Graph Theory

Dr. Irfan Yousuf
Department of Computer Science (New Campus)
UET, Lahore

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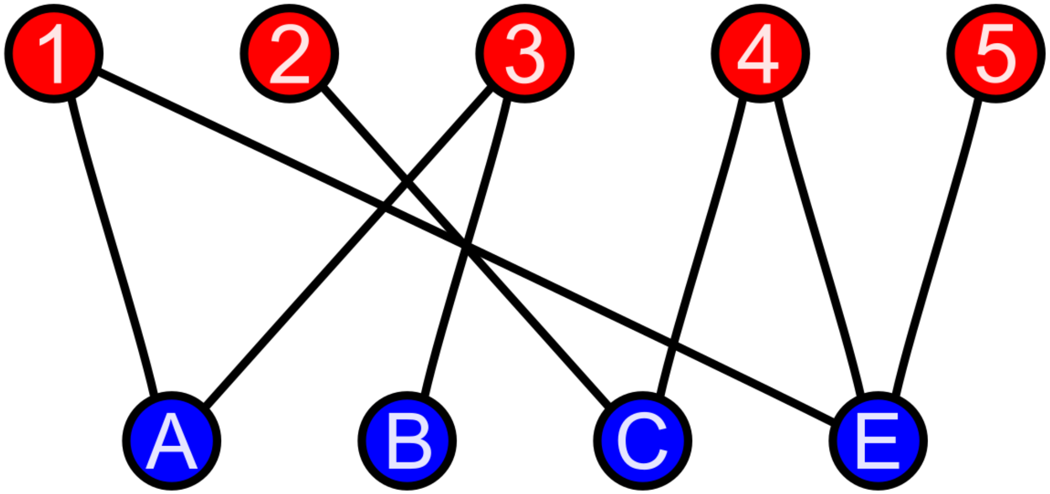
Outline

- Matchings in Bipartite graphs

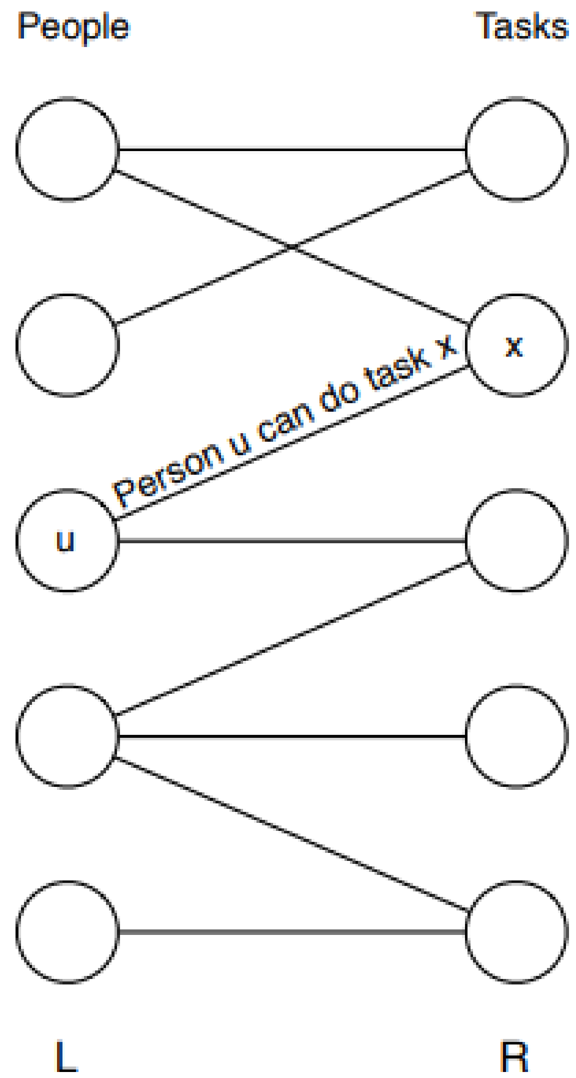
Bipartite Graphs

- A graph $G = (V, E)$ is bipartite if the vertex set V can be partitioned into two sets A and B (the bipartition) such that no edge in E has both endpoints in the same set of the bipartition.



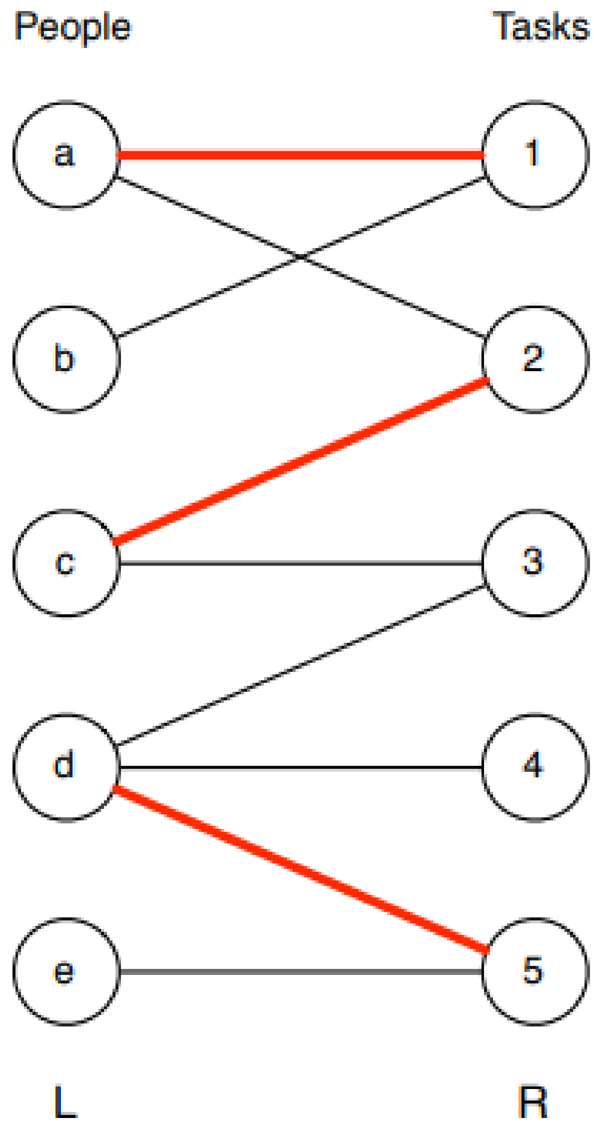
Bipartite Graphs

- Suppose we have a set of people L and set of jobs R.
- Each person can do only some of the jobs.
- Can model this as a bipartite graph $\longrightarrow$



Bipartite Matching

- A **matching** gives an assignment of people to tasks.
- Want to get as many tasks done as possible.
- So, want a **maximum matching**: one that contains as many edges as possible.



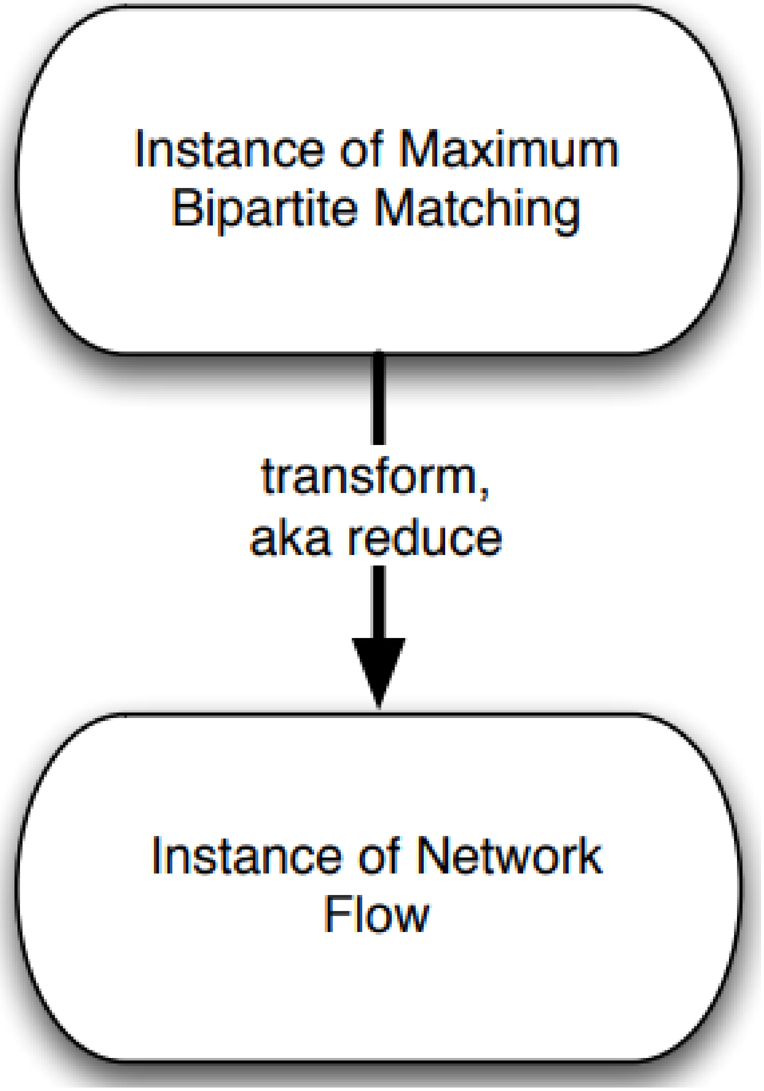
Maximum Bipartite Matching

Maximum Bipartite Matching

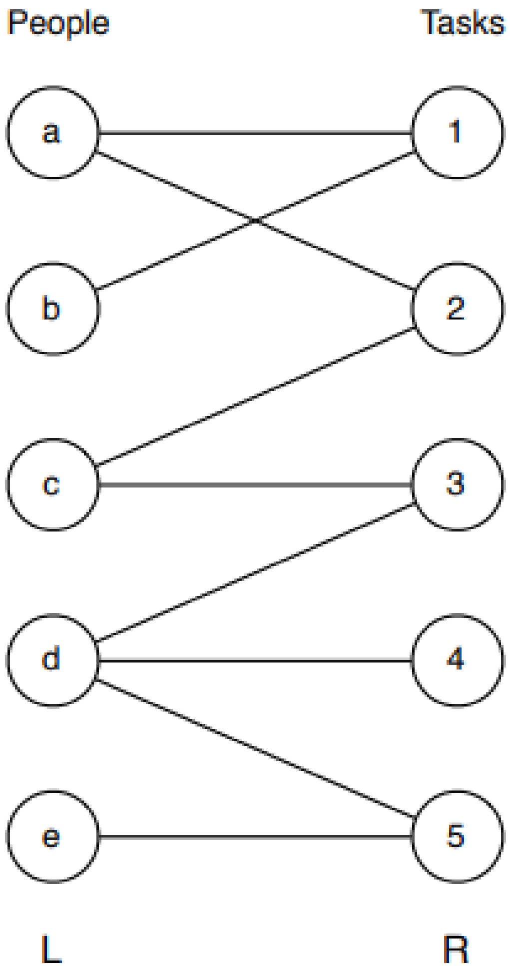
Given a bipartite graph $G = (A \cup B, E)$, find an $S \subseteq A \times B$ that is a matching and is as large as possible.

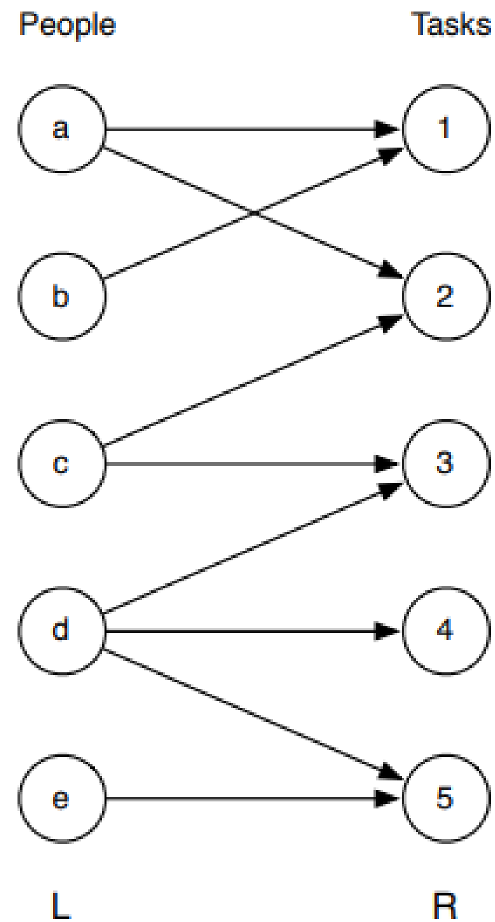
Maximum Bipartite Matching

- Given an instance of bipartite matching,
- Create an instance of network flow.
- Where the solution to the network flow problem can easily be used to find the solution to the bipartite matching.

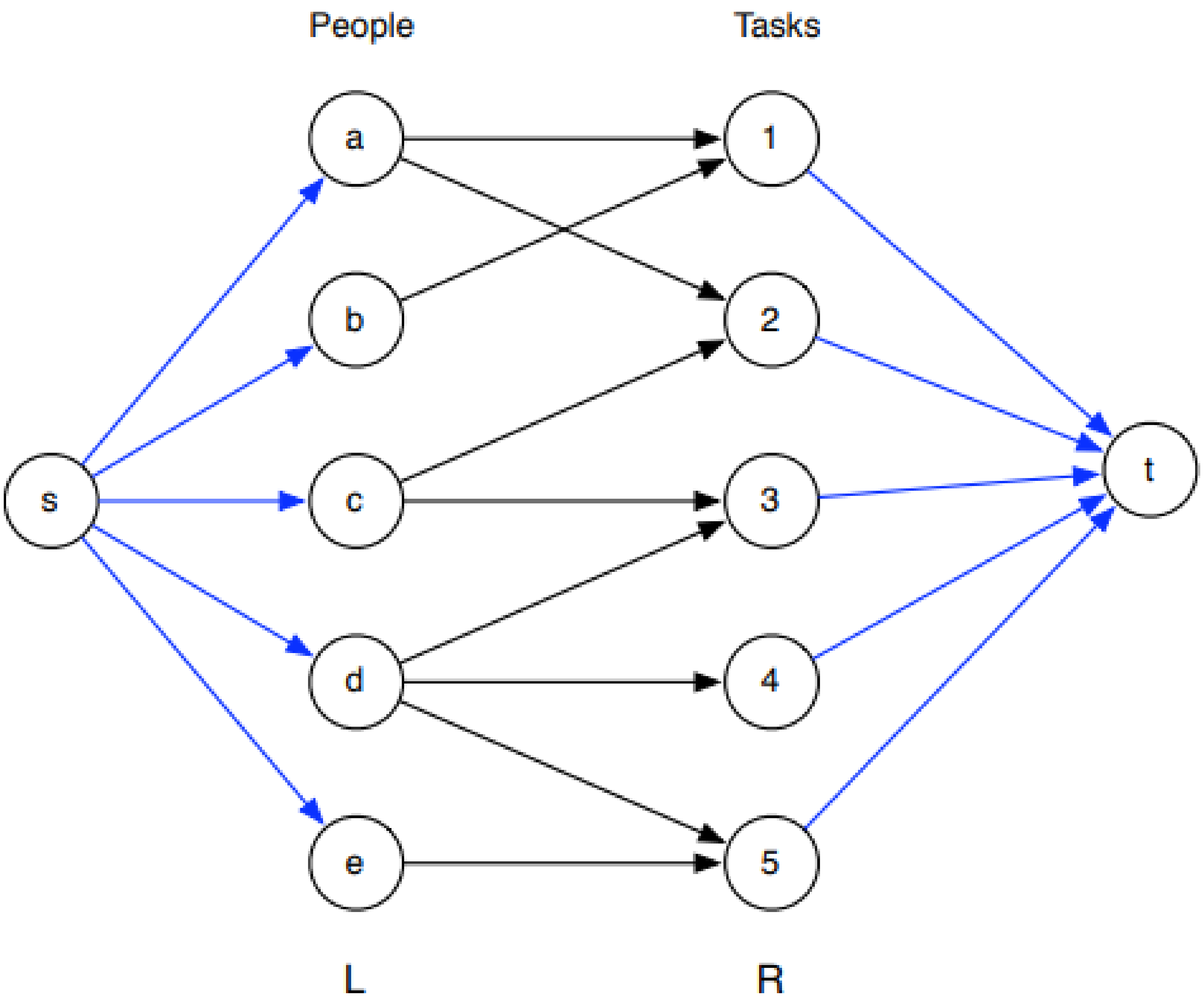


Maximum Bipartite Matching

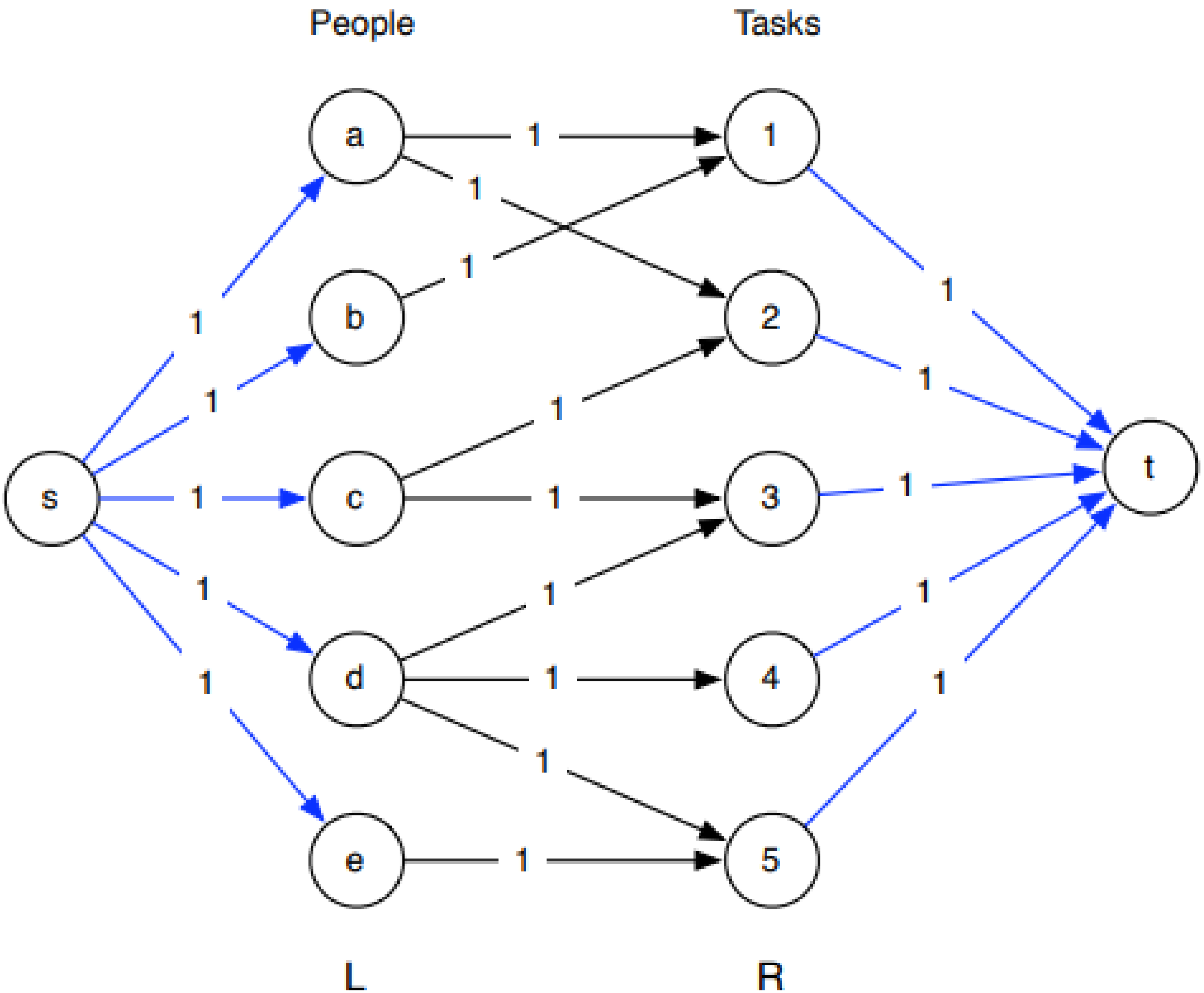




Maximum Bipartite Matching

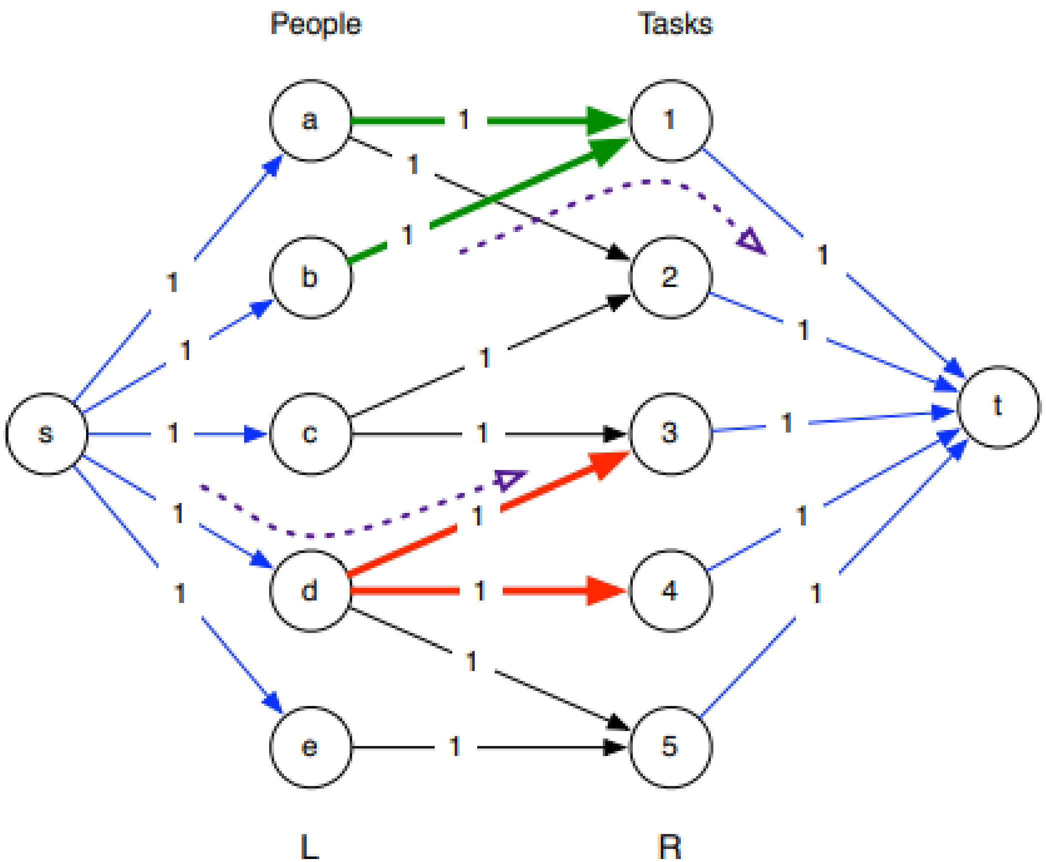


Maximum Bipartite Matching



Maximum Bipartite Matching

We can choose at most one edge leaving any node in A.
We can choose at most one edge entering any node in B.

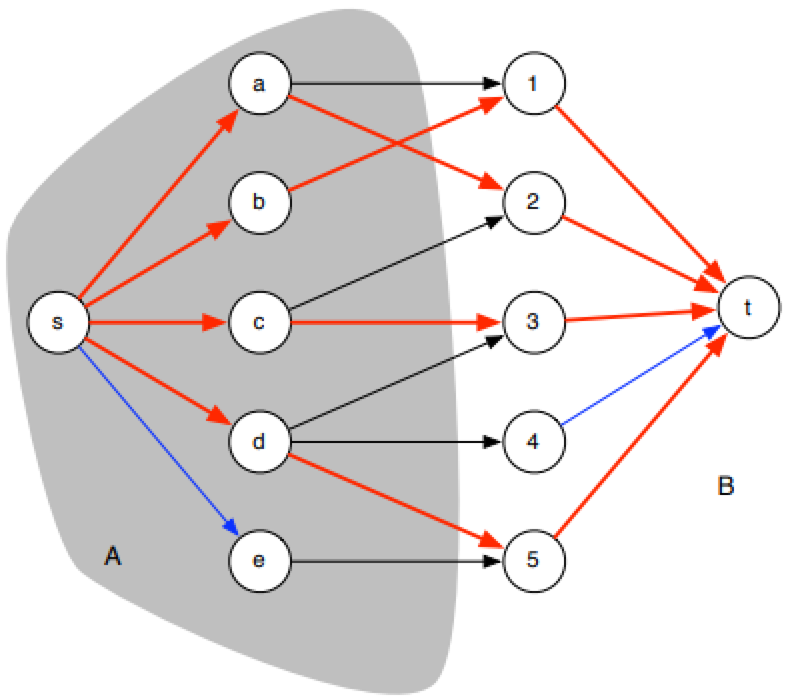


If we chose more than 1, we couldn't have balanced flow.

Maximum Bipartite Matching

- If there is a matching of k edges, there is a flow f of value k.

- If there is a flow f of value k, there is a matching with k edges.



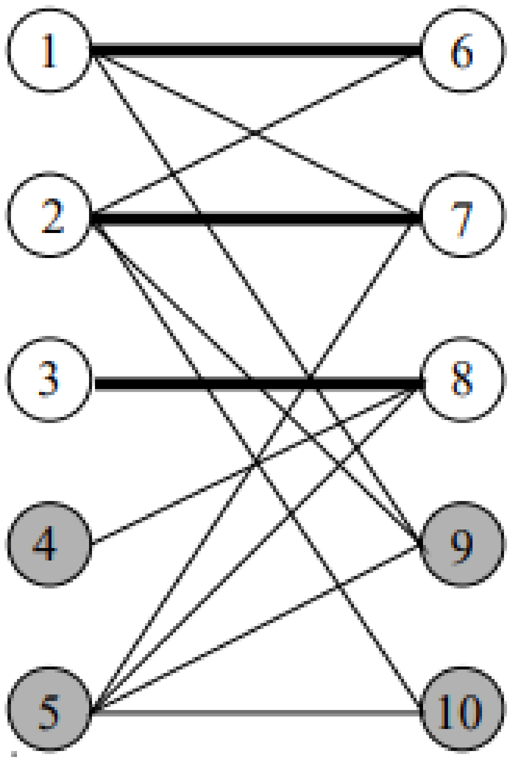
Maximum Bipartite Matching

1. Given bipartite graph $G = (A \cup B, E)$, direct the edges from A to B.
2. Add new vertices s and t.
3. Add an edge from s to every vertex in A.
4. Add an edge from every vertex in B to t.
5. Make all the capacities 1.
6. Solve maximum network flow problem on this new graph G'.

The edges used in the maximum network flow will correspond to the largest possible matching!

Maximum Bipartite Matching

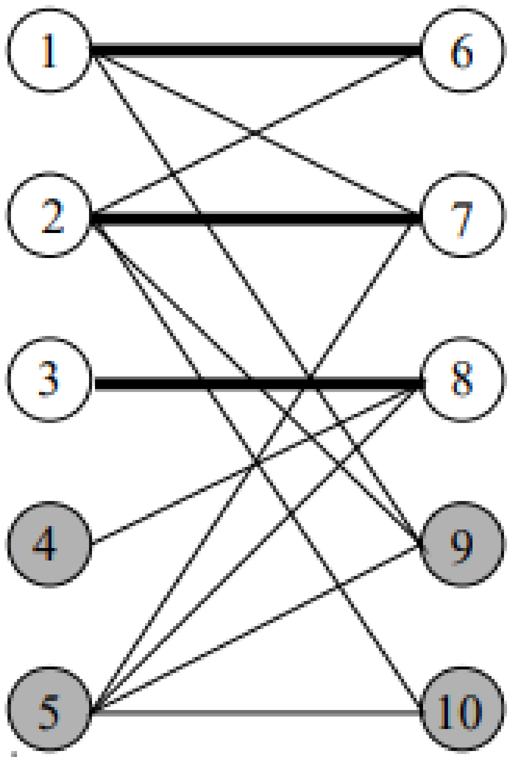
An **alternating path with respect to** M *is a path that alternates between edges in* M *and edges in* $E - M$.



4-8-3
6-1-7-2
5-7-2-6-1-9

Maximum Bipartite Matching

An **augmenting path with respect to** *M is an alternating path in which the first and last vertices are exposed.*



5-7-2-6-1-9
9-1-6-2-7-5

Maximum Bipartite Matching

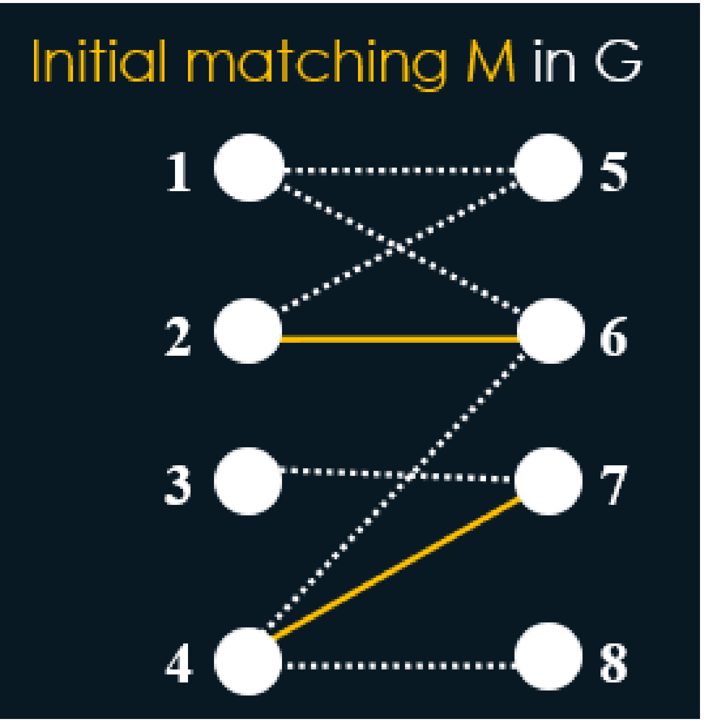
- An augmenting path with respect to M which contains k edges of M must also contain exactly k + 1 edges not in M.

- The two endpoints of an augmenting path must be on different sides of the bipartition

Maximum Bipartite Matching

Theorem: *A matching M is maximum if and only if there are no augmenting paths with respect to M.*

Maximum Bipartite Matching



Augmenting Path:

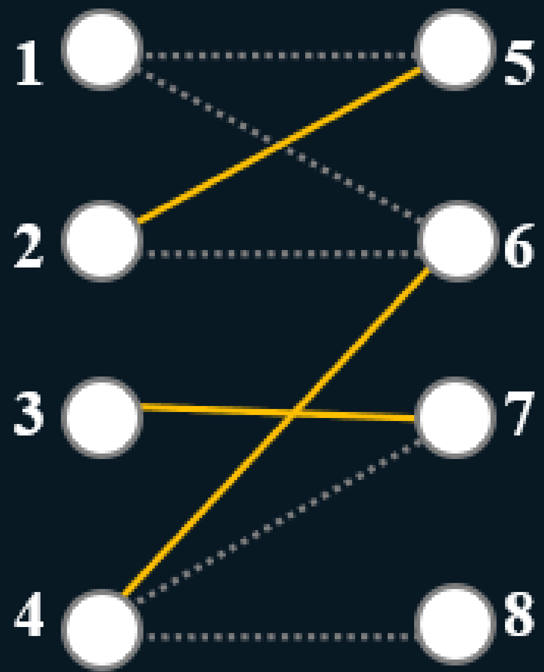
p = ((5,2), (2,6), (6,4), (4,7), (7,3))

Maximum Bipartite Matching

Augmenting path

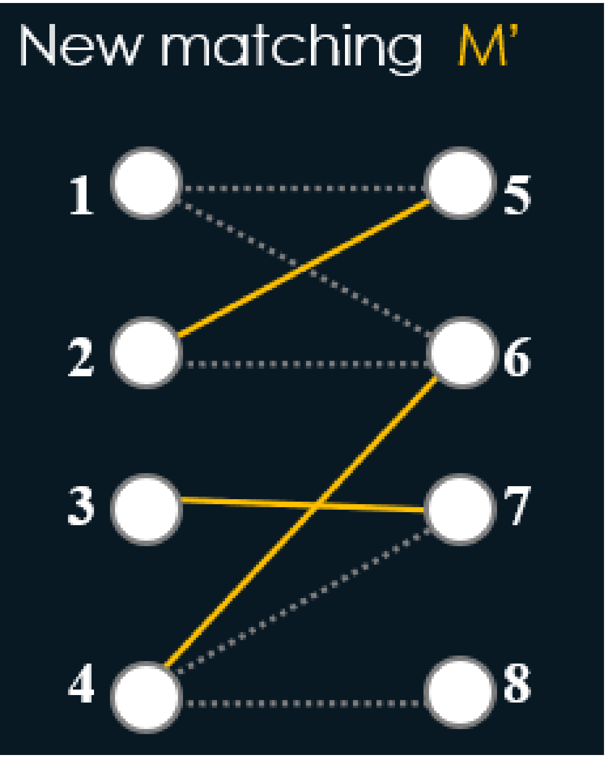
p = ((5,2), (2,6), (6,4), (4,7), (7,3))

New matching $M' = P \oplus M = (P \cup M) - (P \cap M)$



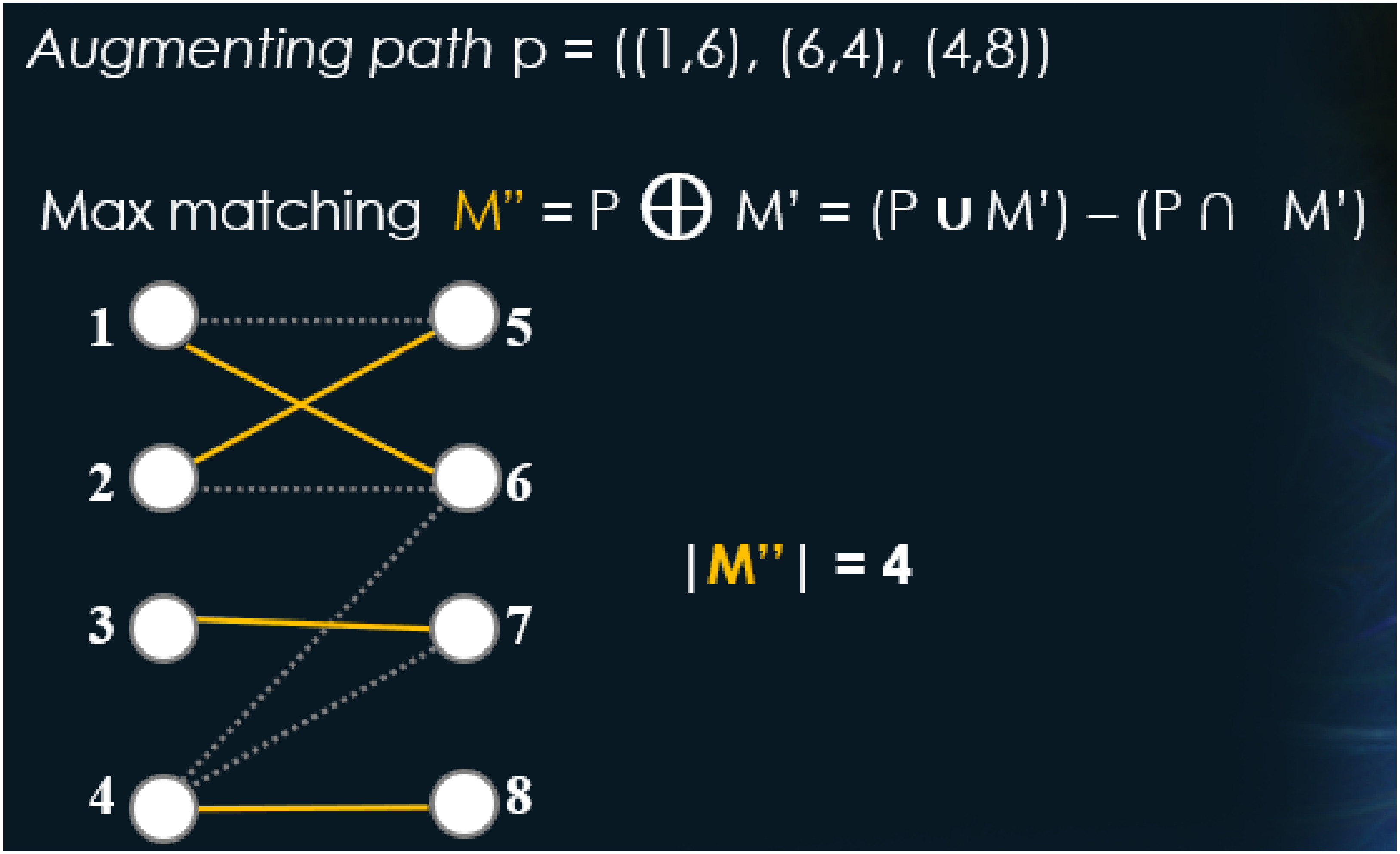
$|P \oplus M| = 3$

Maximum Bipartite Matching

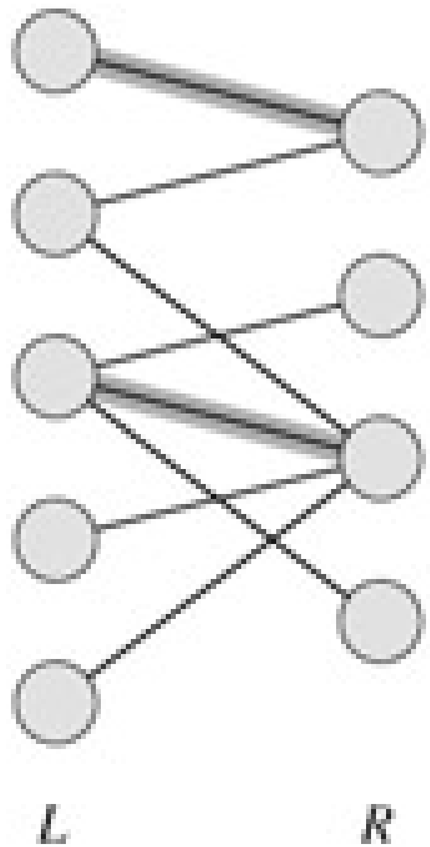


Augmenting Path:
p = ((1,6), (6,4), (4,8))

Maximum Bipartite Matching



Maximum Bipartite Matching



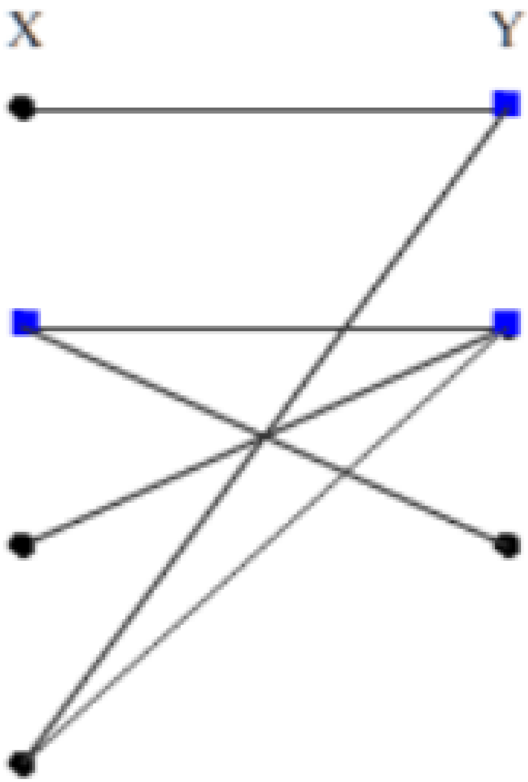
Konig Theorem

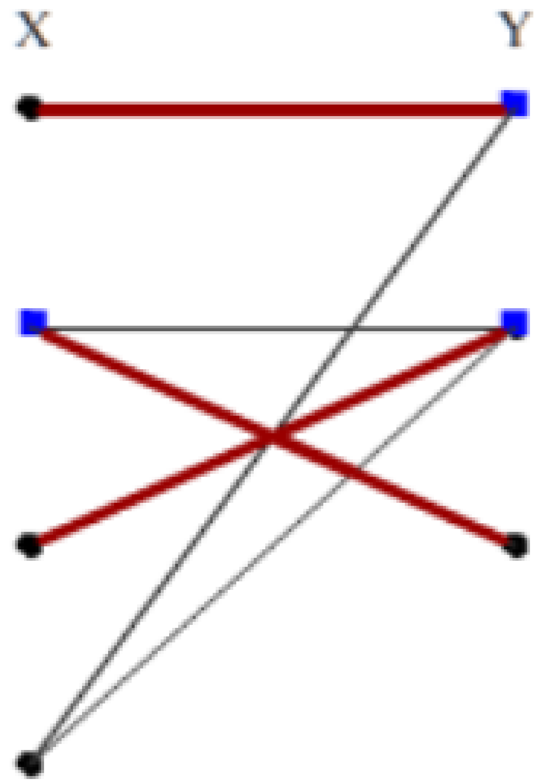
For any bipartite graph, the maximum size of a matching is equal to the minimum size of a vertex cover.

Given a bipartite graph $G = (V, E)$*:*

$$\min_{\textit{vertex cover } C} |C| = \max_{\textit{matchings } M} |M|$$

Konig Theorem





Summary

- Matchings in Bipartite graphs