Exercise Sheet 9

These exercises cover part of chapter 5. Full answers will be published on moodle.

1. Suppose that $X_1, \ldots, X_n$ are independent random variables, each uniformly distributed on $(0, \theta)$. Let $\overline{X} = \{\sum_{i=1}^n X_i\}/n$ and $X_{(n)} = \max\{X_i, i = 1, \ldots, n\}$.

   Find the expectations of $\overline{X}$ and $X_{(n)}$ in terms of θ.
   `Hint:` For the expectation of $X_{(n)}$ you will first need to find the pdf of $X_{(n)}$.
   Hence propose two unbiased estimators of θ based on each of $\overline{X}$ and $X_{(n)}$. Compare the mean square errors of these estimators. Which estimator would you prefer and why?

2. Let $X_1, \ldots, X_n$ be independent and identically distributed $N(\theta, \sigma^2)$ random variables, where the variance σ^2 is known but the mean θ is unknown. Obtain the Cramér–Rao lower bound for the variance of any unbiased estimator of θ. Find an unbiased estimator of θ that achieves this lower bound.

3. Let $X_1, \ldots, X_n$ be independent, identically distributed Poisson random variables with mean θ. Use the method of moments to find an estimator of θ.

   Determine the bias of this estimator and compare its variance with the Cramér–Rao lower bound.

4. Let $X_1, \ldots, X_n$ be a random sample from a $\mathrm{Gam}(\alpha, \lambda)$ distribution, where both α and λ are unknown. Show that the method of moments estimator of α is given by $\hat{\alpha} = 1/c^2$, where $c = S/\bar{X}$ is the sample coefficient of variation and $S^2 = \sum_i (X_i - \bar{X})^2/n$.