CCM Technical Report
March 4, 2021

The ultimate guide to Maxwell layer potentials on the sphere

In this note, we consolidate analytic expressions for various maxwell layer potentials on the sphere. In the following, $P_{nm}(t)$ are the associated legendre polynomials of degree n and order m, $Y_{nm}(\theta,\phi)$, are the spherical harmonics of degree n and order m given by

$$Y_{nm}(\theta,\phi) = \sqrt{2n+1}\frac{(n-|m|)!}{(n+|m|)!}P_{n|m|}(\cos\theta)e^{im\phi}\,. \tag{0.1}$$

Let $\boldsymbol{Y}_{nm}(\theta,\phi), \boldsymbol{\Psi}_{nm}(\theta,\phi), \boldsymbol{\Phi}_{nm}(\theta,\phi)$ denote the vector spherical harmonics given by

$$\begin{aligned}
\boldsymbol{Y}_{nm}(\theta,\phi) &= Y_{nm}(\theta,\phi)\begin{bmatrix}\sin\theta\cos\phi\\ \sin\theta\sin\phi\\ \cos\theta\end{bmatrix}\,,\\
\boldsymbol{\Psi}_{nm}(\theta,\phi) &= -P'_{n|m|}(\cos\theta)\sin\theta\begin{bmatrix}\cos\theta\cos\phi\\ \cos\theta\sin\phi\\ -\sin\theta\end{bmatrix} + \frac{imY_{nm}(\theta,\phi)}{\sin(\theta)}\begin{bmatrix}-\sin\phi\\ \cos\phi\\ 0\end{bmatrix}\,,\\
\boldsymbol{\Psi}_{nm}(\theta,\phi) &= \hat{\boldsymbol{r}}\times\boldsymbol{\Psi}_{nm}(\theta,\phi)\,.
\end{aligned} \tag{0.2}$$

Let $j_n(z), h_n(z)$ denote the spherical Bessel and spherical Hankel functions of degree n. Let Ω be the interior of the unit sphere $\partial\Omega$ denote it's boundary. Finally, let $\mathcal{S}_k[\sigma]$ denote the Helmholtz single layer potential given by

$$\mathcal{S}_k[\sigma](\boldsymbol{x}) = \int_{\partial\Omega}\frac{e^{ik|\boldsymbol{x}-\boldsymbol{y}|}}{|\boldsymbol{x}-\boldsymbol{y}|}\sigma(\boldsymbol{y})dS_{\boldsymbol{y}}\,. \tag{0.3}$$

For any $\boldsymbol{x}\in\Omega$, let (r,θ,ϕ), denote it's spherical coordinates. Then the following equalities hold:

$$\begin{aligned}
\mathcal{S}_k[\boldsymbol{\Psi}_{nm}](\boldsymbol{x}) &= in(n+1)\left(\frac{j_n(kr)}{kr}\left(h_n(k)+kh_n'(k)\right)+j_n'(k)h_n(k)\right)\boldsymbol{Y}_{nm}(\theta,\phi)+\\
&\quad i\left(\left(j_n'(kr)+\frac{j_n(kr)}{kr}\right)\left(h_n(k)+kh_n'(k)\right)+n(n+1)h_n(k)\frac{j_n(kr)}{kr}\right)\boldsymbol{\Psi}_{nm}(\theta,\phi)\,,\\
\mathcal{S}_k[\boldsymbol{\Phi}_{nm}](\boldsymbol{x}) &= ikj_n(kr)h_n(k)\boldsymbol{\Phi}_{nm}(\theta,\phi)\,,\\
\nabla\mathcal{S}_k[Y_{nm}](\boldsymbol{x}) &= ik^2h_n(k)j_n'(kr)\boldsymbol{Y}_{nm}(\theta,\phi)+ikh_n(k)\frac{j_n(kr)}{r}\boldsymbol{\Psi}_{nm}(\theta,\phi)\,,\\
\nabla\times\mathcal{S}_k[\boldsymbol{\Psi}_{nm}](\boldsymbol{x}) &= -ikj_n(kr)\left(h_n(k)+kh_n'(k)\right)\boldsymbol{\Phi}_{nm}(\theta,\phi)\,,\\
\nabla\times\mathcal{S}_k[\boldsymbol{\Phi}_{nm}](\boldsymbol{x}) &= -in(n+1)kh_n(k)\frac{j_n(kr)}{r}\boldsymbol{Y}_{nm}(\theta,\phi)-ih_n(k)k^2\left(j_n'(kr)+\frac{j_n(kr)}{kr}\right)\boldsymbol{\Psi}_{nm}(\theta,\phi)\,,\\
\nabla\times\nabla\times\mathcal{S}_k[\boldsymbol{\Psi}_{nm}](\boldsymbol{x}) &= in(n+1)k\frac{j_n(kr)}{r}\left(h_n(k)+kh_n'(k)\right)\boldsymbol{Y}_{nm}(\theta,\phi)+\\
&\quad i\left(k^2j_n'(kr)+k\frac{j_n(kr)}{r}\right)\left(h_n(k)+kh_n'(k)\right)\boldsymbol{\Psi}_{nm}(\theta,\phi)\,,\\
\nabla\times\nabla\times\mathcal{S}_k[\boldsymbol{\Phi}_{nm}](\boldsymbol{x}) &= ik^3j_n(kr)h_n(k)\boldsymbol{\Phi}_{nm}(\theta,\phi)\,.
\end{aligned} \tag{0.4}$$