Exercise Sheet 6

A: Warm-up question

1. For geometrically distributed variables X and Y we have that $\mathrm{P}(X > n) = (1-p)^n$ (no successes in n trials, or sum the GP). Further, with $Z = \min(X, Y)$ we have

   $$\mathrm{P}(Z > n) = \mathrm{P}(X > n \text{ and } Y > n) = \mathrm{P}(X > n)\mathrm{P}(Y > n)$$

   since X, Y are independent. Thus,

   $$\mathrm{P}(Z > n) = (1-p)^{2n}\,.$$

   If we set $q = (1-p)^2$ we have $\mathrm{P}(Z > n) = q^n$ which has the same structure as the corresponding probability for a geometric variable if the parameter is $\tilde{p} = 1 - q = 1 - (1-p)^2 = p(2-p)$.
   Intuition: Z counts the number of trials required until the first success occurs, either in the X-sequence or the Y-sequence of trials. But the probability of a success is $p + p - p^2 = p(2-p)$, by the addition rule.

B: questions handed in

1. (a) i.

   $$\mathbb{E}[X^3] \approx 4$$

   ii.

   $$\mathbb{E}[X^3] = ... = 6$$

   (b) i.

   $$\mathbb{E}[W + Z] = ... = \frac{\pi^2}{24}$$

   $$\mathbb{E}[\sin(W + Z)] \approx 0.794$$

   ii.

   $$\mathbb{E}[\phi(W)\psi(Z)] = ... = \mathbb{E}[\phi(W)]\mathbb{E}[\psi(Z)]$$

   iii. Using the previous result, the computation is fairly striaghtforward if one remembers trigonometric addition theorems.

   $$\mathbb{E}[\sin(W + Z)] = ... = 2\mathbb{E}[\sin W]\mathbb{E}[\cos W],$$

   $$\mathbb{E}[\sin(W + Z)] \approx 0.811$$

2. (a) Use of the iterated conditional expectation formula is probably the easiest way forward especially since expectation of Y is given:

$$\mathbb{E}[X] = ... = 1/2$$

(b)

$$M_X(s) = ... = 2\frac{e^s - 1 - s}{s^2}$$

for any $s \in \mathbb{R}\ \{0\}$ as well as $M_x(0) = 1$, of course.

(c) Differentiation will have to use the quotient rule leading to somewhat lengthy expressions:

$$\frac{dM_X(s)}{ds} = \frac{-4e^s + 2se^s + 2s + 4}{s^3}$$

$$\mathbb{E}[X] = ... = 1/3$$

3. (a)

$$f_{X_1}(x) = \frac{k}{\lambda^k} x^{k-1} e^{-(x/\lambda)^k} \text{ on } x > 0$$

$$\mathbb{E}[X_1] = ... \overset{z=(x/\lambda)^k}{=} \lambda \int_0^\infty z^{1/k} e^{-z}\, dz = \lambda\Gamma(1 + 1/k),$$

(b)

$$F_1(x) = ... = 1 - e^{-(x/(\lambda n^{-1/k})^k} \text{ on } x > 0$$

(c) We recognise the cdf ... Hence, ... $\mathbb{E}[X_{(1)}] = \lambda n^{-1/k}\Gamma(1 + 1/k)$ which tends to zero as $n \to \infty$.

C: exam practice question

later.