Example 5.1 $X_i \sim N(\mu, \sigma^2)$ i.i.d.

mse of S^2 $\quad mse(S^2; \sigma^2) = (\mathbb{E}[S^2] - \sigma^2)^2 + var(S^2)$

S^2: estimator; σ^2: true value of population parameter

$$= 0^2 + \frac{2\sigma^4}{n-1} = \frac{2\sigma^4}{n-1}$$

Ch.4 (4.3.1)

$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$$

$$\tilde{S}^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2 = \frac{n-1}{n} S^2$$

$mse(\tilde{S}^2; \sigma^2) = ?$

$$\mathbb{E}\tilde{S}^2 = \mathbb{E}\left[\frac{n-1}{n} S^2\right] = \frac{n-1}{n}\mathbb{E}S^2 = \frac{n-1}{n}\sigma^2$$

$$Var\,\tilde{S}^2 = Var\left(\frac{n-1}{n}S^2\right) = \left(\frac{n-1}{n}\right)^2 Var\,S^2 = \left(\frac{n-1}{n}\right)^2 \cdot \frac{2\sigma^4}{n-1}$$

$$mse(\tilde{S}^2; \sigma^2) = (\mathbb{E}\tilde{S}^2 - \sigma^2)^2 + Var\,\tilde{S}^2$$

$$= \left(-\frac{1}{n}\sigma^2\right)^2 + \frac{2(n-1)}{n^2}\sigma^4 = \frac{2n-1}{n^2}\sigma^4 < \frac{2\sigma^4}{n-1}$$

So $mse(\tilde{S}^2; \sigma^2) < mse(S^2; \sigma^2)$.

Example 5.3 $X_i \sim N(\mu, \sigma^2)$

$\theta = \sigma^2$ is the parameter of interest.
(because S_μ^2 estimates σ^2, not σ.)

$$I(\theta) = n \cdot i(\theta) = n \cdot \mathbb{E}_X\left[-\frac{\partial^2}{\partial\theta^2} \log f_X(X;\theta)\right]$$

$$f_X(x;\theta) = \frac{1}{\sqrt{2\pi\theta}} \exp\left(-\frac{1}{2\theta}(x-\mu)^2\right)$$

$$\log f_X(x;\theta) = -\frac{1}{2}\log\theta - \frac{1}{2\theta}(x-\mu)^2 + \text{const.}$$

$\log x^\alpha = \alpha \log x$

$$\frac{\partial}{\partial\theta}\log f_X(x;\theta) = -\frac{1}{2\theta} + \frac{1}{2\theta^2}(x-\mu)^2$$

$$\frac{\partial^2}{\partial\theta^2}\log f_X(x;\theta) = \frac{1}{2\theta^2} - \frac{2}{2\theta^3}(x-\mu)^2$$

$$I(\theta) = n \cdot i(\theta) = n \cdot \mathbb{E}\left[-\left(\frac{1}{2\theta^2} - \frac{1}{\theta^3}(X-\mu)^2\right)\right]$$

$$= -\frac{n}{2\theta^2} + \frac{n}{\theta^3}\underbrace{\mathbb{E}\left[(X-\mu)^2\right]}_{=\theta} = \frac{n}{2\theta^2}$$

So, the Cramér-Rao lower bound says

$$\mathrm{Var}(T) \geq \frac{1}{I(\theta)} = \frac{2\theta^2}{n} = \frac{2\sigma^4}{n}$$

Note: We know $\mathrm{Var}(S_\mu^2) = \frac{2\sigma^4}{n}$, so S_μ^2 achieves the Cramér-Rao lower bound.