

The ultimate guide to layer potentials on the sphere

In this note, we consolidate analytic expressions for various layer potentials on the sphere. In the following, $P_{nm}(t)$ are the associated Legendre polynomials of degree n and order m , $Y_{nm}(\theta, \phi)$, are the spherical harmonics of degree n and order m given by

$$Y_{nm}(\theta, \phi) = \sqrt{2n+1} \sqrt{\frac{(n-|m|)!}{(n+|m|)!}} P_{n|m|}(\cos \theta) e^{im\phi}. \quad (0.1)$$

Let $j_n(z), h_n(z)$ denote the spherical Bessel and spherical Hankel functions of degree n . Let Ω^- be the interior of the unit sphere, $\partial\Omega$ denote it's boundary, and Ω^+ denote the exterior of the unit sphere. For any $\mathbf{x} \in \mathbb{R}^3$, let (r, θ, ϕ) , denote it's spherical coordinates.

Scalar potentials

Let $G_k(\mathbf{x}, \mathbf{y}) = e^{ik|\mathbf{x}-\mathbf{y}|}/(4\pi|\mathbf{x}-\mathbf{y}|)$ denote the Helmholtz Green's function in three dimensions. Let $\mathcal{S}_k[\sigma]$ denote the Helmholtz single layer potential given by

$$\mathcal{S}_k[\sigma](\mathbf{x}) = \int_{\partial\Omega} G_k(\mathbf{x}, \mathbf{y}) \sigma(\mathbf{y}) dS_{\mathbf{y}}, \quad (0.2)$$

and $\mathcal{D}_k[\sigma]$ denote the Helmholtz double layer potential given by

$$\mathcal{D}_k[\sigma](\mathbf{x}) = \int_{\partial\Omega} \nabla_{\mathbf{y}} G_k(\mathbf{x}, \mathbf{y}) \cdot \mathbf{n}(\mathbf{y}) \sigma(\mathbf{y}) dS_{\mathbf{y}}. \quad (0.3)$$

We will use $\mathcal{S}^{\pm}[\sigma]$, and $\mathcal{D}^{\pm}[\sigma]$ to also denote the restrictions to these operators to the boundary where the superscript '+' indicates an exterior limit, and the superscript '-' denotes the interior limit. Let $\mathcal{S}'^{\pm}_k[\sigma]$, and $\mathcal{D}'^{\pm}_k[\sigma]$ denote the limiting values of the Neumann data corresponding to the single and double layer potentials given by

$$\begin{aligned} \mathcal{S}'^{\pm}_k[\sigma](\mathbf{x}) &= \lim_{\substack{\mathbf{z} \rightarrow \mathbf{x} \\ \mathbf{x} \in \Omega^{\pm}}} \left(\mathbf{n}(\mathbf{x}) \cdot \nabla_{\mathbf{x}} \left(\int_{\partial\Omega} G_k(\mathbf{z}, \mathbf{y}) \sigma(\mathbf{y}) dS_{\mathbf{y}} \right) \right), \\ \mathcal{D}'_k[\sigma](\mathbf{x}) &= \text{f.p.} \left(\mathbf{n}(\mathbf{x}) \cdot \nabla_{\mathbf{x}} \left(\int_{\partial\Omega} (\nabla_{\mathbf{y}} G_k(\mathbf{x}, \mathbf{y}) \cdot \mathbf{n}(\mathbf{y})) \sigma(\mathbf{y}) dS_{\mathbf{y}} \right) \right), \end{aligned} \quad (0.4)$$

for $\mathbf{x} \in \partial\Omega$. Finally, let $\mathcal{S}''^{\pm}_k[\sigma]$ denote the limiting value of the second normal derivative of \mathcal{S}_k restricted to $\partial\Omega$ given by

$$\mathcal{S}''^{\pm}_k[\sigma](\mathbf{x}) = \lim_{\substack{\mathbf{z} \rightarrow \mathbf{x} \\ \mathbf{x} \in \Omega^{\pm}}} \left(\mathbf{n}(\mathbf{x}) \cdot \nabla_{\mathbf{x}} \nabla_{\mathbf{x}} \left(\int_{\partial\Omega} G_k(\mathbf{z}, \mathbf{y}) \sigma(\mathbf{y}) dS_{\mathbf{y}} \right) \cdot \mathbf{n}(\mathbf{x}) \right), \quad (0.5)$$

for $\mathbf{x} \in \partial\Omega$, and let $H(\mathbf{x})$ denote the mean curvature.

In a slight abuse of notation, let the principal value/finite part of these operators be denoted by $\mathcal{S}_k[\sigma], \mathcal{D}_k[\sigma], \mathcal{S}'_k[\sigma], \mathcal{D}'_k[\sigma]$, and $\mathcal{S}''_k[\sigma]$ respectively. Of these $\mathcal{D}_k, \mathcal{S}'_k$ are principal value integrals, \mathcal{S}_k is absolutely convergent, and $\mathcal{D}'_k, \mathcal{S}''_k$ are finite part integrals.

Limiting values on the boundary

The layer potentials satisfy the following jump relations. Suppose that $\mathbf{x} \in \Gamma$, then

$$\begin{aligned} \lim_{\substack{\mathbf{y} \rightarrow \mathbf{x} \\ \mathbf{y} \in \Omega^\pm}} \mathcal{S}_k[\sigma](\mathbf{y}) &= S_k[\sigma](\mathbf{x}), \\ \mathcal{D}_k^\pm[\sigma](\mathbf{x}) &= \pm \frac{\sigma(\mathbf{x}_0)}{2} + D_k[\sigma](\mathbf{x}), \\ \mathcal{S}_k^{\prime,\pm}[\sigma](\mathbf{x}) &= \mp \frac{\sigma(\mathbf{x}_0)}{2} + S_k'[\sigma](\mathbf{x}), \\ \mathcal{D}_k^{\prime,\pm}[\sigma](\mathbf{x}) &= D_k'[\sigma](\mathbf{x}), \\ \mathcal{S}_k^{\prime\prime,\pm}[\sigma](\mathbf{x}) &= S_k''[\sigma](\mathbf{x}). \end{aligned} \tag{0.6}$$

The layer potentials have the following limiting values on the boundary, see [?], for example.

$$\begin{aligned} S_k[Y_{nm}] &= ik j_n(k) h_n(k) Y_{nm}(\theta, \phi), \\ \mathcal{D}_k^+[Y_{nm}] &= ik^2 j_n'(k) h_n(k) Y_{nm}(\theta, \phi), \\ \mathcal{D}_k^-[Y_{nm}] &= ik^2 j_n(k) h_n'(k) Y_{nm}(\theta, \phi), \\ D_k[Y_{nm}] &= \frac{ik^2}{2} (j_n(k) h_n'(k) + j_n'(k) h_n(k)) Y_{nm}(\theta, \phi), \\ \mathcal{S}_k^{\prime,+}[Y_{nm}] &= ik^2 j_n(k) h_n'(k) Y_{nm}(\theta, \phi), \\ \mathcal{S}_k^{\prime,-}[Y_{nm}] &= ik^2 j_n'(k) h_n(k) Y_{nm}(\theta, \phi), \\ S_k'[Y_{nm}] &= \frac{ik^2}{2} (j_n(k) h_n'(k) + j_n'(k) h_n(k)) Y_{nm}(\theta, \phi), \\ \mathcal{D}_k^{\prime,\pm} &= D_k'[Y_{nm}] = ik^3 j_n'(k) h_n'(k) Y_{nm}(\theta, \phi). \end{aligned} \tag{0.7}$$

Potentials in the interior and exterior

The value of the potentials in the interior and exterior domain are given by

$$\begin{aligned} \mathcal{S}_k[Y_{nm}](\mathbf{x}) &= \begin{cases} ik h_n(k) j_n(kr) Y_{nm}(\theta, \phi) & r < 1, \\ ik j_n(k) h_n(kr) Y_{nm}(\theta, \phi) & r > 1, \end{cases} \\ \mathcal{D}_k[Y_{nm}](\mathbf{x}) &= \begin{cases} ik^2 h_n'(k) j_n(kr) Y_{nm}(\theta, \phi) & r < 1, \\ ik^2 j_n'(k) h_n(kr) Y_{nm}(\theta, \phi) & r > 1. \end{cases} \end{aligned} \tag{0.8}$$

Solution to various boundary value problems with incident planewaves

We begin by expressing the plane waves in terms of spherical bessel functions,

$$e^{ik\mathbf{x} \cdot \hat{\mathbf{z}}} = e^{ikr \cos \theta} = \sum_{n=0}^{\infty} i^n \sqrt{2n+1} j_n(kr) Y_{n0}(\theta, \phi), \tag{0.9}$$

Note that solutions in interior domains with planewave data are the planewaves themselves since they already satisfy the PDE. However, in exterior domains, they are not radiating/outgoing and hence the scattered field requires some computation. In the following, that the scattered field u is outgoing and satisfies the Sommerfeld radiation condition.

0.0.1 Dirichlet problem

Consider the exterior Dirichlet problem, then

$$\begin{aligned} (\Delta + k^2)u &= 0, \quad \mathbf{x} \in \Omega^+, \\ u &= -e^{ik\mathbf{x} \cdot \hat{\mathbf{z}}}, \quad \mathbf{x} \in \partial\Omega. \end{aligned} \quad (0.10)$$

Then

$$u = \sum_{n=0}^{\infty} i^n \sqrt{2n+1} j_n(k) h_n(kr) Y_{n0}(\theta, \phi). \quad (0.11)$$

Suppose now we use a combined field representation to solve the Dirichlet problem then

$$u = \alpha \mathcal{S}_k[\sigma] + \beta \mathcal{D}_k[\sigma], \quad (0.12)$$

then

$$\sigma(\alpha i k j_n(k) h_n(k) + \beta i k^2) = \quad (0.13)$$

Vector potentials

Let $\mathbf{Y}_{nm}(\theta, \phi)$, $\mathbf{\Psi}_{nm}(\theta, \phi)$, $\mathbf{\Phi}_{nm}(\theta, \phi)$ denote the vector spherical harmonics given by

$$\begin{aligned} \mathbf{Y}_{nm}(\theta, \phi) &= Y_{nm}(\theta, \phi) \begin{bmatrix} \sin \theta \cos \phi \\ \sin \theta \sin \phi \\ \cos \theta \end{bmatrix}, \\ \mathbf{\Psi}_{nm}(\theta, \phi) &= -P'_{n|m|}(\cos \theta) \sin \theta e^{im\phi} \begin{bmatrix} \cos \theta \cos \phi \\ \cos \theta \sin \phi \\ -\sin \theta \end{bmatrix} + \frac{im Y_{nm}(\theta, \phi)}{\sin(\theta)} \begin{bmatrix} -\sin \phi \\ \cos \phi \\ 0 \end{bmatrix}, \\ \mathbf{\Phi}_{nm}(\theta, \phi) &= \hat{\mathbf{r}} \times \mathbf{\Psi}_{nm}(\theta, \phi). \end{aligned} \quad (0.14)$$

Then the following equalities hold for:

$$\mathcal{S}_k[\mathbf{\Psi}_{nm}](\mathbf{x}) = \begin{cases} in(n+1) \left(\frac{j_n(kr)}{kr} (h_n(k) + kh'_n(k)) + j'_n(kr) h_n(k) \right) \mathbf{Y}_{nm}(\theta, \phi) + \\ i \left((j'_n(kr) + \frac{j_n(kr)}{kr}) (h_n(k) + kh'_n(k)) + n(n+1) h_n(k) \frac{j_n(kr)}{kr} \right) \mathbf{\Psi}_{nm}(\theta, \phi) & r < 1, \\ in(n+1) \left(\frac{h_n(kr)}{kr} (j_n(k) + kj'_n(k)) + h'_n(kr) j_n(k) \right) \mathbf{Y}_{nm}(\theta, \phi) + \\ i \left((h'_n(kr) + \frac{h_n(kr)}{kr}) (j_n(k) + kj'_n(k)) + n(n+1) j_n(k) \frac{h_n(kr)}{kr} \right) \mathbf{\Psi}_{nm}(\theta, \phi) & r > 1. \end{cases} \quad (0.15)$$

$$\mathcal{S}_k[\mathbf{\Phi}_{nm}](\mathbf{x}) = \begin{cases} ik j_n(kr) h_n(k) \mathbf{\Phi}_{nm}(\theta, \phi) & r < 1, \\ ik j_n(k) h_n(kr) \mathbf{\Phi}_{nm}(\theta, \phi) & r > 1. \end{cases} \quad (0.16)$$

$$\nabla \mathcal{S}_k[Y_{nm}](\mathbf{x}) = \begin{cases} ik^2 h_n(k) j'_n(kr) \mathbf{Y}_{nm}(\theta, \phi) + ik h_n(k) \frac{j_n(kr)}{r} \mathbf{\Psi}_{nm}(\theta, \phi) & r < 1, \\ ik^2 j_n(k) h'_n(kr) \mathbf{Y}_{nm}(\theta, \phi) + ik j_n(k) \frac{h_n(kr)}{r} \mathbf{\Psi}_{nm}(\theta, \phi) & r > 1. \end{cases} \quad (0.17)$$

$$\nabla \times \mathcal{S}_k[\mathbf{\Psi}_{nm}](\mathbf{x}) = \begin{cases} -ik j_n(kr) (h_n(k) + kh'_n(k)) \mathbf{\Phi}_{nm}(\theta, \phi) & r < 1, \\ -ik h_n(kr) (j_n(k) + kj'_n(k)) \mathbf{\Phi}_{nm}(\theta, \phi) & r > 1. \end{cases} \quad (0.18)$$

$$\nabla \times \mathcal{S}_k[\Phi_{nm}](\mathbf{x}) = \begin{cases} -in(n+1)kh_n(k)\frac{j_n(kr)}{r}\mathbf{Y}_{nm}(\theta, \phi) - ih_n(k)k^2\left(j'_n(kr) + \frac{j_n(kr)}{kr}\right)\mathbf{\Psi}_{nm}(\theta, \phi) & r < 1, \\ -in(n+1)kj_n(k)\frac{h_n(kr)}{r}\mathbf{Y}_{nm}(\theta, \phi) - ij_n(k)k^2\left(h'_n(kr) + \frac{h_n(kr)}{kr}\right)\mathbf{\Psi}_{nm}(\theta, \phi) & r > 1. \end{cases} \quad (0.19)$$

$$\nabla \times \nabla \times \mathcal{S}_k[\Psi_{nm}](\mathbf{x}) = \begin{cases} in(n+1)k\frac{j_n(kr)}{r}(h_n(k) + kh'_n(k))\mathbf{Y}_{nm}(\theta, \phi) + \\ i\left(k^2j'_n(kr) + k\frac{j_n(kr)}{r}\right)(h_n(k) + kh'_n(k))\mathbf{\Psi}_{nm}(\theta, \phi) & r < 1, \\ in(n+1)k\frac{h_n(kr)}{r}(j_n(k) + kj'_n(k))\mathbf{Y}_{nm}(\theta, \phi) + \\ i\left(k^2h'_n(kr) + k\frac{h_n(kr)}{r}\right)(j_n(k) + kj'_n(k))\mathbf{\Psi}_{nm}(\theta, \phi) & r > 1. \end{cases} \quad (0.20)$$

$$\nabla \times \nabla \times \mathcal{S}_k[\Phi_{nm}](\mathbf{x}) = \begin{cases} ik^3j_n(kr)h_n(k)\mathbf{\Phi}_{nm}(\theta, \phi) & r < 1, \\ ik^3h_n(kr)j_n(k)\mathbf{\Phi}_{nm}(\theta, \phi) & r > 1. \end{cases} \quad (0.21)$$

$$\begin{bmatrix} e^{ikr \cos(\theta)} \\ 0 \\ 0 \end{bmatrix} = \sum_{n=1}^{\infty} i^n \sqrt{\frac{2n+1}{n(n+1)}} \left(j_n(kr) \operatorname{Im}(\Phi_{n1}(\theta, \phi)) + \right. \\ \left. i \left(\frac{n(n+1)}{kr} j_n(kr) \operatorname{Re}(\mathbf{Y}_{n1}(\theta, \phi)) + \left(j'_n(kr) + \frac{j_n(kr)}{kr} \right) \operatorname{Re}(\Psi_{n1}(\theta, \phi)) \right) \right) \quad (0.22)$$