Transforming $P(r)$ to $P(\eta)$

The definition of the log-distance ratio of a galaxy is

$$\eta = \log\left(\frac{d}{r}\right), \tag{1}$$

where d represents its observed comoving distance (calculated from the observed redshift) and r represents its true distance. Equivalently, we can write this equation as

$$r = d \cdot 10^{-\eta}. \tag{2}$$

We can treat d as a constant for every galaxy.

Due to Malmquist bias, the probability distribution of the true distance must follow

$$P(r) \propto r^2. \tag{3}$$

By transforming the variables, the probability distribution of the log-distance is given by

$$P(\eta) = P(r)\left|\frac{dr}{d\eta}\right|. \tag{4}$$

From (2), we get

$$\left|\frac{dr}{d\eta}\right| = d \cdot 10^{-\eta} \ln 10\,. \tag{5}$$

By combining (3) and (5), we get

$$P(\eta) = kr^2 \cdot d \cdot 10^{-\eta} \ln 10 = k' \cdot 10^{-3\eta}.$$

Therefore, the prior of η that allows for homogeneous Malmquist bias is

$$P(\eta) \propto 10^{-3\eta}.$$

We can also determine the implication of assuming a flat prior in η to the assumption of the distribution in r. Since $P(\eta) = constant,$ we have

$$P(r) = P(\eta)\left|\frac{d\eta}{dr}\right| = k \cdot \left(\frac{1}{\ln 10}\frac{1}{r}\right).$$

Therefore, assuming a flat prior distribution in η means

$$P(r) \propto \frac{1}{r}.$$