STAT2003: INTRODUCTION TO APPLIED PROBABILITY 2014

Answer ALL questions. Section A carries 40% of the total marks and Section B carries 60%. The relative weights attached to each question are as follows: A1 (10), A2 (10), A3 (10), A4 (10), B1 (30), B2 (30). The numbers in square brackets indicate the relative weights attached to each part question.

Section A

A1 A Markov Chain, $\{X_0, X_1, \dots\}$ with states $\{1, 2, 3, 4\}$ has transition matrix

$$\boldsymbol{P} = \begin{pmatrix} 1/3 & 0 & 2/3 & 0 \\ 0 & 0 & 1/2 & 1/2 \\ 1 & 0 & 0 & 0 \\ 0 & 1/4 & 0 & 3/4 \end{pmatrix}.$$

(a) Find the irreducible classes of intercommunicating states and classify them in terms of positive or null recurrence, transience, periodicity and ergodicity. [3]

(b) Calculate the invariant distribution for this chain. [3]

(c) Calculate μ_1, the mean number of steps for the first return to state 1, given that one starts there. [2]

(d) Calculate $\mathrm{P}(X_4 = 2 \mid X_0 = 2)$. [2]

A2 (a) For a random variable, X, define the probability generating function (pgf) $G_X(s)$. [1]

(b) Suppose that $X_1, \dots, X_N$ are iid random variables with common pgf $G_X(s)$ and that N has pgf $G_N(s)$. Prove that the pgf of $Y = X_1 + \cdots + X_N$ is given by

$$G_Y(s) = G_N(G_X(s)).$$

[4]

(c) The Statistics Department has N seminars per term where $N \sim$ Poisson(6). For the ith seminar the number of people who fall asleep is X_i where $X_i \sim$ Poisson(2) and $X_1, \dots, X_N$ are iid. Derive the pgf of the total number of instances of people falling asleep in seminars during the term and, using that pgf, calculate its mean. HINT: the pgf of a random variable, Z, which has a Poisson distribution with parameter λ is $G_Y(s) = e^{\lambda(s-1)}$. [5]

A3 (a) For each of the following statements about discrete time homogeneous Markov Chains, state whether they are true or false:

(i) $\mathrm{E}(Y \mid X)$ is a function of the random variable Y.

(ii) State i communicates with state j if and only if $p_{ij} > 0$.

(iii) The parameter set is discrete.

(iv) A chain with a null recurrent state must be infinite.

(v) Every chain with a closed ergodic class has an equilibrium distribution.

[5]

(b) For each of the following statements about continuous time homogeneous Markov Processes, state whether they are true or false:

(i) The Poisson process is irreducible.

(ii) For irreducible processes, an equilibrium distribution exists if and only if an invariant one does.

(iii) If an invariant distribution exists then $\boldsymbol{\pi} = \boldsymbol{\pi}\boldsymbol{P}(t)$.

(iv) The forward equations are $\boldsymbol{P}^{'}(t) = \boldsymbol{P}(t)\boldsymbol{Q}$.

(v) All entries in $\boldsymbol{Q}$ are independent of time t.

[5]

A4 Visitors arrive at a tasting session for dark chocolate in a Poisson Process of rate 9 per minute. Independently, visitors arrive at a tasting session for milk chocolate in a Poisson Process of rate 16 per minute. Both sessions open their doors at the same moment.

(a) What is the distribution of the time until the 1st dark chocolate visitor has arrived? [1]

(b) What is the variance of the time until the 6th dark chocolate visitor has arrived? Explain how you reached your answer. [2]

(c) Derive the probability that the first visitor after the doors open is to the milk chocolate session. [3]

(d) Consider the first 10 visitors to both tasting sessions combined. Write down the distribution of the number of visitors to the milk chocolate session, together with its parameters. State any statistical properties you use. [4]

Section B

B1 Hens A, B and C each lay eggs, independently of each other, at most once in any consecutive two day period. If there is a possibility that a hen will lay an egg in a particular day, then it will do so with probability 1/3. Let X_n be the number of eggs laid by the three hens, in total, on day n.

(a) Explain why $X_0, X_1, \ldots$ is a Markov Chain. [2]

(b) Draw the state space diagram for the Markov Chain. [2]

(c) Show that $\mathrm{P}(X_{n+1} = 2 \mid X_n = 0) = 2/9$ and $\mathrm{P}(X_{n+1} = 2 \mid X_n = 2) = 0$. [3]

(d) Write down $\boldsymbol{P}$, the transition matrix for the chain. [4]

(e) Define both the invariant and equilibrium distributions of any discrete time Markov Chain and list the conditions under which they are the same. [4]

(f) The invariant distribution for the chain is $\boldsymbol{\pi} = (27, 27, 9, 1)/64$. What is the long run expected number of eggs that will be laid by the three hens in a day? Justify any assumptions you make. [3]

Each hen, independently of the other hens, spends the night in one of three locations: 1, 2 or 3. The sequence of locations for any hen, $Y_0, Y_1, \ldots,$ forms a Markov Chain with transition matrix (rows and columns in location order)

$$\boldsymbol{P} = \begin{pmatrix} 1/8 & 3/4 & 1/8 \\ 1/2 & 0 & 1/2 \\ 1/3 & 0 & 2/3 \end{pmatrix}.$$

(g) If hen A spends Thursday night in location 1, find the expected number of nights until it next sleeps in the same location. [6]

(h) One Sunday night, the three hens are in different locations. Find the expected number of nights that they will continue sleeping apart. [6]

B2 (a) Fill in the question marks in the following definition of a Poisson Process. Let $N(t)$ be the number of occurrences of an event in $(0, t]$ for which there exists $\lambda > 0$ such that for $h > 0$:

- P(1 event in $(t, t+h]$) =?
- P(0 events in $(t, t+h]$) =?
- The number of events in (t, t + h] is ? of the process in ?.

[4]

Geraint decides to start the UCL unicyclists club. He does not join himself as he has an appalling sense of balance. Members apply for membership in a Poisson Process with parameter of 12 per week. However, each applicant has probability 1/3 of being rejected due to lack of unicycling ability, independently of all other applications and rejections.

(b) Prove that the number of accepted unicyclists follows a Poisson Process with parameter 8 per week. Name the property of Poisson Processes that you have just proved. [6]

(c) Each club member, independently of other members and arrivals, may decide that one wheel is not enough and leave. The probability of any particular member leaving in $(t, t+h]$ is $h/4 + o(h)$. Let $N(t)$, a Markov Chain, be the number of members in the club at time t. Write down its state space and the first 5 rows and columns of its generator matrix, $\boldsymbol{Q}$. [4]

(d) What are the conditions for any irreducible continuous time Markov Chain to have an equilibrium distribution, and how in practice does one find it? Do you think that $N(t)$ has an equilibrium distribution? Give an informal justification. [6]

(e) Write down the first 4 rows and columns of the transition matrix of the embedded jump chain for $N(t)$. If the club has 3 members, what is the name of the distribution of the time until that number changes? What is the mean of that distribution? [6]

(f) Calculate the mean value of the time in weeks from the start of the club until there are two members for the first time. [4]

END OF PAPER