

8

Tree-Based Methods

In this chapter, we describe *tree-based* methods for regression and classification. These involve *stratifying or segmenting the predictor space into a number of simple regions*. In order to make a prediction for a given observation, we typically use the mean or the mode of the training observations in the region to which it belongs. Since *the set of splitting rules used to segment the predictor space can be summarized in a tree*, these types of approaches are known as *decision tree* methods.

decision tree

Tree-based methods are simple and useful for interpretation. However, they typically are not competitive with the best supervised learning approaches, such as those seen in Chapters 6 and 7, in terms of prediction accuracy. Hence in this chapter we also introduce *bagging, random forests*, and *boosting*. Each of these approaches involves producing multiple trees which are then combined to yield a single consensus prediction. We will see that combining a large number of trees can often result in dramatic improvements in prediction accuracy, at the expense of some loss in interpretation.

8.1 The Basics of Decision Trees

Decision trees can be applied to both regression and classification problems. We first consider regression problems, and then move on to classification.

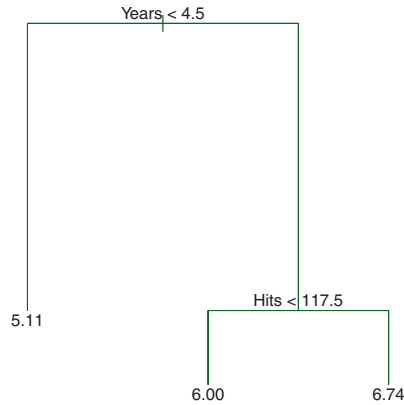


FIGURE 8.1. For the **Hitters** data, a regression tree for predicting the log salary of a baseball player, based on the number of years that he has played in the major leagues and the number of hits that he made in the previous year. At a given internal node, the label (of the form $X_j < t_k$) indicates the left-hand branch emanating from that split, and the right-hand branch corresponds to $X_j \geq t_k$. For instance, the split at the top of the tree results in two large branches. The left-hand branch corresponds to **Years**<4.5, and the right-hand branch corresponds to **Years**>=4.5. The tree has two internal nodes and three terminal nodes, or leaves. The number in each leaf is the mean of the response for the observations that fall there.

8.1.1 Regression Trees

In order to motivate *regression trees*, we begin with a simple example.

regression
tree

Predicting Baseball Players' Salaries Using Regression Trees

We use the **Hitters** data set to predict a baseball player's **Salary** based on **Years** (the number of years that he has played in the major leagues) and **Hits** (the number of hits that he made in the previous year). We first remove observations that are missing **Salary** values, and log-transform **Salary** so that its distribution has more of a typical bell-shape. (Recall that **Salary** is measured in thousands of dollars.)

Figure 8.1 shows a regression tree fit to this data. It consists of a series of splitting rules, starting at the top of the tree. The top split assigns observations having **Years**<4.5 to the left branch.¹ The predicted salary

¹Both **Years** and **Hits** are integers in these data; the **tree()** function in **R** labels the splits at the midpoint between two adjacent values.

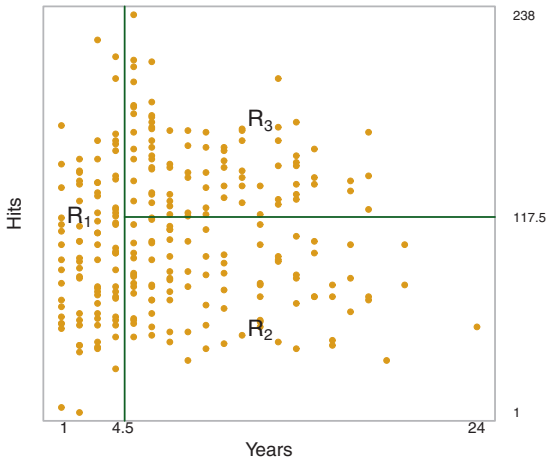


FIGURE 8.2. The three-region partition for the **Hitters** data set from the regression tree illustrated in Figure 8.1.

for these players is given by the mean response value for the players in the data set with $\text{Years} < 4.5$. For such players, the mean log salary is 5.107, and so we make a prediction of $e^{5.107}$ thousands of dollars, i.e. \$165,174, for these players. Players with $\text{Years} \geq 4.5$ are assigned to the right branch, and then that group is further subdivided by **Hits**. Overall, the tree stratifies or segments the players into three regions of predictor space: players who have played for four or fewer years, players who have played for five or more years and who made fewer than 118 hits last year, and players who have played for five or more years and who made at least 118 hits last year. These three regions can be written as $R_1 = \{X \mid \text{Years} < 4.5\}$, $R_2 = \{X \mid \text{Years} \geq 4.5, \text{Hits} < 117.5\}$, and $R_3 = \{X \mid \text{Years} \geq 4.5, \text{Hits} \geq 117.5\}$. Figure 8.2 illustrates the regions as a function of **Years** and **Hits**. The predicted salaries for these three groups are $\$1,000 \times e^{5.107} = \$165,174$, $\$1,000 \times e^{5.999} = \$402,834$, and $\$1,000 \times e^{6.740} = \$845,346$ respectively.

In keeping with the *tree* analogy, the regions R_1 , R_2 , and R_3 are known as *terminal nodes* or *leaves of the tree*. As is the case for Figure 8.1, decision trees are typically drawn *upside down*, in the sense that the leaves are at the bottom of the tree. The points along the tree where the predictor space is split are referred to as *internal nodes*. In Figure 8.1, the two internal nodes are indicated by the text $\text{Years} < 4.5$ and $\text{Hits} < 117.5$. We refer to the segments of the trees that connect the nodes as *branches*.

We might interpret the regression tree displayed in Figure 8.1 as follows: **Years** is the most important factor in determining **Salary**, and players with less experience earn lower salaries than more experienced players. Given that a player is less experienced, the number of hits that he made in the previous year seems to play little role in his salary. But among players who

terminal
node
leaf
internal node
branch

have been in the major leagues for five or more years, the number of hits made in the previous year does affect salary, and players who made more hits last year tend to have higher salaries. The regression tree shown in Figure 8.1 is likely an over-simplification of the true relationship between **Hits**, **Years**, and **Salary**. However, it has advantages over other types of regression models (such as those seen in Chapters 3 and 6): it is easier to interpret, and has a nice graphical representation.

Prediction via Stratification of the Feature Space

We now discuss the process of building a regression tree. Roughly speaking, there are two steps.

1. We **divide the predictor space**—that is, the set of possible values for X_1, X_2, \dots, X_p —**into J distinct and non-overlapping regions**, R_1, R_2, \dots, R_J .
2. **For every observation that falls into the region R_j** , we make the same prediction, which is simply the **mean of the response values for the training observations in R_j** .

For instance, suppose that in Step 1 we obtain two regions, R_1 and R_2 , and that the response mean of the training observations in the first region is 10, while the response mean of the training observations in the second region is 20. Then for a given observation $X = x$, if $x \in R_1$ we will predict a value of 10, and if $x \in R_2$ we will predict a value of 20.

We now elaborate on Step 1 above. How do we construct the regions R_1, \dots, R_J ? In theory, the regions could have any shape. However, we choose to divide the predictor space into high-dimensional rectangles, or *boxes*, for simplicity and for ease of interpretation of the resulting predictive model. **The goal is to find boxes R_1, \dots, R_J that minimize the RSS**, given by

$$\sum_{j=1}^J \sum_{i \in R_j} (y_i - \hat{y}_{R_j})^2, \quad (8.1)$$

where \hat{y}_{R_j} is the mean response for the training observations within the j th box. Unfortunately, **it is computationally infeasible to consider every possible partition of the feature space into J boxes**. For this reason, we take a **top-down, greedy approach** that is known as **recursive binary splitting**. The approach is **top-down** because it begins at the **top of the tree** (at which point all observations belong to a single region) and then successively splits the predictor space; each split is indicated via two new branches further down on the tree. **It is greedy because at each step of the tree-building process, the best split is made at that particular step**, rather than looking ahead and picking a split that will lead to a better tree in some future step.

recursive
binary
splitting

In order to perform recursive binary splitting, we first select the predictor X_j and the cutpoint s such that splitting the predictor space into the regions $\{X|X_j < s\}$ and $\{X|X_j \geq s\}$ leads to the greatest possible reduction in RSS. (The notation $\{X|X_j < s\}$ ~~means the region of predictor space in which X_j takes on a value less than s .~~) That is, we consider all predictors X_1, \dots, X_p , and all possible values of the cutpoint s for each of the predictors, and then choose the predictor and cutpoint such that the resulting tree has the lowest RSS. In greater detail, for any j and s , we define the pair of half-planes

$$R_1(j, s) = \{X|X_j < s\} \text{ and } R_2(j, s) = \{X|X_j \geq s\}, \quad (8.2)$$

and we seek the value of j and s that minimize the equation

$$\sum_{i: x_i \in R_1(j, s)} (y_i - \hat{y}_{R_1})^2 + \sum_{i: x_i \in R_2(j, s)} (y_i - \hat{y}_{R_2})^2, \quad (8.3)$$

where \hat{y}_{R_1} is the mean response for the training observations in $R_1(j, s)$, and \hat{y}_{R_2} is the mean response for the training observations in $R_2(j, s)$. Finding the values of j and s that minimize (8.3) can be done quite quickly, especially when the number of features p is not too large.

Next, we repeat the process, looking for the best predictor and best cutpoint in order to split the data further so as to minimize the RSS within each of the resulting regions. However, this time, instead of splitting the entire predictor space, we split one of the two previously identified regions. We now have three regions. Again, we look to split one of these three regions further, so as to minimize the RSS. The process continues until a stopping criterion is reached; for instance, we may continue until no region contains more than five observations.

Once the regions R_1, \dots, R_J have been created, we predict the response for a given test observation using the mean of the training observations in the region to which that test observation belongs.

A five-region example of this approach is shown in Figure 8.3.

Tree Pruning

The process described above may produce good predictions on the training set, but is likely to overfit the data, leading to poor test set performance. This is because the resulting tree might be too complex. A smaller tree with fewer splits (that is, fewer regions R_1, \dots, R_J) might lead to lower variance and better interpretation at the cost of a little bias. One possible alternative to the process described above is to build the tree only so long as the decrease in the RSS due to each split exceeds some (high) threshold. This strategy will result in smaller trees, but is too short-sighted since a seemingly worthless split early on in the tree might be followed by a very good split—that is, a split that leads to a large reduction in RSS later on.

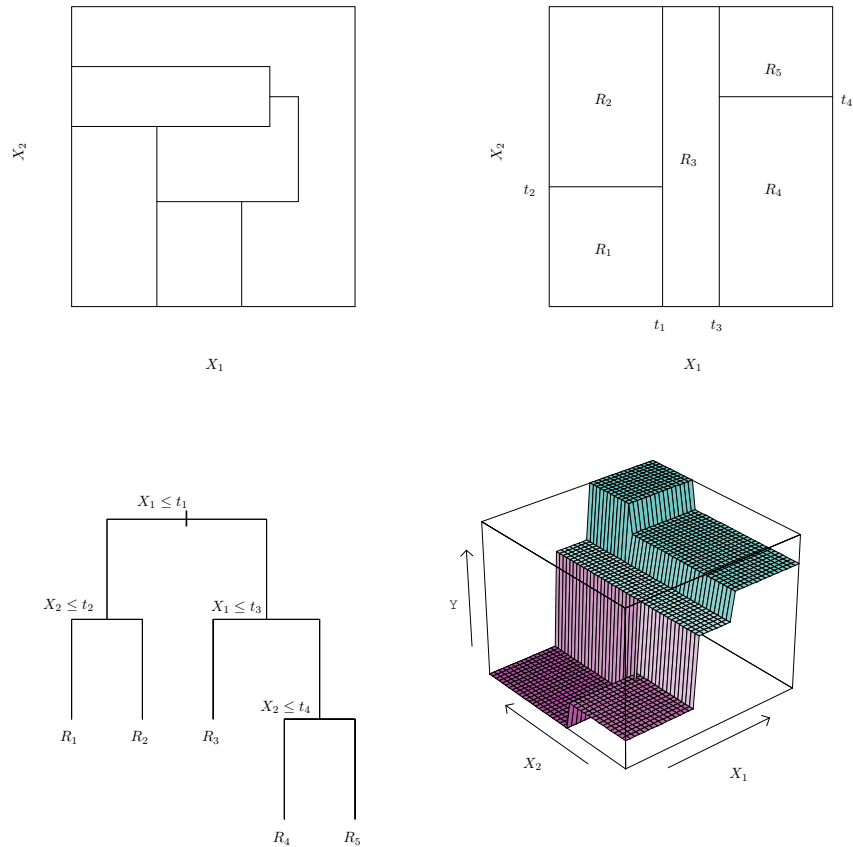


FIGURE 8.3. Top Left: A partition of two-dimensional feature space that could not result from recursive binary splitting. Top Right: The output of recursive binary splitting on a two-dimensional example. Bottom Left: A tree corresponding to the partition in the top right panel. Bottom Right: A perspective plot of the prediction surface corresponding to that tree.

Therefore, a better strategy is to grow a very large tree T_0 , and then prune it back in order to obtain a subtree. How do we determine the best way to prune the tree? Intuitively, our goal is to select a subtree that leads to the lowest test error rate. Given a subtree, we can estimate its test error using cross-validation or the validation set approach. However, estimating the cross-validation error for every possible subtree would be too cumbersome, since there is an extremely large number of possible subtrees. Instead, we need a way to select a small set of subtrees for consideration.

prune
subtree

Cost complexity pruning—also known as *weakest link pruning*—gives us a way to do just this. Rather than considering every possible subtree, we consider a sequence of trees indexed by a nonnegative tuning parameter α .

cost
complexity
pruning
weakest link
pruning

Algorithm 8.1 *Building a Regression Tree*

1. Use **recursive binary splitting** to **grow a large tree on the training data, stopping only when each terminal node has fewer than some minimum number of observations.**
2. **Apply cost complexity pruning to the large tree in order to obtain a sequence of best subtrees, as a function of α .**
3. Use **K-fold cross-validation** to choose $\hat{\alpha}$. That is, divide the training observations into K folds. For each $k = 1, \dots, K$:
 - (a) Repeat Steps 1 and 2 on all but the k th fold of the training data.
 - (b) Evaluate the mean squared prediction error on the data in the left-out k th fold, as a function of α .

Average the results for each value of α , and pick α to minimize the average error.
4. **Return the subtree from Step 2 that corresponds to the chosen value of $\hat{\alpha}$.**

For each value of α there corresponds **a subtree $T \subset T_0$** such that

$$\sum_{m=1}^{|T|} \sum_{i: x_i \in R_m} (y_i - \hat{y}_{R_m})^2 + \alpha |T| \quad (8.4)$$

is **as small as possible**. Here $|T|$ indicates **the number of terminal nodes of the tree T** , R_m is the rectangle (i.e. the subset of predictor space) **corresponding to the m th terminal node**, and \hat{y}_{R_m} is the predicted response **associated with R_m** —that is, the mean of the training observations in R_m . **The tuning parameter α controls a trade-off between the subtree's complexity and its fit to the training data.** When $\alpha = 0$, then the subtree T will simply equal T_0 , because then (8.4) just measures the training error. However, as α increases, there is a price to pay for having a tree with many terminal nodes, and so the quantity (8.4) will tend to be minimized for a smaller subtree. Equation 8.4 is reminiscent of the lasso (6.7) from Chapter 6, in which a similar formulation was used in order to control the complexity of a linear model.

It turns out that **as we increase α from zero in (8.4), branches get pruned from the tree in a nested and predictable fashion**, so obtaining the whole sequence of subtrees as a function of α is easy. **We can select a value of $\hat{\alpha}$ using a validation set or using cross-validation.** We then return to the full data set and obtain the subtree corresponding to α . This process is summarized in Algorithm 8.1.

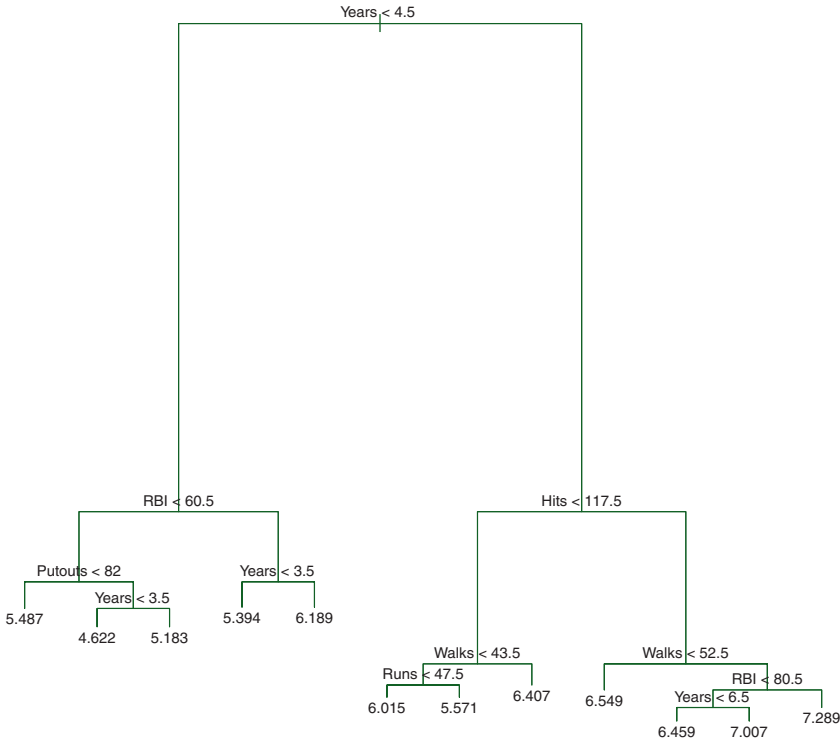


FIGURE 8.4. Regression tree analysis for the **Hitters** data. The unpruned tree that results from top-down greedy splitting on the training data is shown.

Figures 8.4 and 8.5 display the results of fitting and pruning a regression tree on the **Hitters** data, using nine of the features. First, we randomly divided the data set in half, yielding 132 observations in the training set and 131 observations in the test set. We then built a large regression tree on the training data and varied α in (8.4) in order to create subtrees with different numbers of terminal nodes. Finally, we performed six-fold cross-validation in order to estimate the cross-validated MSE of the trees as a function of α . (We chose to perform six-fold cross-validation because 132 is an exact multiple of six.) The unpruned regression tree is shown in Figure 8.4. The green curve in Figure 8.5 shows the CV error as a function of the number of leaves,² while the orange curve indicates the test error. Also shown are standard error bars around the estimated errors. For reference, the training error curve is shown in black. The CV error is a reasonable approximation of the test error: the CV error takes on its

²Although CV error is computed as a function of α , it is convenient to display the result as a function of $|T|$, the number of leaves; this is based on the relationship between α and $|T|$ in the original tree grown to all the training data.

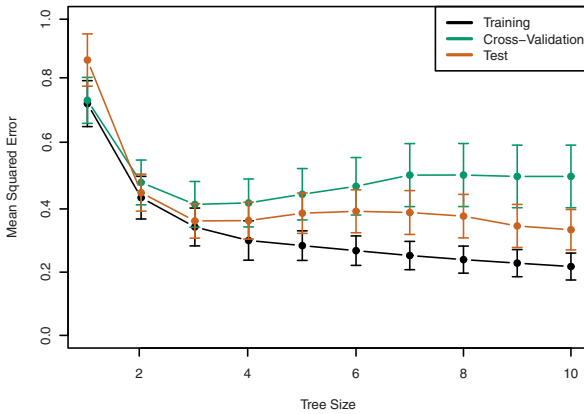


FIGURE 8.5. Regression tree analysis for the **Hitters** data. The training, cross-validation, and test MSE are shown as a function of the number of terminal nodes in the pruned tree. Standard error bands are displayed. The minimum cross-validation error occurs at a tree size of three.

minimum for a three-node tree, while the test error also dips down at the three-node tree (though it takes on its lowest value at the ten-node tree). The pruned tree containing three terminal nodes is shown in Figure 8.1.

8.1.2 Classification Trees

A *classification tree* is very similar to a regression tree, except that it is used to predict a qualitative response rather than a quantitative one. Recall that for a regression tree, the predicted response for an observation is given by the mean response of the training observations that belong to the same terminal node. In contrast, for a classification tree, **we predict that each observation belongs to the most commonly occurring class of training observations in the region to which it belongs**. In interpreting the results of a classification tree, we are often interested not only in the class prediction corresponding to a particular terminal node region, but also in the *class proportions* among the training observations that fall into that region.

classification
tree

The task of growing a classification tree is quite similar to the task of growing a regression tree. Just as in the regression setting, we use recursive binary splitting to grow a classification tree. However, in the classification setting, RSS cannot be used as a criterion for making the binary splits. A natural alternative to RSS is the *classification error rate*. Since **we plan to assign an observation in a given region to the most commonly occurring class of training observations in that region**, the classification error rate is simply the fraction of the training observations in that region that do not belong to the most common class:

classification
error rate

$$E = 1 - \max_k(\hat{p}_{mk}). \quad (8.5)$$

Here \hat{p}_{mk} represents the proportion of training observations in the m th region that are from the k th class. However, it turns out that classification error is not sufficiently sensitive for tree-growing, and in practice two other measures are preferable.

The *Gini index* is defined by

$$G = \sum_{k=1}^K \hat{p}_{mk}(1 - \hat{p}_{mk}), \quad (8.6)$$

Gini index

a measure of total variance across the K classes. It is not hard to see that the Gini index takes on a small value if all of the \hat{p}_{mk} 's are close to zero or one. For this reason the Gini index is referred to as a measure of node purity—a small value indicates that a node contains predominantly observations from a single class.

An alternative to the Gini index is *entropy*, given by

$$D = - \sum_{k=1}^K \hat{p}_{mk} \log \hat{p}_{mk}. \quad (8.7)$$

entropy

Since $0 \leq \hat{p}_{mk} \leq 1$, it follows that $0 \leq -\hat{p}_{mk} \log \hat{p}_{mk}$. One can show that the entropy will take on a value near zero if the \hat{p}_{mk} 's are all near zero or near one. Therefore, like the Gini index, the entropy will take on a small value if the m th node is pure. In fact, it turns out that the Gini index and the entropy are quite similar numerically.

When building a classification tree, either the Gini index or the entropy are typically used to evaluate the quality of a particular split, since these two approaches are more sensitive to node purity than is the classification error rate. Any of these three approaches might be used when pruning the tree, but the classification error rate is preferable if prediction accuracy of the final pruned tree is the goal.

Figure 8.6 shows an example on the *Heart* data set. These data contain a binary outcome *HD* for 303 patients who presented with chest pain. An outcome value of *Yes* indicates the presence of heart disease based on an angiographic test, while *No* means no heart disease. There are 13 predictors including *Age*, *Sex*, *Chol* (a cholesterol measurement), and other heart and lung function measurements. Cross-validation results in a tree with six terminal nodes.

In our discussion thus far, we have assumed that the predictor variables take on continuous values. However, decision trees can be constructed even in the presence of qualitative predictor variables. For instance, in the *Heart* data, some of the predictors, such as *Sex*, *Thal* (Thallium stress test), and *ChestPain*, are qualitative. Therefore, a split on one of these variables amounts to assigning some of the qualitative values to one branch and

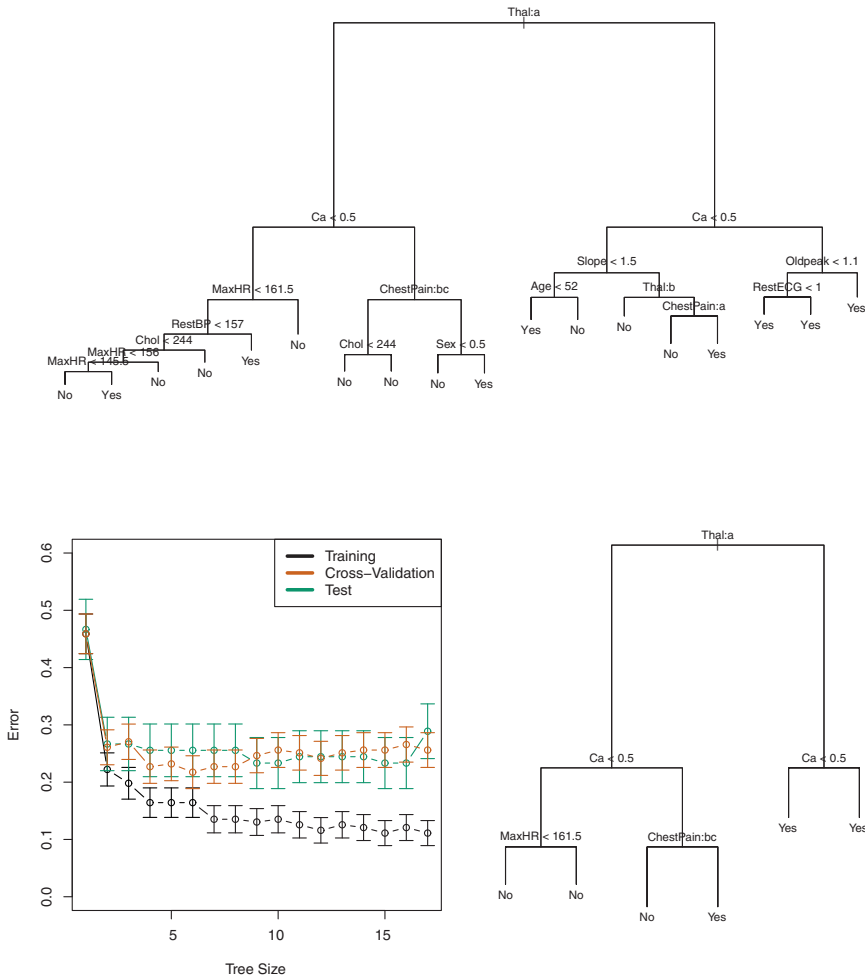


FIGURE 8.6. Heart data. Top: The unpruned tree. Bottom Left: Cross-validation error, training, and test error, for different sizes of the pruned tree. Bottom Right: The pruned tree corresponding to the minimal cross-validation error.

assigning the remaining to the other branch. In Figure 8.6, some of the internal nodes correspond to splitting qualitative variables. For instance, the top internal node corresponds to splitting **Thal**. The text **Thal:a** indicates that the left-hand branch coming out of that node consists of observations with the first value of the **Thal** variable (normal), and the right-hand node consists of the remaining observations (fixed or reversible defects). The text **ChestPain:bc** two splits down the tree on the left indicates that the left-hand branch coming out of that node consists of observations with the second and third values of the **ChestPain** variable, where the possible values are typical angina, atypical angina, non-anginal pain, and asymptomatic.

Figure 8.6 has a surprising characteristic: some of the splits yield two terminal nodes that have the *same predicted value*. For instance, consider the split `RestECG<1` near the bottom right of the unpruned tree. Regardless of the value of `RestECG`, a response value of `Yes` is predicted for those observations. **Why, then, is the split performed at all? The split is performed because it leads to increased node purity.** That is, all 9 of the observations corresponding to the right-hand leaf have a response value of `Yes`, whereas 7/11 of those corresponding to the left-hand leaf have a response value of `Yes`. Why is node purity important? Suppose that we have a test observation that belongs to the region given by that right-hand leaf. Then we can be pretty certain that its response value is `Yes`. In contrast, if a test observation belongs to the region given by the left-hand leaf, then its response value is probably `Yes`, but we are much less certain. Even though the split `RestECG<1` does not reduce the classification error, it improves the Gini index and the entropy, which are more sensitive to node purity.

8.1.3 Trees Versus Linear Models

Regression and classification trees have a very different flavor from the more classical approaches for regression and classification presented in Chapters 3 and 4. In particular, linear regression assumes a model of the form

$$f(X) = \beta_0 + \sum_{j=1}^p X_j \beta_j, \quad (8.8)$$

whereas regression trees assume a model of the form

$$f(X) = \sum_{m=1}^M c_m \cdot 1_{(X \in R_m)} \quad (8.9)$$

where R_1, \dots, R_M represent a partition of feature space, as in Figure 8.3.

Which model is better? It depends on the problem at hand. If the relationship between the features and the response is well approximated by a linear model as in (8.8), then an approach such as linear regression will likely work well, and will outperform a method such as a regression tree that does not exploit this linear structure. If instead there is a highly non-linear and complex relationship between the features and the response as indicated by model (8.9), then decision trees may outperform classical approaches. An illustrative example is displayed in Figure 8.7. The relative performances of tree-based and classical approaches can be assessed by estimating the test error, using either cross-validation or the validation set approach (Chapter 5).

Of course, other considerations beyond simply test error may come into play in selecting a statistical learning method; for instance, in certain settings, prediction using a tree may be preferred for the sake of interpretability and visualization.

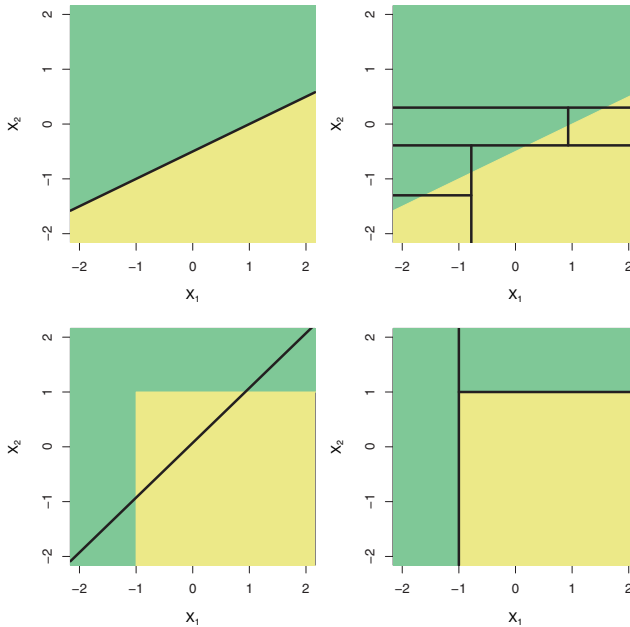


FIGURE 8.7. Top Row: A two-dimensional classification example in which the true decision boundary is linear, and is indicated by the shaded regions. A classical approach that assumes a linear boundary (left) will outperform a decision tree that performs splits parallel to the axes (right). Bottom Row: Here the true decision boundary is non-linear. Here a linear model is unable to capture the true decision boundary (left), whereas a decision tree is successful (right).

8.1.4 Advantages and Disadvantages of Trees

Decision trees for regression and classification have a number of advantages over the more classical approaches seen in Chapters 3 and 4:

- ▲ Trees are very easy to explain to people. In fact, they are even easier to explain than linear regression!
- ▲ Some people believe that decision trees more closely mirror human decision-making than do the regression and classification approaches seen in previous chapters.
- ▲ Trees can be displayed graphically, and are easily interpreted even by a non-expert (especially if they are small).
- ▲ Trees can easily handle qualitative predictors without the need to create dummy variables.

- ▼ Unfortunately, trees generally do not have the same level of predictive accuracy as some of the other regression and classification approaches seen in this book.
- ▼ Additionally, trees can be very non-robust. In other words, a small change in the data can cause a large change in the final estimated tree.

However, by aggregating many decision trees, using methods like *bagging*, *random forests*, and *boosting*, the predictive performance of trees can be substantially improved. We introduce these concepts in the next section.

8.2 Bagging, Random Forests, Boosting

Bagging, random forests, and boosting use trees as building blocks to construct more powerful prediction models.

8.2.1 Bagging

The bootstrap, introduced in Chapter 5, is an extremely powerful idea. It is used in many situations in which it is hard or even impossible to directly compute the standard deviation of a quantity of interest. We see here that the bootstrap can be used in a completely different context, in order to improve statistical learning methods such as decision trees.

The decision trees discussed in Section 8.1 suffer from *high variance*. This means that if we split the training data into two parts at random, and fit a decision tree to both halves, the results that we get could be quite different. In contrast, a procedure with *low variance* will yield similar results if applied repeatedly to distinct data sets; linear regression tends to have low variance, if the ratio of n to p is moderately large. *Bootstrap aggregation*, or *bagging*, is a general-purpose procedure for reducing the variance of a statistical learning method; we introduce it here because it is particularly useful and frequently used in the context of decision trees.

bagging

Recall that given a set of n independent observations Z_1, \dots, Z_n , each with variance σ^2 , the variance of the mean \bar{Z} of the observations is given by σ^2/n . In other words, *averaging a set of observations reduces variance*. Hence a natural way to reduce the variance and hence increase the prediction accuracy of a statistical learning method is to take many training sets from the population, build a separate prediction model using each training set, and average the resulting predictions. In other words, we could calculate $\hat{f}^1(x), \hat{f}^2(x), \dots, \hat{f}^B(x)$ using B separate training sets, and average them in order to obtain a single low-variance statistical learning model,