Exercise Sheet 5

The exercise sheet is subdivided into three parts:

- Part A contains warm-up questions you should do in your own time.
- You should solve all questions of Part B and put your solutions in the locker in the undergraduate common room in the Department of Statistical Science (Room number 117) assigned to your tutor group by the deadline stated on the exercise sheet. Don't forget to mark your solutions with your name and student number.

  Reminder: If you miss the deadline for reasons outside your control, for example illness or bereavement, you ***must*** submit a claim for extenuating circumstances, normally within a week of the deadline. Your home department will advise you of the appropriate procedures. For Statistical Science students, the relevant information is on the DOSSSH Moodle page.

  Your tutor will then mark one question selected at random after you handed in your solutions (mark counts towards ICA) and will return your marked solutions to you at the next tutorial at which the solutions to the part B questions will be discussed. Marks available for each part question are indicated in square brackets, e.g. [3]. Very succinct solutions to these questions will usually be made available on Moodle at 9:15am on Thursday mornings, so don't hand in late!
- Solutions to Part C questions will be made available in time for exam revision. These questions will not be discussed at the tutorials.

A: Warm-up questions

1. X takes values 1, 2, 3, 4 with probabilities $\frac{1}{10}, \frac{1}{5}, \frac{3}{10}, \frac{2}{5}$ and $Y = (X-2)^2$.

   (i) Find $\mathbb{E}(Y)$ and $\text{var}(Y)$ using the formulae for $\mathbb{E}_X[\phi(X)]$.

   (ii) Calculate the pmf of Y and use it to calculate $\mathbb{E}_Y(Y)$ and $\text{Var}_Y(Y)$ directly.

2. The random variable X has pdf $f_X(x) = e^{-x}$ on $(0, \infty)$. Obtain the pdf of $Y = e^X$.

B: Answers to hand in by Thursday, 19th November, 9am

1. Let the continuous random variables X and Y have joint pdf

$$f_{X,Y}(x, y) = C \exp(-\sqrt{x^2 + y^2})$$

   for $x, y \in \mathbb{R}$, where $C > 0$ is a normalization constant. Consider the transformed random variables R and φ taking values in $[0, \infty)$ and $[0, 2\pi)$ respectively, where $X = R\cos\varphi$ and $Y = R\sin\varphi$.

   (a) Show that the joint pdf of R and φ is given by

$$f_{R,\varphi}(r, \varphi) = Cre^{-r} \text{ on } r > 0,\ \varphi \in [0, 2\pi).$$

   and determine the value of C. [4]

   (b) Explain why R and φ are independent. `Hint:` factorisation theorem. [2]

   (c) Name the distributions that R and φ follow and provide any parameter values required. `Hint:` consult the appendix of the lecture notes. [2]

   (d) Compute $\mathbb{E}[X]$ and $\mathbb{E}[XY]$. `Hint:` independence of R and φ may help. [4]

   (e) Compute $\text{corr}(X, Y)$. [1]

   (f) Compute the expectation of R. [2]

2. Let $U \sim \text{Unif}(-\alpha, \alpha)$ follow the uniform distribution on the interval $(-\alpha, \alpha)$ for some parameter $\alpha > 0$ and consider the transformed random variable $X = \sin U$. Note that you will need some trigonometric identities such as $\cos(\arcsin(x)) = \sqrt{1-x^2}$ in this question, so it may be helpful to revise these.
   (a) Sketch the arcsin function. Remember it maps $[-1,1]$ to $[-\pi/2, \pi/2]$. [2]
   (b) In the case $\alpha = \pi/2$, compute the pdf of X. [3]
   (c) In the case $\alpha = 2\pi$, compute the pdf of X. [4]
   (d) Compute the cdf of X for the two preceding cases. [2]
   (e) In the same two cases, compute the expectation and variance of X. `Hint:` it may be easier to integrate with respect to U rather than X. [4]

3. Let N_1, N_2, N_3 follow a trinomial distribution with success probabilities $p_1 = 0.3$, $p_2 = 0.4$ and $p_3 = 0.3$ and sample size $n \in \mathbb{N}$.
   (a) For $n = 5$, what is the probability that N_1 and N_2 will both equal their expected value, i.e. the probability that $N_1 = \mathbb{E}[N_1]$ and $N_2 = \mathbb{E}[N_2]$ happen together? [3]
   (b) For a sample size of $n = 5$, what is the probability that $0.6\mathbb{E}[N_1] \leq N_1 \leq 1.4\mathbb{E}[N_1]$ and $0.6\mathbb{E}[N_2] \leq N_2 \leq 1.4\mathbb{E}[N_2]$ happen together? [4]
   (c) If you were to answer the preceding part with a changed sample size of $n = 100$, how many probability contributions would you need to sum up? **NB:** you do *not* need to actually calculate the probability - just calculate how many contributions there would be. [4]
   (d) For an arbitrary sample size $n \in \mathbb{N}$, find a bivariate normal distribution for the random variables X and Y such that $\mathbb{E}[X] = \mathbb{E}[N_1]$, $\mathbb{E}[Y] = \mathbb{E}[N_2]$ and $\text{Var}(X) = \text{Var}(N_1)$, $\text{Var}(Y) = \text{Var}(N_2)$ as well as $\text{Cov}(X, Y) = \text{Cov}(N_1, N_2)$. Write down the final answer in the form $\begin{pmatrix} X \\ Y \end{pmatrix} \sim N(\mu, \Sigma)$ for mean vector $\mu \in \mathbb{R}^2$ and covariance matrix $\Sigma \in \mathbb{R}^{2\times 2}$ specified in terms of $n \in \mathbb{N}$. [4]

C: Exam Practice Questions

1. Suppose that X and Y are independent gamma variables having parameters (n, θ) and (m, θ) respectively. Show $Z = X/(X+Y)$ has probability density given by
$$f_Z(z) = \frac{z^{n-1}(1-z)^{m-1}\Gamma(n+m)}{\Gamma(n)\Gamma(m)}, \qquad \text{on } z \in (0,1).$$
[**Hint:** consider the pair (Z, W), where $W = X + Y$.]

2. Suppose $f_{X_1,X_2}(x_1, x_2) = e^{-x_1-x_2}$ for $x_1, x_2 > 0$. Obtain the joint density of $Y_1 = \frac{1}{2}(X_1 + X_2)$ and $Y_2 = \frac{1}{2}(X_1 - X_2)$. (Be careful finding the range of (y_1, y_2).) Hence write down the marginal distribution of Y_1.

3. Let X_1 and X_2 be independent $\text{Exp}(\lambda)$, and let $Y_1 = X_1 + X_2$ and $Y_2 = X_1/X_2$. Show that Y_1 and Y_2 are independent and find their densities.