A Short Introduction to Stochastic Processes

Chapter 2 of notes

STAT2003 / STAT3102

Contents

- Formal definition: stochastic process.
- The Markov property.

Definition - stochastic process

Definition

A *stochastic process* is a collection of random variables $\{X_t, t \in T\}$ taking values in the *state space* S.

T is the *parameter set* or *index set*.

T often represents time.

Notation - stochastic processes

Notation

Discrete-time processes: $\{X_t, t \in T\}$, *where* $T = \{0, 1, 2, \ldots\}$.

Continuous-time processes: $\{X(t), t \in T\}$, *where* $T = \mathbb{R}_+$ *(i.e.* $t \geqslant 0$*).*

Continuous-time processes can change their value/state ('jump') at *any* instant of time.

Discrete-time processes can only do this at a discrete set of time points.

Remark

For discrete-time processes when we use the word 'time' we mean 'NUMBER OF TRANSITIONS/STEPS'.

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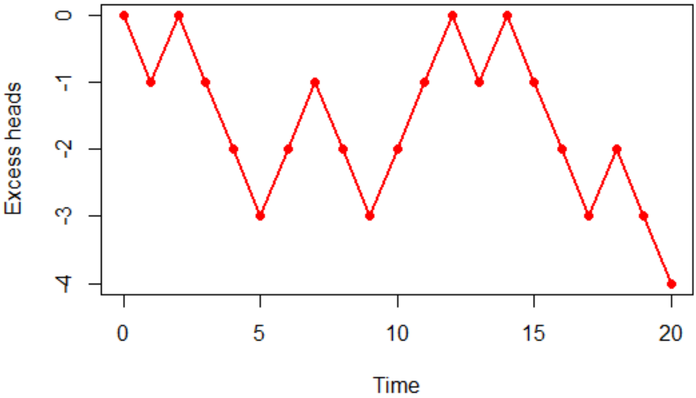
Examples of a stochastic process

How many more heads than tails?

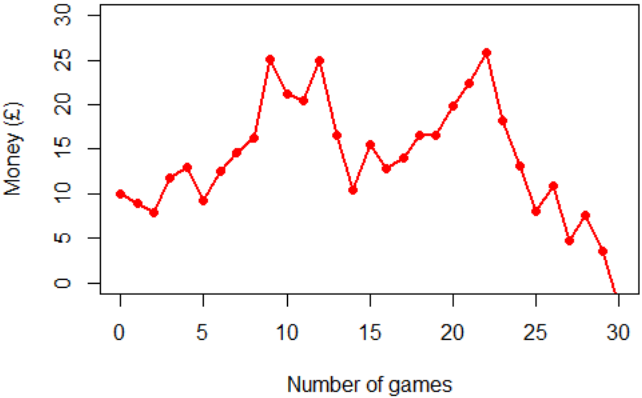
DISCRETE time and DISCRETE state space



Examples of a stochastic process

Gambler's winnings and losses after each bet.

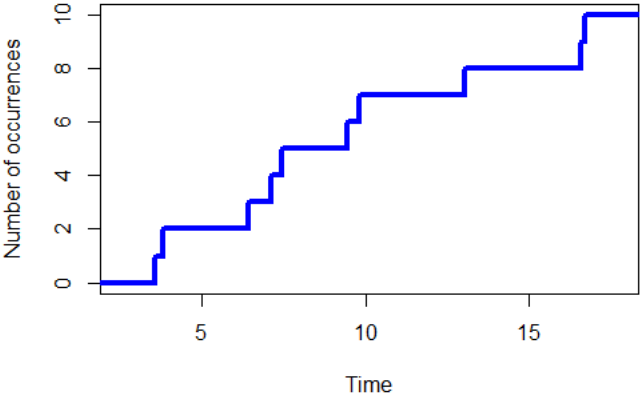
DISCRETE time and CONTINUOUS state space



Examples of a stochastic process

Arrivals/ number of events (Poisson process).

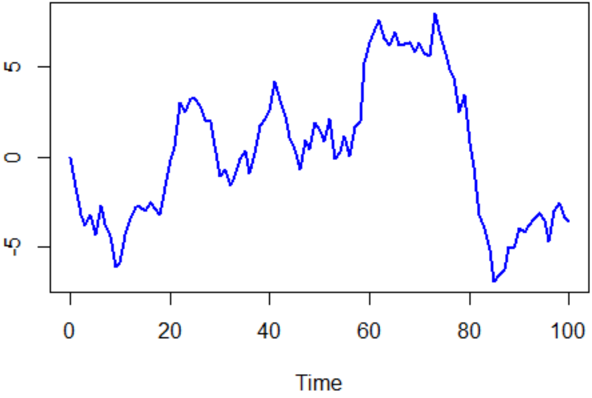
CONTINUOUS time and DISCRETE state space



Examples of a stochastic process

Brownian motion

CONTINUOUS time and CONTINUOUS state space



The Markov property

IMPORTANT CONCEPT FOR REMAINDER OF COURSE

Definition

A stochastic process $\{X_t, t \in T\}$ is a Markov process if, for any sequence of times $0 < t_1 < \cdots < t_n < t_{n+1}$ and any $n \geqslant 0$,

$$P(X_{t_{n+1}} = j \mid X_{t_n} = i_n, \cdots, X_{t_0} = i_0) = P(X_{t_{n+1}} = j \mid X_{t_n} = i_n),$$

for any $j, i_0, \ldots, i_n$ in S.

"Given the state of the process at time t_n, the future of the process is independent of its past."

The Markov property

Let

- X_n be the present state;
- A be a past event (involving $X_0, \ldots, X_{n-1}$);
- B be a future event (involving $X_{n+1}, X_{n+2}, \ldots$.)

The Markov property (MP) states that:

"given the present state (X_n), the future B is conditionally independent of the past A".

Markov property and independence

Notation

$$B \perp\!\!\!\perp A \mid X_n.$$

Therefore,

$$P(A \cap B \mid X_n = i) = P(A \mid X_n = i)\, P(B \mid X_n = i),$$

and

$$P(B \mid X_n = i,\, A) = P(B \mid X_n = i),$$

for any $n \geqslant 0$*, any* i *and any* A *and* B*.*

An important note!

$$P(X_{t_{n+1}} = j \mid \mathbf{X_{t_n}} = \mathbf{i_n}, \cdots, X_{t_0} = i_0) = P(X_{t_{n+1}} = j \mid X_{t_n} = i_n).$$

We can only use the Markov property if the most recent information gives us **specific** information about the location of the process.

Can we use the Markov property to simplify:

$$P(X_{t_{n+1}} = j \mid \mathbf{X_{t_n}} = \mathbf{i_n} \text{ or } \mathbf{j_n}, \cdots, X_{t_0} = i_0)?$$

Why is the Markov property useful?

- The Markov property is a strong independence assumption.
- Useful because it simplifies probability calculations.
- Enables joint probabilities to be expressed as a product of simple conditional probabilities.

For example, if $\{X_n, n = 0, 1, 2, ...\}$ is a Markov process:

$$\begin{aligned}
&\mathrm{P}(X_0 = 1, X_1 = 2, X_2 = 0) \\
=&\mathrm{P}(X_2 = 0 \mid X_1 = 2, X_0 = 1)\,\mathrm{P}(X_1 = 2, X_0 = 1) \\
=&\mathrm{P}(X_2 = 0 \mid X_1 = 2)\,\mathrm{P}(X_1 = 2 \mid X_0 = 1)\,\mathrm{P}(X_0 = 1),
\end{aligned}$$

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\end{aligned}
$$

Simplifying joint probabilities

When the Markov property holds!

If $\{X_n, n = 0, 1, 2, ...\}$ is a Markov process, show that

$$\mathrm{P}(X_0 = i_0, \ldots, X_n = i_n) = \prod_{k=1}^{n} \mathrm{P}(X_k = i_k \mid X_{k-1} = i_{k-1})\, \mathrm{P}(X_0 = i_0).$$

How can I tell if a process is Markov?

1. Without using any mathematics, deduce that the process is/ isn't Markov logically.
2. Checking (mathematically) that the Markov property holds for all states in the state space at all times.

How can I tell if a process is Markov?

Checking mathematically

Mainly used when we suspect that the Markov property does not hold by finding a **counter example** which does not satisfy the Markov property.

That is, find one set of states a, b, c and perhaps d such that **one** of the following holds:

$$P(Y_{n+1} = a | Y_n = b, Y_{n-1} = c) \neq P(Y_{n+1} = a | Y_n = b)$$
$$P(Y_{n+1} = a | Y_n = b, Y_{n-1} = c) \neq P(Y_{n+1} = a | Y_n = b, Y_{n-1} = d)$$

Why would either of these show that the Markov property does not hold?

Tips for finding a counter example

If $\{X_n, n = 0, 1, 2, ...\}$ is a Markov process, and $\{Y_n, n = 0, 1, 2, ...\}$ is derived from $\{X_n, n = 0, 1, 2, ...\}$ in some way, is $\{Y_n, n = 0, 1, 2, ...\}$ also a Markov process?

- Is the function in question bijective (one-to-one)? If it is, then the new process will also be Markov.
- If the function in question is not bijective, then the new process may, or may not be Markov. In this case, if you are trying to prove that the new process is *not* Markov, a good strategy is to exploit the fact that you do not have uniqueness.

Tips for finding a counter example

If the function in question is not bijective, then the new process may, or may not, be Markov. In this case, if you are trying to prove that the new process is not Markov, a good strategy is to exploit the fact that you do not have uniqueness.

When trying to find an example of one of the following:

$$P(Y_{n+1} = a | Y_n = b, Y_{n-1} = c) \neq P(Y_{n+1} = a | Y_n = b)$$
$$P(Y_{n+1} = a | Y_n = b, Y_{n-1} = c) \neq P(Y_{n+1} = a | Y_n = b, Y_{n-1} = d)$$

choose the state b for the process Y so that it prompts ambiguity in the state of the original process, X.

Examples

Let X_i, $i = 1, 2, 3, ...$ i.i.d. Bernoulli(0.3) random variables. Are either of the following modifications of X_i, $i = 1, 2, 3, ...$ a Markov process?

1. $Y_n = X_n + X_{n-1}$
2. $Z_n = X_n^2$