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The ultimate guide to layer potentials on the sphere

In this note, we consolidate analytic expressions for various layer potentials on the sphere. In the following, $P_{nm}(t)$ are the associated Legendre polynomials of degree n and order m, $Y_{nm}(\theta,\phi)$, are the spherical harmonics of degree n and order m given by

$$Y_{nm}(\theta,\phi) = \sqrt{2n+1}\frac{(n-|m|)!}{(n+|m|)!}P_{n|m|}(\cos\theta)e^{im\phi}\,. \tag{0.1}$$

Let $j_n(z), h_n(z)$ denote the spherical Bessel and spherical Hankel functions of degree n. Let Ω^- be the interior of the unit sphere, $\partial\Omega$ denote it's boundary, and Ω^+ denote the exterior of the unit sphere. For any $\boldsymbol{x}\in\mathbb{R}^3$, let (r,θ,ϕ), denote it's spherical coordinates.

Scalar potentials

Let $G_k(\boldsymbol{x},\boldsymbol{y}) = e^{ik|\boldsymbol{x}-\boldsymbol{y}|}/(4\pi|\boldsymbol{x}-\boldsymbol{y}|)$ denote the Helmholtz Green's function in three dimensions Let $\mathcal{S}_k[\sigma]$ denote the Helmholtz single layer potential given by

$$\mathcal{S}_k[\sigma](\boldsymbol{x}) = \int_{\partial\Omega} G_k(\boldsymbol{x},\boldsymbol{y})\sigma(\boldsymbol{y})dS_{\boldsymbol{y}}\,, \tag{0.2}$$

and $\mathcal{D}_k[\sigma]$ denote the Helmholtz double layer potential given by

$$\mathcal{D}_k[\sigma](\boldsymbol{x}) = \int_{\partial\Omega} \nabla_{\boldsymbol{y}} G_k(\boldsymbol{x},\boldsymbol{y})\cdot\boldsymbol{n}(\boldsymbol{y})\sigma(\boldsymbol{y})dS_{\boldsymbol{y}}\,. \tag{0.3}$$

We will use $\mathcal{S}^{\pm}[\sigma]$, and $\mathcal{D}^{\pm}[\sigma]$ to also denote the restrictions to these operators to the boundary where the superscript '$+$' indicates an exterior limit, and the superscript '$-$' denotes the interior limit. Let $\mathcal{S}_k^{',\pm}[\sigma]$, and $\mathcal{D}_k^{',\pm}[\sigma]$ denote the limiting values of the Neumann data corresponding to the single and double layer potentials given by

$$\begin{aligned}
\mathcal{S}_k^{',\pm}[\sigma](\boldsymbol{x}) &= \lim_{\substack{\boldsymbol{z}\to\boldsymbol{x}\\ \boldsymbol{x}\in\Omega^{\pm}}} \left(\boldsymbol{n}(\boldsymbol{x})\cdot\nabla_{\boldsymbol{x}}\left(\int_{\partial\Omega} G_k(\boldsymbol{z},\boldsymbol{y})\sigma(\boldsymbol{y})\,dS_{\boldsymbol{y}}\right)\right)\,,\\
\mathcal{D}_k'[\sigma](\boldsymbol{x}) &= \text{f.p.}\left(\boldsymbol{n}(\boldsymbol{x})\cdot\nabla_{\boldsymbol{x}}\left(\int_{\partial\Omega}\left(\nabla_{\boldsymbol{y}} G_k(\boldsymbol{x},\boldsymbol{y})\cdot\boldsymbol{n}(\boldsymbol{y})\right)\sigma(\boldsymbol{y})\,dS_{\boldsymbol{y}}\right)\right)\,,
\end{aligned} \tag{0.4}$$

for $\boldsymbol{x}\in\partial\Omega$. Finally, let $\mathcal{S}_k^{'',\pm}[\sigma]$ denote the limiting value of the second normal derivative of $\mathcal{S}_k$ restricted to $\partial\Omega$ given by

$$\mathcal{S}_k^{'',+}[\sigma](\boldsymbol{x}) = \lim_{\substack{\boldsymbol{z}\to\boldsymbol{x}\\ \boldsymbol{x}\in\Omega^{\pm}}} \left(\boldsymbol{n}(\boldsymbol{x})\cdot\nabla_{\boldsymbol{x}}\nabla_{\boldsymbol{x}}\left(\int_{\partial\Omega} G_k(\boldsymbol{z},\boldsymbol{y})\sigma(\boldsymbol{y})\,dS_{\boldsymbol{y}}\right)\cdot\boldsymbol{n}(\boldsymbol{x})\right)\,, \tag{0.5}$$

for $\boldsymbol{x}\in\partial\Omega$, and let $H(\boldsymbol{x})$ denote the mean curvature.

In a slight abuse of notation, let the principal value/finite part of these operators be denoted by $S_k[\sigma], D_k[\sigma], S_k'[\sigma], D_k'[\sigma]$, and $S_k''[\sigma]$ respectively. Of these D_k, S_k' are principal value integrals, S_k is absolutely convergent, and D_k', S_k'' are finite part integrals.

Limiting values on the boundary

The layer potentials satisfy the following jump relations. Suppose that $\boldsymbol{x} \in \Gamma$, then

$$\begin{aligned}
\lim_{\substack{\boldsymbol{y}\to\boldsymbol{x}\\ \boldsymbol{y}\in\Omega^\pm}} \mathcal{S}_k[\sigma](\boldsymbol{y}) &= S_k[\sigma](\boldsymbol{x})\,,\\
\mathcal{D}_k^{\pm}[\sigma](\boldsymbol{x}) &= \pm\frac{\sigma(\boldsymbol{x}_0)}{2} + D_k[\sigma](\boldsymbol{x})\,,\\
\mathcal{S}_k^{',\pm}[\sigma](\boldsymbol{x}) &= \mp\frac{\sigma(\boldsymbol{x}_0)}{2} + S_k'[\sigma](\boldsymbol{x})\,,\\
\mathcal{D}_k^{',\pm}[\sigma](\boldsymbol{x}) &= D_k'[\sigma](\boldsymbol{x})\,,\\
\mathcal{S}_k^{'',\pm}[\sigma](\boldsymbol{x}) &= S_k''[\sigma](\boldsymbol{x})\,.
\end{aligned} \tag{0.6}$$

The layer potentials have the following limiting values on the boundary, see [**?**], for example.

$$\begin{aligned}
S_k[Y_{nm}] &= ikj_n(k)h_n(k)Y_{nm}(\theta,\phi)\,,\\
\mathcal{D}_k^{+}[Y_{nm}] &= ik^2j_n'(k)h_n(k)Y_{nm}(\theta,\phi)\,,\\
\mathcal{D}_k^{-}[Y_{nm}] &= ik^2j_n(k)h_n'(k)Y_{nm}(\theta,\phi)\,,\\
D_k[Y_{nm}] &= \frac{ik^2}{2}\left(j_n(k)h_n'(k) + j_n'(k)h_n(k)\right)Y_{nm}(\theta,\phi)\,,\\
\mathcal{S}_k^{',+}[Y_{nm}] &= ik^2j_n(k)h_n'(k)Y_{nm}(\theta,\phi)\,,\\
\mathcal{S}_k^{',-}[Y_{nm}] &= ik^2j_n'(k)h_n(k)Y_{nm}(\theta,\phi)\,,\\
S_k'[Y_{nm}] &= \frac{ik^2}{2}\left(j_n(k)h_n'(k) + j_n'(k)h_n(k)\right)Y_{nm}(\theta,\phi)\,,\\
\mathcal{D}_k^{',\pm} = D_k'[Y_{nm}] &= ik^3j_n'(k)h_n'(k)Y_{nm}(\theta,\phi)\,.
\end{aligned} \tag{0.7}$$

Potentials in the interior and exterior

The value of the potentials in the interior and exterior domain are given by

$$\begin{aligned}
\mathcal{S}_k[Y_{nm}](\boldsymbol{x}) &= \begin{cases} ikh_n(k)j_n(kr)Y_{nm}(\theta,\phi) & r<1\,,\\ ikj_n(k)h_n(kr)Y_{nm}(\theta,\phi) & r>1\,,\end{cases}\\
\mathcal{D}_k[Y_{nm}](\boldsymbol{x}) &= \begin{cases} ik^2h_n'(k)j_n(kr)Y_{nm}(\theta,\phi) & r<1\,,\\ ik^2j_n'(k)h_n(kr)Y_{nm}(\theta,\phi) & r>1\,.\end{cases}
\end{aligned} \tag{0.8}$$

Solution to various boundary value problems with incident planewaves

We begin by expressing the plane waves in terms of spherical bessel functions,

$$e^{ik\boldsymbol{x}\cdot\hat{\boldsymbol{z}}} = e^{ikr\cos\theta} = \sum_{n=0}^{\infty} i^n\sqrt{2n+1}\,j_n(kr)Y_{n0}(\theta,\phi)\,, \tag{0.9}$$

Note that solutions in interior domains with planewave data are the planewaves themselves since they already satisfy the PDE. However, in exterior domains, they are not radiating/outgoing and hence the scattered field requires some computation. In the following, that the scattered field u is outgoing and satisfies the Sommerfeld radiation condition.

0.0.1 Dirichlet problem

Consider the exterior Dirichlet problem, then

$$\begin{aligned} (\Delta + k^2)u = 0\,, \quad \boldsymbol{x} \in \Omega^+, \\ u = -e^{ik\boldsymbol{x}\cdot\hat{\boldsymbol{z}}}\,, \quad \boldsymbol{x} \in \partial\Omega\,. \end{aligned} \tag{0.10}$$

Then

$$u = \sum_{n=0}^{\infty} i^n \sqrt{2n+1} j_n(k) h_n(kr) Y_{n0}(\theta,\phi). \tag{0.11}$$

Suppose now we use a combined field representation to solve the Dirichlet problem then

$$u = \alpha \mathcal{S}_k[\sigma] + \beta \mathcal{D}_k[\sigma]\,, \tag{0.12}$$

then

$$\sigma(\alpha i k j_n(k) h_n(k) + \beta i k^2) = \tag{0.13}$$

Vector potentials

Let $\boldsymbol{Y}_{nm}(\theta,\phi), \boldsymbol{\Psi}_{nm}(\theta,\phi), \boldsymbol{\Phi}_{nm}(\theta,\phi)$ denote the vector spherical harmonics given by

$$\begin{aligned} \boldsymbol{Y}_{nm}(\theta,\phi) &= Y_{nm}(\theta,\phi) \begin{bmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{bmatrix}, \\ \boldsymbol{\Psi}_{nm}(\theta,\phi) &= -P'_{n|m|}(\cos\theta)\sin\theta e^{im\phi} \begin{bmatrix} \cos\theta\cos\phi \\ \cos\theta\sin\phi \\ -\sin\theta \end{bmatrix} + \frac{imY_{nm}(\theta,\phi)}{\sin(\theta)} \begin{bmatrix} -\sin\phi \\ \cos\phi \\ 0 \end{bmatrix}, \\ \boldsymbol{\Phi}_{nm}(\theta,\phi) &= \hat{\boldsymbol{r}} \times \boldsymbol{\Psi}_{nm}(\theta,\phi)\,. \end{aligned} \tag{0.14}$$

Then the following equalities hold for:

$$\mathcal{S}_k[\boldsymbol{\Psi}_{nm}](\boldsymbol{x}) = \begin{cases} in(n+1)\left(\frac{j_n(kr)}{kr}\left(h_n(k) + kh'_n(k)\right) + j'_n(kr)h_n(k)\right)\boldsymbol{Y}_{nm}(\theta,\phi) + \\ i\left(\left(j'_n(kr) + \frac{j_n(kr)}{kr}\right)\left(h_n(k) + kh'_n(k)\right) + n(n+1)h_n(k)\frac{j_n(kr)}{kr}\right)\boldsymbol{\Psi}_{nm}(\theta,\phi) & r < 1\,, \\ in(n+1)\left(\frac{h_n(kr)}{kr}\left(j_n(k) + kj'_n(k)\right) + h'_n(kr)j_n(k)\right)\boldsymbol{Y}_{nm}(\theta,\phi) + \\ i\left(\left(h'_n(kr) + \frac{h_n(kr)}{kr}\right)\left(j_n(k) + kj'_n(k)\right) + n(n+1)j_n(k)\frac{h_n(kr)}{kr}\right)\boldsymbol{\Psi}_{nm}(\theta,\phi) & r > 1\,. \end{cases} \tag{0.15}$$

$$\mathcal{S}_k[\boldsymbol{\Phi}_{nm}](\boldsymbol{x}) = \begin{cases} ikj_n(kr)h_n(k)\boldsymbol{\Phi}_{nm}(\theta,\phi) & r < 1\,, \\ ikj_n(k)h_n(kr)\boldsymbol{\Phi}_{nm}(\theta,\phi) & r > 1\,. \end{cases} \tag{0.16}$$

$$\nabla\mathcal{S}_k[Y_{nm}](\boldsymbol{x}) = \begin{cases} ik^2h_n(k)j'_n(kr)\boldsymbol{Y}_{nm}(\theta,\phi) + ikh_n(k)\frac{j_n(kr)}{r}\boldsymbol{\Psi}_{nm}(\theta,\phi) & r < 1\,, \\ ik^2j_n(k)h'_n(kr)\boldsymbol{Y}_{nm}(\theta,\phi) + ikj_n(k)\frac{h_n(kr)}{r}\boldsymbol{\Psi}_{nm}(\theta,\phi) & r > 1\,. \end{cases} \tag{0.17}$$

$$\nabla\times\mathcal{S}_k[\boldsymbol{\Psi}_{nm}](\boldsymbol{x}) = \begin{cases} -ikj_n(kr)\left(h_n(k) + kh'_n(k)\right)\boldsymbol{\Phi}_{nm}(\theta,\phi) & r < 1\,, \\ -ikh_n(kr)\left(j_n(k) + kj'_n(k)\right)\boldsymbol{\Phi}_{nm}(\theta,\phi) & r > 1\,. \end{cases} \tag{0.18}$$

$$\nabla\times\mathcal{S}_k[\mathbf{\Phi}_{nm}](\boldsymbol{x}) = \begin{cases} -in(n+1)kh_n(k)\frac{j_n(kr)}{r}\boldsymbol{Y}_{nm}(\theta,\phi) - ih_n(k)k^2\left(j_n'(kr)+\frac{j_n(kr)}{kr}\right)\mathbf{\Psi}_{nm}(\theta,\phi) & r<1\,, \\ -in(n+1)kj_n(k)\frac{h_n(kr)}{r}\boldsymbol{Y}_{nm}(\theta,\phi) - ij_n(k)k^2\left(h_n'(kr)+\frac{h_n(kr)}{kr}\right)\mathbf{\Psi}_{nm}(\theta,\phi) & r>1\,. \end{cases} \tag{0.19}$$

$$\nabla\times\nabla\times\mathcal{S}_k[\mathbf{\Psi}_{nm}](\boldsymbol{x}) = \begin{cases} \begin{aligned} & in(n+1)k\tfrac{j_n(kr)}{r}\left(h_n(k)+kh_n'(k)\right)\boldsymbol{Y}_{nm}(\theta,\phi) + \\ & i\left(k^2j_n'(kr)+k\tfrac{j_n(kr)}{r}\right)\left(h_n(k)+kh_n'(k)\right)\mathbf{\Psi}_{nm}(\theta,\phi) \end{aligned} & r<1\,, \\ \begin{aligned} & in(n+1)k\tfrac{h_n(kr)}{r}\left(j_n(k)+kj_n'(k)\right)\boldsymbol{Y}_{nm}(\theta,\phi) + \\ & i\left(k^2h_n'(kr)+k\tfrac{h_n(kr)}{r}\right)\left(j_n(k)+kj_n'(k)\right)\mathbf{\Psi}_{nm}(\theta,\phi) \end{aligned} & r>1\,. \end{cases} \tag{0.20}$$

$$\nabla\times\nabla\times\mathcal{S}_k[\mathbf{\Phi}_{nm}](\boldsymbol{x}) = \begin{cases} ik^3j_n(kr)h_n(k)\mathbf{\Phi}_{nm}(\theta,\phi) & r<1\,, \\ ik^3h_n(kr)j_n(k)\mathbf{\Phi}_{nm}(\theta,\phi) & r>1\,. \end{cases} \tag{0.21}$$

$$\begin{bmatrix} e^{ikr\cos(\theta)} \\ 0 \\ 0 \end{bmatrix} = \sum_{n=1}^{\infty} i^n\sqrt{\frac{2n+1}{n(n+1)}}\Bigg(j_n(kr)\,\mathrm{Im}\left(\Phi_{n1}(\theta,\phi)\right) + \\ i\left(\frac{n(n+1)}{kr}j_n(kr)\,\mathrm{Re}\left(\boldsymbol{Y}_{n1}(\theta,\phi)\right) + \left(j_n'(kr)+\frac{j_n(kr)}{kr}\right)\mathrm{Re}\left(\Psi_{n1}(\theta,\phi)\right)\right)\Bigg) \tag{0.22}$$