Please put in your tutor's postbox in Room 117 by 12pm Thursday March 17th.

1. A machine is either working (in state 1) or being repaired (in state 0). Repair times are independent exponentially distributed random variables with expectation $1/\alpha$. Working times between breakdowns are independent exponentially distributed random variables with expectation $1/\beta$, and are independent of the repair times. Let $X(t)$ be the state of the machine at time t. Initially the machine is working, i.e. $X(0) = 1$.

   (a) Find the matrix Q of transition rates for the continuous time Markov chain $\{X(t); t \geq 0\}$.

   (b) Find the invariant distribution $\underline{\pi} = (\pi_0, \pi_1)$ of the process.

   (c) Is this also the equilibrium distribution of the process? Explain your answer.

   (d) Find the forward equations for $p_{1j}(t) = P(X(t) = j|X(0) = 1)$ for $j = 0, 1$.

   (e) Using the forward equation for $p_{10}(t)$, obtain a differential equation for $p_{10}(t)$ in terms of $p_{10}(t)$ **only**.
   HINT: What must $p_{10}(t) + p_{11}(t)$ be?

   (f) Show that this differential equation is solved by

   $$p_{10}(t) = \frac{\beta}{\beta + \alpha}(1 - \exp(-(\alpha + \beta)t).$$

   (g) By considering the limit of $p_{10}(t)$ as $t \to \infty$, find the limiting probability that the machine is working, and check that this agrees with your answer for (b).

2. Men and women arrive at a shop, forming independent Poisson processes of rates α and β per hour respectively. Let M_1 be the time until the first male customer arrives and let W_1 be the time until the first female customer arrives.

   (a) Calculate the probability $P(M_1 < W_1)$ that the first customer is male.

   (b) Let N be the number of male customers that arrive before the first female customer. Using $P(M_1 < W_1)$ and the lack-of-memory property of the exponential distribution, *or* by conditioning on the arrival time W_1 of the first female customer, find the probability distribution of N.

   (c) Find the distribution of the time $Z = \min(M_1, W_1)$ until the first customer arrives.
   HINT: First calculate $P(Z > z)$.

   (d) Exactly one female customer arrived in the first hour (note: nothing is said here about how many male customers arrived). Let T be the time at which the first customer arrived. Find $P(T > t)$. Hence, evaluate $E[T]$.
   HINTS:

   * To calculate $P(T > t)$: if you know how many female customers arrived in the first hour, what is the distribution of the time at which she arrived?
   * To calculate $E[T]$: use the result that if T is a random variable on $[0, \infty)$ and $f_T(t) = 0$ for $t > a$ some fixed a, then $E[T] = \int_0^a P(T > t)dt$.

Challenges

Questions in this section will not be covered in the tutorial. If you think you have solved the questions or would like some help, do post to the discussion forum.

A Poisson Process in n dimensions (assume we are dealing with Euclidean space and distances) with parameter λ can be defined (roughly) as a set of events (ie locations) in n dimensional space such that:

1. In any region of volume V, the number of events in that region has a Poisson distribution with parameter λV

2. The numbers of events in two regions that do not overlap are independent of each other.

For $n = 2$, consider any point in the plane and let the random variable X be the distance to the nearest event. Prove that

(a) $P(X > x) = e^{-\lambda \pi x^2}$

(b) $E[X] = \frac{1}{\sqrt{2\pi}}$.

What are the corresponding answers for $n = 3$?