Example 5.7 $\quad X_i \sim Ber(p)$, i.i.d., $i=1,\dots,n \quad p\in[0,1]=\Theta$

$$\mathcal{L}(p) = \prod_{i=1}^{n} P(X_i = x_i; p)$$

Observations $x_1, x_2, \dots, x_n$, estimate p. $x_i = 1$ for success, 0 for failure.

$$= \prod_{i=1}^{n} \begin{cases} p & \text{if } x_i = 1 \\ 1-p & \text{if } x_i = 0 \end{cases}$$

$$= \prod_{i=1}^{n} p^{x_i}\cdot(1-p)^{1-x_i} = \cancel{p}\left(\prod_{i=1}^{n} p^{x_i}\right)\cdot\left(\prod_{i=1}^{n}(1-p)^{1-x_i}\right)$$

if $p \notin \{0,1\}$

$$= p^{\sum_{i=1}^{n} x_i}\,(1-p)^{\sum_{i=1}^{n}(1-x_i)}$$

$$= (1-p)^n \left(\frac{p}{1-p}\right)^{\sum_{i=1}^{n} x_i}, \quad S := \sum_{i=1}^{n} x_i$$

$$= (1-p)^n \cdot \left(\frac{p}{1-p}\right)^{S}$$

$$\ell(p) = \log \mathcal{L}(p) = n \log(1-p) + S \cdot \log\frac{p}{1-p}$$

$= \log p - \log(1-p)$

$$\frac{d}{dp}\ell(p) = \frac{-n}{1-p} + S\left(\frac{1}{p} + \frac{1}{1-p}\right) \overset{!}{=} 0$$

$$\Leftrightarrow \frac{S}{n} = \frac{1}{\left(\frac{1}{p}+\frac{1}{1-p}\right)\cdot(1-p)} = \frac{1}{\frac{1-p}{p}+1} = \frac{1}{1/p} = p.$$

candidate: $\hat{p}_{ML} = \frac{1}{n}\cdot S$

$$\frac{d^2}{dp^2}\ell(p) = \frac{-n}{(1-p)^2} + S\left(\frac{-1}{p^2} + \frac{1}{(1-p)^2}\right)$$

$$= \frac{1}{(1-p)^2}\cdot(-n+S) - \frac{S}{p^2} < 0,$$ so $\hat{p}_{ML}$ is a local maximum.

Check the boundaries:

$\cancel{\mathcal{L}(0) = (1-0)^n \cdot \left(\frac{0}{1-0}\right)^S}$

$$\boxed{\mathcal{L}(1) = \prod_{i=1}^{n} \begin{cases} 1 & \text{if } x_i = 1 \\ 0 & \text{if } x_i = 0 \end{cases} = \begin{cases} 0 & \text{if } S < n \\ 1 & \text{if } S = n \end{cases}}$$

So, for $S = 0$, $\hat{p} = 0$ is the maximum likelihood estimate.

for $S = n$, $\hat{p} = 1$ is the maximum likelihood estimate

so $\mathcal{L}(0) = 0$ unless $S \in \{0, n\}$

$$\boxed{\mathcal{L}(0) = \prod_{i=1}^{n} \begin{cases} 1 & \text{if } x_i = 0 \\ 0 & \text{if } x_i = 1 \end{cases} = \begin{cases} 0 & \text{if } S > 0 \\ 1 & \text{if } S = 0 \end{cases}}$$