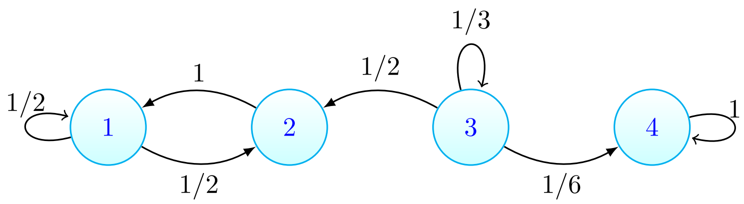
The focus for these exercises is on manipulating transition matrices and probability vectors. The main comment I have is that you must make sure that you justify why a process is Markov thoroughly - it is no good re-writing the definition of the Markov property. You must say *why the Markov property holds in the context of the question.*

1. State space diagram



   (a) If the chain starts in state 4, it must stay there.

   (b) If the chain starts in state 1, it will remain in $\{1, 2\}$.

   (c) The three-step transition matrix is

$$P^{(3)} = \begin{pmatrix} 5/8 & 3/8 & 0 & 0 \\ 3/4 & 1/4 & 0 & 0 \\ 5/12 & 11/36 & 1/27 & 13/54 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

   Reading off the relevant probability from this matrix, we have that:

$$P(X_3 = 1|X_0 = 1) = 5/8$$

2. X_n is the coin chosen for the nth flip, not heads or tails. Heads or tails are only used to determine which coin is chosen. State-space is $\{1, 2\}$, [coin number].

   (a) The state space is $\{1, 2\}$. This is a Markov chain, as the coin we choose next only depends on the outcome of the last flip (which in itself depends on which coin was used). Notice here that I have put the explanation in context. You do not need to write very much, just as long as you explain why the Markov property holds.
   The transition matrix is

$$P \;=\; \begin{array}{c} \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \begin{pmatrix} 0.4 & 0.6 \\ 0.8 & 0.2 \end{pmatrix} \end{array}.$$

   (b) From the question

$$\underline{p}^{(0)} = (0.5, 0.5)$$

   and then

$$\begin{aligned} \underline{p}^{(1)} &= \underline{p}^{(0)} P \\ &= (0.6, 0.4) \\ \underline{p}^{(2)} &= \underline{p}^{(1)} P \\ &= (0.56, 0.44), \end{aligned}$$

   (c) $P(X_2 = 1) = 0.56$.

   (d) $P(X_3 = 1, X_2 = 1) = 0.224$ by using an application of Bayes's theorem.

3. If $Z_n = 0$ then so must Z_{n+1} be, irrespective of what happened with earlier random variables. If $Z_n = 1$, then all earlier random variables must also be 1 and so, for Z_{n+1}, conditioning on Z_n is equivalent to conditioning on $\{Z_n, \ldots, Z_0\}$. Thus, this is a Markov chain.

   The transition matrix is:

$$P = \begin{pmatrix} 1 & 0 \\ 1-\alpha & \alpha \end{pmatrix}.$$

An incorrect argument

Many of you said that 'Z is a one-to-one transformation of the Markov process X, and therefore Z must be a Markov chain'.

Firstly, the transformation applied is not one-to-one. If it were, however, you need to **show** that there is a one-to-one mapping (and not just state this!). Had you taken time to write down such a mapping in this case, it would have been clear that this is not one-to-one:

- If $Z_n = 1$ then $X_n = 1$.
- If $Z_n = 0$, then X_n could be either 0 or 1.

Therefore, the 'one-to-one' argument is invalid here.

(b) $P(Z_n = 1) = P(X_n = 1, X_{n-1} = 1, \ldots, X_0 = 1) = \alpha^n p$, by using the result on page 25 of the notes.

As always, come to an office hour if anything is unclear!