Exercise Sheet 4

A: Warm-up questions

1. From the specification of the bivariate normal distribution for (X, Y), we can read off: $\mu_1 = 0$, $\mu_2 = 1$, $\sigma_1 = 1$, $\sigma_2 = 2$, $\rho = \frac{1}{2}$.

   Therefore, the joint density is easy to write down using a formula from the lecture notes:
$$f(x_1, x_2) = \frac{1}{2\pi\sqrt{3}} \exp\left[-\frac{2}{3}\left\{x_1^2 - \frac{x_1(x_2-1)}{2} + \frac{(x_2-1)^2}{4}\right\}\right]$$
   The marginal distributions are straightforward, again using results from section 1.5.2 of the lecture notes: X_1 is $N(0, 1)$, X_2 is $N(1, 4)$. Finally, the conditional distributions are given as follows: $X_2|X_1 = x_1$ is $N(x_1 + 1,\, 3)$, $X_1|X_2 = x_2$ is $N(\frac{1}{4}(x_2 - 1),\, \frac{3}{4})$

2. (a) You can get rid of all those zeros by multiplying the two vecctors by $1/1000$, so you only have to solve
$$\begin{pmatrix} 5 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 5 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 10 \\ 0 \\ 2 \end{pmatrix}$$
   I admit that this is a rather simple example and the simplification isn't all that significant but at least here it's easy to see exactly what's going on – this will change when handling multivariate normal distributions.

   (b)
$$\begin{aligned}
\frac{5}{7}\begin{pmatrix} 5 & 1 & 0 \\ 1 & 3 & 0 \\ 1 & 7 & 7 \end{pmatrix} - \frac{5}{7}\begin{pmatrix} 5 & 1 & 1 \\ 1 & 3 & -7 \\ 0 & 0 & 0 \end{pmatrix}^T &= \frac{5}{7}\begin{pmatrix} 5 & 1 & 0 \\ 1 & 3 & 0 \\ 1 & 7 & 7 \end{pmatrix} - \frac{5}{7}\begin{pmatrix} 5 & 1 & 0 \\ 1 & 3 & 0 \\ 1 & -7 & 0 \end{pmatrix} \\
&= \frac{5}{7}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 14 & 7 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 10 & 5 \end{pmatrix}
\end{aligned}$$
   Note that here it was advantageous to carry out the matrix algebra first and exploit all those cancellations before multiplying by $5/7$. The resulting matrix is not symmetric, i.e. $A = A^T$ does not hold, so it cannot be a covariance matrix. (Remember: covariance is symmetric, i.e. $cov(X, Y) = cov(Y, X)$.)

B: Hand-in questions

1. (a) The case of bag 1 is the easiest: since all letters in that bag score one point, the conditional expected value is one, $\mathbb{E}[T|B = 1] = 1$.

There are 17 letters in bag 2 altogether, whence

$$\mathbb{E}[T|B=2] = ... = \frac{38}{17} \approx 2.235$$

There are 15 letters in bag 3 altogether, whence

$$\mathbb{E}[T|B=3] = ... = \frac{81}{15} \approx 5.4$$

The interpretation is that ...

(b) The variance in bag one is again easy: since all letters have the same number of points, the variance will be zero.
... from which it follows that

$$\mathrm{Var}(T|B=2) \approx 0.886$$
$$\mathrm{Var}(T|B=3) \approx 5.04$$

The interpretation is that ...

(c) The marginal expectation is

$$\mathbb{E}[T] = ... \approx 3.612$$

and this stands for ...
The marginal variance is

$$\mathrm{Var}(T) = ... \approx 6.183$$

Again, this is the variance of the number of points ...

2. (a) The sketch for should show a triangle with corners $(-1,0), (1,0), (\alpha,1)$.
The three lines are $y = 0$, $x = -1 + (\alpha+1)y$ and $x = 1 + y(\alpha - 1)$ (or equivalent expressions of y in terms of x, both are fine).
For $\alpha = -1$, ... Therefore, we expect negative covariance.
For $\alpha = 0$, ..., so we expect zero covariance here.
For $\alpha = 1$,... we expect positive covariance here.

(b) Assuming that $y \in [0,1]$, we obtain:

$$f_Y(y) = ... = 2(1-y) \,\text{on}\, y \in [0,1].$$

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dy$$

Now, the integrand will only be positive once the point (x,y) ...

$$f_X(x) = ... = \frac{x+1}{\alpha+1}$$

and a similar result holds for $\alpha < x \leq 1$:

$$f_X(x) = ... = \frac{x-1}{\alpha-1}$$

The expected values are obtained by integration using the given marginal densities:

$$\mathbb{E}[Y] = ... = \frac{1}{3}$$
$$\mathbb{E}[X] = ... = \frac{\alpha}{3}$$

(c)

$$\mathbb{E}[XY] = ... = \frac{\alpha}{6}$$

The covariance is therefore

$$\mathrm{Cov}(X, Y) = ... = \frac{\alpha}{18}$$

which has the same sign as α and therefore agrees with the observations made in part (a).

Section C

later