Exercise Sheet 3

A: Warm-up questions

1. $\mathrm{E}(X) = \mathrm{E}(Y) = 7/12$, $\mathrm{var}(X) = \mathrm{var}(Y) = 11/144$, $\mathrm{E}(XY) = 1/3$, thus $\mathrm{Corr}(X, Y) = -1/11 \approx -0.09$. Everything by straightforward integration – no special tricks required, details below:
   $\mathrm{corr}(X, Y) = \mathrm{cov}(X, Y)/\sqrt{\mathrm{var}(X)\mathrm{var}(Y)}$, where $\mathrm{var}(X) = \mathrm{E}(X^2) - \{\mathrm{E}(X)\}^2$ *etc.* Now,
   $$\mathrm{E}(X) = \int_0^1 \int_0^1 x(x+y)\ \mathrm{d}x\ \mathrm{d}y = [x^3/3 + x^2/4]_0^1 = 7/12 = \mathrm{E}(Y) \text{ (by symmetry)}$$
   and
   $$\mathrm{E}(X^2) = \int_0^1 \int_0^1 x^2(x+y)\ \mathrm{d}x\ \mathrm{d}y = [x^4/4 + x^3/6]_0^1 = 5/12 = \mathrm{E}(Y^2) \text{ (by symmetry)}.$$
   Thus, $\mathrm{var}(X) = \mathrm{var}(Y) = 11/144$. To compute the covariance we also need:
   $$\mathrm{E}(XY) = \int_0^1 \int_0^1 xy(x+y)\ \mathrm{d}x\ \mathrm{d}y = [x^3/6 + x^2/6]_0^1 = 1/3.$$
   Use $\mathrm{cov}(X, Y) = \mathrm{E}(XY) - \mathrm{E}(X)\mathrm{E}(Y) = -1/144$. Inserting everything in the above equation for the correlation gives $\mathrm{corr}(X, Y) = -1/11 \approx -0.09$, which is a very small negative correlation.

2. Marginal densities: $f_1(x_1) = 4\int_0^1 x_1 x_2 dx_2 = 2x_1,\ 0 \le x_1 \le 1$
   $f_2(x_2) = 4\int_0^1 x_1 x_2 dx_1 = 2x_2,\ 0 \le x_2 \le 1$
   Conditionals:
   $$f_{1|2}(x_1|x_2) = \begin{cases} 2x_1 & \text{if } \ 0 \le x_1 \le 1,\ 0 < x_2 \le 1 \\ 0 & \text{if } \ (x_1 < 0 \ \text{ or } x_1 > 1) \text{ and } 0 < x_2 \le 1 \\ \text{undefined} & \text{if } \ x_2 \le 0 \text{ or } x_2 > 1 \end{cases}$$
   $$f_{2|1}(x_2|x_1) = \begin{cases} 2x_2 & \text{if } \ 0 \le x_2 \le 1,\ 0 < x_1 \le 1 \\ 0 & \text{if } \ (x_2 < 0 \ \text{ or } x_2 > 1) \text{ and } 0 < x_1 \le 1 \\ \text{undefined} & \text{if } \ x_1 \le 0 \text{ or } x_1 > 1 \end{cases}$$
   Means: $\mathrm{E}(X_1) = 4\int_0^1 x_1^2 x_2 dx_1 = 2/3$, $\mathrm{E}(X_2) = 4\int_0^1 x_1 x_2^2 dx_2 = 2/3$
   Variances: $\mathrm{var}(X_1) = \mathrm{E}(X_1^2) - \{\mathrm{E}(X_1)\}^2 = 1/18$, $\mathrm{var}(X_2) = 1/18$
   Correlation: $\mathrm{corr}(X_1, X_2) = 0$ since X_1 and X_2 are independent (since $f(x_1, x_2) = f_1(x_1)f_2(x_2)$)

B: Answers to hand in

1. (a) The integration is less challenging than the example in the lectures because integration boundaries come from a rectangle:
   $$f_X(x) = ... = \frac{\alpha+1}{\alpha} e^{-x}\left(-e^{-\alpha x} + 1\right) \text{ on } x > 0$$

(b) ..., so including the necessary case distinction the result is

$$f_{Y|X}(y|x) = \begin{cases} \frac{\alpha x \exp(-\alpha x y)}{1-e^{-\alpha x}} & \text{if } x > 0,\ y \in (0,1) \\ 0 & \text{if } x > 0,\ y \in (-\infty, 0] \cup [1, \infty) \\ \text{undefined} & \text{if } x \leq 0 \end{cases},$$

where $\alpha > 0$ is assumed. In the case $\alpha = 0$, the conditional distribution is simply

$$f_{Y|X}(y|x) = \begin{cases} 1 & \text{if } x > 0,\ y \in (0,1) \\ 0 & \text{if } x > 0,\ y \in (-\infty, 0] \cup [1, \infty) \\ \text{undefined} & \text{if } x \leq 0 \end{cases},$$

(c) The conditional expectation is obtained by integration by parts which is an important integation technique to revise.

$$\mathbb{E}_{Y|X}[Y|X = x] = ... = \frac{1}{\alpha x} - \frac{1}{e^{\alpha x} - 1}$$

holds in the case $\alpha > 0$. In the case $\alpha = 0$, the expectation is $1/2$ because the distribution is uniform as shown above.

2. (a) The expected value of X_i is given by the standard binomial expectation available from the appendix of the lecture notes if not remembered, $\mathbb{E}[X_i] = 10p$.

   ...

   Hence $f(n) = 10pn$ does the job.

   (b) To compute this covariance:

$$Cov(Y_n, Y_m) = ... = 10p(1-p)n$$

   (c) Low definite indices are chosen such that those students who get confused by the sums in part (b) have a fighting chance of working out part of (c).
   $\gamma_{1,2} = 10p(1-p)$. $10p(1-p) = 0 \Rightarrow p \in \{0, 1\}$
   For the extreme values we have $P(Y_n = 10n) = p$ and $P(Y_n = 0) = 1 - p$ whence the joint probabilities are the product of the marginal probabilities: $P(Y_n = 10n, Y_m = 10m) = p = p^2 = P(Y_n = 10n)P(Y_m = 10m)$ and $P(Y_n = 0, Y_m = 0) = 1 - p = (1-p)^2 = P(Y_n = 0)P(Y_m = 0)$ (note carefully that $p^2 = p$ and $(1-p)^2 = 1 - p$ hold for $p \in \{0, 1\}$ but not generally, of course).

3. This is the experiment I actually performed during Thursday's (22 Oct) lecture. The events B_1, B_2, B_3 are exactly the events of the corresponding questions on sheet 2 to get chosen for marking, hence I'm hoping students will be interested in computing their probabilities, and there may be an opportunity to discuss a little modelling. For the record, the observed lengths were $l = 26.1$cm and $l_1 = 12.4$cm.

   Note carefully that at this point in the lecture course, students have not seen transformation of continuous, let alone bivariate random variables. Therefore, solutions based on transformation formulae are not expected (and would be quite difficult to compute, I suspect).

(a) The option hinted at in the question is

$$\mathbb{E}[L_1] = ... \overset{TOK}{=} ... = 13\text{cm}$$

One alternative way is to exploit independence of U and L to simply write $\mathbb{E}[UL] = \mathbb{E}[L]\mathbb{E}[U] = 26\text{cm} \cdot \frac{1}{2}$.

(b) Using the brand new iterated conditional variance formula the solution is

$$\text{Var}(L_1) = ... = \frac{169.09}{3}\text{cm}^2 \approx 56.363\text{cm}^2$$

A solution is also possible (and maybe even sligthly shorter) exploiting independence and this should also attract full marks if done correctly which includes that in lectures, only $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$ was pointed out for independent X and Y – nothing has been said about variances of products of independent random variables, so all variances will have to be expressed as expectations.

(c) ..., whence $L_1 \sim \text{Unif}(-0.003, 26.003)$. The approximate probabilites are thus $P(B_2) \approx 38.5\%$, $P(B_3) \approx 38.5\%$ and $P(B_1) \approx 23.1\%$.

C: Exam Practice Questions

later