# SENG440 Embedded Systems

## – Lesson 112: Matrix Diagonalization –

**Mihai SIMA**

**msima@ece.uvic.ca**

Academic Course

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# Disclaimer

The purpose of this course is to present general techniques and concepts for the analysis, design, and utilization of embedded systems. The requirements of any real embedded system can be intimately connected with the environment in which the embedded system is deployed. The presented design examples should not be used as the full design for any real embedded system.

# Lesson 112: Matrix Diagonalization

# Introduction

- Matrix diagonalization is required in many domains
  - Wireless communications
  - Control applications
- Singular Value Decomposition (SVD) achieves diagonalization
- It is difficult to calculate it fast
- SVD can be parallelized to a certain extend
- It poses numerical stability issues (especially in integer arithmetic)

 

# Singular Value Decomposition

- The Singular Value Decomposition (SVD) of a $n \times n$ matrix $M$ is:

$$M = U \Sigma V^T$$

  where $\Sigma$ is a diagonal matrix of singular values, and $U$ and $V$ are orthogonal / unitary matrices when $M$ is real / complex-valued
- Carl Gustav Jacob Jacobi proposed a method to calculate the SVD
- The **Jacobi method** systematically reduces the off-diagonal elements to zero by applying a sequence of plane rotations to $M$ to transform it into $\Sigma$
- Several sweeps over the entire matrix $M$ may be necessary
- Within each sweep, the matrix elements need to be paired and appropriate rotations needs to be calculated. The $n \times n$ matrix is partitioned in $n/2 \times n/2$ blocks, each block being a $2 \times 2$ matrix.

# Singular Value Decomposition – Jacobi method

- Consider the following real-valued matrix $M$:

$$M = \begin{pmatrix} m_{00} & \dots & m_{0i} & \dots & m_{0j} & \dots & m_{0n} \\ \vdots & & \vdots & & \vdots & & \vdots \\ m_{i0} & \dots & m_{ii} & \dots & m_{ij} & \dots & m_{in} \\ \vdots & & \vdots & & \vdots & & \vdots \\ m_{j0} & \dots & m_{ji} & \dots & m_{jj} & \dots & m_{jn} \\ \vdots & & \vdots & & \vdots & & \vdots \\ m_{n0} & \dots & m_{ni} & \dots & m_{nj} & \dots & m_{nn} \end{pmatrix}$$

- Choose $(i, j)$ such that $|m_{ij}|$ is the maximum non-diagonal element
- For the following $2 \times 2$ matrix, the elements $m_{ij}$ and $m_{ji}$ are forced to zero

$$\begin{pmatrix} m_{ii} & m_{ij} \\ m_{ji} & m_{jj} \end{pmatrix}$$

- Propagate the computation along the rows $i$ and $j$ and columns $i$ and $j$

# Singular Value Decomposition – the core operation I

- The basic operation is the two-sided rotation of each $2 \times 2$ matrix

$$R(\theta_l) \begin{pmatrix} m_{ii} & m_{ij} \\ m_{ji} & m_{jj} \end{pmatrix} R(\theta_r)^T = \begin{pmatrix} \Psi_1 & 0 \\ 0 & \Psi_2 \end{pmatrix}$$

  where $\theta_l$ and $\theta_r$ are the left and right rotation angles, respectively.
- The rotation matrices have the following forms:

$$R(\theta_l) = \begin{pmatrix} \cos\theta_l & -\sin\theta_l \\ \sin\theta_l & \cos\theta_l \end{pmatrix} \qquad R(\theta_r) = \begin{pmatrix} \cos\theta_r & -\sin\theta_r \\ \sin\theta_r & \cos\theta_r \end{pmatrix}$$

- Two issues need to be addressed:
  - Calculation of the rotation angles
  - Performing the rotations

# Singular Value Decomposition – the core operation II

- The direct two-angle method calculates $\theta_l$ and $\theta_r$ by computing the inverse tangents of the data elements of $M$:

$$\theta_{\mathrm{SUM}} = \theta_r + \theta_l = \arctan\left(\frac{m_{ji} + m_{ij}}{m_{jj} - m_{ii}}\right)$$

$$\theta_{\mathrm{DIFF}} = \theta_r - \theta_l = \arctan\left(\frac{m_{ji} - m_{ij}}{m_{jj} + m_{ii}}\right)$$

- The two angles, $\theta_l$ and $\theta_r$, can be separated from the sum and difference results and applied to the two-sided rotation module to diagonalize $M$

# Cyclic Jacobi method

- The Jacobi method requires at each step the scanning of $n(n-1)/2$ numbers for one of maximum modulus
  - This can be time consuming for large matrices
- **Cyclic Jacobi method**: select the pairs $(i, j)$ in some cyclic order
- The following order is called cyclic-by-rows:
  $1-2, 1-3, \ldots, 1-n, 2-3, \ldots, 2-n, 3-4, \ldots, (n-1)-n$
- More than one sweep is likely to be needed
- Although some on-diagonal energy may go off-diagonal at some iterations, the process converges in a few sweeps
- <u>There is no needed to completely vanish an off-diagonal element!</u>
  - The off-diagonal energy moves toward the main diagonal
  - The magnitude of the off-diagonal elements approaches zero

# Singular Value Decomposition – example I

- The matrix for which SVD is to be calculated is $M$

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{U} \cdot \underbrace{\begin{pmatrix} 31 & 77 & -11 & 26 \\ -42 & 14 & 79 & -53 \\ -68 & -10 & 45 & 90 \\ 34 & 16 & 38 & -19 \end{pmatrix}}_{M} \cdot \underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{V^T}$$

- In the initial state of the algorithm $\Sigma = M$ and $U = V = I$
- Plane rotations will be applied to diagonalize $M$
- These plane rotations will be incorporated into $U$ and $V$

# Singular Value Decomposition – example II

- Cyclic Jacobi method is used with the following six iterations: $1-2$, $1-3$, $1-4$, $2-3$, $2-4$, $3-4$ (this constitutes one sweep)
- The calculations for the first sweep are presented in the appendix
- The evolution of the diagonalization process over four sweeps is shown in the next slide
- Observe that the off-diagonal energy accumulates on the main diagonal
  - The magnitudes of the off-diagonal elements decrease
  - The magnitudes of the diagonal elements increase, which creates difficulties in calculating $M$ in fixed-point arithmetic
- The orthogonal matrices $U$ and $V$ are shown in the appendix
  - Their elements range from -1.0 to 1.0 $\rightarrow$ this guarantees stability in calculating $U$ and $V$ in fixed-point arithmetic

 

# Singular Value Decomposition – multiple sweeps

$$M = \begin{pmatrix} 31 & 77 & -11 & 26 \\ -42 & 14 & 79 & -53 \\ -68 & -10 & 45 & 90 \\ 34 & 16 & 38 & -19 \end{pmatrix}$$

$$M_1 = \begin{pmatrix} 100.529 & -15.789 & 8.876 & 29.798 \\ -24.869 & 104.453 & 2.662 & 9.100 \\ -11.624 & -0.992 & -110.708 & 0 \\ 2.764 & -0.250 & 0 & 37.710 \end{pmatrix}$$

$$M_2 = \begin{pmatrix} 85.636 & 1.354 & 0.696 & 0.160 \\ -4.998 & 126.299 & 0.027 & 0 \\ -1.134 & 0.190 & -110.873 & 0 \\ -0.265 & 0 & 0 & 34.008 \end{pmatrix}$$

$$M_3 = \begin{pmatrix} 85.570 & 0.009 & 0 & 0 \\ 0 & 126.429 & 0 & 0 \\ 0 & 0 & -110.905 & 0 \\ 0 & 0 & 0 & 34.008 \end{pmatrix}$$

$$\Sigma = \begin{pmatrix} 85.570 & 0 & 0 & 0 \\ 0 & 126.429 & 0 & 0 \\ 0 & 0 & -110.905 & 0 \\ 0 & 0 & 0 & 34.008 \end{pmatrix}$$

- The original matrix is $M$
- The matrices $M_1$, $M_2$, and $M_3$ are the matrices at the end of sweeps 1, 2, and 3, respectively
- At the end of Sweep 4 the matrix is completely diagonalized; therefore, it is labeled $\Sigma$
- The algorithm converges fast
- The negative diagonal elements can be forced positive through a simple multiplication
- A final permutation can order the singular values

# Singular Value Decomposition – operation count I

- **Calculation of the rotation angles requires**:
  - The evaluation of $\arctan$
- $\arctan$ is a transcendental function
  - Is series expansion appropriate to evaluate $\arctan$?
  - Are there alternatives to a series expansion?
- **Performing the rotations requires**:
  - The evaluation of $\cos$ and $\sin$
  - Matrix multiplication
- $\cos$ and $\sin$ are transcendental functions
  - Is series expansion appropriate to evaluate $\cos$ and $\sin$?
  - Are there alternatives to a series expansion?
- Can matrix multiplication be implemented with standard instructions?
  - Multiply-and-ACcumulate (MAC) operations are good candidates
  - SIMD operations are good candidates, since the code can be parallelized

# Singular Value Decomposition – operation count II

- In a serial computer, the **calculation of the rotation angles** and **performing the rotations** are both expensive tasks
- Specialized processors, with architectural support for DSP and SIMD operations, speed up matrix multiplication, but the calculation of trigonometric functions remains difficult
- An Application-Specific Instruction set Processor (ASIP), which provides architectural support for trigonometric functions would be an excellent choice, but such a processor is not readily available
- A good understanding of the SVD algorithm is required in order to simplify the calculation of the trigonometric functions
  - There is no needed to completely vanish an off-diagonal element as long as the off-diagonal energy keeps moving toward the main diagonal $\rightarrow$ incomplete rotations should be simpler to implement, but more sweeps might me needed

# How to calculate arctan$(x)$? I

- The function $\arctan : (-\infty, +\infty) \longrightarrow \left(-\frac{\pi}{2}, +\frac{\pi}{2}\right)$
  - A domain like $(-\infty, +\infty)$ should be avoided
    - Calculate $\arctan(x)$ when $|x| \leq 1$
    - Calculate $\text{arccot}(x)$ when $|x| > 1$ and adjust the angle accordingly
- In C using floating point:

```
#include <math.h>
...
int x, y, angle;

if (x > y)
  angle = arctan( y/x);          /* arctan() returns a float */
else
  angle = PI/2 - arctan( x/y); /* arctan() returns a float */
```

# How to calculate $\arctan(x)$? II

- Integer / fixed-point arithmetic is required!
  - C standard library (**math.h**): $\arctan()$ is a floating-point function
  - "/" is not a good option to divide integers
  - $\pi$ is a fractional number
- How to implement $\arctan()$ in integer / fixed-point arithmetic?
  - Taylor series expansion about a point – approximation good for 1 point

$$\arctan(x) = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \dots$$

  - Tchebishev polynomial – approximation good for an interval (homework)
  - Piecewise linear approximation with three middle points:

$$\arctan(x) = \begin{cases} 0.644\,x + 0.142 & \text{if } 0.5 < x \le 1.0, \\ 0.928\,x & \text{if } -0.5 \le x \le 0.5, \\ 0.644\,x - 0.142 & \text{if } -1.0 \le x < -0.5. \end{cases}$$

# $\arctan(x)$ – piecewise linear approximation I

- Piecewise linear approximation using fractional numbers

$$\arctan(x) = \begin{cases} 0.644\,x + 0.142 & \text{if } 0.5 < x \leq 1.0, \\ 0.928\,x & \text{if } -0.5 \leq x \leq 0.5, \\ 0.644\,x - 0.142 & \text{if } -1.0 \leq x < -0.5. \end{cases}$$

- Consider 12-bit signed integers
  Conversion from fractional representation to integer representation
  - 1.0 is represented as $2^{11}$ (in fact, as $2^{11} - 1$)
  - 0.928 is represented as $1900 = 76\mathrm{C}_\mathrm{h}$
  - 0.644 is represented as $1319 = 527_\mathrm{h}$
  - 0.142 is represented as $291 = 123_\mathrm{h}$
  - 0.5 is represented as $1024 = 400_\mathrm{h}$
  - $x$ is represented as $X = 2^{11}\,x$

# $\arctan(x)$ – piecewise linear approximation II

- Piecewise linear approximation using integer arithmetic

$$\arctan(X) = \begin{cases} 1319\,X + 291 & \text{if } 1024 < X \leq 2048, \\ 1900\,X & \text{if } -1024 \leq X \leq 1024, \\ 1319\,X - 291 & \text{if } -2048 \leq X < -1024. \end{cases}$$

- $\arctan(X)$ is a signed integer ranging $-1,350,947 \cdots +1,350,947$, which in hex is $-149\mathrm{D}23_{\mathrm{h}} \cdots +149\mathrm{D}23_{\mathrm{h}}$
  - How many bits are needed to represent $\arctan(X)$?
- Questions to be posed:
  - What is the computing time in an all-software implementation versus in a hardware implementation?
  - What is the precision of the piecewise linear approximation when integer / fixed-point arithmetic is used?
- The same considerations can be extended to sine and cosine

# $\arctan(x)$ – piecewise linear approximation III

- Calculating the arctangent (that is, the rotation angles) with low accuracy amounts with a partial rotation, which slightly slows the convergence of the SVD algorithm
- With what accuracy should sine and cosine be calculated?
  - In any case with equal accuracy, or the rotation will no longer be a rotation
- Low (but equal) accuracy for sine and cosine amounts with a partial rotation, which is acceptable as discussed
- Maybe a better idea would be to restrict the rotation angles to a set of predefined angles, whose arctangent, sine, and cosine functions can be either easily calculated or prestored in memory
  - The student(s) doing the SVD project should consult the student(s) doing the CORDIC project

# Jacobi method – side effects

- It works fine with rectangular matrices, too.
- If the matrix is symmetric, the algorithm finds the eigenvalues.
- The SVD of a complex matrix can be calculated in a similar way
  - This is beyond the course scope
- Matrix triangularization can be achieved with one-side rotations
  - Upper triangularization with left-side rotations
  - Lower triangularization with right-side rotations
- The algorithm can be parallelized, which will increase the speed of the algorithm implementation
  - SIMD (vector) operations are good candidates for this

# Matrix diagonalization – project requirements I

- Generate a square matrix of integers (12-bit wordlength, for example)
- Consider the piecewise linear approximation for $\arctan$, $\sin$, and $\cos$, and determine the maximum error for three middle points
- Implement piecewise linear approximation using integer arithmetic for $\sin$, $\cos$, and $\arctan$ in:
  - software (write C routines)
  - horizontal firmware with two issue slots
  - custom hardware (write VHDL/Verilog)
- Define a new instruction that will return the trigonometric function
  - You must comply with the ARM architecture (you can have at most two arguments and one result per instruction call)

# Matrix diagonalization – project requirements II

- Rewrite the high-level code and instantiate the new instruction
  - Use assembly inlining
- Diagonalize a square matrix using piecewise linear approximation of trigonometric functions and estimate:
  - the performance improvement of hardware-based solution versus software-based solution
  - the performance improvement of a 2-issue slot firmware-based solution versus software-based solution
- Estimate the penalty for the hardware solution
  - Determine the number of gates needed to implement the new instruction

# Appendix – SVD numerical example

- The matrix for which SVD is to be calculated is $M$

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{U} \cdot \underbrace{\begin{pmatrix} 31 & 77 & -11 & 26 \\ -42 & 14 & 79 & -53 \\ -68 & -10 & 45 & 90 \\ 34 & 16 & 38 & -19 \end{pmatrix}}_{M} \cdot \underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{V^T}$$

- In the initial state of the algorithm $\Sigma = M$ and $U = V = I$
- Plane rotations will be applied to diagonalize $M$
- These plane rotations will be incorporated into $U$ and $V$

 

# SVD – Iteration 1, Sweep 1: pair $(1-2)$ is selected I

- Elements on Rows and Columns 1 and 2 are selected to form a $2 \times 2$ matrix
- The left and right rotation angles, $\theta_{l,12}$ and $\theta_{r,12}$ are calculated

$$\underbrace{\begin{pmatrix} \cos\theta_{l,12} & -\sin\theta_{l,12} \\ \sin\theta_{l,12} & \cos\theta_{l,12} \end{pmatrix}}_{R(\theta_{l,12})} \cdot \underbrace{\begin{pmatrix} 31 & 77 \\ -42 & 14 \end{pmatrix}}_{M(1\&2,1\&2)} \cdot \underbrace{\begin{pmatrix} \cos\theta_{r,12} & \sin\theta_{r,12} \\ -\sin\theta_{r,12} & \cos\theta_{r,12} \end{pmatrix}}_{R(\theta_{r,12})^T} = \begin{pmatrix} 83.067 & 0 \\ 0 & 44.157 \end{pmatrix}$$

$$\left.\begin{aligned} \theta_{\mathrm{SUM},12} &= \theta_{r,12} + \theta_{l,12} = \arctan\left(\frac{-42+77}{14-31}\right) = -1.11860 \text{ rad} \\ \theta_{\mathrm{DIFF},12} &= \theta_{r,12} - \theta_{l,12} = \arctan\left(\frac{-42-77}{14+31}\right) = -1.20930 \text{ rad} \end{aligned}\right\} \Rightarrow \begin{cases} \theta_{l,12} = \phantom{-}0.04531 \text{ rad} \\ \theta_{r,12} = -1.16400 \text{ rad} \end{cases}$$

$$\begin{aligned} \cos\theta_{l,12} &= 0.99897 & \cos\theta_{r,12} &= \phantom{-}0.39571 \\ \sin\theta_{l,12} &= 0.04530 & \sin\theta_{r,12} &= -0.91838 \end{aligned}$$

# SVD – Iteration 1, Sweep 1: pair $(1-2)$ is selected II

- The orthogonal matrix $U_{12}$ is built with the left rotation angle, $\theta_{l,12}$
- The orthogonal matrix $V_{12}$ is built with the right rotation angle, $\theta_{r,12}$
- The left and right rotations are applied to $M$ to give $M'$
- The matrices $U_{12}$ and $V_{12}$ are incorporated into $U$ and $V$ to give $U'$ and $V'$
- $U$, $M$, and $V$ are updated with $U'$, $M'$, and $V'$, to be used in the next iteration

$$U_{12} = \begin{pmatrix} \cos\theta_{l,12} & -\sin\theta_{l,12} & 0 & 0 \\ \sin\theta_{l,12} & \cos\theta_{l,12} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad V_{12} = \begin{pmatrix} \cos\theta_{r,12} & -\sin\theta_{r,12} & 0 & 0 \\ \sin\theta_{r,12} & \cos\theta_{r,12} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$U \cdot M \cdot V^T = \underbrace{U \cdot U_{12}^T}_{U'} \cdot \underbrace{U_{12} \cdot M \cdot V_{12}^T}_{M'} \cdot \underbrace{V_{12} \cdot V^T}_{V'^T}$$

$$U = U' \qquad M = M' \qquad V = V'$$

# SVD – Iteration 1, Sweep 1: pair $(1-2)$ is selected III

- The matrices $U$, $M$, and $V$ at the end of iteration $(1-2)$ are shown below
- The off-diagonal energy accumulates on the main diagonal
  - The magnitudes of the off-diagonal elements decrease
  - The magnitudes of the diagonal elements increase
- The elements of the orthogonal matrices $U$ and $V$ range from -1.0 to 1.0

$$U = \begin{pmatrix} 0.99897 & 0.04530 & 0 & 0 \\ -0.04530 & 0.99897 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad V = \begin{pmatrix} 0.39571 & -0.91838 & 0 & 0 \\ 0.91838 & 0.39571 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$M = \begin{pmatrix} 83.067 & 0 & -14.567 & 28.374 \\ 0 & 44.157 & 78.421 & -51.768 \\ -36.092 & 58.492 & 45 & 90 \\ 28.148 & -24.893 & 38 & -19 \end{pmatrix}$$

# SVD – Iteration 2, Sweep 1: pair $(1-3)$ is selected I

- Elements on Rows and Columns 1 and 3 are selected to form a $2 \times 2$ matrix
- The left and right rotation angles, $\theta_{l,13}$ and $\theta_{r,13}$ are calculated

$$\underbrace{\begin{pmatrix} \cos\theta_{l,13} & -\sin\theta_{l,13} \\ \sin\theta_{l,13} & \cos\theta_{l,13} \end{pmatrix}}_{R(\theta_{l,13})} \cdot \underbrace{\begin{pmatrix} 83.067 & -14.567 \\ -36.092 & 45 \end{pmatrix}}_{M(1\&3,1\&3)} \cdot \underbrace{\begin{pmatrix} \cos\theta_{r,13} & \sin\theta_{r,13} \\ -\sin\theta_{r,13} & \cos\theta_{r,13} \end{pmatrix}}_{R(\theta_{r,13})^T} = \begin{pmatrix} 83.067 & 0 \\ 0 & 44.157 \end{pmatrix}$$

$$\left.\begin{array}{l} \theta_{\mathrm{SUM},13} = \theta_{r,13} + \theta_{l,13} = \arctan\left(\dfrac{-36.092 - 14.567}{45 - 83.067}\right) = 0.92638 \text{ rad} \\ \theta_{\mathrm{DIFF},13} = \theta_{r,13} - \theta_{l,13} = \arctan\left(\dfrac{-36.092 + 14.567}{45 + 83.067}\right) = -0.16652 \text{ rad} \end{array}\right\} \Rightarrow \begin{cases} \theta_{l,13} = 0.54645 \text{ rad} \\ \theta_{r,13} = 0.37793 \text{ rad} \end{cases}$$

$$\begin{array}{ll} \cos\theta_{l,13} = 0.85438 & \cos\theta_{r,13} = 0.92869 \\ \sin\theta_{l,13} = 0.51966 & \sin\theta_{r,13} = 0.37085 \end{array}$$

# SVD – Iteration 2, Sweep 1: pair $(1-3)$ is selected II

- The orthogonal matrix $U_{13}$ is built with the left rotation angle, $\theta_{l,13}$
- The orthogonal matrix $V_{13}$ is built with the right rotation angle, $\theta_{r,13}$
- The left and right rotations are applied to $M$ to give $M'$
- The matrices $U_{13}$ and $V_{13}$ are incorporated into $U$ and $V$ to give $U'$ and $V'$
- $U$, $M$, and $V$ are updated with $U'$, $M'$, and $V'$, to be used in the next iteration

$$U_{13} = \begin{pmatrix} \cos\theta_{l,13} & 0 & -\sin\theta_{l,13} & 0 \\ 0 & 1 & 0 & 0 \\ \sin\theta_{l,13} & 0 & \cos\theta_{l,13} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad V_{13} = \begin{pmatrix} \cos\theta_{r,13} & 0 & -\sin\theta_{r,13} & 0 \\ 0 & 1 & 0 & 0 \\ \sin\theta_{r,13} & 0 & \cos\theta_{r,13} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$U \cdot M \cdot V^T = \underbrace{U \cdot U_{13}^T}_{U'} \cdot \underbrace{U_{13} \cdot M \cdot V_{13}^T}_{M'} \cdot \underbrace{V_{13} \cdot V^T}_{V'^T}$$

$$U = U' \qquad M = M' \qquad V = V'$$

# SVD – Iteration 2, Sweep 1: pair $(1-3)$ is selected III

- The matrices $U$, $M$, and $V$ at the end of iteration $(1-3)$ are shown below
- The off-diagonal energy accumulates on the main diagonal
  - The magnitudes of the off-diagonal elements decrease
  - The magnitudes of the diagonal elements increase
- The elements of the orthogonal matrices $U$ and $V$ range from -1.0 to 1.0

$$U = \begin{pmatrix} 0.85350 & 0.04530 & 0.51912 & 0 \\ -0.03870 & 0.99897 & -0.02354 & 0 \\ -0.51966 & 0 & 0.85438 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad V = \begin{pmatrix} 0.36749 & -0.91837 & 0.14675 & 0 \\ 0.85289 & 0.39571 & 0.34058 & 0 \\ -0.37085 & 0 & 0.92869 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$M = \begin{pmatrix} 96.616 & -30.396 & 0 & -22.527 \\ -29.083 & 44.157 & 72.829 & -51.768 \\ 0 & 49.975 & 33.248 & 91.638 \\ 12.049 & -24.893 & 45.729 & -19 \end{pmatrix}$$

# SVD – Iteration 3, Sweep 1: pair $(1-4)$ is selected I

- Elements on Rows and Columns 1 and 4 are selected to form a $2 \times 2$ matrix
- The left and right rotation angles, $\theta_{l,14}$ and $\theta_{r,14}$ are calculated

$$\underbrace{\begin{pmatrix} \cos\theta_{l,14} & -\sin\theta_{l,14} \\ \sin\theta_{l,14} & \cos\theta_{l,14} \end{pmatrix}}_{R(\theta_{l,14})} \cdot \underbrace{\begin{pmatrix} 96.616 & -22.527 \\ 12.049 & -19 \end{pmatrix}}_{M(1\&4,1\&4)} \cdot \underbrace{\begin{pmatrix} \cos\theta_{r,14} & \sin\theta_{r,14} \\ -\sin\theta_{r,14} & \cos\theta_{r,14} \end{pmatrix}}_{R(\theta_{r,14})^T} = \begin{pmatrix} 100.529 & 0 \\ 0 & -15.561 \end{pmatrix}$$

$$\left.\begin{aligned} \theta_{\mathrm{SUM},14} = \theta_{r,14} + \theta_{l,14} = \arctan\left(\frac{12.049 - 22.527}{-19 - 96.616}\right) = 0.09039 \text{ rad} \\ \theta_{\mathrm{DIFF},14} = \theta_{r,14} - \theta_{l,14} = \arctan\left(\frac{12.049 + 22.527}{-19 + 96.616}\right) = 0.41908 \text{ rad} \end{aligned}\right\} \Rightarrow \begin{cases} \theta_{l,14} = -0.16435 \text{ rad} \\ \theta_{r,14} = \quad 0.25473 \text{ rad} \end{cases}$$

$$\begin{aligned} \cos\theta_{l,14} &= \quad 0.98653 & \cos\theta_{r,14} &= 0.96773 \\ \sin\theta_{l,14} &= -0.16361 & \sin\theta_{r,14} &= 0.25199 \end{aligned}$$

# SVD – Iteration 3, Sweep 1: pair $(1-4)$ is selected II

- The orthogonal matrix $U_{14}$ is built with the left rotation angle, $\theta_{l,14}$
- The orthogonal matrix $V_{14}$ is built with the right rotation angle, $\theta_{r,14}$
- The left and right rotations are applied to $M$ to give $M'$
- The matrices $U_{14}$ and $V_{14}$ are incorporated into $U$ and $V$ to give $U'$ and $V'$
- $U$, $M$, and $V$ are updated with $U'$, $M'$, and $V'$, to be used in the next iteration

$$U_{14} = \begin{pmatrix} \cos\theta_{l,14} & 0 & 0 & -\sin\theta_{l,14} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sin\theta_{l,14} & 0 & 0 & \cos\theta_{l,14} \end{pmatrix} \qquad V_{14} = \begin{pmatrix} \cos\theta_{r,14} & 0 & 0 & -\sin\theta_{r,14} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sin\theta_{r,14} & 0 & 0 & \cos\theta_{r,14} \end{pmatrix}$$

$$U \cdot M \cdot V^T = \underbrace{U \cdot U_{14}^T}_{U'} \cdot \underbrace{U_{14} \cdot M \cdot V_{14}^T}_{M'} \cdot \underbrace{V_{14} \cdot V^T}_{V'^T}$$

$$U = U' \qquad M = M' \qquad V = V'$$

# SVD – Iteration 3, Sweep 1: pair $(1-4)$ is selected III

- The matrices $U$, $M$, and $V$ at the end of iteration $(1-4)$ are shown below
- The off-diagonal energy accumulates on the main diagonal
  - The magnitudes of the off-diagonal elements decrease
  - The magnitudes of the diagonal elements increase
- The elements of the orthogonal matrices $U$ and $V$ range from -1.0 to 1.0

$$U = \begin{pmatrix} 0.84200 & 0.04530 & 0.51912 & -0.13964 \\ -0.03818 & 0.99897 & -0.02354 & 0.00633 \\ -0.51265 & 0 & 0.85438 & 0.08502 \\ 0.16361 & 0 & 0 & 0.98653 \end{pmatrix} \quad V = \begin{pmatrix} 0.35563 & -0.91837 & 0.14675 & 0.09260 \\ 0.82536 & 0.39571 & 0.34058 & 0.21492 \\ -0.35889 & 0 & 0.92869 & -0.09345 \\ -0.25199 & 0 & 0 & 0.96773 \end{pmatrix}$$

$$M = \begin{pmatrix} 100.529 & -34.059 & 7.482 & 0 \\ -15.099 & 44.157 & 72.829 & -57.426 \\ -23.092 & 49.974 & 33.248 & 88.681 \\ 0 & -19.585 & 45.113 & -15.560 \end{pmatrix}$$

# SVD – Iteration 3, Sweep 1: pair $(2-3)$ is selected I

- Elements on Rows and Columns 2 and 3 are selected to form a $2 \times 2$ matrix
- The left and right rotation angles, $\theta_{l,23}$ and $\theta_{r,23}$ are calculated

$$\underbrace{\begin{pmatrix} \cos\theta_{l,23} & -\sin\theta_{l,23} \\ \sin\theta_{l,23} & \cos\theta_{l,23} \end{pmatrix}}_{R(\theta_{l,23})} \cdot \underbrace{\begin{pmatrix} 44.157 & 72.829 \\ 49.974 & 33.248 \end{pmatrix}}_{M(2\&3,2\&3)} \cdot \underbrace{\begin{pmatrix} \cos\theta_{r,23} & \sin\theta_{r,23} \\ -\sin\theta_{r,23} & \cos\theta_{r,23} \end{pmatrix}}_{R(\theta_{r,23})^T} = \begin{pmatrix} 101.998 & 0 \\ 0 & -21.289 \end{pmatrix}$$

$$\left.\begin{array}{l} \theta_{\mathrm{SUM},23} = \theta_{r,23} + \theta_{l,23} = \arctan\left(\dfrac{49.974+72.829}{33.248-44.157}\right) = -1.48220\ \mathrm{rad} \\[2ex] \theta_{\mathrm{DIFF},23} = \theta_{r,23} - \theta_{l,23} = \arctan\left(\dfrac{49.974-72.829}{33.248+44.157}\right) = -0.28710\ \mathrm{rad} \end{array}\right\} \Rightarrow \begin{cases} \theta_{l,23} = -0.59755\ \mathrm{rad} \\ \theta_{r,23} = -0.88465\ \mathrm{rad} \end{cases}$$

$$\begin{aligned} \cos\theta_{l,23} &= \phantom{-}0.82672 & \cos\theta_{r,23} &= \phantom{-}0.63356 \\ \sin\theta_{l,23} &= -0.56262 & \sin\theta_{r,23} &= -0.77369 \end{aligned}$$

# SVD – Iteration 3, Sweep 1: pair $(2-3)$ is selected II

- The orthogonal matrix $U_{23}$ is built with the left rotation angle, $\theta_{l,23}$
- The orthogonal matrix $V_{23}$ is built with the right rotation angle, $\theta_{r,23}$
- The left and right rotations are applied to $M$ to give $M'$
- The matrices $U_{23}$ and $V_{23}$ are incorporated into $U$ and $V$ to give $U'$ and $V'$
- $U$, $M$, and $V$ are updated with $U'$, $M'$, and $V'$, to be used in the next iteration

$$U_{23} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta_{l,23} & -\sin\theta_{l,23} & 0 \\ 0 & \sin\theta_{l,23} & \cos\theta_{l,23} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad V_{23} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta_{r,23} & -\sin\theta_{r,23} & 0 \\ 0 & \sin\theta_{r,23} & \cos\theta_{r,23} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$U \cdot M \cdot V^T = \underbrace{U \cdot U_{23}^T}_{U'} \cdot \underbrace{U_{23} \cdot M \cdot V_{23}^T}_{M'} \cdot \underbrace{V_{23} \cdot V^T}_{V'^T}$$

$$U = U' \qquad M = M' \qquad V = V'$$

# SVD – Iteration 3, Sweep 1: pair $(2-3)$ is selected III

- The matrices $U$, $M$, and $V$ at the end of iteration $(2-3)$ are shown below
- The off-diagonal energy accumulates on the main diagonal
  - The magnitudes of the off-diagonal elements decrease
  - The magnitudes of the diagonal elements increase
- The elements of the orthogonal matrices $U$ and $V$ range from -1.0 to 1.0

$$U = \begin{pmatrix} 0.84200 & 0.32951 & 0.40368 & -0.13964 \\ -0.03818 & 0.81263 & -0.58150 & 0.00633 \\ -0.51265 & 0.48069 & 0.70633 & 0.08502 \\ 0.16361 & 0 & 0 & 0.98653 \end{pmatrix} \quad V = \begin{pmatrix} 0.35563 & -0.46831 & 0.80351 & 0.09260 \\ 0.82536 & 0.51421 & -0.09038 & 0.21492 \\ -0.35889 & 0.71852 & 0.58838 & -0.09345 \\ -0.25199 & 0 & 0 & 0.96773 \end{pmatrix}$$

$$M = \begin{pmatrix} 100.529 & -15.790 & 31.091 & 0 \\ -25.475 & 101.998 & 0 & 2.419 \\ -10.595 & 0 & -21.289 & 105.623 \\ 0 & 22.495 & 43.735 & -15.560 \end{pmatrix}$$

# SVD – Iteration 3, Sweep 1: pair $(2-4)$ is selected I

- Elements on Rows and Columns 2 and 4 are selected to form a $2 \times 2$ matrix
- The left and right rotation angles, $\theta_{l,24}$ and $\theta_{r,24}$ are calculated

$$\underbrace{\begin{pmatrix} \cos\theta_{l,24} & -\sin\theta_{l,24} \\ \sin\theta_{l,24} & \cos\theta_{l,24} \end{pmatrix}}_{R(\theta_{l,24})} \cdot \underbrace{\begin{pmatrix} 101.998 & 2.419 \\ 22.495 & -15.560 \end{pmatrix}}_{M(2\&4,2\&4)} \cdot \underbrace{\begin{pmatrix} \cos\theta_{r,24} & \sin\theta_{r,24} \\ -\sin\theta_{r,24} & \cos\theta_{r,24} \end{pmatrix}}_{R(\theta_{r,24})^T} = \begin{pmatrix} 104.453 & 0 \\ 0 & -15.716 \end{pmatrix}$$

$$\left.\begin{array}{l} \theta_{\mathrm{SUM},24} = \theta_{r,24} + \theta_{l,24} = \arctan\left(\dfrac{22.495 + 2.419}{-15.560 - 101.998}\right) = -0.20884 \text{ rad} \\ \theta_{\mathrm{DIFF},24} = \theta_{r,24} - \theta_{l,24} = \arctan\left(\dfrac{22.495 - 2.419}{-15.560 + 101.998}\right) = \phantom{-}0.22822 \text{ rad} \end{array}\right\} \Rightarrow \left\{\begin{array}{l} \theta_{l,24} = -0.21853 \text{ rad} \\ \theta_{r,24} = \phantom{-}0.00969 \text{ rad} \end{array}\right.$$

$$\begin{array}{ll} \cos\theta_{l,24} = \phantom{-}0.97622 & \cos\theta_{r,24} = 0.99995 \\ \sin\theta_{l,24} = -0.21679 & \sin\theta_{r,24} = 0.00969 \end{array}$$

# SVD – Iteration 3, Sweep 1: pair $(2-4)$ is selected II

- The orthogonal matrix $U_{24}$ is built with the left rotation angle, $\theta_{l,24}$
- The orthogonal matrix $V_{24}$ is built with the right rotation angle, $\theta_{r,24}$
- The left and right rotations are applied to $M$ to give $M'$
- The matrices $U_{24}$ and $V_{24}$ are incorporated into $U$ and $V$ to give $U'$ and $V'$
- $U$, $M$, and $V$ are updated with $U'$, $M'$, and $V'$, to be used in the next iteration

$$U_{24} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta_{l,24} & 0 & -\sin\theta_{l,24} \\ 0 & 0 & 1 & 0 \\ 0 & \sin\theta_{l,24} & 0 & \cos\theta_{l,24} \end{pmatrix} \qquad V_{24} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta_{r,24} & 0 & -\sin\theta_{r,24} \\ 0 & 0 & 1 & 0 \\ 0 & \sin\theta_{r,24} & 0 & \cos\theta_{r,24} \end{pmatrix}$$

$$U \cdot M \cdot V^T = \underbrace{U \cdot U_{24}^T}_{U'} \cdot \underbrace{U_{24} \cdot M \cdot V_{24}^T}_{M'} \cdot \underbrace{V_{24} \cdot V^T}_{V'^T}$$

$$U = U' \qquad M = M' \qquad V = V'$$

# SVD – Iteration 3, Sweep 1: pair $(2-4)$ is selected III

- The matrices $U$, $M$, and $V$ at the end of iteration $(2-4)$ are shown below
- The off-diagonal energy accumulates on the main diagonal
  - The magnitudes of the off-diagonal elements decrease
  - The magnitudes of the diagonal elements increase
- The elements of the orthogonal matrices $U$ and $V$ range from -1.0 to 1.0

$$U = \begin{pmatrix} 0.84200 & 0.29140 & 0.40368 & -0.20776 \\ -0.03818 & 0.79467 & -0.58150 & -0.16999 \\ -0.51265 & 0.48769 & 0.70633 & -0.02121 \\ 0.16361 & 0.21387 & 0 & 0.96306 \end{pmatrix} \quad V = \begin{pmatrix} 0.35563 & -0.46831 & 0.80351 & 0.08806 \\ 0.82536 & 0.51211 & -0.09038 & 0.21989 \\ -0.35889 & 0.71939 & 0.58838 & -0.08648 \\ -0.25199 & -0.00938 & 0 & 0.96769 \end{pmatrix}$$

$$M = \begin{pmatrix} 100.529 & -15.789 & 31.091 & -0.153 \\ -24.869 & 104.453 & 9.481 & 0 \\ -10.595 & -1.024 & -21.289 & 105.618 \\ 5.523 & 0 & 42.694 & -15.716 \end{pmatrix}$$

# SVD – Iteration 3, Sweep 1: pair $(3-4)$ is selected I

- Elements on Rows and Columns 3 and 4 are selected to form a $2 \times 2$ matrix
- The left and right rotation angles, $\theta_{l,34}$ and $\theta_{r,34}$ are calculated

$$\underbrace{\begin{pmatrix} \cos\theta_{l,34} & -\sin\theta_{l,34} \\ \sin\theta_{l,34} & \cos\theta_{l,34} \end{pmatrix}}_{R(\theta_{l,34})} \cdot \underbrace{\begin{pmatrix} -21.289 & 105.618 \\ 42.694 & -15.716 \end{pmatrix}}_{M(3\&4,3\&4)} \cdot \underbrace{\begin{pmatrix} \cos\theta_{r,34} & \sin\theta_{r,34} \\ -\sin\theta_{r,34} & \cos\theta_{r,34} \end{pmatrix}}_{R(\theta_{r,34})^T} = \begin{pmatrix} -110.708 & 0 \\ 0 & 37.710 \end{pmatrix}$$

$$\left.\begin{aligned} \theta_{\mathrm{SUM},34} &= \theta_{r,34} + \theta_{l,34} = \arctan\left(\frac{42.694+105.618}{-15.716+21.289}\right) = 1.53320 \text{ rad} \\ \theta_{\mathrm{DIFF},34} &= \theta_{r,34} - \theta_{l,34} = \arctan\left(\frac{42.694-105.618}{-15.716-21.289}\right) = 1.03920 \text{ rad} \end{aligned}\right\} \Rightarrow \begin{cases} \theta_{l,34} = 0.24703 \text{ rad} \\ \theta_{r,34} = 1.28620 \text{ rad} \end{cases}$$

$$\cos\theta_{l,34} = 0.96964 \qquad \cos\theta_{r,34} = 0.28076$$
$$\sin\theta_{l,34} = 0.24452 \qquad \sin\theta_{r,34} = 0.95978$$

# SVD – Iteration 3, Sweep 1: pair $(3-4)$ is selected II

- The orthogonal matrix $U_{34}$ is built with the left rotation angle, $\theta_{l,34}$
- The orthogonal matrix $V_{34}$ is built with the right rotation angle, $\theta_{r,34}$
- The left and right rotations are applied to $M$ to give $M'$
- The matrices $U_{34}$ and $V_{34}$ are incorporated into $U$ and $V$ to give $U'$ and $V'$
- $U$, $M$, and $V$ are updated with $U'$, $M'$, and $V'$, to be used in the next iteration

$$U_{34} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\theta_{l,34} & -\sin\theta_{l,34} \\ 0 & 0 & \sin\theta_{l,34} & \cos\theta_{l,34} \end{pmatrix} \qquad V_{34} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\theta_{r,34} & -\sin\theta_{r,34} \\ 0 & 0 & \sin\theta_{r,34} & \cos\theta_{r,34} \end{pmatrix}$$

$$U \cdot M \cdot V^T = \underbrace{U \cdot U_{34}^T}_{U'} \cdot \underbrace{U_{34} \cdot M \cdot V_{34}^T}_{M'} \cdot \underbrace{V_{34} \cdot V^T}_{V'^T}$$

$$U = U' \qquad M = M' \qquad V = V'$$

# SVD – Iteration 3, Sweep 1: pair $(3-4)$ is selected III

- The matrices $U$, $M$, and $V$ at the end of iteration $(3-4)$ are shown below
- The off-diagonal energy accumulates on the main diagonal
  - The magnitudes of the off-diagonal elements decrease
  - The magnitudes of the diagonal elements increase
- The elements of the orthogonal matrices $U$ and $V$ range from -1.0 to 1.0

$$U = \begin{pmatrix} 0.84200 & 0.29140 & 0.44223 & -0.10274 \\ -0.03818 & 0.79467 & -0.52228 & -0.30702 \\ -0.51265 & 0.48769 & 0.69007 & 0.15214 \\ 0.16361 & 0.21387 & -0.23549 & 0.93383 \end{pmatrix} \quad V = \begin{pmatrix} 0.35563 & -0.46831 & 0.14108 & 0.79592 \\ 0.82536 & 0.51211 & -0.23642 & -0.02501 \\ -0.35889 & 0.71939 & 0.24820 & 0.54044 \\ -0.25199 & -0.00938 & -0.92876 & 0.27169 \end{pmatrix}$$

$$M = \begin{pmatrix} 100.529 & -15.789 & 8.876 & 29.798 \\ -24.869 & 104.453 & 2.662 & 9.100 \\ -11.624 & -0.992 & -110.708 & 0 \\ 2.764 & -0.250 & 0 & 37.710 \end{pmatrix}$$

# Singular Value Decomposition – multiple sweeps

$$M = \begin{pmatrix} 31 & 77 & -11 & 26 \\ -42 & 14 & 79 & -53 \\ -68 & -10 & 45 & 90 \\ 34 & 16 & 38 & -19 \end{pmatrix}$$

$$M_1 = \begin{pmatrix} 100.529 & -15.789 & 8.876 & 29.798 \\ -24.869 & 104.453 & 2.662 & 9.100 \\ -11.624 & -0.992 & -110.708 & 0 \\ 2.764 & -0.250 & 0 & 37.710 \end{pmatrix}$$

$$M_2 = \begin{pmatrix} 85.636 & 1.354 & 0.696 & 0.160 \\ -4.998 & 126.299 & 0.027 & 0 \\ -1.134 & 0.190 & -110.873 & 0 \\ -0.265 & 0 & 0 & 34.008 \end{pmatrix}$$

$$M_3 = \begin{pmatrix} 85.570 & 0.009 & 0 & 0 \\ 0 & 126.429 & 0 & 0 \\ 0 & 0 & -110.905 & 0 \\ 0 & 0 & 0 & 34.008 \end{pmatrix}$$

$$\Sigma = \begin{pmatrix} 85.570 & 0 & 0 & 0 \\ 0 & 126.429 & 0 & 0 \\ 0 & 0 & -110.905 & 0 \\ 0 & 0 & 0 & 34.008 \end{pmatrix}$$

- The original matrix is $M$
- The matrices $M_1$, $M_2$, and $M_3$ are the matrices at the end of sweeps 1, 2, and 3, respectively
- At the end of Sweep 4 the matrix is completely diagonalized; therefore, it is labeled $\Sigma$
- The algorithm converges fast
- The negative diagonal elements can be forced positive through a simple multiplication
- A final permutation can order the singular values

# References


- Gene H. Golub and Charles F. van Loan, *Matrix Computations*, Johns Hopkins University Press, fourth edition, 2012.
- Lloyd N. Trefethen and David Bau, III, *Numerical Linear Algebra*, Society for Industrial and Applied Mathematics, 1997.
- James W. Demmel, *Applied Numerical Linear Algebra*, Society for Industrial and Applied Mathematics, 1997.
- Carl Meyer, *Matrix Analysis and Applied Linear Algebra*, Society for Industrial and Applied Mathematics, 2001.
- Professor Richard P. Brent (publications on matrix factorization) `http://web.comlab.ox.ac.uk/oucl/work/richard.brent/`

# Questions, feedback

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# Notes I

# Notes II

# Notes III

# Notes IV

# Project Specification Sheet

- **Student name:** ..........................................
- **Student ID:** ..............................................
- **Function to be implemented:** ..............................
- **Argument range:** ..........................................
- **Wordlength:** ..............................................