Example 5.8 $X_i \sim Exp(\lambda)$, i.i.d.

$\lambda \in \Theta = (0, \infty)$.

$f_X(x) = \lambda \exp(-\lambda x)$ on $x > 0$.

therefore, we may assume that all observations $x_i$ are positive.

$$\mathcal{L}(\lambda) = \prod_{i=1}^{n} f_X(x_i; \lambda) = \prod_{i=1}^{n} \lambda \exp(-\lambda x_i) = \lambda^n \exp\left(-\lambda \sum_{i=1}^{n} x_i\right)$$

Again, $S := \sum_{i=1}^{n} x_i$, in fact, we can write

$$\mathcal{L}(\lambda) = \prod_{i=1}^{n} f_X(x_i; \lambda) = \lambda^n \exp(-\lambda S),$$

so we only need to know $S$, not each individual $x_i$.

Hence, $S$ is called a sufficient statistic.

$$\ell(\lambda) = n \log \lambda - \lambda S \qquad \frac{d\ell}{d\lambda} = \frac{n}{\lambda} - S \overset{!}{=} 0$$

candidate: $\hat{\lambda} = \left(\frac{1}{n} S\right)^{-1}$

check: $\frac{d^2\ell}{d\lambda^2} = -\frac{n}{\lambda^2} < 0$, so $\hat{\lambda}$ is a local maximum.

Check the boundary: i) $\lambda \downarrow 0$ $\lim_{\lambda \downarrow 0} \ell(\lambda) = -\infty$.

ii) $\lambda \uparrow \infty$ $\lim_{\lambda \uparrow \infty} (n \log \lambda - \lambda S) = -\infty$

log $\lambda$ grows more slowly than $\lambda$ as $\lambda \uparrow \infty$.