STAT3001/STATM012/STATG012 Statistical Inference

Week 7: Problem Sheet 1

1. Consider a population consisting of measurements 0,2 and 4 and described by the following probability distribution

x	0	2	4
$p(x)$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$

   (a) List all the possible samples of size $n = 2$ measurements that can be selected from this population, specifying the probability of selecting each sample.

   (b) Determine the sampling distribution of the sample mean $\bar{X}$.

2. In each of the following cases, $X_1, \ldots, X_n$ are independent and identically distributed with the same distribution as X.

   (a) $X \sim N(\theta, \theta)$ where $\theta > 0$. Find a *single* sufficient statistic for the unknown parameter θ, based on observation of $X_1, \ldots, X_n$.

   Show that when $n = 1$ both x and x^2 are single sufficient statistics for θ. Hence deduce that, in general, when there is a single unknown parameter a single sufficient statistic is not necessarily minimal sufficient.

   (b) X has density $p(x; \theta)$ given by

$$p(x; \theta) = k\theta^k x^{k-1} \exp\{-(\theta x)^k\} \text{ for } x > 0$$

   and $p(x; \theta) = 0$ otherwise, where k is known. Find a single sufficient statistic, based on observation of $X_1, \ldots, X_n$, for the unknown parameter θ.

   (c) $X \sim \text{Uniform}[\theta, \theta + 1]$ where $\theta > 0$. Find a pair of statistics, based on observation of $X_1, \ldots, X_n$, that are jointly sufficient for the unknown parameter θ.

3. Suppose that $X_1, \ldots, X_n$ are independent random variables, where X_i has a gamma distribution with known index a_i and unknown parameter θ. Find a single sufficient statistic for θ.

4. Let θ be the proportion of defective items in a large manufactured lot. Assume that the prior distribution of θ is given by:

$$\pi(0.1) = 0.7, \qquad \pi(0.2) = 0.3$$

Suppose we select 8 items at random from the lot and observe 2 defective items. Determine the posterior distribution of θ.