Report on "On the discretization of Laplace's equation with Neumann boundary conditions on polygonal domains" by Hoskins and Rachh

March 3, 2020

This manuscript extends a recent line of work on handling the singularities in the density of the standard boundary integral equation formulation of Laplace's equation that arise at the corners of polygonal domains, through the use of specialized, high-order quadratures, which are computed using analytic expansions of the density. Previous work in this direction has difficulties directly applying this strategy to Neumann problems, since the singularities are significantly worse here than in the Dirichlet case. The current work makes the incredibly interesting observation that one can instead use the adjoint of a discretization to the Dirichlet problem to obtain a solution that is accurate in a weak sense. Furthermore, pointwise accuracy in a region arbitrarily close to a corner can be obtained through a novel local refinement process. The relevant theory for justifying both that this approach yields a density that can be accurately interpolated away from the corners and that the refinement process can be done in a completely local manner is developed. Furthermore, the approach is illustrated through exceptionally thorough and impressive numerical results.

In my opinion, this paper certainly should be published, due to the very high quality of the work and good presentation. However, the manuscript in its current state is missing a few details that I believe are of crucial interest to potential readers and is confusing in a few places. I suspect that a couple of these missing details are present elsewhere in the literature, but I still believe that it would be valuable to include them here for the purpose of producing a relatively self-contained work. Furthermore, there are several suspected typos and notational inconsistencies that should be fixed to maximize readability. When these are resolved, I think this will be a very strong paper.

Remarks on content and readability:

1. p. 2. The last sentence of the first paragraph mentions that the disadvantage of compression-based approaches to quadrature is cost in 3D. Given that these approaches have already been used in 3D problems, unlike the work in the current paper, I think that as a minimum there should be a couple of brief remarks somewhere about how the method here would extend and scale to 3D.

2. p. 6. On a first reading, Theorem 6 is not easy to digest. Could you add a few comments to provide some intuition for this result, and also a comment on the connection of the result on an open wedge to the case of polygons? Do we expect the expansions to be exactly the same?

3. p. 8. I think it is extremely important to explicitly discuss how large we expect K to be in section 4.1.1, especially how it relates to the accuracy a user might desire. I realize the relevant information is likely in [14], but it is of sufficient importance to the current work to make this clear by either stating a result or computational experience.

4. p. 11. The theory as developed considers only the exterior Neumann problem. I'm guessing a similar duality relationship holds with the interior Neumann problem and the exterior Dirichlet problem, and the procedure carries over straightforwardly (indeed, you consider an interior Neumann problem in the numerical results). For the sake of completeness, could you add a brief remark on whether or not the procedure is exactly the same for the interior Neumann problem, and state if there are any special considerations that are different from the exterior Neumann case?

5. p. 13. I find the choice of notation in Theorem 8 to be very poor. In particular A_0 and f_0 are used right above in (56) to denote the local problem after the corner has been refined (A_0 is of size $P + 2M \times P + 2M$). Then in the statement of the theorem, these variables are used to denote the analogous variables on just the corner panel of the unrefined problem (A_0 is of size $P \times P$), and then in the proof of the result these variables are used to refer to the original corner panel as well as the adjacent panels (A_0 is of size $P + 2M \times P + 2M$). Also, I am not sure why $\widetilde{f}_0$ is not underlined, since it is a vector. Could you please fix this notation?

6. p. 19. In the same vein as comment 3, I find the absence of any information on conditioning to be a bit startling, since one of the main appeals of working with boundary integral equations is the ability

to form well-conditioned discretizations. Again, I am aware that this falls squarely into the scope of existing work (again, probably [14]), but find this also sufficiently important to at least summarize in the current work.

7. p. 19. Even for a problem with 22k dofs, 105 iterations of a Krylov method seems like a lot, especially if the problem is well-conditioned. I can't help but wonder if this is an artifact of the fact that your FMM tolerance is the same as your iterative method tolerance. If you set the latter to 1e-14 or 1e-13, does the number of iterations decrease drastically?

8. p. 21-22. I find the content of Appendix B difficult to understand for a number of reasons. The motivation for such a result is clear, but I'm not sure I totally understand the conclusion. In particular, you want to show that using the adjoint discretization one can interpolate the density on a panel far from the corner. That makes sense. Your approach to showing this is to use the relation (B.1) to find another equivalent integral equation, which you then show can be solved using existing information accurately.

   Is the claim that to actually interpolate the density on such a panel, one would need to solve another Dirichlet problem, and it does NOT suffice to just interpolate the value of σ obtained on the GL nodes when solving the original problem? Please clarify this point.

   There are these other issues I'm having with it:

   (a) On your assumptions about the discretization, 1 and 3 have an easy-to-understand interpretation. It's unclear to me what 2 is really assuming about the discretization, given that it seems to mostly to be introducing notation. Of course, I can imagine conditions where ρ_0 is less than the width of some panels, but is there some deeper restriction?

   (b) Your assumption that $\rho_0 > 1$ and your later statement that the rate at which a polynomial interpolant will converge is $C\rho_0^{-M}$ is a bit awkward, since these choices are specific for a function on an interval of length 2.

   (c) The second paragraph on p. 22 is very unclear to me. I don't even understand the first sentence as to why $k(s,t)$ would be identically 0 on the same segment as $\gamma([s_1, s_2])$, or what is really even meant by "segment" here. Furthermore, it is mentioned that $k(s,t)$ when considered as a function of $t \in \mathbb{C}$ has a singularity at $\gamma(s)$. Given the current setup wouldn't the singularity be at s? I would recommend rewriting this entire paragraph to be more transparent.

Small fixes and typos:

1. The "Significance and novelty" section needs another pass through:

   (a) First sentence. solution $\to$ solutions

   (b) Third sentence. boundary equation $\to$ boundary integral equation, expansion terms $\to$ expansion in terms

   (c) Fourth sentence. of he $\to$ of the

   (d) Fifth sentence. I think the interval should be $(-1/2, 0)$, as stated in the abstract.

   (e) Eight sentence: point wise $\to$ pointwise

   (f) Ninth sentence: the corner $\to$ a corner, sub-problems $\to$ subproblems

2. Highlights:

   (a) First point. discretization to $\to$ discretization of

   (b) Second and third points. As written, it's not clear to me how these are different. I would merge these and instead make the third bullet about the high accuracy you are able to achieve.

3. Throughout the whole document, "arc length", "arc-length" and "arclength" are used. Please stick to one for consistency.

4. p. 1. study field enhancements $\to$ study of field enhancements

5. p. 4. There's a non-bold ν in Def 3.2.
6. p. 4. Extra right parenthesis in (19).
7. p. 4. In (24), should $\nu(x_0)$ in RHS by $\nu(y)$?
8. p. 4. Although Theorem 1 is a fairly standard result in the BIE community, I would still include a pointer to a reference (probably [15]).
9. p. 4. Don't the signs in (20) and (23) depend on which normal is taken? Since ν in Def 3.2 is the inward normal, you might want to make this clear. Also, τ isn't currently assumed to be positive.
10. p. 4. In Theorem 1, perhaps include the continuity of the normal derivative of the double layer potential for completeness.
11. p. 4. Also in Def 3.3 is the fact $\gamma(t_0) = x_0$ relevant?
12. p. 5. You change notation from 3.1 to 3.2 from the single and double layer operators acting on functions defined on the boundaries to operators acting on functions defined on the interval $[0, L]$. While this is a standard abuse of notation, I think it's still worth explicitly saying.
13. p. 5. In Theorems 2,3,4, and 5, $L^2[0, L] \to L^2([0, L])$ to make it consistent with elsewhere.
14. p. 5. Add "for all $s \in [0, L)$" after (28), (30), and (31).
15. p. 5. Put dt after the integral in statement of Theorem 4.
16. p. 5. Is there a reason the boundary data for the Neumann case in Theorems 4 and 5 is complex-valued unlike the Dirichlet data?
17. p. 6. In the first paragraph in section 3.3, ν is not bold.
18. p. 6. The notation for the exponents in (36) defined in (37) and (38) seems bad to me, in light of the fact that σ and ν have such consistent, unrelated meanings. Could you use different letters?
19. p. 7. End of Theorem 7: $\rho \to \sigma$.
20. p. 8. At the top, insert "and" between $f(s_i)$ and the other term.
21. p. 8. Should the $-\sigma(s_i)$ term be divided by 2?
22. p. 8. In the first paragraph of 4.1.1, I feel like there is a mixing of terminology for $[0, L]$ and Γ. While the notions of length are the same, saying Γ can be separated into intervals seems awkward to me.
23. p. 8. Perhaps define "nested Gauss-Legendre panels".
24. p. 8. Does your discretization of μ include 0 i.e. is it a GL discretization of $[1/2, 50]$ union $\{0\}$?
25. p. 8. Re-using the variable N in 4.1.1 is a bit annoying.
26. p. 8. In the middle of 4.1.1, I think "right singular vectors" $\to$ "left singular vectors".
27. p. 9. Should s_j in (45) be s_i? Same for (46).
28. p. 9. In (47), should t_j be s_j?
29. p. 9. In the paragraph after Remark 4.2, perhaps "the solution of which" $\to$ "the scaled solution of".
30. p. 9. In Definition 4.1, the type of interpolant should be specified (e. g. the set of functions that can be accurately interpolated using a global polynomial interpolant on a set of points is very different than that corresponding to piecewise local polynomials).
31. p. 9. Empty citation after Def 4.1

32. p. 9. It seems like for the most part you are using underlined letters for vectors. Is there a good reason that the vector f is bold?

33. p. 9. Also at Theorem 7, it might be good to point out that $\widetilde{\sigma}$ needs to be scaled back before interpolating.

34. p. 10. In Lemma 1, S_ϵ seems to refer to an arbitrary subspace when the notation was defined for a specific subspace in Def 4.1. Perhaps change S_ϵ to just S.

35. p. 10. The use of * to denote the adjoint as opposed to ^{T} seems inconsistent with the notation in Prop 3.1.

36. p. 10. In Cor 5.1, again why bold?

37. p. 11. ie. $\to$ i.e. for consistency (twice)

38. p. 11. Throughout, you have been using Γ to refer to the boundary of Ω. On this page, however, you introduce the notation $\partial\Omega$, which I think only appears here. Could you make this consistent?

39. p. 11. Final line: $\gamma[-\delta, \delta] \to \gamma([-\delta, \delta])$.

40. p. 12. The period at the beginning of line 5 looks strange. Is there an extraneous space?

41. p. 12. Second paragraph: $\mu \in 0 \cup [1/2, 50] \to \mu \in \{0\} \cup [1/2, 50]$.

42. p. 12. Same paragraph, is the relevant interval $(-1/2, 1/2)$ or should it be $(-1/2, 0)$?

43. p. 12. Prop 5.1: "Suppose that f be a" $\to$ "Suppose that f is a".

44. p. 12. Prop 5.1: Again with the bold f.

45. p. 12. Prop 5.1: You've defined the parametrization to go from $[-\delta, L-\delta]$ and are now in the statement talking about the interval $(-2\delta, \delta)$.

46. p. 12. Are the coefficients for the σ term in (54) and (55) correct?

47. p. 12. (55) should end with a comma, not a period

48. p. 13. Empty citation in the first paragraph.

49. p. 13. ie. $\to$ i.e.

50. p. 13. The K_0 and Q_0 notation, as far as I can tell, is only used in (63). Could you perhaps avoid introducing this extra notation?

51. p. 13. In the statement of Theorem 8, $\mu = 0, 1/2 - 40 \to \mu \in \{0\} \cup [1/2, 50]$ for consistency (note both the right endpoint and the notation)?

52. p. 14. Second sentence after proof of Theorem 8: "as its, adjoint" $\to$ "as its adjoint".

53. p. 14. Could you please verify the coefficient in front of the σ terms in (64) and (64) are correct?

54. p. 14. Penultimate paragraph: "which it" $\to$ "which each".

55. p. 15. The numbering on Remarks and 1 and 2 is off.

56. p. 16. In "scattering problem whose right hand side", I think replacing "right hand side" with boundary conditions is more natural.

57. p. 16. Periods need to be inserted after (68) and (69).

58. p. 18. The sentence before section 6.2 begins does not accurately describe the figure it refers to.

59. p. 19. Specify which iterative method you are using.

60. p. 20. In the sentences above Theorem 9, C is never used and the assumptions on r are redundant because these are in the statement of the theorem anyways.

61. p. 20. Statement of theorem: ofΓ $\to$ of Γ.

62. p. 20. In (A.1), $f(s) \to f(\gamma(s))$.

63. p. 21. The second sentence of the caption of Figure B.6 doesn't make any sense and certainly does not illustrate the condition you want it to.

64. p. 21. In paragraph above (B.1), "Neumann problem, let σ" $\to$ "Neumann problem, and let σ".

65. p. 21. In same paragraph, "right hand side" $\to$ "right-hand side" for consistency

66. p. 21. In (B.2), add a comma after $f(s)$.

67. p. 22. At the top: "Dirichlet" $\to$ "Dirichlet problem".

68. p. 22. Beginning of penultimate paragraph, "of the of the".

69. p. 22. Last paragraph: "piecewise analytic ," $\to$ "piecewise analytic,".

70. References:

    (a) [4],[11],[17],[18] "laplace" $\to$ "Laplace".

    (b) I think [15] is a book, so all letters in the title should be capitalized.