THE CONNECTED COMPONENTS OF THE CHARACTER VARIETY OF THE FOUR-PUNCTURED SPHERE

MASON BARNES
ADVISED BY PROFESSOR SARA MALONI

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1. Introduction

In this paper we study the geometry of hyperbolizable oriented surfaces with punctures. The term hyperbolizable means the surfaces admits local charts into hyperbolic space $\mathbb{H}$ with transition maps given by hyperbolic isometries. There are many possible hyperbolic structures that can be given to a surface and the Teichmüller Space of the surface is the moduli space of all such such distinct structures.

The Teichmüller Space can also be viewed as the set of maps from the fundamental group $\pi_1(S)$ into $\mathrm{PSL}(2,\mathbb{R})$ satisfying certain conditions (discrete and faithful), up to conjugation by elements in $\mathrm{PSL}(2,\mathbb{R})$. It is natural to consider larger spaces of homomorphisms from $\pi_1(S)$ into $\mathrm{PSL}(2,\mathbb{R})$ (such maps are called *representations*) up to conjugation by elements in $\mathrm{PSL}(2,\mathbb{R})$. This is the motivation behind the character variety $\mathfrak{X}(S)$, the set of representations from $\pi_1(S)$ into $\mathrm{PSL}(2,\mathbb{R})$ up to conjugation in $\mathrm{PSL}(2,\mathbb{R})$.

In this paper, we will study topological, geometrical, and dynamical properties of such character varieties in the case of punctured surfaces. The motivation for these questions comes from the study of the fundamental group for closed surfaces.

Closed case: We define the $\mathrm{PSL}(2,\mathbb{R})$ character variety for a closed and oriented surface S_g of genus g to be

$$\mathfrak{X}(S_g) = \mathrm{Hom}(\pi_1(S_g), PSL(2,\mathbb{R}))/\mathrm{PSL}(2,\mathbb{R}).$$

For each map $\rho\colon \pi_1(S) \to \mathrm{PSL}(2,\mathbb{R})$, one can define the Euler class $e(\rho)$ as a value in $\mathbb{Z}$ and this does not depend on the conjugacy class of ρ. Milnor [12] and Wood [16] showed that $|e(\rho)| \leq 2g-2$, the character class of the surface. Goldman [3] proved that $|e(\rho)| = 2g-2$ if and only if the representation is discrete and faithful. In particular, this means that the set of maps satisfying $|e(\rho)| = 2g-2$ are the Teichmüller components $\mathfrak{T}(S) \sqcup \mathfrak{T}(\overline{S})$, where $\overline{S}$ is the surface S with the opposite orientation. Goldman [3] also showed that the connected components of the character variety $\mathfrak{X}(S)$ are given by the relative Euler class.

The group of symmetries of a surface, called the Mapping Class Group, is defined as

$$\mathrm{Mod}(S) \coloneqq \mathrm{Homeo}^+(S, \partial S)/\mathrm{Homeo}_0(S))$$

where $\mathrm{Homeo}^+(S)$ is the group of orientation preserving homeomorphisms of S and $\mathrm{Homeo}_0(S)$ is the group of homeomorphisms S which are homotopically equivalent to the identity. $Mod(S)$ acts on the the character variety as follows: given $[f] \in Mod(S)$ and $\rho \in \mathfrak{X}(S_g)$, the action of f on ρ is given by the induced group-automorphism f_*^{-1} on $\pi_1(S)$.

Fricke proved that that the Mapping Class Group acts properly discontinuously on the Teichmüller components, see for example [11]. Goldman conjectured [4] that the action of the Mapping Class Group on all the other connected components of the character variety is ergodic or “chaotic”. Goldman's conjecture is in stark contrast with Fricke's finding for the Teichmüller components. A geometric question of Bowditch [2] asks: is it true that almost-all totally hyperbolic representations are Fuchsian? Recall that a representation is called totally hyperbolic if the trace of the image of every simple closed curve is strictly bigger than 2, which is a characterization of hyperbolic elements of $\mathrm{PSL}(2,\mathbb{R})$. These conjectures are still open in most cases but they have been proven in the case $g = 2$ by Marché and Wolff [10] [9]. The case of Euler class 0 is more complicated because of hyperelliptic involution, see Marché and Wolff for more details. Furthermore, Marché and Wolff also showed that a positive answer to Bowditch's question implies a positive answer to Goldman's question.

Punctured case: In order to have a similar behavior for punctured hyperbolizable surfaces, we introduce the notion of type-preserving representations. A representation $\rho\colon \pi_1(S_{g,n}) \to \mathrm{PSL}(2,\mathbb{R})$ is *type-preserving* if it sends peripheral curves to parabolic elements in $\mathrm{PSL}(2,\mathbb{R})$. Recall that a peripheral curve is one which is freely homotopic to a circle about a puncture. We define the type-preserving character variety $\mathfrak{X}(S)$ to be the set of conjugacy classes of type-preserving representations:

$$\mathfrak{X}(S_{g,n}) = \{\, p\colon \pi_1(S_{g,n}) \to \mathrm{PSL}(2,\mathbb{R}) \mid \text{type-preserving}\,\}/\mathrm{PSL}(2,\mathbb{R}).$$

The above discussion can be generalized to type-preserving character varieties. In this paper, we will focus on three open questions for punctured surfaces. The first is a topological question due to Kashaev which asks about the connected components of $\mathfrak{X}(S_{g,n})$. Kashaev [6] conjectures that the connected components are characterized by the Euler class of a representation (as Goldman showed for the character variety of closed surfaces) along with another invariance, the the signs of the punctures. This is an assignment of signs $\{+1,-1\}$ to each puncture. The second, a geometric question of Bowditch asks: are almost-all type-preserving representation sending all non-peripheral loops to hyperbolic elements Fuchsian? The third, a dynamical question due to Goldman, wonders about the action of the mapping class group on the connected components of the character variety. For punctured surfaces, the mapping class group can be defined as

$$\mathrm{Mod}(S) \coloneqq \mathrm{Homeo}^+(S,\partial S)/\mathrm{Homeo}_0(S,\partial S).$$

where $\mathrm{Homeo}^+(S,\partial S)$ is the group of orientation preserving homeomorphisms fixing the boundary of S point-wise and $\mathrm{Homeo}_0(S,\partial S)$ is the group of orientation preserving homeomorphisms homotopically equivalent to the identity.

Goldman conjectures that the action on the non-Teichmüller components of $\mathfrak{X}(S)$ is ergodic. Just as with Bowditch, this question can be modified to the context of type-preserving representations $\mathfrak{X}(S)$.

We will investigate Tian Yang's [17] answers to these questions specifically in the case of the four punctured sphere $S_{0,4}$. In this case, Kashaev's conjecture for the connected components of $\mathfrak{X}(S)$ turns out to be true, Goldman's conjuecture about the chaotic group action on non-external components is true as well, but Bowditch's question turns out to be false for certain connected components with Euler class ± 1. We will follow the format of a similar paper in the case of the thrice punctured projective plane by Maloni, Palesi, and Yang [8].

Plan for the paper: At the start, we introduce hyperbolic space $\mathbb{H}$ and explore some of the properties of this space. The symmetries and isometries of hyperbolic space will be neatly classified. We will then discuss what it means for a surface to have hyperbolic structure and find a nice "closed form" for such hyperbolic surfaces. Finally, we will discuss the Teichmüller Space of a surface. We will see that this space corresponds with a certain set of homomorphisms which will provide us information about the possible hyperbolic structures of a surface.

We will then discuss the Mapping Class Group $\mathrm{Mod}(S)$ of a surface which formalizes the notion of boundary-preserving and orientation-preserving self-homeomorphisms as described in the discussion of the Goldman conjecture. We will consider some specific examples of the Mapping Class Group and give a brief explanation of how the Mapping Class Group acts on the Teichmüller Space of the surface in a nice, properly discontinuous way.

In the final section, we will discuss the type-preserving Character Variety of a punctured surface. We will see that the Teichmüller Space is contained in the Character Variety. Specifically, Teichmüller Space can be seen as a connected component of the Character Variety. We will then explore some conjectures about the action of the Mapping Class Group on the Character Variety. Will the Mapping Class Group act as nicely on the entire Character Variety as it does on just the Teichmüller Space component? We end the paper with Tian Yang's [17] answers to these conjectures in the specific case of the four-punctured sphere, written in the format of Maloni, Palesi, and Yang's paper [8].

2. Hyperbolic Geometry

2.1. **Models of Hyperbolic Space.** Hyperbolic Space is the space with constant Gaussian curvature -1. There are many reasons to consider hyperbolic space. For one, any surface with negative Euler characteristic will have a hyperbolic geometric structure. Therefore hyperbolic geometry is very important for the study of surfaces. Another reason is the historical problem of Euclid's parallel postulate. Many mathematicians made attempts to derive Euclid's parallel postulate from Euclid's other axioms of geometry. (Recall that Euclid's parallel or fifth postulate is the fact that given a line L and a point P not on L, there is a unique line through P not intersecting L). Hyperbolic geometry does not satisfy the parallel postulate but does satisfy Euclid's other axioms.

Definition 2.1. There are two commonly used models of hyperbolic space.

(1) The Upper-Half Plane $\mathbb{H}$ is the set
$$\{\, x+iy \in \mathbb{C} \mid y > 0 \,\}$$
together with the metric
$$ds = \frac{\sqrt{dx^2+dy^2}}{y}.$$
(2) The Poincaré Disk $\mathbb{D}$ is the set
$$\{\, x+iy \in \mathbb{C} \mid |x+iy| < 1 \,\}$$
together with the metric
$$ds = \frac{2\sqrt{dx^2+dy^2}}{1-x^2-y^2}.$$

Note that both ds above are Riemannian metrics. This allows us to define lengths and angles in $\mathbb{H}$ and $\mathbb{D}$ in a well-defined way.

In this paper, we will be focusing on the upper-half plane model $\mathbb{H}$ of hyperbolic space.

Theorem 2.2. *$\mathbb{H}$ and $\mathbb{D}$ have constant curvature -1.*

Given a metric $ds^2 = g_1 dx^2 + g_2 dy^2$ on a surface, the Gaussian curvature is calculated by the following formula [13]:
$$K = \frac{1}{\sqrt{g_1 g_2}} \left(\frac{\partial}{\partial x} \left(\frac{1}{\sqrt{g_1}} \frac{\partial \sqrt{g_2}}{\partial x} \right) + \frac{\partial}{\partial y} \left(\frac{1}{\sqrt{g_2}} \frac{\partial \sqrt{g_1}}{\partial y} \right) \right).$$
In the upper-half plane model $\mathbb{H}$,
$$ds^2 = \frac{1}{y^2}(dx^2+dy^2).$$
On the other hand, in the disk model $\mathbb{D}$,
$$ds^2 = \frac{4}{(1-x^2-y^2)^2}(dx^2+dy^2).$$

Using the above formula, we calculate the Gaussian curvature of $\mathbb{H}$ and $\mathbb{D}$ to be -1. Therefore both $\mathbb{H}$ and $\mathbb{D}$ model hyperbolic space.

We now turn to an immediate application of these metrics by using a familiar tool from Calculus:

Definition 2.3. Let $\gamma(t) = x(t) + iy(t) \colon [0,1] \to \mathbb{H}$ be a curve. Then the length of $\gamma(t)$ is given by

$$\ell(\gamma) := \int_0^1 \frac{1}{y(t)} \sqrt{x'(t)^2 + y'(t)^2}\ dt.$$

Similarly, let $\alpha(t) = x(t) + iy(t) \colon [0,1] \to \mathbb{D}$ be a curve. Then the length of $\alpha(t)$ is given by

$$\ell(\alpha) := \int_0^1 \frac{2}{1 - x(t)^2 - y(t)^2} \sqrt{x'(t)^2 + y'(t)^2}\ dt.$$

Note the differences in the hyperbolic arc length formulas compared to the Euclidean arc length formula. If $\beta(t) = (x(t), y(t)) \colon [0,1] \to \mathbb{R}^2$ is a curve then the length of $\beta(t)$ is given by

$$\ell(\beta) := \int_0^1 \sqrt{x'(t)^2 + y'(t)^2}\ dt.$$

In particular, the position of a curve in either $\mathbb{H}$ or $\mathbb{D}$ "distorts" the length of the curve. In $\mathbb{H}$, a segment close to the real axis will be much longer than in $\mathbb{R}^2$. On the other hand, lines near the center of $\mathbb{D}$ more closely match their Euclidean length while arcs near $\partial\mathbb{D}$ are longer than they appear.

Definition 2.4. We define the distance between $x, y \in \mathbb{H}$ to be

$$d(x,y) = \inf \ell(\gamma),$$

where the infimum is considered over all possible paths $\gamma \colon [0,1] \to \mathbb{H}$ such that $\gamma(0) = x$ and $\gamma(1) = y$.

One can verify that the pair $(\mathbb{H}, d)$ is a metric space. In the exact same way, distance can be defined in the disk model $\mathbb{D}$.

A *geodesic* is a path between two points with minimal arc length. In Euclidean space, this would be the straight line between two points.

Proposition 2.5. *Vertical lines are geodesics in* $\mathbb{H}$.

Proof. Let $a + bi, a + ci \in \mathbb{H}$ be two points with the same real part, and let $\gamma(t) = x(t) + iy(t) \colon [0,1] \to \mathbb{H}$ be any path with $\gamma(0) = a + bi, \gamma(1) = a + ci$. Then

$$\ell(\gamma) = \int_0^1 \frac{1}{y(t)} \sqrt{x'(t)^2 + y'(t)^2}\ dt \geq \int_0^1 \frac{1}{y(t)} \sqrt{y'(t)^2}\ dt = \ell(a + iy(t)).$$

In particular, paths with constant real part minimize arc length. We conclude that the vertical line between $a + bi$ and $a + ci$ are geodesics.

Knowing this geodesic, we can explicitly solve for $d(a+bi, a+ci)$. Let $\alpha(t) = a + i(b + (c-b)t)$ be a path tracing out the vertical line from $a+bi$ to $a+ci$. We have

$$d(a+bi, a+ci) = \ell(\alpha) = \int_0^1 \frac{1}{b+(c-b)t}\sqrt{(c-b)^2}\, dt = \int_b^c \frac{1}{t}\, dt = \log\left(\frac{c}{b}\right).$$

□

2.2. **Möbius Transformations.** We will now shift our attention to symmetries of hyperbolic space which will be neatly classified into three distinct types of transformations. In order to discuss the symmetries of $\mathbb{H}$ or $\mathbb{D}$ however, we first need to discuss the symmetries of a larger space which contains both of these models as subsets: the Riemann Sphere.

Definition 2.6. The Riemann Sphere $\widehat{\mathbb{C}}$ is the one point compactification of $\mathbb{C}$. We add one extra point, ∞, so that $\lim_{|z|\to 0} \frac{1}{z} = \infty$. The topology of $\widehat{\mathbb{C}}$ is induced by $\mathbb{C}$.

The Riemann Sphere is compact as it includes the one missing limit point of $\mathbb{C}$. The Riemann Sphere can be imagined as a sphere by "wrapping" the complex plane into a ball so that 0 lies at the south pole and the point ∞ lies at the north pole. One can give an explicit formula for this construction via stereographic projection.

Noting that $\mathbb{H}$ and $\mathbb{D}$ are contained in $\widehat{\mathbb{C}}$ as subsets, our understanding of the symmetries of $\widehat{\mathbb{C}}$ allows us to understand the symmetries of $\mathbb{H}$ and $\mathbb{D}$. In particular, we will study the symmetries of $\widehat{\mathbb{C}}$ which preserve $\mathbb{H}$ and $\mathbb{D}$ set-wise. For now, we turn toward an important family of symmetries of $\widehat{\mathbb{C}}$.

Definition 2.7 (Möbius maps)**.** A Möbius map is a map $f\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ such that $\forall z \in \mathbb{C}$, $f(z) = \dfrac{az+b}{cz+d}$ where $a, b, c, d \in \mathbb{C}$ and $ad - bc \neq 0$, and $f(\infty) = \frac{a}{c}$.

The decision for $f(\infty) = \frac{a}{c}$ is because of the fact that

$$\lim_{|z|\to\infty} \frac{az+b}{cz+d} = \lim_{|z|\to\infty} \frac{a+\frac{b}{z}}{c+\frac{d}{z}} = \frac{a}{c}.$$

Furthermore, we note that $f(\frac{-d}{c}) = \infty$. After evaluating a Möbius map at $\frac{-d}{c}$, the numerator must be nonzero (as we require $ad - bc \neq 0$) and the denominator is zero. Therefore, we can take a limit to see that $f(\frac{-d}{c}) = \infty$.

Möbius maps are invertible with inverse given by

$$z \to \frac{dz-b}{-cz+a}.$$

One can further check that the composition of two Möbius maps is again a Möbius map. Therefore, the set of Möbius maps under function composition is a group, which we denote temporarily denote by Möb.

Proposition 2.8. *Möb* $\cong PSL(2, \mathbb{C})$.

Proof. We notice that in the definition of a Möbius Map above, we can equivalently require the stronger condition that $ad-bc=1$ by scaling all of the coefficients by a common factor $k \in \mathbb{C}$:

$$\frac{az+b}{cz+d} = \frac{k}{k} * \frac{az+b}{cz+d} = \frac{kaz+kb}{kcz+kd} = \frac{a'z+b'}{c'z+d'}.$$

By choosing $k = \frac{1}{\sqrt{ad-bc}}$, the new coefficients of the transformation give $a'd' - b'c' = 1$. Therefore we can always further restrict Möbius maps to a form in which the coefficients satisfy $ad - bc = 1$.

Consider the map $F\colon SL(2,\mathbb{C}) \to Mob$ given by:

$$F\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{az+b}{cz+d}$$

One can check that F is a group homomorphism. Note that

$$\frac{az+b}{cz+d} = z \iff cz^2 + dz - az - b = 0 \ \forall z$$

which gives that

$$\mathrm{Ker}(F) = \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{C}) \mid c = b = 0,\ d = a \}$$

Note that as $ad-bc=1$ for all matrices in $\mathrm{SL}(2,\mathbb{C})$, we get that $a=d=1$ or $a=d=-1$, so $\ker(F) = \pm I$. To see that F is surjective, take any Möbius map $\dfrac{az+b}{cz+d}$ represented by coefficients $a,b,c,d \in \mathbb{C}$ such that $ad-bc=1$ which is possible by the discussion above. Then $F\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \dfrac{az+b}{cz+d}$.

By the first isomorphism theorem, $Mob \cong \mathrm{SL}(2,\mathbb{C})/\pm\{I\} =: \mathrm{PSL}(2,\mathbb{C})$. We call $\mathrm{PSL}(2,\mathbb{C})$ the projective special linear group. □

Theorem 2.9. *For any two ordered triples $(z_1,z_2,z_3),(w_1,w_2,w_3) \in \widehat{\mathbb{C}}^3$ there exists a unique Möbius map T with $T(z_i) = w_i$. In other words, a Möbius map is determined by where it sends three points.*

Proof. First consider the Möbius map

$$T(z) = \frac{z-z_1}{z-z_2} * \frac{z_3-z_2}{z_3-z_1} \text{ where } z_1,z_2,z_3 \in \mathbb{C}$$

We have that $T(z_1) = 0, T(z_2) = \infty, T(z_3) = 1$, so we can always find a Möbius map that sends any three points to $0,\infty,1$.

Let $(z_1,z_2,z_3),(w_1,w_2,w_3) \in \widehat{\mathbb{C}}^3$. By the above we can find Möbius maps T_1,T_2 such that

$$T_1(z_1) = 0, T_1(z_2) = \infty, T_1(z_3) = 1$$
$$T_2(w_1) = 0, T_2(w_2) = \infty, T_2(w_3) = 1$$

Then $T_2^{-1} \circ T_1$ is a Möbius map which sends the ordered triple $(z_1,z_2,z_3) \mapsto (0,\infty,1) \mapsto (w_1,w_2,w_3)$. Therefore we can always find a Möbius map that sends any three points to any other three points.

To show this map is unique, we first claim that any Möbius map fixing the points $0, \infty, 1$ must be the identity. Suppose $\exists\ W \in \mathrm{Aut}(\widehat{\mathbb{C}})$ such that

$$W(0) = 0, W(\infty) = \infty, W(1) = 1$$

Then

$$\begin{aligned} W(0) &= \frac{b}{d} = 0 \implies b = 0, \\ W(\infty) &= \frac{a}{c} = \infty \implies c = 0, \\ W(1) &= \frac{az}{d} = 1 \implies ad = 1 \implies a = d = 1 \text{ or } a = d = -1. \end{aligned}$$

Therefore $W(z) = z$ is the identity.

To prove uniqueness, suppose there was another Möbius map $F, F \neq T_2^{-1} \circ T_1$ with the property that $F(z_1) = w_1, F(z_2) = w_2, F(z_3) = w_3$. Then by the above, $T_2 \circ F \circ T_1^{-1}$ sends the ordered triple $(0, \infty, 1) \mapsto (z_1, z_2, z_3) \mapsto (w_1, w_2, w_3) \mapsto (0, \infty, 1)$. Therefore, $T_2 \circ F \circ T_1^{-1}$ fixes $(0, \infty, 1)$, so $T_2 \circ F \circ T_1^{-1} = id \implies F = T_2^{-1} \circ T_1$. Therefore $T_2^{-1} \circ T_1$ is the unique Möbius map sending $(z_1, z_2, z_3) \mapsto (w_1, w_2, w_3)$. □

Recall that a group action is free if the action is injective, and a group action is transitive if $\forall x, y \in X\ \exists\ g \in G$ such that $gx = y$. The proof above shows that Möbius maps act freely and transitively on $\widehat{\mathbb{C}}^3$.

Theorem 2.10. *Möbius maps are angle-preserving*

Recall from Complex Analysis that analytic maps are angle-preserving where the derivative is nonzero[15]. We will first show that Möbius maps are analytic on $\widehat{\mathbb{C}}$ and then show they have nonvanishing derivative.

Proof. Möbius maps are a composition of entire maps. Recall from Complex Analysis that an entire map is everywhere analytic.

(1) $az + b$ is entire as it is a polynomial.
(2) $\dfrac{1}{z}$ is entire on $\widehat{\mathbb{C}}$ as $\infty = \lim \frac{1}{z_n}$.
(3) By the above two facts, $\dfrac{1}{cz+d}$ is entire as the composition of entire maps is entire.
(4) Finally, $\dfrac{az+b}{cz+d}$ is entire as multiplication of entire functions is entire.

The derivative of any Möbius map f is nonzero on $\mathbb{C}$

$$f'(z) = \frac{ad - bc}{(cz+d)^2} \neq 0$$

To check that f is conformal at ∞, we calculate

$$\lim_{|z|->\infty} f'\left(\frac{1}{z}\right) = \lim_{|z|->\infty} \frac{1}{\left(\frac{c}{z} + d\right)^2} = \frac{1}{d^2} \neq 0.$$

Therefore Möbius maps are angle-preserving. □

If we think of the symmetries of a space as differentiable angle-preserving bijections, we have shown that Möbius maps are symmetries of $\widehat{\mathbb{C}}$. We will now prove the fact that Möbius maps are the only symmetries of $\widehat{\mathbb{C}}$.

Theorem 2.11. *$Aut(\widehat{\mathbb{C}}) \cong PSL(2, \mathbb{C})$*

Define $\mathrm{Aut}(\widehat{\mathbb{C}})$ to be the group of all analytic angle-preserving bijective maps of $\widehat{\mathbb{C}}$ under function composition. We have already shown that Möbius maps are contained in $\mathrm{Aut}(\widehat{\mathbb{C}})$. The following proof shows that all automorphisms of $\widehat{\mathbb{C}}$ are Möbius maps.

Proof. We will show that all bijective angle-preserving maps are Möbius maps. This immediately gives that $\mathrm{Aut}(\hat{\mathbb{C}}) \cong P\mathrm{SL}(2, \mathbb{C})$ by Proposition 2.8.

Let $f \in \mathrm{Aut}(\widehat{\mathbb{C}})$. As f is bijective, $\exists\ z_1, z_2, z_3 \in \widehat{\mathbb{C}}$ such that $f(0) = z_1, f(1) = z_2, f(\infty) = z_3$. Let T be the unique Möbius Transformation such that $T(z_1) = 0, T(z_2) = 1, T(z_3) = \infty$, and take $g = T \circ f$.

Then g is the composition of conformal and analytic maps on $\widehat{\mathbb{C}}$ fixing the points $0, 1, \infty$. Consider the map $\frac{g(z)}{z}$ which is analytic on $\widehat{\mathbb{C}}$.

We will show that $\frac{g(z)}{z}$ is bounded on $\widehat{\mathbb{C}}$ by showing it is bounded in closed disks centered at 0 and ∞. By taking large enough closed disks, this will show that $\frac{g(z)}{z}$ is bounded in $\mathbb{C}$.

First, note that

$$\lim_{z \to 0} \frac{g(z)}{z} = \lim_{z \to 0} \frac{g(z) - g(0)}{z - 0} = g'(0).$$

Therefore, $\frac{g(z)}{z}$ is bounded on closed disks about 0 as continuous functions are bounded on compact sets.

To show $\frac{g(z)}{z}$ is bounded in closed disks centered at ∞, first consider the map $G(w) = \frac{1}{g(\frac{1}{w})}$, a composition of conformal and analytic maps in $\widehat{\mathbb{C}}$.

We then get

$$\lim_{z \to \infty} \frac{g(z)}{z} = \frac{1}{\lim_{z \to \infty} \dfrac{z}{g(z)}},$$

where

$$\begin{aligned} \lim_{z \to \infty} \frac{z}{g(z)} &= \lim_{z \to \infty} \frac{\frac{1}{g(z)}}{\frac{1}{z}} \\ &= \lim_{w \to 0} \frac{\frac{1}{g(\frac{1}{w})} - \frac{1}{\frac{1}{0}}}{w - 0} \\ &= G'(0). \end{aligned}$$

Therefore, $\frac{g(z)}{z}$ is bounded on closed disks about ∞ as continuous maps are bounded on compact sets. It follows that g is bounded on $\mathbb{C}$ by taking large enough disks covering $\widehat{\mathbb{C}}$. Then by Liouville's Theorem, $\frac{g(z)}{z}$ is constant on $\mathbb{C}$ which then means that $\frac{g(z)}{z} = c$ for some $c \in \mathbb{C}$. As $\frac{g(z)}{z}$ fixes 1, it follows that $g = id$.

As $g = f \circ T$ and the inverse of T is unique, we conclude that $f = T^{-1}$ which is a Möbius map. Therefore every automorphism of $\widehat{\mathbb{C}}$ is a Möbius map. $\square$

2.3. **Isometries of $\mathbb{H}$.** Now that we understand the symmetries of $\widehat{\mathbb{C}}$, we can turn our attention toward the symmetries and isometries of $\mathbb{H}$.

Definition 2.12 (Automorphisms of Hyperbolic Space)**.** We define the automorphisms of $\mathbb{H}$ as

$$\mathrm{Aut}(\mathbb{H}) = \{\, f \in \mathrm{Aut}(\widehat{\mathbb{C}}) \mid f(\mathbb{H}) = \mathbb{H} \,\}.$$

Theorem 2.13. *$Aut(\mathbb{H}) \cong PSL(2, \mathbb{R})$*

Proof. We will prove that the angle-preserving analytic bijections of $\mathbb{H}$ are exactly the Möbius maps with real coefficients.

Let $f(z) = \dfrac{az+b}{cz+d} \in \mathrm{Aut}(\widehat{\mathbb{C}})$ be a Möbius map such that $a, b, c, d \in \mathbb{R}$ and $ad - bc = 1$. We would like to show that f fixes $\mathbb{H}$ set-wise. We use Im to denote the imaginary part of a complex number. Then

$$\mathrm{Im}\left(\frac{az+b}{cz+d}\right) = \mathrm{Im}\left(\frac{(az+b)\overline{(cz+d)}}{|cz+d|^2}\right).$$

The denominator is a positive real number. Check the numerator

$$\begin{aligned}\mathrm{Im}((az+b)\overline{(cz+d)}) &= \mathrm{Im}((az+b)(c\overline{z}+d)) \\ &= \mathrm{Im}(ac|z|^2 + adz + bc\overline{z} + bd) \\ &= ad\,\mathrm{Im}(z) - bc\,\mathrm{Im}(z) \\ &= \mathrm{Im}(z) > 0\end{aligned}$$

Therefore $f(\mathbb{H}) \subset \mathbb{H} \implies f(\mathbb{H}) = \mathbb{H}$ as Möbius maps are bijective. So Möbius maps with real coefficients are automorphisms of $\mathbb{H}$.

To show the other direction, let $G : \mathrm{PSL}(2, \mathbb{R}) \to \mathrm{Aut}(\mathbb{H})$ be the restriction of the isomorphism $F\colon P\mathrm{SL}(2, \mathbb{C}) \to \mathrm{Aut}(\widehat{\mathbb{C}})$ from Theorem 2.11. G is well-defined by the first part of the proof. G is also injective as it is the restriciton of an injective map, so it is left to show G is surjective.

Let $g = \dfrac{az+b}{cz+d} \in \mathrm{Aut}(\mathbb{H})$. First, note that $g(\overline{\mathbb{H}}) \subseteq \overline{g(\mathbb{H})} = \overline{\mathbb{H}}$. As $g(\mathbb{H}) \subset \mathbb{H}$, then g must fix $\partial\mathbb{H} = \mathbb{R} \cup \{\infty\}$.

Therefore, $g(0) = \frac{b}{d}$, $g(\infty) = \frac{a}{c} \in \mathbb{R} \cup \{\infty\}$. Furthermore, $g^{-1}(0) = \frac{-b}{a}$, $g^{-1}(\infty) = \frac{-d}{c} \in \mathbb{R} \cup \{\infty\}$ by applying the same logic to the inverse map.

As $ad - bc = 1$, then a, b, c, d cannot all be 0. We will show the case that $a \neq 0$; the other cases are similar.

We break into two further subcases. The first subcase is that either b or c are nonzero. In this case, either $\frac{d}{a} = \frac{d}{b} * \frac{b}{a}$ or $\frac{d}{a} = \frac{d}{c} * \frac{c}{a}$. Therefore $\frac{d}{a}$ is a real number. Then we can rewrite

$$\frac{az+b}{cz+d} = \frac{z + \frac{b}{a}}{\frac{c}{a}z + \frac{d}{a}} \in G(\mathrm{PSL}(2, \mathbb{R})).$$

The second subcase is that both b and c are zero. Then we get

$$\frac{az+b}{cz+d} = \frac{a}{d}z \in G(\mathrm{PSL}(2, \mathbb{R})).$$

In either case, $\dfrac{az+b}{cz+d}$ can be represented as a real-valued matrix in $\mathrm{PSL}(2, \mathbb{R})$. Again, the cases that b, c, or d are non-zero are similar. $\square$

Theorem 2.14. *Möbius maps in* $Aut(\mathbb{H})$ *are orientation-preserving isometries.*

An isometry of $\mathbb{H}$ is a map $f : \mathbb{H} \to \mathbb{H}$ such that

$$d(z, w) = d(f(z), f(w)) \; \forall z, w \in \mathbb{H}$$

and so that $\det(D(f)) \neq 0$ where D is the Jacobian matrix of f viewed as a function $\mathbb{R}^2 \to \mathbb{R}^2$. We call the isometry orientation preserving if $\det(D(f)) > 0$. Similarly, an isometry is orientation reversing if $\det(D(f)) < 0$. We define $\mathrm{Isom}^+(\mathbb{H})$ to be the set of orientation preserving isometries of $\mathbb{H}$ and $\mathrm{Isom}^-(\mathbb{H})$ to be the set of orientation-reversing isometries of $\mathbb{H}$. Finally, $\mathrm{Isom}(\mathbb{H})$ will denote the set of all isometries of $\mathbb{H}$.

We will prove that maps in $\mathrm{Aut}(\mathbb{H})$ preserve arc length and are therefore isometries. As Möbius maps are angle preserving, we will conclude that they are orientation preserving isometries of $\mathbb{H}$. In fact, We will later show that all orientation preserving isometries of $\mathbb{H}$ are Möbius maps.

Proof. Let $\gamma = (x(t), y(t)) : [0, 1] \to \mathbb{H}$ be a curve and $T(z) = \dfrac{az+b}{cz+d}$ be a Möbius map. Then the arc length of $T(\gamma)$ is computed as

$$\begin{aligned}
\ell(T(\gamma)) &= \int_0^1 \frac{1}{\mathrm{Im}(T(\gamma(t)))} \sqrt{\left(\frac{d}{dt}\mathrm{Re}(T(\gamma(t)))\right)^2 + \left(\frac{d}{dt}\mathrm{Im}(T(\gamma(t)))\right)^2} \, dt \\
&= \int_0^1 \frac{1}{\mathrm{Im}(T(\gamma(t)))} \left| \frac{d}{dt} T(\gamma(t)) \right| \, dt.
\end{aligned}$$

Note that

$$\frac{d}{dt}T(\gamma(t)) = \frac{d}{dt}\frac{a\gamma(t)+b}{c\gamma(t)+d} = \frac{ad-bc}{(c\gamma(t)+d)^2}\gamma'(t)$$

by the chain rule. Furthermore, note that

$$\mathrm{Im}(T(\gamma(t))) = \frac{(ad-bc)y(t)}{|c\gamma(t)+d|^2}$$

by use of the calculation at the beginning of Theorem 2.14. Then

$$\begin{aligned}\int_0^1 \frac{1}{\operatorname{Im}(T(\gamma(t)))}\left|\frac{d}{dt}T(\gamma(t))\right| \, dt &= \int_0^1 \frac{|c\gamma(t)+d|^2}{(ad-bc)y(t)}\frac{(ad-bc)|\gamma'(t)|}{|c\gamma(t)+d|^2}\, dt \\ &= \int_0^1 \frac{1}{y(t)}\sqrt{x'(t)^2+y'(t)^2}\, dt \\ &= \ell(\gamma).\end{aligned}$$

Therefore Möbius maps are orientation-preserving isometries. □

Proposition 2.15. *$Isom^+(\mathbb{H}) \cong PSL(2,\mathbb{R})$.*

Proof. We already know that $\mathrm{PSL}(2,\mathbb{R}) \cong \mathrm{Aut}(\mathbb{H}) \subset \mathrm{Isom}^+(\mathbb{H})$ as Möbius maps preserve arc-length. In fact, we also already know that $\mathrm{Isom}^+(\mathbb{H}) \subset \mathrm{Aut}(\mathbb{H}) \cong \mathrm{PSL}(2,\mathbb{R})$ as these maps are bijective, differentiable, and angle-preserving (determinant of Jacobian matrix is positive). Therefore $\mathrm{Isom}^+(\mathbb{H}) \cong \mathrm{PSL}(2,\mathbb{R})$. □

On the other hand, one can show that the group of all isometries $\mathrm{Isom}(\mathbb{H}) \cong \mathrm{PGL}(2,\mathbb{R})$. However, in this paper, we will only consider orientation-preserving isometries of $\mathbb{H}$ as we are only considering orientable surfaces in later sections.

We now apply the above understanding of orientation preserving isometries in order to classify all possible geodesics in hyperbolic space.

Theorem 2.16. *The geodesics in hyperbolic space are:*

(1) Vertical lines in $\mathbb{H}$;
(2) Semi-circles centered on the real axis in $\mathbb{H}$.

In order to classify all geodesics in hyperbolic space, we will start with Proposition 2.5 which states that vertical lines in $\mathbb{H}$ are geodesics. We will then apply isometries to find the other geodesics.

Proof. Let P and Q be two points in $\mathbb{H}$ with different real part, so that there is not a vertical line joining the two points. Consider the unique (Euclidean) circle passing through P and Q such that the center of the circle lies on the real axis. Call this center point $X \in \mathbb{R}$ and let $r \in \mathbb{R}^+$ be the radius. Let $\gamma(t) = r\cos(t) + X + ir\sin(t)$ parameterize this circle, $t \in [-\pi, \pi]$.

Denote the points where the circle intersects the real axis by $a = X - r$ and $b = X + r$. Then consider the Möbius Transformation $T = \dfrac{z-b}{z-a} \in \mathrm{Aut}(\mathbb{H})$. Composing with $\gamma(t)$,

$$T(\gamma(t)) = \frac{r\cos(t)+X+ir\sin(t)-b}{r\cos(t)+X+ir\sin(t)-a} = \frac{i2r^2\sin(t)}{|r\cos(t)+r+ir\sin(t)|^2}$$

which is contained in the vertical axis $i\mathbb{R}$.

In particular, $T(\gamma)$ minimizes the distance between $T(P)$ and $T(Q)$ by Proposition 2.5. As Möbius maps are isometries, it follows that γ minimizes the distance betwen P and Q. Therefore, geodesics in $\mathbb{H}$ are either vertical lines or arcs of semi-circles centered on the real axis. $\square$

We can now see why the parallel postulate fails in hyperbolic space. Consider the vertical line $i\mathbb{R}$ and any point disjoint from this line. Then there are an infinitely many semi-circles centered on the real axis passing through the point which do not intersect $i\mathbb{R}$.

We now turn our attention toward another application of our understanding of geodesics.

Definition 2.17 (Hyperbolic Triangles)**.** A triangle in hyperbolic space is formed by three geodesic segments which intersect each other at one point. We will show that the area of a triangle in hyperbolic space is given by the difference between π and the sum of the three angles of the triangle. This means that we cannot find triangles in hyperbolic space that have different areas and the same set of angles. This is different from Euclidean space where congruent triangles can have different areas.

We also define an **ideal triangle** to be a type of hyperbolic triangle where the three geodesics intersect each other only on $\partial\mathbb{H}$. This means that in an ideal triangle, the angle between any two sides is 0 degrees.

Proposition 2.18 (Area of a Hyperbolic Triangle)**.** *Let α, β, γ be the three angles of a hyperbolic triangle T. Then the area of T is*

$$A(T) \coloneqq \pi - (\alpha + \beta + \gamma).$$

Proof. The area formula for a region of hyperbolic space is defined by the double integral

$$A(C) = \iint_C \frac{1}{y^2} dxdy.$$

Note the similarity to the arc-length formula. This means that isometries of $\mathbb{H}$ preserve area just as they preserve arc length.

This proof is done in three steps:

(1) The area of a triangle with angles 0, α, and $\pi/2$ is $\pi/2 - \alpha$.

Consider the triangle T_0 with vertices at i, ∞, and $e^{i\alpha}$. Then T_0 has angles 0, α, and $\pi/2$.

$$A(T_0) = \int_0^{\cos(\alpha)} \int_{\sqrt{1-x^2}}^{\infty} \frac{1}{y^2} dydx = \int_0^{\cos(\alpha)} \frac{1}{\sqrt{1-x^2}} dx = \frac{\pi}{2} - \alpha.$$

Now let T be any triangle with angles 0, α, and $\pi/2$. As isometries of $\mathbb{H}$ act on ordered triples of points transitively (Theorem 2.9) and fix $\partial\mathbb{H}$ set-wise (check Theorem 2.13), there is an orientation-preserving isometry sending the three vertices of T to i, ∞, and $e^{i\alpha}$. Such a map is angle-preserving and area-preserving, hence any ideal triangle has area π.

(2) The area of a triangle with angles 0, α, and β is $\pi - (\alpha + \beta)$.

Let T_1 be a triangle with angles 0, α, and β. We can break T_1 into two triangles with angles $0, \pi/2, \alpha$ and $0, \pi/2, \beta$. Using (1) and linearity of integration, we get that $A(T_1) = \pi - (\alpha + \beta)$.

(3) The area of a triangle with angles α, β, and γ is $\pi - (\alpha + \beta + \gamma)$.

Let ABC be a triangle with angles α, β, and γ and vertices ABC. Extend the side AC until it meets $\partial\mathbb{H}$ at a point X. This line forms two new triangles ABX and BCX such that $A(ABC) = A(ABX) - A(BCX)$. We then use the results above to compute that $A(ABC) = \pi - (\alpha + \beta + \gamma)$. □

We now turn toward classifying the isometries of $\mathbb{H}$.

Remark 2.19. Let $\dfrac{az+b}{cz+d} \in \mathrm{Aut}(\mathbb{H})$. We would like to find the fixed points of this transformation. Note that

$$\frac{az+b}{cz+d} = z \iff cz^2 + (d-a)z - b = 0.$$

By the quadratic formula, the solutions to this equation are

$$z = \frac{a - d \pm \sqrt{(d-a)^2 + 4bc}}{2c} = \frac{a - d \pm \sqrt{(a+d)^2 - 4}}{2c}.$$

While the trace of a matrix in $\mathrm{PSL}(2,\mathbb{R})$ is not well-defined, the square of the trace is well-defined.

We classify the maps in $\mathrm{Aut}(\mathbb{H}) - \{I\}$ into three distinct categories. In this classification we can either use the trace, the fixed points, or the conjugation class:

- A **parabolic** transformation is a map in $\mathrm{Aut}(\mathbb{H})$ with trace ± 2. As $\sqrt{(a+d)^2 - 4} = 0$, Parabolic transformations have one fixed point in $\mathbb{R} \cup \{\infty\}$. By conjugating a parabolic transformation by any map in $\mathrm{Aut}(\mathbb{H})$ that sends its unique fixed point to infinity, we see that all parabolic maps are conjugate to a map of the form $z + n$ for some $n \in \mathbb{R} - \{0\}$. In this way, we see that parabolic transformations are analogous to Euclidean translations.
- A **hyperbolic transformation** is a map with trace in the range $(-\infty, -2) \cup (2, \infty)$. These maps are conjugate to a map of the form $a^2 z$ where $a \in \mathbb{R}_+ - \{1\}$. Therefore, these maps can be seen as analogs of Euclidean hometheties (expansions/contractions). A hyperbolic transformation satisfies $\sqrt{(a+d)^2 - 4} > 0$. Therefore hyperbolic transformations have two fixed points in $\mathbb{R} \cup \{\infty\}$.
- An **elliptic transformation** is a Möbius map in $\mathrm{Aut}(\mathbb{H})$ with trace in the range $(-2, 2)$. Elliptic transformations are conjugate to a map of the form $\begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}$ which tells us that the trace is $2\cos(\theta)$. We can view elliptic transformations as analogs of Euclidean rotations. In this case, $\sqrt{(a+d)^2 - 4}$ is imaginary, so the fixed point of an elliptic transformation lies in $\mathbb{H}$.

3. Hyperbolic Structure on Surfaces and Teichmüller Space

We will now turn our attention toward building the tools necessary for characterizing surfaces with hyperbolic structure. We explore discrete subgroups of $\mathrm{Aut}(\mathbb{H})$, also known as Fuchsian groups. Discrete subgroups are special types of subgroups which induce a tessellation of hyperbolic space by polygons (also known as fundamental domains). This tessellation will allow us to construct a surface with a hyperbolic structure. Moreover, we will show that any complete surface with hyperbolic structure can be viewed as one of these surfaces defined via Fuchsian groups.

3.1. **Fuchsian Groups.**

Definition 3.1. (Fuchsian Groups) $\mathrm{SL}(2,\mathbb{R})$ can be viewed as a subset of $\mathbb{R}^4$. This allows us to give a topology to $\mathrm{SL}(2,\mathbb{R})$ viewed as solutions to the equation $ad-bc=1$ in $\mathbb{R}^4$. Similarly, we can give a topology to $\mathrm{PSL}(2,\mathbb{R})$ as the quotient of $\mathrm{SL}(2,\mathbb{R})$ with respect to the map $(a,b,c,d)\mapsto(-a,-b,-c,-d)$. Matrix multiplication and inversion are continuous, so it follows that $\mathrm{SL}(2,\mathbb{R})$ and $\mathrm{PSL}(2,\mathbb{R})$ are Lie groups.

A discrete subgroup of a Lie group is a subgroup that can be viewed as a discrete space with respect to the topology of the Lie group. Recall that a space is discrete if and only if it contains no limit points. A **Fuchsian** group is defined as a discrete subgroup of $\mathrm{PSL}(2,\mathbb{R})$.

Theorem 3.2 (Conditions for a Fuchsian Group)**.** *For $G\leqslant SL(2,\mathbb{R})$, the following are equivalent:*

(1) G has no limit points in G.
(2) G has no limit points in $SL(2,\mathbb{R})$.
(3) Id is an isolated point of G.

Proof. We will first prove the implication (1) $\implies$ (2) by the contrapositive. Suppose that there exists some non-constant sequence $g_n\to g\in\mathrm{SL}(2,\mathbb{R})$. Then the sequence $g_ng_{n+1}^{-1}\to g*g^{-1}=id$ is a non-constant sequence converging to $id\in G$. But then G has a limit point in G.

Note that we have proved that if G does not have a limit point, then it also does not have a limit point in the larger space $\mathrm{SL}(2,\mathbb{R})$, which is not a property of metric spaces in general.

Next, (2) $\implies$ (3) is clear as $id\in\mathrm{SL}(2,\mathbb{R})$, and therefore is isolated if G has no limit points in $\mathrm{SL}(2,\mathbb{R})$.

Finally, we show the implication (3) $\implies$ (1). As id is isolated, take an open neighborhood U of id such that $G\cap U=\{id\}$. Then gU is an open set (as multiplication by g is a homeomorphism), and $G\cap gU=\{g\}$. In particular, g is isolated, so G has no limit points. □

Definition 3.3. A group acts properly discontinuously on $\mathbb{H}$ if for any compact set $K\subset\mathbb{H}, gK\cap K=\emptyset$ for all but finitely many $g\in G$.

We will show that Fuchsian groups act properly discontinuously on $\mathbb{H}$. Moreover, we will show that Fuchsian groups are the only subgroups of

$\mathrm{SL}(2,\mathbb{R})$ that act properly discontinuously on $\mathbb{H}$. First, we will consider some nice conditions for proper discontinuity.

Theorem 3.4 (Proper Discontinuity). *For $G \leqslant SL(2,\mathbb{R})$, the following are equivalent:*

(1) *G does not act properly discontinuously on $\mathbb{H}$.*
(2) *There is a $z \in \mathbb{H}$ such that the G-orbit of z has a limit point.*
(3) *Every G-orbit of $z \in \mathbb{H}$ has a limit point provided that z is not fixed by all $g \in G$.*

Proof. First, we show (1) $\implies$ (2). Suppose there exists compact $K \subset \mathbb{H}$ such that $g_n K \cap K \neq \emptyset$ for infinitely many distinct $g_n \in G$. Then there exists points $x_n \in K$ such that $g_n x_n \in K$. By compactness, there is a subsequence z_n of x_n with $z_n \to w \in K$.

By sequence convergence, for large enough n,

$$d(z_n, w) < 1 \implies d(g_n z_n, g_n w) < 1$$

as g_n is an isometry in $\mathrm{Aut}(\mathbb{H})$. Furthermore, by compactness, $K \subseteq B_r(w)$, a closed ball, for some $r \in \mathbb{R}^+$. Therefore, as each $g_n z_n \in K$, $g_n w \in B_{r+1}(w)$. By compactness, there is a subsequence of $g_n w$ converging in $B_{r+1}(w)$. In particular, the G-orbit of w has a limit point.

Next, we show (2) $\implies$ (3). Suppose there exists $a \in \mathbb{H}$ such that the G-orbit of a has a limit point. Then there exist distinct points $g_n a \to b \in \mathbb{H}$. Now let $z \in \mathbb{H}$ be any other point, and consider the collection $g_n z$.

Then

$$d(g_n z, z) \leq d(g_n z, g_n a) + d(g_n a, b) + d(b, z) = d(z, a) + d(g_n a, b) + d(b, z)$$

As $d(g_n a, b)$ is bounded (it is a convergent sequence), then $d(g_n z, z)$ is also bounded. In particular, $g_n z$ is contained in some closed ball centered at z, so by compactness there is a convergent subsequence of the $g_n z$ in this closed ball. If this subsequence is non-constant, then the orbit of z contains a limit point.

However in the case that this subsequence is constant, we get that $g_{m'} z = g_{n'} z$ for all $g_{m'}, g_{n'}$ in the subsequence. In particular, we get that there exist infinitely many elements of the form $g_{m'} g_{n'}^{-1}$ with the property that $g_{m'} g_{n'}^{-1} z = z$. As $z \in \mathbb{H}$ and each $g_{m'} g_{n'}^{-1}$ fixes z, then $g_{m'} g_{n'}^{-1}$ must all be distinct elliptic transformations which share the same fixed point. In particular, each is a rotation about z by a distinct angle θ. However, this can only happen for at most one point: given any other $w \in \mathbb{H}$, each distinct $g_{m'} g_{n'}^{-1} w$ is a distinct point lying on some circle centered at the fixed point z, and therefore by compactness there exists a convergent subsequence of the $g_{m'} g_{n'}^{-1} w$ in the circle. In particular, w contains a limit point.

In conclusion, if the G-orbit of one point in $\mathbb{H}$ has a limit point, then the G-orbit of every point in $\mathbb{H}$ has a limit point, besides possibly one point which is the shared fixed point of an infinite family of elliptic elements in G.

Finally, we show (3) $\implies$ (1). Let $z \in \mathbb{H}$ be a point such that the G-orbit of z has a limit point. Then there is a sequence $g_n z \to w \in \mathbb{H}$. Consider a closed ball B centered at w which also contains z. By sequence convergence, B contains infinitely many $g_n z$. Therefore, $g_n z \in g_n B \cap B$ for infinitely many g_n. In particular, $g_n B \cap B \neq \emptyset$ for infinitely many g_n and so G is not properly discontinuous. $\square$

Theorem 3.5. *$G \leqslant SL(2,\mathbb{R})$ acts properly discontinuously on $\mathbb{H} \iff G$ is discrete*

Proof. We first prove the forward direction: every subgroup of $\mathrm{PSL}(2,\mathbb{R})$ that acts properly discontinosly on $\mathbb{H}$ is discrete. We prove this direction by the contrapositive. Suppose $G \leqslant \mathrm{SL}(2,\mathbb{R})$ is not discrete. Then there exists a sequence $g_n \to id \in G$ by discreteness (See Theorem 3.2 (3)). Now choose any $z \in \mathbb{H}$ such that z is not a fixed point of $g_m g_n^{-1}$ for all m, n. This is possible as there are uncountably many points in $\mathbb{H}$ (as opposed to the countably many $g_m g_n^{-1}$). This means that $g_m z \neq g_n z$ for all m, n, so in particular the sequence $g_n z \to idz = z$ is composed of distinct points. Therefore z is a limit point of the G-orbit of z, and by the conditions for proper discontinuity (Theorem 3.4), G is not properly discontinuous.

Next we prove the converse: if G is a discrete subgroup of $\mathrm{SL}(2,\mathbb{R})$, then G acts properly discontinuously on $\mathrm{SL}(2,\mathbb{R})$.

Let $K \subset \mathbb{H}$ be a compact set. Note that as K is compact, it is closed and bounded. In particular, K is contained in some closed ball B centered at i with radius $R > 0$. Note that for any $g \in G$, if $gK \cap K \neq \emptyset$ then also $gB \cap B \neq \emptyset$ as $K \subset B$. In particular, it is sufficient to show that $gB \cap B \neq \emptyset$ for finitely many $g \in G$ to show that G acts properly discontinuously.

Moreover, if $gB \cap B \neq \emptyset$, then there exists $x, y \in B$ such that $x = gy$. This gives that

$$d(i, gi) \leq d(i, gy) + d(gy + gi) = d(i, x) + d(i, y) \leq 2R$$

where we use the fact that $x = gy$ and that g is an isometry. Therefore, it is sufficient to show that there are only finitely many $g \in G$ such that $d(i, gi) \leq 2R$. It is equivalent to show that there are only finitely $g \in G$ such that $gi \in B^*$ where $B^* \subset \mathbb{H}$ is the closed ball centered at i of radius $2R$. We proceed by showing that Möbius maps with this property are both compact and discrete in $\mathrm{SL}(2,\mathbb{R})$ and therefore finite.

Consider the set $E = \{\, T \in \mathrm{SL}(2,\mathbb{R}) \mid T(i) \in B^* \,\}$. As the topology of $\mathrm{SL}(2,\mathbb{R})$ comes from $\mathbb{R}^4$, it is enough to show that E is closed and bounded. To see that E is closed, consider the continuous map $f(T) = T(i)$, a function of the four entries a, b, c, d of the matrix T. Then $E = f^{-1}(B^*)$ and is therefore closed.

Next we need to show that E is bounded. First note that as B^* is bounded, the modulus of all elements in B^* is bounded by some $M_1 \in \mathbb{R}^+$. Also, as B^* is closed, there is a positive Euclidean distance between the real axis and B^*, so that the imaginary part of elements in $B^* \geq M_2$. In particular, for any $T \in E$, $\frac{|ai+b|}{|ci+d|} < M_1$ and $\mathrm{Im}\left(\frac{ai+b}{ci+d}\right) = \frac{\mathrm{Im}(i)}{|ci+d|^2} \geq M_2$ where the equality comes from remembering the formula for the imaginary part of a Möbius Transformation given in Theorem 2.13. Simplifying further, $|ci+d| < \sqrt{\frac{1}{M_2}}$ and $|ai+b| < M_1\sqrt{\frac{1}{M_2}}$. In particular, $|a|, |b|, |c|, |d|$ are bounded, which gives that E is bounded.

Next, consider the set $E \cap G$ which is the set of $g \in G$ such that $d(i, gi) \leq 2R$. $E \cap G$ is discrete in $\mathrm{SL}(2, \mathbb{R})$ as G is discrete. However, if $E \cap G$ had infinitely many points, then by compactness of E there would be a sequence of points in $E \cap G$ converging in E. In particular, $E \cap G$ would not be discrete by the conditions for discrete subgroups (Theorem 3.2). Therefore $E \cap G$ must be finite, and so there are only finitely many $g \in G$ such that $d(i, gi) \leq 2R$. □

Definition 3.6 (Elementary Groups)**.** There are four types of simpler Fuchsian groups. They are

1. $< T >$ where T is parabolic;
2. $< T >$ where T is elliptic;
3. $< T >$ where T is hyperbolic;
4. $< S, T >$ where S is hyperbolic and T is elliptic. (It can be shown that $< S, T >$ is a dihedral group).

3.2. **Fundamental Domains.** The intuition for a properly discontinuous action on a space is that points are sent far away from each other by the group elements. In this section, we will discuss fundamental domains which are regions in $\mathbb{H}$ which, under the properly discontinuous action of a Fuchsian group, tesselate $\mathbb{H}$.

Definition 3.7. Let $G \leqslant \mathrm{SL}(2, \mathbb{R})$ be a discrete subgroup. A fundamental domain of G is a set $R \subset \mathbb{H}$ satisfying:

1. $gR \cap R = \emptyset \ \forall g \in G - \{id\}$,
2. $\bigcup_{g \in G} g\overline{R} = \mathbb{H}$,
3. R is the interior of a convex set with ∂R a countable union of geodesics,
4. For any compact $K \subset \mathbb{H}$, there are only finitely many $g \in G$ such that $gR \cap K \neq \emptyset$.

We can think of a fundamental domain as a single puzzle piece which, under the action of elements of G, tile $\mathbb{H}$ in a nice, non-overlapping manner. Fundamental domains of a Fuchsian group are very much non-unique and there are a few different methods for constructing them. In this paper, we provide the Dirichlet Domain method.

In order to use this method, we need to show that for any two points in $\mathbb{H}$ there exists a unique perpendicular bisector. To see this, map z and z' to points on the imaginary axis of $\mathbb{H}$ by an isometry so that the geodesic path between z and z' maps to a vertical line (this is possible as Möbius maps act on ordered triples of $\widehat{\mathbb{C}}$ transitively). It is clear then that the only possible geodesic bisector is a semi-circle centered at the origin intersecting the vertical line halfway (with respect to hyperbolic distance). Mapping back by the inverse of this isometry, we get a unique perpindicular bisector between two points.

Example 3.8 (Dirichlet Domain)**.** Let G be a discrete subgroup. Pick any $z \in \mathbb{H}$. For every $g \in G$, consider the unique geodesic path from z to gz. There is a unique perpendicular bisector of this line. This bisector cuts $\mathbb{H}$ into two "half-planes", and we define H_g to be the half-plane that contains z. More explicitly, $H_g = \{\, w \in \mathbb{H} \mid d(z,w) < d(gz,w) \,\}$.

Then the Dirchlet Domain of G is

$$R_z := \bigcap_{g \in G - \{id\}} H_g = \{\, w \in \mathbb{H} \mid d(z,w) < d(gz,w) \ \forall g \in G - \{id\} \,\}.$$

We will show R_z is a fundamental domain of G (which depends on the initially chosen point z).

Proposition 3.9. *Given G Fuchsian and a point $z \in \mathbb{H}$, the Dirichlet Domain R_z centered at z is a Fundamental Domain for the action of G on $\mathbb{H}$.*

Proof. First we show that $hR_z = R_{hz}$. A point

$$\begin{aligned} hx \in hR_z &\iff x \in R_z \\ &\iff d(x,z) < d(x,gz) \ \forall g \in G \\ &\iff d(hx,hz) < d(hx,hgz) \ \forall g \in G - \{id\} \\ &\iff d(hx,hz) < d(hx,(hgh^{-1})hz) \ \forall hgh^{-1} \in G - \{id\}. \end{aligned}$$

Therefore $hx \in hR_z \iff hx \in R_{hz}$.

We will now prove that R_z satisfies the properties of a fundamental domain.

• We prove the first property of fundamental domains: that $gR_z \cap R_z = R_{gz} \cap R_z = \emptyset$. Suppose that $y \in R_{gz} \cap R_z$. Then $d(y,z) < d(y,gz)$ as $y \in R_z$ but also $d(y,gz) < d(y,g^{-1}gz) = d(y,gz)$ as $y \in R_{gz}$. This is a contradiction, so y cannot exist.

• To show $\bigcup_{g\in G} g\overline{R_z} = \mathbb{H}$, we show that for any $q \in \mathbb{H}$, there is an element of G mapping q into $\overline{R_z}$. Let B be a closed ball about q containing at least one orbit point $g_0 z$ for $g_0 \in G$. As G acts properly discontinuously on $\mathbb{H}$ by Theorem 3.5, there are only finitely many orbit points of z in B. Therefore, choose the orbit point $kz \in B$ closest to q, that is

$$d(q,kz) = \min_{g\in G} d(q,gz).$$

We will show that $k^{-1}q \in \overline{R_z}$ by showing that the line from z to $k^{-1}q$ is almost entirely contained in R_z.

Consider the unique geodesic from z to $k^{-1}q$. For any point P on this geodesic path (besides the endpoint $k^{-1}q$) and $\forall g \in G - \{id\}$

$$d(z, P) + d(P, k^{-1}q) = d(z, k^{-1}q) = d(kz, q) \leq d(q, gz) = d(k^{-1}q, k^{-1}gz)$$

as kz is the closest orbit point to z (besides z itself).

Then $d(P, z) \leq d(k^{-1}q, k^{-1}gz) - d(P, k^{-1}q) \leq d(P, k^{-1}gz)$ $\forall g \in G - \{id\}$. As any element of G-{id} can be written as $k^{-1}g$ for some $g \in G$, we have nearly shown that $P \in R_z$. We just need to show that equality cannot hold. If $d(P, z) = d(P, k^{-1}q) + d(P, k^{-1}gz)$ for some $g \in G$ (so that the inequalities above are actually equalities for some g), then P must be on a geodesic path from $k^{-1}q$ to $k^{-1}gz$. However, we have already assumed that P is on the geodesic path from z to $k^{-1}q$. These paths intersect exactly at the endpoint $k^{-1}q$, however we assumed that P is not this endpoint. So equality cannot hold, and $P \in R_z$. Now, take a sequence of points in the interior of the geodesic path joining z to $k^{-1}q$ that converge to $k^{-1}q$. All such points are in R_z, so $k^{-1}q \in \overline{R_z}$.

• Next we skip to proving the fourth property that for any compact $K \subset \mathbb{H}$ there are only finitely many $g \in G$ such that $gK \cap K \neq \emptyset$. As K is bounded, it is contained in some closed ball B of radius $r > 0$ centered at z. In particular, if $gR_z \cap B \neq \emptyset$ for only finitely many g, then also $gR_z \cap K \neq \emptyset$ for only finitely many g. Let $w \in gR_z \cap B$ so that $d(z, w) < r$. We have

$$d(gz, z) \leq d(w, gz) + d(w, z) \leq d(w, z) + d(w, z) = 2r$$

where the second inequality comes from $w \in gR_z = R_{gz}$. In particular, we have that $w \in gR_z \cap B$ only if gz is in the closed ball of radius $2r$ about z. By proper discontinuity, only finitely many such gz exist, so only finitely many such gR_z intersect B.

• Finally, we prove that R_z is the interior of a convex polygon with countable sides in $\mathbb{H}$. Note that R_z is defined as the intersection of convex sets, so R_z is convex.

We would like to see that ∂R_z is formed by a countable union of geodesic segments. Note that by definition of R_z as an intersection of half-planes with geodesic boundary, ∂R_z is formed by a union of geodesic segments each contained in some ∂H_g. To see that there are at most countably many such geodesic sides forming ∂R_z, it is enough to show that for any open ball B intersecting R_z, there are at most countably many perpendicular bisectors intersecting B (because a countable union of countable sets is countable).

Let B be an open ball of radius $r > 0$ such that $B \cap R_z \neq \emptyset$. Suppose $w \in B$ such that w lies on the perpendicular bisector of the line between z and gz for some $g \in G$. This can only happen if $d(z, gz) \leq d(z, w) + d(w, gz) = 2d(w, z) \leq 2r$ by definition of perpendicular bisector. By proper discontinuity of G, this happens for only finitely many $g \in G$. Therefore only finitely many perpendicular bisectors (∂H_g) intersect B. Therefore that ∂R_z is formed by a union of countable geodesic segments.

• The last thing to check is that R_z is open. Again it is enough to show that for any open ball, $B \cap R_z$ is open. We already know that B contains finitely many ∂H_g, so it follows that $R_z \cap B$ can be written as a finite intersection of these H_g with B, which are open as H_g is open. Therefore, R_z is the interior of a convex polygon, which concludes the proof that R_z is a fundamental domain. □

Proposition 3.10 (Dirichlet Domain for $PSL(2,\mathbb{Z})$)**.** *The Dirichlet domain for $PSL(2,\mathbb{Z})$ is the region*

$$R = \{\, z \in \mathbb{H} \mid |Re(z)| < \frac{1}{2}, |z| > 1 \,\}$$

Proof. We calculate the Dirichlet domain for the Fuchsian group $PSL(2,\mathbb{Z}$ centered at 2i. Let D denote this Dirichlet domain. It is well known that $SL(2,\mathbb{Z})$ is generated by the matrices $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ so we consider the image of $2i$ under the transformations $A = z+1, B = \frac{-1}{z}$. Then $A(2i) = 2i+1$ and $A^{-1}(2i) = 2i-1$. Consider the half-planes of points closer to $2i$ than to $A(2i)$ and $A^{-1}(2i)$. These half-planes intersect on $|Re(z)| < \frac{1}{2}$. Next, $B(2i) = \frac{i}{2}$. The points closer to $2i$ then $\frac{i}{2}$ is given by the points satisfying $|z| > 1$. It is clear that $D \subseteq R$ as R is given by an intersection of just a few of the half-planes which determine the Dirichlet domain.

To show the other direction, we note that for any $z \notin D$, $\exists g \in G - \{id\}$ such that $gz \in D$ by the definition of a fundamental domain. Therefore, if $R \neq D$, then there exists some point $w \in R$ such that $gw \in D$ for some $g \neq id$. However, such a point cannot exist. Suppose in order to reach a contradiction that a point w as above exists. Then $gw \in D \subseteq R$. If we let $gw = \dfrac{aw+b}{cw+d}$, then the imaginary part $\mathrm{Im}(gw) = \dfrac{\mathrm{Im}(w)}{|cw+d|^2}$. As $gz \in R$,

$$|cw+d|^2 = c^2|w^2| + 2|cd|\,\mathrm{Re}(w) + d^2 > c^2 - |cd| + d^2 = (|c| - |d|)^2 + |cd| \geq 1$$

The last inequality comes from the fact that c and d are integers which cannot both be zero. It follows that the $\mathrm{Im}(gz) < \mathrm{Im}(z)$. However, we can apply the exact same argument in reverse by considering the map g^{-1} taking gz to z. This results in the inequality $\mathrm{Im}(z) < \mathrm{Im}(gz) < \mathrm{Im}(z)$ which is a contradiction.

Therefore, R must be the entire Dirichlet domain D. □

3.3. **Hyperbolic Structure and Uniformization Theorem.** We now have the required tools to understand surfaces with hyperbolic structure. In this paper, we are only considering orientable surfaces. We will first show that the action of discrete groups on $\mathbb{H}$ can create a hyperbolic surface $\mathbb{H}/G$. We will then show a partial converse through the Uniformization Theorem: complete surfaces with hyperbolic structure are isometric to $\mathbb{H}/G$ for some Fuchsian group G. Note that from this point forward we will use the notion

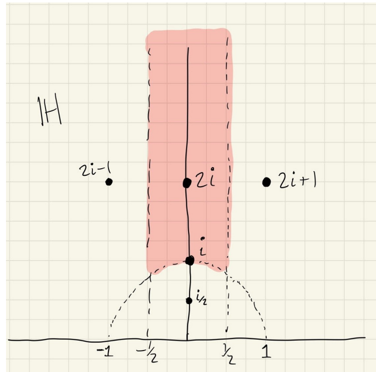

FIGURE 1. The Dirichlet Domain for $\mathrm{PSL}(2,\mathbb{Z})$ centered at $2i$ is highlighted in light red.

of fundamental groups and covering spaces from Algebraic Topology. For a treatment of those topics, see for example Hatcher [5].

Definition 3.11. Let S be an orientable surface. A hyperbolic structure on S is a collection of coordinate charts $(\varphi_\alpha, U_\alpha)$ where U_α is open in S and $\varphi_\alpha\colon U_\alpha \to \mathbb{H}$ such that:

(1) $\varphi\colon U_\alpha \to \varphi(U_\alpha)$ is a homeomorphism,
(2) $\bigcup_\alpha U_\alpha = S$,
(3) for every connected component $C \subset U_\alpha \cap U_\beta$ the transition maps $g_c\colon \varphi_\beta(C) \to \varphi_\alpha(C)$ are orientation preserving isometries of $\mathbb{H}$ in $\mathrm{PSL}(2,\mathbb{R})$

Proposition 3.12. *Let G be a Fuchsian group acting freely on $\mathbb{H}$. Then $\mathbb{H}/G$ is a hyperbolic surface.*

Proof. Let G be a discrete subgroup of $\mathrm{PSL}(2,\mathbb{R})$ acting freely on $\mathbb{H}$. Define $\mathbb{H}/G := \mathbb{H}/\sim$ where $x \sim y$ when $\exists\ g \in G$ such that $gx = y$. In particular, $\mathbb{H}/G$ is the space of G-orbits of $\mathbb{H}$. We can give $\mathbb{H}/G$ the quotient topology.

Let $x \in \mathbb{H}$. Let $B_x \subset \mathbb{H}$ be an open ball such that $B_x \cap gB_x = \emptyset\ \forall g \in G$ which is possible as G acts freely and properly discontinuously on $\mathbb{H}$. (Note that for any closed ball, only finitely many gB_x intersect B_x. Furthermore, $x \notin gB_x$ for any $g \in G$ as G acts freely on $\mathbb{H}$. So we can choose a small enough open ball about x that does not intersect any of the gB_x). Define U_x to be the image of B_x under the quotient $\mathbb{H} \to \mathbb{H}/G$. Note that the preimage of U_x under the quotient map is a disjoint union of open balls in $\mathbb{H}$, each centered at gx for some $g \in G$. In particular, U_x is open in $\mathbb{H}/G$ with respect to the quotient topology.

For each $[x] \in \mathbb{H}/G$, define charts (U_x, ϕ_x) where ϕ_x is the homeomorphism between U_x and $B_x \subset \mathbb{H}$. The collection of $U_z\ \forall z \in \mathbb{H}$ clearly covers $\mathbb{H}/G$. Also note from this construction that $\mathbb{H}/G$ is Hausdorff, as for any G-orbits $[z_1], [z_2] \in U_x \subset \mathbb{H}/G$, we can map U_x homeomorphically to B_x, find disjoint open sets about $\phi_x([z_1]), \phi_x([z_2])$ by using the fact that $\mathbb{H}$ is

Hausdorff, and then map these sets homeomorphically to open sets in $\mathbb{H}/G$. The fact that $\mathbb{H}/G$ is T_2 follows from the fact that $\mathbb{H}$ is T_2. Therefore $\mathbb{H}/G$ is a manifold covered with coordinate charts mapping into $\mathbb{H}$. It is left to show that transition maps are orientation preserving isometries of $\mathbb{H}$.

Suppose U_x and U_y have nonempty intersection. Fix a connected component C of $U_x \cap U_y$ and consider the sets $\phi_x(C) \subset \phi_x(U_x)$ and $\phi_y(C) \subset \phi_y(U_y)$ in $\mathbb{H}$. We want to show that there exists an isometry $g \in \mathrm{PSL}(2, \mathbb{R})$ such that $g\phi_x(C) = \phi_y(C)$. Fix any point $[z] \in C$. Then

$$\phi_x([z]) = g_x z \in \phi_x(C) \subset \phi_x(U_x) \text{ and } \phi_y([z]) = g_y z \in \phi_y(C) \subset \phi_y(U_y)$$

for some $g_x, g_y \in \mathrm{PSL}(2, \mathbb{R})$. Therefore, there is an isometry $g_y g_x^{-1}$ which sends $g_y z$ to $g_x z$. We will show that $g_y g_x^{-1} \phi_x(C) = \phi_y(C)$.

First, let $[w] \neq [z] \in U_x \cap U_y$ be any other point. Then $\phi_x([w]) = h_x w$ and $\phi_y([w]) = h_y w$ where $h_x, h_y \in \mathrm{PSL}(2, \mathbb{R})$. We will show that $g_y g_x^{-1} h_x w = h_y w \in \phi_y(C)$. Consider a path γ from $g_x z$ to $h_x w$. We inductively construct an open covering of γ as follows. Let O_1 be an open ball centered at $g_x z$ such that the open ball $g_y g_x^{-1} O_1$ centered at $g_y z$ is contained in $\phi_y(C)$. Inductively define the open ball O_i to be an open ball centered at some point in $\gamma \cap O_{i-1}$ such that $O_i \nsubseteq O_{i-1}$ and $g_y g_x^{-1} O_i \subseteq \phi_y(C)$. Furthermore, we can require that $O_i \cap O_j = \emptyset \; \forall j \neq i-1$. By compactness, we can find finitely many such $O_1, ..., O_n$ which cover γ. Then $g_y g_x^{-1}(O_1 \cup ... \cup O_n) \subseteq \phi_y(C)$ and therefore so is γ. In particular, we have that $g_y^{-1} g_x h_x w \in \phi_y(C) \subseteq \phi_y(U_y)$. As ϕ_y is a homeomorphism, then $g_y g_x^{-1} h_x w = h_y w$. As the action of G is free, $g_y g_x^{-1} = h_x^{-1} h_y$. This holds for any point $w \in C$. Therefore $g_y g_x^{-1} \phi_x(C) \subseteq \phi_y(C)$. To show the other direction, we can repeat the exact same argument in reverse using the isometry $g_x g_y^{-1}$ and a path γ' between $g_y z$ and $h_y w$.

Finally, we will define a distance function on $\mathbb{H}/G$. For $[x], [y] \in \mathbb{H}/G$, consider

$$d([x], [y]) = \min_{x' \in [x], y' \in [y]} d_{\mathbb{H}}(x', y')$$

where $d_{\mathbb{H}}$ is is the distance function on $\mathbb{H}$.

As G has no limit points in $\mathbb{H}$ (by Theorem 3.4), the distance between $[x]$ and $[y]$ is non-zero. One can check that d satisfies all the requirements for a distance function. □

We will soon see that the Uniformization Theorem gives a partial converse to the above: every complete hyperbolic surface is of the form $\mathbb{H}/G$ for some discrete, torsion-free $G \leqslant \mathrm{PSL}(2, \mathbb{R})$. However before we can complete this proof, we must first describe a global definition of a hyperbolic structure using a developing map and a holonomy map.

Definition 3.13 (Developing Map)**.** Let S be a path-connected, locally path-connected, and semi-locally simply connected hyperbolic surface. Fix a base point $x_0 \in S$ and a chart $U_0 \ni x_0$. Then there exists a universal cover

$\widetilde{S}$ of S given by the set of all homotopy classes of paths $[0,1] \to S$ beginning at x_0. [5]

For any $[\alpha] \in \widetilde{S}$ and a base chart $U_0 \ni x_0$, α can be covered by finitely many charts (U_i, Φ_i) in S by compactness, say $1 \leq i \leq n$. We can choose these charts such that $U_i \cap U_j \neq \emptyset$ only when $|i-j| = 1$. Whenever there is an overlap of charts $U_i \cap U_{i+1} \neq \emptyset$, the transition map $\Phi_i \Phi_{i+1}^{-1} \colon \Phi_i(U_i \cap U_{i+1}) \to \Phi_i(U_i \cap U_{i+1})$ is given by an isometry $g_i \in \mathrm{PSL}(2, \mathbb{R})$ by the hyperbolic structure on S. We can then piece-wise construct a path $\widetilde{\alpha} \subset \mathbb{H}$ such that $\widetilde{\alpha}|_{U_1} = \Phi_1(\alpha)$ and $\widetilde{\alpha}|_{U_i} = g_1 ... g_{i-1}\ \Phi_i(\alpha)$. The path $\widetilde{\alpha} \colon [0,1] \to \mathbb{H}$ is well-defined in $U_i \cap U_{i+1}$ as $\Phi_i(U_i \cap U_{i+1}) = g_i \Phi_{i+1}(U_i \cap U_{i+1})$ for each $1 \leq i \leq n-1$.

We call $\widetilde{\alpha}$ the developing image of α, and we define the **developing map** D to be the endpoint $\widetilde{\alpha}(1)$ of $\widetilde{\alpha}$ in $\mathbb{H}$. That is, we have the map

$$\begin{aligned} D \colon \widetilde{S} &\to \mathbb{H} \\ [\alpha] &\mapsto \widetilde{\alpha}(1). \end{aligned}$$

The developing map is smooth as it is constructed as a composition of a path, a chart homeomorphism, an isometry, and the evaluation function.

We can check that D is well-defined for any other choice of charts (V_j, Ψ_j) which cover α $(1 \leq j \leq m)$ up to choice of the initial chart V_1 and for any other representative $\beta \in [\alpha]$.

First, consider the case where α can be covered by two different coverings of charts $\{U_1, U_2\}$ and $\{U_1, V_2\}$ such that $V_2 \cap U_2$ is connected. Let D be the developing map according to the covering $\{U_1, U_2\}$ and D^* be the developing map according to the covering $\{U_1, V_2\}$. We have that $D_{U_2 \cap \alpha} = g_1 \Phi_2(\alpha)$ and $D^*_{V_2 \cap \alpha} = g_1^* \Psi_2(\alpha)$. Let h be the isometry corresponding to the overlap $U_2 \cap V_2$. Then h and $g_1 \circ g_1^{*-1}$ agree on $\Phi_1(U_1) \cap \Phi_2(U_2) \cap \Psi_2(V_2)$ which is an open set and therefore must contain at least three points. As isometries are determined by where they send three points, we have that $h = g_1 \circ g_1^{*-1}$. This gives the following

$$\begin{aligned} D^*|_{U_2 \cap V_2 \cap \alpha} &= g_1^{*-1} \circ \Psi_1|_{U_2 \cap V_2 \cap \alpha} \\ &= g_1^{-1} h \circ \Psi_1|_{U_2 \cap V_2 \cap \alpha} \\ &= g_1^{-1} \circ \Phi_1|_{U_2 \cap V_2 \cap \alpha} \\ &= D|_{U_2 \cap V_2 \cap \alpha}. \end{aligned} \tag{1}$$

From the above equalities along with the fact that both covers share the same inital chart, D and D^* are the same.

Next, consider any two homotopic curves γ_1 and γ_2 covered by different families of charts as in the definition of the developing map above. Let $H \colon I \times I \to S$ be the homotopy between γ_1 and γ_2. There are finitely many charts covering $H(I \times I)$ by compactness. Therefore, there exists a partition of $I \times I$ into pieces $[s_k, s_{k+1}] \times [t_l, t_{l+1}]$ such that $H([s_k, s_{k+1}] \times [t_l, t_{l+1}])$ is entirely contained inside a chart for each $1 \leq k \leq N, 1 \leq t \leq M$. Then the

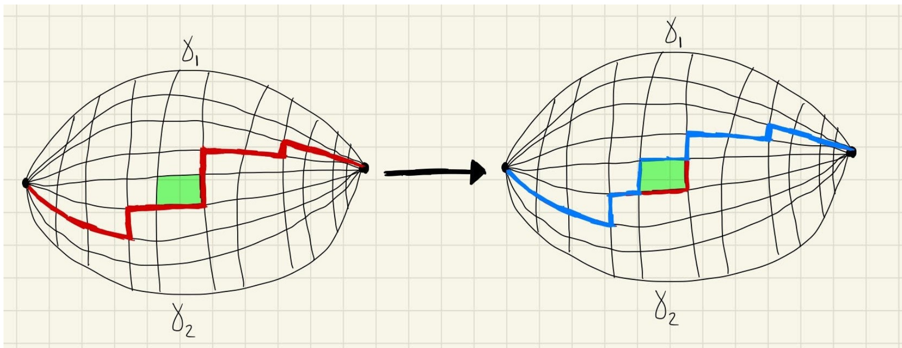

FIGURE 2. A "before and after" picture of the homotopy $H_{a,b}$. The region $H([s_a, s_{a+1}] \times [t_b, t_{b+1}])$ is in green.

homotopy between γ_1 and γ_2 can be thought of as several homotopies $H_{a,b}$ in succession, each restricted to the region $H([s_a, s_{a+1}] \times [t_b, t_{b+1}])$ for each $1 \leq a \leq N, 1 \leq b \leq M$. For each $H_{a,b}$, we can inductively apply (1) to see that the developing map is well-defined.

Therefore the developing map is unique up to the choice of a basepoint and the initial chart for the basepoint. These choices are equivalent to precomposition of the developing map by some isometry (consider the overlap isometry between two different possible choices of initial chart or the composition of transition maps coming from a covering of a path between two different choices of basepoint).

Definition 3.14 (Holonomy Homomorphism)**.** Let S be a path-connected, locally path-connected, and semi-locally simply connected hyperbolic surface. Let $[\alpha] \in \pi_1(S, x_0)$. Then

$$D([\alpha]) = \widetilde{\alpha}(1) = g_1 ... g_{n-1} \Phi_n(\alpha(1)) = g\Phi_1(x_0)$$

for some $g_1, ..., g_n \in \mathrm{PSL}(2, \mathbb{R})$. As the developing map is well-defined for any choice of charts covering α, we can choose charts covering α such that the the final chart $U_n = U_1$, the initial chart.

The **holonomy homoromorphism** is defined to be

$$\rho\colon \pi_1(S, x_0) \to \mathrm{PSL}(2, \mathbb{R})$$
$$[\alpha] \mapsto g,$$

where g is the isometry such that $D([\alpha]) = g\Phi_1(x_0)$.

To check this map is a homomorphism, consider two loops α and β based at x_0 and set $\rho([\alpha]) = g_\alpha$ and $\rho([\beta]) = g_\beta$. If $\mathfrak{X}_1$ is the open covering of α used to define $D([\alpha])$ and $\mathfrak{X}_2$ is the open covering of β used to define $D([\beta])$, then the concatenation $\alpha * \beta$ is covered by $\mathcal{U}_1 \cup \mathcal{U}_2$. Tracing out the transition maps of these covers, we get that $\alpha * \beta$ maps to $g_\alpha g_\beta$.

Furthermore, ρ is injective. Let $[\alpha] \in \ker(\rho)$. Then $D([\alpha]) = \widetilde{\alpha}(1) = \phi_1(x_0) = \widetilde{\alpha}(0)$. Therefore $\widetilde{\alpha}$ is a loop in $\mathbb{H}$. As $\mathbb{H}$ is simply connected, it follows that $\widetilde{\alpha}$ is null-homotopic. We can pull this homotopy back via charts to see that α is null-homotopic. Therefore $[\alpha] = id$.

Note that p is well defined up to conjugation by isometries in $\mathrm{PSL}(2,\mathbb{R})$. By the discussion above, changing the basepoint in S conjugates the developing map D by isometries in $\mathrm{PSL}(2,\mathbb{R})$. On the other hand, choosing a different chart postcomposes the developing map by an isometry in $\mathrm{PSL}(2,\mathbb{R})$. This is equivalent to conjugating the holonomy homomorphism ρ by isometries in $\mathrm{PSL}(2,\mathbb{R})$.

The developing map and holonomy homomorphism pair (D,ρ) define a global geometric structure on the surface. The definition of this pair depends just on the choice of basepoint and the initial chart about the basepoint. If we change this choice, the pair (D,ρ) will change by the action of $\mathrm{PSL}(2,\mathbb{R})$:

$$A \cdot (D,\rho) = (AD, A\rho A^{-1}).$$

for each $A \in \mathrm{PSL}(2,\mathbb{R})$. We can use these constructions to show that every complete hyperbolic surface is of the form $\mathbb{H}/G$ for some Fuchsian group G. This fact will be an immediate corollary of the Hopf-Rinow Theorem which uses the developing map to prove that the universal cover of every complete hyperbolic surface is $\mathbb{H}$. Before, we prove the Hopf-Rinow theorem, we need the following lemma from Algebraic Topology.

Lemma 3.15. *A surjective local homeomorphism with the path lifting property is a covering map.*

Proof. We prove the claim that surjective local homeomomrphisms satisfying the path lifting property are covering maps. Let X, Y be path-connected, locally path-connected, and semi-locally simply connected spaces. Let $f\colon X \to Y$ be a surjective local homeomorphism satisfying the path-lifting property and let $p : \widetilde{Y} \to Y$ be the universal cover of Y. Let $x_0 \in X, y_0 = f(x_0) \in Y$, and $\widetilde{y_0} \in p^{-1}(y)$ be basepoints of X, Y, and $\widetilde{Y}$ respectively.

First, we will define a map $P\colon \widetilde{Y} \to X$ such that $f \circ P = p$. Given any $\widetilde{y} \in \widetilde{Y}$, let α be a path from $\widetilde{y_0}$ to $\widetilde{y}$. Then $p \circ \alpha$ is a path from y_0 to $p(\widetilde{y}) \in Y$. As f satisfies the path lifting property, $p \circ \alpha$ lifts to a path from x_0 to some $x \in X$. Define $P(\widetilde{y}) = x$. To see that P is well-defined, let β be another path between $\widetilde{y_0}$ and $\widetilde{y}$. Then $\alpha * \beta^{-1}$ is nullhomotopic as $\widetilde{Y}$ is simply connected. In particular, we can contract this loop to some loop γ such that γ lies in some open set mapping homeomorphically into a neighborhood of y. Furthermore, we can choose this open neighborhood such that it maps homeomorphically to a neighborhood of x using the local homeomorphism property of f. It follows that the lift of $p \circ \beta(1) = p \circ \alpha(1)$ by lifting the homotopy rel endpoints between $p \circ \alpha * \beta^{-1}$ and $p \circ \gamma$ to a unique homotopy rel endpoints in X using the path-lifting property of X.

Now we want to find the evenly covered neighborhoods in Y by open sets in X to prove the claim. Let $y \in Y$ and let U be a neighborhood of y such that $U_i \subset \widetilde{Y}$ are disjoint open sets mapping homeomorphically to U. Note

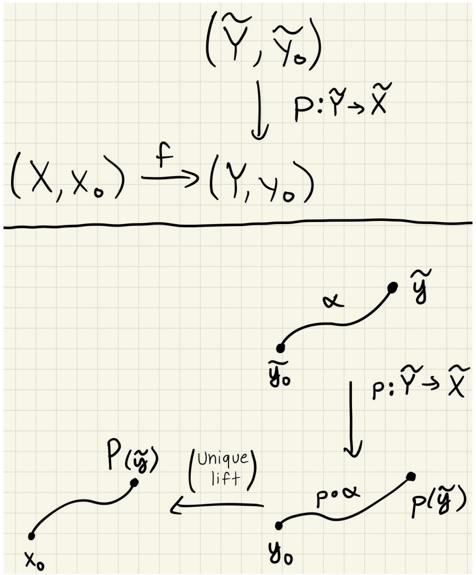

FIGURE 3. Definition of P in Lemma 3.15.

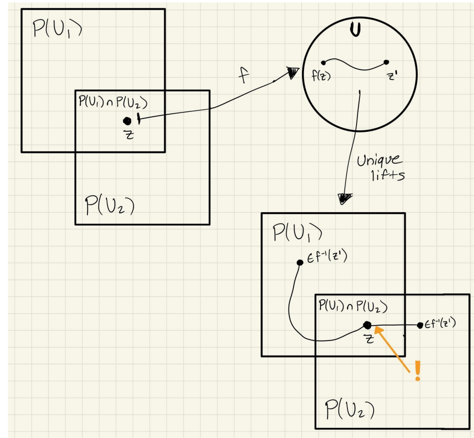

FIGURE 4. We can apply the path lifting property to $P(U_i)$ in Lemma 3.15.

that $(f \circ P)(U_i) \cong p(U_i) \cong U$ as f and p are local homeomorphisms. It is only left to show that the sets $P(U_i)$ are disjoint. Let $z \in P(U_i) \cap P(U_j)$. Then $f(z) \in U$. Let $z' \in U$ be any other point. Then there exists a path from $f(z)$ to z' contained in U as U is path-connected. We can lift this path to a unique path based at $f^{-1}(z') \in P(U_i)$ and a unique path based at $f^{-1}(z') \in P(U_j)$. However, both of these lifts intersect at $z \in f^{-1}(f(z))$ which is contained in both $P(U_i)$ and $P(U_j)$. Then $P(U_i) = P(U_j)$ by uniqueness of lifts. Therefore for any two $P(U_i)$ and $P(U_j)$, the sets $P(U_i)$ and $P(U_j)$ are either equal or disjoint. $\square$

Theorem 3.16 (Hopf-Rinow Theorem for Hyperbolic Surfaces)**.** *Let S be a connected, path-connected, and semi-locally simply connected hyperbolic surface and let $\widetilde{S}$ be its universal cover. The following are equivalent:*

(1) The developing map $D : \widetilde{S} \to \mathbb{H}$ is a surjective covering map. As $\widetilde{S}$ is the universal cover of S, it follows that $\widetilde{S} \cong \mathbb{H}$.
(2) S is a complete metric space.
(3) $\widetilde{S}$ is a complete metric space.

Proof. First, we define distance functions on S and $\widetilde{S}$.

Let $\alpha\colon [0,1] \to S$ be a path. Cover α with finitely many charts $(U_1, \phi_1), ..., (U_n, \phi_n)$. Define $\alpha_1, ..., \alpha_n$ to be paths such that $\alpha_i \subset U_i$ and the concatenation $\alpha_1 * ... * \alpha_n = \alpha$. We define the arc-length of α to be

$$\ell(\alpha) = \ell_{\mathbb{H}}(\phi_1(\alpha_1) * g_1\phi_2(\alpha_2) * ... * g_1...g_{n-1}\phi_n(\alpha_n))$$

where $\ell_{\mathbb{H}}$ is the arc-length of a curve in $\mathbb{H}$ and each g_k are transition maps, $1 \le k \le n-1$. For $x, y \in S$, define

$$d_S(x,y) = \inf_{\alpha} \ell(\alpha)$$

where α is any path from x to y.

We define the distance function on $\widetilde{S}$ in the exact same way by giving $\widetilde{S}$ a hyperbolic structure induced by S. Let (U_i, ϕ_i) be a collection of charts covering S and V_j be a collection of evenly covered open sets covering S. Then the collection $(U_i \cap V_j, \phi_i|_{U_i \cap V_j})$ is a collection of evenly covered charts covering S. These sets can be homeomorphically pulled back to open sets in $\widetilde{S}$ by the covering map. It follows that the collection $p^{-1}(U_i \cap V_j)$ defines a hyperbolic structure on $\widetilde{S}$ by a composition of homemorphisms $\widetilde{S} \to S \to \mathbb{H}$.

For this proof, let $p \colon \tilde{S} \to S$ be the covering map.

$(1 \implies 2)$. We will show that every closed and bounded subset of S is compact. This is enough to prove (2): any Cauchy sequence $(x_n)_{n \in \mathbb{N}}$ is bounded, so there exists some closed ball containing x_n $\forall n \in \mathbb{N}$. If this closed ball were compact then there is a convergent subsequence, so the Cauchy sequence x_n must also converge to the limit of this subsequence.

• First we show that D is a homeomorphism. As both $\widetilde{S}$ and $\mathbb{H}$ are simply connected and because universal covers are unique up to homeomorphism [5], then $\widetilde{S} \cong \mathbb{H}$.

• Next we show that D is a local isometry. Let $[\alpha]$ and $[\beta]$ be two homotopy classes of paths in $\tilde{S}$ in the same chart $(O, \varphi \circ p)$. In particular, this means that the points $\alpha(1)$ and $\beta(1)$ lie in the same chart φ in S. Given any path γ between $[\alpha]$ and $[\beta]$, $\ell(\gamma) = \ell(\varphi \circ p(\gamma))$. Therefore, $d([\alpha], [\beta]) = d(\alpha(1), \beta(1))$.

We wish to show that $d(D([\alpha]),\ D([\beta])) = d(\alpha(1), \beta(1))$. Let $\widetilde{\alpha}$ and $\widetilde{\beta}$ be lifts of α and β to $\tilde{S}$ such that $\widetilde{\alpha}(1) = [\alpha]$ and $\widetilde{\beta}(1) = [\beta]$. As $[\alpha]$ and $[\beta]$ are in the same path component in the pre-image of an evenly covered neighborhood, it follows that $\widetilde{\alpha}(0) = \widetilde{\beta}(0) = \widetilde{x_0} \in p^{-1}(x_0)$. We then have that $\widetilde{\alpha} * \gamma$ is a path starting at $\widetilde{x_0}$ and ending at $[\beta]$. By uniqueness of lifts, it follows that $\widetilde{\beta} = \widetilde{\alpha} * \gamma$. We therefore have that

$$D([\alpha]) = g_1 ... g_n \varphi(\alpha(1))$$

and

$$D([\beta]) = D([\alpha * p(\gamma)]) = g_1 ... g_n \varphi(\alpha * p(\gamma(1))) = g_1 ... g_n \varphi(\beta(1)).$$

The second inequality follows from $p(\gamma)$ being contained in a single chart and the last equality comes from $\widetilde{\alpha} * \gamma$ being a lift of β.

• We now show that closed and bounded sets in $\widetilde{S}$ are compact. Let A be a closed and bounded set of $\widetilde{S}$. As D is a homeomorphism, then $D(A)$ is closed. As A is bounded, A can be covered by finitely many charts which are each bounded. As D is an isometry when restricted to these charts, their image is also bounded. So $D(A)$ is closed and bounded in $\mathbb{H}$ and therefore compact. it follows that $D^{-1}(A) = A$ is compact.

• Finally, we show that closed and bounded sets in S are compact. Let $x \in S$ and let B be a closed ball centered at x with radius $r > 0$. We will prove that B is compact by showing that B is contained in a compact set. Let $y \in B$ be any other point. Then there exists a path β from x to y with length less than r. Lift this path to a path $\widetilde{\beta}$ based at $\widetilde{x} \in p^{-1}(x)$ and ending at some $\widetilde{y}$ in $\widetilde{S}$. Remember that lifts preserve length, so if $\widetilde{B}$ is a closed ball centered at $\widetilde{x}$ with radius $r > 0$ (which is compact by above), then $\widetilde{y} \in \widetilde{B}$. We therefore get that $y \in p(\widetilde{B})$, the continuous image of a compact set. As y is an arbitrary point in B, we get that $B \subset p(\widetilde{B})$. In particular, B is a closed subset of a compact set and therefore compact.

($2 \implies 3$) Let $\widetilde{x}, \widetilde{y}$ be two points in $\widetilde{S}$. Let α be a path between x and y. Then $p(\alpha)$ is some path between $p(x)$ and $p(y)$. By definition, $d(p(x), p(y)) \leq \ell(p(\alpha))$, but $\ell(p(\alpha)) = \ell(\alpha)$ as lifts preserve length. Therefore $d(p(x), p(y)) \leq d(x, y)$.

Now let $(x_n)_{n \in \mathbb{N}}$ be a Cauchy sequence in $\widetilde{S}$. Then $p(x_n)_{n \in \mathbb{N}}$ is Cauchy in S by the above. As S is metrically complete, $p(x_n)$ converges to some $x_0 \in S$. Take an open ball B centered at x_0 and contained within an evenly covered chart of x_0. The lift $p^{-1}(B)$ is a disjoint union of open balls isometric to B. All but finitely many $p(x_n)$ are contained in B, so all but finitely many x_n are contained in $p^{-1}(B)$. Furthermore, as x_n is Cauchy, all but finitely many x_n lie in the same path component of $p^{-1}(B)$. The closure of this path component is compact, so x_n converges.

($3 \implies 1$)

For this last direction, we need to prove a lemma about surjective local homeomorphisms. We will then show that if $\tilde{S}$ is complete, the developing map D is a surjective local homeomorphism and apply Lemma 3.15.

Let U be an open set in S. Let $\widetilde{U}$ be the set of homotopy classes of paths of the form $[\gamma * \alpha]$ where γ is a path from the basepoint of S to some point in U and α is contained in U (we use $*$ to denote the concatenation operation). By construction of the topology of $\widetilde{S}$, $\widetilde{U}$ is open. Furthermore, the set $\widetilde{U}$ maps homeomorphically to U by the covering map.

• First, we show that D is a local homeomorphism. Let $[\gamma] \in \widetilde{S}$. Let $(U_1, \phi_1), ..., (U_n, \phi_n)$ be evenly covered charts covering γ. Note that $\phi_n(U_n)$ contains $D([\gamma]) = \widetilde{\alpha}(1)$. Consider the corresponding open set $\widetilde{U_n}$ in $\widetilde{S}$ as described above. For every $[\gamma] * [\alpha] \in \widetilde{U_n}$, the path $\gamma * \alpha$ is covered by $(U_1, \phi_1), ..., (U_n, \phi_n)$. As ϕ_n is a homeomorphism from S to $\mathbb{H}$, it follows that restricting D to $\widetilde{U_n}$ is a homeomorphism onto its image. In other words, D is a local homeomorphism. Furthermore, ϕ_n is an isometry onto its image by definition of the distance function on S, so it follows that $D|_{\widetilde{U_n}}$ is an isometry onto its image. In other words, D is a local isometry.

• Next, we show that D has the path lifting property. Let $\beta\colon [0,1] \to \mathbb{H}$ be a path such that $\beta(0) \in \mathrm{Im}(D)$, where Im here denotes the image of the function. Fix $\widetilde{s_0} \in D^{-1}(\beta(0))$. We will show that β has a unique lift $\widetilde{\beta}$ in $\widetilde{S}$ based at $\widetilde{\beta}(0) := \widetilde{s_0} \in D^{-1}(\beta(0))$. We inductively define $\widetilde{\beta}$ as follows:

Suppose $t \in [0,1)$ is such that $\beta|_{[0,t]}$ can be uniquely lifted to $\widetilde{\beta}|_{[0,t]}$. Then choose an open neighborhood O about $\widetilde{\beta}(t)$ such that $D|_O$ is an isometry which is possible as D is a local isometry. Then for some $\epsilon > 0$, $\beta((t-\epsilon, t+\epsilon)) \subset D(O)$. To extend $\widetilde{\beta}$, we define $\widetilde{\beta}(s) = D^{-1}(\beta(s))$ for each $s \in (t, t+\epsilon)$. As $D|_O$ is bijective, it follows that this definition of $\widetilde{\beta}$ is unique. Next, we want to extend $\widetilde{\beta}$ to the point $t+\epsilon$ so we can inductively apply the result above. Pick a sequence of points t_n converging to $t+\epsilon$ from the left. As α is continuous, $\alpha(t_n)$ converges in $\mathbb{H}$. In particular, $\alpha(t_n)$ is a Cauchy sequence. All but finitely many $\alpha(t_n) \in D(O)$ and so all but finitely many $D^{-1}(\alpha)(t_n) \in O$. As D is a local isometry, it follows that $D^{-1}(\alpha)(t_n)$ is a Cauchy sequence. We assumed that $\widetilde{S}$ is complete, so $D^{-1}(\alpha)(t_n)$ converges to some $w \in \widetilde{S}$. Define $\widetilde{\beta}(t+\epsilon) = w$. Note that $D(w)$ must be $\alpha(t+\epsilon)$, so $\widetilde{\beta}$ still lifts β. This definition of $\widetilde{\beta}$ is unique as limits are unique.

• Next, we see that D is surjective. Let $y \in \mathbb{H}$ and $x \in Im(D)$. Consider a path joining x and y. Lift this path to a path in $\widetilde{S}$ as above. It follows that $y \in Im(D)$, so D is surjective. □

A corollary of the Hopf-Rinow Theorem for hyperbolic manifolds is that every complete surface S with hyperbolic structure is isomorphic to $\mathbb{H}/G$ for some torsion-free Fuchsian group G. This result can be thought of as a partial converse to Proposition 3.12 or as a partial classification of surfaces with hyperbolic structure. First, we define a map that will allows us to relate a hyperbolic surface with a subgroup of $\mathrm{PSL}(2,\mathbb{R})$.

Theorem 3.17 (Uniformization Theorem). *Let S be a complete hyperbolic surface. Then $S \cong \mathbb{H}/G$ for some Fuchsian group G.*

Proof. Let S be a complete hyperbolic surface and let $x_0 \in S$ be a basepoint. Let $\rho\colon \pi_1(S) \to \mathrm{PSL}(2,\mathbb{R})$ be the holonomy homomorphism as given in Definition 3.14 and let $G = \mathrm{Im}(\rho)$. Remember that ρ is injective, so G is an isomorophic copy of $\pi_1(S, x_0)$ in $\mathrm{PSL}(2,\mathbb{R})$. We will first show that G is discrete and then show that S is isometric to $\mathbb{H}/G$.

Let $p\colon \mathbb{H} \to \mathbb{H}/G$ be the projection map and let x_0 be the basepoint of S. Recall that the group of deck transformations $G(\widetilde{S}) \cong \pi_1(S, x_0)$. Furthermore, as S is complete, we know by the Hopf-Rinow Theorem for Hyperbolic Surfaces that $\widetilde{S} \cong \mathbb{H}$. Finally, we recall that the group of deck transformations $G(\widetilde{S})$ acts on $\widetilde{S}$ as a covering space action. By using the isomorphism $\rho\colon \pi_1(S, x_0) \to G$ and the homeomorphism $D\colon \widetilde{S} \to \mathbb{H}$, we see that the action of G on $\mathbb{H}$ is a covering space action. Properly discontinuous actions are weaker than covering space actions [5]. In particular, G acts on

$\mathbb{H}$ properly discontinuously and so G is discrete. In particular, this means that $\mathbb{H}/G$ admits hyperbolic structure as in Proposition 3.12.

Let $x \in S$. Let α be a path from x_0 to x. We define the map $\overline{D}\colon S \to S$ given by:

$$\overline{D}(x) \coloneqq p(D([\alpha]))$$

Note that while the developing map is well-defined for homotopy classes of paths, two different paths from x to x_0 are not necessarily homotopic. Therefore it is necessary to check that $\overline{D}$ is well-defined.

Let α and β be two different paths based at x_0 and ending at x. Let (U, ϕ) be a chart containing x. Then $D([\alpha]) = g_1, ..., g_{n-1}\phi(\alpha(1)) = g\phi(\alpha(1))$ and similarly $D([\beta]) = h_1, ..., h_{m-1}\phi(\beta(1)) = h\phi(\alpha(1))$ for some finite collection of $g_i, h_j \in \mathrm{PSL}(2, \mathbb{R})$. Then the concatenation $\overline{\alpha} * \beta$ is a loop in $\pi_1(S, x_0)$ where the notation $\overline{\alpha}$ denotes the reversed path $\alpha(1-t)$. Furthermore, we can cover $\overline{\alpha} * \beta$ with the same charts used to cover α and β as above to get that $D([\beta * \overline{\alpha}](1)) = h_1...h_{m-1}g_{n-1}^{-1}...g_1^{-1}\phi(\beta * \overline{\alpha}(1)) = hg^{-1}\phi(x_0)$. In particular, $hg^{-1} \in G = \mathrm{Im}(\rho)$. Then, $D([\alpha])$ and $D([\beta])$ are in the same G-orbit, so $p(D([\alpha]) = p(D([\beta]))$.

Next, we check that $\overline{D}$ is an isometry by proving that $\overline{D}$ is a bijective local isometry. We first note that $\overline{D}$ is a local isometry. Let $z \in S$. Let (U, ϕ) be a chart about z. Then $\phi(U)$ is an open neighborhood of $\phi(z)$ in $\mathbb{H}$. As $p\colon \mathbb{H} \to \mathbb{H}/G$ is a covering map, there exists neighborhoods V about $\phi(z)$ and W about $p(\phi(z))$ such that $p(V) \cong W$. Moreover, if $r > 0$ is the closest distance between any distinct orbit points of $\phi(z)$ (which is well-defined as G is discrete), we can take $V = V \cap B_{\frac{r}{3}}(\phi(z))$ small enough so that, by the definition of distance in $\mathbb{H}/G$ as in Proposition 3.12, we get an isometry from V to W. Therefore $\overline{D}$ is a local isometry. The pullback of W is a disjoint union of open sets, so this also proves that $\overline{D}$ is a covering map.

Finally, in order to see that $\overline{D}$ is bijective, we first note that it is a covering map with evenly covered neighborhoods given by the local isometry as above. We show that $\overline{D}$ is a one-sheeted cover in order to prove injectivity. Let $[y] \in \mathbb{H}/G$. The the preimage of $[y]$ under the quotient map $p : \mathbb{H} \to \mathbb{H}/G$ is the G-orbit of y. Recall that G is the image of the holonomy homomorphism. As $\widetilde{S}$ is complete, we have that the developing map D is a homeomorphism. So we can pull the G-orbit of $[y]$ back to $\widetilde{S}$ resulting in a discrete subgroup of $\widetilde{S}$ which are all in the same orbit of the action of $\pi_1(S)$. This is called the ρ-equivariance of D: $\rho(\gamma \cdot D(\widetilde{s})) = \rho(\gamma) \cdot D(\widetilde{s})$ where $\gamma \in \pi_1(S, x_0)$ and $\widetilde{s} \in \widetilde{S}$. In particular, the pull-back of the G-orbit of x in $\widetilde{S}$ is the preimage of some point $s \in S$ by the covering map $\widetilde{S} \to S$ which is exactly given by the set of homotopy classes of paths on S from x_0 to s. The map $\overline{D}$ is hence a one-sheeted covering map and therefore bijective. Injective local isometries are isometries, so $\overline{D}$ is an isometry between $\widetilde{H}/G$ and S for some Fuchsian group G.

To see that local bijective isometries are isometries, consider two points a and b on S and a path γ between a and b of minimum length. Cover the path γ with charts over which $\overline{D}$ is an isometry. Then $\overline{D}(\gamma)$ is some path between $\overline{D}(a)$ and $\overline{D}(b)$, so $d(\overline{D}(a), \overline{D}(b)) \leq d(a, b)$. To see the equality, apply the same argument to the inverse of $\overline{D}$. $\square$

The above constructions may seem complicated and abstract, but Poincaré gives us an easy way to generate lots of examples of hyperbolic surfaces. We state the following without proof but provide the following references for further information [13], [1].

Let R be a finite-sided convex polygon in $\mathbb{H}$. We call the isometries $G^* = \{g_1, ..., g_r\}$ which map one side of R onto to another side of R side pairings. Then each vertex is associated to a **vertex cycle**, an ordered list of vertices of R where each vertex is mapped to the consecutive one by some side pairing. Each vertex is associated to an angle θ of the polygon R and therefore each vertex cycle is associated to an angle sum (the sum of each angle associated to each vertex of the vertex cycle).

A vertex has propery (*) if:

(1) The vertex is in $\mathbb{H}$, the angle sum of the vertex cycle is $\frac{2\pi}{p}$ for some $p \in \mathbb{N}$, and the product of the side-pairings $h_1, ..., h_n$ which generate the vertex cycle satisfy the **vertex relation** $(h_1 ... h_n)^p = id$.
(2) If the vertex lies on $\partial\mathbb{H}$, then the vertex relation $h_1 ... h_n$ is a parabolic isometry.

We have the following theorem by Poincaré:

Theorem 3.18 (Poincaré Theorem)**.** *Let R be as above. If each vertex of R satisfies (*) then*

(1) G^* generates a Fuchsian group G;
(2) R is a fundamental domain for the G action;
(3) $G = < g_1, ..., g_r |$ vertex relations $>$

We can use Poincaré's theorem to generate Fuchsian groups which define hyperbolic surfaces.

Definition 3.19 (Euler Characteristic)**.** The Euler characteristic of a surface, $\chi(S)$, is defined as the alternating sum $v - e + f$ where v is the number of vertices, e is the number of edges, and f is the number of faces of a cell-structure on S.

If the Euler characteristic of a closed surface S is negative then the surface admits a hyperbolic structure. The proof of this fact is not hard, but it is slightly technical. The idea is to notice that a surface with negative Euler characteristic can be partitioned by curves such that, after cutting along these curves, the surface becomes a disjoint union of pairs of pants, which are closed surfaces with three boundary components. Pairs of pants have hyperbolic structure defined by the lengths of the peripheral curves about each boundary component. In particular, these lengths define a hyperbolic hexagon on the interior of the pair of pants which gives the hyperbolic structure. We can pull this structure back to S by the pants decomposition to get a hyperbolic structure on S. [11]

3.4. **Teichmüller Space.** Now that we have classified all complete hyperbolic surfaces, we can now consider the moduli space of all possible hyperbolic structures on a surface.

Definition 3.20 (Marked Hyperbolic Structure)**.** Let S be a compact, smooth, and orientable surface. A marked hyperbolic structure on S is a pair (X, f) where X is a hyperbolic structure of the form $\mathbb{H}/\Gamma$ where Γ is a discrete subgroup of $\mathrm{PSL}(2, \mathbb{R})$ and $f : S \to X$ is a diffeomorphism. The function f is called a marking, and the pair (X, f) is called a marked hyperbolic structure.

Definition 3.21 (Teichmüller Space)**.** The Teichmüller Space of a hyperbolizable surface S, denoted $\mathfrak{T}(S)$, is the space of marked hyperbolic structures of S up to the relation

$$(X_1, f_1) \sim (X_2, f_2) \iff \exists \text{ isometry } i : X_1 \to X_2 \text{ st. } i \circ f_1 \cong f_2$$

where $\cong$ here denotes homotopy equivalence.

As of right now, this is just a set. However, we can show that this set is in bijection to a particular space of homomorphisms. Using this correspondence, we can define a topology on $\mathfrak{T}(S)$. First, we turn to an example which will help us define this topology on closed surfaces with genus $g \geq 2$.

Example 3.22 ($\mathfrak{T}(T^2) = \mathbb{H}$)**.**

Let T^2 be the torus. The space T^2 does not have hyperbolic structure; it has Euclidean structure (notice that the universal cover is $\mathbb{R}^2$). However, we can define $\mathfrak{T}(T^2)$ as the set of Euclidean structures on T^2 such that the area of T^2 (viewed as the square $\mathbb{R}^2/\mathbb{Z}^2$ in the usual way) is one. This example will generalize nicely to general closed surfaces of genus $g \geq 2$ which do have hyperbolic structure.

We note that we do not have to worry about scaling to unit area for surfaces with hyperbolic structure. Recall from Proposition 2.18 that the area of a hyperbolic triangle depends only on the angle sum. We can apply this fact to a triangulation of a hyperbolic surface to see that the area of the surface is fixed.

This proof is completed in two steps. In the first step, we show $\mathfrak{T}(T^2)$ is in bijection with a particular equivalence class of objects known as marked lattices. In the second step, we show that marked lattices are in bijection with $\mathbb{H}$.

Proof. Step 1:

A lattice is a discrete subgroup $\bigwedge \leqslant (\mathbb{R}^2, +)$ such that $\bigwedge \cong \mathbb{Z}^2$. The area of $\bigwedge$ is defined to be the area of any fundamental domain $\mathbb{R}^2/\bigwedge$. We can view $\bigwedge$ as a $\mathbb{R}$-vector space where the generators of $\bigwedge$ are linearly independent vectors spanning $\mathbb{R}^2$. There is a unique $\lambda > 0$ such that, after scaling, the area of the fundamental domain is 1 (λ is given by the determinant of a matrix representing the quadrilateral $\mathbb{R}^2/\bigwedge$). A lattice is marked if we associate generators of $\bigwedge$ as ordered bases for the $\mathbb{R}$-span of the generators.

We show $\mathfrak{T}(T^2) = \{\text{marked lattices}\}/\sim$ where two marked lattices are equivalent if they differ by a map $z \to cz$ for $c > 0$ or by an Euclidean isometry.

Let a, b be the usual generators of $\pi_1(T^2)$. For any marked lattice $\bigwedge$, the associated ordered (vector-space) basis is a basis of $\pi_1(\mathbb{R}^2/\bigwedge)$. We scale so that $\mathbb{R}^2/\bigwedge$ has area 1. Let F be the unique linear map taking the standard (vector-space) basis of $\mathbb{R}^2$ to the associated ordered basis of $\mathbb{R}^2/\bigwedge$; this map is a marking from $T^2 \to \mathbb{R}^2/\bigwedge$, and the pair $[(\mathbb{R}^2/\bigwedge, F)] \in \mathfrak{T}(T^2)$.

On the other hand, let $[(X, \phi)] \in \mathfrak{T}(T^2)$. Then the universal cover $\widetilde{X}$ is isometric to $\mathbb{R}^2$ by a Euclidean analog to the result of 3.17. The group of deck transformations of $\widetilde{X} \cong \mathbb{R}^2$ (which we can view as $\pi_1(X)$) is a lattice generated by integer translations of the lifts of the generators of $\pi_1(X)$ based at 0.

These identifications are inverses. The resulting marked T^2 structure from the first inclusion $[(\mathbb{R}^2/\bigwedge, F)] \in \mathfrak{T}(T^2)$ has deck transformation group $\bigwedge$. On the other hand, the resulting marked lattice structure given by the deck transformation group of $\widetilde{X}$ in the second inclusion defines a Euclidean unit-area surface diffeomorphic to T^2 via the unique linear map which sends the standard basis of $\mathbb{R}^2$ to the generators of the deck transformations.

Step 2:

Let $\bigwedge$ be a marked lattice in $\mathbb{R}^2 \cong \mathbb{C}$. The generators of $\bigwedge$ can be thought of as linearly independent vectors in $\mathbb{R}^2$ which can in turn be thought of as complex numbers. So $\bigwedge$ is associated to two complex numbers (y, w), and we can rotate and scale $\bigwedge$ so that $(y, w) = (1, z)$ for some $z \in \mathbb{C}$. Rotation (a Euclidean isometry) and scaling does not change the equivalence class of $\bigwedge$ by the definition of the equivalence relation. The choice $(1, z)$ is not unique to $\bigwedge$ as one could also choose the points $(1, \overline{z})$ to represent the reflection of $\bigwedge$ across the real axis which is in the same equivalence class of $\bigwedge$ (reflections are Euclidean isometries). Therefore, we identify $\bigwedge$ with whichever z or $\overline{z}$ is in the upper-half plane.

The above identification is a bijection. Surjectivity follows from the fact that for any $k \in \mathbb{H}$, the marked lattice generated by the vectors associated to 1 and k will map to $k \in \mathbb{H}$. To see injectivity, suppose two marked lattice $\bigwedge_1$ and $\bigwedge_2$ were associated to the same $c \in \mathbb{H}$. Then there is a pair of rotation and scale maps from the generators of $\bigwedge_1$ to $(1, c)$ and a pair of rotation and scale maps from the generators of $\bigwedge_2$ to $(1, c)$. Therefore, the generators of $\bigwedge_1$ can be taken to the generators of $\bigwedge_2$ by a combination of rotation and scale maps. In particular, $\bigwedge_1$ and $\bigwedge_2$ are in the same equivalency class. $\square$

We can use the correspondence above to give the topology of $\mathbb{H}$ to $\mathfrak{T}(T^2)$. We will now use a similar method to find a topology of the Teichmüller Spaces for closed surfaces with genus $g \geq 2$ (which have hyperbolic structure).

Theorem 3.23 (Topology of $\mathfrak{T}(S_g)$)**.** *Let S_g be a closed, smooth, and oriented surface with genus $g \geq 2$. There is a bijection between*

$$\mathfrak{T}(S_g) \text{ and } \{\, \rho : \pi_1(S_g) \to PSL(2,\mathbb{R}) \mid \rho \text{ is discrete and faithful} \}/\sim,$$

where $\rho_1 \sim \rho_2 \iff \rho_1([\gamma]) = A\rho_2([\gamma])A^{-1}$ for every $[\gamma] \in \pi_1(S_g)$ and where $A \in PSL(2,\mathbb{R})$.

We denote $DF(\pi_1(S), \mathrm{PSL}(2,\mathbb{R}))/\mathrm{PSL}(2,\mathbb{R})$ to be the set

$$\{\, \rho : \pi_1(S_g) \to \mathrm{PSL}(2,\mathbb{R}) \mid \rho \text{ is discrete and faithful} \,\}/\sim .$$

Remark 3.24. Here discrete means the image of ρ is discrete, and faithful means that ρ is injective. Using this correspondence, we can topologize $\mathfrak{T}(S)$ using the compact-open topology of the the set of maps $\pi_1(S) \to \mathrm{PSL}(2,\mathbb{R})$. Also, we remark that the following proof does not discern punctures and genus. We can consider a surface with n punctures to be a surface with $+n$ genus for the purposes of this proof.

Proof. Let $[(X, \phi)] \in \mathfrak{T}(S_g)$. The map ϕ_* identifies $\pi_1(S_g)$ with $\pi_1(X)$. Then the holonomy homomorphism $\rho\colon \pi_1(S_g) \to \mathrm{PSL}(2,\mathbb{R})$ is discrete and faithful. Furthermore ρ is well defined up to conjugation in $\mathrm{PSL}(2,\mathbb{R})$ by Definition 3.14. If we chose another representative $(X, \phi') \in [(X, \phi)]$, then ϕ' is in the same homotopy class as ϕ, so the induced map $\phi'_*\colon \pi_1(X) \to \pi_1(S_g)$ is equivalent to the map ϕ_*. Therefore the choice of holonomy homomorphism is well-defined for classes $[X, \phi] \in \mathfrak{T}(S_g)$.

On the other hand, let $p \in DF(\pi_1(S), \mathrm{PSL}(2,\mathbb{R}))/\mathrm{PSL}(2,\mathbb{R})$ be a discrete and faithful homomorphism. Recall that a free and properly discontinuous action on a Hausdorff space is a covering space action. The group $p(\pi_1(S_g))$ is discrete and therefore acts properly discontinuously on $\mathbb{H}$ by Theorem 3.5. Furthrmore, $p(\pi_1(S_g))$ also acts freely on $\mathbb{H}$. Assume to reach a contradiction that $p(\pi_1(S_g))$ did not act freely. Then there is some non-identity Möbius map in $p(\pi_1(S_g))$ that fixes a point. In particular, this map must be an elliptic transformation. However, we recall that $\pi_1(S_g)$ does not have torsion (the only group relations are given by the product of commutators of the usual generators) and therefore no maps in the image $p(\pi_1(S_g))$ have torsion. It follows that any elliptic transformation in $p(\pi_1(S_g))$ must rotate by an irrational angle, but this would violate the discreteness of p. So $p(\pi_1(S_g))$ acts freely on $\mathbb{H}$ and therefore the action of $p(\pi_1(S_g))$ on $\mathbb{H}$ is a covering space action.

Then consider the space $X = \mathbb{H}/p(\pi_1(S_g))$ which is diffeomorphic to S_g and has fundamental group $\pi_1(S_g)$. We can construct a homomorphism $p_*\colon \pi_1(S_g) \to \pi_1(X)$ by first mapping elements of $\pi_1(S_g)$ to $p(\pi_1(S_g))$ (which are deck transformations of X) and then identifying these deck transformations with the elements of $\pi_1(X)$. As both X and S_g are $K(\pi_1(S_g), 1)$ surfaces, the homomorphism p_* is induced by a unique homotopy class of maps between S_g and X. This homotopy class must contain a diffeomorphism [14] and we define this diffeomorphism to be the marking $\phi\colon S_g \to X$.

If we chose a map p' conjugate to p, this results in a diffeomorphic space X' with fundamental group conjugate to $\pi_1(X)$. In other words, X' is the same space as X but with a different choice of basepoint and therefore in the same class as $X \in \mathfrak{T}(S_g)$. $\square$

This correspondence explains why $\mathfrak{T}(S_g)$ is referred to as a "space" and serves as a starting point for further analysis of the hyperbolic structures of a closed surface, possibly with punctures.

4. Mapping Class Group

We now turn our attention to the Mapping Class Group associated to a surface. This may seem like a detour from the running theme of hyperbolic geometry, however the mapping class group has a particular interaction with Teichmüller Space which makes it very relevant to the discussion so far.

Definition 4.1. Let S be a surface. Let $\mathrm{Homeo}^+(S, \partial S)$ be the group of orientation preserving homeomorphisms $S \to S$ which fix the boundary ∂S point-wise. If S has any punctures, we will also require $\mathrm{Homeo}^+(S, \partial S)$ to fix the punctures point-wise. Let $\mathrm{Homeo}_0(S, \partial S)$ be the group of homeomorphisms $S \to S$ which are homotopically equivalent to the identity. The mapping class group of S, $\mathrm{Mod}(S)$, is defined as:

$$\mathrm{Mod}(S) := \mathrm{Homeo}^+(S, \partial S)/\mathrm{Homeo}_0(S, \partial S).$$

The choice to fix punctures is sometimes called the pure mapping class group. However, we will not be making a distinction between the mapping class group and the pure mapping class group in this paper.

Proposition 4.2. *$Mod(D^2) \cong \{id\}$*

The following proof is known as the Alexander Lemma.

Proof. Let D^2 be the closed disk. Let $f \in \mathrm{Homeo}^+(D^2, \partial D^2)$. Consider the map $F\colon D^2 \times I \to D^2$ given by

$$F(x,t) := \begin{cases} (1-t)f(\dfrac{x}{1-t}), & \text{if } 0 \leq |x| < 1-t \\ x, & \text{if } 1-t \leq |x| \leq 1 \end{cases}$$

F(x, t) is continuous as $(1-t)f(\dfrac{x}{1-t})$ and x is continuous for all $(x,t) \in D^2 \times I - \{1\}$ and the two functions agree at time $1-t$.

Furthermore, $F(x, 0) = f(x)$ and $F(x, 1) = id$, so the above defines a homotopy between arbitrary $f \in \mathrm{Homeo}^+(D^2, \partial D^2)$ and id_{D^2}. Therefore $\mathrm{Mod}(D^2)$ is trivial. $\square$

Proposition 4.3. *$Mod(A) \cong \mathbb{Z}$*

Proof. Let A be the annulus. Let $[f] \in \mathrm{Mod}(A)$. Let $\delta\colon [0,1] \to A$ be a straight line between the two boundary components of A. Because f fixes the boundary of A point-wise, there exists a loop $[f(\delta) * \overline{\delta}] \in \pi_1(A) \cong \mathbb{Z}$. We use $\overline{\delta}$ to denote the path $\delta(1-t)$ and $*$ is the concatenation operation.

Define $\Psi\colon \mathrm{Mod}(A) \to \mathbb{Z}$ so that $\Psi([f]) = [f(\delta) * \overline{\delta}] \in \pi_1(A) \cong \mathbb{Z}$. This map is well defined as any map homotopic to f maps to a loop in the same homotopy class in $\pi_1(A)$.

Before we show Ψ is an isomorphism, note that the universal cover of $A \cong \mathbb{R} \times [0,1]$ is given by $\mathbb{R} \times [0,1]$ and that integer translations of the open unit square $(0,1) \times (0,1)$ map homemomorphically to A via the covering map. This follows from the fact that the universal cover of S^1 is given by $\mathbb{R}$.

First we show Ψ is surjective. Consider the matrix

$$M_n\colon = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \in \mathrm{SL}(2, \mathbb{R}).$$

As M_n fixes $\mathbb{R} \times [0,1]$ set-wise, M_n induces a map φ_n on A. Note that $M_n(\delta)$ passes through n consecutive integer translations of the unit square which means that $\varphi_n(\delta) * \overline{\delta}$ has winding number n. In particular, $\Psi(\phi_n) = [\varphi_n(\delta) * \overline{\delta}] = n \in \mathbb{Z}$, so Ψ is surjective.

Next we show Ψ is injective. Let $[\phi] \in \ker(\Psi)$. Then $\phi(\delta) * \delta^{-1}$ is nullhomotopic. In particular, $\phi(\delta) * \delta^{-1}$ is homotopic to $\delta * \delta^{-1}$ which gives that $\phi(\delta)$ is homotopic to δ. We can extend the homotopy between $\phi(\delta)$ and δ to the entire annulus resulting in a map $\overline{\phi} \in [\phi]$ where $\overline{\phi}$ fixes δ. Cutting the annulus along δ results in a disk. As $\overline{\phi}$ fixes the boundary of this disk, it follows that $\overline{\phi}$ induces a map in $\mathrm{Mod}(D^2)$. By Proposition 4.2, this map is homotopic to the identity on D^2. As $\overline{\phi}$ also fixes $\delta([0,1]) = A - D^2$, $\overline{\phi}$ is homotopic to the identity, so $\ker(\Psi) = \{id\}$. $\square$

Proposition 4.4. *Mod*$(T^2) \cong SL(2, \mathbb{R})$

Proof. Let $T^2 \cong S^1 \times S^1$ be the torus. Let $[f] \in \mathrm{Mod}(S^1 \times S^1)$. Then the representative f induces an isomorphism $f_* \coloneqq \pi_1(S^1 \times S^1) \to \pi_1(S^1 \times S^1)$ as f is a homeomorphism. Because $\pi_1(S^1 \times S^1) \cong \mathbb{Z}^2$, we can view f_* as a matrix in $\mathrm{GL}(2, \mathbb{R})$.

The loops in $\pi_1(S^1 \times S^1) \cong \mathbb{Z}^2$ corresponding to the generators $(1,0)$ and $(0,1)$ are simple closed curves intersecting at one point. As f_* is an isomorphism, then $f_*(1,0)$ and $f_*(0,1)$ are also simple closed curves intersecting at one point. We will show that homotopy classes of simple closed curves in $S^1 \times S^1$ correspond to $(p,q) \in \mathbb{Z}^2$ such that $\gcd(p,q) \neq 1$ or $(p,q) = (\pm 1, 0)$ or $(p,q) = (0, \pm 1)$. This will allow us to find a “closed form” for f_*.

Let α be a simple closed curve. We will now view the torus $S^1 \times S^1$ as $\mathbb{R}^2 \backslash \mathbb{Z}^2$. Lift α to a path in $\mathbb{R}^2$ starting at $(0,0)$ which therefore ends at some $(p,q) \in \mathbb{Z}^2$. This path is homotopic to the straight line between $(0,0)$ and (p,q). If this straight line intersects some point of $\mathbb{Z}^2$ other than at the endpoints, then the projection of this line to $\mathbb{R}^2 \backslash \mathbb{Z}^2$ would not represent a simple closed curve on the torus. This can only occur if $\gcd(p,q) = 1$ or $(p,q) = (\pm 1, 0)$ or $(p,q) = (0, \pm 1)$.

In particular, as $f_*(1,0)$ and $f_*(0,1)$ are simple closed curves

$$f_* = \begin{pmatrix} p & p' \\ q & q' \end{pmatrix}$$

where $\gcd(p,q) = 1$ and $\gcd(p',q') = 1$.

We now show that f_* actually maps to transformations in $SL(2,\mathbb{Z})$. We will do this in two steps: first we define an automorphism of the torus which will simplify the problem; then we will then consider the induced map of this automorphism on the fundamental group of the torus which will allow us to relate the number of times $f_*((1,0))$ and $f_*((0,1))$ intersect (just once) with the determinant of f_*.

As $\gcd(p,q) = 1$, there exists $a, b \in \mathbb{Z}$ such that $ap+bq = 1$. Then consider the matrix $B = \begin{pmatrix} a & b \\ -q & p \end{pmatrix} \in SL(2,\mathbb{Z})$. B defines an automorphism of $\mathbb{R}^2 \backslash \mathbb{Z}^2$ by restricting B to the unit square and composing with the projection. Now consider the induced map $B_*\colon \pi_1(\mathbb{R}^2\backslash\mathbb{Z}^2) \to \pi_1(\mathbb{R}^2\backslash\mathbb{Z}^2)$. Given any γ in $\pi_1(\mathbb{R}^2\backslash\mathbb{Z}^2)$, lift γ to a path $\widetilde{\gamma}\colon [0,1] \to \mathbb{R}^2$. Then $B(\widetilde{\gamma})$ has endpoints at $(0,0)$ and $B(\widetilde{\gamma(1)})$. In particular, we associate γ to the point $\widetilde{\gamma(1)} \in \mathbb{Z}^2$ and we associate $B(\gamma)$ with the point $B(\widetilde{\gamma(1)}) \in \mathbb{Z}^2$. Therefore, we get that the induced map on the fundamental group $B_* = B$.

We can apply B_* to the columns of the matrix f_*. We get that

$$B_*(p',q') = (ap' + bq', pq' - p'q).$$

Now it is easy to see that the curves $B_*(p,q) = (1,0)$ and $B_*(p',q')$ intersect $pq' - p'q$ many times by looking at the second entries of the ordered pairs. As B is an isomorphism, it preserves the number of times that (p,q) and (p',q') intersect, so it follows that $\det(f_*) = pq' - p'q = 1$.

Define the map $\Psi \coloneqq \mathrm{Mod}(S^1 \times S^1) \to SL(2,\mathbb{Z})$ by sending $[f] \in \mathrm{Mod}(S^1 \times S^1)$ to the corresponding $f_* \in SL(2,\mathbb{Z})$. Note that Ψ is well defined as homotopic maps induce the same map on $\pi_1(S^1 \times S^1)$. We will show Ψ is an isomorphism.

First, we show Ψ is surjective. Let $M \in SL(2,\mathbb{Z})$. This transformations defines an automorphism f_M on $S^1 \times S^1 \cong \mathbb{R}^2\backslash\mathbb{Z}^2$. But then $\Psi([f_M]) = f_{M*} = M$ by the same reasoning that $B_* = B$ above. Therefore Ψ is surjective.

Next, we show Ψ is injective. Let $[\phi] \in \ker(\Psi)$. Then $\phi_* = id$. In particular, if we set α to be the simple closed curve of $\mathbb{R}^2\backslash\mathbb{Z}^2$ corresponding to $(1,0) \in \mathbb{Z}^2$, then $\phi_*([\alpha]) = \phi(\alpha)$ is homotopic to α. Then we can find a map $\overline{\phi} \in [\phi]$ which fixes α by extending the homotopy from $\phi(\alpha)$ to α to the whole torus.

Cutting along α results in an annulus A. As $\overline{\phi}$ fixes α, it follows that $\overline{\phi}$ induces a map on $\mathrm{Mod}(A) \cong \mathbb{Z}$. Then if we let $\beta \in \pi_1(\mathbb{R}^2\backslash\mathbb{Z}^2)$ be the curve corresponding to $(0,1) \in \mathbb{Z}^2$, then β and $\overline{\phi(\beta)}$ are homotopic arcs of the annulus, again as $\phi_* = id$. In particular, this means that $\overline{\phi(\beta)}$ wraps around the annulus zero times, so the map on A induced by $\overline{\phi}$ is homotopic to the identity on A by Proposition 4.3. But then $\overline{\phi}$ also fixes $\alpha = S^1 \times S^1 - \{A\}$, so $\overline{\phi}$ is homotopic to the identity. □

Proposition 4.5. *Mod*$(S_{0,4}) \cong PSL(2,\mathbb{Z})$

We will obtain $S_{0,4}$ from $S^1 \times S^1$ in order to use the previous example to compute Mod$(S_{0,4})$. Note that we require all maps in Homeo$^+(S_{0,4}, \delta S_{0,4})$ to fix the four punctures.

We will view the torus as a square with opposite sides identified (which is similar to the perspective of viewing the torus as $\mathbb{R}^2/\mathbb{Z}^2$ in the previous example). Consider the map j given by a rotation about the center of the square by an angle of π. This map is called the hyperlliptic involution, and can be thought of as flipping the torus upside down in R^3 about the x-axis. Note that j has 4 fixed points: the center of the square is fixed by j, the vertices of the square are all identified with each other and and fixed by j, the points halfway along the vertical edges of the square are identified and are fixed by j, and the points halfway along the horizontal edges of the square are identified and fixed by j. When viewing j as a rotation of $T^2 \subset \mathbb{R}^3$, these are the four points of T^2 which lie on the axis of rotation. We see that quotient T^2/j identifies two halves of the torus such that the quotient is a deformed cylinder shape where the boundary components of the cylinder are squished to a point (if we think of the half-torus as a cylinder, the two boundary components homeomorphic to S^1 project to an interval).

In particular, we can view $S_{0,4}$ as $(S^1 \times S^1)/j$ where the four punctures are given by the four points fixed by j. This will allow us to compute Mod$(S_{0,4})$ by using the result from Proposition 4.4.

Let $[f] \in$ Mod$(S_{0,4})$. By the identification above, f lifts to two orientation preserving homeomorphisms of the torus: $\overline{f}$ and $j \circ \overline{f}$. Define $\Psi\colon$ Mod$(S_{0,4}) \to PSL(2,\mathbb{Z})$ given by $\Psi([f]) = \pm[\overline{f}]$ where $[\overline{f}] \in \mathrm{SL}(2,\mathbb{R})$ is the linear transformation induced on the fundamental group as in Proposition 4.4. This map is independent of choice of lift as both $\overline{f}$ and $j \circ \overline{f}$ induce the same matrices up to multiplication by $-I \in SL(2,\mathbb{Z})$ and hence are identified in $PSL(2,\mathbb{Z})$.

To see that Ψ is surjective, note that any matrix in $\mathrm{SL}(2,\mathbb{R})$ corresponds to exactly one map in Mod$(S^1 \times S^1)$ by Proposition 4.4. Therefore, a map in $\mathrm{PSL}(2,\mathbb{R})$ corresponds to a pair of maps in Mod$(S^1 \times S^1)$ which differ by composition by j. These maps descend to the same map in Mod$(S_{0,4}$ which map by Ψ to our original matrix in $\mathrm{PSL}(2,\mathbb{R})$.

Before we show that Ψ is injective, we note that the homotopy classes of simple closed curves of $S_{0,4}$ are in bijection with the homotopy classes of simple closed curves of $S^1 \times S^1$. We know from Proposition 4.4 that simple closed curves in $S^1 \times S^1$ are in correspondence to $(p,q) \in \mathbb{Z}^2 \cong \pi_1(S^1 \times S^1)$ where p and q are relatively prime. The homotopy class (p,q) is represented by the curve which winds around the $S^1 \times S^1$ p-times but is slightly homotoped to be injective, and then is twisted q times in a direction orthogonal to the curve p. We picture such a curve below. The generators

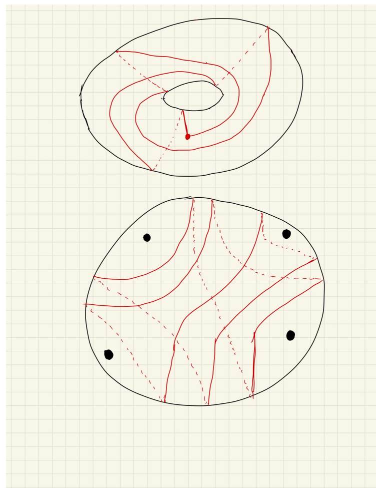

FIGURE 5. The curve $(p, q) = (3, 2)$ displayed on T^2 and $S_{0,4}$.

of $\pi_1(S^1 \times S^1)$ descend to two simple closed curves which intersect each other at one point. It follows that we can reconstruct the (p, q) in the context of $S_{0,4}$ by taking p copies of one generator (slightly homotoped to be injecive), and then twisted in the direction of the other generator q times. The correspondence goes in the other direction as well, so we have a bijection between homotopy classes of simple closed curves in $S_{0,4}$ and $S^1 \times S^1$.

To see that Ψ is injective, let $[g] \in \ker(\Psi)$. Then $\Psi(g) = \pm I$. One pair of lifts of g is $([id_{S^1\times S^1}][j])$. The map $[id_{S^1\times S^1}]$ fixes the generators α and β of $\pi_1(S^1 \times S^1)$. Let $\overline{\alpha}$ and $\overline{\beta}$ be the images of the curves under the quotient to $S_{0,4}$. The curves $\overline{\alpha}$ and $\overline{\beta}$ partition $S_{0,4}$ into four disks, each containing one puncture. Then $[g]$ fixes the homotopy classes of both $\overline{\alpha}$ and $\overline{\beta}$ point-wise as the lift $[id_{S^1\times S^1}]$ does by bijection of homotopy classes of simple closed curves between $S_{0,4}$ and $S^1 \times S^1$, and $[g]$ fixes each of the four disks set-wise as $[g]$ fixes one point (the puncture) in each disk. Therefore by Proposition 4.2, $[g]$ is homotopic to the identity on each disk which means that $[g] = [id_{S_{0,4}}]$.

Note that if we did not require the mapping class group to fix punctures point-wise, then a map in $\mathrm{PSL}(2, \mathbb{R})$ would correspond to a few different maps which differ by not only a composition by j but also by maps which possibly permute the punctures. However, in our definition of the mapping class group, we do not have to worry about this possibility.

Theorem 4.6 (Fricke's Theorem)**.** *The Mapping Class Group acts on Teichmüller Space Properly Discontinuously.*

This proof is long and technical, so we opt to briefly outline the ideas and provide a reference for further reading. Let $S_{g,n}$ be a hyperbolizable surface with genus g and with n punctures. Let $\phi \in \mathrm{Mod}(S_{g,n})$. There is an induced map $\phi_*\colon \pi_1(S_{g,n}) \to \pi_1(S_{g,n})$. Therefore given a discrete and faithful representation $\rho \in \mathfrak{T}(S_{g,n})$, we define $\phi \cdot \rho = \rho \circ \phi_*$.

Fricke's theorem consists of defining a metric on Teichmüller Space using quasi-conformal maps and proves the proper discontinuity of the action using non-trival properties of quasi-conformal maps. The proof also uses the non-trivial Alexander method. The full argument is given in [11].

5. Character Varieties and the Four-Punctured Sphere

We now turn toward studying the character variety $\mathfrak{X}(S)$ of a punctured surface. The character variety is a natural generalization of Teichmüller Space for hyperbolizable surfaces. In this section of the paper, we will look at a few open questions about character varieties punctured surfaces and then answer these questions in the specific case of the four-punctured sphere.

5.1. **Type-Preserving Representations.**

Definition 5.1 (Type-Preserving Representation)**.** Let $S_{g,n}$ be a closed surface of genus g and n punctures. A representation $p\colon \pi_1(S_{g,n}) \to \mathrm{PSL}(2,\mathbb{R})$ is type-preserving if peripheral elements in $\pi_1(S_{g,n})$ are mapped to parabolic transformations.

Recall that a peripheral simple closed curve in $\pi_1(S_{g,n})$ is a loop that goes around a boundary component (in this case, the punctures of the surface) and that parabolic transformations are isometries of hyperbolic space fixing one point on the boundary of $\mathbb{H}$.

Remark 5.2. Note that in the case of the four punctured sphere $S_{0,4}$, peripheral loops are the ones which are freely homotopic to a circle about one of the four punctures. On the other hand, the twice punctured torus $S_{1,2}$ has only two free homotopy classes of peripheral curves around each puncture. The spaces $S_{0,4}$ and $S_{1,2}$ have the same fundamental group and therefore correspond to the same $\mathrm{PSL}(2,\mathbb{R})$ representations. However, the set of type-preserving representations for the above surfaces will be different as these surfaces have a different number of punctures.

Definition 5.3 (Pseudo-Developing Map)**.** Let $S_{g,n}$ be a surface and let ρ be a representation. Then there is an associated map $\widetilde{S_{g,n}} \to \mathbb{H}$ called the pseudo-developing map D_ρ.

The map D_ρ is defined to behave similarly to the developing map D from definition 3.14. In particular, D_ρ is defined to be a piece-wise smooth ρ-equivariant map which means that for each $\widetilde{x} \in \tilde{S}$ and $\gamma \in \pi_1(S_{g,n})$

$$D_\rho(\gamma \cdot \tilde{x}) = \rho(\gamma) \cdot D_\rho(\tilde{x}).$$

The pseudo-developing map D_ρ is only required to be piece-wise smooth while the developing map D was smooth. We are considering the pair ρ (which is not necessarily discrete and faithful) and the associated map D_ρ

as a generalization of the relationship between the holonomy homomorphism and the developing map as seen in the proof of the Uniformization Theorem.

Note that if ρ_0 is a type-preserving representation, then if γ is an arc with endpoints at the punctures of $S_{g,n}$, ρ_0 sends peripheral curves about the endpoints of γ to certain a parabolic elements. Then the curve $D_{\rho_0}(\gamma)$ is an arc in $\mathbb{H}$ joining the two fixed points of the parabolic elements.

Definition 5.4 (Relative Euler Class)**.** The relative Euler class for a representation ρ is defined as

$$e(\rho) = \frac{1}{2\pi}\int_{S_{g,n}} D_\rho^*\omega.$$

where $\omega = \frac{1}{y^2}dxdy$ is the area form of $\mathbb{H}$ and $D_\rho^*\omega$ is the pull-back of this form to $\widetilde{S_{g,n}}$ [17]. As D_ρ is ρ-equivariant, the area form descends to $S_{g,n}$.

The map D_ρ does not differ in the conjugacy class of ρ so neither does $e(\rho)$. The Milnor-Wood inequality is a well-known result [12], [16], [7] giving the bound

$$|e(p)| \leq 2g - 2$$

where g is the genus of the surface. In the case of $S_{0,4}$, a representation ρ can have three different possible values of relative Euler class (-1, 0, or +1).

Definition 5.5 (Character Variety of Closed, Orientable Surfaces)**.** We define the $\mathrm{PSL}(2,\mathbb{R})$ character variety for an oriented surface $S_{g,n}$ with puncture to be

$$\mathfrak{X}(S_{g,n}) = \{\, p\colon \pi_1(S_{g,n}) \to \mathrm{PSL}(2,\mathbb{R}) \mid \text{type-preserving} \,\}/\mathrm{PSL}(2,\mathbb{R}).$$

In words, this is the set of type-preserving homomorphisms from $\pi_1(S_{g,n})$ into $\mathrm{PSL}(2,\mathbb{R})$ up to conjugation in $\mathrm{PSL}(2,\mathbb{R})$. We can give this space a topology via the compact-open topology and quotient topology.

In general, $\mathfrak{X}(S_g)$ may not be Hausdorff, so we will further quotient $\mathfrak{X}(S_g)$ so that if the closure of two orbits intersect nontrivially, the two orbits are identified in $\mathfrak{X}(S_g)$. There are more nuanced methods of "Hausdorffization" which can be performed on $\mathfrak{X}(S_{g,n})$ (namely, the geometric invariant theory ("GIT") quotient), but they are out of the scope of this paper.

Remark 5.6 (Connected Components of Character Variety)**.** Note that in Theorem 3.23 we identified the Teichmüller Space $\mathfrak{T}(S_g)$ (remember we made no distinction between genus and puncture in that proof) with the set of discrete and faithful homomorphisms from $\pi_1(S_g) \to \mathrm{PSL}(2,\mathbb{R})$ up to conjugation by $\mathrm{PSL}(2,\mathbb{R})$. In particular, $\mathfrak{T}(S_g) \subset \mathfrak{X}(S_g)$.

Definition 5.7 (Horocycle)**.** A horocycle is a Euclidean circle in $\mathbb{H} \cup \partial\mathbb{H}$ tangent to one point on $\partial\mathbb{H}$.

Definition 5.8 (Decorated Character Variety)**.** Let $S_{g,n}$ be an oriented surface of genus g with n punctures. Let $\rho\colon \pi_1(S_{g,n}) \to \mathrm{PSL}(2,\mathbb{R})$ be a type-preserving representation. Let v_0 be a puncture and let γ be a peripheral curve in $\pi_1(S_{g,n})$ freely homotopic to a circle about v_0. Then γ is mapped by ρ to a parabolic element by the type-preserving property. This parabolic element has one fixed point on $\partial\mathbb{H}$. We choose a $p(\pi_1(S_g))$ invariant horocycle about the fixed point. A decoration d for the map p is an assignment of such a horocycle to each puncture. A pair (p_1, d_1) and (p_2, d_2) are identified if $p_1 = Ap_2A^{-1}$ and $d_1 = A\dot{d}_2$ for some $A \in \mathrm{PSL}(2, \mathbb{R})$. The quotient space of these decorated type-preserving representations (p, d) under the action above is called the decorated character variety $\mathfrak{X}^d(S_g)$.

Definition 5.9 (Ergodic Group Action)**.** Let G be a group acting on a measure space (X, Σ, m) by measure-preserving transformations (so that $m(g \cdot A) = m(A)$ for $A \in \Sigma$). The action is said to be ergodic if

$$g \cdot A = A \implies m(A) = 0 \text{ for } A \in \Sigma.$$

For more information, see [7].

Conjecture 5.10 (Goldman Conjecture)**.** *The Mapping Class Group acts ergodically on the non-external components of* $\mathfrak{X}(S_{g,n})$.

Goldman's conjecture [4] was originally posed in the context of the character variety of a closed surface $\mathfrak{X}(S_{g,0})$. However, the question can be modified for the punctured-case by considering the type-preserving character variety of a punctured surface $\mathcal{X}(S_{g,n})$.

We have already seen in section 4 that $\mathrm{Mod}(S_g) \curvearrowright \mathfrak{T}(S_g)$ properly discontinuously. One might then wonder how $\mathrm{Mod}(S_g)$ acts on the larger space $\mathfrak{X}(S_g)$. Goldman suggests that while $\mathrm{Mod}(S_g)$ acts in a simple and controlled manner on the external components of $\mathfrak{X}(S_g)$ (properly discontinuously), the action of $\mathrm{Mod}(S_g)$ on the non-external components behaves in the exact opposite way. In particular, Goldman conjectures that the action of $\mathrm{Mod}(S_g)$ on the non-external components of a character variety is ergodic.

The following conjecture is closely related to the Goldman conjecture when stated for punctured surfaces.

Question 5.11 (Bowditch Question)**.** *Is it true that every non- elementary type-preserving representation of a punctured surface group that sends every non-peripheral simple closed curve to a hyperbolic element must be Fuchsian? [2]*

Recall that a Fuchsian group G is elementary if it is of the form $G =< T >$ where $< T >$ is elliptic, parabolic, or hyperbolic, or if G is a dihedral group generated by an elliptic transformation and a hyperbolic transformation (Definition 3.6). A non-elementary representation is a representation with non-elementary image in $\mathrm{PSL}(2, \mathbb{R})$. By punctured surface group, Bowditch is referring to the fundamental group of a punctured surface.

The reason for considering non-elementary (elementary subgroups are simple, cyclic, etc. and we want to ignore these edge cases, and elementary has measure zero)

Definition 5.12 (Sign of a Puncture)**.** Let $S_{g,n}$ be a surface with puncture. Then for a fixed puncture v_0 of $S_{g,n}$ and any representation ρ, ρ associates a parabolic transformation with this puncture, by the definition of type-preserving. By the classification of hyperbolic isometries (Remark 2.19), every parabolic transformation is conjugate to a Möbius map of the form $z + n$ for some $n \in \mathbb{R}$. Such n cannot be zero as parabolic transformations are not the identity. If n is positive, then we say that ρ assigns a positive sign (+1) to v_0. If n is negative, then we say that ρ assigns a negative sign (-1) to v_0.

Conjecture 5.13 (Kashaev Conjecture)**.** *Let S be an oriented hyperbolzi-able surface with punctures $x_1, ..., x_n$. Let $\mathfrak{X}_{k,s}(S_g)$ be the space of type-preserving representations with relative Euler class k satisfying the property that the signs of the punctures are given by $s \in \{+1, -1\}^n$. Then $\mathfrak{X}_{k,s}(S_g)$ is connected if nonempty. In other words, the signs of punctures and the Euler class characterizes the connected components. [6]*

Goldman showed that the connected components for the character variety $\mathfrak{U}(S_g)$ of a closed surface is given by the relative Euler class [3]. The conjecture posed by Kashaev is the analog in the punctured case: the connected components will be given not only by the realtive Euler class but also by the signs of the punctures associated to each representation.

We now set out to answer both the Kashaev Conjecture and Bowditch Question in the specific case of the four-punctured sphere $S_{0,4}$.

5.2. **Combinatorics of $S_{0,4}$.** We now turn our attention to the specific case of the four-punctured sphere. We will first put coordinates on $\mathfrak{X}^d(S_{0,4})$. These coordinates will descend to a special open and dense subsets of $\mathfrak{X}(S_{0,4})$ defined by triangulations on $S_{0,4}$. These open and dense subsets have easily understood connected components. We will then study the effect of a switching to a different triangulation (which equivalently results in a switch to a different open and dense subset). Finally, these switches will reveal the connected component of $\mathfrak{X}(S_{0,4})$. We follow the argument of Tian Yang [17] and present the argument in the style of a related paper by Maloni, Palesi, and Yang [8] on the type-preserving representations of the thrice-punctured projective plane.

Definition 5.14 (Admissable Triangulations of $S_{0,4}$)**.** An ideal arc is an arc between two punctures (possibly going back to the same puncture). An ideal arc γ in $S_{0,4}$ is called p-admissable if the endpoints of $D_p(\widetilde{\gamma})$ are distinct, where $\widetilde{\gamma}$ is any any lift of γ to the universal cover $\widetilde{S}$.

An ideal triangulation of S is a disjoint union of ideal arcs such that the complement of these arcs is a disjoint union of ideal triangles. Such a triangluation is called p-admissable if each of the ideal arcs defining the triangulation is also p-admissable.

A theorem of Kashaev [6] is that the set

$$\mathfrak{X}_\Delta(S_{g,n}) = \{\, [p] \in \mathfrak{X}(S) \mid \Delta \text{ is } p\text{-admissable} \,\}$$

is open and dense in $\mathfrak{X}(S_{g,n})$. Furthermore, there exists finitely many ideal triangulations $\Delta_1, ..., \Delta_n$ of $S_{g,n}$ such that the sets $\mathfrak{X}_{\Delta_i}(S_{g,n})$ cover the character variety $\mathfrak{X}(S)$. [17]

Definition 5.15 (Kashaev Coordinates for the Decorated Character Variety)**.** Let S be a hyperbolizable surface with punctures and let Δ be a ρ-admissable ideal triangulation of S. Let E be the edge set of Δ and T be the ideal triangle set of Δ. Let D_ρ be the pseudo-developing map. We can then construct coordinates for the decorated pair (ρ, d) via this ideal triangulation.

For any edge $e \in E$, we choose some lift $\widetilde{e}$ and map this to $\mathbb{H}$ via the pseudo-developing map D_ρ. As e is a ρ-admissible ideal arc, the distinct endpoints of $D_\rho(\widetilde{e})$ in $\partial\mathbb{H}$ have associated horocycles H_1 and H_2 given by the decoration d. We define $l(e)$ to be the signed distance between H_1 and H_2 which means that $l(e) > 0$ when H_1 and H_2 are disjoint and $l(e) \leq 0$ otherwise.

Then we define the λ-length of e as

$$\lambda(e) = \exp(\frac{l(e)}{2}).$$

The value $\lambda(e)$ does not depend on the choice of lift.

Similarly, for any $t \in T$, let (v_1, v_2, v_3) be the three vertices of t, ordered in such a way that agrees with the positive orientation of t induced by the orientation of S. Let $\widetilde{t}$ be a lift of t to the universal cover $\widetilde{S}$ and $\widetilde{v_i}$ be the corresponding lifts of the vertices. Then D_ρ sends the $\widetilde{v_i}$ to three distinct points on $\partial\mathbb{H}$ which define an ideal triangle in $\mathbb{H}$. Note that on the Riemann sphere $\widehat{\mathbb{C}}$, the set $\partial\mathbb{H}$ is actually a circle which we can give an orientation. Now consider the tuple $(D_\rho(\widetilde{v_1}), D_p(\widetilde{v_2}), D_p(\widetilde{v_3}))$. If these three vertices are ordered in such a way that agrees with the positive orientation of $\partial\mathbb{H}$, then we say that the sign of t is positive and write $\epsilon(t) = 1$. On the other hand, if $(D_p(\widetilde{v_1}), D_p(\widetilde{v_2}), D_p(\widetilde{v_3}))$ do not agree with the positive orientation of $\partial\mathbb{H}$, then we say that the sign of t is negative and write $\epsilon(t) = -1$.

The relative Euler class can then be computed using the signs of the triangles.

$$e(\rho) = \frac{1}{2\pi}\int_{S_{g,n}} D_\rho^*\omega = \frac{1}{2\pi}\sum_{t\in T}\epsilon(t)\pi = \frac{1}{2}\sum_{t\in T}\epsilon(t).$$

using the fact that the area of an ideal triangle is π by Proposition 2.18. The notion $\epsilon(t)$ tells us exactly when the orientation of the triangle is reversed when pulling back from $\mathbb{H}$.

Definition 5.16 (λ-lengths)**.** Let S be a surface with punctures and let δ be an ideal triangulation of S with edge set E. Let (ρ, d) be a decorated ρ-admissible representation. We define

$$\begin{aligned} \lambda \colon E &\to \mathbb{R}^+ \\ e &\mapsto \lambda(e) \end{aligned}$$

where $\lambda(e)$ is given as above.

Definition 5.17 (Sign of Triangles)**.** Let S be a surface with punctures and let δ be an ideal triangulation of S with ideal triangle set T. Let (ρ, d) be a decorated ρ-admissible representation. We define

$$\begin{aligned} \epsilon\colon T &\to \{+1, -1\} \\ t &\mapsto \epsilon(t) \end{aligned}$$

where $\epsilon(t)$ is given as above.

Definition 5.18 (Diagonal Switch)**.** Let S be a hyperbolizable surface with punctures. A key operation that can be performed on an ideal triangulation of S is called a diagonal switch. The operation is as follows:

Take any two adjacent triangles in the ideal triangulation (denote as t_1 and t_2). These triangles share one edge (denote as e); the other four edges outline a quadrilateral (denote as q). Note that e is an edge between two diagonal (i.e. non-adjacent) vertices of q. We can switch to a new ideal triangulation by replacing e with the edge between the diagonal vertices of q which are not incident to e, in particular the only other possible diagonal edge of q. Call this edge e'. This new ideal triangulation is equal to the original with a difference of two triangles and one edge. Denote the two new triangles as t_1' and t_2'.

Kashaev computed the result of a diagonal switch on the Kashaev coordinates as follows:

Theorem 5.19 (Diagonal Switches and Kashaev Coordinates)**.** *Let t_1 and t_2 be adjacent triangles as above. Let e be the shared edge. Let t_1' and t_2' be the new adjacent triangles after the diagonal switch and e' be the new shared edge after the diagonal switch. Let $e_{i,j}$ be the edge shared between t_i and t_j' for $i, j \in \{1, 2\}$.*

If $\epsilon(t_1) = \epsilon(t_2)$,

$$\epsilon(t_1') = \epsilon(t_2') = \epsilon(t_1) = \epsilon(t_2) \text{ and } \lambda(e') = \frac{\lambda(e_{1,1})\lambda(e_{2,2}) + \lambda(e_{1,2})\lambda(e_{2,1})}{\lambda(e)}.$$

If $\epsilon(t_1) \neq \epsilon(t_2)$, then $\lambda(e_{1,1})\lambda(e_{2,2}) - \lambda(e_{1,2})\lambda(e_{2,1}) \neq 0$. Therefore we separate into two further subcases.

If $\epsilon(t_1) \neq \epsilon(t_2)$ and $\lambda(e_{1,1})\lambda(e_{2,2}) - \lambda(e_{1,2})\lambda(e_{2,1}) < 0$,

$$\epsilon(t_1') = \epsilon(t_1),\ \epsilon(t_2') = \epsilon(t_2), \text{ and } \lambda(e') = \frac{\lambda(e_{1,2})\lambda(e_{2,1}) - \lambda(e_{1,1})\lambda(e_{2,2})}{\lambda(e)}.$$

In the case that $\lambda(e_{1,1})\lambda(e_{2,2}) - \lambda(e_{1,2})\lambda(e_{2,1}) > 0$,

$$\epsilon(t_1') = \epsilon(t_2),\ \epsilon(t_2') = \epsilon(t_1), \text{ and } \lambda(e') = \frac{\lambda(e_{1,1})\lambda(e_{2,2}) - \lambda(e_{1,2})\lambda(e_{2,1})}{\lambda(e)}.$$

In fact, any two ideal triangulations of S are separated by finitely many diagonal switches. Therefore, if Δ and Δ' are different ρ-admissible triangulations, then the above result allows us to relate the Kashaev coordinates of the decorated representation (ρ, d) given by the two different triangulations. [17]

Let S be a hyperbolizable surface with punctures. Let ρ be a type-preserving representation. We now examine the combinatorics of the ρ image of loops on S.

Fix an ideal triangulation of S. Let γ be an immersed closed curve on S. Homotopy the curve γ so that, in any triangle t, γ is a simple arc joining two different sides of t. Fix a triangle t which γ intersects. Let e_1 be the edge at which γ enters t. Let e_2 be the edge immediately clockwise of e_1 (to the left of e_1) and let e_3 be the edge immediately counter-clockwise of e_1 (to the right of e_1). Note that in any triangle t, γ either exits at e_2 (left-hand turn) or e_3 (right-hand turn).

If γ makes a left-hand turn at t, define

$$M(t) = \begin{pmatrix} \lambda(e_1) & \epsilon(t)\lambda(e_3) \\ 0 & \lambda(e_2) \end{pmatrix}$$

If γ makes a right-hand turn at t, define

$$M(t) = \begin{pmatrix} \lambda(e_2) & 0 \\ \epsilon(t)\lambda(e_3) & \lambda(e_1) \end{pmatrix}$$

Theorem 5.20 (Trace of Immersed Closed Curve)**.** *The following theorem is a result of Sun and Yang. [17]*

If γ is a simple closed curve that intersects edges $e_1, ..., e_m$ and respectively triangles $t_1, ..., t_m$ where each e_i is the shared edge between adjacent triangles t_i and t_{i-1} where i is taken modulo m, we then have the following formula:

$$|tr(p([\gamma])| = \frac{|tr(M(t_1)...M(t_m))|}{\lambda(e_1)...\lambda(e_m)}$$

We will be able to use the above trace formula in order to answer the Kashaev and Bowditch Conjectures in the case of the 4-punctured sphere.

Definition 5.21 (Tetrahedral Triangulations)**.** A tetrahedral triangulation of the four-punctured sphere $S_{0,4}$ is an ideal triangulation with edge and vertex set given by the edge and vertex set of the 1-skeleton of a tetrahedron. In particular, associate the four punctures of $S_{0,4}$ with the four vertices of a tetrahedron. Then ideal arcs on $S_{0,4}$ correspond to the edges of the tetrahedron.

In the specific case of $S_{0,4}$, let $v_1, ..., v_4$ be the four vertices and t_i be the triangle disjoint from vertex v_i. Let $e_{i,j}$ be the edge between v_i and $v_j, 1 \leq i, j \leq 4$. Edges incident to disjoint sets of vertices are called opposite. (For example, $e_{1,2}$ and $e_{3,4}$ are opposite).

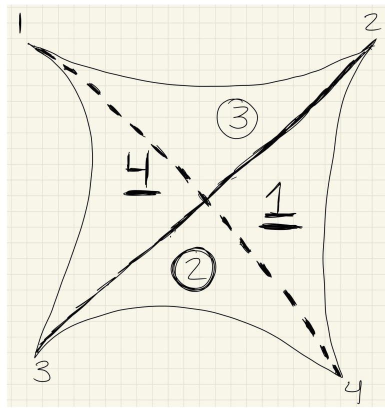

FIGURE 6. An image of a tetrahedral triangulation. The circled triangles are on the "back" while the underlined triangles are on the "front".

A non-peripheral simple closed curve is said to be distinguished if it does not intersect a pair of opposite edges, but intersects every other edge of the tetrahedral triangulation exactly once. It is a quick check that every non-peripheral simple closed curve is distinguished in some tetrahedral triangulation.

To see this, note that there are three homotopy classes of distinguished curves on $S_{0,4}$ given by the natural choice of curve parallel to an opposite pair. These three distinct classes of distinguished curves intersect each other in two points. On the other hand, given any three non-peripheral simple closed curves which intersect each other at 2 points, we can define a teterahedral triangulation where ideal arcs are given by opposite pairs constructed to be parallel to the three given curves. Therefore tetrahedral triangulations are in correspondence with triples of non-peripheral simple closed curves that mutually intersect in two points.

Definition 5.22 (Simultaneous Diagonal Switch)**.** A simultaneous diagonal switch is a move between two different tetrahedral triangulations of $S_{0,4}$. The operation is given by performing diagonal switches at opposite pairs of edges. The effect of this operation is to change the homotopy class of the distinguished curve corresponding to the opposite pair that is switched. There are three possible diagonal switches which will be denoted as S_x, S_y, and S_z corresponding to the three different opposite pairs of curves. These diagonal switches correspond to the curves γ_x, γ_y, γ_z as pictured.

Note that S_x performs two diagonal switches at the edges $e_{3,4}$ and $e_{1,2}$, S_y performs two diagonal switches at the edges $e_{2,4}$ and $e_{1,3}$, and S_z performs two diagonal switches at the edges $e_{2,3}$ and $e_{1,4}$.

Definition 5.23 (Farey Complex and Farey Tree)**.** The Farey complex is a graph with vertex set $\mathbb{Q} \cup \infty$ (the extended reduced fractions where $\frac{1}{0} = \infty$

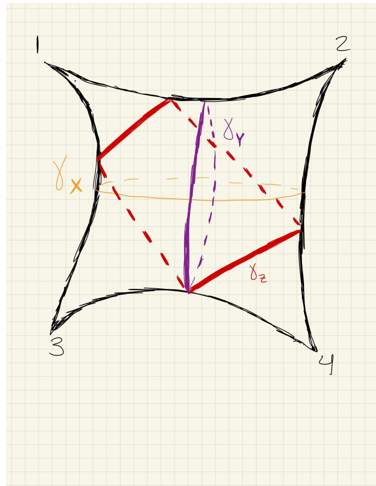

FIGURE 7. The three distinguished curves of a tetrahedral triangulation.

and $\frac{0}{1} = 0$).The vertices are typically arranged in a circle. An edge is drawn between $\frac{p}{q}$ and $\frac{p'}{q'}$ if $pq' - p'q = 1$. Note that the Farey complex defines a triangle between the vertices $\frac{p}{q}$, $\frac{p'}{q'}$, and $\frac{p+p'}{q+q'}$.

The Farey tree is constructed from the Farey complex by placing a vertex at the center of each triangle within the Farey complex. Edges are drawn between vertices corresponding to adjacent triangles in the Farey complex.

Theorem 5.24. *There is a correspondence between tetrahedral triangulations and the Farey tree*

Proof. The vertices of the Farey complex are given by the non-peripheral simple closed curves of $S_{0,4}$. We connect two non-peripheral simple closed curves by an edge if they intersect minimally (in two points). Triangles in the Farey complex correspond to three non-peripheral simple closed curves which mutually intersect each other in two points. To see this, note that the simple closed curves associated to the vertices of the triangle must intersect each other minimally as they are each connected by the edges of the triangle. Therefore, the vertices of the Farey tree correspond to a triple of non-peripheral simple closed curves that intersect minimally which in turn corresponds to a specific tetrahedral triangulation. The edges of the Farey tree represent a simultaneous diagonal switch between two different tetrahedral triangulations. To see this, notice that adjacent triangles in the Farey diagram share two vertices. Therefore, an edge represents changing one of the three defining non-peripheral simple closed curves of the tetrahedral triangulation. $\square$

We now turn our attention toward parameterizing $\mathfrak{X}(S_{0,4})$. Let Δ be a tetrahedral triangulation of $S_{0,4}$ and consider the set

$$\mathfrak{X}_\Delta(S_{0,4}) = \{ [p] \in \mathfrak{X}(S_{0,4}) \mid \Delta \text{ is } p\text{-admissable} \}$$

which is open and dense by Definition 5.11.

We will parameterize $\mathfrak{X}_\Delta(S_{0,4})$ using products of certain λ-lengths from the Kashaev coordinates. Specifically, if $\rho \in \mathfrak{X}_\Delta(S)$, then $[(\rho, d)]$ is associated to a particular $\lambda\colon E \to \mathbb{R}^+$ and $\epsilon\colon T \to \{+1, -1\}^4$. Denote the set of λ-lengths for $[(p, d)]$ as $\mathbb{R}^E_{>0}$. Furthermore, $[(p, d)]$ is also associated with a set of horocycles for each vertex of Δ given by the decoration d. In particular, these horocylces have a defined radius, and so for each vertex we assign a real number according to the radius of the horocyle. Denote this set as $\mathbb{R}^V_{>0}$. We would like to "forget" these decorations when passing the Kashaev coordinates of the decorated character variety $\mathfrak{X}^d_\Delta(S)$ down to coordinates in $\mathfrak{X}_\Delta(S)$ via projection. We do this by defining an action of $\mathbb{R}^V_{>0}$ on $\mathbb{R}^E_{>0}$ which allows us to safely project down the Kashaev coordinates.

We define an action of $\mathbb{R}^V_{>0}$ on $\mathbb{R}^E_{>0}$. Let $u \in \mathbb{R}^V_{>0}$ and $\lambda \in \mathbb{R}^E_{>0}$. Define

$$(u \cdot \lambda)(e_{i,j}) = u(v_i) * \lambda(e_{i,j}) * u(v_j)$$

which is just scalar multiplication of each λ-length and the radii of the horocycles corresponding to the vertices of the edges.

We will show that the quotient of $\mathbb{R}^E_{>0}$ by this action is diffeomorphic to a more concrete space. Namely, consider the set of triples of positive real numbers $\mathbb{R}^3_{>0}$ and the action of $\mathbb{R}_{>0}$ on $\mathbb{R}^3_{>0}$ given by

$$r \cdot (a, b, c) = (ra, rb, rc)$$

The quotient of $\mathbb{R}^3_{>0}$ by the above action of $\mathbb{R}_{>0}$ will be diffeomorphic to the quotient $\mathbb{R}^E_{>0}/\mathbb{R}^V_{>0}$.

Theorem 5.25 (Tetrahedral Coordinates for Character Variety of $S_{0,4}$)**.** *There is a map $\phi\colon \mathbb{R}^E_{>0} \to \mathbb{R}^3_{>0}$ which induces a diffeomorphism $\phi_*\colon \mathbb{R}^E_{>0}/\mathbb{R}^V_{>0} \to \mathbb{R}^3_{>0}/\mathbb{R}_{>0}$. In particular, ϕ_* is a paramaterization of $\mathfrak{X}_\Delta(S)$ without decorations.*

Proof. For each $\lambda \in \mathbb{R}^E_{>0}$ define $X_1 = \lambda(e_{1,2})\lambda(e_{3,4})$, $X_2 = \lambda(e_{1,3})\lambda(e_{2,4})$, and $X_3 = \lambda(e_{1,4})\lambda(e_{2,3})$. We call X_1, X_2, X_3 the tetrahedral coordinates of $[p] \in \mathfrak{X}_\Delta(S)$. Define the map

$$\begin{aligned} \phi\colon \mathbb{R}^E_{>0} &\to \mathbb{R}^3_{>0} \\ (\lambda(e_{1,2}), ..., \lambda(e_{3,4})) &\mapsto (X_1, X_2, X_3) \end{aligned}$$

First, we check that ϕ_* is well-defined. We have the following

$$\begin{aligned}\phi(u\cdot\lambda) &= \phi(u(v_1)\lambda(e_{1,2})u(v_2), ..., u(v_3)\lambda(e_{3,4})u(v_4))\\ &= (\prod_{i=1}^{4} u(v_i)X_1, \prod_{i=1}^{4} u(v_i)X_2, \prod_{i=1}^{4} u(v_i)X_3)\\ &= \prod_{i=1}^{4} u(v_i)(X_1, X_2, X_3)\end{aligned}$$

Therefore the induced map ϕ_* is well-defined.

Next, we check surjectivity. Let $(a, b, c) \in \mathbb{R}^3$. Then

$$\phi(\sqrt{a}, \sqrt{b}, \sqrt{c}, \sqrt{c}, \sqrt{b}, \sqrt{a}) = (a, b, c)$$

so ϕ is surjective.

Finally, we check injectivity. Suppose $\phi(\lambda') = r \cdot \phi(\lambda)$ for some $\lambda, \lambda' \in \mathbb{R}^E_{>0}, r \in \mathbb{R}_{>0}$. We want to show that $\lambda' = u \cdot \lambda$. For each puncture define the quantity

$$v_i(\lambda) = \left(\prod_{j\neq i} \lambda(e_{i,j})^2\right)\left(\prod_{k,k'\neq i} \lambda(e_{k,k'})\right).$$

Then consider decorations of the punctures given by $u(v_i) = r^{-4}\ \dfrac{v_i(\lambda')^{\frac{1}{2}}}{v_i(\lambda)^{\frac{1}{2}}}$.

We have that

$$\begin{aligned}u(v_i) * \lambda(e_{i,j}) * u(v_j) &= r^{-4}\ \frac{v_i(\lambda')^{\frac{1}{2}} * v_j(\lambda')^{\frac{1}{2}}}{v_i(\lambda)^{\frac{1}{2}} * v_j(\lambda)^{\frac{1}{2}}}\lambda(e_{i,j})\\ &= r^{-4}\ \frac{(\lambda'(e_{i,j})^2 * r^8 X_1^2X_2^2X_3^2X_2X_3)^{\frac{1}{2}}}{(\lambda(e_{i,j})^2 * X_1^2X_2^2X_3^2X_2X_3)^{\frac{1}{2}}}\ \lambda(e_{i,j})\\ &= \lambda'(e_{i,j}).\end{aligned}$$

The second equality comes from the fact that the triple

$$(\lambda'(e_{1,2})\lambda'(e_{3,4}), \lambda'(e_{1,3})\lambda'(e_{2,4}), \lambda'(e_{1,4}\lambda'(e_{2,3}))) = (rX_1, rX_2, rX_3)$$

as $\phi(\lambda') = r\cdot\phi(\lambda)$ and re-writing the products $v_i(\lambda')*v_j(\lambda')$ and $v_i(\lambda)*v_j(\lambda)$ in terms of X_1, X_2, X_3 when possible in order to simplify the expression.

By construction, ϕ_* and the inverse ϕ_*^{-1} is differentiable and so ϕ_* is a diffeomorphism. ☐

Furthermore, notice that we can also see that the set $\mathbb{R}^3_{>0}/\mathbb{R}_{>0}$ with the simplex

$$T = \{\, (x_1, x_2, x_3) \in \mathbb{R}^3_{>0} \mid x_1 + x_2 + x_3 = 1 \,\}$$

To see this identification, note that we can consider $\mathbb{R}^3_{>0}/\mathbb{R}_{>0}$ as the set of all open rays in $\mathbb{R}^3_{>0}$ beginning from the origin. Each ray corresponds to a point on the simplex T given by the intersection of the ray with T. For the rest of this paper, we will consider X_1, X_2, and X_3 as points on the simplex T.

5.3. **Kashaev Conjecture of $S_{0,4}$.** We now have the necessary tools to answer the Kashaev Conjecture in the case of $S_{0,4}$. In this section, we apply the above results to calculate the connected components of the type-preserving character variety $\mathfrak{X}(S_{0,4})$ as discovered by Tian Yang [17].

Theorem 5.26 (Peripheral Curves in Tetrahedral Triangulation)**.** *There are four free homotopy classes of peripheral curves in $S_{0,4}$. Denote these curves $\gamma_1, ..., \gamma_4$ such that γ_i is freely homotopic to a circle about the puncture v_i. Let p be a type-preserving representation. Then up to conjugation by elements in $PSL(2,\mathbb{R})$, the image of each γ_i under p are*

(1) $p([\gamma_1]) = \pm \begin{pmatrix} 1 & \frac{1}{\lambda(e_{4,3})\lambda(e_{2,3})}(\epsilon(t_2)X_1 + \epsilon(t_3)X_2 + \epsilon(t_4)X_3 \\ 0 & 1 \end{pmatrix}$

(2) $p([\gamma_2]) = \pm \begin{pmatrix} 1 & \frac{1}{\lambda(e_{1,3})\lambda(e_{3,4})}(\epsilon(t_1)X_1 + \epsilon(t_3)X_3 + \epsilon(t_4)X_2) \\ 0 & 1 \end{pmatrix}$

(3) $p([\gamma_3]) = \pm \begin{pmatrix} 1 & \frac{1}{\lambda(e_{1,2})\lambda(e_{2,3})}(\epsilon(t_1)X_3 + \epsilon(t_2)X_2 + \epsilon(t_3)X_1) \\ 0 & 1 \end{pmatrix}$

(4) $p([\gamma_3]) = \pm \begin{pmatrix} 1 & \frac{1}{\lambda(e_{1,4})\lambda(e_{2,4})}(\epsilon(t_1)X_2 + \epsilon(t_2)X_3 + \epsilon(t_4)X_1) \\ 0 & 1 \end{pmatrix}$

Proof. By Theorem 5.14 from Sun and Yang, we can calculate the trace of free homotopy classes of certain curves by computing matrix multiplications corresponding to the left-hand and right-hand turns that the curve makes with respect to some ideal triangulation Δ.

If we homotope the peripheral curves γ_i tight enough, γ_i intersects three different triangles for a given tetrahedral triangulation. This is clear because three of the ideal triangles in a given tetrahedral triangulation have a vertex at v_i while the ideal triangle t_i does not have a vertex at v_i.

We calculate $p(\gamma_1), ..., p(\gamma_4)$ using the formula found by Sun and Yang. In particular, this formula relates the matrices $p(\gamma_1), ..., p(\gamma_4) \in \mathrm{PSL}(2,\mathbb{R})$ with the λ-lengths and signs of triangles associated to p.

First, we need to check the triangles that each peripheral curve intersects, the order at which it intersects them, and whether the peripheral curve makes a right or left turn.

Without loss of generality, we can choose representatives of $[\gamma_1], ..., [\gamma_4]$ which always make left-hand turns through the ideal triangles and we can choose which ideal triangle at which the representative loop should start.

We do the calculation for $[\gamma_1]$. The calculations for $[\gamma_2], [\gamma_3], [\gamma_4]$ are similar.

Set γ_1 to pass through ideal triangles t_4 then t_3 then t_2.

We compute

$$
\begin{aligned}
p([\gamma_1]) &= \pm \begin{pmatrix} \lambda(e_{2,4}) & \epsilon(t_4)\lambda(e_{1,4}) \\ 0 & \lambda(e_{3,4}) \end{pmatrix} * \begin{pmatrix} \lambda(e_{3,4}) & \epsilon(t_3)\lambda(e_{1,3}) \\ 0 & \lambda(e_{2,3}) \end{pmatrix} * \begin{pmatrix} \lambda(e_{2,3}) & \epsilon(t_2)\lambda(e_{1,2}) \\ 0 & \lambda(e_{2,4}) \end{pmatrix} \\
&= \pm \begin{pmatrix} 1 & \frac{1}{\lambda(e_{4,3})\lambda(e_{2,3})}(\epsilon(t_2)\lambda(e_{1,2})\lambda(e_{3,4}) + \epsilon(t_3)\lambda(e_{1,3})\lambda(e_{2,4}) + \epsilon(t_4)\lambda(e_{1,4})\lambda(e_{2,3})) \\ 0 & 1 \end{pmatrix} \\
&= \pm \begin{pmatrix} 1 & \frac{1}{\lambda(e_{4,3})\lambda(e_{2,3})}(\epsilon(t_2)X_1 + \epsilon(t_3)X_2 + \epsilon(t_4)X_3) \\ 0 & 1 \end{pmatrix}.
\end{aligned}
$$

The second and third inequalities use the fact that scaling all entries of a matrix is equivalent to conjugation by some diagonal matrix and hence does not change the class of p in $\mathfrak{X}(S_{0,4})$. The other three calculations are similar. $\square$

For a given p and each $1 \leq i \leq 4$, $p(v_i)$ is a parabolic transformation. Therefore we must have that the top-right term is nonzero. Theorem 5.26 relates the tetrahedral coordinates and the signs of triangles to the top-right term of each $p(v_i), 1 \leq i \leq 4$. In particular, if we are given the signs of each ideal triangle in Δ, then we obtain four different inequalities involving the tetrahedral coordinates X_1, X_2, and X_3. This idea will be formally applied in the result below.

First, define the sets

$$\mathfrak{X}_k(S_{0,4}) = \{\, \rho \in \mathfrak{X}(S_{0,4}) \mid e(\rho) = k \,\}$$

for $k = -1, 0, 1$. Furthermore, define

$$\mathfrak{X}_\Delta(S_{0,4}, \epsilon) = \{\, \rho \in \mathfrak{X}_\Delta(S_{0,4}) \mid \text{the signs of the triangles are given by } \epsilon \,\}$$

for $\epsilon \in \{+1, -1\}^4$.

Corollary 5.27. *We have shown that the tetrahedral coordinates X_1, X_2, and X_3 can be identified with a simplex T. Define the following subsets*

$$
\begin{aligned}
T_{i,j} &= \{\, (x_1, x_2, x_3) \in T) \mid x_k \neq x_i + x_j, k \neq i, j \,\} \textit{ for } \{i, j\} \subset \{1, 2, 3\} \\
T_* &= \{\, (x_1, x_2, x_3) \in T) \mid x_1 \neq x_2 + x_3, x_2 \neq x_1 + x_3, x_3 \neq x_1 + x_2 \,\}
\end{aligned}
$$

We have the following identifications

(1) $\mathfrak{X}_0(S_{0,4}) \cap \mathfrak{X}_\Delta(S_{0,4}) \cong \bigsqcup_{\{i,j\} \subset \{1,2,3\}} T_{i,j} \sqcup T_{i,j}$

(2) $\mathfrak{X}_1(S_{0,4}) \cap \mathfrak{X}_\Delta(S_{0,4}) \cong \bigsqcup_{i=1}^4 T_*$

(3) $\mathfrak{X}_{-1}(S_{0,4}) \cap \mathfrak{X}_\Delta(S_{0,4}) \cong \bigsqcup_{i=1}^4 T_*$

Proof. Fix a p-admissable tetrahedral triangulation Δ. We have that

$$\mathfrak{X}_\Delta(S_{0,4}) = \bigsqcup_{\epsilon \in \{\pm 1\}^4} \mathfrak{X}_\Delta(S_{0,4}, \epsilon)$$

as a result of Theorem 5.25. We now consider how this set looks under the intersection of $X_k(S_{0,4})$ for possible k.

(1) Case $e(p) = 0$:

By the equality $2e(p) = \sum_{i=1}^{4} \epsilon(t_i)$, we must have in this case that two triangles have postive sign and two triangles have negative sign.

Let $\epsilon_{i,j} \in \{+1,-1\}^4$ be the element with 1s at the i^{th} and j^{th} entries and -1s at the other entries. Then $(\epsilon(t_1), \epsilon(t_2), \epsilon(t_3), \epsilon(t_4)) = \epsilon_{i,j}$ for some $\{i,j\} \subset \{1,2,3,4\}$. In particular,

$$\mathfrak{X}_0(S_{0,4}) \cap \mathfrak{X}_\Delta(S_{0,4}) = \bigsqcup_{\{i,j\}\subset\{1,2,3,4\}} \mathfrak{X}_\Delta(S_{0,4}, \epsilon_{i,j})$$

By the images of the peripheral curves computed above, we obtain four different inequalities involving the signs of the triangles and the tetrahedral coordinates given by the top-right entry of the image of each peripheral curve being non-zero by the type-preserving property of p:

$$\begin{aligned}
v_1 &: \epsilon(t_2)X_1 + \epsilon(t_3)X_2 + \epsilon(t_4)X_3 \neq 0 \\
v_2 &: \epsilon(t_1)X_1 + \epsilon(t_3)X_3 + \epsilon(t_4)X_2 \neq 0 \\
v_3 &: \epsilon(t_1)X_3 + \epsilon(t_2)X_2 + \epsilon(t_3)X_1 \neq 0 \\
v_4 &: \epsilon(t_1)X_2 + \epsilon(t_2)X_3 + \epsilon(t_4)X_1) \neq 0
\end{aligned}$$

After substituting the signs of each triangle, we obtain the following

$$\begin{aligned}
\mathfrak{X}_\Delta(S_{0,4}, \epsilon_{1,2}) &\cong T_{2,3} \cong \mathfrak{X}_\Delta(S_{0,4}, \epsilon_{3,4}) \\
\mathfrak{X}_\Delta(S_{0,4}, \epsilon_{1,3}) &\cong T_{1,3} \cong \mathfrak{X}_\Delta(S_{0,4}, \epsilon_{2,4}) \\
\mathfrak{X}_\Delta(S_{0,4}, \epsilon_{1,4} &\cong T_{1,2} \cong \mathfrak{X}_\Delta(S_{0,4}, \epsilon_{2,3})
\end{aligned}$$

(2) Case $e(p) = 1$:

By the equality $2e(p) = \sum_{i=1}^{4} \epsilon(t_i)$, we must have in this case that three triangles have positive sign and one triangle has negative sign.

Let $\epsilon_i \in \{-1,+1\}^4$ such that the i^{th} entry is negative and the other entires are positive. Then $(\epsilon(t_1), \epsilon(t_2), \epsilon(t_3), \epsilon(t_4)) = \epsilon_i$ for some $i \in \{1,2,3,4\}$. In particular,

$$\mathfrak{X}_1(S_{0,4}) \cap \mathfrak{X}_\Delta(S_{0,4}) = \bigsqcup_{i=1}^{4} \mathfrak{X}_\Delta(S_{0,4}, \epsilon_i)$$

However, for any $1 \leq i \leq 4$, the inequalities coming from the images of the peripheral curves after substituting the values of the signs of the triangles gives

$$\mathfrak{X}_\Delta(S_{0,4}, \epsilon_i) \cong T_*$$

(3) Case $e(p) = -1$:

This case is very similar to (2). $\square$

Recall that the Kashaev Conjecture states that the connected components of a punctured character variety are determined by the sets of representations with both the same relative Euler class and the same assignment of signs to each puncture. Furthermore, recall that the sign of the puncture v_i is defined as the sign of the top-right term of the parabolic $p(\gamma_i)$ when written in standard upper-diagonal form (Definition 5.12). Theorem 5.26

relates the signs of triangles with the signs of punctures. In particular, if we are given the signs which p gives to each ideal triangle in Δ, then we know the signs which p gives to each puncture v_i. We apply Theorem 5.26 and Corollary 5.27 to obtain the next result.

Corollary 5.28. *Let p be a type-preserving representation and let $u_p(v_i)$ be the sign assigned to the puncture v_i by p. Define the set*

$$\mathfrak{X}_{k,s}(S_{0,4}) = \{\, p \in \mathfrak{X}_k(S_{0,4}) \mid (u_p(v_1), u_p(v_2), u_p(v_3), u_p(v_4)) = s \in \{+1,-1\}^4 \,\}$$

Let $s_{i,j} = \epsilon_{i,j}, s_i = \epsilon_i$ as above and let $s^+, s^- \in \{+1,-1\}^4$ be the elements with all positive entries and negative entries respectively. Then the following are equalities

(1) $\mathfrak{X}_0(S_{0,4}) = \bigsqcup_{\{i,j\}\subset\{1,2,3,4\}} \mathfrak{X}_{0,s_{i,j}}(S_{0,4})$

(2) $\mathfrak{X}_1(S_{0,4}) = \bigsqcup_{i=1}^4 \mathfrak{X}_{1,s_i}(S_{0,4}) \bigsqcup X_{1,s^+}(S_{0,4})$

(3) $\mathfrak{X}_{-1}(S_{0,4}) = \bigsqcup_{i=1}^4 \mathfrak{X}_{-1,-s_i}(S_{0,4}) \bigsqcup X_{-1,s^-}(S_{0,4})$

Proof. Because $\mathfrak{X}_{k,s}(S_{0,4}) \cap \mathfrak{X}_\Delta(S_{0,4})$ is open and dense in $\mathfrak{X}_{k,s}(S_{0,4})$, it follows that $\mathfrak{X}_{k,s}(S_{0,4})$ is only nonempty when $\mathfrak{X}_{k,s}(S_{0,4}) \cap \mathfrak{X}_\Delta(S_{0,4})$ is also nonempty. In particular, we reduce to checking the signs of representations within the context of some fixed p-admissible triangulation Δ.

(1) Case $e(p) = 0$:

By corollary 5.27, we have that $(\epsilon(t_1), \epsilon(t_2), \epsilon(t_3), \epsilon(t_4)) = \epsilon_{i,j}$ for some $\{i,j\} \subset \{1,2,3,4\}$. We have the following four equations given by the images of the peripheral curves under p

$$\begin{aligned} v_1 &: \epsilon(t_2)X_1 + \epsilon(t_3)X_2 + \epsilon(t_4)X_3 \neq 0 \\ v_2 &: \epsilon(t_1)X_1 + \epsilon(t_3)X_3 + \epsilon(t_4)X_2 \neq 0 \\ v_3 &: \epsilon(t_1)X_3 + \epsilon(t_2)X_2 + \epsilon(t_3)X_1 \neq 0 \\ v_4 &: \epsilon(t_1)X_2 + \epsilon(t_2)X_3 + \epsilon(t_4)X_1 \neq 0 \end{aligned}$$

Note that exactly two of these equations (corresponding to some vertices v_{k_1}, v_{k_2}) contain both terms $\epsilon(t_i)$ and $\epsilon(t_j)$. It follows that v_{k_1}, v_{k_2} have the same sign. Similarly, the other two vertices v_{k_3}, v_{k_4} must share the same sign, and the sign between v_{k_1} and v_{k_3} are opposite.

Therefore, it follows that $(u(v_1), u(v_2), u(v_3), u(v_4))$ must be given by either ϵ_{k_1,k_2} or ϵ_{k_3,k_4}.

(2) Case $e(p) = 1$:

By corollary 5.27, we have that

$(\epsilon(t_1), \epsilon(t_2), \epsilon(t_3), \epsilon(t_4)) = \epsilon_i$ for some $i \in \{1,2,3,4\}$ and we have the inequalties

$$X_1 \neq X_2 + X_3,\ X_2 \neq X_1 + X_3,\ X_3 \neq X_1 + X_2$$

Suppose that $X_{k_1} > X_{k_2} + X_{k_3}$ for some $\{k_1,k_2,k_3\} = \{1,2,3\}$. Then $X_{k_1} > X_{k_2}, X_{k_3}$ as tetrahedral coordinates are positive numbers. Note that by the equations given by the images of the peripheral curves under p that exactly one vertex weights the large weight X_{k_1} by a factor of -1. This one vertex will have a negative sign and the other vertices will have a positive sign. In this case, $(u(v_1), u(v_2), u(v_3), u(v_4)) = \epsilon_{k_1}$.

On the other hand, if $X_1 < X_2 + X_3, X_2 < X_1 + X_3, X_3 < X_1 + X_2$ then $u(v_i)$ must be positive for each i; the negative term will always be less than the sum of the two positive terms so $(u(v_1), u(v_2), u(v_3), u(v_4)) = s^+$.

(3) Case $e(p) = -1$:

The result is similar to case (2). □

Finally, we will show that each of the above disjoint components are path-connected. In particular, the above is a complete characterization of the connected components of $\mathfrak{X}(S_{0,4})$. However, before we carry out this proof, we need to first take a look at simultaneous diagonal switches performed on a p-admissible triangulation. We will need to know whether the resultant tetrahedral triangulation of the simultaneous diagonal switch will also be p-admissible, and, if so, how the tetrahedral coordinates of p will change when viewed in the context of the new triangulation. This result directly follows from the change of Kashaev Coordinates formula from Theorem 5.19.

Fix a p-admissible tetrahedral triangulation Δ and let Δ' be the tetrahedral triangulation obtained as a result of performing a simultaneous diagonal switch on Δ. If Δ' is also p-admissible, let λ' be the corresponding Kashaev lengths of p with respect to Δ' and let X_1', X_2', X_3' be the resulting tetrahedral coordinates corresponding to p in $\mathfrak{X}_{\Delta'}(S_{0,4})$.

Lemma 5.29 (Switching Between p-admissible Triangulations via Diagonal Switches)**.** *We have the following formulae in the case that $e(p) = 0$; without loss of generality suppose triangles 1 and 2 have positive sign in this case (so triangles 3 and 4 have negative sign):*

(1) If Δ' is a result of S_x on Δ, then Δ' is p-admissible $\iff X_2 \neq X_3$ and the tetrahedral coordinates are given by

$$(X_1, X_2, X_3) = \left(\frac{|X_2 + X_3|^2}{X_1}, X_2, X_3\right)$$

(2) If Δ' is a result of S_y on Δ, then Δ' is p-admissible $\iff X_1 \neq X_3$ and the tetrahedral coordinates are given by

$$(X_1, X_2, X_3) = \left(X_1, \frac{|X_1 - X_3|^2}{X_2}, X_3\right)$$

(3) If Δ' is a result of S_z on Δ, then Δ' is p-admissible $\iff X_1 \neq X_2$ and the tetrahedral coordinates are given by

$$(X_1, X_2, X_3) = \left(X_1, X_2, \frac{|X_1 - X_2|^2}{X_3}\right)$$

We have the following formulae in the case that $e(p) = \pm 1$; without loss of generality suppose $e(p) = 1$ and that triangle 1 has negative sign so that the other triangles have positive sign:

(1) If Δ' is a result of S_x on Δ, then Δ' is p-admissible $\iff X_2 \neq X_3$ and the tetrahedral coordinates are given by

$$(X_1, X_2, X_3) = \left(\frac{|X_2^2 - X_3^2|}{X_1}, X_2, X_3\right)$$

(2) If Δ' is a result of S_y on Δ, then Δ' is p-admissible $\iff X_1 \neq X_3$ and the tetrahedral coordinates are given by

$$(X_1, X_2, X_3) = \left(X_1, \frac{|X_1^2 - X_3^2|}{X_2}, X_3\right)$$

(3) If Δ' is a result of S_z on Δ, then Δ' is p-admissible $\iff X_1 \neq X_2$ and the tetrahedral coordinates are given by

$$(X_1, X_2, X_3) = \left(X_1, X_2, \frac{|X_1^2 - X_2^2|}{X_3}\right)$$

Proof. We show the case that Δ' is obtained by the operation S_x on Δ. The other subcases follow by symmetry.

(1) Case $e(p) = 0$:

We compute the case of S_x. The others are similar.

In this case, two triangles have positive sign and two triangles have negative sign. Suppose triangles 1 and 2 have the same sign. Then S_x performs a simultaneous diagonal switch at $e_{1,2}$ and $e_{3,4}$. By the diagonal switch formula for Kashaev Coordinates (Theorem 5.19), it follows that Δ' is ρ-admissibile. The new λ' lengths are the same as the old ones, besides two updates on the opposite pair associated to S_x. This means that only the coordinate X_1' is changed following the operation S_x. We compute

$$\lambda'(e_{1,2}) = \frac{X_1 + X_3}{\lambda(e_{1,2})}$$

and

$$\lambda'(e_{3,4}) = \frac{X_1 + X_3}{\lambda(e_{3,4})}$$

Multiplying the new λ-lengths gives the desired result.

(2) Case $e(p) = \pm 1$:

We compute the case of S_x. The others are similar.

In this case, one triangle has negative sign and the others have positive sign. Suppose triangles 1 has negative sign. Then S_x performs a simultaneous diagonal switch at $e_{1,2}$ and $e_{3,4}$. By the diagonal switch formula for Kashaev Coordinates (Theorem 5.19), it follows that Δ' is ρ-admissibile when $X_2 \neq X_3$. The new λ' lengths are the same as the old ones, besides two updates on the opposite pair associated to S_x. This means that only the coordinate X_1' is changed following the operation S_x. We compute

$$\lambda'(e_{1,2}) = \frac{X_1 - X_3}{\lambda(e_{1,2})}$$

and
$$\lambda'(e_{3,4}) = \frac{X_1 + X_3}{\lambda(e_{3,4})}$$
Multiplying the new λ-lengths gives the desired result. $\square$

Now that we know the effect of diagonal switches on the Tetrahedral Coordinates of a representation p, we are ready to prove the Kashaev Conjecture in the case of $S_{0,4}$.

Theorem 5.30 (Connected Components of $\mathfrak{X}(S_{0,4})$)**.** *The disjoint components outlined in Corollary 5.28 are path-connected. In particular, the Kashaev Conjecture holds in the $S_{0,4}$ case.*

Proof. Fix a tetrahedral triangulation Δ. To show connectedness of $\mathfrak{X}_{k,s}(S_{0,4})$, it is enough to show that any two points p, q in the subset $\mathfrak{X}_{k,s}(S_{0,4}) \cap \mathfrak{X}_\Delta(S_{0,4})$ can be connected with a path, as any points outside of $\mathfrak{X}_{k,s}(S_{0,4}) \cap \mathfrak{X}_\Delta(S_{0,4})$ are joined by a path to $\mathfrak{X}_{k,s}(S_{0,4}) \cap \mathfrak{X}_\Delta(S_{0,4})$ by the dense property.

In this proof, we calculate that $\mathfrak{X}_{0,s_{1,2}}(S_{0,4}), \mathfrak{X}_{0,s_1}(S_{0,4})$, and $\mathfrak{X}_{0,s^+}(S_{0,4})$ are all connected. The other possible $\mathfrak{X}_{k,s}(S_{0,4})$ are connected by similar, symmetric arguments.

(1) $\mathfrak{X}_{0,s_{1,2}}(S_{0,4})$ is connected:

Note that $\mathfrak{X}_{0,s_{1,2}}(S_{0,4}) \cap \mathfrak{X}_\Delta(S_{0,4}) = \mathfrak{X}_\Delta(S_{0,4}, \epsilon_{3,4}) \cong T_{2,3}$ by Theorem 5.26 and Corollary 5.27. Specifically, by Corollary 5.27, we know that the equations corresponding to the punctures with positive sign have two tetrahedral coordinates weighted by a $+1$.

Now partition $T_{2,3}$ into its two path-components:
$$T_{2,3}^+ = \{\, (x_1, x_2, x_3) \in T_{1,2} \mid x_1 > x_2 + x_3 \,\}$$
$$T_{2,3}^- = \{\, (x_1, x_2, x_3) \in T_{i,j} \mid x_1 < x_2 + x_3 \,\}$$
If $p, q \in \mathfrak{X}_{0,s_{1,2}}(S_{0,4}) \cap \mathfrak{X}_\Delta(S_{0,4})$ are both in $T_{2,3}^+$ or $T_{2,3}^-$, then there is a path between the two points in $T_{2,3}$ which pulls back to a path in $\mathfrak{X}_{0,s_{1,2}}(S_{0,4})$ by the diffeomorphism ϕ_*.

Therefore, it is only left to show that representations which map to different path components of $T_{2,3}$ are joined by a path in $\mathfrak{X}_{0,s_{1,2}}(S_{0,4})$. Clearly, these elements are not joined by a path in $\mathfrak{X}_\Delta(S_{0,4}, \epsilon_{3,4}) \cong T_{2,3}$. However, we can perhaps change our perspective to a different triangulation Δ' such that there exist representations from each distinct path component of $\mathfrak{X}_\Delta(S_{0,4}, \epsilon_{3,4})$ which are actually in the same path component of $\mathfrak{X}_{\Delta'}(S_{0,4}, \epsilon_{3,4})$. This would give us the desired result that any two representations in the open and dense set $\mathfrak{X}_\Delta(S_{0,4}, \epsilon_{i,j})$ can be connected by a path.

In particular, consider the representations $p = (a, b, c) \in T_{2,3}^+$ and $q = (a', b', c') \in T_{2,3}^-$ such that $a' > b'$ (at least some such points exists in $T_{2,3}^-$). We will show that we can switch to a different triangulation Δ' such that p and q can be joined by a path in $\mathfrak{X}_{0,s_{1,2}}(S_{0,4}) \cap \mathfrak{X}_{\Delta'}(S_{0,4})$ which proves the claim. We have that $a > b + c$ as $p \in T_{2,3}^+$ which means $a > b$ as c is positive.

In particular, $a \neq b$ and $a' \neq b'$. Therefore by Lemma 5.29, the diagonal switch S_z will result in a triangulation Δ' which is both p-admissible and q-admissible. Furthermore, the new coordinates for p and q with respect to the set $\mathfrak{X}_{\Delta'}(S_{0,4})$ are given by $p = (a, b, \frac{|a-b|^2}{c})$ and $q = (a', b', \frac{|a'-b'|^2}{c'})$. We know that the signs of triangles are preserved by simultaneous diagonal switches so $p, q \in \mathfrak{X}_{\Delta'}(S_{0,4}, \epsilon_{3,4}) \cong T_{2,3}$. Note that we have $b + \frac{|a-b|^2}{c} < b + \frac{|a-b|^2}{|a-b|} = b + |a-b| = a$. On the other hand, we know that $0 < a' - b' < c'$, which means that $|a' - b'|^2 < c'(a' - b')$ and therefore $b' + \frac{|a'-b'|}{c'} < a'$. In fact, this means that with respect to Δ', both p and q are in $T^+_{2,3}$ and therefore in the same path-component of $\mathfrak{X}_{\Delta'}(S_{0,4}, \epsilon_{3,4})$. Therefore there must be a path between p and q in $\mathfrak{X}_{0,s_{1,2}}(S_{0,4})$.

(2) $\mathfrak{X}_{1,s_1}(S_{0,4})$ is connected:

Partition T_* into its path-components:

$$\begin{aligned}
T^+_{2,3} &= \{ (x_1, x_2, x_3) \in T_* \mid x_1 > x_2 + x_3 \} \\
T^+_{1,3} &= \{ (x_1, x_2, x_3) \in T_* \mid x_2 > x_1 + x_3 \} \\
T^+_{1,2} &= \{ (x_1, x_2, x_3) \in T_* \mid x_3 > x_1 + x_2 \} \\
T_c &= \{ (x_1, x_2, x_3) \in T_* \mid x_1 < x_2 + x_3, x_2 < x_1 + x_3, x_3 < x_1 + x_2 \}
\end{aligned}$$

Note that $\mathfrak{X}_{1,s_1}(S_{0,4}) \cap \mathfrak{X}_{\Delta}(S_{0,4}) \cong T^+_{2,3} \cup T^+_{1,3} \cup T^+_{1,2}$ by Theorem 5.26 and Corollary 5.27. This is because representations in $T^+_{i,j}$ satisfy the condition that $X_k > X_i + X_j$ which is exactly when the representations are contained in the open, dense subset $\mathfrak{X}_{1,s_1}(S_{0,4}) \cap \mathfrak{X}_{\Delta}(S_{0,4}, \epsilon(t_j))$ for some $j \in \{2, 3, 4\}$.

If $p, q \in \mathfrak{X}_{1,s_1}(S_{0,4}) \cap \mathfrak{X}_{\Delta}(S_{0,4})$ are both in the same path-component of T_* with respect to the Δ tetrahedral coordinates, then there must be a path between p and q in $\mathfrak{X}_{1,s_1}(S_{0,4})$. Therefore it remains to show that if p and q are in different path-components of T_* with respect to Δ tetrahedral coordinates that they can still be connected by a path.

Suppose $p = (a, b, c) \in T^+_{2,3}$ and $q = (a', b', c') \in T^+_{1,3}$. We have that $a \neq b$ and $a' \neq b'$, therefore by Lemma 5.29, the diagonal switch S_z will result in a triangulation Δ' which is both p-admissible and q-admissible. Furthermore, the new coordinates for p and q with respect to the set $\mathfrak{X}_{\Delta'}(S_{0,4})$ are given by $p = (a, b, \frac{|a^2-b^2|}{c})$ and $q = (a', b', \frac{|a'^2-b'^2|}{c'})$. We have that $a - b > c$ which means $\frac{|a^2-b^2|}{c} = \frac{|a+b||a-b|}{c} > a + b$. The exact same goes for the Δ' coordinates for q. In particular, p and q are in the same connected component when considered under the tetrahedral coordinates of Δ' and therefore there is a path between p and q in $\mathfrak{X}_{1,s_1}(S_{0,4})$.

If $r = (a'', b'', c'') \in T^+_{1,2}$ with respect to the triangulation Δ, then a similar argument shows that there is a path between r and p as well as r and q using simultaneous diagonal switches.

(3) $\mathfrak{X}_{1,s^+}(S_{0,4})$ is connected

Finally, we have that $\mathfrak{X}_{1,s^+}(S_{0,4}) \cong \bigsqcup_{i=1}^4 T_c$ by Theorem 5.26 and Corollary 5.27. This is because the signs of the punctures are all positive if and only if $X_1 < X_2 + X_3, X_2 < X_1 + X_3, X_3 < X_1 + X_2$ by Corollary 5.21. Note that this condition is met for some elements of $\mathfrak{X}_\Delta(S_{0,4}, \epsilon(t_i))$ for each $i \in \{1, 2, 3, 4\}$.

If $p, q \in \mathfrak{X}_{1,s^+}(S_{0,4}) \cap \mathfrak{X}_\Delta(S_{0,4})$ assign the same signs to the ideal triangles of Δ, then they are in the same connected component T_c and therefore there must be a path between them in $\mathfrak{X}_{1,s^+}(S_{0,4})$.

Choose $p = (a, b, c) \in T_c$ such that $a > b$ and $q = (a', b', c') \in T_c$ such that $b' > a'$, and such that p, q are in different connected components of $\mathfrak{X}_{1,s^+}(S_{0,4}) \cap \mathfrak{X}_\Delta(S_{0,4})$ (i.e. p and q assign different signs to the ideal triangles of Δ). Let Δ' be the tetrahedral triangulation given by the diagonal switch S_z applied to Δ. Note that $a \neq b$ and $a' \neq b'$, so Δ' is both p-admissible and q-admissible. With respect to the tetrahedral coordinates of Δ', $p = (a, b, \frac{|a^2-b^2|}{c})$ and $q = (a', b', \frac{|a'^2-b'^2|}{c'})$ by Lemma 5.29. We note that p and q assign the same signs of triangles with respect to the triangulation Δ'. As $p \in T^c$, we have the three inequalities

$$\begin{aligned} a &< b + c \\ b &< a + c \\ c &< a + b \end{aligned}$$

which we can use to see following three inequalities

$$\frac{|a^2 - b^2|}{c} = \frac{|a + b||a - b|}{c} < a + b \iff a - b < c$$

$$a < b + \frac{|a^2 - b^2|}{c} \iff 1 < \frac{|a + b|}{c} \iff c < a + b,$$

$$b < a + \frac{a^2 - b^2}{c}.$$

The last inequality simply comes from the fact that we required $b < a$ when choosing p. We obtain very similar results for the representation q. In particular, p and q are in the same connected component when considered under the tetrahedral coordinates of Δ' and therefore there is a path between p and q in $\mathfrak{X}_{1,s^+}(S_{0,4})$. □

5.4. **Bowditch and Goldman Conjectures of $S_{0,4}$.** We state without proof the Bowditch and Goldman Conjectures (as discussed in Conjecture 5.10 and Question 5.11) answered in the case of $S_{0,4}$ as found by Tian Yang [17].

Theorem 5.31 (Bowditch Question on $S_{0,4}$)**.** *We have the following:*

1. *Bowditch's question is true on the connected components of $\mathfrak{X}_0(S_{0,4})$,*
2. *Bowditch's question is true on the connected components $\mathfrak{X}_{s^+}(S_{0,4})$ and $\mathfrak{X}_{s^-}(S_{0,4})$,*
3. *Bowditch's question is false on the connected components $\mathfrak{X}_{1,s_i}(S_{0,4})$ and $\mathfrak{X}_{1,-s_i}(S_{0,4})$ for $1 \leq i \leq 4$.*

The Bowditch question is false for some maps with relative Euler class ± 1. In other words, the surface $S_{0,4}$ is a counter-example to the Bowditch question. However, the answer to the Bowditch question is true for maps with relative Euler class 0 [17].

For the connected components where Bowditch's question is true, we can directly apply Yang's [17] trace formula for simple closed curves in order to determine when the image of a simple closed curve under a representation is hyperbolic.

The counter-example proof for the components $\mathfrak{X}_{s^+}(S_{0,4})$ and $\mathfrak{X}_{s^-}(S_{0,4})$ is called a "trace-reduction" argument. It can be shown that for non-Fuchsian type-preserving representations, the trace of the image of any loop in $\pi_1(S_{g,n})$ is bounded by Fuchsian type-preserving representations. Yang uses an algorithm to iteratively find type-representations which send elements to matrices with smaller and smaller trace. Yang proves that this algorithm can be used to find type-preserving representations with trace in $(-2, 2)$; in particular, these are elliptic elements which contradict Bowditch claim.

Theorem 5.32 (Goldman Conjecture in the 4-puncture Sphere Case)**.** *Goldman's conjecture holds on $S_{0,4}$.*

The Goldman conjecture is true in case of $S_{0,4}$ [17]. Recall that Marché and Wolff proved that an affirmative answer to Bowditch's question implies an affirmative answer to the Goldman conjecture [10] [9]. However, even though the Bowditch question not true for type-preserving representations with $e(p) = \pm 1$, we still have an affirmative answer to the Goldman conjecture on $S_{0,4}$.

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