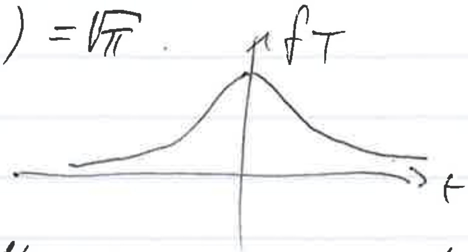
Example 4.4 $\quad T = \dfrac{Z}{\sqrt{U/\nu}} \qquad Z \sim N(0,1), \quad U \sim \chi^2_\nu, \quad Z \perp\!\!\!\perp U.$

$$f_T(t) = \frac{\Gamma((\nu+1)/2)}{\Gamma(\nu/2)} \frac{1}{\sqrt{\nu\pi}} \left(1+\frac{t^2}{\nu}\right)^{-(\nu+1)/2}, \quad t \in \mathbb{R}.$$

For $\nu = 1$: $\quad f_T(t) = \dfrac{\Gamma(1)}{\Gamma(\frac{1}{2})} \dfrac{1}{\sqrt{\pi}} (1+t^2)^{-1}$

$\Gamma(n) = (n-1)!$ for any $n \in \mathbb{N}$, $\quad \Gamma(\frac{1}{2}) = \sqrt{\pi}$.

$$f_T(t) = \frac{1}{\pi(1+t^2)}$$



$$\mathbb{E}T = \int_0^\infty t \cdot f_T(t)\,dt + \int_{-\infty}^0 t \cdot f_T(t)\,dt = \infty - \infty \quad \text{not defined.}$$

$T_1 \approx$ Cauchy distribution (see Ch. 2)

Note: $\mathbb{E}(U/\nu) = 1$, $\operatorname{Var}(U/\nu) = 2\nu/\nu^2 = 2/\nu \longrightarrow 0$ as $\nu \to \infty$.

STAT3101 only: $\nu \to \infty$: we expect $T \longrightarrow N(0,1)$

$$\lim_{\nu\to\infty} \frac{\Gamma(\frac{\nu+1}{2})}{\Gamma(\frac{\nu}{2})\sqrt{\frac{\nu}{2}}} \frac{1}{\sqrt{2\pi}} \left(1+\frac{t^2}{\nu}\right)^{-(\nu+1)/2}$$

$$= \lim_{\nu\to\infty} \frac{\Gamma(\dots)}{\Gamma(\dots)\sqrt{\dots}} \frac{1}{\sqrt{2\pi}} \left(1+\frac{t^2}{\nu}\right)^{-\nu/2} \cdot \underbrace{\left(1+\frac{t^2}{\nu}\right)^{-1/2}}_{\to 1 \text{ as } \nu\to\infty}$$

$$= \lim_{\nu\to\infty} \frac{\Gamma(\frac{\nu+1}{2})}{\Gamma(\frac{\nu}{2})\sqrt{\frac{\nu}{2}}} \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{\left(1+\frac{t^2}{\nu}\right)^\nu}}$$

Use $\forall a_n \to a$: $\lim_{n\to\infty} \left(1+\frac{a_n}{n}\right)^n = \exp(a)$ (cf. proof of CLT)

$$= \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{\exp(t^2)}} \underbrace{\lim_{\nu\to\infty} \frac{\Gamma(\frac{\nu+1}{2})}{\Gamma(\frac{\nu}{2})\sqrt{\frac{\nu}{2}}}}_{=1} \quad \text{(theory of special functions)}$$

$$= \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}t^2\right) \quad \text{which is the pdf of } N(0,1).$$