Graph Theory

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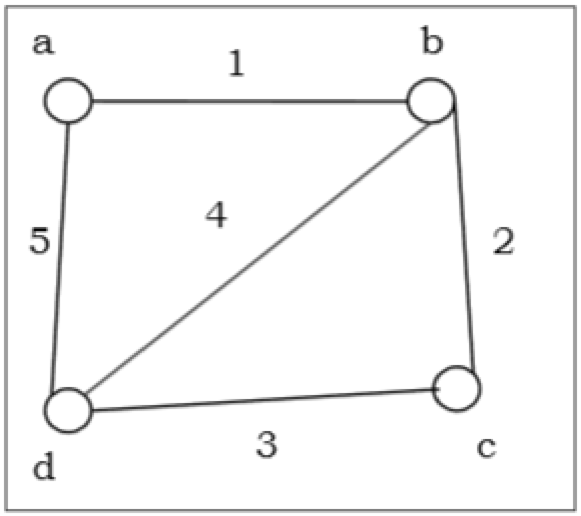
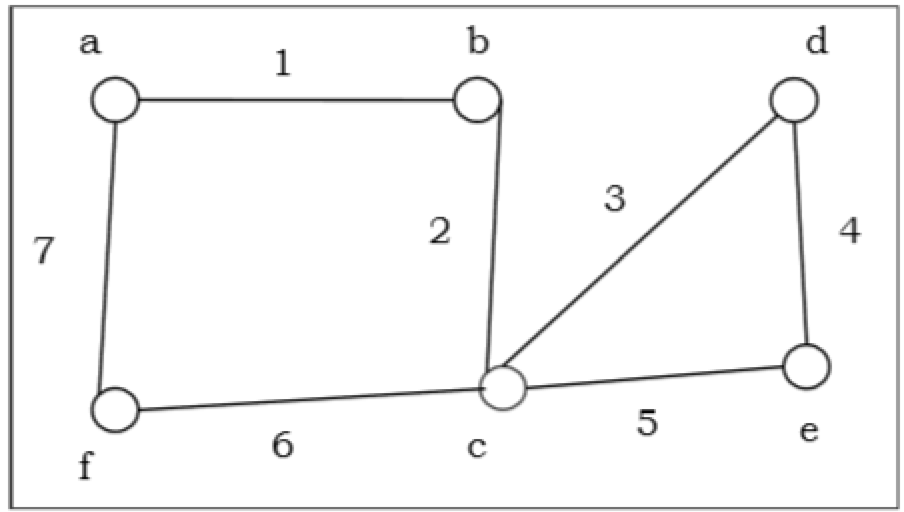
Outline

- Eulerian and Hamiltonian Graphs

Euler Graph

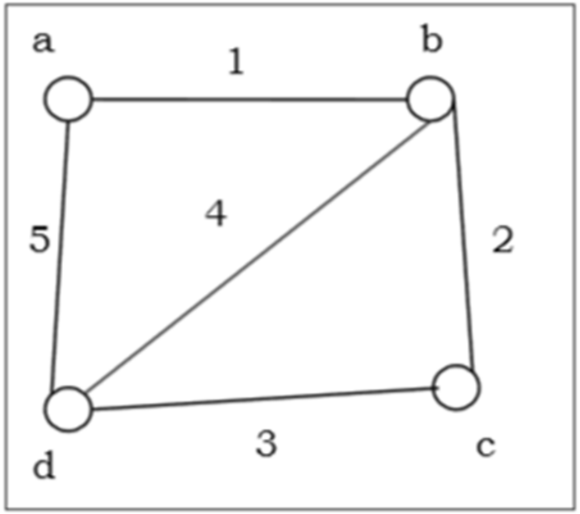
- A connected graph G is called an Euler graph, if there is a closed trail (or circuit) which includes every edge of the graph G.

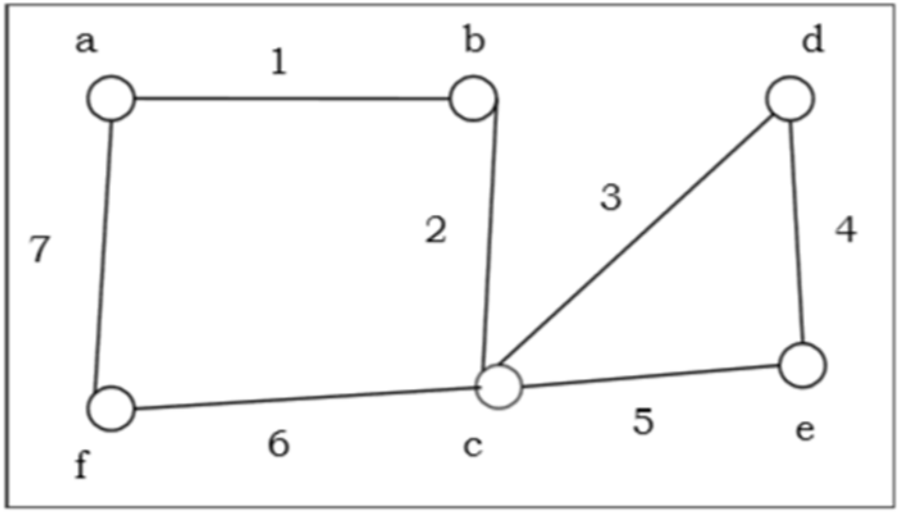




Euler Graph

- Any connected graph is called as an Euler Graph if and only if all its vertices are of even degree.



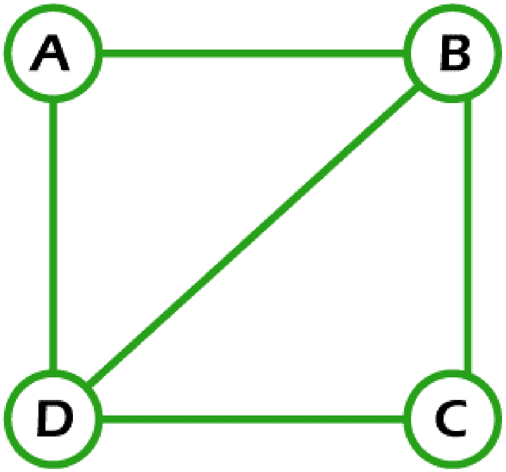


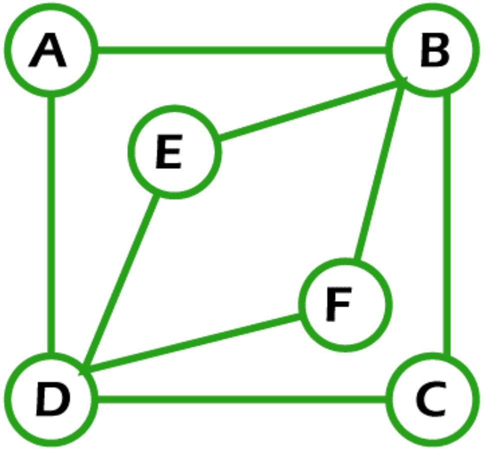
Euler Path

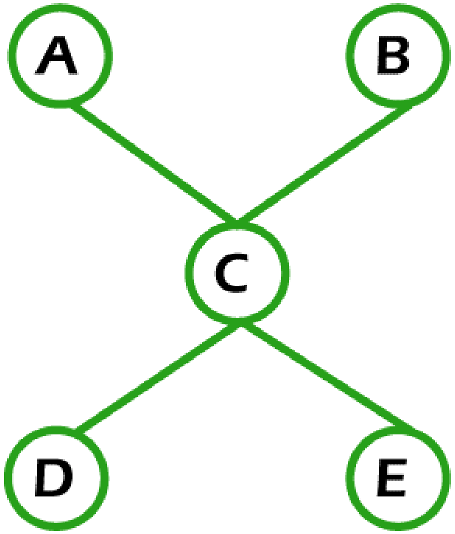
- We can also call the Euler path as Euler walk or Euler Trail.

- If there is a connected graph with a trail that has all the edges of the graph, then that type of trail will be known as the **Euler trail**.
- If there is a connected graph, which has a walk that passes through each and every edge of the graph only once, then that type of walk will be known as the **Euler walk**.
- An **Euler path** is a path that uses every edge of a graph exactly once.

Euler Path







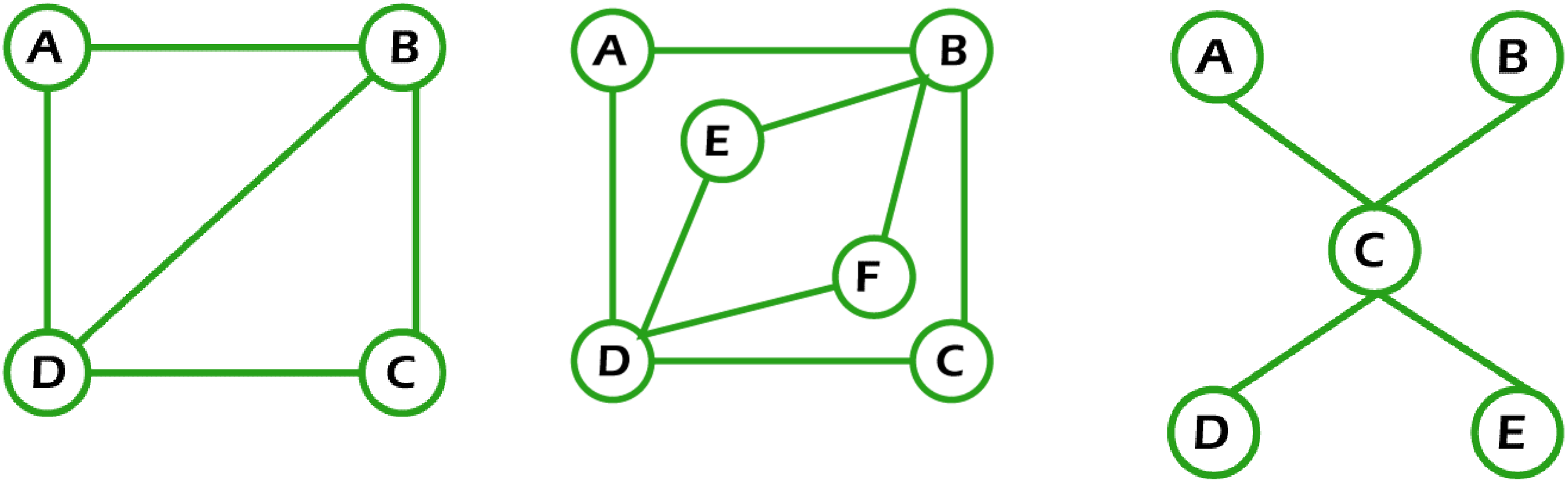
A connected graph has a Euler path if and only if the number of its vertices of odd degree is ≤ 2.

Euler Circuit

- Euler circuit is also known as Euler Cycle or Euler Tour.

- If there exists a Circuit in the connected graph that contains all the edges of the graph, then that circuit is called as an Euler circuit.
- If there exists a walk in the connected graph that starts and ends at the same vertex and visits every edge of the graph exactly once with or without repeating the vertices, then such a walk is called as an Euler circuit.
- An Euler trail that starts and ends at the same vertex is called as an Euler circuit.

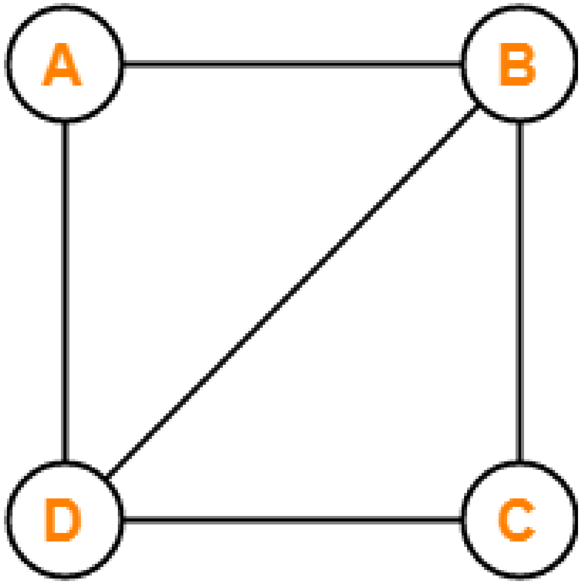
Euler Circuit



A graph will contain an Euler circuit if and only if all its vertices are of even degree.

Semi-Euler Graph

- If a connected graph contains an Euler trail but does not contain an Euler circuit, then such a graph is called as a semi-Euler graph.



Semi-Euler Graph

Euler Graph: Notes

- If the graph is connected and contains an Euler circuit, then it is an Euler graph.
- If all its vertices are of even degree, then graph contains an Euler circuit otherwise not.
- If the graph is connected and contains an Euler trail (not Euler Circuit), then graph is a semi-Euler graph otherwise not.
- If the number of vertices with odd degree are at most 2, then graph contains an Euler trail otherwise not.

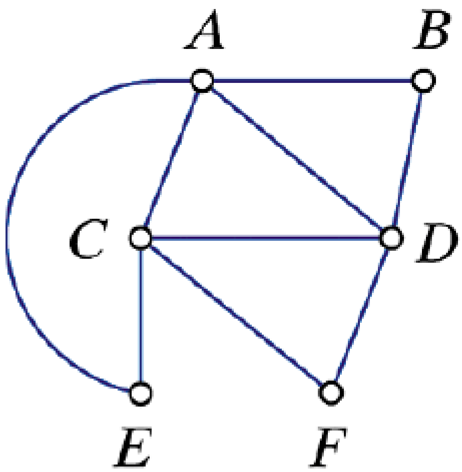
How to find Euler Path or Circuit?

- Fleury’s algorithm finds an Euler circuit or an Euler path in a connected graph.
- Fleury's algorithm is an elegant but inefficient algorithm.
- It proceeds by repeatedly removing edges from the graph in such way, that the graph remains Eulerian.

Fleury’s Algorithm

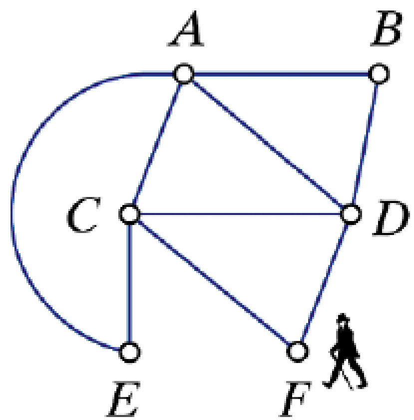
- **Preliminaries.** Make sure that the graph is connected and either (1) has no odd vertices (circuit) or (2) has just two odd vertices (path).
- **Start.** Choose a starting vertex. [In case (1) this can be any vertex; in case (2) it must be one of the two *odd* vertices.]
- **Intermediate steps.** At each step, if you have a choice, *don’t choose a bridge of the yet-to-be-traveled part* of the graph. However, if you have only one choice, take it.
- **End.** When you can’t travel any more, the circuit (path) is complete. [In case (1) you will be back at the starting vertex; in case (2) you will end at the other odd vertex.]

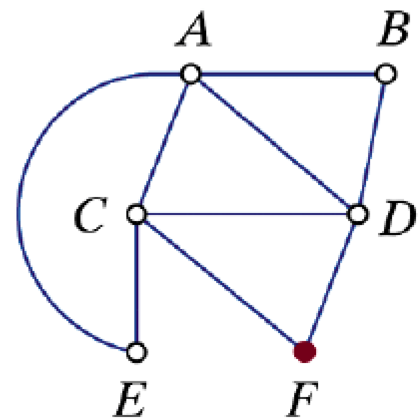
Fleury's Algorithm



Fleury’s Algorithm

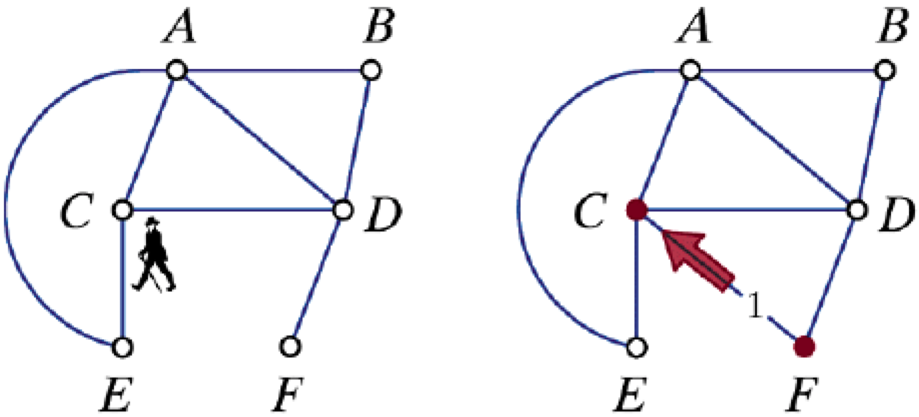
Start: We can pick any starting point we want. Let’s say we start at *F*.





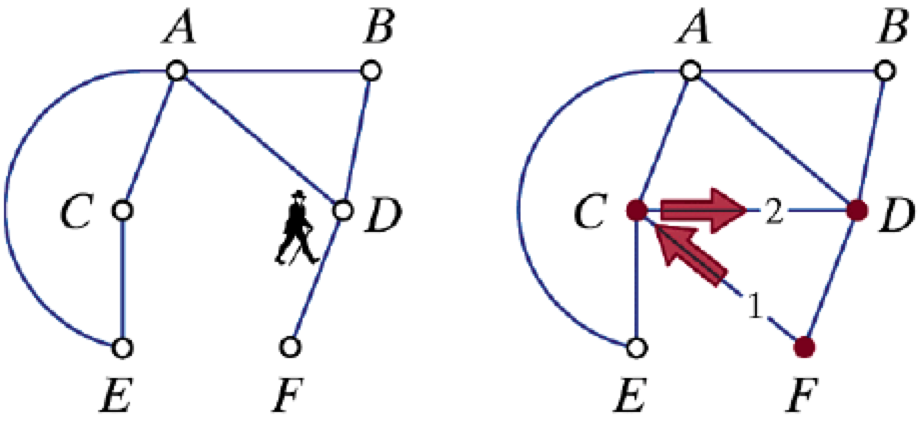
Fleury's Algorithm

Step 1: Travel from F to C.
(Could have also gone from F to D.)



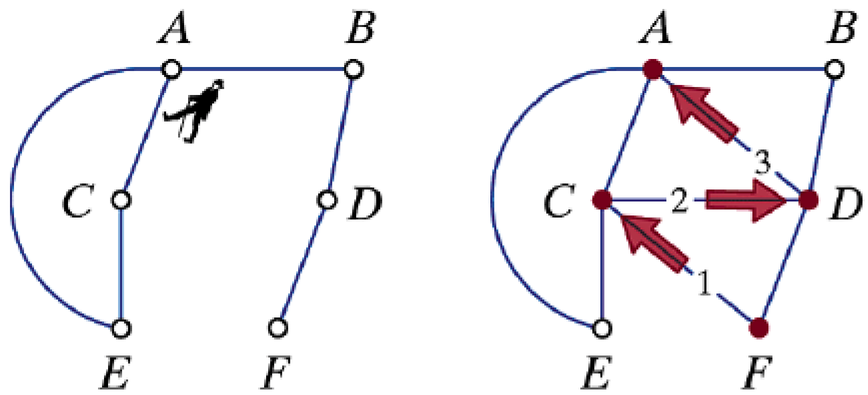
Fleury's Algorithm

Step 2: Travel from C to D.
(Could have also gone to A or to E.)



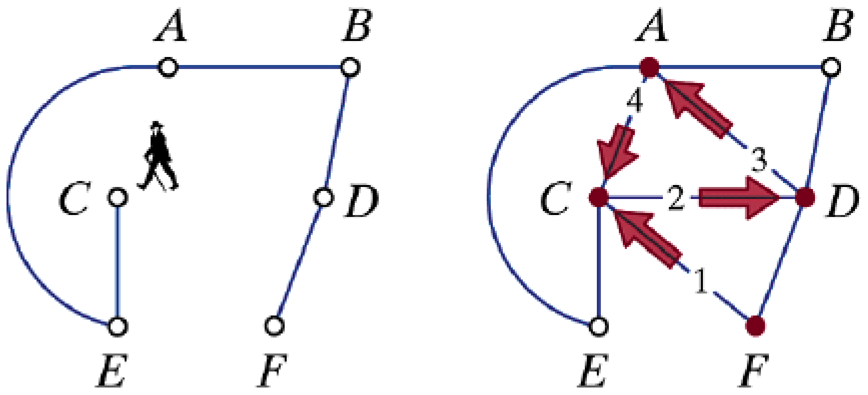
Fleury’s Algorithm

Step 3: Travel from D to A.
(Could have also gone to B but not to F—DF is a bridge!)



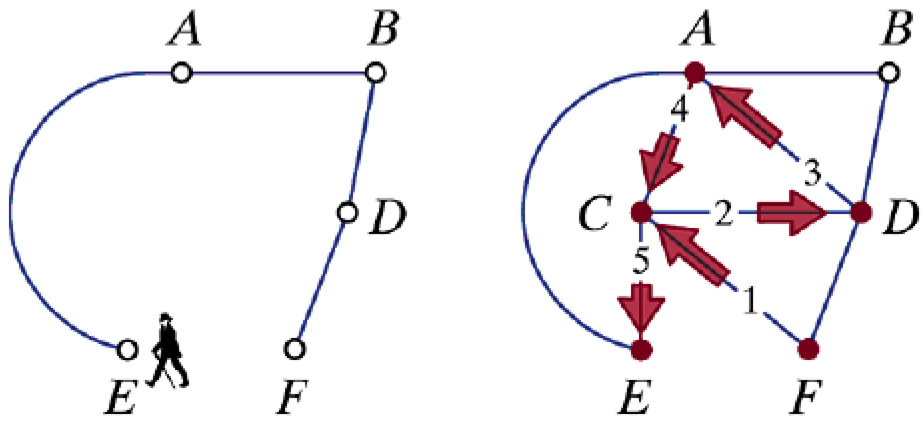
Fleury's Algorithm

Step 4: Travel from A to C.
(Could have also gone to E but not to B—AB is a bridge!)



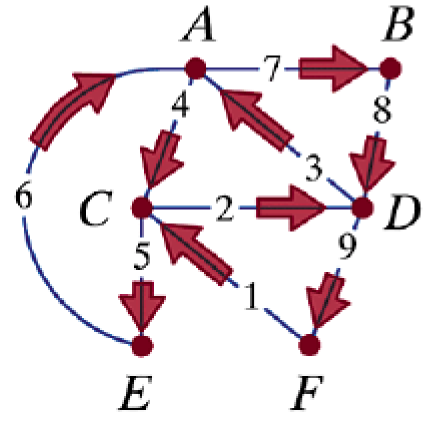
Fleury's Algorithm

Step 5: Travel from *C* to *E*.
(There is no choice!)



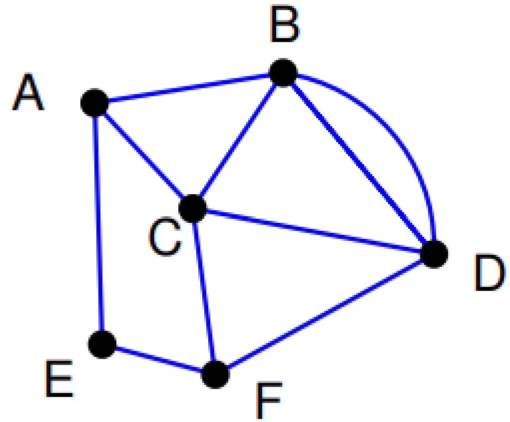
Fleury's Algorithm

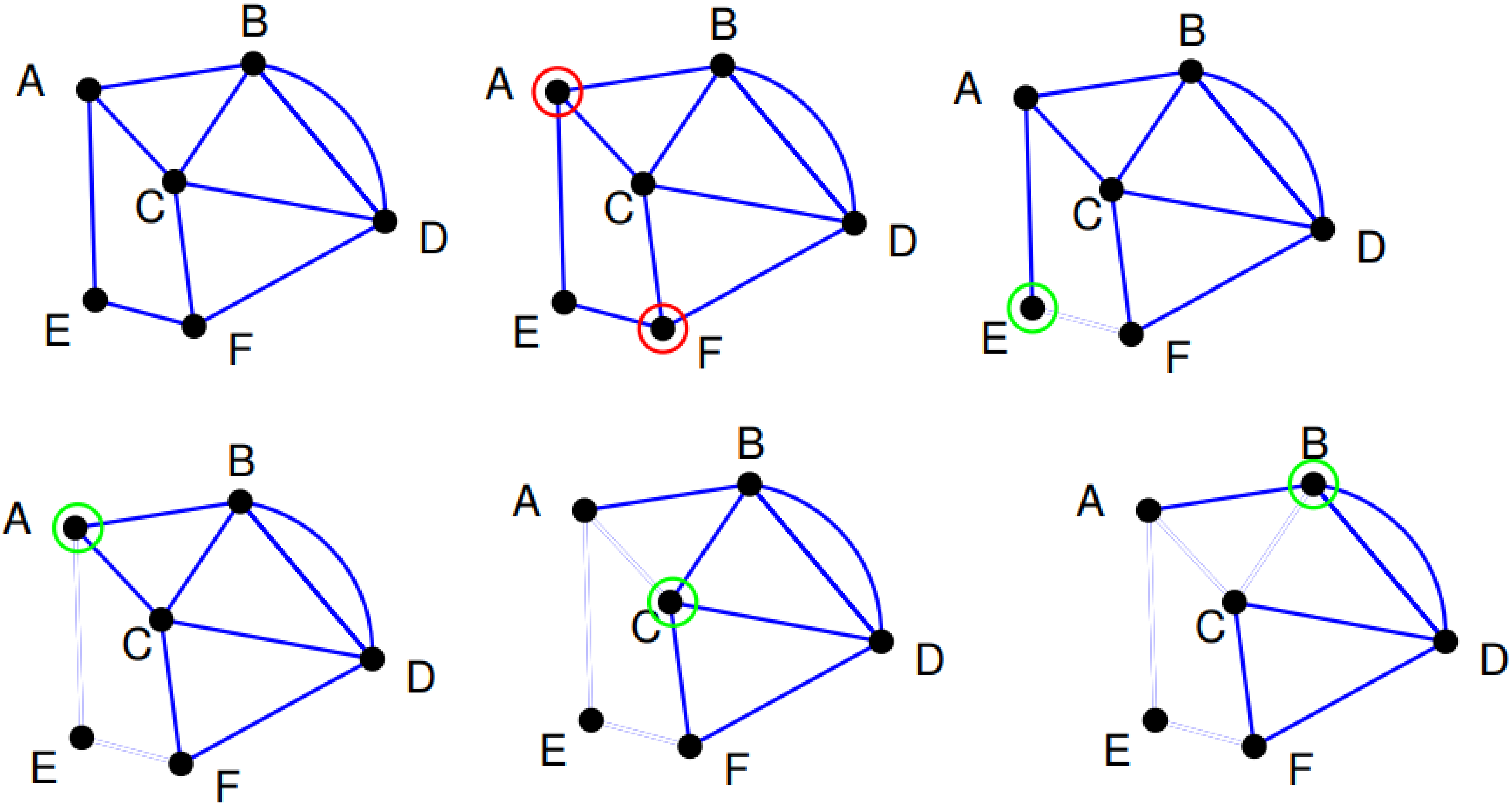
Steps 6, 7, 8, and 9: Only one way to go at each step.



Fleury’s Algorithm

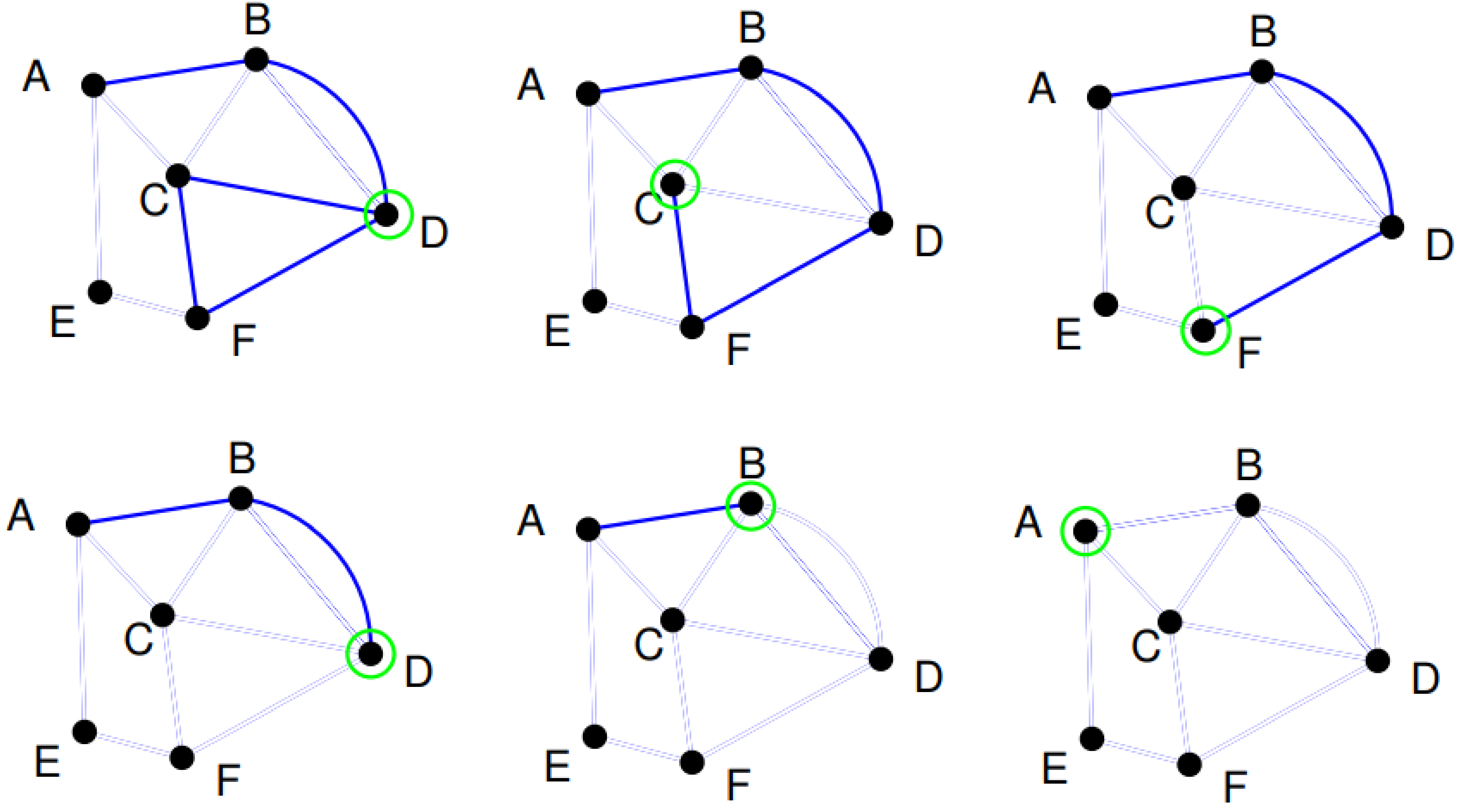
Find an Euler circuit in the graph.





Up until this point, the choices didn't matter.

But now, crossing the edge BA would be a mistake, because we would be stuck there.



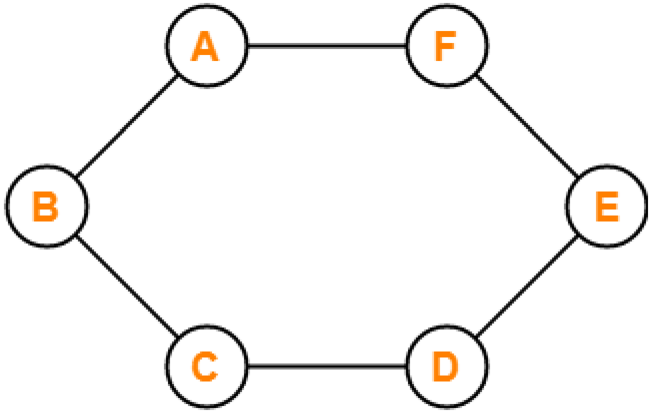
Euler Path: FEACBDCFDBA

Applications of Euler Graphs

- Euler paths and circuits can be used to solve many practical problems such as finding a path or circuit that traverses each
  - street in a neighborhood,
  - road in a transportation network,
  - connection in a utility grid,
  - link in a communications network.
- Other applications are found in the
  - layout of circuits,
  - network multicasting,
  - molecular biology, where Euler paths are used in the sequencing of DNA.

Hamiltonian Graph

- A connected graph G is called Hamiltonian graph if there is a cycle which includes **every vertex** of G and the cycle is called **Hamiltonian cycle**.



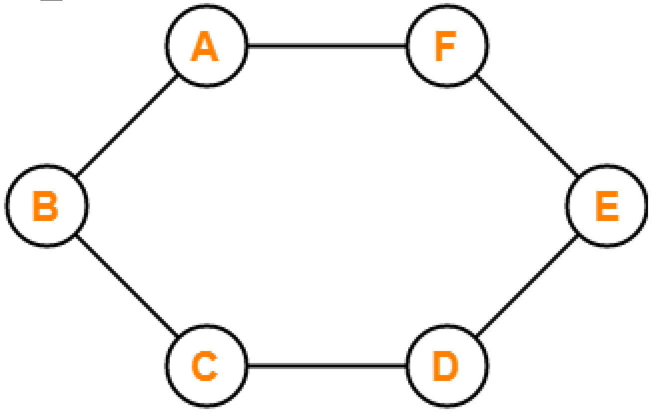
Example of Hamiltonian Graph

Hamiltonian Graph

- **Dirac's Theorem:** A simple graph with n vertices ($n \geq 3$) is Hamiltonian if every vertex has degree n/2 or greater.

- **Ore's Theorem:** A simple graph with n vertices ($n \geq 3$) is Hamiltonian if, for every pair of non-adjacent vertices, the sum of their degrees is n or greater.

Hamiltonian Graph



Example of Hamiltonian Graph

$$\forall(\text{non-adjacent vertices pair } v, u)(\deg(v) + \deg(w) \geq n)$$
$$\Rightarrow \text{Graph is Hamiltonian}$$

But this does not imply the reverse of it, that means,

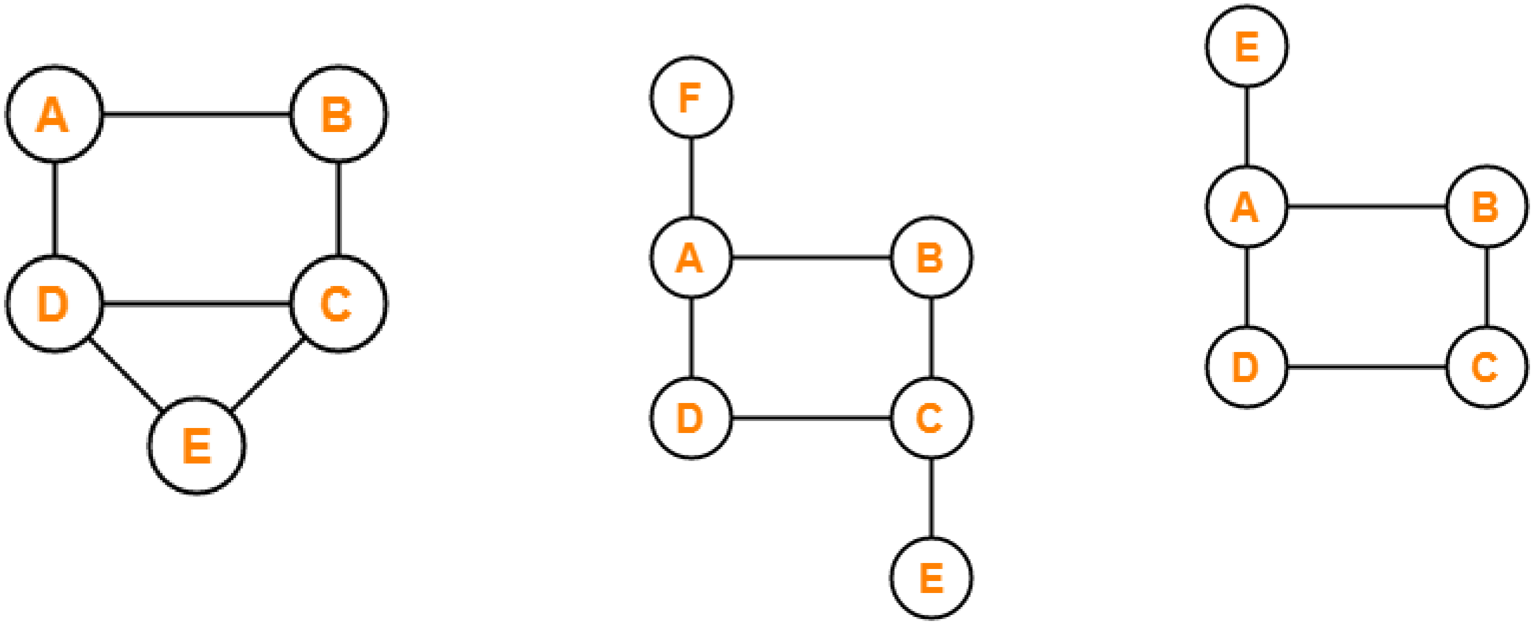
$$\text{Graph is Hamiltonian} \not\Rightarrow$$
$$\forall(\text{non-adjacent vertices pair } v, u)(\deg(v) + \deg(w) \geq n)$$

Hamiltonian Path

- If there exists a walk in the connected graph that visits every vertex of the graph exactly once without repeating the edges, then such a walk is called as a Hamiltonian path.

- If there exists a Path in the connected graph that contains all the vertices of the graph, then such a path is called as a Hamiltonian path.

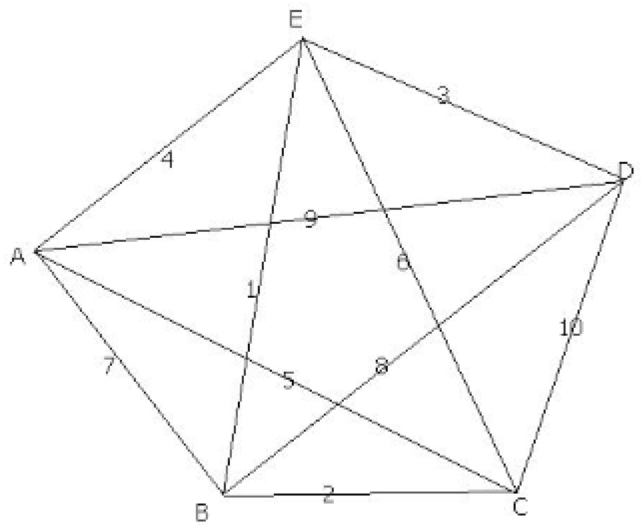
Hamiltonian Path

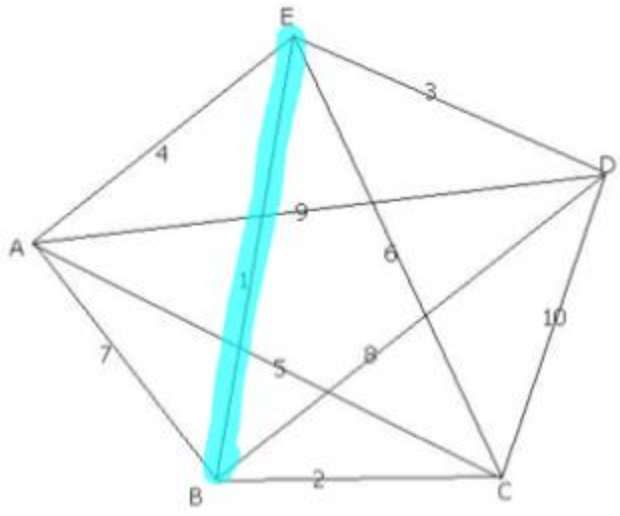


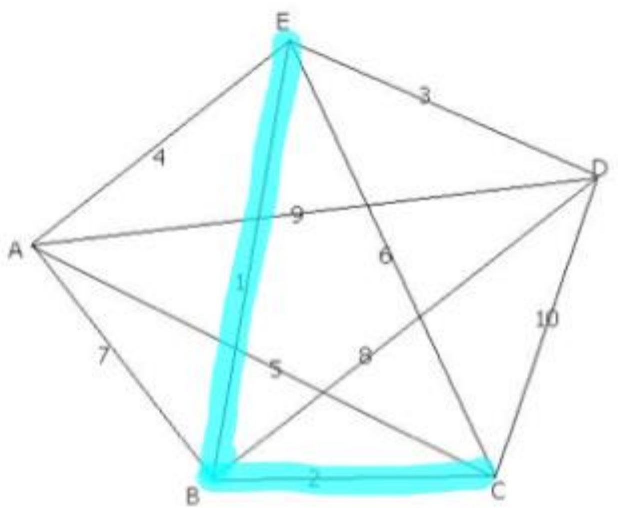
How to find Hamiltonian Cycle / Circuit?

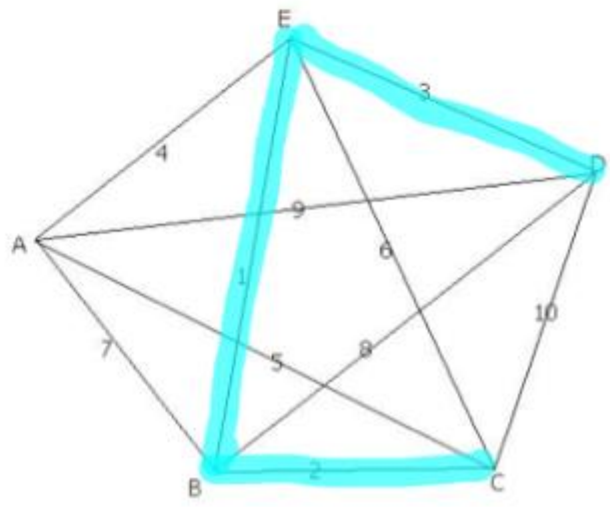
- Cheapest Link Algorithm
- Pick the link with the smallest weight first (if there is a tie, randomly pick one). Mark the corresponding edge in color.
- Pick the next cheapest link and mark the corresponding edge in color.
- Continue picking the cheapest link available. Mark the corresponding edge in color except when **a)** it closes a circuit or **b)** it results in three edges coming out of a single vertex.
- When there are no more vertices to link, close the red circuit.

How to find Hamiltonian Cycle / Circuit?

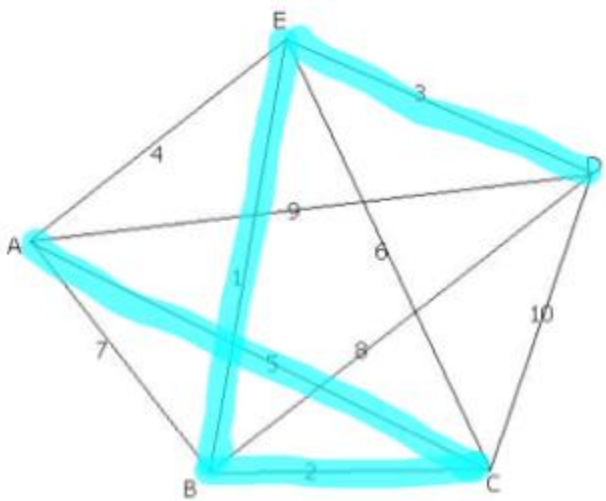


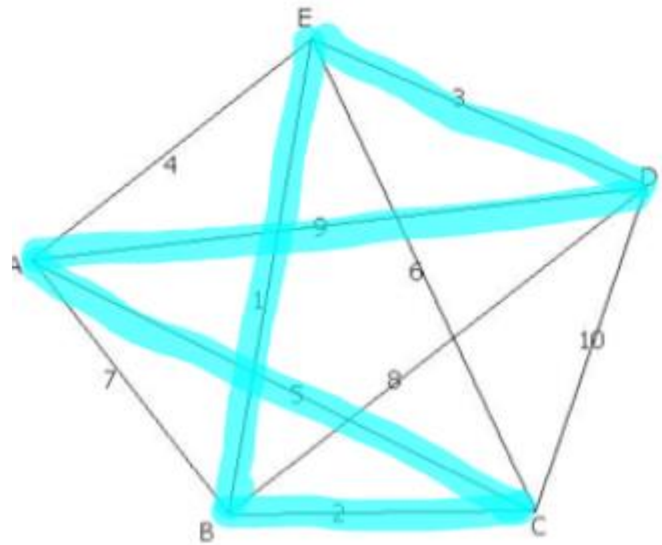






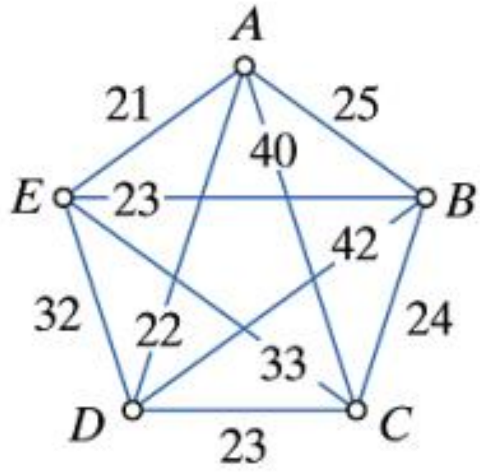
How to find Hamiltonian Cycle / Circuit?



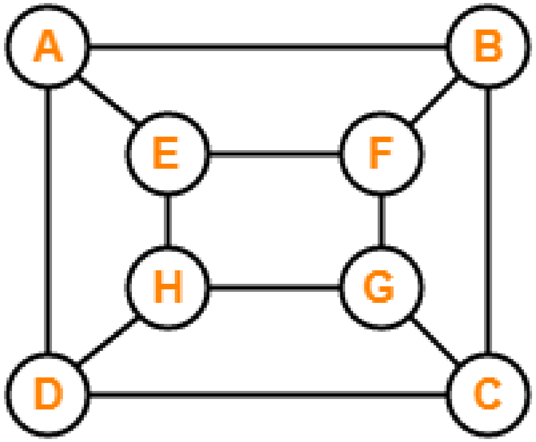


The solution is ACBEDA or ADEBCA with total weight of 20 miles.

How to find Hamiltonian Cycle / Circuit?



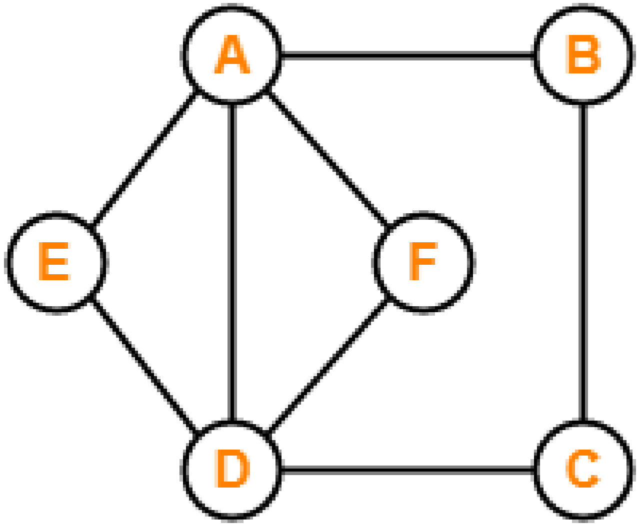
Euler or Hamiltonian?



The graph contains both a Hamiltonian path (ABCDHGFE) and a Hamiltonian circuit (ABCDHGFEA).

It is not an Euler graph.

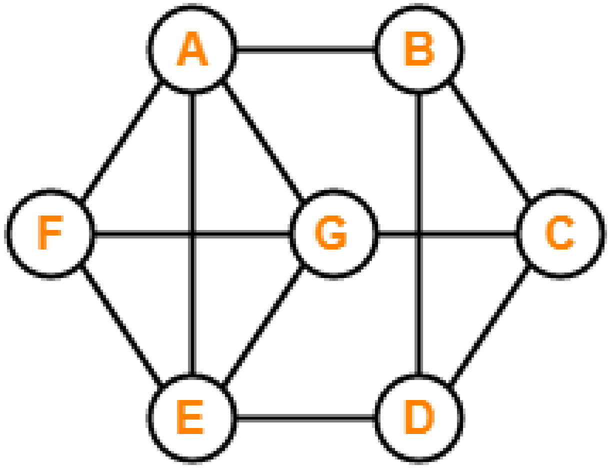
Euler or Hamiltonian?



- No Hamiltonian path nor it contains a Hamiltonian circuit.

- It is an Euler graph.

Euler or Hamiltonian?



The graph contains both a Hamiltonian path (ABCDEFG) and a Hamiltonian circuit (ABCDEFGA).

It is not an Euler graph.

Euler vs. Hamiltonian?

Property	Euler	Hamilton
Repeated visits to a given node allowed?	Yes	No
Repeated traversals of a given edge allowed?	No	No
Omitted nodes allowed?	No	No
Omitted edges allowed?	No	Yes

Summary

- Euler Graphs
- Hamiltonian Graphs