Please put in your tutor's postbox in Room 117 by 12pm Thursday February 4th.

1. Consider the Markov chain $\{X_n : n \geq 0\}$ with state space $S = \{1, 2, 3, 4, 5, 6, 7\}$ and transition matrix

$$P = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1/2 & 0 & 0 & 1/2 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 & 1/4 & 0 & 1/4 \\ 0 & 0 & 0 & 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 1/2 & 1/4 & 0 & 1/4 \\ 0 & 0 & 0 & 1/2 & 1/4 & 0 & 1/4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

   (a) Find the irreducible classes of intercommunicating states and classify them in terms of positive or null recurrence, transience and periodicity.

   (b) Calculate $P(X_1 = 5, X_2 = 5, X_3 = 7|X_0 = 3)$, stating any properties that you use.

   (c) Given that the chain starts in state 5, state the distribution of the time the chain stays in state 5 before leaving.

   (d) Given that the chain starts in state 5, what is the distribution of the states the process goes to when it eventually leaves state 5?

2. Let $\{X_n : n \geq 0\}$ be a Markov chain with state space $\{0, 1, 2, 3\}$ and transition matrix

$$P = \begin{pmatrix} 1/4 & 0 & 1/2 & 1/4 \\ 0 & 1/5 & 0 & 4/5 \\ 0 & 1 & 0 & 0 \\ 1/3 & 1/3 & 0 & 1/3 \end{pmatrix}.$$

   (a) Suppose that $X_0 = 0$. Compute the expected time (number of steps) until the chain reaches state 3 for the first time.

   (b) Again assuming that $X_0 = 0$, compute the expected number of times that the chain visits state 1 before reaching state 3 for the first time.

   Note the difference between (a) and (b): for the first one, you set a counter that increases by one at each time point until you reach state 3. For the second, you set a counter that increases by one only when you visit state 1 [before visiting state 3, of course!]. You might find it useful to watch the 'basic skills storyline' on Moodle (section on first step decomposition) before attempting these questions.

   A new process $\{Z_n : n \geq 0\}$ is defined by $Z_n = 0$ if $X_n = 0$ or 1, and $Z_n = X_n$ if $X_n = 2$ or 3.

   (c) Find $P(Z_{n+1} = 2|Z_n = 0, Z_{n-1} = 2)$.

   (d) Find $P(Z_{n+1} = 2|Z_n = 0, Z_{n-1} = 3)$.

   (e) Is $Z_0, Z_1, Z_2, \ldots$ a Markov chain? Why?

   HINT: to find these probabilities, think what the equivalent value for X_n would be. For example, if $Z_n = 2$ then we must have that $X_n = 2$. By doing this, you can use the probabilities in the transition matrix for $\{X_n; n \geq 0\}$.

Challenges

Questions in this section will not be covered in the tutorial. If you think you have solved the questions or would like some help, do post to the discussion forum.

1. Let $\{X_0, X_1, \ldots\}$ and $\{Y_0, Y_1, \ldots\}$ be Markov Chains, independent from each other. Define $Z_i = X_i + Y_i$, $i = 0, 1, 2, \ldots$. What conditions (if any) are sufficient for $\{Z_0, Z_1, \ldots\}$ to be a Markov Chain? Are they also necessary? Can this result be generalised to the situation where $Z_i = f(X_i, Y_i)$ for any bivariate function f? How about where $Z_i = g(X_j, Y_k)$ for any bivariate function g where $j, k \leq i$?