Iterated Conditional Expectation

$$\mathbb{E}_Y[\mathbb{E}_{X|Y}[X|Y]] \underset{\substack{\uparrow \\ \text{def. of} \\ \text{expectation}}}{=} \sum_{y_j} \mathbb{E}[X|Y=y_j]\, p_Y(y_j)$$

$$= \sum_{y_j} \sum_{x_i} x_i\, p_{X|Y}(x_i|y_j)\, p_Y(y_j)$$

$$= \sum_{y_j} \sum_{x_i} x_i\, p_{X,Y}(x_i, y_j)$$

need mathematical argument

$$= \sum_{x_i} \sum_{y_j} x_i\, p_{X,Y}(x_i, y_j)$$

$$= \sum_{x_i} x_i \sum_{y_j} p_{X,Y}(x_i, y_j) = \sum_{x_i} x_i\, p_X(x_i)$$

$$= \mathbb{E}[X]. \quad \square$$

Example 1.20

$$\mathbb{E}[R] = \mathbb{E}_N(\mathbb{E}_{R|N}[R|N]]$$
$$= \mathbb{E}_N[N\pi] = \pi\, \mathbb{E}_N[N]$$
$$= \pi \cdot \lambda.$$

TOK: $f(y_j) := \mathbb{E}_{X|Y}[\phi(Y)\psi(X,Y) \mid Y=y_j]$

$$= \sum_{x_i} \phi(y_j)\psi(x_i, y_j)\, p_{X|Y}(x_i|y_j)$$

$$= \phi(y_j) \sum_{x_i} \psi(x_i, y_j)\, p_{X|Y}(x_i|y_j)$$

$$= \phi(y_j)\, \mathbb{E}_{X|Y}[\psi(X, y_j) | Y = y_j]$$

This holds for all possible values y_j. Therefore

$$f(Y) = \mathbb{E}_{X|Y}[\phi(Y)\psi(X,Y)|Y] = \phi(Y)\mathbb{E}_{X|Y}[\psi(X,Y)|Y]. \quad \square$$