The Balancing Act: Finding an Ideal Environmental Policy in a World of Tradeoffs

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Abstract

We consider the control problem of optimally taxing a firm to reduce pollution. We use a state equation generated from economic principles and solve a simple cost functional. We then explore variations in our problem by considering a time-dependent natural abatement rate, an alternate, "green" production possibility, and a more realistic discrete policy requirement. In most settings, optimal policy results in a steady state level of both taxes and pollution. We find that in environments where pollution abates quickly, lower tax rates are optimal. Initial conditions do not affect final steady state outcomes.

1 Background

In recent years, environmental policy has become a focal point in public economics. Many government entities seek to mitigate climate change through public policy. While governments may not be able to directly control how businesses run, they can influence their production decisions by adjusting incentives.

Pollution is a primary example of what economists call a "negative externality". A negative externality occurs when a firm seeking to maximize profit does not consider what impact their activity will have on their surroundings. As Thomas Helbling of the International Monetary Fund put it, negative externalities are when "the social—that is, total—costs of production are larger than the private costs" [1]. Governments seek to "internalize" externalities by penalizing firms that pollute; often they do so through taxes or fines to offending companies. For example, some governments have implemented "carbon pricing" where CO_2 emissions are taxed [2].

For this paper, we will use the following example: consider a factory that makes an arbitrary product, generating pollution in the process. The local government seeks to impose a per-unit tax on this factory, thereby making production more expensive and reducing the amount produced, leading to reduced pollution. However, the government recognizes that there are negative consequences when the tax level is too high. For example, the factory could fail, and high taxes are politically unpopular because the high resulting prices anger voters. Therefore, the government seeks a tax rate that mitigates pollution but still keeps tax rates low. This balancing act can be modeled through an optimal control problem. In the following sections, we will derive the state equations from economic principles, construct a simple cost functional, and explore a few variations to this problem.

2 Mathematical Representation

2.1 Economic Derivation of the Problem

We derive our state equations from economic principles as follows. We assume that the firm is a monopoly which produces a good at zero marginal cost (both assumptions could be relaxed and give similar results). In producing the good, the firm releases pollution as a byproduct, which the government is interested in regulating. Pollution is measured in units of production, so that producing one unit of the good produces one unit of pollution. Consumers buy the good according to the following demand function:

$$Q = a - bP$$

Where Q is the quantity demanded, P is the price, and a and b are parameters characterizing the linear demand function. The firm faces costs of the form:

$$C = u \cdot Q$$

Where u is the per-unit tax level. That means the firm has the following profit function π, which it will maximize:

$$\pi = Q(P - u) = (a - bP)(P - u)$$

$$\rightarrow P^* = \frac{a + bu}{2b}$$

$$\rightarrow Q^* = \frac{a - bu}{2}$$

This implies that price increases linearly as the tax rate increases, while quantity produces decreases linearly. These equations indicate that both the firm and the consumers suffer when taxes go up - firm profits and consumer surplus are decreasing in tax rate.

In our simulations, we let $a = 2$ and $b = 1$ unless other stated. This has the desirable property that with no taxes ($u = 0$), the price is 1, the quantity produced is 1, and the firm profits are 1. This gives the following equations:

$$P^* = 1 + \frac{u}{2}$$

$$Q^* = 1 - \frac{u}{2}$$

$$\dot{x} = -\beta x + 1 - \frac{u}{2}$$

2.2 Simplest Problem

We assume that pollution is added to the environment only by producing this particular good. Pollution dissipates gradually over time, either by being absorbed by the environment or dispersed to areas that policy makers do not care about. Then we model pollution as

$$\dot{x} = -\beta x(t) + Q = -\beta x(t) + \frac{a - bu(t)}{2},\ x(0) = x_0$$

so that the derivative of pollution is decreasing in the tax rate.

In these equations, x is the level of pollution, β is the dissipation rate, and Q is the quantity produced, where we substitute in values of the control depending on the production process. We assume that pollution begins at a fixed level x_0. Most of our simulations assume that β is constant, however, we also explore situations in which β varies over time.

Our cost functional is given as follows:

$$J[u] = \int_0^{t_f} [x(t)^2 + \alpha u(t)^2 - \frac{1}{1000} log(\frac{a^2}{b^2} - u^2)]\, dt$$

- It would be reasonable to include the price consumers pay rather than the tax level, but because that price is $\frac{a+bu}{2}$, it is linear in u, so including u is sufficient.

- α is a parameter representing how much the government values low prices compared to low levels of pollution.

- The $-\frac{1}{1000} log(\frac{a^2}{b^2} - u^2)$ term is added as a soft restriction. We want $u(t) \in [0, \frac{a}{b})$, otherwise the quantity produced becomes negative, but imposing hard constraints creates a KKT problem which is difficult to solve. $-\frac{1}{1000} log(\frac{a^2}{b^2} - u^2)$ invalidates values of $u \geq \frac{a^2}{b^2}$, but is inconsequential when u is not near $\frac{a}{b}$.

- While values of $u(t) < 0$ are technically possible in this model, it is strongly discouraged by the cost functional and state equations. If $u(t) < 0$, $x(t)$ increases quickly, increasing the cost value. A negative tax rate is similar to a subsidy.

2.3 Variations of the Problem

We now consider three variations to this problem.

2.3.1 Time-Dependent Abatement Rates

One variation to consider is what happens to the ideal tax rates when the natural rate of abatement β changes over time (such as the seasonal inversion in Utah during the winter). Using the same system of equations as before, we can now choose a value of $\beta(t)$ that is time dependent.

For the sake of numerical stability, we modified our cost functional by replacing the softmax term $-\frac{1}{1000}\log(\frac{a^2}{b^2}-u^2)$ with the alternate term $\left(\frac{bu(t)}{a}\right)^{10}$. This leaves us with the cost functional

$$J[u]=\int_0^{t_f}\Big(x(t)^2+\alpha u(t)^2+(\frac{bu(t)}{a})^{10}\Big)dt$$

Although any particular choice of $\beta(t)$ is to some extent arbitrary, for the sake of this analysis, we will consider two example choices of $\beta(t)$ that illustrate some interesting dynamics:

$$\beta_1(t)=\frac{6}{5}+\frac{3}{5}\sin(2\pi t)$$
$$\beta_2(t)=\sqrt{1+t}+\frac{3}{5}\sin(2\pi t)-\frac{1}{5}$$

Both of these functions are periodic with a period of 1 year, which represents the seasonal trends that we might see. $\beta_1(t)$ is seasonal in nature without any long-term trend. In contrast, $\beta_2(t)$ includes both a seasonal as well as a gradual long-term component. One way to think about $\beta_2(t)$ is as a situation where conservation efforts lead to more tree growth that boosts abatement rates over time.

2.3.2 Two Production Methods

We also model a situation in which the firm has two different methods to produce the good. Customers are indifferent between the two means of production, so the demand function is unchanged. The first method of production is identical to the simple case, while the second method produces no pollution; however, it has a cost function of the form:

$$C_2=\frac{Q_2^2}{2}$$

The quadratic cost function indicates the production of each unit becomes gradually more expensive as more units are produced. This setting might be appropriate if the firm has multiple ways of powering their production, for instance, gasoline and solar. Gasoline is assumed to be cheap but polluting, as in the first production process. Solar power does not pollute, and is initially inexpensive. However, the more energy is required the more expensive it is to acquire given the difficulty in building more solar panels, dependence on the weather, etc. This is reflected in the quadratic cost function of the second production process. This results in the firm using as much solar energy as is feasible before switching over to gasoline to power their production once it becomes cheaper.

Assume that the tax u only applies to the polluting method. Basic analysis indicates that given a tax level u, the firm will produce using the clean method until it becomes cheaper to pay the tax. If taxes are low ($u<\frac{2}{3}$), we have

$$P^* = 1 + \frac{u}{2}, \quad Q_1^* = 1 - \frac{3u}{2}, \quad Q_2^* = u$$

Otherwise, if $u > \frac{2}{3}$, it is not reasonable to use the polluting process at all:

$$P^* = \frac{4}{3}, \quad Q_1^* = 0, \quad Q_2^* = \frac{2}{3}$$

With these two production methods, the new state equation is

$$\dot{x} = -\beta x + \max(0, 1 - \frac{3u}{2})$$

The two production process case has a cost functional of:

$$\int_0^{t_f} [x(t)^2 + \alpha \min(\frac{4}{3}, 1 + \frac{u(t)}{2})^2] dt$$

We can omit the $-\frac{1}{1000}\log(4-u^2)$ term because there is no benefit to setting taxes above $\frac{2}{3}$. $\min(\frac{4}{3}, 1 + \frac{u}{2})$ is the price, which enters the cost functional directly.

2.3.3 Discrete Tax Policy

In reality, the government is unlikely to change its tax policy continuously. Instead, the government must forecast the appropriate tax level without knowing future quantity produced, and once that tax level is set, it cannot change it for a given time period. In our model, the government decides the optimal control at each time period and applies that control uniformly for the next time unit. Then the system runs for a one time unit without their being able to make alterations. Then at the next decision point the government commits to a new tax level given current levels of pollution. Everything else about this situation is the same as the simplest problem, including the cost functional, state equation, and parameters.

3 Solution

3.1 Simplest Problem

In the simplest problem setup, our Hamiltonian is the following:

$$H = p(t)\Big(- \beta x(t) + \frac{a - bu(t)}{2}\Big) - x(t)^2 - \alpha u(t)^2 + \frac{1}{1000}\log(\frac{a^2}{b^2} - u^2)$$

Pontryagin's Maximum Principles gives us the following differential equations for $x(t)$ and $p(t)$:

$$\dot{x}(t) = \frac{\partial H}{\partial p} = -\beta x(t) + \frac{a - bu(t)}{2}, \; x(0) = x_0$$

$$\dot{p}(t) = -\frac{\partial H}{\partial x} = 2x(t) + \beta p(t),\ p(t_f) = 0$$

The $p(t_f) = 0$ condition comes from the fact that $x(t_f)$ is not constrained. Deriving the Hamlitonian with respect to u, we get the following implicit equation for the optimal $\tilde{u}$:

$$-\frac{b}{2}p(t) - 2\alpha u(t) - \frac{1}{1000}\frac{2u(t)}{\frac{a^2}{b^2} - u(t)^2} = 0$$

The implicit solution for $\tilde{u(t)}$ can be found using a root-finding method. We then solved for the state (pollution) and costate using solve_bvp. Since H is concave with respect to $u(t)$ (See Appendix), we know that $\tilde{u}(t)$ is a global maximizer of the Hamiltonian at time t for specific values of $x(t)$ and $p(t)$.

3.2 Time-Dependent Abatement Rates

Since this seasonal variation approach is based on the simplest problem, it looks quite similar to it. In fact, both the state and costate evolution equations are the same (although the value of β is now time-dependent). The one difference comes from the softmax regularization term which makes the Hamiltonian into the following:

$$H = p(t)\Big(-\beta(t)x(t) + \frac{a - bu(t)}{2}\Big) - x(t)^2 - \alpha u(t)^2 - \Big(\frac{bu(t)}{a}\Big)^{10}$$

We then get the first order condition:

$$\frac{\partial H}{\partial u} = -\frac{bp(t)}{2} - 2\alpha u(t) - 10\frac{b}{a}\Big(\frac{bu(t)}{a}\Big)^9 = 0$$

which we can then solve implicitly to find $\tilde{u}(t)$. Like the simplest problem above, H is concave with respect to $u(t)$, so $\tilde{u}(t)$ is a global maximizer.

3.3 Two Processes

The Hamiltonian becomes:

$$H(t) = p(t)(-\beta x + \max(0, 1 - \frac{3u(t)}{2})) - x^2 - \alpha \min(\frac{4}{3}, 1 + \frac{u(t)}{2})^2$$

Which has a piecewise first order condition:

$$\frac{\partial H}{\partial u} = \frac{-3}{2}p(t)\chi_{u(t)\leq\frac{2}{3}} - \alpha u(t)\chi_{u(t)\leq\frac{2}{3}}$$

Which we solve in python by using minimum and maximum functions. Note that because the derivative is zero for values of $u \geq \frac{2}{3}$ that this problem is very sensitive to initial conditions, and must be initialized with values below $\frac{2}{3}$

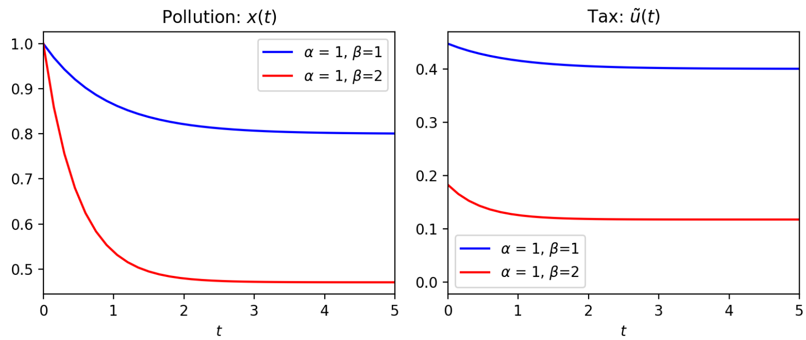

Figure 1: Variation in β

3.4 Discrete

This solution is identical to the simplest problem, except the solution is iterated many times with evolving starting conditions.

4 Interpretation

For each solution, we used a fixed end time, but the solution approaches a steady state. As the end time approaches, the solution changes to adjust to the endpoint condition $p(t_f) = 0$. As t_f grows, the steady state lasts for longer and longer. Therefore, to approximate an infinite time-horizon problem, we set $t_f = 15$ but only consider the first several time periods, after which the solution approaches a steady state.

4.1 Simplest Problem

Figures 1 and 2 show our results for $x_0 = 1$, $a = 2$, $b = 1$, and $t_f = 15$. Figure 1 shows two values of β with a fixed α, and Figure 2 shows two values of α with a fixed β.

- For each choice of parameters, the optimal tax $\tilde{u}$(t) reaches a steady solution, and then dips to 0 when t_f approaches (this is not pictured in the graph). This is because as we approach the final time, the tax $\tilde{u}$(t) affects the pollution $x(t)$ less, so there is more leeway in minimizing $x(t)$. Additionally, since $p(t_f) = 0$, $u(t_f) = 0$ is a viable solution.
- The optimal tax $\tilde{u}$ is different for each parameter choice. With our choice of a = 2 and b = 1, it varies between 0 and 2. The optimal tax is the highest when β is low. If α and β are both low, there are numerical stability issues.

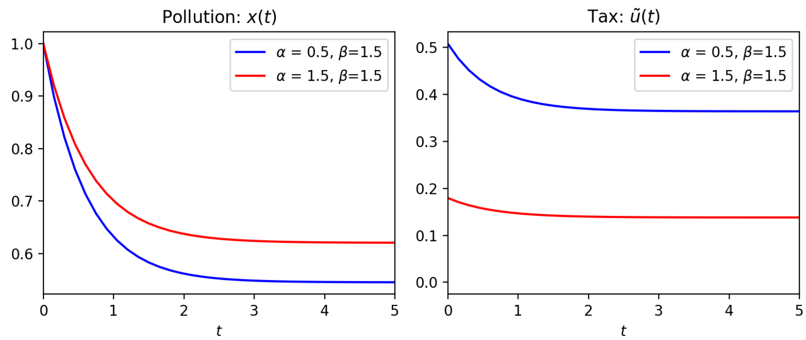

Figure 2: Variation in α

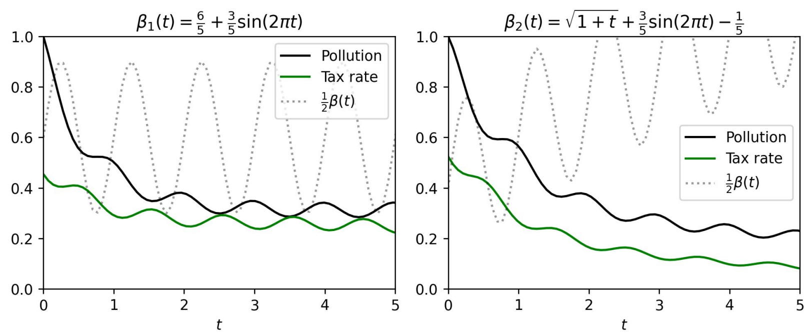

Figure 3: Optimal tax rate and resulting pollution when abatement changes over time.

- The optimal pollution level also achieves a steady solution, which depends heavily on the parameters α and β. A lower value for α will make the pollution go down close to zero while a low β will push taxes up.

4.2 Time-Dependent Abatement Rates

In Figure 3, the plot on the left shows us an example of an ideal tax scenario under the natural abatement rate $\beta_1(t)$ changes seasonally without trending in either direction (consider the inversion here in Utah). The plot on the right considers a situation where $\beta_2(t)$ varies seasonally while also trending upwards over time (perhaps conservation efforts are boosting the number of trees in the area).

Note in both cases how the ideal tax rate rises immediately before the troughs of $\beta(t)$ in anticipation of how pollution will remain for longer during those troughs.

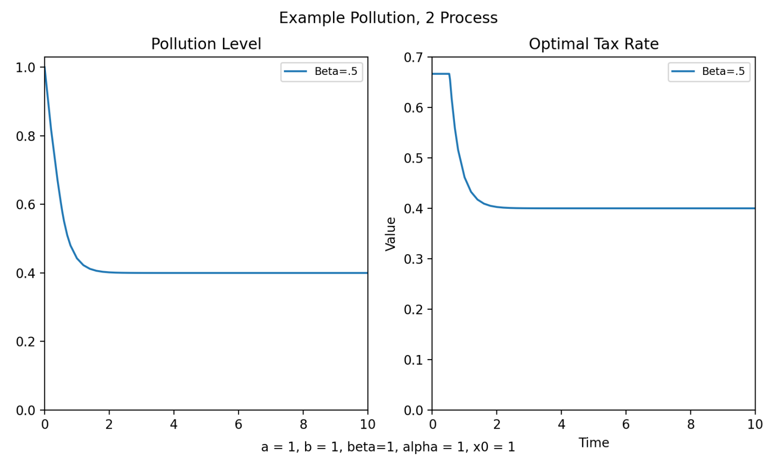

Figure 4: Optimal tax rate and resulting pollution, Discrete

4.3 Two Processes

Figure 4 shows a simple problem involving the two production processes. The solution tends towards a steady state as before, but the behavior is slightly more complicated. There are three separate stages in the optimal tax rate. First, the tax rate is at the maximum value of $\frac{2}{3}$ which leads to 0 new pollution. Then it gradually transitions to a steady state tax level at which the new production exactly balances the abatement rate β, then stays at that level for all time afterwards.

In the appendix, Figures 6, 7, and 8 show the results of varying β, α, and x_0 while all other parameters remain fixed. They all maintain roughly the same three stages as the first example. Higher values of beta result in higher steady states and faster convergence to the steady state. Presumably this is because the trade off between pollution and taxation is not as harsh in this setting, so with higher values of natural abatement the government can afford a lower tax rate. Higher values of α result in the same steady state solution, with some slight hiccups in the rate of convergence. Different values of x_0 have no effect on the steady state, but they do affect the optimal tax rate before reaching a steady state. If x_0 is high, then starting taxes remain at $\frac{2}{3}$ for an extended period before dropping to the steady state. Conversely, if x_0 is small then taxes can be low to allow for briefly lower prices before converging to the steady state.

These results have some significant differences with the simplest problem and the discrete solution, because the tradeoff between taxation and pollution is fundamentally different in this problem. Note that we also allow the simulations to run for longer, since it frequently takes more time periods to converge to the steady state.

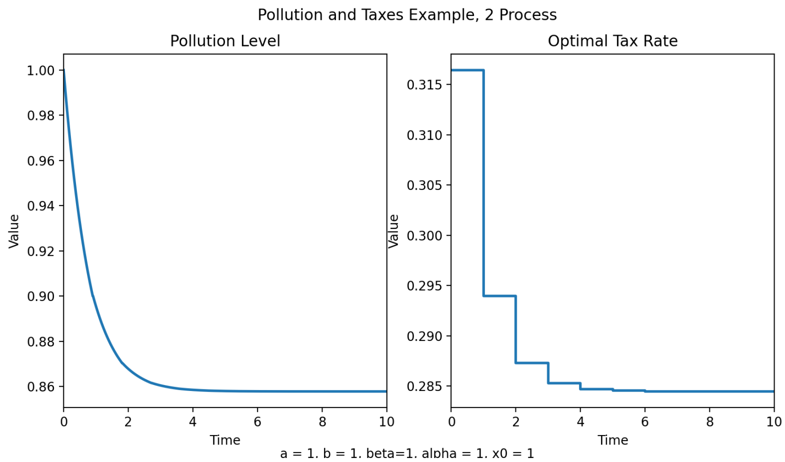

Figure 5: Optimal tax rate and resulting pollution, Discrete

4.4 Discrete

Figure 5 shows the results for a simple simulation of the discrete case. The results are qualitatively similar to the simplest problem, with the solutions converging to a steady state. In the appendix, figures 9, 10, and 11 show the effects of manipulating β, α, and x_0

As β increases the steady state pollution and tax rate both decrease, representing more favorable environmental conditions. As α increases the steady state tax decreases and the steady state pollution level increases, reflecting the trade off between taxes and pollution. Finally, x_0 does not affect the steady state levels of taxes or pollution, only the transition to reach them.

5 Conclusion

5.1 Extensions

Ultimately, we have considered only very simple models to explore optimal taxation of pollutants. More complicated extensions of the model might include one or more of the following:

1. Discounting. It is reasonable to assume that people are more sensitive to the present prices and pollution levels than future ones. Including the term $e^{-\rho t}$ in the cost functional would capture this. This could alter the steady state depending on α to front load benefits. Pollution and prices may have different discount rates, which could result in a different transition to the steady state or no steady state.

2. Multiple goods and multiple pollutants. The government may be faced with multiple goods, multiple production processes, and multiple pollu-

tants. There may be nonlinear relationships between the production equations and pollution state equations.

3. Endpoint costs. The government may set a particular target for pollution levels to be achieved after a certain number of years, t_f. We can model this by including an endpoint cost of the form $\phi(x(t_f)) = (\max(x(t_f) - X, 0))^2$ where X is the target level of pollution. This is incorporated into the prior models by changing the endpoint condition to require that $p(t_f) = -\frac{d\phi}{dx(t_f)}$

4. Uncertainty. The government must forecast the appropriate tax level without knowing future quantity produced. Moreover, the firm must commit to future production without knowing what the eventual tax rate will be. We would also introduce a randomness to the price and quantity produced to model the uncertainty inherent in production and sales. Once the firm and government have made their policy decisions, the system would run for one time period without their being able to make alterations. Then at the next decision point they must commit to new levels of production and taxes given how the environment has changed.

5.2 Implications

These results constitute a general framework to examine long-term economic policy when applied to environmental sustainability.

We caution readers against reading too far into these results. Many times, policy choices differentially impact the involved parties in serious ways, and this is a simple model incapable of reflecting the diversity of actors and consequences of environmental policy. It also does not allow any scope for technological improvements which could increase or decrease pollution. Finally, the units used in this example are not relevant to any particular pollutant and production process, and serve only to illustrate the dynamics of the problem.

Beyond a simple empirical question, these choices constitute a moral, ethical, and political decision that cannot be solved purely by way of an algorithm. This model can inform the reader, but it cannot decisively determine the "right" answer to the challenge of balancing the damage done by pollution and the drawbacks of restrictive economic policy.

This paper does give some insight into the optimal taxation of pollutants. For instance, with the preferences in our cost functional the optimal solution usually approaches a steady state where prices and pollution levels are fixed. If the environment absorbs pollution more quickly (β is higher), then lower tax rates are optimal because they keep prices low at minimal cost to the environment. Moreover, in this model the initial conditions are relatively inconsequential to the final outcomes under optimal policy. However, lower initial levels of pollution lead to better outcomes in the short term, indicating that there are consequences to delaying the optimal policy. Indeed, when pollution levels were high it was always optimal to set a high tax rate at first to control pollution before relaxing the tax rate.

References

[1] Helbling, Thomas. ”What Are Externalities?”, International Monetary Fund. Accessed from https://www.imf.org/external/pubs/ft/fandd/2010/12/basics.htm

[2] The World Bank. ”Carbon Pricing Dashboard”. Accessed from https://carbonpricingdashboard.worldbank.org/what-carbon-pricing

6 Appendix

6.1 Proof of Hamiltonian Concavity

Here we work with the Hamiltonian of our simplest problem:

$$G(u) := H = p(t)\Big(-\beta x(t) + \frac{a - bu(t)}{2}\Big) - x(t)^2 - \alpha u(t)^2 + \frac{1}{1000} log(\frac{a^2}{b^2} - u(t)^2)$$

We note that the sum of concave functions are themselves concave. By Jensen's inequality, for any $\lambda \in [0,1]$ we have $G(\lambda u_1 + (1-\lambda)u_2)$
$= p\Bigg(-\beta x + \frac{1}{2}\big(a - b(\lambda u_1 + (1-\lambda))u_2\big)\Bigg) - x^2 - \alpha\big(\lambda u_1 + (1-\lambda)u_2\big)^2$
$+ \frac{1}{1000}\log(\frac{a^2}{b^2} - (\lambda u_1 + (1-\lambda)u_2)^2)$
$\geq p\Bigg(-\beta x + \frac{1}{2}\big(a - b(\lambda u_1 + (1-\lambda))u_2\big)\Bigg) - x^2 - \alpha\big(\lambda u_1 + (1-\lambda)u_2\big)^2$
$+ \frac{1}{1000}\log(\frac{a^2}{b^2} - \lambda u_1^2 + (1-\lambda)u_2^2)$
$\geq \lambda p\Bigg(-\beta x + \frac{1}{2}\big(a - bu_1\big)\Bigg) + (1-\lambda)p\Bigg(-\beta x + \frac{1}{2}\big(a - bu_2\big)\Bigg) - \lambda x^2 - (1-\lambda)x^2$
$-\lambda\alpha u_1^2 - (1-\lambda)\alpha u_2^2 + \frac{\lambda}{1000}\log(\frac{a^2}{b^2} - u_1^2) + \frac{1-\lambda}{1000}\log(\frac{a^2}{b^2} - u_2^2) = \lambda G(u_1) + (1-\lambda)G(u_2).$

Thus, at any time t, we have that H is concave with respect to $u(t)$. The proof for the variable $\beta(t)$ problem is similar.

6.2 Additional Figures

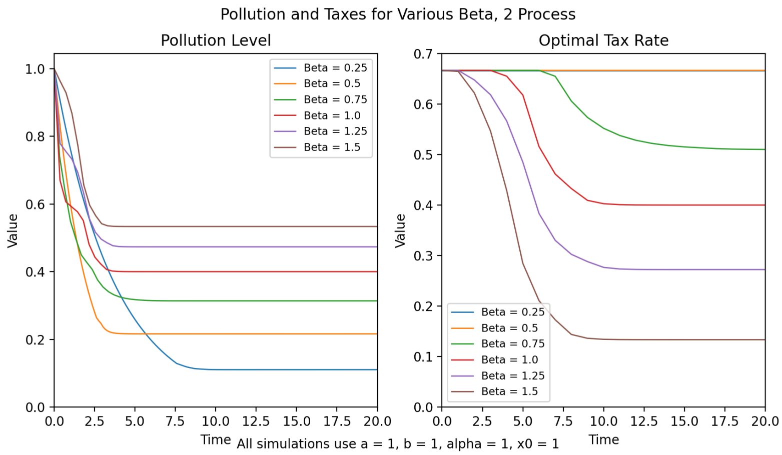

Figure 6: Optimal tax rate and resulting pollution with various values of β, 2 Process

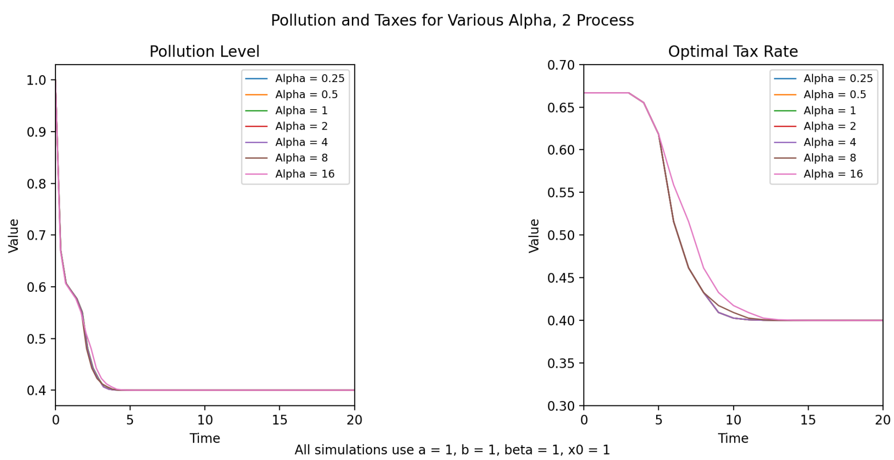

Figure 7: Optimal tax rate and resulting pollution with various values of α, 2 Process

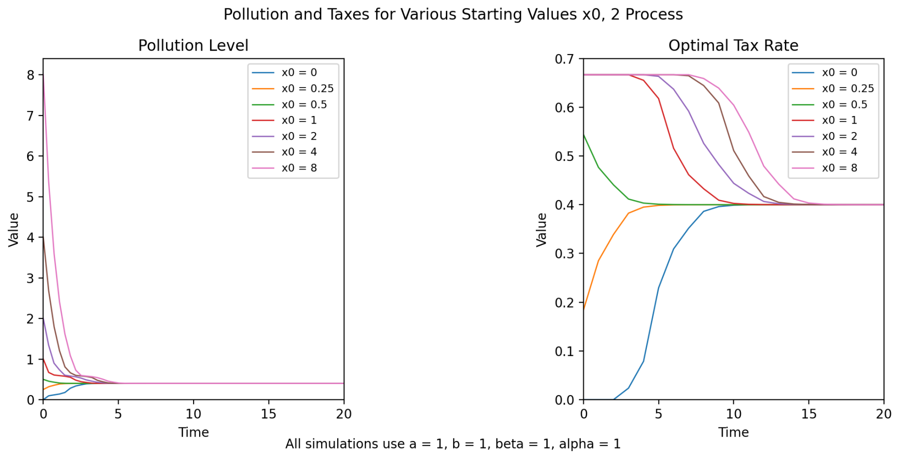

Figure 8: Optimal tax rate and resulting pollution with various values of x_0, 2 Process

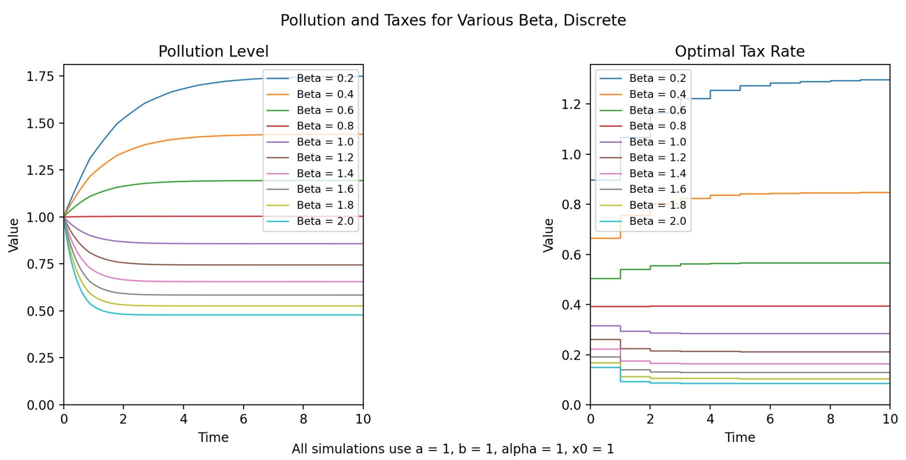

Figure 9: Optimal tax rate and resulting pollution with various values of β, Discrete

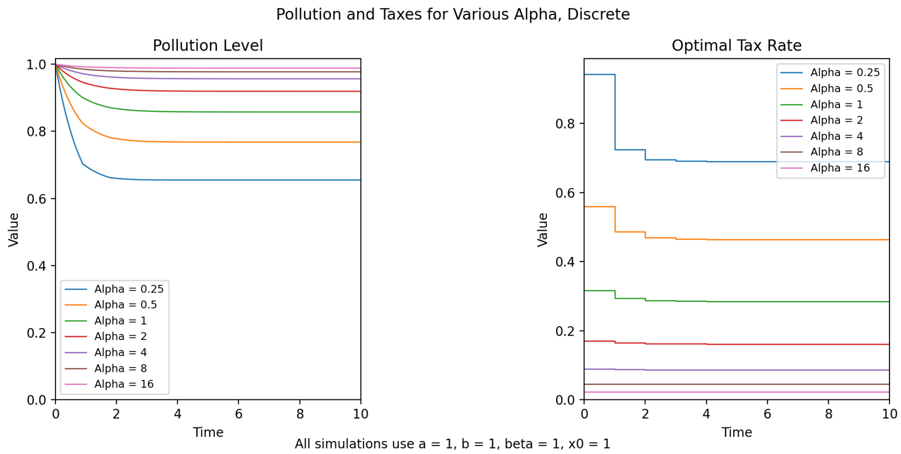

Figure 10: Optimal tax rate and resulting pollution with various values of α, Discrete

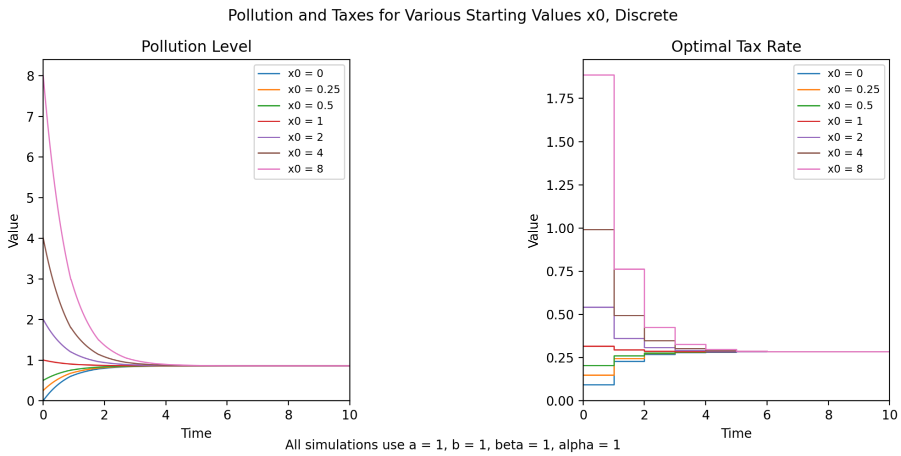

Figure 11: Optimal tax rate and resulting pollution with various values of x_0, Discrete