Exercise Sheet 7

A: Warm-up questions

1.

$$G(z) = \sum_{r=1}^{\infty} z^r \{\pi(1-\pi)^{r-1}\} = \pi z \sum_{r=1}^{\infty} \{(1-\pi)z\}^{r-1} = \pi z\{1-(1-\pi)z\}^{-1}.$$

$G^{'}(z) = \pi\{1-(1-\pi)z\}^{-2}$ so $\mu = G^{'}(1) = 1/\pi$
$G^{''}(z) = 2\pi(1-\pi)\{1-(1-\pi)z\}^{-3}$ so $G^{''}(1) = 2(1-\pi)/\pi^2$
Therefore $\sigma^2 = G^{''}(1) + G^{'}(1) - \{G^{'}(1)\}^2 = (1-\pi)/\pi^2$.

2. The mgf of $\log X$ is $\mathrm{E}(e^{s\log X}) = \mathrm{E}(X^s) = \{\Gamma(\nu)\}^{-1}\lambda^{\nu}\int_0^{\infty} x^{s+\nu-1}e^{-\lambda x}dx$
$= \lambda^{-s}\Gamma(s+\nu)/\Gamma(\nu)$ [substitute $u = \lambda x$].

B: questions handed in

1. (a) This is a standard application of the result ... derived in lectures. ... to obtain that $Y_1 \sim N(0, 1/2)$.

   (b) This mimicks the derivation of the above standard result closely:

$$M_Y(s) \overset{definition}{=} ... = \exp\left((A^T s)^T\mu + \frac{1}{2}(A^T s)^T\Sigma(A^T s)\right)$$
$$= \exp\left(s^T A\mu + \frac{1}{2}s^T A\Sigma A^T s\right)$$

   ...
   Finally, it remains to insert the numbers for A and Σ which yields $Y \sim N\left(0, \begin{pmatrix} 1/2 & 0 \\ 0 & 6 \end{pmatrix}\right)$

   (c) ... Therefore, the right solutions are

$$R \in \left\{\begin{pmatrix} \sqrt{2} & -1/\sqrt{2} \\ 0 & \sqrt{3/2} \end{pmatrix}, \begin{pmatrix} -\sqrt{2} & 1/\sqrt{2} \\ 0 & \sqrt{3/2} \end{pmatrix}, \begin{pmatrix} \sqrt{2} & -1/\sqrt{2} \\ 0 & -\sqrt{3/2} \end{pmatrix}, \begin{pmatrix} -\sqrt{2} & 1/\sqrt{2} \\ 0 & -\sqrt{3/2} \end{pmatrix}\right\}$$

   (d) ...

$$\lambda \in \{1, 3\}$$

   ... The diagonalization of Σ is then

$$\Sigma = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$$

   To show the required properties first note that ...

(e) My random numbers are

$$(x_1, x_2) = (1.22259, -0.597286)$$

..., so the sample is

$$(y_1, y_2) = ... = \begin{pmatrix} 1.729003 \\ -1.596025 \end{pmatrix}$$

Similarly, setting $Y = SX$ does the trick and the result is

$$(y_1, y_2) = ... = \begin{pmatrix} 1.888711 \\ -1.263407 \end{pmatrix}$$

2. (a) i. Since $S - 36$ is $\text{Bin}(36, 0.5)$, its mean and variance are 18 and 9 respectively. Hence $\mathbb{E}[S] = 54$ and $\text{Var}(S) = 9$, so the normal approximation of the binomial delivers

$$P(S \in [49, 55]) = ... \approx 0.5827$$

ii. The continuity correction was introduced in STAT1005 (at the latest), it basically says that since $P(S + c < 56)) = P(S < 56)$ for all $c \in [0, 1)$, we should approximate the probability by selecting c in the middle, i.e. $c = 1/2$. This leads to

$$P(S \in [49, 55]) = P(48.5 < S \leq 55.5) \approx ... \approx 0.6581$$

iii. Without continuity correction, the result is not a very good approximation (not even the first digit is correct). With continuity correction, the result is accurate to three decimals.

(b) This question shows that even the contuinity correction cannot help when the CLT is grossly misunderstood.

i.

$$P(\bar{X} \leq 0) = ... \approx \Phi(-\sqrt{n\frac{\delta}{1-\delta}})$$

ii. The continuity correction needs to interpolate between 0 and the next possible result, which is α/n. Hence we obtain:

$$P(\bar{X} \leq 0) = ... \approx \Phi(\frac{1/2n - \delta}{\sqrt{\delta(1-\delta)/n}})$$

iii. Setting $\alpha = \sqrt{\frac{n^3}{n-1}}$ ensures that all X_i have unit variance. Also, the expectation tends to one as $n \to \infty$, so there are no singularities in the expectation or variance.

In the non-continuity corrected case we get

$$\lim_{n\to\infty} \Phi(-\sqrt{n\frac{\delta}{1-\delta}})... = \Phi(-1) \approx 0.15866$$

In the continuity-corrected case we get

$$\lim_{n\to\infty} \Phi(\frac{1/2n - \delta}{\sqrt{\delta(1-\delta)/n}}) = ... = \Phi(-1/2) \approx 0.3085$$

Comparing to the given true value of $\lim_{n\to\infty} P(\bar{X} = 0) = 1/e \approx 0.3679$, this is still about 6 percentage points off, in spite of the sample size having been sent to infinity.
The explanation of the problem lies in the fact that ...