Important types of continuous time processes

Poisson processes and birth-death processes

Chapter 5 of notes

STAT2003 / STAT3102

Contents

- Poisson processes;
- Birth-death processes.

The Poisson process

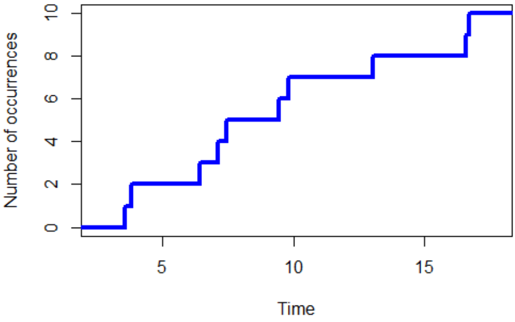
Very simple but important Markov process which is the building block for more complex processes.

Counts the number of events in $(0, t]$. Informally, events arrive

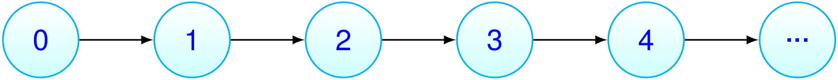
- one at a time;
- independently;
- randomly;
- uniformly, i.e. at a constant rate, λ, per unit time.

Poisson processes usually denoted $\{N(t), t \geqslant 0\}$. [$N(t)$ tells us how many events have occurred up to time t.]

The Poisson process



State space diagram



Formal definition of the Poisson process

Definition

Let $N(t)$ denote the number of occurrences of some event in $(0, t]$ for which there exists $\lambda > 0$ such that for $h > 0$:

(i) $P(1 \text{ event in } (t, t+h]) = \lambda h + o(h)$;

(ii) $P(\text{no events in } (t, t+h]) = 1 - \lambda h + o(h)$;

(iii) The number of events in $(t, t+h]$ is independent of the process in $(0, t]$.

Then $\{N(t), t \geqslant 0\}$ is a Poisson process of rate λ.

Note: $\{N(t), t \geqslant 0\}$ is a continuous-time stochastic process with state space $S = \{0, 1, 2, \ldots\}$.

Questions

1. Is the process in continuous time?
2. Is the process Markov?

The Poisson process satisfies the Markov property

Want to show that the past is independent of the future, given the present:

$$\begin{aligned}
&\mathrm{P}(N(t_{n+1}) = i_{n+1} \mid N(t_n) = i_n, \ldots, N(t_1) = i_1) \\
=&\mathrm{P}(N(t_{n+1}) = i_{n+1} \mid N(t_n) = i_n)
\end{aligned}$$

Consider times $t_1 < t_2 < \cdots < t_n < t_{n+1}$. Then

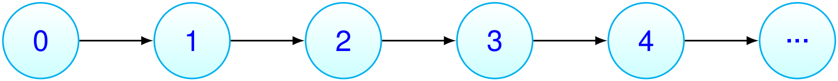
$$\begin{aligned}
&\mathrm{P}(N(t_{n+1}) = i_{n+1} \mid N(t_n) = i_n, \ldots, N(t_1) = i_1) \\
=&\mathrm{P}(i_{n+1} - i_n \text{ events in } (t_n, t_{n+1}] \mid N(t_n) = i_n, \ldots, N(t_1) = i_1) \\
=&\mathrm{P}(i_{n+1} - i_n \text{ events in } (t_n, t_{n+1})) \qquad \text{by (iii)} \\
=&\mathrm{P}(N(t_{n+1}) = i_{n+1} \mid N(t_n) = i_n).
\end{aligned}$$

More questions

1. Jump chain of the Poisson process?
2. Holding time in each state (i.e. how long between events)?
3. Generator matrix?
4. Useful properties?
5. Long-term behaviour?

The jump chain

For the Poisson process, the jump chain is very simple.

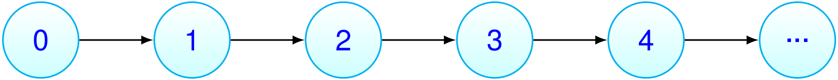


If we are currently in state i, we *must* move to state $(i+1)$ next.

Think about this: the Poisson process counts the number of events. If we have seen i events (‘we are in state i’), then we move state when we see the next event (i.e. have seen $(i+1)$ events in total, so we are in state $(i+1)$).

The jump chain

For the Poisson process, the jump chain is very simple.



The transition matrix of the jump chain is therefore

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix}$$

Holding time

How long between events

We know from general results on continuous-time Markov processes:

Holding time is exponential. Rate of leaving state i, q_i?

Remember that $q_i = q_{i,i+1}$ and $q_{ij} = 0$ if $j \neq i + 1$. Why?

Holding time

How long between events

We know from general results on continuous-time Markov processes:

Holding time is exponential. Rate of leaving state i, q_i?

If our Poisson process is of rate λ, then the rate at which the process leaves state i is λ for all i.

Also see this from previous result: $p_{ii}(h) = 1 - q_i h + o(h)$.
From the definition of a Poisson process:
$P(\text{no events in } (0, h]) = 1 - \lambda h + o(h)$.
Comparing the two: $q_i = \lambda$.

The generator matrix

Using these rates, we know that

$$Q = \begin{pmatrix} -\lambda & \lambda & 0 & 0 & 0 & \cdots \\ 0 & -\lambda & \lambda & 0 & 0 & \cdots \\ 0 & 0 & -\lambda & \lambda & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix}$$

What is the distribution of $N(t)$?

For fixed time t.

What is the distribution of the number of events by time t?

- Need to find $\mathrm{P}(N(t) = j)$ in the hope of recognising the form of its pmf!
- Notice: $\mathrm{P}(N(t) = j) = \mathrm{P}(j \text{ events in } (0, t]) = p_{0j}(t)$.

Kolmogorov's forward equations tell us that:

$$\begin{aligned} p'_{0j}(t) &= \lambda p_{0,j-1}(t) - \lambda p_{0j}(t) \\ p'_{00}(t) &= -\lambda p_{00}(t) \end{aligned}$$

The solution

Need an initial condition to solve the forward equations, say $N(0) = 0$:

$$p_{0k}(0) = \mathrm{P}(N(0) = k) = \begin{cases} 1 & \text{if } k = 0 \\ 0 & \text{for } k = 1, 2, 3, \ldots, \end{cases}$$

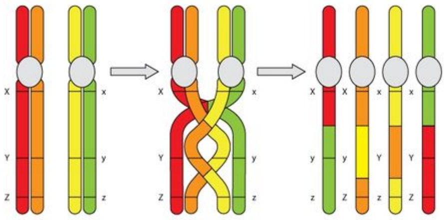
Solve to find:

$$\boxed{p_{0k}(t) = \mathrm{P}(N(t) = k) = \frac{(\lambda t)^k \, \mathrm{e}^{-\lambda t}}{k!}, \quad k = 0, 1, 2, \ldots,}$$

That is, we have shown that

$N(t) \sim \text{Poisson}(\lambda t)$.

Haldane's model for genetic recombination



- Crossovers are points at which chromosome which goes to the gamete switches from being a copy of either one of the parent's chromosomes to being a copy of the other.
- Haldane assumed that crossovers occur as a Poisson process.
- If genetic distance is measured in Morgans, the rate of the Poisson process is 1 [per Morgan].

Haldane's model for genetic recombination

Denote by $N(t)$ the number of crossovers that occur along a chromosome of length t.

(i) For $t > 0$, name the distribution of $N(t)$ and state its mean and variance.

The longest chromosome, chromosome 1, is approximately 3 Morgans in length.

(ii) What is the probability that there are no crossovers on chromosome 1?

(iii) What is the expected number of crossovers on chromosome 1?

(iv) Given that the *first* crossover occurs along the first 0.5 Morgans of the chromosome, find the probability that the crossover was located in the first 0.2 Morgans.

The distribution of the number of events

- **The number of events in $(s, s+t]$ has a Poisson(λt) distribution (for all s).**

  $$\mathrm{P}(k \text{ events in } (s, s+t]) = \mathrm{P}(k \text{ events in } (0, t]) = \frac{(\lambda t)^k \mathrm{e}^{-\lambda t}}{k!}.$$

  Stationary increments: distribution of the number of events in an interval does not depend on when the interval starts.

- **If $(a, b]$ and $(c, d]$ are non-overlapping intervals, then the number of events in $(a, b]$ is independent of the numbers of events in $(c, d]$).**
  This follows from (iii): independent increments.

The distribution of time to events

- **Time T_1 to first event has $T_1 \sim$ exp(λ).**

$$\mathrm{P}(T_1 > t) = \mathrm{P}(N(t) = 0) = \mathrm{e}^{-\lambda t}, \quad t > 0.$$

Therefore,

$$F_{T_1}(t) = \mathrm{P}(T_1 \leqslant t) = 1 - \mathrm{e}^{-\lambda t} \qquad \text{and} \qquad f_{T_1}(t) = \lambda\, \mathrm{e}^{-\lambda t}, t > 0.$$

- **Time to the rth event has gamma(r, λ) distribution.**
  Let S_r be the time to the rth event. Then

$$S_r = T_1 + \cdots + T_r.$$

This is the sum of r i.i.d. exp(λ) random variables, which has a gamma(r, λ) distribution.

The distribution of time between events

Time between successive events is $\exp(\lambda)$.

Let T_2 be the time between the 1st and 2nd events. Find the marginal distribution of T_2:

$$\begin{aligned}
P(T_2 > t) &= \int_0^\infty P(T_2 > t \mid T_1 = s) f_{T_1}(s)\,\mathrm{d}s \\
&= \int_0^\infty P(\text{no events in } (s, s+t])\lambda\,\mathrm{e}^{-\lambda s}\,\mathrm{d}s \\
&= \int_0^\infty \mathrm{e}^{-\lambda t}\,\lambda\,\mathrm{e}^{-\lambda s}\,\mathrm{d}s \\
&= \mathrm{e}^{-\lambda t}.
\end{aligned}$$

Therefore, $T_2 \sim \exp(\lambda)$.

The distribution of time between events

Times between successive events are independent.

Let T_2 be the time between the 1st and 2nd events. Show that T_1 and T_2 are independent:

$$\begin{aligned}
\mathrm{P}(T_1 > v, T_2 > t) &= \int_v^\infty \mathrm{P}(T_2 > t \mid T_1 = u) f_{T_1}(u)\, \mathrm{d}u \\
&= \int_v^\infty \mathrm{P}(\text{no events in } (u, u+t])\, \lambda\, \mathrm{e}^{-\lambda u}\, \mathrm{d}u \\
&= \int_v^\infty \mathrm{e}^{-\lambda t}\, \lambda\, \mathrm{e}^{-\lambda u}\, \mathrm{d}u \\
&= \mathrm{e}^{-\lambda t} \mathrm{e}^{-\lambda v} = \mathrm{P}(T_2 > t)\mathrm{P}(T_1 > v).
\end{aligned}$$

Therefore, T_1 and T_2 are independent. Repeating the argument for (T_2, T_3), (T_3, T_4), ... gives the result.

Time from an arbitrary point to the next event

The time from an arbitrary time point t to the next event is an $\exp(\lambda)$ random variable.

Follows from lack-of-memory of the exponential distribution.

Let T_1 be the time to the next event and T_2 be the time from this event until the following event. Then

$$\mathrm{P}(0 \text{ events in } (t, t+h]) = \mathrm{P}(T_1 > h) = \mathrm{e}^{-\lambda h} = 1 - \lambda h + o(h).$$

$$\mathrm{P}(\geqslant 1 \text{ event in } (t, t+h]) = \mathrm{P}(T_1 < h) = 1 - \mathrm{e}^{-\lambda h} = \lambda h + o(h).$$

$$\mathrm{P}(\geqslant 2 \text{ events in } (t, t+h]) = \mathrm{P}(T_1 + T_2 < h) = \int_0^h \lambda^2 x \, \mathrm{e}^{-\lambda x} \, \mathrm{d}x = o(h)$$

Therefore, $\mathrm{P}(1 \text{ event in } (t, t+h]) = \lambda h + o(h)$.

Distribution of arrival times of events

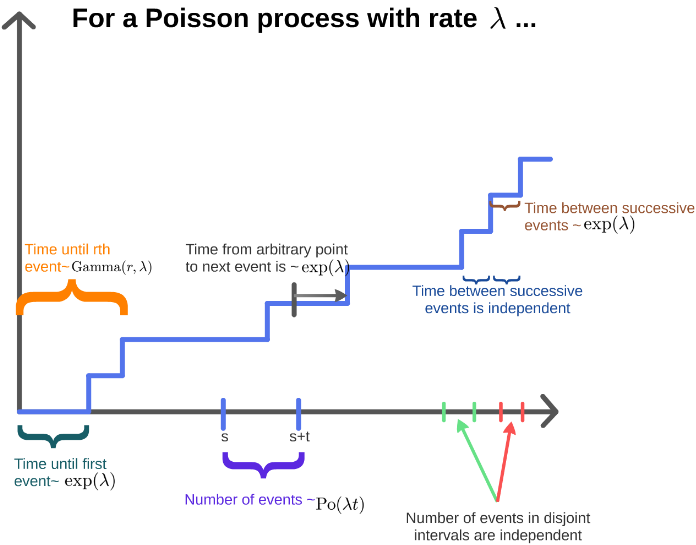
The (unordered) set of arrival times in (0,t] are i.i.d., each of which uniformly distributed on (0,t).

That is, given that exactly k events occur in $(0, t]$, the distribution of the set of (unordered) arrival times $(U_1, \ldots, U_k)$ of k is

$$\boxed{U_1, \ldots, U_{N(t)} \mid N(t) = k \overset{\text{i.i.d.}}{\sim} U(0, t).}$$

This makes sense: the k events are randomly scattered in the interval $(0, t]$.

Pictorial representation of a PP



Long term behaviour of a Poisson process

Discussion point:

How do you think the Poisson process behaves in the long run?

Without trying to solve the relevant equation, does the Poisson process have:

- An invariant distribution?
- An equilibrium distribution?

Long term behaviour of a Poisson process

Discussion point:
How do you think the Poisson process behaves in the long run?

Try to solve $\underline{\pi}Q = \underline{0}$, where $\underline{\pi} = (\pi_0, \pi_1, \pi_2, ...)$:

$$\begin{aligned}\pi_0 &= 0\\ \pi_1 &= 0\\ \pi_2 &= 0\\ \pi_3 &= 0\\ &...\end{aligned}$$

This is a valid vector, but is not a probability distribution.

Therefore an invariant distribution does not exist.

Therefore an equilibrium distribution cannot exist.

Superposition of Poisson processes.

The addition of two independent Poisson processes is a Poisson process.

- $\{N_1(t), t \geqslant 0\}$ and $\{N_2(t), t \geqslant 0\}$ are independent Poisson processes;
- Rates λ_1 and λ_2 respectively.

Then $N(t) = N_1(t) + N_2(t)$ for all $t \geqslant 0$ is a Poisson process of rate $(\lambda_1 + \lambda_2)$.

Show that $N(t)$ satisfies the three requirements of a Poisson process:

(i) P(1 event in $(t, t+h]) = (\lambda_1 + \lambda_2)h + o(h)$.

(ii) P(0 events in $(t, t+h)) = 1 - (\lambda_1 + \lambda_2)h + o(h)$.

(iii) Independent increments. This holds because it holds for each process individually and the two process are independent.

Thinning of a Poisson process.

A *thinned* Poisson process is still a Poisson process.

- $\{N(t), t \geqslant 0\}$ is a Poisson process of rate λ.
- Each event of the process is deleted w.p. $1 - p$ and kept w.p. p independently of all other events in the process.
- Let $\{M(t), t \geqslant 0\}$ be the process containing only the events which are kept.

Then $\{M(t), t \geqslant 0\}$ is a Poisson process of rate $p\lambda$.

Show that $M(t)$ satisfies the three requirements of a Poisson process:

(i) P(1 event in $(t, t+h]) = p\lambda h + o(h)$.

(ii) P(0 events in $(t, t+h]) = 1 - p\lambda h + o(h)$.

(iii) Deleting events doesn't affect the independent increments property because events are deleted independently of each other.

Example STAT2003 exam 2011

In a certain town at time $t = 0$ there are no bears. Brown bears begin to arrive according to a Poisson process of rate β. Also, gray bears arrive independently according to a Poisson process of rate γ.

(i) Name the distribution of the total number of bears in the town at time $t > 0$.

(ii) Find the probability that no new bears will arrive in the town during the time period $t \in [1, 2] \cup [3, 5]$.

(iii) Show that the probability that the first bear to arrive is brown is $\beta/(\beta + \gamma)$.

Example STAT2003 exam 2014

Visitors arrive at a tasting session for dark chocolate in a Poisson process of rate 9 per minute. Independently, visitors arrive at a tasting session for milk chocolate in a Poisson process of rate 16 per minute. Both sessions open their doors at the same moment.

(a) What is the distribution of the time until the 1st dark chocolate visitor has arrived?

(b) What is the variance of the time until the 6th dark chocolate visitor has arrived? Explain how you reached your answer.

(c) Derive the probability that the first visitor after the doors open is to the milk chocolate session.

(d) Consider the first 10 visitors to both tasting sessions combined. Write down the distribution of the number of visitors to the milk chocolate session, together with its parameters. State any statistical properties you use.

Example: STAT2003 exam 2011.

Cars arrive at a small petrol station according to a Poisson process at rate λ per minute. There is one petrol pump at the station, and space for two more cars to wait. An arriving car immediately gets served if the pump is free, or joins a queue if the station is not full; otherwise it leaves the station. The service time at the pump follows an exponential distribution with mean $1/\mu$ minutes. Service times are independent of each other and of the arrival process. Let $N(t)$ be the number of cars at the station at time t.

Write down the state-space S and the generator matrix Q of transition rates for the continuous-time Markov chain $N(t)$.

Example: STAT2003 exam 2011.

You are given that the invariant distribution of the Markov chain, π, is such that $\pi_i = (\lambda/\mu)^i \pi_0$ for $i = 1, 2, 3$, and that $\lambda/\mu = 2$.

(i) Find the long-run proportion of arriving cars that find the station full and depart.

(ii) Find the expected number of cars at the petrol station in the long run.

(iii) What is the long-run probability that a car finds the pump free, *given* that it stops at the station?

Birth and death processes

This is an important class of continuous-time Markov chains in which jumps are always one step up ('birth') or one step down ('death').

Remark

Births and deaths occur singly and independently of each other. The state space is $S = \{0, 1, 2, \ldots\}$.

Remark

The states correspond to the number of individuals in the population. For example, if a birth-death process is in state j at time t, then there are j individuals in the population at time t.

Birth and death processes

Important questions, and how to answer them

- Probability of a birth (death) in a small time interval?

  Rates, q_{ij}.

- Probability that the next event is a birth? A death?

  Jump chain

- How long must we wait until the next event?

  Holding time

- How do these processes behave in the long run?

  Are birth-death processes irreducible?

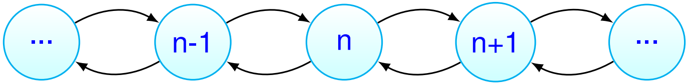
- Will the population ever die out (become extinct)?

  Long-run behaviour.

Transition rates

If at time t the size of the population is n, then next transition is:

- to state $(n+1)$ (a birth) OR
- to state $(n-1)$ (a death).



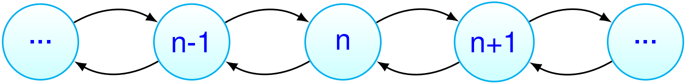
Notation:

Birth rate: λ_n	IF the next event is a birth, this is the rate at which we leave state n and enter state $(n+1)$
Death rate: μ_n	IF the next event is a birth, this is the rate at which we leave state n and enter state $(n-1)$.

Transition rates

If at time t the size of the population is n, then next transition is:

- to state $(n+1)$ (a birth) OR
- to state $(n-1)$ (a death).



Notation:

Birth rate: λ_n
Death rate: μ_n

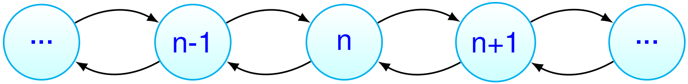
Obviously we must have $\mu_0 = 0$. Why?

Note: birth and death rates can depend on the population size at the time.

Transition rates

If at time t the size of the population is n, then next transition is:

- to state $(n+1)$ (a birth) at rate λ_n OR
- to state $(n-1)$ (a death) at rate μ_n.



Different ways of asking the same type of question:

- At what rate do we *leave state n*?
- Distribution of the holding time in state n?
- How long do we wait until the next ‘event’ (which will be a birth or death)?

Holding time

The (general) birth and death process

- At what rate do we *leave state n*?
- Distribution of the holding time in state n?
- How long do we wait until the next 'event' (which will be a birth or death)?

Suppose that $\{X(t), t \geqslant 0\}$ is a birth-death process and $X(t) = n$. From general results in last section:

Time (T) until the next 'event' will be...

... exponential with rate $(\lambda_n + \mu_n)$.

$$T \sim \exp(\lambda_n + \mu_n)$$

From rates to the generator matrix

The (general) birth and death process

The generator matrix of transition rates is

$$Q = \begin{pmatrix} -\lambda_0 & \lambda_0 & 0 & 0 & 0 & 0 \\ \mu_1 & -(\lambda_1 + \mu_1) & \lambda_1 & 0 & 0 & 0 \\ 0 & \mu_2 & -(\lambda_2 + \mu_2) & \lambda_2 & 0 & 0 \\ 0 & 0 & \mu_3 & -(\lambda_3 + \mu_3) & \lambda_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix}$$

$\lambda_0, \lambda_1, \lambda_2, \ldots$ are the birth rates and $\mu_1, \mu_2, \ldots$ are the death rates.

Some important special cases

(i) IMMIGRATION at rate α: $\lambda_n = \alpha$, $n = 0, 1, 2 \ldots$.

(ii) LINEAR BIRTH at rate λ per individual (that is, every individual alive at time t has probability $\lambda h + o(h)$ of giving birth to a new individual in $(t, t+h]$):

$$\lambda_n = n\lambda, \qquad n = 0, 1, 2, \ldots.$$

(iii) EMIGRATION at rate β: $\mu_n = \beta$, $n = 1, 2, \ldots$.

(iv) LINEAR DEATH at rate μ per individual (that is, every individual alive at time t has probability $\mu h + o(h)$ of dying in $(t, t+h]$):

$$\mu_n = n\mu, \qquad n = 1, 2, \ldots.$$

Some important special cases

We could define a process that has (i), (ii), (iii) and (iv), that is,

$$\lambda_n = \alpha + n\lambda, \;\; n = 0, 1, 2, \ldots, \quad \text{and} \quad \mu_n = \beta + n\mu, \;\; n = 1, 2, \ldots,$$

although it would be difficult to study this process algebraically.

Notes

Remark

A Poisson process of rate λ is a pure immigration process, i.e.,

$$\lambda_n = \lambda, \;\; n = 0, 1, \dots \quad \text{and} \quad \mu_n = 0, \;\; n = 1, 2, \dots .$$

Therefore, immigration at rate α is equivalent to a Poisson process of rate α of arrivals.

Notes

Remark

A pure death process has $\lambda_n = 0$, for $n = 0, 1, 2, \ldots$.

Recall that the time until the next jump is $\exp(-q_i)$.

Under pure linear death the time until the next jump is $\exp(n\mu)$.

This makes sense because the time until a given individual dies is $\exp(\mu)$ and all individuals are independent of each other.

If $T_i \overset{\text{i.i.d.}}{\sim} \exp(\mu)$, $i = 1, \ldots, n$, then $\min(T_1, \ldots, T_n) \sim \exp(n\mu)$

Notes

Remark

A pure birth process has $\mu_n = 0$, for $n = 1, 2, \ldots$.

(... and we can make similar statements to pure linear death with 'death' replaced by 'birth').

Generator matrix

Immigration-emigration process

Generator matrix for the immigration-emigration process is:

$$
\begin{array}{cc}
 & \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & \ldots \end{array} \\
Q = & \begin{pmatrix}
-\lambda & \lambda & 0 & 0 & 0 & \cdots \\
\mu & -(\lambda+\mu) & \lambda & 0 & 0 & \cdots \\
0 & \mu & -(\lambda+\mu) & \lambda & 0 & \cdots \\
0 & 0 & \mu & -(\lambda+\mu) & \lambda & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots &
\end{pmatrix}
\end{array}
$$

The holding time T_i in state i is given by

$$
T_i \sim \begin{cases} \exp(\lambda) & \text{for } i = 0 \\ \exp(\lambda+\mu) & \text{for } i = 1, 2, \ldots \end{cases}
$$

Generator matrix

Linear birth- emigration process

Generator matrix for the linear birth- emigration process is:

$$
\begin{array}{cc}
 & \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & \ldots \end{array} \\
Q = & \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & \cdots \\
\mu & -(\lambda+\mu) & \lambda & 0 & 0 & \cdots \\
0 & \mu & -(2\lambda+\mu) & 2\lambda & 0 & \cdots \\
0 & 0 & \mu & -(3\lambda+\mu) & 3\lambda & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots &
\end{pmatrix}
\end{array}
$$

The holding time T_i in state i is given by

$$T_i \sim \text{exponential}(i\lambda + \mu) \text{ for } i = 1, 2, \ldots$$

What happens when the process reaches state 0?

Jump chain

The (general) birth and death process

What does the jump chain for the general birth-death process look like?

Suppose that $\{X(t), t \geqslant 0\}$ is a birth-death process and $X(t) = n$. From general results in last section:

Next 'event' will be...

- ... a birth, with probability $\lambda_n/(\lambda_n + \mu_n)$.
- ... a death, with probability $\mu_n/(\lambda_n + \mu_n)$.

Jump chain

The (general) birth and death process

What does the jump chain for the general birth-death process look like?

Suppose that $\{X(t), t \geqslant 0\}$ is a birth-death process and $X(t) = n$. From general results in last section:

Transition matrix for jump chain is therefore:

$$P = \begin{pmatrix} ? & ? & 0 & 0 & 0 & \cdots \\ \frac{\mu_1}{\lambda_1+\mu_1} & 0 & \frac{\lambda_1}{\lambda_1+\mu_1} & 0 & 0 & \cdots \\ 0 & \frac{\mu_2}{\lambda_2+\mu_2} & 0 & \frac{\lambda_2}{\lambda_2+\mu_2} & 0 & \cdots \\ 0 & 0 & \frac{\mu_3}{\lambda_3+\mu_3} & 0 & \frac{\lambda_3}{\lambda_3+\mu_3} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix}$$

Example of a jump chain

Linear birth-death

The transition matrix of the discrete-time embedded jump chain of this process is given by

$$P_{LBD} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ \frac{\mu}{\lambda+\mu} & 0 & \frac{\lambda}{\lambda+\mu} & 0 & 0 & \cdots \\ 0 & \frac{\mu}{\lambda+\mu} & 0 & \frac{\lambda}{\lambda+\mu} & 0 & \cdots \\ 0 & 0 & \frac{\mu}{\lambda+\mu} & 0 & \frac{\lambda}{\lambda+\mu} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix}$$

Example of a jump chain

Immigration - emigration process

The transition matrix of the jump chain is given by

$$P_{IE} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \cdots \\ \frac{\mu}{\lambda+\mu} & 0 & \frac{\lambda}{\lambda+\mu} & 0 & 0 & \cdots \\ 0 & \frac{\mu}{\lambda+\mu} & 0 & \frac{\lambda}{\lambda+\mu} & 0 & \cdots \\ 0 & 0 & \frac{\mu}{\lambda+\mu} & 0 & \frac{\lambda}{\lambda+\mu} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix}$$

Example of a jump chain

Linear birth - emigration process

The transition matrix of the jump chain is given by

$$P_{LBE} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ \frac{\mu}{\lambda+\mu} & 0 & \frac{\lambda}{\lambda+\mu} & 0 & 0 & \cdots \\ 0 & \frac{\mu}{2\lambda+\mu} & 0 & \frac{2\lambda}{2\lambda+\mu} & 0 & \cdots \\ 0 & 0 & \frac{\mu}{3\lambda+\mu} & 0 & \frac{3\lambda}{3\lambda+\mu} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix}$$

Probability of birth or death in a small interval

The (general) birth and death process

Recall that for $i \neq j$,

$$p_{ij}(h) = q_{ij}h + o(h).$$

Here,

$$q_{n,n+1} = \lambda_n \quad \text{and} \quad q_{n,n-1} = \mu_n.$$

Therefore:

- the probability of a birth in $(t, t+h]$ is $\lambda_n h + o(h)$.
- the probability of a death in $(t, t+h]$ is $\mu_n h + o(h)$.

Transition probabilities in a small interval

The (general) birth and death process

Let $p_{ij}(h) = \mathrm{P}(X(t+h) = j \mid X(t) = i)$. Then

$$\begin{aligned} p_{n,n+1}(h) &= \{\lambda_n h + o(h)\}\{1 - \mu_n h + o(h)\} + o(h) \\ &= \lambda_n h + o(h) \end{aligned}$$

$$\begin{aligned} p_{n,n-1}(h) &= \{1 - \lambda_n h + o(h)\}\{\mu_n h + o(h)\} + o(h) \\ &= \mu_n h + o(h) \end{aligned}$$

$$\begin{aligned} p_{nn}(h) &= \{1 - \lambda_n h + o(h)\}\{1 - \mu_n h + o(h)\} + o(h) \\ &= 1 - (\lambda_n + \mu_n)h + o(h). \end{aligned}$$

$$p_{nm}(h) = o(h), \qquad \text{if } m \notin \{n-1, n, n+1\}.$$

Example: probabilities over small intervals

The linear birth-death process

Here we have:

- LINEAR BIRTH at rate λ per individual (that is, every individual alive at time t has probability $\lambda h + o(h)$ of giving birth to a new individual in $(t, t+h]$):

$$\lambda_n = n\lambda, \qquad n = 0, 1, 2, \ldots.$$

- LINEAR DEATH at rate μ per individual (that is, every individual alive at time t has probability $\mu h + o(h)$ of dying in $(t, t+h]$):

$$\mu_n = n\mu, \qquad n = 1, 2, \ldots.$$

Example: probabilities over small intervals

The linear birth-death process

Population size at time t is n (in state n at time t):
Probability of a birth in $(t, t+h]$ is $\lambda_n h + o(h)$.
Probability of a death in $(t, t+h]$ is $\mu_n h + o(h)$.

In $(t, t+h]$ each individual alive at time t has:

- probability $\lambda h + o(h)$ of giving birth to a new individual;
- probability $\mu h + o(h)$ of dying,

where $\lambda, \mu > 0$.

Assumptions:

- Probabilities **do not depend on the age of the individual.**
- All individuals are **independent** of each other.

Example: probabilities over small intervals

The linear birth-death process

Population size at time t is n (in state n at time t):
Probability of a birth in $(t, t+h]$ is $\lambda_n h + o(h)$.
Probability of a death in $(t, t+h]$ is $\mu_n h + o(h)$.

In $(t, t+h]$ each individual alive at time t has:

- probability $\lambda h + o(h)$ of giving birth to a new individual;
- probability $\mu h + o(h)$ of dying,

where $\lambda, \mu > 0$.

In $(t, t+h]$, with population of size n at time t,

- the probability of a birth is $\lambda n h + o(h)$
- the probability of a death is $\mu n h + o(h)$

where $\lambda, \mu > 0$.

If the population size at time t is n...

	I-E process	I-E with linear birth
Birth rate		
Death rate		
$P(Birth \in (t, t+h])$		
$P(Death \in (t, t+h])$		
Q		

Transition probabilities - summary

The (general) birth and death process

Let $X(t)$ denote the size of a population at time $t \geqslant 0$. Suppose that $\{X(t), t \geqslant 0\}$ is a continuous-time Markov chain and that if $X(t) = n$ then

- the probability of a birth in $(t, t+h]$ is $\lambda_n h + o(h)$;
- the probability of a death in $(t, t+h]$ is $\mu_n h + o(h)$;
- the probability of 2 or more births or deaths in $(t, t+h]$ is $o(h)$.

Equilibrium distribution

The (general) birth and death process

Recall: this amounts to deciding whether the probability that the (general) birth and death process will be in state j at time t converges to a limiting value that is independent of the initial state.

We study

$$\boxed{\lim_{t\to\infty} p_{ij}(t)}$$

Do all birth-death processes have an equilibrium distribution?

Limiting result - Recall

Suppose that $\{X(t), t \geqslant 0\}$ is an irreducible continuous-time Markov chain.

(i) If there exists an invariant distribution $\underline{\pi}$ then it is unique and $p_{ij}(t) \to \pi_j$ as $t \to \infty$ for all $i, j \in S$.
   - That is $\underline{\pi}$ is the equilibrium distribution of the chain.

(ii) If there is no invariant distribution then $p_{ij}(t) \to 0$ as $t \to \infty$ for all $i, j \in S$.
   - No equilibrium distribution exists.

Are all birth-death processes irreducible?

No. When a process is not irreducible, cannot guarantee that an invariant distribution is an equilibrium distribution.

Which birth-death processes are irreducible?

	Linear birth	Immigration	Linear birth with Immigration
Linear death			
Emigration			
Linear death with emigration			

If irreducible - existence of invariant distribution implies existence of equilibrium distribution.
If NOT irreducible - cannot yet tell if equilibrium distribution exists...

Find an INVARIANT distribution

The (general) birth and death process

This gives

$$\boxed{-\lambda_0\,\pi_0 + \mu_1\,\pi_1 = 0}$$

$$\boxed{\lambda_{j-1}\,\pi_{j-1} - (\lambda_j + \mu_j)\,\pi_j + \mu_{j+1}\,\pi_{j+1} = 0, \quad j = 1, 2, \ldots.}$$

The first equation gives

$$\pi_1 = \frac{\lambda_0}{\mu_1}\,\pi_0.$$

Solving the second set of equations iteratively gives

$$\pi_j = \frac{\lambda_{j-1}\cdots\lambda_0}{\mu_j\cdots\mu_1}\,\pi_0. \qquad j = 2, 3, \ldots. \tag{1}$$

Find an INVARIANT distribution

The (general) birth and death process

$$\boxed{\pi_1 = \frac{\lambda_0}{\mu_1}\,\pi_0; \qquad \pi_j = \frac{\lambda_{j-1}\cdots\lambda_0}{\mu_j\cdots\mu_1}\,\pi_0. \qquad j = 2, 3, \ldots.}$$

If $\underline{\pi}$ is to be a probability distribution it must satisfy $\sum_{j=0}^{\infty} \pi_j = 1$:

$$\pi_0 \left(1 + \sum_{j=1}^{\infty} \frac{\lambda_{j-1}\cdots\lambda_0}{\mu_j\cdots\mu_1} \right) = 1,$$

$$\left(1 + \sum_{j=1}^{\infty} \frac{\lambda_{j-1}\cdots\lambda_0}{\mu_j\cdots\mu_1} \right)^{-1} = \pi_0. \tag{2}$$

Find an INVARIANT distribution

The (general) birth and death process

Remark

The probabilities $\pi_1, \pi_2, \ldots$ are given by (1).

These equations define a probability distribution if and only if the sum in (2) is finite. But is it an equilibrium distribution?*

If this sum is divergent an invariant distribution (and therefore an equilibrium distribution) cannot exist.

Note: If the state space S is finite, then the sum in (2) must be finite.

*Recall that if the process is irreducible, if the invariant distribution exists then so does the equilibrium (and the two distributions are the same).

Example - the linear birth death process

This process is not irreducible. Does it have an equilibrium?

Under certain circumstances, the linear birth death process does have an equilibrium distribution.

Which circumstances, though?

Example - the linear birth death process

Recall that for the linear birth death process, the generator matrix is:

$$Q = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \cdots \\ \mu & -(\lambda+\mu) & \lambda & 0 & 0 & \cdots \\ 0 & 2\mu & -2(\lambda+\mu) & 2\lambda & 0 & \cdots \\ 0 & 0 & 3\mu & -3(\lambda+\mu) & 3\lambda & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix}$$

Find the equilibrium distribution

When does the linear birth-death process have an equilibrium distribution?

Note:

- ⋆ 0 is an absorbing state.
  - If the process enters state 0 the birth rate is 0 and the population becomes extinct.
- ⋆ If we search for an invariant distribution, by solving $\underline{\pi}\, Q_{LBD} = \underline{0}$, we obtain a solution $\underline{\pi} = (1, 0, 0, \ldots)$.
  - Invariant distribution: population becomes extinct
- ⋆ This Markov chain is *not* irreducible
  - this is because state 0 does not communicate with any other state;
  - therefore cannot conclude that $\underline{\pi}$ is an equilibrium distribution.

Under what circumstances is $\underline{\pi}$ an equilibrium distribution?

Some remarks

Under what circumstances is $\underline{\pi}$ an equilibrium distribution?

Under which conditions does the population die out with probability 1?

- This is similar to examples we saw in the discrete-time part of the course.
- State 0 is positive recurrent.
- We will see that if $\mu > 0$ the probability that the chain is absorbed into the state 0 is positive.
- Therefore all the other states of the process are transient (return is not certain).

Probability of extinction

linear birth-death

To understand when the invariant distribution $\underline{\pi}$ is also an equilibrium distribution for a linear birth-death process...

Under what circumstances does the population become extinct with probability 1?

Under what circumstances is it certain that the chain will reach the ergodic state 0?

Elegant way of solving this is via **GENERATING FUNCTIONS.**

(If a process has a strictly positive immigration rate, can the population ever be considered extinct?)

Probability of extinction

linear birth-death

Let

$$G(s,t) = \mathrm{E}\left[s^{X(t)}\right] = \sum_{n=0}^{\infty} s^n \mathrm{P}(X(t) = n).$$

It can be shown that

$$G(s,t) = \left(\frac{\mu(1-s) - (\mu - \lambda s)\mathrm{e}^{-(\lambda-\mu)t}}{\lambda(1-s) - (\mu - \lambda s)\mathrm{e}^{-(\lambda-\mu)t}}\right)^{X(0)} \quad \lambda \neq \mu,$$

$$G(s,t) = \left(\frac{\lambda t(1-s) + s}{\lambda t(1-s) + 1}\right)^{X(0)} \quad \lambda = \mu.$$

Probability of extinction

linear birth-death

Now,

$$G(0,t) = \sum_{n=0}^{\infty} 0^n \, \mathrm{P}(X(t) = n) = \mathrm{P}(X(t) = 0),$$

so that

$$\boxed{G(0,t) = \begin{cases} \left(\dfrac{\mu - \mu\, \mathrm{e}^{-(\lambda-\mu)t}}{\lambda - \mu\, \mathrm{e}^{-(\lambda-\mu)t}} \right)^{X(0)} & \text{if } \lambda \neq \mu \\[2ex] \left(\frac{\lambda t}{1+\lambda t} \right)^{X(0)} & \text{if } \lambda = \mu \end{cases}}$$

is the probability that extinction occurs at or before time t.

Probability of extinction

linear birth-death

If we let $t \to \infty$ we get

$$P(\text{eventual extinction}) = \begin{cases} 1 & \text{if } \lambda \leqslant \mu \\ (\mu/\lambda)^{X(0)} & \text{if } \lambda > \mu. \end{cases}$$

If the birth rate does not exceed the death rate the population is certain to become extinct eventually.

So, $\underline{\pi} = (1, 0, 0, \ldots)$ is an equilibrium distribution if $\lambda \leqslant \mu$.

Probability of extinction

linear birth-death

It can be shown (using $G(s, t)$) that

$$\mathrm{E}[X(t)] = X(0)\,\mathrm{e}^{(\lambda-\mu)t}.$$

So, if $\lambda > \mu$ the expected population size increases to ∞ as $t \to \infty$.

However, extinction is still possible and so the population size either increases without limit or becomes 0.

No equilibrium distribution!

Exam 2008 - question B3

The population of an island evolves according to an immigration-death process. Specifically, the population evolves according to the following rules.

People immigrate onto the island in a Poisson process of rate α per year. Each person on the island at time t has probability $\mu h + o(h)$ of dying in the time interval $(t, t+h]$ years. There is no emigration from the island and there are no births.

Immigrations and deaths are independent and the probability of 2 or more events (immigrations or deaths) in $(t, t+h]$ is $o(h)$. The parameters α and μ are positive.

Let $N(t)$ denote the number of people on the island at time t.

Exam 2008 - question B3

(a) Write down the state space S and the generator matrix Q of transition rates of the continuous-time Markov chain $\{N(t), t \geqslant 0\}$.

[4]

(b) Let $\{Y_n, n = 0, 1, 2, \ldots\}$ be the embedded discrete-time jump chain of $\{N(t)\}$.

(i) Write down the transition matrix P of $\{Y_n\}$.

[2]

(ii) Let T_i be the holding time in state i, that is, the time spent in state i on a visit to state i. State the distribution of T_i, for $i = 0, 1, 2, \ldots$.

[2]

(c) Suppose that $N(0) = 0$. Calculate the expected time until there are 2 people on the island.

[4]

Exam 2008 - question B3

(d) You are given that, if $N(0) = 0$, the probability generating function $G(s,t) = \mathrm{E}\left(s^{N(t)}\right)$ of $N(t)$ is given by

$$G(s,t) = \exp\left\{\frac{\alpha}{\mu}\left(1 - \mathrm{e}^{-\mu t}\right)(s-1)\right\}.$$

(i) Use $G(s,t)$ to infer the distribution of $N(t) \mid N(0) = 0$.

[2]

(ii) Hence, or otherwise, infer the equilibrium distribution of $\{N(t)\}$.

[2]

Exam 2008 - question B3

(e) Suppose now that the island has an active volcano. When the volcano erupts the whole population of the island is killed, reducing the population size to zero. Eruptions occur in a Poisson process of rate ϕ per year. Otherwise the population of the island evolves according to the immigration-death process described above.
Using your answer to (d)(i), or otherwise, calculate the expected number of people killed by the next eruption to occur after time 0, given that $N(0) = 0$.

[4]

Exam 2011 - question B3

An isolated species of mammals evolves on an island according to a linear birth-death Markov chain. In particular, we assume that each mammal on the island at time t has probability $\mu h + o(h)$ of dying in the time interval $(t, t+h]$, and probability $\lambda h + o(h)$ of giving birth to a baby mammal during $(t, t+h]$, with all involved events being independent of each other.

Exam 2011 - question B3

(i) Write down the state space S and generator matrix Q of the Markov chain.

[3]

(ii) Given that at time t there are N mammals on the island, find the probability that there will be at least one death or birth in the time interval $(t, t+\delta]$, for $\delta > 0$.

[3]

(iii) At time $T > 0$ a comet falls on the island. As a result, the mammals cannot reproduce any more and the death rate for each mammal increases from μ to $a + \mu$ for some $a > 0$. If there were $N_T > 0$ living mammals at the time of impact T, what is the expected amount of time until the species becomes extinct?

[4]

Exam 2012 - question B3

Consider a birth-death Markov process $\{N(t)\}$, such that within the time period $(t, t+h]$ the probability of a death is $\mu h + o(h)$, whereas the probability of a birth is $\lambda h + o(h)$ if $N(t)$ is an even number $(0, 2, 4, \ldots)$ or $2\lambda h + o(h)$ if it is an odd number. All involved events are assumed to be independent of each other.

Exam 2012 - question B3

i. Write down the state space S and the generator matrix Q of the Markov chain.

[3]

ii. Assume that currently $N(0) = 1$. Name the distribution for the amount of time until the size of the population will change for the first time. Find the probability of the event that the above time will be less than 1 *and* that the change will be due to a birth.

[3]

iii. Solving $\pi Q = 0$ gives $\pi_{2j} = (\frac{\lambda}{\mu})^{2j} 2^j \pi_0$ and $\pi_{2j+1} = (\frac{\lambda}{\mu})^{2j+1} 2^j \pi_0$ for $j \geq 0$. State a necessary and sufficient condition on λ/μ so that the process $N(t)$ has an equilibrium distribution.

[4]