Exercise Sheet 1

The exercise sheet is subdivided into three parts:

- Part A contains warm-up questions you should do in your own time.
- You should solve all questions of Part B and put your solutions in the locker in the undergraduate common room in the Department of Statistical Science (Room number 117) assigned to your tutor group by the deadline stated on the exercise sheet. Don't forget to mark your solutions with your name and student number.

  Reminder: If you miss the deadline for reasons outside your control, for example illness or bereavement, you ***must*** submit a claim for extenuating circumstances, normally within a week of the deadline. Your home department will advise you of the appropriate procedures. For Statistical Science students, the relevant information is on the DOSSSH Moodle page.

  Your tutor will then mark one question selected at random after you handed in your solutions (mark counts towards ICA) and will return your marked solutions to you at the next tutorial at which the solutions to the part B questions will be discussed. Marks available for each part question are indicated in square brackets, e.g. [3]. Very succinct solutions to these questions will usually be made available on Moodle at 9:15am on Thursday mornings, so don't hand in late!
- Solutions to Part C questions will be made available in time for exam revision. These questions will not be discussed at the tutorials.

A: Warm-up questions: attempt these before your first tutorial

1. The sample space for an experiment is $\Omega = \{\omega_1, \dots, \omega_5\}$, where $\mathrm{P}(\omega_1) = \frac{2}{16}$, $\mathrm{P}(\omega_2) = \mathrm{P}(\omega_3) = \mathrm{P}(\omega_4) = \frac{3}{16}$ and $\mathrm{P}(\omega_5) = \frac{5}{16}$. The events E, F and G are defined as

   $$E = \{\omega_1, \omega_2, \omega_3\}; \qquad F = \{\omega_1, \omega_2, \omega_4\}; \qquad G = \{\omega_1, \omega_3, \omega_4\}.$$

   Show that $\mathrm{P}(E \cap F \cap G) = \mathrm{P}(E)\mathrm{P}(F)\mathrm{P}(G)$. Are E, F and G (mutually) independent?

2. Consider two independent fair dice. In a turn-based board game, if the sum of scores on the dice of the player show less than six, the player has to pause a number of turns which is given by six minus the sum of scores; otherwise (i.e. if the sum of scores is at least six) she can continue playing right away. What is the expected number of turns she will have to pause for?

B: Answers to hand in by Thursday, 15 October, 9am

1. Three independent fair dice are thrown. The dice are modelled using a sample space Ω, the outcomes ω in which are written as the triples of scores (d_1, d_2, d_3) shown by the three dice. Let A denote the event that all dice show the same number. Let B_1, B_2, B_3 denote the events that the first, second and third die respectively shows the number 6.

   (a) Using correct mathematical notation, write down Ω such that you avoid using even a single word (i.e. write down only mathematical symbols). [3]

   (b) Show that A and B_1 are independent. [3]

   (c) Show that B_1, B_2, B_3 are independent. [5]

   (d) Decide whether A, B_1, B_2, B_3 are independent and justify your decision. [4]

2. Let X follow the Poisson distribution with parameter $\mu > 0$. Consider the transformed random variable $Y = X!$ (note that $0! = 1$).

   (a) Write down the pmf p_Y of Y. [3]

   (b) Sketch the cdf F_Y of Y. In addition to the usual labelling of the axes, mark the first three possible outcomes of Y on the horizontal axis as well as the first three non-zero probabilities on the vertical axis clearly and compute their values in terms of μ. [6]

   (c) Compute the expected value of Y. For which values of μ is it finite? **Marks:6**

3. Given an uncertain future loss L which is modelled as a random variable with cdf F_L , the *value at risk* (VaR) at confidence level α is defined as

$$\text{VaR}_\alpha(L) = \inf\{l \in \mathbb{R} | F_L(l) \geq \alpha\}.$$

   Intuitively, it denotes the loss threshold that is exceeded with probability $1 - \alpha$.

   (a) Compute the VaR at confidence level $\alpha \in (0, 1)$ when the distribution of the loss L_1 is given via the pdf $f_{L_1}(l) = \begin{cases} (l+1)^{-2} & \text{if } l \geq 0 \\ 0 & \text{otherwise} \end{cases}$. [5]

   (b) Compute the VaR at confidence level $\alpha \in (0, 1)$ when the distribution of the loss L_2 is given via the pdf $f_{L_2}(l) = \begin{cases} 2le^{-l^2} & \text{if } l \geq 0 \\ 0 & \text{otherwise} \end{cases}$. [5]

   (c) Note that VaR and variance are two very different things. To see this, give an example of a simple change in the distribution of L_2 that leads to the loss L_3 such that $\text{Var}(L_2) = \text{Var}(L_3)$ but $\text{VaR}_\alpha(L_2) < \text{VaR}_\alpha(L_3)$ for all $\alpha \in (0, 1)$, explaining clearly why the latter two relations hold for your example. [5]

C: Exam Practice Questions

1. X takes values 0, 1, 2, 3 with probabilities $\frac{1}{4}, \frac{1}{5}, \frac{3}{10}, \frac{1}{4}$. Compute (as fractions) $\mathbb{E}(X), \mathbb{E}(2X+3), \text{Var}(X)$ and $\text{Var}(2X+3)$. Is X a discrete or a continuous random variable? Justify your answer and sketch its cdf.

2. X has density function $f(x) = kx(1-x)$ on $(0, 1)$, $f(x) = 0$ elsewhere. Calculate k, F_X and sketch $F_X(x)$. Compute the mean and variance of X, and $\text{P}(0.3 \leq X \leq 0.6)$.