Notes

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1 Shape Optimization

We are interested in maximizing the following scale-invariant quantity:

$$\phi(\Omega) = \frac{\|\nabla u\|_\infty}{\|u\|_2\sqrt{\lambda}},$$

where λ, u depending on Ω are the first eigenfunction/eigenvalue of $-\Delta$ equipped with Dirichlet boundary condition.

Example 1. Consider the family of rectangles $\{\mathcal{R}_\alpha\}$ parametrized by $\alpha \in \mathbb{R}^+$. By symmetry, we might assume without loss of generality that $\alpha \in [1, \infty)$. By translational invariance, we assume the four vertices are $(0,0), (0,1), (\alpha, 1), (\alpha, 0)$. Separation of variables tells us that

$$u(x,y) = \sin(\pi x)\sin(\pi y/\alpha), \qquad \lambda = \pi^2 + \frac{\pi^2}{\alpha^2} = \pi^2\frac{\alpha^2+1}{\alpha^2}.$$

Recall the fact that ∇u achieves maximum on the boundary. By symmatry, it suffices to consider the two sides where one variable vanish:

$$|\nabla u(x,0)| = |\partial_y u(x,0)| = \frac{\pi}{\alpha}|\sin(\pi x)|, \qquad |\nabla u(0,y)| = |\partial_x u(0,y)| = \pi|\sin(\pi y/\alpha)|.$$

This, together with the fact that $\alpha \geq 1$, implies $\|\nabla u\|_\infty = \pi$ and occurs at the midpoint of either of the vertical sides of the rectangle. Computing $\|u\|_2$ is again straight forward:

$$\|u\|_2^2 = \left(\int_0^1 \sin(\pi x)^2 dx\right)\left(\int_0^\alpha \sin(\pi y/\alpha)^2 dy\right) = \alpha\left(\int_0^1 \sin(\pi x)^2 dx\right)^2 = \frac{\alpha}{4}.$$

Combining the above,

$$\phi(\mathcal{R}_\alpha) = \frac{\pi}{\sqrt{\frac{\alpha}{4}\cdot\frac{\pi^2(\alpha^2+1)}{\alpha^2}}} = \sqrt{2}\cdot\sqrt{\frac{2\alpha}{\alpha^2+1}} \leq \sqrt{2},$$

where equality holds if and only if $\alpha = 1$ (i.e a square.)