Graph Theory

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Outline

- Planar Graphs

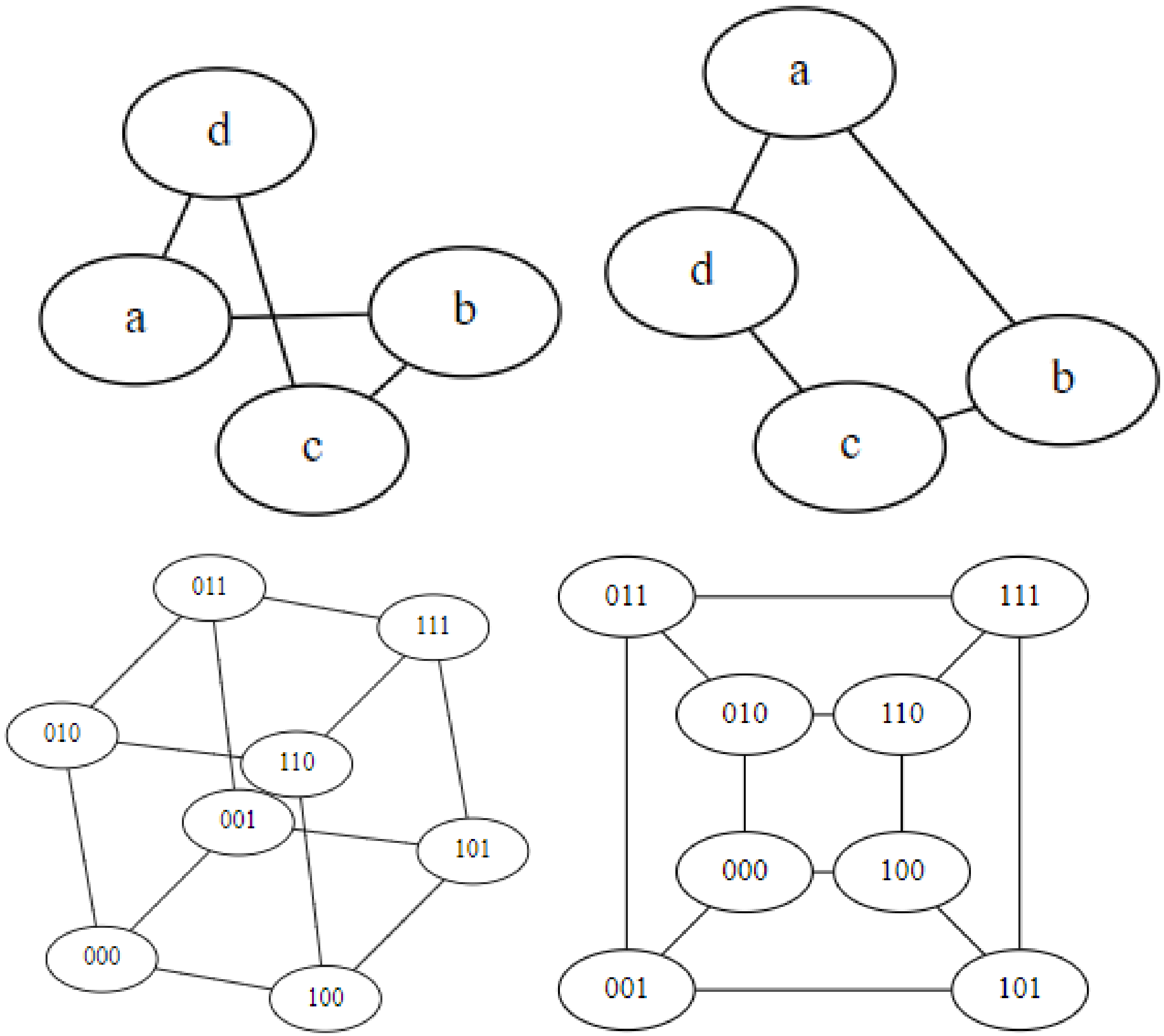
Planar Graphs

- A planar graph is a graph that can be embedded in the plane, i.e., it can be drawn on the plane in such a way that its edges intersect only at their endpoints.

- A graph that can be drawn on a plane without edges crossing is called planar.

- Such a drawing is called a plane graph or planar embedding of the graph.

Planar Graphs

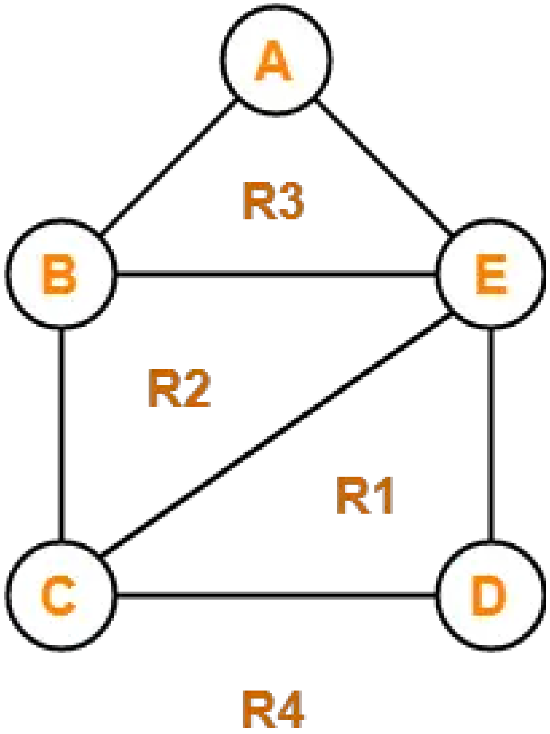


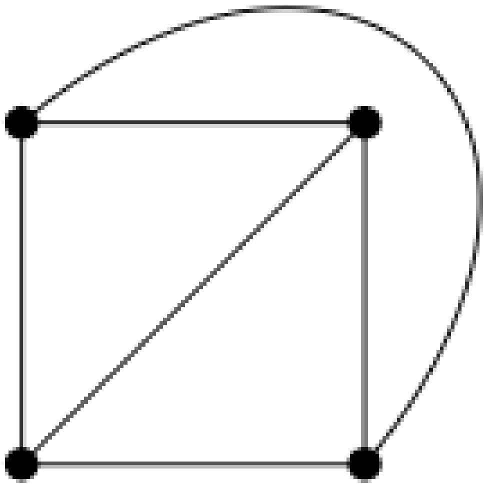
Why Planar Graphs?

- Sometimes, it's really important to be able to draw a graph without crossing edges.
- Connecting utilities (electricity, water, natural gas) to houses. If we can keep from crossing those lines, it will be safer and easier to install.
- Connecting components on a circuit board: the connections on a circuit board cannot cross. If we can connect them without resorting to another layer of traces, it will be cheaper to produce.
- Subway / Railway system: if subway lines need to cross, we've got some serious engineering to do.

Planar Graph: Definitions & Properties

- When a connected graph can be drawn without any edges crossing, it is called planar. When a planar graph is drawn in this way, it divides the plane into **regions called faces or regions**.

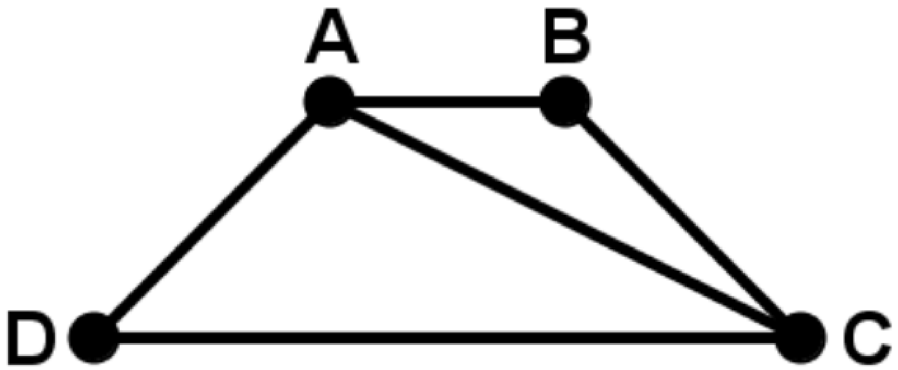


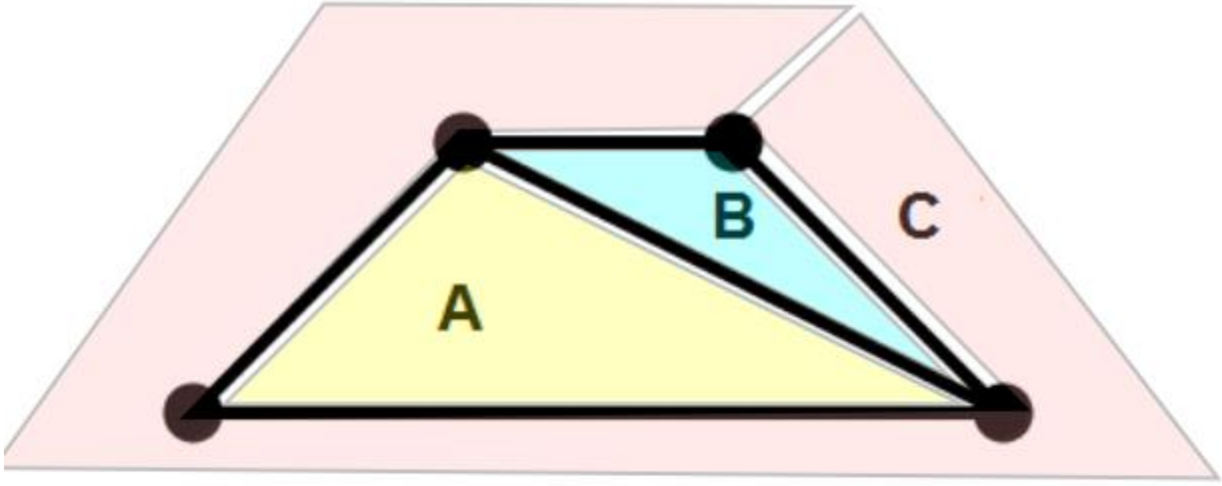


Planar Graph: Definitions & Properties

- The regions enclosed by the planar graph are called **interior faces** of the graph.
- The region surrounding (outside) the planar graph is called the **exterior face** of the graph.
- When we say **faces of the graph** we mean BOTH the interior AND the exterior faces. We usually denote the number of faces of a planar graph by f.

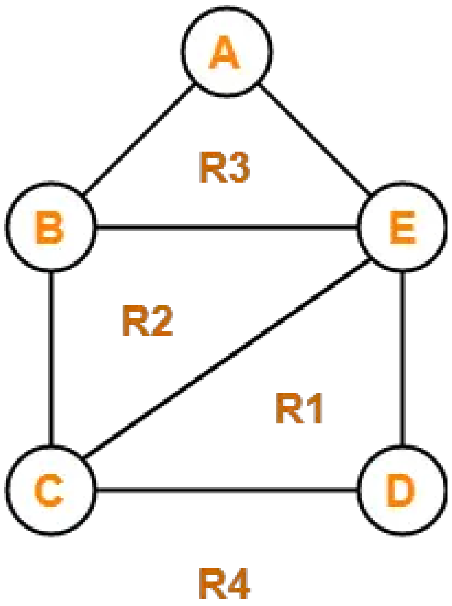
Planar Graph: Definitions & Properties





Planar Graph: Definitions & Properties

The number of edges bordering a particular face is called the degree of the face.

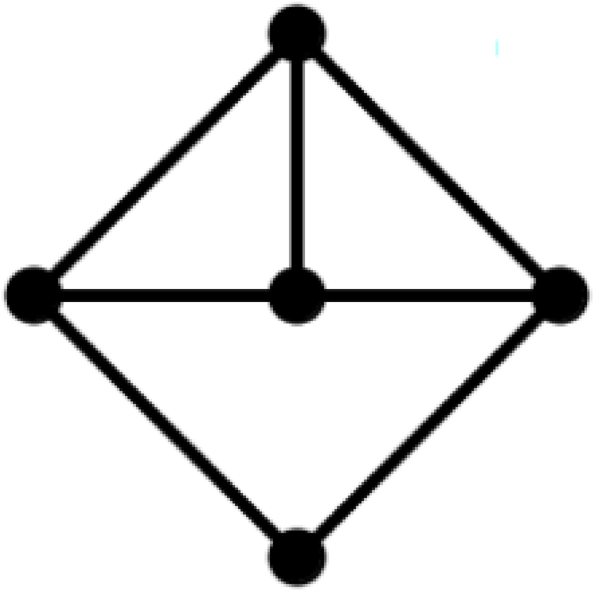


Each region has some degree associated with it given as-

- Degree of Interior region = Number of edges enclosing that region
- Degree of Exterior region = Number of edges exposed to that region

- Degree (R1) = 3
- Degree (R2) = 3
- Degree (R3) = 3
- Degree (R4) = 5

Planar Graph: Definitions & Properties

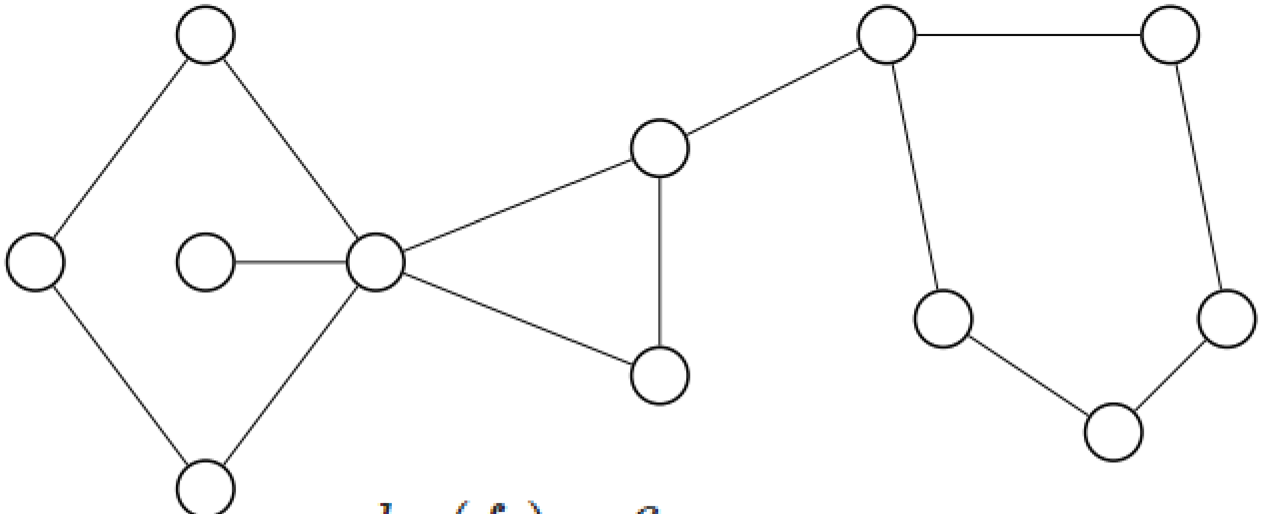


The interior faces A and B each have degree 3.

The interior face C has degree 4.

The exterior face D has degree 4.

Planar Graph: Definitions & Properties



$$deg(f_1) = 6$$

$$deg(f_2) = 3$$

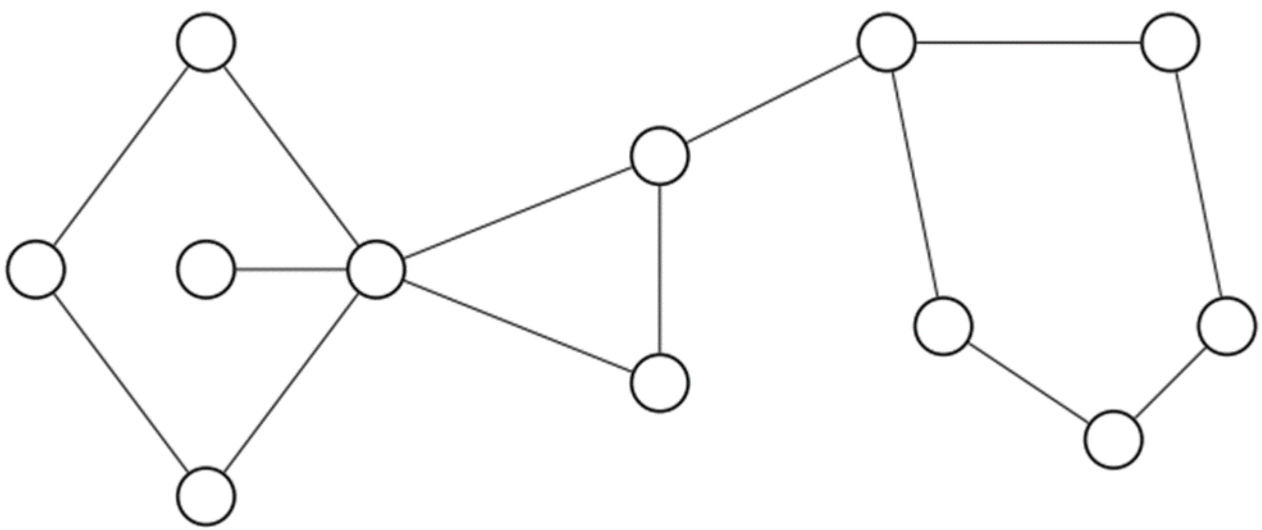
$$deg(f_3) = 5$$

$$deg(f_4) = 14$$

Planar Graph: Euler's Formula

For a connected (one-piece) planar graph with v vertices, e edges, and f faces,

$$v - e + f = 2$$



Planar Graph: Definitions & Properties

A connected planar graph has 24 vertices and 30 faces. How many edges does the graph have?

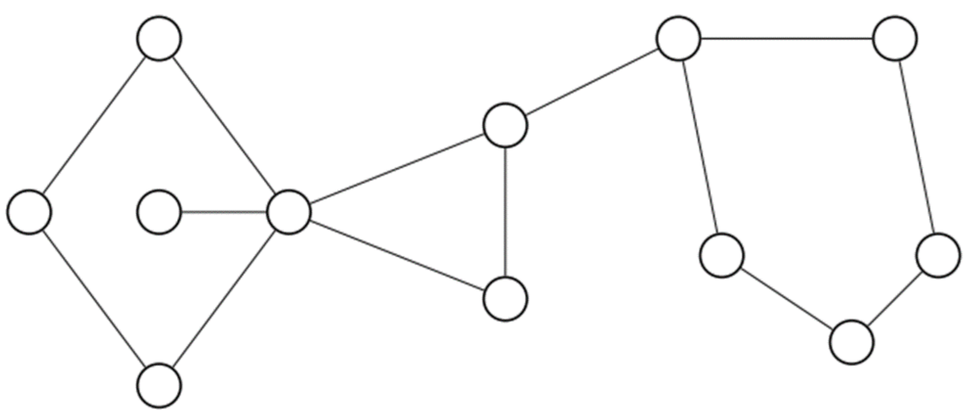
Planar Graph: Definitions & Properties

If G is a planar graph with k components, then-

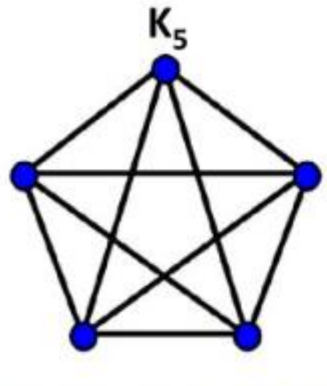
$$r = e - v + (k + 1)$$

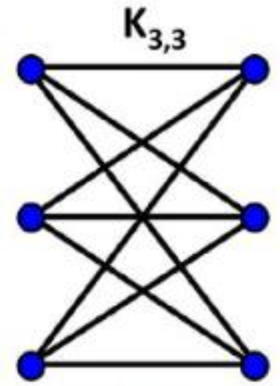
Planar Graph: Definitions & Properties

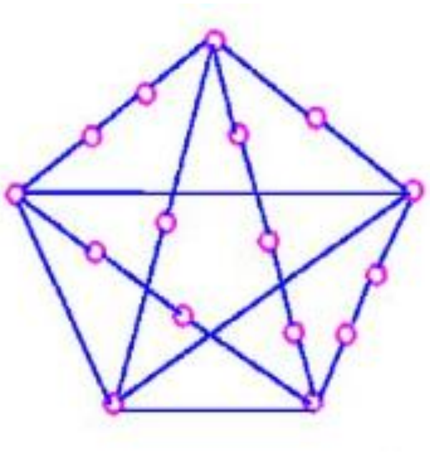
- If you **add up the degrees** of every vertex and **divide by 2**, you get the **number of edges**.
- If you **add up the degrees** of every face and **divide by 2**, you get the **number of edges**.
- Conversely, if you take the **number of edges and multiply by 2**, you get the **sum of degrees of the vertices or faces**.

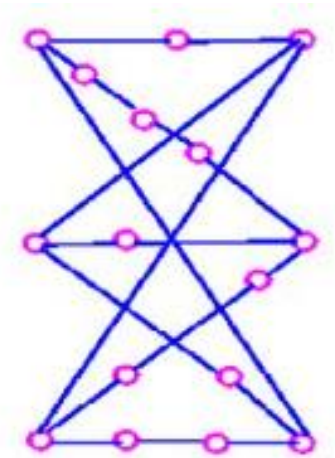


Planarity Testing









Subdivision

Planarity Testing

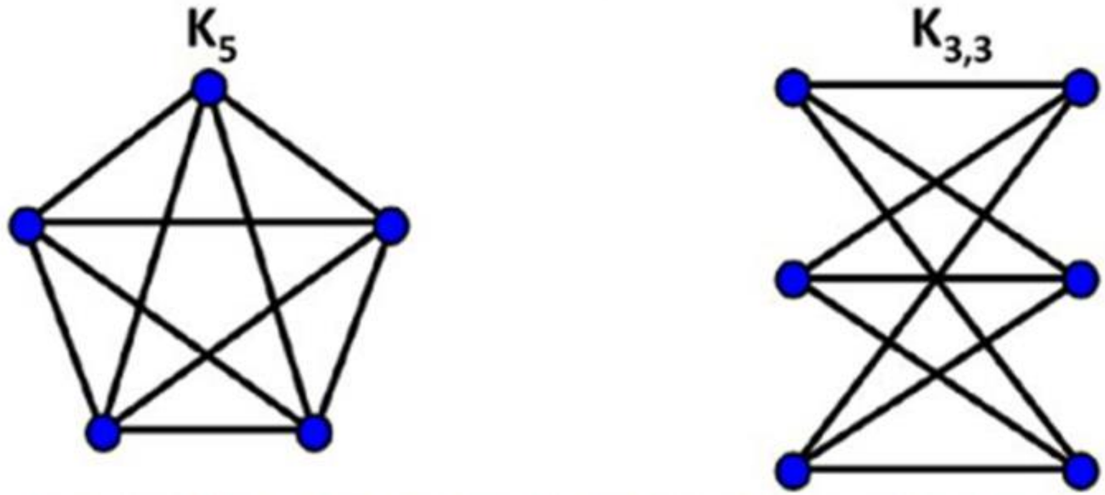
- **Kuratowski's theorem** is a mathematical characterization of planar graphs.

- It states that a finite graph is planar if and only if it does not contain a subgraph that is a subdivision of K_5 (the complete graph on five vertices) or of $K_{3,3}$ (a complete bipartite graph on six vertices).
- Such a graph is called homeomorphic to K_5 or $K_{3,3}$.

Planarity Testing

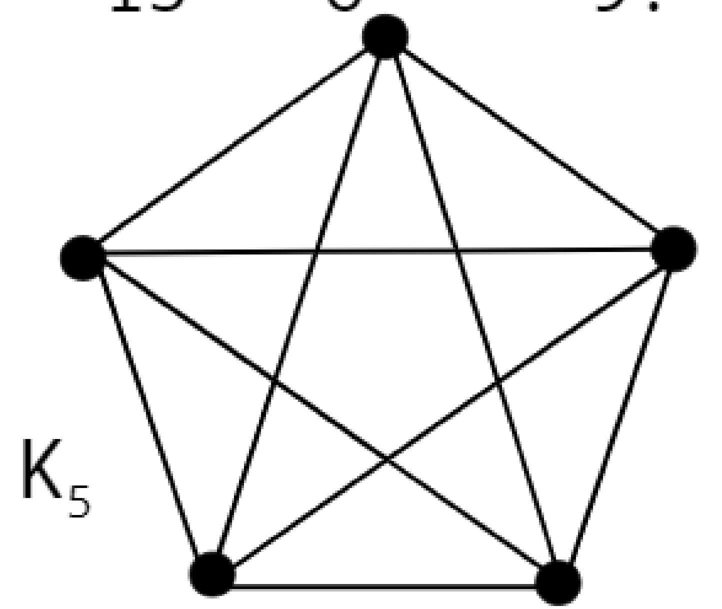
If G *is a simple planar graph such that* $n \geq 3$, *then we*

$$m \leq 3n - 6$$



Planarity Testing

- K_5 has 5 vertices and 10 edges.
- We see that $v \geq 3$.
- So, if K_5 is planar, it must be true that $e \leq 3v - 6$.
- $3v - 6 = 3*5 - 6 = 15 - 6 = 9$.
- So e must be ≤ 9.
- But $e = 10$.
- So, K_5 is nonplanar.

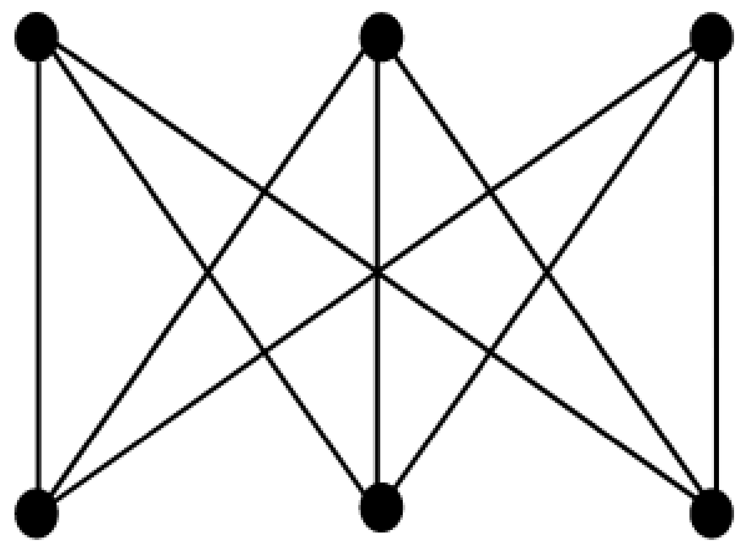


Planarity Testing

- If a connected planar simple graph has e edges and v vertices with $v \leq 3$ and no circuits of length 3, then $e \leq 2v - 4$.

Planarity Testing

- $K_{3,3}$ has 6 vertices and 9 edges.
- Obviously, v ≥ 3 and there are no circuits of length 3.
- If $K_{3,3}$ were planar, then $e \leq 2v - 4$ would have to be true.
- $2v - 4 = 2*6 - 4 = 8$
- So e must be ≤ 8.
- But $e = 9$.
- So $K_{3,3}$ is nonplanar.



$K_{3,3}$

Planarity Testing

For a planar graph having v vertices and e edges, the following holds:

- If $v \geq 3$ then $e \leq 3v - 6$;
- If $v \geq 3$ and there are no cycles of length 3, then $e \leq 2v - 4$.

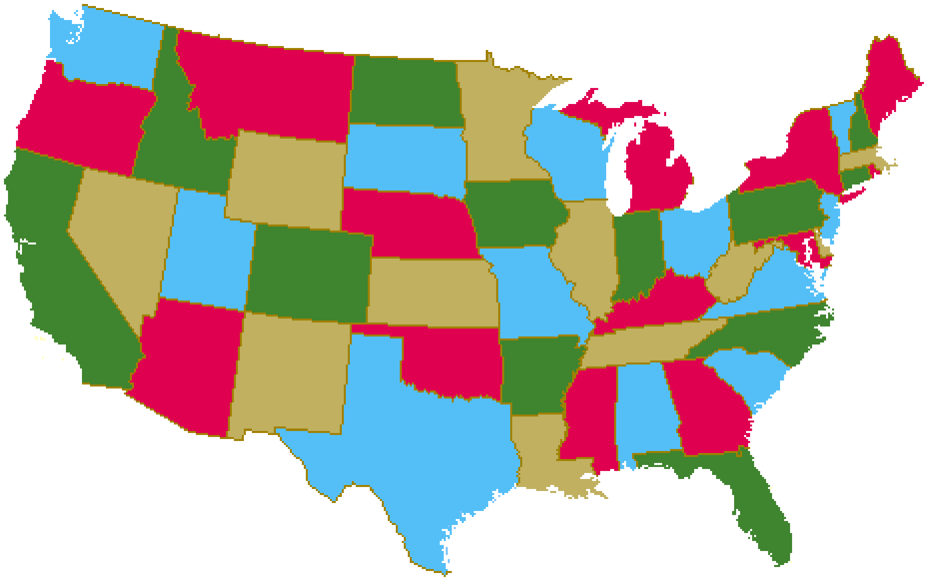
Four-Color Theorem

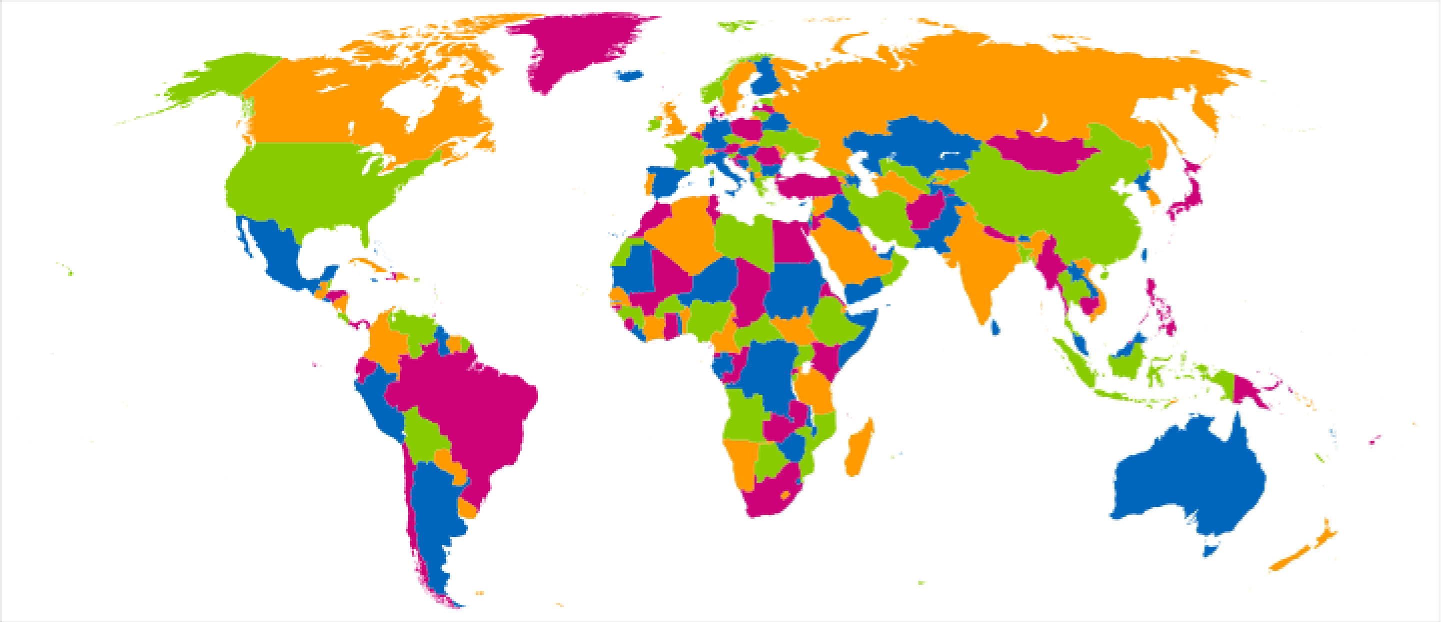
- In mathematics, the four color theorem, or the four color map theorem, states that no more than four colors are required to color the regions of any map so that no two adjacent regions have the same color.
- Adjacent means that two regions share a <u>common boundary</u> curve segment, not merely a corner where three or more regions meet.
- In graph-theoretic terms, the theorem states that for loopless planar graph G, its chromatic number is $X(G) \leq 4$

Four-Color Theorem: History

- The four-color theorem was conjectured in 1852 and proved in 1976 by Wolfgang Haken and Kenneth Appel at the University of Illinois with the aid of a computer program that was thousands of lines long and took over 1200 hours to run.
- Since that time, a collective effort by interested mathematicians has been under way to check the program.
- So far the only errors that have been found are minor and were easily fixed. Many mathematicians accept the theorem as true.

Four-Color Theorem





Summary

- Planar Graphs