Example uniform distribution

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Suppose abilities are drawn from the uniform distribution with parameters $a > 0$ and $b > a$:

$$a_i \sim \text{Uniform}(a, b). \tag{1}$$

Identification

The $G^{-1}(.)$ function for $n = 2$ and $\alpha = 1$ is then:

$$G(\hat{a}) \equiv \hat{a}F(\hat{a}) = \hat{a}\frac{(\hat{a} - a)}{b - a} \tag{2}$$

Hence, the inverse $G^{-1}(\cdot)$ is

$$G^{-1}(u) = \frac{1}{2}(b - a)\left(-\frac{\sqrt{a^2 - 4au + 4bu}}{a - b} - \frac{a}{a - b}\right) \tag{3}$$

Important properties are:

1. $G(\cdot)$ is monotonic increasing in the parameter b (the upper limit of the uniform distribution).
2. $G(\cdot)$ is monotonic increasing in the parameter a (the lower limit of the distribution).
3. By the chain of derivatives, also the inverse $G^{-1}(\cdot)$ is monotonic increasing in the parameters a and b.

Our proof shows that when $G()$ is increasing monotonic in the distribution, then the model is *identifiable.* Let's check this result here.

By using $\theta = (a, b)$, the binomial probability $p(x, \theta)$ is then:

$$p(x, \theta) = \frac{b - G^{-1}(x, a, b)}{b - a} \tag{4}$$

I need to verify that there are no a, b combinations that give the same probability for any x, x' in X.

Estimation

The likelihood for $l = 1, ..., L$ iid rooms characterized by the vector x_l with Y_l participants is

$$\text{Likelihood} = \prod_{l=1}^{L} p(x_l, \theta)^{Y_l}[1 - p(x_l, \theta)]^{N_l - Y_l} \tag{5}$$

the binomial term can be ignored since it does not contain the parameters. The log-likelihood is then

$$\text{log-likelihood} = \sum_{l=1}^{L} Y_l \log p(x_l, \theta) + (N_l - Y_l) \log(1 - p(x_l, \theta)). \tag{6}$$

First order conditions:

$$\sum_{l=1}^{L} \left[\frac{Y_l}{p(x_l, \theta)} - \frac{(N_l - Y_l)}{(1 - p(x_l, \theta))} \right] \frac{\partial p(x_l, \theta)}{\partial \theta} = 0. \tag{7}$$

That is:

$$\sum_{l=1}^{L} \left[\frac{Y_l - N_l p(x_l, \theta)}{p(x_l, \theta)(1 - p(x_l, \theta))} \right] \frac{\partial p(x_l, \theta)}{\partial \theta} = 0. \tag{8}$$

Second order conditions:

$$\begin{aligned} &\sum_{l=1}^{L} \left[\frac{Y_l - N_l p(x_l, \theta)}{p(x_l, \theta)(1 - p(x_l, \theta))} \right] \frac{\partial^2 p(x_l, \theta)}{\partial \theta^2} + \\ &+ \left[-\frac{Y_l}{p(x_l, \theta)^2} + \frac{(N_l - Y_l)}{(1 - p(x_l, \theta)^2)} \right] \left(\frac{\partial p(x_l, \theta)}{\partial \theta} \right)^2 \leq 0. \end{aligned} \tag{9}$$

References