Example 4.3

Show that $\bar{X}$ and U_i are independent by computing their covariance:

$U_i = X_i - \bar{X}$

$$\mathrm{Cov}(U_i, \bar{X}) = \mathrm{Cov}\left(X_i - \bar{X}, \frac{1}{n}\sum_{k=1}^{n} X_k\right)$$

$$= \mathrm{Cov}\left(X_i - \frac{1}{n}\sum_{j=1}^{n} X_j, \frac{1}{n}\sum_{k=1}^{n} X_k\right)$$

$$= \mathrm{Cov}\left(X_i, \frac{1}{n}\sum_k X_k\right) - \frac{1}{n^2}\mathrm{Cov}\left(\sum_{j=1}^{n} X_j, \sum_k X_k\right)$$

$$= \sum_{k=1}^{n}\frac{1}{n}\mathrm{Cov}(X_i, X_k) - \frac{1}{n^2}\sum_{j,k=1}^{n}\mathrm{Cov}(X_j, X_k)$$

$$= \frac{1}{n}\mathrm{Cov}(X_i, X_i) - \frac{1}{n^2}\sum_{j=1}^{n}\mathrm{Cov}(X_j, X_j)$$

$$= \frac{1}{n}\mathrm{Var}(X_i) - \frac{1}{n^2}\sum_j \mathrm{Var}(X_j)$$

$$= \frac{1}{n}\underbrace{\cancel{\mathrm{Var}(X_i)}}_{\sigma^2} - \frac{1}{n^2}\cdot n \cdot \sigma^2$$

$$= 0$$

$\mathrm{Cov}(X-Y, Z) = \mathrm{Cov}(X,Z) - \mathrm{Cov}(Y,Z)$

$\mathrm{Cov}(aX, Y) = a\,\mathrm{Cov}(X,Y)$

$\mathrm{Cov}(X,X) = \mathrm{Var}(X)$

Since $\bar{X}$ and U_i are jointly normal, ~~the fact that~~ and have covariance zero, they are independent.

<u>NB</u>: We have not shown that $(\bar{X}, U_1, \ldots, U_n)$ are independent. To do this, we ~~then~~ would have to show that the covariance matrix

$$M_{ij} = \mathrm{Cov}(Y_i, Y_j), \quad Y_i = \begin{cases} \bar{X} & \text{if } i = 1 \\ U_{i-1} & \text{if } i > 1 \end{cases}$$

is diagonal.