# Exercise Sheet 5

## A: Warm-up questions

1. (i) $\mathrm{E}(Y) = \sum(x-2)^2 p_X(x) = \frac{1}{10} + 0 + \frac{3}{10} + \frac{8}{5} = 2$,
   $\mathrm{var}(Y) = \mathrm{E}(Y^2) - \{\mathrm{E}(Y)\}^2 = \sum(x-2)^4 p_X(x) - \{\mathrm{E}(Y)\}^2 = \frac{14}{5}$.
   (ii) $p_y(y) = \frac{1}{5}, \frac{2}{5}, \frac{2}{5}$ for $y = 0, 1, 4$
   $\mathrm{E}(Y) = \sum y p_Y(y) = 2$, $\mathrm{var}(Y) = \sum y^2 p_Y(y) - \{\mathrm{E}(Y)\}^2 = \frac{14}{5}$.

2. $x = \log y, dx/dy = y^{-1}$ so $f_Y(y) = f_X(x)|dx/dy| = y^{-2}$, $y > 1$

3. Inverse transformation is $x = \pm y^{1/2}$, $0 < y \leq 1$ (two values of $x$ for each $y-$value) and $x = y^{1/2}$, $1 < y \leq 4$ (one-to-one in this region). Also $|dx/dy| = y^{-1/2}/2$ in all cases. Therefore
   if $0 < y \leq 1$: $f_Y(y) = \{\frac{2}{9}(1 + y^{1/2}) + \frac{2}{9}(1 - y^{1/2})\}\frac{1}{2}y^{-1/2} = \frac{2}{9}y^{-1/2}$;
   if $1 < y \leq 4$: $f_Y(y) = \{\frac{2}{9}(1 + y^{1/2})\}\frac{1}{2}y^{-1/2} = \frac{1}{9}(1 + y^{-1/2})$

## B: Hand-in questions

1. 

2. (a) The sketch should show the arcsine function with axes labeled, going through $(0, 0)$ and with slope tending to $\infty$ at $-\pi/2$ and $\pi/2$.

   (b)
   $$f_X(x) = ... = \frac{1}{\pi\sqrt{1-x^2}} \text{ on } x \in (-1, 1)$$

   (c) The more general transformation formula is needed here and it's worth taking the time to spell out exactly which probabilities are added up.

   $$f_X(x) = ...$$
   $$= \frac{1}{4\pi}\left(|\cos(\arcsin(x))|^{-1} + |\cos(\arcsin(x))|^{-1} + |-\cos(\arcsin(x))|^{-1} + |-\cos(\arcsin(\right.$$

   ...the given trigonometric identity to find
   $$f_X(x) = \frac{1}{\pi\sqrt{1-x^2}} \text{ on } x \in (-1, 1)$$

   (d) The cdf is
   $$F_X(x) = ... = \begin{cases} 0 & \text{if } x < -1 \\ \frac{\arcsin(x)}{\pi} + \frac{1}{2} & \text{if } -1 \leq x \leq 1 \\ 1 & \text{if } x > 1 \end{cases}$$

(e)

$$\mathbb{E}[X] = ... = 0$$
$$\mathbb{E}[X^2] = ... = 1/2$$

whence the variance is $\text{Var}(X) = 1/2$.

3. (a) The expected values are $\mathbb{E}[N_1] = np_1 = 5 \times 0.3 = 1.5$, $\mathbb{E}[N_2] = np_2 = 2$. ... the probability of this outcome is zero.

   (b)

$$P(0.6\mathbb{E}[N_1] \leq N_1 \leq 1.4\mathbb{E}[N_1],\, 0.6\mathbb{E}[N_2] \leq N_2 \leq 1.4\mathbb{E}[N_2]) = ... = \frac{162}{625}$$

   (c) For $n = 100$, the event of interest would be $18 \leq N_1 \leq 42$, $24 \leq N_2 \leq 56$, and since all combinations of these values for $N_1$ and $N_2$ are possible (since $42 + 56 = 98 \leq 100$), the number of probability contributions would be $25 \times 33 = 825$.

   (d) Inserting the expected values already derived then yields $\mu = (0.3n, 0.4n)^T$ and $\Sigma = \begin{pmatrix} 0.21n & 0.12n \\ 0.12n & 0.24n \end{pmatrix}$.

# Section C

later.

1. The joint density of $X$ and $Y$ is given by

$$f_{XY}(x, y) = \frac{\theta^n x^{n-1} e^{-\theta x}}{\Gamma(n)} \frac{\theta^m y^{m-1} e^{-\theta y}}{\Gamma(m)}$$

on $x, y > 0$.
We want to find the density of $X/(X + Y)$. In order to use the transformation theorem we define $W = X + Y$ (other choices are possible), find the joint density of $Z, W$ and finally integrate out $W$.
The inverse transformations are: $X = ZW$ and $Y = W - ZW$. The ranges are: $0 < Z < 1$ and $W > 0$. The Jacobian of the inverse transformation is

$$\left| \frac{\partial(x, y)}{\partial(z, w)} \right| = \begin{vmatrix} w & z \\ -w & 1 - z \end{vmatrix} = w > 0\,.$$

Thus the joint density of $Z, W$ is given by

$$f_{ZW}(z, w) = \frac{\theta^{n+m} (zw)^{n-1} (w - zw)^{m-1} e^{-\theta w} \cdot w}{\Gamma(n)\Gamma(m)}$$

on $0 < z < 1$ and $w > 0$.
To find the distribution of $Z$ alone we integrate out $W$. Notice that the above density factorises (and the ranges are independent), so $Z$ and $W$ are independent. The factor for $Z$ is

$$g(z) = \frac{z^{n-1}(1-z)^{m-1}}{\Gamma(n)\Gamma(m)}$$

and

$$f_Z(z) = g(z)\int_0^\infty \theta^{n+m} w^{n+m-1} e^{-\theta w}\ \mathrm{d}w \overset{u=\theta w}{=} g(z) \underbrace{\int_0^\infty u^{n+m-1} e^{-u}\ \mathrm{d}u}_{\Gamma(n+m)} = g(z)\Gamma(n+m)$$

which yields the desired density of $Z$.
Note: $W \sim \text{Gamma}(n+m, \theta)$ with density

$$f_W(w) = \frac{\theta^{n+m} w^{n+m-1} e^{-\theta w}}{\Gamma(n+m)}$$

(Recall: sums of Gammas are also Gamma.)

2. Transformation is $\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}(X_1+X_2) \\ \frac{1}{2}(X_1-X_2) \end{pmatrix}$, so $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} y_1+y_2 \\ y_1-y_2 \end{pmatrix}$ with Jacobian $\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2$.
$x_1 > 0,\, x_2 > 0 \Leftrightarrow y_1+y_2 > 0,\, y_1-y_2 > 0 \Leftrightarrow y_1 > 0, |y_2| < y_1$ (draw a sketch in the $y_1 - y_2$ plane to see this).
Therefore $f_{Y_1,Y_2}(y_1,y_2) = e^{-2y_1}|-2| = 2e^{-2y_1}$ for $y_1 > 0, |y_2| < y_1$.
$f_{Y_1}(y_1) = \int_{-y1}^{y1} 2e^{-2y_1}\, dy_2 = 4y_1e^{-2y_1}$, $y_1 > 0$. (Thus $Y_1 \sim \text{Gam}(2,2)$.)
Alternatively, express the joint density as $f_{Y_1,Y_2}(y_1,y_2) = \{4y_1e^{-2y_1}\}\{(2y_1)^{-1}\}$, from which it is clear that $Y_1 \sim \text{Gam}(2,2)$ (and $Y_2|Y_1 = y_1 \sim \text{U}(-y_1,y_1)$).

3. Transformation is $\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = \begin{pmatrix} X_1+X_2 \\ X_1/X_2 \end{pmatrix}$, so $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} y_1y_2/(y_2+1) \\ y_1/(y_2+1) \end{pmatrix}$ with Jacobian $\begin{vmatrix} y_2/(y_2+1) & y_1/(y_2+1)^2 \\ 1/(y_2+1) & -y_1/(y_2+1)^2 \end{vmatrix} = -y_1/(y_2+1)^2$.
Here $x_1 > 0,\, x_2 > 0 \Leftrightarrow y_1 > 0,\, y_2 > 0$. Hence
$f_{Y_1,Y_2}(y_1,y_2) = \lambda^2 \exp[-\lambda\{(y_1y_2/(y_2{+}1)){+}(y_1/(y_2{+}1))\}]\,|-y_1/(y_2{+}1)^2|$ on $(0,\infty)^2$.
Either integrate wrt to $y_2$ (for the marginal of $Y_1$) and wrt to $y_1$ (for the marginal of $Y_2$) or just note the factorisation
$f_{Y_1,Y_2}(y_1,y_2) = (\lambda^2 y_1 e^{-\lambda y_1})(1+y_2)^{-2} = f_{Y_1}(y_1)f_{Y_2}(y_2)$
$\Rightarrow$ $Y_1$ and $Y_2$ are independent with these densities.

4. The distribution functions are

$$F_1(x) = \begin{cases} 0, & x \le 0 \\ x, & 0 < x \le 1 \\ 1, & x > 1 \end{cases}$$

$$F_2(x) = \begin{cases} 0, & x \le 0 \\ 1-e^{-x}, & x > 0 \end{cases}$$

For $Y = \max(X_1, X_2)$ it follows that $F_Y(y) = \mathrm{P}(Y \leq y) = \mathrm{P}(\max(X_1, X_2) \leq y)$ which is in turn $= \mathrm{P}(X_1 \leq y)\mathrm{P}(X_2 \leq y)$ (because of independence). Thus,

$$F_Y(y) = \begin{cases} 0, & y \leq 0 \\ y(1-e^{-y}), & 0 < y \leq 1 \\ 1-e^{-y}, & y > 1 \end{cases} .$$

The expectation is given by integration:

$$\begin{aligned} \mathrm{E}(Y) &= \int_0^1 y(1-e^{-y}+ye^{-y})\,\mathrm{d}y + \int_1^\infty ye^{-y}\,\mathrm{d}y \\ &= \frac{1}{2} - (1+2e^{-1}) + (2-5e^{-1}) + 2e^{-1} \\ &= 1.5 - e^{-1} \end{aligned}$$

(use integration by parts).

5. First, note that $\mu_X = 15$ and $\sigma_X^2 = 25/3$ due to the properties of a uniform distribution.
With $\phi(X) = Y = 1/X$ we have $\phi'(x) = -1/x^2$ and $\phi''(x) = 2/x^3$. Thus, we get the following approximations (using second order and first order Taylor series approximations about $\mu_X$ for mean and variance, respectively):

$$\mathrm{E}(Y) \approx \phi(\mu_X) + \frac{1}{2}\phi''(\mu_X)\sigma_X^2 = \frac{1}{15} + \frac{1}{2}\frac{2}{15^3}\frac{25}{3} = \frac{28}{405} = 0.0691$$

and

$$\mathrm{var}(Y) \approx \phi'(\mu_X)^2\sigma_X^2 = \frac{1}{15^4}\frac{25}{3} = \frac{1}{6075} = 0.000165.$$

The exact results are obtained by integration:

$$\mathrm{E}(Y) = \int_{10}^{20} \frac{1}{x}\frac{1}{10}\,\mathrm{d}x = \frac{1}{10}[\log 20 - \log 10] = 0.0693$$

$$\mathrm{E}(Y^2) = \int_{10}^{20} \frac{1}{x^2}\frac{1}{10}\,\mathrm{d}x = \frac{1}{10}\left[\frac{1}{10} - \frac{1}{20}\right] = \frac{1}{200} = 0.005$$

so that $\mathrm{var}(Y) = 0.005 - (0.0693)^2 = 0.000198$. The approximation for the mean is good up to the third dp and the one for the variance is good up to the fourth dp.