Example 3.7 $\quad X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \sim N\left(\begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 & 0{,}7 \\ 0{,}7 & 1 \end{pmatrix} \right)$

$$\mathbb{E}[X_1 X_2] = Cov(X_1, X_2) + \mathbb{E}[X_1]\,\mathbb{E}[X_2] = 0{,}7 + 2 \cdot 1 = \underline{\underline{2{,}7}}$$

$$M_X(s_1, s_2) = \exp\left(s^T \mu + \frac{1}{2} s^T \Sigma s\right), \; s \in \mathbb{R}^2, \; \mu = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \; \Sigma = \begin{pmatrix} 1 & 0{,}7 \\ 0{,}7 & 1 \end{pmatrix}$$

$$= \exp\left((s_1, s_2) \begin{pmatrix} 2 \\ 1 \end{pmatrix} + \frac{1}{2} (s_1, s_2) \begin{pmatrix} 1 & 0{,}7 \\ 0{,}7 & 1 \end{pmatrix} \begin{pmatrix} s_1 \\ s_2 \end{pmatrix} \right)$$

$$= \exp\left(2s_1 + s_2 + \frac{1}{2} s_1^2 + \frac{1}{2} s_1 s_2 \cdot 0{,}7 + \frac{1}{2} s_2\, 0{,}7 s_1 + \frac{1}{2} s_2^2 \right)$$

$$= \exp\left(2s_1 + s_2 + \frac{1}{2} s_1^2 + 0{,}7 s_1 s_2 + \frac{1}{2} s_2^2 \right)$$

$$\frac{\partial M_X}{\partial s_2}(s_1, s_2) = \frac{\partial}{\partial s_2} \exp\left(2s_1 + s_2 + \frac{1}{2} s_1^2 + 0{,}7 s_1 s_2 + \frac{1}{2} s_2^2 \right)$$

$$= \exp\left(2s_1 + s_2 + \frac{1}{2} s_1^2 + 0{,}7 s_1 s_2 + \frac{1}{2} s_2^2 \right) \cdot \left(1 + 0{,}7 s_1 + s_2 \right)$$

$$\frac{\partial^2}{\partial s_1 \partial s_2} M_X(s_1, s_2) = \exp\left(2s_1 + s_2 + \frac{1}{2} s_1^2 + 0{,}7 s_1 s_2 + \frac{1}{2} s_2^2 \right) \left(2 + s_1 + 0{,}7 s_2 \right) \cdot \left(1 + 0{,}7 s_1 + s_2 \right) \cancel{\left(1 + 0{,}7 s_1 + s_2\right)}$$

$$+ \exp\left(2s_1 + s_2 + \frac{1}{2} s_1^2 + 0{,}7 s_1 s_2 + \frac{1}{2} s_2^2 \right) \cdot 0{,}7$$

$$= \exp\left(2s_1 + s_2 + \frac{1}{2} s_1^2 + 0{,}7 s_1 s_2 + \frac{1}{2} s_2^2 \right) \cdot \left[(2 + s_1 + 0{,}7 s_2)(1 + 0{,}7 s_1 + s_2) + 0{,}7 \right]$$

$$\mathbb{E}[X_1 X_2] = \frac{\partial^2}{\partial s_1 \partial s_2} M_X(0, 0) = \left. \frac{\partial^2 M_X}{\partial s_1 \partial s_2} \right|_{s=0}$$

$$= \exp\left(2 \cdot 0 + 0 + \frac{1}{2} 0^2 + 0{,}7 \cdot 0 \cdot 0 + \frac{1}{2} 0^2 \right) \cdot \left[2 \cdot 1 + 0{,}7 \right]$$

$$= \exp(0) \cdot 2{,}7 = \underline{\underline{2{,}7}}$$

$$\mathbb{E}[X_1 X_2] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2 \underbrace{f_{X_1, X_2}(x_1, x_2)}_{\text{pdf of bivariate normal}} \, dx_1 \, dx_2$$