Please put in your tutor's postbox in Room 117 by 12pm Thursday March 3rd.

The purpose of this exercise sheet is to:

(i) Give you more practice in handling discrete time chains (though you will not get the tutorial for this sheet until after the ICA);

(ii) Make sure you understand how to handle $o(h)$;

(iii) Show some *very* useful properties of the exponential distribution.

1. Consider a Markov chain $\{X_n, n = 0, 1, 2, ...\}$ with state-space $S = \{1, 2, 3, 4, 5\}$ and transition matrix

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 1/2 & 0 \\ 1/2 & 0 & 0 & 1/2 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1/3 & 0 & 1/3 & 0 & 1/3 \end{pmatrix}$$

   (a) Find the irreducible classes of intercommunicating states and classify them in terms of positive or null recurrence, transience, periodicity and ergodicity.

   (b) Calculate $P(X_{10} = 5|X_0 = 2)$.

   (c) Does the Markov chain X_n have an equilibrium distribution? Justify your answer.

   (d) Without doing any calculations, state with a reason:

      (i) The mean recurrence time to state 5, if the chain starts in state 5.

      (ii) The 5th element, π_5, of all invariant distributions $\underline{\pi}$.

   (e) Suppose that $X_0 = 5$. Find expected time until the chain is in a closed class for the first time.

   (f) Let $Z_0 = 0$ and for $n = 1, 2, \ldots$, define $Z_n = |X_n - X_{n-1}|$.

      (i) Write down the state space of the stochastic process $\{Z_n, n = 0, 1, 2, \ldots\}$.

      (ii) Is Z_n a Markov chain? Either justify that it is, or give an example of where the Markov property breaks down.

2. Suppose that the functions $f(\cdot)$ and $g(\cdot)$ are both $o(h)$. Which of the following functions are also $o(h)$? **JUSTIFY** your answers.

   (a) $f(\cdot) + g(\cdot)$;

   (b) $cf(\cdot)$ for some constant $c > 0$;

   (c) $f(\cdot)g(\cdot)$.

3. If X_i, $i = 1, 2, 3, ...$ are independent exponential random variables with $X_i \sim \text{exponential}(\lambda_i)$, show that:

   (a) $P(X_1 < X_2) = \lambda_1/(\lambda_1 + \lambda_2)$;
   Hint: You've actually seen how to show this result in an example, both in lectures and in your course notes...

   (b) $P(X_j = \min\{X_1, X_2, ..., X_k\}) = \lambda_j/(\lambda_1 + \lambda_2 + ... + \lambda_k)$ for any $j \in \{1, ..., k\}$.

Challenges

Questions in this section will not be covered in the tutorial. If you think you have solved the questions or would like some help, do post to the discussion forum.

Think about the following procedure for shuffling a deck of n cards. Starting with any initial ordering, a card is randomly chosen (each card is equally likely to be chosen), and is placed on top of the deck. Repeat this procedure indefinitely. Show that, in the limit, the deck is perfectly shuffled in the sense that the resultant ordering is equally likely to be any of the $n!$ possible orderings.