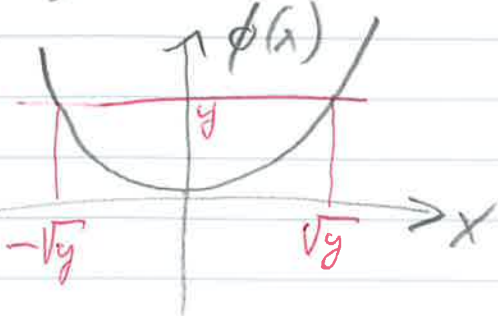
Example 4.1 $X \sim \mathcal{N}(0,1)$, $\phi(x) = x^2$

$X \in \mathbb{R}$

$Y = \phi(X) = X^2$. $Y \geq 0$



$\frac{dy}{dx} = 2x$

$$f_Y(y) = \sum_{x:\,\phi(x)=y} f_X(x) \cdot \left|\frac{dy}{dx}\right|^{-1}$$

$$= f_X(-\sqrt{y}) \cdot (2\sqrt{y})^{-1} + f_X(\sqrt{y}) \cdot (2\sqrt{y})^{-1}$$

$$= \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}(-\sqrt{y})^2\right) \cdot \frac{1}{2\sqrt{y}} + \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}(\sqrt{y})^2\right) \cdot \frac{1}{2\sqrt{y}}$$

$$= \frac{1}{\sqrt{2} \cdot \sqrt{\pi}} \cdot y^{-\frac{1}{2}} \cdot \exp(-y/2), \quad y > 0$$

Compare to Gam(α, λ) pdf: $\dfrac{\lambda^\alpha z^{\alpha-1} e^{-\lambda z}}{\Gamma(\alpha)}$

$\alpha = \frac{1}{2}$, $\lambda = \frac{1}{2}$

NB: $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, $\Gamma(n) = (n-1)!$ for $n \in \mathbb{N}$.

Example 4.2 $U = \sum_{i=1}^{\nu} X_i^2$, $X_i \sim \mathcal{N}(0,1)$ i.i.d.

$X_i^2 \sim \text{Gam}\left(\frac{1}{2}, \frac{1}{2}\right)$

$$\mathbb{E}U = \mathbb{E}\sum_{i=1}^{\nu} X_i^2 = \sum_{i=1}^{\nu} \mathbb{E}X_i^2 = \nu$$

$$\mathbb{E}X_i^2 = \frac{1/2}{1/2} = 1$$

$$\operatorname{Var} U = \operatorname{Var}\left(\sum_{i=1}^{\nu} X_i^2\right) = \sum_{i=1}^{\nu} \operatorname{Var}(X_i^2) = \sum_{i=1}^{\nu} 2 = 2\nu$$

X_i independent!

$$\operatorname{Var}(X_i^2) = \frac{1/2}{(1/2)^2} = 2$$