Dirichlet discretization of Neumann problems

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1 Introduction

2 Mathematical preliminaries

2.1 Boundary value problems

Consider a polygonal domain $\Omega \subset \mathbb{R}^2$ with boundary Γ. Let $\gamma : [0, L] \to \Gamma$ be an arclength parametrization, and $\nu(t)$ denote the outward-pointing normal to Γ at $\gamma(t)$. Let $\omega \in \mathbb{C} \setminus \{0\}$ in the first quadrant of the complex plane denote the Helmholtz wave number.

1. The interior Dirichlet problem for Laplace's equation:

$$\Delta u(\boldsymbol{x}) = 0, \qquad \boldsymbol{x} \in \Omega, \tag{1}$$
$$u(\gamma(t)) = f(t), \quad t \in [0, L]. \tag{2}$$

2. The interior Dirichlet problem for Helmholtz's equation:

$$(\Delta + \omega^2) u(\boldsymbol{x}) = 0, \qquad \boldsymbol{x} \in \Omega, \tag{3}$$
$$u(\gamma(t)) = f(t), \quad t \in [0, L]. \tag{4}$$

3. The exterior Dirichlet problem for Laplace's equation:

$$\Delta u(\boldsymbol{x}) = 0, \qquad \boldsymbol{x} \in \mathbb{R}^2 \setminus \Omega, \tag{5}$$
$$u(\gamma(t)) = f(t), \quad t \in [0, L], \tag{6}$$
$$u(\boldsymbol{x}) = O(1), \quad |\boldsymbol{x}| \to \infty. \tag{7}$$

4. The exterior Dirichlet problem for Helmholtz's equation:

$$\Delta u(\boldsymbol{x}) = 0, \qquad x \in \mathbb{R}^2 \setminus \Omega, \tag{8}$$
$$u(\gamma(t)) = f(t), \quad t \in [0, L], \tag{9}$$
$$\sqrt{|\boldsymbol{x}|} \left| \partial_{|\boldsymbol{x}|} u - i\omega u(\boldsymbol{x}) \right| \to 0, \qquad |\boldsymbol{x}| \to \infty. \tag{10}$$

5. The interior Neumann problem for Laplace's equation:

$$\Delta u(\boldsymbol{x}) = 0, \qquad \boldsymbol{x} \in \Omega, \tag{11}$$
$$\nabla u(\gamma(t)) \cdot \boldsymbol{\nu}(t) = f(t), \quad t \in [0, L], \tag{12}$$
$$\int_0^L f(t)dt = 0. \tag{13}$$

6. The interior Neumann problem for Helmholtz's equation:

$$(\Delta + \omega^2)u(\boldsymbol{x}) = 0, \qquad \boldsymbol{x} \in \Omega, \tag{14}$$
$$\nabla u(\gamma(t)) \cdot \boldsymbol{\nu}(t) = f(t), \quad t \in [0, L]. \tag{15}$$

7. The exterior Neumann problem for Laplace's equation:

$$\Delta u(\boldsymbol{x}) = 0, \qquad \boldsymbol{x} \in \mathbb{R}^2 \setminus \Omega, \tag{16}$$
$$\nabla u(\gamma(t)) \cdot \boldsymbol{\nu}(t) = f(t), \quad t \in [0, L], \tag{17}$$
$$\left| u(\boldsymbol{x}) + \left(\frac{1}{2\pi} \int_0^L f(t)dt \right) \log |\boldsymbol{x}| \right| \to 0, \quad |\boldsymbol{x}| \to \infty. \tag{18}$$

8. The exterior Neumann problem for Helmholtz's equation:

$$\Delta u(\boldsymbol{x}) = 0, \qquad x \in \mathbb{R}^2 \setminus \Omega, \tag{19}$$
$$\nabla u(\gamma(t)) \cdot \boldsymbol{\nu}(t) = f(t), \quad t \in [0, L], \tag{20}$$
$$\sqrt{|\boldsymbol{x}|} \left| \partial_{|\boldsymbol{x}|} u - i\omega u(\boldsymbol{x}) \right| \to 0, \qquad |\boldsymbol{x}| \to \infty. \tag{21}$$

Remark 2.1. *The existence and uniqueness of solutions to the above equations is a classical result. The interior Dirichlet and Neumann problems for Helmholtz equation have a countable set of real ω's (which only accumulate at ∞) for which the PDE has a non-unique solution.*

3 Boundary integral equations

A classical way of solving the interior and exterior Dirchlet and Neumann Laplace and Helmholtz boundary value problems given above is to convert them to boundary integral equations. Before describing this reduction we first define the single and double layer potential operators and summarize their relevant properties.

3.1 Layer potentials

Definition 3.1. *Given a density σ defined on Γ the single-layer potential is defined by*

$$\mathcal{S}_\omega[\sigma](\boldsymbol{y}) = \int_\Gamma G_\omega(\boldsymbol{x}, \boldsymbol{y})\sigma(\boldsymbol{x})dS_{\boldsymbol{x}}\,, \tag{22}$$

where

$$G_\omega(\boldsymbol{x},\boldsymbol{y}) = \begin{cases} \frac{i}{4}H_0(\omega|\boldsymbol{x}-\boldsymbol{y}|) & \omega \neq 0\,, \\ -\frac{1}{2\pi}\log|\boldsymbol{x}-\boldsymbol{y}| & \omega = 0. \end{cases} \tag{23}$$

Here H_0 is the Hankel function of the first kind of order 0. Similarly, the double-layer potential is defined via the formula

$$\mathcal{D}_\omega[\sigma](\boldsymbol{y}) = \int_\Gamma \boldsymbol{\nu}(\boldsymbol{x})\cdot\nabla_{\boldsymbol{x}}G_\omega(\boldsymbol{x},\boldsymbol{y})\sigma(\boldsymbol{x})dS_{\boldsymbol{x}}, \tag{24}$$

where with some abuse of notation we denote the normal to Γ at $\boldsymbol{x}$ by $\boldsymbol{\nu}(\boldsymbol{x})$.

Remark 3.1. *The layer potentials $\mathcal{S}_0[\sigma]$ and $\mathcal{D}_0[\omega]$ are the layer potentials for Laplace's equation.*

Definition 3.2. *For $\boldsymbol{x}\in\Gamma$ we define the kernel $K_\omega(\boldsymbol{x},\boldsymbol{y})$ by*

$$K_\omega(\boldsymbol{x},\boldsymbol{y}) = \boldsymbol{\nu}(\boldsymbol{x})\cdot\nabla_{\boldsymbol{x}}G_\omega(\boldsymbol{x},\boldsymbol{y}), \tag{25}$$

where $\nu(\boldsymbol{x})$ is the normal to Γ at $\boldsymbol{x}$. It is often convenient to work instead with the parametrization of K_ω which we will denote by $k_\omega : [0,L]\times k[0,L]\to\mathbb{R}$ and is defined by

$$k_\omega(s,t) = K_\omega(\gamma(s),\gamma(t)). \tag{26}$$

The following theorems describe the behaviour of the single and double layer potentials in the vicinity of the boundary curve Γ.

Theorem 1. *Suppose the point $\boldsymbol{x}$ approaches a point $\boldsymbol{x}_0=\gamma(t_0)$ from the inside along a path such that*

$$-1+\alpha < \frac{\boldsymbol{x}-\boldsymbol{x}_0}{\|\boldsymbol{x}-\boldsymbol{x}_0\|}\cdot\gamma'(t_0) < 1-\alpha \tag{27}$$

for some $\alpha>0$. Then

$$\lim_{\boldsymbol{x}\to\boldsymbol{x}_0}\mathcal{S}_\omega[\sigma](\boldsymbol{x}) = \mathcal{S}_\omega[\sigma](\boldsymbol{x}_0) \tag{28}$$

$$\lim_{\boldsymbol{x}\to\boldsymbol{x}_0}\mathcal{D}_\omega[\rho](\boldsymbol{x}) = \mathcal{D}_\omega[\rho](\boldsymbol{x}_0) - \rho(\boldsymbol{x}_0) \tag{29}$$

$$\lim_{\boldsymbol{x}\to\boldsymbol{x}_0}\frac{\mathrm{d}}{\mathrm{d}\tau}\bigg|_{\tau=0}\mathcal{S}_\omega[\rho](\boldsymbol{x}+\tau\boldsymbol{\nu}(t_0)) = \frac{\mathrm{d}}{\mathrm{d}\tau}\bigg|_{\tau=0}\mathcal{S}_\omega[\rho](\boldsymbol{x}_0+\tau\boldsymbol{\nu}(t_0)) + \rho(\boldsymbol{x}_0). \tag{30}$$

Similarly, if $\boldsymbol{x}$ approaches a point $\boldsymbol{x}_0=\gamma(t_0)$ from the outside then

$$\lim_{\boldsymbol{x}\to\boldsymbol{x}_0}\mathcal{S}_\omega[\sigma](\boldsymbol{x}) = \mathcal{S}_\omega[\sigma](\boldsymbol{x}_0) \tag{31}$$

$$\lim_{\boldsymbol{x}\to\boldsymbol{x}_0}\mathcal{D}_\omega[\rho](\boldsymbol{x}) = \mathcal{D}_\omega[\rho](\boldsymbol{x}_0) + \rho(\boldsymbol{x}_0) \tag{32}$$

$$\lim_{\boldsymbol{x}\to\boldsymbol{x}_0}\frac{\mathrm{d}}{\mathrm{d}\tau}\bigg|_{\tau=0}\mathcal{S}_\omega[\rho](\boldsymbol{x}+\tau\boldsymbol{\nu}(t_0)) = \frac{\mathrm{d}}{\mathrm{d}\tau}\bigg|_{\tau=0}\mathcal{S}_\omega[\rho](\boldsymbol{x}_0+\tau\boldsymbol{\nu}(t_0)) - \rho(\boldsymbol{x}_0). \tag{33}$$

We define the following operator which arises in the study of Neumann boundary value problems.

Definition 3.3. *Let $\mathcal{S}_\omega$ be the single-layer potential operator. Let $\boldsymbol{\nu} \cdot \nabla \mathcal{S}_\omega$ denote its normal derivative restricted to Γ. In particular, for $\boldsymbol{y} \in \Gamma$,*

$$\boldsymbol{\nu}(\boldsymbol{y}) \cdot \nabla \mathcal{S}_\omega[\rho](\boldsymbol{y}) = \frac{\mathrm{d}}{\mathrm{d}\tau}\bigg|_{\tau=0} \mathcal{S}_\omega[\rho](\boldsymbol{y} + \tau \boldsymbol{\nu}(t_0)), \tag{34}$$

where $\gamma(t_0) = \boldsymbol{y}$.

The following proposition relates the normal derivative of the single-layer operator to the double-layer operator. Its proof follows directly from the definitions.

Proposition 3.1. *Let $\mathcal{S}_\omega, \mathcal{D}_\omega : L^2(\Gamma) \to L^2(\Gamma)$ be defined as above. Let $\boldsymbol{\nu} \cdot \nabla \mathcal{S}_\omega$ denote the normal derivative of $\mathcal{S}_\omega$ in the sense of the previous definition. Then $\boldsymbol{\nu} \cdot \nabla \mathcal{S}_\omega = \mathcal{D}_\omega^T$ where T denotes the adjoint operator with respect to the inner product*

$$< f, g >= \int_0^L f(\gamma(t)) g(\gamma(t)) dt\,. \tag{35}$$

In particular, for all $\rho, \sigma \in L^2(\Gamma)$

$$\mathcal{D}_\omega[\sigma](\gamma(t)] = \int_0^L k_\omega(s,t)\, \sigma(\gamma(s))\, \mathrm{d}s \tag{36}$$

and

$$\boldsymbol{\nu}(\gamma(t)) \cdot \nabla \mathcal{S}_\omega[\rho](\gamma(t)] = \int_0^L k_\omega(t,s)\, \rho(\gamma(s))\, \mathrm{d}s. \tag{37}$$

3.2 Reduction of boundary value problems

In this section we describe the conversion of the Laplace boundary value problems (interior Dirichlet, exterior Dirichlet, interior Neumann, and exterior Neumann) to second-kind integral equations.

Theorem 2 (Interior Dirichlet problem for the Laplace and Helmholtz equations)**.** *Let $f : [0, L] \to \mathbb{C}$ be in $L^2[0, L]$. If ω is not an Dirichlet resonance of Ω, then there exists a unique $\sigma \in L^2[0, L]$ which satisfies*

$$f(s) = -\sigma(s) + \int_0^L k_\omega(t,s)\, \sigma(t)\, \mathrm{d}t, \tag{38}$$

Moreover, the solution to the interior Dirichlet problem for Laplace/Helmeholtz equation with boundary data f is given by $u(\boldsymbol{y}) = \mathcal{D}_\omega[\sigma](\boldsymbol{y})$ for all $\boldsymbol{y} \in \Omega$.

Theorem 3 (Exterior Dirichlet problem for the Laplace equation)**.** *Let $f : [0, L] \to \mathbb{C}$ be in $L^2[0, L]$. Then there exists a unique $\sigma \in L^2[0, L]$ which satisfies*

$$f(s) = \sigma(s) + \int_0^L (k_0(t, s) + 1)\, \sigma(t)\, \mathrm{d}t, \tag{39}$$

for all $s \in [0, L]$. Moreover, the solution to the exterior Dirichlet problem for Laplace's equation with boundary data f is given by $u(\boldsymbol{y}) = \mathcal{D}_0[\sigma](\boldsymbol{y}) + \int_0^L \sigma(t)dt$ for all $\boldsymbol{y} \in \mathbb{R}^2 \setminus \Omega$.

Theorem 4 (Exterior Dirichlet problem for the Helmholtz equation)**.** *Let $f : [0, L] \to \mathbb{C}$ be in $L^2[0, L]$. Suppose ω is not an interior Neumann resonance of Ω. Then there exists a unique $\sigma \in L^2[0, L]$ which satisfies*

$$f(s) = \sigma(s) + \int_0^L k_\omega(t, s)\, \sigma(t)\, \mathrm{d}t, \tag{40}$$

for all $s \in [0, L]$. Moreover, the solution to the exterior Dirichlet problem for Helmholtz's equation with boundary data f is given by $u(\boldsymbol{y}) = \mathcal{D}_\omega[\sigma](\boldsymbol{y})$ for all $\boldsymbol{y} \in \mathbb{R}^2 \setminus \Omega$.

Theorem 5 (Interior Neumann problem for the Laplace equation)**.** *Let $f : [0, L] \to \mathbb{C}$ be in $L^2[0, L]$ and satisfy $\int_0^L f(t) = 0$. Then there exists a unique $\rho \in L^2[0, L]$ which satisfies*

$$f(s) = \rho(s) + \int_0^L (k_0(s, t) + 1)\, \sigma(t)\, \mathrm{d}t, \tag{41}$$

Moreover, the solution to the interior Neumann problem for Laplace equation with boundary data f is given by $u(\boldsymbol{y}) = \mathcal{S}_0[\rho](\boldsymbol{y})$ for all $\boldsymbol{y} \in \Omega$.

Theorem 6 (Interior Neumann problem for the Helmholtz equation)**.** *Let $f : [0, L] \to \mathbb{C}$ be in $L^2[0, L]$. Suppose that ω is not an interior Neumann resonance for Ω. Then there exists a unique $\rho \in L^2[0, L]$ which satisfies*

$$f(s) = \rho(s) + \int_0^L k_\omega(s, t)\, \sigma(t)\, \mathrm{d}t, \tag{42}$$

Moreover, the solution to the interior Neumann problem for the Helmholtz equation with boundary data f is given by $u(\boldsymbol{y}) = \mathcal{S}_\omega[\rho](\boldsymbol{y})$ for all $\boldsymbol{y} \in \Omega$.

Theorem 7 (Exterior Neumann problem for the Laplace and Helmholtz equations)**.** *Let $f : [0, L] \to \mathbb{C}$ be in $L^2[0, L]$. Suppose that ω is not an interior Dirichlet resonance for Ω. Then there exists a unique $\rho \in L^2[0, L]$ which satisfies*

$$f(s) = \rho(s) + \int_0^L k_\omega(s, t)\, \sigma(t)\, \mathrm{d}t, \tag{43}$$

Moreover, the solution to the interior Neumann problem for the Laplace ($\omega = 0$) and Helmholtz equation with boundary data f is given by $u(\boldsymbol{y}) = \mathcal{S}_\omega[\rho](\boldsymbol{y})$ for all $\boldsymbol{y} \in \Omega$.

Remark 3.2. *Even though the exterior Dirichlet and Neumann problems for Helmholtz equations are well-posed partial differential equations, the corresponding integral equations obtained by setting $u = \mathcal{D}_\omega[\sigma](\boldsymbol{y})$ for the Dirichlet problem and $u = \mathcal{S}_\omega[\rho](\boldsymbol{y})$ are known to have a countable set of spurious resonances at the interior Neumann and Dirichlet resonances respectively. These spurious resonances can be eliminated by using a combined field integral representation for u instead. We defer the extension of our method to the combined field representation to section (ENTER section number here).*

3.3 Corner expansions

In the remainder of this section we assume Γ is an open wedge with sides of length one and interior angle $\pi\alpha$ with $0 < \alpha < 2$. Let $\gamma : [-1, 1] \to \Gamma$ be an arc length parametrization of Γ and $\nu : [-1, 1] \to \mathbb{R}^2$ be the inward-pointing normal to Γ. The following theorem gives an explicit representation of solutions near corners.

Theorem 8 ([?]). *Suppose that $0 < \alpha < 2$ and that N is a positive integer. Let $\lceil\cdot\rceil$ and $\lfloor\cdot\rfloor$ denote the ceiling and floor functions, respectively, and define $\overline{L}$, $\underline{L}$, $\overline{M}$, and $\underline{M}$ by the following formulas*

$$\overline{L} = \left\lceil \frac{\alpha N}{2} \right\rceil, \tag{44}$$

$$\underline{L} = \left\lfloor \frac{\alpha N}{2} \right\rfloor, \tag{45}$$

$$\overline{M} = \left\lceil \frac{(2-\alpha) N}{2} \right\rceil, \tag{46}$$

$$\underline{M} = \left\lfloor \frac{(2-\alpha) N}{2} \right\rfloor. \tag{47}$$

Suppose further that ρ is defined via the formula

$$\begin{aligned} \rho(t) =& b_0 + \sum_{i=1}^{\overline{L}} b_i |t|^{\frac{2i-1}{\alpha}} + \sum_{i=1}^{\underline{M}} b_{\overline{L}+i} |t|^{\frac{2i}{2-\alpha}} (\log|t|)^{\sigma_{N,\alpha}(i)} \\ &+ \sum_{i=1}^{\overline{M}} c_i \mathrm{sgn}(t) |t|^{\frac{2i-1}{2-\alpha}} + \sum_{i=1}^{\underline{L}} c_{\overline{M}+i} \mathrm{sgn}(t) |t|^{\frac{2i}{\alpha}} (\log|t|)^{\nu_{N,\alpha}(i)} \end{aligned} \tag{48}$$

where $b_0, b_1, \ldots, b_N$ and $c_1, c_2, \ldots, c_N$ are arbitrary real numbers and the functions $\sigma_{\alpha,N}(i)$ and $\nu_{\alpha,N}(i)$ are defined as follows

$$\sigma_{N,\alpha}(i) = \begin{cases} 1 & \text{if } \frac{2i}{2-\alpha} = \frac{2j-1}{\alpha} \text{ for some } j \in \mathbb{Z},\ 1 \le j \le \left\lceil \frac{\alpha N}{2} \right\rceil \\ 0 & \text{otherwise,} \end{cases} \tag{49}$$

$$\nu_{N,\alpha}(i) = \begin{cases} 1 & \text{if } \frac{2i}{\alpha} = \frac{2j-1}{2-\alpha} \text{ for some } j \in \mathbb{Z},\ 1 \le j \le \left\lceil \frac{(2-\alpha)N}{2} \right\rceil \\ 0 & \text{otherwise.} \end{cases} \tag{50}$$

If g is defined by

$$f(t) = \rho(s) + \int_{-1}^{1} k(t, s) \rho(t)\, \mathrm{d}t. \tag{51}$$

then there exist sequences of real numbers β_0, β_1, ... and γ_0, γ_1, ... such that

$$f(t) = \sum_{n=0}^{\infty} \beta_n |t|^n + \sum_{n=0}^{\infty} \gamma_n \mathrm{sgn}(t) |t|^n, \tag{52}$$

for all $-1 \leq t \leq 1$. *Conversely, suppose that* f *has the form (37). Suppose further that* N *is an arbitrary positive integer. Then, for all angles* $\pi\alpha$ *there exist unique real numbers* $b_0, b_1, \ldots, b_N$ *and* $c_0, c_1, \ldots, c_N$ *such that* ρ, *defined by (33), solves equation (36) to within an error* $\mathrm{O}(t^{N+1})$.

Remark 3.1. *A similar result holds for the case where the identity term in (36) is replaced by its negative; the change in sign corresponds to replacing the boundary integral equation for the exterior Dirichlet problem with the boundary integral equation corresponding to interior Dirichlet problem. Similar expansions also hold for both the exterior and interior Neumann problems, in which case the singular powers are obtained by subtracting one from the singular powers arising in the Dirichlet problem.*

The following Corollary characterizes of the solutions to the Dirichlet and Neumann boundary integral equations in the vicinity of a corner.

Corollary 3.1. *Let* Γ *be the boundary of a polygonal region and suppose one of its corners has interior angle* $\pi\alpha$ *where* $\alpha \in (0, 2)$. *Let* $\gamma : (-\delta, \delta) \to \mathbb{R}^2$ *be an arclength parametrization of* Γ *in the vicinity of the corner, with* $\gamma(0)$ *coinciding with the corner. If the boundary data,* f, *is analytic on either side of the corner then there exist unique real numbers* $b_0, b_1, \ldots, b_N$ *and* $c_0, c_1, \ldots, c_N$ *such that the density,* ρ, *defined by (33) satisfies the interior Dirichlet boundary integral equation to within an error* $\mathrm{O}(t^{N+1})$ *for* t *within* δ *of the corner. For the Neumann problems the representation is the same with the powers in the expansion reduced by one.*

4 Numerical preliminaries

In this section we summarize the numerical tools which are necessary for the main result. In particular we summarize the method for discretizing the boundary integral equation for the Dirichlet problem described in [], which uses the expansion in Theorem 6.

4.1 Discretization of the Dirichlet problem

In this section we sketch an algorithm for solving the interior Dirichlet boundary integral equation using a Nyström method; the exterior Dirichlet boundary integral equation can be discretized in a similar way. See [] for a thorough description of the method. We begin by constructing a discretization of the boundary Γ with nodes $s_1, \ldots, s_N$, and weights $w_1, \ldots, w_N$, which enable interpolation of the left- and right-hand sides of the boundary integral equation

$$f(t) = -\sigma(t) + \int_0^L k(t, s)\, \sigma(t)\, \mathrm{d}t \tag{53}$$

with precision ϵ. We proceed by enforcing equality at the discretization nodes, which yields the system of equations

$$f(s_i)\sqrt{w_i} = -\pi\sigma(s_i)\sqrt{w_i} + \sqrt{w_i} \int_0^L K(\gamma(s_i), \gamma(t))\, \sigma(t)\, \mathrm{d}t, \quad i = 1, \ldots, N. \tag{54}$$

Scaling by the square root of the weights is equivalent to solving the problem in the L^2 sense, and results in discretized operators with condition numbers which are close to those of the original physical systems [**?**].

The boundary Γ is separated into intervals which are at least some distance δ (measured in terms of arclength) away from a corner and intervals of length 2δ centered about each corner. The former are discretized using a standard smooth quadrature rule such as nested Gauss-Legendre quadrature while the latter are discretized using a custom set of interpolation nodes constructed in the following way.

First, all functions of the form x^μ, $\mu \in \{0\} \cup [1/2, 40]$, $x \in [0, 1]$ are discretized using nested Gauss-Legendre panels in x and a single Gauss-Legendre panel in μ. This creates a $N \times M$ matrix where N denotes the number of spatial discretization points r_i and M denotes the number of μ_j chosen. M and N are increased until it is guaranteed that using Lagrange interpolation from the nested discretization the function x^μ can be interpolated to within an L^2 error less than ϵ on the interval $[0, 1/2]$ for any μ in the specified range. A singular value decomposition is then performed on the $N \times M$ matrix. Let K denote the number of singular values greater than ϵ. The right singular vectors correspond to discretizations of an orthonormal set of functions $\phi_1, \ldots, \phi_K$ such that x^μ is in the span of $\phi_1, \ldots, \phi_K$ to within an accuracy of ϵ.

Finally, a set of interpolation points x_j, $j = 1, \ldots, K$ and quadrature weights w_j, $j = 1, \ldots, K$ are chosen for $\phi_1, \ldots, \phi_K$ such that the matrix $U_{ij} = \phi_i(x_j)\sqrt{x_j}$ is well-conditioned. In practice suitable interpolation points can be obtained by using the roots of ϕ_{K+1} and calculating the corresponding weights by solving a linear system. The corresponding discretization nodes and weights for the corner-containing intervals of Γ are obtained by suitable translations and scalings of $\{x_j\}$ and $\{w_j\}$.

Once the discretization has been constructed it is necessary to construct appropriate quadrature for the integrals appearing in equation (39). When s_i and t do not belong to the same corner panel (in particular when either is not itself contained in a corner panel) then the weights and nodes associated with the discretization can be used as the quadrature rule. When s_i corresponds to a corner panel special care must be taken. Instead, using an algorithm for generating generalized Gaussian quadratures [], quadrature nodes are chosen which integrate

$$\int_0^\delta k(t, s_j)\tilde{\phi}_j(t)\,\mathrm{d}t \tag{55}$$

where for ease of exposition we assume that the corner panel corresponds to $(-\delta, \delta)$ in the parametrization with $t = 0$ corresponding to the corner itself. Moreover, it is assumed that s_j lies in the half of a corner panel parametrized by $(-\delta, 0)$ and $\tilde{\phi}_j$ is a suitably scaled and translated copy of the singular function obtained in the discretization step.

Remark 4.1. *Due to scale invariance, it suffices to compute quadratures for*

$$\int_0^{\frac{1}{2}} k(t, -x_j)\phi_j(t)\,\mathrm{d}t, \tag{56}$$

where x_j was one of the original discretization nodes generated on the interval $[0, 1/2]$.

Remark 4.2. *By interpolating from the discretization nodes to these quadrature nodes we obtain a set of weights $\tilde{w}_{i,j}$ such that if $s_1, \ldots, s_{2K}$ correspond to the discretization of a corner parametrized by $(-\delta, \delta)$ with 0 corresponding to the corner then*

$$\left| \int_{-\delta}^{\delta} k(t, s_i)\, \tilde{\phi}_m(t)\, \mathrm{d}t - \sum_{j=1}^{2K} \tilde{w}_{ij} \tilde{\phi}_m(t_j) \right| < \epsilon \tag{57}$$

for all $i = 1, \ldots, 2K$ and $m = 1, \ldots, K$.

After all the quadratures have been constructed the result is an $N \times N$ linear system the solution of which gives an approximation to σ sampled at the discretization nodes.

Definition 4.1. *Let $S_\epsilon \subset L^2([0, L])$ denote the set of functions which can be interpolated from their values at the N discretization nodes to any point in $[0, L]$ with an L^2 accuracy of ϵ. That is to say that for $f \in S_\epsilon$ if $\tilde{f} : [0, L] \to \mathbb{R}$ denotes the function obtained by interpolating using the values $f(s_1), \ldots, f(s_N)$ then $\|f - \tilde{f}\|_{L^2} < \epsilon$.*

The results of this algorithm are summarized in the following theorem (see []).

Theorem 9. *Let A be the $N \times N$ matrix obtained by discretizing the interior Dirichlet problem in the preceding manner. In particular if $f \in S_\epsilon$ is piecewise analytic and $\mathbf{f} = (\sqrt{w_1} f(s_1), \ldots, \sqrt{w_N} f(s_N))^T$ then*

$$\underline{\sigma} = A^{-1} \mathbf{f} \tag{58}$$

can be interpolated to a function $\tilde{\sigma}$ which is within ϵ of the true density σ in an L^2-sense.

4.2 Discretization of the Neumann problem

In principle a similar method could be employed to discretize the Neumann boundary integral equations. Unfortunately, the singular nature of the powers (the smallest in the expansion given in Theorem 6 lies in the range $(-1/2, 0)$) makes it difficult to produce universal discretizations and quadratures which work for large ranges of angles. When the above method is run on these problems, discretization nodes tend to accumulate close to the corner (within 10^{-14}). Apart from posing certain numerical challenges, it also makes the task of finding suitable quadrature formulae difficult. Instead a different set of discretization nodes and a different set of quadrature nodes must be constructed for each angle, drastically increasing the precomputation cost.

Finally, in many applications one already has a discretization of the Dirichlet problem. For example, when considering Laplace transmission problems or triple junction problems one has to solve two decoupled boundary integral equations: one of them a Dirichlet-type boundary integral equation with the diagonal term scaled and the other a Neumann-type boundary integral equation with the identity term scaled (see [] and [] for example). In such cases it is convenient to reuse the Dirichlet discretization for the Neumann problem.

5 Numerical apparatus

5.1 Adjoint method

The following lemma relates the discretization of the inverse of an operator to the adjoint of the discretization of its inverse. Its proof follows directly from the definition of the adjoint and is omitted.

Lemma 1. *Suppose $A : L^2([0, L]) \to L^2([0, L])$ is a bounded invertible operator and that A_ϵ is an operator such that*

$$\left|\langle f, A^{-1}g\rangle - \langle f, A_\epsilon^{-1}g\rangle\right| \leq \epsilon\|f\|\|g\|, \tag{59}$$

for all f and g in some subspace $S \subset L^2([0, L])$. Here $\langle\cdot,\cdot\rangle$ denotes the inner product on $L^2([0, L])$ and $\|\cdot\|$ denotes the norm for $L^2([0, L])$. Then, for all functions f and g in S

$$\left|\langle f, (A^{-1})^* g\rangle - \langle f, \left(A_\epsilon^{-1}\right)^* g\rangle\right| \leq \epsilon\|f\|\|g\| \tag{60}$$

where * denotes the adjoint.

The following corollary follows immediately from the previous result.

Corollary 5.1. *Let A be the $N \times N$ matrix obtained by discretizing the interior Dirichlet problem and S_ϵ be the collection of functions defined in Definition 4.1. Then for all functions $f, g \in S_\epsilon$*

$$\left|\langle \mathbf{g}, (A^T)^{-1}\mathbf{f}\rangle - \int_0^L g(t)\,\sigma(t)\,\mathrm{d}t\right| < \epsilon\|f\|\,\|g\|, \tag{61}$$

where $\mathbf{f}, \mathbf{g}$ are the discretizations of f and g scaled by the square roots of the discretization weights, and σ is the solution to the exterior Neumann problem with boundary data f.

Hence a discretization of the Neumann problem can be obtained simply by taking the adjoint of the Dirichlet problem. The resulting density σ obtained is accurate in a weak sense, ie. its inner products against functions in S_ϵ are accurate to within an error of ϵ.

We conclude this section with a few remarks.

Remark 5.1. *We observe that if the solution to the boundary value problem is being calculated at a point $\boldsymbol{y} \in \mathbb{R}^2 \setminus \Omega$ more than a panel length away from the boundary curve Γ then the Neumann density σ obtained using the above result will give an accuracy of ϵ, ie. the function $K(\boldsymbol{y}, \gamma(t)) \in S_\epsilon$. Thus accurate values of the solution in the far-field can be obtained almost immediately.*

Remark 5.2. *Similarly, if the point $\boldsymbol{y} \in \mathbb{R}^2 \setminus \Omega$ at which the solution to the Neumann boundary value problem is to be calculated lies close to a smooth panel then the density σ near that point can be interpolated to a finer set of quadrature points and the value of $u(\boldsymbol{y})$ can once again be obtained to precision ϵ. We note, however, that in general the density in the vicinity of a corner cannot be interpolated accurately. This follows from the fact that the interpolation scheme constructed is only guaranteed to interpolate the powers arising in the Dirichlet problem accurately near the corner. The Neumann problem expansions contain negative powers which are not contained in this set and hence are not interpolated accurately.*

5.2 Weak corner re-solving

In this section we address the problem highlighted in the previous one; namely, the accurate evaluation of the solution to the exterior Neumann problem in the vicinity of a corner. Our approach is based on the observation that the contribution from the density on the boundary outside of a sufficiently small neighborhood of the corner is smooth when evaluated in the vicinity of the corner. This allows us to convert the problem of evaluating the potential near the corner (given the approximation to the density obtained using the adjoint approach described in the previous section) into a purely local one. In particular, we rediscretize only a small neighborhood of the corner which in turn allows us to evaluate the potential arbitrarily close to the corner to within a small factor of machine precision.

In the following we assume that we are given a discretization of the interior Dirichlet boundary integral equation (**??**) with nodes $x_1, \ldots, x_N$ and corresponding weights $w_1, \ldots, w_N$. In particular, we assume that the discretization nodes are obtained by subdividing the boundary into panels. Those panels which contain a vertex are discretized using a custom discretization scheme (see Section **??**) while the remaining panels are discretized using a standard smooth quadrature rule (such as Gauss-Legendre or Chebyshev nodes). In the following we assume that an M-point Gauss-Legendre quadrature rule is used. Additionally, we denote the discretization of the interior Dirichlet operator (using the custom quadratures described in Section **??**) by A. Let $\underline{f} = (f_1, \ldots, f_N)^T$ where $f_i = f(x_i)\sqrt{w_i}$ and $f : \partial\Omega \to \mathbb{R}$ is the right-hand side of the exterior Neumann problem. Finally, let $\underline{\sigma}$ be the approximation to the density (scaled by the square roots of the weights) obtained by solving the linear system

$$A^T \underline{\sigma} = \underline{f}. \tag{62}$$

For notational convenience we let $\gamma : [-\delta, L - \delta] \to \partial\Omega$ be a counterclockwise arclength parametrization of $\partial\Omega$ such that $\gamma(0)$ corresponds to a vertex and $\gamma[-\delta, \delta]$ corresponds to a corner panel.

For a panel $\gamma([s_1, s_2])$ with discretization nodes $x_i, \ldots, x_{i+M}$ corresponding to a Gauss-Legendre panel the density is smooth and thus it is expected to be well-represented in the basis of Legendre polynomials (shifted and scaled to the interval $[s_1, s_2]$). Hence standard interpolation techniques can be used to obtain an accurate approximation to the density σ on the interval $s_1 \leq s \leq s_2$ (see [] for a detailed description of this process and quantitative estimates on the accuracy of the result).

add remark about ratio of size of panel to distance from the corner

For corner panels the nodes were constructed to enable stable interpolation of densities s^μ, $\mu \in 0 \cup [1/2, 40]$, on the interval $s \in (-\delta, \delta)$ - assuming for simplicity that the corner is at 0 and the panel is of length 2δ. As mentioned above, the density is expected to contain terms of the form s^μ for some finite collection of μ in the interval $(-1/2, 1/2)$, and hence will not in general be stably interpolable on the interval $(-\delta, \delta)$. However, it is possible to use the density obtained using (47) to construct a sequence of nested problems in the neighborhood of the corner, the solutions of which enable accurate interpolation of the density arbitrarily close to the vertex. The number of these problems depends only on the distance of the closest

evaluation point to the corner. In particular, if r is the smallest distance of an evaluation point from the corner then only $\log_2 r/\delta$ levels are required. Each problem involves the solution of a small linear system (typically less than 100×100) and as such can be performed quickly. Furthermore, we note that the algorithm can be easily parallelized to treat multiple corners concomitantly.

We begin with the following proposition, the proof of which follows immediately from the definition of the kernel k and is omitted.

Proposition 5.1. *Suppose that f be a piecewise-analytic function in S_ϵ and $\underline{\sigma} = (A^T)^{-1}\mathbf{f}$ is the approximation to the Neumann density obtained using the adjoint of the discretization for the interior Dirichlet boundary integral equation. Then for all $t \in (-\delta, \delta)$*

$$h(t) = \sum_{i=M+1}^{N} k(t, s_i)\sqrt{w_i}\,\underline{\sigma}_i \tag{63}$$

is an analytic function of t for all $t \in (-\delta, \delta)$.

In light of this we consider the following integral equation

$$-\sigma(s) + \int_{-\delta}^{\delta} k(s,t)\sigma(t)\,\mathrm{d}t = f(s) - h(s), \quad -\delta \leq s \leq \delta. \tag{64}$$

We note that the solution to (49) is equal to the solution of (**??**) restricted to the interval $[-\delta, \delta]$. Taking the adjoint of (49) we obtain

$$-\sigma(s) + \int_{-\delta}^{\delta} k(t,s)\sigma(t)\,\mathrm{d}t = f(s) - h(s), \quad -\delta \leq s \leq \delta. \tag{65}$$

which is a Dirichlet boundary integral equation for a wedge with a piecewise analytic right-hand side. In particular, we can discretize the operator using the method summarized in the previous section. Specifically, we subdivide the interval $[-\delta, \delta]$ into three subintervals $I_0 = [-\delta, \delta/2]$, $L_0 = [-\delta/2, \delta/2]$, and $J_0 = [\delta/2, \delta]$. On I_0 and J_0 we place standard Gauss-Legendre discretization nodes, while on L_0 we use the custom discretization scheme for corners, outlined in Section **??** (see [] for a detailed description of the method). Let $\underline{f}_0$ denote the right-hand side of (49) evaluated at these discretization nodes and scaled by the square roots of the corresponding weights. Let A_0 be the discretization of the interior Dirichlet problem operator (ie. the operator acting on σ on the left-hand side of (50)). We note that due to the scale invariance of Laplace's equation for polygonal domains the portion of A_0 corresponding to the self-interaction of L_0 is a submatrix of the original matrix A. All other blocks can be generated using the discretization nodes as quadrature nodes.

The analysis of the previous section then shows that if $\underline{\sigma}_0$ is the solution of the equation

$$A_0^T \underline{\sigma}_0 = \underline{f}_0 \tag{66}$$

then $\underline{\sigma}_0$ gives a weak solution to the integral equation (49), i.e. for any function g which is analytic on $[-\delta, 0]$ and $[0, \delta]$ the inner product $\langle g, \sigma \rangle$ can be calculated to precision ϵ using the

solution $\underline{\sigma}_0$. Moreover, since the true density σ is smooth on I_0 and J_0 the Gauss-Legendre discretization allows accurate interpolation of the density on those regions.

This can be represated to obtain an interpolable approximation to the density on $L_0 = [-\delta/2, \delta/2]$. In particular, we consider the restriction of the exterior Neumann integral equation, as well as its, adjoint to the interval L_0. For the right-hand side we use the original right-hand side f minus the contribution from the remainder of the domain. In particular, if we define

$$h_1(s) = \sum_{x_i \in I_0, J_0} k(s, x_i)\sigma_0^{(i)}\sqrt{w_i} \tag{67}$$

then σ restricted to the interval L_0 satisfies

$$-\sigma(s) + \int_{-\delta/2}^{\delta/2} k(s,t)\sigma(t)\,\mathrm{d}t = f(s) - h(s) - h_1(s), \quad -\delta/2 \leq s \leq \delta/2. \tag{68}$$

The corresponding adjoint equation is given by

$$-\sigma(s) + \int_{-\delta/2}^{\delta/2} k(t,s)\sigma(t)\,\mathrm{d}t = f(s) - h(s) - h_1(s), \quad -\delta/2 \leq s \leq \delta/2. \tag{69}$$

Once again, we divide L_0 into three intervals I_1, L_1, and J_1 and discretize each interval as before. After solving the corresponding discretization of (53) using the adjoint of the discretization of the integral operator appearing in (54) we obtain a weak solution of σ on the interval L_0 which can be interpolated on I_1, and J_1 to within precision ϵ.

This process can be repeated an arbitrary number of times to yield a sequence of solutions $\underline{\sigma}_j$, $j = 0, 1, 2, \ldots$ together with corresponding intervals $I_0, I_1, \ldots$ and $J_0, J_1, \ldots$ on which it can be interpolated.

Note that if $\boldsymbol{x}$ is a point a distance r away from the corner then after $J = 1 + \log_2 r/d$ such subdivisions $\boldsymbol{x}$ will be at least twice the corner panel length away from the corner. Thus $K(\boldsymbol{x}, \cdot)$ will be smooth when restricted to the corner panel $[-\delta/2^J, \delta/2^J]$ and hence will be integrated accurately using the corner panel discretization nodes and weights.

6 Numerical results

7 Conclusion and future work