

From Functional Nondeterministic Transducers to Deterministic Two-Tape Automata

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Abstract

The question whether $P = NP$ revolves around the discrepancy between active production and mere verification by Turing machines. In this paper, we examine the analogous problem for finite transducers and automata. Every nondeterministic finite transducer defines a binary relation associating input words with output words that are computed and accepted by the transducer. Functional transducers are those for which the relation is a function. We characterize functional nondeterministic transducers whose relations can be verified by a deterministic two-tape automaton, show how to construct such an automaton if one exists, and prove the undecidability of the criterion.

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Related Version The results presented in this paper were also formally proved in Isabelle/HOL, a theorem prover using higher-order logic, to guarantee their correctness. The formalization is available at <https://github.com/mraszyk/icalp20>.

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1 Introduction

One of the simplest computation models is that of a finite automaton reading its input word from left to right while deterministically updating its state according to a transition function; once the input word has been read completely, it is either accepted or rejected depending on the state in which the automaton ends up. We refer to such an automaton as a deterministic finite automaton (DFA). For a nondeterministic finite automaton (NFA), the transition function may permit no or multiple transitions from a given state; the input word is then accepted if it is accepted for any of the possible choices of transitions. The set of words accepted by an automaton A is its recognized language $L(A)$. For any machine model M , there is a set $\mathcal{L}(M)$ of the languages recognized by such machines. It is well-known that deterministic and nondeterministic automata both recognize the same class of *regular* languages [7], i.e., nondeterminism does not add any computational power to the finite automaton model.

Transducers are a generalization of automata. A transducer still reads the given input word symbol by symbol, but additionally computes a separate output word while doing so. During each transition reading one input symbol, the transducer produces a possibly empty sequence of output symbols. The final output word is obtained by concatenating the outputs



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$$\begin{array}{ccc}
& \mathcal{L}(2\text{t-DFA}) & \subsetneq \mathcal{L}(\text{NFT}) \\
& \cup & \cup \\
\mathcal{L}(\text{DFT}) & \subsetneq \mathcal{L}(\text{f-2t-DFA}) & \subsetneq \mathcal{L}(\text{f-NFT})
\end{array}$$

■ **Figure 1** The hierarchy of languages recognized in the different machine models. This paper proves the strict inclusion of $\mathcal{L}(\text{f-2t-DFA})$ into $\mathcal{L}(\text{f-NFT})$ and examines the relation between the underlying models more closely.

of all transitions. The language accepted by a transducer, called *transduction*, is thus a relation between input and output words. For a deterministic finite transducer (DFT), this relation is always a partial function. A transduction by a nondeterministic finite transducer (NFT) is not necessarily functional, however. If it is, we call the NFT functional (f-NFT).

Another way to generalize finite automata is the multi-tape model. Here, an automaton has multiple tapes, each with one head that can read the word written on the tape from left to right. There are several different but equivalent ways to model multi-tape automata, most notably the Rabin-Scott model and the Turing machine model. In the Rabin-Scott model, only one of the heads is reading a symbol and advancing to the next one in each step; the current state determines on which tape this happens. The computation is either accepted or rejected once all heads have reached the end of the tape. In the Turing machine model, in contrast, the multi-tape automaton reads all symbols at the current head positions simultaneously and can then move forward any subset of the heads in each step. In this paper, we will only examine automata with two tapes (2t-DFAs). A functional two-tape automaton (f-2t-DFA) is one that recognizes a functional relation.

We remark that both multi-tape automata and transducers can be extended to the more powerful *two-way* model that allows the heads to move not only forward but also back, from right to left. In this paper we solely consider the one-way model, however.

The main motivation underlying all computation models is to study their computational power, i.e., the set of languages that machines can recognize in each model. It is also interesting to characterize the languages that can be recognized in one given model but not the other. For instance, DFTs are less expressive than f-NFTs. Béal and Carton define the so-called *twinning* property and show that having this property is both a necessary and sufficient condition for an f-NFT to have an equivalent DFT [1].

Given a DFT, we can construct an equivalent 2t-DFA that simply simulates the DFT on the first tape and checks whether the output of the DFT matches the content of the second tape. Since a DFT is inherently functional, we obtain an f-2t-DFA. Formally, we get that $\mathcal{L}(\text{DFT}) \subseteq \mathcal{L}(\text{f-2t-DFA})$. To see that this inclusion is proper, we consider the relation that maps any binary string w to $0^{|w|}$ if the last symbol of w is 0 and to $1^{|w|}$ otherwise. It is easy to construct an f-2t-DFA that verifies this relation as follows: The automaton advances its two heads synchronously, always comparing the current output symbol with the previous one. If there is a discrepancy at any point, the automaton rejects; otherwise, the output word is guaranteed to be uniform. If the two heads reach the end of the input and output word at the same time and the last two symbols match, then the automaton accepts; otherwise, it rejects. A DFT cannot compute this relation, however, because it cannot output any symbol until it knows the last symbol of the word on the first tape, and by that time, the transducer cannot recall the length of the input word which equals the number of copies of the last symbol that the transducer needs to produce.

Furthermore, given a (functional) 2t-DFA, we can construct an equivalent (functional) NFT that nondeterministically guesses its output and verifies it by simulating the 2t-DFA.

Formally, we can conclude that $\mathcal{L}(\text{f-2t-DFA}) \subseteq \mathcal{L}(\text{f-NFT})$ and $\mathcal{L}(\text{2t-DFA}) \subseteq \mathcal{L}(\text{NFT})$. We will illustrate why both of these inclusions are in fact strict when discussing Example 10. An overview of the language classes and their inclusions is given in Figure 1.

Fischer and Rosenberg [5] have shown that, given a 2t-NFA, it is undecidable whether there is an equivalent 2t-DFA. They prove this by a reduction from the Post correspondence problem. Their proof crucially relies on nonfunctional 2t-NFAs, however. In this paper, we extend the undecidability result to f-NFTs and f-2t-DFAs.

1.1 Related Work

The comparative expressiveness of various transducer and automaton models has been a subject of intensive study ever since these machine models were introduced. Besides the mentioned results comparing DFTs to NFTs and 2t-DFAs to 2t-NFAs, Rabin and Scott [7] proved that two-way automata are as powerful as one-way automata for a single tape but recognize strictly more language with multiple tapes. Furthermore, they prove that the deterministic and nondeterministic multi-tape models are equivalent if the heads are moving synchronously on all tapes. If the heads are asynchronous, however, it is even undecidable whether a multi-tape NFA has an equivalent multi-tape DFA [5].

Filiot et al. [4] showed that it is decidable whether a two-way DFT can be transformed into an equivalent one-way DFT and provide a characterization of the two-way DFTs for which this is possible. It is known that two-way DFTs compute exactly those relations that are expressible as monadic second-order logic string transductions [3]. Yao and Rivest have shown that increasing the number of heads on a tape yields a strict hierarchy of languages [10]; see also the survey by Holzer et al. [6]. A new result by Raszyk et al. [8] proves that DFTs with multiple heads are strictly more expressive than f-NFTs. Asking whether multi-head two-way DFAs can simulate two-head one-way NFAs is equivalent to the famous question whether $L = NL$ [9].

1.2 Our Contributions

We compare the computational power of 2t-DFAs and f-NFTs. To this end, we describe a subclass of f-NFTs characterizing the relations computable by a 2t-DFA, i.e., given an f-NFT, there is an equivalent 2t-DFA if and only if the f-NFT lies in said subclass. Finally, we show this membership question to be undecidable.

We have formally proved the main results of this paper (Theorems 8 and 9 and Claims 14 and 15) as well as Examples 7 and 10 in the Isabelle proof assistant [2], ensuring their correctness. We also provide pen-and-paper proofs for all of our results.

2 Preliminaries

We formally describe the automaton model and transducer model considered in this paper and then introduce some useful terminology, in particular our core notion of bounded trailing.

2.1 Two-Tape Deterministic Finite Automaton

There are many equivalent ways of extending the one-tape automaton model to multiple tapes. We base our definition on the common Turing machine model.

► **Definition 1.** A two-tape deterministic finite automaton (2t-DFA), is a septuple $A = (Q, \Sigma, \Gamma, \sqcup, \delta, q_0, F)$ consisting of

- 125 ■ finite, nonempty sets Q , Σ , and Γ , the set of states and the input and output alphabet,
- 126 ■ a blank symbol $\sqcup \notin \Sigma \cup \Gamma$,
- 127 ■ a transition function $\delta: Q \times (\Sigma \cup \{\sqcup\}) \times (\Gamma \cup \{\sqcup\}) \rightarrow Q \times \{N, R\} \times \{N, R\}$ satisfying

$$(q, \sigma, \gamma) \mapsto (q', m_1, m_2)$$
- 128 ■ $\sigma = \sqcup \implies m_1 = N$ (i.e., the first head cannot move past the end delimiter \sqcup),
- 129 ■ $\gamma = \sqcup \implies m_2 = N$ (i.e., the second head cannot move past the end delimiter \sqcup), and
- 130 ■ $m_1 = R \vee m_2 = R$ (i.e., at least one head advances in every step), and
- 131 ■ an initial state $q_0 \in Q$ and a set of accepting states $F \subseteq Q$.

132 The computation of a 2t-DFA proceeds as follows. There are two tapes that we call the
 133 *input* and *output* tape; the former contains a word $a \in \Sigma^*$, the latter a word $u \in \Gamma^*$. Both
 134 words are delimited by the blank symbol \sqcup at the end. The two tapes have one reading
 135 head each, which we refer to as the *input* and *output* head. The input and output head are
 136 initially positioned on the first symbol of a and u , respectively. Depending on the current
 137 state of the automaton and what symbols the two heads are reading, either the input or
 138 output head or both advance by one symbol in each step. As soon as a head reaches the
 139 blank symbol delimiting a word, it cannot move any further. The computation ends when
 140 both heads have reached the blank symbol. A word pair (a, u) is in the language accepted
 141 by the automaton if and only if the computation on this pair of words ends in an accepting
 142 state. A *configuration* consists of the positions of the two heads and the current state. We
 143 write $q \xrightarrow[a]{u} q'$ if the automaton can start in a state q and end up in a state q' by reading a
 144 word a with the input head and u with the output head.

145 2.2 Nondeterministic Finite Transducer

146 We now formally define a nondeterministic finite transducer.

147 ► **Definition 2.** A nondeterministic finite transducer (NFT) is a sextuple $T = (Q, \Sigma, \Gamma, \delta,$
 148 $q_0, F)$, where Q , Σ , Γ , q_0 , and F are defined as for a 2t-DFA in Definition 1, but the
 149 transition function δ now is a function that maps each pair $(q, \sigma) \in Q \times \Sigma$ of a state and
 150 input symbol to a finite subset of $Q \times \Gamma^*$, describing the nondeterministic transition options.

151 The computation of an NFT proceeds just like the computation of an NFA except that
 152 the NFT produces in each step a possible empty sequence of output symbols. The outputs
 153 from the single steps are concatenated to obtain the final output of the complete computation.
 154 Using the image of a two-tape machine, we can also view the computation of an NFT as
 155 reading the word on the input tape symbol-wise while writing to the initially empty output
 156 tape, appending each step's output. We write $q \xrightarrow[a]{u} q'$ if the transducer can go from a state
 157 q to a state q' with the input head reading $a \in \Sigma^*$ and the output head producing $u \in \Gamma^*$.

158 The computation ends once the entire input word has been read, i.e., when the input head
 159 has reached the blank symbol. If the transducer is in an accepting state at this moment, then
 160 the word pair (a, u) on the two tapes is in the relation $L(T)$ computed by T . If the transition
 161 function does not offer any option for a step and thus forcibly ends the computation before
 162 the blank symbol on the input tape is reached, then this computation does not contribute to
 163 the relation $L(T)$.

164 A binary relation $R \subseteq \Sigma^* \times \Gamma^*$ is *functional* if and only if every $a \in \Sigma^*$ is associated with
 165 at most one $u \in \Gamma^*$, that is, $\forall a \in \Sigma^*: |\{u \in \Gamma^* \mid (a, u) \in R\}| \leq 1$.

166 ► **Definition 3.** An NFT T is a *functional nondeterministic finite transducer* (f-NFT) if
 167 $L(T)$ is a functional relation.

2.3 Terminology

We will use the following two notions many times in our proofs.

Definition 4 (Shortcut guarantee g). *Let an NFT $T = (Q, \Sigma, \Gamma, \delta, q_0, F)$ be given. Let $Q' \subseteq Q$ denote the set of all useful states, that is, states that are part of at least one accepting computation. For every $q \in Q'$, let $g_q = \min\{|x| \mid \exists u \in \Gamma^*, f \in F: q \xrightarrow{u} f\}$ denote the length of a shortest word x that leads from q into an accepting state. We call $g(T) = \max_{q \in Q'} g_q$ the shortcut guarantee of T . If T has no accepting computation, we formally define $g(T) = \infty$.*

Definition 5 (Output speed s). *Let an NFT $T = (Q, \Sigma, \Gamma, \delta, q_0, F)$ be given. We call $s(T) = \max\{|\gamma| \mid \exists q, q' \in Q, \sigma \in \Sigma: (q', \gamma) \in \delta(q, \sigma)\}$ the output speed of T . Note that $s(T)$ is well-defined since δ only maps to finite subsets of $Q \times \Gamma^*$.*

Finally, we introduce our notion that characterizes f-NFTs having an equivalent f-2t-DFA.

Definition 6 (Bounded trailing). *An NFT $T = (Q, \Sigma, \Gamma, \delta, q_0, F)$ has bounded trailing if*

$$\exists t \in \mathbb{N}: \quad \forall f_1, f_2 \in F, \quad q_1, q_2 \in Q, \quad a, b_1, b_2 \in \Sigma^*, \quad u, v, w_1, w_2 \in \Gamma^*:$$

$$q_0 \xrightarrow{uv} q_1 \xrightarrow{b_1} f_1 \quad \wedge \quad q_0 \xrightarrow{u} q_2 \xrightarrow{vw_2} f_2 \quad \implies \quad |v| \leq t$$

Otherwise, we say that T has unbounded trailing.

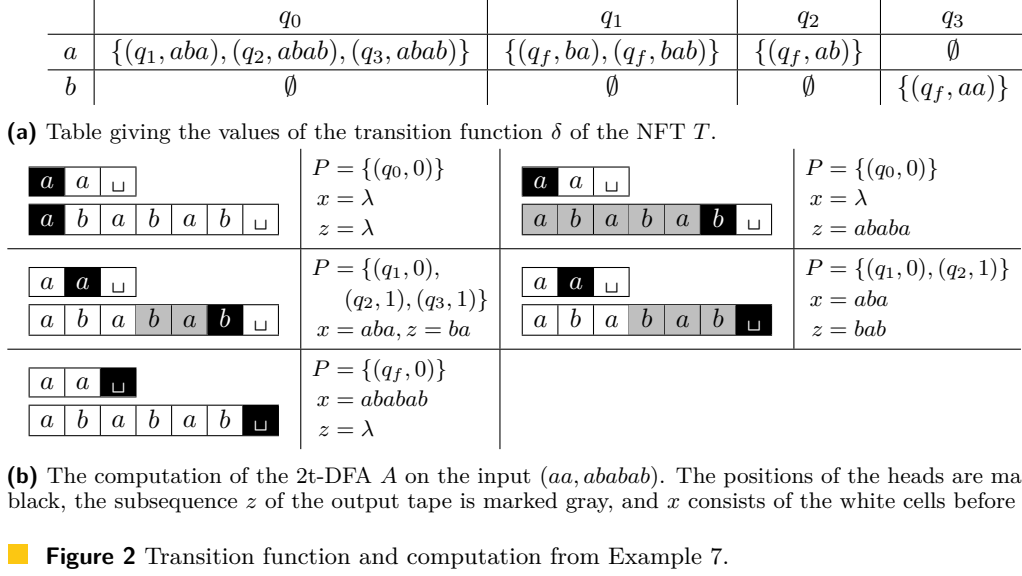
The following three sections prove bounded trailing to be a necessary and sufficient condition for an f-NFT to have an equivalent 2t-DFA and that bounded trailing is undecidable.

3 Bounded Trailing is Sufficient

In this section, we show that any NFT that has bounded trailing can be transformed into an equivalent 2t-DFA. Let T be an NFT with a trailing bound $t \in \mathbb{N}$. We construct an equivalent 2t-DFA A that simulates all nondeterministic computations of transducer T that are compatible with the output seen so far. Automaton A uses its states to maintain a subword z of the output word with the following property. For the currently read prefix p of the input word, there is a prefix x of the output word such that for any computation of T that starts in the initial state q_0 of T , reads p , reaches a state q of T , and produces an output w consistent with the given output tape, we can write w as xy for a prefix y of z . In other words: Whatever prefix p of its input word automaton A has read at the moment, there is an x such that every computation of T on p consistent with A 's output word has the form $q_0 \xrightarrow{p} q$ for some prefix y of z .

Automaton A stores in its current state a representation of each such computation of T , namely the pair $(q, |y|)$. We show that this is feasible with a finite set of states by maintaining a subword z of length at most $r = s + t$, where s and t are T 's output speed and trailing bound, respectively. Initially, z is empty and the set P of pairs (q, n) stored in A 's state contains only a single pair, $(q_0, 0)$, where q_0 is T 's initial state. This reflects the fact that the only computation of T on the empty prefix of the input tape keeps T in its initial state and produces no output. If $L(T) = \emptyset$, then A immediately rejects in its initial state.

Automaton A now proceeds as follows. As long as the length of the subsequence z stays below r and the output tape has not been fully read, the next symbol on the output tape is read and appended to z . Moreover, A removes from the set P all representations of computations that are no longer consistent with the extended subsequence z of the output



tape. If the set P becomes empty, then automaton A rejects, because there is no computation of T consistent with the given input and output tape. Otherwise, A determines the minimum m such that $(q, m) \in P$ for any q and drops the first m output symbols from the subsequence z . This corresponds to cutting off from z a prefix of length m and appending it to x . Note that this is sound because none of the stored computations end before outputting the new x . This way, A maintains the invariant that there is some state q of T such that $(q, 0) \in P$.

Once the length of the subsequence z becomes r or the output tape has been fully read, automaton A reads in the next symbol from the input tape and then simulates every single step that is nondeterministically possible for every single stored computation and updates the set P accordingly to a new P' . The mentioned invariant guarantees that $(q, 0) \in P$ for some state q of T . The fact that T has bounded trailing with a trailing bound t implies that $n \leq t$ holds for every pair $(q, n) \in P$. Hence, performing one further nondeterministically possible step continuing T 's computation represented by $(q, n) \in P$ yields a pair $(q', n') \in P'$ that satisfies $n' \leq t + s$ since s is the longest output that T can produce while reading a single symbol. This is just within the length limit $r = s + t$ that we set for z . As before, automaton A rejects if the set P' becomes empty, because this means there is no computation of T consistent with the given input and output tape. Otherwise, automaton A performs on P' the normalization described in the previous paragraph to obtain a new set P that maintains the invariant that there is some q' for which $(q', 0) \in P$.

Finally, if both the input and output tape have been fully read, automaton A accepts if and only if there is some $q \in F$ with $(q, |z|) \in P$, i.e., an accepting state q of T that some computation of T arrives at after producing an output that matches xz until the very end, meaning that the output is equal to the content of the output tape.

► **Example 7.** We consider the NFT T with the set of states $Q = \{q_0, q_1, q_2, q_3, q_f\}$, initial state $q_0 \in Q$, the set of accepting states $F = \{q_f\}$, and the transition function δ given in Figure 2a. We can check that the transduction computed by T is $L(T) = \{(aa, ababa), (aa, ababab), (ab, ababaa)\}$ and that the trailing of T is bounded by $t = 1$.

The computation of automaton A on the input-output pair $(aa, ababab)$ is summarized in Figure 2b. We now describe this computation in detail. Because $L(T) \neq \emptyset$, the initial state of automaton A is storing $z = \lambda$, where λ denotes the empty word, and $P = \{(q_0, 0)\}$. The

length of the longest output that T can produce while reading one input symbol is $s = 4$. Hence, the maximum length of the subsequence z maintained in A 's state is $r = s + t = 5$.

Because the output tape consists of six symbols, the first $r = 5$ of them are read and appended to z . Since all output words in $L(T)$ start with the same five symbols, the only pair in P , namely $(q_0, 0)$, stays consistent with all of them while successively reading the first five output symbols and appending them to z . After this, A 's state is storing $z = ababa$ and $P = \{(q_0, 0)\}$.

Now that the length of z is $5 = r$, the first symbol a from the input tape is read. All nondeterministic steps from $\delta(q_0, a) = \{(q_1, aba), (q_2, abab), (q_3, abab)\}$ are consistent with z , which leads to $P' = \{(q_1, 3), (q_2, 4), (q_3, 4)\}$. The minimum m for which $(q, m) \in P'$ for any q is $m = 3$. After performing the normalization of P' with this m , the state of automaton A is storing $z = ba$ and $P = \{(q_1, 0), (q_2, 1), (q_3, 1)\}$.

Since the length of z is now $2 < r$, automaton A reads the next output symbol b and appends it to z , which thus becomes $z = bab$. Then A removes the pair $(q_3, 1)$ from the set P because this pair is no longer consistent with the extended subsequence $z = bab$ since the only possible transition from q_3 produces the output aa , which is inconsistent with the suffix ab of $z = bab$. The remaining two pairs $(q_1, 0)$ and $(q_2, 1)$ are still consistent with the extended $z = bab$. As $(q_1, 0)$ stays in P , no normalization need be performed.

Now that the end of the output tape has been reached, A reads in the next symbol a from the input tape despite $|z| = 3 < r$. Performing one step that reads a on every pair in P yields $P' = \{(q_f, 3)\}$. Note that $(q_f, 2) \notin P'$ because it is not consistent with b , the last symbol of z . After one more normalization, A 's state is storing $z = \lambda$ and $P = \{(q_f, 0)\}$.

Finally, A has reached the end of both the input and output tape, thus it checks whether $(q, |z|) \in P$ for some $q \in F$. As $(q_f, 0) \in P$, $q_f \in F$, $|z| = 0$, A accepts the input-output pair $(aa, ababab) \in L(T)$.

We conclude this section by formally stating its main result.

► **Theorem 8.** *Any NFT with bounded trailing has an equivalent 2t-DFA.*

4 Bounded Trailing is Necessary

In this section, we prove the reverse of Theorem 8 for the functional case.

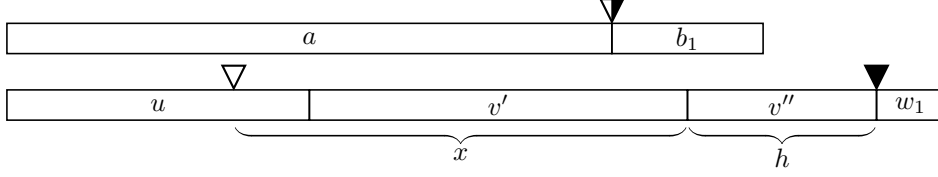
► **Theorem 9.** *Any f-NFT with an equivalent 2t-DFA has bounded trailing.*

Proof. Let an NFT T and a 2t-DFA A with $L(T) = L(A)$ be given. We assume that T has unbounded trailing and contradict the functionality of T , thus proving the theorem. We may assume without loss of generality that A moves exactly one head in each step because any step moving both heads at once can be simulated by two steps moving only one head at a time using one additional intermediate state.

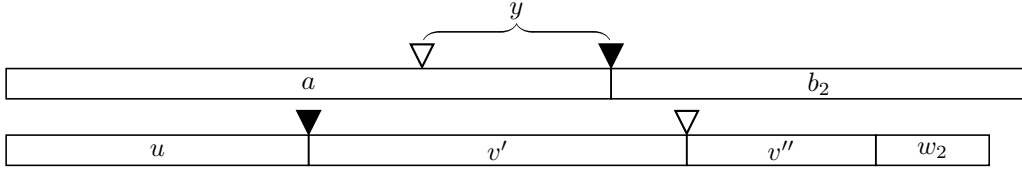
Denote the state sets of transducer T and automaton A by Q_T and Q_A , respectively. Let g and s be the transducer's shortcut guarantee and output speed, respectively. Finally, we define a *homestretch length* $h = (g + 1) \cdot (|Q_A| + 1)$ and a *trail length* minimum $t = |Q_T| \cdot |Q_A| \cdot s \cdot (h + (g + 1)s + 1)$. The reason for choosing exactly these values will become clear during the proof. For now, we note that they depend only on the given transducer T and the automaton A , hence they are constant within this proof.

Since T has unbounded trailing, there are accepting computations $q_0 \xrightarrow{a}{uv} q_1 \xrightarrow{b_1}{w_1} f_1$ and $q_0 \xrightarrow{a}{u} q_2 \xrightarrow{b_2}{vw_2} f_2$ such that $|v| > t$. We split the *trail* v into a prefix v' of length $|v| - h$ and the remaining *homestretch* v'' of length h . We consider A 's computation on the

input-output prefix (a, uv') . The two-tape automaton model ensures that both heads of A will eventually reach the end of a and uv' , respectively. Depending on which one does so first, we distinguish two cases.



(a) Case 1: Automaton A 's head positions when the input head has just reached the end of a and transducer T 's head positions after the computation $q_0 \xrightarrow{a/uv} q_1$.



(b) Case 2: Automaton A 's head positions when the output head has just reached the end of uv' and transducer T head positions after computing $q_0 \xrightarrow{a/u} q_2$.

■ **Figure 3** Two different input-output pairs in the relation $L(A) = L(T)$. The black and white triangles represent the heads of transducer T and automaton A , respectively.

Case 1: Input head is first In this case, we consider the input-output pair (ab_1, uvw_1) ; see Figure 3a and recall that $v = v'v''$. We begin by showing why we may assume without loss of generality that $|b_1| \leq g$. For this, we consider transducer T 's configuration after the computation $q_0 \xrightarrow{a/uv} q_1$; the corresponding head positions are indicated by the black triangles in Figure 3a. The shortcut guarantee g ensures the existence of a word pair $(b, w) \in \Sigma^* \times \Gamma^*$ and a final state $f \in F$ such that $q_1 \xrightarrow{b/w} f$ and $|b| \leq g$; we can therefore substitute b , w , and f for b_1 , w_1 , and f_1 .

Now, we consider automaton A 's computation on the same input-output pair (ab_1, uvw_1) . Let x denote the suffix of uv' that has not yet been read by A when the input head has just reached the end of a ; see the white triangles in Figure 3a for the automaton's head positions. The input head has only b_1 left to read but the output head all of $xv''w_1$. We have already established that $|b_1| \leq g$ using the shortcut guarantee and we know that $|xv''w_1| \geq |v''| = h$ since v'' is the homestretch. In each step, A advances exactly one head by exactly one symbol, and a head does not move anymore once it has reached the blank symbols at the end of the given word pair. Thus there are at most g movements of the input head left but at least h by the output head. We call a step in which the input head moves an *input step* and a step in which the output head moves an *output step*. The input steps split the remaining computation into at most $g + 1$ sequences of consecutive output steps. Since there are at least h output steps, there is at least one sequence of $h/(g + 1)$ uninterrupted output steps. Because $h/(g + 1) > |Q_A|$ there are within that sequence at least two different output steps leading A into the same state. Choosing any two such steps, we can cut out the nonempty part of the output word that starts at the position of the output head immediately after the first step and ends with the symbol at the position of the output head just before the second step. This results in an accepting computation, with the word on the input tape unmodified. Hence A associates two different output words, uvw_1 and some shorter one, with the same input word ab_1 , contradicting the functionality of $L(A) = L(T)$.

Case 2: Output head is first In this case, the output head reaches the end of uv' before the input head has finished reading a . This remains true for A 's computation on the input-output pair (ab_2, uvw_2) ; see Figure 3b. Let y be the still unread suffix of a at the moment when the output head reaches the end of uv' . At this point, the input head still has to read yb_2 and the output head $v''w_2$. In the following paragraph we will establish an upper bound on the length of the remaining output $v''w_2$.

The homestretch length $|v''| = h$ is already fixed. The length of w_2 can be bounded by combining the shortcut guarantee g and the output speed s as follows. Consider the transducer's computation $q_2 \xrightarrow{b_2/vw_2} f_2$. Since the output head can write at most s symbols in a single step, we can cut off from b_2 a prefix b' such that $q_2 \xrightarrow{b'/vw'} q'$ for some prefix w' of w_2 with length $|w'| < s$ and a state q' . Denote by b'' and w'' the remaining suffixes such that we have $b_2 = b'b''$ and $w_2 = w'w''$. The state q' is useful as evidenced by the accepting computation $q_0 \xrightarrow{a/u} q_2 \xrightarrow{b'/vw'} q' \xrightarrow{b''/w''} f_2$. By the definition of g , we can therefore shortcut this computation by substituting $q' \xrightarrow{b''/w''} f'$ with $|b''| \leq g$ for its last part. From this we immediately obtain $|w''| \leq g \cdot s$ since the output speed s tells us how many symbols the transducer can output at most when reading one input symbol. Replacing b_2 by $b'b''$ and w_2 by $w'w''$ if necessary, we can thus assume without loss of generality that $|w_2| \leq |w'w''| < s + g \cdot s = (g+1)s$. This finally gives us the desired upper bound $|v''w_2| < h + (g+1)s$ on the length of the remaining output word that A still has to read together with the input word b_2 .

We continue to examine the computation $q_2 \xrightarrow{b_2/vw_2} f_2$ by transducer T ; our focus now lies on b_2 instead of w_2 , however. In every computation step the transducer reads one input symbol and produces an output of length at most s . Thus we know that there are at least $|vw_2|/s$ computation steps during which T transduces one symbol from b_2 into a nonempty output; we call these steps and the corresponding head positions *productive*. Using the trail length bound $|v| > t$, we can see that the number of productive steps during the transduction of b_2 is at least $|vw_2|/s > t/s$.

Switching to the automaton computation, we now know that in the case under consideration A has an output suffix of length at most $h + (g+1)s$ left to read. As before we use the fact that only one of the two heads can move in each step, but now in order to split the input steps into at most $h + (g+1)s + 1$ consecutive sequences of steps in which only the input head advances through yb_2 , with the output head standing still. Since transducer T has over t/s productive steps during the transduction of b_2 —as established in the previous paragraph—there has to be, among the up to $h + (g+1)s + 1$ sequences of consecutive input steps by A , at least one during which the input head passes more than $(t/s)/(h + (g+1)s + 1) \geq |Q_A| \cdot |Q_T|$ of T 's productive positions. Thus there inevitably are two different productive positions among these such that the corresponding two steps advancing the input head from these positions both lead automaton A and transducer T into the same state from Q_A and Q_T , respectively. We cut out from the input ab_2 the part that starts immediately after the first of these two positions and ends with the symbol at the second position. From uvw_2 we remove the nonempty part that is produced by the transducer during the corresponding productive steps. We thus obtain two shortened words $d \in \Sigma^*$ and $w \in \Gamma^*$ such that T can transduce d into w and accept, while the computation of automaton A with the new input word d and the old output word uvw_2 on its tapes will still be accepting. Hence two different output words w and uvw_2 are associated to the same input word d by the relation $L(A) = L(T)$, which yields the desired contradiction and concludes the proof. ◀

► **Example 10.** We present an illustrative example of an f-NFT with unbounded trailing.

359 According to Theorem 9, there is no equivalent f-2t-DFA, which implies that the inclusion
 360 $\mathcal{L}(\text{f-2t-DFA}) \subseteq \mathcal{L}(\text{f-NFT})$ is proper; see Figure 1.

361 The transducer's alphabets are $\Sigma = \Gamma = \{0, 1\}$ and its relation is $\{(0^i 10^j 10, 0^i) \mid i, j \in$
 362 $\mathbb{N}\} \cup \{(0^i 10^j 11, 0^j) \mid i, j \in \mathbb{N}\}$. This union contains exactly the pairs of the form $(0^i 10^j 1\beta, 0^k)$,
 363 where β is a bit valued 0 or 1 and $k = i$ if $\beta = 0$ and $k = j$ if $\beta = 1$. This relation is easily
 364 computed by an f-NFT that nondeterministically guesses β , then copies either 0^i or 0^j to
 365 the output tape, and finally accepts or rejects depending on whether the guess was correct.

366 No f-2t-DFA can compute this relation, however, for the following intuitive reason.
 367 Consider an input-output pair $(0^i 10^j 1\beta, 0^k)$ with sufficiently large i, j , and k . By the time
 368 the input head reaches the first 1, the output head has either read most of 0^k already or
 369 has a large part of it still lying ahead. In the first case, the automaton may have potentially
 370 checked whether $k = i$, but cannot remember how many zeroes of 0^k the output head has
 371 already passed making it impossible to check whether $k = j$ if the input ends with $\beta = 1$.
 372 In the second case, the automaton can potentially still check whether $k = j$, but cannot
 373 remember how many zeroes the input head has already passed, i.e., the value of i , making it
 374 impossible to check whether $k = i$ if the input ends with $\beta = 0$. Therefore, the automaton
 375 fails on inputs with $\beta = 1$ in the first case and on inputs with $\beta = 0$ in the second case.

376 To show formally that the described transducer has unbounded trailing we use the
 377 variable names of Definition 6. For any given $t \in \mathbb{N}$, we can choose $u = b_1 = w_1 = w_2 = \lambda$,
 378 where λ denotes the empty word, $a = v = 0^t$, $b_1 = 10^t 10$, and $b_2 = 10^t 11$. This yields two
 379 computations showing that the trailing bound must be at least t , namely

$$380 \quad q_0 \xrightarrow{0^t} q_1 \xrightarrow{\frac{10^t 10}{\lambda}} f_1 \text{ and } q_0 \xrightarrow{\frac{0^t}{\lambda}} q_2 \xrightarrow{\frac{10^t 11}{0^t}} f_2.$$

381 5 Bounded Trailing is Undecidable

382 In this section, we prove that determining whether an f-NFT has bounded trailing is
 383 undecidable. This is achieved by reducing the halting problem on the empty input, which is
 384 known to be undecidable, to the problem of determining whether an f-NFT has bounded
 385 trailing. We present a reduction via a third problem, namely determining whether a Turing
 386 machine reaches infinitely many configurations on the empty input.

387 Undecidable Problems about Turing Machines

388 We begin by formally defining our standard model of a deterministic Turing machine with a
 389 single tape that is unbounded in both directions.

390 ► **Definition 11.** A Turing machine is a sextuple $M = (Q, \Gamma, \sqcup, q_0, F, \delta)$, where

- 391 ■ Q is a finite, non-empty set of states,
- 392 ■ Γ is a finite, non-empty set of alphabet symbols,
- 393 ■ $\sqcup \in \Gamma$ is the blank symbol,
- 394 ■ $q_0 \in Q$ is the initial state,
- 395 ■ $\delta : (Q \setminus F) \times \Gamma \rightarrow Q \times (\Gamma \setminus \{\sqcup\}) \times \{L, R\}$ is a (partial) transition function, and
- 396 ■ $F \subseteq Q$ is the set of accepting states.

397 A *configuration* of a Turing machine consists of its current state, the content of the
 398 tape, and the position of the head on the tape. We will only consider the computations
 399 of a Turing machine on the empty input, the initial configuration thus always consists of
 400 the initial state q_0 , a tape containing only blank symbols, and the head scanning one of
 401 them. A configuration is called *accepting* if its current state q is accepting, i.e., if $q \in F$.

A configuration is called *halting* if it is accepting or the transition function is undefined for the current state $q \in Q \setminus F$ and the symbol $a \in \Gamma$ currently scanned by the head. If a configuration is not halting, the next configuration reached in one step of the Turing machine's computation is obtained by updating the current state, writing a non-blank symbol to the tape's cell scanned by the head, and moving the head either one cell to the left or one cell to the right.

Any configuration reached during the Turing machine's computation on the empty input consists of a finite contiguous sequence of non-blank symbols and the position of the head scanning either any symbol within this sequence or one of the two blank symbols delimiting it. Hence, we can represent a configuration of the Turing machine as a finite sequence of cells of two types: In every configuration there is exactly one cell of the first type, namely the one currently scanned by the head. In our representation, this type of cell contains some potentially blank symbol and the current state. The second type of cells contains a non-blank symbol only. The initial configuration, for example, is represented by a single cell containing the blank symbol and the initial state—recall that we consider only the computation on the empty input.

A Turing machine halts on the empty input if it reaches a halting configuration during its computation starting in the initial configuration. The undecidability of determining whether a halting configuration can be reached is well known. We show that it is also undecidable whether a given Turing machine reaches infinitely many different configurations during its computation on the empty input.

► **Lemma 12.** *The problem of determining whether a Turing machine reaches infinitely many different configurations during its computation on the empty input is undecidable.*

Proof. We prove the lemma by contradiction. Suppose that there is an algorithm A_∞ that decides for every given Turing machine M whether it reaches infinitely many different configurations on the empty input. We will use A_∞ to design an algorithm A_H that decides for any Turing machine M whether it reaches a halting configuration on the empty input. The latter problem is known to be undecidable, yielding the desired contradiction.

Given a Turing machine M , algorithm A_H first invokes A_∞ to decide whether M reaches infinitely many different configurations on the empty input. If it does, then A_H outputs “No” because reaching a halting configuration implies reaching only finitely many configurations in total. Otherwise, A_H simulates M 's deterministic computation on the empty input step by step, remembering all configurations, until either a halting or a previously encountered configuration is reached. Then A_H outputs “Yes” in the former case and “No” in the latter. ◀

Because both the set of states and the alphabet are finite, the set of all configurations of a bounded length is necessarily finite. It follows that a Turing machine M reaches infinitely many configurations if and only if it reaches configurations of arbitrary length.

► **Corollary 13.** *The problem of determining whether a Turing machine reaches configurations of arbitrary length on the empty input is undecidable.*

Reduction to Bounded Trailing

We show how to construct from a given deterministic Turing machine M an f-NFT $T(M)$ such that T has unbounded trailing if and only if M reaches configurations of arbitrary length.

A valid input for T is a sequence of M 's configurations followed by one of two special symbols that we call *mode indicators*. The configurations are represented by finite sequences

c_1	c_2	c_3	\cdots	c_k	copy			c_1	c_2	c_3	\cdots	c_k	step
c_1	c_2	c_3	\cdots	c_k				c_0	c'_1	c'_2	\cdots	c'_{k-1}	c'_k

(a) The two types of valid inputs for T and the corresponding accepted outputs. Each c_i with $i > 0$ encodes an arbitrary configuration of M , c_0 is its initial configuration, and c'_i is c_i 's successor configuration.

c_0	c_1	c_2	\cdots	c_k	copy			c_0	c_1	c_2	\cdots	c_k	step
c_0	c_1	c_2	\cdots	c_k				c_0	c_1	c_2	\cdots	c_k	c_{k+1}

(b) Two accepting computation patterns of T that make unbounded trailing inevitable if M reaches arbitrarily long configurations. Here, c_0 is still the initial configuration of M on the empty input, but c_{i+1} is now the successor configuration of c_i , under a single computation step of M .

■ **Figure 4** Accepting computation patterns of transducer T , which depends on Turing machine M .

of cells as described in the previous section and separated from each other by a dedicated symbol not occurring anywhere else. The two mode indicators are represented by a cell containing either of the two words in $\{\text{copy}, \text{step}\}$. The transducer T starts its computation by nondeterministically guessing the mode indicator and then operates in the corresponding mode described below. If the guess turns out to be wrong or if the input is invalid in any way, the computation is aborted and the transducer rejects the input.

Copy Mode Copy the given input to the output tape symbol by symbol while moving the heads synchronously, omitting only the mode indicator in the end.

Step Mode In the first step, output M 's entire fixed initial configuration c_0 while reading and remembering the first input symbol. Then read from the input sequence one configuration c after the other. While reading c , compute and output the successor configuration c' that M reaches from c in one computation step. Of course, this all assumes that such a configuration c' exists; otherwise, T aborts the computation as it does for invalid inputs.

The two types of input-output pairs after an accepting computation are depicted in Figure 4a.

To see that T can in fact realize these computations, observe that the changes necessary to turn a configuration c into its successor configuration c' do, on the one hand, only depend on the single type-one cell of c containing the currently scanned symbol and the current state and, on the other hand, only affect the immediate proximity of this cell.

Hence, the successor configuration c' can indeed be computed from c by a finite transducer that essentially is still copying each configuration symbol by symbol, but with a buffer of three cells, which allows it to modify the configuration in the right place to produce the successor configuration. Transducer T can be effectively computed from any given Turing machine M ; we omit the technicalities.

We can easily check that the described transducer T is functional as follows. On the one hand, it rejects all invalid inputs anyway. On a valid input, on the other hand, T takes only a single nondeterministic decision to choose the operating mode and then accepts for only one of the two choices, depending on the mode indicator. We state the functionality formally.

▷ **Claim 14.** Transducer T is functional, i.e., T is an f-NFT.

We will now sketch the proof of the crucial connection between the length of M 's configurations and T 's trailing.

▷ **Claim 15.** The f-NFT T has unbounded trailing if and only if Turing machine M reaches configurations of arbitrary length during its computation on the empty input.

The transducer T takes only one nondeterministic decision during any computation, namely to operate in either copy or step mode, the rest of the computation is deterministic.

According to Definition 6, the only way for any trailing to occur is therefore a pair of two computations, one in copy mode and one in step mode, producing consistent output words. One of these two output words is a prefix of the other; we call it the common prefix of the two computations. See Figure 4b for an example of the arising situation. In the next paragraph we argue why the configurations within the common prefix represent a valid computation of M and why the maximal trail length is equal to the length of the longest configuration, up to a constant.

Since transducer T always starts by outputting the initial configuration c_0 in step mode, this has to be the first configuration on the output tape in copy mode as well. In copy mode, T can only write this configuration c_0 to the beginning of the output tape if c_0 is also the first configuration on the input tape. The step mode behavior now ensures that the second configuration on the output tape is the successor of c_0 , call it c_1 . Iterating this argument, we see that the common prefix indeed contains a valid computation c_0, c_1, \dots, c_k , where each configuration is the successor of the previous one under one computation step of M . Finally, the trail is always as long as the configuration that is currently being read, up to the size of the three cells contained in the buffer. Thus the maximal trail length is indeed always equal to the length of the longest configuration occurring in a computation of M on the empty input word up to constant.

Finally, we state the main result of this section.

► **Theorem 16.** *Whether a given f-NFT has bounded trailing is an undecidable problem.*

Proof. We prove the theorem by contradiction. Suppose that there is an algorithm A deciding whether an f-NFT has bounded trailing. We construct an algorithm A_∞ deciding whether a Turing machine M reaches configurations of arbitrary length. The latter problem is undecidable by Corollary 13, yielding the contradiction.

Given a Turing machine M , algorithm A_∞ constructs the f-NFT T and uses algorithm A to decide whether T has bounded trailing. If it does, then A_∞ outputs “No,” otherwise it outputs “Yes.” The correctness of algorithm A_∞ follows from Claim 15. ◀

6 Conclusion

We have shown f-NFTs to be more powerful than f-2t-DFAs, characterized f-NFTs having an equivalent f-2t-DFA as those with bounded trailing, proved that it is undecidable whether an f-NFT has bounded trailing, and showed how to construct the f-2t-DFA if it is possible. The undecidability of transforming a 2t-NFA into an equivalent 2t-DFA has been known for the nonfunctional case already, albeit without the characterization and without construction of the deterministic automaton [5]. The picture painted by these results is still not complete, however, even if we restrict our attention to the question of decidability exclusively, ignoring the characterization and construction. The undecidability in the nonfunctional case was proved using 2t-NFAs whose relations map every input word to infinitely many output words. Thus, it still remains open whether the undecidability still holds for NFTs that are not functional but still finite-valued.

The results of this paper already show that bounded trailing is sufficient for an equivalent 2t-DFA to exist for any NFT, finite-valued ones in particular. We conjecture that our proof of the necessity of this condition can be expanded to cover the finite-valued case with more technical work. Together with the established undecidability of bounded trailing in the functional and hence also finite-valued case, this would prove that our undecidable criterion of bounded trailing still holds for finite-valued NFTs, thus closing this remaining gap.

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