Example 2.6 $f_{X,Y}(x,y) = xy e^{-(x+y)}$, $x>0$, $y>0$ 06/11/15

$$f_Z(z) = \int_{-\infty}^{\infty} f_{X,Y}(x, z-x)\,dx = \int_0^z x(z-x) e^{-(x+z-x)}\,dx$$

$$y>0 \Rightarrow z-x>0 \Rightarrow z>x$$

$$= e^{-z}\int_0^z xz - x^2\,dx = e^{-z}\left[\frac{z}{2}x^2 - \frac{1}{3}x^3\right]_0^z$$

$$= \frac{z^3}{6} e^{-z} \sim Gam(4,1)$$

Note $f_{X,Y}(x,y) = \begin{cases} xe^{-x} & \text{if } x>0 \\ 0 & \text{oth.} \end{cases} \cdot \begin{cases} ye^{-y} & \text{if } y>0 \\ 0 & \text{oth} \end{cases}$

thus, $X \perp\!\!\!\perp Y$. Whenever $X \sim Gam(2,1)$

$Y \sim Gam(2,1)$

$X+Y \sim Gam(4,1)$

More generally (to be shown - Ch. 3)

$$X \sim Gam(\alpha_1, \beta),\ Y \sim Gam(\alpha_2, \beta),\ X \perp\!\!\!\perp Y$$

$$\Rightarrow X+Y \sim Gam(\alpha_1 + \alpha_2, \beta)$$