The focus for these exercises is on the use of conditioning and PGFs - both are used later in the course. The main comment I have is that you must define random variables appropriately. See examples below of what is, and what isn't, OK.

Question 1
This was generally well answered. Please note the following:

- Though many of you defined a random variable to denote the time until the prisoner is free correctly, some of you incorrectly defined the random variable describing the door chosen initially. The correct way of doing this is as follows: let C denote the door the prisoner chooses first. Now we can see immediately that the random variable C can take the values 1, 2 or 3. Instead of using C, some of you defined a random variable C_i with $i = 1, 2, 3$ to be the door the prisoner chooses first, and wrote (where D is the time until freedom, say):

$$E[E[D|C_i]] = 0.5E[D|C_1] + 0.3E[D|C_2] + 0.2E[D|C_3].$$

Think: what is C_i? What values does it take? What are you actually conditioning on? In the expression $E[T|C_1]$ for example, you need to equate C_1 to a number for this to make sense. Do this every time you define a random variable: check which values it takes and whether this is what you wanted.

- Be careful when computing $E[D^2]$. Many of you wrote, for example,

$$E[D^2|C = 1] = 0.5E[2 + D^2].$$

This is incorrect: we need $(2 + D)^2$ rather than $(2 + D^2)$.

(a) The key here was to use iterated conditional expectation. Letting D denote the number of days until the prisoner is free, and C the first door chosen by the prisoner (C can take on the values 1, 2 or 3) and you should have

$$E[D] = E[E[D|C]] = 9.5 \text{ days}.$$

(b) Again, apply the iterated conditional expectation. You should get

$$E[D^2] = E[E[D^2|C]] = 204 \text{ days}.$$

The variance is easily calculable from there.

Question 2
The same techniques as in question 1 need to be employed, and the same comment about defining random variables appropriately applies. You will find that instead of applying the iterated conditional expectation formula once (like for question 1), you will need to apply it twice: the first time conditioning on the first door he chooses, and if necessary (i.e. if he didn't choose the correct door to begin with), then condition on the second door he chooses.

Now define C_i to be door chosen on the prisoner's ith attempt (note the difference between this C_i and the C_i that many of you defined for question 1: here, the subscript i denotes the attempt number, and not the door number, and will be equated to a number in the computations below).

Then

$$\begin{aligned} E[D] &= E[E[D|C_1]] \\ &= \sum_{i=1}^{3} E[D|C_1 = i]P(C_1 = i). \end{aligned}$$

You will need to calculate each $E[D|C_1 = i]$ for each i by applying iterated conditional expectation once again, for example if $i = 1$:

$$E[D|C_1 = 1] = E\left[E[D|C_1 = 1, C_2]\right].$$

Note that I am now conditioning on C_2, the second door chosen. You should get that $E[D] = 2.5$ days.

Question 3

Though you may not like using PGFs, it is the quickest and cleanest route to answering this question correctly (and is less prone to error too).

The argument given is incomplete because we have not proven that resulting distribution is still Poisson, let alone its parameter. We may not jump to this conclusion without justifying it rigorously.

Let N be the number of fish caught in a day, and K the number of fish taken home in a day so that $N \sim \text{Poi}(30)$ and $K \mid N = n \sim \text{Bin}(n, 1/3)$. **It is important to note that the assumption we make** is that the probability of throwing out any particular fish is 2/3, independently of any other fish caught and of the number caught.

PGF approach:

$$G_K(s) = E[s^K] = E[E[s^K|N]].$$

Now, if you use the assumption that $K \mid N = n \sim \text{Bin}(n, 1/3)$, and substitute the relevant PGF for the *random variable* $(K \mid N = n)$, then you get that

$$G_K(s) = \exp(10(s-1)),$$

which is the PGF of a Poisson(10) distribution with the usual probabilities. This is not an altogether surprising answer. You can also do this 'the long way' by calculating $P(K = k)$. This would get you the same answer but is long and prone to mistakes.

As always, you are welcome to come to one of my office hours if you would like to discuss any of these questions in more detail.