Exercise Sheet 8

The exercise sheet is subdivided into three parts:

- Part A contains warm-up questions you should do in your own time.
- You should solve all questions of Part B and put your solutions in the locker in the undergraduate common room in the Department of Statistical Science (Room number 117) assigned to your tutor group by the deadline stated on the exercise sheet. Don't forget to mark your solutions with your name and student number.

  Reminder: If you miss the deadline for reasons outside your control, for example illness or bereavement, you ***must*** submit a claim for extenuating circumstances, normally within a week of the deadline. Your home department will advise you of the appropriate procedures. For Statistical Science students, the relevant information is on the DOSSSH Moodle page.

  Your tutor will then mark one question selected at random after you handed in your solutions (mark counts towards ICA) and will return your marked solutions to you at the next tutorial at which the solutions to the part B questions will be discussed. Marks available for each part question are indicated in square brackets, e.g. [3]. Very succinct solutions to these questions will usually be made available on Moodle at 9:15am on Thursday mornings, so don't hand in late!
- Solutions to Part C questions will be made available in time for exam revision. These questions will not be discussed at the tutorials.

A: Warm-up questions

1. Let $X, Y, Z \sim N(0,1)$ be independent. What is the distribution of $X+1$? What is the distribution of $(X+1)+(Y+1)+(Z+1)$? What are the distributions of X^2 and of $X^2+Y^2+Z^2$? Have you seen the distribution of $(X+1)^2$ before?

B: Answers to hand in by Thursday, 10th December, 9am

1. Let $X_1, X_2, X_3, \ldots$ be independent standard normal random variables. Let $\alpha \in \mathbb{R}$ and $n \in \mathbb{N}$ be parameters and define $Y_i = (\alpha + X_i)^2$ and $T = \sum_{i=1}^n (\alpha + X_i)^2$.

   (a) Compute the expectation and variance of T. You may use without proof that $\mathbb{E}[X_1^3] = 0$ and $\mathbb{E}[X_1^4] = 3$. [4]

   (b) Show that the mgf of Y_1 is given by

   $$M_{Y_1}(s) = \frac{\exp\left(\frac{\alpha^2 s}{1-2s}\right)}{\sqrt{1-2s}} \qquad \text{on } s < 1/2$$

   and hence deduce the mgf of T. [6]

   (c) Now let $M \sim \text{Poi}(n\alpha^2/2)$ be another independent random variable. Let

   $$Z = \sum_{i=1}^{n+2M} X_i^2$$

   (note that the number of terms in the sum is now a random variable). State the distribution of Z conditional on $M = m \in \mathbb{N} \cup \{0\}$, and write down the mgf of this distribution. Hence find the moment generating function of Z and use it to identify the distribution of Z as one of the distributions whose mgf has been calculated before. [5]

2. (a) For $n \in \mathbb{N}$, let $X_1, \ldots, X_n$ be independent and identically distributed following a normal distribution with mean μ and variance $\sigma^2 < \infty$. Compute the expectation and variance of $S_\mu^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \mu)^2$ justifying your calculations carefully. You are required to carry out the calculations and explain your reasoning clearly at each step: do not merely quote the result and sketch derivation from p77 of the lecture notes. You may, however, use known properties of the chi-squared family of distributions. [5]

   (b) Instead of being fixed, let the number of samples n itself be a random variable following the distribution given by $n = N + 1$ where $N \sim \text{Poi}(\lambda)$ with $\lambda > 0$. Conditionally on N, let $X_1, \ldots, X_{N+1}$ be independent and identically distributed following an $N(\mu, \sigma^2)$ distribution with $\sigma^2 < \infty$. Compute the expectation and variance of $S_\mu^2 = \frac{1}{N+1}\sum_{i=1}^{N+1}(X_i - \mu)^2$ in this case. [10]

C: Exam Practice Question

1. Consider three independent random variables X_1, X_2, X_3 and suppose that $X_i \sim N(i, i^2)$, $i = 1, 2, 3$. Using only these three random variables, give an example of a transformation that has

   (a) a χ_3^2–distribution.

   (b) a $F_{1,2}$–distribution.

   (c) a t_2–distribution.