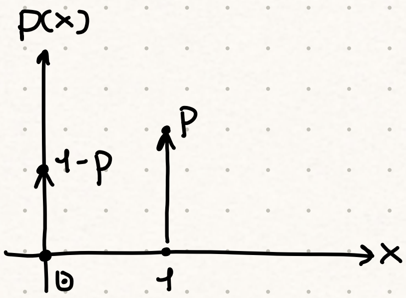
$X \sim \text{Bernoulli}(p)$, a **Bernoulli random variable**

$X \in \{0, 1\}$

$p \in [0, 1]$

$X$ **models a coin toss**

The **PDF** of $X$ is $p(x) = \mathbb{P}[X = x]$




Let $X_1, X_2, \ldots, X_n \overset{iid}{\sim} \text{Bernoulli}(p)$

$$Y = X_1 + X_2 + \ldots + X_n$$
$$= \sum_{i=1}^{n} X_i$$

**i.i.d.**
Independent
Identically
Distributed

$Y \in \{0, 1, 2, \ldots, n\}$

$Y$ models the **sum of $n$ coin tosses**

$Y \sim \text{Binomial}(n, p)$ is a binomial random variable

What's the PDF of $Y$?

$$p(y) = \mathbb{P}[Y = y] = ?$$

$$= \mathbb{P}[\text{If I throw } n \text{ coins, I get } y \text{ heads}]$$

$$= \underbrace{p^y}_{\text{probability of getting } y \text{ heads in the first } y \text{ throws}} \underbrace{(1-p)^{n-y}}_{\text{probability of getting } n-y \text{ tails in the next } n-y \text{ throws}} \binom{n}{y}$$

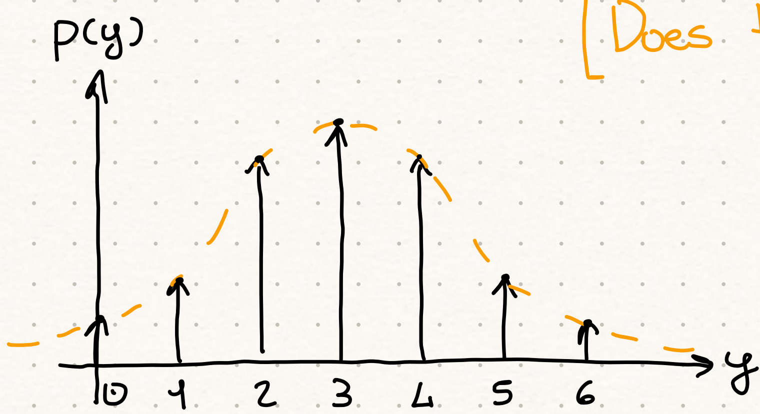
Accounting for all possible combinations of head, tails

probability of getting $y$ heads in the first $y$ throws

probability of getting $n-y$ tails in the next $n-y$ throws

Let's draw this PDF...

[Does this look familiar?]




What's $Z = Y_1 + Y_2$ ?

where $Y_1 \sim \text{Binomial}(n_1, p)$

$Y_2 \sim \text{Binomial}(n_2, p)$

$$p(z) = \mathbb{P}[Z = z] = ?$$

$$= \mathbb{P}\left[\begin{array}{l}\text{If I throw } n_1 + n_2 \text{ coins,} \\ \text{I get } z = y_1 + y_2 \text{ heads}\end{array}\right]$$

$$= \sum_{y=0}^{z} p_{Y_1}(y)\, p_{Y_2}(z-y)$$

$$\triangleq [p_{Y_1} * p_{Y_2}](y)$$

This is the discrete convolution of $p_{Y_1}, p_{Y_2}$

Let's compute it!

$$p_Z(z) = [p_{Y_1} * p_{Y_2}](z)$$

$$= \sum_{y=0}^{z} p_{Y_1}(y)\, p_{Y_2}(z-y)$$

$$= \sum_{y=0}^{z} \binom{n_1}{y} p^y (1-p)^{n_1-y} \binom{n_2}{z-y} p^{z-y} (1-p)^{n_2-z+y}$$

$$= \underbrace{\sum_{y=0}^{z} \binom{n_1}{y}\binom{n_2}{z-y}}_{\downarrow \text{ By Vandermonde's identity}} p^{\cancel{y}+z-\cancel{y}} (1-p)^{n_1-\cancel{y}+n_2-z+\cancel{y}}$$

$$= \binom{n_1+n_2}{z} p^z (1-p)^{(n_1+n_2)-z}$$

but this looks like the PDF of a new Binomial...

$$p_Z(z) = [p_{Y_1} * p_{Y_2}](z) = \binom{n_1+n_2}{z} p^z (1-p)^{(n_1+n_2)-z}$$

Then $Z = Y_1 + Y_2 \sim \text{Binomial}(n_1+n_2, p)$

or, in other words ...

$$\text{Bin}(n_1, p) + \text{Bin}(n_2, p) = \text{Bin}(n_1+n_2, p)$$

or ...

$$p(z; n_1, p) * p(z; n_2, p) = p(z; n_1+n_2, p)$$

The convolution of Binomials is a Binomial