Ex 8 B1

$X_1, X_2, X_3, \dots$ iid $N(0,1)$, $\alpha \in \mathbb{R}$.

$Y_i := (\alpha + X_i)^2$, $T := \sum_{i=1}^n (\alpha + X_i)^2$ non-central χ^2-distribution.

(a)

$$\mathbb{E}T = \sum_{i=1}^n \mathbb{E}Y_i = \sum_{i=1}^n \mathbb{E}(\alpha^2 + 2\alpha X_i + X_i^2) = \sum_{i=1}^n (\alpha^2+1) = n(1+\alpha^2)$$

$$\mathrm{Var}\,T \underset{\uparrow}{=} \sum_{i=1}^n \mathrm{Var}(Y_i) = \sum_{i=1}^n \mathrm{Var}(\alpha^2 + 2\alpha X_i + X_i^2) = \sum_{i=1}^n \mathrm{Var}(2\alpha X_i + X_i^2)$$

Y_i indep.

$\mathrm{Var}\,X = \mathbb{E}X^2 - (\mathbb{E}X)^2$

$$= \sum_{i=1}^n \left(\mathbb{E}[4\alpha^2 X_i^2 + 4\alpha X_i^3 + X_i^4] - (\mathbb{E}[X_i^2])^2\right)$$

$$= \sum_{i=1}^n (4\alpha^2 + 0 + 3 - 1) = 2n(2\alpha^2 + 1)$$

$\mathbb{E}[X_i^3] = 0$ from question

$\mathbb{E}[X_i^4] = 3$

(b)

pdf of $X_1 \sim N(0,1)$

$$M_{Y_1}(s) = \mathbb{E}[\exp(sY_1)] = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(s(\alpha+x)^2 - \frac{1}{2}x^2\right) dx$$

$$= \frac{1}{\sqrt{2\pi}} \int \exp\left(-\left(\tfrac{1}{2} - s\right)x^2 + 2s\alpha x + s\alpha^2\right) dx$$

$$= \frac{1}{\sqrt{2\pi}} \int \exp\left(-\left(\tfrac{1}{2} - s\right)\left[x^2 - \frac{2}{\frac{1}{2}-s}\alpha s x\right] + s\alpha^2\right) dx$$

$$= \frac{1}{\sqrt{2\pi}} \int \exp\left(-\left(\tfrac{1}{2} - s\right)\left[x - \frac{s\alpha}{\frac{1}{2}-s}\right]^2 + \frac{\alpha^2 s^2}{\frac{1}{2}-s} + s\alpha^2\right) dx$$

$$= \exp\left(\frac{\alpha^2 s^2}{\frac{1}{2}-s} + s\alpha^2\right) \frac{1}{\sqrt{1-2s}} \underbrace{\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi/(1-2s)}} \exp\left(-\frac{1}{2\frac{1}{1-2s}}\left[x - \frac{s\alpha}{\frac{1}{2}-s}\right]^2\right) dx}_{\text{pdf of } N\left(\frac{s\alpha}{\frac{1}{2}-s}, \frac{1}{1-2s}\right)}$$

$$= \frac{1}{\sqrt{1-2s}} \exp\left(\alpha^2 s \cdot \left(\frac{s}{\frac{1}{2}-s} + 1\right)\right)$$

$$= \frac{1}{\sqrt{1-2s}} \exp\left(\frac{\alpha^2 s}{1-2s}\right) \quad \text{q.e.d.}, \quad s < \frac{1}{2}.$$