Example 1.24

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \begin{cases} \frac{2x}{y^2} & \text{if } 0 \le x \le y \le 1 \text{ and } y > 0 \\ 0 & \text{if } 0 < y < 1 \text{ and } (x < 0 \text{ or } x > y) \\ \text{undefined} & \text{if } y \le 0 \text{ or } y \ge 1 . \end{cases}$$

Reminder: $f_{X,Y}(x,y) = 8xy$ if $0 \le x \le y \le 1$

$f_Y(y) = 4y^3$ if $y \in (0,1)$

$$\mathbb{E}[X|Y=y] \underset{y \in (0,1)}{=} \int_{-\infty}^{\infty} x \, f_{X|Y}(x|y) \, dx = \int_0^y x \cdot \frac{2x}{y^2} \, dx = \frac{2}{3} y .$$

$$\mathbb{E}[X] = \mathbb{E}_Y[\mathbb{E}_{X|Y}[X|Y]] = \mathbb{E}_Y[\tfrac{2}{3} Y]$$

$$= \int_0^1 \frac{2}{3} y \cdot 4y^3 \, dy = \frac{8}{15} .$$

Other ways: $\mathbb{E}[X] = \int_{-\infty}^{\infty} x \cdot f_X(x) \, dx$

$$\mathbb{E}[X] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x \cdot f_{X,Y}(x,y) \, dx \, dy = \int_{-\infty}^{\infty} x \underbrace{\int_{-\infty}^{\infty} f_{X,Y}(x,y) \, dy}_{f_X(x)} \, dx$$