STAT2003/STAT3102 In Course Assessment

Friday 6th March 2015, 3pm

Answer ALL questions. Time allowed 45 minutes.

Overall, this ICA was very well answered, with many of you getting high grades. However, many marks were lost due to carelessness - either in not reading the question properly or in using a random variable without defining it first. Feedback on each part question is given below.

1. **(28 marks)**

   Consider the Markov chain $\{X_n : n = 0, 1, 2 \ldots\}$ with state space $S = \{1, 2, 3, 4, 5, 6, 7\}$ and transition matrix

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1/3 & 0 & 2/3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2/3 & 1/3 & 0 \\ 0 & 2/3 & 0 & 1/3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/3 & 2/3 \\ 0 & 0 & 0 & 0 & 0 & 1/4 & 3/4 \end{pmatrix}.$$

   (a) Find the irreducible classes of intercommunicating states of $\{X_n\}$ and classify them in terms of positive or null recurrence, transience, periodicity and ergodicity. Also state whether each class is closed or not. [8]

   $\{1\}$ - transient, aperiodic (by definition), not ergodic, not closed.
   $\{2, 3, 4, 5\}$ - transient, period 2, not ergodic, not closed.
   $\{6, 7\}$ - positive recurrent, aperiodic, ergodic, closed.

   Some of you lost marks, disappointingly, due to not getting the right irreducible classes - a common mistake was to split $\{2, 3, 4, 5\}$ into two classes. Another common mistake was to say that state 1 had period 0 or ∞. In the notes, we explicitly define a state which cannot communicate with itself to have period 1. I also accepted those who classified the period of $\{1\}$ as undefined (some texts define it like this).

   (b) State $\underline{p}^{(3)}$ given that $X_0 = 1$. [2]

   No calculation required: $(0, 1/3, 0, 2/3, 0, 0, 0)$

   Most of you succeeded in getting this right, but some were careless with their notation. The consequence of this was that they in fact calculated $\underline{p}^{(4)}$ instead of $\underline{p}^{(3)}$. Note that no calculation was required here - you can deduce $\underline{p}^{(3)}$ from the state diagram directly.

(c) Calculate $P(X_5 = 2|X_0 = 1)$. [2]

Only two possible paths: 1,2,3,2,3,2 or 1,2,3,4,5,2, each with probability 1/3 and 2/3 respectively.

$$P(X_5 = 2|X_0 = 1) = 1/9 + 8/27 = 11/27$$

Again, carelessness here cost marks! Some of you didn't spot the second possible path.

(d) Use first step decomposition to find the probability that the chain ever visits state 5, given that it starts in state 1. [8]

Since the chain starts in state 1, it must travel via states 2 and 3. Therefore equivalent to requiring the probability that the chain ever visits state 5, given that it starts in state 3. Use first step decomposition yields that the probability is 2/3.

The majority did very well indeed here.

(e) State why this Markov chain has an equilibrium distribution. [2]

Finite number of transient states and one closed class (which is ergodic). Therefore an equilibrium distribution exists.

I awarded marks for saying 'guaranteed to enter the single closed class' or similar, but something to this effect must have been stated - many of you omitted this!

(f) What proportion of time do we expect the chain to spend in state 6 in the long run? [4]

Look for the equilibrium distribution, in particular we want π_6. Must have $\underline{\pi} = (0, 0, 0, 0, 0, \pi_6, \pi_7)$ as states 1, 2, 3, 4, 5 are transient, and $\pi_6 + \pi_7 = 1$. Solving yields $\pi_6 = 3/11$.

Several of you got confused here. I did not ask for an expected time, but the *proportion* of time - a good sanity check is to make sure that your answer to this question is between 0 and 1. Also spot the 'long-run' which should remind you of the equilibrium distribution. Another common mistake was to use the geometric distribution. This is wrong on two counts - this doesn't give the long run proportion and in any case only applies to transient states.

(g) What is the expected amount of time until the chain first returns to state 6, given that it started in state 6? [2]

$E[T_{66}] = 11/3$.

Look at the main limit theorem in section 2.9 of the course notes to see that we needed only to invert π_6 from part (f)- in particular, no need for first-step decomposition! The fact that there were only two marks for this question should have implied that a simple method exists. Also, another sanity check - if you got an answer less than 1, then something has definitely gone wrong. You need at least one step to return to state 6, so this expectation should be at least 1! As in part (f), the geometric distribution cannot be used here as this applies only to transient states, and, in any case, computes the number of hits on a state rather than return time.

2. **(22 marks)**

Consider the following Markov chain with states $\{1,2,3,4,5\}$ defined by

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1/4 & 1/2 & 1/4 & 0 & 0 \\ 0 & 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 1/6 & 5/6 \\ 0 & 0 & 0 & 1/2 & 1/2 \end{pmatrix}.$$

(a) Find all closed classes of this Markov chain. [2]

$\{1\}$ and $\{4,5\}$

Some of you didn't read the question properly and stated all the irreducible classes. Be careful!

(b) Does this Markov chain have an equilibrium distribution? Why? [2]

No - more than one closed class.

I'm pleased that the vast majority answered this question correctly.

(c) Calculate the expected time until the Markov chain is absorbed into a closed class, given the start distribution $\underline{p}^{(0)} = (0,1,0,0,0)$. [8]

Apply first step decomposition (twice) to yield the expected time is 10/3

Marks were lost here for two reasons: failing to define the random variable used, and again carelessness in using probabilities/ adding steps/ solving the last equation. Make sure that you know exactly when to add 1 to an expectation (some of you seemed to do this at random which is concerning). If you're not happy with doing this, I suggest you come and see me as soon as possible.

(d) Now consider a modified process, Z_n, where $Z_n = |X_n - 3|$ and let $Z_0 = 0$. Is Z_n a Markov chain? If it is, explain why. Otherwise, find an example to show that it cannot be a Markov chain. [10]

[**HINT**: it might be useful to consider the long term behaviour of X_n, and deduce how Z_n must behave in the long run.]

As a couple of you spotted, I wasn't crystal clear on what X_n was here. The upside to this is that I've been **very** generous in awarding marks...

This is not a Markov chain as the long-term behaviour is either it is absorbed in $\{2\}$ or absorbed in $\{1,2\}$ which violates assumptions of a Markov process. Saying that was enough, or alternatively you could concoct any counter example you want, e.g.

$$P(Z_{n+1} = 2 | Z_n = 2, X_n \text{ absorbed in } \{1\}) = 1$$
$$P(Z_{n+1} = 2 | Z_n = 2, X_n \text{ absorbed in } \{4,5\}) = \frac{1}{2}$$

Lots of you spotted the long term behaviour of X, saying that it ends up in $\{1\}$ or $\{4,5\}$, then said that if it ends up in $\{1\}$ then Z is a Markov chain taking the value 2 always, and if it ends up in $\{4,5\}$ then Z is a Markov chain which lives in states 1 and 2. Then you concluded that this is a Markov chain... Your reasoning was correct, but conclusion wrong, since Z depends on what happened in the past (i.e. into which class X was absorbed, and hence whether Z is in state 2 forever more or moves freely between states 1 and 2).

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