Races or Tournaments: Identification & Estimation

Andrea Blasco (`ablasco@fas.harvard.edu`)

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Identification

In this section, we discuss the identification problem, which consists in the ability to use the observed data to identify one or more of the parameters of the contest model presented in Section **??**. In particular, we will discuss conditions necessary to identify parameters of the cost function and the ability distribution, also called the "primitives" of the model, and how these vary across different competition modes: race, tournament, and tournament with minimum-entry requirements.

Our observations come from the results of the experimental challenge described in Section **??**. For each room $(l = 1, ..., L)$, we observe the number Y_l of participating competitors among N_l registered competitors who have been assigned to to some treatment τ_l. The results are reported in Table 1.

The following model is assumed: the number of participating competitors Y_l at a room $x_l = c(\tau_l, z_l)$, where τ_l is the treatment randomly assigned to the room and z_l is a vector of additional room characteristics, is distributed as a binomial variable with parameters N_l and $p(x_l, \theta)$ where $p(x, \theta)$ describes the probability for a competitor to participate in the challenge having been assigned to a room described by x. The model is then the following:

$$Y_l \sim \text{Binomial}(N_l, p(x_l, \theta)) \quad \text{for each } l = 1, ..., L. \qquad (1)$$

In binomial linear models, the probability function $p(x, \theta)$ is related to a linear prediction through a one-to-one transformation called the link function (see xxxx).

In the MS theoretical approach, our preliminary analysis of the model showed that the predictor is not generally linear and it is modeled as follows:

$$p(x_l, \theta) = 1 - F_\theta(\hat{a}) \qquad (2)$$

where F_θ is a parametric distribution function of a latent ability variable that is privately observed by each player with θ being a vector of distributional parameters and $\hat{a}$ being the marginal type of the room.

In each room, the marginal type is then determined by the follow-

Room name	Treatment	Registrants	Participants
Group 1A	Race	9	2
Group 1B	Race	10	2
Group 1C	Race	10	5
Group 1D	Race	10	1
Group 1E	Race	15	4
Group 1F	Race	15	3
Group 1G	Race	15	4
Group 1H	Race	15	5
Group 2A	Tournament	10	3
Group 2B	Tournament	10	5
Group 2C	Tournament	10	5
Group 2D	Tournament	10	2
Group 2E	Tournament	15	5
Group 2F	Tournament	15	4
Group 2G	Tournament	15	4
Group 2H	Tournament	15	5
Group 3A	Reserve	10	1
Group 3B	Reserve	10	3
Group 3C	Reserve	10	3
Group 3D	Reserve	10	2
Group 3E	Reserve	15	4
Group 3F	Reserve	15	6
Group 3G	Reserve	15	3
Group 3H	Reserve	15	5

Table 1: Participating competitors by room and treatment group. Registered competitors signed up to the challenge and were randomly assigned to non-overlapping rooms. Each room was then randomly assigned to a treatment. A participant is a registered competitor who has made at least one submission during the eight-day submission period of the challenge.

ing zero profit condition:

$$R(\hat{a}) = c_a(\hat{a})c_y(\underline{y})c_t(\bar{t}) \tag{3}$$

where the $R(\hat{a})$ is the expected revenue of the marginal type must be equal to the costs. The expected revenue is

$$R(\hat{a}) = \alpha F_{(n-1:n-1)}(\hat{a}) + (1-\alpha)[1 - F_{(n-1:n-1)}(\hat{a})]F_{(n-2:n-2)}(\hat{a}). \tag{4}$$

The left hand side of the above equation describes the value of winning the first prize α times the probability of the marginal player $\hat{a}$ being the winner. The second term is the value of winning the second prize $1 - \alpha$ times the probability of the marginal player being the second ranked competitor.

Because abilities are drawn at random and independently, the distribution of the order statistics on the left hand side depends on the distribution F_θ and, hence, it is known up to the parameters θ that we want to estimate from the data.

Also, the cost function of the marginal type is evaluated at the target and the deadline, which are determined by the room characteristics x (e.g., the problem, whether there's a deadline or not, whether's there's a minimum entry requirement or not). We assume the following linear relationship:

$$C(\underline{y}, \bar{t}) = \beta x_l \tag{5}$$

That is, the costs is linear in the the room characteristics x_l with the parameters β that we also want to estimate from the data.

If we divide all sides by the ability cost c_a the left hand side of (3), then we use the following definition:

$$G(\hat{a}, F_\theta) \equiv c_a(\hat{a})^{-1} R(\hat{a}) \tag{6}$$

where the function $G(\cdot, F_\theta)$ is strictly monotone in the marginal type, as we showed in the discussion of the model. Because monotone in its first argument, the function is also invertible. This leads to the following

$$\hat{a} = G^{-1}(\beta x_l, F_\theta). \tag{7}$$

Using (**??**) with (2) leads to:

$$p(x_l, \omega) = 1 - F_\theta(G^{-1}(\beta x_l, F_\theta)) \tag{8}$$

with $\omega = (\theta, \beta)$.

Example with the Uniform distribution

Suppose $\alpha = 1$ to simplify computations. Imagine that the distribution F_θ is uniform on $(0, \theta)$ and the cost ability $c_a(x) = 1/x$. Then the zero profit condition (3) becomes

$$\left(\frac{\hat{a}}{\theta}\right)^{n-1} \hat{a} = \beta x_l \tag{9}$$

Solving for the marginal type gives:

$$\hat{a} = \left(\beta x_l \theta^{n-1}\right)^{1/n} \tag{10}$$

So, the probability of participation (2) becomes

$$p(x_l, \theta) = 1 - F_\theta(\hat{a}) \tag{11}$$

$$= 1 - \frac{\left(\beta x_l \theta^{n-1}\right)^{1/n}}{\theta} \tag{12}$$

$$= 1 - \left(\frac{\beta x_l}{\theta}\right)^{1/n}. \tag{13}$$

Because it is non-linear in its parameters, the model is not identifiable [need proof]. However, we can re-parametrize the model with $\beta/\theta \equiv \eta$ to make the parameter η identifiable.
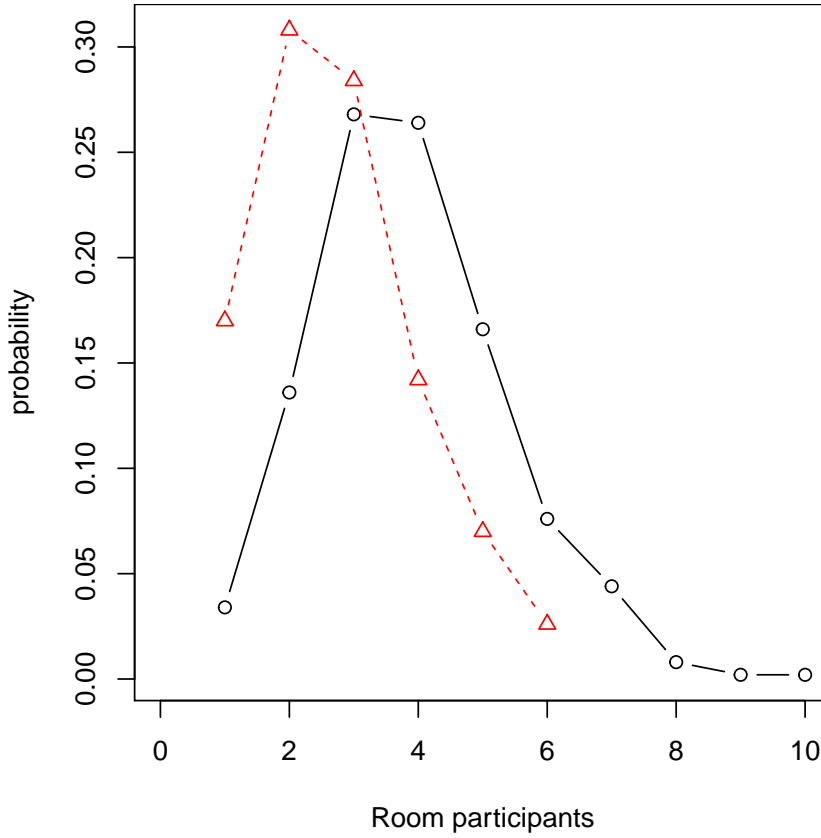

Figure 1: Simulated distribution of participants with uniform distribution. Parameters are $\eta = 0.1$ and $n = 15$. Data simulated 500 times. Dashed curve had higher costs ($x_l = 1.5$) than the solid curve ($x_l = 0.5$).

Example with Beta distribution

In this example, we assume abilities are drawn from a lognormal distribution with parameters μ and σ.

Estimation

Our aim is to estimate the parameters θ and β and to evaluate whether costs are different between the three competition modes under study: race, tournament, tournament + requirement.

It is natural to estimate the parameters θ by maximum likelihood because the ability distribution (and therefore the distribution of the outcomes) is known. The estimation criterion used here is the maximization of the *deviance*.

The deviance is:

$$D(\theta) = -2 \sum Y \log(\frac{Np(x,\theta)}{Y}) + (N - Y) \log(\frac{N - Np}{N - Y}). \qquad (14)$$

Note that the deviance is a function of the likelihood (see xxx pg. xxx).

```
# Define likelihood
structreg <- function(x, y, wt = rep(1, length(y)), intercept = T, start = rep(0, p), ...) {
    fmin <- function(beta, X, y, w) {
        # Suppose uniform with 10 competitors
        p <- 1 - (beta[1] * X)^(1/10)  # Function of parameters
        -sum(2 * w * ifelse(y, log(p), log(1-p)))
    }
    gmin <- function(beta, X, y, w) {
        # Gradient here!
    }
    if(is.null(dim(x))) dim(x) <- c(length(x), 1)
    dn <- dimnames(x)[[2]]
    if(!length(dn)) dn <- paste("Var", 1:ncol(x), sep="")
    p <- ncol(x) + intercept
    if(intercept) {x <- cbind(1, x); dn <- c("(Intercept)", dn)}
    if(is.factor(y)) y <- (unclass(y) != 1)
#   fit <- optim(start, fmin, gmin, X = x, y = y, w = wt, method = "BFGS", ...)
    fit <- optim(0.5, fmin, X = x, y = y, w = wt, method = "BFGS", ...)
    names(fit$par) <- dn
    cat("\nCoefficients:\n"); print(fit$par)
    cat("\nResidual Deviance:", format(fit$value), "\n")
    cat("\nConvergence message:", fit$convergence, "\n")
    invisible(fit)
}


# Create data
n <-10
n.rooms <- 500
```

```
X <- rchisq(n.rooms, df=3)
eta <- 0.05
p <- function(eta, X) {
    prob <- 1 - (eta * X)^{1/n}
    ifelse(prob>0, prob, 0.05)
}
x.P <- p(eta, X)
y <- sapply(x.P, function(x) rbinom(n=1, size=n, prob=x))

structreg(x=X, y=y)

fn <- function(beta) {
    1 - (beta * 1.25)^(1/n)
}
curve(fn(x))
```

More examples

Suppose F is lognormal(μ, σ), then the zero profit condition is:

$$\left[\frac{1}{2} + \frac{1}{2} \left(\frac{\log \hat{a} - \mu}{\sqrt{2}\sigma} \right) \right] = a^{-1/(n-1)} K_l \qquad (15)$$

```
# xxx
zeroprofit <- function(x, mu, sigma, n) {
    x * plnorm(x, meanlog=mu, sdlog=sigma)^(n-1)
}
for (i in 3:10)
    curve(zeroprofit(x, i, 10, 10), add=TRUE, lty=i)
```

References