Discrete Time Markov Chains
Part 2

Chapter 3 of notes (section 3.2)

STAT2003 / STAT3102

Contents

- Communicating classes;
- Recurrence and transience;
  - Positive recurrence and null recurrence.

Further questions

How do we study:

- long-run probabilistic behaviour of the chain?
- the probability that some future event of interest happens?

Our tools so far are:

- the transition matrix, P;
- the initial distribution, $\underline{p}^{(0)}$,

from which we can deduce a multitude of other quantities of interest.

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Classification of Markov chains

- We are able to use some general results on the behaviour of Markov chains to help answer important questions.
- In order to use this theory we need to classify each state of the chain
  - Classify each state as being one of a number of types;
  - Hence classify the type of Markov chain that we have.

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How do we classify Markov chains?

The first step in classifying Markov chains is to split the state space into non-overlapping groups (classes) of states.

Remark

We will see later on that states in the **same class** are of the **same type**.

This simplifies the problem of classifying all the states in the chain because we only need to classify **one** state in each class.

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Example

$$S = \{1, 2, 3, 4\}, \qquad P = \begin{pmatrix} 1/2 & 1/2 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1/2 & 1/3 & 1/6 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Draw a diagram to summarise the possible moves which can be made in one step (a state diagram).

Communicating states

Definition

State i communicates with state j (we write $i \to j$) if

$$p_{ij}^{(n)} > 0 \qquad \text{for some } n \geqslant 0,$$

that is, starting from state i, it is possible that the chain will eventually hit state j.

So i communicates with j if there is a path from i to j.

Communicating states

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For the Markov chain with the above one-step transition matrix, answer the following:

(a) Does state 1 communicate with state 2?

(b) Does state 4 communicate with state 3?

(c) Does state 1 communicate with state 4?

Intercommunicating states

Definition

States i and j intercommunicate (we write $i \leftrightarrow j$) if $i \to j$ and $j \to i$.

Note that this means that i communicates with j, AND that j communicates with i.

So i intercommunicates with j if there is a path from i to j AND a path from j to i.

Intercommunicating states

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For the Markov chain above, which states intercommunicate?

Properties of intercommunicating states

(i) $i \leftrightarrow i$ for all states i.

(ii) if $i \leftrightarrow j$ then $j \leftrightarrow i$.

(iii) if $i \leftrightarrow j$ and $j \leftrightarrow k$ then $i \leftrightarrow k$.

(i)+(ii)+(iii) mean that $\leftrightarrow$ is an **equivalence relation** on S.

Creating classes of states

IDEA: Split state space S into non-overlapping classes of intercommunicating states.

Remark

- Within a particular class, **every pair of states intercommunicate**;
- Pairs of states in **different** classes do not intercommunicate;
- Each class is called an **irreducible class** of states.
- If there is only one irreducible class, then the Markov chain is said to be **irreducible**.

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Example

Find the irreducible classes for the Markov chain described by:

$$S = \{1, 2, 3, 4\}, \qquad P = \begin{pmatrix} 1/2 & 1/2 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1/2 & 1/3 & 1/6 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Recurrence and Transience

Let $\{X_n\}$ be a Markov chain with state space S and let $i \in S$. Let

$$\begin{aligned} f_i =& \mathrm{P}(\text{ever return to } i | X_0 = i) \\ =& \mathrm{P}(X_n = i \text{ for some } n \geqslant 1 | X_0 = i). \end{aligned}$$

Definition

If $f_i = 1$ then i is **recurrent** (eventual return to state i is certain).

If $f_i < 1$ then i is **transient** (eventual return to state i is uncertain).

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Recurrence and Transience

Classify each state as **recurrent** or **transient** for each of the following Markov chains

1. $S = \{0, 1, 2\}$, $$P = \begin{pmatrix} 1 & 0 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \end{pmatrix}.$$

2. $S = \{0, 1, 2, 3\}$, $$P = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

How often does the chain return?

If we start in state i, how many times does the chain pass through state i subsequently?

Given that we start in state i, look at two cases:

- i is transient (i.e. not guaranteed of returning at all);
- i is recurrent (i.e. we return with probability 1).

.

How often does the chain return?

(i) If i is **transient**

$$\boxed{f_i = \text{P}(\text{ever return to } i | X_0 = i)}$$

Let N be the number of returns to i (including the return from the fact that $X_0 = i$). Then

$$\begin{aligned}\text{P}(N = n \mid X_0 = i) &= \text{P}(\text{return } n-1 \text{ times, then never return} \mid X_0 = i) \\ &= f_i^{n-1}(1 - f_i),\end{aligned}$$

So, given $X_0 = i$, N has a geometric distribution with parameter $1 - f_i$. So N is finite with probability 1 and

$$\mathrm{E}(N \mid X_0 = i) = \frac{1}{1 - f_i} \quad (< \infty).$$

How often does the chain return?

(ii) If i is **recurrent**

With probability 1 the chain will eventually return to state i.

By **time homogeneity** and the **Markov property**, the chain ‘starts afresh’ on return to the initial state, so that state i will eventually be visited again (with probability 1).

Repeating the argument shows that the chain will **return to i infinitely often** (with probability 1).

We also have

$$\mathrm{E}(N \mid X_0 = i) \text{ is infinite.}$$

First passage times

Let

$$T_{ii} = \min\{n \geqslant 1 : X_n = i \mid X_0 = i\}.$$

Definition

T_{ii} is the **first passage time** from state i to itself, that is, the number of steps until the chain first returns to state i given that it starts in state i.

If the chain never returns to state i we set $T_{ii} = \infty$.

First passage times

$$\boxed{f_i = \mathrm{P}(\text{ever return to } i | X_0 = i)}$$

Note that:

$$\begin{aligned} f_i &= \mathrm{P}(\text{ever return to } i \mid X_0 = i) = \mathrm{P}(T_{ii} < \infty) \\ 1 - f_i &= \mathrm{P}(\text{never return to } i \mid X_0 = i) = \mathrm{P}(T_{ii} = \infty). \end{aligned}$$

So:

- If i is recurrent then $\mathrm{P}(T_{ii} < \infty) = 1$;
  - So T_{ii} is finite with probability 1.
- If i is transient then $\mathrm{P}(T_{ii} < \infty) < 1$.
  - Equivalently $\mathrm{P}(T_{ii} = \infty) = 1 - \mathrm{P}(T_{ii} < \infty) > 0$;
  - So T_{ii} is infinite with positive probability;
  - $\mu_i = \mathrm{E}(T_{ii})$ must be infinite.

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Properties of recurrent/transient states

	State i is **recurrent**	State i is **transient**
$P(\text{ever return to } i \mid X_0 = i)$	$f_i = 1$	$f_i < 1$
$P(\text{never return to } i \mid X_0 = i)$	$1 - f_i = 0$	$1 - f_i > 0$
$(\text{number of hits on } i \mid X_0 = i)$	∞ (w.p. 1)	$< \infty$ (w.p. 1)
$\mathrm{E}\left[\text{number of hits on } i \mid X_0 = i\right]$	∞	$\frac{1}{1-f_i} < \infty$
$\mathrm{P}(T_{ii} < \infty)$	$f_i = 1$	$f_i < 1$
$\mathrm{P}(T_{ii} = \infty)$	$1 - f_i = 0$	$1 - f_i > 0$

A final note on recurrence

Definition

Recall that the first passage time (also called the *recurrence time*) T_{ii} of a state i is the number of steps until the chain first returns to state i given that it starts in state i.

Let $\mu_i = \mathrm{E}(T_{ii})$ be the *mean recurrence time* of state i.

We have already seen that if i is transient then $\mu_i = \infty$.

We **didn't** calculate $\mu_i = \mathrm{E}(T_{ii})$ for a recurrent state. All we said is that T_{ii} must be finite with probability 1.

In fact, we can have that $\mu_i = \mathrm{E}(T_{ii})$ is infinite for a recurrent state.

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A final note on recurrence

There are two types of recurrent state.

Definition

A recurrent state i is

positive recurrent	if	$\mu_i < \infty$	(that is, μ_i is finite),
null recurrent	if	$\mu_i = \infty$	(that is, μ_i is infinite).

A final note on recurrence

At first glance it may seem strange that a state i can be null recurrent:

- return to state i is certain,
- expected time (number of steps) to return to i is infinite.

In other words,

- the random variable T_{ii} is finite with probability 1,
- the mean of T_{ii}, $\mathrm{E}(T_{ii})$, is infinite.

Such states *do* exist and we will see examples later in the course.

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A toy example to show that this can occur...

Suppose that state i is recurrent. Then we know that

$$f_i = P(\text{ever return to } i | X_0 = i) = 1.$$

Suppose that

$$P(\text{first return to state } i \text{ at time } k | X_0 = i) = \frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}.$$

This gives

$$\begin{aligned} &P(\text{ever return to } i | X_0 = i) \\ &= \sum_{k=1}^{\infty} P(\text{first return to state } i \text{ at time } k | X_0 = i) \\ &= \sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{k+1} \right) = 1 \end{aligned}$$

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But what about $E[T_{ii}]$ in this case?

$$
\begin{aligned}
E[T_{ii}] &= \sum_{k=1}^{\infty} kP(\text{first return to } i \text{ at time } k | X_0 = i) \\
&= \sum_{k=1}^{\infty} k\left(\frac{1}{k} - \frac{1}{k+1}\right) \\
&= \sum_{k=1}^{\infty} \frac{1}{k+1} = \infty
\end{aligned}
$$

So we have a recurrent state with expected time of first return being infinite.

Periodicity

Definition

The *period* of a state i is the greatest common divisor of the set of integers $n \geqslant 1$ such that $p_{ii}^{(n)} > 0$, that is,

$$d_i = \gcd\left\{n \geqslant 1 : p_{ii}^{(n)} > 0\right\}.$$

- If $d_i = 1$ then state i is *aperiodic*.
- If $p_{ii}^{(n)} = 0$ for all $n \geqslant 1$ (i.e. return to state i is impossible) then, by convention, state i is also said to be aperiodic.

Class properties

A *class property* is a property which is **shared by all the states in an irreducible class.** If it can be shown that one of the states in a class has a certain class property then all the states in the irreducible class must also have that same property.

Class properties

Theorem

Recurrence/transience is a class property.

Remark

The theorem tells us that a pair of intercommunicating states are either both recurrent or both transient. That is, recurrence/transience is a class property.

It is not possible to have a mixture of recurrent and transient states in an irreducible class.

For example:

- If we show that state i is recurrent, say, and $i \leftrightarrow j$ then j is also recurrent.
- If we show that state i is recurrent, say, and j is in the same irreducible class, then j is also recurrent.

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Exercise

Find **and classify** the irreducible classes of these Markov chains.

1. $S = \{1, 2, 3, 4\}, \qquad P = \begin{pmatrix} 1/2 & 1/2 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1/2 & 1/3 & 1/6 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$
2. $S = \{0, 1, 2\}, \qquad P = \begin{pmatrix} 1 & 0 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \end{pmatrix}.$

Null recurrence is a class property

Sketch proof

Step 1 Suppose that state i is null recurrent.

Step 2 Choose state j s.t. $i \leftrightarrow j$ with $p_{ij}^{(n)} > 0$, $p_{ji}^{(m)} > 0$.

Step 3 Write $p_{ii}^{(m+k+n)} \geqslant p_{ij}^{(n)} p_{jj}^{(k)} p_{ji}^{(m)}$.

Step 4 Show that if state i is null recurrent then state j must be either transient or null recurrent*.

Step 5 Recurrence is a class property, so state j cannot be transient.

Step 6 State j must be null recurrent.

Null recurrence is a class property

To help with the proof (step 4)*

As $n \to \infty$,

if i is aperiodic, positive recurrent then $p_{ii}^{(n)} \to \frac{1}{\mu_i} > 0$

if i is periodic, positive recurrent then $p_{ii}^{(nd)} \to \frac{d}{\mu_i} > 0$

if i is null recurrent then $p_{ii}^{(n)} \to 0$

if i is transient then $\sum_{n=0}^{\infty} p_{ii}^{(n)} < \infty$ and $p_{ii}^{(n)} \to 0$

Positive recurrence is a class property

Proof:

- Suppose that $i \leftrightarrow j$ and that i is positive recurrent.
- Then j must be recurrent (because recurrence is a class property).
- If j is null recurrent then i is also null recurrent.
- This is a contradiction.
- Hence j must be positive recurrent.

$\square$

Periodicity is a class property

Sketch proof

- Let i and j be states with $i \leftrightarrow j$;
- Let d_i be the period of i and d_j be the period of j;
- Let n and m be such that $p_{ij}^{(n)} > 0$ and $p_{ji}^{(m)} > 0$.
- Let k be such that $p_{ii}^{(k)} > 0$.

Step 1 Show that $d_j \mid k$, i.e. d_j is a divisor of k.

Step 2 Show that $d_j \leqslant d_i$.

Step 3 Reversing the roles of i and $j \Rightarrow d_i \leqslant d_j$.

Step 4 Conclude that $d_i = d_j$.

Class properties - summary

Periodicity

Recurrence / Transience

Positive recurrence

Null recurrence

An important result

Theorem

It is not possible for all states in a finite state space Markov chain to be transient.

- Suppose all states are transient and $S = \{0, 1, \ldots, M\}$.
- A transient state will be visited only a finite number of times.
- So for each state i there exists a time T_i after which i will never be visited again.
- Therefore, after time $T = \max\{T_0, \ldots, T_M\}$ no state will be visited again.
- BUT the Markov chain must be in some state. Therefore we have a contradiction.
- Therefore, at least one state must be recurrent.

An important result

Therefore, *all* states in a finite, irreducible Markov chain are **recurrent** (because recurrence is a class property).

[That is, finite and irreducible $\Rightarrow$ recurrent.]

*Remember that an **irreducible Markov chain** is one in which there is only ONE communicating class.*

Finite irreducible Markov chains are not null recurrent

Proof:

- Let $p_j^{(n)} := \mathrm{P}(X_n = j)$, and suppose chain is null recurrent.
- **It can be shown** that $p_j^{(n)} \to 0$ as $n \to \infty$ for all $j \in S$.

We know that for all n,

$$\sum_{j \in S} p_j^{(n)} = 1 \quad \text{and so} \quad \lim_{n\to\infty} \sum_{j \in S} p_j^{(n)} = 1.$$

But S is finite so if the chain is null recurrent

$$\lim_{n\to\infty} \sum_{j \in S} p_j^{(n)} = \sum_{j \in S} \lim_{n\to\infty} p_j^{(n)} = 0. \qquad \text{CONTRADICTION!}$$

- So Markov chain cannot be null recurrent.
- Since this Markov chain cannot be transient, all states in a finite irreducible Markov chain are positive recurrent.

Finite Markov chains cannot contain any null recurrent states

Proof:

- Suppose that the chain contains a state i which is null recurrent.
- Then, since null recurrence is a class property, state i is in some irreducible finite closed class of null recurrent states.
- This is not possible by last result.
- So a finite Markov chain cannot contain any null recurrent states.

□

Closed classes and absorbing states

Definition

A class C of intercommunicating states is **closed** if $p_{ij} = 0$ for all $i \in C$ and $j \notin C$, that is, once the chain gets into C, it never leaves C.

If a single state forms a closed class then it is called an **absorbing state**.

Examples

$$S = \{1, 2, 3, 4\}, \qquad P = \begin{pmatrix} 1/2 & 1/2 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1/2 & 1/3 & 1/6 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

- Irreducible classes?
- Any closed classes?
- Any absorbing states?

Absorption and recurrence

A **finite closed class** must be **recurrent**

Once entered, the class behaves like a finite irreducible Markov chain.

An **absorbing state** is **positive recurrent**

Its mean recurrence time is 1.

An irreducible class C of recurrent states is closed

Proof:

- Suppose that C is not closed.
- Then there exist $i \in C$ and $j \notin C$ such that $i \to j$.
- But $j \not\to i$; otherwise i and j would intercommunicate and j would be in C.
- Thus, there is positive probability that the chain leaves i and never returns to i.
- This means that i is transient.
- This is a contradiction, as $i \in C$ and C is recurrent.
- An irreducibe class C of recurrent states must therefore be closed.

$\square$

Example- Inheritance of X-linked genes

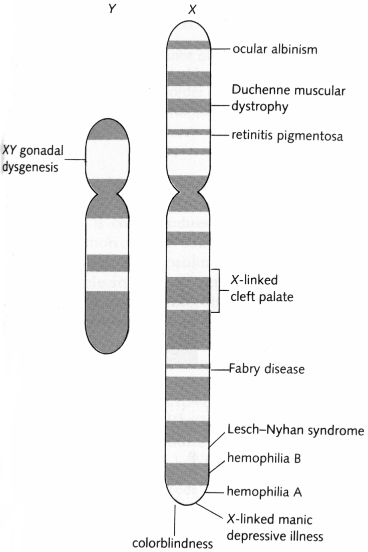
Mate two individuals initially.

At succeeding generations, choose randomly a male and a female from same litter and mate them.

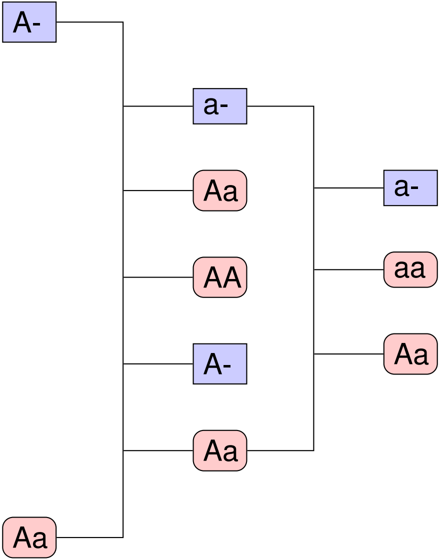
Genotype of individuals (two possible alleles: **A** or **a**):

Males: **A-** or **a-**

Females: **AA** or **Aa** or **aa**



Pedigree (family tree)



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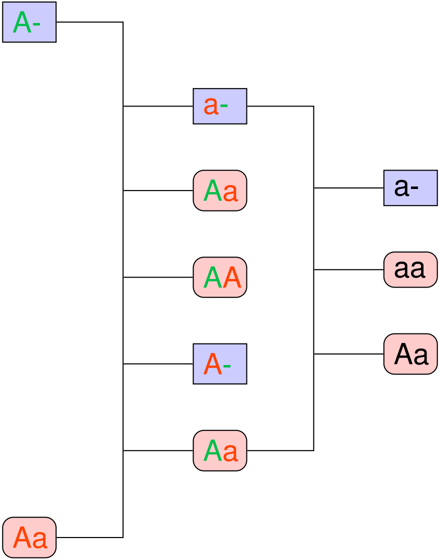

Example- Inheritance of X-linked genes

Genotype pairs (**States**)
A- $\times$ AA
A- $\times$ Aa
A- $\times$ aa
a- $\times$ AA
a- $\times$ Aa
a- $\times$ aa

- Let X_0 be the genotype pair of the original two individuals (generation 0).
- Let X_1 be the genotype pair of the randomly chosen male and female in generation 1.
- ...
- Let X_n be the genotype pair of the randomly chosen male and female in generation n.

Then $\{X_n : n \geqslant 0\}$ is a Markov chain.

Homework: satisfy yourself that this is the case.

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- Let X_0 be the genotype pair of the original two individuals (generation 0).
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- ...
- Let X_n be the genotype pair of the randomly chosen male and female in generation n.

Then $\{X_n : n \geqslant 0\}$ is a Markov chain.

Homework: satisfy yourself that this is the case.

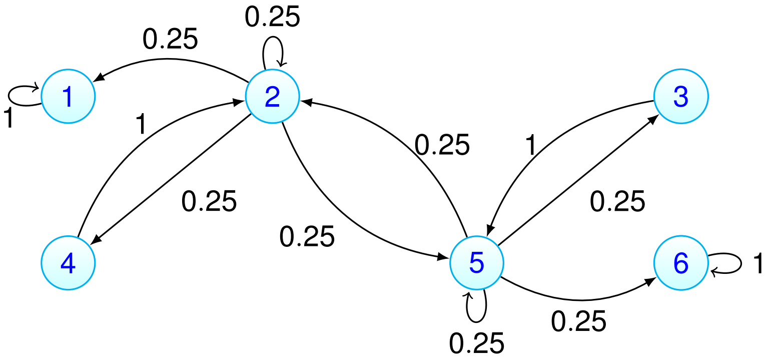
Example- Inheritance of X-linked genes

Genotype pair	State
A- $\times$ AA	1
A- $\times$ Aa	2
A- $\times$ aa	3
a- $\times$ AA	4
a- $\times$ Aa	5
a- $\times$ aa	6

$$P = \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \end{matrix} \\ \begin{pmatrix} 1 & - & - & - & - & - \\ 1/4 & 1/4 & - & 1/4 & 1/4 & - \\ - & - & - & - & 1 & - \\ - & 1 & - & - & - & - \\ - & 1/4 & 1/4 & - & 1/4 & 1/4 \\ - & - & - & - & - & 1 \end{pmatrix} \end{array}$$

- Irreducible classes?
- Any closed classes?
- Any absorbing states?
- Invariant distribution?
- Equilibrium distribution?

Example- Inheritance of X-linked genes



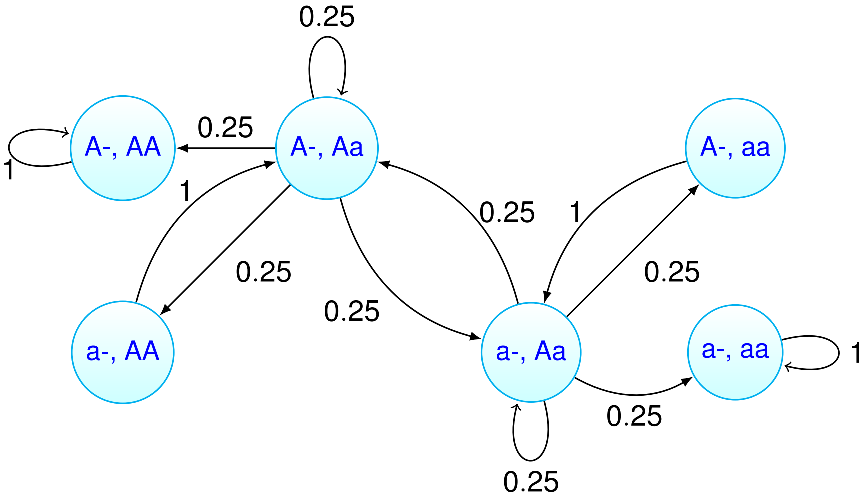
- Irreducible classes?
- Any closed classes?
- Any absorbing states?
- Invariant distribution?
- Equilibrium distribution?

Calculating probabilities and expectations

Use our tools to calculate various probabilities and expectations on Markov chains with absorbing states.

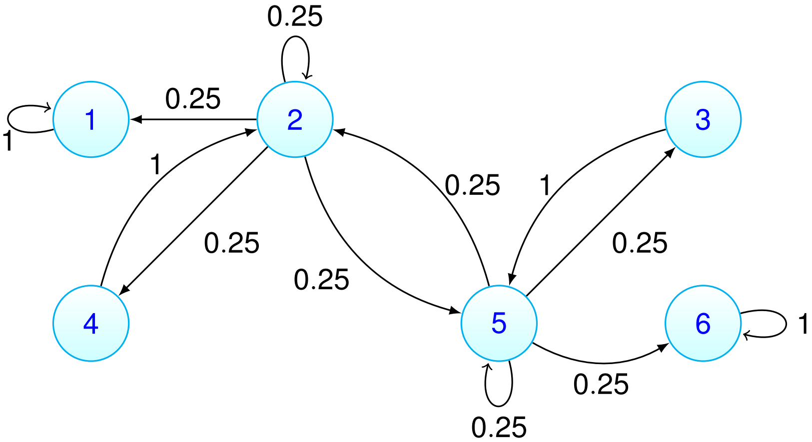
Main tool: FIRST STEP DECOMPOSITION

Probability of absorption (extinction)



Suppose that $X_0 = \{\text{A-, Aa}\}$. Find the probability that the chain is eventually absorbed in $\{\text{A-, AA}\}$.

Probability of absorption (extinction)

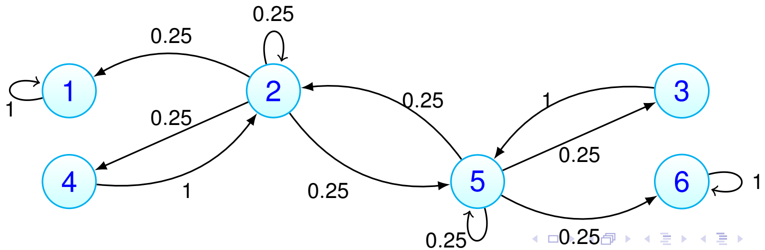


Suppose that $X_0 = 2$. Find the probability that the chain is eventually absorbed in state 1.

Probability of absorption (extinction)

Let $P_2 = P(\text{absorbed in } 1 | X_0 = 2)$. Then:

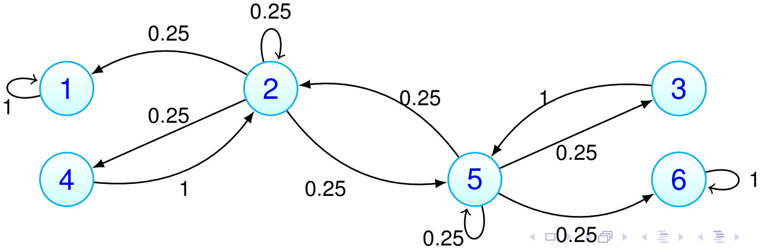
$$
\begin{aligned}
P_2 =& \frac{1}{4} P(\text{absorbed in } 1 | X_1 = 1, X_0 = 2) \\
&+ \frac{1}{4} P(\text{absorbed in } 1 | X_1 = 2, X_0 = 2) \\
&+ \frac{1}{4} P(\text{absorbed in } 1 | X_1 = 4, X_0 = 2) \\
&+ \frac{1}{4} P(\text{absorbed in } 1 | X_1 = 5, X_0 = 2)
\end{aligned}
$$



Probability of absorption (extinction)

Let $P_2 = P(\text{absorbed in } 1|X_0 = 2)$. Then:

$$
\begin{aligned}
P_2 = &\frac{1}{4} \\
&+\frac{1}{4}P(\text{absorbed in } 1|X_1 = 2, X_0 = 2) \\
&+\frac{1}{4}P(\text{absorbed in } 1|X_1 = 4, X_0 = 2) \\
&+\frac{1}{4}P(\text{absorbed in } 1|X_1 = 5, X_0 = 2)
\end{aligned}
$$



Probability of absorption (extinction)

Let $P_2 = P(\text{absorbed in } 1 | X_0 = 2)$. Then:

$$
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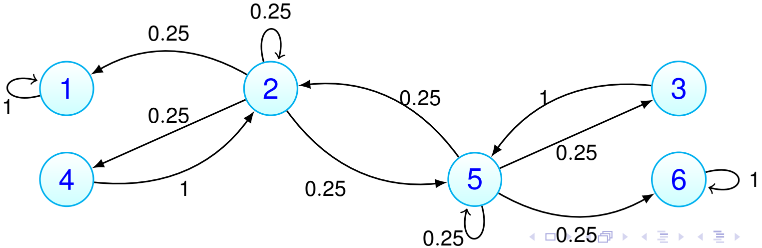
P_2 = & \frac{1}{4} \\
& + \frac{1}{4} P_2 \\
& + \frac{1}{4} P(\text{absorbed in } 1 | X_1 = 4, X_0 = 2) \\
& + \frac{1}{4} P(\text{absorbed in } 1 | X_1 = 5, X_0 = 2)
\end{aligned}
$$



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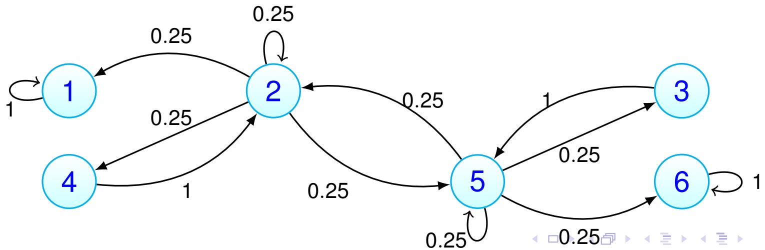
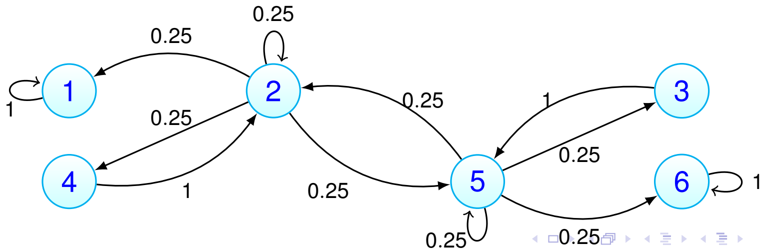
& + \frac{1}{4} P_2 \\
& + \frac{1}{4} P(\text{absorbed in } 1 | X_1 = 5, X_0 = 2)
\end{aligned}
$$



Probability of absorption (extinction)

Let $P_2 = P(\text{absorbed in } 1 | X_0 = 2)$. Then:

$$\begin{aligned} P_2 = & \frac{1}{4} \\ & + \frac{1}{4} P_2 \\ & + \frac{1}{4} P_2 \\ & + \frac{1}{4} P(\text{absorbed in } 1 | X_1 = 5) \end{aligned}$$



Probability of absorption (extinction)

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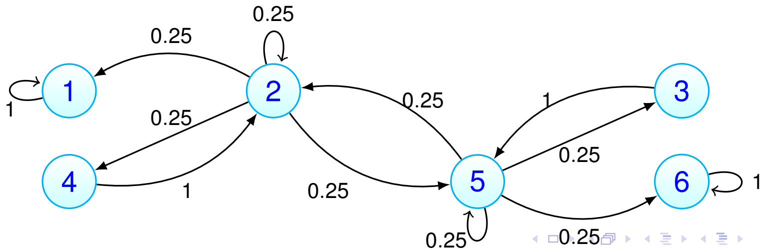

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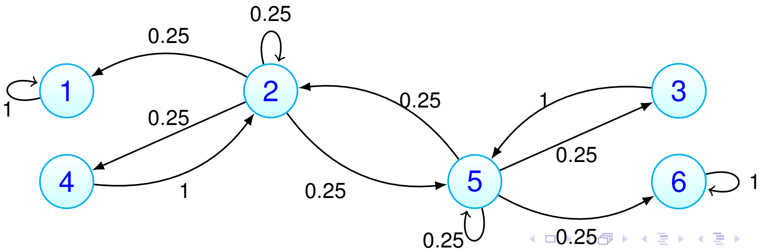
& + \frac{1}{4} P(\text{absorbed in } 1 | X_0 = 5)
\end{aligned}
$$



Probability of absorption (extinction)

Let

$$P_2 = P(\text{absorbed in } 1 | X_0 = 2)$$
$$P_5 = P(\text{absorbed in } 1 | X_0 = 5) = \frac{1}{2} P_2$$

Then:

$$\begin{aligned}
P_2 &= \frac{1}{4} + \frac{1}{2} P_2 + \frac{1}{4} P(\text{absorbed in } 1 | X_0 = 5) \\
&= \frac{1}{4} + \frac{1}{2} P_2 + \frac{1}{4} P_5 \\
&= \frac{1}{4} + \frac{1}{2} P_2 + \frac{1}{4} \left(\frac{1}{2} P_2 \right) \\
&= \frac{1}{4} + \frac{5}{8} P_2
\end{aligned}$$

Solving gives $P_A = 2/3$. (And so $P_B = 1/3$.)

Probability of absorption (extinction)

Let

$$P_2 = P(\text{absorbed in } 1 | X_0 = 2)$$

$$P_5 = P(\text{absorbed in } 1 | X_0 = 5) = \frac{1}{2} P_2$$

Then:

$$\begin{aligned} P_2 =& \frac{1}{4} + \frac{1}{2} P_2 + \frac{1}{4} P(\text{absorbed in } 1 | X_0 = 5) \\ =& \frac{1}{4} + \frac{1}{2} P_2 + \frac{1}{4} P_5 \\ =& \frac{1}{4} + \frac{1}{2} P_2 + \frac{1}{4} \left(\frac{1}{2} P_2 \right) \\ =& \frac{1}{4} + \frac{5}{8} P_2 \end{aligned}$$

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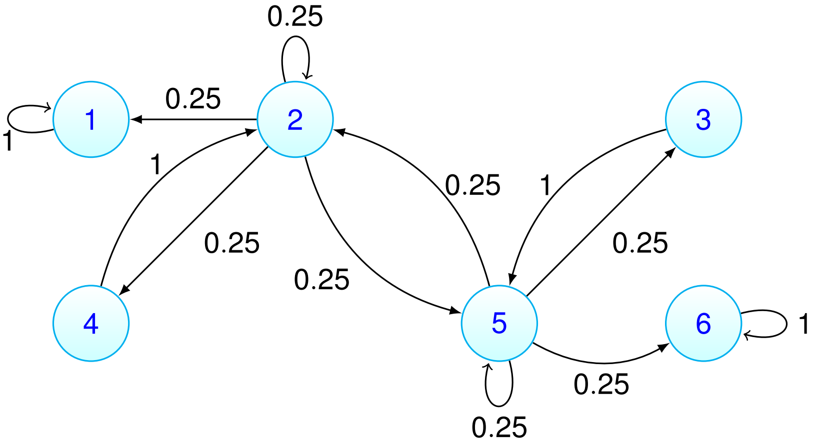
Probability of absorption (extinction)

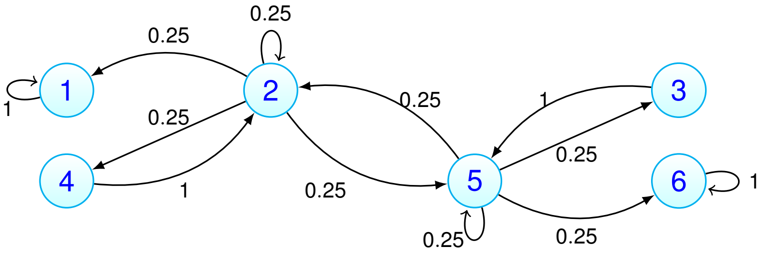


Suppose that $\underline{p}^{(0)} = (0, 1/6, 0, 1/3, 1/2, 0)$. Find the probability that the chain is eventually absorbed in state 1.

Probability of absorption (extinction)

We have already established that:

$$P_2 := P(\text{absorbed in } 1 | X_0 = 2) = 2/3$$
$$P_5 := P(\text{absorbed in } 1 | X_0 = 5) = 1/3$$

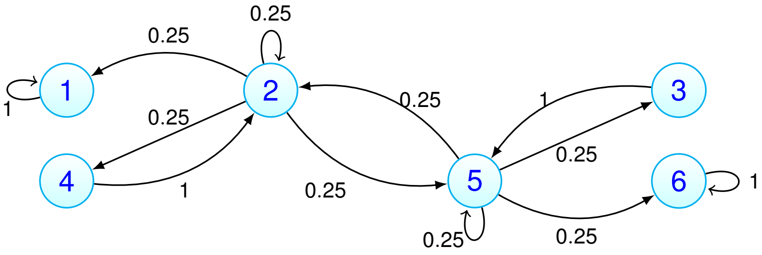


Therefore $P(\text{absorbed in } 1 | X_0 = 4) = 2/3$. (WHY?)

Probability of absorption (extinction)

We have already established that:

$$P_2 := P(\text{absorbed in } 1 | X_0 = 2) = 2/3$$
$$P_5 := P(\text{absorbed in } 1 | X_0 = 5) = 1/3$$



Therefore $P(\text{absorbed in } 1 | X_0 = 4) = 2/3$. (WHY?)

Probability of absorption (extinction)

Let

$$P_C = P(\text{absorbed in } 1 | X_0 \in \{2,4,5\})$$
$$\underline{p}^{(0)} = (0, 1/6, 0, 1/3, 1/2, 0)$$

Now:

$$\begin{aligned}
P_C =& \frac{P(\text{absorbed in } 1, X_0 \in \{2,4,5\})}{P(X_0 \in \{2,4,5\})} \\
=& P(\text{absorbed in } 1, X_0 \in \{2,4,5\}) \\
=& P(\text{absorbed in } 1, X_0 = 2) \\
+& P(\text{absorbed in } 1, X_0 = 4) \\
+& P(\text{absorbed in } 1, X_0 = 5)
\end{aligned}$$

Probability of absorption (extinction)

Let

$$P_C = P(\text{absorbed in } 1 | X_0 \in \{2,4,5\})$$
$$\underline{p}^{(0)} = (0, 1/6, 0, 1/3, 1/2, 0)$$

Now:

$$\begin{aligned}
P_C =& \frac{P(\text{absorbed in } 1, X_0 \in \{2,4,5\})}{P(X_0 \in \{2,4,5\})} \\
=& P(\text{absorbed in } 1, X_0 \in \{2,4,5\}) \\
=& P(\text{absorbed in } 1 | X_0 = 2) P(X_0 = 2) \\
+& P(\text{absorbed in } 1 | X_0 = 4) P(X_0 = 4) \\
+& P(\text{absorbed in } 1 | X_0 = 5) P(X_0 = 5)
\end{aligned}$$

Probability of absorption (extinction)

Let

$$P_C = P(\text{absorbed in } 1 | X_0 \in \{2, 4, 5\})$$
$$\underline{p}^{(0)} = (0, 1/6, 0, 1/3, 1/2, 0)$$

Now:

$$\begin{aligned} P_C =& \frac{P(\text{absorbed in } 1, X_0 \in \{2, 4, 5\})}{P(X_0 \in \{2, 4, 5\})} \\ =& P(\text{absorbed in } 1, X_0 \in \{2, 4, 5\}) \\ =& P_2 \times P(X_0 = 2) \\ +& P_2 \times P(X_0 = 4) \\ +& P_5 \times P(X_0 = 5) \end{aligned}$$

Probability of absorption (extinction)

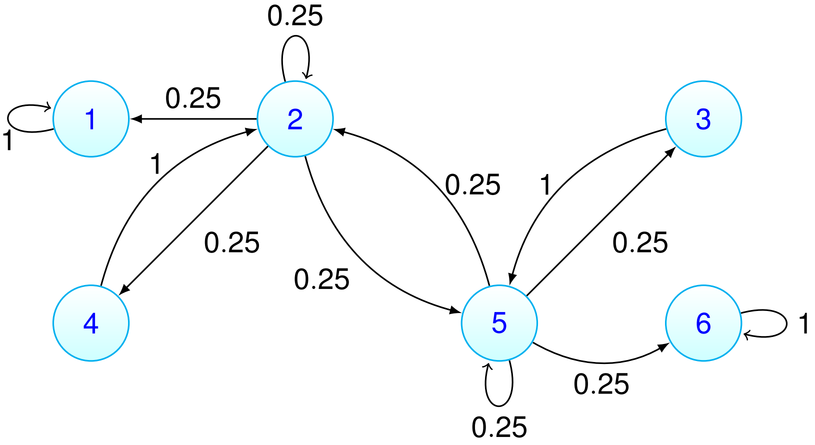
Let

$$P_C = P(\text{absorbed in } 1 | X_0 \in \{2, 4, 5\})$$
$$\underline{p}^{(0)} = (0, 1/6, 0, 1/3, 1/2, 0)$$

Working all this out, gives $P_C = 1/2$.

Probability of absorption (extinction)

Homework challenge

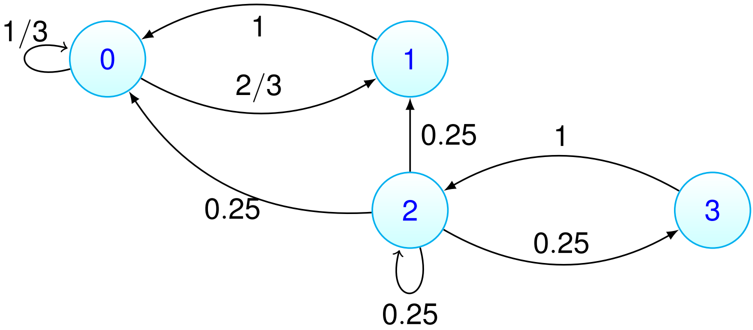


Suppose that $\underline{p}^{(0)} = (0, 1/6, 0, 1/3, 1/2, 0)$ and that the chain is absorbed in state 1. What is the probability that $X_0 = 2$?

Expected time to absorption

One closed class

$$S = \{0, 1, 2, 3\}, \qquad P = \begin{pmatrix} 1/3 & 2/3 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1/4 & 1/4 & 1/4 & 1/4 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$



If $X_0 = 2$, find the expected time until absorption into $\{0, 1\}$.

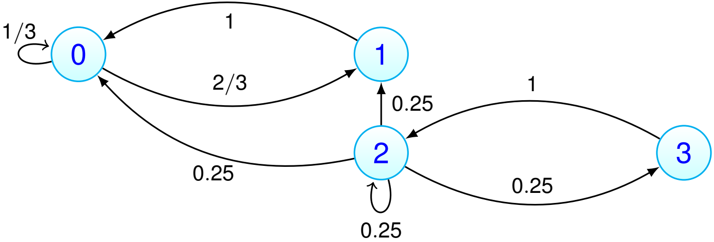
Expected time to absorption

One closed class

Let T_i denote the time to absorption into $\{0, 1\}$, given $X_0 = i$.

We want to compute $E[T_2]$.

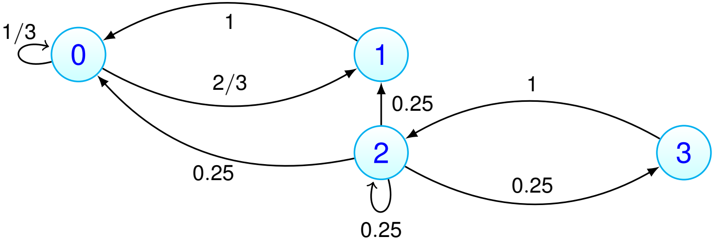
Note that $E[T_0] = E[T_1] = 0$.



Expected time to absorption

One closed class

$$
\begin{aligned}
E[T_2] =& E\left[E[T_2|X_1]\right] \\
=& \frac{1}{4}E[1+T_0] + \frac{1}{4}E[1+T_1] \\
&+ \frac{1}{4}E[1+T_2] + \frac{1}{4}E[1+T_3]
\end{aligned}
$$



Expected time to absorption

One closed class

$$
\begin{aligned}
E[T_2] =& E\left[E[T_2|X_1]\right] \\
=& \frac{1}{4} + \frac{1}{4} \\
&+ \frac{1}{4} + \frac{1}{4}E[T_2] + \frac{1}{4} + \frac{1}{4}E[T_3]
\end{aligned}
$$

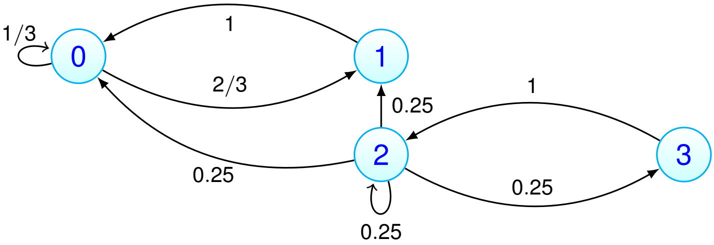
$$
\begin{tikzcd}
0 \arrow[loop left, "1/3"] \arrow[rr, bend right, "2/3"'] & & 1 \arrow[ll, bend right, "1"'] & & \\
& & 2 \arrow[u, "0.25"'] \arrow[llu, bend left, "0.25"] \arrow[loop below, "0.25"] \arrow[rr, bend right, "0.25"'] & & 3 \arrow[ll, bend right, "1"']
\end{tikzcd}
$$

Expected time to absorption

One closed class

Let T denote the time to absorption into $\{0, 1\}$, given $X_0 = 2$.

$$E[T_2] = E\left[E[T_2|X_1]\right]$$
$$= 1 + \frac{1}{4}E[T_2] + \frac{1}{4}E[T_3]$$

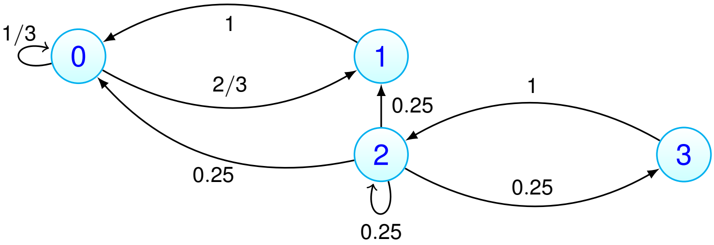


Expected time to absorption

One closed class

Let T denote the time to absorption into $\{0, 1\}$, given $X_0 = 2$.

$$
\begin{aligned}
E[T_2] =& E\left[E[T_2|X_1]\right] \\
=& 1 + \frac{1}{4}E[T_2] + \frac{1}{4}(1 + E[T_2])
\end{aligned}
$$



Expected time to absorption

One closed class

So this gives:

$$E[T_2] = \frac{5}{4} + \frac{1}{2}E[T_2]$$

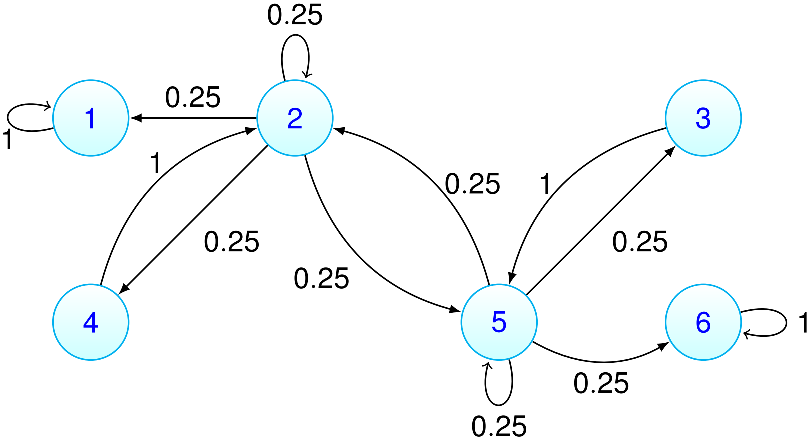
Solving:

$$E[T_2] = 2.5$$

Expected time to absorption

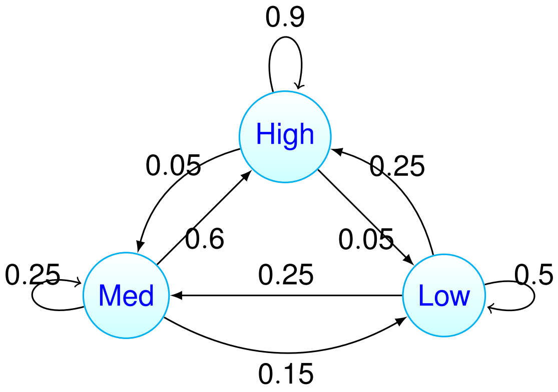
Homework - Two closed classes (messy but easy)

Find the expected time to absorption in one or other of the two closed classes, starting from state 2.



First passage times to non-absorbing states

HIV progression



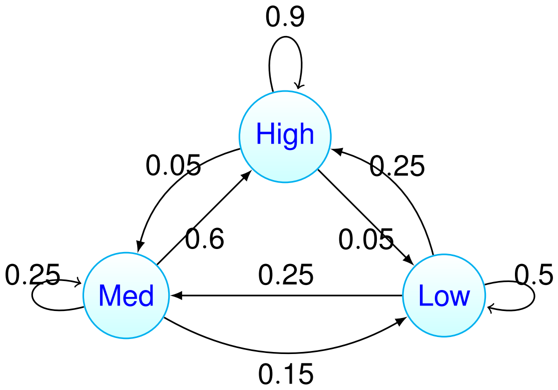
Find the expected first passage time from state ‘High’ to state ‘Low’.

First passage times to non-absorbing states

HIV progression

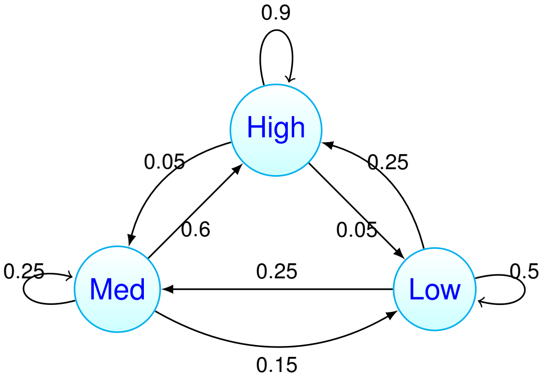
Let T_i denote the time the process first enters state 'Low', given it starts in state i (H, M or L).

We want $E[T_H]$.



First passage times to non-absorbing states

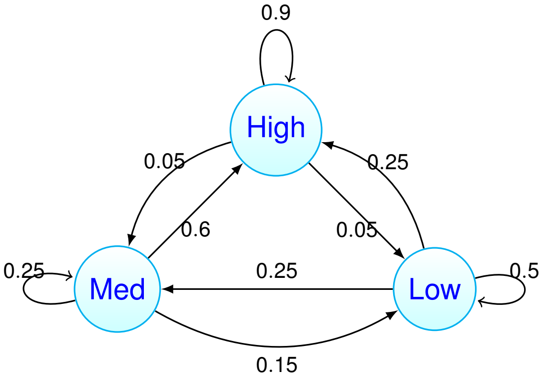
HIV progression



$$E[T_H] = E\left[E[T_H|X_1]\right] = ?$$

First passage times to non-absorbing states

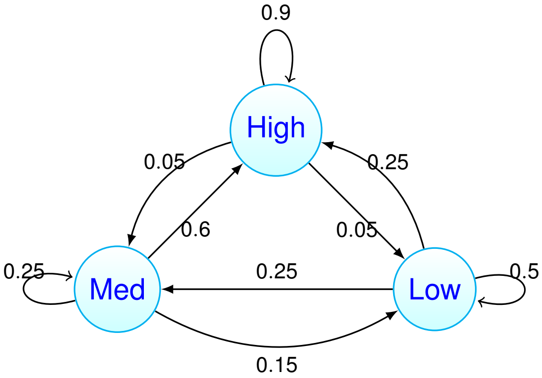
HIV progression



$$
\begin{aligned}
E[T_H] = E\left[E[T_H|X_1]\right] =& 0.05 + 0.9\left(1 + E[T_H]\right) + 0.05\left(1 + E[T_M]\right) \\
=& 1 + 0.9E[T_H] + 0.05E[T_M]
\end{aligned}
$$

First passage times to non-absorbing states

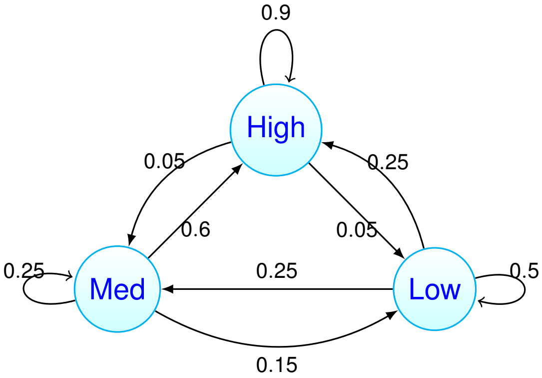
HIV progression



$$E[T_M] = 4/3 + 0.8E[T]$$

First passage times to non-absorbing states

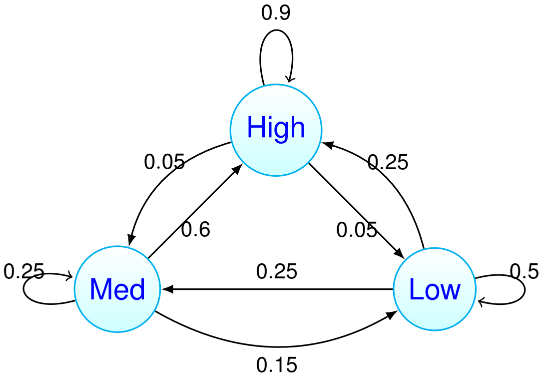
HIV progression



$$
\begin{aligned}
E[T_H] =& 1 + 0.9E[T_H] + 0.05E[T_M] \\
=& 1 + 0.9E[T_H] + 0.05\,(4/3 + 0.8E[T_H])
\end{aligned}
$$

First passage times to non-absorbing states

HIV progression



$$E[T] = 160/9$$

Decomposition of the states of a Markov chain

Putting all the previous results together shows that the state space S of a Markov chain can be decomposed as

$$S = T \cup C_1 \cup C_2 \cup \cdots ,$$

where

- T is a set of transient states;
- $C_1, C_2, \ldots$ are irreducible closed classes of recurrent states.

Decomposition of the states of a Markov chain

Note the following:

1. Each C_i is either null recurrent or positive recurrent;
2. All states in a particular C_i have the same period;
3. Different C_is can have different periods;
4. If $X_0 \in C_i$ then the chain stays in C_i forever.
5. If $X_0 \in T$ then either:
   (a) the chain always stays in T; or
   (b) the chain is eventually absorbed in one of the C_is (where it stays forever).

Decomposition of the states of a Markov chain

Consequences of having a finite state space

- Impossible for the chain to remain forever in the (finite) set T of transient states.
- At least one state must be visited infinitely often and so there must be at least one recurrent state.

If S is finite, there are no null recurrent states. Therefore:

(a) There must be at least one positive recurrent state.

(b) If also irreducible, then it must be positive recurrent.

(c) A finite closed irreducible class must be positive recurrent.