Example 3.6 $f_X(x) = \frac{1}{|\det 2\pi\Sigma|^{\frac{1}{2}}} \exp\left(-\frac{1}{2}(x-\mu)^T\Sigma^{-1}(x-\mu)\right)$.

$e^a e^b = e^{a+b}$

$$s \in \mathbb{R}^n$$

$$M_X(s) = \mathbb{E}[e^{s^T X}] = \int \frac{1}{|\det 2\pi\Sigma|^{\frac{1}{2}}} \exp\Big(\underbrace{-\frac{1}{2}(x-\mu)^T\Sigma^{-1}(x-\mu) + s^T x}_{=:Q(x)}\Big)dx.$$

$s^T x = x^T s$

$$Q(x) := -\frac{1}{2}x^T\Sigma^{-1}x + \frac{1}{2}\mu^T\Sigma^{-1}x + \frac{1}{2}x^T\Sigma^{-1}\mu - \frac{1}{2}\mu^T\Sigma^{-1}\mu + s^T x$$

$ax^2 + bx = a\left(x^2 + \frac{b}{a}x\right)$

$$= -\frac{1}{2}x^T\Sigma^{-1}x + \frac{1}{2}\left(\mu^T\Sigma^{-1}x + s^T x\right) + \frac{1}{2}\left(x^T\Sigma^{-1}\mu + x^T s\right) - \frac{1}{2}\mu^T\Sigma^{-1}\mu$$

$$s^T x = s^T\left(\Sigma\Sigma^{-1}x\right) = \left(\Sigma^T s\right)^T\Sigma^{-1}x = (\Sigma s)^T\Sigma^{-1}x$$

(used $\Sigma = \Sigma^T$, i.e. Σ is symmetric because it's a covariance matrix)

$$= -\frac{1}{2}x^T\Sigma^{-1}x + \frac{1}{2}(\mu + \Sigma s)^T\Sigma^{-1}x + \frac{1}{2}x^T\Sigma^{-1}(\mu + \Sigma s) - \frac{1}{2}\mu^T\Sigma^{-1}\mu$$

$$= -\frac{1}{2}(x - \mu - \Sigma s)^T\Sigma^{-1}(x - \mu - \Sigma s) + \frac{1}{2}(\mu + \Sigma s)^T\Sigma^{-1}(\mu + \Sigma s) - \frac{1}{2}\mu^T\Sigma^{-1}\mu$$

$$= -\frac{1}{2}(x - \mu - \Sigma s)^T\Sigma^{-1}(x - \mu - \Sigma s) + \frac{1}{2}(\Sigma s)^T\Sigma^{-1}(\Sigma s) + \frac{1}{2}\mu^T\Sigma^{-1}\Sigma s + \frac{1}{2}(\Sigma s)^T\Sigma^{-1}\mu$$

$$= -\frac{1}{2}(x - \mu - \Sigma s)^T\Sigma^{-1}(x - \mu - \Sigma s) + \frac{1}{2}s^T\Sigma s + \mu^T s.$$

$$M_X(s) = \int \underbrace{\frac{1}{|\det 2\pi\Sigma|^{\frac{1}{2}}} \exp\left(-\frac{1}{2}(x - \mu - \Sigma s)^T\Sigma^{-1}(x - \mu - \Sigma s)\right)}_{\text{pdf of } \mathcal{N}(\mu + \Sigma s,\ \Sigma)} \cdot \exp\left(\frac{1}{2}s^T\Sigma s + \mu^T s\right)dx$$

$$= \exp\left(\frac{1}{2}s^T\Sigma s + \mu^T s\right).$$