Lecture 13

Disjoint-Set Data Structure (contd.)

Source: Introduction to Algorithms, CLRS

Disjoint-Sets as Trees

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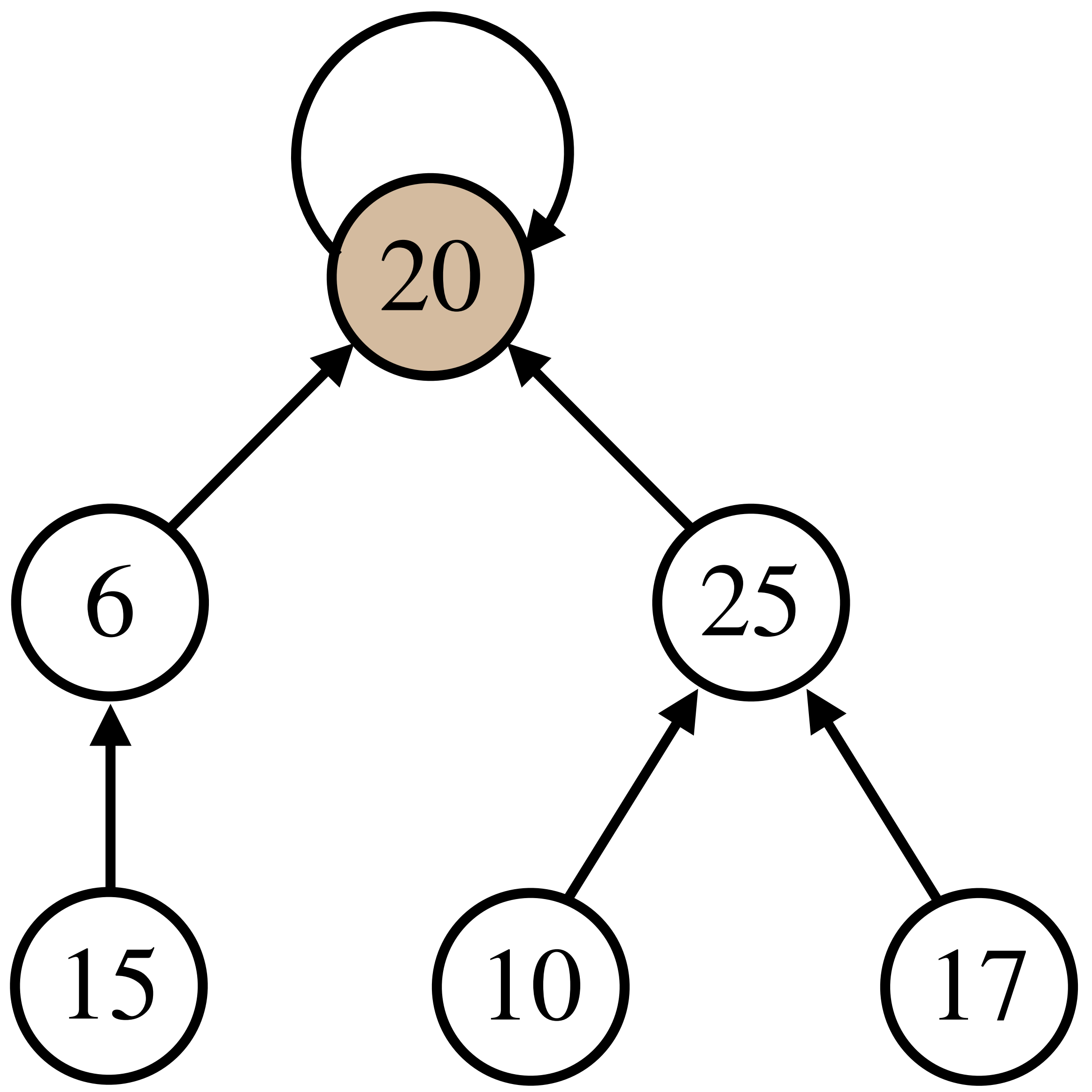

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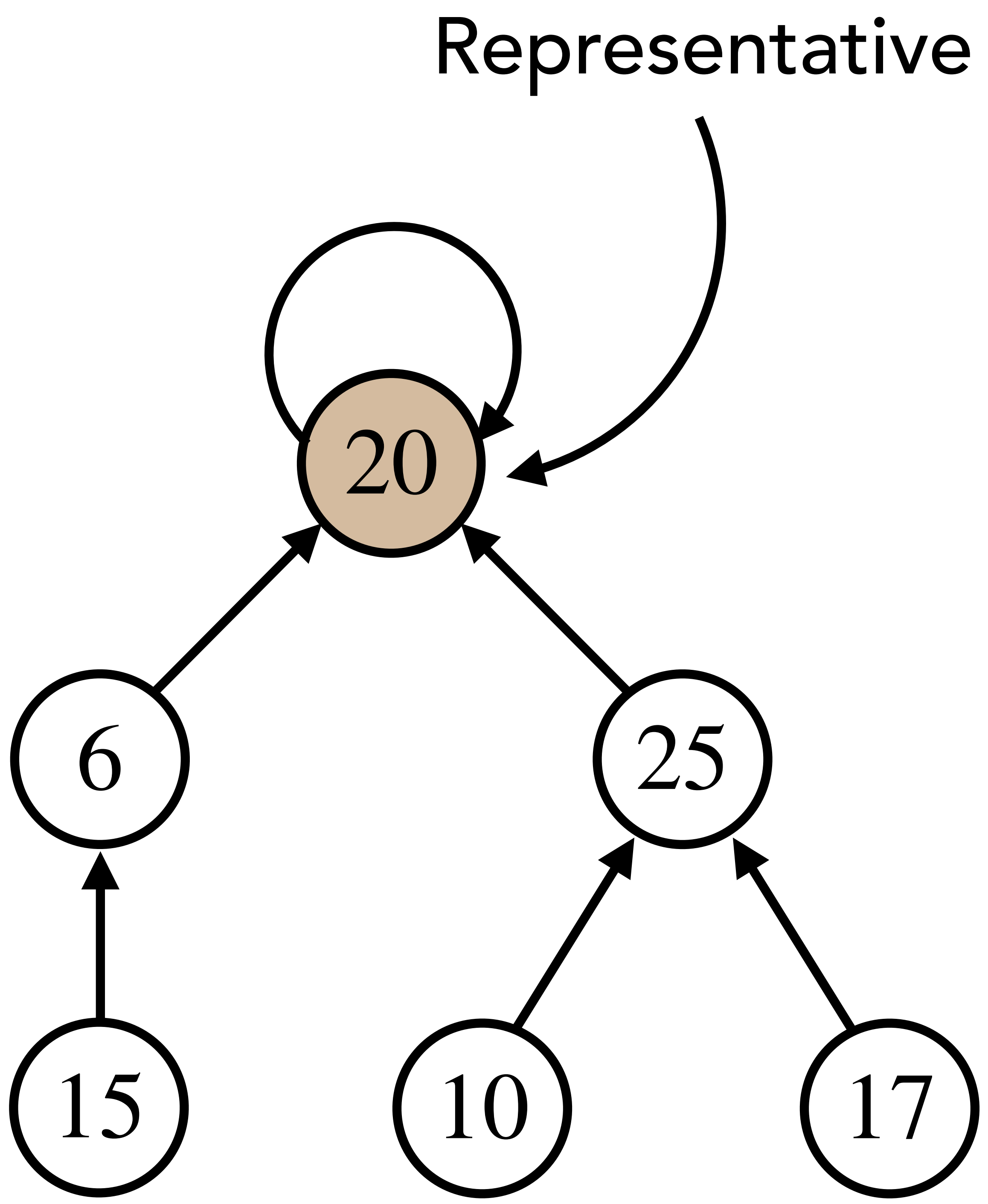

Union on Disjoint-Sets as Trees

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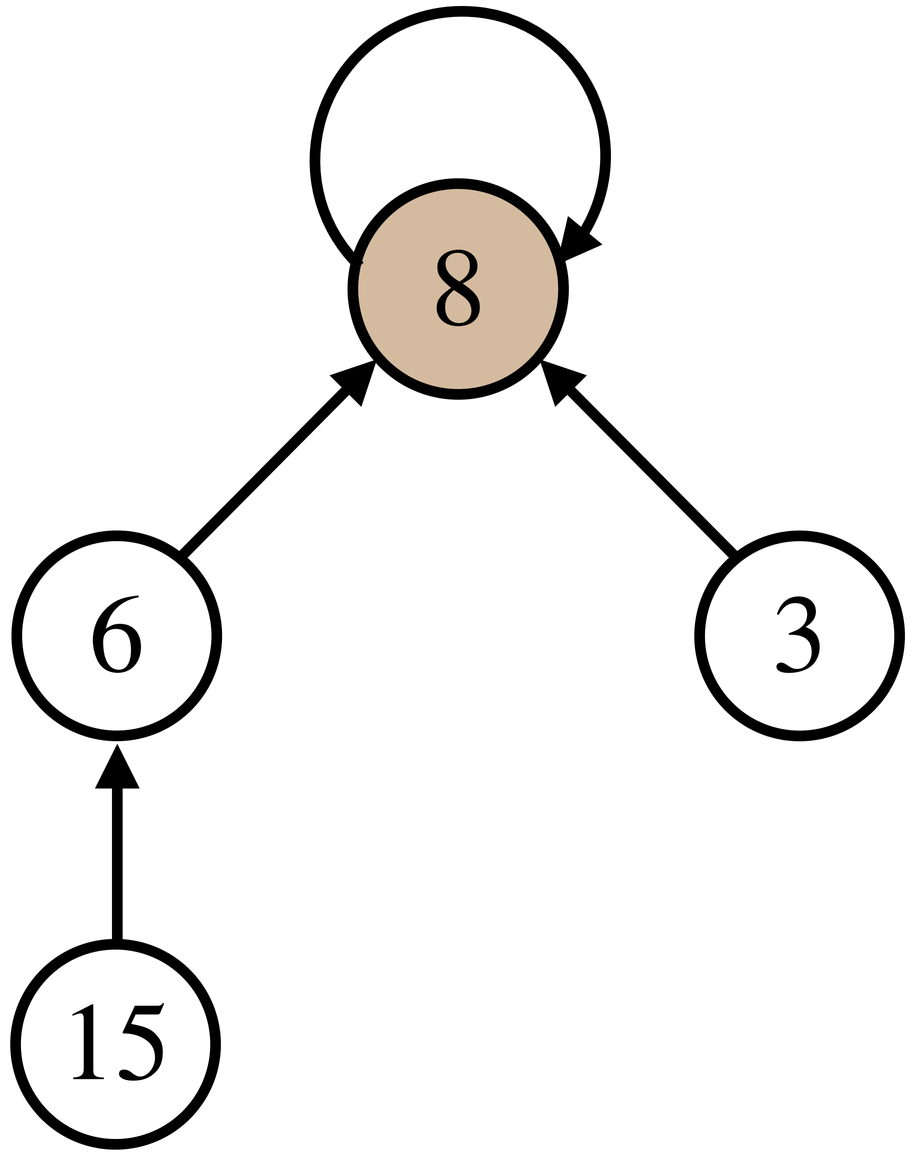

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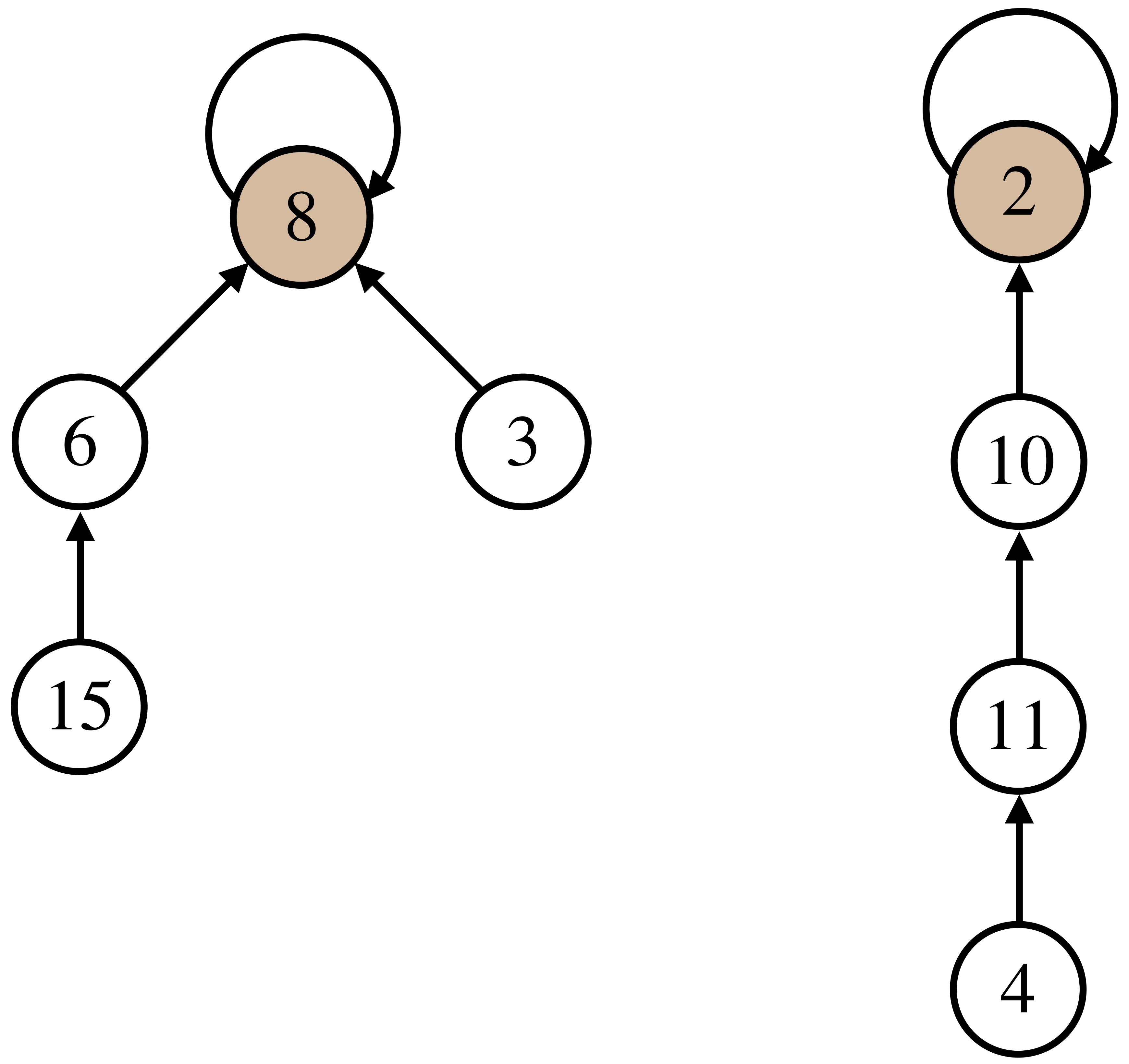

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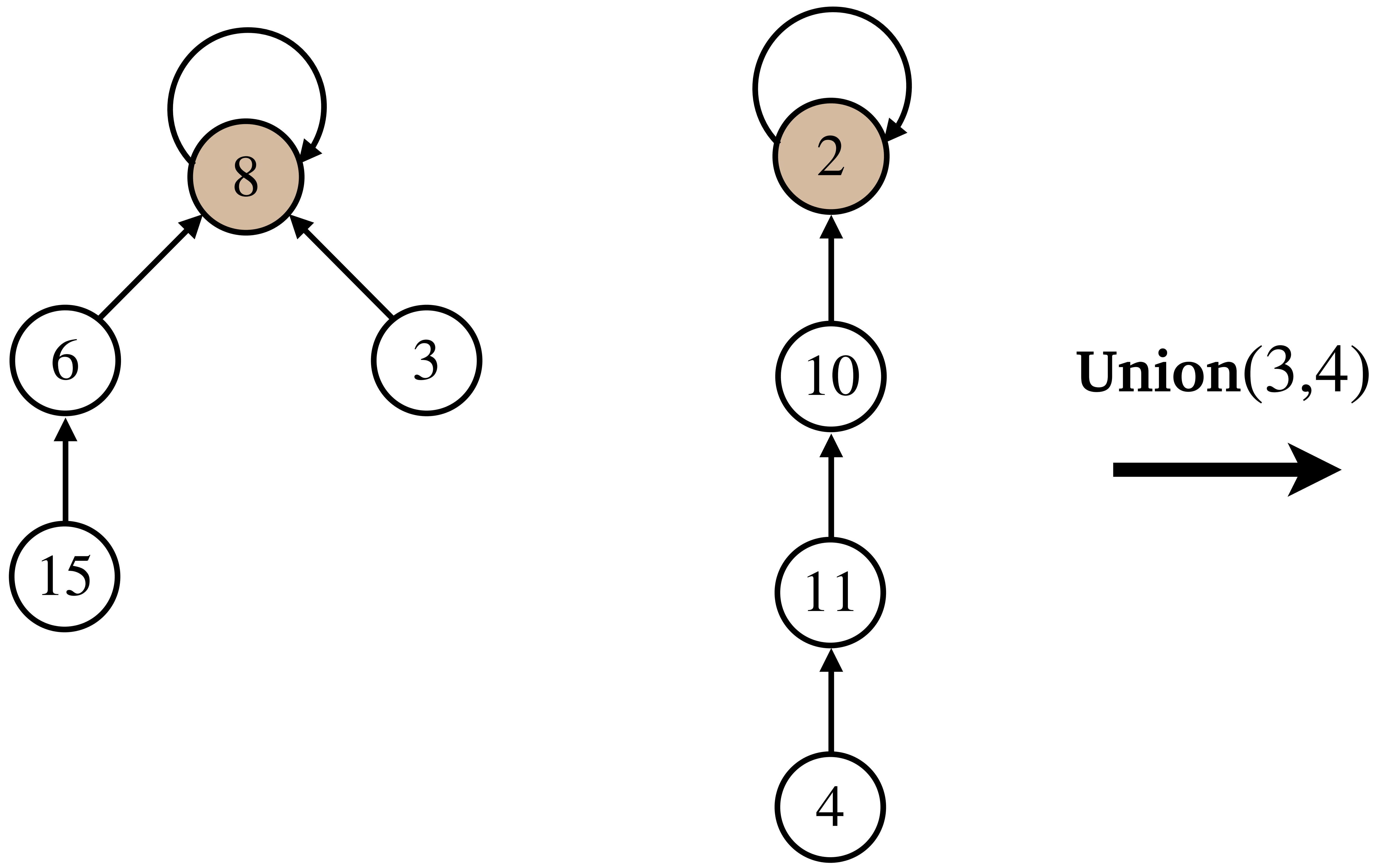

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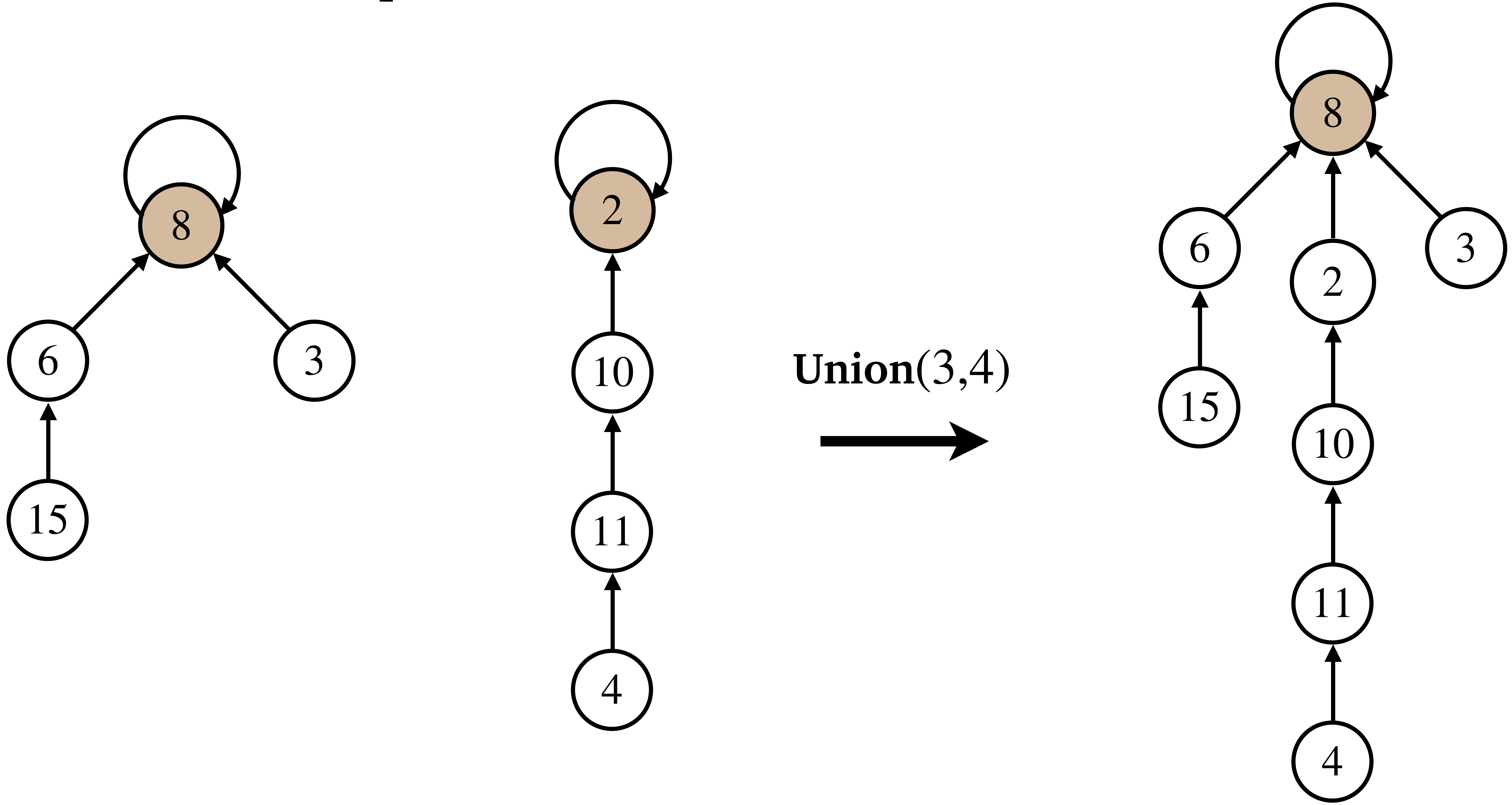

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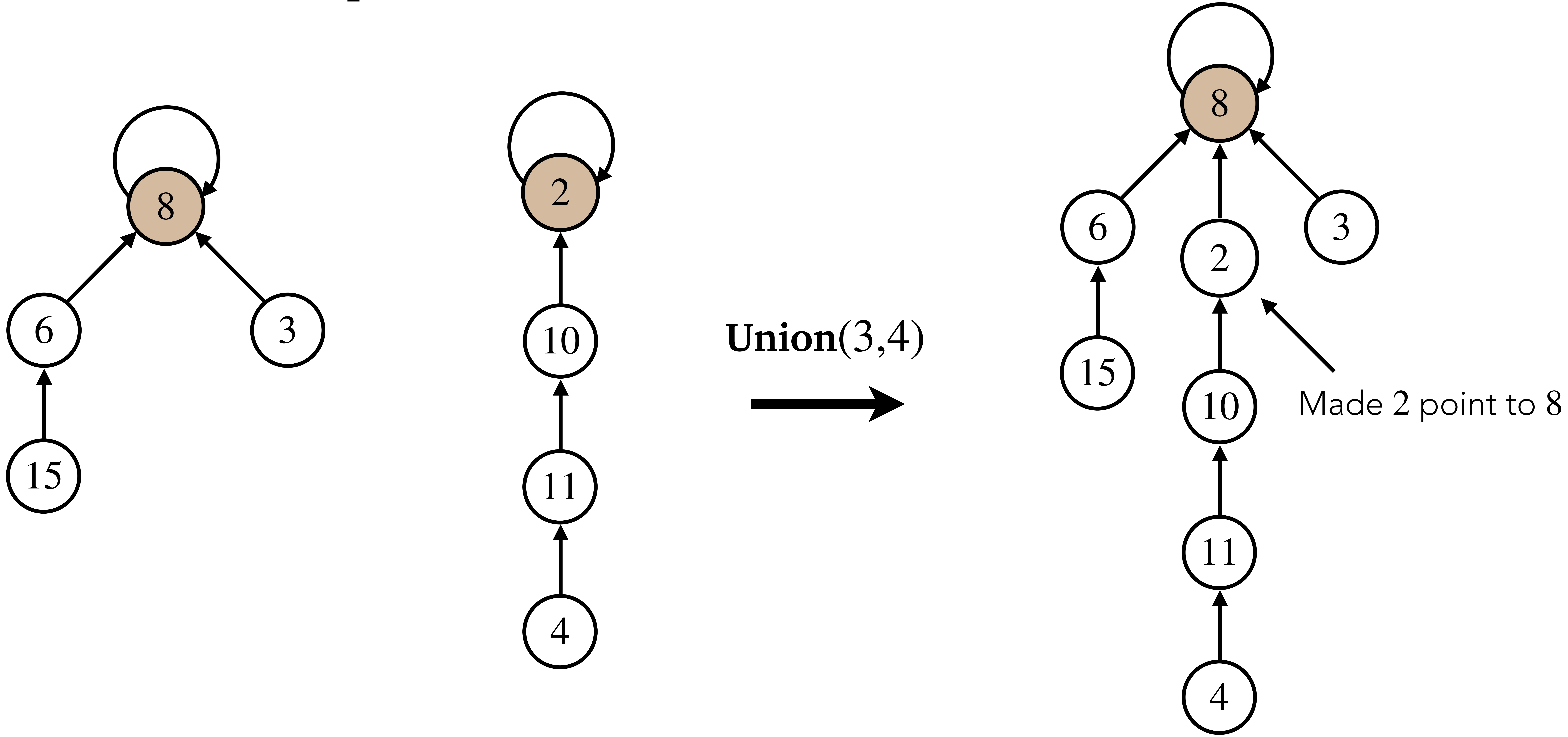

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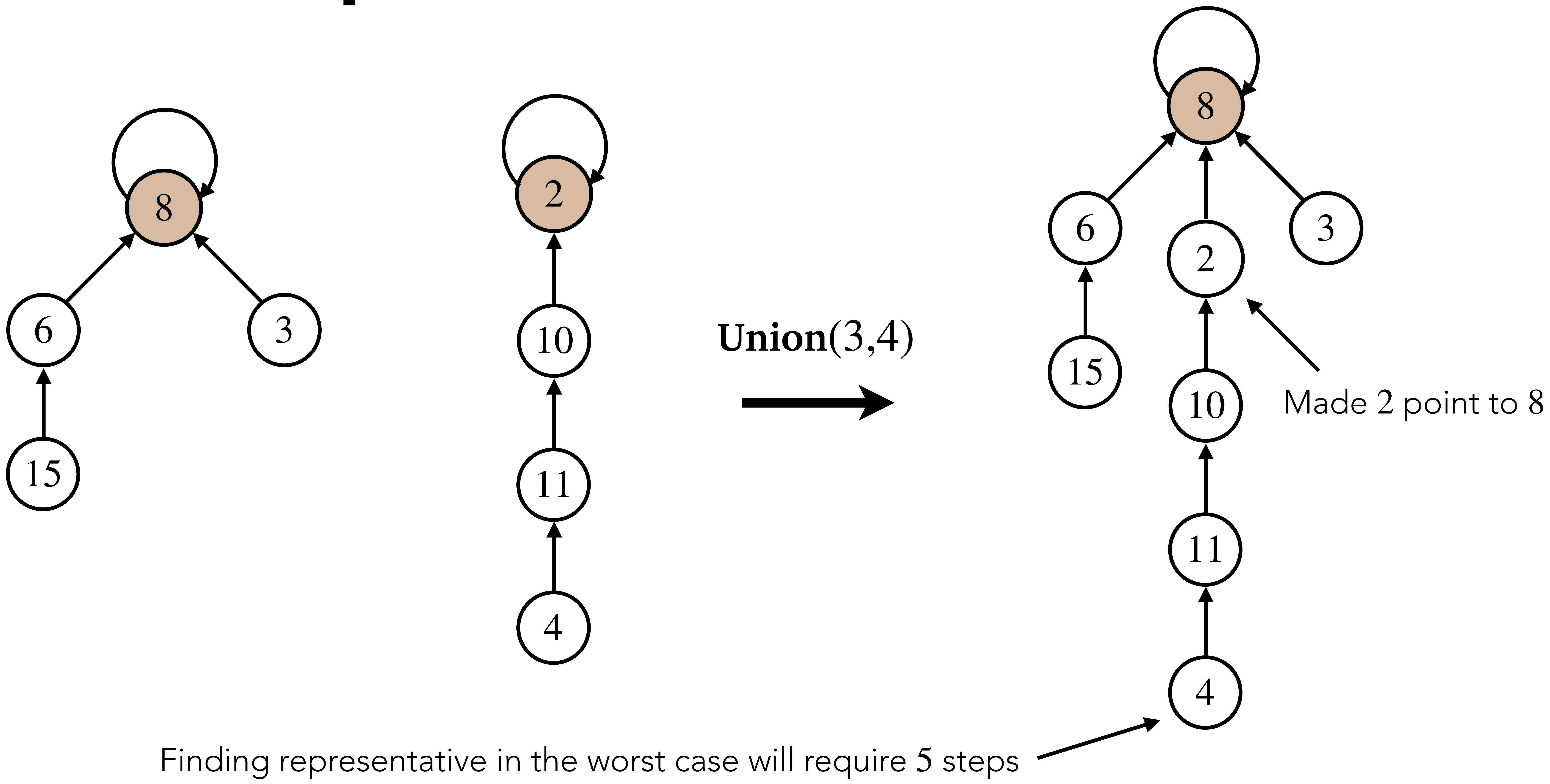

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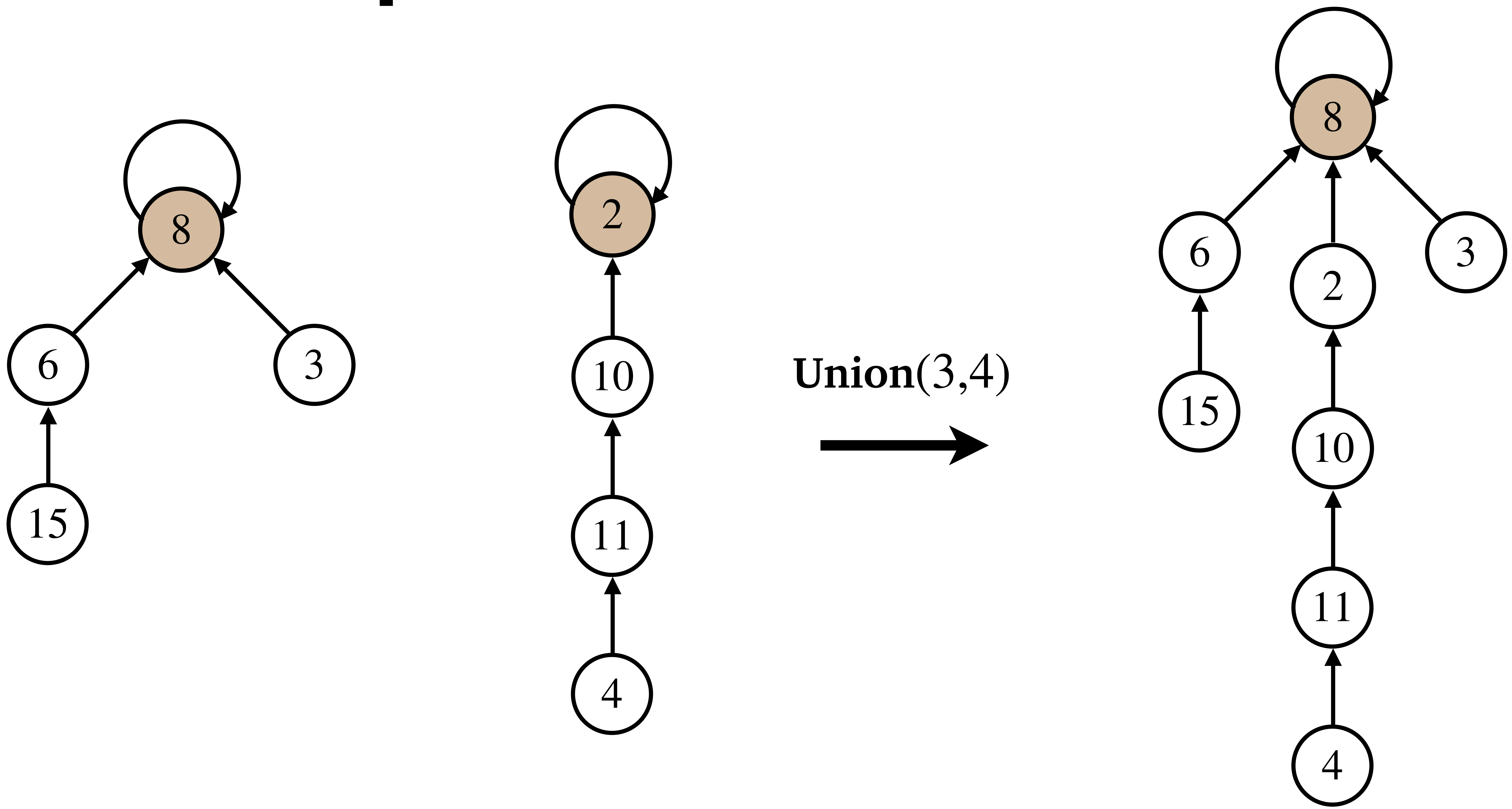
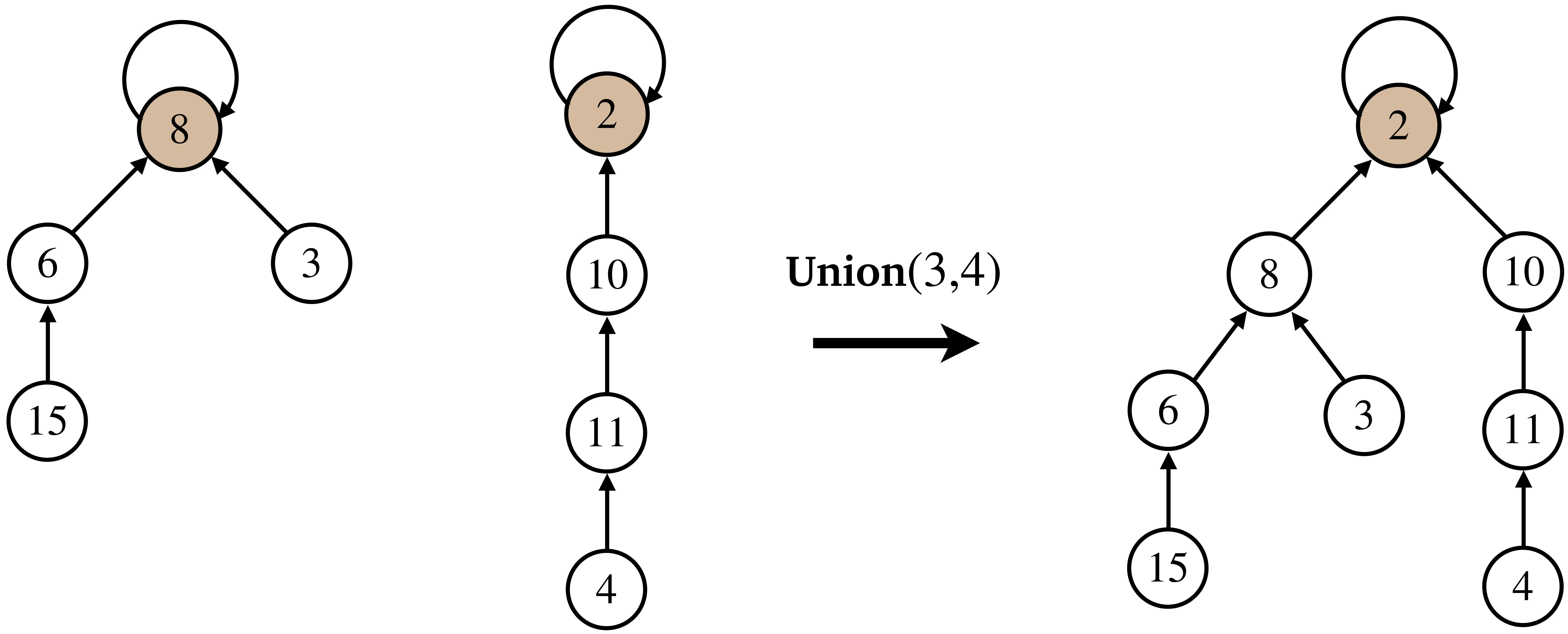


Shouldn't root with smaller height point to root with larger height?

Union on Disjoint-Sets as Trees



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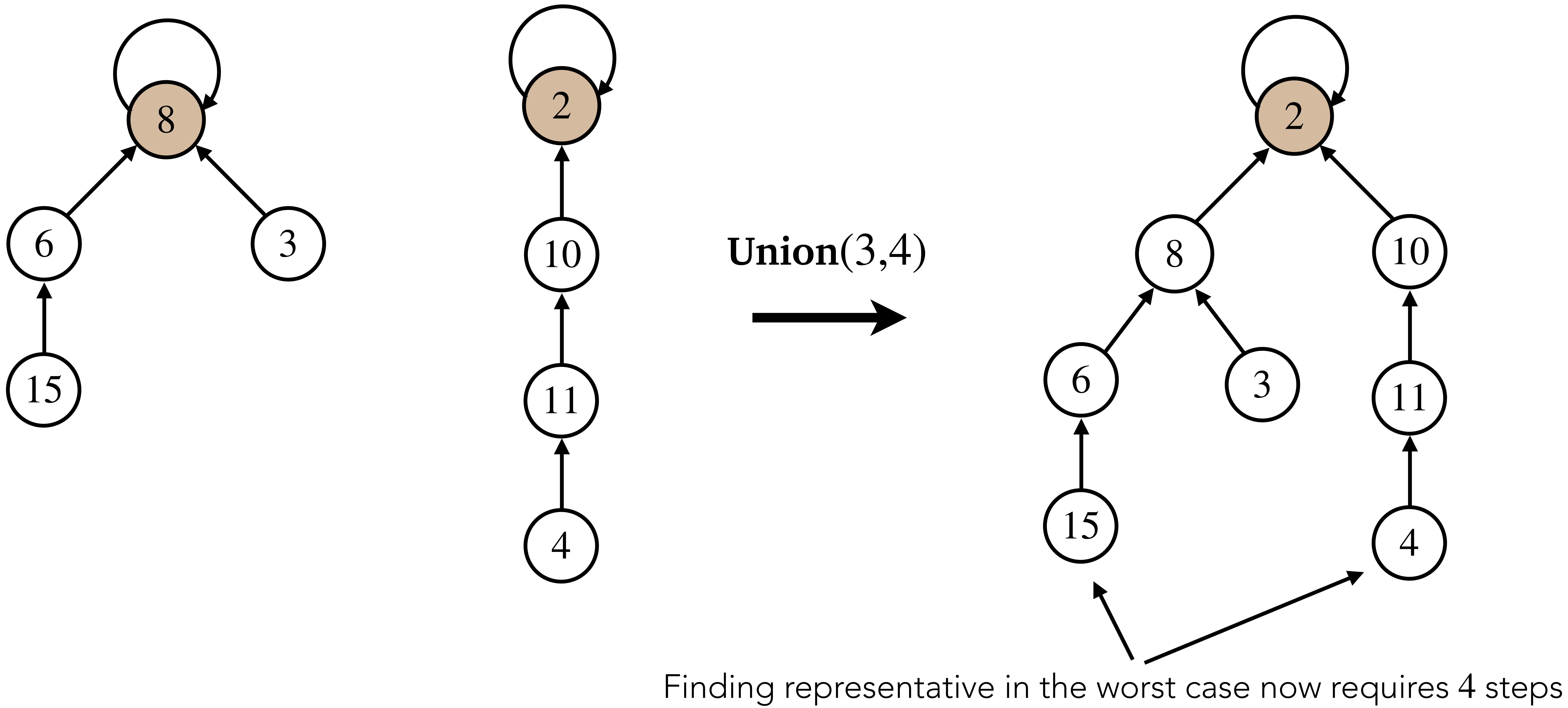

Union on Disjoint-Sets as Trees using Rank

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Rank starts with 0

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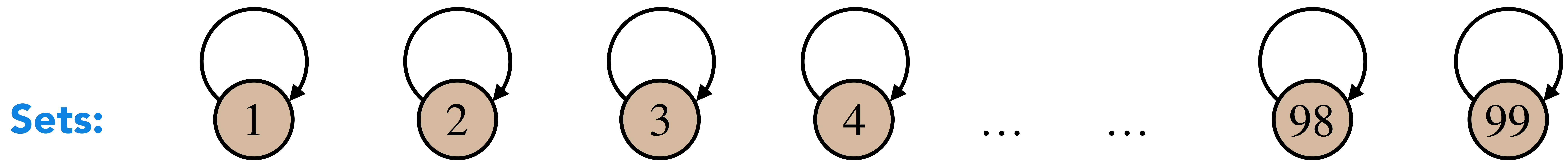

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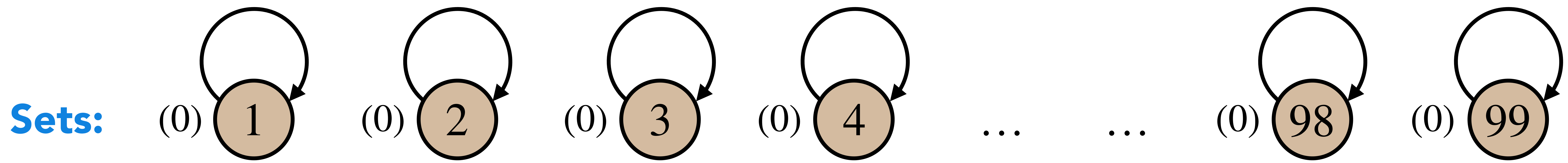

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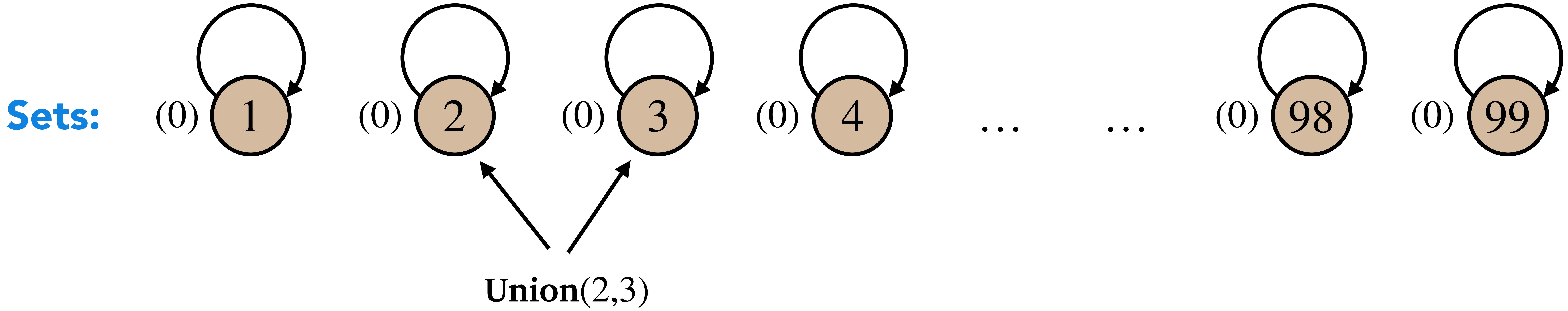

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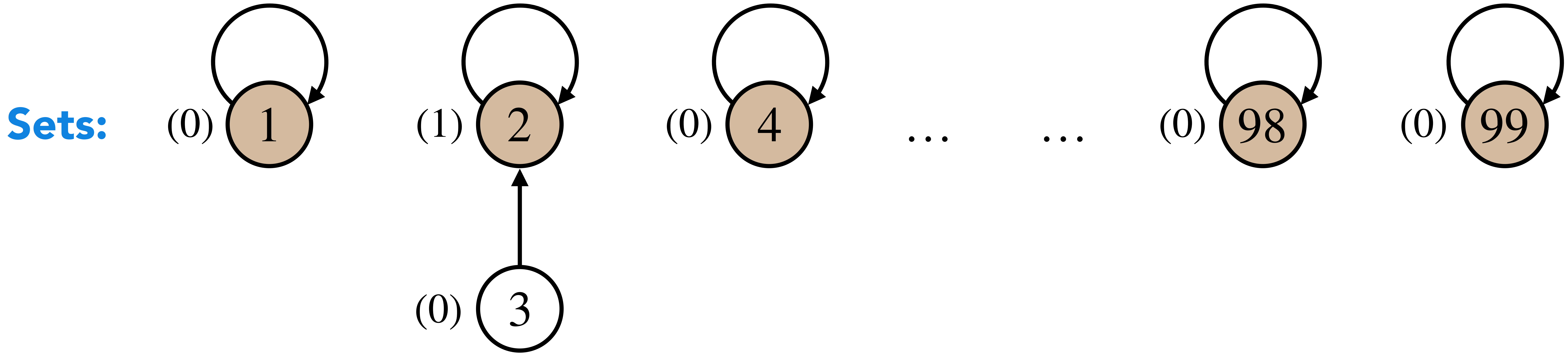

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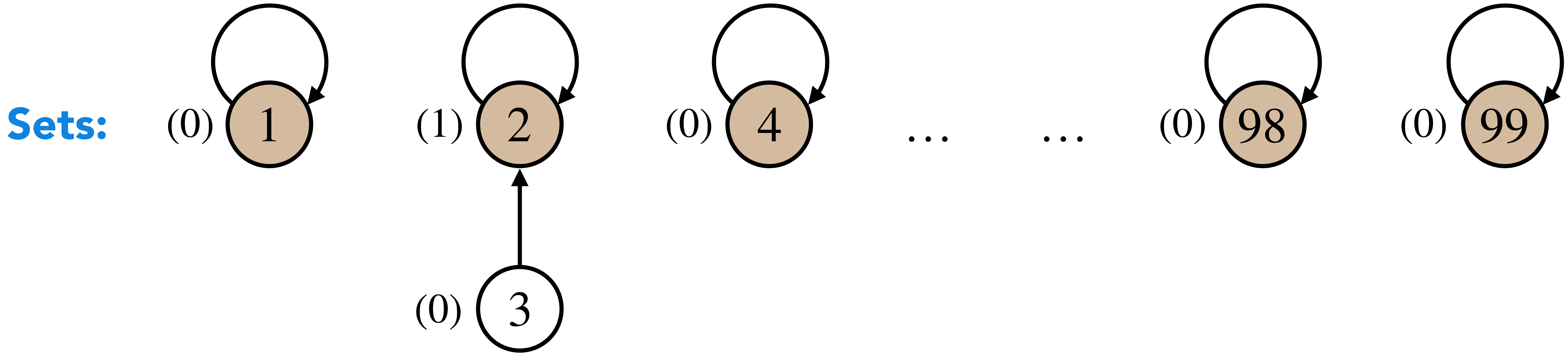

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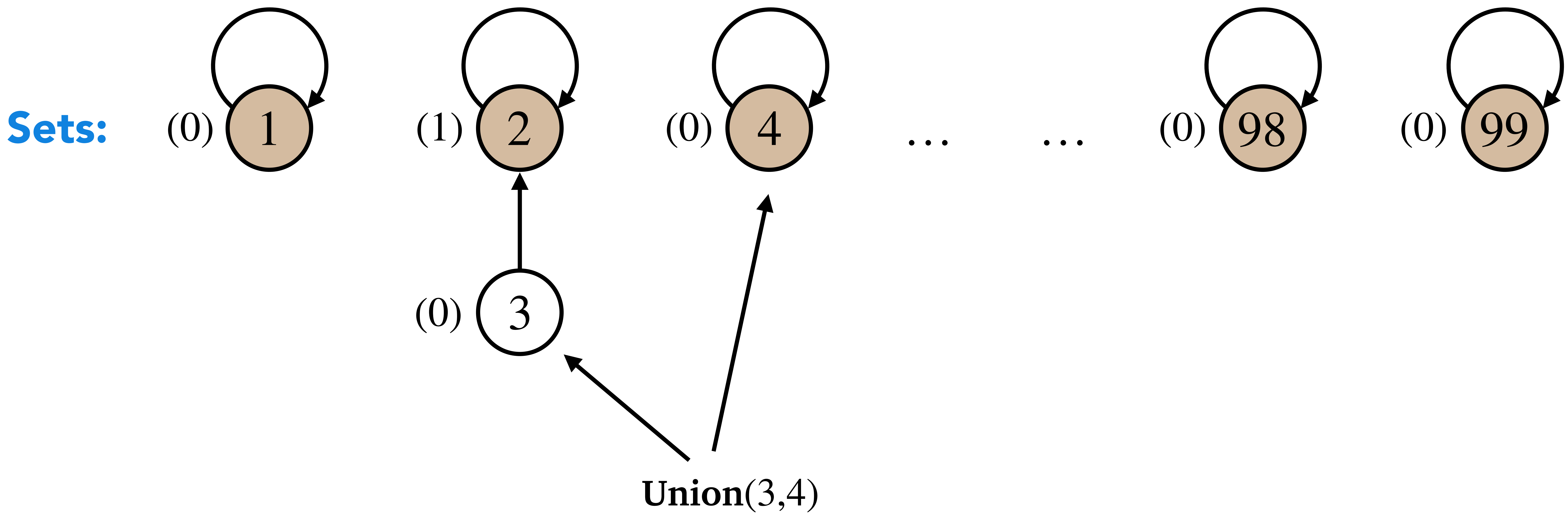

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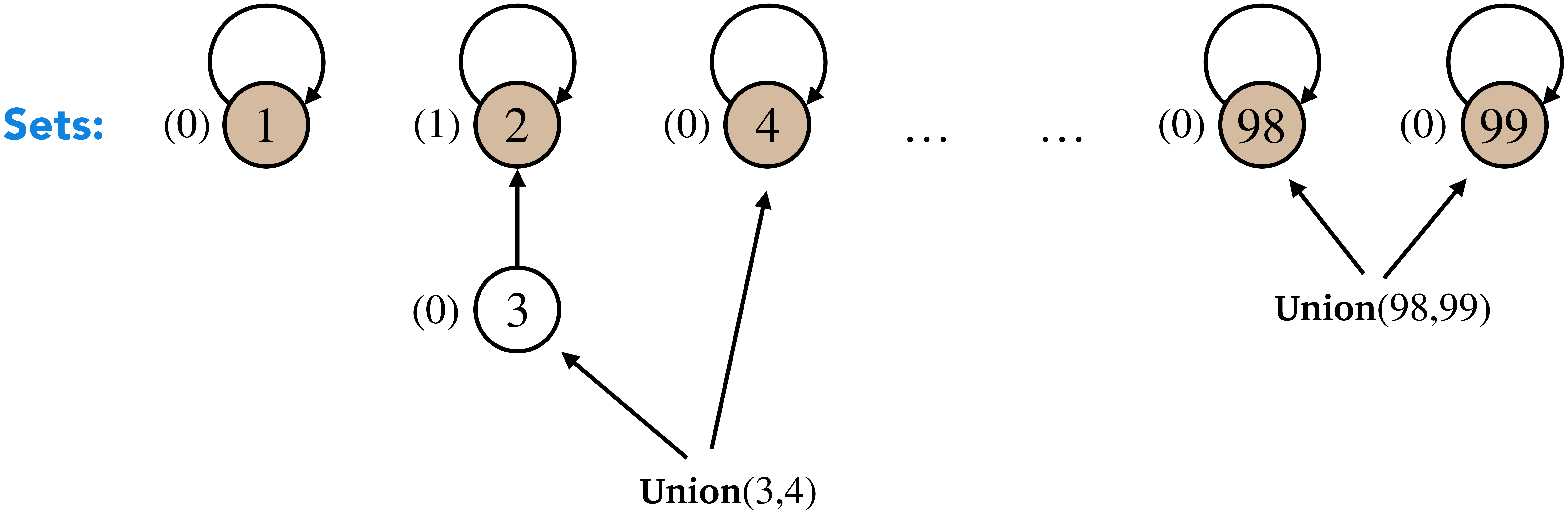

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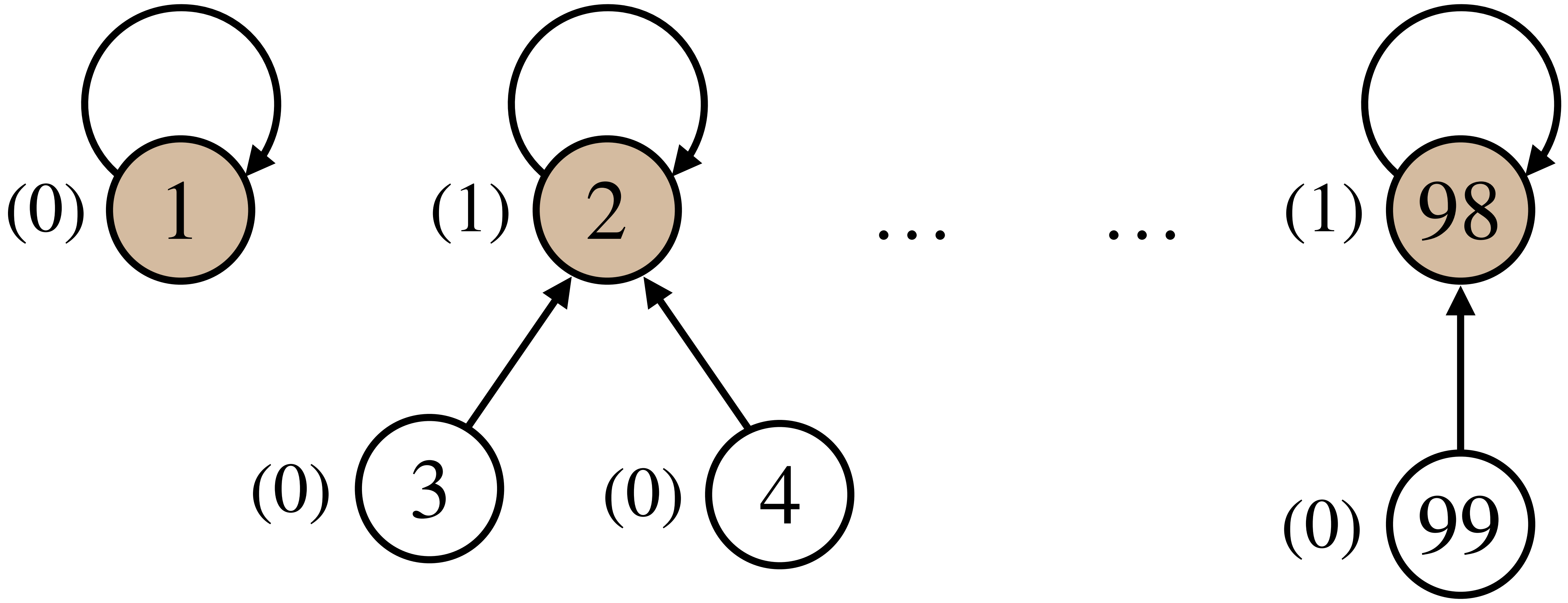

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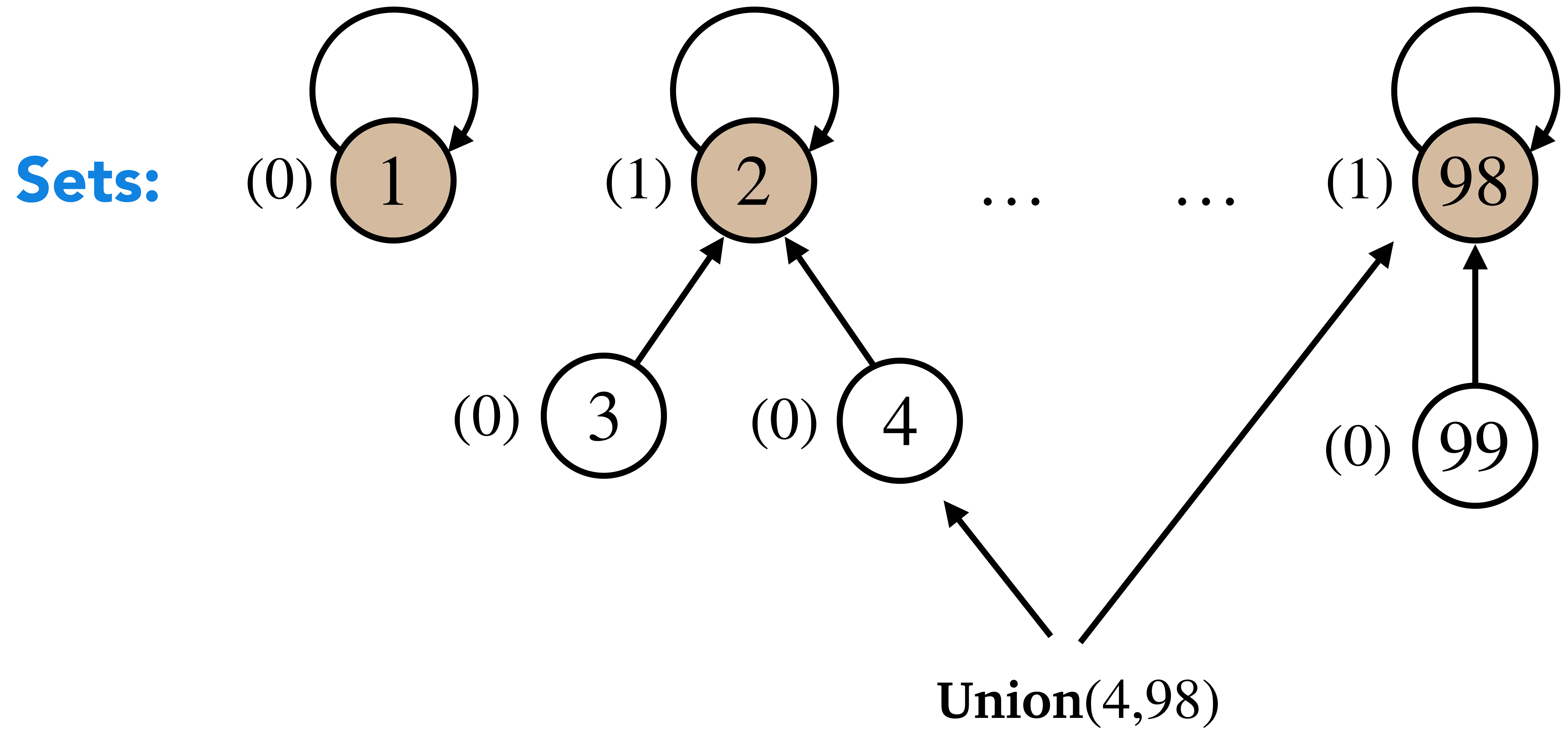

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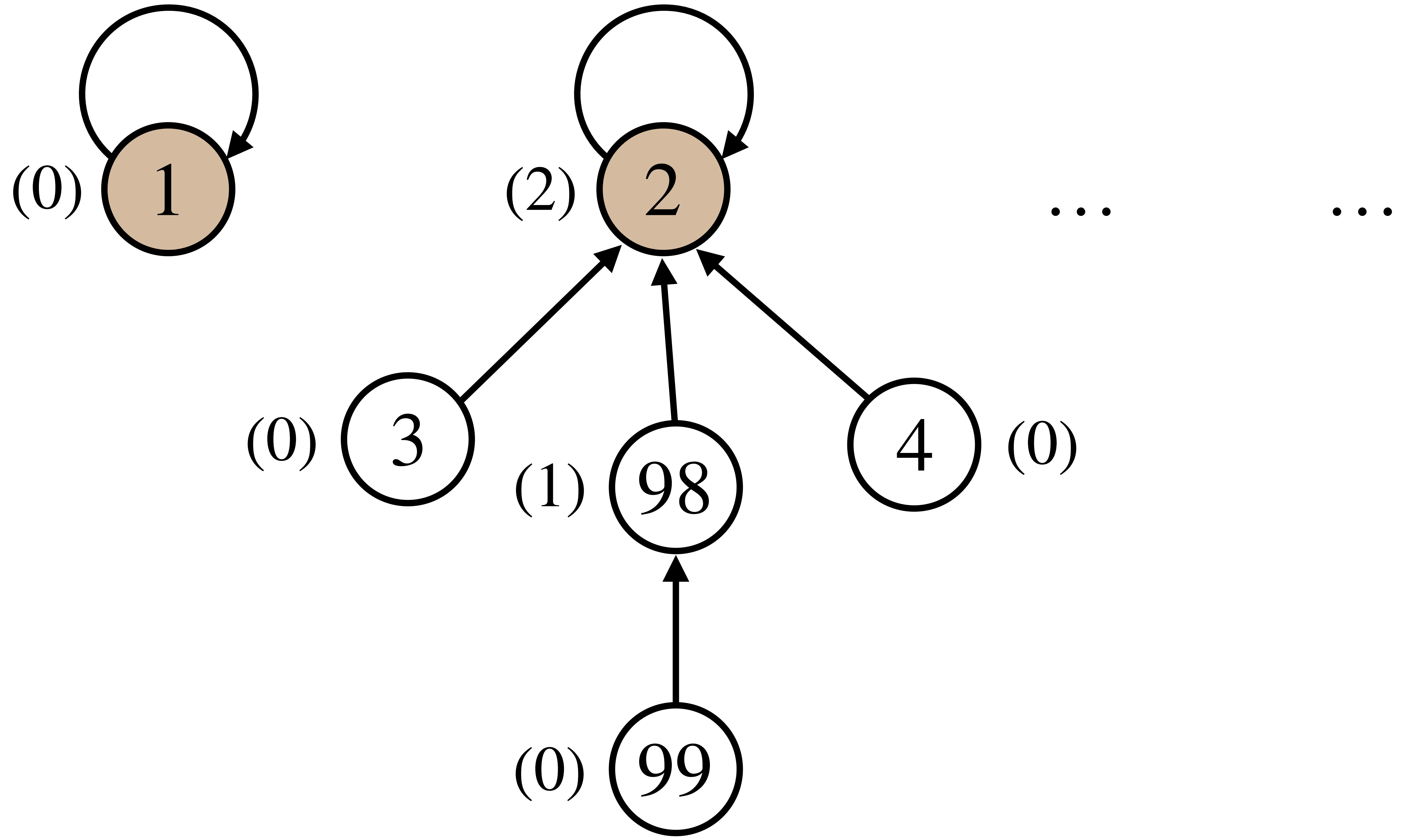

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6. **else**
7. $x\,.\,p = y$
8. $y\,.\,rank = y\,.\,rank + 1$

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assume x and y are in different sets

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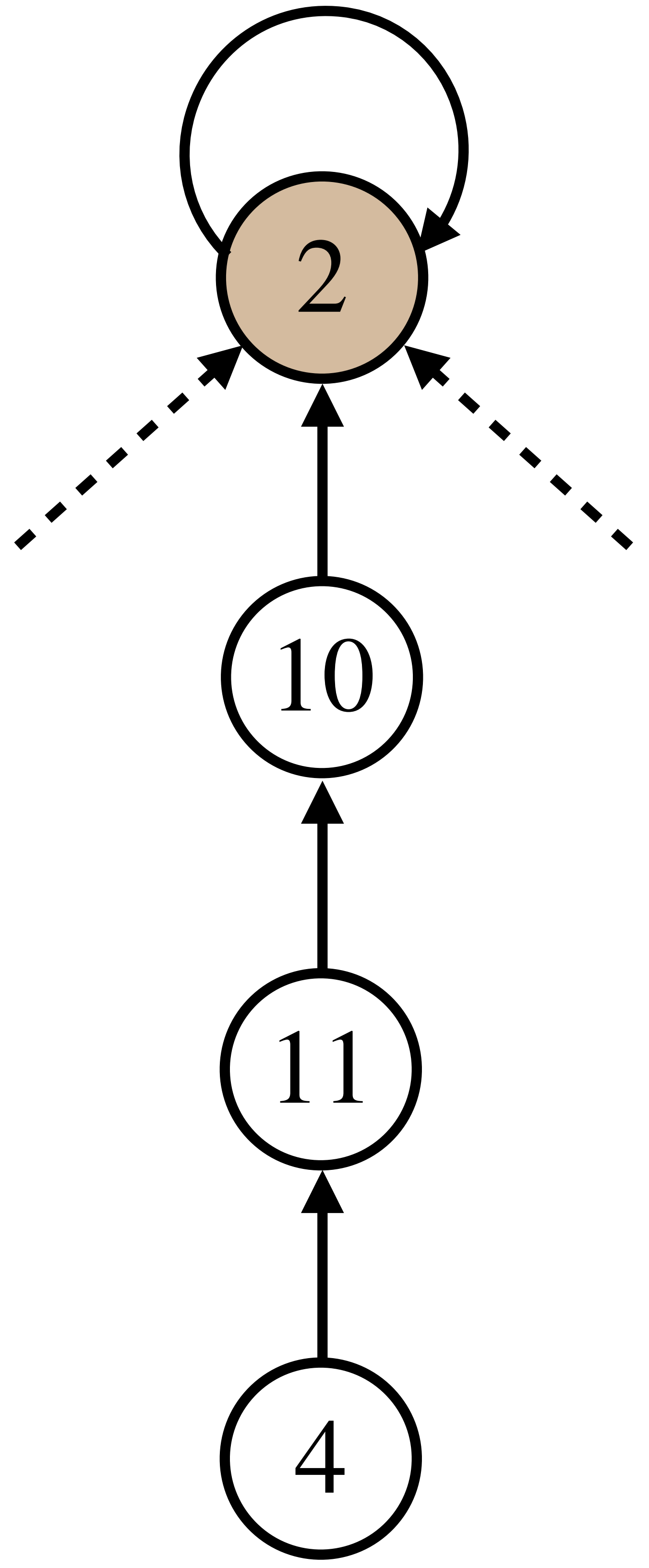

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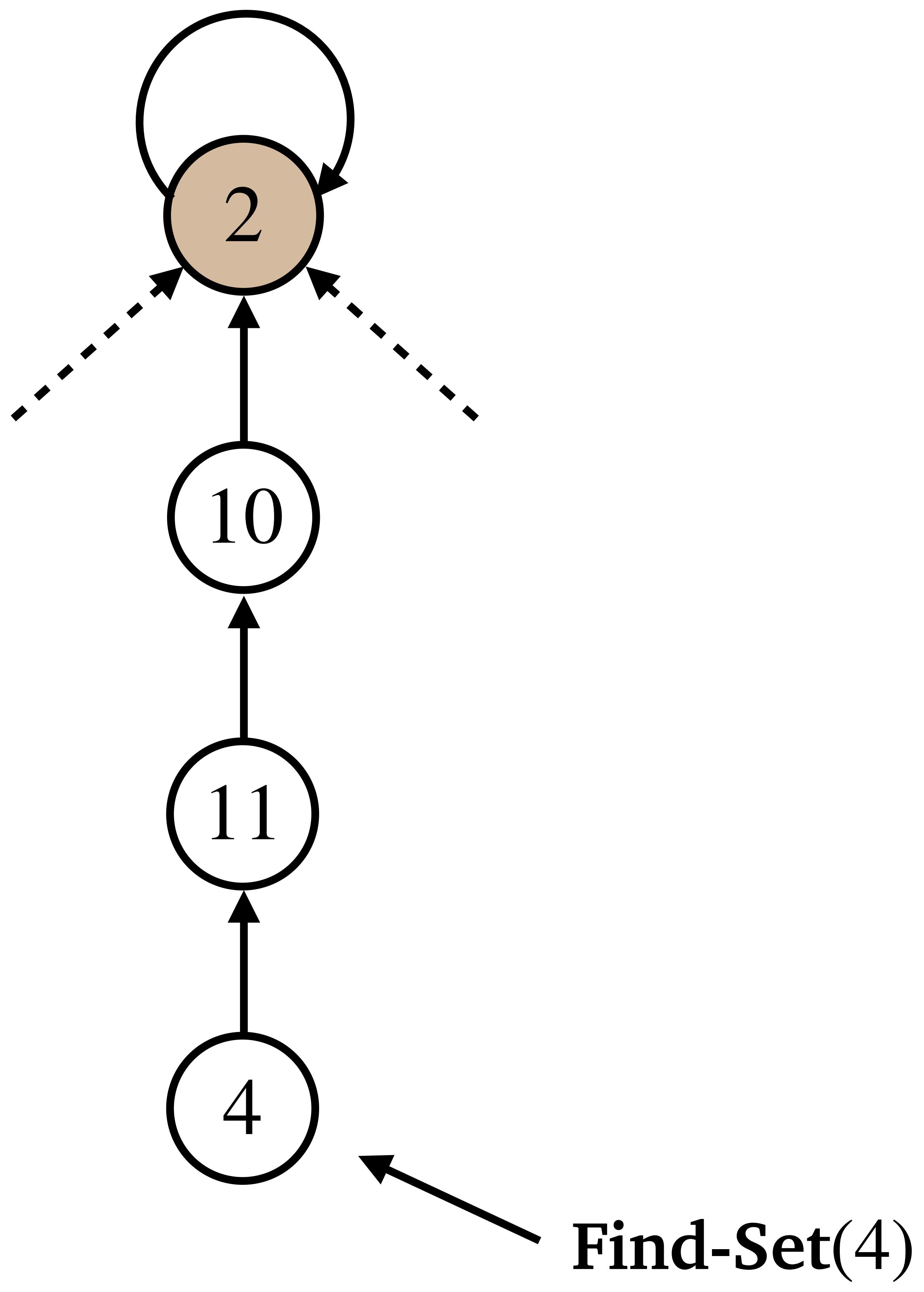

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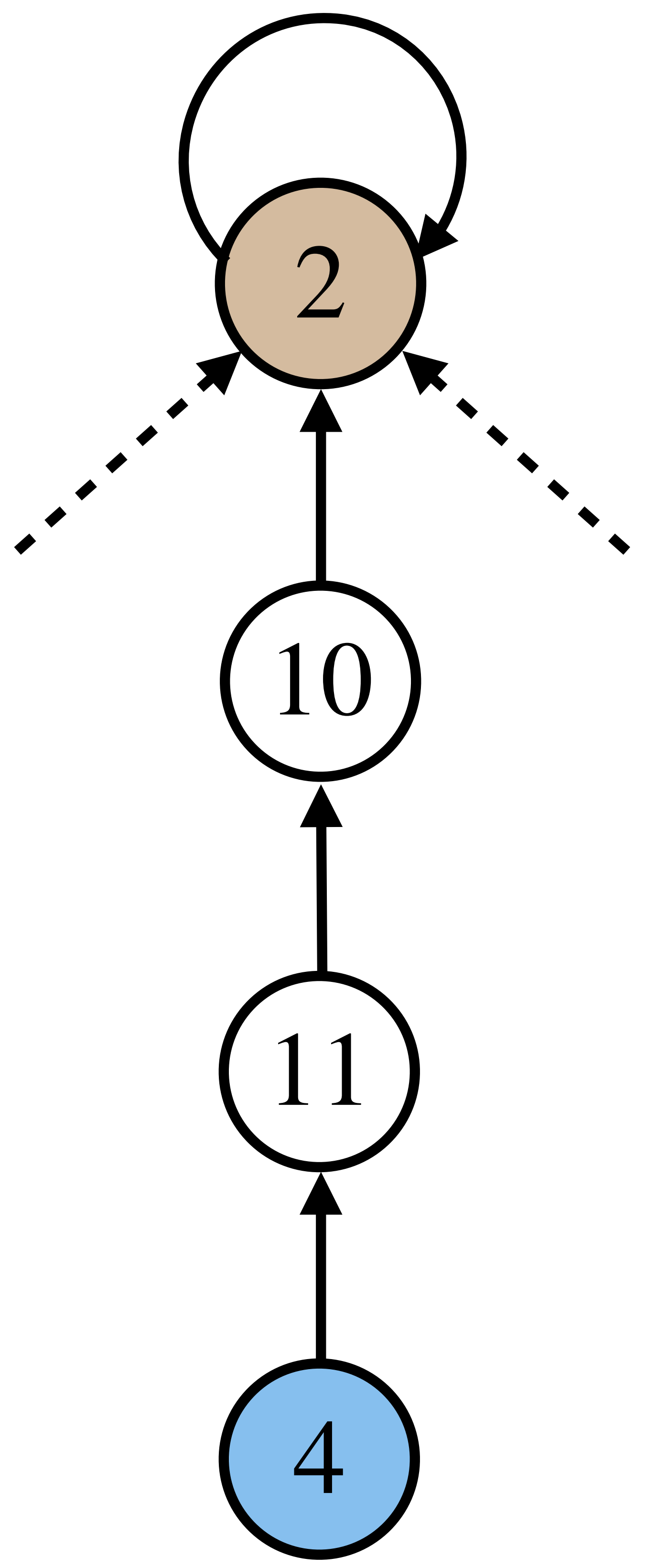

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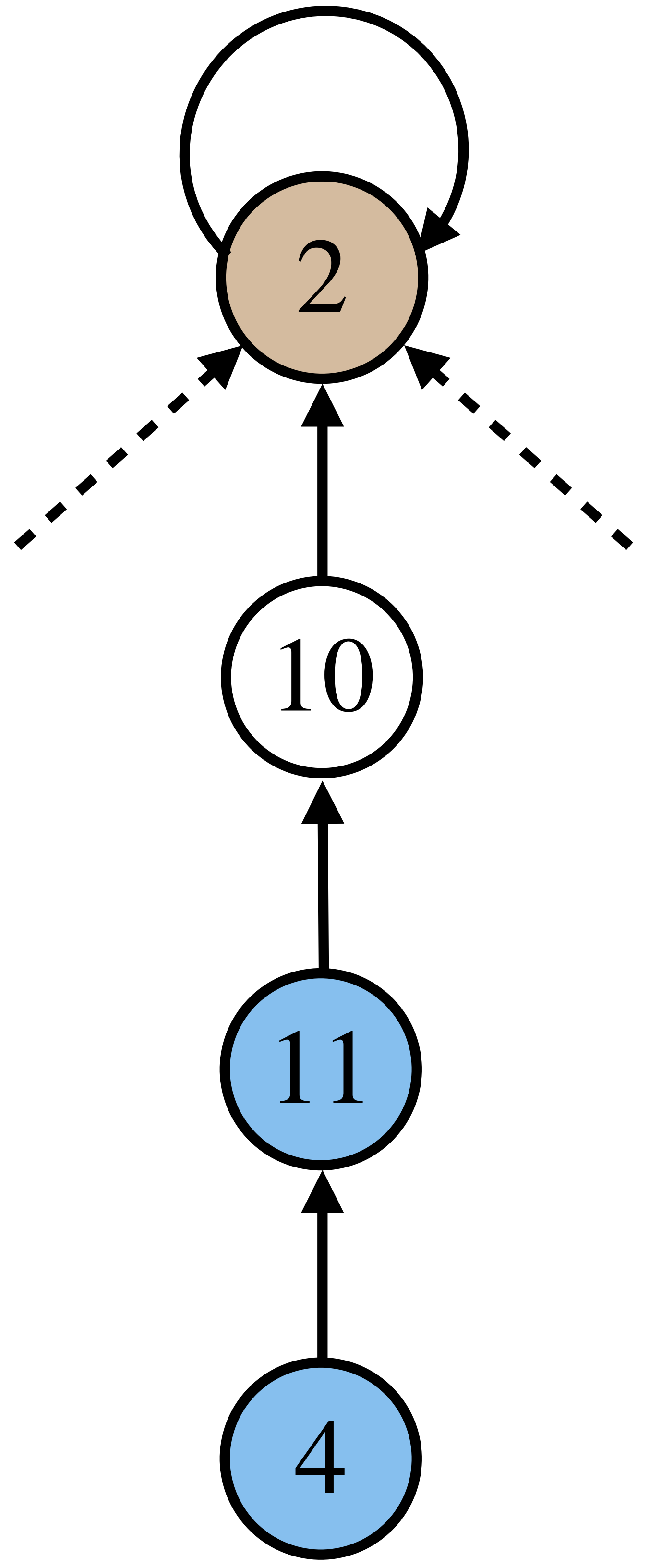

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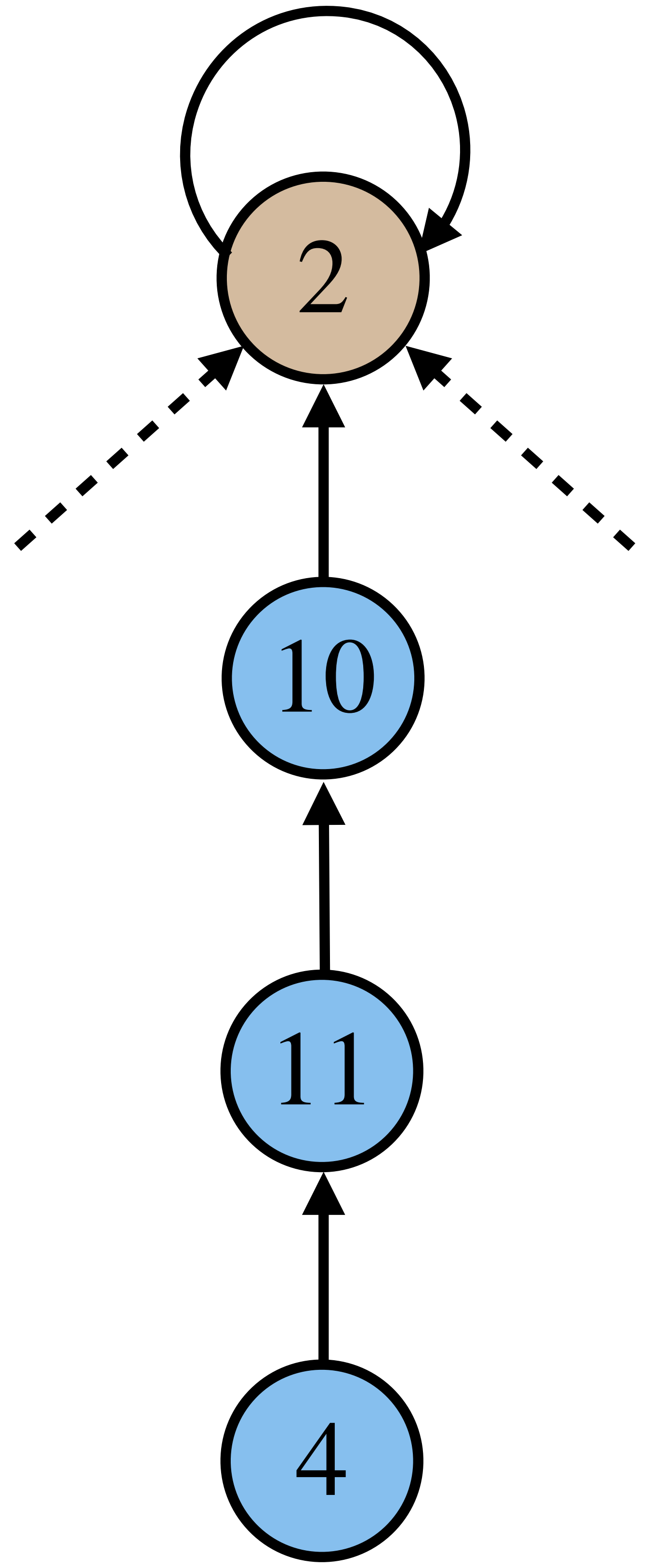

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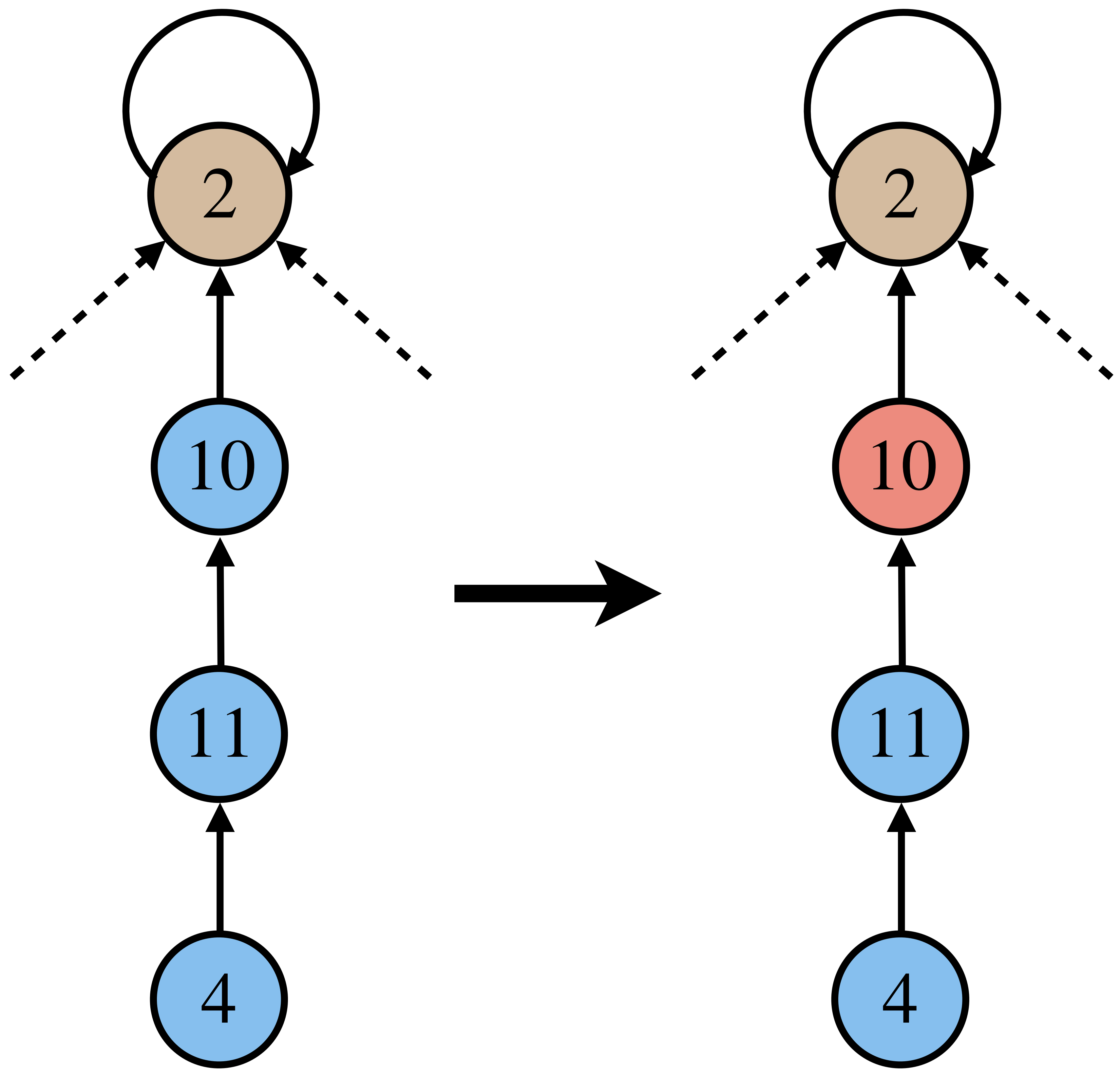

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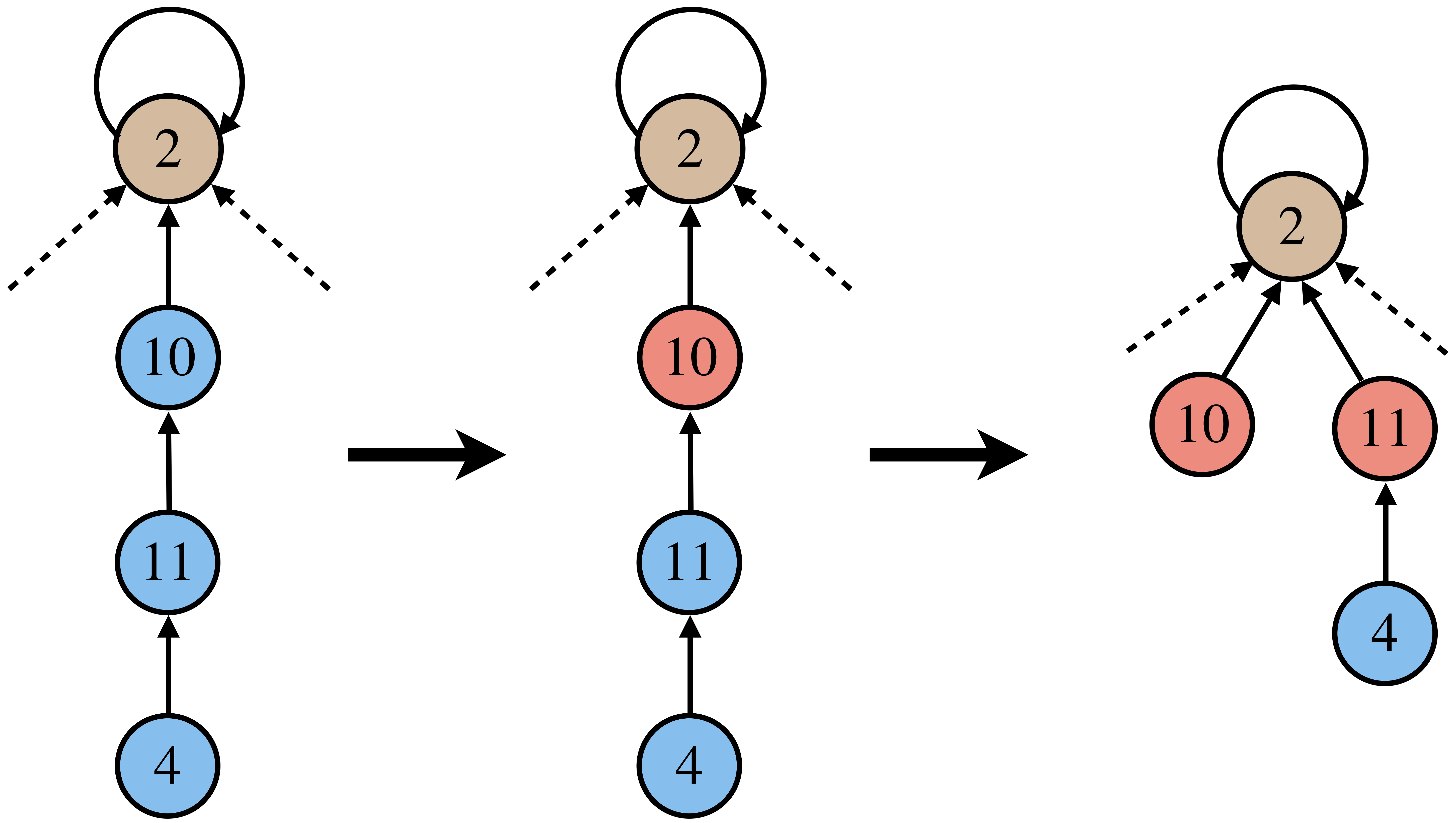

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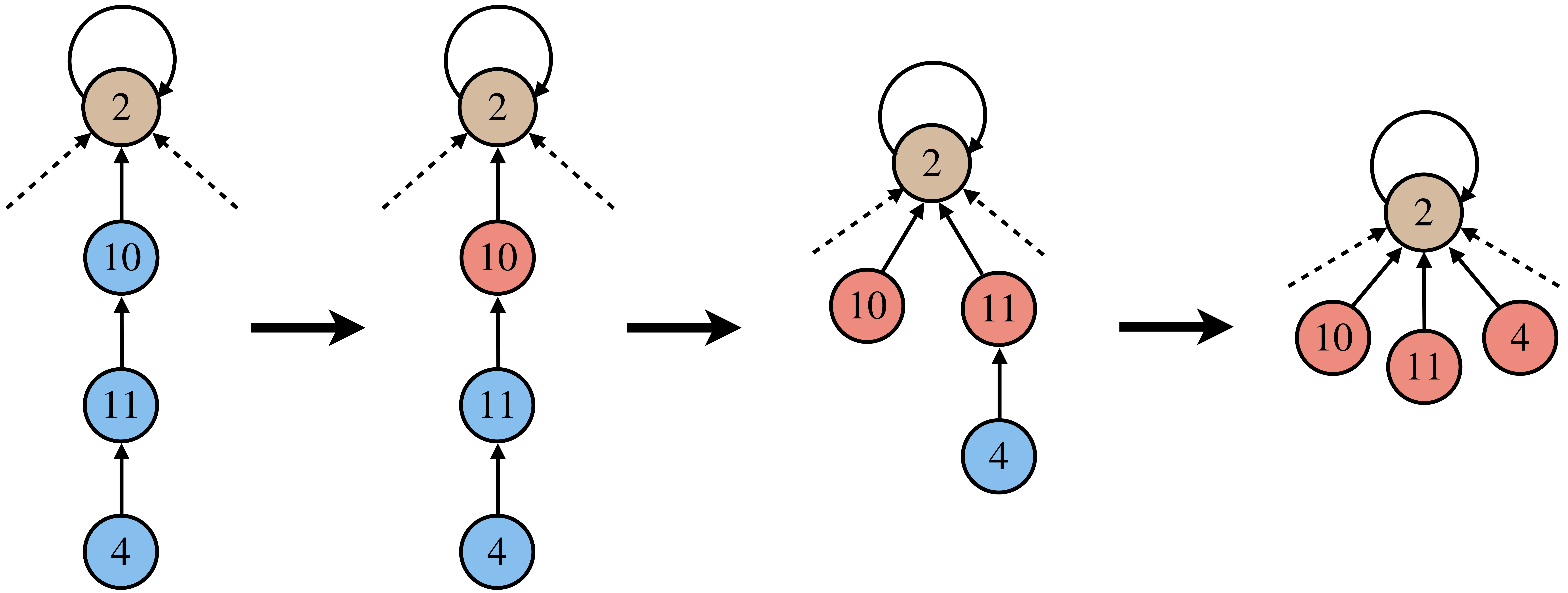

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