Please put in your tutor's postbox in Room 117 by 12pm Thursday March 10th.

1. A factory produces racing bikes. The first stage is to assemble the bike on the factory floor, which takes an exponentially distributed length of time with mean 10 hours. Once assembled, the bike is then immediately inspected.

   If the bike passes inspection (which happens at a rate of 1 per hour), then the bike is shipped to its new owner and does not return to the factory floor or for inspection. If it does not pass inspection (which happens at a rate of 0.5 per hour) then it is sent back to the factory floor to be reassembled.

   The distribution of time it takes to reassemble is the same as for the original assembly. This transfer between assembly and inspection will continue indefinitely until the bike passes inspection.

   Let X_t be the status of a bike at time t, and assume that $\{X_t; t \geq 0\}$ is a continuous time Markov chain with state space, S, given by:

   $$S = \{\text{'being assembled or reassembled', 'being inspected', 'shipped'}\}.$$

   You may find it useful to draw a state-space diagram before answering the following questions!

   (a) Write down the generator matrix of the process $\{X_t; t \geq 0\}$.

   (b) Find the invariant distribution of the process $\{X_t; t \geq 0\}$. Explain why it makes sense that this is the only invariant distribution.

   (c) Write down the transition matrix of the jump chain of the process $\{X_t; t \geq 0\}$.

   (d) What is the probability that a bike currently being inspected will pass (and so be shipped to its new owner)?

2. Suppose that $\{X_t; t \geq 0\}$ is a continuous time Markov chain with state space $S = \{1, 2, 3, 4\}$ and generator matrix, Q, given by:

   $$Q = \begin{pmatrix} -4 & 4 & 0 & 0 \\ 1 & -5 & 1 & 3 \\ 0 & 0 & -5 & 5 \\ 0 & 0 & 2 & -2 \end{pmatrix}.$$

   (a) Draw a state space diagram that describes how the process X_t moves from state to state.

   (b) The process has been in state 2 for t units of time. State the distribution of the further time that the process will stay in state 2 before leaving.

   (c) For very, very small $h > 0$ and any $s \geq 0$, give the following probabilities (using the $o(h)$ notation):

      (i) $P(X_h = 4 | X_0 = 2)$;

      (ii) $P(X_h = 4 | X_0 = 1)$ [the answer to this is not zero];

      (iii) $P(X_{s+h} = 4 | X_s = 2)$;

      (iv) $P(X_h = 4 | X_0 = 4)$.

   (d) State Kolmogorov's forward equation for $p_{43}(t)$ (i.e. the 'pair' (4,3)).
   Note: this is asking for ONE of the 16 differential equations, so the answer is NOT $P'(t) = P(t)Q$. Use the formula for the individual equations instead!

   (e) **Show that** (i.e. no need to solve differential equations!) the solution to Kolmogorov's forward equation you give in part (d) is

   $$p_{43}(t) = \frac{2}{7}\left(1 - \exp(-7t)\right).$$

   HINT: notice that you can replace $p_{44}(t) = 1 - p_{43}(t)$. Why? In addition, you also know what $p_{42}(t)$ is too (look at your state space diagram!).

Challenges

Questions in this section will not be covered in the tutorial. If you think you have solved the questions or would like some help, do post to the discussion forum.

It has been stated in the lectures that the individual equations from $\underline{0} = \underline{\pi}Q$ will always have one equation that is redundant, i.e. at least one equation is a linear combination of the other equations. Prove this statement. This is the continuous time equivalent of the question asked in a previous homework.