Races and Tournaments

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JEL Codes: D02, J4, L2, M5

1. Tournament Game Definition

■ Consider an n-player simultaneous tournament ($n \geq 2$). Each player draws an ``ability'', $a_i \in [0, m]$, where $m < 1$. Ability is distributed randomly according to distribution F. The distribution has continuous support with density f and is twice continuously differentiable. The distribution F is common knowledge while a player's ability is private information.

Player i chooses a costly bid, $q_i \in \mathbf{R}_+$.

$$
\begin{aligned}
c(a,q,t) &= \delta(a)\gamma(q)\eta(t) > 0 \\
c(a,0,t) &= 0 \\
c_a(a,q,t) &= \gamma(q)\eta(t)\frac{\partial\delta}{\partial a}(a) < 0 \\
c_q(a,q,t) &= \delta(a)\eta(t)\frac{\partial\gamma}{\partial q}(q) > 0 \\
c_t(a,q,t) &= \delta(a)\gamma(q)\frac{\partial\eta}{\partial t}(t) < 0
\end{aligned}
$$

$$
\begin{aligned}
c_{aq}(a,q,t) = c_{qa}(a,q,t) &= \eta(t)\frac{\partial\delta}{\partial a}(a)\frac{\partial\gamma}{\partial q}(q) < 0 \\
c_{at}(a,q,t) = c_{ta}(a,q,t) &= \gamma(q)\frac{\partial\delta}{\partial a}(a)\frac{\partial\eta}{\partial t}(t) > 0 \\
c_{qt}(a,q,t) = c_{tq}(a,q,t) &= \delta(a)\frac{\partial\gamma}{\partial q}(q)\frac{\partial\eta}{\partial t}(t) < 0
\end{aligned}
$$

$p < n$ prizes are awarded to the p bidders with the highest quality bids in rank order *at time* T^{max}. Bids are

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not observed by other players. The value of the prizes is strictly decreasing, $V_1 > V_2 > \ldots V_p$. Ties are broken randomly. Players are risk-neutral.

2. Tournament Equilibrium

■ Note that submitting a bid prior to T^{max} is strictly dominated by submitting the same bid at T^{max}. Since costs fall monotonically in time and signaling is not possible, later is always strictly preferred.

Assume that there exists a symmetric equilibrium bid function $q(a)$, strictly monotonically increasing in ability. Note that strategies mapping from ability to bids induce a distribution on bids. Denote the distribution arising form the symmetric equilibrium as G. Let r_i be the bid rank of the i^{th} player. Let

$$P_{j,n}(b) = \Pr\{r_i = j\} = \binom{n-1}{j-1}(1 - G(q))^{j-1}G(q)^{n-j}.$$

Since q is strictly monotone by assumption, there exists an inverse function $a(q)$ mapping abilities from bids. Then

$$P_{j,n}(q) = P_{j,n}(a) = \binom{n-1}{j-1}(1 - F(a(q)))^{j-1}F(a(q))^{n-j}.$$

The probability of winning prize j is closely related to order statistics. Let $F_{j,n}$ be the distribution of the j^{th} *lowest* value from a sample of n random variables distributed according to F. $F_{j,n}$ can be written as

$$F_{j,n} = \sum_{k=j}^{n}\binom{n}{k}F^k(1-F)^{n-k}.$$

The pdf of $F_{j,n}$ can be written as

$$f_{j,n} = \frac{n!}{(j-1)!(n-j)!}F^{j-1}(1-F)^{n-j}f.$$

It follows that

$$\begin{aligned}
P_{j,n} &= \binom{n-1}{j-1}(1-F)^{j-1}F^{n-j} = F_{n-j,n-1} - F_{n-j+1,n-1} \\
\frac{\partial P_{j,n}}{\partial a} &= f_{n-j,n-1} - f_{n-j+1,n-1}
\end{aligned}$$

Assume all other players are using the strategies defined by $q(a)$. Then the expected payoff for player i for bid q is

$$\pi(q) = \sum_{j=1}^{p} P_{j,n}(a(q))V_j - \delta(a_i)\eta(T^{max})\gamma(q)$$

FOCs imply that

$$\frac{\partial \pi}{\partial q}(q) = \sum_{j=1}^{p} \frac{\partial P_{j,n}}{\partial a}(a(q))\frac{\partial a}{\partial q}(q)V_j - \delta(a_i)\eta(T^{max})\frac{\partial \gamma}{\partial q}(q) = 0$$

Let $V_{p+1} = 0$ and $\Delta V_j = V_j - V_{j+1}$. Then the sum can be rearranged.

$$\begin{aligned}
\sum_{j=1}^{p} \Delta V_j f_{n-j,n-1}(a(q))\frac{\partial a}{\partial q}(q) - \delta(a_i)\eta(T^{max})\frac{\partial \gamma}{\partial q}(q) = 0 \\
\sum_{j=1}^{p} \Delta V_j f_{n-j,n-1}(a(q))\frac{\partial a}{\partial q}(q) &= \delta(a_i)\eta(T^{max})\frac{\partial \gamma}{\partial q}(q) \\
[\delta(a_i)\eta(T^{max})]^{-1}\sum_{j=1}^{p} \Delta V_j f_{n-j,n-1}(a(q))\frac{\partial a}{\partial q}(q) &= \frac{\partial \gamma}{\partial q}(q) \\
[\delta(a_i)\eta(T^{max})]^{-1}\sum_{j=1}^{p} \Delta V_j f_{n-j,n-1}(a(q)) &= \frac{\partial \gamma}{\partial q}(q)\frac{\partial a}{\partial q}(q)^{-1} \\
[\delta(a_i)\eta(T^{max})]^{-1}\sum_{j=1}^{p} \Delta V_j f_{n-j,n-1}(a(q)) &= \frac{\partial \gamma}{\partial q}(q)\frac{\partial q}{\partial a}(a(q)).
\end{aligned}$$

If q is a symmetric equilibrium, then the FOC must hold for all values of a_i.

$$\begin{aligned}
[\delta(a)\eta(T^{max})]^{-1}\sum_{j=1}^{p} \Delta V_j f_{n-j,n-1}(a(q)) &= \frac{\partial \gamma}{\partial q}(q(a))\frac{\partial q}{\partial a}(a(q)) \\
[\delta(a)\eta(T^{max})]^{-1}\sum_{j=1}^{p} \Delta V_j f_{n-j,n-1}(a(q)) &= \frac{\partial}{\partial a}\left[\gamma(q(a))\right].
\end{aligned}$$

Then the FOC is a separated differential equation and a closed form of q can be recovered.

$$\begin{aligned}
\frac{\partial}{\partial a}\left[\gamma(q(a))\right] &= [\delta(a)\eta(T^{max})]^{-1}\sum_{j=1}^{p}\Delta V_j f_{n-j,n-1}(a(q)) \\
\int_0^a \frac{\partial}{\partial a}\left[\gamma(q(z))\right]dz &= \int_0^a [\delta(z)\eta(T^{max})]^{-1}\sum_{j=1}^{p}\Delta V_j f_{n-j,n-1}(z)dz \\
\gamma(q(a)) &= \int_0^a [\delta(z)\eta(T^{max})]^{-1}\sum_{j=1}^{p}\Delta V_j f_{n-j,n-1}(z)dz - \gamma(q(0)) \\
\gamma(q(a)) &= \int_0^a [\delta(z)\eta(T^{max})]^{-1}\sum_{j=1}^{p}\Delta V_j f_{n-j,n-1}(z)dz \\
q(a) &= \gamma^{-1}\left\{\int_0^a [\delta(z)\eta(T^{max})]^{-1}\sum_{j=1}^{p}\Delta V_j f_{n-j,n-1}(z)dz\right\} \\
q(a) &= \gamma^{-1}\left\{\frac{1}{\eta(T^{max})}\int_0^a [\delta(z)]^{-1}\sum_{j=1}^{p}\Delta V_j f_{n-j,n-1}(z)dz\right\} \\
q(a) &= \gamma^{-1}\left\{\frac{1}{\eta(T^{max})}\sum_{j=1}^{p}\Delta V_j\int_0^a \delta(z)^{-1}f_{n-j,n-1}(z)dz\right\} \\
q(a) &= \gamma^{-1}\left\{\sum_{j=1}^{p}\frac{\Delta V_j}{\eta(T^{max})}\int_0^a \delta(z)^{-1}f_{n-j,n-1}(z)dz\right\}
\end{aligned}$$

3. Race Definition

■ Consider an n-player simultaneous tournament ($n \geq 2$). Each player draws an ``ability'', $a_i \in [0, m]$, where $m < 1$. Ability is distributed randomly according to distribution F. The distribution has continuous support with density f and is twice continuously differentiable. The distribution F is common knowledge while a player's ability is private information.

Player i chooses a costly bid, $q_i \in \mathbf{R}_+$.

$$\begin{aligned}
c(a,q,t) &= \delta(a)\gamma(q)\eta(t) > 0 \\
c(a,0,t) &= 0 \\
c_a(a,q,t) &= \gamma(q)\eta(t)\frac{\partial\delta}{\partial a}(a) < 0 \\
c_q(a,q,t) &= \delta(a)\eta(t)\frac{\partial\gamma}{\partial q}(q) > 0 \\
c_t(a,q,t) &= \delta(a)\gamma(q)\frac{\partial\eta}{\partial t}(t) < 0
\end{aligned}$$

$$\lim_{t\to\infty} \eta(t) = \underline{\eta} > 0$$

$$c_{aq}(a,q,t) = c_{qa}(a,q,t) = \eta(t)\frac{\partial\delta}{\partial a}(a)\frac{\partial\gamma}{\partial q}(q) < 0$$
$$c_{at}(a,q,t) = c_{ta}(a,q,t) = \gamma(q)\frac{\partial\delta}{\partial a}(a)\frac{\partial\eta}{\partial t}(t) > 0$$
$$c_{qt}(a,q,t) = c_{tq}(a,q,t) = \delta(a)\frac{\partial\gamma}{\partial q}(q)\frac{\partial\eta}{\partial t}(t) < 0$$

$p < n$ prizes are awarded to the p bidders who submit the earliest in rank order with bid quality *at least* Q. Bids are not observed by other players. The value of the prizes is strictly decreasing, $V_1 > V_2 > \ldots V_p$. Ties are broken randomly. Players are risk-neutral.

4. Race Equilibrium

■ Since costs are strictly increasing in q, submitting a bid quality larger than Q is strictly dominated by submitting the same bid at Q. A positive bid with quality below Q will be rejected leading to a zero prize. Since positive bids carry positive costs, any positive bid below Q is strictly dominated by bidding 0. Hence, the only equilibrium bid qualities are Q and 0.

Assume that there exists a symmetric equilibrium timing function $t(a)$, strictly monotonically decreasing in ability. Note that strategies mapping from ability to timing induce a distribution on timings. Denote the distribution arising form the symmetric equilibrium as G. Let r_i be the timing rank of the i^{th} player. Let

$$P_{j,n}(b) = \Pr\{r_i = j\} = \binom{n-1}{j-1}(1-G(q))^{j-1}G(q)^{n-j}.$$

Since t is strictly monotone by assumption, there exists an inverse function $a(t)$ mapping abilities from timing. Then

$$P_{j,n}(t) = P_{j,n}(a) = \binom{n-1}{j-1}(1-F(a(t)))^{j-1}F(a(t))^{n-j}.$$

Assume all other players are using the strategies defined by $t(a)$. Then the expected payoff for player i, biding Q, for timing t is

$$\pi(t) = \sum_{j=1}^{p} P_{j,n}(a(t))V_j - \delta(a_i)\eta(t)\gamma(Q)$$

FOCs imply that

$$\frac{\partial \pi}{\partial t}(t) = \sum_{j=1}^{p} \frac{\partial P_{j,n}}{\partial a}(a(t))\frac{\partial a}{\partial t}(t)V_j - \delta(a_i)\gamma(Q)\frac{\partial \eta}{\partial t}(t) = 0$$

Let $V_{p+1} = 0$ and $\Delta V_j = V_j - V_{j+1}$. Then the sum can be rearranged.

$$\begin{aligned}
\sum_{j=1}^{p} \Delta V_j f_{n-j,n-1}(a(t))\frac{\partial a}{\partial t}(t) - \delta(a_i)\gamma(Q)\frac{\partial \eta}{\partial t}(t) = 0 \\
\sum_{j=1}^{p} \Delta V_j f_{n-j,n-1}(a(t))\frac{\partial a}{\partial t}(t) &= \delta(a_i)\gamma(Q)\frac{\partial \eta}{\partial t}(t) \\
\frac{1}{\delta(a_i)\gamma(Q)}\sum_{j=1}^{p} \Delta V_j f_{n-j,n-1}(a(t)) &= \frac{1}{\frac{\partial a}{\partial t}(t)}\frac{\partial \eta}{\partial t}(t) \\
\frac{1}{\delta(a_i)\gamma(Q)}\sum_{j=1}^{p} \Delta V_j f_{n-j,n-1}(a(q)) &= \frac{\partial t}{\partial a}(a(t))\frac{\partial \eta}{\partial t}(t)
\end{aligned}$$

If t is a symmetric equilibrium, then the FOC must hold for all values of a_i.

$$\begin{aligned}
\frac{1}{\delta(a)\gamma(Q)}\sum_{j=1}^{p} \Delta V_j f_{n-j,n-1}(a) &= \frac{\partial t}{\partial a}(a)\frac{\partial \eta}{\partial t}(t) \\
\frac{1}{\delta(a)\gamma(Q)}\sum_{j=1}^{p} \Delta V_j f_{n-j,n-1}(a) &= \frac{\partial}{\partial a}\left[\eta(t(a))\right] \\
\eta(t(a)) &= \int_{\underline{a}}^{a} \frac{1}{\delta(z)\gamma(Q)}\sum_{j=1}^{p} \Delta V_j f_{n-j,n-1}(z)dz + \eta(t(\underline{a})) \\
t(a) &= \eta^{-1}\left\{\int_{\underline{a}}^{a} \frac{1}{\delta(z)\gamma(Q)}\sum_{j=1}^{p} \Delta V_j f_{n-j,n-1}(z)dz + \eta(t(\underline{a}))\right\}
\end{aligned}$$

Asserting the outside option constraint.

$$\sum_{j=1}^{p} P_{j,n}(\underline{a})V_j - \delta(\underline{a})\eta(t(\underline{a}))\gamma(Q) = 0$$

$$\sum_{j=1}^{p} P_{j,n}(\underline{a})V_j = \delta(\underline{a})\eta(t(\underline{a}))\gamma(Q)$$

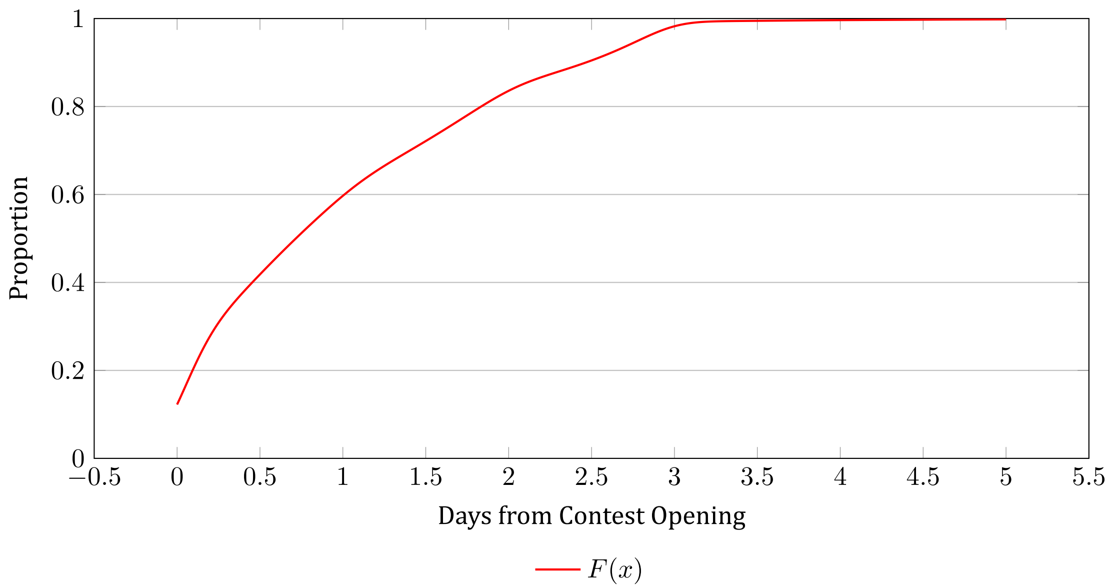
$$\sum_{j=1}^{p} P_{j,n}(\underline{a})V_j \geq \delta(\underline{a})\underline{\eta}\gamma(Q)$$

$$\frac{1}{\delta(\underline{a})\gamma(Q)} \sum_{j=1}^{p} P_{j,n}(\underline{a})V_j \geq \underline{\eta}$$

5. Empirical Analysis

■

FIGURE 1
CDF OF RELATIVE REGISTRATION TIMES



☐ **Registration Time.**

☐ **Development Ratings and Algorithm Ratings.**

	Development Ratings	Algorithm Ratings
Minimum	159	198
Maximum	2482	3067
Mean	1283	1229
corr	0.1135	

FIGURE 2
SCATTERPLOT OF ALGORITHM AND DEVELOPMENT RATINGS

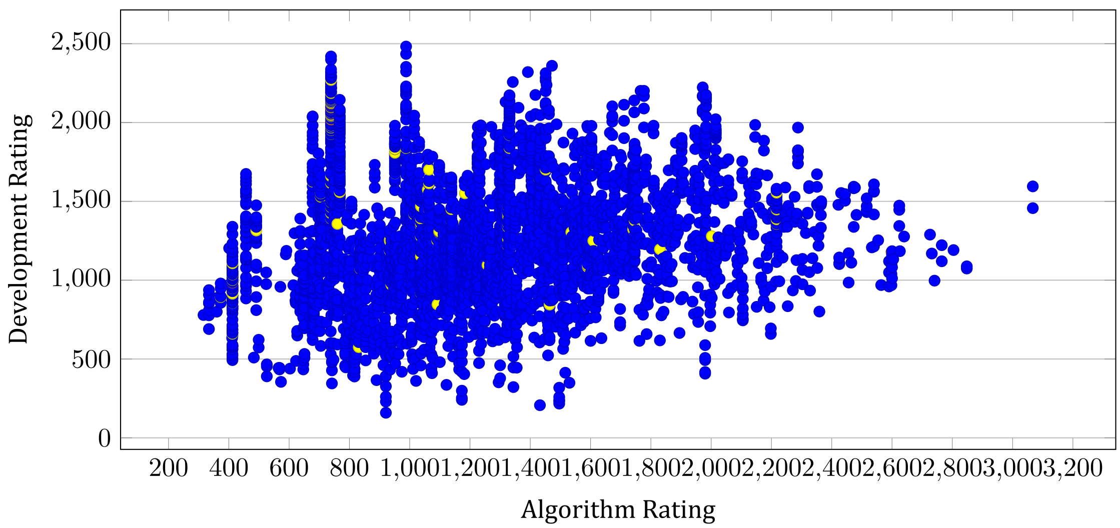


Figure 3
Nonparametric Fit using Development Rating as Ability

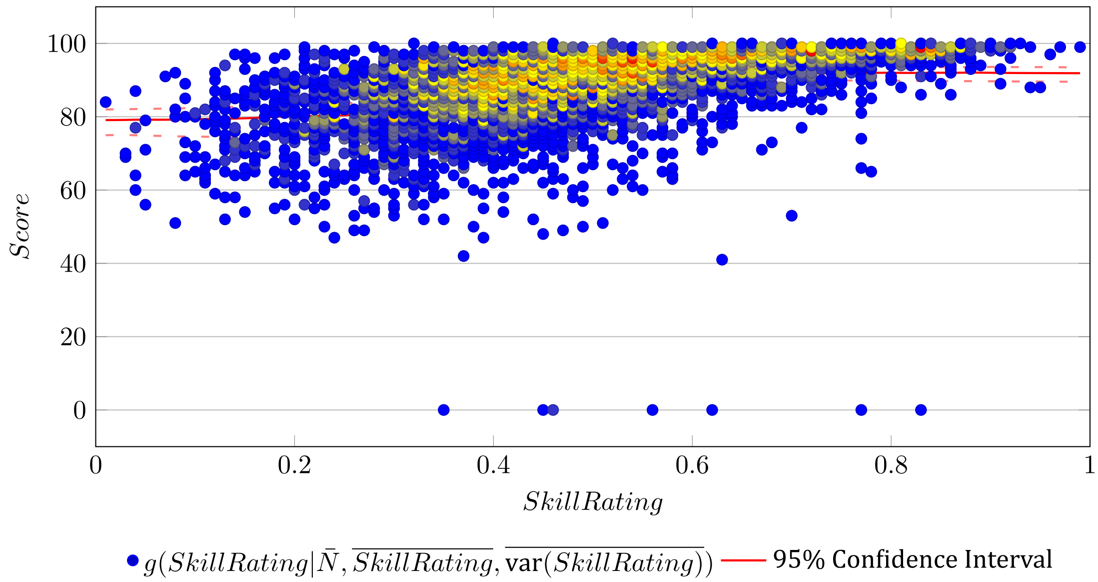


Figure 4
Nonparametric Fit using Algorithm Rating as Ability

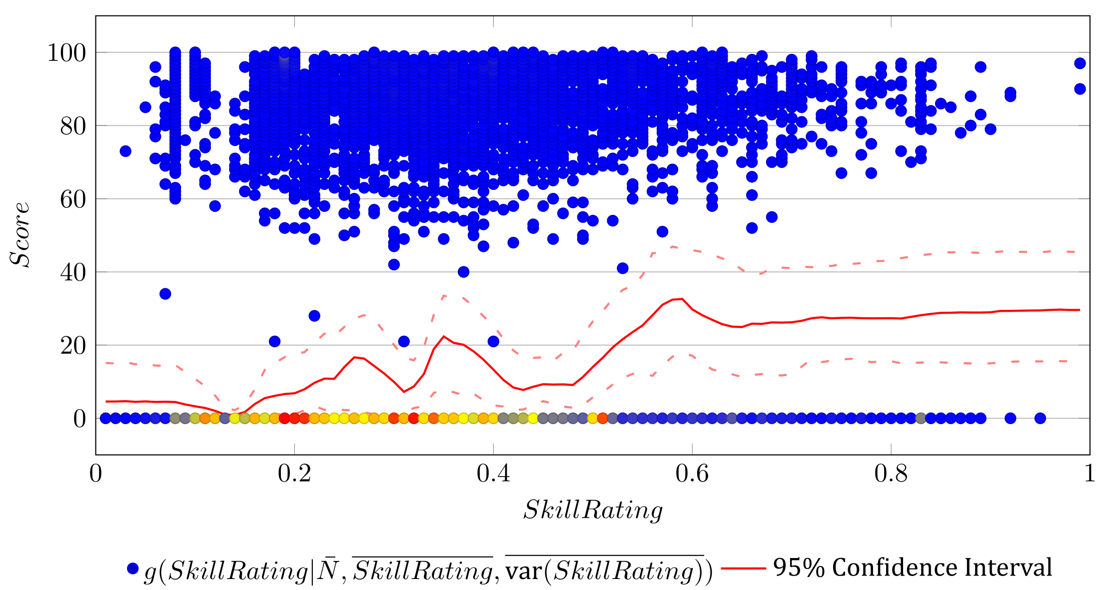


Figure 5
Nonparametric Fit with Discontinuity using Algorithm Rating as Ability

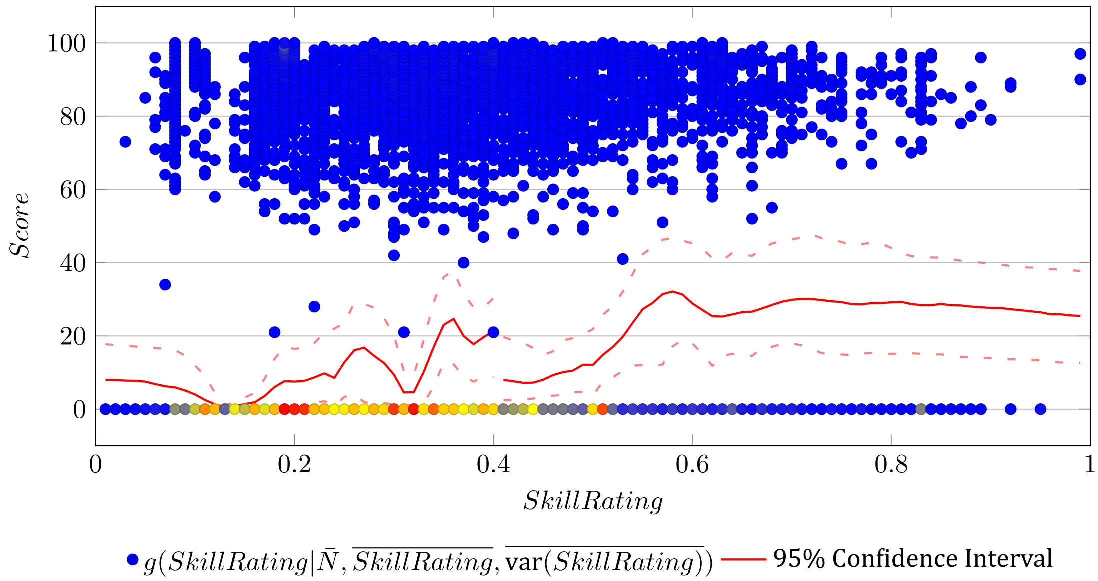


Figure 6
Nonparametric Fit using Max of Algorithm and Development Ratings as Ability

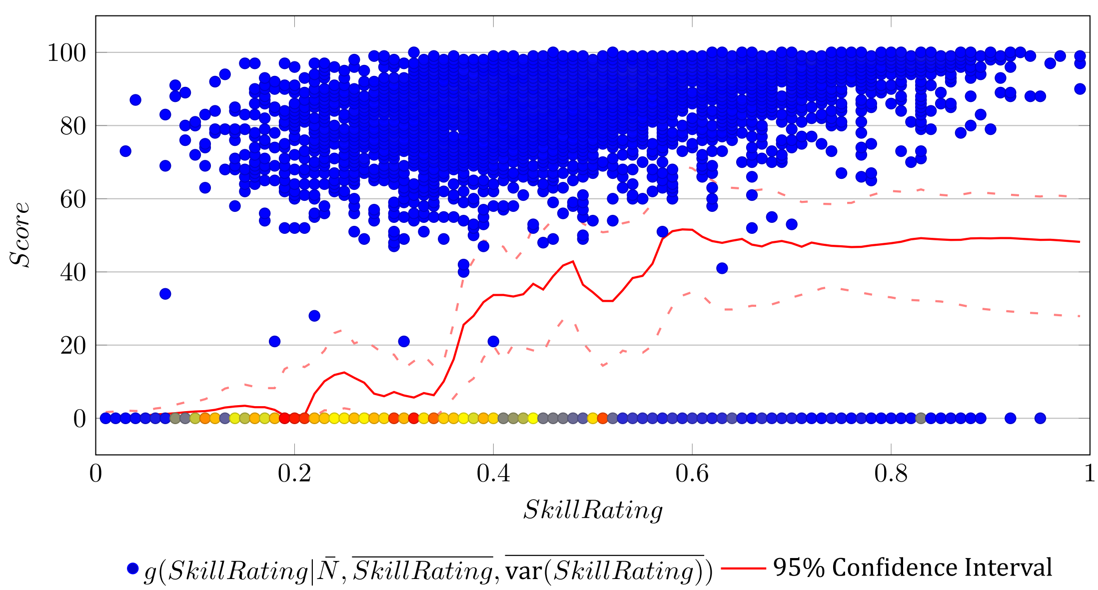


Figure 7

Nonparametric Fit with Discontinuity using Max of Algorithm and Development Ratings as Ability