Assignment #9

① find the eigenvalues and eigenspaces for $\begin{pmatrix} 2 & -3 \\ 3 & 2 \end{pmatrix}$.

Since this is a special case where a 2×2 matrix has same entry on the diagonal and has same number on the other spo
we know ;

$$\lambda = a \pm b ;$$

which concludes,

$$\lambda = 2+3i \, , \, 2-3i .$$

② find the eigenvalues and eigenspace for $\begin{pmatrix} 7 & -5 \\ 1 & 3 \end{pmatrix}$.

$$\det \begin{pmatrix} 7-\lambda & -5 \\ 1 & 3-\lambda \end{pmatrix} = 0$$

$$\Rightarrow (7-\lambda)(3-\lambda)+5 = 0$$

$$\Rightarrow \lambda^2 - 10\lambda + 26 \quad \therefore \text{using quadratic equation}$$

$$\Rightarrow \lambda = 5+i \, , \, 5-i$$

Finding eigenspace :-

$$\therefore [A-(5+i)I]V = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 7-(5+i) & -5 \\ 1 & 3-(5+i) \end{pmatrix} = \left(\begin{array}{cc|c} 2-i & -5 & 0 \\ 1 & -2-i & 0 \end{array}\right)$$

$$\therefore (-2+i)R_2 + R_1 \to R_1$$

$$\Rightarrow (2-i)(-2+i) = 5 \qquad \Rightarrow 5-5=0$$

$$\Rightarrow \left(\begin{array}{cc|c} 0 & 0 & 0 \\ 1 & -2-i & 0 \end{array}\right)$$

$$\Rightarrow x_1 = 2+i\,x^2 \, , \quad x_2 = x^2 \Rightarrow \alpha \begin{pmatrix} 2+i \\ 1 \end{pmatrix}$$

So,

$$E_{5+i} = \left\{ \alpha \begin{pmatrix} 2+i \\ 1 \end{pmatrix} \middle| \, \alpha \in \mathbb{C} \right\}$$

$$\therefore E_{5-i} = \left\{ \alpha \begin{pmatrix} 2-i \\ 1 \end{pmatrix} \middle| \, \alpha \in \mathbb{C} \right\}$$

③ let $A = \begin{pmatrix} 3 & -\sqrt{3} \\ \sqrt{3} & 3 \end{pmatrix}$. The transformation $x \to Ax$ applies both a scaling and a rotation to $\mathbb{R}^2$. Use the eigenvalues of A to describe both the scaling and the rotation.

Since this is a special case :- $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ matrix, the eigenvalues are ;

$$\lambda = 3+\sqrt{3}i \, , \, 3-\sqrt{3}i$$

$$\Rightarrow 3+\sqrt{3}i = r(\cos(\theta) + \sqrt{3}i\sin(\theta))$$

$$r = \sqrt{3^2+(\sqrt{3})^2} \Rightarrow \sqrt{9+3} \Rightarrow 2\sqrt{3}$$

$$\tan(\theta) = \frac{\sqrt{3}}{3} \, , \quad \theta = \pi/6$$

④ let $A = \begin{pmatrix} 3 & -3 \\ 1 & 1 \end{pmatrix}$. find an invertible matrix P and a matrix $C = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ so that $A = PCP^{-1}$.

$$\det \begin{pmatrix} 3-\lambda & -3 \\ 1 & 1-\lambda \end{pmatrix} = 0$$

$$\Rightarrow (3-\lambda)(1-\lambda)+3 = 0$$

$$\Rightarrow \lambda^2 - 4\lambda + 6 = 0$$

$$\Rightarrow \lambda = 2+\sqrt{2}i \, , \, 2-\sqrt{2}i$$

Using $\lambda = 2-\sqrt{2}i$:-

$\therefore [A-(2-\sqrt{2}i)I_2]V = 0$ for $V \neq 0$

$$= \begin{pmatrix} 1+\sqrt{2}i & -3 \\ 1 & -1+\sqrt{2}i \end{pmatrix} V = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$= \left(\begin{array}{cc|c} 1+\sqrt{2}i & -3 & 0 \\ 1 & -1+\sqrt{2}i & 0 \end{array}\right)$$

$\therefore (-1-\sqrt{2}i)R_2 + R_1 \to R_1$

$= (-1-\sqrt{2}i)(-1+\sqrt{2}i) = 3$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 1 & -1+\sqrt{2}i & 0 \end{pmatrix}$$

$V_1 = 1-\sqrt{2}i \quad \circ \; V_2 = V_2$

$\alpha = \begin{pmatrix} 1-\sqrt{2}i \\ 1 \end{pmatrix}, \alpha \in \mathbb{R}.$

Using $v = \begin{pmatrix} 1-\sqrt{2}i \\ 1 \end{pmatrix}$ and taking $\alpha = 1$

$Re(V) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad Im(V) = \begin{pmatrix} -\sqrt{2} \\ 0 \end{pmatrix}$

With $C = \begin{pmatrix} 2 & -\sqrt{2} \\ \sqrt{2} & 2 \end{pmatrix}$ and $P = \begin{pmatrix} 1 & -\sqrt{2} \\ 1 & 0 \end{pmatrix}$ we get $A = PCP^{-1}$.

(5) Let $A = \begin{pmatrix} 7 & 11 & 20 & 17 \\ -20 & -40 & -86 & -74 \\ 0 & -5 & -10 & -10 \\ 10 & 28 & 60 & 53 \end{pmatrix}$. find an invertible matrix P and a block diagonal matrix $C = \begin{pmatrix} a & -b & 0 & 0 \\ b & a & 0 & 0 \\ 0 & 0 & c & -d \\ 0 & 0 & d & c \end{pmatrix}$ so that $A = PCP^{-1}$.

Eigenvalue of matrix A :-

$\lambda_1 = 3-i$

$\lambda_2 = 3+i$

$\lambda_3 = 2-5i$

$\lambda_4 = 2+5i$

eigenvectors of $\lambda = 3-i$:- $V = \begin{pmatrix} -1/2 \\ i/2 \\ \frac{-3-i}{4} \\ 1 \end{pmatrix}$

eigenvectors of $\lambda = 2-5i$:- $w = \begin{pmatrix} \frac{1-i}{2} \\ -i/2 \\ \frac{-3+i}{4} \\ 1 \end{pmatrix}$

$C = \begin{pmatrix} 3 & -1 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 0 & 0 & 2 & -5 \\ 0 & 0 & 5 & 2 \end{pmatrix}$, $\therefore P = [Re(V) \; Im(V) \;\; Re(w) \; Im(w)]$

$$P = \begin{pmatrix} -1/2 & 0 & 1/2 & -1 \\ 2 & 1 & 2 & -1 \\ -3/4 & -1 & -3/4 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$$

⑥ $x = \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}$ and $y = \begin{pmatrix} +2 \\ -3 \\ 2 \end{pmatrix}$. Compute $\left(\frac{xy}{\|y\|^2}\right) y$.

$$x = \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}, \quad y = \begin{pmatrix} 2 \\ -3 \\ 2 \end{pmatrix}$$

$$x \cdot y = -1(2) + 2(-3) + 3(2)$$
$$= -2$$

$$\|y\| = \sqrt{2^2 + (-3)^2 + 2^2} = \sqrt{4+9+4} = \sqrt{17}$$

$$\|y\|^2 = (\sqrt{17})^2 = 17$$

We have $\left(\frac{-2}{17}\right) y = \begin{matrix} -2/17(2) \\ -2/17(-3) \\ -2/17(2) \end{matrix} \quad \begin{pmatrix} -4/17 \\ 6/17 \\ -4/17 \end{pmatrix}$

⑦ find a unit vector in the direction of $\begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}$.

$$\left\| \alpha \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix} \right\| = 1$$

$$\alpha = \frac{1}{\left\| \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix} \right\|}$$

$$= \sqrt{(-1)^2 + 2^2 + 3^2} = \sqrt{14}$$

$$\alpha = 1/\sqrt{14}$$

Using $\alpha \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -1/\sqrt{14} \\ 1/\sqrt{7} \\ 3/\sqrt{14} \end{pmatrix}$

⑧ find a unit vector in the direction of $\begin{pmatrix} +2 \\ -3 \\ 2 \end{pmatrix}$.

$$\left\| \alpha \begin{pmatrix} 2 \\ -3 \\ 2 \end{pmatrix} \right\| = 1$$

$$\alpha = \frac{1}{\left\| \begin{pmatrix} 2 \\ -3 \\ 2 \end{pmatrix} \right\|}$$

$$\left\| \begin{pmatrix} 2 \\ -3 \\ 2 \end{pmatrix} \right\| = \sqrt{2^2 + (-3)^2 + 2^2} = \sqrt{17}$$

$$\alpha = \frac{1}{\sqrt{17}}$$

Using $\alpha \begin{pmatrix} 2 \\ -3 \\ 2 \end{pmatrix} = \begin{pmatrix} 2/\sqrt{17} \\ -3/\sqrt{17} \\ 2/\sqrt{17} \end{pmatrix}$

⑨ find the distance from $\begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}$ to $\begin{pmatrix} 2 \\ -3 \\ 2 \end{pmatrix}$.

$$\begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix} - \begin{pmatrix} 2 \\ -3 \\ 2 \end{pmatrix} = \begin{pmatrix} -3 \\ 5 \\ 1 \end{pmatrix}$$

$$\left\| \begin{pmatrix} -3 \\ 5 \\ 1 \end{pmatrix} \right\| = \sqrt{(-3)^2 + 5^2 + 1^2} = \sqrt{9+25+1} = \sqrt{35}$$

⑩ Determine whether the vectors $u = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$ and $v = \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix}$ are orthogonal.

$u \cdot v$ are orthogonal if only $u \cdot v = 0$

$$\begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix}$$

$\Rightarrow 2(2) + (-3)(3) + (1)(5)$

$\Rightarrow 0$; yes they are orthogonal.

(14) Determine whether $\left\{\begin{pmatrix}5\\-4\\0\\3\end{pmatrix}, \begin{pmatrix}-4\\1\\-3\\8\end{pmatrix}, \begin{pmatrix}3\\3\\5\\-1\end{pmatrix}\right\}$ is an orthogonal set.

$\left\{\begin{pmatrix}5\\-4\\0\\3\end{pmatrix}, \begin{pmatrix}-4\\1\\-3\\8\end{pmatrix}, \begin{pmatrix}3\\3\\5\\-1\end{pmatrix}\right\}$ $\therefore$ u, v, w are orthogonal if $u \perp v$ (ie $u \cdot v = 0$), $u \perp w$ (ie $u \cdot w = 0$), $v \perp w$ (ie $v \cdot w = 0$).

$\Rightarrow u \cdot v = 5(-4) + (-4)(1) + 0(-3) + 3(8) = 0$
$\Rightarrow u \cdot w = 5(3) + (-4)(3) + 0(5) + 3(-1) = 0$
$\Rightarrow v \cdot w = (-4)(3) + (1)(3) + (-3)(5) + 8(-1) = -32$

$\therefore$ $v \cdot w$ is not an orthogonal vector, which proves that this is not an orthogonal set.

(15) Show that $B = \left\{\begin{pmatrix}1\\0\\-1\end{pmatrix}, \begin{pmatrix}1\\\sqrt{2}\\1\end{pmatrix}, \begin{pmatrix}1\\-\sqrt{2}\\1\end{pmatrix}\right\}$ is an orthogonal set.

$\left\{\begin{pmatrix}1\\0\\-1\end{pmatrix}, \begin{pmatrix}1\\\sqrt{2}\\1\end{pmatrix}, \begin{pmatrix}1\\-\sqrt{2}\\1\end{pmatrix}\right\}$ $\therefore$ u, v, w are orthogonal if $u \perp v$ (i.e $u \cdot v = 0$), $u \perp w$ (ie $u \cdot w = 0$), $v \perp w$ ($v \cdot w = 0$).

$\Rightarrow u \cdot v = (1)(1) + 0(\sqrt{2}) + (-1)(1) = 0$
$\Rightarrow u \cdot w = (1)(1) + (0)(-\sqrt{2}) + (-1)(1) = 0$
$\Rightarrow v \cdot w = (1)(1) + (\sqrt{2})(-\sqrt{2}) + (1)(1) = 0$

$\therefore$ since $u \perp v$, $u \perp w$ and $v \perp w$ are all equals to 0, this set is orthogonal.

(16) Show that $B = \left\{\begin{pmatrix}1\\1\\-1\end{pmatrix}, \begin{pmatrix}2\\3\\5\end{pmatrix}\right\}$ is an orthogonal set.

$\left\{\begin{pmatrix}1\\1\\-1\end{pmatrix}, \begin{pmatrix}2\\3\\5\end{pmatrix}\right\}$ $\therefore$ This set is orthogonal if $u \perp v$. (ie $u \cdot v = 0$).

$\Rightarrow u \cdot v = (1)(2) + (1)(3) + (-1)(5) = 0$

$\therefore$ since $u \perp v$ (ie $u \cdot v = 0$), this set is orthogonal.

(17) find the orthogonal projection of the vector $x = \begin{pmatrix}1\\1\\2\end{pmatrix}$ onto span B where B is the set in problem #16.

$\text{Span } B = \left\{\begin{pmatrix}1\\1\\-1\end{pmatrix}, \begin{pmatrix}2\\3\\5\end{pmatrix}\right\}$, $x = \begin{pmatrix}1\\1\\2\end{pmatrix}$

$\text{Proj}_B\, x = \text{Proj}_{v_1} x + \text{Proj}_{v_2} x$

$= \frac{x \cdot v_1}{v_1 \cdot v_1} v_1 + \frac{x \cdot v_2}{v_2 \cdot v_2} v_2$

$x v_1 = (1)(1) + (1)(1) + (-1)(2) = 0$
$x v_2 = (1)(2) + (1)(3) + (2)(5) = 15$
$v_1 \cdot v_1 = (1)(1) + (1)(1) + (-1)(-1) = 3$
$v_2 v_2 = (2)(2) + (3)(3) + (5)(5) = 38$

$\therefore \text{Proj}_B\, x = \frac{0}{3} + \frac{15}{38}\begin{pmatrix}2\\3\\5\end{pmatrix}$

$= \begin{pmatrix}15/19\\45/38\\75/38\end{pmatrix}$

(18) Write $\begin{pmatrix}3\\3\\5\\-1\end{pmatrix}$ as a sum of two vectors, where one vector is

span $\left\{\begin{pmatrix}5\\-4\\0\\3\end{pmatrix}\right\}$ and the other vector is orthogonal to $\begin{pmatrix}5\\-4\\0\\3\end{pmatrix}$.

$y = \begin{pmatrix}3\\3\\5\\-1\end{pmatrix}, \quad u = \begin{pmatrix}5\\-4\\0\\3\end{pmatrix}$

$\Rightarrow y \cdot u = (3)(5) + (3)(-4) + (5)(0) + (-1)(3) = 0$

$\Rightarrow u \cdot u = (5)(5) + (-4)(-4) + (0)(0) + (3)(3) = 50$

$\therefore y = \text{Proj}_u\, y + (y - \text{Proj}_u\, y)$

$\text{Proj}_u\, y = \left(\frac{y \cdot u}{u \cdot u}\right) u \Rightarrow \left(\frac{0}{50}\right)\begin{pmatrix}5\\-4\\0\\3\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\end{pmatrix}$

$y \cdot \text{Proj}_u\, y = \begin{pmatrix}3\\3\\5\\-1\end{pmatrix} - \begin{pmatrix}0\\0\\0\\0\end{pmatrix} = \begin{pmatrix}3\\3\\5\\-1\end{pmatrix}$

$\therefore y = \begin{pmatrix}0\\0\\0\\0\end{pmatrix} + \begin{pmatrix}3\\3\\5\\-1\end{pmatrix}$ $\quad\therefore y =$ in span $\{u\}$ + orthogonal

(19) Find the distance from $\begin{pmatrix}1\\1\\-1\end{pmatrix}$ to the line given by span $\left\{\begin{pmatrix}2\\3\\5\end{pmatrix}\right\}$.

$\begin{pmatrix}1\\1\\1\end{pmatrix}$, span $\left\{\begin{pmatrix}2\\3\\5\end{pmatrix}\right\}$

distance $= \sqrt{(2-1)^2 + (3-1)^2 + (5-(-1))^2}$

$= \sqrt{1^2 + 2^2 + 6^2}$

distance $= \sqrt{41}$

(20) You showed earlier that $\left\{\begin{pmatrix}1\\0\\-1\end{pmatrix}, \begin{pmatrix}1\\\sqrt{2}\\1\end{pmatrix}, \begin{pmatrix}1\\-\sqrt{2}\\1\end{pmatrix}\right\}$ is an orthogonal set.

Normalize the vectors in this set to create an orthonormal set.

$\left\{\frac{V_1}{\|V_1\|}, \frac{V_2}{\|V_2\|}, \frac{V_3}{\|V_3\|}\right\}$ $\therefore$ let $V_1 = \begin{pmatrix}1\\0\\-1\end{pmatrix}, V_2 = \begin{pmatrix}1\\\sqrt{2}\\1\end{pmatrix}, V_3 = \begin{pmatrix}1\\-\sqrt{2}\\1\end{pmatrix}$

$\|V_1\| = \sqrt{1^2 + 0^2 + (-1)^2} = \sqrt{2}$

$\|V_2\| = \sqrt{1^2 + (\sqrt{2})^2 + 1^2} = 2$

$\|V_3\| = \sqrt{1^2 + (-\sqrt{2})^2 + 1^2} = 2$

$\therefore \left\{\begin{pmatrix}1/\sqrt{2}\\0\\-1/\sqrt{2}\end{pmatrix}, \begin{pmatrix}1/2\\1/\sqrt{2}\\1/2\end{pmatrix}, \begin{pmatrix}1/2\\-1/\sqrt{2}\\1/2\end{pmatrix}\right\}$

This is an orthonormal set.