General continuous time Markov processes

Sections 4.3 - 4.5 of notes

STAT2003 / STAT3102

Contents

- Revision of definitions;
- Holding time;
- The jump chain;
- Chapman-Kolmogorov again;
- Kolmogorov's differential equations;
- Limiting behaviour.

Continuous-time Markov chains

We consider stochastic processes $\{X(t), t \geqslant 0\}$ with countable or finite state space S.

Definition

$\{X(t), t \geqslant 0\}$ is a *continuous-time Markov chain* if, for all $t \geqslant 0$, $0 \leqslant t_0 < t_1 < \cdots < t_n < s$ and for all states $i, j, i_0, i_1, \ldots, i_n \in S$

$$\mathrm{P}(X(s+t) = j \mid X(s) = i, X(t_n) = i_n, \ldots, X(t_0) = i_0)$$

$$= \mathrm{P}(X(s+t) = j \mid X(s) = i),$$

that is, the continuous-time process $\{X(t), t \geqslant 0\}$ satisfies the *Markov property*.

Transition probabilities

The **transition probabilities** are

$$p_{ij}(t) = \mathrm{P}(X(t) = j \mid X(0) = i),$$

and for each $t \geqslant 0$ we have a **transition matrix** $\mathrm{P}(t)$.

If $S = \{0, 1, 2, \ldots\}$ this will be of the form

$$P(t) = \begin{pmatrix} p_{00}(t) & p_{01}(t) & p_{02}(t) & \cdots \\ p_{10}(t) & p_{11}(t) & p_{12}(t) & \cdots \\ p_{20}(t) & p_{21}(t) & p_{22}(t) & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix}$$

Transition probabilities

Remark

- For continuous time processes, we don't have "1-step" transitions.
- Instead we have transition probabilities, and associated transition matrices, for a particular time t.
- These $p_{ij}(t)$ are interpreted as 'the probability that the process is in state j at time t, given that the process is in state i at time zero'.
- Remember that t doesn't have to be an integer for continuous time processes.

Fundamental assumptions

For each t,

$$p_{ij}(t) \geqslant 0 \qquad \text{and} \qquad \sum_j p_{ij}(t) = 1.$$

When $t = 0$,

$$p_{ij}(0) = P(X(0) = j \mid X(0) = i) \;=\; \begin{cases} 1 & \text{if } j = i, \\ 0 & \text{if } j \neq i, \end{cases}$$

so $\mathrm{P}(0) = I$ (the identity matrix).

Time homogeneity

The concept of 'time homogeneity' extends to continuous time processes, too.

Definition

If

$$P(X(s+t) = j \mid X(s) = i) = P(X(t) = j \mid X(0) = i)$$

for all s, t and all $i, j \in S$, then $\{X(t)\}$ is *time-homogeneous*

Remark

- Only the time elapsed is important, and not the times themselves.
- In the above, for example, the elapsed time t is important in calculating the probability while s is irrelevant in time-homogeneous processes.

Breaking down a continuous time process

Holding times

How long does the process stay in a particular state?

How long is the process *held* in a state.

Equivalent to looking at the rate at which the *process leaves a state*.

The jump chain

The sequence of states which the process visits.

Don't care about *when* the process visits the states, just in which *order* it visits them.

How long do we stay in a state?

The *holding* time

Suppose that the chain starts in state i.

It stays there for a random length of time known as the **holding** time.

The holding time is **exponentially distributed** with some parameter q_i.

$$\boxed{\{\text{Length of stay in state } i \mid \text{start in state } i\} \sim \exp(q_i).}$$

So the **mean holding time** in state i is $1/q_i$.

How long do we stay in a state?

The *holding* time

$$\boxed{\{\text{Length of stay in state } i \mid \text{start in state } i\} \sim \exp(q_i).}$$

So the **mean holding time** in state i is $1/q_i$.

Remark

Can think of q_i as the rate at which we leave state i, since the length of time we stay in a state is equivalent to when we leave the state.

Remark

The holding times are independent of each other.

Why is the distribution of the holding time exponential?

Let $T(i)$ be the holding time in state i.

Strategy:

1. Deduce that the distribution of $T(i)$ is memoryless, so that for $u, s \geqslant 0$,

$$P(T(i) > u + s | T(i) > u) = P(T(i) > s).$$

2. Then conclude that the distribution of $T(i)$ must be exponential.

The events $\{T(i) > u\}$ and $\{X(t) = i \text{ for all } 0 \leqslant t \leqslant u\}$ are identical for any $u \geqslant 0$.

Why is the distribution of the holding time exponential?

$$P(T(i) > s + t | T(i) > s)$$

$\downarrow$ Equivalent statements

$$= P(X(u) = i \text{ for all } 0 \leqslant u \leqslant t + s \,|\, X(u) = i \text{ for all } 0 \leqslant u \leqslant s)$$

$\downarrow$ Using information from the conditioning

$$= P(X(u) = i \text{ for all } s \leqslant u \leqslant t + s \,|\, X(u) = i \text{ for all } 0 \leqslant u \leqslant s)$$

$\downarrow$ Markov property

$$= P(X(u) = i \text{ for all } s \leqslant u \leqslant t + s \,|\, X(s) = i)$$

$\downarrow$ Time homogeneity

$$= P(X(u) = i \text{ for all } 0 \leqslant u \leqslant t \,|\, X(0) = i)$$

$\downarrow$ Equivalent statements

$$= P(T(i) > t)$$

Why is the distribution of the holding time exponential?

We've shown that

$$P(T(i) > s + t | T(i) > s) = P(T(i) > t)$$

and so the distribution of $T(i)$ is exponential.

The distribution of $T(i)$ (i.e. the holding time in state i) is exponential!

Holding time and rate of leaving a state

The parameter of the holding time in state i, q_i can be thought of as the rate at which we leave state i:

'How long does the process stay in a state?'
is equivalent to
'When does the process leave the state?'

The jump chain of a continuous-time Markov chain

The ‘embedded Markov chain’

Record the sequence of states that the chain visits, without worrying about how long the process stays in each state. This is a DISCRETE TIME MARKOV CHAIN!

The process leaves state i at rate q_i.

At which point it enters another state, say j.

Define rate at which the process enters state j from i to be q_{ij}.

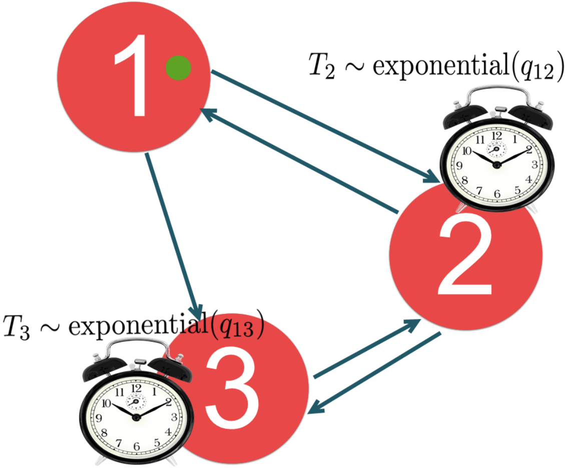
Can use exactly the same argument to show that IF the process enters state j next, then the time is does so, T_j, is

$$T_j \sim \text{exponential}(q_{ij}).$$

The jump chain of a continuous-time Markov chain

The 'embedded Markov chain'

Leave state 1 at time t is equivalent to $\min\{T_2, T_3\} = t$.



The jump chain of a continuous-time Markov chain

The 'embedded Markov chain'

In general, leave state i at time t is equivalent to $\min_{j \neq i}\{T_j\} = t$.

1. Time at which we leave state i, $T(i)$, is exponential(q_i).
2. If the process is in state i currently and goes to state j next, then the time at which it leaves i to go to j, T_j is exponential(q_{ij}).

For a Markov chain with state space $S = \{1, ..., k\}$, we have:

$$T(i) = \min\{T_1, ..., T_{i-1}, T_{i+1}, ..., T_k\}$$

How are q_i and q_{ij} related?

Important result about exponential random variables

Let $Y_1, ..., Y_n$ be independent random variables such that $Y_i \sim$ exponential(λ_i). Then:

- The distribution of the minimum of these random variables is also exponential, with parameter $(\lambda_1 + ... + \lambda_n)$.
- The probability that Y_k is the minimum of these random variables can be shown to be

$$\frac{\lambda_k}{\lambda_1 + ... + \lambda_n}$$

How are q_i and q_{ij} related?

Since

$$T(i) \sim \text{exponential}(q_i);$$
$$T(i) = \min\{T_1, ..., T_{i-1}, T_{i+1}, ..., T_k\};$$
$$T_j \sim \text{exponential}(q_{ij});$$
$$T_j \text{ are independent random variables;}$$

The result on previous slide indicates that:

$$T(i) \sim \text{exponential}\left(\sum_{j \neq i} q_{ij}\right)$$

Must have:

$$q_i = \sum_{j \neq i} q_{ij}$$

How are q_i and q_{ij} related?

Summary

$T(i) \sim \text{exponential}(q_i)$;

$T(i) = \min\{T_1, ..., T_{i-1}, T_{i+1}, ..., T_k\}$;

$T_j \sim \text{exponential}(q_{ij})$;

T_j are independent random variables;

$$q_i = \sum_{j \neq i} q_{ij}$$

Probability of moving to state j next

What's the probability that from state i we go to state j?

Since the time until the alarm clock for state j rings is exponential with parameter q_{ij}, then

$$
\begin{aligned}
p_{ij} :=& P\left(\text{from state } i,\ \text{move to state } j\right) \\
=& P\left(\text{ alarm for state } j \text{ rings first} \mid \text{currently in state } i\right) \\
=& P\left(T_j < T_k \text{ for } k \neq j \mid \text{currently in state } i\right) \\
=& P\left(T_j = \min\{T_k\} \mid \text{currently in state } i\right) \\
=& \frac{q_{ij}}{\sum_{k \neq i} q_{ik}} =: \frac{q_{ij}}{q_i}
\end{aligned}
$$

Probability is given by $p_{ij} = q_{ij}/q_i$.

Remarks about the jump chain

Remark

1. Suppose that the continuous-time Markov chain has an **absorbing state**, state i. If the chain enters state i then it will stay there forever. In this case we would **set** $p_{ii} = 1$**, rather than using the equation given above**.

2. The jump chain can be used to **answer questions which involve the states that the chain enters** but not the times at which they are entered.

Examples of using jump chains

1. The jump chain could be used to calculate the probability that $\{X(t), t \geqslant 0\}$ ever enters a particular state.
2. We could use the jump chain to calculate the probability that a population ever becomes extinct, that is, the chain ever enters state 0. We'll see this later when we study birth-death processes.

Example - transition matrix of a jump chain

Suppose that a continuous time Markov chain has state-space $S = \{1, 2, 3, 4\}$, and that q_{ij} denotes the rate at which the chain moves from state i to state j (for $j \neq i$). Let

$$q_i = \sum_{j \neq i} q_{ij} \qquad \text{for } i = 1, 2, 3, 4.$$

The transition matrix of its jump chain is

$$P = \begin{pmatrix} 0 & q_{12}/q_1 & q_{13}/q_1 & q_{14}/q_1 \\ q_{21}/q_2 & 0 & q_{23}/q_2 & q_{24}/q_2 \\ q_{31}/q_3 & q_{32}/q_3 & 0 & q_{34}/q_3 \\ q_{41}/q_4 & q_{42}/q_4 & q_{43}/q_4 & 0 \end{pmatrix}$$

Diagonal is ALWAYS ZERO. Check that all rows sum to 1!

Transition probabilities

Reminder!

BIG problem: no "smallest time" until the next transition. Instead, we can jump to another state at *any* time.

Transition probabilities $p_{ij}(t)$, for states i and j ($j \neq i$) (*a function of* t).

PROBLEM: $p_{ij}(t)$ are messy and hard to work with.

AIM: Dispose of the need to work with transition probabilities!

Strategy

For disposing of need to work with $p_{ij}(t)$

1. Study transition probabilities $p_{ij}(h)$ over a *very small* period of time of length h.
2. Use this to develop further tools:
   - The continuous time version of the Chapman-Kolmogorov equations;
   - Kolmogorov's forward equations;
   - Kolmogorov's backward equations
3. This will show us that another quantity (easier to work with) can replace need to work directly with $p_{ij}(t)$.

Transition probabilities

How could we get from state i to j in a small interval of time?

Two broad options:

1. Go directly from state i to state j, with the transition occurring sometime in the interval $[0, h]$.
2. Go from state i to state j via another state k, with $k \neq i, j$, (or via more than one other state) with all transitions occurring within the interval $[0, h]$.

We will see that only the first option is possible:

P(more than one transition in a very short time interval)≈ 0.

Transition probabilities

ONE transition in a SMALL interval

Start in state i. Probability of one transition in $[0, h]$ is

$$P(T(i) < h) = 1 - \exp(-q_i h) = q_i h + o(h).$$

Probability of one transition in $[0, h]$ and that the transition is to state $j \neq i$ is

$$p_{ij}(h)$$

$$= P(T(i) < h \mid \text{ move from } i \text{ to } j \text{ in the interval } [0,h]) P(\text{ move from } i \text{ to } j \text{ in the interval } [0,h])$$

$$= P(T(i) < h) P(\text{move from } i \text{ to } j \text{ in the interval } [0,h])$$

$$= (1 - \exp(-q_i h)) \frac{q_{ij}}{q_i}$$

$$= (q_i h + o(h)) \frac{q_{ij}}{q_i} = q_{ij} h + o(h)$$

Note: the rates q_{ij} and q_i make an appearance here!

Transition probabilities

TWO transitions in a SMALL interval

Probability of two transitions in $[0, h]$:

$\sum_{k\neq i} P(T(i)+T(k)<h|$ move from i to k first$)P($move from i to k first$)=\sum_{k\neq i} o(h)q_{ik}=o(h)$,

Probability of two transitions in $[0, h]$ and chain at time h is in state j:

$$P(\text{two transitions in } [0,h] \text{ and that the chain at time } h \text{ is in state } j) \leqslant P(\text{two transitions in } [0,h]) = o(h).$$

Similarly, we can show that for more than two transitions,

$$P(\text{more than two transition in } [0,h]) = o(h).$$

Transition probabilities

Over a SMALL interval

Starting in state i:

$$P(\text{one transition in } [0,h]) = q_i h + o(h)$$
$$P(\text{two or more transitions in } [0,h]) = o(h).$$

Putting this together, to find probability of no transitions,

$$\begin{aligned} p_{ii}(h) &= P(\text{no transitions in } [0,h] | X(0) = i) \\ &= 1 - P(\text{one transition in } [0,h] | X(0) = i) - P(\text{two transitions in } [0,h] | X(0) = i) - \ldots \\ &= 1 - q_i h + o(h) \end{aligned}$$

Transition probabilities

Over a SMALL interval

In summary, in a very small time interval of length h, AT MOST one transition will occur.

The probability of moving DIRECTLY from state i to state j in the small time interval $[0, h]$ is

$$p_{ij}(h) = q_{ij}h + o(h).$$

The probability being in state i at time h, given that the chain starts in state i, is

$$p_{ii}(h) = 1 - q_i h + o(h)$$

and that this occurs because we have no transitions during this interval.

Transition probabilities

Over a SMALL interval

Putting this together we have that

$$p_{ij}(h) = \begin{cases} 1 - q_i h + o(h) & \text{for } i = j \\ q_{ij} h + o(h) & \text{for } i \neq j. \end{cases}$$

The transition probabilities rely on:

- the rate at which we leave state i, q_i ;
- the rate at which we go to state j from state i, q_{ij}.

Chapman- Kolmogorov in continuous time

Transitions over any length of time

The Chapman-Kolmogorov equations in continuous time are

$$\boxed{P(s+t) = P(s)\,P(t).} \tag{1}$$

(Proof as in discrete time: condition on the state at time s. Makes use of the transition probabilities we derived over small intervals.)

We also have

$$\boxed{\underline{p}^{(t)} = \underline{p}^{(0)} P(t),}$$

where $\underline{p}^{(t)} = (p_0^{(t)}, p_1^{(t)}, \ldots)$, so that $\underline{p}^{(0)}$ is the distribution of X_0, the initial distribution.

Chapman- Kolmogorov in continuous time

Transitions over any length of time

Useful because they are used in proving two vital results:

1. Kolmogorov's forward equations;
2. Kolmogorov's backward equations.

These are similar results, derived by looking at the same problem from different angles.

They are the key to disposing of the need to work with $p_{ij}(t)$!

Kolmogorov's forward equations

The Kolmogorov forward differential equations (KFDEs) are

$$p'_{ij}(t) = -p_{ij}(t)\, q_j + \sum_{\substack{k \in S \\ k \neq j}} p_{ik}(t)\, q_{kj}$$

one for each pair (i, j).

Note: these equations rely on:

- the rate at which we leave state i, q_i ;
- the rate at which we go to state j from state i, q_{ij}.

This gives a plausible way to compute $p_{ij}(t)$ from the transition rates q, by solving the differential equations!

Kolmogorov's forward equations

Matrix version

Write the Kolmogorov forward equation in matrix notation which encodes the KDFEs for ALL pairs (i, j):

$$P'(t) = P(t)\,Q,$$

Q is a matrix of rates.

For a continuous time Markov chain with state space $S = \{1, 2, 3, ...\}$,

$$Q = \begin{bmatrix} -q_1 & q_{12} & q_{13} & \cdots \\ q_{21} & -q_2 & q_{23} & \cdots \\ q_{31} & q_{32} & -q_3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

Why are the forward equations useful?

$$P'(t) = P(t)\,Q,$$

Now have a way of computing $P(t)$ from the *rates*, by solving the differential equation.

e.g. to compute $p_{ij}(t)$ I can solve the differential equation

$$p'_{ij}(t) = -p_{ij}(t)\,q_j + \sum_{\substack{k\in S\\ k\neq j}} p_{ik}(t)\,q_{kj}$$

subject to the initial condition $p_{ij}(0) = 0$.

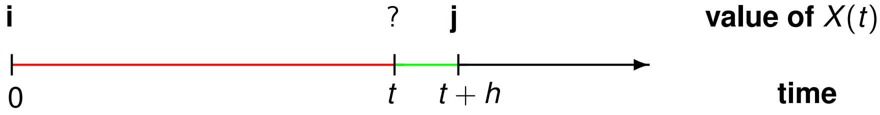
Instead of using $P(t)$ to study the behaviour of a continuous time Markov chain, we can use Q instead.

This is because we now know that $P(t)$ and Q are directly related via the forward equations.

Deriving the forward equations

How do we prove the forward equations?

- We take an initial time point (0)
- Take a further two time points (t and $t+h$) close together.



Deriving the forward equations

C-K equations: $p_{ij}(t+h) = \sum_{k\in S} p_{ik}(t)p_{kj}(h)$.

For small $h > 0$,

$$p_{ij}(h) = \begin{cases} 1 - q_i h + o(h) & \text{for } i = j \\ q_{ij} h + o(h) & \text{for } i \neq j. \end{cases}$$

Substituting this in and separating the case $k = j$ from $k \neq j$,

$$\boxed{p_{ij}(t+h) = p_{ij}(t) - hp_{ij}(t)q_j + h\sum_{\substack{k\in S\\ k\neq j}} p_{ik}(t)\, q_{kj} + o(h).}$$

Deriving the forward equations

$$p_{ij}(t+h) = p_{ij}(t) - hp_{ij}(t)q_j + h \sum_{\substack{k \in S \\ k \neq j}} p_{ik}(t)\, q_{kj} + o(h).$$

Now subtract $p_{ij}(t)$ from each side, and divide each side by h:

$$\frac{p_{ij}(t+h) - p_{ij}(t)}{h} = -p_{ij}(t)q_j + \sum_{\substack{k \in S \\ k \neq j}} p_{ik}(t)\, q_{kj} + \frac{o(h)}{h}$$

Letting $h \downarrow 0$ gives

$$p'_{ij}(t) = -p_{ij}(t)q_j + \sum_{\substack{k \in S \\ k \neq j}} p_{ik}(t)\, q_{kj}, \quad \text{for all } i, j \in S,$$

which shows that the KFDEs hold.

The forward equations

Exercise: check that

$$p'_{ij}(t) = -p_{ij}(t)q_j + \sum_{\substack{k\in S\\ k\neq j}} p_{ik}(t)\, q_{kj}, \qquad \text{and} \qquad P'(t) = P(t)\, Q$$

are equivalent.

Kolmogorov's backward equations

The Kolmogorov backward differential equations (KBDEs) are

$$p'_{ij}(t) = -q_i \, p_{ij}(t) + \sum_{\substack{k \in S \\ k \neq j}} q_{ik} \, p_{kj}(t)$$

where this depends on

- the rate at which we leave state i, q_i ;
- the rate at which we go to state j from state i, q_{ij}.

Compare the forward and backward equations!

Kolmogorov's backward equations

Matrix version

In matrix notation,

$$P'(t) = Q\,P(t),$$

Q is a matrix of rates, and is the same as the 'Q' in the forward equations!

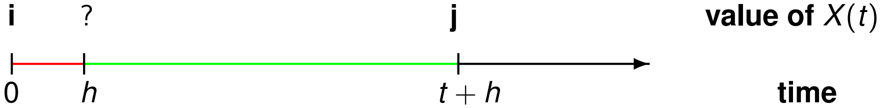
For a continuous time Markov chain with state space $S = \{1, 2, 3, ...\}$,

$$Q = \begin{bmatrix} -q_1 & q_{12} & q_{13} & \cdots \\ q_{21} & -q_2 & q_{23} & \cdots \\ q_{31} & q_{32} & -q_3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

Deriving the backward equations

How do we prove the backward equations?

- We take two initial time points (0 and h) close together.
- Take a later single time point ($t + h$).



Compare this to the set-up for deriving the forward equations. Proceed as for the forward equations (look in the notes for details).

Solving Kolmogorov's equations

Forward equations: $P'(t) = P(t)Q$

Backward equations: $P'(t) = QP(t)$

Solving either of these equations subject to the initial condition $P(0) = I$ gives

$$\boxed{P(t) = \sum_{n=0}^{\infty} \frac{t^n Q^n}{n!} = \exp(tQ).}$$

For small h, this gives:

$$P(h) = I + hQ + o(h)$$

Notice that:

$$Q = \left.\frac{\mathrm{d}P(t)}{\mathrm{d}t}\right|_{t=0} = P'(0).$$

The generator matrix

The rates play an important role, and determine the transition matrix $P(t)$. The matrix of rates in the KFDE and KBDE, Q, is known as the generator matrix.

For a continuous time Markov chain with state space $S = \{1, 2, 3, ...\}$,

$$Q = \begin{bmatrix} -q_1 & q_{12} & q_{13} & \cdots \\ q_{21} & -q_2 & q_{23} & \cdots \\ q_{31} & q_{32} & -q_3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

What properties does Q possess?

Properties of Q

$$Q = \begin{bmatrix} -q_1 & q_{12} & q_{13} & \cdots \\ q_{21} & -q_2 & q_{23} & \cdots \\ q_{31} & q_{32} & -q_3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

The elements of the matrix Q will satisfy:

- Off-diagonal elements are $\geqslant 0$, e.g. (i,j)th element is q_{ij}, for $i \neq j$;
- Diagonal elements are negative, e.g. $-q_i \leqslant 0$;
- Rows of Q sum to 0! Why?

How do these compare with the properties of the one-step transition matrix for discrete time Markov chains?

Transition in continuous time

Discrete time process **specified by:**	**Continuous time process** **specified by:**
Initial distribution, $\underline{p}^{(0)}$ Transition matrix, P	Initial distribution, $\underline{p}^{(0)}$ Generator matrix, Q

In continuous time, the initial distribution, $\underline{p}^{(0)}$ and the matrix Q specify the process exactly and we can compute all that we're interested in using these quantities.

How does the process behave in the long run?

Follow same thinking as for discrete time processes:

- Find invariant distributions, if any exist;
- Find an equilibrium distribution, if it exists.
- Find general result for when these exist?

Invariant distributions

We also have **invariant distributions** for continuous time Markov chains.

Definition

The probability row vector $\underline{\pi}$ is an invariant distribution of a continuous-time Markov chain $\{X(t), t \geqslant 0\}$ with transition matrix $P(t)$ if

$$\underline{\pi} = \underline{\pi}\, P(t), \qquad \text{for all } t \geqslant 0.$$

Using $P(t) = \exp(tQ)$ gives

$$\begin{aligned} \underline{\pi}\, \exp(tQ) &= \underline{\pi} \\ \underline{\pi}\left(1 + tQ + \frac{t^2Q^2}{2!} + \cdots\right) &= \underline{\pi} \\ t\underline{\pi}Q + \frac{t^2\underline{\pi}Q^2}{2!} + \cdots &= \underline{0}, \end{aligned}$$

which holds for all $t \geqslant 0$ if and only if $\underline{\pi}Q = \underline{0}$.

Invariant distributions

So $\underline{\pi} = \underline{\pi} P(t)$ is equivalent to $\underline{\pi} Q = \underline{0}$.

If $\underline{p}^{(0)} = \underline{\pi}$ then $\underline{p}^{(t)} = \underline{\pi}$ for all $t \geqslant 0$.

That is, if we start in $\underline{\pi}$ we stay there.

*Easier to solve $\underline{\pi} Q = \underline{0}$ than $\underline{\pi} = \underline{\pi} P(t)$!

Equilibrium distributions

An equilibrium distribution

$$\underline{\pi} = \{\pi_j, j \in S\}$$

exists if

$$p_{ij}(t) \to \pi_j \text{ as } t \to \infty \text{ for each } j \in S,$$

where $\underline{\pi}$ is a probability distribution that does not depend on the initial state i.

Equilibrium distributions

... and the forward equations

Suppose that such an equilibrium distribution exists. Then

$$p_{ij}(t) \to \pi_j \qquad \Rightarrow \qquad p'_{ij}(t) \to 0 \qquad \text{as } t \to \infty.$$

The **forward equations** give

$$p'_{ij}(t) = -p_{ij}(t)q_j + \sum_{\substack{k \in S \\ k \neq j}} p_{ik}(t)\, q_{kj}.$$

We expect, as $t \to \infty$:
LHS $\to 0$ and RHS $\to -\pi_j q_j + \sum_{\substack{k \in S \\ k \neq j}} \pi_k\, q_{kj}$.

Equivalent to $\underline{\pi} Q = \underline{0}$

An equilibrium distribution is an invariant distribution of $\{X(t), t \geqslant 0\}$.

Equilibrium distributions

... and the backward equations

Note: the **backward equations** give

$$\underline{0} = -q_i\pi_j + \sum_{\substack{k\in S\\ k\neq j}} q_{ik}\pi_j = \pi_j\left(-q_i + \sum_{\substack{k\in S\\ k\neq j}} q_{ik}\right) = \underline{0},$$

and are **not helpful.**

Why?

Equilibrium distributions

Though the KFDEs and KBDEs look very similar, it is sometimes the case that only one of these is useful in solving a particular problem.

To find the connection between the invariant and equilibrium distributions, the KFDEs were useful but the KBDEs were not.

In other cases, which we will not consider in this course, the backward equations prove to be useful while the forward equations are not.

A limit theorem for continuous-time Markov chains

To help us find a general result for existence of an equilibrium distribution, we need:

Definition

A continuous-time Markov chain is *irreducible* if, for every i and j in S, $p_{ij}(t) > 0$ for some t. That is, all states in the chain intercommunicate.

Limiting result

Suppose that $\{X(t), t \geqslant 0\}$ is an irreducible continuous-time Markov chain.

(i) If there exists an invariant distribution $\underline{\pi}$ then it is unique and $p_{ij}(t) \to \pi_j$ as $t \to \infty$ for all $i, j \in S$.
   - That is $\underline{\pi}$ is the equilibrium distribution of the chain.

(ii) If there is no invariant distribution then $p_{ij}(t) \to 0$ as $t \to \infty$ for all $i, j \in S$.
   - No equilibrium distribution exists.

Remarks

- For *continuous-time* Markov chains we do not need to worry about periodicity.
- If the chain is positive recurrent then such an invariant distribution does exist. Therefore, if an irreducible Markov chain has no invariant distribution it must be either null recurrent or transient.
- It does not follow that only irreducible processes can have an equilibrium distribution.