Solutions to Exercise Sheet 9

Please try and solve the questions yourself first before reading the solutions.

Really, once you know the solutions, it may seem easier than it actually is...

Seriously.

1. Let $X_1, \ldots, X_n$ be iid from $U(0, \theta)$ and consider $\overline{X}$ as well as $X_{(n)}$ as bases for estimators for θ. Check whether they are unbiased:

$$\mathrm{E}(\overline{X}) = \mathrm{E}(X_i) = \frac{\theta}{2},$$

so that $T_1 = 2\overline{X}$ is unbiased for θ. This seems plausible since $\overline{X}$ gives the 'middle' of the data so that the upper bound should be two times $\overline{X}$.
To determine the expectation of $X_{(n)}$, first find its cdf and pdf. We have $F_n(x) = \mathrm{P}(\max(X_1, \ldots, X_n) \leq x) = \prod_i \mathrm{P}(X_i \leq x)$ so that

$$F_n(x) = \begin{cases} 0, & x \leq 0 \\ \left(\frac{x}{\theta}\right)^n, & 0 < x \leq \theta \\ 1, & x > \theta \end{cases}$$

The density is therefore $f_n(x) = \frac{nx^{n-1}}{\theta^n}$ on $0 < x < \theta$ (see lecture notes for distribution of sample maximum). Therefore the expectation is given by

$$\mathrm{E}(X_{(n)}) = \int_0^\theta x f_n(x)\ \mathrm{d}x = \int_0^\theta \frac{nx^n}{\theta^n}\ \mathrm{d}x = \frac{n\theta}{n+1}.$$

This implies that $T_2 = \frac{(n+1)X_{(n)}}{n}$ is unbiased for θ. Again, this is plausible because n observations from a uniform distribution are expected to divide the interval $[0, \theta]$ into $n+1$ equal parts so that the upper bound will approximately be $X_{(n)} + X_{(n)}/n$.
Since both estimators are unbiased their mean squared errors are equal to their variances. For the first we find

$$\mathrm{mse}(T_1) = \mathrm{var}(2\overline{X}) = 4\mathrm{var}(X_i)/n = \frac{\theta^2}{3n}.$$

The variance of $X_{(n)}$ is found as follows:

$$\mathrm{E}(X_{(n)}^2) = \int_0^\theta \frac{nx^{n+1}}{\theta^n}\ \mathrm{d}x = \frac{n\theta^2}{n+2}$$

so that $\mathrm{var}(X_{(n)}) = \mathrm{E}(X_{(n)}^2) - \{\mathrm{E}(X_{(n)})\}^2 = \frac{n\theta^2}{(n+2)(n+1)^2}$. This gives

$$\mathrm{mse}(T_2) = \mathrm{var}\left(\frac{n+1}{n} X_{(n)}\right) = \frac{(n+1)^2}{n^2} \frac{n\theta^2}{(n+2)(n+1)^2} = \frac{\theta^2}{n(n+2)}.$$

For samples of size at least two, the second mse is smaller (since $n(n+2) > 3n$ for all $n \in \mathbb{N} \setminus \{1\}$) so that T_2 is to be preferred.

2. The density of X is

$$f(x; \theta) = (2\pi\sigma^2)^{-1/2} \exp\{-\frac{1}{2\sigma^2}(x-\theta)^2\}$$

with log-density

$$l \equiv \log f(x; \theta) = -\frac{1}{2}\log(2\pi\sigma^2) - \frac{1}{2\sigma^2}(x-\theta)^2.$$

Therefore
$$\frac{\partial l}{\partial \theta} = -\frac{1}{\sigma^2}(x-\theta) \quad \text{and} \quad \frac{\partial^2 l}{\partial \theta^2} = -\frac{1}{\sigma^2}$$
so that $i(\theta) = \frac{1}{\sigma^2}$. The Cramér-Rao lower bound is therefore $\text{var}(T) \geq \frac{\sigma^2}{n}$. Consider the 'obvious' estimator $T = \bar{X}$. Then $\text{E}(T) = \theta$ and $\text{var}(T) = \frac{\sigma^2}{n}$ so $T = \bar{X}$ is the best unbiased estimator of θ.

3. Since $\text{E}(X) = \theta$ for $\text{Poi}(\theta)$, the method of moments simply gives
$$\hat{\theta} = \overline{X}$$
Since $\text{E}(\hat{\theta}) = \theta$ the estimator is unbiased. Its variance is $\text{var}(\hat{\theta}) = \theta/n$ (variance of the sample mean).
The Cramér–Rao bound is derived as follows:
$$\log f(x;\theta) = x\log\theta - \theta - \log(x!)$$
$$\frac{\partial}{\partial\theta}\log f(x;\theta) = \frac{x}{\theta} - 1 \quad \frac{\partial^2}{\partial\theta^2}\log f(x;\theta) = -\frac{x}{\theta^2}$$
so that the Fisher information is $i(\theta) = -\text{E}(-X/\theta^2) = 1/\theta$. The Cramér–Rao bound is therefore θ/n.
This shows that $\hat{\theta} = \overline{X}$ is unbiased and attains the lower variance bound in the Poisson case.

4. If $X \sim \text{Gam}(\alpha, \lambda)$ then the mean and variance of X are $\text{E}(X) = \alpha/\lambda$, $\text{var}(X) = \alpha/\lambda^2$. The first two moments are therefore α/λ and $\alpha/\lambda^2 + (\alpha/\lambda)^2 = \alpha(1+\alpha)/\lambda^2$. Equate these to the sample moments $\bar{X}$ and $\sum_i X_i^2/n$ respectively to give two equations in the two unknowns α and λ. The first equation gives $\hat{\lambda} = \hat{\alpha}/\bar{X}$. Substituting this into the second equation gives $(1+\hat{\alpha})\bar{X}^2 = \hat{\alpha}\sum_i X_i^2/n$. Solving this equation for $\hat{\alpha}$, noting that $\sum_i X_i^2/n - \bar{X}^2 = \sum_i (X_i - \bar{X})^2/n$, gives $\hat{\alpha} = \bar{X}^2/S^2$ as required.