Exercise Sheet 6

The exercise sheet is subdivided into three parts:

- Part A contains warm-up questions you should do in your own time.
- You should solve all questions of Part B and put your solutions in the locker in the undergraduate common room in the Department of Statistical Science (Room number 117) assigned to your tutor group by the deadline stated on the exercise sheet. Don't forget to mark your solutions with your name and student number.

  Reminder: If you miss the deadline for reasons outside your control, for example illness or bereavement, you ***must*** submit a claim for extenuating circumstances, normally within a week of the deadline. Your home department will advise you of the appropriate procedures. For Statistical Science students, the relevant information is on the DOSSSH Moodle page.

  Your tutor will then mark one question selected at random after you handed in your solutions (mark counts towards ICA) and will return your marked solutions to you at the next tutorial at which the solutions to the part B questions will be discussed. Marks available for each part question are indicated in square brackets, e.g. [3]. Very succinct solutions to these questions will usually be made available on Moodle at 9:15am on Thursday mornings, so don't hand in late!
- Solutions to Part C questions will be made available in time for exam revision. These questions will not be discussed at the tutorials.

A: Warm-up question

1. Let X and Y be independent, identically distributed variables having a geometric distribution with parameter p, so that

   $$\mathrm{P}(X = k) = p(1-p)^{k-1}, k = 1, 2, \dots,$$

   and let $Z = \min(X, Y)$.

   Find $\mathrm{P}(X > n)$, deduce $\mathrm{P}(Z > n)$, and hence explain why you can infer that Z has a geometric distribution with parameter $p(2-p)$. Give an intuitive interpretation of this result.

B: Answers to hand in by Thursday, 26th November, 9am

1. (a) Let X follow an exponential distribution with parameter $\lambda = 1$. Denote the mean value and variance of X by $\mu = 1$ and $\sigma^2 = 1$ respectively.

      i. Compute the approximate expectation of X^3 using the 2nd order moment approximation $\mathbb{E}[\phi(X)] \approx \phi(\mu) + \frac{1}{2}\phi''(\mu)\sigma^2$ with $\phi(x) = x^3$. Sketch the graph of ϕ as well as the approximating function obtained from 2nd order Taylor approximation about μ, i.e. about $x = 1$. [4]

      ii. Compute $\mathbb{E}[X^3]$ exactly, e.g. by integration or by using the mgf, and discuss the direction of the deviation of the approximation computed above from the true value. [3]

   (b) Let W and Z be independent both following a $U(0, \pi/2)$ distribution.

      i. Compute $\mathbb{E}[\sin(W + Z)]$ using the 2nd order moment approximation. `Hint:` write $V = W + Z$. What are $\mathbb{E}[V]$ and $\mathrm{Var}(V)$? [3]

      ii. For functions $\phi(\cdot)$ and $\psi(\cdot)$, write down the definition of $\mathbb{E}[\phi(W)\psi(Z)]$ as an integral. By considering the form of this integral, explain carefully why $\mathbb{E}[\phi(W)\psi(Z)] = \mathbb{E}[\phi(W)]\mathbb{E}[\psi(Z)]$. [2]

iii. Compute $\mathbb{E}[\sin(W + Z)]$ exactly. How accurate is the 2nd-order moment approximation from part (b)i compared to the one in part (a)? `Hint:` Remembering trigonometric identities will help you use the result from part (b)ii here. [3]

2. Let Y follow the distribution described by the pdf $f_Y(y) = 2y$ on $(0, 1)$. You may use without proof that $\mathbb{E}[Y] = 2/3$. Conditionally on $Y = y$, X follows a uniform distribution on $(0, y)$.

   (a) Compute $\mathbb{E}[X]$ and $\mathbb{E}[X/Y]$. [5]

   (b) Compute the mgf $M_X(\cdot)$ of X. `Hint:` the mgf *is* an expectation. [6]

   (c) Using differentiation, obtain the expectation of X from the mgf computed above. `Hint:` you may need to use l'Hôpital's rule to evaluate the derivative. [4]

3. Let $X_1, X_2, \ldots$ be independent all following the distribution with cdf $F_X(x) = 1 - e^{-\lambda^{-k}x^k}$ on $(0, \infty)$, where $\lambda > 0$ and $k > 0$ are parameters.

   (a) Show that $\mathbb{E}[X_1] = \lambda\Gamma(1 + 1/k)$, remembering that $\Gamma(t) = \int_0^\infty x^{t-1}e^{-x}\,dx$ by definition. [5]

   (b) Compute the cdf F_1 of the sample minimum $X_{(1)}$ of $X_1, X_2, \ldots, X_n$, where $n \in \mathbb{N}$. [4]

   (c) Compute the expected value of the sample minimum, $\mathbb{E}[X_{(1)}]$, and show that it tends to zero as the sample size n tends to infinity. Give an intuitive explanation of why this convergence is expected. [6]

C: Exam Practice Question

1. Suppose that X_1 and X_2 are independent random variables, where X_1 has a continuous uniform distribution on the interval [0,1], and X_2 is exponentially distributed with mean 1. Let $Y = \max\{X_1, X_2\}$.

   Write down the distribution functions of X_1 and X_2 and hence determine the distribution function of Y. Show that $\mathrm{E}(Y) = 3/2 - 1/e$.

2. Suppose that X is uniform on $(10, 20)$. Obtain approximations to the mean and variance of $Y = 1/X$. Compare these approximations to the exact values.