ON THE KASHAEV CONJECTURE OF FOUR- AND FIVE-PUNCTURED SPHERES

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Contents

1. Introduction

In this paper, we study the geometry of hyperbolizable oriented surfaces with punctures. The term hyperbolizable means the surfaces admit local charts into hyperbolic space $\mathbb{H}^2$ with transition maps given by hyperbolic isometries. There are many possible hyperbolic structures that can be given to a surface and the Teichmüller Space of the surface is the moduli space of all such such distinct structures.

The Teichmüller Space can also be viewed as the set of conjugacy classes of maps from the fundamental group $\pi_1(S)$ into $\mathrm{PSL}(2,\mathbb{R})$ satisfying certain conditions. Specifically, the maps are injective with image being a discrete subgroup of $\mathrm{PSL}(2,\mathbb{R})$. It is natural to consider, more generally, the space of homomorphisms from $\pi_1(S)$ into $\mathrm{PSL}(2,\mathbb{R})$ (such maps are called *representations*) up to conjugation by elements in $\mathrm{PSL}(2,\mathbb{R})$. This is the motivation behind the character variety $\mathfrak{X}(S)$, the set of representations from $\pi_1(S)$ into $\mathrm{PSL}(2,\mathbb{R})$ up to conjugation in $\mathrm{PSL}(2,\mathbb{R})$.

In this paper, we will study topological, geometrical, and dynamical properties of such character varieties in the case of punctured surfaces. The motivation for these questions comes from the study of the fundamental group for closed surfaces.

Closed case: We define the $\mathrm{PSL}(2,\mathbb{R})$ character variety for a closed and oriented surface S_g of genus g to be

$$\mathfrak{X}(S_g) = \mathrm{Hom}(\pi_1(S_g), \mathrm{PSL}(2,\mathbb{R}))/\mathrm{PSL}(2,\mathbb{R}).$$

For each map $\rho\colon \pi_1(S) \to \mathrm{PSL}(2,\mathbb{R})$, one can define the Euler class $e(\rho)$ as a value in $\mathbb{Z}$ and this does not depend on the conjugacy class of ρ. Milnor [13] and Wood [15] showed that $|e(\rho)| \leq 2g-2 = |\chi(S_g)|$ where $\chi(S_g)$ is the Euler characteristic. Goldman [3] proved that $|e(\rho)| = 2g-2$ if and only if the representation is discrete and faithful. In particular, this means that the set of maps satisfying $|e(\rho)| = 2g-2$ are the Teichmüller components $\mathfrak{T}(S)\sqcup\mathfrak{T}(\overline{S})$, where $\overline{S}$ is the surface S with the opposite orientation. Goldman [3] also showed that the connected components of the character variety $\mathfrak{X}(S)$ are determined by the Euler class.

The group of symmetries of a surface, called the Mapping Class Group, is defined as

$$\mathrm{Mod}(S) \coloneqq \mathrm{Homeo}^+(S,\partial S)/\mathrm{Homeo}_0(S))$$

where $\mathrm{Homeo}^+(S)$ is the group of orientation preserving homeomorphisms of S and $\mathrm{Homeo}_0(S)$ is the group of homeomorphisms S which are homotopically equivalent to the identity. $\mathrm{Mod}(S)$ acts on the the character variety as follows: given $[f] \in \mathrm{Mod(S)}$ and $\rho \in \mathfrak{X}(S_g)$, the action of f on ρ is given by the induced group-automorphism f_*^{-1} on $\pi_1(S)$.

Fricke proved that that the Mapping Class Group acts properly discontinuously on the Teichmüller components, see for example [12]. Goldman conjectured [4] that the action of the Mapping Class Group on all the other connected components of the character variety is ergodic or "chaotic". Goldman's conjecture is in stark contrast with Fricke's finding for the Teichmüller components. A geometric question of Bowditch [1] asks: is it true that almost-all totally hyperbolic representations are Fuchsian? Recall that a representation is called totally hyperbolic if the trace of the image of every simple closed curve is strictly bigger than 2, which is a characterization of hyperbolic elements of $\mathrm{PSL}(2,\mathbb{R})$. These conjectures are still open in most cases but they have been proven in the case $g = 2$ by Marché and Wolff [11] [10]. The case of Euler class 0 is more complicated because of hyperelliptic involution, see Marché and Wolff for more details. Furthermore, Marché and Wolff also showed that a positive answer to Bowditch's question implies a positive answer to Goldman's conjecture.

Punctured case: Define $S_{g,n}$ to be the closed surface of genus g with n punctures. In order to have a similar behavior for punctured hyperbolizable surfaces, we introduce the notion of type-preserving representations. A representation $\rho\colon \pi_1(S_{g,n}) \to \mathrm{PSL}(2,\mathbb{R})$ is *type-preserving* if it sends peripheral curves to parabolic elements in $\mathrm{PSL}(2,\mathbb{R})$. Recall that a peripheral curve is one which is freely homotopic to a circle about a puncture. We define the type-preserving character variety $\mathfrak{X}(S)$ to be the set of conjugacy classes of type-preserving representations:

$$\mathfrak{X}(S_{g,n}) = \{\, \rho\colon \pi_1(S_{g,n}) \to \mathrm{PSL}(2,\mathbb{R}) \mid \text{type-preserving} \,\}/\mathrm{PSL}(2,\mathbb{R}).$$

The above discussion can be generalized to type-preserving character varieties. However, in this paper, we will focus on an open question specific to punctured surfaces. It is a topological question due to Kashaev which asks about the connected components of $\mathfrak{X}(S_{g,n})$. Kashaev [7] conjectures that the connected components are characterized by the Euler class of a representation (as Goldman showed for the character variety of closed surfaces) along with another invariant, the "signs of the punctures", which is an assignment of signs $\{+1,-1\}$ to each puncture.

We will investigate Tian Yang's [16] answer to this questions specifically in the case of the four punctured sphere $S_{0,4}$. In this case, Kashaev's conjecture for the connected components of $\mathfrak{X}(S)$ turns out to be true. In total, there are 16 non-Teichmüller connected components in $\mathfrak{X}(S_{0,4})$. We will follow the format of a similar paper in the case of the thrice punctured projective plane by Maloni, Palesi, and Yang [9].

Next, we will prove a new result on the number of connected components in the case of the five punctured sphere $S_{0,5}$, again verifying Kashaev's conjecture. There are 64 non-Teichmüller connected components in $\mathfrak{X}_{S_{0,5}}$.

Plan for the paper: At the start, we introduce the background we need for the rest of the thesis. In particular, we will recall hyperbolic space $\mathbb{H}^2$ and explore some of the properties of this space. The isometries of hyperbolic space will be neatly classified. We will then discuss what it means for a surface to have hyperbolic structure and find a nice "closed form" for such hyperbolic surfaces. Finally, we will discuss the Teichmüller Space of a surface.

We will then discuss the Mapping Class Group $\mathrm{Mod}(S)$ of a surface, which formalizes the notion of boundary-preserving and orientation-preserving self-homeomorphisms as described in the discussion of the Goldman conjecture. We will consider some specific examples of the Mapping Class Group and give a brief explanation of how the Mapping Class Group acts on the Teichmüller Space of the surface in a nice, properly discontinuous way.

In the following sections, we will discuss the Character Variety of both closed and punctured surfaces. We will see that the Teichmüller Space can be seen as a connected component of the Character Variety. We then provide Tian Yang's [16] answer to the Kashaev Conjecture in the specific case of the four-punctured sphere, written in the format of Maloni, Palesi, and Yang's paper [9]. Finally, we will examine the Kashaev Conjecture in the case of $S_{0,5}$. In particular, we characterize the connected components of $\mathfrak{X}(S_{0,5})$ and verify the Kashaev Conjecture in this case.

2. Background

The background of this paper follows Caroline Series' [14] notes on hyperbolic geometry.

2.1. Models of Hyperbolic Space.

There are many reasons to consider hyperbolic space. For one, any surface with negative Euler characteristic can have a hyperbolic geometric structure. Therefore, hyperbolic geometry is very important for the study of surfaces. In this paper, we will mostly focus on the upper-half plane model of hyperbolic space.

Definition 2.1. There are two commonly used models of hyperbolic space.

(1) The Upper-Half Plane $\mathbb{H}^2$ is the set

$$\{\, x + iy \in \mathbb{C} \mid y > 0 \,\},$$

together with the metric

$$ds = \frac{\sqrt{dx^2 + dy^2}}{y}.$$

(2) The Poincaré Disk $\mathbb{D}$ is the set

$$\{\, x + iy \in \mathbb{C} \mid |x + iy| < 1 \,\},$$

together with the metric

$$ds = \frac{2\sqrt{dx^2 + dy^2}}{1 - x^2 - y^2}.$$

Note that both ds above are Riemannian metrics. This allows us to define lengths and angles in $\mathbb{H}^2$ and $\mathbb{D}$ in a well-defined way.

We now turn to an immediate application of these metrics by using the length of curve formula from Differential Geometry.

Definition 2.2. Let $\gamma(t) = x(t) + iy(t) \colon [0,1] \to \mathbb{H}^2$ be a curve. Then the length of $\gamma(t)$ is given by

$$\ell(\gamma) := \int_0^1 \frac{1}{y(t)} \sqrt{x'(t)^2 + y'(t)^2}\; dt.$$

Similarly, let $\alpha(t) = x(t) + iy(t) \colon [0,1] \to \mathbb{D}$ be a curve. Then the length of $\alpha(t)$ is given by

$$\ell(\alpha) := \int_0^1 \frac{2}{1 - x(t)^2 - y(t)^2} \sqrt{x'(t)^2 + y'(t)^2}\; dt.$$

Note the differences in the hyperbolic arc length formulas compared to the Euclidean arc length formula. In particular, a segment close to the real axis will be much longer in $\mathbb{H}^2$ than in $\mathbb{R}^2$.

Definition 2.3. We define the distance between $x, y \in \mathbb{H}^2$ to be

$$d(x, y) = \inf \ell(\gamma),$$

where the infimum is considered over all possible paths $\gamma\colon [0,1] \to \mathbb{H}^2$ such that $\gamma(0) = x$ and $\gamma(1) = y$.

One can verify that the pair $(\mathbb{H}^2, d)$ is a metric space. In the exact same way, distance can be defined in the disk model $\mathbb{D}$.

A *geodesic* is a path between two points which has minimal arc length and is parameterized by arc length. One can use the definition above and prove directly the following.

Proposition 2.4. *Vertical lines are geodesics in* $\mathbb{H}^2$.

2.2. **Möbius Transformations.** In this section, we recall that the symmetries of hyperbolic space are classified into three distinct types of transformations. In order to discuss the symmetries of $\mathbb{H}^2$ or $\mathbb{D}$ however, we will first discuss the Riemann Sphere.

Definition 2.5. The Riemann Sphere $\widehat{\mathbb{C}}$ is the one point compactification of $\mathbb{C}$. The topology of $\widehat{\mathbb{C}}$ is induced by $\mathbb{C}$. One can give an explicit formula for this construction via stereographic projection.

Since $\mathbb{H}^2 \subset \widehat{\mathbb{C}}$, our understanding of the symmetries of $\widehat{\mathbb{C}}$ allows us to understand the symmetries of $\mathbb{H}^2$.

Definition 2.6 (Möbius maps)**.** A Möbius map is a map $f\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ such that $\forall z \in \mathbb{C},\ f(z) = \dfrac{az+b}{cz+d}$ where $a, b, c, d \in \mathbb{C}, ad - bc \neq 0$, and $f(\infty) = \frac{a}{c}$. When $cz + d = 0$, we define $f(z) = \infty$.

Möbius maps are invertible with inverse given by the map

$$z \mapsto \frac{dz - b}{-cz + a}.$$

The set of Möbius maps under function composition is a group, which we denote temporarily by Möb. One can prove the following proposition by noticing that the composition of Möbius maps behaves like matrix multiplication. This gives rise to a homomorphism $F\colon \mathrm{SL}(2, \mathbb{C}) \to$ Möb with $\ker(F) = \{\pm I\}$.

Proposition 2.7. *Möb* $\cong PSL(2, \mathbb{C})$.

This identification is given by the map

$$\frac{az+b}{cz+d} \mapsto \left[\frac{1}{ad-bc}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right].$$

Theorem 2.8. *For any two ordered triples* $(z_1, z_2, z_3), (w_1, w_2, w_3) \in \widehat{\mathbb{C}}^3$*, there exists a unique Möbius map* T *with* $T(z_i) = w_i$*. In other words, a Möbius map is uniquely determined by where it sends three distinct points of* $\mathbb{H}^2$*.*

Recall that a group action is free if the action is injective, and a group action is transitive if $\forall x, y \in X \; \exists \; g \in G$ such that $gx = y$. The theorem above tells us that Möbius maps act freely and transitively on distinct tuples in $\widehat{\mathbb{C}}^3$.

Theorem 2.9. *Möbius maps are angle-preserving.*

In particular, the above theorem gives us that

$$\text{Möb} \subseteq \text{Aut}(\hat{\mathbb{C}}) \coloneqq \{\, f \colon \hat{\mathbb{C}} \to \hat{\mathbb{C}} \mid f \text{ is analytic, angle-preserving, and bijective} \,\}.$$

In fact, one can prove that every $f \in \text{Aut}(\hat{\mathbb{C}})$ is a Möbius map using the fact that Möbius maps act freely and transitively on distinct ordered triples in $\hat{\mathbb{C}}^3$ along with Liouville's theorem. For more details, see Series [14].

Theorem 2.10. *$Aut(\widehat{\mathbb{C}}) \cong PSL(2, \mathbb{C})$.*

2.3. **Isometries of $\mathbb{H}^2$.** We now turn our attention toward the symmetries and isometries of $\mathbb{H}^2$. We define the automorphisms of $\mathbb{H}^2$ as

$$\text{Aut}(\mathbb{H}^2) = \{\, f \in \text{Aut}(\widehat{\mathbb{C}}) \mid f(\mathbb{H}^2) = \mathbb{H}^2 \,\}.$$

An *isometry* of $\mathbb{H}^2$ is a map $f \colon \mathbb{H}^2 \to \mathbb{H}^2$ such that

$$d(z, w) = d(f(z), f(w)) \; \forall z, w \in \mathbb{H}^2$$

and so that $\det(D(f)) \neq 0$, where D is the Jacobian matrix of f viewed as a function $\mathbb{R}^2 \to \mathbb{R}^2$. We call an isometry *orientation preserving* if $\det(D(f)) > 0$. Otherwise, we say the isometry is *orientation reversing.* We define $\text{Isom}^+(\mathbb{H}^2)$ to be the set of orientation preserving isometries of $\mathbb{H}^2$ and $\text{Isom}^-(\mathbb{H}^2)$ to be the set of orientation-reversing isometries of $\mathbb{H}^2$. Finally, $\text{Isom}(\mathbb{H}^2)$ will denote the set of all isometries of $\mathbb{H}^2$.

Theorem 2.11. *Möbius maps in $Aut(\mathbb{H}^2)$ are orientation-preserving isometries.*

Proposition 2.12. *$Isom^+(\mathbb{H}^2) \cong Aut(\mathbb{H}^2) \cong PSL(2, \mathbb{R})$.*

On the other hand, one can show that the group of all isometries $\text{Isom}(\mathbb{H}^2) \cong \text{PGL}(2, \mathbb{R})$. However, we will only consider orientation-preserving isometries of $\mathbb{H}^2$ as we will focus on orientable surfaces in later sections of this paper. We can now use this result to classify the geodesics in $\mathbb{H}^2$.

Theorem 2.13. *The geodesics in hyperbolic space are:*

(1) Vertical lines in $\mathbb{H}^2$;
(2) Semi-circles centered on the real axis in $\mathbb{H}^2$.

The above theorem can be seen by examining the image of known geodesics (vertical lines in $\mathbb{H}^2$) under maps in $\text{Isom}^+(\mathbb{H}^2)$. With this classification of geodesics, we may now define hyperbolic triangles.

Definition 2.14 (Hyperbolic Triangles)**.** A triangle in hyperbolic space is formed by three geodesic segments which intersect each other at one point. In fact, given a hyperbolic triangle T with interior angles α, β and γ, the area is given by the quantity

$$A(T) = \pi - \alpha - \beta - \gamma.$$

This means that we cannot find triangles in hyperbolic space that have different areas and the same set of angles. In particular, unlike Euclidean geometry, there are no similar triangles in hyperbolic geometry.

We define an *ideal triangle* to be a type of hyperbolic triangle where the three geodesics intersect each other only on $\partial\mathbb{H}^2 = \mathbb{R} \cup \{\infty\}$. This means that in an ideal triangle, the angle between any two sides is 0 degrees.

We now turn toward classifying the isometries of $\mathbb{H}^2$.

Remark 2.15. Let $\frac{az+b}{cz+d} \in \mathrm{Aut}(\mathbb{H}^2)$. We would like to find the fixed points of this transformation. Note that

$$\frac{az+b}{cz+d} = z \iff cz^2 + (d-a)z - b = 0.$$

By the quadratic formula, the solutions to this equation are

$$z = \frac{a - d \pm \sqrt{(d-a)^2 + 4bc}}{2c} = \frac{a - d \pm \sqrt{(a+d)^2 - 4}}{2c}.$$

While the trace of a matrix in $\mathrm{PSL}(2, \mathbb{R})$ is not well-defined, the square of the trace is well-defined.

We classify the maps in $\mathrm{Aut}(\mathbb{H}^2)\backslash\{I\}$ into three distinct categories. In this classification, we can either use the trace, the fixed points, or the conjugation class:

• A **parabolic** transformation is a map in $\mathrm{Aut}(\mathbb{H}^2)$ with trace ± 2. As $\sqrt{(a+d)^2 - 4} = 0$, parabolic transformations have one fixed point in $\mathbb{R} \cup \{\infty\}$. By conjugating a parabolic transformation by any map in $\mathrm{Aut}(\mathbb{H}^2)$ that sends its unique fixed point to infinity, we see that all parabolic maps are conjugate to a map of the form $z + n$ for some $n \in \mathbb{R} - \{0\}$. In this way, we see that parabolic transformations are analogous to "Euclidean translations".

• A **hyperbolic transformation** is a map with trace in the range $(-\infty, -2) \cup (2, \infty)$. These maps are conjugate to a map of the form $a^2 z$ where $a \in \mathbb{R}_+ - \{1\}$. Therefore, these maps can be seen as analogs of Euclidean hometheties (expansions/contractions). A hyperbolic transformation satisfies $\sqrt{(a+d)^2 - 4} > 0$. Therefore, hyperbolic transformations have two fixed points in $\mathbb{R} \cup \{\infty\}$.

• An **elliptic transformation** is a Möbius map in $\mathrm{Aut}(\mathbb{H}^2)$ with trace in the range $(-2, 2)$. Elliptic transformations are conjugate to a map of the form $\begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}$, which tells us that the trace is $2\cos(\theta)$. We can view elliptic transformations as analogs of Euclidean rotations. In this case, $\sqrt{(a+d)^2-4}$ is imaginary, so the fixed point of an elliptic transformation lies in $\mathbb{H}^2$.

Definition 2.16 (Horocycle)**.** A horocycle is a Euclidean circle in $\mathbb{H}^2 \cup \partial\mathbb{H}$ tangent to $\partial\mathbb{H}^2$. Horocycles tangent to the fixed point of a parabolic transformation are fixed set-wise by that parabolic transformation.

We will now turn our attention toward building the tools necessary for characterizing surfaces with hyperbolic structure.

2.4. **Fuchsian Groups.** In this section, we will explore discrete subgroups of $\mathrm{Aut}(\mathbb{H}^2)$, also known as Fuchsian groups. These groups will induce a tessellation of hyperbolic space by polygons known as fundamental domains. Moreover, we will show that any complete surface with hyperbolic structure is induced by Fuchsian subgroups.

View $\mathrm{SL}(2,\mathbb{R})$ as a subset of $\mathbb{R}^4$. This allows us to give a topology to $\mathrm{SL}(2,\mathbb{R})$, viewed as solutions to the equation $ad-bc=1$ in $\mathbb{R}^4$. Similarly, we can give a topology to $\mathrm{PSL}(2,\mathbb{R})$ as the quotient of $\mathrm{SL}(2,\mathbb{R})$ with respect to the map $(a,b,c,d) \mapsto (-a,-b,-c,-d)$. Matrix multiplication and inversion are continuous, so it follows that $\mathrm{SL}(2,\mathbb{R})$ and $\mathrm{PSL}(2,\mathbb{R})$ are Lie groups.

A discrete subgroup of a Lie group is a subgroup that can be viewed as a discrete space with respect to the topology of the Lie group. Recall that a space is discrete if and only if it contains no limit points. A *Fuchsian* group is defined as a discrete subgroup of $\mathrm{PSL}(2,\mathbb{R})$.

Theorem 2.17 (Conditions for a Fuchsian Group)**.** *For $G \leqslant SL(2,\mathbb{R})$, the following are equivalent:*

(1) G has no limit points in G.
(2) G has no limit points in $SL(2,\mathbb{R})$.
(3) Id is an isolated point of G.

Definition 2.18. A group acts properly discontinuously on $\mathbb{H}^2$ if for any compact set $K \subset \mathbb{H}^2$, we have that $gK \cap K = \emptyset$ for all but finitely many $g \in G$.

Fuchsian groups act properly discontinuously on $\mathbb{H}^2$. In fact, Fuchsian groups are the only subgroups of $\mathrm{SL}(2,\mathbb{R})$ that act properly discontinuously on $\mathbb{H}^2$. In order to see this, we recall equivalent conditions for proper discontinuity.

Theorem 2.19 (Proper Discontinuity)**.** *For $G \leqslant SL(2,\mathbb{R})$, the following are equivalent:*

(1) G does not act properly discontinuously on $\mathbb{H}^2$.
(2) There is a $z \in \mathbb{H}^2$ such that the G-orbit of z has a limit point.
(3) Every G-orbit of $z \in \mathbb{H}^2$ has a limit point provided that z is not fixed by all $g \in G$.

Using the above characterizations of Fuchsian groups and proper discontinuity, we have the following nice result of subgroups of $\mathrm{SL}(2,\mathbb{R})$.

Theorem 2.20. *$G \leqslant SL(2,\mathbb{R})$ acts properly discontinuously on $\mathbb{H}^2 \iff G$ is discrete.*

Definition 2.21 (Elementary Groups)**.** There are four special types of Fuchsian groups. They are

(1) $< T >$ where T is parabolic;
(2) $< T >$ where T is elliptic;
(3) $< T >$ where T is hyperbolic;
(4) $< S, T >$ where S is hyperbolic and T is elliptic. (It can be shown that $< S, T >$ is a dihedral group).

2.5. **Fundamental Domains.** In this section, we will recall fundamental domains which are regions in $\mathbb{H}^2$ which, under the properly discontinuous action of a Fuchsian group, tesselate $\mathbb{H}^2$.

Definition 2.22. Let $G \leqslant \mathrm{SL}(2,\mathbb{R})$ be a discrete subgroup. A fundamental domain of G is an open set $R \subset \mathbb{H}^2$ satisfying:

(1) $gR \cap R = \emptyset \ \forall g \in G - \{id\}$,
(2) $\bigcup_{g \in G} g\overline{R} = \mathbb{H}^2$,
(3) R is the interior of a convex set with ∂R a countable union of geodesics,
(4) For any compact $K \subset \mathbb{H}^2$, there are only finitely many $g \in G$ such that $gR \cap K \neq \emptyset$.

A fundamental domain for a Fuchsian group is very much non-unique and there are different methods for constructing them. In this paper, we will provide one way to find a fundamental domain for a Fuchsian group.

Example 2.23 (Dirichlet Domain)**.** Let G be a discrete subgroup. Pick any $z \in \mathbb{H}^2$. For every $g \in G$, consider the unique geodesic path from z to gz. There is a unique perpendicular bisector of this line. This bisector cuts $\mathbb{H}^2$ into two "half-planes", and we define H_g to be the half-plane that contains z. More explicitly, $H_g = \{\, w \in \mathbb{H}^2 \mid d(z,w) < d(gz,w)\,\}$.

Then, the *Dirchlet Domain* of G, centered at $z \in \mathbb{H}^2$, is defined to be

$$R_z := \bigcap_{g \in G - \{id\}} H_g = \{\, w \in \mathbb{H}^2 \mid d(z,w) < d(gz,w) \ \forall g \in G - \{id\}\,\}.$$

One can then show that R_z is a fundamental domain of G (depending on the initial $z \in \mathbb{H}^2$).

Proposition 2.24. *Given G Fuchsian and a point $z \in \mathbb{H}^2$, the Dirichlet Domain R_z centered at z is a Fundamental Domain for the action of G on $\mathbb{H}^2$.*

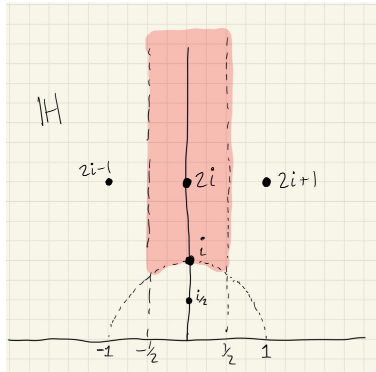

FIGURE 1. The Dirichlet Domain for $\mathrm{PSL}(2,\mathbb{Z})$ centered at $2i$ is highlighted in light red.

Proposition 2.25 (Dirichlet Domain for $PSL(2,\mathbb{Z})$)**.** *The Dirichlet domain for $PSL(2,\mathbb{Z})$ is the region*

$$R = \{\, z \in \mathbb{H}^2 \mid |Re(z)| < \frac{1}{2}, |z| > 1 \,\}.$$

2.6. **Hyperbolic Structure and Uniformization Theorem.** We will turn our attention to surfaces with hyperbolic structure. In this paper, we will only consider orientable surfaces. We will first recall that the action of discrete groups on $\mathbb{H}^2$ defines a hyperbolic surface $\mathbb{H}^2/G$. We will then state a partial converse via the Uniformization Theorem: complete surfaces with hyperbolic structure are isometric to $\mathbb{H}^2/G$ for some Fuchsian group G.

Definition 2.26. Let S be an orientable surface. A *hyperbolic structure* on S is a collection of coordinate charts $(\varphi_\alpha, U_\alpha)$ where U_α is open in S and $\varphi_\alpha \colon U_\alpha \to \mathbb{H}^2$ such that:

(1) $\varphi \colon U_\alpha \to \varphi(U_\alpha)$ is a homeomorphism,
(2) $\bigcup_\alpha U_\alpha = S$,
(3) for every connected component $C \subset U_\alpha \cap U_\beta$ the transition maps $g_c \colon \varphi_\beta(C) \to \varphi_\alpha(C)$ are the restriction of orientation preserving isometries of $\mathbb{H}^2$ in $\mathrm{PSL}(2,\mathbb{R})$.

One can then show the following.

Proposition 2.27. *Let G be a Fuchsian group acting freely on $\mathbb{H}^2$. Then, $\mathbb{H}^2/G$ is a hyperbolic surface.*

Next, we recall the global definition of a hyperbolic structure using the notion of developing map and holonomy map.

Definition 2.28 (Developing Map)**.** Let S be a hyperbolic surface. Fix a base point $x_0 \in S$ and a chart $U_0 \ni x_0$. Then there exists a universal cover $\widetilde{S}$ of S given by the set of all homotopy classes of paths $[0,1] \to S$ beginning at x_0 (Hatcher, [5]).

For any $[\alpha] \in \widetilde{S}$ and a base chart $U_0 \ni x_0$, α can be covered by finitely many charts (U_i, Φ_i) in S by compactness, say $1 \leq i \leq n$. We can choose these charts such that $U_i \cap U_j \neq \emptyset$ only when $|i-j| = 1$. Whenever there is an overlap of charts $U_i \cap U_{i+1} \neq \emptyset$, the transition map $\Phi_i \Phi_{i+1}^{-1} \colon \Phi_i(U_i \cap U_{i+1}) \to \Phi_i(U_i \cap U_{i+1})$ is given by an isometry $g_i \in \mathrm{PSL}(2, \mathbb{R})$ by the hyperbolic structure on S. We can then piece-wise construct a path $\widetilde{\alpha} \subset \mathbb{H}^2$ such that $\widetilde{\alpha}|_{U_1} = \Phi_1(\alpha)$ and $\widetilde{\alpha}|_{U_i} = g_1 ... g_{i-1}\ \Phi_i(\alpha)$. The path $\widetilde{\alpha} \colon [0,1] \to \mathbb{H}^2$ is well-defined in $U_i \cap U_{i+1}$ as $\Phi_i(U_i \cap U_{i+1}) = g_i \Phi_{i+1}(U_i \cap U_{i+1})$ for each $1 \leq i \leq n-1$.

We call $\widetilde{\alpha}$ the developing image of α, and we define the *developing map* D to be the endpoint $\widetilde{\alpha}(1)$ of $\widetilde{\alpha}$ in $\mathbb{H}^2$. That is, we have the map

$$\begin{aligned} D \colon \widetilde{S} &\to \mathbb{H}^2 \\ [\alpha] &\mapsto \widetilde{\alpha}(1). \end{aligned}$$

We can check that D is well-defined up to choice of the basepoint and the initial chart about the basepoint. That is, for any other choice of representative β for the class $[\alpha]$ and any other choice of charts (V_j, Ψ_j) which cover β (for each $1 \leq j \leq m$), the result will be the same.

One can show the developing map is smooth as it is constructed as a composition of a path, a chart homeomorphism, an isometry, and the evaluation function.

Definition 2.29 (Holonomy Homomorphism)**.** Let S be a path-connected, locally path-connected, and semi-locally simply connected hyperbolic surface. Let $[\alpha] \in \pi_1(S, x_0)$. Then

$$D([\alpha]) = \widetilde{\alpha}(1) = g_1 ... g_{n-1} \Phi_n(\alpha(1))$$

for some $g_1, ..., g_n \in \mathrm{PSL}(2, \mathbb{R})$. As the developing map is well-defined for any choice of charts covering α, we can choose charts covering α such that the the final chart $U_n = U_1$, the initial chart.

The *holonomy homoromorphism* is defined to be

$$\begin{aligned} \rho \colon \pi_1(S, x_0) &\to \mathrm{PSL}(2, \mathbb{R}) \\ [\alpha] &\mapsto g_1 ... g_{n-1}. \end{aligned}$$

One can check that the map ρ is injective.

The developing map and holonomy homomorphism pair (D, ρ) define a global geometric structure on the surface. The definition of this pair depends just on the choice of basepoint and the initial chart about the basepoint. These choices are equivalent to precomposition of the developing map by some isometry (consider the transition map for two different possible choices of initial chart or the composition of transition maps coming from a covering of a path between two different choices of basepoint). If we change this choice, the pair (D, ρ) will change by the action of $\mathrm{PSL}(2, \mathbb{R})$:

$$A \cdot (D, \rho) = (AD, A\rho A^{-1}),$$

for each $A \in \mathrm{PSL}(2,\mathbb{R})$. These constructions show that every complete hyperbolic surface is of the form $\mathbb{H}^2/G$ for some Fuchsian group G, an immediate corollary of the Hopf-Rinow Theorem.

Theorem 2.30 (Hopf-Rinow Theorem for Hyperbolic Surfaces)**.** *Let S be a connected, path-connected, and semi-locally simply connected hyperbolic surface and let $\widetilde{S}$ be its universal cover. The following are equivalent:*

(1) The developing map $D : \widetilde{S} \to \mathbb{H}^2$ is a surjective covering map.
(2) S is a complete metric space.
(3) $\widetilde{S}$ is a complete metric space.

This result can be thought of as a partial classification of surfaces with hyperbolic structure. Note that as $\widetilde{S}$ is the universal cover of S, it follows that $\widetilde{S} \cong \mathbb{H}^2$ when S is a complete metric space. In particular, $S \cong \mathbb{H}^2/G$ for some torsion-free Fuchsian group G.

Theorem 2.31 (Uniformization Theorem)**.** *Let S be a complete hyperbolic surface. Then $S \cong \mathbb{H}^2/G$ for some Fuchsian group G.*

Remark 2.32 (Euler Characteristic)**.** The Euler characteristic of a surface, $\chi(S)$, is defined as the alternating sum $v - e + f$ where v is the number of vertices, e is the number of edges, and f is the number of faces of a cell-structure on S. For a closed surface S_g, the Euler characteristic of the surface is $2g - 2$. We remark that a surface is hyperbolizable if the Euler characteristic is negative.

2.7. **Teichmüller Space.** Now that we have classified all complete hyperbolic surfaces, we can now consider the moduli space of all possible hyperbolic structures on a surface.

Definition 2.33 (Marked Hyperbolic Structure)**.** Let S be a compact, smooth, and orientable surface. A marked hyperbolic structure on S is a pair (X, f), where X is a hyperbolic structure of the form $\mathbb{H}^2/\Gamma$ for Γ a discrete subgroup of $\mathrm{PSL}(2,\mathbb{R})$ and $f\colon S \to X$ is a diffeomorphism. The function f is called a marking, and the pair (X, f) is called a marked hyperbolic structure.

Definition 2.34 (Teichmüller Space)**.** The Teichmüller Space of a hyperbolizable surface S, denoted $\mathfrak{T}(S)$, is the space of marked hyperbolic structures of S up to the relation

$$(X_1, f_1) \sim (X_2, f_2) \iff \exists \text{ isometry } i : X_1 \to X_2 \text{ st. } i \circ f_1 \cong f_2,$$

where $\cong$ here denotes homotopy equivalence.

As of right now, this is just a set. However, we can show that this set is in bijection to a particular space of homomorphisms. Using this correspondence, we can define a topology on $\mathfrak{T}(S)$.

Theorem 2.35 (Topology of $\mathfrak{T}(S_g)$)**.** *Let S_g be a closed, smooth, and oriented surface with genus $g \geq 2$. There is a bijection between*

$$\mathfrak{T}(S_g) \text{ and } \{\, \rho : \pi_1(S_g) \to PSL(2,\mathbb{R}) \mid \rho \text{ is discrete and faithful} \,\}/\sim,$$

where $\rho_1 \sim \rho_2 \iff \rho_1([\gamma]) = A\rho_2([\gamma])A^{-1}$ for every $[\gamma] \in \pi_1(S_g)$, and where $A \in PGL(2,\mathbb{R})$.

We denote $DF(\pi_1(S), \mathrm{PSL}(2,\mathbb{R}))/\mathrm{PSL}(2,\mathbb{R})$ to be the set

$$\{\, \rho : \pi_1(S_g) \to \mathrm{PSL}(2,\mathbb{R}) \mid \rho \text{ is discrete and faithful} \,\}/\sim .$$

Remark 2.36. Here, discrete means the image of ρ is discrete, and faithful means that ρ is injective. Using this correspondence, we can topologize $\mathfrak{T}(S)$ using the compact-open topology of the the set of maps $\pi_1(S) \to \mathrm{PSL}(2,\mathbb{R})$. We remark that the above result will also hold for punctured surfaces.

2.8. **Mapping Class Group.** We now recall the definition of the Mapping Class Group of a surface and its relationship with Teichmüller Space. The following section follows Farb-Margalit's *A Primer on Mapping Class Groups* [12].

Definition 2.37. Let S be a surface. Let $\mathrm{Homeo}^+(S, \partial S)$ be the group of orientation preserving homeomorphisms $S \to S$ which fix the boundary ∂S point-wise. If S has any punctures, we will also require $\mathrm{Homeo}^+(S, \partial S)$ to fix the punctures point-wise. Let $\mathrm{Homeo}_0(S, \partial S)$ be the group of homeomorphisms $S \to S$ which are homotopically equivalent to the identity. The mapping class group of S, $\mathrm{Mod}(S)$, is defined as:

$$\mathrm{Mod}(S) \coloneqq \mathrm{Homeo}^+(S, \partial S)/\mathrm{Homeo}_0(S, \partial S).$$

The choice to fix punctures is sometimes called the pure mapping class group. However, we will not be making a distinction between the mapping class group and the pure mapping class group in this paper.

The mapping class group of the four-puncture sphere is given by $\mathrm{PSL}(2,\mathbb{Z})$.

Example 2.38. $\mathrm{Mod}(S_{0,4}) \cong PSL(2,\mathbb{Z})$.

A very important result is the following:

Theorem 2.39 (Fricke's Theorem)**.** *[12] The Mapping Class Group acts on Teichmüller Space Properly Discontinuously.*

Let $S_{g,n}$ be a hyperbolizable surface with genus g and with n punctures. Let $\phi \in \mathrm{Mod}(S_{g,n})$. There is an induced map $\phi_* \colon \pi_1(S_{g,n}) \to \pi_1(S_{g,n})$. Therefore, given a discrete and faithful representation $\rho \in \mathfrak{T}(S_{g,n})$, we define $\phi \cdot \rho = \rho \circ \phi_*$.

Fricke's theorem consists of defining a metric on Teichmüller Space using quasi-conformal maps and proves the proper discontinuity of the action using non-trival properties of quasi-conformal maps. The proof also uses the Alexander method. The full argument is given in [12].

3. Character Varieties

We now study the character variety $\mathfrak{X}(S)$ of a surface. The notion of character variety is a natural generalization of Teichmüller Space viewed as the set of discrete and faithful representations $\pi_1(S) \to \mathrm{PSL}(2,\mathbb{R})$. In this section of the paper, we will first look at the character variety of closed surfaces and then generalize to punctured surfaces. We discuss several open questions on the character variety of punctured surfaces.

3.1. **Character Variety of Closed Surfaces.**

Definition 3.1 (Character Variety of Closed, Orientable Surfaces)**.** We define the $\mathrm{PSL}(2,\mathbb{R})$ character variety for a closed, hyperbolizable, and oriented surface S_g to be

$$\mathfrak{X}(S_g) = \{p\colon \pi_1(S_g) \to \mathrm{PSL}(2,\mathbb{R})\}/\mathrm{PSL}(2,\mathbb{R}).$$

In words, this is the set of homomorphisms from $\pi_1(S_{g,n})$ into $\mathrm{PSL}(2,\mathbb{R})$ up to conjugation in $\mathrm{PSL}(2,\mathbb{R})$. We can give this space a topology via the algebraic topology and quotient topology. In general, using this definition we see that the set $\mathfrak{X}(S_g)$ may not be Hausdorff, so we will further quotient $\mathfrak{X}(S_g)$ so that if the closure of two orbits in $\mathrm{PSL}(2,\mathbb{R})$ intersect nontrivially, the two orbits are identified in $\mathfrak{X}(S_g)$. There are more methods of "Hausdorffization" which can be performed on $\mathfrak{X}(S_g)$ (namely, the geometric invariant theory ("GIT") quotient), but they are out of the scope of this paper. Each of the three definitions differ by a set of measure zero. For the purposes of this paper, we will consider the definition of $\mathfrak{X}(S_g)$ as given above.

Remark 3.2 (Teichmüller Space and the Character Variety)**.** Recall that we identify the Teichmüller Space $\mathfrak{T}(S_g)$ with the set of discrete and faithful homomorphisms from $\pi_1(S_g) \to \mathrm{PSL}(2,\mathbb{R})$ up to conjugation by $\mathrm{PGL}(2,\mathbb{R})$, as recalled in 2.35. Then $\mathfrak{T}(S_g)$ breaks into two components in $\mathfrak{X}(S_g)$. In particular, $\mathfrak{T}(S_g) \subset \mathfrak{X}(S_g)$. Furthermore, we remark that the three definitions for the character variety above agree over irreducible representations, and we also have that representations in the Teichüller component are irreducible. So we have that $\mathfrak{T}(S_g) \subset \mathfrak{X}(S_g)$ when considering the other definitions as well.

3.1.1. *Connected Components of the Character Variety (closed surface).*

In this section, we will state a classification of the connected components of the character variety of a closed surface. To carry out this classification, for each $[\rho] \in \mathfrak{X}(S_g)$, we can associate a pseudo-developing map that generalizes the construction of the developing map in the Uniformization Theorem (2.31). We will use the notion of pseudo-developing map to define a particular integer which will index the connected components of the character variety.

Definition 3.3 (Pseudo-Developing Map)**.** Let S_g be a surface and let ρ be a representation. A map $D_\rho\colon \widetilde{S_g} \to \mathbb{H}^2$ is called a pseudo-developing map if D_ρ is piece-wise smooth and ρ-equivariant. A map is *ρ-equivariant* if for each $\widetilde{x} \in \tilde{S}$ and $\gamma \in \pi_1(S_{g,n})$

$$D_\rho(\gamma \cdot \tilde{x}) = \rho(\gamma) \cdot D_\rho(\tilde{x}).$$

It is a consequence of Kashaev that such a map exists for surfaces [6], [2]. Using the pseudo-developing map, we can define the notion of relative Euler class for a representation.

Definition 3.4 (Relative Euler Class)**.** The relative Euler class for a representation ρ is defined as

$$\begin{aligned} e\colon \mathfrak{X}(S_g) &\to \mathbb{Z} \\ [\rho] &\mapsto \frac{1}{2\pi} \int_{S_{g,n}} D_\rho^* \omega, \end{aligned}$$

where $\omega = \frac{1}{y^2} dxdy$ is the area form of $\mathbb{H}^2$ and $D_\rho^*\omega$ is the pull-back of this form to $\widetilde{S_g}$ [16]. As D_ρ is ρ-equivariant, the area form descends to $S_{g,n}$.

The map D_ρ does not differ in the conjugacy class of ρ so neither does $e(\rho)$. The Milnor-Wood inequality is a well-known result [13], [15], [8] giving the bound

$$|e(p)| \leq 2g - 2,$$

where g is the genus of the surface. In the case of S_2, for example,

$$e(\rho) \in \{-2, -1, 0, 1, 2\}.$$

Theorem 3.5 (Connected Components of $\mathfrak{X}(S_g)$, (Goldman, [3]))**.** *Let S_g be a closed, oriented, and hyperbolizable surface. Then the connected components of $\mathfrak{X}(S_g)$ are indexed by the Euler class. In particular, the connected components of $\mathfrak{X}(S_g)$ are exactly*

$$\mathfrak{X}(S_g) = \bigsqcup_{k \in \{2-2g, .., 2g-2\}} \{\, [\rho] \in \mathfrak{X}(S_g) \mid e(p) = k \,\},$$

where k varies over the possible values of $e(\rho)$.

Theorem 3.6 (Extremal Components of $\mathfrak{X}(S_g)$)**.** *There are two distinguished components of $\mathfrak{X}(S_g)$ [3]:*

- *The connected component with Euler class $2g - 2$ is $\mathfrak{T}(S_g)$.*
- *The connected component with Euler class $2 - 2g$ is $\mathfrak{T}(\overline{S_g})$, where $\overline{S_g}$ is the surface S_g endowed with the opposite orientation.*

3.2. **Character Variety of Punctured Surfaces.** In this section, we will generalize the definitions and theorems regarding the character variety of closed surfaces to the case of punctured surfaces. One problem in generalizing $\mathfrak{X}(S_g)$ to punctured surfaces is that many punctured surfaces have the same fundamental group (e.g. $S_{0,4}$ and $S_{1,2}$). In order to see behaviors similar to the ones described above in the closed surface case, we need to introduce the following definition.

Definition 3.7 (Type-Preserving Representation)**.** Let $S_{g,n}$ be a closed surface of genus g and n punctures. A representation $\rho\colon \pi_1(S_{g,n}) \to \mathrm{PSL}(2,\mathbb{R})$ is *type-preserving* if the peripheral elements in $\pi_1(S_{g,n})$ are mapped to parabolic transformations.

Recall that a peripheral simple closed curve in $\pi_1(S_{g,n})$ is a loop that goes around a boundary component (in this case, the punctures of the surface) and that parabolic transformations are non-trivial isometries of hyperbolic space fixing exactly one point on the boundary of $\mathbb{H}^2$. Note that two distinct punctured surfaces with the same fundamental group will have distinct peripheral curves. In this way, we can capture the differences between distinct punctured surfaces by considering their type-preserving representations.

Definition 3.8 (Character Variety of Punctured, Orientable Surfaces)**.** We define the $\mathrm{PSL}(2,\mathbb{R})$ character variety for a hyperbolizable, oriented surface $S_{g,n}$ with puncture to be

$$\mathfrak{X}(S_{g,n}) = \{\, p\colon \pi_1(S_{g,n}) \to \mathrm{PSL}(2,\mathbb{R}) \mid \text{type-preserving}\,\}/\mathrm{PSL}(2,\mathbb{R}).$$

Note that this definition is vacuous if there are no punctures and so generalizes the definition of the character variety for closed surfaces from the previous section. In particular, we modify the definition of the character variety in the closed surface case by requiring all representations to be type-preserving.

3.2.1. *Kashaev Conjecture.* In the case of punctured surfaces, it is no longer true in general that the set

$$\mathfrak{X}_k(S_{g,n}) \coloneqq \{\, [\rho] \in \mathfrak{X}(S_{g,n}) \mid \mathrm{e}(\rho) = k \,\}$$

is connected. Therefore, we have to define a new invariant in order to classify the connected components of $\mathfrak{X}(S_{g,n})$.

Definition 3.9 (Sign of a Puncture)**.** Let $S_{g,n}$ be a surface with puncture and let $[\rho] \in \mathfrak{X}(S_{g,n})$. Let $[\gamma]$ be the class of peripheral curves about the puncture v_0 of $S_{g,n}$. Then ρ associates a parabolic transformation with this puncture by the definition of type-preserving. By the classification of hyperbolic isometries, every parabolic transformation is conjugate to a Möbius map of the form $z + t$ for some $t \in \mathbb{R} \setminus \{0\}$. In fact, the $\mathrm{PSL}(2,\mathbb{R})$-conjugacy class of the parabolic transformation is determined by the sign of t. If t is positive, then we say that ρ assigns a positive sign (+1) to v_0. If t is negative, then we say that ρ assigns a negative sign (-1) to v_0.

Conjecture 3.10 (Kashaev Conjecture)**.** *Let $S_{g,n}$ be an oriented hyperbolzi-able surface with puncture. Let $\mathfrak{X}_{k,s}(S_{g,n})$ be the space of type-preserving representations with relative Euler class k satisfying the property that the signs of the punctures are given by $s \in \{+1,-1\}^n$. Then $\mathfrak{X}_{k,s}(S_{g,n})$ is connected if nonempty. In other words, the signs of punctures and the Euler class characterizes the connected components. [7]*

Goldman showed that the connected components for the character variety $\mathfrak{X}(S_g)$ of a closed surface is given by the relative Euler class [3]. The conjecture posed by Kashaev is the analog in the punctured case: the connected components will be given by the signs of the punctures associated to each representation, along with the Euler class.

3.3. **Decorated Character Varieties and their Coordinates.** We will now discuss a method of Kashaev for putting coordinates on the character variety of a punctured surface. First, we define the decorated character variety of a punctured surface. Next, we define coordinates over the decorated character variety. These coordinates will then descend to the character variety of a punctured surface.

Definition 3.11 (Decorated Character Variety)**.** Let $S_{g,n}$ be a hyperbolizable and oriented surface with puncture. Let $[\rho] \in \mathfrak{X}(S_{g,n})$. Let v_0 be a puncture of $S_{g,n}$ and let γ be a peripheral curve in $\pi_1(S_{g,n})$ freely homotopic to a circle about v_0. Then γ is mapped by ρ to a parabolic element by the type-preserving property. This parabolic element has one fixed point on $\partial\mathbb{H}^2$. We choose a $\rho(\pi_1(S_g))$ invariant horocycle about the fixed point. A decoration d for the map ρ is an assignment of such a horocycle to each puncture. A pair (ρ_1, d_1) and (ρ_2, d_2) are identified if $\rho_1 = A\rho_2 A^{-1}$ and $d_1 = A\dot{d}_2$ for some $A \in \mathrm{PSL}(2, \mathbb{R})$. The quotient space of these decorated type-preserving representations $[(\rho, d)]$ under the action above is called the decorated character variety $\mathfrak{X}^d(S_{g,n})$.

Definition 3.12 (Ideal Arc)**.** Let $S_{g,n}$ be a hyperbolizable and oriented surface with puncture. An *ideal arc* is an arc which begins and ends at punctures of $S_{g,n}$. An ideal arc γ is called ρ-admissable if the endpoints of $D_\rho(\widetilde{\gamma})$ are distinct, where $\widetilde{\gamma}$ is any any lift of γ to the universal cover $\widetilde{S}$.

Definition 3.13 (Admissable Triangulations of $S_{g,n}$)**.** An ideal triangulation of S is a disjoint union of ideal arcs such that the complement of these arcs is a disjoint union of ideal triangles. Such a triangluation is called ρ-admissable if each of the ideal arcs defining the triangulation is ρ-admissable.

Theorem 3.14. *A theorem of Kashaev [6] is that the set*

$$\mathfrak{X}_\Delta(S_{g,n}) = \{\, [p] \in \mathfrak{X}(S) \mid \Delta \textit{ is } p\textit{-admissable} \,\}$$

is open and dense in $\mathfrak{X}(S_{g,n})$. Furthermore, there exists finitely many ideal triangulations $\Delta_1, ..., \Delta_n$ of $S_{g,n}$ such that the sets $\mathfrak{X}_{\Delta_i}(S_{g,n})$ cover the character variety $\mathfrak{X}(S)$. [16]

Fix an ideal triangulation Δ on a hyperbolizable and oriented surface $S_{g,n}$. Kashaev uses the notion of ideal triangulations in order to put coordinates on the set

$$\mathfrak{X}^d_\Delta(S_{g,n}) \colon = \{\, [(\rho, d)] \in \mathfrak{X}^d(S_{g,n}) \mid \Delta \text{ is } \rho - \text{admissible} \,\}.$$

Definition 3.15 (Kashaev Coordinates [6])**.** Let $S_{g,n}$ be a hyperbolizable, oriented surface with puncture. Let $[(\rho, d)] \in \mathfrak{X}^d(S_{g,n})$ and let Δ be a ρ-admissable ideal triangulation of S. Let E be the edge set of Δ and T be the ideal triangle set of Δ. Let D_ρ be a pseudo-developing map.

λ-lengths: For each edge $e \in E$, we choose any lift $\widetilde{e}$ which maps to $\mathbb{H}^2$ via the pseduo-developing map D_ρ. As e is a ρ-admissible ideal arc, $D_\rho(\widetilde{e})$ has distinct endpoints in $\partial\mathbb{H}^2$ which in turn have associated horocycles H_1 and H_2 given by the decoration d. We define $l(e)$ to be the signed distance between H_1 and H_2. In particular, $l(e) > 0$ when H_1 and H_2 are disjoint and $l(e) \leq 0$ otherwise. Then we define the λ-length of e as

$$\begin{aligned} \lambda \colon E &\to \mathbb{R}^+ \\ e &\mapsto \exp\left(\frac{l(e)}{2}\right). \end{aligned}$$

Note that value of $\lambda(e)$ does not depend on the choice of lift or the choice of D_ρ.

The sign of a triangle: Similarly, for any $t \in T$, let (v_1, v_2, v_3) be the three vertices of t, ordered in such a way that agrees with the positive orientation of t induced by the orientation of S. Let $\widetilde{t}$ be a lift of t to the universal cover $\widetilde{S}$ and $\widetilde{v_i}$ be the corresponding lifts of the vertices. Then D_ρ sends the $\widetilde{v_i}$ to three distinct points on $\partial\mathbb{H}^2$ which define an ideal triangle in $\mathbb{H}^2$. Note that on the Riemann sphere $\widehat{\mathbb{C}}$, the set $\partial\mathbb{H}^2$ is a circle which we can give an orientation. Now consider the tuple $(D_\rho(\widetilde{v_1}), D_p(\widetilde{v_2}), D_p(\widetilde{v_3}))$. If these three points are ordered in such a way that agrees with the positive orientation of $\partial\mathbb{H}^2$, then we say that the sign of t is positive and write $\epsilon(t) = 1$. On the other hand, if $(D_p(\widetilde{v_1}), D_p(\widetilde{v_2}), D_p(\widetilde{v_3}))$ do not agree with the positive orientation of $\partial\mathbb{H}^2$, then we say that the sign of t is negative and write $\epsilon(t) = -1$. We call

$$\begin{aligned} \epsilon \colon T &\to \{+1, -1\} \\ t &\mapsto \epsilon(t) \end{aligned}$$

the sign of the triangles given by ρ.

All together, the pair

$$(\lambda(e_1), ..., \lambda(e_E), \epsilon(t_1), ..., \epsilon(t_T)) \in \mathbb{R}^E \times \{\pm 1\}^T$$

are the Kashaev coordinates for the pair $[(\rho, d)] \in \mathfrak{X}^d(S_{g,n})$ with respect to the triangulation Δ.

Remark 3.16 (Euler Class and Signs of Triangles)**.** The relative Euler class can be computed using the signs of the triangles since

$$e(\rho) = \frac{1}{2\pi}\int_{S_{g,n}} D_\rho^*\omega = \frac{1}{2\pi}\sum_{t\in T}\epsilon(t)\pi = \frac{1}{2}\sum_{t\in T}\epsilon(t).$$

using the fact that the area of an ideal triangle is π (2.14) and that the function $\epsilon(t)$ tells us exactly when the orientation of the triangle is reversed when pulling back from $\mathbb{H}^2$.

Let $[(\rho, d)] \in \mathfrak{X}^d(S_{g,n})$ and suppose Δ_1 and Δ_2 are both ρ-admissible triangles on $S_{g,n}$. Both Δ_1 and Δ_2 endow the pair $[(\rho, d)]$ with Kashaev coordinates. In the following sections, we characterize how the coordinates for the pair $[(\rho, d)]$ may change when considered under different triangulations.

Definition 3.17 (Diagonal Switch)**.** Let $S_{g,n}$ be a hyperbolizable, oriented surface with puncture and let Δ be an ideal triangulation. Let E be the edge set of Δ and let T be the triangle set of Δ. Let $t_1, t_2 \in T$ be adjacent triangles sharing the edge e. The other four edges of t_1 and t_2 define a quadrilateral q such that e is an edge between two diagonal (i.e. non-adjacent) vertices of q. Construct a new diagonal edge e' between the other pair of diagonal (i.e. non-adjacent) vertices of q. A diagonal switch at edge e is the operation of replacing $e \in E$ with the new edge e'. The result of a diagonal switch on Δ is a new triangulation which differs only on two triangles and one edge.

Kashaev computed the result of a diagonal switch on the Kashaev coordinates as follows:

Theorem 3.18 (Diagonal Switches and Kashaev Coordinates)**.** *Let t_1 and t_2 be adjacent triangles as above. Let e be the shared edge. Let t'_1 and t'_2 be the new adjacent triangles after the diagonal switch and let e' be the new edge shared by t'_1 and t'_2 after the diagonal switch. Let $e_{i,j}$ be the edge shared between t_i and t'_j for $i, j \in \{1, 2\}$.*

If $\epsilon(t_1) = \epsilon(t_2)$ then,

$$\epsilon(t'_1) = \epsilon(t'_2) = \epsilon(t_1) = \epsilon(t_2) \text{ and } \lambda(e') = \frac{\lambda(e_{1,1})\lambda(e_{2,2}) + \lambda(e_{1,2})\lambda(e_{2,1})}{\lambda(e)}.$$

If $\epsilon(t_1) \neq \epsilon(t_2)$, then $\lambda(e_{1,1})\lambda(e_{2,2}) - \lambda(e_{1,2})\lambda(e_{2,1}) \neq 0$. Therefore we separate into two further subcases.

- *If $\epsilon(t_1) \neq \epsilon(t_2)$ and $\lambda(e_{1,1})\lambda(e_{2,2}) - \lambda(e_{1,2})\lambda(e_{2,1}) < 0$,*

$$\epsilon(t'_1) = \epsilon(t_1),\ \epsilon(t'_2) = \epsilon(t_2), \text{ and } \lambda(e') = \frac{\lambda(e_{1,2})\lambda(e_{2,1}) - \lambda(e_{1,1})\lambda(e_{2,2})}{\lambda(e)}.$$

- *In the case that $\lambda(e_{1,1})\lambda(e_{2,2}) - \lambda(e_{1,2})\lambda(e_{2,1}) > 0$,*

$$\epsilon(t'_1) = \epsilon(t_2),\ \epsilon(t'_2) = \epsilon(t_1), \text{ and } \lambda(e') = \frac{\lambda(e_{1,1})\lambda(e_{2,2}) - \lambda(e_{1,2})\lambda(e_{2,1})}{\lambda(e)}.$$

In fact, any two ideal triangulations of S are separated by finitely many diagonal switches. Therefore, if Δ_1 and Δ_2 are different ρ-admissible triangulations, then the above result allows us to relate the Kashaev coordinates of the decorated representation $([\rho], \mathrm{d})$ defined using different triangulations [16].

3.3.1. *Image of Simple Closed Curves.* Let $S_{g,n}$ be a hyperbolizable, oriented surface with puncture. In the following section, we explore a result of Sun and Yang on the image of simple closed curves under a representation $\rho \in \mathfrak{X}(S_{g,n})$.

Fix an ideal triangulation Δ on $S_{g,n}$. Let γ be an immersed closed curve on S. Homotopy the curve γ so that, in any triangle t, γ is a simple arc joining two different sides of t. Fix a triangle t which γ intersects. Let e_1 be the edge at which γ enters t. Let e_2 be the edge at which γ leaves t and let e_3 be the other edge.

If γ makes a left-hand turn at t, define

$$M(t) = \begin{pmatrix} \lambda(e_1) & \epsilon(t)\lambda(e_3) \\ 0 & \lambda(e_2) \end{pmatrix}.$$

If γ makes a right-hand turn at t, define

$$M(t) = \begin{pmatrix} \lambda(e_2) & 0 \\ \epsilon(t)\lambda(e_3) & \lambda(e_1) \end{pmatrix}.$$

Theorem 3.19 (Trace of Immersed Closed Curve, Sun and Yang [16])**.** *If γ is a simple closed curve that intersects edges $e_1, ..., e_m$ and respectively triangles $t_1, ..., t_m$ where each e_i is the shared edge between adjacent triangles t_i and t_{i-1} where i is taken modulo m, we then have the following formula for $\rho(\lceil\gamma\rceil)$ up to conjugation in $PSL(2,\mathbb{R})$:*

$$\rho(\lceil\gamma\rceil) \cong \frac{M(t_1)...M(t_m)}{\lambda(e_1)...\lambda(e_m)}.$$

Sun and Yang originally used this result in order to quantify the trace of the ρ-image of simple closed curves in $\pi_1(S_{g,n})$, but it will be useful to us to consider the entire ρ-image of simple closed curves in the following sections.

4. Type-Preserving Representations of $S_{0,4}$

In this section, we answer the Kashaev Conjecture in the specific case of the four-punctured sphere $S_{0,4}$ following Yang [16]. First, we will define a specific class of ideal triangulations on $S_{0,4}$ called tetrahedral triangulations. We will use these triangulations to find the Kashaev coordinates of $\mathfrak{X}^d_\Delta(S_{0,4})$. These coordinates on the decorated character variety will define coordinates for $\mathfrak{X}_\Delta(S_{0,4})$, and we will also analyze the change of coordinates between $\mathfrak{X}_\Delta(S_{0,4})$ and $\mathfrak{X}_{\Delta'}(S_{0,4})$ for different tetrahedral triangulations Δ and Δ'. We will then analyze the connected components of $\mathfrak{X}_\Delta(S_{0,4})$. Ultimately, these combinatorics will reveal the connected components of $\mathfrak{X}(S_{0,4})$. We follow the argument of Tian Yang [16] and present the argument in the style of a related paper by Maloni, Palesi, and Yang [9] on the type-preserving representations of the thrice-punctured projective plane.

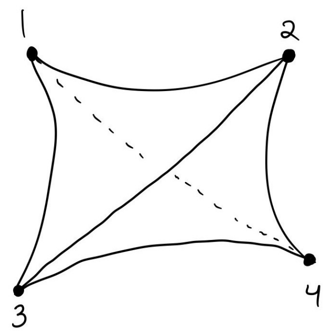

FIGURE 2. An image of a tetrahedral triangulation.

4.1. **Tetrahedral Triangulations.**

Definition 4.1 (Tetrahedral Triangulations)**.** A tetrahedral triangulation on the four-punctured sphere $S_{0,4}$ is an ideal triangulation with edge and vertex set identical to the edge and vertex set of the 1-skeleton of a tetrahedron. In particular, associate the four punctures of $S_{0,4}$ with the four vertices of a tetrahedron. Then ideal arcs on $S_{0,4}$ correspond to the edges of the tetrahedron.

Definition 4.2 (Opposite Pairs)**.** Let Δ be a tetrahedral triangulation on $S_{0,4}$. Let E be the triangle set of Δ and let T be the triangle set of Δ. Let $v_1, ..., v_4$ be the four punctures of $S_{0,4}$ and let $t_i \in T$ be the triangle disjoint from vertex v_i. Let $e_{i,j} \in E$ be the edge between v_i and $v_j, 1 \leq i, j \leq 4$. Edges incident to disjoint sets of vertices are called *opposite pairs*. For example, $e_{1,2}$ and $e_{3,4}$ are opposite pairs.

A non-peripheral simple closed curve is said to be *distinguished* if it does not intersect a pair of opposite edges, but intersects every other edge of the tetrahedral triangulation exactly once. It is a quick check that every non-peripheral simple closed curve is distinguished in some tetrahedral triangulation.

To see this, note that there are three homotopy classes of distinguished curves on $S_{0,4}$ given by the natural choice of curve parallel to an opposite pair. We define these classes of curves as γ_x, γ_y, and γ_z. These three distinct classes of distinguished curves intersect each other in two points. On the other hand, given any three non-peripheral simple closed curves which intersect each other at 2 points, we can define a teterahedral triangulation where ideal arcs are given by opposite pairs constructed to be parallel to the three given curves. Therefore tetrahedral triangulations are in bijective correspondence with triples of non-peripheral simple closed curves that mutually intersect in two points.

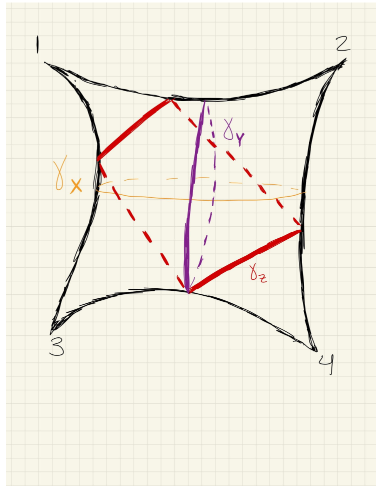

FIGURE 3. The three distinguished curves of a tetrahedral triangulation.

Definition 4.3 (Farey Complex and Farey Tree)**.** The Farey complex is a graph with vertex set $\mathbb{Q} \cup \infty$. The vertices are typically arranged in a circle. An edge is drawn between $\frac{p}{q}$ and $\frac{p'}{q'}$ if $pq' - p'q = 1$. Note that the Farey complex defines a triangle between the vertices $\frac{p}{q}$, $\frac{p'}{q'}$, and $\frac{p+p'}{q+q'}$.

The Farey tree is constructed from the Farey complex by placing a vertex at the center of each triangle defined by the Farey complex. Edges are drawn between vertices corresponding to adjacent triangles in the Farey complex.

Definition 4.4 (Simultaneous Diagonal Switch)**.** A simultaneous diagonal switch is a move between two different tetrahedral triangulations of $S_{0,4}$ by performing a diagonal switch at both edges in an opposite pair. There are three possible simultaneous diagonal switches which will be denoted as S_x, S_y, and S_z corresponding to the three different opposite pairs of curves. The effect of this operation is to change the homotopy class of the distinguished curve corresponding to the opposite pair that is switched. These diagonal switches correspond to the curves γ_x, γ_y, γ_z as pictured. We note that S_x performs two diagonal switches at the edges $e_{3,4}$ and $e_{1,2}$, S_y performs two diagonal switches at the edges $e_{2,4}$ and $e_{1,3}$, and S_z performs two diagonal switches at the edges $e_{2,3}$ and $e_{1,4}$.

Theorem 4.5. *There is a correspondence between tetrahedral triangulations and the Farey tree*

Proof. The Farey complex corresponds to the curve complex of $S_{0,4}$. Their identification depends on a choice of a pair of curves intersecting minimally, as well as which curve is identified with $\frac{1}{0}$ or $\frac{0}{1}$, see Farb-Margalit [12] for more details. In particular, the vertices of the Farey complex are in bijective correspondence with the non-peripheral simple closed curves of $S_{0,4}$. We connect two non-peripheral simple closed curves by an edge if they intersect minimally (in two points). Triangles in the Farey complex correspond to three non-peripheral simple closed curves which mutually intersect each other in two points. To see this, note that the simple closed curves associated to the vertices of the triangle must intersect each other minimally as they are each connected by an edge in the Farey complex. Therefore, the vertices of the Farey tree correspond to a triple of non-peripheral simple closed curves that intersect each other minimally. This in turn corresponds to a specific tetrahedral triangulation as described above. The edges of the Farey tree represent a simultaneous diagonal switch between two different tetrahedral triangulations. To see this, notice that adjacent triangles in the Farey diagram share two vertices. Therefore, an edge represents changing one of the three non-peripheral simple closed curves which define the tetrahedral triangulation. □

We now turn our attention toward parameterizing $\mathfrak{X}(S_{0,4})$. Let Δ be a tetrahedral triangulation of $S_{0,4}$ and consider the set

$$\mathfrak{X}_\Delta(S_{0,4}) = \{\, [\rho] \in \mathfrak{X}(S_{0,4}) \mid \Delta \text{ is } \rho\text{-admissable}\,\}.$$

We will parameterize $\mathfrak{X}_\Delta(S_{0,4})$ using products of certain λ-lengths from the Kashaev coordinates. Specifically, if $[\rho] \in \mathfrak{X}_\Delta(S)$, then $[(\rho, d)]$ is associated to a particular $\lambda\colon E \to \mathbb{R}^+$ and $\epsilon\colon T \to \{+1, -1\}^4$. Denote the set of λ-lengths for $[(\rho, d)]$ as $\mathbb{R}^E_{>0}$. Furthermore, $[(\rho, d)]$ is also associated with a set of horocycles for each vertex of Δ given by the decoration d. In particular, these horocylces have a defined radius, and so for each vertex we assign a real number according to the radius of the horocyle. Denote this set as $\mathbb{R}^V_{>0}$.

We define an action of $\mathbb{R}^V_{>0}$ on $\mathbb{R}^E_{>0}$. Let $u \in \mathbb{R}^V_{>0}$ and $\lambda \in \mathbb{R}^E_{>0}$. Define

$$(u \cdot \lambda)(e_{i,j}) = u(v_i) \cdot \lambda(e_{i,j}) \cdot u(v_j),$$

which is just scalar multiplication of each λ-length and the radii of the horocycles corresponding to the vertices of the edges.

We will show that the quotient of $\mathbb{R}^E_{>0}$ by this action is diffeomorphic to a more concrete space. Namely, the set of triples of positive real numbers $\mathbb{R}^3_{>0}$ under the action of $\mathbb{R}_{>0}$ on $\mathbb{R}^3_{>0}$ given by

$$r \cdot (a, b, c) = (ra, rb, rc).$$

Theorem 4.6 (Tetrahedral Coordinates for $\mathfrak{X}_\Delta(S_{0,4})$)**.** *There is a map $\phi\colon \mathbb{R}^E_{>0} \to \mathbb{R}^3_{>0}$ which induces a diffeomorphism $\phi_*\colon \mathbb{R}^E_{>0}/\mathbb{R}^V_{>0} \to \mathbb{R}^3_{>0}/\mathbb{R}_{>0}$.*

Proof. For each $\lambda \in \mathbb{R}^E_{>0}$ define $X_1 = \lambda(e_{1,2})\lambda(e_{3,4})$, $X_2 = \lambda(e_{1,3})\lambda(e_{2,4})$, and $X_3 = \lambda(e_{1,4})\lambda(e_{2,3})$. We call X_1, X_2, X_3 the tetrahedral coordinates of $[p] \in \mathfrak{X}_\Delta(S)$. Define the map

$$\begin{gathered}\phi\colon \mathbb{R}^E_{>0} \to \mathbb{R}^3_{>0} \\ (\lambda(e_{1,2}), ..., \lambda(e_{3,4})) \mapsto (X_1, X_2, X_3).\end{gathered}$$

This induces the map

$$\begin{aligned}\phi_*\colon \mathbb{R}^E_{>0}/\mathbb{R}^V_{>0} &\to \mathbb{R}^3_{>0}/\mathbb{R}_{>0}\\ [\lambda] &\mapsto [(X_1, X_2, X_3)].\end{aligned}$$

First, we check that ϕ_* is well-defined. We have the following:

$$\begin{aligned}\phi(u \cdot \lambda) &= \phi(u(v_1)\lambda(e_{1,2})u(v_2), ..., u(v_3)\lambda(e_{3,4})u(v_4))\\ &= (\prod_{i=1}^{4} u(v_i)X_1, \prod_{i=1}^{4} u(v_i)X_2, \prod_{i=1}^{4} u(v_i)X_3)\\ &= \prod_{i=1}^{4} u(v_i)(X_1, X_2, X_3).\end{aligned}$$

Therefore the induced map ϕ_* is well-defined.

Next, we check surjectivity. Let $(a, b, c) \in \mathbb{R}^3$. Then

$$\phi(\sqrt{a}, \sqrt{b}, \sqrt{c}, \sqrt{c}, \sqrt{b}, \sqrt{a}) = (a, b, c),$$

so ϕ is surjective.

Finally, we check injectivity. Suppose $\phi(\lambda') = r \cdot \phi(\lambda)$ for some $\lambda, \lambda' \in \mathbb{R}^E_{>0}, r \in \mathbb{R}_{>0}$. We want to show that $\lambda' = u \cdot \lambda$. For each puncture define the quantity

$$v_i(\lambda) = \left(\prod_{j\neq i} \lambda(e_{i,j})^2\right) \left(\prod_{k,k'\neq i} \lambda(e_{k,k'})\right).$$

Then consider decorations of the punctures given by $u(v_i) = r^{-4}\ \dfrac{v_i(\lambda')^{\frac{1}{2}}}{v_i(\lambda)^{\frac{1}{2}}}$.

We have that

$$\begin{aligned}u(v_i) \cdot \lambda(e_{i,j}) \cdot u(v_j) &= r^{-4}\ \frac{v_i(\lambda')^{\frac{1}{2}} * v_j(\lambda')^{\frac{1}{2}}}{v_i(\lambda)^{\frac{1}{2}} * v_j(\lambda)^{\frac{1}{2}}}\lambda(e_{i,j})\\ &= r^{-4}\ \frac{(\lambda'(e_{i,j})^2 * r^8 X_1^2 X_2^2 X_3^2 X_2 X_3)^{\frac{1}{2}}}{(\lambda(e_{i,j})^2 * X_1^2 X_2^2 X_3^2 X_2 X_3)^{\frac{1}{2}}}\ \lambda(e_{i,j})\\ &= \lambda'(e_{i,j}).\end{aligned}$$

The second equality comes from the fact that the triple

$$(\lambda'(e_{1,2})\lambda'(e_{3,4}), \lambda'(e_{1,3})\lambda'(e_{2,4}), \lambda'(e_{1,4})\lambda'(e_{2,3})) = (rX_1, rX_2, rX_3)$$

as $\phi(\lambda') = r \cdot \phi(\lambda)$, and re-writing the products $v_i(\lambda')v_j(\lambda')$ and $v_i(\lambda)v_j(\lambda)$ in terms of X_1, X_2, X_3 when possible in order to simplify the expression.

By construction, ϕ_* and the inverse ϕ_*^{-1} is differentiable and so ϕ_* is a diffeomorphism. □

We can identify the set $\mathbb{R}^3_{>0}/\mathbb{R}_{>0}$ with the simplex

$$T = \{\, (x_1, x_2, x_3) \in \mathbb{R}^3_{>0} \mid x_1 + x_2 + x_3 = 1 \,\}$$

To see this identification, note that we can consider $\mathbb{R}^3_{>0}/\mathbb{R}_{>0}$ as the set of all open rays in $\mathbb{R}^3_{>0}$ beginning from the origin. Each ray corresponds to a point on the simplex T given by the intersection of the ray with T. For the rest of this paper, we will consider X_1, X_2, and X_3 as points on the simplex T.

4.2. **Kashaev Conjecture of $S_{0,4}$.** We now have the necessary tools to answer the Kashaev Conjecture in the case of $S_{0,4}$. In this section, we apply the above results to calculate the connected components of the type-preserving character variety $\mathfrak{X}(S_{0,4})$ as discovered by Tian Yang [16].

Theorem 4.7 (Peripheral Curves in Tetrahedral Triangulation)**.** *There are four free homotopy classes of peripheral curves in $S_{0,4}$. Denote these curves $\gamma_1, ..., \gamma_4$ such that γ_i is freely homotopic to a circle about the puncture v_i. Let ρ be a type-preserving representation. Then up to conjugation by elements in $PSL(2,\mathbb{R})$, the image of each γ_i under ρ are*

(1) $\rho([\gamma_1]) \cong \pm \begin{pmatrix} 1 & (\epsilon(t_2)X_1 + \epsilon(t_3)X_2 + \epsilon(t_4)X_3) \\ 0 & 1 \end{pmatrix}$,

(2) $\rho([\gamma_2]) \cong \pm \begin{pmatrix} 1 & (\epsilon(t_1)X_1 + \epsilon(t_3)X_3 + \epsilon(t_4)X_2) \\ 0 & 1 \end{pmatrix}$,

(3) $\rho([\gamma_3]) \cong \pm \begin{pmatrix} 1 & (\epsilon(t_1)X_3 + \epsilon(t_2)X_2 + \epsilon(t_3)X_1) \\ 0 & 1 \end{pmatrix}$,

(4) $\rho([\gamma_4]) \cong \pm \begin{pmatrix} 1 & (\epsilon(t_1)X_2 + \epsilon(t_2)X_3 + \epsilon(t_4)X_1) \\ 0 & 1 \end{pmatrix}$.

Proof. Let Δ be a ρ-admissible tetrahedral triangulation. By Theorem 3.19 from Sun and Yang, we can calculate the conjugacy class of the image of the free homotopy classes of certain curves by computing matrix multiplications corresponding to the left-hand and right-hand turns which the curve makes with respect to Δ.

If we homotope the peripheral curves γ_i tight enough, γ_i intersects three different triangles for a given tetrahedral triangulation. This is clear because three of the ideal triangles in a given tetrahedral triangulation have a vertex at v_i while the ideal triangle t_i does not have a vertex at v_i, for each $1 \leq i \leq 4$.

We calculate $[\rho(\gamma_1)], ..., [\rho(\gamma_4)]$ using the Sun and Yang trace formula. In particular, this formula relates $[\rho(\gamma_1)], ..., [\rho(\gamma_4)]$ with the λ-lengths and signs of triangles associated to ρ.

First, we need to check the triangles that each peripheral curve intersects and whether the peripheral curve makes a right or left turn. Without loss of generality, we can choose representatives of $\lceil\gamma_1\rceil, ..., \lceil\gamma_4\rceil$ which always make left-hand turns through each ideal triangle and we can choose the ideal triangle at which the representative loop should start. We do the calculation for $\lceil\gamma_1\rceil$. The other calculations are symmetric.

Set γ_1 to the peripheral curve which makes left-hand turns through ideal triangles t_4 then t_3 then t_2.

We compute

$$\begin{aligned}
\rho([\gamma_1]) &\cong \pm \begin{pmatrix} \lambda(e_{2,4}) & \epsilon(t_4)\lambda(e_{1,4}) \\ 0 & \lambda(e_{3,4}) \end{pmatrix} * \begin{pmatrix} \lambda(e_{3,4}) & \epsilon(t_3)\lambda(e_{1,3}) \\ 0 & \lambda(e_{2,3}) \end{pmatrix} * \begin{pmatrix} \lambda(e_{2,3}) & \epsilon(t_2)\lambda(e_{1,2}) \\ 0 & \lambda(e_{2,4}) \end{pmatrix} \\
&\cong \pm \begin{pmatrix} 1 & \frac{1}{\lambda(e_{4,3})\lambda(e_{2,3})}(\epsilon(t_2)\lambda(e_{1,2})\lambda(e_{3,4}) + \epsilon(t_3)\lambda(e_{1,3})\lambda(e_{2,4}) + \epsilon(t_4)\lambda(e_{1,4})\lambda(e_{2,3})) \\ 0 & 1 \end{pmatrix} \\
&\cong \pm \begin{pmatrix} 1 & \frac{1}{\lambda(e_{4,3})\lambda(e_{2,3})}(\epsilon(t_2)X_1 + \epsilon(t_3)X_2 + \epsilon(t_4)X_3) \\ 0 & 1 \end{pmatrix} \\
&\cong \pm \begin{pmatrix} 1 & (\epsilon(t_2)X_1 + \epsilon(t_3)X_2 + \epsilon(t_4)X_3) \\ 0 & 1 \end{pmatrix}.
\end{aligned}$$

The second equality uses the fact that scaling all entries of a matrix is equivalent to conjugation by some diagonal matrix and hence does not change the class of ρ in $\mathfrak{X}(S_{0,4})$. □

For a given $\rho \in \mathfrak{X}(S_{0,4})$ and each $1 \leq i \leq 4$, $\rho(v_i)$ is a parabolic transformation. In particular, if we are given the signs of each ideal triangle in Δ, then we obtain four different inequalities involving the tetrahedral coordinates X_1, X_2, and X_3.

First, define the sets

$$\mathfrak{X}_k(S_{0,4}) = \{\, \rho \in \mathfrak{X}(S_{0,4}) \mid e(\rho) = k \,\}$$

for $k = -1, 0, 1$. Furthermore, define

$$\mathfrak{X}_\Delta(S_{0,4}, \epsilon) = \{\, [\rho] \in \mathfrak{X}_\Delta(S_{0,4}) \mid \text{the signs of the triangles are given by } \epsilon \,\}$$

for $\epsilon \in \{+1, -1\}^4$.

We have shown that the tetrahedral coordinates X_1, X_2, and X_3 can be identified with a simplex T. Define the following subsets

$$\begin{aligned}
T_{i,j} &= \{\, (x_1, x_2, x_3) \in T \mid x_k \neq x_i + x_j, k \neq i, j \,\} \text{ for } \{i, j\} \subset \{1, 2, 3\}, \\
T_* &= \{\, (x_1, x_2, x_3) \in T \mid x_1 \neq x_2 + x_3, x_2 \neq x_1 + x_3, x_3 \neq x_1 + x_2 \,\}.
\end{aligned}$$

Corollary 4.8. *We have the following identifications*

(1) $\mathfrak{X}_0(S_{0,4}) \cap \mathfrak{X}_\Delta(S_{0,4}) \cong \bigsqcup_{\{i,j\}\subset\{1,2,3\}} T_{i,j} \bigsqcup_{\{i,j\}\subset\{1,2,3\}} T_{i,j}$,

(2) $\mathfrak{X}_1(S_{0,4}) \cap \mathfrak{X}_\Delta(S_{0,4}) \cong \bigsqcup_{i=1}^4 T_*$,

(3) $\mathfrak{X}_{-1}(S_{0,4}) \cap \mathfrak{X}_\Delta(S_{0,4}) \cong \bigsqcup_{i=1}^4 T_*$.

Proof. Fix a ρ-admissable tetrahedral triangulation Δ. We have that

$$\mathfrak{X}_\Delta(S_{0,4}) = \bigsqcup_{\epsilon\in\{\pm1\}^4} \mathfrak{X}_\Delta(S_{0,4}, \epsilon).$$

We now consider how this set looks under the intersection of $X_k(S_{0,4})$ for possible k.

(1) Case $e(\rho) = 0$:

By the relation $2e(\rho)=\sum_{i=1}^{4}\epsilon(t_i)$ (3.16), we must have in this case that two triangles have postive sign and two triangles have negative sign. Let $\epsilon_{i,j}\in\{+1,-1\}^4$ be the element with 1s at the i^{th} and j^{th} entries and -1s at the other entries. In particular we have that

$$\mathfrak{X}_0(S_{0,4})\cap\mathfrak{X}_\Delta(S_{0,4})=\bigsqcup_{\{i,j\}\subset\{1,2,3,4\}}\mathfrak{X}_\Delta(S_{0,4},\epsilon_{i,j}).$$

Using the ρ-images of the peripheral curves computed above, we obtain four different inequalities involving the signs of the triangles and the tetrahedral coordinates given by the top-right entry being non-zero:

$$\begin{aligned}
&v_1\colon \epsilon(t_2)X_1+\epsilon(t_3)X_2+\epsilon(t_4)X_3\neq 0,\\
&v_2\colon \epsilon(t_1)X_1+\epsilon(t_3)X_3+\epsilon(t_4)X_2\neq 0,\\
&v_3\colon \epsilon(t_1)X_3+\epsilon(t_2)X_2+\epsilon(t_3)X_1\neq 0,\\
&v_4\colon \epsilon(t_1)X_2+\epsilon(t_2)X_3+\epsilon(t_4)X_1)\neq 0.
\end{aligned}$$

We substitute the possible signs of each triangle to obtain

$$\begin{aligned}
\mathfrak{X}_\Delta(S_{0,4},\epsilon_{1,2})&\cong T_{2,3}\cong\mathfrak{X}_\Delta(S_{0,4},\epsilon_{3,4}),\\
\mathfrak{X}_\Delta(S_{0,4},\epsilon_{1,3})&\cong T_{1,3}\cong\mathfrak{X}_\Delta(S_{0,4},\epsilon_{2,4}),\\
\mathfrak{X}_\Delta(S_{0,4},\epsilon_{1,4})&\cong T_{1,2}\cong\mathfrak{X}_\Delta(S_{0,4},\epsilon_{2,3}).
\end{aligned}$$

(2) Case $e(\rho)=1$:

By the equality $2e(\rho)=\sum_{i=1}^{4}\epsilon(t_i)$, we must have in this case that three triangles have positive sign and one triangle has negative sign. Let $\epsilon_i\in\{-1,+1\}^4$ such that the i^{th} entry is negative and the other entries are positive. In particular, we have

$$\mathfrak{X}_1(S_{0,4})\cap\mathfrak{X}_\Delta(S_{0,4})=\bigsqcup_{i=1}^{4}\mathfrak{X}_\Delta(S_{0,4},\epsilon_i).$$

Using the ρ-images of the peripheral curves computed above, we obtain four different inequalities involving the signs of the triangles and the tetrahedral coordinates given by the top-right entry being non-zero:

$$\begin{aligned}
&v_1\colon \epsilon(t_2)X_1+\epsilon(t_3)X_2+\epsilon(t_4)X_3\neq 0,\\
&v_2\colon \epsilon(t_1)X_1+\epsilon(t_3)X_3+\epsilon(t_4)X_2\neq 0,\\
&v_3\colon \epsilon(t_1)X_3+\epsilon(t_2)X_2+\epsilon(t_3)X_1\neq 0,\\
&v_4\colon \epsilon(t_1)X_2+\epsilon(t_2)X_3+\epsilon(t_4)X_1)\neq 0.
\end{aligned}$$

For any $1\leq i\leq 4$, substituting the possible signs of the triangles into the inequalities above gives

$$\mathfrak{X}_\Delta(S_{0,4},\epsilon_i)\cong T_*$$

(3) Case $e(\rho)=-1$:

This result is similar to case (2), so we will not repeat the discussion. $\square$

Let $[\rho] \in \mathfrak{X}(S_{g,n})$. Recall that ρ assigns a sign to the puncture v_i given by the sign of the parabolic transformation $\rho(\gamma_i)$. We may use the formula for $\rho(\gamma_i)$ (Theorem 4.7) in order to compute the possible signs of the punctures.

Let $s_{i,j} \in \{+1,-1\}$ be the element with 1's in the ith and jth position and -1's elswhere. Let $s_i \in \{+1,-1\}$ be the element with -1 is in the ith position and 1's elsewhere. Let $s^+, s^- \in \{+1,-1\}^4$ be the elements with all positive entries and negative entries respectively.

Corollary 4.9. *Let ρ be a type-preserving representation and let $u_\rho(v_i)$ be the sign assigned to the puncture v_i by ρ. Define the set*

$$\mathfrak{X}_{k,s}(S_{0,4}) = \{\, \rho \in \mathfrak{X}_k(S_{0,4}) \mid (u_\rho(v_1), u_\rho(v_2), u_\rho(v_3), u_\rho(v_4)) = s \in \{+1,-1\}^4 \,\}$$

Then we have the following

(1) $\mathfrak{X}_0(S_{0,4}) = \bigsqcup_{\{i,j\}\subset\{1,2,3,4\}} \mathfrak{X}_{0,s_{i,j}}(S_{0,4})$

(2) $\mathfrak{X}_1(S_{0,4}) = \bigsqcup_{i=1}^4 \mathfrak{X}_{1,s_i}(S_{0,4}) \bigsqcup X_{1,s^+}(S_{0,4})$

(3) $\mathfrak{X}_{-1}(S_{0,4}) = \bigsqcup_{i=1}^4 \mathfrak{X}_{-1,-s_i}(S_{0,4}) \bigsqcup X_{-1,s^-}(S_{0,4})$

Proof. Let Δ be a tetrahedral triangulation. Because $\mathfrak{X}_{k,s}(S_{0,4}) \cap \mathfrak{X}_\Delta(S_{0,4})$ is open and dense in $\mathfrak{X}_{k,s}(S_{0,4})$, it follows that $\mathfrak{X}_{k,s}(S_{0,4})$ is nonempty exactly when $\mathfrak{X}_{k,s}(S_{0,4}) \cap \mathfrak{X}_\Delta(S_{0,4})$ is nonempty. In particular, we reduce to checking the signs of representations within the context of some fixed ρ-admissible triangulation Δ.

(1) Case $e(\rho) = 0$:

In this case, we must have two triangles with positive sign and two triangles with negative sign. We have the following four equations given by the images of the peripheral curves under ρ

$$\begin{aligned}
v_1 &: \epsilon(t_2)X_1 + \epsilon(t_3)X_2 + \epsilon(t_4)X_3 \neq 0,\\
v_2 &: \epsilon(t_1)X_1 + \epsilon(t_3)X_3 + \epsilon(t_4)X_2 \neq 0,\\
v_3 &: \epsilon(t_1)X_3 + \epsilon(t_2)X_2 + \epsilon(t_3)X_1 \neq 0,\\
v_4 &: \epsilon(t_1)X_2 + \epsilon(t_2)X_3 + \epsilon(t_4)X_1 \neq 0.
\end{aligned}$$

Note that exactly two of these equations (corresponding to some vertices v_{k_1}, v_{k_2}) contain both terms $\epsilon(t_i)$ and $\epsilon(t_j)$. It follows that v_{k_1}, v_{k_2} have the same sign. By the same argument, the other two vertices v_{k_3}, v_{k_4} must share the same sign, and the sign between v_{k_1} and v_{k_3} are opposite. Therefore, it follows that the signs of the punctures $(u(v_1), u(v_2), u(v_3), u(v_4))$ must be given by either s_{k_1,k_2} or s_{k_3,k_4}, for $\{k_1, k_2\} \subseteq \{1,2,3,4\}$.

(2) Case $e(\rho) = 1$:

In this case, we have that one triangle is negative while the other triangles are positive. Label this triangle t_i. Furthermore, we have the inequalities

$$X_1 \neq X_2 + X_3,\ X_2 \neq X_1 + X_3,\ X_3 \neq X_1 + X_2$$

We now have two cases to compute.

∘ For the first case, suppose that $X_{k_1} > X_{k_2} + X_{k_3}$ for some $\{k_1, k_2, k_3\} = \{1, 2, 3\}$. Then $X_{k_1} > X_{k_2}$ and $X_{k_1} > X_{k_3}$ as tetrahedral coordinates are positive numbers. By examining the ρ-image of the peripheral curves

$$\begin{aligned}
v_1 &: \epsilon(t_2)X_1 + \epsilon(t_3)X_2 + \epsilon(t_4)X_3 \neq 0,\\
v_2 &: \epsilon(t_1)X_1 + \epsilon(t_3)X_3 + \epsilon(t_4)X_2 \neq 0,\\
v_3 &: \epsilon(t_1)X_3 + \epsilon(t_2)X_2 + \epsilon(t_3)X_1 \neq 0,\\
v_4 &: \epsilon(t_1)X_2 + \epsilon(t_2)X_3 + \epsilon(t_4)X_1 \neq 0,
\end{aligned}$$

we note that exactly one vertex weights the largest tetrahedral coordinate X_{k_1} by a factor of $-1 = \epsilon(t_i)$. This vertex will be negative while the others will be positive. In particular, $(u(v_1), u(v_2), u(v_3), u(v_4)) = s_j$ for some $1 \leq j \leq 4$.

∘ In the second case, suppose

$$X_1 < X_2 + X_3, X_2 < X_1 + X_3, \text{ and } X_3 < X_1 + X_2.$$

Then $u(v_i)$ must be positive for each i since the negative term will always be less than the sum of the two positive terms. In particular, $(u(v_1), u(v_2), u(v_3), u(v_4)) = s^+$.

(3) Case $e(\rho) = -1$:
The result is similar to case (2). □

Finally, we will show that each of the above disjoint components are path-connected. In particular, the above description is a complete characterization of the connected components of $\mathfrak{X}(S_{0,4})$. However, before we carry out this proof, we first need to consider the effect of a simultaneous diagonal switch on a ρ-admissible tetrahedral triangulation. We will need to know whether the resultant tetrahedral triangulation of the simultaneous diagonal switch is also be ρ-admissible, and, if so, how the tetrahedral coordinates of $[\rho]$ will change when viewed in the context of the new triangulation. This result directly follows from the change of Kashaev Coordinates formula (Theorem 3.18).

Fix a ρ-admissible tetrahedral triangulation Δ and let Δ' be the tetrahedral triangulation obtained as a result of performing a simultaneous diagonal switch on Δ. If Δ' is also ρ-admissible, let λ' be the corresponding Kashaev lengths of ρ with respect to Δ' and let X_1', X_2', X_3' be the resulting tetrahedral coordinates corresponding to ρ in $\mathfrak{X}_{\Delta'}(S_{0,4})$.

Lemma 4.10 (Simultaneous Diagonal Switch and Tetrahedral Coordinates if $\mathrm{e}(\rho) = 0$)**.** *We have the following formulae in the case that $e(\rho) = 0$ (suppose without loss of generality that the signs of the triangles are given by $\epsilon_{1,2}$):*

(1) If Δ' is a result of S_x on Δ, then Δ' is ρ-admissible $\iff X_2 \neq X_3$ and the tetrahedral coordinates are given by

$$(X_1, X_2, X_3) = \left(\frac{|X_2 + X_3|^2}{X_1}, X_2, X_3\right).$$

(2) If Δ' is a result of S_y on Δ, then Δ' is ρ-admissible $\iff X_1 \neq X_3$ and the tetrahedral coordinates are given by

$$(X_1, X_2, X_3) = \left(X_1, \frac{|X_1 - X_3|^2}{X_2}, X_3 \right).$$

(3) If Δ' is a result of S_z on Δ, then Δ' is ρ-admissible $\iff X_1 \neq X_2$ and the tetrahedral coordinates are given by

$$(X_1, X_2, X_3) = \left(X_1, X_2, \frac{|X_1 - X_2|^2}{X_3} \right).$$

Lemma 4.11 (Simultaneous Diagonal Switch and Tetrahedral Coordinates if $e(\rho) = 1$)**.** *We have the following formulae in the case that $e(\rho) = 1$ (suppose without loss of generality the signs of the triangles are given by ϵ_1):*

(1) If Δ' is a result of S_x on Δ, then Δ' is ρ-admissible $\iff X_2 \neq X_3$ and the tetrahedral coordinates are given by

$$(X_1, X_2, X_3) = \left(\frac{|X_2^2 - X_3^2|}{X_1}, X_2, X_3 \right).$$

(2) If Δ' is a result of S_y on Δ, then Δ' is ρ-admissible $\iff X_1 \neq X_3$ and the tetrahedral coordinates are given by

$$(X_1, X_2, X_3) = \left(X_1, \frac{|X_1^2 - X_3^2|}{X_2}, X_3 \right).$$

(3) If Δ' is a result of S_z on Δ, then Δ' is ρ-admissible $\iff X_1 \neq X_2$ and the tetrahedral coordinates are given by

$$(X_1, X_2, X_3) = \left(X_1, X_2, \frac{|X_1^2 - X_2^2|}{X_3} \right).$$

The case that $e(\rho) = -1$ is similar to the case $e(\rho) = 1$.

Proof. (1) Case $e(\rho) = 0$:

We have two further cases to check.

∘ Subcase 1: First, suppose Δ' is obtained by the operation S_x on Δ.

In this case, two triangles have positive sign and two triangles have negative sign. Suppose without loss of generality that triangles 1 and 2 have the same sign. Then S_x performs a simultaneous diagonal switch at $e_{1,2}$ and $e_{3,4}$. By the diagonal switch formula for Kashaev Coordinates (3.18), it follows that Δ' is ρ-admissibile $\iff X_2 \neq X_3$. The new λ' lengths are the same as the old ones, besides two updates on the opposite pair associated to S_x. This means that only the coordinate X_1' is changed following the operation S_x. We compute

$$\lambda'(e_{1,2}) = \frac{X_2 + X_3}{\lambda(e_{1,2})}$$

and

$$\lambda'(e_{3,4}) = \frac{X_2 + X_3}{\lambda(e_{3,4})}.$$

Multiplying the new λ-lengths gives the desired result.

$\circ$ Subcase 2: On the other hand, the cases of S_y and S_z are symmetric to each other, but different from the case calculated above. This is because the two diagonal switches performed above are between triangles with the same sign, while the diagonal switches in the case of S_y and S_z are carried out between triangles with different signs.

If Δ' is obtained by the operation S_y on Δ, we have that Δ' is ρ-admissible $\iff X_1 \neq X_3$. Only the coordinate X_2' is changed following the operation S_y and we compute

$$\lambda'(e_{13}) = \frac{|X_1 - X_3|}{\lambda(e_{13})}$$

and

$$\lambda'(e_{24}) = \frac{|X_1 - X_3|}{\lambda_{e_24}}$$

(2) Case $e(\rho) = \pm 1$: We show the case that Δ' is obtained by the operation S_x on Δ. The other subcases follow by symmetry.

In this case, one triangle has negative sign and the others have positive sign. Suppose triangle 1 has negative sign. Then S_x performs a simultaneous diagonal switch at $e_{1,2}$ and $e_{3,4}$. By the diagonal switch formula for Kashaev Coordinates (3.18), it follows that Δ' is ρ-admissibile $\iff X_2 \neq X_3$. The new λ' lengths are the same as the old ones, besides two updates on the opposite pair associated to S_x. This means that only the coordinate X_1' is changed following the operation S_x. We compute

$$\lambda'(e_{1,2}) = \frac{X_2 - X_3}{\lambda(e_{1,2})}$$

and

$$\lambda'(e_{3,4}) = \frac{X_2 + X_3}{\lambda(e_{3,4})}$$

Multiplying the new λ-lengths gives the desired result. $\square$

Now that we know the effect of diagonal switches on the Tetrahedral Coordinates of a representation $[\rho]$, we are ready to prove the Kashaev Conjecture in the case of $S_{0,4}$.

Theorem 4.12 (Connected Components of $\mathfrak{X}(S_{0,4})$)**.** *The disjoint components outlined in Corollary 4.9 are path-connected. In particular, the Kashaev Conjecture holds in the $S_{0,4}$ case.*

Proof. Fix a tetrahedral triangulation Δ. To show connectedness of $\mathfrak{X}_{k,s}(S_{0,4})$, it is enough to show that any $p, q \in \mathfrak{X}_{k,s}(S_{0,4}) \cap \mathfrak{X}_\Delta(S_{0,4})$ can be connected with a path since $\mathfrak{X}_\Delta(S_{0,4})$ is dense. We will show that $\mathfrak{X}_{0,s_{1,2}}(S_{0,4}), \mathfrak{X}_{0,s_1}(S_{0,4})$, and $\mathfrak{X}_{0,s^+}(S_{0,4})$ are all connected. The other possible $\mathfrak{X}_{k,s}(S_{0,4})$ are connected by similar, symmetric arguments.

(1) $\mathfrak{X}_{0,s_{1,2}}(S_{0,4})$ is connected:

Let Δ be a tetrahedral triangulation and let $p, q \in \mathfrak{X}_{0,s_{1,2}}(S_{0.4}) \cap \mathfrak{X}_\Delta(S_{0,4})$. Let (X_1, X_2, X_3) be the tetrahedral coordinates for p and let ϵ be the signs of the triangles given by p. Let (X_1', X_2', X_3') be the tetrahedral coordinates for q and let ϵ' be the signs of the triangles given by q. Since the signs of punctures given by p is $s_{1,2}$, we must have that either $\epsilon = \epsilon_{1,2}$ and $X_1 > X_2 + X_3$ or $\epsilon = \epsilon_{3,4}$ and $X_1 < X_2 + X_3$ (Corollary 4.9). This similarly holds for q. Partition $T_{2,3}$ into its two path-components

$$T_{2,3}^+ = \{ (x_1, x_2, x_3) \in T_{1,2} \mid x_1 > x_2 + x_3 \}$$
$$T_{2,3}^- = \{ (x_1, x_2, x_3) \in T_{i,j} \mid x_1 < x_2 + x_3 \}.$$

If the tetrahedral coordinates for $p, q \in \mathfrak{X}_\Delta(S_{0,4})$ are both in the same connected component (either $T_{2,3}^+$ or $T_{2,3}^-$), we can find a path between p and q by pulling back with the diffeomorphism ϕ_*.

Therefore, it is only left to show that representations which map to different path components of $T_{2,3}$ are joined by a path in $\mathfrak{X}_{0,s_{1,2}}(S_{0,4})$. To do this, we will find two elements that lie in different path components of $\mathfrak{X}_\Delta(S_{0,4})$ but lie in the same path component of $\mathfrak{X}_{\Delta'}(S_{0,4})$, for some other tetrahedral triangulation Δ'.

In particular, consider the representations $p = (a, b, c) \in T_{2,3}^+$ and $q = (a', b', c') \in T_{2,3}^-$ such that $a' > b'$ (at least some such points exists in $T_{2,3}^-$). We have that $a > b + c$ as $p \in T_{2,3}^+$ which means $a > b$ as c is positive. In particular, $a \neq b$ and $a' \neq b'$. Therefore by Lemma 4.10, the diagonal switch S_z will result in a triangulation Δ' which is both p-admissible and q-admissible. Furthermore, the new coordinates for p and q with respect to the set $\mathfrak{X}_{\Delta'}(S_{0,4})$ are given by $p = (a, b, \frac{|a-b|^2}{c})$ and $q = (a', b', \frac{|a'-b'|^2}{c'})$. Furthermore, $\epsilon = \epsilon' = \epsilon_{12}$ with respect to the triangulation Δ'. Note that we have $b + \frac{|a-b|^2}{c} < b + \frac{|a-b|^2}{|a-b|} = b + |a - b| = a$. On the other hand, we know that $0 < a' - b' < c'$, which means that $|a' - b'|^2 < c'(a' - b')$ and therefore $b' + \frac{|a'-b'|}{c'} < a'$. In fact, this means that with respect to Δ', both p and q are in $T_{2,3}^+$ and therefore in the same path-component of $\mathfrak{X}_{\Delta'}(S_{0,4})$. Therefore there must be a path between p and q in $\mathfrak{X}_{0,s_{1,2}}(S_{0,4})$.

(2) $\mathfrak{X}_{1,s_1}(S_{0,4})$ is connected:

Let Δ be a tetrahedral triangulation and let $p, q \in \mathfrak{X}_{0,s_{1,2}}(S_{0.4}) \cap \mathfrak{X}_\Delta(S_{0,4})$. Let (X_1, X_2, X_3) be the tetrahedral coordinates for p and let ϵ be the signs of the triangles given by p. Let (X_1', X_2', X_3') be the tetrahedral coordinates for q and let ϵ' be the signs of the triangles given by q. Since the signs of punctures given by p is s_1, we must have that either $\epsilon = \epsilon_2$ and $X_1 > X_2 + X_3$, $\epsilon = \epsilon_3$ and $X_2 > X_1 + X_3$, or $\epsilon = \epsilon_4$ and $X_3 > X_1 + X_2$ (Corollary 4.9). This similarly holds for q. Partition T_* into its path-components:

$$T_{2,3}^+ = \{ (x_1, x_2, x_3) \in T_* \mid x_1 > x_2 + x_3 \},$$
$$T_{1,3}^+ = \{ (x_1, x_2, x_3) \in T_* \mid x_2 > x_1 + x_3 \},$$
$$T_{1,2}^+ = \{ (x_1, x_2, x_3) \in T_* \mid x_3 > x_1 + x_2 \},$$
$$T_c = \{ (x_1, x_2, x_3) \in T_* \mid x_1 < x_2 + x_3, x_2 < x_1 + x_3, x_3 < x_1 + x_2 \}.$$

If the tetrahedral coordinates for $p, q \in \mathfrak{X}_\Delta(S_{0,4})$ are both in the same connected component (either $T^+_{2,3}$, $T^+_{1,3}$, or $T^+_{1,2}$), we can find a path between p and q by pulling back with the diffeomorphism ϕ_*.

Therefore, it is only left to show that representations which map to different path components of T_* are joined by a path in $\mathfrak{X}_{0,s_1}(S_{0,4})$. To do this, we will find two elements that lie in different path components of $\mathfrak{X}_\Delta(S_{0,4})$ but lie in the same path component of $\mathfrak{X}_{\Delta'}(S_{0,4})$, for some other tetrahedral triangulation Δ'.

Suppose $p = (a, b, c) \in T^+_{2,3}$ and $q = (a', b', c') \in T^+_{1,3}$. We have that $a \neq b$ and $a' \neq b'$, therefore by Lemma 4.10, the diagonal switch S_z will result in a triangulation Δ' which is both p-admissible and q-admissible. Furthermore, the new coordinates for p and q with respect to the set $\mathfrak{X}_{\Delta'}(S_{0,4})$ are given by $p = (a, b, \frac{|a^2-b^2|}{c})$ and $q = (a', b', \frac{|a'^2-b'^2|}{c'})$. We have that $a - b > c$ which means $\frac{|a^2-b^2|}{c} = \frac{|a+b||a-b|}{c} > a + b$. The signs of the triangles are given by $\epsilon = \epsilon_4$. The exact same goes for q. In particular, p and q are in the same connected component when considered under the tetrahedral coordinates of Δ' and therefore there is a path between p and q in $\mathfrak{X}_{1,s_1}(S_{0,4})$. Finding a path between elements in $T^+_{2,3}$ and $T^+_{1,2}$ is a symmetric argument.

(3) $\mathfrak{X}_{1,s^+}(S_{0,4})$ is connected

Let Δ be a tetrahedral triangulation and let $p, q \in \mathfrak{X}_{0,s^+}(S_{0.4}) \cap \mathfrak{X}_\Delta(S_{0,4})$. Let (X_1, X_2, X_3) be the tetrahedral coordinates for p and let ϵ be the signs of the triangles given by p. Let (X'_1, X'_2, X'_3) be the tetrahedral coordinates for q and let ϵ' be the signs of the triangles given by q. Since the signs of punctures given by p is s^+, we must have that $X_1 < X_2 + X_3$, $X_2 < X_1 + X_3$, and $X_3 < X_1 + X_2$ and $\epsilon = \epsilon_j$ for some $1 \leq j \leq 4$. This similarly holds for q.

Therefore, it is only left to show that representations which map to different path components (i.e. assign different signs to the ideal triangles of Δ) are joined by a path in $\mathfrak{X}_{0,s^+}(S_{0,4})$. To do this, we will find two elements that lie in different path components of $\mathfrak{X}_\Delta(S_{0,4})$ but lie in the same path component of $\mathfrak{X}_{\Delta'}(S_{0,4})$, for some other tetrahedral triangulation Δ'.

Choose $p = (a, b, c) \in T_c$ such that $a > b$ and $\epsilon = \epsilon_2$. Choose $q = (a', b', c') \in T_c$ such that $b' > a'$ and $\epsilon' = \epsilon_3$. Let Δ' be the tetrahedral triangulation given by the diagonal switch S_z applied to Δ. Note that $a \neq b$ and $a' \neq b'$, so Δ' is both p-admissible and q-admissible. With respect to the tetrahedral coordinates of Δ', $p = (a, b, \frac{|a^2-b^2|}{c})$ and $q = (a', b', \frac{|a'^2-b'^2|}{c'})$ by Lemma 4.10. We note that p and q have signs of triangles given by ϵ_4 with respect to the triangulation Δ'. As $p \in T^c$, we have the three inequalities

$$a < b + c$$
$$b < a + c$$
$$c < a + b$$

which we can use to see following three inequalities

$$\frac{|a^2-b^2|}{c} < a+b$$
$$a < b + \frac{|a^2-b^2|}{c}$$
$$b < a + \frac{a^2-b^2}{c}.$$

The first inequality comes from the relation $a-b<c$. The second inequality is a result of the relation $c<a+b$. Finally, the last inequality simply comes from the fact that we required $b<a$ when choosing p. We obtain very similar results for the representation q. In particular, p and q are in the same connected component when considered under the tetrahedral coordinates of Δ' and therefore there is a path between p and q in $\mathfrak{X}_{1,s^+}(S_{0,4})$. □

5. Type-Preserving Representations of $S_{0,5}$

In this section, we answer the Kashaev Conjecture in the specific case of the five-punctured sphere $S_{0,5}$. First, we will define a certain type of ideal triangulation on $S_{0,5}$, and we will see that the Kashaev coordinates of $\mathfrak{X}^d_\Delta(S_{0,5})$ will descend to $\mathfrak{X}_\Delta(S_{0,5})$. We will then analyze the connected components of $\mathfrak{X}_\Delta(S_{0,5})$. In order to do so, we will have to see how different signs of the triangles give different signs for the punctures. This will show which signs of the punctures will appear. We will consider how certain diagonal switch operations on Δ will change the coordinates. Ultimately, these combinatorics will reveal the connected components of $\mathfrak{X}(S_{0,5})$. We present the argument in the style of a related paper by Maloni, Palesi, and Yang [9] on the type-preserving representations of the thrice-punctured projective plane.

5.1. **Pyramidal Triangulations.**

We start by defining the class of triangulations we will use. These are triangulations isomorphic to the pictures in Figure 4.

Definition 5.1 (Pyramidal Triangulation)**.** Label the five punctures of $S_{0,5}$ as v_1, v_2, v_3, v_A, v_B and connect them with edges as in Figure 4. Note that every pair of vertices is connected with an edge besides the special vertex pair (v_A, v_B). The two "special" vertices with valence 3 are called v_A and v_B, while the three "non-special" vertices have valence 4 and are called v_1, v_2, and v_3. Label the edges $e_{ij} = e_{ji}$ where i and j are the indices of the incident vertices. The triangles are labeled T_{ijk} where i, j and k are the vertices of the triangle. This triangulation is called a Pyramidal triangulation because of the resemblance to the edge set of a right pyramid. One can also view this triangulation as a "double tetrahedron" with distinguished vertices v_A and v_B connected to a base triangle formed by vertices v_1, v_2, and v_3.

We now define which switches we consider.

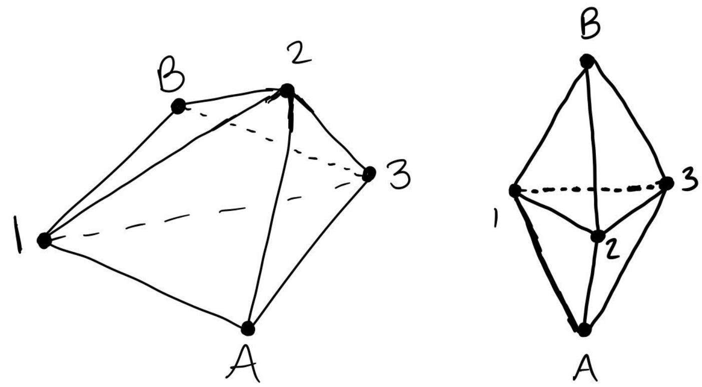

FIGURE 4. Pyramidal Triangulation (On the left, viewed as a pyramid. On the right, viewed as a “double tetrahedron”.)

Definition 5.2 (Pyramidal Switches with Relabeling)**.** Let $\sigma = (123)$ be the 3-cycle permutation. For each $i \in \{1, 2, 3\}$, set $j = \sigma(i)$. Define the pyramidal switch S_{ij} to be the diagonal switch at e_{ij}, along with a vertex re-labeling given by swapping the indices of v_A and v_i, as well as the indices of v_B and v_j. This vertex re-labeling follows the permutation $(Ai)(Bj)$.

For example, S_{12} is defined as a diagonal switch at edge e_{12}. We then re-label the vertices of the pyramidal triangulation by switching the labels of v_1 and v_A as well as the labels v_2 and v_B. The vertex re-labeling is given by the permutation $(1A)(2B)$. The operations are pictured in Figure 5.1.

The purpose of re-labeling the vertices after each switch is to keep track of the special pair of vertices (v_A, v_B) which do not share an edge. Furthermore, re-labeling the vertices after each switch allows one to compose multiple pyramidal switches in a row. This last fact is especially convenient when automating a series of pyramidal switches with a computer. Note that performing a pyramidal switch on a pyramidal triangulation returns a new pyramidal triangulation.

Remark 5.3. The cyclic order of the vertices v_1, v_2, v_3 may change after doing a switch. Is this a problem? Furthermore, is the graph connected?

We now turn toward finding a system of coordinates for $\mathfrak{X}_\Delta(S_{0,5})$. We define an action of $\mathbb{R}^V_{>0}$ on $\mathbb{R}^E_{>0}$. Let $u \in \mathbb{R}^V_{>0}$ and $\lambda \in \mathbb{R}^E_{>0}$. Define

$$(u \cdot \lambda)(e_{i,j}) = u(v_i) \cdot \lambda(e_{i,j}) \cdot u(v_j)$$

which is just scalar multiplication of each λ-length and the radii of the horocycles corresponding to the vertices of the edges.

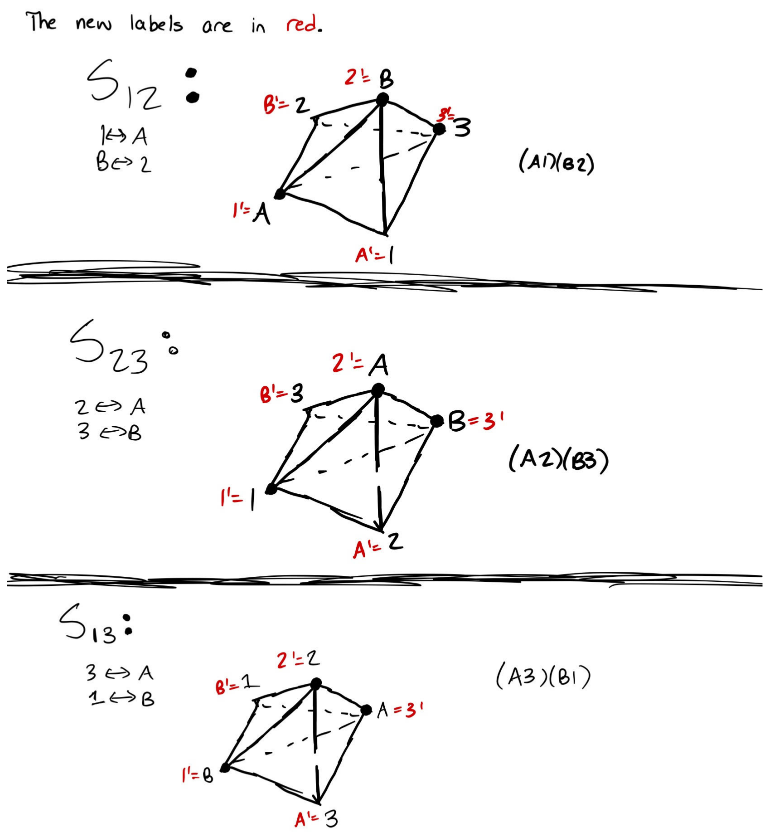

Figure 5. All three pyramidal switches are performed on a pyramidal triangulation.

We will show that the quotient of $\mathbb{R}^E_{>0}$ by this action is diffeomorphic to a more concrete space. Namely, consider the set of triples of positive real numbers $\mathbb{R}^3_{>0}$ and the action of $\mathbb{R}_{>0}$ on $\mathbb{R}^3_{>0}$ given by

$$r \cdot (a, b, c) = (ra, rb, rc).$$

We have that $\mathbb{R}^E_{>0}/\mathbb{R}^V_{>0} \cong (\mathbb{R}^3_{>0}/\mathbb{R}_{>0}) \times (\mathbb{R}^3_{>0}/\mathbb{R}_{>0})$.

Theorem 5.4 (Pyramidal Coordinates for $\mathfrak{X}_\Delta(S_{0,5})$)**.** *There is a map*

$$\phi\colon \mathbb{R}^E_{>0} \to \mathbb{R}^6_{>0}$$

which induces a diffeomorphism

$$\phi_*\colon \mathbb{R}^E_{>0}/\mathbb{R}^V_{>0} \to (\mathbb{R}^3_{>0}/\mathbb{R}_{>0}) \times (\mathbb{R}^3_{>0}/\mathbb{R}_{>0}).$$

Proof. For each $\lambda \in \mathbb{R}^E_{>0}$ define

$$\begin{aligned} X_1 &= \lambda(e_{23})\lambda(e_{A1}), \\ X_2 &= \lambda(e_{13})\lambda(e_{A2}), \\ X_3 &= \lambda(e_{12})\lambda(e_{A3}), \\ Y_1 &= \lambda(e_{23})\lambda(e_{B1}), \\ Y_2 &= \lambda(e_{13})\lambda(e_{B2}), \\ Y_3 &= \lambda(e_{12})\lambda(e_{B3}). \end{aligned}$$

We call $(X_1, X_2, X_3, Y_1, Y_2, Y_3)$ the *tetrahedral coordinates* of $[p] \in \mathfrak{X}_\Delta(S_{0,5})$. Define the map

$$\begin{aligned} \phi \colon \mathbb{R}^E_{>0} &\to \mathbb{R}^6_{>0} \\ \lambda &\mapsto (X_1, X_2, X_3, Y_1, Y_2, Y_3). \end{aligned}$$

This induces the map

$$\begin{aligned} \phi_* \colon \mathbb{R}^E_{>0}/\mathbb{R}^V_{>0} &\to \mathbb{R}^3_{>0}/\mathbb{R}_{>0} \\ [\lambda] &\mapsto [(X_1, X_2, X_3, Y_1, Y_2, Y_3)]. \end{aligned}$$

First, we check that ϕ_* is well-defined. We have the following

$$\phi(\mu \cdot \lambda) = (r_1X_1, r_1X_2, r_1X_3, r_2Y_1, r_2Y_2, r_2Y_3) = (r_1, r_2) \cdot \phi(\lambda),$$

where $r_1 = \mu(v_1)\mu(v_2)\mu(v_3)\mu(v_A)$ and $r_2 = \mu(v_1)\mu(v_2)\mu(v_3)\mu(v_B)$. Therefore the induced map ϕ_* is well-defined.

Next, we show surjectivity of ϕ^*. Given $(a, b, c, d, e, f) \in (\mathbb{R}^{>0})^6$, set

$$\begin{aligned} \lambda(e_{A1}) &= b^{\frac{1}{2}}e^{-\frac{1}{2}},\ \lambda(e_{B1}) = b^{-\frac{1}{2}}e^{\frac{1}{2}} \\ \lambda(e_{A2}) &= c^{\frac{1}{2}}f^{-\frac{1}{2}},\ \lambda(e_{B2}) = c^{-\frac{1}{2}}f^{\frac{1}{2}} \\ \lambda(e_{A3}) &= a^{\frac{1}{2}}d^{-\frac{1}{2}},\ \lambda(e_{B3}) = a^{-\frac{1}{2}}d^{\frac{1}{2}} \\ \lambda(e_{12}) = a^{\frac{1}{2}}d^{\frac{1}{2}},&\ \lambda(e_{23}) = b^{\frac{1}{2}}e^{\frac{1}{2}},\ \lambda(e_{13}) = c^{\frac{1}{2}}f^{\frac{1}{2}} \end{aligned}$$

so that $(X_1, X_2, X_3, Y_1, Y_2, Y_3) = (a, b, c, d, e, f)$.

Finally, to show the injectivity of ϕ^*, we need to show that for any $\lambda, \Lambda \in \mathbb{R}^E_{>0}$, we have

$$\phi^*(\lambda) = \phi^*(\Lambda)$$

only when $\lambda = \mu \cdot \Lambda$. In particular, we need to find the expression for μ which makes this expression true. So suppose $\phi(\lambda) = (r_1, r_2) \cdot \phi(\Lambda) = (r_1X_1, r_1X_2, r_1X_3, r_2Y_1, r_2Y_2, r_2Y_3)$. We define $\lambda_{ij} = \lambda(e_{ij})$. Consider the following decoration of the vertices

$$\mu(v_1) = (r_1r_2)^{-\frac{3}{4}} \frac{\lambda_{12}\lambda_{13}\lambda_{A3}^{\frac{1}{4}}\lambda_{B3}^{\frac{1}{4}}\lambda_{A2}^{\frac{1}{4}}\lambda_{B2}^{\frac{1}{4}}\lambda_{A1}^{\frac{1}{4}}\lambda_{B1}^{\frac{1}{4}}}{\Lambda_{12}\Lambda_{13}\Lambda_{A3}^{\frac{1}{4}}\Lambda_{B3}^{\frac{1}{4}}\Lambda_{A2}^{\frac{1}{4}}\Lambda_{B2}^{\frac{1}{4}}\Lambda_{A1}^{\frac{1}{4}}\Lambda_{B1}^{\frac{1}{4}}}$$

$$\mu(v_2) = (r_1r_2)^{-\frac{3}{4}} \frac{\lambda_{12}\lambda_{23}\lambda_{A3}^{\frac{1}{4}}\lambda_{B3}^{\frac{1}{4}}\lambda_{A2}^{\frac{1}{4}}\lambda_{B2}^{\frac{1}{4}}\lambda_{A1}^{\frac{1}{4}}\lambda_{B1}^{\frac{1}{4}}}{\Lambda_{12}\Lambda_{23}\Lambda_{A3}^{\frac{1}{4}}\Lambda_{B3}^{\frac{1}{4}}\Lambda_{A2}^{\frac{1}{4}}\Lambda_{B2}^{\frac{1}{4}}\Lambda_{A1}^{\frac{1}{4}}\Lambda_{B1}^{\frac{1}{4}}}$$

$$\mu(v_3) = (r_1r_2)^{-\frac{3}{4}} \frac{\lambda_{13}\lambda_{23}\lambda_{A3}^{\frac{1}{4}}\lambda_{B3}^{\frac{1}{4}}\lambda_{A2}^{\frac{1}{4}}\lambda_{B2}^{\frac{1}{4}}\lambda_{A1}^{\frac{1}{4}}\lambda_{B1}^{\frac{1}{4}}}{\Lambda_{13}\Lambda_{23}\Lambda_{A3}^{\frac{1}{4}}\Lambda_{B3}^{\frac{1}{4}}\Lambda_{A2}^{\frac{1}{4}}\Lambda_{B2}^{\frac{1}{4}}\Lambda_{A1}^{\frac{1}{4}}\Lambda_{B1}^{\frac{1}{4}}}$$

$$\mu(v_A) = r_1^{-2} \frac{\lambda_{A1}\lambda_{A2}\lambda_{A3}\lambda_{12}^{\frac{1}{2}}\lambda_{13}^{\frac{1}{2}}\lambda_{23}^{\frac{1}{2}}}{\Lambda_{A1}\Lambda_{A2}\Lambda_{A3}\Lambda_{12}^{\frac{1}{2}}\Lambda_{13}^{\frac{1}{2}}\Lambda_{23}^{\frac{1}{2}}}$$

$$\mu(v_B) = r_2^{-2} \frac{\lambda_{B1}\lambda_{B2}\lambda_{B3}\lambda_{12}^{\frac{1}{2}}\lambda_{13}^{\frac{1}{2}}\lambda_{23}^{\frac{1}{2}}}{\Lambda_{B1}\Lambda_{B2}\Lambda_{B3}\Lambda_{12}^{\frac{1}{2}}\Lambda_{13}^{\frac{1}{2}}\Lambda_{23}^{\frac{1}{2}}}.$$

We will show that $\lambda_{12} = \mu \cdot \Lambda_{12}$ and $\lambda_{A3} = \mu \cdot \Lambda_{A3}$. The other cases follow similarly by symmetry. We have

$$\begin{aligned}
\mu(v_1)\Lambda_{12}\mu(v_2) &= (r_1r_2)^{-\frac{3}{2}} \frac{\lambda_{12}^2\lambda_{13}\lambda_{23}\lambda_{A3}^{\frac{1}{2}}\lambda_{B3}^{\frac{1}{2}}\lambda_{A2}^{\frac{1}{2}}\lambda_{B2}^{\frac{1}{2}}\lambda_{A1}^{\frac{1}{2}}\lambda_{B1}^{\frac{1}{2}}}{\Lambda_{12}^2\Lambda_{13}\Lambda_{23}\Lambda_{A3}^{\frac{1}{2}}\Lambda_{B3}^{\frac{1}{2}}\Lambda_{A2}^{\frac{1}{2}}\Lambda_{B2}^{\frac{1}{2}}\Lambda_{A1}^{\frac{1}{2}}\Lambda_{B1}^{\frac{1}{2}}}\Lambda_{12} \\
&= (r_1r_2)^{-\frac{3}{2}} \frac{\lambda_{12}(r_1X_1)^{\frac{1}{2}}(r_1X_2)^{\frac{1}{2}}(r_1X_3)^{\frac{1}{2}}(r_2Y_1)^{\frac{1}{2}}(r_2Y_2)^{\frac{1}{2}}(r_2Y_3)^{\frac{1}{2}}}{\Lambda_{12}X_1^{\frac{1}{2}}X_2^{\frac{1}{2}}X_3^{\frac{1}{2}}Y_1^{\frac{1}{2}}Y_2^{\frac{1}{2}}Y_3^{\frac{1}{2}}}\Lambda_{12} \\
&= \lambda_{12}.
\end{aligned}$$

and also

$$\begin{aligned}
\mu(v_A)\Lambda_{A3}\mu(v_3) &= (r_1^{\frac{-11}{4}}r_2^{\frac{-3}{4}}) \frac{(\lambda_{A1}\lambda_{A2}\lambda_{A3}\lambda_{12}^{\frac{1}{2}}\lambda_{13}^{\frac{1}{2}}\lambda_{23}^{\frac{1}{2}})(\lambda_{13}\lambda_{23}\lambda_{A3}^{\frac{1}{4}}\lambda_{B3}^{\frac{1}{4}}\lambda_{A2}^{\frac{1}{4}}\lambda_{B2}^{\frac{1}{4}}\lambda_{A1}^{\frac{1}{4}}\lambda_{B1}^{\frac{1}{4}})}{(\Lambda_{13}\Lambda_{23}\Lambda_{A3}^{\frac{1}{4}}\Lambda_{B3}^{\frac{1}{4}}\Lambda_{A2}^{\frac{1}{4}}\Lambda_{B2}^{\frac{1}{4}}\Lambda_{A1}^{\frac{1}{4}}\Lambda_{B1}^{\frac{1}{4}})}\Lambda_{A3} \\
&= (r_1^{\frac{-11}{4}}r_2^{\frac{-3}{4}}) \frac{(r_1X_1)(r_1X_2)(r_1X_3)^{\frac{1}{4}}(r_2Y_3)^{\frac{1}{4}}(r_1X_2)^{\frac{1}{4}}(r_2Y_2)^{\frac{1}{4}}(r_1X_1)^{\frac{1}{4}}(r_2Y_1)^{\frac{1}{4}}}{X_1X_2X_3^{\frac{1}{4}}Y_3^{\frac{1}{4}}X_2^{\frac{1}{4}}Y_2^{\frac{1}{4}}X_1^{\frac{1}{4}}Y_1^{\frac{1}{4}}}\lambda_{A3} \\
&= \lambda_{A3}.
\end{aligned}$$

Therefore ϕ_* is injective. By construction, ϕ_* and the inverse ϕ_*^{-1} is differentiable and so ϕ_* is a diffeomorphism. □

A crucial theorem will be the expression of the conjugacy class of the peripheral curves in terms of the pyramidal coordinates and the signs of the triangles. In particular, we note that the pyramidal coordinates appear in the top-right entry of a matrix representative for the ρ-images of the peripheral curves on $S_{0,5}$.

Theorem 5.5 (Peripheral Curves in Pyramidal Triangulation)**.** *Let $\rho \in \mathfrak{X}_\Delta(S_{0,5})$. There are five free homotopy classes of peripheral curves in $S_{0,5}$. Denote these curves $\gamma_A, \gamma_B, \gamma_1, \gamma_2, \gamma_3$ so that γ_i is freely homotopic to a circle about the puncture v_i. Let ρ be a type-preserving representation. Then up to conjugation by elements in $PSL(2,\mathbb{R})$, the image of each γ_i under ρ are*

(1) $\rho([\gamma_A]) \cong \pm \begin{pmatrix} 1 & \epsilon(t_{A23})X_1 + \epsilon(t_{A13})X_2 + \epsilon(t_{A12})X_3 \\ 0 & 1 \end{pmatrix}$,

(2) $\rho([\gamma_B]) \cong \pm \begin{pmatrix} 1 & \epsilon(t_{B23})Y_1 + \epsilon(t_{B13})Y_2 + \epsilon(t_{B12})Y_3 \\ 0 & 1 \end{pmatrix}$,

(3) $\rho([\gamma_1]) \cong \pm \begin{pmatrix} 1 & X_1(\epsilon(t_{B13})Y_3 + \epsilon(t_{B12})Y_2) + Y_1(\epsilon(t_{A13})X_3 + \epsilon(t_{A12})X_2) \\ 0 & 1 \end{pmatrix}$,

(4) $\rho([\gamma_2]) \cong \pm \begin{pmatrix} 1 & X_2(\epsilon(t_{B23})Y_3 + \epsilon(t_{B12})Y_1) + Y_2(\epsilon(t_{A23})X_3 + \epsilon(t_{A12})X_1) \\ 0 & 1 \end{pmatrix}$,

(5) $\rho([\gamma_3]) \cong \pm \begin{pmatrix} 1 & X_3(\epsilon(t_{B23})Y_2 + \epsilon(t_{B13})Y_1) + Y_3(\epsilon(t_{A23})X_2 + \epsilon(t_{A13})X_1) \\ 0 & 1 \end{pmatrix}$.

Proof. This is a direct result of Sun and Yang's trace formula, Theorem 3.19. □

In particular, the sign which ρ associates to a puncture is exactly the sign of the expression coming from the top-right entry of the corresponding matrix representation above. Therefore, given the signs of the triangles, we can compute the possible signs of the punctures.

Finally, we note the following effect of pyramidal switches on the kashaev coordinates.

Lemma 5.6 (Effect of Pyramidal Switch on the Kashaev Coordinates)**.** *Let Δ be a tetrehdral triangulation. The λ lengths and signs of triangles of Δ are changed by pyramidal triangulations in the following way:*

(1) Suppose Δ' is the triangulation as a result of S_{12}.
- *In the case $\epsilon(t_{A12}) = \epsilon(t_{B12})$, we have that*

$$\lambda(e_{AB}) = \frac{\lambda(e_{A1})\lambda(e_{B2}) + \lambda(e_{A2})\lambda(e_{B1})}{\lambda(e_{12})},$$

and $\epsilon(t_{AB1}) = \epsilon(t_{AB2}) = \epsilon(t_{A12})$.
- *In the case $\epsilon(t_{A12}) \neq \epsilon(t_{B12})$, we break into further cases.*
If $\lambda(e_{A1})\lambda(e_{B2}) - \lambda(e_{A2})\lambda(e_{B1}) > 0$, then

$$\lambda(e_{AB}) = \frac{\lambda(e_{A1})\lambda(e_{B2}) - \lambda(e_{A2})\lambda(e_{B1})}{\lambda(e_{12})},$$

and $\epsilon(t_{AB1}) = \epsilon(t_{B12})$, $\epsilon(t_{AB2}) = \epsilon(t_{A12})$.
If $\lambda(e_{A1})\lambda(e_{B2}) - \lambda(e_{A2})\lambda(e_{B1}) < 0$, then

$$\lambda(e_{AB}) = \frac{\lambda(e_{A2})\lambda(e_{B1}) - \lambda(e_{A1})\lambda(e_{B2})}{\lambda(e_{12})},$$

and $\epsilon(t_{AB1}) = \epsilon(t_{A12})$, $\epsilon(t_{AB2}) = \epsilon(t_{B12})$.

We note that according to the vertex re-labeling operation, e_{AB} is then re-labeled as e_{12}. Furthermore, t_{AB1} is re-labeled as t_{A12} and t_{AB2} is re-labeled as t_{B12}.

Therefore, the new pyramidal coordinates are given by

$$X_1' = \frac{X_1Y_3}{e_{23}e_{12}}, X_2' = \frac{X_3Y_1}{e_{23}e_{12}}, X_3' = \frac{|X_1Y_2 \pm X_2Y_1|}{e_{12}e_{23}}$$

$$Y_1' = \frac{X_2Y_3}{e_{12}e_{13}}, Y_2' = \frac{X_3Y_2}{e_{12}e_{13}}, Y_3' = \frac{|X_1Y_2 \pm X_2Y_1|}{e_{12}e_{13}}$$

(2) *Suppose Δ' is the triangulation as a result of S_{13}.*

- *In the case $\epsilon(t_{A13}) = \epsilon(t_{B13})$, we have that*

$$\lambda(e_{AB}) = \frac{\lambda(e_{A1})\lambda(e_{B3}) + \lambda(e_{A3})\lambda(e_{B1})}{\lambda(e_{13})},$$

and $\epsilon(t_{AB1}) = \epsilon(t_{AB3}) = \epsilon(t_{A13})$.

- *In the case $\epsilon(t_{A13}) \neq \epsilon(t_{B13})$, we break into further cases. If $\lambda(e_{A1})\lambda(e_{B3}) - \lambda(e_{A3})\lambda(e_{B1}) > 0$, then*

$$\lambda(e_{AB}) = \frac{\lambda(e_{A1})\lambda(e_{B3}) - \lambda(e_{A3})\lambda(e_{B1})}{\lambda(e_{13})},$$

and $\epsilon(t_{AB1}) = \epsilon(t_{A13})$, $\epsilon(t_{AB3}) = \epsilon(t_{B13})$.

If $\lambda(e_{A1})\lambda(e_{B3}) - \lambda(e_{A3})\lambda(e_{B1}) < 0$, then

$$\lambda(e_{AB}) = \frac{\lambda(e_{A3})\lambda(e_{B1}) - \lambda(e_{A1})\lambda(e_{B3})}{\lambda(e_{13})},$$

and $\epsilon(t_{AB1}) = \epsilon(t_{B13})$, $\epsilon(t_{AB3}) = \epsilon(t_{A13})$.

We note that according to the vertex re-labeling operation, e_{AB} is then re-labeled as e_{13}. Furthermore, t_{AB1} is re-labeled as t_{B13} and t_{AB3} is re-labeled as t_{A13}.

Therefore, the new pyramidal coordinates are given by

$$X_1' = \frac{X_2Y_3}{e_{12}e_{13}}, X_2' = \frac{|X_1Y_3 \pm X_3Y_1|}{e_{13}e_{12}}, X_3' = \frac{X_3Y_2}{e_{13}e_{12}}$$

$$Y_1' = \frac{X_2Y_1}{e_{23}e_{13}}, Y_2' = \frac{|X_1Y_3 \pm X_3Y_1|}{e_{23}e_{13}}, Y_3' = \frac{X_1Y_2}{e_{23}e_{13}}$$

(3) *Suppose Δ' is the triangulation as a result of S_{23}.*

- *In the case $\epsilon(t_{A23}) = \epsilon(t_{B23})$, we have that*

$$\lambda(e_{AB}) = \frac{\lambda(e_{A2})\lambda(e_{B3}) + \lambda(e_{A3})\lambda(e_{B2})}{\lambda(e_{23})},$$

and $\epsilon(t_{AB2}) = \epsilon(t_{AB3}) = \epsilon(t_{A23})$.

- *In the case $\epsilon(t_{A23}) \neq \epsilon(t_{B23})$, we break into further cases.*

If $\lambda(e_{A2})\lambda(e_{B3}) - \lambda(e_{A3})\lambda(e_{B2}) > 0$*, then*

$$\lambda(e_{AB}) = \frac{\lambda(e_{A2})\lambda(e_{B3}) - \lambda(e_{A3})\lambda(e_{B2})}{\lambda(e_{23})},$$

and $\epsilon(t_{AB2}) = \epsilon(t_{B23})$*,* $\epsilon(t_{AB3}) = \epsilon(t_{A23})$*.*

If $\lambda(e_{A2})\lambda(e_{B3}) - \lambda(e_{A3})\lambda(e_{B2}) < 0$*, then*

$$\lambda(e_{AB}) = \frac{\lambda(e_{A3})\lambda(e_{B2}) - \lambda(e_{A2})\lambda(e_{B3})}{\lambda(e_{23})},$$

and $\epsilon(t_{AB2}) = \epsilon(t_{A23})$*,* $\epsilon(t_{AB3}) = \epsilon(t_{B23})$*.*

We note that according to the vertex re-labeling operation, e_{AB} *is then re-labeled as* e_{23}*. Furthermore,* t_{AB2} *is re-labeled as* t_{A23} *and* t_{AB3} *is re-labeled as* t_{B23}*.*

Therefore, the new pyramidal coordinates are given by

$$X_1' = \frac{|X_2Y_3 \pm X_3Y_2|}{e_{13}e_{23}}, X_2' = \frac{X_2Y_1}{e_{13}e_{23}}, X_3' = \frac{X_1Y_2}{e_{13}e_{23}}$$

$$Y_1' = \frac{|X_2Y_3 \pm X_3Y_2|}{e_{12}e_{23}}, Y_2' = \frac{X_3Y_1}{e_{12}e_{23}}, Y_3' = \frac{X_1Y_3}{e_{12}e_{23}}$$

Proof. This is a direct application of Kashaev's diagonal switch formula, Theorem 3.18. □

We will now compute the connected components of $\mathfrak{X}(S_{0,5})$ using the above pyramidal switches and the relationship between the signs of the triangles and the signs of the punctures.

5.2. **The Connected Components of $\mathfrak{X}_0(S_{0,5})$.** Let Δ be a pyramidal triangulation. In this section, we fix $e(\rho) = 0$ and classify the connected components of $\mathfrak{X}_0(S_{0,5})$. In particular, let $u \in \{+1, -1\}^V$. Define the set

$$\mathfrak{X}_{k,s}(S_{0,5}) = \{\, \rho \in \mathfrak{X}_k(S_{0,5}) \mid \rho \text{ associates the sign } u(v_i) \text{ to } v_i \,\}.$$

We will show that the connected components of $\mathfrak{X}_0(S_{0,5})$ are of the form

$$\mathfrak{X}_0(S_{0,5}) = \bigsqcup_{s \in \{+1,-1\}^5} \mathfrak{X}_{0,s}(S_{0,5}).$$

However, not every possible signs of punctures are possible, so we first check when $\mathfrak{X}_{0,s}(S_{0,5})$ is non-empty.

For $\rho \in \mathfrak{X}_\Delta(S_{0,5})$, the signs of the punctures are given by the sign of the following expressions

$$\begin{aligned}
v_A &: \epsilon(t_{A23})X_1 + \epsilon(t_{A13})X_2 + \epsilon(t_{A12})X_3,\\
v_B &: \epsilon(t_{B23})Y_1 + \epsilon(t_{B13})Y_2 + \epsilon(t_{B12})Y_3,\\
v_1 &: X_1(\epsilon(t_{B13})Y_3 + \epsilon(t_{B12})Y_2) + Y_1(\epsilon(t_{A13})X_3 + \epsilon(t_{A12})X_2),\\
v2 &: X_2(\epsilon(t_{B23})Y_3 + \epsilon(t_{B12})Y_1) + Y_2(\epsilon(t_{A23})X_3 + \epsilon(t_{A12})X_1),\\
v3 &: X_3(\epsilon(t_{B23})Y_2 + \epsilon(t_{B13})Y_1) + Y_3(\epsilon(t_{A23})X_2 + \epsilon(t_{A13})X_1),
\end{aligned}$$

by Theorem 5.5. In the case $\rho \in \mathfrak{X}_0(S_{0,5}) \cap \mathfrak{X}_\Delta(S_{0,5})$, we have from Theorem 3.16 that

$$\sum_{t \in \Delta} \epsilon(t) = 0.$$

In particular, ρ assigns a value of $+1$ to three triangles of Δ and -1 to the other three triangles of Δ.

Define $\epsilon_A^+, \epsilon_{Aij} \in \{+1,-1\}^{\{T_{A12},T_{A13},T_{A23}\}}$ so that

$$\begin{aligned} \epsilon_A^+(T_{Aij}) &= 1 \text{ for all } \{i,j\} \subseteq \{1,2,3\}, \\ \epsilon_{Aij}(T_{Aij}) &= -1, \\ \epsilon_{Aij}(T_{Akl}) &= 1 \text{ for all } \{k,l\} \neq \{i,j\}. \end{aligned}$$

Similarly, define $\epsilon_B^+, \epsilon_{Bij} \in \{+1,-1\}^{\{T_{B12},T_{B13},T_{B23}\}}$ so that

$$\begin{aligned} \epsilon_B^+(T_{Bij}) &= 1 \text{ for all } \{i,j\} \subseteq \{1,2,3\}, \\ \epsilon_{Bij}(T_{Bij}) &= -1, \\ \epsilon_{Bij}(B_{Bkl}) &= 1 \text{ for all } \{k,l\} \neq \{i,j\}. \end{aligned}$$

We also define the notation $\epsilon_A^- := -\epsilon_A^+$ and $\epsilon_B^- := -\epsilon_B^+$. Then ρ gives signs of triangles according to one of the following tuples

$$(\epsilon_A^+, \epsilon_B^-), (\epsilon_A^-, \epsilon_B^+),$$

$$(\epsilon_{Aij}, -\epsilon_{Bij}), (-\epsilon_{Aij}, \epsilon_{Bij}),$$

$$(\epsilon_{Aij}, -\epsilon_{Bik}), (-\epsilon_{Aij}, \epsilon_{Bik}),$$

where $\{i,j\}, \{i,k\} \subseteq \{1,2,3\}$ such that $j \neq k$.

We also need to introduce notation for the signs of the punctures. Define $s_+, s_A, s_B \in \{+1,-1\}^{\{A,B\}}$ so that

$$s_+(A) = 1, s_+(B) = 1,$$

$$s_A(A) = 1, s_A(B) = -1,$$

$$s_B(A) = -1, s_B(B) = 1.$$

Also, define $s_i, s_+ \in \{+1,-1\}^{\{1,2,3\}}$ so that

$$s_+(i) = 1 \text{ for all } i \in \{1,2,3\},$$

$$s_i(i) = -1 \text{ and } s_i(j) = 1 \text{ for all } j \neq i \in \{1,2,3\}.$$

Theorem 5.7. *Fix a pyramidal triangulation Δ. We have the following characterization of the possible signs of punctures when $e(\rho) = 0$:*

(1) *Let ρ assign signs to the triangles according to $(\epsilon_A^+, -\epsilon_B^+)$. Then ρ gives signs to the punctures according to $(s_A, \pm s_t)$ for some $1 \leq t \leq 3$.*
(2) *Let ρ assign signs to the triangles according to $(\epsilon_B^+, -\epsilon_A^+)$. Then ρ gives signs to the punctures according to $(s_B, \pm s_t)$ for some $1 \leq t \leq 3$.*

(3) *Let ρ assign signs to the triangles according to $(\epsilon_{Aij}, -\epsilon_{Bij})$ or $(-\epsilon_{Aij}, \epsilon_{Bij})$ for any $\{i, j\} \subseteq \{1, 2, 3\}$. Then ρ gives signs to the punctures according to either*
   - $(s_A, \pm s_t)$ *for some* $1 \leq t \leq 3$,
   - $(s_B, \pm s_t)$ *for some* $1 \leq t \leq 3$,
   - $\pm(s_+, -s_k)$ *for* $i \neq k \neq j$,

(4) *Let ρ assign signs to the triangles according to either $(\epsilon_{Aij}, -\epsilon_{Bik})$ where $\{i, j\} \neq \{i, k\}$. Then ρ gives signs to the punctures according to either*
   - $\pm(s_+, -s_t)$ *where* $t \in \{j, k\}$,
   - $\pm(s_+, -s_+)$,
   - $\pm(s_A, s_j)$,
   - $\pm(s_B, -s_k)$.

(5) *Let ρ assign signs to the triangles according to either $(-\epsilon_{Aij}, \epsilon_{Bik})$ where $\{i, j\} \neq \{i, k\}$. Then ρ gives signs to the punctures according to either*
   - $\pm(s_+, -s_t)$ *where* $t \in \{j, k\}$,
   - $\pm(s_+, -s_+)$,
   - $\pm(-s_A, s_j)$,
   - $\pm(s_B, s_k)$.

Proof. Due to symmetry, there are three distinct cases to check.

• **Case** $(\epsilon_A^+, \epsilon_B^-)$.

In this case, the signs of the punctures are given by the signs of the equations

$$\begin{aligned}
v_A &: X_1 + X_2 + X_3, \\
v_B &: -Y_1 - Y_2 - Y_3, \\
v_1 &: X_1 v_B + Y_1 v_A, \\
v_2 &: X_2 v_B + Y_2 v_A, \\
v_3 &: X_3 v_B + Y_3 v_A.
\end{aligned}$$

We see that v_A is always positive while v_B is always negative. Furthermore, we have the relation $v_1 + v_2 + v_3 = 0$. Therefore, the sign of the punctures v_1, v_2, and v_3 cannot all be the same. Therefore, there are a total of six possibilities for the signs of the punctures $(s_A, \pm s_1), (s_A, \pm s_2)$, or $(s_A, \pm s_3)$.

We note that the case $(\epsilon_A^-, \epsilon_B^+)$ is similar. In this case, the signs of the punctures will be $(s_B, \pm s_1), (s_B, \pm s_2)$, or $(s_B, \pm s_3)$.

• **Case** $(\epsilon_{A12}, -\epsilon_{B12})$.

In this case, the signs of the punctures are given by the signs of the equations

$$\begin{aligned} v_A &: X_1 + X_2 - X_3, \\ v_B &: Y_3 - Y_1 - Y_2, \\ v_1 &: -X_1 v_B - Y_1 v_A, \\ v_2 &: -X_2 v_B - Y_2 v_A, \\ v_3 &: X_3 v_B + Y_3 v_A. \end{aligned}$$

We note that when the sign of v_A and v_B are the same, then the sign of v_1, v_2, and v_3 are determined by the sign of v_A and v_B. On the other hand, we note that we have the relation $v_1 + v_2 + v_3 = 0$, which means that when v_A and v_B have different sign, the sign of v_1, v_2, and v_3 cannot all be the same. Therefore, there are a total of fourteen possibilities for the signs of the punctures given by $\pm(s_+, -s_3)$, $\pm(s_A, \pm s_1)$, $\pm(s_A, \pm s_2)$, or $\pm(s_A, \pm s_3)$.

• **Case** $(\epsilon_{A12}, -\epsilon_{B13})$**:**

In this case, the signs of the punctures are given by the signs of the equations

$$\begin{aligned} v_A &: X_1 + X_2 - X_3, \\ v_B &: Y_2 - Y_1 - Y_3, \\ v_1 &: -X_1 v_B - Y_1 v_A, \\ v_2 &: X_2 v_B - Y_2 v_A, \\ v_3 &: -X_3 v_B + Y_3 v_A. \end{aligned}$$

We note that we have the relation $v_3 = v_1 - v_2$. Furthermore, the signs of the punctures v_A and v_B partially determine the signs of v_1, v_2 and v_3.

∘ If v_A and v_B are both positive, then v_1 must be negative. By the relation $v_3 = v_1 - v_2$, if v_2 is positive, then v_3 must be negative. On the other hand, if v_2 is negative, then v_3 can have any sign. In particular, if v_A and v_B are positive then the signs of the punctures must be $(s_+, -s_2), (s_+, -s_3)$ or (s_+, s_-).

∘ If v_A and v_B are both negative, then by a similar reasoning we have that that the signs of the punctures must be $(s_-, s_2), (s_-, s_3)$, or (s_-, s_+).

∘ If v_A is positive and v_B is negative, then v_2 must be negative and v_3 must be positive. By the relation $v_3 = v_1 - v_2$, we see that v_1 can have any sign. Therefore the signs of the punctures must be (s_A, s_2) or $(s_A, -s_3)$.

∘ If v_A is negative and v_B is positive, then by a similar reasoning we must have the signs of the punctures must be $(s_B, -s_2)$ or (s_B, s_3).

Therefore, there are ten possibilities for the sign of the punctures in this case given by $\pm(s_A, s_2)$, $\pm(s_A, s_3)$, $\pm(s_+, -s_2)$, $\pm(s_+, -s_3)$ or $\pm(s_+, s_-)$. □

In total, there are twenty possible signs of punctures which may be assigned by a representation $[\rho] \in \mathfrak{X}_0(S_{0,5})$. We will now show that the sign of the punctures determines the connected component a representation $[\rho] \in \mathfrak{X}_0(S_{0,5})$.

Theorem 5.8. *The following subsets of* $\mathfrak{X}_0(S_{0,5})$ *are connected:*

- $\mathfrak{X}_{0,\pm(s_A,\pm s_i)}(S_{0,5})$,
- $\mathfrak{X}_{0,\pm(s_+,-s_i)}(S_{0,5})$,
- $\mathfrak{X}_{0,\pm(s_+,s_-)}(S_{0,5})$,

for $i \in \{1,2,3\}$.

Proof. Many of these cases are symmetric, and many reduce to other cases following a single pyramidal switch. In total, there are 2 cases to check.

To show the connectedness of $\mathfrak{X}_{k,s}(S_{0,5})$, we will show that any $p, q \in \mathfrak{X}_{k,s}(S_{0,5})$ can be connected with a path in $\mathfrak{X}_{k_s}(S_{0,5})$.

Let Δ be a pyramidal triangulation. Let $(X_1, X_2, X_3, Y_1, Y_2, Y_3)$ be the pyramidal coordinates for p and let ϵ be the signs of triangles for p. Let $(X_1', X_2', X_3', Y_1', Y_2', Y_3')$ be the pyramidal coordinates for q and let ϵ' be the signs of triangles for q. If $\epsilon = \epsilon'$, then p and q lie in the same path component of $\mathfrak{X}_\Delta(S_{0,5}, \epsilon)$. When $\epsilon \neq \epsilon'$, we show that there exists a pyramidal triangulation Δ' such that p and q are in the same path component of $\mathfrak{X}_{\Delta'}(S_{0,5})$.

Furthermore, note that if a representation gives signs of punctures $(s_+, -s_1)$ with respect to Δ, then following the pyramidal switch S_{13}, the signs of punctures for that representation are now given by (s_A, s_1). In particular, we can reduce the connectivity of $\mathfrak{X}_{0,(s_+,-s_1)}(S_{0,5})$ to the calculation in case 1 below.

Also, by multiplying the signs of each puncture by -1 and the signs of each triangle by -1, we see that the case $(s_B, -s_1)$ is symmetric to the case (s_A, s_1). By a similar trick, the case (s_-, s_1) is symmetric to the case $(s_+, -s_1)$. Therefore, every possible signs of punctures besides (s_+, s_-) reduces to case 1 below.

In the following 2 cases, we will switch which signs of punctures we will check. Within the subcases, we are considering different triangle signs for p and q.

• **Case 1:** $\mathfrak{X}_{0,(s_A,s_1)}(S_{0,5})$ **is connected.**

Let $p, q \in \mathfrak{X}_{0,(s_A,s_1)}(S_{0,5})$. The possible signs of triangles for p and q are

$$\epsilon, \epsilon' \subseteq \{(\epsilon_A^+, \epsilon_B^-), \pm(\epsilon_{Aij}, -\epsilon_{Bij}), \pm(\epsilon_{Alk}, -\epsilon_{Blt})\}$$

for $\{i,j\} \subseteq \{1,2,3\}$ and $\{l,k\} \neq \{l,t\} \subseteq \{1,2,3\}$ and $l \neq 1$. Up to symmetry, there are three subcases to check.

◦ Subcase 1.1: Suppose $\epsilon = (\epsilon_{A12}, -\epsilon_{B12})$ and $\epsilon' = (\epsilon_A^+, \epsilon_B^-)$.

We note that the pyramidal coordinates for any representation in $\mathfrak{X}_\Delta(S_{0,5}, (\epsilon_{A12}, \epsilon_{B12}))$ giving signs of punctures (s_A, s_1) are subject to the inequalities

$$v_1 : -X_1 v_B - Y_1 v_A < 0$$
$$v_2 : -X_2 v_B - Y_2 v_A > 0$$

Therefore, we have the relation

$$\frac{X_1}{Y_1} > \frac{-v_A}{v_B} > \frac{X_2}{Y_2}.$$

This tells us that $X_1 Y_2 - X_2 Y_1 > 0$.

Similarly, q is subject to the inequalities

$$v_1' : X_1' v_B' + Y_1' v_A' < 0$$
$$v_2' : X_2' v_B' + Y_2' v_A' > 0$$

which tells us that $X_1' Y_2' - X_2' Y_1' > 0$.

Let Δ' be the resulting pyramidal triangulation from the pyramidal switch S_{12}. With respect to Δ', both p and q give signs of triangles $(\epsilon_{A23}, -\epsilon_{B13})$. Furthermore, both p and q give signs of punctures (s_B, s_2) with respect to Δ', so p and q lie in the same path component of $\mathfrak{X}_{\Delta'}(S_{0,5}, (\epsilon_{A23}, -\epsilon_{B13}))$.

∘ Subcase 1.2: Suppose $\epsilon = (\epsilon_{A12}, -\epsilon_{B12})$ and $\epsilon' = (\epsilon_{A12}, -\epsilon_{B23})$.

In this case, p is subject to the inequalities

$$v_A : X_1 + X_2 - X_3 > 0,$$
$$v_B : Y_3 - Y_1 - Y_2 < 0,$$
$$v_1 : -X_1 v_B - Y_1 v_A < 0,$$
$$v_2 : -X_2 v_B - Y_2 v_A > 0,$$
$$v_3 : X_3 v_B + Y_3 v_A > 0.$$

Therefore, we may assume p to have pyramidal coordinates

$$X_1 = \frac{4}{7}, X_2 = \frac{1}{14}, X_3 = \frac{5}{14}, Y_1 = \frac{5}{7}, Y_2 = \frac{1}{7}, Y_3 = \frac{1}{7}$$

which satisfies the above 5 constraints as well as $X_1 Y_3 - X_3 Y_1 > 0$.

Let Δ' be the resulting pyramidal triangulation from the pyramidal switch S_{13}. With respect to Δ', p gives signs of triangles $(\epsilon_{A12}, -\epsilon_{B12})$ and q gives signs of triangles by either $(\epsilon_A^+, \epsilon_B^-)$ or $(\epsilon_{A13}, -\epsilon_{B13})$. In either case, p and q will be connected after the switch by reducing to subcase 1.

∘ Subcase 1.3: Suppose $\epsilon = (\epsilon_{A12}, -\epsilon_{B12})$ and $\epsilon' = (-\epsilon_{A23}, \epsilon_{B23})$.

For the same argument as before, we may choose p such that $X_3 Y_1 - X_1 Y_3 > 0$ (by choosing $Y_1 > Y_2 + Y_3$). We may choose q such that $X_1' Y_3' - X_3' Y_1' > 0$ by a symmetric argument as well.

Let Δ' be the resulting pyramidal triangulation from the pyramidal switch S_{13}. With respect to Δ', p gives signs of triangles $(\epsilon_{A23}, -\epsilon_{B23})$ and q gives signs of triangles $(\epsilon_{A23}, -\epsilon_{B23})$. Both p and q give signs of punctures (s_A, s_1) with respect to Δ'. Therefore p and q lie in the same path component of $\mathfrak{X}_{\Delta'}(S_{0,5}, (\epsilon_{A23}, -\epsilon_{B23}))$.

• **Case 2: $\mathfrak{X}_{0,(s_+,s_-)}(S_{0,5})$ is connected.**

Let $p, q \in \mathfrak{X}_{0,(s_A,-s_1)}(S_{0,5})$. We must have that

$$\epsilon, \epsilon' \subseteq \{\pm(\epsilon_{Alk}, -\epsilon_{Blt})\}$$

for $\{i, j\} \subseteq \{1, 2, 3\}$ and $\{l, k\} \neq \{l, t\} \subseteq \{1, 2, 3\}$.

We have three subcases to check.

∘ Subcase 2.1: Suppose $\epsilon = (\epsilon_{A23}, -\epsilon_{B13})$ and $\epsilon' = (-\epsilon_{A13}, \epsilon_{B23})$.

We have that p satisfies the inequalities

$$v_1 : -X_1 v_B + Y_1 v_A < 0,$$
$$v_2 : X_2 v_B - Y_2 v_A < 0.$$

Therefore, we get the relation

$$\frac{X_1}{Y_1} > \frac{v_A}{v_B} > \frac{X_2}{Y_2}$$

so $X_1Y_2 - X_2Y_1 > 0$. A similar argument shows that $X_2'Y_1' - X_1'Y_2' > 0$.

Let Δ' be the resulting pyramidal triangulation from the pyramidal switch S_{12}. With respect to Δ', p and q both give signs of triangles $(\epsilon_{A12}, -\epsilon_{B12})$. Also, the signs of the punctures with respect to Δ' is given by (s_-, s_3) after the vertex re-labeling. Therefore p and q are connected.

∘ Subcase 2.2: Suppose $\epsilon = (\epsilon_{A23}, -\epsilon_{B13})$ and $\epsilon' = (-\epsilon_{A23}, \epsilon_{B13})$.

By the signs of the punctures and the signs of the triangles, we must have that $X_1Y_2 - X_2Y_1 > 0$. Furthermore, we must have that $X_1'Y_2' - X_2'Y_1' > 0$.

Let Δ' be the resulting pyramidal triangulation from the pyramidal switch S_{12}. With respect to Δ', p gives signs of triangles $(\epsilon_{A12}, -\epsilon_{B12})$ and q gives signs of triangles $(-\epsilon_{A12}, \epsilon_{B12})$. Also, the signs of the punctures with respect to Δ' is given by (s_-, s_3) after the vertex re-labeling. Therefore, we reduce to case 1.

∘ Subcase 2.3: Suppose $\epsilon = (\epsilon_{A23}, -\epsilon_{B13})$ and $\epsilon' = (\epsilon_{A12}, -\epsilon_{B23})$.

Above, we already verified that $X_1Y_2 - X_2Y_1 > 0$ in this case. The resulting signs of triangles for q after the switch S_{12} is forced regardless of signs of punctures.

Let Δ' be the resulting pyramidal triangulation from the pyramidal switch S_{12}. With respect to Δ', p gives signs of triangles $(\epsilon_{A12}, -\epsilon_{B12})$ and q gives signs of triangles $(-\epsilon_{A13}, \epsilon_{B12})$. Also, the signs of the punctures with respect to Δ' is given by (s_-, s_3) after the vertex re-labeling. Therefore, we again reduce to case 1. □

The above proof can be visualized by creating a graph with vertex set given by the possible signs of the triangles and edges drawn when two signs of triangles differ by a single pyramidal switch as seen in Figure 5.2.

5.3. **The Connected Components of $\mathfrak{X}_1(S_{0,5})$.** In this section, we show that the path components of $\mathfrak{X}_1(S_{0,5})$ are of the form

$$\mathfrak{X}_1(S_{0,5}) = \bigsqcup_{s \in \{+1,-1\}^5} \mathfrak{X}_{1,s}(S_{0,5}).$$

However, $\mathfrak{X}_{1,s}(S_{0,5})$ is empty for some signs of punctures, so we need to check which signs of punctures are possible.

We use the same notation for signs of triangles and signs of punctures as defined in the previous section.

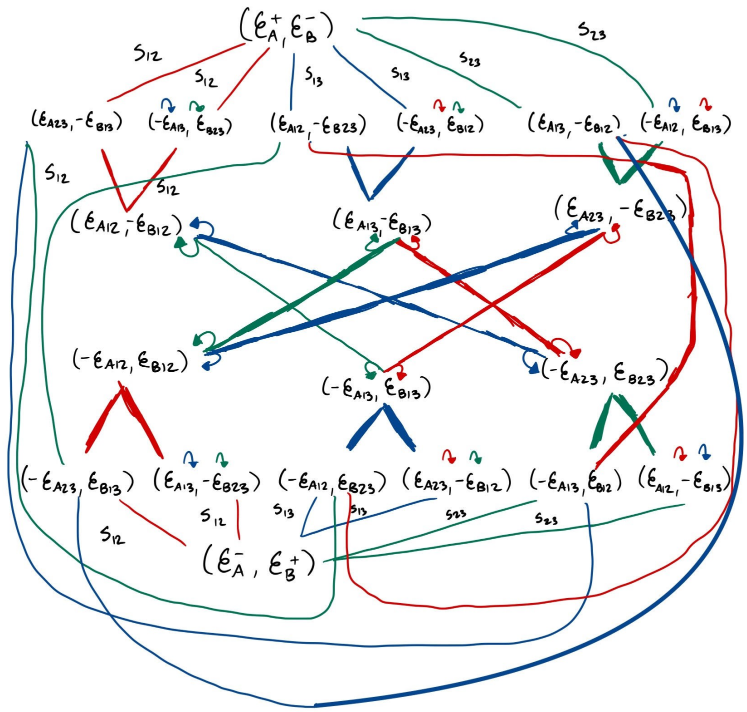

FIGURE 6. Pictured are the possible triangle signs in the case $e(\rho) = 0$ and how to switch between them.

The possible signs of triangles in the case $e(\rho) = 1$ are now given by the tuples

$$\pm\,(\epsilon_A^+, -\epsilon_{Bij})$$
$$(\epsilon_{Aij}, \epsilon_{Bij})$$
$$(\epsilon_{Aij}, \epsilon_{Bik})$$

where $\{i,j\},\{i,k\} \subseteq \{1,2,3\}$ such that $j \neq k$.

Theorem 5.9. *Fix a pyramidal triangulation Δ. Up to symmetry, we have the following characterization of the possible signs of punctures when $e(\rho)$=1:*

(1) Let ρ assign signs to the triangles according to $(\epsilon_A^+, -\epsilon_{Bij})$. Then ρ gives signs to the punctures according to either
- (s_A, s_+), (s_A, s_t),
- (s_+, s_+), (s_+, s_t), $(s_+, -s_t)$,

for each $1 \leq t \leq 3$.

(2) Let ρ assign signs to the triangles $(\epsilon_{Aij}, \epsilon_{Bij})$ for any $\{i,j\} \subseteq \{1,2,3\}$. Set $t = \{1,2,3\} \backslash \{i,j\}$. Then ρ gives signs to the punctures according to either
- (s_-, s_+),

- (s_+, s_+), (s_+, s_i), (s_+, s_j), *or* $(s_+, -s_t)$,
- (s_A, s_+), (s_A, s_i), (s_A, s_j),
- (s_B, s_+), (s_B, s_i), (s_B, s_j).

(3) Let ρ assign signs to the triangles $(\epsilon_{Aij}, \epsilon_{Bik})$ where $\{i, j, k\} = \{1, 2, 3\}$ and $j \neq k$. Then ρ gives signs to the punctures according to either
- $(s_+, s_+), (s_+, \pm s_t)$ *for each* $1 \leq 2 \leq 3$,
- (s_A, s_+), (s_A, s_i), (s_A, s_j)
- (s_B, s_+), (s_B, s_i), (s_B, s_k)
- (s_-, s_+).

Proof. Due to symmetry, there are three distinct cases to check.

- **Case** $(\epsilon_A^+, -\epsilon_{B12})$**.**

In this case, the signs of the punctures are given by the signs of the equations

$$\begin{aligned}
v_A &: X_1 + X_2 + X_3, \\
v_B &: Y_3 - Y_2 - Y_1, \\
v_1 &: X_1(Y_2 - Y_3) + Y_1(X_3 + X_2), \\
v_2 &: X_2(Y_1 - Y_3) + Y_2(X_3 + X_1), \\
v_3 &: X_3(-Y_1 - Y_2) + Y_3(X_1 + X_2).
\end{aligned}$$

We have the relation

$$v_1 + v_2 + v_3 = 2(X_1Y_2 + X_2Y_1) > 0.$$

Therefore, v_1, v_2, and v_3 cannot all be negative. We also have that v_A is always positive, and v_B could be positive or negative.

Finally, we have the following calculations for $v_B < 0$:

$$\begin{aligned}
v_1 &= X_1(Y_2 - Y_3) + Y_1(X_3 + X_2) \\
&= X_1(Y_2 - Y_3) + Y_1 - X_1Y_1 \\
&= X_1(Y_2 - Y_3 - Y_1) + Y_1 \\
&> -2X_1Y_1 + Y_1 \\
&= Y_1(X_2 + X_3 - X_1)
\end{aligned}$$

using the relation that $Y_2 - Y_3 > Y_1$ by v_B and the fact that $X_1 + X_2 + X_3 = 1$. As a result, we have that if $v_1 < 0$, then $X_1 > X_2 + X_3$.

On the other hand, we also have

$$\begin{aligned}
v_2 &= X_2(Y_1 - Y_3) + Y_2(X_1 + X_3) \\
&= X_2(Y_1 - Y_3) + Y_2 - Y_2X_2 \\
&= X_2(Y_1 - Y_3 - Y_2) + Y_2 \\
&> -2X_2Y_2 + Y_2 \\
&= Y_2(X_1 + X_3 - X_2)
\end{aligned}$$

using the relation that $Y_2 - Y_1 > Y_3$ by v_B and the fact that $X_1 + X_2 + X_3 = 1$. As a result, we have that if $v_2 < 0$, then $X_2 > X_1 + X_3$. However, only one of these possibilities may occur, so v_1 and v_2 cannot both be negative.

- **Case** $(\epsilon_{A12}, \epsilon_{B12})$**.**

In this case, the signs of the punctures are given by the signs of the equations

$$\begin{aligned} v_A &: X_1 + X_2 + X_3, \\ v_B &: Y_1 + Y_2 - Y_3, \\ v_1 &: -X_1 v_B - Y_1 v_A + 2X_1Y_1, \\ v_2 &: -X_2 v_B - Y_2 v_A + 2X_2Y_2, \\ v_3 &: X_3(Y_1 + Y_2) + Y_3(X_1 + X_2). \end{aligned}$$

We note that $v_3 > 0$. Furthermore, when both v_A and v_B are negative, we have that v_1 and v_2 must be positive.

Finally, in the case $v_A > 0$ and $v_B < 0$, we have the calculation

$$\begin{aligned} v_1 + v_2 &= X_1Y_3 - X_1Y_2 + Y_1X_3 - Y_1X_2 + X_2Y_3 - X_2Y_1 + Y_2X_3 - Y_2X_1 \\ &= X_3(Y_1 + Y_2) + Y_3(X_1 + X_2) - 2(X_2Y_1 + X_1Y_2) \\ &> 2(X_3(Y_1 + Y_2) - X_2Y_1 - X_1Y_2) \\ &= 2(Y_1(X_3 - X_2) + Y_2(X_3 - X_1)) \\ &> 2(X_1Y_1 + Y_2X_1). \end{aligned}$$

Therefore, v_1 and v_2 are both not negative. Similarly, when $v_B < 0$ and $v_A > 0$, we also have that v_1 and v_2 cannot both be negative.

- **Case** $(\epsilon_{A12}, \epsilon_{B13})$**.**

In this case, the signs of the punctures are given by the signs of the equations

$$\begin{aligned} v_A &: X_1 + X_2 + X_3, \\ v_B &: Y_1 + Y_3 - Y_2, \\ v_1 &: -X_1 v_B - Y_1 v_A + 2X_1Y_1, \\ v_2 &: X_2 v_B - Y_2 v_A + 2X_2Y_2, \\ v_3 &: -X_3 v_B + Y_3 v_A + 2X_3Y_3. \end{aligned}$$

We have the relation $v_1 + v_2 + v_3 > 0$. Therefore, we cannot have v_1, v_2, and v_3 all positive.

∘ When v_A and v_B are positive, v_1, v_2, and v_3 can have any sign besides all negative.

∘ When v_A is positive and v_B is negative, then v_3 must be positive.

Furthermore, we have the calculation

$$\begin{aligned} v_1 + v_2 &= X_1(Y_2 - Y_3) + Y_1(X_3 - X_2) + X_2(Y_3 + Y_1) + Y_2(X_3 - X_1) \\ &> X_1(Y_2 - Y_3) + X_1Y_1 + X_2(Y_3 + Y_1) + X_2Y_2 \\ &= X_1(Y_2 + Y_1 - Y_3) + X_2(Y_1 + Y_2 + Y_3) \qquad > 0. \end{aligned}$$

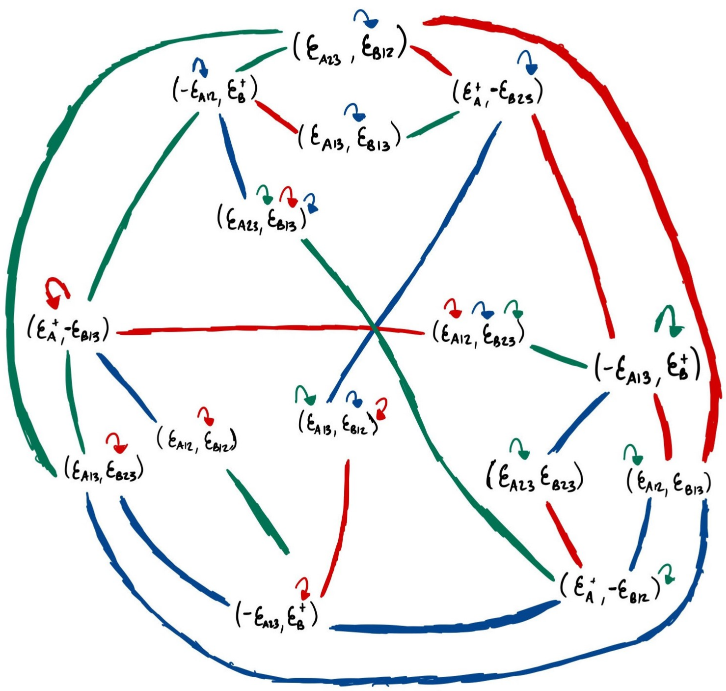

FIGURE 7. Pictured are the possible triangle signs in the case $e(\rho) = 1$ and how to switch between them.

So v_1 and v_2 cannot both be negative.

∘ When v_B is positive and v_A is negative, then v_2 must be positive. Furthermore, we have the calculation

$$\begin{aligned} v_1 + v_3 &= X_1(Y_2 - Y_3) + Y_1(X_3 - X_2) + X_3(Y_2 - Y_1) + Y_3(X_2 + X_1) \\ &> X_1(Y_2 - Y_3) + X_1Y_1 + X_3Y_3 + Y_3(X_2 + X_1) \\ &= X_1(Y_2 + Y_1 - Y_3) + Y_3(X_1 + X_2 + X_3) & > 0. \end{aligned}$$

Therefore v_1 and v_3 cannot both be negative.

∘ Finally, when v_A and v_B are both negative, then v_1, v_2, and v_3 must be positive. This is due to the fact that

$$\begin{aligned} v_1 &= X_1(Y_2 - Y_3) + Y_1(X_3 - X_2), \\ v_2 &= X_2(Y_1 + Y_3) + Y_2(X_3 - X_1), \\ v_3 &= X_3(Y_2 - Y_1) + Y_3(X_1 + X_2), \end{aligned}$$

where v_A and v_B give that $X_3 > X_1 + X_2$ and $Y_2 > Y_1 + Y_3$. □

In total, there are 16 different possible signs of the punctures for representations in the Euler class 1 component. We will now prove that the signs of the punctures determine the connected component of a representation in $\mathfrak{X}_1(S_{0,5})$.

Theorem 5.10. *The following subsets of* $\mathfrak{X}_1(S_{0,5})$ *are connected:*

- $\mathfrak{X}_{1,(s_+,s_+)}(S_{0,5})$,
- $\mathfrak{X}_{1,(s_+,s_i)}(S_{0,5})$, $\mathfrak{X}_{1,(s_A,s_+)}(S_{0,5})$, $\mathfrak{X}_{1,(s_B,s_+)}(S_{0,5})$,
- $\mathfrak{X}_{1,(s_A,s_i)}(S_{0,5})$, $\mathfrak{X}_{1,(s_B,s_i)}(S_{0,5})$, $\mathfrak{X}_{1,(s_+,-s_i)}(S_{0,5})$, $\mathfrak{X}_{1,(s_-,s_+)}(S_{0,5})$,

for each $i \in \{1,2,3\}$.

Proof. Many of these cases are symmetric, and many reduce to other cases following a single pyramidal switch. In total, there are 3 cases to check.

Fix a pyramidal triangulation Δ. To show the connectedness of $\mathfrak{X}_{k,s}(S_{0,5})$, it is enough to show that any $p, q \in \mathfrak{X}_{k,s}(S_{0,5})$ can be connected with a path.

Let $(X_1, X_2, X_3, Y_1, Y_2, Y_3)$ be the pyramidal coordinates for p and let ϵ be the signs of triangles for p. Let $(X_1', X_2', X_3', Y_1', Y_2', Y_3')$ be the pyramidal coordinates for q and let ϵ' be the signs of triangles for q. If $\epsilon = \epsilon'$, then p and q lie in the same path component of $\mathfrak{X}_\Delta(S_{0,5}, \epsilon)$. When $\epsilon \neq \epsilon'$, we show that there exists a pyramidal triangulation Δ' such that p and q are in the same path component of $\mathfrak{X}_{\Delta'}(S_{0,5})$.

In the following 3 cases, we will switch which signs of punctures we will check. Within the subcases, we are considering different triangle signs for p and q.

• **Case 1:** $\mathfrak{X}_{1,(s_+,s_+)}(S_{0,5})$ **is connected.**

Let $p, q \in \mathfrak{X}_{1,(s_+,s_+)}(S_{0,5})$. In fact, (s_+, s_+) is a possible sign for the punctures for any possible signs of triangles. Up to symmetry, we have 5 subcases to check for connectivity of $\mathfrak{X}_{1,(s_+,s_+)}(S_{0,5})$.

∘ Subcase 1.1: Suppose $\epsilon = (\epsilon_A^+, -\epsilon_{B13})$ and $\epsilon' = (\epsilon_{A23}, \epsilon_{B12})$.

We have that p satisfies the inequalities

$$\begin{aligned}
v_A &: X_1 + X_2 + X_3 > 0,\\
v_B &: Y_2 - Y_1 - Y_3 > 0,\\
v_1 &: X_1(Y_3 - Y_2) + Y_1(X_3 + X_2) > 0,\\
v_2 &: Y_2 - X_2 > 0,\\
v_3 &: X_3(Y_1 - Y_2) + Y_3(X_2 + X_1).
\end{aligned}$$

There is a representation with pyramidal coordinates given by

$$(X_1 = \frac{1}{4}, X_2 = \frac{5}{12}, X_3 = \frac{1}{3}, Y_1 = \frac{1}{4}, Y_2 = \frac{1}{2}, Y_3 = \frac{1}{4})$$

which satisfies the above inequalities as well as the relation $X_3Y_2 - X_2Y_3 > 0$. As p lies in the same path component as this representation, without loss of generality suppose p has these pyramidal coordinates.

Similarly, q satisfies the inequalities

$$\begin{aligned}
v_A &: X_2' + X_3' - X_1' > 0,\\
v_B &: Y_2' + Y_1' - Y_3' > 0,\\
v_1 &: X_1'(Y_3' - Y_2') + Y_1'(X_3' + X_2') > 0,\\
v_2 &: X_2'(Y_3' - Y_1') + Y_2'(X_1' - X_3') > 0,\\
v_3 &: X_3'(Y_1' + Y_2') + Y_3'(X_1' - X_2').
\end{aligned}$$

There is a representation with pyramidal coordinates given by

$$(X_1' = \frac{5}{12}, X_2' = \frac{1}{3}, X_3' = \frac{1}{4}, Y_1' = \frac{1}{6}, Y_2' = \frac{3}{8}, Y_3' = \frac{11}{24}$$

which satisfies the above inequalities as well as the relation $X_2'Y_3' - X'3Y_2' > 0$. Again q lies in the same path component as this representation, so suppose q has these pyramidal coordinates.

Let Δ' be the resulting pyramidal triangulation from the pyramid switch S_{23}. With respect to Δ', both p and q give signs of triangles $(\epsilon_{A13}, \epsilon_{B23})$ and signs of punctures (s_+, s_+), so p and q are connected.

∘ Subcase 1.2: Suppose $\epsilon = (\epsilon_{A23}, \epsilon_{B13})$ and $\epsilon' = (\epsilon_A^+, -\epsilon_{B13})$.

We have already verified that q can give signs of triangles $(\epsilon_{A13}, \epsilon_{B23})$ after the switch S_{23} in subcase 1.1. Furthermore, we note that there is a representation in the same path component as p with pyramidal coordinates

$$(X_1 = \frac{11}{24}, X_2 = \frac{3}{8}, X_3 = \frac{1}{6}, Y_1 = \frac{1}{3}, Y_2 = \frac{5}{12}, Y_3 = \frac{1}{4}$$

which we can verify satisfies the inequalities coming from the signs of punctures and furthermore satisfies the relation $X_2Y_3 - X_3Y_2 > 0$.

Let Δ' be the resulting pyramidal triangulation from the pyramid switch S_{23}. With respect to Δ', p gives signs of triangles $(\epsilon_A^+, -\epsilon_{B12})$ while q gives signs of triangles $(\epsilon_{A13}, \epsilon_{B23})$. Both p and q give signs of punctures (s_+, s_+). We reduce to a case that is symmetric to subcase 1.1.

∘ Subcase 1.3: $\epsilon = (\epsilon_{A23}, \epsilon_{B13})$ and $\epsilon' = (-\epsilon_{A12}, \epsilon_B^+)$.

We note that p satisfies the equations

$$\begin{aligned}
v_A &: X_3 + X_2 - X_1 > 0,\\
v_B &: Y_3 + Y_1 - Y_2 > 0,\\
v_1 &: X_1(Y_2 - Y_3) + Y_1(X_2 + X_3),\\
v_2 &: X_2(Y_3 + Y_1) + Y_2(X_1 - X_3),\\
v_3 &: X_3(Y_2 - Y_1) + Y_3(X_1 - X_2).
\end{aligned}$$

Note that the representation given by the pyramidal coordinates

$$(X_1 = \frac{11}{24}, X_2 = \frac{1}{6}, X_3 = \frac{3}{8}, Y_1 = \frac{1}{3}, Y_2 = \frac{5}{12}, Y_3 = \frac{1}{4}$$

satisfies the above inequalities, so without loss of generality we may assume these are the coordinates for p. Furthermore, we have that $X_3Y_1 - X_1Y_3 > 0$.

Furthermore, there is a representation in the same path component as q which satisfies the relation $X_1'Y_3' - X_3'Y_1' > 0$. The argument is similar to the one presented in subcase 1.1.

Let Δ' be the resulting pyramidal triangulation from the pyramid switch S_{13}. With respect to Δ', both p and q give signs of triangles $(-\epsilon_{A12}, \epsilon_B^+)$, and both p and q give signs of punctures (s_+, s_+). Therefore p and q are connected by a path.

∘ Subcase 1.4: Let $\epsilon = (\epsilon_{A13}, \epsilon_{B13})$ and $\epsilon' = (\epsilon_{A23}, \epsilon_{B13})$.

The resultant signs of triangles after the pyramidal switch S_{12} is forced in both cases, regardless of the sign of $X_1Y_2 - X_2Y_1$ and $X_1'Y_2' - X_2'Y_1'$.

Let Δ' be the resulting pyramidal triangulation from the pyramid switch S_{12}. With respect to Δ', p gives signs of triangles $(-\epsilon_{A12}, \epsilon_B^+)$ while q gives signs of triangles $(\epsilon_{A23}, \epsilon_{B13})$. Both p and q give signs of punctures (s_+, s_+). We reduce to subcase 1.3.

∘ Subcase 1.5: Let $\epsilon = (\epsilon_A^+, -\epsilon_{B23})$ and $\epsilon' = (\epsilon_{A23}, \epsilon_{B13})$.

We have already verified the sign of $X_3Y_2 - X_2Y_3$ as this case is symmetric to the one in subcase 1.1. The signs of triangles for q is always forced.

Let Δ' be the resulting pyramidal triangulation from the pyramid switch S_{12}. With respect to Δ', p gives signs of triangles $(\epsilon_{A23}, \epsilon_{B12})$ while q gives signs of triangles $(\epsilon_{A23}, \epsilon_{B13})$. Both p and q give signs of punctures (s_+, s_+). By subcases 1.1 and 1.2, we know that p and q are connected.

• **Case 2: $\mathfrak{X}_{1,(s_+,s_1)}(S_{0,5})$ is connected.**

Let $p, q \in \mathfrak{X}_{1,(s_+,s_1)}(S_{0,5})$. Note that (s_+, s_1) appears with every possible signs of triangles besides $(\epsilon_{A23}, \epsilon_{B23})$. Therefore, we are able to follow a very similar process as case 1 as far as which switches to consider, but we need to make sure that we are still able to give the same signs of triangles after the switch. Furthermore, the connectivity of the signs of the punctures (s_A, s_+) and (s_B, s_+) both reduce to the case (s_+, s_i) for some $i \in \{1, 2, 3\}$. This is because if $\rho \in \mathfrak{X}_\Delta(S_{0,5})$ gives signs of punctures (s_A, s_+), then after pyramidal switch S_{12}, ρ gives signs of punctures (s_+, s_2) with respect to Δ'. Therefore, we can reduce to case 2 computed below. Similar for (s_B, s_+).

∘ Subcase 2.1: Suppose $\epsilon = (\epsilon_A^+, -\epsilon_{B13})$ and $\epsilon' = (\epsilon_{A23}, \epsilon_{B12})$.

We have that p satisfies the inequalities

$$\begin{aligned}
v_A &: X_1 + X_2 + X_3 > 0,\\
v_B &: Y_2 - Y_1 - Y_3 > 0,\\
v_1 &: X_1(Y_3 - Y_2) + Y_1(X_3 + X_2) < 0,\\
v_2 &: Y_2 - X_2 > 0,\\
v_3 &: X_3(Y_1 - Y_2) + Y_3(X_2 + X_1).
\end{aligned}$$

There is a representation with pyramidal coordinates given by

$$(X_1 = \frac{2}{3}, X_2 = \frac{1}{4}, X_3 = \frac{1}{12}, Y_1 = \frac{1}{4}, Y_2 = \frac{7}{12}, Y_3 = \frac{1}{6})$$

which satisfies the above inequalities as well as the relation $X_3Y_2-X_2Y_3 > 0$. As p lies in the same path component as this representation, without loss of generality suppose p has these pyramidal coordinates.

Similarly, q satisfies the inequalities

$$\begin{aligned} v_A &: X'_2 + X'_3 - X'_1 > 0, \\ v_B &: Y'_2 + Y'_1 - Y'_3 > 0, \\ v_1 &: X'_1(Y'_3 - Y'_2) + Y'_1(X'_3 + X'_2) < 0, \\ v_2 &: X'_2(Y'_3 - Y'_1) + Y'_2(X'_1 - X'_3) > 0, \\ v_3 &: X'_3(Y'_1 + Y'_2) + Y'_3(X'_1 - X'_2). \end{aligned}$$

There is a representation with pyramidal coordinates given by

$$(X'_1 = \frac{11}{24}, X'_2 = \frac{1}{3}, X'_3 = \frac{5}{24}, Y'_1 = \frac{3}{23}, Y'_2 = \frac{12}{23}, Y'_3 = \frac{8}{23})$$

which satisfies the above inequalities as well as the relation $X'_2Y'_3 - X'3Y'_2 > 0$. Again q lies in the same path component as this representation, so suppose q has these pyramidal coordinates.

Let Δ' be the resulting pyramidal triangulation from the pyramid switch S_{23}. With respect to Δ', both p and q give signs of triangles $(\epsilon_{A13}, \epsilon_{B23})$ and signs of punctures (s_B, s_+), so p and q are connected.

$\circ$ Subcase 2.2: Suppose $\epsilon = (\epsilon_{A23}, \epsilon_{B13})$ and $\epsilon' = (\epsilon_A^+, -\epsilon_{B13})$.

We have already verified that q can give signs of triangles $(\epsilon_{A13}, \epsilon_{B23})$ after the switch S_{23} in subcase 2.1. Furthermore, by a similar argument as the one presented in subcase 2.1, we note that there is a representation in the same path component as p which satisfies the inequalities coming from the signs of punctures and furthermore satisfies the relation $X_2Y_3 - X_3Y_2 > 0$.

Let Δ' be the resulting pyramidal triangulation from the pyramid switch S_{23}. With respect to Δ', p gives signs of triangles $(\epsilon_A^+, -\epsilon_{B12})$ while q gives signs of triangles $(\epsilon_{A13}, \epsilon_{B23})$. Both p and q give signs of punctures (s_+, s_1). We reduce to a case that is symmetric to subcase 2.1.

$\circ$ Subcase 2.3: $\epsilon = (\epsilon_{A23}, \epsilon_{B13})$ and $\epsilon' = (-\epsilon_{A12}, \epsilon_B^+)$.

We note that p satisfies the equations

$$\begin{aligned} v_A &: X_3 + X_2 - X_1 > 0, \\ v_B &: Y_3 + Y_1 - Y_2 > 0, \\ v_1 &: X_1(Y_2 - Y_3) + Y_1(X_2 + X_3), \\ v_2 &: X_2(Y_3 + Y_1) + Y_2(X_1 - X_3), \\ v_3 &: X_3(Y_2 - Y_1) + Y_3(X_1 - X_2). \end{aligned}$$

In fact, we have that

$$v_2 + v_3 - v_1 = X_3Y_1 - X_1Y_3$$

is positive, as the signs of v_1, v_2, and v_3 are given by s_1.

Similarly, q satisfies the equations

$$v'_2 + v'_3 - v'_1 = X'_3Y'_1 - X'_1Y'_3,$$

which tells us that $X'_3Y'_1 - X'_1Y'_3$ is positive for q.

Let Δ' be the resulting pyramidal triangulation from the pyramid switch S_{13}. With respect to Δ', both p and q give signs of triangles $(-\epsilon_{A12}, \epsilon_B^+)$, and both p and q give signs of punctures (s_A, s_+). Therefore p and q are connected by a path.

∘ Subcase 2.4: Let $\epsilon = (\epsilon_{A13}, \epsilon_{B13})$ and $\epsilon' = (\epsilon_{A23}, \epsilon_{B13})$.

The resultant signs of triangles after the pyramidal switch S_{12} is forced in both cases, regardless of the sign of $X_1Y_2 - X_2Y_1$ and $X'_1Y'_2 - X'_2Y'_1$.

Let Δ' be the resulting pyramidal triangulation from the pyramid switch S_{12}. With respect to Δ', p gives signs of triangles $(-\epsilon_{A12}, \epsilon_B^+)$ while q gives signs of triangles $(\epsilon_{A23}, \epsilon_{B13})$. Both p and q give signs of punctures (s_B, s_+). We reduce to subcase 2.3.

∘ Subcase 2.5: Let $\epsilon = (\epsilon_A^+, -\epsilon_{B23})$ and $\epsilon' = (\epsilon_{A23}, \epsilon_{B13})$.

We have already verified the sign of $X_3Y_2 - X_2Y_3$ as this case is symmetric to the one in subcase 1.1. The signs of triangles for q is always forced.

Let Δ' be the resulting pyramidal triangulation from the pyramid switch S_{12}. With respect to Δ', p gives signs of triangles $(\epsilon_{A23}, \epsilon_{B12})$ while q gives signs of triangles $(\epsilon_{A23}, \epsilon_{B13})$. Both p and q give signs of punctures (s_B, s_+). By subcases 2.1 and 2.2, we know that p and q are connected.

• **Case 3: $\mathfrak{X}_{1,(s_+,-s_1)}(S_{0,5})$ is connected.**

Let $p, q \in \mathfrak{X}_{1,(s_+,-s_1)}(S_{0,5})$. The signs of punctures $(s_+, -s_1)$ can appear with every possible signs of triangles besides $(\epsilon_{A12}, \epsilon_{B12})$ and $(\epsilon_{A13}, \epsilon_{B13})$. However, we need to use new strategies with the switches in order to connect $(s_+, -s_1)$. This is because much more of the inequalities are forced to occur in this case. Furthermore, we note that (s_-, s_+) reduces to the case $(s_+, -s_3)$ after the switch S_{12}. Similarly, the case (s_A, s_i) and (s_B, s_i) reduce to cases symmetric to case 3 following switches, for each $i \in \{1, 2, 3\}$.

∘ Subcase 3.1: Suppose $\epsilon = (\epsilon_A^+, -\epsilon_{B13})$ and $\epsilon' = (\epsilon_{A23}, \epsilon_{B12})$.

In this case, p satisfies the inequality

$$v_3 = X_3Y_1 - X_3Y_2 + Y_3X_1 + Y_3X_2 < 0.$$

Re-arranging, we see that $X_3Y_2 - X_2Y_3 > 0$.

Similarly, q satisfies the inequality

$$v'_3 = X'_3Y'_2 + X'_3Y'_1 + Y'_3X'_1 - X'_2Y'_3 < 0.$$

Re-arranging, we see that $X'_2Y'_3 - X'_3Y'_2 > 0$.

Therefore, after the switch S_{13}, both p and q gives signs of triangles $(\epsilon_{A13}, \epsilon_{B23})$ with respect to Δ'. Furthermore, p and q both give signs of punctures (s_B, s_2). Therefore, p and q are connected by a path.

∘ Subcase 3.2: Suppose $\epsilon = (\epsilon_{A23}, \epsilon_{B13})$ and $\epsilon' = (-\epsilon_{A12}, \epsilon_B^+)$.

In this case, p satisfies the relation

$$v_2 + v_3 - v_1 = 2(X_1Y_3 - X_3Y_1)$$

which is negative as the signs of v_1, v_2, and v_3 are given by $-s_1$.

Similarly, q satisfies the relation

$$v_2' + v_3' - v_1' = 2(X_3'Y_1' - X_1'Y_3')$$

which is again negative.

Therefore, after the switch S_{13}, both p and q gives signs of triangles $(-\epsilon_{A12}, \epsilon_B^+)$ with respect to Δ'. Furthermore, p and q both give signs of punctures (s_B, s_2). Therefore, p and q are connected by a path.

◦ Subcase 3.3: Suppose $\epsilon = (\epsilon_{A23}, \epsilon_{B12})$ and $\epsilon' = (-\epsilon_B^+)$.

In this case, p satisfies the relation

$$v_2 = (X_3Y_2 - X_2Y_3) + (X_1Y_3 - X_3Y_1) < 0.$$

By subcase 3.1, however, we know that $X_3Y_2 - X_3Y_3$ is positive. Therefore, $X_1Y_3 - X_3Y_1$ must be negative.

On the other hand, q satisfies the relation

$$v_2' + v_3' - v_1' = 2(X_1'Y_2' - X_2'Y_1').$$

Therefore, after the switch S_{12}, both p and q gives signs of triangles $(\epsilon_A^+, -\epsilon_{B23})$ with respect to Δ'. Furthermore, p and q both give signs of punctures (s_A, s_3). Therefore, p and q are connected by a path.

◦ Subcase 3.4: Suppose $\epsilon = (\epsilon_A^+, \epsilon_{B13})$ and $\epsilon' = (\epsilon_A^+, -\epsilon_{B12})$.

We have already seen in subcase 3.1 that p satisfies the relation

$$X_3Y_2 - X_2Y_3 > 0,$$

After the switch S_{23}, p gives signs of triangles $(\epsilon_{A13}, \epsilon_{B23})$ while q gives signs of triangles either $(\epsilon_A^+, -\epsilon_{B12})$ or $(\epsilon_{A23}, \epsilon_{B13})$. Regardless of the signs of triangles of q, by reasoning symmetric to subcases 3.1, 3.2, and 3.3, we have that p and q are connected.

◦ Subcase 3.5: Suppose $\epsilon = (\epsilon_{A23}, \epsilon_{B23})$ and $\epsilon' = (\epsilon_{A12}, \epsilon_{B23})$.

The signs of the triangles for both p and q following the switch S_{12} is forced regardless of the signs of $X_1Y_2 - X_2Y_1$ and $X_1'Y_2' - X_2'Y_1'$.

After the switch S_{12}, p gives signs of triangles $(-\epsilon_{A13}, \epsilon_B^+)$ while q gives signs of triangles either $(\epsilon_{A12}, -\epsilon_{B23})$. Both p and q give signs of punctures $(s_+, -s_1)$. By subcases 3.1, 3.2, 3.3, and 3.4, we have that p and q are connected. □

5.4. **The Connected Components of $\mathfrak{X}_2(S_{0,5})$.** In this section, we show that the path components of $\mathfrak{X}_2(S_{0,5})$ are of the form

$$\mathfrak{X}_2(S_{0,5}) = \bigsqcup_{s\in\{+1,-1\}^5} \mathfrak{X}_{2,s}(S_{0,5}).$$

However, $\mathfrak{X}_{2,s}(S_{0,5})$ is empty for some signs of punctures, so we first need to check which signs of punctures are possible.

We use the same notation for signs of triangles and signs of punctures as defined in the previous section.

The possible signs of triangles in the case $e(\rho) = 2$ are now given by the tuples

$$\pm(\epsilon_A^+, \epsilon_{Bij})$$

where $\{i, j\} \subseteq \{1, 2, 3\}$.

Finally, we note that when defining the pyramidal switch S_{ij} where $j = (123)i$, we re-labeled the vertices of the pyramidal triangulation according to the permuation $(Ai)(Bj)$.

For the following proof, given the same i, j as above, we introduce the pyramidal switch S_{ij}^* to be be a diagonal switch at e_{ij} followed by vertex re-labeling given by the permutation $(Aj)(Bi)$. We need these 3 other pyramidal switches to complete the proof of connectivity in the $e(\rho) = 2$ case.

Theorem 5.11. *Fix a pyramidal triangulation Δ. Up to symmetry, we have the following characterization of the possible signs of punctures when $e(\rho)$=2:*

(1) Let ρ assign signs to the triangles according to $(\epsilon_A^+, \epsilon_{Bij})$. Then ρ gives signs to the punctures according to either
- $(s_+, s_i), (s_+, s_j)$,
- (s_A, s_+),
- (s_+, s_+).

Proof. All cases are symmetric to the case $(\epsilon_A^+, \epsilon_{B12})$. In this case, the signs of the punctures are given by the inequalities

$$\begin{aligned} v_A &: X_1 + X_2 + X_3, \\ v_B &: Y_1 + Y_2 - Y_3, \\ v_1 &: -X_1 v_B - Y_1 v_A, \\ v_2 &: -X_2 v_B + Y_1 v_A, \\ v_3 &: X_3 v_B + Y_3 v_A. \end{aligned}$$

If $v_B > 0$, then $v_3 > 0$. Furthermore, we have that $v_1 + v_2 - v_3 = 0$, so v_1 and v_2 cannot both be negative.

If $v_B > 0$, then $v_1 > 0$ and $v_2 > 0$, but this also means $v_3 > 0$ by the relation $v_1 + v_2 - v_3 = 0$. $\square$

In total, there are six possible signs of the punctures for representations in the Euler class 2 component. We will now prove that the signs of the punctures determine the connected component of a representation in $\mathfrak{X}_2(S_{0,5})$.

Theorem 5.12. *The following subsets of $\mathfrak{X}_2(S_{0,5})$ are connected:*

(1) $\mathfrak{X}_{2,(s_+,s_i)}(S_{0,5})$,
(2) $\mathfrak{X}_{2,(s_A,s_+)}(S_{0,5})$,
(3) $\mathfrak{X}_{2,(s_B,s_+)}(S_{0,5})$,
(4) $\mathfrak{X}_{2,(s_+,s_+)}(S_{0,5})$,

for each $i \in \{1, 2, 3\}$.

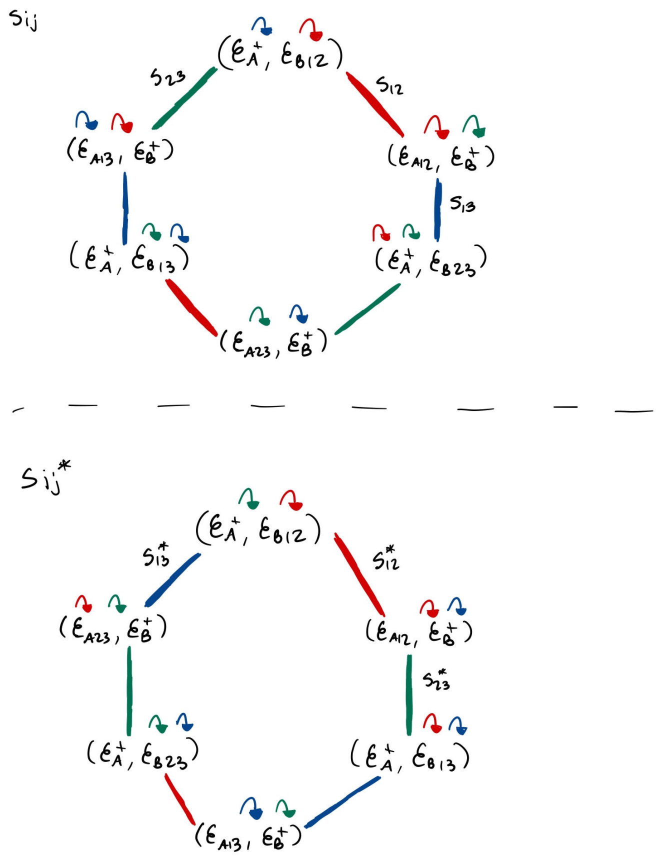

FIGURE 8. Pictured are the possible triangle signs in the case $e(\rho) = 2$ and how to switch between them.

Proof. Fix a pyramidal triangulation Δ. To show the connectedness of $\mathfrak{X}_{k,s}(S_{0,5})$, it is enough to show that any $p, q \in \mathfrak{X}_{k,s}(S_{0,5})$ can be connected with a path.

Let $(X_1, X_2, X_3, Y_1, Y_2, Y_3)$ be the pyramidal coordinates for p and let ϵ be the signs of triangles for p. Let $(X_1', X_2', X_3', Y_1', Y_2', Y_3')$ be the pyramidal coordinates for q and let ϵ' be the signs of triangles for q. If $\epsilon = \epsilon'$, then p and q lie in the same path component of $\mathfrak{X}_\Delta(S_{0,5}, \epsilon)$. When $\epsilon \neq \epsilon'$, we show that there exists a pyramidal triangulation Δ' such that p and q are in the same path component of $\mathfrak{X}_{\Delta'}(S_{0,5})$.

Up to symmetry, there are only 2 cases to check. In the following 2 cases, we will switch which signs of punctures we will check. Within the subcases, we are considering different triangle signs for p and q.

Furthermore, we note that if a representation gives signs of punctures (s_A, s_+) or (s_B, s_+), then following any pyramidal switch, the new signs of punctures with respect to Δ' will given by (s_+, s_i) for some $i \in \{1, 2, 3\}$, and therefore these cases reduce to the calculation below.

• **Case 1: $\mathfrak{X}_{2,(s_+,s_1)}$ is connected.**

Let $p, q \in \mathfrak{X}_{2,(s_+,s_1)}$. The possible signs of the triangles for p and q are

$$(\epsilon_A^+, \epsilon_{B12}), (\epsilon_A^+, \epsilon_{B13}), (\epsilon_{A12}, \epsilon_B^+), (\epsilon_{A13}, \epsilon_B^+).$$

There are two subcases to check.

◦ Subcase 1.1: Suppose $\epsilon = (\epsilon_A^+, \epsilon_{B12})$ and $\epsilon' = (\epsilon_{A12}, \epsilon_B^+)$.

In this case, p satisfies the relation

$$v_1 = X_1Y_3 - X_1Y_2 + Y_1X_3 + Y_1X_2 < 0.$$

Re-arranging, we see that $X_1Y_2 - X_2Y_1 > 0$.

On the other hand, q satisfies the relation

$$v_1' = X_1'Y_3' + X_3'Y_1' + X_1'Y_2' - X_2'Y_1' < 0.$$

Re-arranging, we see that $X_2Y_1 - X_1Y_2 > 0$.

Let Δ' be the resulting pyramidal triangulation from the pyramid switch S_{12}. With respect to Δ', both p and q give signs of triangles $(\epsilon_{A12}, \epsilon_B^+)$ and both p and q give signs of punctures (s_B, s_+). Therefore p and q are connected.

◦ Subcase 1.2: Suppose $\epsilon = (\epsilon_A^+, \epsilon_{B12})$ and $\epsilon' = (\epsilon_A^+, \epsilon_{B13})$.

In this case, the signs of the triangles for p after the switch S_{23}^* are the same regardless of the sign of the quantity $X_2Y_3 - X_3Y_2$. Similarly, the signs of the triangles for q after the switch S_{23}^* are the same regardless of the sign of the quantity $X_2'Y_3' - X_3'Y_2'$.

Let Δ' be the resulting pyramidal triangulation from the pyramid switch S_{23}^*. With respect to Δ', p gives signs of triangles $(\epsilon_A^+, \epsilon_{B12})$ while q gives signs of triangles $(\epsilon_{A12}, \epsilon_B^+)$. Both p and q give signs of punctures (s_+, s_1). Therefore p and q are connected by reducing to subcase 1.1.

• **Case 2: $\mathfrak{X}_{2,(s_+,s_+)}$ is connected.**

Let $p, q \in \mathfrak{X}_{2,(s_+,s_+)}$. Every possible signs of triangles can appear with signs of punctures (s_+, s_+). There are two subcases to check. In this case, with the punctures all positive, we can choose which which direction we would like to switch, as we will see below.

◦ Subcase 2.1: Suppose $\epsilon = (\epsilon_A^+, \epsilon_{B12})$ and $\epsilon' = (\epsilon_{A12}, \epsilon_B^+)$.

Consider the representation

$$(X_1 = \frac{1}{3}, X_2 = \frac{1}{3}, X_3 = \frac{1}{3}, Y_1 = \frac{1}{4}, Y_2 = \frac{1}{3}, Y_3 = \frac{5}{12}).$$

This representation gives the same signs of punctures and same signs of triangles as p, and therefore lies in the same path component as p. Therefore without loss of generality, let the above be the pyramidal coordinates for p. We note that $X_1Y_2 - X_2Y_1 > 0$.

Consider the representation

$$(X_1' = \frac{1}{3}, X_2' = \frac{1}{4}, X_3' = \frac{5}{12}, Y_1 = \frac{1}{3}, Y_2 = \frac{1}{3}, Y_3 = \frac{1}{3}).$$

This representation gives the same signs of punctures and same signs of triangles as q, and therefore lies in the same path component as q. Therefore without loss of generality, let the above be the pyramidal coordinates for q.

We note that $X_1'Y_2' - X_2'Y_1' > 0$.

Let Δ' be the resulting pyramidal triangulation from the pyramid switch S_{12}^*. With respect to Δ', both p and q give signs of triangles $(\epsilon_{A12}, \epsilon_B^+)$. Both p and q give signs of punctures (s_+, s_+). Therefore p and q are connected by a path.

∘ Subcase 2.2: Suppose $\epsilon = (\epsilon_A^+, \epsilon_{B12})$ and $\epsilon' = (\epsilon_{A13}, \epsilon_B^+)$.

By a symmetric argument to subcase 2.1, we can find representations p and q giving signs of triangles $(\epsilon_A^+, \epsilon_{B12})$ and $(\epsilon_{A13}, \epsilon_B^+)$ respectively which both give signs of triangles $(\epsilon_{A13}, \epsilon_B^+)$ under the switch S_{23}^*. □

We note that the cases $e(\rho) = -1$ and $e(\rho) = -2$ are symmetric to $e(\rho) = 1$ and $e(\rho) = 2$ respectively. Therefore, in total, we have

$$20 + 2 * 16 + 2 * 6 = 64$$

non-Teichmüller connected components in $\mathfrak{X}(S_{0,5})$.

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