AFRICAN INSTITUTE FOR MATHEMATICAL SCIENCES

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Course: Operation Research

Assignment Number: 2
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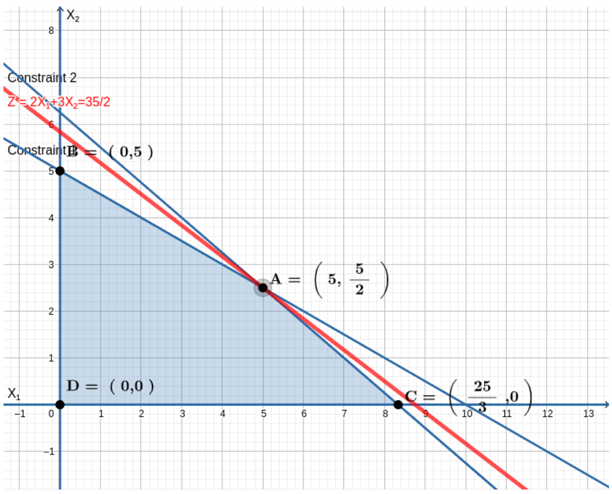
Topic: Integer Programming

(a) Using the Branch-and-bound method

$$\begin{aligned} \max \quad & z = 2x_1 + 3x_2 \\ \text{s.t.} \quad & x_1 + 2x_2 \leq 10 \\ & 3x_1 + 4x_2 \leq 25 \\ & x_1, x_2 \geq 0, x_1, x_2 \quad \text{integers} \end{aligned}$$

Solving the relax version of the problem graphically yields



The following feasible solutions $(0,5), (5, \frac{5}{2}), (\frac{25}{3}, 0), (0,0)$ are spotted. But the point $(5, \frac{5}{2})$ gives the optimal solution , thus, $x_1^* = 5$, $x_2^* = 2.5$ and $z^* = 17.5$

Since, the original problem is an integer problem, we need to branch at $x_2^* = 2.5$ by splitting into two sub-problems such that $x_2 \leq 2$ and $x_2 \geq 3$,we then have the following sub-problems

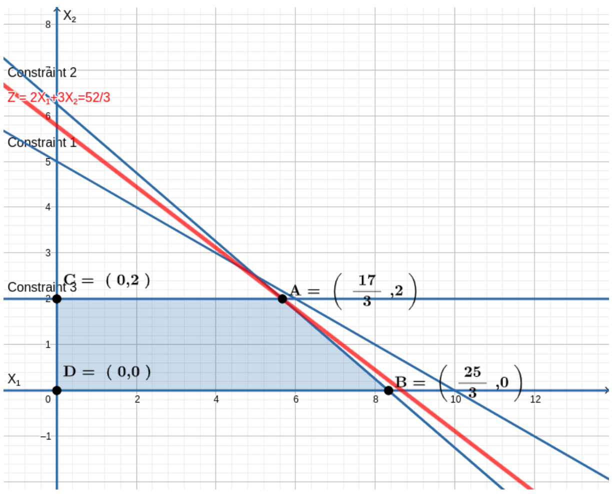
(i)

$$\begin{aligned} \max \quad & z = 2x_1 + 3x_2 \\ \text{s.t.} \quad & x_1 + 2x_2 \leq 10 \\ & 3x_1 + 4x_2 \leq 25 \\ & x_2 \leq 2 \\ & x_1, x_2 \geq 0, x_1, x_2 \end{aligned}$$

(ii)

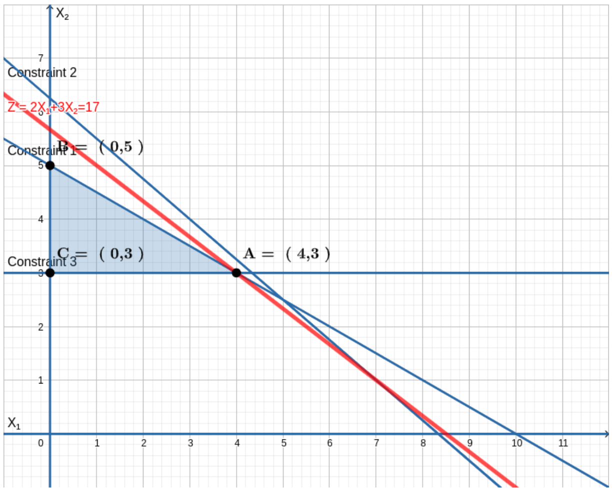
$$\begin{aligned} \max \quad & z = 2x_1 + 3x_2 \\ \text{s.t.} \quad & x_1 + 2x_2 \leq 10 \\ & 3x_1 + 4x_2 \leq 25 \\ & x_2 \geq 3 \\ & x_1, x_2 \geq 0, x_1, x_2 \end{aligned}$$

Solving (i) graphically gives



It can be seen that the optimal point here is $x_1^* = 5.7$, $x_2^* = 2$ and $z^* = 17.3$. Since x_1 is not an integer, We therefore need to branch at x_1 again.

Solving (ii) graphically gives



The feasible solutions are $(0,3),(0,5) and (4,3)$ but the optimal point is $(4,3)$ with $z* = 17$ .For this we stop here since the solutions are integers.

Now, to branch again at $x_1 = 5.7$, with $x_1 \leq 5$ and $x_1 \geq 6$ we therefore formulate the following problems

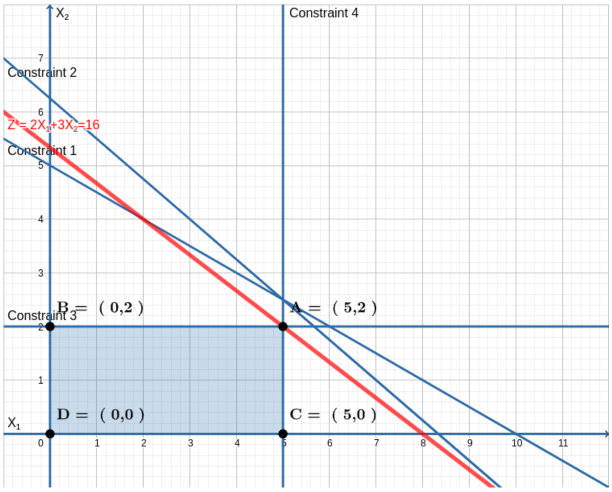
(iii)

$$\begin{aligned} \max \quad & z = 2x_1 + 3x_2 \\ \text{s.t.} \quad & x_1 + 2x_2 \leq 10 \\ & 3x_1 + 4x_2 \leq 25 \\ & x_2 \leq 2 \\ & x_1 \leq 5 \\ & x_1, x_2 \geq 0, x_1, x_2 \end{aligned}$$

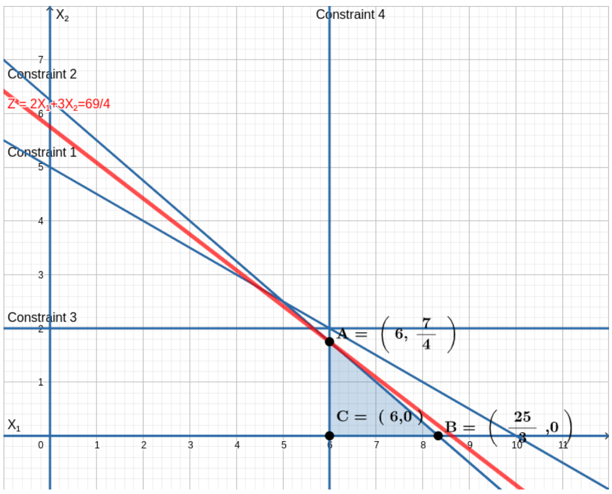
(iv)

$$\begin{aligned} \max \quad & z = 2x_1 + 3x_2 \\ \text{s.t.} \quad & x_1 + 2x_2 \leq 10 \\ & 3x_1 + 4x_2 \leq 25 \\ & x_2 \leq 2 \\ & x_1 \geq 6 \\ & x_1, x_2 \geq 0, x_1, x_2 \end{aligned}$$

Now , solving (iii) graphically gives



It is obvious that the optimal solution is $(5, 2)$ with $z^* = 16$ and this an integer solution. Now solving problem (iv) graphically gives



The feasible solutions are $(6, 0), (\frac{25}{3}, 0), (6, \frac{7}{4})$ with the optimal solution $(6, \frac{7}{4})$ and $z^* = 17.25$. Here $x_2 = 1.75$ is decimal , we should branch with $x_2 \leq 1$ and $x_2 \geq 2$. We therefore formulate the following sub-problems

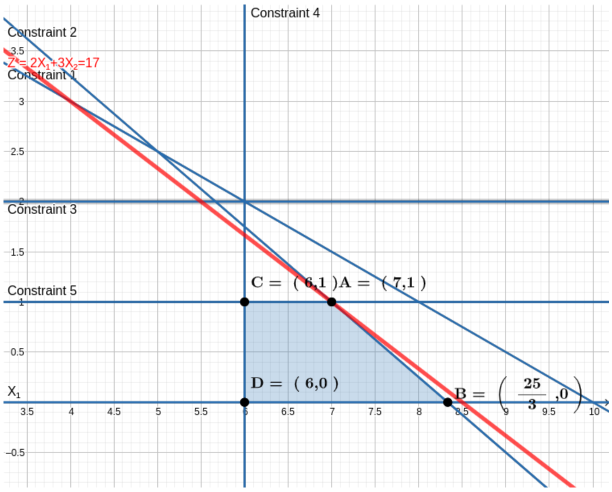
(v)

$$
\begin{aligned}
\max \quad & z = 2x_1 + 3x_2 \\
\text{s.t.} \quad & x_1 + 2x_2 \leq 10 \\
& 3x_1 + 4x_2 \leq 25 \\
& x_2 \leq 2 \\
& x_1 \geq 6 \\
& x_2 \leq 1 \\
& x_1, x_2 \geq 0, x_1, x_2
\end{aligned}
$$

(vi)

$$\begin{aligned} \max \quad & z = 2x_1 + 3x_2 \\ \text{s.t.} \quad & x_1 + 2x_2 \leq 10 \\ & 3x_1 + 4x_2 \leq 25 \\ & x_2 \leq 2 \\ & x_1 \geq 6 \\ & x_2 \geq 2 \\ & x_1, x_2 \geq 0, x_1, x_2 \end{aligned}$$

Solving problem (v) graphically gives



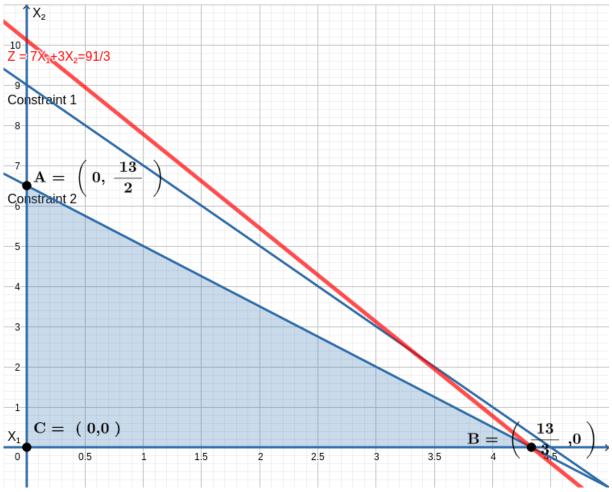
The feasible solutions are $(6,0), (\frac{25}{3}, 0), (6,1), (7,1)$ with the optimal solution $(7,1)$ and $z^* = 17$.

It can be obviously seen that problem (vi) is infeasible. Therefore, the optimal solution to the original Integer problem is $x_1 = 7$, $x_2 = 1$ and max $z = 17$

(b) Given that

$$\begin{aligned} \max \quad & z = 7x_1 + 3x_2 \\ \text{s.t.} \quad & 2x_1 + x_2 \leq 9 \\ & 3x_1 + 2x_2 \leq 13 \\ & x_1, x_2 \geq 0, x_1, x_2 \quad \text{integers} \end{aligned}$$

Solving the relax version of the problem graphically gives



The feasible solutions are $(0,0), (0, \frac{13}{2}), (\frac{13}{3}, 0)$ with an optimal solution $(\frac{13}{3}, 0)$ and $z^* = 30.3$. Since x_1 is not an integer, we branch using $x_1 \leq 4$ and $x_1 \geq 5$. We have the following sub-problems

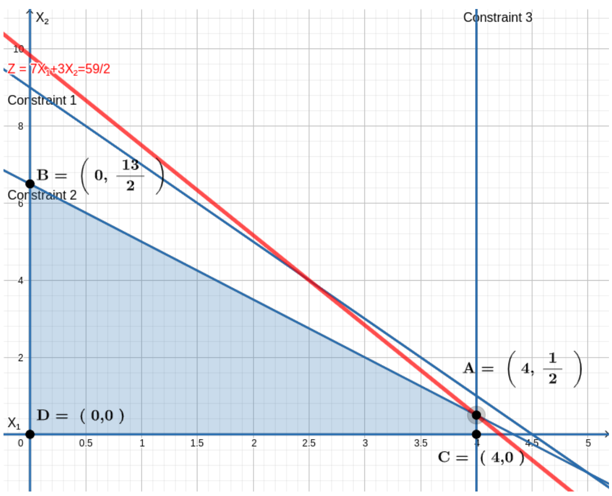
(vii)

$$\begin{aligned} \max \quad & z = 7x_1 + 3x_2 \\ \text{s.t.} \quad & 2x_1 + x_2 \leq 9 \\ & 3x_1 + 2x_2 \leq 13 \\ & x_1 \leq 4 \\ & x_1, x_2 \geq 0, x_1, x_2 \end{aligned}$$

(viii)

$$\begin{aligned} \max \quad & z = 7x_1 + 3x_2 \\ \text{s.t.} \quad & 2x_1 + x_2 \leq 9 \\ & 3x_1 + 2x_2 \leq 13 \\ & x_1 \geq 5 \\ & x_1, x_2 \geq 0, x_1, x_2 \end{aligned}$$

Solving problem (vii)graphically gives



The feasible solutions are $(0,0), (0, \frac{13}{2}), (4, \frac{1}{2}), (4,0)$ with an optimal solution $(4, 0.5)$ and $z^* = 29.5$.
It can be seen problem (viii) is infeasible. We therefore proceed to branch at $x_2 = 0.5$. Using $x_2 \leq 0$ and $x_2 \geq 1$ we formulate the following sub-problems

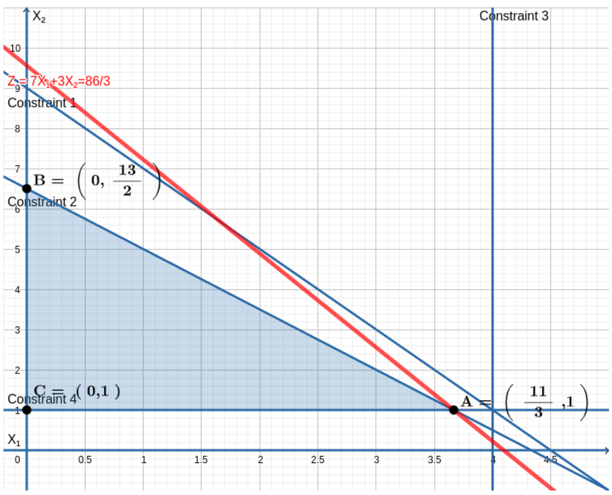
(ix)

$$\begin{aligned} \max \quad & z = 7x_1 + 3x_2 \\ \text{s.t.} \quad & 2x_1 + x_2 \leq 9 \\ & 3x_1 + 2x_2 \leq 13 \\ & x_1 \leq 4 \\ & x_2 \leq 0 \\ & x_1, x_2 \geq 0, x_1, x_2 \end{aligned}$$

(x)

$$\begin{aligned} \max \quad & z = 7x_1 + 3x_2 \\ \text{s.t.} \quad & 2x_1 + x_2 \leq 9 \\ & 3x_1 + 2x_2 \leq 13 \\ & x_1 \leq 4 \\ & x_2 \geq 1 \\ & x_1, x_2 \geq 0, x_1, x_2 \end{aligned}$$

It can be seen that problem (ix) is infeasible, we then solve problem (x) graphically and obtain



The optimal solution is $(\frac{11}{3}, 1)$ with $z^* = 28$, we need to branch at $x_1 = 3.7$ since it is not an integer. Using $x_1 \leq 3$ and $x_1 \geq 4$ we formulate the following sub-problems

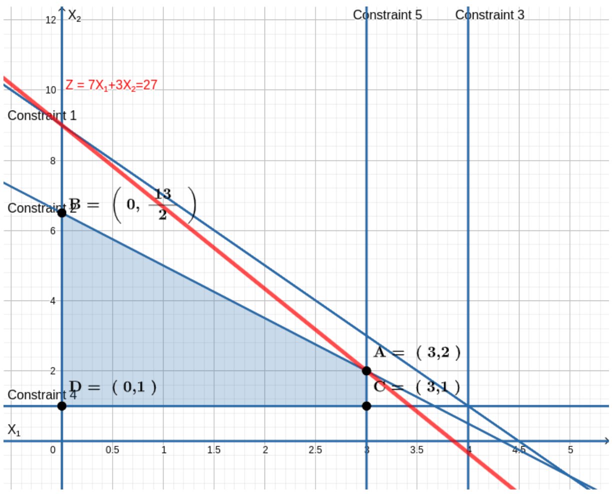
(xi)

$$\begin{aligned} \max \quad & z = 7x_1 + 3x_2 \\ \text{s.t.} \quad & 2x_1 + x_2 \leq 9 \\ & 3x_1 + 2x_2 \leq 13 \\ & x_1 \leq 4 \\ & x_2 \geq 1 \\ & x_1 \leq 3 \\ & x_1, x_2 \geq 0, x_1, x_2 \end{aligned}$$

(xii)

$$\begin{aligned} \max \quad & z = 7x_1 + 3x_2 \\ \text{s.t.} \quad & 2x_1 + x_2 \leq 9 \\ & 3x_1 + 2x_2 \leq 13 \\ & x_1 \leq 4 \\ & x_2 \geq 1 \\ & x_1 \geq 4 \\ & x_1, x_2 \geq 0, x_1, x_2 \end{aligned}$$

Solving problem (xi) graphically gives



The optimal solution is $(3, 2)$ with $z^* = 27$.

It is obvious that problem (xii) is infeasible, we therefore stop here and conclude that the optimal solution to the original integer problem is $x_1 = 3$, $x_2 = 2$ and max $z = 27$

Topic: Sequence problems

2. Given that

Books	Printing time	Binding time
1	35	80
2	130	100
3	60	90
4	30	60
5	90	30
6	120	20

The first minimum time is 20 and is in the second column, then we put job 6 in the last. The next minimum is 30 in both columns, then we put job 4 in the first and job 5 next to the last. Now the minimum time is 35, we put job 1 next to job 4. The next minimum is 60 in the first colum, we then put job 3 next to job 1 and finally we put job 2 to complete the sequence. Thus, the optimal sequence is

4	1	3	2	5	6

To find the minimum total processing time and the idle times :

Books	Printing time		Binding time		Idle Binding
	Time In	Time Out	Time In	Time Out	
4	0	30	30	90	30
1	30	65	90	170	0
3	65	125	170	260	0
2	125	255	260	360	0
5	255	345	360	390	0
6	345	465	465	485	75

From the table above, it can be seen that the total elapsed time = 485 minutes.
Idle time for Binding = 105 minutes.
Idle time for Printing = 20 minutes.

3. Given that

Tasks	M_1	M_2	M_3
A	4	5	7
B	9	3	6
C	8	2	8
D	5	4	12
E	8	1	6
F	9	3	5
G	7	5	11

We observe that $\min M_{i3} = \max M_{i2}$ where $i = 1, ..., 7$. We therefore create two fictitious machines F_1 and F_2 such that $F_1 = M_{i1} + M_{i2}$ and $F_2 = M_{i2} + M_{i3}$. We have

Tasks	F_1	F_2
A	9	12
B	12	9
C	10	10
D	9	16
E	9	7
F	12	8
G	12	16

The minimum processing time is 7 in column F_2, we then place task "E" in the last. The next minimum is 8 also in column F_2, we then place task "F" next to task "E". The next minimum is 9 in both columns, we then place task "A" in the first position to the left and task "B" in the last. The next is again 9 we place task "D" next to task A. The next minimum is 10 and it appears in both columns at the same task we then place task "C" next to task "D". Now we fill the last space with task "G" to complete the sequence. Thus the optimal sequence is

A	D	C	G	B	F	E

The total elapsed time and the idle times are found as follows

Tasks	M_1		M_2			M_3		
	Time In	Time Out	Time In	Time Out	$Idle for M_2$	Time In	Time Out	$Idle for M_3$
A	0	4	4	9	4	9	16	9
D	4	9	9	13	0	16	28	0
C	17	17	19	4	0	28	36	0
G	17	24	24	29	5	36	47	0
B	24	33	33	36	4	47	53	0
F	33	42	42	45	6	53	58	0
E	42	50	50	51	5	58	64	0

From the table above, we can find the following:
Total elapsed time = 64 hours
Idle time for $M_1 = 14$ hours
Idle time for $M_2 = 41$ hours
Idle time for$M_3 = 9$ hours

5. (a) Given that

Items	Cutting processing time	Sewing processing time
1	6	3
2	8	7
3	4	6
4	5	4
5	7	10
6	6	6
7	14	9

The minimum time is 3 in the second column , we then place item "1" in the last position. The next minimum is 4 in both columns, we then place item "3" and "4" in the first and last positions respectively. The next minimum is 6 in both columns at same item, we place item "6" next to item "3" . The next minimum is 7 in both columns, we then place item "5" and "2" in the left and right postions respectively. Now, only item "7" is left which is then placed to complete the sequence. Thus the optimal sequence is

3	6	5	7	2	4	1

(b) Suppose the third stage is added such that

Items	Cutting processing time	Sewing processing time	Pressing and packing time
1	6	3	10
2	8	7	12
3	4	6	11
4	5	4	13
5	7	10	12
6	6	6	10
7	14	9	11

We create two fictitious stages F_1 and F_2 such that $F_1 =$ cutting + sewing processing time and $F_2 =$ sewing processing time + Pressing and packing time we then have

Item	F_1	F_2
1	9	13
2	15	19
3	10	17
4	9	17
5	17	22
6	12	16
7	23	20

The minimum time is 9 in both columns, the corresponding values in the second column determines the first item to be positioned. Thus , we place item "1" in the first position then item "4" next to it. The next minimum value is 10 in the first column , we then place item "3" in the left next to item "4". The next minimum value is 12 in the first column , we then place item "6" in the left next to item "3". The next minimum value is 15 in the first column , we then place item "2" in the left next to item "6". The next minimum value is 17 in the first column , we then place item "5" in the left next to item "2". Now we fill in the remaining position with item "7" to complete the sequence. Thus the optimal sequence is

1	4	3	6	2	5	7

To find the minimum processing time , let CP, SP and PP represent the cutting processing time, sewing processing time and pressing and packing processing time respectively. Let I, O, $Idle_s$ $Idle_p$ represent time in, time out, idle time for sewing and idle time for pressing respectively. Then we obtain the following table

Item	CP		SP			PP		
	I	O	I	O	$Idle_s$	I	O	$Idle_p$
1	0	6	6	9	6	9	19	9
4	6	11	11	15	2	19	32	0
3	11	15	15	21	0	32	43	0
6	15	21	21	27	0	43	53	0
2	21	29	29	36	2	53	65	0
5	29	36	36	46	0	65	77	0
7	36	50	50	59	4	74	88	0

Minimum total processing time = 88 hours
Idle time for Pressing= 9 hours
Idle time for Sewing= 14 +(88-59)=43 hours
Idle time for Cutting= 88-50=38 hours

Topic: Duality

6.1 (a)

$$\begin{aligned}
\max \quad & z' = 3w_1 - 4w_2 + 2w_3 + 10w_4 \\
\text{s.t.} \quad & w_1 + 2w_3 + w_4 \leq 3 \\
& -2w_1 + w_2 - 3w_3 = 2 \\
& 3w_1 + 3w_2 - 7w_3 = -3 \\
& 4w_1 + 4w_2 - 4w_3 \geq 4 \\
& w_1 \leq 0, w_2 \geq 0, w_3 \quad \text{free}, w_4 \leq 0
\end{aligned}$$

(b)

$$\begin{aligned}
\min \quad & z' = 3v_1 + 4v_2 + 2v_3 \\
\text{s.t.} \quad & v_1 + 6v_2 + v_3 \geq 3 \\
& 3v_1 - v_2 + 2v_3 \geq 2 \\
& v_1 \geq 0, v_2 \quad \text{free}, v_3 \geq 0
\end{aligned}$$

6.2 Given the primal and the dual , the matrix A of the coefficient in the primal can be written as

$$A = \begin{pmatrix} 2 & 1 & 1 \\ c & 0 & 1 \\ 0 & 1 & 2 \end{pmatrix}$$

and the matrix B of coefficient in the dual can be written as

$$B = \begin{pmatrix} 2 & -5 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

Recall that $B = A^T$

$$\Rightarrow \begin{pmatrix} 2 & c & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 2 & -5 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

$\Rightarrow c = -5$
Also a=d.

Given that $x = \begin{pmatrix} 0 \\ 11 \\ -1 \end{pmatrix}$ and $v = \begin{pmatrix} -1 \\ 8 \\ 0 \end{pmatrix}$ are the optimal primal and dual solutions respectively.

Since min $\quad z =$ max $\quad w$, then $3(0) - (11) + a(-1) = 10(-1) - (8) + 5(0)$. Simplifying this gives $a = 7$, therefore, $d = 7$. Now, the nature of the third constraint in the primal determines the restriction on the third decision variable in the dual . That is, b is equivalent to e. Substituting the optimal primal solution in the right hand side of the third constraint gives $9 \quad b \quad 5$ this is true only if b is equivalent to " $\geq$ " therefore e is also $\geq$.