Example 5.12 $X_i \sim Ber(p)$, $p \in (0,1)$

From Example 5.7: $\ell(p) = n \log(1-p) + S \cdot \log \frac{p}{1-p}$

$$S = \sum_{i=1}^{n} x_i .$$

$$\frac{d^2}{dp^2} \ell(p) = \frac{1}{(1-p)^2}(-n+S) - \frac{S}{p^2}$$

Fisher information $I(p) = \mathbb{E}_X\left[-\frac{d^2}{dp^2}\ell(p;X)\right]$

$$= \mathbb{E}_X\left[-\frac{S-n}{(1-p)^2} + \frac{S}{p^2}\right]$$

$\hookrightarrow X = (X_1, X_2, \dots, X_n)$, $S = \sum_{i=1}^{n} X_i \sim Bin(n,p)$

$$= \frac{n}{(1-p)^2} + \mathbb{E}[S] \cdot \left(\frac{-1}{(1-p)^2} + \frac{1}{p^2}\right) \quad \Rightarrow \mathbb{E}S = np.$$

$= \dots = \frac{n}{(1-p)p}$ So, the CRLB says $Var\,T \geq \frac{1}{I(p)} = \frac{(1-p)p}{n}$.

Compare $\hat{p}_{ML} = \frac{1}{n}S$, so $Var(\hat{p}_{ML}) = \frac{1}{n^2} Var\,S = \frac{np(1-p)}{n^2} = \frac{(1-p)p}{n}$.

So, $\hat{p}_{ML}$ attains the CRLB.

Asymptotically $\hat{p}_{ML} \overset{approx}{\sim} N\left(p, \frac{p(1-p)}{n}\right)$, so

$\left(\hat{p}_{ML} - 2\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}, \hat{p}_{ML} + 2\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}\right)$ is a 95%-confidence interval for p.