Exercise Sheet 8

A: warm-up question

1. $X + 1 \sim N(1, 1)$, $X + Y + Z \sim N(3, 3)$, $X^2 \sim \chi^2_1$, $X^2 + Y^2 + Z^2 \sim \chi^2_3$. $(X + 1)^2$ does not follow any of the distributions used in STAT2001 so far.

B: questions handed in

1. (a)

$$\mathbb{E}[Y] = ... = \sum_{i=1}^{n}(\alpha^2 + 1) = n(1 + \alpha^2)$$

$$\mathrm{Var}(Y) = ... = 2n(2\alpha^2 + 1)$$

(b)

$$\begin{aligned} M_{Y_1}(s) &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(s(\alpha + y)^2 - \frac{1}{2}y^2\right) dy \\ &= ... \\ &= \exp\left(\frac{\alpha^2 s^2}{1/2 - s} + s\alpha^2\right) \sqrt{\frac{1}{1 - 2s}} \int \frac{1}{\sqrt{2\pi/(1 - 2s)}} \exp\left(-\frac{1}{2\frac{1}{1-2s}}\left[y - \frac{s\alpha}{1/2 - s}\right]^2\right) \end{aligned}$$

Now the final integral can be recognized as the density of a $N(\frac{s\alpha}{1/2-s}, \frac{1}{1-2s})$ which integrates to one. Therefore the result follows after one more step of algebra to simplify the exponent.

$$M_Y(s) = ... = (1 - 2s)^{-n/2} \exp\left(\frac{n\alpha^2 s}{1 - 2s}\right)$$

(c)

$$M_Z(s) = ... = (1 - 2s)^{-n/2} \exp\left(n\alpha^2 s/(1 - 2s)\right) \qquad \text{on } s < 1/2$$

which is precisely ...

2. (a) The result was merely stated in the lecture but succinct step-by-step instructions are available from the lecture notes for those who read them. Since $X_i \sim N(\mu, \sigma^2)$, it follows that ... Since the X_i are independent, variances add up just like expectations, so ... Multiplying by $\frac{\sigma^2}{n}$, we thus get $\mathbb{E}[S^2_\mu] = \frac{\sigma^2}{n}n = \sigma^2$ and $\mathrm{Var}(S^2_\mu) = \left(\frac{\sigma^2}{n}\right)^2 2n = \frac{2\sigma^4}{n}$.

(b)

$$\mathbb{E}[S_\mu^2] = ... = \mathbb{E}_N[\sigma^2] = \sigma^2,$$

$$\mathrm{Var}(S_\mu^2) = ... = \frac{2\sigma^4}{\lambda}\left(1 - e^{-\lambda}\right)$$

C: Exam Practice Question

1. (a) $\sum_i^3 (X_i/i - 1)^3 \sim \chi_3^2$.

   (b) $\frac{2(X_1-1)^2}{(\frac{1}{2}X_2-1)^2+(\frac{1}{3}X_3-1)^2} \sim F_{1,2}$.

   (c) $\frac{\sqrt{2}(X_1-1)}{\sqrt{(\frac{1}{2}X_2-1)^2+(\frac{1}{3}X_3-1)^2}} \sim t_2$.