Discrete Time Markov Chains
Part 3

Chapter 3 of notes (section 3.3 of notes)

STAT2003 / STAT3102

Contents

Limiting behaviour (how will the Markov chain behave in the long run?);

- Invariant distributions
- The equilibrium distribution
- When does the equilibrium distribution exist?

Long-run behaviour

The general idea

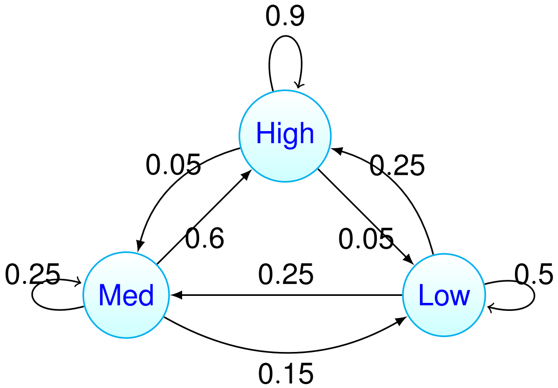
This is the crux of the first half of the course: understanding a Markov chain's behaviour in the long run (*limiting* behaviour).

> Does the distribution of states, $\underline{p}^{(n)}$, settle down (i.e. converge) as $n \to \infty$. If it does, what is this limit?

To study this, need to classify the Markov chain in terms of:

(a) Irreducible classes;
(b) Transience and positive/null recurrence;
(c) Periodicity;

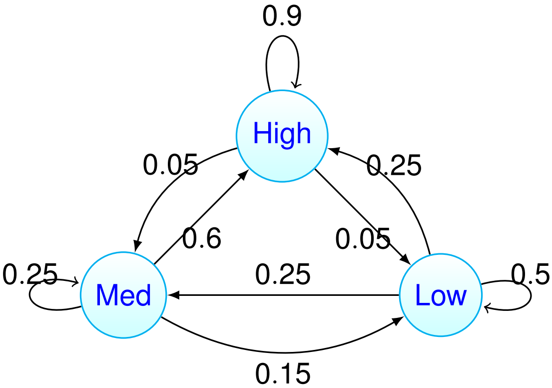
HIV progression



How does $P^{(n)} = P^n$ change as n increases?

$$P^2 = \begin{bmatrix} 0.85 & 0.07 & 0.0775 \\ 0.7275 & 0.13 & 0.1425 \\ 0.5 & 0.2 & 0.3 \end{bmatrix}$$

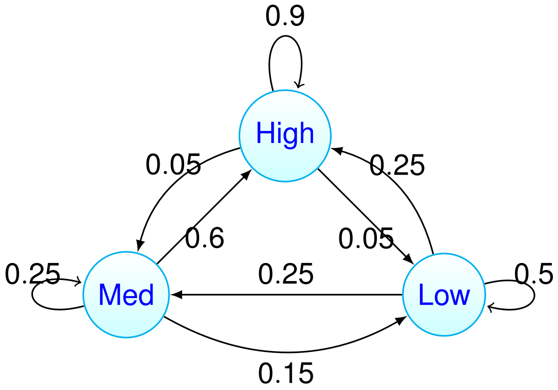
HIV progression - continued



How does $P^{(n)} = P^n$ change as n increases?

$$P^3 = \begin{bmatrix} 0.83 & 0.08 & 0.09 \\ 0.77 & 0.10 & 0.13 \\ 0.65 & 0.15 & 0.20 \end{bmatrix}$$

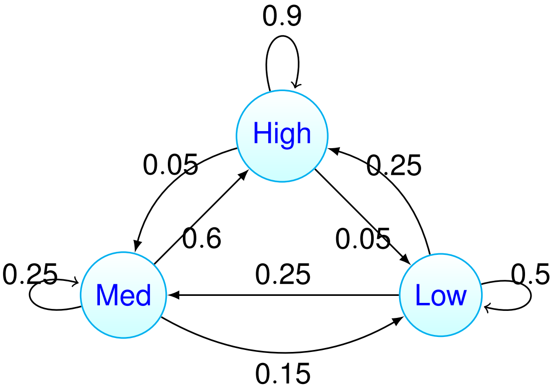
HIV progression - continued



How does $P^{(n)} = P^n$ change as n increases?

$$P^6 = \begin{bmatrix} 0.81 & 0.09 & 0.10 \\ 0.80 & 0.09 & 0.11 \\ 0.78 & 0.10 & 0.12 \end{bmatrix}$$

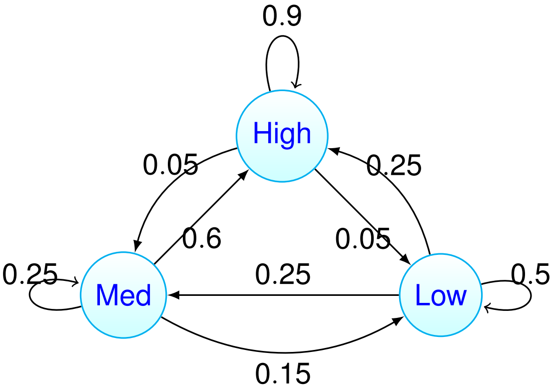
HIV progression - continued



How does $P^{(n)} = P^n$ change as n increases?

$$P^{10} = \begin{bmatrix} 0.80 & 0.09 & 0.11 \\ 0.80 & 0.09 & 0.11 \\ 0.80 & 0.09 & 0.11 \end{bmatrix}$$

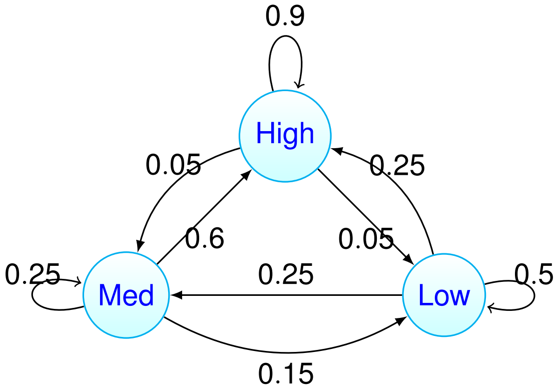
HIV progression - continued



How does $P^{(n)} = P^n$ change as n increases?

$$P^{20} = \begin{bmatrix} 0.80 & 0.09 & 0.11 \\ 0.80 & 0.09 & 0.11 \\ 0.80 & 0.09 & 0.11 \end{bmatrix}$$

HIV progression - continued



How does $P^{(n)} = P^n$ change as n increases?

$$P^{100} = \begin{bmatrix} 0.80 & 0.09 & 0.11 \\ 0.80 & 0.09 & 0.11 \\ 0.80 & 0.09 & 0.11 \end{bmatrix}$$

HIV progression - continued

How does $P^{(n)} = P^n$ change as n increases?

$$P^{100} = \begin{bmatrix} 0.80 & 0.09 & 0.11 \\ 0.80 & 0.09 & 0.11 \\ 0.80 & 0.09 & 0.11 \end{bmatrix}$$

So, taking $n = 100$,

$$\mathrm{P}(X_{100} = \text{High} \mid X_0 = \text{High}) = 0.80, \qquad [\underline{p}^{(0)} = (1,0,0)]$$

$$\mathrm{P}(X_{100} = \text{High} \mid X_0 = \text{Med}) = 0.80. \qquad [\underline{p}^{(0)} = (0,1,0)]$$

$$\mathrm{P}(X_{100} = \text{High} \mid X_0 = \text{Low}) = 0.80. \qquad [\underline{p}^{(0)} = (0,0,1)]$$

HIV progression - continued

Now suppose that $\underline{p}^{(0)} = (0.3, 0.2, 0.5)$. We have:

$$\underline{p}^{(1)} = \underline{p}^{(0)}P = (0.52, 0.19, 0.29)$$

$$\underline{p}^{(2)} = \underline{p}^{(1)}P = (0.65, 0.15, 0.20)$$

$$\vdots$$

$$\underline{p}^{(100)} = \underline{p}^{(0)}P^{100} = (0.80, 0.09, 0.11)$$

Now suppose that $\underline{p}^{(0)} = (0, 0.5, 0.5)$:

$$\underline{p}^{(100)} = (0.80, 0.09, 0.11).$$

HIV progression - continued

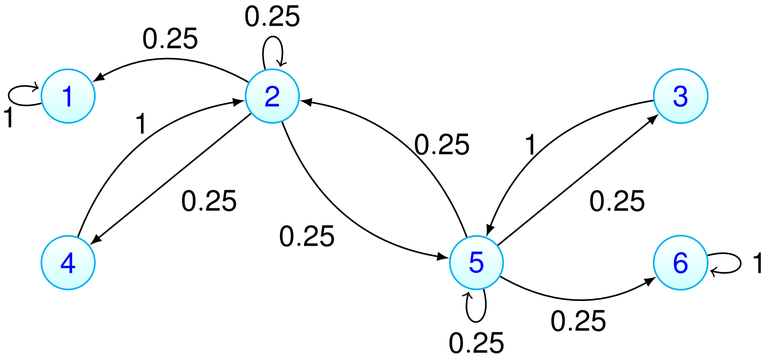
In summary:

$\underline{p}^{(0)}$	n=1	...	n=100
(1, 0, 0)	(0.90, 0.05, 0.05)	...	(0.80, 0.09, 0.11)
(0, 1, 0)	(0.60, 0.25, 0.15)	...	(0.80, 0.09, 0.11)
(0, 0, 1)	(0.25, 0.25, 0.50)	...	(0.80, 0.09, 0.11)
(0.3, 0.2, 0.5)	(0.52, 0.19, 0.29)	...	(0.80, 0.09, 0.11)
(0, 0.5, 0.5)	(0.43, 0.25, 0.32)	...	(0.80, 0.09, 0.11)

What do you conclude about the long-run behaviour of this Markov process?

Inheritance of X-linked genes

What about the long-term behaviour in this case?



Inheritance of X-linked genes

$$P = \begin{pmatrix} 1 & - & - & - & - & - \\ 1/4 & 1/4 & - & 1/4 & 1/4 & - \\ - & - & - & - & 1 & - \\ - & 1 & - & - & - & - \\ - & 1/4 & 1/4 & - & 1/4 & 1/4 \\ - & - & - & - & - & 1 \end{pmatrix}$$

$$\lim_{n\to\infty} P^{(n)} = \begin{pmatrix} 1 & - & - & - & - & - \\ 2/3 & - & - & - & - & 1/3 \\ 1/3 & - & - & - & - & 2/3 \\ 2/3 & - & - & - & - & 1/3 \\ 1/3 & - & - & - & - & 2/3 \\ - & - & - & - & - & 1 \end{pmatrix}$$

Inheritance of X-linked genes

In summary:

Start distribution of chain	Limit of distribution of states
(1, 0, 0, 0, 0, 0)	(1, 0, 0, 0, 0, 0)
(0, 1, 0, 0, 0, 0)	(2/3, 0, 0, 0, 0, 1/3)
(0, 0, 1, 0, 0, 0)	(1/3, 0, 0, 0, 0, 2/3)
(0, 0, 0, 1, 0, 0)	(2/3, 0, 0, 0, 0, 1/3)
(0, 0, 0, 0, 1, 0)	(1/3, 0, 0, 0, 0, 2/3)
(0, 0, 0, 0, 0, 1)	(0, 0, 0, 0, 0, 1)
(0.1, 0.2, 0.2, 0.2, 0.2, 0.1)	(1/2, 0, 0, 0, 0, 1/2)

What do you conclude about the long-run behaviour of this Markov process?

Invariant distributions

Long run behaviour in **specific** cases

The probability row vector $\underline{\pi}$ is an invariant distribution of a discrete-time process $\{X_t, t \in T\}$ if

$$\boxed{\underline{\pi} = \underline{\pi}\, P.}$$

Suppose that such a vector exists and that $\underline{p}^{(0)} = \underline{\pi}$. Then:

$$\begin{aligned} \underline{p}^{(1)} &= \underline{p}^{(0)}\, P = \underline{\pi}\, P = \underline{\pi}. \\ \underline{p}^{(2)} &= \underline{p}^{(1)}\, P = \underline{\pi}\, P = \underline{\pi}, \quad \text{etc.} \end{aligned}$$

In fact

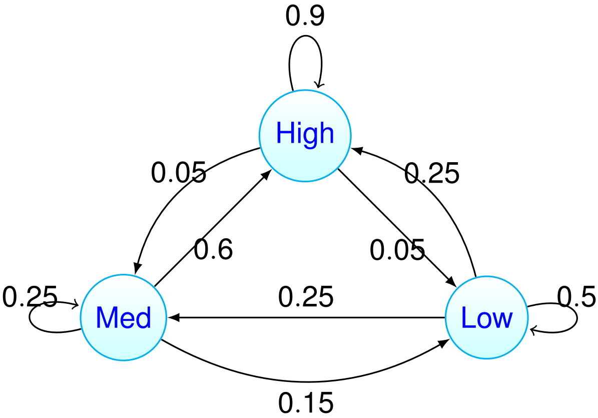
$$\underline{p}^{(n)} = \underline{\pi}, \quad \text{for } n = 1, 2, 3, \ldots$$

Invariant distributions

So, if $\underline{p}^{(0)} = \underline{\pi}$ then $\underline{p}^{(n)} = \underline{\pi}$ for all $n \in \{1, 2, 3, \ldots\}$.

That is, if we **start** in $\underline{\pi}$ we **stay** there, hence the name 'invariant'.

Invariant distributions



Exercise:
Find the invariant distribution for the Markov chain in the HIV progression example.

Invariant distributions

Find the invariant distribution, $\underline{\pi}$ by solving:

$$\underline{\pi} = \underline{\pi} P = \underline{\pi} \begin{bmatrix} 0.9 & 0.05 & 0.05 \\ 0.6 & 0.25 & 0.15 \\ 0.25 & 0.25 & 0.5 \end{bmatrix}$$

Gives three equations:

$$\begin{aligned} \pi_1 &= 0.9\pi_1 + 0.6\pi_2 + 0.25\pi_3 \\ \pi_2 &= 0.05\pi_1 + 0.25\pi_2 + 0.25\pi_3 \\ \pi_3 &= 0.05\pi_1 + 0.15\pi_2 + 0.5\pi_3 \end{aligned}$$

One equation is redundant! Also use $\pi_1 + \pi_2 + \pi_3 = 1$, then:
$\pi_1 = 0.8035714, \pi_2 = 0.08928571, \pi_3 = 0.1071429$.
That is: $\underline{\pi} \simeq (0.80, 0.09, 0.11)$.
Compare with earlier findings on this example!

Is there always an invariant distribution?

- If the state space S is finite, an invariant distribution always exists.
- If S is infinite then an invariant distribution need not exist.
  - See exercises 3, question 3(b)
- If an invariant distribution does exist, it may not be unique.
  - There could be more than one vector $\underline{\pi}$ that satisfies $\underline{\pi} = \underline{\pi}P$.

The equilibrium distribution

Definition

A probability row vector $\underline{\pi} = \{\pi_j, \, j \in S\}$ is an equilibrium distribution for the discrete-time Markov chain $\{X_n\}$ if

$$\underline{p}^{(n)} \longrightarrow \underline{\pi} \quad \text{as} \quad n \longrightarrow \infty,$$

independently of the initial distribution $\underline{p}^{(0)}$.

That is,

$$\pi_j = \lim_{n\to\infty} p_{ij}^{(n)},$$

for all $i, j \in S$.

Long-run behaviour of a Markov chain

Let $\underline{\pi}_e = \lim_{n\to\infty} \underline{p}^{(n)}$.

- $\underline{\pi}_e$ is the **equilibrium distribution**.
- Since $\underline{p}^{(n)} = \underline{p}^{(n-1)}P$, then as $n \to \infty$,

$$\underline{\pi}_e = \underline{\pi}_e P,$$

- So $\underline{\pi}_e$ is an **invariant distribution** and we shall now use $\underline{\pi}$.

Solving $\underline{\pi} = \underline{\pi}P$ is usually easier than calculating $\lim\limits_{n\to\infty} \underline{p}^{(n)}$.

Invariant and equilibrium distributions

Concept of an invariant distribution and an equilibrium distribution are closely related but they are **not the same thing.**

- An equilibrium distribution must also be an invariant distribution.
- An invariant distribution is not necessarily an equilibrium distribution.

$$\mathit{EQUILIBRIUM} \quad \begin{array}{c} \Longrightarrow \\ \not\Longleftarrow \end{array} \quad \mathit{INVARIANT}$$

Invariant and equilibrium distributions

What exactly is the difference?

- INVARIANT distribution $\underline{\pi}$:
  - If the chain (starts in / gets to) $\underline{\pi}$ it stays there forever.
  - Distribution of chain remains the same.
- EQUILIBRIUM distribution $\underline{\pi}$:
  - If the chain is run for long enough (i.e. forever) its probabilistic behaviour settles down to that of $\underline{\pi}$,
  - This is **regardless of the state in which the chain starts**.

Invariant and equilibrium distributions

Notice:

- No invariant distribution $\Rightarrow$ no equilibrium distribution.
- Exactly 1 invariant distribution $\Rightarrow$ there may or may not be an equilibrium distribution.
- Two or more invariant distributions $\Rightarrow$ no equilibrium distribution.

Invariant and equilibrium distributions

IF we know that the equilibrium distribution exists, we can find it by solving $\underline{\pi} = \underline{\pi} P$ (i.e. find the invariant distribution).

This only applies when we know that an equilibrium distribution exists* - then the equilibrium is the (unique) invariant distribution.

* - we will work towards a general theory that tells us exactly which Markov chains have an equilibrium distribution.

Do all Markov chains have equilibrium distributions?

- What will happen to the Markov chains below in the 'long-run'?
- Will this depend on the state in which the chain started?

(a)

$$S = \{0, 1\}, \qquad P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

(b)

$$S = \{0, 1\}, \qquad P = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Later: how to determine whether a given Markov chain has an equilibrium distribution.

What are we aiming for?

Ultimate aim: identify which Markov chains have an equilibrium distribution.

We will need the following ideas (already covered) in order to do this:

- irreducibility;
- recurrence (including transience);

Summary so far: Invariant and equilibrium distributions

- An **invariant distribution** satisfies

$$\underline{\pi} = \underline{\pi} P.$$

i.e. 'If the chain starts in $\underline{\pi}$ (or gets to $\underline{\pi}$) it stays in $\underline{\pi}$'.

- An **equilibrium distribution** satisfies

$$\underline{\pi} = \lim_{n \to \infty} \underline{p}^{(n)}.$$

i.e. 'If the chain is run for long enough then the probability distribution of X_n is (approximately) given by $\underline{\pi}$ (regardless of $\underline{p}^{(0)}$)'.

Ergodicity

Definition

An *ergodic* state is one that is both positive recurrent and aperiodic.

Find and classify the irreducible classes of the following Markov chain.

$$S = \{0, 1, 2, 3, 4\}, \qquad P = \begin{pmatrix} 1/2 & 1/2 & 0 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1/4 & 0 & 1/4 & 0 & 1/2 \end{pmatrix}.$$

Equilibrium

When does the equilibrium distribution exist?

Answer depends on *ergodicity*.

Ergodic Markov chain

Proof will be omitted

Let $\{X_n\}$ be an irreducible ergodic Markov chain. Without loss of generality we assume that $S = \{0, 1, 2, \dots\}$. Then

(a) there is a unique probability row vector $\underline{\pi} = (\underline{\pi}_0, \pi_1, \pi_2, \dots)$ satisfying

$$\underline{\pi} = \underline{\pi} P ,$$

(b) $p_{ij}^{(n)} \to \pi_j$ as $n \to \infty$, for all $i, j \in S$,

(c) $\underline{\pi}$ satisfies

$$\pi_j = \frac{1}{\mu_j}, \quad \text{for all } j \in S ,$$

where μ_j is the mean recurrence time of state j.

Remarks

There is a unique probability row vector $\underline{\pi}$ s.t. $\underline{\pi} = \underline{\pi} P$

- This can be strengthened to

  An irreducible, aperiodic MC has an invariant distribution

  $\Updownarrow$

  The chain is positive recurrent

- $\underline{\pi}$ is then unique and satisfies $\pi_j = 1/\mu_j$, $j \in S$.

Provides alternative way to show that a Markov chain is positive recurrent: if the chain is

1. irreducible and aperiodic,
2. has an invariant distribution (i.e., $\exists \underline{\pi}$ s.t. $\underline{\pi} = \underline{\pi} P$)

then it must be positive recurrent.

Remarks

$$\boxed{p_{ij}^{(n)} \;\to\; \pi_j \quad \text{as } n \to \infty, \text{ for all } i, j \in S\,,}$$

1. The n-step transition matrix, $P^{(n)}$,

$$\lim_{n\to\infty} P^{(n)} = \begin{pmatrix} \pi_0 & \pi_1 & \pi_2 & \cdots \\ \pi_0 & \pi_1 & \pi_2 & \cdots \\ \pi_0 & \pi_1 & \pi_2 & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix},$$

2. Implies that $\underline{\pi}$ is the equilibrium distribution of the chain (since $\underline{p}^{(n)} = \underline{p}^{(0)} P^{(n)}$ tends to $\underline{\pi}$ as $n \to \infty$).

Remarks

The following interpretations of $\underline{\pi}$ are important.

(i) π_j is the limiting probability that the chain is in state j at time n.

(ii) It can be shown that π_j is equal to the long-run proportion of time that the chain spends in state j.

Examples

What can you say about the following Markov chains?

(a) $S = \{ R, F \}, \quad P = \begin{pmatrix} 0.6 & 0.4 \\ 0.5 & 0.5 \end{pmatrix}.$

(b) $S = \{0, 1\}, \quad P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$

(c) $S = \{0, 1\}, \quad P = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$

Are we there yet?

Can we tell when a Markov chain will have an equilibrium distribution?

We only know that **if** we have an irreducible and ergodic [positive recurrent, aperiodic] Markov chain, then an equilibrium distribution exists.

Are we there yet?

Explore other possibilities...

1. What happens if we assume irreducibility and positive recurrence, but not aperiodicity?
2. What happens if we assume that all states are null recurrent or transient?
3. What happens when the chain is not irreducible, but instead has two or more closed classes?
4. What happens when the chain has a finite number of transient states, and a single closed class?

Remove aperiodicity

An irreducible, positive recurrent, Markov chain which is not aperiodic ($d > 1$) **does not have** an equilibrium distribution.

Let state j have mean recurrence time μ_j. Construct a new process $\{Y_n\}$ with $Y_n = X_{nd}$.

- $\{Y_n, n=0,1,2,\ldots\}$ is an ergodic Markov chain
- In this new chain state j has mean recurrence time μ_j/d.

From the Main Limit Theorem it follows that

$$\mathrm{P}(Y_n=j \mid Y_0=j) \to \frac{d}{\mu_j} \text{ as } n\to\infty,$$

$$\textit{or} \quad \mathrm{P}(X_{nd}=j \mid X_0=j) = p_{jj}^{(nd)} \to \frac{d}{\mu_j} \text{ as } n\to\infty.$$

However, $p_{jj}^{(n)} = 0$ for $n \notin \{0, d, 2d, 3d, \ldots\}$.

Hence $\{X_n\}$ has no equilibrium distribution.

All states null recurrent / transient.

A Markov chain with only null recurrent or transient states **does not have** an equilibrium distribution.

Suppose that a Markov chain consists only of transient and/or null-recurrent states. Then all states have $\mu_j = \infty$. It follows that for all i, j

$$p_{ij}^{(n)} \rightarrow \frac{1}{\mu_j} = 0 \text{ as } n \rightarrow \infty,$$

that is, all limits exist but are zero. This limit is not a proper probability distribution so no equilibrium distribution exists.

More than one closed class

A Markov chain with more than one closed class **does not have** an equilibrium distribution.

Suppose that a Markov chain contains 2 or more closed classes, $C_1, C_2, \ldots$.
If, for example, class C_1 is ergodic then for all $j \in C_1$

$$\begin{cases} p_{ij}^{(n)} \to \frac{1}{\mu_j}, \text{ as } n \to \infty, & \text{if } i \in C_1 \\ p_{ij}^{(n)} = 0, \text{ for all } n, & \text{if } i \in C_2. \end{cases}$$

Hence there is no equilibrium distribution (the limit depends on the initial state).

Single closed class

A Markov chain with finite number of transient states and one closed class **may have** an equilibrium distribution.

Chain eventually absorbed into the single closed class, C.

C is ergodic then there is an equilibrium distribution:

$$\pi_j = \frac{1}{\mu_j} \text{ if } j \in C.$$

For a transient state $\pi_j = 0$ and $\mu_j = \infty$, so

$$\pi_j = \frac{1}{\mu_j} \quad \text{for all } j.$$

If C is not ergodic then no equilibrium distribution exists.

Summary

When does an equilibrium distribution exist?

(a) More than 1 closed class $\Rightarrow$ no equilibrium distribution.
- The long-term behaviour of the chain depends on $\underline{p}^{(0)}$.

(b) No closed classes (i.e. all states are transient) $\Rightarrow$ no equilibrium distribution.
- Since $\underline{p}^{(n)} \to \underline{0}$ as $n \to \infty$.

(c) Exactly one closed class C.
Is it certain that the Markov chain will eventually be absorbed into C?
- If “no” then no equilibrium distribution exists (long-term behaviour depends on $\underline{p}^{(0)}$).
- If “yes” then an equilibrium distribution exists if and only if C is ergodic.

Summary

Can we now tell when a Markov chain will have an equilibrium distribution?

Corollary

An equilibrium distribution exists if and only if there is an ergodic class C and the Markov chain is certain to be absorbed into C eventually, wherever it starts.

Don't forget...

Equilibrium versus invariant distribution

- It is possible for a Markov chain to **have an invariant distribution but not have an equilibrium distribution.**
- It is possible for a Markov chain to have **more than one invariant distribution but not have an equilibrium distribution.**
- It is possible for a Markov chain to have **neither an invariant distribution nor an equilibrium distribution.**

Exercise - does an equilibrium distribution exist?

Explain if not, and find it if it does

$$S = \{0, 1, 2, 3\}, \qquad P = \begin{pmatrix} 1/3 & 2/3 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1/4 & 1/4 & 1/4 & 1/4 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

$$S = \{0, 1, 2, 3\}, \qquad P = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1/4 & 1/4 & 1/4 & 1/4 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Challenge

Three cards, labelled A,B and C, are placed in a row.

- The left or the right hand card is chosen randomly.
- This card is then placed between the other two.
- Process is repeated; successive choices are independent.

1. Find the transition matrix for the six-state Markov chain of successive orders $X_1, X_2, \ldots$ of the cards.
2. Is the chain irreducible, aperiodic?
3. Find the two-step transition matrix.
4. If the initial order is ABC, find approximately the probability that after $2n$ changes, where n is large, the order is ABC.