**) from another direction. The cylinder itself is not a contradiction. The different shadows happen because you are looking from different directions and each shadow throws away information.

Quantum measurements are similar: different measurement setups are like different "shadows" of the same underlying quantum state.

Can we look at a photon "mid-flight"? If you do *nothing* in the middle, you only have a math description that tells you the chances of different detector outcomes later. If you try to look in the middle, you must interact with it (like poking a soap bubble to see where it is). That interaction can change what happens later.

So: "mid-flight" does not mean "it becomes unknowable." It means you must say what measurement you use, and that measurement can change the result.

3.2 Graduate-level map (definitions in one place)

If you want the precise version immediately, here is the minimal toolkit used throughout:

- **States:** density operators $\rho \geq 0$ with $\mathrm{Tr}(\rho) = 1$ on a Hilbert space $\mathcal{H}$.
- **Time evolution:** $\rho(t) = U(t, t_0)\rho(t_0)U^\dagger(t, t_0)$.

- **Measurements:** POVMs $\{E_k\}$ with $E_k \geq 0$ and $\sum_k E_k = I$; outcome probabilities $p(k) = \mathrm{Tr}(\rho E_k)$.
- **Measurement backaction:** instruments/Kraus operators $\{M_k\}$ with $E_k = M_k^\dagger M_k$ and $\rho \mapsto M_k \rho M_k^\dagger / p(k)$.
- **Optical modes:** single-mode ladder operators $a, a^\dagger$ with $[a, a^\dagger] = 1$; number operator $N = a^\dagger a$; wavepacket modes $a_f = \int d\omega\, f(\omega) a(\omega)$.
- **On/off detection:** threshold POVM $\Pi_{\text{off}} = |0\rangle\langle 0|$, $\Pi_{\text{on}} = I - |0\rangle\langle 0|$, and the standard inefficient model $\Pi_{\text{off}}^{(\eta)} = \sum_{n \geq 0} (1-\eta)^n |n\rangle\langle n|$.
- **Interference vs. which-path:** coherence is carried by off-diagonal terms (e.g. ρ_{AB} in the path basis) and is reduced by entanglement or intermediate measurement; quantitatively summarized by visibility–distinguishability tradeoffs (e.g. Englert's inequality [7]).
- **Planck time:** $t_P = \sqrt{\hbar G / c^5}$ (dimensional scale), and heuristic arguments (e.g. [8, 9]) suggesting that probing $\Delta t \ll t_P$ pushes required energies toward the Planck scale, where gravitational backreaction cannot be neglected.

3.3 Superposition: explicit definitions (math, physics, and operational)

This paper uses the word *superposition* in its standard quantum-mechanics sense. Because it is commonly misused in popular discussions, we make the definition explicit in three complementary ways.

Definition 1 (linear combination of state vectors). Let $\mathcal{H}$ be a complex Hilbert space. If $|\psi\rangle, |\phi\rangle \in \mathcal{H}$ are state vectors, then any vector

$$|\chi\rangle = \alpha|\psi\rangle + \beta|\phi\rangle \qquad (\alpha, \beta \in \mathbb{C})$$

is a **superposition** (linear combination) of $|\psi\rangle$ and $|\phi\rangle$. If $|\chi\rangle$ is to represent a physical *pure state*, it is normalized ($\langle\chi|\chi\rangle = 1$), and physical states are rays: $|\chi\rangle$ and $e^{\mathrm{i}\theta}|\chi\rangle$ represent the same pure state (global phase is unobservable).

Definition 2 (superposition in a chosen basis and Born rule meaning). Fix an orthonormal basis $\{|i\rangle\}$ of $\mathcal{H}$. Any pure state can be expanded as

$$|\psi\rangle = \sum_i c_i |i\rangle, \qquad \sum_i |c_i|^2 = 1.$$

This is a superposition of basis states with complex amplitudes c_i. If one measures in the $\{|i\rangle\}$ basis (projective measurement with $P_i = |i\rangle\langle i|$), the Born rule gives

$$p(i) = \langle\psi|P_i|\psi\rangle = |c_i|^2.$$

The *relative phases* between the c_i (e.g. the phase of $c_i c_j^*$) do not affect $p(i)$ in this basis but *do* affect statistics in other measurement contexts (e.g. interference measurements), which is the operational content of coherence.

Definition 3 (density-matrix / coherence definition; superposition vs. mixture). The pure state $|\psi\rangle$ corresponds to $\rho = |\psi\rangle\langle\psi|$. In the basis $\{|i\rangle\}$,

$$\rho = \sum_{i,j} c_i c_j^* \, |i\rangle\langle j|.$$

The **off-diagonal terms** ($i \neq j$) are the **coherences**. They are the mathematical signature of superposition *relative to that basis.*

By contrast, a classical probabilistic mixture over the same basis states,

$$\rho_{\text{mix}} = \sum_i p_i |i\rangle\langle i|, \qquad p_i \geq 0, \ \sum_i p_i = 1,$$

has *no* off-diagonal terms in that basis. Many measurements (including any measurement diagonal in the $\{|i\rangle\}$ basis) cannot distinguish ρ from ρ_{mix} if they share the same diagonal elements, but interference-type measurements can.

Operational criterion (how to "detect" superposition). In practice, one says a system exhibits superposition between alternatives $\{|i\rangle\}$ if there exists a measurement context (often a complementary basis or an interferometric recombination) whose outcome probabilities depend on relative phases and thus on off-diagonal elements of ρ in the $\{|i\rangle\}$ basis. Equivalently: superposition is operationally manifested by **interference**.

3.4 Elementary constructions (same idea, two difficulty levels)

This subsection gives a few *elementary constructions* that work as:

- short stories a fifth grader can follow, and
- small mathematical models (projections/POVMs) a graduate student can check.

3.4.1 Three polarizers: adding a filter can increase light

Kid version. Imagine you have two pairs of sunglasses. If you rotate one pair sideways compared to the other, almost no light gets through. Now the weird part: if you put a *third* pair of sunglasses in between, tilted halfway, *some* light can get through again. Adding a filter can increase the light!

This is like the cylinder shadow: the middle sunglasses changes the "question" being asked of the light.

Math version (a clean projection example). Model polarization as a two-dimensional Hilbert space spanned by horizontal/vertical states $|H\rangle, |V\rangle$. A perfect polarizer at angle θ transmits the state

$$|\theta\rangle := \cos\theta \, |H\rangle + \sin\theta \, |V\rangle$$

and is represented by the projector $P_\theta = |\theta\rangle\langle\theta|$. If the input is $|H\rangle$, the probability to pass a θ polarizer is

$$p = \langle H|P_\theta|H\rangle = |\langle\theta|H\rangle|^2 = \cos^2\theta,$$

which is Malus's law at the single-photon level.

With two crossed polarizers (0° then 90°), the pass probability is 0. With three polarizers (0° then 45° then 90°), the pass probability is

$$\cos^2(45°)\ \cos^2(45°) = \frac{1}{4}.$$

The increase occurs because the intermediate projection changes the state.

3.4.2 Two paths (interferometer): why "peeking" changes the outcome

Kid version. Think of a maze with two paths that split and then join again. If you do not peek, the light can act like a wave and the two routes can "match up" so that the photon always exits the same door. If you peek to see which path it took, the peek is a physical interaction and it can ruin the matching, so the exits become more random.

Math version (coherence vs. which-path record). Represent the two paths by orthonormal states $|A\rangle, |B\rangle$ and consider

$$|\psi\rangle = \frac{|A\rangle + e^{\mathrm{i}\phi}|B\rangle}{\sqrt{2}}.$$

Interference at recombination depends on the off-diagonal term ρ_{AB} of $\rho = |\psi\rangle\langle\psi|$. If the path becomes correlated with environment states $|E_A\rangle, |E_B\rangle$ with overlap $\gamma = \langle E_B|E_A\rangle$, the interference term is reduced by $|\gamma|$, and output probabilities take the form

$$P = \frac{1}{2}\left[1 + |\gamma|\cos(\phi - \arg\gamma)\right],$$

with full visibility at $|\gamma| = 1$ and no interference at $|\gamma| = 0$.

3.4.3 On/off vs. phase: why a clicker cannot read "hidden wave-ness"

Kid version. Suppose you only have a simple clicker that tells you "something arrived" or "nothing arrived." Two very different situations can look the same to a clicker:

- Sometimes there is a little bit of light every time.
- Other times, it is *either* no light or a full flash, but mixed randomly.

To tell these apart you need a fancier measurement than a click/no-click box.

Math version. In the $|0\rangle, |1\rangle$ basis (vacuum vs. one-photon), the superposition $|\psi\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$ and the mixture $\rho_{\mathrm{mix}} = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1|$ have the same number populations but different coherence. Any on/off POVM diagonal in number (e.g. $\Pi_{\mathrm{off}}^{(\eta)}$) depends only on populations and cannot distinguish them, while quadrature/homodyne statistics (which depend on off-diagonal terms) can.

3.5 Glossary: kid words $\leftrightarrow$ physics words

This table is meant to make the paper self-contained even if you have never seen the formal terms before.

Kid words	Physics words	Meaning (in this paper)
"Click / no click"	Threshold detector; POVM	A measurement with two outcomes (on/off).
"On/off"	$\Pi_{\mathrm{on}}, \Pi_{\mathrm{off}}$; POVM elements	The operators that compute click probabilities.
"A photon in flight"	Evolving state $\rho(t)$	The mathematical object that predicts future measurement statistics.
"Peeking"	Which-path measurement; environment record	Any interaction that makes path information available and reduces coherence.
"Two roads"	Path basis $\{\lvert A\rangle, \lvert B\rangle\}$	A two-dimensional subspace describing interferometer arms.
"Wave matching"	Interference; phase coherence	Output depends on off-diagonal terms like ρ_{AB}.
"Special mix"	Superposition (pure state)	A coherent combination of possibilities with phase relations.
"Random mix"	Mixed state (density matrix)	Classical uncertainty or entanglement-induced mixture (no coherence).
"Different shadows"	Different measurement bases / projections	Different experimental questions about the same underlying state.
"Changing the question changes the answer"	Context dependence / incompatibility	Different POVMs probe different observables; you cannot demand all answers at once.
"Planck time"	$t_P = \sqrt{\hbar G/c^5}$	A natural quantum-gravity scale, not automatically a smallest tick.

4 Planck units and Planck time

4.1 Dimensional analysis derivation

We seek a time scale constructed from $\hbar$, G, and c. Write

$$t_P \propto \hbar^a G^b c^d.$$

Using base dimensions:

$$[\hbar] = \mathrm{M}\,\mathrm{L}^2\,\mathrm{T}^{-1}, \quad [G] = \mathrm{L}^3\,\mathrm{M}^{-1}\,\mathrm{T}^{-2}, \quad [c] = \mathrm{L}\,\mathrm{T}^{-1}.$$

We require T^1 overall. Compute:

- Mass exponent: $a - b = 0 \Rightarrow a = b$.
- Length exponent: $2a + 3b + d = 0 \Rightarrow 2a + 3a + d = 0 \Rightarrow d = -5a$.
- Time exponent: $-a - 2b - d = 1 \Rightarrow -a - 2a - (-5a) = 1 \Rightarrow 2a = 1 \Rightarrow a = b = \frac{1}{2}$.

Thus $d = -\frac{5}{2}$ and

$$t_P = \sqrt{\frac{\hbar G}{c^5}}.$$

4.2 Numerical value and associated scales

Using (approximately):

$$\hbar \approx 1.054571817 \times 10^{-34}\,\mathrm{J\,s}, \quad G \approx 6.67430 \times 10^{-11}\,\mathrm{m^3\,kg^{-1}\,s^{-2}}, \quad c = 2.99792458 \times 10^{8}\,\mathrm{m\,s^{-1}},$$

we obtain:

$$t_P \approx 5.39 \times 10^{-44}\,\mathrm{s}.$$

Related Planck units:

$$\ell_P = \sqrt{\frac{\hbar G}{c^3}} \approx 1.62 \times 10^{-35}\,\mathrm{m}, \quad m_P = \sqrt{\frac{\hbar c}{G}} \approx 2.18 \times 10^{-8}\,\mathrm{kg},$$

$$E_P = m_P c^2 = \sqrt{\frac{\hbar c^5}{G}} \approx 1.96 \times 10^{9}\,\mathrm{J} \approx 1.22 \times 10^{19}\,\mathrm{GeV}.$$

Note the identity $E_P = \hbar/t_P$.

4.3 What Planck time does (and does not) imply

- **Does imply:** a natural scale where the dimensionless combination of couplings suggests quantum-gravity effects may be important.
- **Does not (by itself) imply:** that time is fundamentally discrete in steps of t_P, or that physical systems must "update" at that cadence, or that any particular "on/off" predicate must flip on Planck timescales.

4.4 Heuristic "minimum time" from quantum localization + gravity (careful)

A standard heuristic combines:

1. To operationally probe time resolution Δt, one needs frequency components with $\Delta\omega \gtrsim 1/\Delta t$ (Fourier limitation).
2. Quanta at angular frequency ω have energy $\sim \hbar\omega$, so resolving Δt pushes energies toward $E \sim \hbar/\Delta t$ (order-of-magnitude).
3. Concentrating energy E within a region of size $R \sim c\Delta t$ yields an associated Schwarzschild radius
$$r_s = \frac{2GE}{c^4}.$$

Requiring $r_s \lesssim R$ (avoid immediate horizon formation in this crude model) gives:

$$\frac{2GE}{c^4} \lesssim c\Delta t.$$

Insert $E \sim \hbar/\Delta t$:

$$\frac{2G\hbar}{c^4\,\Delta t} \lesssim c\Delta t \;\Rightarrow\; \Delta t^2 \gtrsim \frac{2\hbar G}{c^5} \;\Rightarrow\; \Delta t \gtrsim \sqrt{2}\,t_P.$$

Interpretation: this is a plausibility argument that sub-Planckian operational time resolution may require energies at which gravitational backreaction becomes non-negligible. It is not a derivation of time discreteness or a theorem of quantum theory alone.

5 Quantized optical modes: "what is a photon?"

In quantum optics, a "photon" is most cleanly defined as an excitation of a **mode** of the electromagnetic field. A "mode" is not unique: it depends on boundary conditions (cavity/waveguide), filtering, pulse shaping, detection mode-matching, etc.

5.1 Single-mode field algebra

For a single bosonic mode:

$$[a, a^\dagger] = 1, \quad N = a^\dagger a.$$

The Fock basis $\{|n\rangle\}_{n=0}^{\infty}$ satisfies:

$$N|n\rangle = n|n\rangle, \quad a|n\rangle = \sqrt{n}\,|n-1\rangle, \quad a^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle.$$

5.2 Canonical states

- **Vacuum:** $|0\rangle$.
- **Number (Fock) state:** $|n\rangle$ with definite photon number n.
- **Coherent state:** $|\alpha\rangle$ defined by $a|\alpha\rangle = \alpha|\alpha\rangle$. Expansion:

  $$|\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle.$$

  Photon number distribution:

  $$P(n) = |\langle n|\alpha\rangle|^2 = e^{-|\alpha|^2} \frac{|\alpha|^{2n}}{n!},$$

  i.e. Poisson with mean $\mu = |\alpha|^2$.
- **Thermal state:** diagonal in number basis with Bose–Einstein distribution. If mean photon number is $\bar{n}$:

  $$\rho_{\text{th}} = \sum_{n=0}^{\infty} \frac{\bar{n}^n}{(1+\bar{n})^{n+1}} |n\rangle\langle n|.$$

5.3 Wavepacket modes (time/frequency localized photons)

For a continuum of frequency modes with operators $a(\omega)$ satisfying

$$[a(\omega), a^\dagger(\omega')] = \delta(\omega - \omega'),$$

define a normalized wavepacket mode $f(\omega)$ with $\int d\omega\, |f(\omega)|^2 = 1$ and

$$a_f = \int d\omega\, f(\omega)\, a(\omega), \quad [a_f, a_f^\dagger] = 1.$$

Then $|1_f\rangle = a_f^\dagger|0\rangle$ is "one photon in mode f." Crucially, different choices of f define different "photon number" questions.

6 What does "on/off" mean? Explicit measurement models

6.1 Photon number projective measurement vs. on/off detection

An ideal number-resolving measurement uses projectors $\Pi_n = |n\rangle\langle n|$. An **ideal on/off detector** (threshold detector) distinguishes only:

- "off" = vacuum ($n = 0$),
- "on" = any nonzero photon number ($n \geq 1$).

The corresponding POVM elements are:

$$\Pi_{\text{off}} = |0\rangle\langle 0|, \quad \Pi_{\text{on}} = I - |0\rangle\langle 0|.$$

For a state ρ, predicted probabilities:

$$p_{\text{off}} = \text{Tr}(\rho\, \Pi_{\text{off}}), \quad p_{\text{on}} = \text{Tr}(\rho\, \Pi_{\text{on}}) = 1 - p_{\text{off}}.$$

6.2 Inefficiency and dark counts (standard quantum-optics model)

Let $\eta \in [0, 1]$ be detection efficiency. A common model treats each photon as independently detected with probability η. Then the probability of **no click** given n photons is $(1-\eta)^n$. This corresponds to the POVM:

$$\Pi_{\text{off}}^{(\eta)} = \sum_{n=0}^{\infty} (1-\eta)^n\, |n\rangle\langle n|, \quad \Pi_{\text{on}}^{(\eta)} = I - \Pi_{\text{off}}^{(\eta)}.$$

Add a dark-count probability p_d per detection window by mixing with a classical "false click":

$$p_{\text{off}} = (1 - p_d)\, \text{Tr}(\rho\, \Pi_{\text{off}}^{(\eta)}), \quad p_{\text{on}} = 1 - p_{\text{off}}.$$

6.3 Immediate consequence: phases can be invisible to on/off

Because $\Pi_{\text{off}}^{(\eta)}$ is diagonal in the number basis, any relative phase between $|0\rangle$ and $|1\rangle$ in a superposition $\alpha|0\rangle + \beta|1\rangle$ will not affect $p_{\text{on/off}}$. To see that phase, one needs a different measurement (e.g. homodyne).

7 Scenarios with explicit calculations

7.1 Vacuum vs. one-photon Fock state

Let $\rho = |0\rangle\langle 0|$. With ideal on/off:

$$p_{\text{off}} = 1, \quad p_{\text{on}} = 0.$$

With inefficiency and dark counts:

$$\text{Tr}(\rho\, \Pi_{\text{off}}^{(\eta)}) = (1-\eta)^0 = 1 \Rightarrow p_{\text{on}} = 1 - (1-p_d) \cdot 1 = p_d.$$

So even vacuum can "click" due to dark counts.

Now take $\rho = |1\rangle\langle 1|$. Then

$$\text{Tr}(\rho\, \Pi_{\text{off}}^{(\eta)}) = (1-\eta)^1 = 1 - \eta \Rightarrow p_{\text{on}} = 1 - (1-p_d)(1-\eta) = p_d + (1-p_d)\eta.$$

For $p_d = 0$, $p_{\text{on}} = \eta$: a one-photon state does not guarantee a click unless $\eta = 1$.

7.2 The clearest "on/off superposition": $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

Let $|\alpha|^2 + |\beta|^2 = 1$ and $\rho = |\psi\rangle\langle\psi|$. Because $\Pi_{\text{off}}^{(\eta)}$ is diagonal in $|n\rangle$,

$$\mathrm{Tr}(\rho\,\Pi_{\text{off}}^{(\eta)}) = |\alpha|^2(1-\eta)^0 + |\beta|^2(1-\eta)^1 = |\alpha|^2 + (1-\eta)|\beta|^2 = 1 - \eta|\beta|^2.$$

Thus (with $p_d = 0$):

$$p_{\text{on}} = \eta|\beta|^2.$$

With ideal detection $\eta = 1$, $p_{\text{on}} = |\beta|^2$.

7.2.1 On/off cannot see coherence; homodyne (quadrature) can (explicit calculation)

The on/off POVM depends only on the populations ρ_{nn}. This means a coherent superposition and an incoherent mixture can have identical on/off statistics. Define the mixture

$$\rho_{\text{mix}} := |\alpha|^2|0\rangle\langle 0| + |\beta|^2|1\rangle\langle 1|.$$

Then for either $\rho = |\psi\rangle\langle\psi|$ or ρ_{mix},

$$p_{\text{on}} = 1 - \mathrm{Tr}(\rho\,\Pi_{\text{off}}^{(\eta)}) = \eta|\beta|^2.$$

To see the difference, consider a homodyne (quadrature) measurement. Define the dimensionless quadrature operator

$$X := \frac{a + a^\dagger}{\sqrt{2}},$$

with eigenstates $|x\rangle$ satisfying $X|x\rangle = x|x\rangle$. In the x-representation, the first two harmonic-oscillator wavefunctions are:

$$\psi_0(x) := \langle x|0\rangle = \pi^{-1/4}e^{-x^2/2}, \qquad \psi_1(x) := \langle x|1\rangle = \pi^{-1/4}\sqrt{2}\,x\,e^{-x^2/2}.$$

For the pure state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$,

$$\langle x|\psi\rangle = \alpha\psi_0(x) + \beta\psi_1(x) = \pi^{-1/4}e^{-x^2/2}\left(\alpha + \sqrt{2}\,x\,\beta\right),$$

so the probability density is

$$P_\psi(x) = |\langle x|\psi\rangle|^2 = \pi^{-1/2}e^{-x^2}\left(|\alpha|^2 + 2x^2|\beta|^2 + 2\sqrt{2}\,x\,\mathrm{Re}(\alpha^*\beta)\right).$$

For the mixture,

$$P_{\text{mix}}(x) = \mathrm{Tr}(\rho_{\text{mix}}|x\rangle\langle x|) = \pi^{-1/2}e^{-x^2}\left(|\alpha|^2 + 2x^2|\beta|^2\right),$$

which lacks the linear "interference" term proportional to $\mathrm{Re}(\alpha^*\beta)$. Thus coherence (and relative phase) is operationally accessible in quadrature statistics even though it is invisible to on/off detection. In practice one measures a phase-rotated quadrature $X_\varphi = (ae^{-\mathrm{i}\varphi} + a^\dagger e^{\mathrm{i}\varphi})/\sqrt{2}$ by varying the local-oscillator phase φ.

7.3 Coherent state $|\alpha\rangle$: Poisson counting $\Rightarrow$ closed-form click probability

For $\rho = |\alpha\rangle\langle\alpha|$ with mean photon number $\mu = |\alpha|^2$,

$$\mathrm{Tr}(\rho\,\Pi_{\mathrm{off}}^{(\eta)}) = \sum_{n=0}^{\infty}(1-\eta)^n\,P(n) = \sum_{n=0}^{\infty}(1-\eta)^n\,e^{-\mu}\frac{\mu^n}{n!}.$$

Recognize the exponential series:

$$\mathrm{Tr}(\rho\,\Pi_{\mathrm{off}}^{(\eta)}) = e^{-\mu}\sum_{n=0}^{\infty}\frac{((1-\eta)\mu)^n}{n!} = e^{-\mu}\,e^{(1-\eta)\mu} = e^{-\eta\mu}.$$

So (with dark counts p_d):

$$p_{\mathrm{on}} = 1-(1-p_d)e^{-\eta\mu}.$$

7.4 Thermal state ρ_{th}: click probability and heavy tails

For $\rho = \rho_{\mathrm{th}}$ with mean $\bar{n}$,

$$\mathrm{Tr}(\rho\,\Pi_{\mathrm{off}}^{(\eta)}) = \sum_{n=0}^{\infty}(1-\eta)^n\,\frac{\bar{n}^n}{(1+\bar{n})^{n+1}} = \frac{1}{1+\bar{n}}\sum_{n=0}^{\infty}\left(\frac{(1-\eta)\bar{n}}{1+\bar{n}}\right)^n.$$

This is a geometric series with ratio $r = \frac{(1-\eta)\bar{n}}{1+\bar{n}}$, so

$$\mathrm{Tr}(\rho\,\Pi_{\mathrm{off}}^{(\eta)}) = \frac{1}{1+\eta\bar{n}}.$$

Thus (with $p_d = 0$):

$$p_{\mathrm{on}} = 1-\frac{1}{1+\eta\bar{n}} = \frac{\eta\bar{n}}{1+\eta\bar{n}}.$$

7.5 Mach–Zehnder interferometer: single-photon interference vs. which-path information

We model two spatial modes (arms) A and B with creation operators $a^\dagger, b^\dagger$.

7.5.1 50/50 beamsplitter transformation (one convention)

Let the first beamsplitter implement:

$$U_{\mathrm{BS}}^\dagger a\,U_{\mathrm{BS}} = \frac{a+\mathrm{i}b}{\sqrt{2}},\quad U_{\mathrm{BS}}^\dagger b\,U_{\mathrm{BS}} = \frac{\mathrm{i}a+b}{\sqrt{2}}.$$

Input state: one photon in mode a, vacuum in b:

$$|\psi_{\mathrm{in}}\rangle = |1\rangle_a|0\rangle_b = a^\dagger|0\rangle.$$

After the first beamsplitter:

$$|\psi_1\rangle = U_{\mathrm{BS}}|\psi_{\mathrm{in}}\rangle = \left(\frac{a^\dagger - \mathrm{i}b^\dagger}{\sqrt{2}}\right)|0\rangle = \frac{|1,0\rangle - \mathrm{i}|0,1\rangle}{\sqrt{2}}.$$

Apply a phase shift ϕ in arm B: $b^\dagger \mapsto e^{\mathrm{i}\phi} b^\dagger$:

$$|\psi_\phi\rangle = \frac{|1,0\rangle - \mathrm{i}e^{\mathrm{i}\phi}|0,1\rangle}{\sqrt{2}}.$$

After the second identical beamsplitter, define output modes:

$$c = \frac{a + \mathrm{i}b}{\sqrt{2}}, \quad d = \frac{\mathrm{i}a + b}{\sqrt{2}} \Rightarrow a = \frac{c - \mathrm{i}d}{\sqrt{2}}, \quad b = \frac{-\mathrm{i}c + d}{\sqrt{2}}.$$

Using

$$a^\dagger = \frac{c^\dagger + \mathrm{i}d^\dagger}{\sqrt{2}}, \quad b^\dagger = \frac{\mathrm{i}c^\dagger + d^\dagger}{\sqrt{2}},$$

we have

$$|\psi_\phi\rangle = \frac{1}{\sqrt{2}}\left(a^\dagger - \mathrm{i}e^{\mathrm{i}\phi}b^\dagger\right)|0\rangle = \frac{1}{\sqrt{2}}\left(\frac{c^\dagger + \mathrm{i}d^\dagger}{\sqrt{2}} - \mathrm{i}e^{\mathrm{i}\phi}\frac{\mathrm{i}c^\dagger + d^\dagger}{\sqrt{2}}\right)|0\rangle.$$

Simplify the bracket:

$$\frac{1}{\sqrt{2}}\left(\frac{c^\dagger + \mathrm{i}d^\dagger}{\sqrt{2}} + \frac{e^{\mathrm{i}\phi}c^\dagger - \mathrm{i}e^{\mathrm{i}\phi}d^\dagger}{\sqrt{2}}\right) = \frac{1}{2}\left((1 + e^{\mathrm{i}\phi})c^\dagger + \mathrm{i}(1 - e^{\mathrm{i}\phi})d^\dagger\right).$$

Thus:

$$|\psi_{\text{out}}\rangle = \frac{1 + e^{\mathrm{i}\phi}}{2}|1\rangle_c|0\rangle_d + \frac{\mathrm{i}(1 - e^{\mathrm{i}\phi})}{2}|0\rangle_c|1\rangle_d.$$

Therefore detection probabilities:

$$P(c) = \left|\frac{1 + e^{\mathrm{i}\phi}}{2}\right|^2 = \frac{1 + \cos\phi}{2} = \cos^2\left(\frac{\phi}{2}\right),$$

$$P(d) = \left|\frac{1 - e^{\mathrm{i}\phi}}{2}\right|^2 = \frac{1 - \cos\phi}{2} = \sin^2\left(\frac{\phi}{2}\right).$$

7.5.2 Which-path information as dephasing: interference disappears

Suppose the path becomes entangled with an environment/pointer state $|E_A\rangle, |E_B\rangle$:

$$|\Psi\rangle = \frac{|1,0\rangle|E_A\rangle - \mathrm{i}|0,1\rangle|E_B\rangle}{\sqrt{2}}.$$

If $\langle E_A|E_B\rangle = 0$, tracing out the environment yields:

$$\rho_{AB} = \mathrm{Tr}_E(|\Psi\rangle\langle\Psi|) = \frac{1}{2}\left(|1,0\rangle\langle 1,0| + |0,1\rangle\langle 0,1|\right),$$

with no coherence. Propagating this mixture through the phase shifter and second beamsplitter yields $P(c) = P(d) = \frac{1}{2}$, independent of ϕ.

7.5.3 Partial which-path information: overlap controls visibility and yields a quantitative tradeoff

Let $\gamma := \langle E_B | E_A \rangle$, with $0 \le |\gamma| \le 1$, and include a phase shift ϕ in arm B:

$$|\Psi_\phi\rangle = \frac{|1,0\rangle|E_A\rangle - \mathrm{i}e^{\mathrm{i}\phi}|0,1\rangle|E_B\rangle}{\sqrt{2}}.$$

Tracing out the environment yields

$$\rho_{AB} = \frac{1}{2}\Big(|1,0\rangle\langle 1,0| + |0,1\rangle\langle 0,1| + \mathrm{i}\gamma e^{-\mathrm{i}\phi}|1,0\rangle\langle 0,1| - \mathrm{i}\gamma^* e^{\mathrm{i}\phi}|0,1\rangle\langle 1,0|\Big).$$

Using $c = (a + \mathrm{i}b)/\sqrt{2}$ so that

$$n_c = c^\dagger c = \frac{a^\dagger a + b^\dagger b + \mathrm{i}a^\dagger b - \mathrm{i}b^\dagger a}{2},$$

one obtains

$$P(c) = \frac{1}{2}\left[1 + |\gamma|\cos(\phi - \arg\gamma)\right],$$

so the fringe visibility is $V = |\gamma|$.

The best possible which-path guess from the environment record is a binary state discrimination problem between $|E_A\rangle$ and $|E_B\rangle$ with equal priors. Helstrom's bound gives [6]

$$P^\star_{\mathrm{succ}} = \frac{1}{2}\left(1 + \sqrt{1 - |\langle E_A|E_B\rangle|^2}\right) = \frac{1}{2}\left(1 + \sqrt{1 - |\gamma|^2}\right).$$

Define distinguishability $D := 2P^\star_{\mathrm{succ}} - 1 = \sqrt{1 - |\gamma|^2}$. Therefore

$$V^2 + D^2 = 1$$

in this pure-state model (more general models give $V^2 + D^2 \le 1$).

7.6 Time-bin superposition: "early or late" is not classical ignorance

Let $|E\rangle$ and $|L\rangle$ be orthonormal time-bin modes. Consider

$$|\psi\rangle = \frac{|E\rangle + e^{\mathrm{i}\theta}|L\rangle}{\sqrt{2}}.$$

Measurement in the $\{|E\rangle, |L\rangle\}$ basis yields each with probability 1/2, independent of θ. But recombination (measuring in the $|\pm\rangle = (|E\rangle \pm |L\rangle)/\sqrt{2}$ basis) yields:

$$P(+) = \left|\frac{1 + e^{\mathrm{i}\theta}}{2}\right|^2 = \cos^2\left(\frac{\theta}{2}\right), \qquad P(-) = \sin^2\left(\frac{\theta}{2}\right).$$

7.7 Emitter–field entanglement: "photon on/off" can be conditional

Consider

$$|\Psi\rangle = \frac{|e\rangle|0\rangle + |g\rangle|1\rangle}{\sqrt{2}}.$$

Tracing out the atom yields

$$\rho_{\text{field}} = \text{Tr}_{\text{atom}}(|\Psi\rangle\langle\Psi|) = \frac{1}{2}\left(|0\rangle\langle 0| + |1\rangle\langle 1|\right).$$

An on/off detector with efficiency η and $p_d = 0$ predicts:

$$p_{\text{on}} = \text{Tr}(\rho_{\text{field}}\,\Pi_{\text{on}}^{(\eta)}) = \frac{\eta}{2}.$$

Conditioning on measuring the atom prepares definite field states $|0\rangle$ or $|1\rangle$; thus “photon on/off” can be a conditional statement relative to correlated degrees of freedom.

7.8 Mode-basis dependence: a definite photon in one mode can be a superposition in another

Let a_u, a_v be annihilation operators for orthonormal modes u, v and define rotated modes:

$$a_\pm = \frac{a_u \pm a_v}{\sqrt{2}}.$$

Then $a_u^\dagger = (a_+^\dagger + a_-^\dagger)/\sqrt{2}$. Acting on vacuum:

$$|1_u, 0_v\rangle = a_u^\dagger|0\rangle = \frac{1}{\sqrt{2}}\left(a_+^\dagger + a_-^\dagger\right)|0\rangle = \frac{|1_+, 0_-\rangle + |0_+, 1_-\rangle}{\sqrt{2}}.$$

So “one photon present” is mode-dependent; any binary on/off claim must specify the detector’s mode.

8 Where the “contradictions” come from (and how superposition resolves them)

The recurring pattern is:

- one measurement context asks a yes/no question about a particular basis (e.g., “which path?”, “which time bin?”, “is $n = 0$?”),
- another context asks about interference/coherence in a complementary basis.

Attempting to assign simultaneous, context-independent truth values to both sets of questions leads to apparent contradictions. Quantum theory avoids contradiction by representing the system by a state (vector or density operator) that can contain coherent superpositions, predicting outcomes via Born-rule probabilities $\text{Tr}(\rho\,\Pi_i)$, and allowing measurement interactions (and entanglement with environments) to change which coherences are accessible.

9 Time resolution, bandwidth, and the Planck scale (operational framing)

9.1 “Time window” vs. “photon present”: what a detector really does

A real detector defines a detection **mode** (filtering + mode overlap), a detection **window** of duration T, and a POVM (often thresholding). Thus “photon on in a Planck-time slice” requires a physically realizable detector whose response has support on that timescale *and* couples to the relevant ultra-broadband modes.

9.2 "Photon in flight": what an intermediate-time statement actually means

The phrase "examine the photon mid-flight" is only meaningful once it is translated into an operational question. In standard quantum theory, between a preparation at time t_0 and a later measurement at time t_1, the system is represented by a state that evolves unitarily:

$$\rho(t) = U(t,t_0)\,\rho(t_0)\,U^\dagger(t,t_0).$$

Counterfactual vs. intervening measurement. If one does *not* insert any apparatus at an intermediate time $t \in (t_0, t_1)$, then statements like "the photon was *really* on/off at time t" are counterfactual: they refer to an outcome of a measurement that was not performed. What is well-defined are probabilities for *hypothetical* measurements one *could* perform at time t.

Intermediate measurement as a POVM (and its backaction). To "look" mid-flight, one must couple the field to a device during some time window and thereby implement a POVM $\{E_k\}$ at time t. The Born rule gives

$$p(k;t) = \mathrm{Tr}\big(\rho(t)\,E_k\big).$$

If the measurement is actually performed, the post-measurement state conditioned on outcome k is updated by Kraus operators $\{M_k\}$ (with $E_k = M_k^\dagger M_k$):

$$\rho(t) \;\longrightarrow\; \rho_k(t) = \frac{M_k\,\rho(t)\,M_k^\dagger}{p(k;t)}.$$

This *changes* the statistics of later measurements at t_1. In interferometers, for example, even weak which-path probes reduce coherence and thus reduce visibility (quantitatively captured by overlap factors like γ and inequalities like $V^2 + D^2 \le 1$).

An explicit "mid-flight probe" model: unsharp which-path measurement. It is useful to see the disturbance quantitatively in the simplest nontrivial setting: a single-photon in a two-path superposition. Let $\{|A\rangle, |B\rangle\}$ denote orthonormal path states (arms of a Mach–Zehnder). Consider the pre-measurement state at time t:

$$\rho(t) = \begin{pmatrix} \rho_{AA} & \rho_{AB} \\ \rho_{BA} & \rho_{BB} \end{pmatrix} \quad \text{in the basis } \{|A\rangle, |B\rangle\}.$$

Now insert a weak which-path monitor in arm B whose *outcomes* are: "no record" ($k=0$) and "path-B record" ($k=1$). One simple instrument realizing this is:

$$M_0 = |A\rangle\langle A| + \sqrt{1-\lambda}\,|B\rangle\langle B|, \qquad M_1 = \sqrt{\lambda}\,|B\rangle\langle B|,$$

with measurement strength $\lambda \in [0,1]$. The POVM elements are

$$E_0 = M_0^\dagger M_0 = |A\rangle\langle A| + (1-\lambda)|B\rangle\langle B|, \qquad E_1 = M_1^\dagger M_1 = \lambda |B\rangle\langle B|.$$

If one *ignores* the intermediate outcome (i.e. does not condition on k), the state evolves under the completely positive trace-preserving map

$$\rho \longrightarrow \rho' = \sum_{k=0,1} M_k \rho M_k^\dagger.$$

A short calculation shows that the populations are unchanged, but the coherence is reduced:

$$\rho'_{AA} = \rho_{AA}, \quad \rho'_{BB} = \rho_{BB}, \quad \rho'_{AB} = \sqrt{1-\lambda}\,\rho_{AB}, \quad \rho'_{BA} = \sqrt{1-\lambda}\,\rho_{BA}.$$

Thus any later interference fringes that depend on ρ_{AB} are suppressed by the factor $\sqrt{1-\lambda}$. In other words, "looking a little" mid-flight means *destroying a little* of the phase coherence that produces interference.

Planck time does not grant noninvasive snapshots. The role of the Planck time in this discussion is not that it provides a privileged intermediate instant at which the photon becomes "more real" or "more inspectable." Rather, to implement a measurement in an ultra-short window Δt one needs an apparatus with correspondingly large bandwidth and strong coupling; for Δt pushed toward t_P, heuristic arguments suggest energies approaching the Planck scale and unavoidable gravitational backreaction. In that regime, the standard quantum-optics instrument models above (and even the fixed-background notion of "a photon traveling in spacetime") may cease to apply, but this does not amount to a free pass to assign definite mid-flight properties without backaction.

So is the photon "only mathematically represented" in flight? Between measurements, the state $\rho(t)$ is indeed a mathematical object—but it is not "mere" math: it encodes experimentally testable predictions for any measurement you choose to implement at time t or later. What fails is the classical move of treating $\rho(t)$ as a hidden record of a definite trajectory or a definite on/off property independent of measurement context.

9.3 Fourier-limited pulses: explicit bandwidth–duration relation

For a transform-limited pulse with temporal envelope $g(t)$ and spectrum $\tilde{g}(\omega)$, rms widths satisfy

$$\Delta t\,\Delta\omega \geq \frac{1}{2},$$

with equality for Gaussians. If $\Delta t = t_P$, then $\Delta\omega \gtrsim 1/(2t_P) \sim 10^{43}\,\mathrm{s}^{-1}$, corresponding to energies per quantum of order $\hbar\Delta\omega \sim \hbar/t_P = E_P$.

9.4 Why this does not imply "photons flip on/off each t_P"

The reasoning says that probing such short times pushes one to Planckian energies and gravitational backreaction, so naive quantum-optics models likely fail. It does *not* say that physical fields evolve in discrete ticks, that a photon has an intrinsic binary on/off variable that updates at t_P, or that standard quantum superposition ceases to apply.

10 Critical physicist remarks (caveats and common pitfalls)

This section collects the kinds of objections, clarifications, and "what exactly do you mean?" questions that a skeptical physicist will ask. The intent is not to weaken the operational claims above, but to keep them precise: the core results are about *measurement-defined* questions (POVMs) applied to *mode-defined* degrees of freedom.

10.1 Planck time: dimensional scale vs. operational cutoff

- **Dimensional analysis is not a theorem.** The combination $t_P = \sqrt{\hbar G/c^5}$ is the unique time scale built from $(\hbar, G, c)$, but that alone does not prove a minimum time step or a discrete clock in nature.
- **The black-hole argument is heuristic.** The estimate $\Delta t \gtrsim t_P$ comes from combining Fourier bandwidth ideas with a classical Schwarzschild radius and a crude localization scale. It is a plausibility bound about when gravitational backreaction cannot be ignored, not an experimentally established limit. Related arguments appear in classic measurement-limit discussions [8, 9].
- **Be explicit about what is being localized.** "Resolving a time interval" can mean (i) resolving the arrival time of a click, (ii) estimating a phase in a frequency standard, (iii) constraining a time parameter in a Hamiltonian, etc. These are different operational tasks with different resource scalings (estimation theory matters).

10.2 Time–energy "uncertainty": avoid overclaiming

- **Time is not an operator in standard nonrelativistic QM.** The familiar Robertson relation $\Delta A\,\Delta B \geq \frac{1}{2}|\langle [A, B]\rangle|$ applies to pairs of operators. "Energy–time uncertainty" is subtler and is best framed in terms of (i) Fourier bandwidth of wavepackets, (ii) dynamical timescales set by Hamiltonians, or (iii) bounds from parameter estimation. Conflating these with a commutator relation is a common source of sloppy claims.
- **Bandwidth is the clean statement here.** In this paper, the $\Delta t\,\Delta\omega \gtrsim 1$ statement is a Fourier-limited wavepacket relation (or a statement about signal processing), not a fundamental operator uncertainty principle.

10.3 Photons are mode excitations; localization is subtle

- **A photon is not an object with a position operator like a massive particle.** In relativistic quantum theory, the notion of a "particle" is tied to a mode decomposition; sharply localized photon states and a universally agreed photon position observable are subtle topics. Operationally, experiments access mode overlaps and field correlation functions.
- **Mode definitions are part of the measurement.** Our $a_f = \int d\omega\, f(\omega)a(\omega)$ construction makes this explicit: the question "is there a photon?" is always "is there a photon in *this* mode that my apparatus couples to?" Changing filters, apertures, local-oscillator shapes, cavity boundary conditions, or timing gates changes the effective mode and thus changes what "on/off" means.

10.4 Detector models: POVMs are effective, device-dependent descriptions

- **On/off POVMs are idealizations.** Real detectors have timing jitter, dead time, afterpulsing, saturation, wavelength dependence, and finite bandwidth. All of these modify the effective POVM. The $(1-\eta)^n$ "no-click" model is a useful baseline, not a universal truth.
- **A "click" is not the same as "one photon."** A threshold detector cannot number-resolve; multi-photon states can produce the same "on" outcome as single-photon states. Conversely, a single-photon state can fail to click due to losses. Interpreting click/no-click as a

literal microscopic particle counter is a common category error in discussions that aim to be foundational.

- **Vacuum is not "nothing."** Field vacuum has fluctuations and nontrivial correlations; what matters operationally is whether your detector responds above its noise floor in a specified window. "Dark counts" are a classical manifestation of this: even in the absence of signal photons, detectors can click.

10.5 Interference vs. which-path: complementarity is quantitative, and definitions matter

- **Equality requires idealizations.** The clean relation $V^2 + D^2 = 1$ holds in the pure-state model with equal arm weights and the specific distinguishability definition used here. More realistic situations (mixed environments, unequal splitting, additional noise) yield $V^2+D^2 \leq 1$; a standard reference is Englert's inequality [7].

- **Which-path information is about accessibility.** It is possible for path information to be *in principle* present in some environment degree of freedom but not practically retrievable, or to be "erased" by conditioning on another measurement. This is not paradoxical; it is conditional probability in a larger Hilbert space.

- **Be wary of counterfactual claims.** Statements like "the photon really went through arm A" are not operationally meaningful unless tied to a specific measurement that would establish that fact without destroying the interference being discussed.

10.6 Complementarity, contextuality, and nonlocality are different claims

- **Complementarity:** incompatible measurement contexts (e.g. which-path vs. interference) cannot be simultaneously realized with full sharpness in one experimental configuration.

- **Contextuality:** the outcome assignment for an observable cannot be made independent of the set of compatible observables measured alongside it (Kochen–Specker-type constraints) [10]. Not every "wave/particle" discussion is a contextuality proof.

- **Nonlocality:** Bell-type correlations that cannot be explained by local hidden-variable models. This paper does not attempt a Bell test; it sticks to single-system operational predictions.

10.7 Analogy limits: projection in real space vs. projection in Hilbert space

- **The "square circle" story is an analogy, not a hidden-variable model.** The cylinder example is meant to train the intuition that incompatible "views" can come from information-discarding maps. In quantum mechanics, the relevant "space" is Hilbert space, and measurement is a physical interaction, not merely a change in viewpoint.

- **Not all measurements are projective.** Realistic measurement descriptions are POVMs; projectors are a special ideal case. The operational conclusions here are therefore stated in POVM language wherever possible.

11 Conclusion

1. "Photon on/off" is shorthand for a mode-defined and measurement-defined predicate.
2. With explicit detector POVMs, on/off outcomes are straightforward to compute and show no inconsistency.
3. Apparent contradictions most often come from importing classical assumptions of measurement-independent, simultaneously definite properties for incompatible observables.
4. Planck time is a natural quantum-gravity scale; heuristic arguments suggest operational probes below t_P require Planckian energies, but this does not by itself imply time discreteness or binary "flip" dynamics.

A On/off POVM derivation for inefficiency

If each photon is independently detected with probability η, then with n photons the probability of *no* detection is $(1-\eta)^n$. For a number-diagonal state $\rho = \sum_n p_n |n\rangle\langle n|$,

$$p_{\text{off}} = \sum_{n=0}^{\infty} p_n (1-\eta)^n = \mathrm{Tr}\left(\rho \sum_{n=0}^{\infty} (1-\eta)^n |n\rangle\langle n|\right).$$

Thus the operator in parentheses is the POVM element $\Pi_{\text{off}}^{(\eta)}$.

B Two-mode basis rotation identity

For orthonormal modes u, v, define $a_\pm = (a_u \pm a_v)/\sqrt{2}$. Then

$$a_u^\dagger = \frac{a_+^\dagger + a_-^\dagger}{\sqrt{2}} \Rightarrow a_u^\dagger |0\rangle = \frac{|1_+, 0_-\rangle + |0_+, 1_-\rangle}{\sqrt{2}}.$$

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