Example 5.6 X_i, $i=1,\dots,n$ random sample with population mean $\mu = \mathbb{E}[X_i]$.

Find the least squares estimator:

$$R(\mu) = \sum_{i=1}^{n} (X_i - \mathbb{E}X_i)^2 = \sum_{i=1}^{n} (X_i - \mu)^2$$

$$\frac{dR}{d\mu}(\mu) = \frac{d}{d\mu} \sum_{i=1}^{n} (X_i - \mu)^2 = \sum_{i=1}^{n} 2(X_i - \mu)\cdot(-1) \overset{!}{=} 0$$

$$-2\sum_{i=1}^{n} X_i + 2\sum_{i=1}^{n} \mu = 0 \quad \Big| + 2\sum X_i, \div 2$$

$$n\mu = \sum_{i=1}^{n} \mu = \sum_{i=1}^{n} X_i$$

$$\mu = \frac{1}{n} \sum_{i=1}^{n} X_i$$

Check 2nd derivative: $\frac{d^2R}{d\mu^2} = \frac{d}{d\mu}\left(-2\sum_{i=1}^{n}(X_i - \mu)\right)$

$$= -2\sum_{i=1}^{n}(-1) = 2n > 0.$$

So $\hat{\mu} = \frac{1}{n}\sum_{i=1}^{n} X_i$ is a local ~~maximum~~ minimum.

$R(\mu)$ is twice differentiable (at least), so there are no other local minima. Check the boundaries:

$\lim_{\mu \uparrow \infty} R(\mu) = \infty$. (eventually $\mu > x_i$ for all i)

$\lim_{\mu \downarrow -\infty} R(\mu) = \infty$ (eventually $\mu < x_i$ for all i)

So there are no boundary minima.