Image Restoration Frequency Domain

September 28, 2015

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Image Restoration
Image restoration vs. image enhancement

Image enhancement

- Largely a subjective process
- Prior knowledge about the degradation is not a must (sometimes no degradation is involved)
- Procedures are heuristic and take advantage of the psychophysical aspects of human visual system

Image restoration

- More an objective process
- Images are degraded
- Tires to recover the images by using the knowledge about the degradation

Image degradation

Degradation types

- Additive noise
  - Spatial domain restoration (denoising) techniques are preferred
- Image blur
  - Frequency domain techniques are preferred

Image degradation

Degradation model

$$g(x, y) = h(x, y)^{*} f(x, y) + \eta(x, y)$$

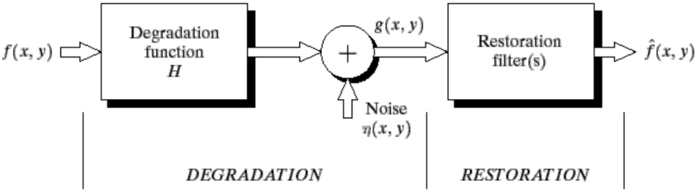
- $f(x, y)$ is the input image free from any degradation
- $g(x, y)$ is the degraded image
- $h(x, y)$ is the degradation function
- $\eta(x, y)$ is additive noise
- $*$ is the convolution operator
- In FD : $G(u, v) = H(u, v)F(u, v) + N(u, v)$

Different cases

$$g(x, y) = f(x, y) + \eta(x, y)$$
$$g(x, y) = f(x, y)^{*} h(x, y)$$
$$g(x, y) = f(x, y)^{*} h(x, y) + \eta(x, y)$$

Image Degradation / Restoration

Degradation/restoration model



Degradation Due to Noise
Noise models

- White noise
- Gaussian noise
- Rayleigh noise
- Erlang (gamma) noise
- Exponential noise
- Uniform noise
- Impulse (salt-and-pepper) noise

Degradation Due to noise
Noise models

White noise

- Autocorrelation function is an impulse function multiplied by a constant

$$a(x,y) = \sum_{s=0}^{N-1}\sum_{t=0}^{M-1} \eta(s,t).\eta(s-x,t-y) = N_0\delta(x,y)$$

- There is no correlation between any two pixels in the noise image
- There is no way to predict the next noise value
- The spectrum of the autocorralation function is a constant (White)

Degradation Due to noise
Noise models

Gaussian noise

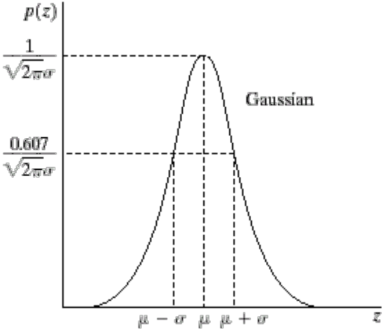
- Noise can be classified according the distribution of the pixel values of the noise image or its normalized histogram
- Gaussian noise is characterized by two parameters, mean (μ) and variance (σ^2)

$$p(z) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(z-\mu)^2/2\sigma^2}$$

- 70 % of z values fall in the range $[(\mu - \sigma), (\mu + \sigma)]$
- 95 % of z values fall in the range $[(\mu - 2\sigma), (\mu + 2\sigma)]$

Degradation Due to noise
Noise models

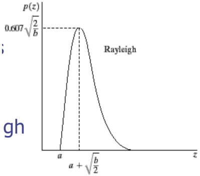
Gaussian noise



Degradation Due to noise
Noise models

Rayleigh noise

$$p(z) = \begin{cases} \frac{2}{b}(z-a)e^{-(z-a)^2/b} & \text{for } z \geq a \\ 0 & \text{for } z < a \end{cases}$$



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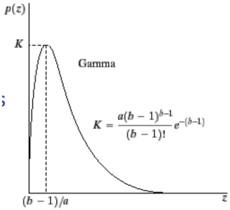
- The mean and variance of this density are given by $\mu = a + \sqrt{(\pi b)/4}$, $\sigma^2 = \frac{b(\pi-4)}{4}$
- a and b can obtained through mean and variance

Degradation Due to noise
Noise models

Gamma (erlang) noise

$$p(z) = \begin{cases} \frac{a^b z^{b-1}}{(b-1)!} e^{-az} & \text{for } z \geq a \\ 0 & \text{for } z < a \end{cases}$$

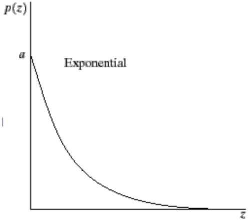
- The mean and variance of this density are given by $\mu = b/a$, $\sigma^2 = b/a^2$
- a and b can obtained through mean and variance



Degradation Due to noise
Noise models

Exponential noise

$$p(z) = \begin{cases} ae^{-az} & \text{for } z \geq a \\ 0 & \text{for } z < a \end{cases}$$



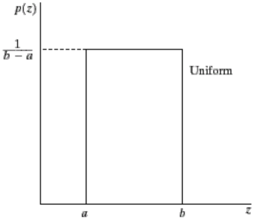
- The mean and variance of this density are given by $\mu = 1/a$, $\sigma^2 = 1/a^2$
- Special case of Erlang *pdf* with $b = 1$

Degradation Due to noise

Noise models

Uniform noise

$$p(z) = \begin{cases} \frac{1}{b-a} & \text{if } a \leq z \geq b \\ 0 & \text{Otherwie} \end{cases}$$



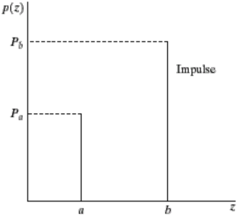
- The mean and variance of this density are given by $\mu = (a+b)/2$, $\sigma^2 = (b-a)^2/12$

Degradation Due to noise
Noise models

Impulse (salt-and-pepper) noise

$$p(z) = \begin{cases} P_a & \text{for } z = a \\ P_b & \text{for } z = b \\ 0 & \text{otherwise} \end{cases}$$

- If either P_a or P_b is zero, the impulse noise is called unipolar
- a and b usually are extreme values because impulse corruption is usually large compared with the strength of the image signal
- It is the only type of noise that can be distinguished from others visually

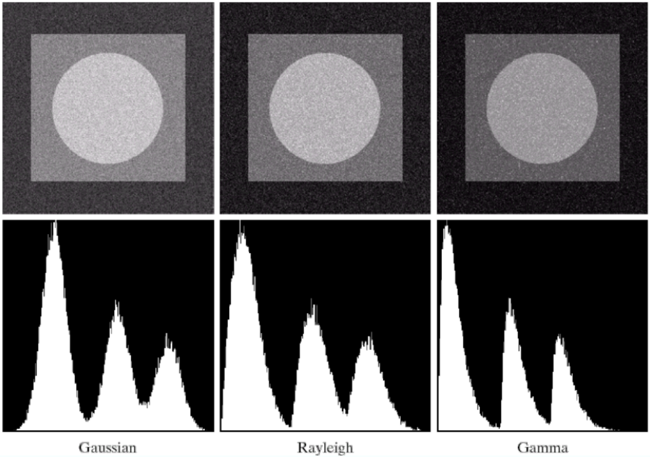


Degradation Due to Noise
Noise models

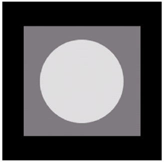
Noise in practice

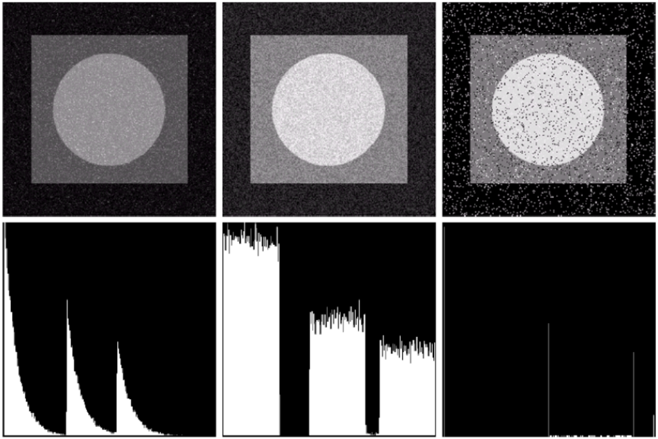
- **Gaussian noise:** electronic circuit noise and sensor noise due to poor illumination or high temperature
- **Rayleigh noise:** range imaging
- **Erlang noise:** laser imaging
- **Impulse noise:** quick transients take place during imaging
- **Uniform noise:** used in simulations

Degradation Due to Noise
Noise example



Degradation Due to Noise
Noise example





Exponential Uniform Salt & Pepper

Degradation Due to Noise
Periodic Noise

- Arises typically from electrical or electromechanical interference during image acquisition
- It can be observed by visual inspection both in the spatial and frequency domain
- But only the spatial domain will be considered
- Parameters can be estimated by inspection of the spectrum

How to estimate the noise *pdf*

- Sensor specifications
- Capture a set of images of plain environment, if imaging sensors are available
- If only noisy images are available, *pdf* parameters can be estimated from small patches of constant regions of the noisy image

Degradation Due to Noise
Estimation of noise parameters

- Commonly, only mean and variance need to be estimated
- Considering a sub-image with plain scene S,

$$\hat{\mu} = \frac{1}{N_s} \sum_{(x_i,y_i)\in S} z(x_i, y_i)$$

$$\hat{\sigma_1^2} = \frac{1}{N_s} \sum_{(x_i,y_i)\in S} (z(x_i, y_i) - \hat{\mu})^2$$

De-Noising

Mean filter

$g(x, y)$ is the corrupted image and $S_{x,y}$ is the mask

- Arithmetic mean filter

$$\hat{f}(x, y) = \frac{1}{mn} \sum_{(s,t)\in S_{x,y}} g(s, t)$$

- Geometric mean filter $\rightarrow$ Trends to preserve more details

$$\hat{f}(x, y) = \left[\prod_{(s,t)\in S_{xy}} g(s, t) \right]^{\frac{1}{mn}}$$

- Harmonic mean filter $\rightarrow$ Works well for salt noise but fails for pepper noise

$$\hat{f}(x, y) = \frac{mn}{\sum_{(x_i,y_i)\in S_{xy}} \frac{1}{g(s,t)}}$$

De-Noising

Mean filter

- Contraharmonic mean filter

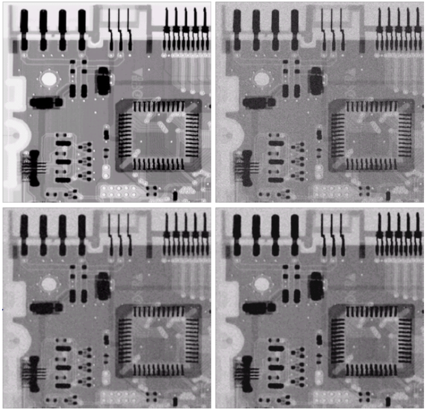
$$\hat{f}(x,y) = \frac{\sum_{(s,t)\in S_{xy}} g(s,t)^{Q+1}}{\sum_{(s,t)\in S_{xy}} g(s,t)^{Q}}$$

  - Q is order of the filter
  - $Q > 0 \rightarrow$ pepper noise
  - $Q < 0 \rightarrow$ salt noise
  - $Q = 0 \rightarrow$ arithmetic mean filter
  - $Q = -1 \rightarrow$ harmonic mean filter

De-Noising

Mean filter

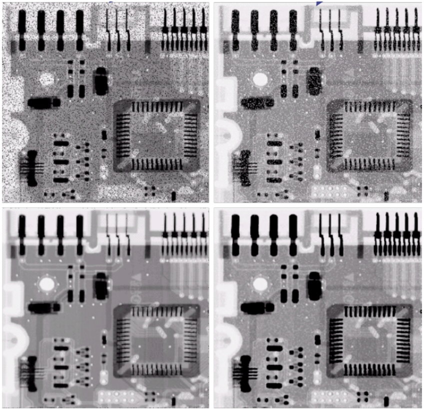
From right to left and up to bottom : original image, corrupted with Gaussian noise, mean filtering, and geometric mean filtering



De-Noising

Mean filter

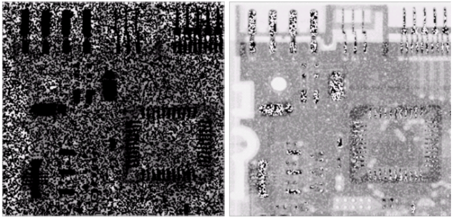
From right to left and up to bottom : corrupted with pepper noise, corrupted with salt noise, Contraharmonic filter $Q = 1.5$, Contraharmonic filter $Q = -1.5$



De-Noising

Mean filter

- Selecting the wrong sign for Contraharmonic filter, from right to left: Contraharmonic filter $Q = -1.5$, Contraharmonic filter $Q = +1.5$



De-Noising

Median filter

$$\hat{f}(x,y) = median_{(s,t)\in S_{xy}} g(s,t)$$

- Median represents the 50^{th} percentile of a ranked set of numbers

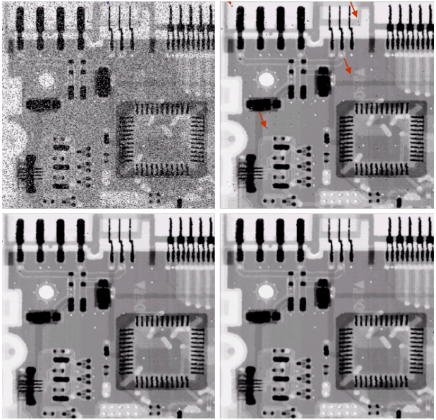
Max and min filter

- Max filter uses the 100^{th} percentile of a ranked set of numbers
  - Good for removing pepper noise
- Min filter uses the 1^{th} percentile of a ranked set of numbers
  - Good for removing salt noise

De-Noising

Median filter example

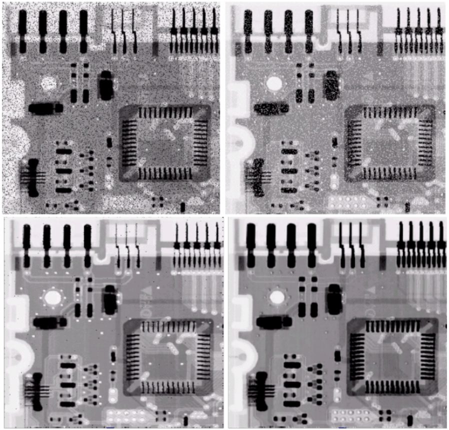
Corrupted image with salt-and-pepper, and one, two and third pass with median filter, respectively



De-Noising

Max and Min filter example

First row: Corrupted images with pepper and salt noise, respectively. Second row: the results of Max and Min filtering, respectively



De-Noising

Midpoint filter

$$\hat{f}(x,y) = \frac{1}{2}\left[\max_{(s,t)\in S_{xy}} g(s,t) + \min_{(s,t)\in S_{xy}} g(s,t)\right]$$

- Works best for noise with symmetric *pdf* like Gaussian or uniform noise

De-Noising

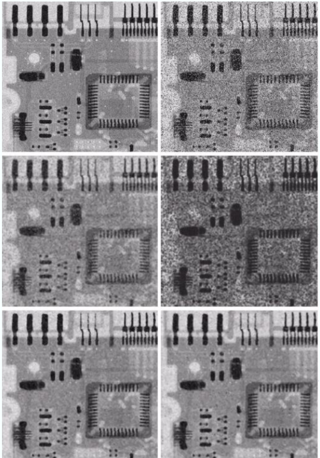
Alpha trimmed mean filter

$$\hat{f}(x,y) = \frac{1}{mn-d} \sum_{(s,t)\in S_{xy}} g_r(s,t)$$

- Takes the mean value of the pixels enclosed by an $m \times n$ mask after deleting the pixels with the $d/2$ lowest and highest gray-level values
- $g_r(s,t)$ represent the remaining $mn-d$ pixels
- Useful while dealing with multiple noise (e.g salt-and-pepper and Gaussian)

De-Noising - example

First column: Corrupted by uniform noise, mean (5×5), and median (5×5), respectively. Second column: Corrupted by uniform and salt-and-pepper noise, geomean(5×5), and alpha-trimmed mean (5×5)



De-Noising
Adaptive Filter

- Filters whose behavior changes based on statistical characteristics of the image.
- Two types of adaptive filter:
  - Adaptive, local noise reduction filter
  - Adaptive median filter

Adaptive, local noise reduction filter

Parameters:

- mean(μ), average gray level
- variance(σ^2), average contrast

Measurements:

- noisy image $g(x, y)$, the variance of noise σ_η^2, local mean m_L in S_{xy}, local variance σ_L^2

De-Noising
Adaptive Filter

Adaptive, local noise reduction filter

$$\hat{f}(x,y) = g(x,y) - \frac{\sigma_\eta^2}{\sigma_L^2}[g(x,y) - m_L]$$

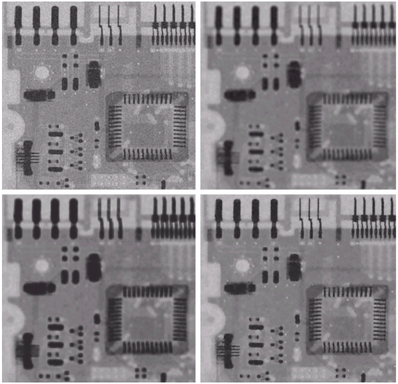
Conditions:

- $\sigma_\eta^2 = 0 \rightarrow$ Zero noise case
  - Return the value of g(x,y)
- $\sigma_L^2 > \sigma_\eta^2$
  - Possible edge and should be preserved
  - Return the value close to g(x,y)
- $\sigma_L^2 = \sigma_\eta^2$
  - When the local area has similar properties with the overall image.
  - Return arithmetic mean value of the pixels in S_{xy}

De-Noising
Adaptive Filter

Adaptive, local noise reduction filter

LRUB: Image corrupted by additive Gaussian nois $(\mu, \sigma^2) = (0, 1000)$, arithmetic mean filter, geometric mean filter, and adaptive noise reduction. Filter size $= 7 \times 7$



De-Noising
Adaptive Filter

Adaptive median filter

- Effective for removing salt and pepper noise
- The density of the impulse noise can not be too high for median filter compare to adaptive median filter
- Notations:
  - Z_{min}, minimum gray value in S_{xy}
  - Z_{max}, maximum gray value in S_{xy}
  - Z_{med}, median gray value in S_{xy}
  - Z_{xy}, gray value of the image at (x, y)
  - S_{max}, maximum allowed size of S_{xy}

De-Noising
Adaptive Filter

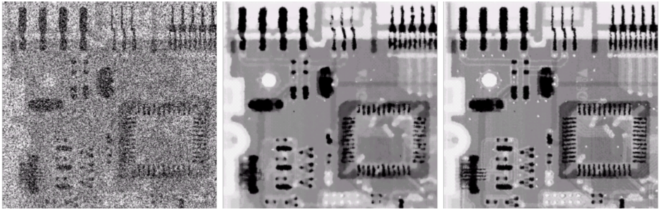
Adaptive median filter

- Two levels of operations:
- Level A, Used to test whether Z_{med} is part of s-and-p noise, if yes, window size is increased
  - $A_1 = Z_{med} - Z_{min}$
  - $A_2 = Z_{med} - Z_{max}$
  - if $A_1 > 0$ & $A_2 < 0 \rightarrow$ Level B
  - else $\rightarrow$ increase S_{xy} size by 2
  - if window size $\leq S_{max}$ repeat Level A else return Z_{xy}
- Level B, Used to test whether Z_{xy} is part of s-and-p noise, if yes, apply regular median filtering
  - $B_1 = Z_{xy} - Z_{min}$
  - $B_2 = Z_{xy} - Z_{max}$
  - if $B_1 > 0$ & $B_2 < 0$, return Z_{xy}
  - else return Z_{med}

De-Noising
Adaptive Filter

Adaptive median filter

Image corrupted with s-and-p noise, median filter (7×7), and adaptive median filtering ($S_{max} = 7$), respectively



Periodic noise reduction
Frequency domain Filtering

- Lowpass and highpass filters for image enhancement
- Bandreject, bandpass and notch filters for periodic noise reduction or removal

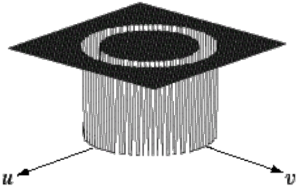
Bandreject filters

- Bamdreject filters remove or attenuate a band of frequencies about the origin of the Fourier transform
- Ideal, Butterworth, Gaussian bandreject filters

Periodic noise reduction

Bandreject Ideal filters

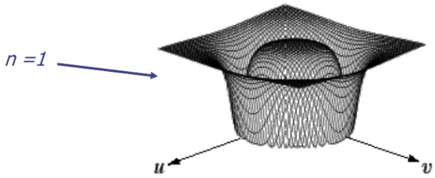
$$H(u,v) = \begin{cases} 1 & \text{if } D(u,v) < D_0 - \frac{W}{2} \\ 0 & \text{if } D_0 - \frac{W}{2} \leq D(u,v) \leq D_0 + \frac{W}{2} \\ 1 & \text{if } D(u,v) > D_0 + \frac{W}{2} \end{cases}$$



Periodic noise reduction

Bandreject Butterworth filters

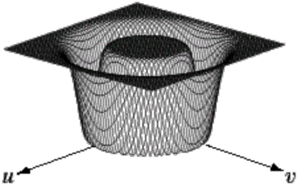
$$H(u,v) = \frac{1}{1 + \left[\frac{D(u,v)W}{D^2(u,v) - D_0^2}\right]^{2n}}$$



Periodic noise reduction

Bandreject Guassian filters

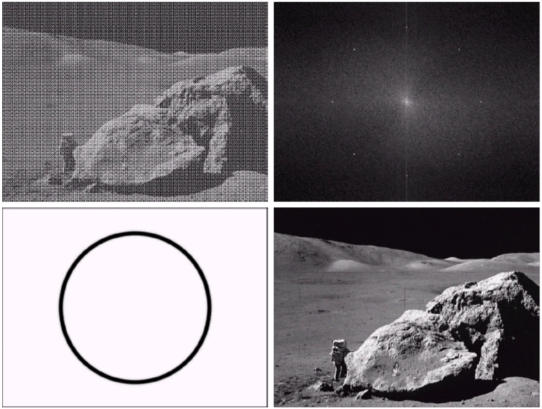
$$H(u,v) = 1 - e^{\frac{-1}{2}\left[\frac{D^2(u,v) - D_0^2}{D(u,v)W}\right]}$$



Periodic noise reduction

Bandreject filter

Corrupted image by sinusoidal noise and filtered image by Butterworth bandrejected filter



Periodic noise reduction

Bandpass filter

Performs opposite of bandreject filter

$$H_{bp}(u, v) = 1 - H_{br}(u, v)$$

Notch filter

- Notch filter ejects frequencies in predefined neighborhoods about a center frequency
- It appears in symmetric pairs about the origin because the Fourier transform of a real valued image is symmetric

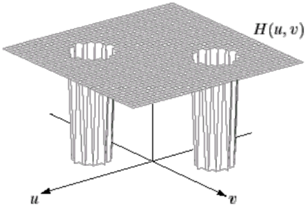
Periodic noise reduction

Ideal notch filter

$$H(u,v) = \begin{cases} 0 & \text{if } D_1(u,v) \leq D_0 \text{or } D_2(u,v) \leq D_0 \\ 1 & \text{otherwise} \end{cases}$$

$$D_1(u,v) = \left[(u - M/2 - u_0)^2 + (u - N/2 - v_0)^2\right]^{1/2}$$

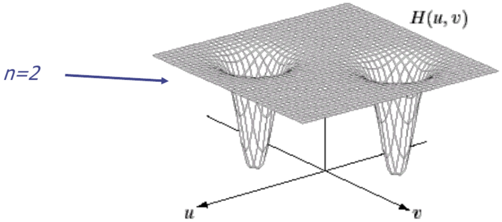
$$D_2(u,v) = \left[(u - M/2 + u_0)^2 + (u - N/2 + v_0)^2\right]^{1/2}$$



Periodic noise reduction

Butterworth notch filter

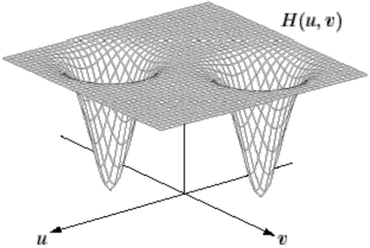
$$H(u,v) = \frac{1}{1 + \left[\frac{D_0^2}{D_1(u,v)D_2(u,v)}\right]^{2n}}$$



Periodic noise reduction

Gaussian notch filter

$$H(u,v) = 1 - e^{\frac{-1}{2}\left[\frac{D_1(u,v)D_2(u,v)}{D_0^2}\right]}$$



Periodic noise reduction

Notch filter

- Notch filter that pass, rather than suppress:

$$H_{np}(u, v) = 1 - H_{nr}(u, v)$$

- If $u_0 = v_0 = 0$, NR filters become highpass filters
- If $u_0 = v_0 = 0$, NP filters become lowpass filters

Periodic noise reduction

Optimum Notch filter

- Minimizing the effects of components not presented in estimation of $\eta(x,y)$, by :

$$\hat{f}(u,v) = g(x,y) - w(x,y)\hat{\eta}(x,y)$$

  $w(x,y)$ is weighted or modulation function
- Here the modulation function is a constant within a neighborhood of size $(2a+1)$ by $(2b+1)$ about a point (x,y)
- We optimize its performance by minimizing the local variance of the restored image at the position (x,y)

$$\sigma^2(x,y) = \frac{1}{(2a+1)(2b+1)} \sum_{s=-a}^{a} \sum_{t=-b}^{b} [\hat{f}(x+s,y+t) - \bar{\hat{f}}(x,y)]^2$$

$$\bar{\hat{f}}(x,y) = \frac{1}{(2a+1)(2b+1)} \sum_{s=-a}^{a} \sum_{t=-b}^{b} \hat{f}(x+s,y+t)$$

Periodic noise reduction

Optimum Notch filter

- Points on or near Edge of the image can be treated by considering partial neighborhoods or by zero padding the borders

$$\sigma^2(x,y) = \frac{1}{(2a+1)(2b+1)} \sum_{s=-a}^{a} \sum_{t=-b}^{b} \{[g(x+s,y+t) - w(x+s,y+t)$$

$$\hat{\eta}(x+s,y+t)] - [\bar{g}(x,y) - \overline{w(x,y)\eta(x,y)}]\}^2$$

- Assumption: $w(x+s,y+t) = w(x,y)$ for $-a \leq s \leq a$ and $-b \leq t \leq b$
  $\Rightarrow \overline{w(x,y)\eta(x,y)} = w(x,y)\bar{\hat{\eta}}(x,y)$

Periodic noise reduction

Optimum Notch filter

$$\Rightarrow \sigma^2(x,y) = \frac{1}{(2a+1)(2b+1)} \sum_{s=-a}^{a} \sum_{t=-b}^{b} \{[g(x+s,y+t)-$$

$$w(x,y)\hat{\eta}(x+s,y+t)] - [\bar{g}(x,y) - w(x,y)\bar{\hat{\eta}}(x,y)]\}^2$$

Periodic noise reduction

Optimum Notch filter

- To minimize $\sigma^2(x,y)$

$$\frac{\partial\sigma^2(x,y)}{\partial w(x,y)} = 0$$

$$\Rightarrow w(x,y) = \frac{\overline{g(x,y)\hat{\eta}(x,y)} - \bar{g}(x,y)\bar{\hat{\eta}}(x,y)}{\bar{\eta^2}(x,y) - \bar{\eta}^2(x,y)}$$

Linear, position-invariant degradation

$$g(x, y) = H[f(x, y)] + \eta(x, y)$$

- In the absence of additive noise:
  - For scalar values of a and b, H is linear, if :

$$H[af_1(x, y) + bf_2(x, y)] = aH[f_1(x, y)] + bH[f_2(x, y)]$$

  - H is position-invariant if:

$$g(x, y) = H[f(x, y)] \Rightarrow H[f(x - \alpha, y - \beta)] = g(x - \alpha, y - \beta)$$

- In the presence of additive noise:
  - if H is position invariant:

$$g(x, y) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(\alpha, \beta) h(x - \alpha, y - \beta) d\alpha d\beta + \eta(x, y)$$

$$g(x, y) = h(x, y)^* f(x, y) + \eta(x, y)$$

$$G(u, v) = H(u, v)F(u, v) + N(u, v)$$

Linear, position-invariant degradation

Summary

- Linearly spatially-invariant degradation system with additive noise:
  - Spatial domain: Convolution of degradation function with image and addition of noise
  - Frequency domain: Product of FT of degradation function with image followed by addition of FT of noise
- Many types of degradation can be approximated by linear, position-invariant processes
- Extensive tools of linear system theory are available
- Degradation are modeled as a result of convolution $\Rightarrow$ Restoration, **image deconvolution** $\Rightarrow$ deconvolution filters

Estimating the degradation function

- Observation
- Experimentation
- Mathematical modeling

Estimation by image observation

- Gather information from the image itself
- Considering a small section of the image with a strong signal content $g_s(x, y)$ construct an un-degradation of this section by using sample gray levels $(\hat{f}_s(x, y))$
- Assuming that the effects of noise is negligible (Strong-signal area):

$$H_s(u, v) = \frac{G_s(u, v)}{\hat{F}_s(u, v)}$$

- Considering the position-invariant assumption, we construct a function $H(u, v)$ on a large scale with the same shape as $H_s(u, v)$.

Estimating the degradation function

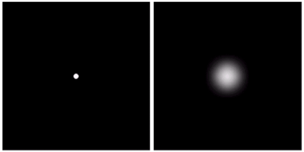
Estimation by experimentation

- Gather information using the image acquisition system (if its available)
- Imaging of impulse response (small dot of light) with the same system, to obtain impulse response of the degradation

$$H(u,v) = \frac{G(u,v)}{A}$$

A is a constant describing the strength of the impulse

- A linear space-invariant system is characterized completely by its impulse response.
  - An impulse of light (right) and Image impulse (degraded, left)



Estimating the degradation function

Estimation by modeling

- E.g: Degradation model proposed for atmospheric turbulence, where k is a constant depending on the nature of turbulence

$$H(u,v) = e^{-k(u^2+v^2)^{5/6}}$$

Estimating the degradation function

Estimation by modeling

- Blurring by linear motion:

$$g(x,y) = \int_0^T f[x - x_0(t), y - y_0(t)]dt$$

$$G(u,v) = F(u,v)\int_0^T e^{-j2\pi[ux_0(t)+vy_0(t)]}dt$$

$$\Rightarrow H(u,v) = \int_0^T e^{-j2\pi[ux_0(t)+vy_0(t)]}dt$$

- if $x_0(t) = at/T$ and $y_0 = 0 \Rightarrow$

$$H(u,v) = \int_0^T e^{-j2\pi uat/T}dt = \frac{T}{\pi ua}\sin(\pi ua)e^{-j\pi ua}$$

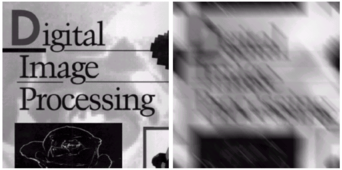
Estimating the degradation function

Estimation by modeling

- if $x_0(t) = at/T$ and $y_0 = bt/T \Rightarrow$

$$H(u,v) = \frac{T}{\pi(ua+vb)} \sin[\pi(ua+vb)] e^{-j\pi(ua+vb)}$$

- Result of motion blur wit $a = b = 0.1$ and $T = 1$



Inverse Filtering

- The simplest form of restoration, direct inverse filtering

$$\hat{F}(u,v) = \frac{G(u,v)}{H(u,v)} = F(u,v) + \frac{N(u,v)}{H(u,v)}$$

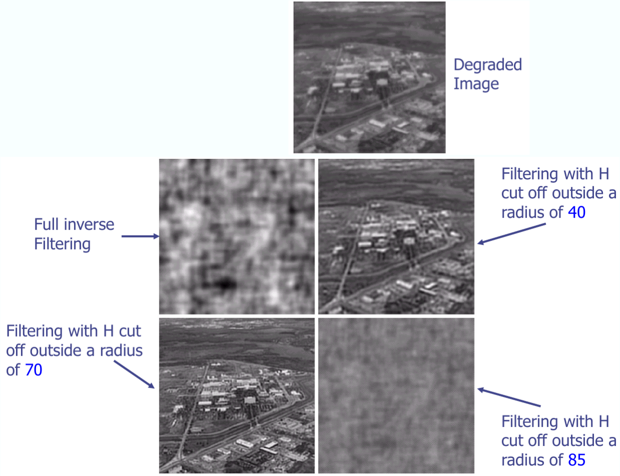
Problems!!

- Since $N(u,v)$ is un-known, even if we know the degradation function, the undegraded image cannot be recovered
- If $H(u,v) \simeq 0$, $N(u,v)/H(u,v)$ dominates the estimate of $\hat{F}(u,v)$

Solutions-1

- Limiting the filter frequencies to value near origin, in order to get around the zero or small value problem

Inverse Filtering



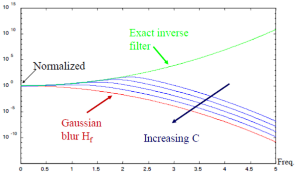
Inverse Filtering

Solutions-2

- If H has zero elements, then at those frequencies the inverse filter does not exist, thus use the following inverting equation:

$$\frac{H^*(u,v)}{H^*(u,v)H(u,v)+C} \approx \begin{cases} H^{-1}(u,v) & \text{for } |H(u,v)|^2 \gg C \\ \frac{1}{c}H^*(u,v) & \text{for } |H(u,v)|^2 \ll c \end{cases}$$

- Scaling the image at high frequencies by $\frac{1}{c}$
- Simple example:

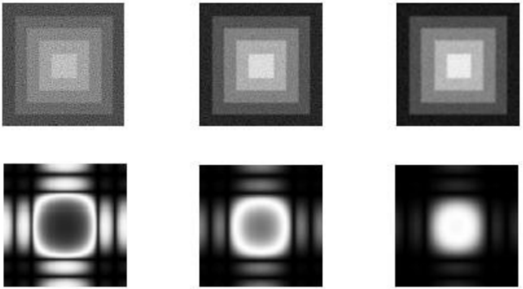


Inverse Filtering

Solution-2

Restored f :

Inverse Filter Used:



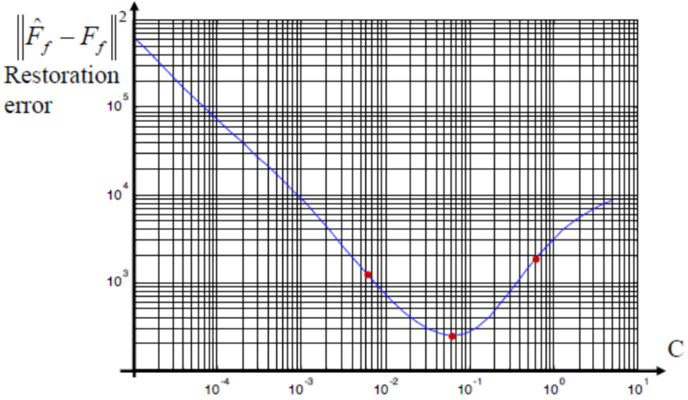
	C=0.0063	C=0.063	C=0.63
$\left\|\hat{F}_f - F_f\right\|^2 \longrightarrow$	Error=1264	Error=247	Error=2046

Larger C suppresses high frequency response in the inverse filter

Inverse Filtering

Solution-2

What is the best choice for C?



It is not a practical way !!

Inverse Filtering

Solution-2

- Drawbacks of this approach:
  - Finding the best C is not
  - C is constant for all the frequencies, maybe it should be frequency dependent
  - For high frequencies, where noise is dominant, the image is amplified by $1/C \gg 1 \Rightarrow$ amplifying the noise
- Different strategies:
  - Same as before based on $H(u, v)$
    - Where $H(u, v)$ is high apply the exact inverse, and where low apply H_f/C
  - Based on $SNR^* H(u, v)$
    - $|H(u, v)|SNR \gg 1 \Rightarrow$ inverse
    - $|H(u, v)|SNR \ll 1 \Rightarrow 0$
  - Signal/Noise $= SNR^2 = \frac{|F(u,v)|^2}{\sigma^2}$

Wiener Filter

- Incorporates both the degradation function and statistical characteristics of noise into the restoration process
- Objective is to find an estimate $\hat{f}$ of the uncorrupted image f, such that mean-square-error (MSR) between then is minimum

$$e^2 = E(f - \hat{f})^2$$

- Assumptions:
  - Noise and image are uncorrelated
  - One or the other has zero mean
  - The gray levels in the estimate are a linear function of the levels in the degradation image
- The minimum of the error function in Frequency domain:

$$\hat{F}(u,v) = [\frac{1}{H(u,v)} \frac{|H(u,v)|^2}{|H(u,v)|^2 + S_\eta(u,v)/S_f(u,v)}]G(u,v)$$

Wiener Filter (minimum mean square error filter)

$$\hat{F}(u,v) = [\frac{1}{H(u,v)}\frac{|H(u,v)|^2}{|H(u,v)|^2 + S_\eta(u,v)/S_f(u,v)}]G(u,v)$$

- $H(u,v)$, degradation function
- $H^*(u,v)$, complex conjugate of $H(u,v)$
- $|H(u,v)|^2 = H^*(u,v)H(u,v)$
- $S_\eta(u,v) = |N(u,v)|^2$, power spectrum of the noise
- $S_f(u,v) = |F(u,v)|^2$, power spectrum of the undegraded image

Wiener Filter (minimum mean square error filter)

- Signal-to-noise ratio (SNR)

$$SNR = \frac{\sum_{u=0}^{M-1}\sum_{v=0}^{N-1}|F(u,v)|^2}{\sum_{u=0}^{M-1}\sum_{v=0}^{N-1}|N(u,v)|^2}$$

- Low noise $\rightarrow$ high SNR , high noise $\rightarrow$ low SNR
- If we consider restored image $\hat{f}(x,y)$ to be **signal** and the difference between this image and the original image, **noise**:

$$SNR = \frac{\sum_{x=0}^{M-1}\sum_{y=0}^{N-1}\hat{f}(x,y)^2}{\sum_{x=0}^{M-1}\sum_{y=0}^{N-1}[f(x,y)-\hat{f}(x,y))]^2}$$

- Closer f and $\hat{f}$, this ration is higher

Wiener Filter (minimum mean square error filter)

- Full inverse filtering, radially limited inverse filtering, and Wiener filtering, respectively