STAT2003: INTRODUCTION TO APPLIED PROBABILITY 2013

Answer ALL questions. Section A carries 40% of the total marks and Section B carries 60%. The relative weights attached to each question are as follows: A1 (10), A2 (10), A3 (10), A4 (10), B1 (30), B2 (30). The numbers in square brackets indicate the relative weights attached to each part question.

Turn Over

Section A

A1 A Markov Chain with states $\{1, 2, 3, 4, 5\}$ has transition matrix

$$\boldsymbol{P} = \begin{pmatrix} 0.4 & 0.6 & 0 & 0 & 0 \\ 0 & 0.2 & 0.4 & 0.4 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0.5 & 0.5 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

(a) Find the irreducible classes of intercommunicating states and classify them in terms of positive or null recurrence, transience, periodicity and ergodicity. [3]

(b) Calculate $\mathrm{P}(X_{n+1} = 2 | X_n = 4, X_{n-1} = 3)$, naming any property you use. [2]

(c) Calculate $\mathrm{P}(X_{n+1} = 2, X_n = 4 | X_{n-1} = 3)$. [2]

(d) Is it true that $\mathrm{P}(X_{n+1} = k | X_n = j, X_{n-1} = i) = \mathrm{P}(X_{n+1} = k, X_n = j | X_{n-1} = i)$ for all values of i, j and k? Justify your answer. [3]

A2 (a) For each of the following statements about discrete time homogeneous Markov Chains, state whether they are true or false:

(i) An irreducible class of recurrent states is closed.

(ii) A finite closed irreducible class must be positive recurrent.

(iii) It is possible to have an equilibrium distribution which is not also an invariant distribution.

(iv) An infinite irreducible ergodic chain has a unique invariant distribution.

(v) An equilibrium distribution exists if and only if there is an ergodic class.

[5]

(b) State, with reasons, whether the Markov Chain with the following transition matrix has

(i) An invariant distribution [2]

(ii) An equilibrium distribution [3]

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$$P = \begin{pmatrix} 0.7 & 0 & 0 & 0 & 0.3 & 0 \\ 0 & 0 & 0.45 & 0 & 0 & 0.55 \\ 0 & 0.2 & 0.4 & 0 & 0 & 0.4 \\ 0.3 & 0.3 & 0 & 0 & 0.4 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.5 & 0.5 & 0 & 0 & 0 \end{pmatrix}.$$

A3 (a) For a function, $f(x)$, define what it means for f to be $o(h)$. [2]

(b) Are the following functions $o(h)$ or not:

(i) $f(x) = x^3$

(ii) $f(x) = x^{1/3}$

(iii) $f(x) = x^3 + x^{1/3}$?

[3]

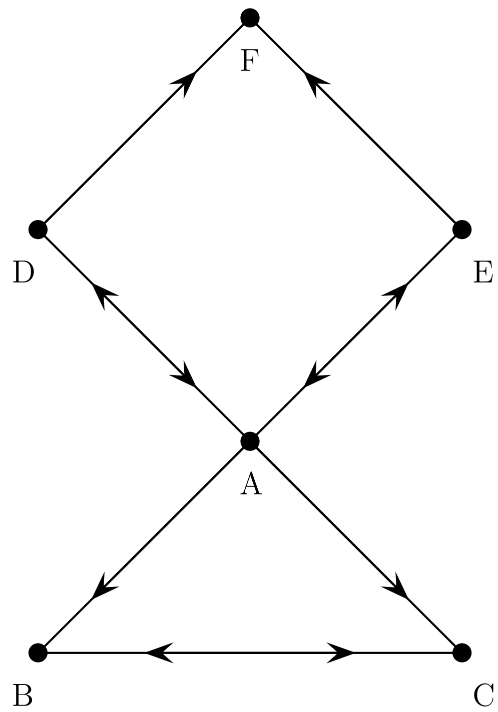
(c) Give the formal definition of a Poisson Process of rate λ in terms of λ and $h > 0$. [3]

(d) Using your answer to part c, for a Poisson Process, show that P(more than one event in $(t, t+h]$) $= o(h)$. [2]

A4 A participant in a game show chooses one of four boxes at random, each with probability 1/4. Box 1 contains £5,000, Box 2 £3,000, Box 3 £1,000 and Box 4 nothing. If she chooses Box 4, the participant leaves the game. Otherwise, she keeps the money in the box chosen, the box is refilled with the same amount and she repeats the process, selecting from among the four boxes at random. What is the expected value of the amount the participant will win by the time she leaves the game? [10]

Section B

B1 Roads connecting six locations in a city are shown in the following diagram, the time taken to cycle along any of the roads being 5 minutes. Chris is a bicycle courier whose office is located at location A. He has a parcel that he needs to deliver to location F. However, he is a law abiding cyclist who will only follow the direction of travel permitted on each road, as indicated by the arrows. Whenever he comes to a junction, he selects any of the routes available to him, including the road on which he arrived, with equal probability. Let $\{L_n : n = 0, 1, 2, \ldots\}$ be the random variables denoting the series of locations that the parcel visits, so that $L_0 = A$.



(a) Treat the diagram as a state space diagram of locations that the parcel might visit during its trip from A to F. Describe the line that needs to be added to represent the fact that once the parcel arrives at F, it will stay there. Define the irreducible classes of locations, specifying which ones are closed. [4]

(b) Is $\{L_n : n = 0, 1, 2, \ldots\}$ a Markov Chain? Explain why. [2]

(c) Write down two different invariant distributions of $\{L_n : n = 0, 1, 2, \ldots\}$ (no calculations are required). [4]

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(d) What is the expected value of the cycling time in minutes that Chris will take to deliver his parcel to F? Justify your answer. [3]

(e) For this part of the question only, assume that Chris only chooses between roads to D and E, when he is at A. Calculate the expected value of his cycling time to F in minutes. [6]

(f) Unfortunately, Chris starts off his journey by taking the road to B and then is stuck cycling between B and C indefinitely. A replacement bicycle courier, Brad, who cycles at the same speed as Chris, is called in to deliver an identical parcel. Brad is prepared to cycle the wrong way down one way streets and chooses roads randomly with equal probability at every junction. Explain why Brad will succeed in delivering the parcel. [3]

(g) What is the expected number of times that Brad will pass Chris on the road between B and C, while he delivers the parcel to F? [8]

B2 The British Space Agency (BSA) is recruiting couples to be part of the first colony on the surface of Jupiter in ten years time. Let the number of couples who apply, pass the aptitude test and arrive on the waiting list be a Poisson Process with parameter $\lambda = 1/3$ per day. Define $\{N(t) : t \geq 0\}$ as the number of couples on the list at time t.

(a) Explain what the lack of memory property for the exponential distribution is. [3]

(b) Specify the distribution, with parameters, of the time between an arbitrary time point, t_1, and the arrival on the list of the fourth couple after t_1, explaining your reasoning. [5]

(c) Given that 10 couples join the waiting list in a period of 30 days, calculate the mean number of couples one would expect to have joined in the first 7 days of that period, justifying your calculation. [3]

(d) The physical and mental status of couples is continuously monitored. If certain indicators (eg body mass index, cholesterol levels) go outside their permitted ranges for either of a couple, the couple is immediately dropped from the waiting list. As a consequence, for $N(t) = n$, the number of couples on the list reduces by one at a rate of $n/6$ per day. Write down the state space, S, of the number of couples. Set out the 5×5 matrix at the top left hand corner of the generator matrix $\boldsymbol{Q}$, for $\{N(t) : t \geq 0\}$. [6]

(e) For this part of the question only, assume that the waiting list can only hold up to 4 couples. Calculate the equilibrium distribution of $\{N(t) : t \geq 0\}$ and show that the long run proportion of couples who apply to go to Jupiter but find the waiting list full is 2/21. [9]

(f) Write down the 5×5 matrix at the top left hand corner of the transition matrix of the embedded jump chain for $\{N(t) : t \geq 0\}$. [4]

END OF PAPER