Please put in your tutor's postbox in Room 117 by 12pm Thursday February 25th (note: after reading week!).

1. Find the irreducible classes of states for the Markov chains with the following transition matrices (all state spaces begin at 0). State whether they are closed or not and classify them in terms of transience, recurrence (positive or null), period and ergodicity. State whether or not an equilibrium distribution exists and, if it does, find it.

(a) $P = \begin{pmatrix} 1/3 & 2/3 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 \\ 1 & 0 & 0 & 0 \end{pmatrix}$

(b) $P = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 \\ 1 & 0 & 0 & 0 \end{pmatrix}$

(c) $P = \begin{pmatrix} 1/6 & 0 & 0 & 1/6 & 2/3 \\ 0 & 1/3 & 2/3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1/4 & 1/4 & 1/2 & 0 \\ 0 & 1/4 & 1/4 & 0 & 1/2 \end{pmatrix}$

(d) $P = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1/3 & 0 & 1/3 & 0 & 1/3 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1/3 & 0 & 1/3 & 1/3 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$

(e) $S = \{0, 1, 2, \ldots\}$, $p_{i0} = 1/4$, $p_{i,i+1} = 3/4$ all i, so that

$$P = \begin{pmatrix} 1/4 & 3/4 & 0 & 0 & \ldots \\ 1/4 & 0 & 3/4 & 0 & \ldots \\ 1/4 & 0 & 0 & 3/4 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Hint for (e): are the states positive recurrent, null recurrent or transient? Think about the distribution of the return times (say to state 0) in this case: can you work out what it would be? From there, you can work out whether the states have a finite or infinite expected return time (only necessary to distinguish between the types of recurrence!).

2. A Markov chain has state space $S = \{0, 1, \ldots, k\}$ with $p + q = 1$, $p > 0$ and $q > 0$, $0 \leq \alpha \leq 1$ and

$$P = \begin{pmatrix} \alpha & 1-\alpha & 0 & 0 & \cdots & 0 & 0 & 0 \\ q & 0 & p & 0 & \cdots & 0 & 0 & 0 \\ 0 & q & 0 & p & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & q & 0 & p \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1-\alpha & \alpha \end{pmatrix}.$$

For what values of α does an equilibrium distribution exist? *Don't* attempt to calculate the equilibrium distribution when it exists!

3. A Markov chain is called *doubly stochastic* if the transition matrix $\mathbf{P} = (p_{ij})$ satisfies $\sum_i p_{ij} = 1$ for all j, i.e. if the sum over each column equals one (in addition to the usual properties of transition matrices).

(a) Explain why in an irreducible, aperiodic, doubly stochastic Markov chain with finite statespace $\{0, 1, 2, \ldots, M-1\}$, an equilibrium distribution π exists. Show that $\pi_i = 1/M$ for all i is that distribution.

[**HINT**: You only need to use the property that the chain is doubly stochastic to calculate π_i, and not to show that the equilibrium distribution exists.]

Let S_n be the sum of the scores obtained on n independent rolls of a fair die. Consider the chain $\{Y_n\}$ where Y_n is the remainder when S_n is divided by 13.

(b) Explain why it is a Markov Chain.

(c) Find its transition matrix.

(d) Find the limiting probability that S_n is a multiple of 13 as n tends to infinity.

[**HINT**: Can you use part (a) of this question to help with this?]

Challenges

Questions in this section will not be covered in the tutorial. If you think you have solved the questions or would like some help, do post to the discussion forum.

1. Give an example of an irreducible transient discrete time Markov Chain.
2. This is harder. Devise an irreducible discrete time Markov Chain whose transition matrix is parameterised (eg a probability might be 2^a and a would be a parameter) so that for different values of the parameters the chain is transient, null recurrent or positive recurrent.