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Example 3.3 $X \sim \mathrm{Gam}(\alpha, \lambda)$, $\alpha, \lambda > 0$.

$$M_X(s) = \mathbb{E}_X[e^{sX}] = \int_0^\infty e^{sx} \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}\, dx$$

$$= \int_0^\infty \frac{\lambda^\alpha x^{\alpha-1}}{\Gamma(\alpha)} e^{-(\lambda-s)x}\, dx = \frac{\lambda^\alpha}{(\lambda-s)^\alpha} \underbrace{\int_0^\infty \frac{(\lambda-s)^\alpha x^{\alpha-1}}{\Gamma(\alpha)} e^{-(\lambda-s)x}\, dx}_{\text{integral of pdf of } \mathrm{Gam}(\alpha, \lambda - s) = 1}$$

$$= \frac{\lambda^\alpha}{(\lambda-s)^\alpha} = \left(\frac{\lambda}{\lambda-s}\right)^\alpha = \left(1 - \frac{s}{\lambda}\right)^{-\alpha}$$

$$\Gamma(t) = \int_0^\infty x^{t-1} e^{-x}\, dx$$

$M_X(s) < \infty$ if $\boxed{s < \lambda.}$

Example 3.4 $X \sim N(\mu, \sigma^2)$

$$M_X(s) = \int_{-\infty}^\infty e^{sx} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}\, dx$$

$$= \int_{-\infty}^\infty \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}\left(x^2 - 2x(\mu + s\sigma^2) + \mu^2\right)\right) dx$$

$$= \int_{-\infty}^\infty \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}\left(x - (\mu + s\sigma^2)\right)^2 - \frac{\mu^2}{2\sigma^2} + \frac{(\mu + s\sigma^2)^2}{2\sigma^2}\right) dx$$

$$= \exp\left(\frac{-\mu^2 + \mu^2 + 2\mu s\sigma^2 + s^2\sigma^4}{2\sigma^2}\right) \underbrace{\int_{-\infty}^\infty \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}\left(x - (\mu + s\sigma^2)\right)^2\right) dx}_{=1 \ (\text{pdf of } N(\mu + s\sigma^2, \sigma^2))}$$

$$= \exp\left(\mu s + \frac{1}{2}\sigma^2 s^2\right)$$

$s \in \mathbb{R}$. Check: $M_X(0) = 1$ ✓

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$\mathbb{E}X = M_X'(0)$

$M_X'(s) = \frac{d}{ds} M_X(s) = \frac{d}{ds} \exp\left(\mu s + \frac{1}{2}\sigma^2 s^2\right)$

$= \exp\left(\mu s + \frac{1}{2}\sigma^2 s^2\right)\left(\mu + \sigma^2 s\right)$

$M_X'(0) = \exp(0) \cdot \mu = \mu = \mathbb{E}[X]$.

$M_X''(s) = \exp\left(\mu s + \frac{1}{2}\sigma^2 s^2\right)\left((\mu + \sigma^2 s)^2 + \sigma^2\right)$

$M_X''(0) = \mu^2 + \sigma^2$

$Var(X) = M_X''(0) - (M_X'(0))^2 = \mu^2 + \sigma^2 - (\mu)^2 = \underline{\underline{\sigma^2}}$

<u>Example 3.5</u>

$X \sim N(\mu, \sigma^2),\ Y = a + bX$

$M_X(s) = \exp\left(\mu s + \frac{1}{2}\sigma^2 s^2\right)$

$M_Y(s) = e^{as} \cdot M_X(bs) = e^{as} e^{\mu b s + \frac{1}{2}\sigma^2 b^2 s^2}$

$= \exp\left((a + \mu b)s + \frac{1}{2}(\sigma b)^2 s^2\right)$

So we have shown $Y \sim N(a + \mu b, \sigma^2 b^2)$

$\mathbb{E}[Y] = \mathbb{E}[a + bX] = a + b\mathbb{E}X = a + b\mu$

$Var(Y) = Var(a + bX) = b^2 Var(X) = b^2\sigma^2$