A Short Introduction to Continuous Time Stochastic Processes

Sections 4.1 and 4.2 of notes

STAT2003 / STAT3102

Contents

- Notation: stochastic processes;
- Little-o notation;
- The exponential distribution and its lack of memory property.

Notation

Discrete-time processes:

$$\{X_t, t \in T\}, \quad \text{where } T = \{0, 1, 2, \ldots\}.$$

Continuous-time processes:

$$\{X(t), t \in T\}, \quad \text{where } T = \mathbb{R}_+ \quad \text{(i.e. } t \geqslant 0\text{)}.$$

Continuous-time processes can change their value/state ('jump') at any instant of time.

Example:
The number of calls received at a call centre by time t.

The difficulty with continuous time processes

No 'smallest' time unit (compare with the discrete time case!).

Notation for the transition probabilities must clearly show the amount of time taken to reach one state from another:

$$p_{ij}(t) = \mathrm{P}(X(t) = j \mid X(0) = i)$$

Store these in a *transition matrix* $\mathrm{P}(t)$. If $S = \{0, 1, 2, \ldots\}$ this will be of the form

$$P(t) = \begin{pmatrix} p_{00}(t) & p_{01}(t) & p_{02}(t) & \cdots \\ p_{10}(t) & p_{11}(t) & p_{12}(t) & \cdots \\ p_{20}(t) & p_{21}(t) & p_{22}(t) & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix}$$

The difficulty with continuous time processes

Transition probabilities formed the basis for studying the behaviour of discrete time Markov processes.

Since transition probabilities are cumbersome in continuous time, would like to dispose of the need to work with them.

...which will require some additional tools...

Little-o notation

Used heavily for continuous time stochastic processes.

Definition

Let $f(x)$ be a function. We define f to be $o(h)$ if

$$\lim_{h\to 0} \frac{f(h)}{h} = 0,$$

That is:

$f(h)$ tends to zero faster than h tends to zero

as $h \to 0$, $f(h) \to 0$ faster than h does.

Example 3.1

- $f(x) = x^2$.

$$\frac{f(h)}{h} = \frac{h^2}{h} = h \to 0 \text{ as } h \to 0.$$

  So x^2 is $o(h)$.

- $f(x) = \sqrt{x}$.

$$\frac{f(h)}{h} = \frac{\sqrt{h}}{h} = \frac{1}{\sqrt{h}} \to \infty \text{ as } h \to 0.$$

  So $\sqrt{x}$ is not $o(h)$.

Exercise (STAT2003 exam, 2013)

(a) For a function, $f(x)$, define what it means for f to be $o(h)$.

(b) Are the following functions $o(h)$ or not:

- (i) $f(x) = x^3$
- (ii) $f(x) = x^{1/3}$
- (iii) $f(x) = x^3 + x^{1/3}$

Memoryless property

A random variable X is said to be memoryless if, for all $s, t \geqslant 0$,

$$P(X > s + t | X > t) = P(X > s)$$

This condition is equivalent to (easy homework!)

$$P(X > s + t) = P(X > s)P(X > t)$$

Question: which distributions are memoryless?

Is the exponential distribution memoryless?

Suppose that $X \sim \exp(\lambda)$. Then for $h > 0$,

$$
\begin{aligned}
\mathrm{P}(X \in (t, t+h] \mid X > t) &= \frac{\mathrm{P}(X \in (t, t+h], X > t)}{\mathrm{P}(X > t)} \\
&= \frac{\mathrm{P}(X \in (t, t+h])}{\mathrm{P}(X > t)} \\
&= \frac{\mathrm{P}(X > t) - \mathrm{P}(X > t+h)}{\mathrm{P}(X > t)} \\
&= \frac{\mathrm{e}^{-\lambda t} - \mathrm{e}^{-\lambda(t+h)}}{\mathrm{e}^{-\lambda t}} \\
&= 1 - \mathrm{e}^{-\lambda h} \\
&= 1 - \left(1 - \lambda h + \frac{(\lambda h)^2}{2!} - \frac{(\lambda h)^3}{3!} + \cdots\right) \\
&= \lambda h + o(h).
\end{aligned}
$$

The lack-of-memory property

We have shown that if $X \sim \exp(\lambda)$,

$$P(X \in (t, t+h] \mid X > t) = 1 - e^{-\lambda h}, \quad \text{for } h > 0.$$

or

$$P(X - t \in (0, h] \mid X > t) = 1 - e^{-\lambda h}, \quad \text{for } h > 0.$$

The right hand side is the cdf of an $\exp(\lambda)$ distribution:

$$X \sim \exp(\lambda) \quad \text{and} \quad (X - t \mid X > t) \sim \exp(\lambda).$$

Thus, if the time until an event has an $\exp(\lambda)$ distribution and we wait t minutes without an event, the further time we have to wait still has an $\exp(\lambda)$ distribution.

The exponential distibution

Simplifying assumption for many continuous-time processes:

assume that certain random variables are exponentially distributed.

Pros:

- Easy to work with;
- Often a good approximation to reality;
- Is *memoryless*

Memoryless property

If the waiting time to be served, X, is exponential with mean 10 minutes:

- the probability of waiting more than 15 minutes to be served is
$$P(X > 15) \approx 0.220$$
- given that you've been waiting more than 10 minutes, what is the probability you wait more than 15 minutes in total?

$$P(X > 15 | X > 10) = P(X > 5) \approx 0.604$$

The exponential distribution does not remember that you've already waited more than 10 minutes

The lack-of-memory property in action

Question:

- Service times at a bank are exponentially and independently distributed with parameter μ.
- One queue feeds two clerks.
- When you arrive, both clerks are busy but the queue is empty.
- You will be served next.

What is the probability that, of the three customers present, you will be last to leave?

The lack-of-memory property in action

Answer:

- When the first customer leaves you are served.
- Your service time T_1 has an $\exp(\mu)$ distribution.
- Lack-of-memory property of the exponential distribution means further service time T_2 of the other customer is also exponential.
- T_1 and T_2 are independent.

$$
\begin{aligned}
\mathrm{P}(\text{you are last to leave}) &= \mathrm{P}(T_1 > T_2) \\
&= \int_0^\infty P(T_1 > T_2 \mid T_2 = t) f_{T_2}(t)\,\mathrm{d}t \\
&= \int_0^\infty \mathrm{P}(T_1 > t)\, f_{T_2}(t)\,\mathrm{d}t \\
&= \int_0^\infty \mathrm{e}^{-\mu t}\ \mu\,\mathrm{e}^{-\mu t}\,\mathrm{d}t = \frac{\mu}{2\mu} = \frac{1}{2}.
\end{aligned}
$$