

Controllers in Reactive Synthesis: A Strategic Perspective

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F.R.S.-FNRS & UMONS – Université de Mons, Belgium

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Journées annuelles du GT Vérif 2024



The talk in one slide

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↪ **That is our topic of the today.**

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Yes, I lied, and I will lie even more. The results I will survey span numerous combinations of

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↪ **I will focus on recent work with marvelous co-authors.**

- 1 Controller synthesis
- 2 Memory
- 3 Randomness
- 4 Beyond Mealy machines

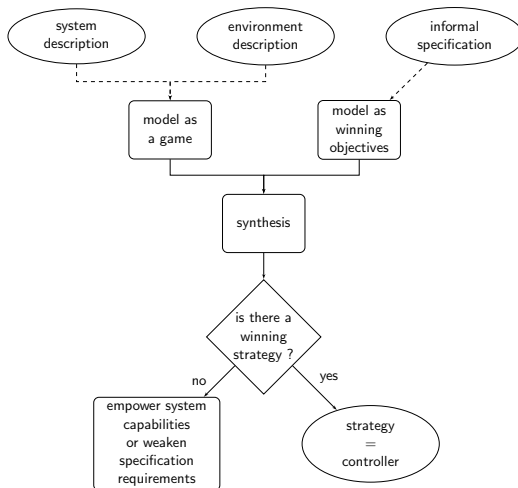
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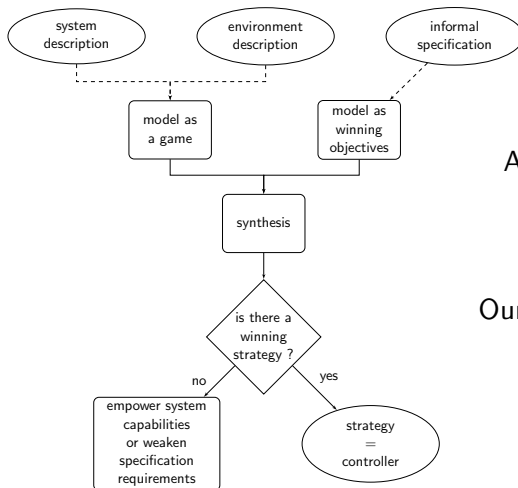
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Controller synthesis: a game-theoretic approach



Controller synthesis: a game-theoretic approach

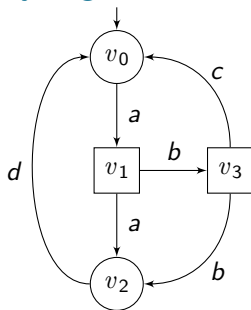


A plethora of models and objectives exist.¹

Our focus here: how complex **strategies** need to be?

¹Randour, "Automated Synthesis of Reliable and Efficient Systems Through Game Theory: A Case Study", 2013; Clarke et al., Handbook of Model Checking, 2018; Fijalkow et al., Games on Graphs, 2023.

Two-player games



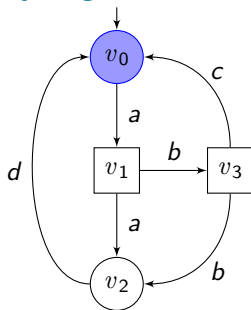
A two-player turn-based finite **arena**
 $\mathcal{A} = (V_{\bigcirc}, V_{\square}, E)$ with no deadlock.

Color function $\mathfrak{c}: E \rightarrow C$.

\hookrightarrow Players move a pebble along the edges creating an infinite **play**.

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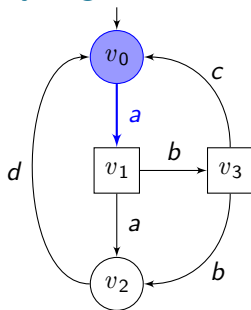
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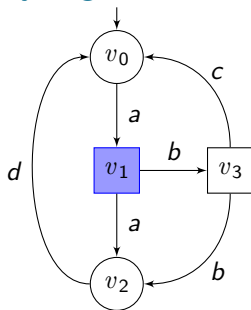
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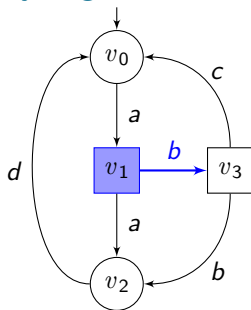
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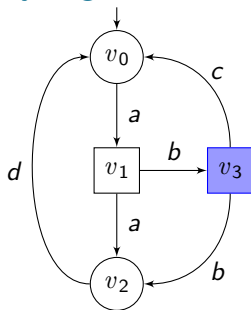
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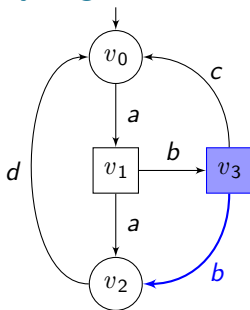
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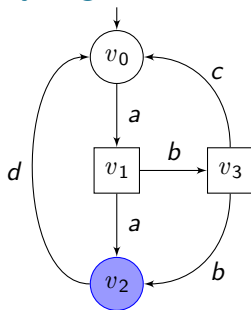
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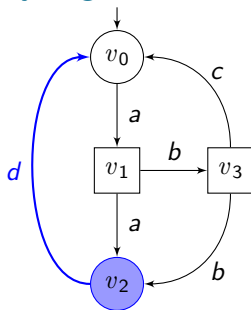
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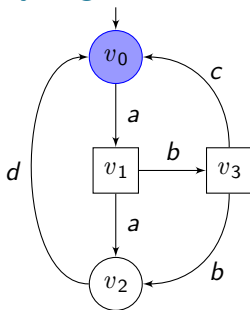
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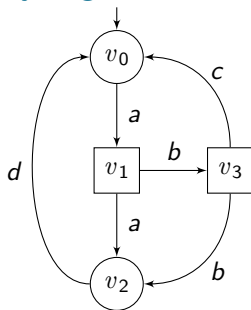
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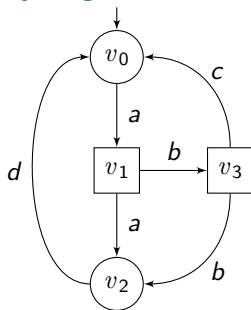
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Sample play: $abbd \dots \in C^{\omega}$

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Usual interpretation

\mathcal{P}_{\bigcirc} (the system to control) tries to satisfy its **specification** while
 \mathcal{P}_{\square} (the environment) tries to prevent it from doing so.

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- 1 A **winning condition**: a set of winning plays that \mathcal{P}_\circ tries to realize. E.g., $\text{Reach}(t) = \{\pi = c_0c_1c_2 \dots \mid t \in \pi\}$, for $t \in C$ a given color, a *reachability* objective.

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- 2 A **payoff function** to optimize, assuming $C \subset \mathbb{Q}$. E.g., the *discounted sum* function, defined as $\text{DS}(\pi) = \sum_{i=0}^{\infty} \gamma^i c_i$ for some discount factor $\gamma \in]0, 1[$.

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- 3 A **preference relation** defines a total preorder over sequences of colors, thus generalizing both previous concepts.

Strategies

Player \mathcal{P}_∇ chooses outgoing edges following a **strategy**

$$\sigma_\nabla: V^* V_\nabla \rightarrow V$$

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Optimal strategies (using a preference relation \sqsubseteq)

A strategy σ_\circ of \mathcal{P}_\circ is optimal if it **guarantees** (i.e., against an optimal adversary \mathcal{P}_\square) a play at least as good as any other strategy σ'_\circ with respect to \sqsubseteq .

MDPs & stochastic games

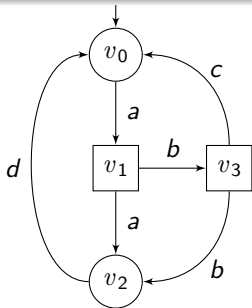
Why?

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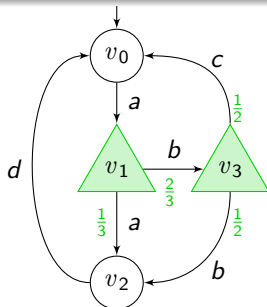
Two-player (deterministic) game.

$$V = V_{\circ} \uplus V_{\square}.$$

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Markov decision process.

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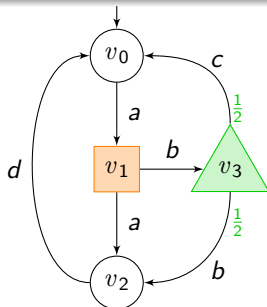
Either \mathcal{P}_{\circ} aims to maximize

- ▷ $\mathbb{P}^{\sigma \circ}[W]$ for some winning condition W ,
- ▷ or $\mathbb{E}^{\sigma \circ}[f]$ for some payoff function f .

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Stochastic game.

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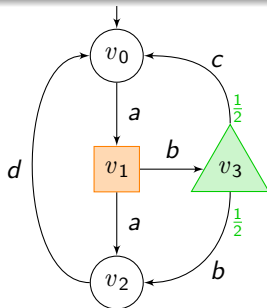
Either \mathcal{P}_{\circ} aims to maximize, against the adversary \mathcal{P}_{\square} ,

- ▷ $\mathbb{P}^{\sigma_{\circ}, \sigma_{\square}}[W]$ for some winning condition W ,
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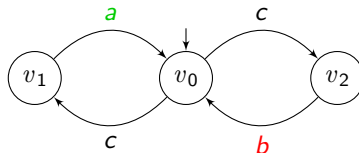
Actions

We often use **actions** instead of stochastic vertices.

Multiple objectives

Combining objectives

Complex objectives arise when **combining** simple objectives, and usually require **more complex** strategies to play optimally.

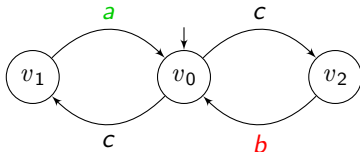


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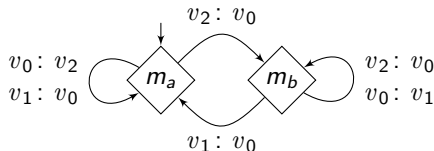
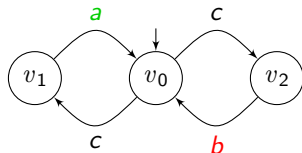
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↪ We are often interested in the **Pareto front**, i.e., all payoff vectors not dominated by another.

Classical representation of strategies: Mealy machines

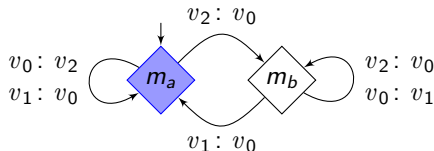
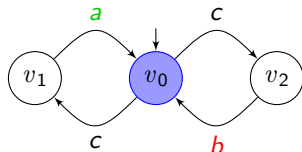


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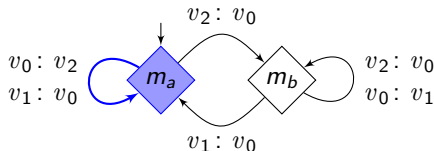
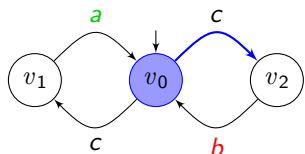


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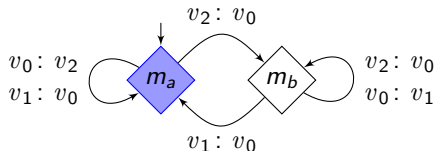
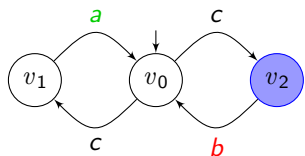


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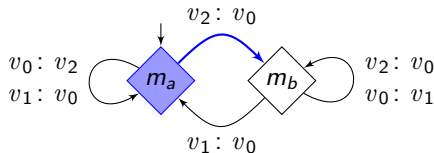
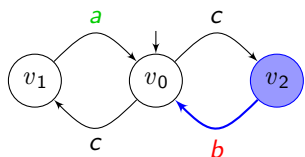


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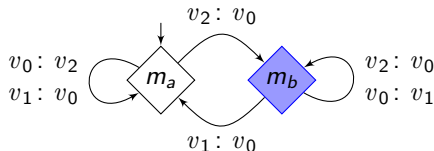
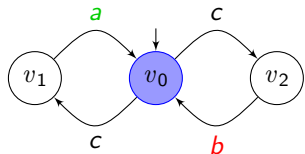


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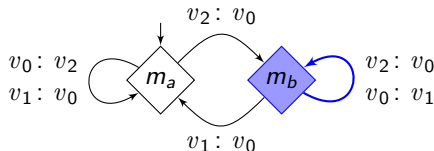
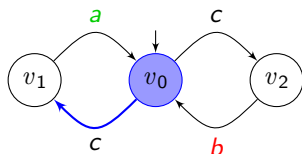


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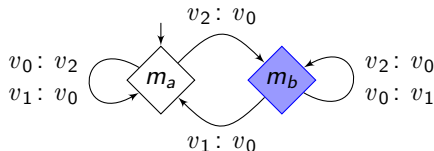
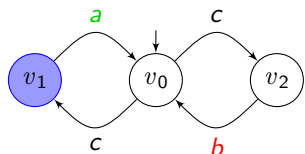


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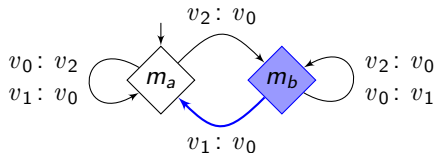
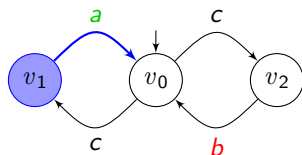


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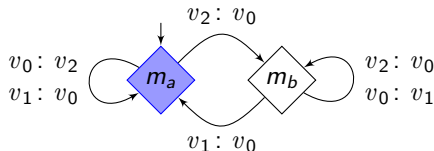
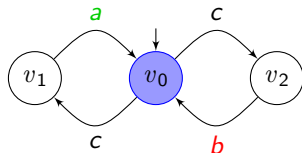


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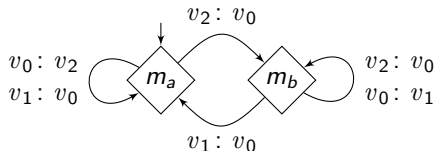
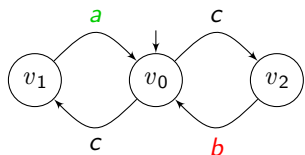


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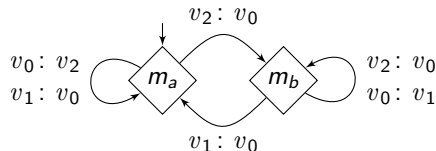


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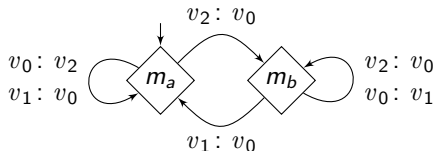
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The ice cream conundrum



This Mealy machine uses **chaotic** (or general) memory: it looks at the actual vertices of the game to update its memory.

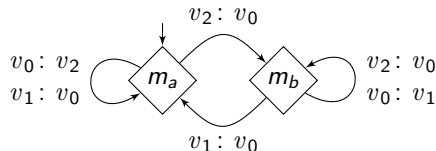
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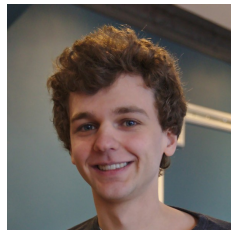
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↪ **We will discuss some of these.**

- 1 Controller synthesis
- 2 Memory
- 3 Randomness
- 4 Beyond Mealy machines

Some amazing co-authors



Section mostly based on joint work with Patricia Bouyer, Stéphane Le Roux, Youssef Oualhadj, and Pierre Vandenhove.²

²Bouyer, Le Roux, et al., "Games Where You Can Play Optimally with Arena-Independent Finite Memory", 2022; Bouyer, Oualhadj, et al., "Arena-Independent Finite-Memory Determinacy in Stochastic Games", 2023; Bouyer, Randour, and Vandenhove, "Characterizing Omega-Regularity through Finite-Memory Determinacy of Games on Infinite Graphs", 2023.

Memoryless strategies

Functions $\sigma_{\nabla}: V_{\nabla} \rightarrow V$.

- ▷ Equivalently, Mealy machines with one state.
- ▷ **Arguably**, the simplest kind of strategies.

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Functions $\sigma_{\nabla}: V_{\nabla} \rightarrow V$.

- ▷ Equivalently, Mealy machines with one state.
- ▷ **Arguably**, the simplest kind of strategies.
- ▷ Sufficient to play optimally for most *single* objectives in (stochastic) games: reachability, parity, mean-payoff, discounted sum, etc.

Starting point of our journey: *deterministic* games

Gimbert and Zielonka's characterization³

Memoryless strategies suffice (for both players) for a preference relation \sqsubseteq iff it is **monotone** and **selective**.

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Memoryless strategies suffice (for both players) for a preference relation \sqsubseteq iff it is **monotone** and **selective**.

Corollary: one-to-two-player lift

If \sqsubseteq is such that

- 1 in all \mathcal{P}_\circ -arenas, \mathcal{P}_\circ has optimal memoryless strategies,
 - 2 in all \mathcal{P}_\square -arenas, \mathcal{P}_\square has optimal memoryless strategies,
- then **both** players have optimal memoryless strategies in all **two-player** arenas.

\implies **Extremely useful as analyzing one-player games (i.e., graphs) is much easier.**

³Gimbert and Zielonka, "Games Where You Can Play Optimally Without Any Memory", 2005.

Handling finite-memory strategies (1/3)

Why?

- ▶ More complex objectives may require **finite** (multi-Büchi) or **infinite** memory (multi-mean-payoff).

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- ▷ More complex objectives may require **finite** (multi-Büchi) or **infinite** memory (multi-mean-payoff).

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Unfortunately, it does not hold.

Handling finite-memory strategies (2/3)

Let $C \subseteq \mathbb{Z}$ and the winning condition for \mathcal{P}_\circ be

$$\overline{TP}(\pi) = \infty \quad \vee \quad \exists^\infty i \in \mathbb{N}, \sum_{i=0}^n c_i = 0$$

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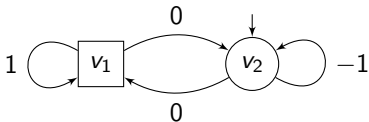
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But the two-player one is not!
 $\Rightarrow \mathcal{P}_\circ$ needs infinite memory to win.

Handling finite-memory strategies (3/3)

A new frontier

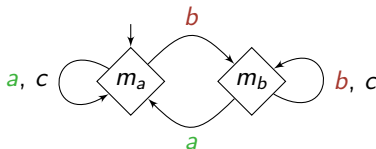
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Example for $C = \{a, b, c\}$ and objective $\text{Büchi}(a) \cap \text{Büchi}(b)$.

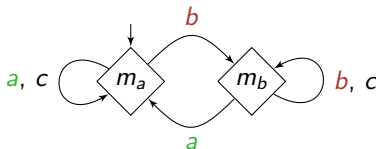


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This memory structure **suffices in all arenas**, i.e., it is always possible to find a suitable α_{next} to build an optimal Mealy machine.

Handling finite-memory strategies (3/3)

A new frontier

We focus on **arena-independent chromatic** memory structures.

Our characterization⁴

We obtain an equivalent to Gimbert and Zielonka's for finite memory:

- 1 a characterization through the concepts of \mathcal{M} -monotony and \mathcal{M} -selectivity,
- 2 a **one-to-two-player lift**.

⁴Bouyer, Le Roux, et al., "Games Where You Can Play Optimally with Arena-Independent Finite Memory", 2022.

Extension to stochastic games

We lift⁵ this result to **pure arena-independent finite-memory** strategies in **stochastic games**:

- 1 characterization based on generalizations of \mathcal{M} -monotony and \mathcal{M} -selectivity,
- 2 **one-to-two-player lift**, from MDPs to stochastic games.

⁵Bouyer, Oualhadj, et al., “Arena-Independent Finite-Memory Determinacy in Stochastic Games”, 2023.

Extension to infinite (deterministic) arenas (1/2)

We consider arenas of arbitrary cardinality and allow infinite branching.

Observation

Memory requirements can be **higher in infinite arenas**: e.g., mean-payoff objectives require infinite memory.

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The case of ω -regular objectives⁶

If a victory condition W is ω -regular, **then** it admits finite-memory optimal strategies in all (infinite) arenas.

⁶Mostowski, "Regular expressions for infinite trees and a standard form of automata", 1985; W. Zielonka, "Infinite games on finitely coloured graphs with applications to automata on infinite trees", 1998.

Extension to infinite (deterministic) arenas (2/2)

The converse⁷

If a **chromatic finite-memory** structure \mathcal{M} suffices for W in all infinite arenas, **then** W is ω -regular.

\Leftrightarrow We build a parity automaton for W , based on \mathcal{M} and \mathcal{S}_W , the *prefix-classifier* of W (recognizing its Myhill-Nerode classes).

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Corollaries

- 1 Game-theoretical characterization of ω -regularity.
- 2 **One-to-two-player lift** for infinite arenas.

⁷Bouyer, Randour, and Vandenhove, “Characterizing Omega-Regularity through Finite-Memory Determinacy of Games on Infinite Graphs”, 2023.

Other criteria and characterizations

There is a plethora of results related to memory (models vary).

Non-exhaustive list:

- ▶ characterizations through universal graphs,⁸
- ▶ tight memory bounds for sub-classes of objectives,⁹
- ▶ criteria for half-positionality,¹⁰
- ▶ one-to-multi-objective lift,¹¹
- ▶ two-to-multi-player lift.¹²

↪ Find more about **chromatic memory** in our survey.¹³

⁸Casares and Ohlmann, “Characterising Memory in Infinite Games”, 2023.

⁹Bouyer, Casares, et al., “Half-Positional Objectives Recognized by Deterministic Büchi Automata”, 2024; Bouyer, Fijalkow, et al., “How to Play Optimally for Regular Objectives?”, 2023; Casares and Ohlmann, “Positional ω -regular languages”, 2024.

¹⁰Aminof and Rubin, “First-cycle games”, 2017.

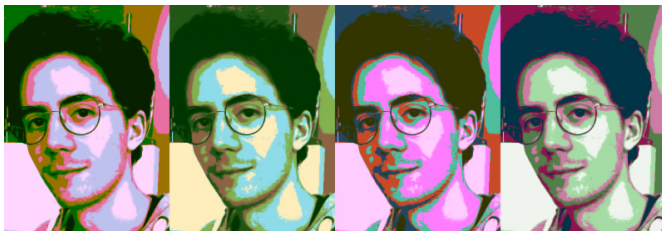
¹¹Le Roux, Pauly, and Randour, “Extending Finite-Memory Determinacy by Boolean Combination of Winning Conditions”, 2018.

¹²Le Roux and Pauly, “Extending Finite Memory Determinacy to Multiplayer Games”, 2016.

¹³Bouyer, Randour, and Vandenhove, “The True Colors of Memory: A Tour of Chromatic-Memory Strategies in Zero-Sum Games on Graphs (Invited Talk)”, 2022.

- 1 Controller synthesis
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The amazing Mr. Main



Section mostly based on (ongoing) joint work with
James C. A. Main.¹⁴

¹⁴Main and Randour, “Different Strokes in Randomised Strategies: Revisiting Kuhn’s Theorem Under Finite-Memory Assumptions”, 2024.

Introducing randomness in strategies (1/2)

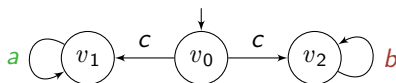
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Introducing randomness in strategies (1/2)

A **pure** strategy is a function $\sigma_{\nabla}: V^* V_{\nabla} \rightarrow V$.

We may need **randomness** to deal with, e.g.,

- ▶ multiple objectives,
- ▶ concurrent games,
- ▶ imperfect information.



$$\text{Objective: } \mathbb{P}^{\sigma \circ}[\text{Reach}(a)] \geq \frac{1}{2} \wedge \mathbb{P}^{\sigma \circ}[\text{Reach}(b)] \geq \frac{1}{2}$$

↪ **Achievable by tossing a coin in v_0 .**

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Several ways of **randomizing** $\sigma_{\nabla}: V^* V_{\nabla} \rightarrow V$:

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Kuhn's theorem¹⁵

All three classes are equivalent in games of **perfect recall**.

↪ **Requires access to infinite memory and infinite support for distributions.**

¹⁵Aumann, "Mixed and Behavior Strategies in Infinite Extensive Games", 1964.

What about finite-memory strategies?

Mealy machine $\mathcal{M} = \{M, m_{\text{init}}, \alpha_{\text{next}}, \alpha_{\text{up}}\}$:

- ▷ M is the set of memory states,
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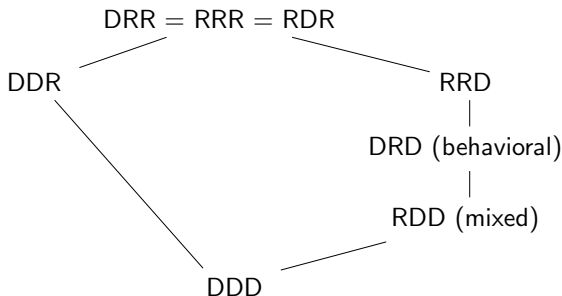
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Stochastic Mealy machine $\mathcal{M} = \{M, \mu_{\text{init}}, \alpha_{\text{next}}, \alpha_{\text{up}}\}$:

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⇒ **Three ways to add randomness: initialization, outputs, and updates.**

Taxonomy¹⁶ (1/2)



Classes XYZ where $X, Y, Z \in \{D, R\}$ where D stands for deterministic and R for random, and

- X characterizes the initialization,
- Y characterizes the next-action function,
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¹⁶Main and Randour, "Different Strokes in Randomised Strategies: Revisiting Kuhn's Theorem Under Finite-Memory Assumptions", 2024.

Taxonomy (2/2)

This taxonomy holds from one-player deterministic games (no collapse) up to concurrent partial-information multi-player games (equivalences hold).

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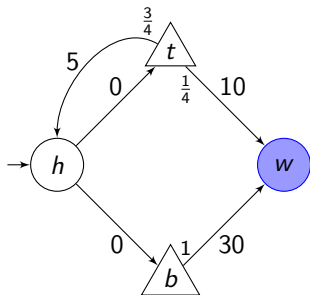
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↪ **Collapses may arise for restricted classes of objectives (WiP).**

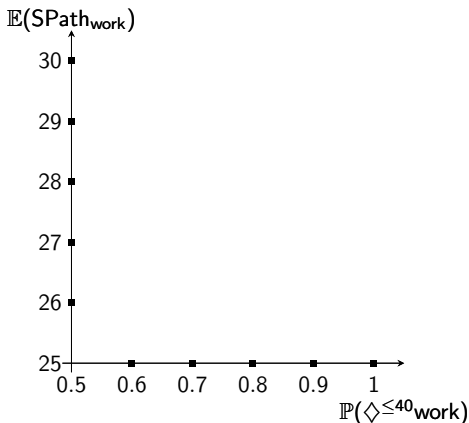
Multi-objectives MDPs (1/2)

We consider **two goals**:

- reaching work under 40 minutes with **high probability**;
- minimizing the **expectancy** of the time to reach work.



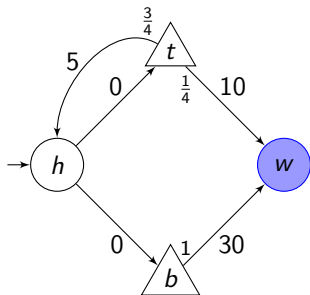
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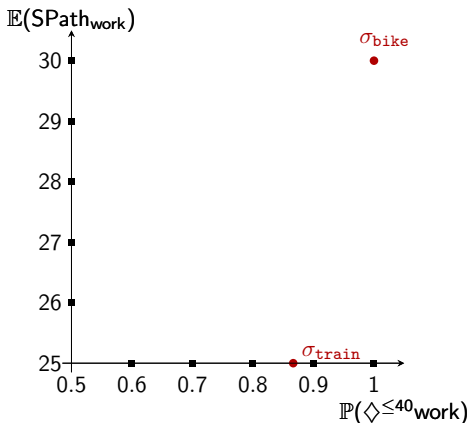
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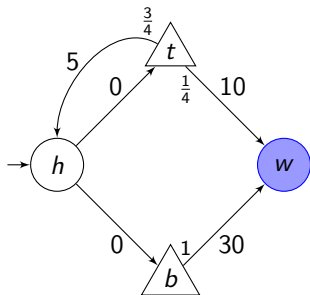
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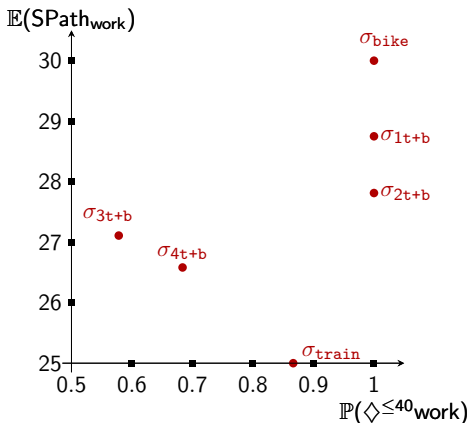
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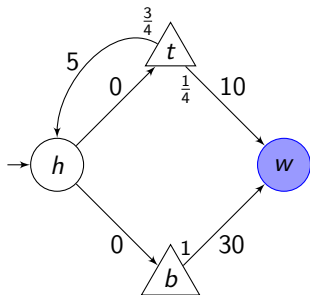
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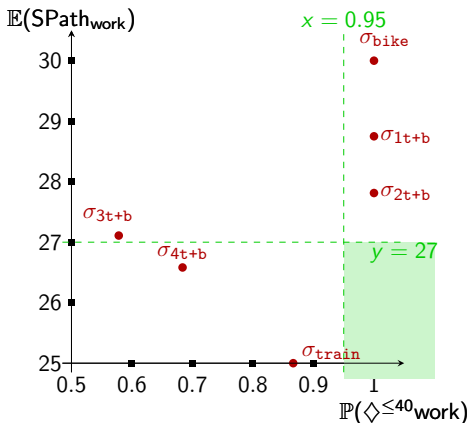
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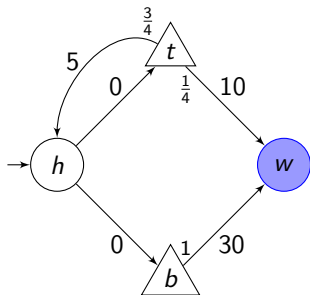
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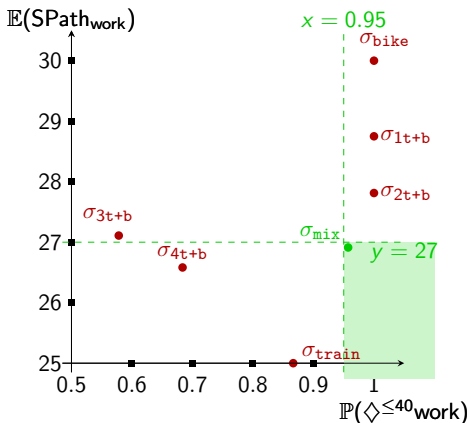
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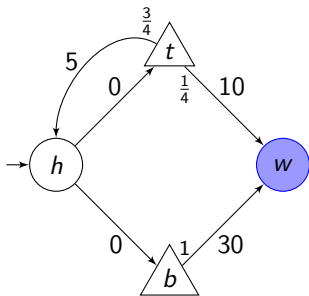
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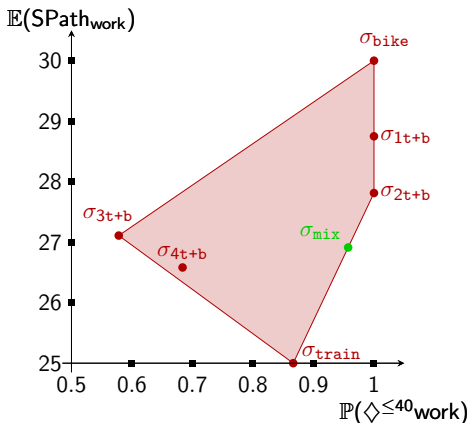
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For *good* payoff functions (\sim expectancy is well-defined),

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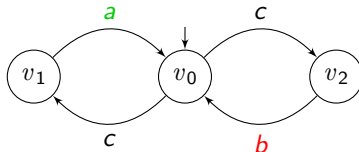
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\implies **RDD-randomization is sufficient in most multi-objective MDPs.**

Trading memory for randomness

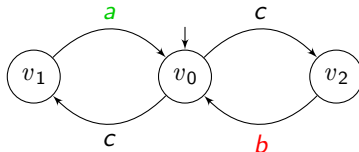
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We need (a two-state) memory to win it with *pure* strategies.

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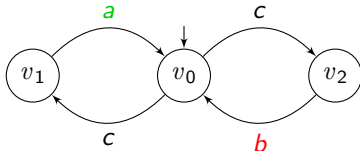


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But a (behavioral) randomized memoryless strategy suffices to win with probability one: playing v_1 and v_2 with non-zero probability ensures it.

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But a (behavioral) randomized memoryless strategy suffices to win with probability one: playing v_1 and v_2 with non-zero probability ensures it.

↪ **Memory can be traded for randomness for some classes of games/objectives.**¹⁷

¹⁷Chatterjee, de Alfaro, and Henzinger, “Trading Memory for Randomness”, 2004; Chatterjee, Randour, and Raskin, “Strategy synthesis for multi-dimensional quantitative objectives”, 2014.

- 1 Controller synthesis
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An incomplete story

Leitmotiv

Simpler strategies are better (for controller synthesis).

An incomplete story

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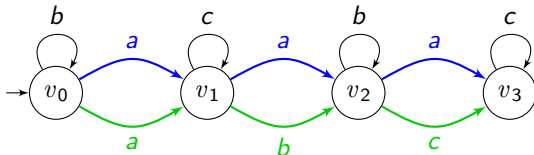
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↪ **Let us question that.**

All memoryless strategies are not created equal

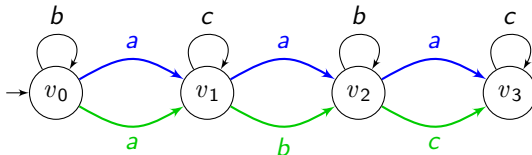
We want to reach v_3 .



Intuitively, the **blue** strategy seems simpler than the **green** one.

All memoryless strategies are not created equal

We want to reach v_3 .

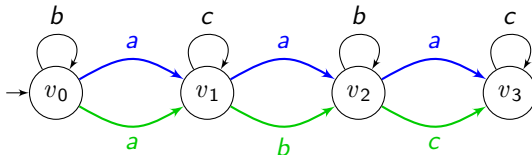


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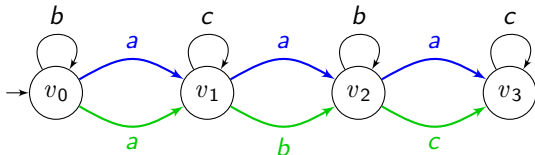


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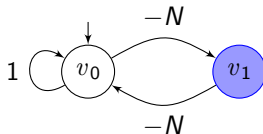
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↪ **Memoryless strategies can already be too large to represent in practice!**

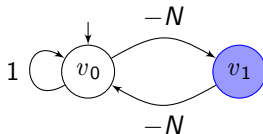
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Multi-objectives games involving payoffs often require **exponential memory**. E.g., **energy-Büchi** objective with $N \in \mathbb{N}$.



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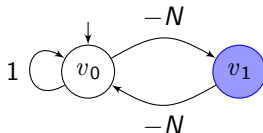
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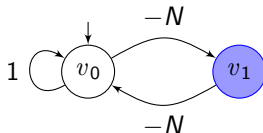


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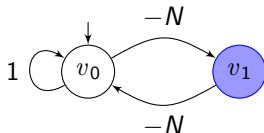
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Hot take

We should explore novel notions of **simplicity**, and consider **alternative representations** of strategies/controllers.

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Hot take

We should explore novel notions of **simplicity**, and consider **alternative representations** of strategies/controllers.

↪ **We quickly survey a few ones in the next slides.**

Structurally-enriched Mealy machines

Idea:

- ▶ Augment Mealy machines with **data structures**: e.g., counters.¹⁸
 - ▶ Avoid “flattening” structural information about the strategy: better understandability and closer to actual controllers.
 - ▶ Link with James’s talk about interval strategies in OC-MDPs.
- ⇒ **Changes our way of thinking which strategies are complex or not.**

¹⁸Blahoudek et al., “Qualitative Controller Synthesis for Consumption Markov Decision Processes”, 2020.

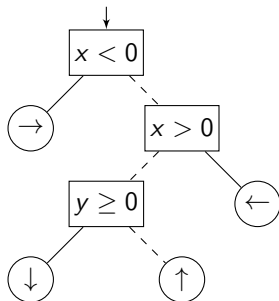
Decision trees

- ▷ Structured state-space (e.g., $\subset \mathbb{Z}^n$) and action-space.
- ▷ Learn a (possibly approximative) decision tree from a given **memoryless** strategy.
- ▷ More understandable and compact than huge action tables.
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Toy example: trying to reach the center (0,0) of a 2D-grid.



instead of

x	y	action
0	1	\downarrow
0	2	\downarrow
...	...	\downarrow
-1	0	\rightarrow
-1	1	\rightarrow
...

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Works well in practice...¹⁹

...starting from a given memoryless strategy.

¹⁹Brazdil, Chatterjee, Chmelik, et al., "Counterexample Explanation by Learning Small Strategies in Markov Decision Processes", 2015; Brazdil, Chatterjee, Kretinsky, et al., "Strategy Representation by Decision Trees in Reactive Synthesis", 2018.

Other alternatives

■ Programmatic representations.

- ▷ Closer to realistic code, understandable.
- ▷ Strongly linked to the input format of the problem (e.g., PRISM code²⁰), hard to generalize.

■ Models inspired by Turing machines.

- ▷ Powerful but hard to work with.
- ▷ Tentative notion of decision speed.²¹

■ Neural networks.

- ▷ Prevalent in RL.
- ▷ Hard to understand and verify.
- ▷ Can be coupled with finite-state-machine abstractions.²²

²⁰Shabadi, Fijalkow, and Matricon, "Theoretical foundations for programmatic reinforcement learning", 2024.

²²Gelderie, "Strategy machines: representation and complexity of strategies in infinite games", 2014.

²²Carr, Jansen, and Topcu, "Verifiable RNN-Based Policies for POMDPs Under Temporal Logic Constraints", 2020.

Wrap-up

Focus

Complexity of strategies in controller synthesis.

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Mealy machines are a powerful tool from a theoretical standpoint.

- ▶ High-level picture w.r.t. **memory** and **randomness**.

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Strategy complexity \neq representation complexity.

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↔ **Many questions are still open!**

Strategy complexity \neq **representation complexity**.

Take-home message

We need a **proper theory of complexity**, and a **toolbox of different representations**.

↔ **Ongoing project ControlléRS.**

Thank you! Any question?