The Optimal Allocation of Prizes in Contests

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We study a contest with multiple, nonidentical prizes. Participants are privately informed about a parameter (ability) affecting their costs of effort. The contestant with the highest effort wins the first prize, the contestant with the second-highest effort wins the second prize, and so on until all the prizes are allocated. The contest's designer maximizes expected effort. When cost functions are linear or concave in effort, it is optimal to allocate the entire prize sum to a single "first" prize. When cost functions are convex, several positive prizes may be optimal. (*JEL* D44, J31, D72, D82)

In 1902, Francis Galton posed the following problem:

> A certain sum, say £100, is available for two prizes to be awarded at a forthcoming competition; the larger one for the first of the competitors, the smaller one for the second. How should the £100 be most suitably divided between the two? What ratio should a first prize bear to that of a second one? Does it depend on the number of competitors, and if so, why?
>
> (*Biometrika*, Vol. 1, 1902.)

In his article Galton proposed a ratio of 3 to 1 to the above question. Since he did not explicitly state what is the designer's goal, the answer is somewhat arbitrary. Nevertheless, Galton's work is important for it pioneered both the scientific literature on contests and the use of the so-called *order statistics*.[1]

Contests are situations in which agents spend resources in order to win one or more prizes. A main feature is that, independently of success, all contestants bear some costs. Most (but not all) of the large economic literature on contests has treated the case where the agents compete in order to win a unique prize.[2] Early applications where the value of the prize to each contestant is common and common knowledge include the rent-seeking model in Gordon Tullock (1980), the sales model in Hal R. Varian (1980),[3] the technological race in Partha Dasgupta (1986), and the all-pay auction in Herve Moulin (1986). A complete characterization of equilibrium be-

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[1] It turns out that three is the limit when the number of contestants (who have normally distributed abilities) goes to infinity of the ratio between (1) the expected difference between the value of the highest-order statistic and the second highest, and (2) the expected difference between the value of the highest-order statistic and the third-highest statistic. For relations involving order statistics and their applications, see Barry C. Arnold and N. Balakrishnan (1989) and Moshe Shaked and George J. Shantikumar (1994).

[2] Applications have been made to rent seeking, lobbying, technological races, political contests, promotions in labor markets, trade wars, military, and biological wars of attrition.

[3] Varian's model of sales is a reversed all-pay auction. The firm setting the lowest sale price gets the prize: the extra demand of informed customers. See also Michael Baye et al. (1992).

havior[4] in the complete-information all-pay auction with one prize is due to Michael R. Baye et al. (1996).

The award of a single prize seems consistent with a general intuition about the efficiency of rewarding only the best (and supposedly ablest) competitor.[5] But, the prevalence of multiple-prize contests is obvious in the real world. Many such contests arise naturally, while others are designed in order to achieve specific goals. Employees spend effort in order to be promoted in organizational hierarchies, which often consist of several types of well-defined positions; students compete for grades in exams; in procurement contests runners-up often serve as "second sources"; in proportional parliamentary systems politicians compete for ranked places on the party's list; athletes compete for gold, silver, and bronze medals, or for monetary prizes; architectural competitions for prominent structures attract several designs by offering substantial monetary prizes;[6] young pianists compete for the first, second, and third prizes in the Rubinstein international competition, etc.

Several papers study complete-information contest models with multiple prizes. Thorsten Broecker's (1990) model of credit markets has several features of an all-pay auction with as many prizes as contestants.[7] Robert Wilson (1979) and James J. Anton and Dennis A. Yao (1992) study split-award auctions where several bidders can win.[8] Derek J. Clark and Christian Riis (1998) analyze contests with multiple identical prizes and compare simultaneous versus sequential designs from the point of view of a revenue-maximizing designer.

Continued interest has been attracted by the technology-inducement prizes offered to individuals or groups who provide best entries in a contest, or who first meet some specified technical goal. Brian D. Wright (1983), Curtis R. Taylor (1995), and Don Fullerton and R. Preston McAfee (1999) focus on various models of research tournaments where a unique prize is awarded. For example, the privately funded Loebner prize is annually awarded to the computer program that is the most "human" in its responses to inquiries. However, other examples belonging to the "best entries" category[9] display multiple prizes: The European Information Technology Society annually awards three grand prizes worth 200,000 euros each[10] for "novel products with high information technologies content and evident market potential"; The Federal Communication Commission's Pioneer Preference Program offered guaranteed slices of spectrum to several companies that developed and implemented innovative communication services and technologies;[11]

In the technological examples above, the designer's goal is not to achieve the highest possible top performance (or some prespecified level of that performance), but rather to induce a general increase of activity in the specific field.[12] Similarly, professors wish to maximize the expected learning effort made by their students, organizers of athletic or artistic competitions often need to maximize average performance (or some related measure) in order

[4] Contrary to earlier erroneous claims, there are many equilibria.

[5] So-called "winner-take-all" contests are the subject of an entertaining book by Robert H. Frank and Philip J. Cook (1995). But many of their examples are in fact multiprize contests.

[6] For example, designers for the German pavilion at the EXPO 2000 in Hannover competed over seven prizes of DM 161,000, DM 119,000, DM 91,000, DM 63,000, DM 49,000, DM 42,000, DM 35,000. In this kind of competition, the actual designer need not be one of the prize winners.

[7] With only two banks, his model is isomorphic to Varian's (1980) model of sales.

[8] Anton and Yao also mention examples such as split home schedules of professional sport franchises wishing to maximize amenities offered by several municipalities.

[9] The most famous example for the "specified technical goal" type is surely the British Longitude Act, issued in 1714, which specified three prizes (£20,000, £15,000, £10,000; these are equivalents of millions of today's dollars) for methods to determine longitude in varying degrees of accuracy (see Dava Sobel, 1996). For more recent examples, such as the Feynman Prize for technical progress in nanotechnology, or the superefficient refrigerator prize, see Patrick Windham (1999).

[10] There are also 20 prizes worth 5,000 euros each.

[11] Three companies were offered pioneer status in 1992. The program probably made sense in an era where spectrum allocation was done by lottery or bureaucratic process. Congress terminated the program in 1997, after the advent of spectrum auctions.

[12] Another well-known example of this type are the early aviation prizes, offered to stimulate the fledgling aeronautic industry. Between the Wright brothers' first flight and 1929, over 50 major prizes were offered by governments, individuals, and corporations. In 1926–1927 alone, Daniel Guggengeim offered more than $2.5 million in prizes.

to thrill audiences, and local authorities organize gardening competitions that improve the community's appearance.

Given the wealth of both one-prize and multiprize examples, it is of interest to offer a rationale for both phenomena in a single, integrated model. Here we address Galton's problem in the following setting: Several risk-neutral agents engage in a contest where multiple prizes with known and common values are awarded. Each contestant i undertakes an observable "effort." The contestant with the highest effort wins the first prize, the contestant with the second-highest effort wins the second prize, and so on until all the prizes are allocated. All contestants (including those that did not win any prize) incur a cost that is a strictly increasing function of their effort. This function is common knowledge. We differentiate among the cases where the cost function is, respectively, linear, concave, or convex in effort. The case of convex cost functions, i.e., where additional effort becomes marginally more and more costly, seems the most important one from the point of view of applications. The cost function of contestant i also depends on a parameter (say "ability") that is *private information* to that player. The main assumption we make is one of separability between ability and effort in the cost function. The function governing the distribution of abilities in the population is common knowledge, and abilities are drawn independently of each other.[13]

On the technical level, the model sketched above is isomorphic to a "private values" all-pay auction with several (potentially unequal) prizes. Equilibrium behavior in an incomplete-information all-pay auction with a unique prize, and the relations to other standard auction procedures have been analyzed by Robert Weber (1985), Arye L. Hillman and John G. Riley (1989), and by Vijay Krishna and John Morgan (1997), who also allow for affiliated signals. Equilibrium uniqueness in frameworks with two players is addressed in Erwin Amann and Wolfgang Leininger (1996) and in Allesandro Lizzeri and Nicola Persico (2000). Jeremy Bulow and Paul Klemperer (1999) study an incomplete-information model of a war of attrition with a fixed number of K identical prizes and $N + K$ contestants. The form of equilibrium behavior depends there on whether the agents continue to bear costs after they drop out of the contest.

In our model each contestant chooses his effort in order to maximize expected utility (given the other competitors' bids and given the values of the different prizes). The goal of the contest designer is to maximize the total expected effort (i.e., the expected sum of the bids) at the contest.[14] The designer can determine the number of prizes having positive value and the distribution of the fixed total prize sum among the different prizes. Related set-ups are analyzed by Amihai Glazer and Refael Hassin (1988) and by Yasar Barut and Dan Kovenock (1998).

Barut and Kovenock (1998) study a multiprize contest with heterogenous prizes.[15] In their symmetric, complete-information environment, these authors show that the revenue-maximizing prize structure allows any combination of $K - 1$ prizes, where K is the number of contestants. (In particular, allocating the entire prize sum to a unique first prize is optimal.)

Our paper is closest in focus to Glazer and Hassin (1988). Besides studying the symmetric equilibria of a multiprize complete-information model, these authors also propose an incomplete-information model that is more general than ours since it allows for both cost functions that are not necessarily separable in ability and effort, and for risk-averse contestants. But this general structure is not easily amenable to anal-

[13] It seems reasonable to assume the existence of a "natural" distribution of abilities, out of which draws are taken. The independence assumption is problematic in some models with endogenous entry. Both assumptions were postulated by Galton who considered independent draws from the normal distribution. Here they allow us to explicitly compute symmetric equilibria by analyzing one differential equation instead of a complex system.

[14] In some situations that fit our model, the designer has other goals. For example, in a lobbying model the contest designer might not be the beneficiary of the "wasteful" lobbying activities, and she might wish to minimize them. Our analysis can be easily extended to other goal functions since we explicitly display bidding equilibria (that are independent of the designer's goal).

[15] In our terminology, players have linear cost functions and the same, common-knowledge ability.

ysis, and, in order to obtain results, the authors further assume a separable and linear cost function such that the lowest ability type has an infinite cost of bidding, a uniform ability distribution, risk-neutral contestants, and no entry fees. With these assumptions, they show that a unique first prize is optimal.

Some of our assumptions (heterogenous, privately informed agents and a deterministic relation between effort and output) should be contrasted with those in the literature pioneered by Edward P. Lazear and Sherwin Rosen (1981).[16] This body of work emphasizes the use of relative compensation schemes in order to extract effort under "moral hazard" conditions. Common assumptions in this literature are that agents are identical, and that observed output is a stochastic function of an unobservable effort. Lazear and Rosen (1981) derive the optimal prize structure in a contest with two workers and two prizes and compare it to optimal piece rates. Using a variation of their model, Krishna and Morgan (1998) assume that the contest designer has a fixed prize purse and study the optimal allocation of the purse among several nonnegative prizes in contests with two, three, or four contestants.

Given the assumptions above, we display symmetric bidding equilibria for any number of prizes and contestants with linear, concave, or convex cost functions. In order to have a less technical exposition, we focus, however, on the designer's problem in the case where she can award two (potentially unequal) prizes, and where there are at least three contestants. Compared to one-prize contests, this case already displays the main ingredient for complexity in bidding.[17] It will become clear that none of our qualitative results changes if we allow for more than two prizes.

We can answer Galton's questions as follows: (1) If the contestants have linear or concave cost functions, it is optimal for the designer to allocate the entire prize sum to a single "first" prize. (2) We give a necessary and sufficient condition ensuring that (at least) two prizes are optimal if the contestants have convex cost functions. Depending on the parameters, the optimal prize structure may involve several equal prizes or different prizes whose ratio can be computed. (3) If the contestants have convex cost functions, several prizes may be optimal even if the contest designer can use instruments (such as entry fees or minimum effort requirements) that exclude types with relatively low abilities.

We now briefly describe the intuition behind our results. The equilibrium effort depends on the contestant's cost (and, hence, on his ability), on the probabilities of winning different prizes, and on the prizes' values.[18] Moreover, the equilibrium effort is increasing in ability. Since a player with higher ability has a higher chance to win the first prize, increasing the value of the first prize by one penny causes an overall increase in equilibrium efforts (and the increase is higher for higher abilities). In contrast, the probability of getting the second prize is not monotone in ability,[19] and therefore the marginal effect of the second prize (and of all other prizes but the first) is ambiguous. The marginal effect of the second prize on the equilibrium-effort function is in fact negative for players with high enough abilities, but it is positive for middle- and low-ability players. Moreover, for contestants with abilities below a certain threshold, the

[16] Their work has been extended in many directions by, among others, Jerry R. Green and Nancy L. Stokey (1983), Barry J. Nalebuff and Joseph E. Stiglitz (1983), and Rosen (1986). Ronald G. Ehrenberg and Michael L. Bognanno (1990) and Charles R. Knoeber and Walter N. Thurman (1994) test several predictions of this theory using observed prize structures in professional golf tournaments, and reward schemes for broiler producers, respectively.

[17] If there is only one prize, or if there are several equal prizes, each contestant perceives two payoff-relevant alternatives: I win a prize, or I win nothing. Hence, bids are determined by the difference in expected payoff between those two alternatives (the same logic applies if there are two unequal prizes but only two contestants). If there are at least two unequal prizes and at least three contestants, each contestant perceives at least three payoff-relevant alternatives (I win the first prize, I win the second prize, ... , I win nothing). Bids are now determined by several differences in expected payoffs.

[18] If there are p prizes, the bid function involves linear combinations of p order statistics. The equilibrium bid for contestants with concave or convex cost functions is obtained by applying the inverse of the cost function to the equilibrium bid for linear cost functions.

[19] For example, both the lowest- and highest-ability types have a zero probability of getting the second prize if there are at least three contestants.

(positive) marginal effect of the second prize is higher than the marginal effect of the first prize, since these types are more likely to get the second prize rather than the first. The relevant variable for a designer who wants to maximize the average (i.e., expected) effort of each contestant becomes the average difference between the marginal effects of the second prize and the first prize, respectively. If the average difference is negative the designer should award only a unique (first) prize. This turns out to be the case for contestants with linear or concave cost functions, no matter what the (common) distribution of abilities is.[20] The opposite can happen only for convex cost functions, where the positive effect of further prizes on "middle-ability" types can more than compensate the decreased effort of the ablest competitors. Two or more prizes may be optimal in this case. The optimal allocation of the prize sum among the several prizes depends then on the number of contestants, the distribution of abilities, and on the form (e.g., curvature) of the cost function. Finally, we show by way of example that, for convex cost functions, the beneficial effect of additional prizes persists even if types of relatively low ability can be excluded by instruments such as entry fees or minimum-effort requirements.

The paper is organized as follows. In Section I we present the contest model with multiple prizes and private information about a parameter (e.g., ability) entering cost functions. In Section II we focus on linear cost functions and we first derive the symmetric equilibrium bid functions. Then we formulate the contest designer's problem and we prove that it is optimal to award a single prize. In Section III we use the result obtained above in order to study the optimal prize structure for contestants with concave and convex cost functions. We illustrate the nontrivial optimal-prize structure in an example with convex cost functions. In Section IV we briefly study the effects of entry fees. In Section V we gather concluding comments. All proofs and most technical derivations appear in Appendices A–C.

[20] A priori it seems that the sign of the average difference will depend on features of the distribution of abilities, such as, say, the relative weight of "middle" types.

I. The Model

Consider a contest where p prizes are awarded. The value of the $j-th$ prize is V_j, where $V_1 \geq V_2 \geq \cdots \geq V_p \geq 0$. The values of the prizes are common knowledge. We assume that $\sum_{i=1}^{p} V_i = 1$. This is just a normalization.

The set of contestants is $\mathcal{K} = \{1, 2, ..., k\}$. Without loss of generality we can assume that $k \geq p$ (i.e., there are at least as many contestants as there are prizes).

In the contest each player i makes a effort x_i. Efforts are undertaken simultaneously. An effort x_i causes a disutility (or cost) denoted by $c_i\gamma(x_i)$, where $\gamma : R_+ \rightarrow R_+$ is a strictly increasing function with $\gamma(0) = 0$, and where $c_i > 0$ is an ability parameter.[21] Note that a *low* c_i means that i has a *high* ability (i.e., lower cost) and vice versa.

The ability (or *type*) of contestant i is private information to i. Abilities are drawn independently of each other from an interval $[m, 1]$ according to the distribution function F which is common knowledge. We assume that F has a continuous density $F' > 0$. In order to avoid infinite bids caused by zero costs, we assume that m, the type with highest possible ability, is strictly positive.[22]

The contestant with the highest effort wins the first prize, V_1. The contestant with the second-highest effort wins the second prize, V_2, and so on until all the prizes are allocated.[23] That is, the payoff of contestant i who has ability c_i, and makes effort x_i is either $V_j - c_i\gamma(x_i)$ if i wins prize j, or $-c_i\gamma(x_i)$ if i does not win a prize.

Each contestant i chooses his effort in order to maximize expected utility (given the other competitors' efforts and given the values of the differ-

[21] The treatment in the case where i's cost function is given by $\delta(c_i)\gamma(x_i)$, with δ strictly monotone increasing, is completely analogous. The main assumption here is the separability of ability and effort.

[22] The case where $m = 0$ can be treated as well, but requires slightly different methods. The choice of the interval $[m, 1]$ is a normalization.

[23] Let $h > 1$ contestants tie for prize j. If $h \leq p - j + 1$, we assume that prizes $j, j+1, \cdots j+h-1$ are randomly allocated among the tied players (each one of them has the same probability of getting each of the prizes). If $h > p - j + 1$, then prizes $j, j+1, \cdots p$, and a total of $h - (p - j + 1)$ zero prizes are randomly allocated among the tied players.

ent prizes). The contest designer determines the number of prizes having positive value and the distribution of the total prize sum among the different prizes in order to maximize the expected value of the sum of the bids $\sum_{i=1}^{k} x_i$ (given the contestants' equilibrium-effort functions).

II. Linear Cost Functions

We assume here that the cost functions are linear in effort, i.e., $\gamma(x) = x$. The next proposition displays the symmetric equilibrium with two prizes and $k \geq 3$ contestants.[24] In Appendix C we provide the general formula for the equilibrium bid functions with $p > 2$ prizes.

PROPOSITION 1: *Assume that there are two prizes, $V_1 \geq V_2 \geq 0$, and $k \geq 3$ contestants. In a symmetric equilibrium,[25] the bid function of each contestant is given by $b(c) = A(c)V_1 + B(c)V_2$, where:*

$$(1) \quad A(c) = (k-1)\int_c^1 \frac{1}{a}(1-F(a))^{k-2} \times F'(a)\, da.$$

$$(2) \quad B(c) = (k-1)\int_c^1 \frac{1}{a}(1-F(a))^{k-3} \times [(k-1)F(a)-1]F'(a)\, da.$$

PROOF:
See Appendix A.

Equilibrium bids are equal to a weighted sum of the values of the two prizes, and the weights differ from type to type according to their respective chances of getting one of the prizes. As Galton anticipated, order statistics and their distributions play a main role in the definitions of the weights $A(c)$ and $B(c)$.

A. *The Designer's Problem*

Let $V_2 = \alpha$ and $V_1 = 1 - \alpha$, where $0 \leq \alpha \leq 1/2$ (since the second prize is smaller than the first). By Proposition 1, each contestant's equilibrium-effort function is given by:

$$(3) \quad b(c) = (1-\alpha)A(c) + \alpha B(c) = A(c) + \alpha(B(c) - A(c)).$$

The average effort of each contestant is given by:

$$(4) \quad \int_m^1 (A(c) + \alpha(B(c) - A(c)))F'(c)\, dc.$$

Since there are k contestants, the seller's problem is:

$$(5) \quad \max_{0 \leq \alpha \leq 1/2} k \int_m^1 (A(c) + \alpha(B(c) - A(c))) \times F'(c)\, dc.$$

The solution to the above problem coincides with the solution to:

$$(6) \quad \max_{0 \leq \alpha \leq 1/2} \alpha \int_m^1 (B(c) - A(c))F'(c)\, dc.$$

The solution to problem (6) is extremely simple: if the definite integral is positive, then the optimal α is 1/2 (i.e., award two equal prizes). Otherwise, the optimal α is zero (i.e., award a unique prize). A "common-sense" conjecture is that the integral's sign will depend on the specific properties of the distribution function. But Lemma 1 in Appendix B shows that the integral in problem (6) is always negative. In other words, the beneficial effect of the second prize

[24] As mentioned in the introduction, if there are only two contestants, the situation is isomorphic to the one where there is a unique prize whose value is equal to the difference between the two prizes. Hence, it is trivially true that awarding a unique prize is optimal.

[25] It is not difficult to show that this is the unique symmetric equilibrium. It is of interest to establish whether the equilibrium remains unique even if asymmetric strategies are considered. Unfortunately, this is an open problem even for the simplest incomplete-information all-pay auction with a continuum of types. The only known results show uniqueness for the two-bidders case (see Amann and Leininger, 1996; Lizzeri and Persico, 2000).

on the effort level of low- and middle-ability types cannot compensate the effort decrease of high-ability types. Hence, the solution to problem (6) must be $\alpha = 0$.[26] Thus we obtain:

PROPOSITION 2: *Assume that the designer can award at most two prizes, $V_1 \geq V_2 \geq 0$, and that there are $k \geq 3$ contestants with linear cost functions. Then it is optimal to allocate the entire prize sum to a single first prize.*

It is important to note that the above result holds even if, a priori, the seller is allowed to award more than two prizes (for the argument, see footnote 37).

The following example illustrates Proposition 2. We use the explicit formulae for the equilibrium-effort functions derived for the case of a uniform distribution of abilities (see Appendix C).

Example 1: Assume that $k = 3$, $m = 1/2$ and $F(a) = 2a - 1$ (i.e., uniform distribution on the interval $[1/2, 1]$). The explicit formulae in Appendix C yield:

$$A(c) = -8 + 8c - 8 \ln c$$

$$B(c) = 16 - 16c + 12 \ln c$$

$$\int_{1/2}^{1} (B(c) - A(c))F'(c)\, dc = -0.137.$$

In Figure 1, the equilibrium-effort curve with two prizes is obtained by taking a certain convex combination of the $A(c)$ and $B(c)$ curves. We have plotted the equilibrium-effort function when there are two equal prizes, $V_1 = V_2 = 1/2$. This nicely illustrates both the resulting effort increase for low-ability types and the effort decrease of the abler ones, relative to the award of one prize. [Note that the curve $A(c)$ describes the equilibrium-effort function when there is a unique prize $V_1 = 1$.]

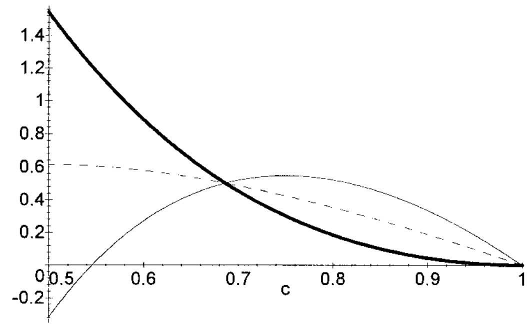

FIGURE 1. THE EQUILIBRIUM-EFFORT CURVE WITH TWO PRIZES

Notes: $A(c)$—thick line; $B(c)$—thin line; $1/2\,(A(c) + B(c))$—dotted line.

III. Concave and Convex Cost Functions

Assume now that a bidder with ability c has a cost function given by $c\gamma$ such that $\gamma(0) = 0$, $\gamma' > 0$. Let $g = \gamma^{-1}$, and observe that $g' > 0$. As mentioned above, from the point of view of applications, the most interesting case seems the one where cost functions are convex, i.e., where subsequent marginal increases in effort become more and more costly. But, using the same techniques, we can easily deal also with the case of concave cost functions. The equilibrium in these two cases is obtained by a simple transformation of the equilibrium strategies we found in the linear case.

PROPOSITION 3: *Assume that there are two prizes, $V_1 \geq V_2 \geq 0$, and $k \geq 3$ contestants. In a symmetric equilibrium, the bid function of each contestant is given by $b(c) = g[A(c)V_1 + B(c)V_2]$, where $A(c)$ and $B(c)$ are defined by equations 1 and 2, respectively.*

PROOF:

See Appendix A.

A. *The Designer's Problem*

Let $V_1 = 1 - \alpha$, and $V_2 = \alpha$, where $0 \leq \alpha \leq 1/2$. Analogous to the case of linear cost functions, the designer's revenue with concave or convex cost functions is given by

[26] As noted in the introduction, we can easily deal with cases where the designer has other goals. For example, if the designer wants to minimize the expected sum of bids, it should award two equal prizes, i.e., $\alpha = 1/2$ is optimal.

$$(7)\quad R(\alpha) = k\int_m^1 g[A(c) + \alpha(B(c) - A(c))] \times F'(c)\,dc$$

and the designer's problem is given by

$$(8)\quad \max_{0\le\alpha\le\frac{1}{2}} k\int_m^1 g[A(c) + \alpha(B(c) - A(c))] \times F'(c)\,dc.$$

Roughly speaking, the main new effect is that, with convex cost functions, the beneficial effect of the second prize on middle- and low-ability players is amplified, while the advantage of having one prize (which strongly motivates high-ability contestants) is decreased. Exactly the opposite occurs for concave cost functions. Since the optimal value of the second prize was zero already in the linear case we first obtain:

PROPOSITION 4: *Assume that the designer can award at most two prizes, $V_1 \ge V_2 \ge 0$, and that there are $k \ge 3$ contestants with concave cost functions. Then it is optimal to allocate the entire prize sum to a single first prize.*

PROOF:
See Appendix B.

The next result finally offers a rationale for the award of several prizes—a feature that, as we have illustrated in the introduction, is prevalent in real-life examples.

PROPOSITION 5: *Assume that the designer can award at most two prizes, $V_1 \ge V_2 \ge 0$, and that there are $k \ge 3$ contestants with convex cost functions. A necessary and sufficient condition for the optimality of two prizes is given by*

$$(9)\quad \int_m^1 (B(c) - A(c))\,g'(A(c))\,F'(c)\,dc > 0.$$

If condition 9 is satisfied,[27] *then it is either optimal to award two prizes $V_1 = 1 - \alpha^*$ and $V_2 = \alpha^*$, where $\alpha^* > 0$ is determined by the equation $R'(\alpha^*) = 0$, or to award two equal prizes, $V_1 = V_2 = 1/2$.*

PROOF:
See Appendix B.

An easy extension of the above proof shows that the optimal value of the second prize will be a (weakly) increasing function of the degree of convexity of the cost function, measured by the familiar Arrow-Pratt coefficient.[28] The basic intuition generalizes also to the case where the designer is not constrained to award two prizes. If the cost function is "convex enough," the optimal prize structure may involve up to $k - 1$ different prizes. It can be also shown that the left side of condition 9 is increasing in the number of competitors. Hence, for a given cost function, it is easier to satisfy the condition if there are more competitors.

Example 2: Let $k = 3$, $m = 1/2$ and $F(a) = 2a - 1$ (i.e., uniform distribution on the interval $[1/2, 1]$). Let the cost function be $c\gamma(x) = cx^2$. We have $\gamma^{-1}(x) = g(x) = x^{1/2}$ and $g'(x) = 1/2\, x^{-1/2}$. Using the results of Example 1 we obtain

$$\int_{1/2}^1 (B(c) - A(c))\,g'(A(c))\,F'(c)$$

$$= \sqrt{2}\int_{1/2}^1 \frac{6 - 6c + 5\ln c}{\sqrt{(-1 + c - \ln c)}}\,dc = 0.19.$$

In Figure 2, note how the beneficial effect of the second price [in the interval where the function $B(c) - A(c)$ is positive] gets amplified by the high values of the function g'. Numerical calculations reveal that $\alpha^* = 1/e$ is an (approximate) solution to the equation $R'(\alpha) = 0$.

[27] Note that the condition involves only primitives of the model: the ability-distribution function, the cost function, and the number of contestants.

[28] This measure of curvature (see Andreu Mas-Colell et al., 1995) is usually used to compare risk attitudes of agents with concave utility functions (risk averters). Of course, analogous properties hold also for convex functions.

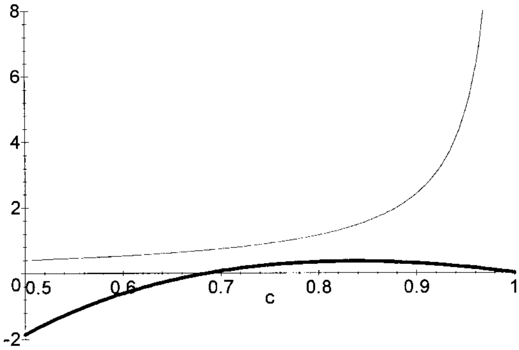

FIGURE 2. THE BENEFICIAL EFFECT OF THE SECOND PRIZE

Notes: $B(c) - A(c)$—thick line; $g'(A(c))$—thin line.

Hence, the optimal prize structure is $V_1 = 1 - 1/e \approx 0.63$, and $V_2 = 1/e \approx 0.37$. The ratio of prizes is $\frac{V_1}{V_2} = e - 1 \approx 1.71$, and the difference $V_1 - V_2$ is about one-quarter of the prize sum.

IV. Entry Fees

Contest organizers often use simple instruments such as entry fees or minimum-effort requirements[29] in order to exclude low-ability agents from the contest.[30]

Let b denote the equilibrium-effort function with linear costs and no entry fee (see Proposition 1), and recall that b was determined using the condition that the type with lowest ability (i.e., $c = 1$) makes no effort (and has zero profit). Clearly such a type will not participate in a contest where an entry fee $E > 0$ has to be paid, and therefore we need to modify here the boundary condition.

Assume then that an entry fee $E > 0$ is imposed. Solving a differential equation which is otherwise analogous to the one in the proof of Proposition 3, we obtain the equilibrium of a two-prize contest where contestants have cost function $c\gamma$ and where the designer imposes an entry fee E: types in an interval $[m, c_E]$ participate[31] and make effort according to the function $b_E(c) = \gamma^{-1}(b(c) - d)$, where $c_E \in [m, 1]$ and $d \geq 0$ are determined by the zero-effort and zero-profit conditions:

$$b_E(c_E) = 0. \tag{10}$$

$$V_1(1 - F(c_E))^{k-1} + (k-1)V_2F(c_E) \times (1 - F(c_E))^{k-2} - c_E b_E(c_E) - E = 0. \tag{11}$$

Since by construction $b_E(c_E) = 0$ and since $b_E(c_E) = 0 \Leftrightarrow \gamma^{-1}(b(c_E) - d) = 0 \Leftrightarrow b(c_E) - d = 0$, we can rewrite the above conditions as:

$$b(c_E) - d = 0. \tag{12}$$

$$V_1(1 - F(c_E))^{k-1} + (k-1)V_2F(c_E)(1 - F(c_E))^{k-2} - E = 0. \tag{13}$$

The two equations above form a system in two unknowns, which can be solved to obtain the equilibrium values for c_E and d.

For linear cost functions, we have shown that a contest designer who cannot use entry fees optimizes by allocating the entire prize sum to a single first prize. An analysis which is similar to the one performed in the seminal studies of Roger B. Myerson (1981) or Riley and William F. Samuelson (1981) shows that, with linear cost functions, a contest with a single first prize and an (optimally set) entry fee[32] is revenue maximizing among all feasible mechanisms.[33]

An interesting tension arises between the

[29] For example, the FCC-organized contest to set the standard for high-definition television was open to anyone with a \$200,000 entry fee (see Taylor, 1995). All professional sport competitions are restricted to athletes or teams that fulfill a certain prespecified standard.

[30] In models with endogenous participation, such instruments can also control for the number of contestants. Having less than the free-entry number of participants has been shown to be optimal in the research contests studied by Taylor (1995) and Fullerton and McAfee (1999). Excluding specific participants can also be beneficial in complete-information models with heterogenous agents—see Baye et al. (1993). In a winner-take-all, all-pay auction with incomplete information, with *ex ante* symmetric players and with a continuum of types, it is never optimal to restrict the number of contestants—this follows by the analysis in Bulow and Klemperer (1996) and by the revenue-equivalence theorem.

[31] We consider here only the nontrivial case where the entry fee is not too large, so that at least some types find it optimal to participate.

[32] This fee depends on the designer's valuation for the prize, on the number of contestants, and on the distribution of abilities.

[33] i.e., mechanisms that are incentive compatible and individually rational.

award of multiple prizes and entry fees (or minimum-effort requirements) since the beneficial effect of additional prizes, i.e., increased bidding by low- and middle-ability types, is reduced if such types are excluded. Hence, one may guess that even with convex cost functions, a second prize becomes superfluous for revenue purposes if entry fees (or minimum bids) can be imposed. We conclude this section by displaying an example showing that this is not the case. Assume that one prize is awarded.

Example 3: Let $k = 3$, $m = 1/2$, $F(a) = 2a - 1$ and $\gamma(x) = x^2$. Let $E \leq 1$. By Examples 1 and 2 and by the above remarks we obtain that, in equilibrium, types in the interval $\left[\frac{1}{2}, 1 - \frac{\sqrt{E}}{2}\right]$ pay the entry fee and make effort according to $b_E(c) = \sqrt{-8 + 8c - 8 \ln c - d}$, where $d = 8 \ln 2 - 4\sqrt{E} - 8 \ln(2 - \sqrt{E})$. A numerical analysis reveals that the designer's payoff decreases as a function of E. Hence, if the designer awards a single prize, the optimal entry fee is zero. But, by Example 2, the designer can do better than that by awarding two prizes.

V. Concluding Comments

We have studied the optimal-prize structure in multiprize contests where risk-neutral players have private information about their abilities. In order to maximize the expected sum of efforts, the designer should organize a winner-take-all contest if contestants have linear or concave cost functions. If the contestants have convex cost functions, then two or more prizes may be optimal.[34] The right proportion between the prizes' values depends then on the number of contestants, on the distribution of abilities in the population, and on the curvature of the cost function.

A relaxation of the risk-neutrality condition introduces a lot of complexity in explicit computations of equilibria. But, assuming that equilibria can be computed, we conjecture that arguments similar to those exhibited here can be used to show the optimality of several prizes for sufficiently risk-averse contestants (even when costs are linear).

Another interesting extension would be the study of several parallel contests (with potentially different prize structures), such that agents can choose where to compete.

It is now time to come back to Francis Galton who explicitly encouraged future research in his concluding remark: "I now commend the subject to mathematicians in the belief that those who are capable, which I am not, of treating it more thoroughly, may find that further investigations will repay trouble in unexpected directions"[35] (Galton, 1902).

Appendix A: Derivation of Equilibria

PROOF OF PROPOSITION 1:

Assume that all contestants in the set $\mathcal{K}\setminus\{i\}$ undertake effort according to the function b, and assume that this function is strictly monotonic and differentiable. Player i's maximization problem reads:

$$\text{(A1)} \quad \max_x [V_1(1 - F(b^{-1}(x)))^{k-1} + (k-1)V_2 F(b^{-1}(x)) \times (1 - F(b^{-1}(x)))^{k-2} - cx].$$

Let y denote the inverse of b. Using strict monotonicity and symmetry, the first-order condition (FOC) is:

$$\text{(A2)} \quad 1 = -(k-1)(V_1 - V_2)y' \frac{1}{y} \times (1 - F(y))^{k-2} F'(y) - (k-1)(k-2)V_2 y' \frac{1}{y} \times F(y)(1 - F(y))^{k-3} F'(y).$$

[34] Bulow and Klemperer (1999) report that Avinash Dixit offers a \$20 prize to the student who continues clapping the longest at the end of his game-theory course. In experimental tests we established that pain is a highly convex function of clapping duration, and therefore we recommend the award of several prizes.

[35] The challenge was immediately picked by the famous statistician Karl Pearson (1902), at that time editor of *Biometrika*. His notes at the end of Galton's article contain a complete solution of the statistical problem posed (calculate the distribution of the difference among two successive order statistics), but do not refer to the original question of prize allocation in contests.

Note that the right-hand side of the FOC is a function of y only.[36]

A contestant with the lowest possible ability $c = 1$ either never wins a prize (if $k > 2$) or wins for sure the second prize (if $k = 2$). Hence, the optimal effort of this type is always zero, and this yields the boundary condition $y(0) = 1$.

Denote

$$\text{(A3)}\quad G(y) = V_1(k-1) \times \int_y^1 \frac{1}{t}(1-F(t))^{k-2}F'(t)\,dt + V_2(k-1)\int_y^1 \frac{1}{t}(1-F(t))^{k-3} \times [1-(k-1)F(t)]F'(t)\,dt.$$

The solution to the differential equation with the boundary condition is given by:

$$\text{(A4)}\quad \int_x^0 dt = -G(y).$$

We obtain that $x = G(y) = G(b^{-1}(x))$, and therefore that $b \equiv G$. Thus, the effort function of every player is given by $b(c) = A(c)V_1 + B(c)V_2$, where:

$$\text{(A5)}\quad A(c) = (k-1)\int_c^1 \frac{1}{a}(1-F(a))^{k-2} \times F'(a)\,da$$

$$B(c) = (k-1)\int_c^1 \frac{1}{a}(1-F(a))^{k-3} \times [(k-1)F(a)-1]F'(a)\,da.$$

We now check that the candidate equilibrium function b is strictly monotonic decreasing (it is clearly differentiable). Note first that

$$\text{(A6)}\quad A'(c) = -(k-1)\frac{1}{c}(1-F(c))^{k-2} \times F'(c) < 0$$

for all $c \in [m, 1)$. We have also

$$\text{(A7)}\quad B'(c) = (k-1)\frac{1}{c}(1-F(c))^{k-3}F'(c) \times [(1-(k-1)F(c)]F'(c).$$

Because $V_1 \geq V_2$ we obtain for all $c \in [m, 1)$:

$$\text{(A8)}\quad \begin{aligned} b'(c) &= A'(c)V_1 + B'(c)V_2 \\ &\leq V_2(A'(c) + B'(c)) \\ &= -V_2(k-1)(k-2)\frac{1}{c}F(c) \\ &\quad \times (1-F(c))^{k-3}F'(c) < 0. \end{aligned}$$

Assuming that all contestants other than i play according to b, we finally need to show that, for any type c of player i, the effort $b(c)$ maximizes the expected utility of that type. The necessary first-order condition is clearly satisfied [since this is how we "guessed" $b(c)$ to start with]. We now show that a sufficient second-order condition (called "pseudo-concavity") is satisfied. Let

$$\pi(x, c) = V_1(1-F(b^{-1}(x)))^{k-1} + (k-1)V_2F(b^{-1}(x)) \times (1-F(b^{-1}(x)))^{k-2} - cx$$

be the expected utility of player i with type c that makes a effort x. We will show that the derivative $\pi_x(c, x)$ is nonnegative if x is smaller than $b(c)$ and nonpositive if x is larger than $b(c)$. As $\pi(x, c)$ is continuous in x, this implies that $\pi(x, c)$ is maximized at $x = b(c)$. Note that

$$\pi_x(x, c) = -(k-1)(V_1 - V_2)\frac{db^{-1}(x)}{dx} \times (1-F(b^{-1}(x)))^{k-2}F'(b^{-1}(x)) - (k-1)(k-2)V_2\frac{db^{-1}(x)}{dx}$$

[36] i.e., this is a differential equation with separated variables.

$$\times F(b^{-1}(x))(1 - F(b^{-1}(x)))^{k-3}$$
$$\times F'(b^{-1}(x)) - c.$$

Let $x < b(c)$, and let $\hat{c}$ be the type who is supposed to bid x, that is $b(\hat{c}) = x$. Note that $\hat{c} > c$ since b is strictly decreasing. Differentiating $\pi_x(x, c)$ with respect to c yields $\pi_{xc}(x, c) = -1 < 0$. That is, the function $\pi_x(x, \cdot)$ is decreasing in c. Since $\hat{c} > c$, we obtain $\pi_x(x, c) \geq \pi_x(x, \hat{c})$.

Since $x = b(\hat{c})$ we obtain by the first-order condition that $\pi_x(x, \hat{c}) = 0$, and therefore that $\pi_x(x, c) \geq 0$ for every $x < b(c)$. A similar argument shows that $\pi_x(x, c) \leq 0$ for every $x > b(c)$.

PROOF OF PROPOSITION 3:

As in the proof of Proposition 1, player i's maximization problem reads:

$$\text{(A9)} \quad \max_x[V_1(1 - F(b^{-1}(x)))^{k-1} + (k-1)V_2F(b^{-1}(x)) \times (1 - F(b^{-1}(x)))^{k-2} - c\gamma(x)].$$

Letting y denote the inverse of b, the FOC is:

$$\text{(A10)} \quad \gamma'(x) = -(k-1)(V_1 - V_2)y'\frac{1}{y} \times (1 - F(y))^{k-2}F'(y) - (k-1)(k-2)V_2y'\frac{1}{y} \times F(y)(1 - F(y))^{k-3}F'(y).$$

Integration and the use of the boundary condition $y(1) = 0$ yield $\gamma(x) = G(y)$, where $G(y)$ is defined exactly as in the proof of Proposition 1 (see equation A3). Hence, $x = \gamma^{-1}(G(y)) = g(G(b^{-1}(x)))$ and $b \equiv g(G)$. The candidate equilibrium-effort function $b(c) = g(A(c)V_1 + B(c)V_2)$ is strictly decreasing since, for all $c \in [m, 1)$, it holds:

$$\text{(A11)} \quad \frac{dg}{dc}(A(c)V_1 + B(c)V_2) = g'(A(c)V_1 + B(c)V_2) \times (A'(c)V_1 + B'(c)V_2) < 0.$$

The last inequality follows because $g' > 0$ by assumption, while $A'(c)V_1 + B'(c)V_2 < 0$ since this is the derivative of the effort function with linear cost functions (see proof of Proposition 1). For the sufficient second-order condition we proceed exactly as in the proof of Proposition 1.

Appendix B: Optimal Prize Allocation

The following technical lemma is important since it is repeatedly used in the proofs of Propositions 2, 4, and 5.

LEMMA 1: *Assume that there are two prizes, $V_1 \geq V_2 \geq 0$, and let $b(c) = A(c)V_1 + B(c)V_2$ be the symmetric equilibrium bid function for $k \geq 3$ contestants having linear cost functions. Then the following properties hold:*

1. $A(1) = B(1) = 0$.
2. $\forall c \in [m, 1)$, $A(c) > 0$, *and* $A'(c) < 0$.
3. *Let c^* be such that* $F(c^*) = \dfrac{1}{k-1}$. *Then* $B'(c^*) = 0$, $B'(c) > 0$ *for all* $c \in [m, c^*)$, *and* $B'(c) < 0$ *for all* $c \in (c^*, 1]$.
4. $|B'(c)| > |A'(c)|$ *for c in a neighborhood of 1.*
5. $B(m) < 0$.
6. *For any $k > 2$, there exists a unique point $c^{**} \neq 1$ such that* $A(c^{**}) = B(c^{**})$.
7. $\int_m^1 (B(c) - A(c))F'(c)\,dc < 0.$[37]

PROOF:

1. This is obvious by definition.
2. $A(c) > 0$ for $c \in [m, 1)$ is obvious by definition. Further we have $A'(c) = -(k-1)\frac{1}{c}(1 - F(c))^{k-2}F'(c) < 0$ for all $c \in [m, 1)$ and $A'(1) = 0$.

[37] In order to prove that one prize is optimal even if the designer can award more than two prizes, it is enough to show that for all s, $2 \leq s \leq p$, it holds that $\int_m^1 \left(\int_c^1 -\frac{1}{a}(F_s'(a) - F_1'(a))\,da\right) dc < 0$ [see formulae (C1) and (C2) below]. The proof here treats the case $s = 2$. The proofs for the cases $s > 2$ are completely analogous.

3. $B'(c) = (k - 1)\frac{1}{c}(1 - F(c))^{k-3}F'(c) \times [(1 - (k - 1)F(c)]$. For c^* such that $F(c^*) = \frac{1}{k-1}$ we obtain $B'(c^*) = 0$. Moreover, $B'(c) > 0$ for all $c \in [m, c^*)$, and $B'(c) < 0$ for all $c \in (c^*, 1)$. Finally, $B'(1) = 0$.
4. For all $c \in [c^*, 1)$ we obtain that

$$\text{(B1)} \quad |B'(c)| - |A'(c)| = -B'(c) + A'(c) = (k - 1)\frac{1}{c}(1 - F(c))^{k-3} \times F'(c)(kF(c) - 2).$$

For $k > 2$ we obtain that $|B'(c)| - |A'(c)|$ is positive for c close enough to 1 (since $kF(c) > 2$ for such types).

5. We have

$$\text{(B2)} \quad B(m) = (k-1)\int_m^{c^*} \frac{1}{a}(1 - F(a))^{k-3} \times [(k-1)F(a) - 1]F'(a)\,da + (k-1)\int_{c^*}^1 \frac{1}{a}(1 - F(a))^{k-3} \times [(k-1)F(a) - 1]F'(a)\,da < (k-1)\int_m^1 (1 - F(a))^{k-3} \times [(k-1)F(a) - 1]F'(a)\,da.$$

The last inequality follows by noting that the integrand in the first integral is negative and that the integrand in the second integral is positive. Thus, if we multiply both integrands by the increasing function $h(a) = a$ we strictly increase the value of the sum of the two integrals. In order to prove that $B(m) < 0$ it is then enough to prove that $\int_m^1 (1 - F(a))^{k-3}[(k-1)F(a) - 1]F'(a)\,da = 0$. By the change of variable $z = F(a)$, we obtain

$$\text{(B3)} \quad \int_m^1 (1 - F(a))^{k-3} \times [(k-1)F(a) - 1]F'(a)\,da = \int_0^1 (1 - z)^{k-3}[(k-1)z - 1]\,dz = 0.$$

6. This follows by combining all properties above.
7. We know that $B(c) - A(c) > 0$ for all $c \in [m, c^{**})$ and that $B(c) - A(c) < 0$ for all $c \in (c^{**}, 1)$. This yields:

$$\text{(B4)} \quad \int_m^1 (B(c) - A(c))F'(c)\,dc = \int_m^{c^{**}} (B(c) - A(c))F'(c)\,dc + \int_{c^{**}}^1 (B(c) - A(c))F'(c)\,dc$$
$$= (k-1)\int_m^{c^{**}} \left[\int_c^1 \frac{(1 - F(a))^{k-3}}{a} \times (kF(a) - 2)F'(a)\,da\right]F'(c)\,dc + (k-1)\int_{c^{**}}^1 \left[\int_c^1 \frac{(1 - F(a))^{k-3}}{a} \times (kF(a) - 2)F'(a)\,da\right]F'(c)\,dc$$

$$< (k-1)\frac{1}{c^{**}}\int_m^{c^{**}}\left[\int_c^1 (1-F(a))^{k-3} \times (kF(a)-2)F'(a)\,da\right]F'(c)\,dc$$

$$+ (k-1)\frac{1}{c^{**}}\int_{c^{**}}^1\left[\int_c^1 (1-F(a))^{k-3} \times (kF(a)-2)F'(a)\,da\right]F'(c)\,dc$$

$$= (k-1)\frac{1}{c^{**}}\int_m^1\left[\int_c^1 (1-F(a))^{k-3} \times (kF(a)-2)F'(a)\,da\right]F'(c)\,dc$$

$$= \frac{1}{c^{**}}\int_0^1\left[\int_v^1 (k-1)(1-z)^{k-3} \times (kz-2)\,dz\right]dv = 0.$$

The last equality follows by the changes of variables $F(a) = z$ and $F(c) = v$.

PROOFS OF PROPOSITIONS 4 AND 5:

Recall that the designer's revenue as a function of the value of the second prize is:

$$\text{(B5)}\quad R(\alpha) = k\int_m^1 g[A(c) + \alpha(B(c) - A(c))]F'(c)\,dc.$$

Note that

$$\text{(B6)}\quad R'(\alpha) = k\int_m^1 (B(c) - A(c))g'[A(c) + \alpha(B(c) - A(c))]F'(c)\,dc$$

and that:

$$\text{(B7)}\quad R''(\alpha) = k\int_m^1 (B(c) - A(c))^2 g''[A(c) + \alpha(B(c) - A(c))]F'(c)\,dc.$$

Observe also that

$$\text{(B8)}\quad \frac{dg'[A(c) + \alpha(B(c) - A(c))]}{dc} = g''[A(c) + \alpha(B(c) - A(c))] \times [(1-\alpha)A'(c) + \alpha B'(c)]$$

and that

$$\text{(B9)}\quad (1-\alpha)A'(c) + \alpha B'(c) < 0$$

since this last term is the derivative of the equilibrium-effort function for contestants having linear cost functions.

Concave Cost Functions.—The cost function of contestant i with ability c, $c\gamma$, has the additional feature that $\gamma'' \leq 0$. Hence $g'' = (\gamma^{-1})'' \geq 0$. By equations (B8) and (B9) we obtain that the positive function $g'[A(c) + \alpha(B(c) - A(c))]$ is decreasing in c. By Lemma 1, there exists a unique point $c^{**} \neq 1$ such that $B(c) < A(c)$ for all $c \in [m, c^{**})$ and $B(c) > A(c)$ for all $c \in (c^{**}, 1]$. This means that all negative terms of the form $B(c) - A(c)$ in the integral defining $R'(\alpha)$ are multiplied by relatively high values of g', while all positive terms $B(c) - A(c)$ are multiplied by relatively lower values. By Lemma 1, we obtain:

$$\text{(B10)}\quad R'(\alpha) = k\int_m^1 g'[A(c) + \alpha(B(c) - A(c))] \times (B(c) - A(c))F'(c)\,dc < 0.$$

Hence, the designer's payoff function has a maximum at $\alpha = 0$, and a single prize is optimal.

Convex Cost Functions.—The cost function $c\gamma$ has the additional feature that $\gamma'' \geq 0$. Hence $g'' = (\gamma^{-1})'' \leq 0$. By equations (B8) and (B9) we obtain that the positive function $g'[A(c) + \alpha(B(c) - A(c))]$ is increasing in c. This means that in the integral defining $R'(\alpha)$ all negative terms of the form $B(c) - A(c)$ are multiplied by relatively low values of g', while all positive terms $B(c) - A(c)$ are multiplied by higher values. Moreover, for all $\alpha \in [0, \frac{1}{2}]$ we have

$$\text{(B11)} \quad R''(\alpha) = k \int_m^1 (B(c) - A(c))^2 \times g''[A(c) + \alpha(B(c) - A(c))]F'(c)\,dc \leq 0.$$

If condition 9 is satisfied we have then:

$$\text{(B12)} \quad R'(0) = k \int_m^1 (B(c) - A(c)) \times g'(A(c))F'(c)\,dc > 0.$$

Hence, the revenue function $R(\alpha)$ cannot have a maximum at $\alpha = 0$. It either has a maximum at α^* such that $R'(\alpha^*) = 0$ or at $\alpha = \frac{1}{2}$.

For the converse, assume that two prizes are optimal. This means that $\alpha = 0$ is not a maximum of $R(\alpha)$. If condition 9 is not satisfied we obtain $R'(0) \leq 0$. Together with $R''(\alpha) \leq 0$ for all $\alpha \in [0, \frac{1}{2}]$ we obtain a contradiction.

APPENDIX C

The Case of a Uniform Distribution of Abilities.—Assume that two prizes are awarded, that costs are linear, and that $F(c) = \frac{1}{1-m}c - \frac{m}{1-m}$, i.e., abilities are uniformly distributed on the interval $[m, 1]$. The expressions in the statement of Proposition 1 become now:

$$\text{(B13)} \quad A(c) = \left(\frac{1}{1-m}\right)^{k-1} (1-k) \times \left(\sum_{s=1}^{k-2} \frac{(1-c)^s}{s} + \ln c\right),$$

$$B(c) = \left(\frac{1}{1-m}\right)^{k-1} (k-1) \times \left[\sum_{s=1}^{k-2} \frac{(1-c)^s}{s} + \ln c + (1-c)^{k-2} + m(k-2) \times \left(\sum_{s=1}^{k-3} \frac{(1-c)^s}{s} + \ln c\right)\right].$$

The Symmetric Equilibrium with p Prizes.— Fix agent i, and let $F_s(a)$, $1 \leq s \leq p$, denote the probability that agent i with type a meets $k - 1$ competitors such that $s - 1$ of them have lower types, and $k - s$ have higher types. Hence F_s is exactly the probability of winning the s'th prize.[38] We have then

$$\text{(C1)} \quad F_s(a) = \frac{(k-1)!}{(s-1)!(k-s)!} \times (1 - F(a))^{k-s}(F(a))^{s-1}.$$

The corresponding derivatives are given by

$$\text{(C2)} \quad F_1'(a) = -(k-1)(1 - F(a))^{k-2}F'(a)$$

[38] Recall that in equilibrium we expect i to bid more than competitors with higher types (lower ability). For the relation between the probabilities F_s and the distribution of differences of successive order statistics (which were in fact Galton's theme), see Chapter 2 in Arnold and Balakrishnan (1989).

and by

$$\text{(C3)}\quad F'_s(a) = \frac{(k-1)!}{(s-1)!(k-s)!} \times (1-F(a))^{k-s-1}(F(a))^{s-2}F'(a) \times [(1-k)F(a) + (s-1)]$$

for $s > 1$. Note that $A(c) = \int_c^1 -\frac{1}{a}F'_1(a)\,da$ and that $B(c) = \int_c^1 -\frac{1}{a}F'_2(a)\,da$. Analogously to the case of two prizes, the equilibrium bid for any number of prizes $p > 2$ and $k \geq p$ contestants with linear cost functions is given by:[39]

$$\text{(C4)}\quad b(c) = \sum_{s=1}^{p} V_s \int_c^1 -\frac{1}{a}F'_s(a)\,da.$$

The equilibrium bid for $p > 2$ prizes, and $k \geq p$ contestants with cost functions of the form $c\gamma(x)$ is given by $b(c) = \gamma^{-1}\left(\sum_{s=1}^{p} V_s \times \int_c^1 -\frac{1}{a}F'_s(a)\,da\right)$, where the $F'_s(a)$ are given in formulas (C2) and (C3).

[39] A proof of monotonicity is more complex here, and it uses stochastic-dominance relations among order statistics (see Shaked and Shantikumar, 1994).

REFERENCES

Amann, Erwin and Leininger, Wolfgang. "Asymmetric All-Pay Auctions with Incomplete Information: The Two-Player Case." *Games and Economic Behavior*, May 1996, *14*(1), pp. 1–18.

Anton, James J. and Yao, Dennis A. "Coordination in Split Award Auctions." *Quarterly Journal of Economics*, May 1992, *107*(2), pp. 681–707.

Arnold, Barry C. and Balakrishnan, N. *Relations, bounds and approximations for order statistics*, Lectures Notes in Statistics 53. New York: Springer-Verlag, 1989.

Barut, Yasar and Kovenock, Dan. "The Symmetric Multiple Prize All-Pay Auction with Complete Information." *European Journal of Political Economy*, November 1998, *14*(4), pp. 627–44.

Baye, Michael R.; Kovenock, Dan and de Vries, Casper G. "It Takes Two to Tango: Equilibria in a Model of Sales." *Games and Economic Behavior*, October 1992, *4*(4), pp. 493–510.

______. "Rigging the Lobbying Process: An Application of the All-Pay Auction." *American Economic Review*, March 1993, *83*(1), pp. 289–94.

______. "The All-Pay Auction with Complete Information." *Economic Theory*, August 1996, *8*(2), pp. 291–305.

Broecker, Thorsten. "Credit-Worthiness Tests and Interbank Competition." *Econometrica*, March 1990, *58*(2), pp. 429–52.

Bulow, Jeremy and Klemperer, Paul. "Auctions versus Negotiations." *American Economic Review*, March 1996, *86*(1), pp. 180–94.

______. "The Generalized War of Attrition." *American Economic Review*, March 1999, *89*(1), pp. 175–89.

Clark, Derek J. and Riis, Christian. "Competition over More Than One Prize." *American Economic Review*, March 1998, *88*(1), pp. 276–89.

Dasgupta, Partha. "The Theory of Technological Competition," in J. Stiglitz and G. Mathewson, eds., *New developments in the analysis of market structure*. Cambridge, MA: MIT Press, 1986, pp. 519–47.

Ehrenberg, Ronald G. and Bognanno, Michael L. "Do Tournaments Have Incentives Effects?" *Journal of Political Economy*, December 1990, *98*(6), pp. 1307–24.

Frank, Robert H. and Cook, Philip J. *The winner-take-all society*. New York: Free Press, 1995.

Fullerton, Don and McAfee, R. Preston. "Auctioning Entry into Tournaments." *Journal of Political Economy*, June 1999, *107*(3), pp. 573–605.

Galton, Francis. "The Most Suitable Proportion Between the Values of First and Second Prizes." *Biometrika*, August 1902, *1*(4), pp. 385–90.

Glazer, Amihai and Hassin, Refael. "Optimal Contests." *Economic Inquiry*, January 1988, *26*(1), pp. 133–43.

Green, Jerry R. and Stokey, Nancy L. "A Comparison of Tournaments and Contracts." *Journal of Political Economy*, June 1983, *91*(3), pp. 349–64.

Hillman, Arye L. and Riley, John G. "Politically Contestable Rents and Transfers." *Economics and Politics*, Spring 1989, *1*(1), pp. 17–39.

Knoeber, Charles R. and Thurman, Walter N. "Testing the Theory of Tournaments: An Empirical Analysis of Broiler Production." *Journal of Labor Economics*, April 1994, *12*(2), pp. 155–79.

Krishna, Vijay and Morgan, John. "An Analysis of the War of Attrition and the All-Pay Auction." *Journal of Economic Theory*, February 1997, *72*(2), pp. 343–62.

______. "The Winner-Take-All Principle in Small Tournaments." *Advances in Applied Microeconomics*, 1998, *7*, pp. 61–74.

Lazear, Edward P. and Rosen, Sherwin. "Rank-Order Tournaments as Optimum Labor Contracts." *Journal of Political Economy*, October 1981, *89*(5), pp. 841–64.

Lizzeri, Alessandro and Persico, Nicola. "Uniqueness and Existence of Equilibrium in Auctions with a Reserve Price." *Games and Economic Behavior*, January 2000, *30*(1), pp. 83–114.

Mas-Colell, Andreu; Whinston, Michael D. and Green, Jerry R. *Microeconomic theory*. New York: Oxford University Press, 1995.

Moulin, Herve. *Game theory for the social sciences*, 2nd Ed. New York: New York University Press, 1986.

Myerson, Roger B. "Optimal Auction Design." *Mathematics of Operations Research*, February 1981, *6*(1), pp. 58–73.

Nalebuff, Barry J. and Stiglitz, Joseph E. "Prizes and Incentives: Towards a General Theory of Compensation and Competition." *Bell Journal of Economics*, Spring 1983, *14*(1), pp. 21–43.

Pearson, Karl. "Note On Francis Galton's Problem." *Biometrika*, August 1902, *1*(4), pp. 390–99.

Riley, John G. and Samuelson, William F. "Optimal Auctions." *American Economic Review*, June 1981, *71*(3), pp. 381–92.

Rosen, Sherwin. "Prizes and Incentives in Elimination Tournaments." *American Economic Review*, September 1986, *76*(4), pp. 701–15.

Shaked, Moshe and Shantikumar, George J. *Stochastic orders and their applications*. San Diego, CA: Academic Press, 1994.

Sobel, Dava. *Longitude*. New York: Penguin Books, 1996.

Taylor, Curtis R. "Digging for Golden Carrots: An Analysis of Research Tournaments." *American Economic Review*, September 1995, *85*(4), pp. 873–90.

Tullock, Gordon. "Efficient Rent-Seeking," in J. Buchanan et al., eds., *Towards a theory of the rent-seeking society*. College Station, TX: Texas A&M University Press, 1980.

Varian, Hal R. "A Model of Sales." *American Economic Review*, September 1980, *70*(4), pp. 651–59.

Weber, Robert. "Auctions and Competitive Bidding," in H. P. Young, ed., *Fair allocation*. Providence, RI: American Mathematical Society, 1985, pp. 143–70.

Wilson, Robert. "Auctions of Shares." *Quarterly Journal of Economics*, November 1979, *93*(4), pp. 675–89.

Windham, Patrick. "A Taxonomy of Technology Prizes and Contests," in *Concerning federally sponsored inducement prizes in engineering and science*. Washington, DC: National Academy Press, 1999, pp. 21–34.

Wright, Brian Davern. "The Economics of Invention Incentives: Patents, Prizes, and Research Contracts." *American Economic Review*, September 1983, *73*(4), pp. 691–707.