

Races and Tournaments

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JEL Codes: D02, J4, L2, M5

1. Tournament Game Definition

■ Consider an n -player simultaneous tournament ($n \geq 2$). Each player draws an "ability", $a_i \in [0, m]$, where $m < 1$. Ability is distributed randomly according to distribution F . The distribution has continuous support with density f and is twice continuously differentiable. The distribution F is common knowledge while a player's ability is private information.

Player i chooses a costly bid, $q_i \in \mathbf{R}_+$.

$$c(a, q, t) = \delta(a)\gamma(q)\eta(t) > 0$$

$$c(a, 0, t) = 0$$

$$c_a(a, q, t) = \gamma(q)\eta(t)\frac{\partial\delta}{\partial a}(a) < 0$$

$$c_q(a, q, t) = \delta(a)\eta(t)\frac{\partial\gamma}{\partial q}(q) > 0$$

$$c_t(a, q, t) = \delta(a)\gamma(q)\frac{\partial\eta}{\partial t}(t) < 0$$

$$c_{aq}(a, q, t) = c_{qa}(a, q, t) = \eta(t)\frac{\partial\delta}{\partial a}(a)\frac{\partial\gamma}{\partial q}(q) < 0$$

$$c_{at}(a, q, t) = c_{ta}(a, q, t) = \gamma(q)\frac{\partial\delta}{\partial a}(a)\frac{\partial\eta}{\partial t}(t) > 0$$

$$c_{qt}(a, q, t) = c_{tq}(a, q, t) = \delta(a)\frac{\partial\gamma}{\partial q}(q)\frac{\partial\eta}{\partial t}(t) < 0$$

$p < n$ prizes are awarded to the p bidders with the highest quality bids in rank order at time T^{max} . Bids are

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not observed by other players. The value of the prizes is strictly decreasing, $V_1 > V_2 > \dots > V_p$. Ties are broken randomly. Players are risk-neutral.

2. Tournament Equilibrium

■ Note that submitting a bid prior to T^{max} is strictly dominated by submitting the same bid at T^{max} . Since costs fall monotonically in time and signaling is not possible, later is always strictly preferred.

Assume that there exists a symmetric equilibrium bid function $q(a)$, strictly monotonically increasing in ability. Note that strategies mapping from ability to bids induce a distribution on bids. Denote the distribution arising from the symmetric equilibrium as G . Let r_i be the bid rank of the i^{th} player. Let

$$P_{j,n}(b) = \Pr \{r_i = j\} = \binom{n-1}{j-1} (1 - G(q))^{j-1} G(q)^{n-j}.$$

Since q is strictly monotone by assumption, there exists an inverse function $a(q)$ mapping abilities from bids.

Then

$$P_{j,n}(q) = P_{j,n}(a) = \binom{n-1}{j-1} (1 - F(a(q)))^{j-1} F(a(q))^{n-j}.$$

The probability of winning prize j is closely related to order statistics. Let $F_{j,n}$ be the distribution of the j^{th} lowest value from a sample of n random variables distributed according to F . $F_{j,n}$ can be written as

$$F_{j,n} = \sum_{k=j}^n \binom{n}{k} F^k (1 - F)^{n-k}.$$

The pdf of $F_{j,n}$ can be written as

$$f_{j,n} = \frac{n!}{(j-1)!(n-j)!} F^{j-1} (1 - F)^{n-j} f.$$

It follows that

$$\begin{aligned} P_{j,n} &= \binom{n-1}{j-1} (1 - F)^{j-1} F^{n-j} = F_{n-j,n-1} - F_{n-j+1,n-1} \\ \frac{\partial P_{j,n}}{\partial a} &= f_{n-j,n-1} - f_{n-j+1,n-1} \end{aligned}$$

Assume all other players are using the strategies defined by $q(a)$. Then the expected payoff for player i for

bid q is

$$\pi(q) = \sum_{j=1}^p P_{j,n}(a(q))V_j - \delta(a_i)\eta(T^{max})\gamma(q)$$

FOCs imply that

$$\frac{\partial \pi}{\partial q}(q) = \sum_{j=1}^p \frac{\partial P_{j,n}}{\partial a}(a(q)) \frac{\partial a}{\partial q}(q) V_j - \delta(a_i)\eta(T^{max}) \frac{\partial \gamma}{\partial q}(q) = 0$$

Let $V_{p+1} = 0$ and $\Delta V_j = V_j - V_{j+1}$. Then the sum can be rearranged.

$$\begin{aligned} \sum_{j=1}^p \Delta V_j f_{n-j,n-1}(a(q)) \frac{\partial a}{\partial q}(q) - \delta(a_i)\eta(T^{max}) \frac{\partial \gamma}{\partial q}(q) &= 0 \\ \sum_{j=1}^p \Delta V_j f_{n-j,n-1}(a(q)) \frac{\partial a}{\partial q}(q) &= \delta(a_i)\eta(T^{max}) \frac{\partial \gamma}{\partial q}(q) \\ [\delta(a_i)\eta(T^{max})]^{-1} \sum_{j=1}^p \Delta V_j f_{n-j,n-1}(a(q)) \frac{\partial a}{\partial q}(q) &= \frac{\partial \gamma}{\partial q}(q) \\ [\delta(a_i)\eta(T^{max})]^{-1} \sum_{j=1}^p \Delta V_j f_{n-j,n-1}(a(q)) &= \frac{\partial \gamma}{\partial q}(q) \frac{\partial a}{\partial q}(q)^{-1} \\ [\delta(a_i)\eta(T^{max})]^{-1} \sum_{j=1}^p \Delta V_j f_{n-j,n-1}(a(q)) &= \frac{\partial \gamma}{\partial q}(q) \frac{\partial q}{\partial a}(a(q)). \end{aligned}$$

If q is a symmetric equilibrium, then the FOC must hold for all values of a_i .

$$\begin{aligned} [\delta(a)\eta(T^{max})]^{-1} \sum_{j=1}^p \Delta V_j f_{n-j,n-1}(a(q)) &= \frac{\partial \gamma}{\partial q}(q(a)) \frac{\partial q}{\partial a}(a(q)) \\ [\delta(a)\eta(T^{max})]^{-1} \sum_{j=1}^p \Delta V_j f_{n-j,n-1}(a(q)) &= \frac{\partial}{\partial a} [\gamma(q(a))]. \end{aligned}$$

Then the FOC is a separated differential equation and a closed form of q can be recovered.

$$\begin{aligned}
\frac{\partial}{\partial a} [\gamma(q(a))] &= [\delta(a)\eta(T^{max})]^{-1} \sum_{j=1}^p \Delta V_j f_{n-j,n-1}(a(q)) \\
\int_0^a \frac{\partial}{\partial a} [\gamma(q(z))] dz &= \int_0^a [\delta(z)\eta(T^{max})]^{-1} \sum_{j=1}^p \Delta V_j f_{n-j,n-1}(z) dz \\
\gamma(q(a)) &= \int_0^a [\delta(z)\eta(T^{max})]^{-1} \sum_{j=1}^p \Delta V_j f_{n-j,n-1}(z) dz - \gamma(q(0)) \\
\gamma(q(a)) &= \int_0^a [\delta(z)\eta(T^{max})]^{-1} \sum_{j=1}^p \Delta V_j f_{n-j,n-1}(z) dz \\
q(a) &= \gamma^{-1} \left\{ \int_0^a [\delta(z)\eta(T^{max})]^{-1} \sum_{j=1}^p \Delta V_j f_{n-j,n-1}(z) dz \right\} \\
q(a) &= \gamma^{-1} \left\{ \frac{1}{\eta(T^{max})} \int_0^a [\delta(z)]^{-1} \sum_{j=1}^p \Delta V_j f_{n-j,n-1}(z) dz \right\} \\
q(a) &= \gamma^{-1} \left\{ \frac{1}{\eta(T^{max})} \sum_{j=1}^p \Delta V_j \int_0^a \delta(z)^{-1} f_{n-j,n-1}(z) dz \right\} \\
q(a) &= \gamma^{-1} \left\{ \sum_{j=1}^p \frac{\Delta V_j}{\eta(T^{max})} \int_0^a \delta(z)^{-1} f_{n-j,n-1}(z) dz \right\}
\end{aligned}$$

3. Race Definition

■ Consider an n -player simultaneous tournament ($n \geq 2$). Each player draws an "ability", $a_i \in [0, m]$, where $m < 1$. Ability is distributed randomly according to distribution F . The distribution has continuous support with density f and is twice continuously differentiable. The distribution F is common knowledge while a player's ability is private information.

Player i chooses a costly bid, $q_i \in \mathbf{R}_+$.

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$$c_q(a, q, t) = \delta(a)\eta(t) \frac{\partial \gamma}{\partial q}(q) > 0$$

$$c_t(a, q, t) = \delta(a)\gamma(q) \frac{\partial \eta}{\partial t}(t) < 0$$

$$\lim_{t \rightarrow \infty} \eta(t) = \underline{\eta} > 0$$

$$\begin{aligned} c_{aq}(a, q, t) &= c_{qa}(a, q, t) = \eta(t) \frac{\partial \delta}{\partial a}(a) \frac{\partial \gamma}{\partial q}(q) < 0 \\ c_{at}(a, q, t) &= c_{ta}(a, q, t) = \gamma(q) \frac{\partial \delta}{\partial a}(a) \frac{\partial \eta}{\partial t}(t) > 0 \\ c_{qt}(a, q, t) &= c_{tq}(a, q, t) = \delta(a) \frac{\partial \gamma}{\partial q}(q) \frac{\partial \eta}{\partial t}(t) < 0 \end{aligned}$$

$p < n$ prizes are awarded to the p bidders who submit the earliest in rank order with bid quality *at least* Q . Bids are not observed by other players. The value of the prizes is strictly decreasing, $V_1 > V_2 > \dots > V_p$. Ties are broken randomly. Players are risk-neutral.

4. Race Equilibrium

■ Since costs are strictly increasing in q , submitting a bid quality larger than Q is strictly dominated by submitting the same bid at Q . A positive bid with quality below Q will be rejected leading to a zero prize. Since positive bids carry positive costs, any positive bid below Q is strictly dominated by bidding 0. Hence, the only equilibrium bid qualities are Q and 0.

Assume that there exists a symmetric equilibrium timing function $t(a)$, strictly monotonically decreasing in ability. Note that strategies mapping from ability to timing induce a distribution on timings. Denote the distribution arising from the symmetric equilibrium as G . Let r_i be the timing rank of the i^{th} player. Let

$$P_{j,n}(b) = \Pr \{r_i = j\} = \binom{n-1}{j-1} (1 - G(b))^{j-1} G(b)^{n-j}.$$

Since t is strictly monotone by assumption, there exists an inverse function $a(t)$ mapping abilities from timing. Then

$$P_{j,n}(t) = P_{j,n}(a) = \binom{n-1}{j-1} (1 - F(a(t)))^{j-1} F(a(t))^{n-j}.$$

Assume all other players are using the strategies defined by $t(a)$. Then the expected payoff for player i ,

bidding Q , for timing t is

$$\pi(t) = \sum_{j=1}^p P_{j,n}(a(t))V_j - \delta(a_i)\eta(t)\gamma(Q)$$

FOCs imply that

$$\frac{\partial \pi}{\partial t}(t) = \sum_{j=1}^p \frac{\partial P_{j,n}}{\partial a}(a(t)) \frac{\partial a}{\partial t}(t) V_j - \delta(a_i)\gamma(Q) \frac{\partial \eta}{\partial t}(t) = 0$$

Let $V_{p+1} = 0$ and $\Delta V_j = V_j - V_{j+1}$. Then the sum can be rearranged.

$$\begin{aligned} \sum_{j=1}^p \Delta V_j f_{n-j,n-1}(a(t)) \frac{\partial a}{\partial t}(t) - \delta(a_i)\gamma(Q) \frac{\partial \eta}{\partial t}(t) &= 0 \\ \sum_{j=1}^p \Delta V_j f_{n-j,n-1}(a(t)) \frac{\partial a}{\partial t}(t) &= \delta(a_i)\gamma(Q) \frac{\partial \eta}{\partial t}(t) \\ \frac{1}{\delta(a_i)\gamma(Q)} \sum_{j=1}^p \Delta V_j f_{n-j,n-1}(a(t)) &= \frac{1}{\frac{\partial a}{\partial t}(t)} \frac{\partial \eta}{\partial t}(t) \\ \frac{1}{\delta(a_i)\gamma(Q)} \sum_{j=1}^p \Delta V_j f_{n-j,n-1}(a(q)) &= \frac{\partial t}{\partial a}(a(t)) \frac{\partial \eta}{\partial t}(t) \end{aligned}$$

If t is a symmetric equilibrium, then the FOC must hold for all values of a_i .

$$\begin{aligned} \frac{1}{\delta(a)\gamma(Q)} \sum_{j=1}^p \Delta V_j f_{n-j,n-1}(a) &= \frac{\partial t}{\partial a}(a) \frac{\partial \eta}{\partial t}(t) \\ \frac{1}{\delta(a)\gamma(Q)} \sum_{j=1}^p \Delta V_j f_{n-j,n-1}(a) &= \frac{\partial}{\partial a} [\eta(t(a))] \\ \eta(t(a)) &= \int_{\underline{a}}^a \frac{1}{\delta(z)\gamma(Q)} \sum_{j=1}^p \Delta V_j f_{n-j,n-1}(z) dz + \eta(t(\underline{a})) \\ t(a) &= \eta^{-1} \left\{ \int_{\underline{a}}^a \frac{1}{\delta(z)\gamma(Q)} \sum_{j=1}^p \Delta V_j f_{n-j,n-1}(z) dz + \eta(t(\underline{a})) \right\} \end{aligned}$$

Asserting the outside option constraint.

$$\sum_{j=1}^p P_{j,n}(\underline{a}) V_j - \delta(\underline{a}) \eta(t(\underline{a})) \gamma(Q) = 0$$

$$\sum_{j=1}^p P_{j,n}(\underline{a}) V_j = \delta(\underline{a}) \eta(t(\underline{a})) \gamma(Q)$$

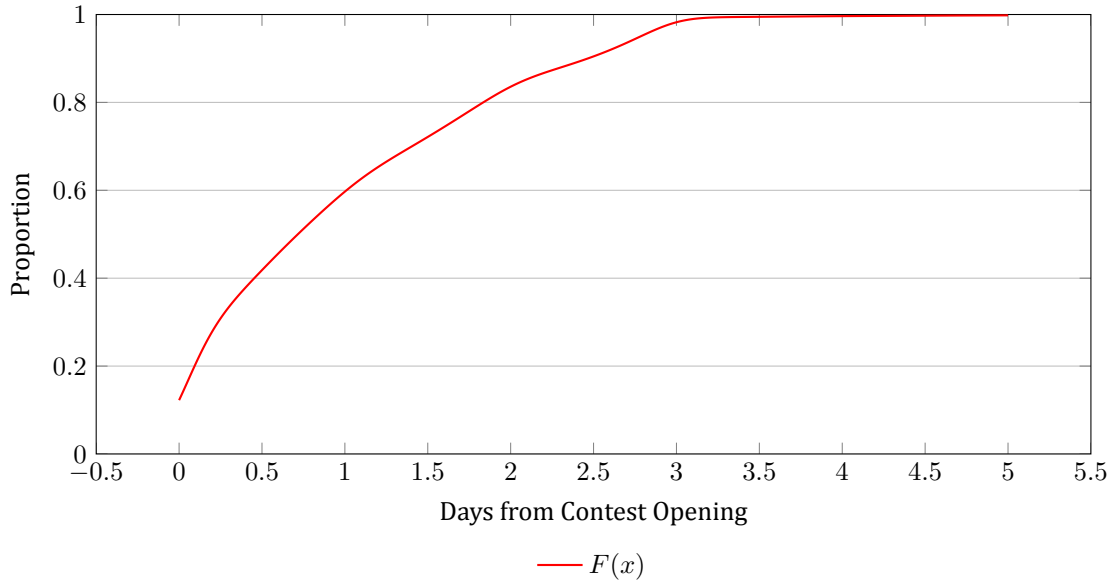
$$\sum_{j=1}^p P_{j,n}(\underline{a}) V_j \geq \delta(\underline{a}) \underline{\eta} \gamma(Q)$$

$$\frac{1}{\delta(\underline{a}) \gamma(Q)} \sum_{j=1}^p P_{j,n}(\underline{a}) V_j \geq \underline{\eta}$$

5. Empirical Analysis

■

FIGURE 1
CDF OF RELATIVE REGISTRATION TIMES



□ **Registration Time.**

□ **Development Ratings and Algorithm Ratings.**

	Development Ratings	Algorithm Ratings
Minimum	159	198
Maximum	2482	3067
Mean	1283	1229
corr	0.1135	

FIGURE 2
SCATTERPLOT OF ALGORITHM AND DEVELOPMENT RATINGS

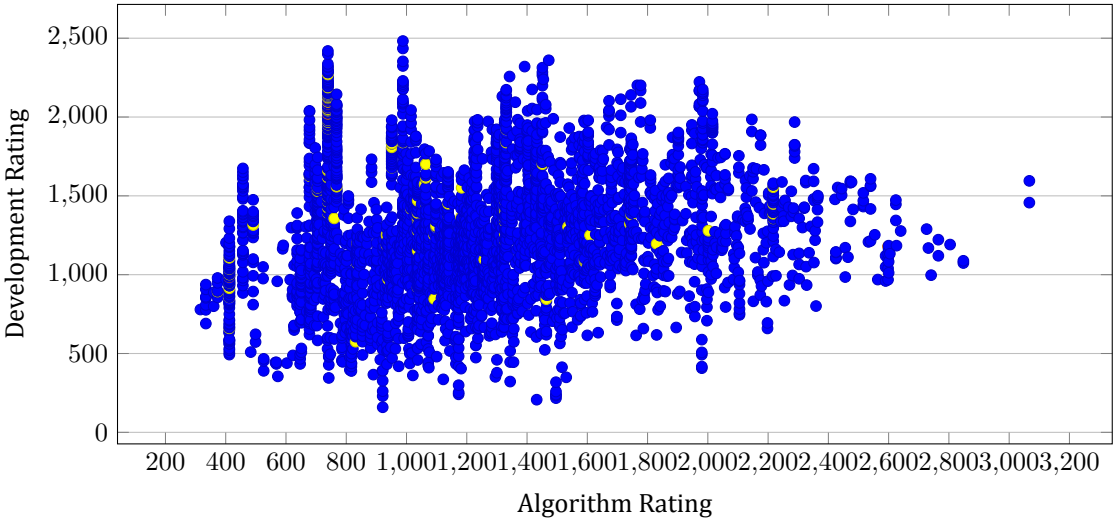


FIGURE 3
NONPARAMETRIC FIT USING DEVELOPMENT RATING AS ABILITY

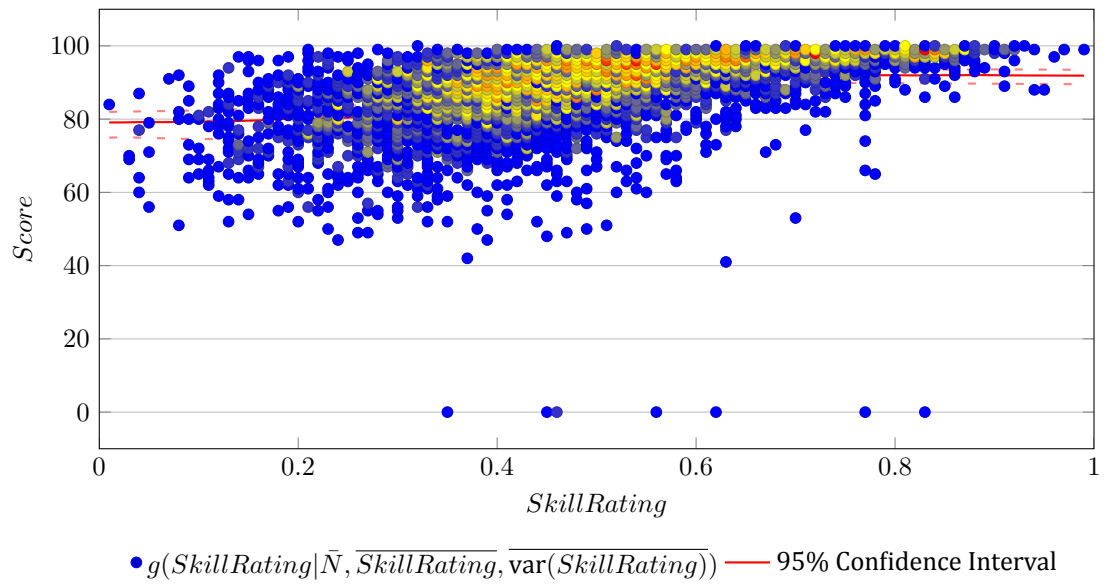


FIGURE 4
NONPARAMETRIC FIT USING ALGORITHM RATING AS ABILITY

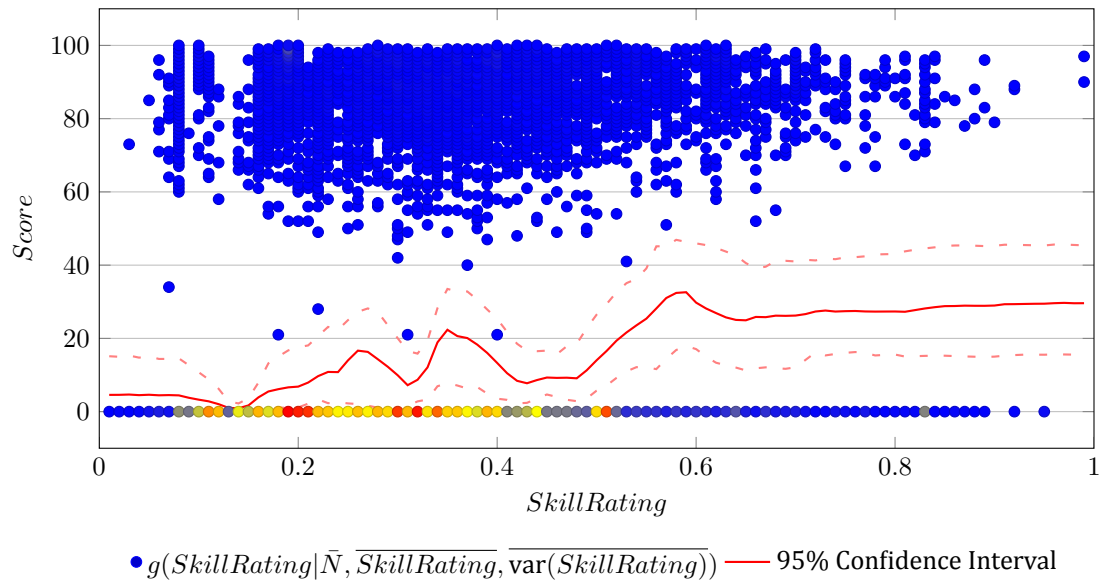


FIGURE 5
NONPARAMETRIC FIT WITH DISCONTINUITY USING ALGORITHM RATING AS ABILITY

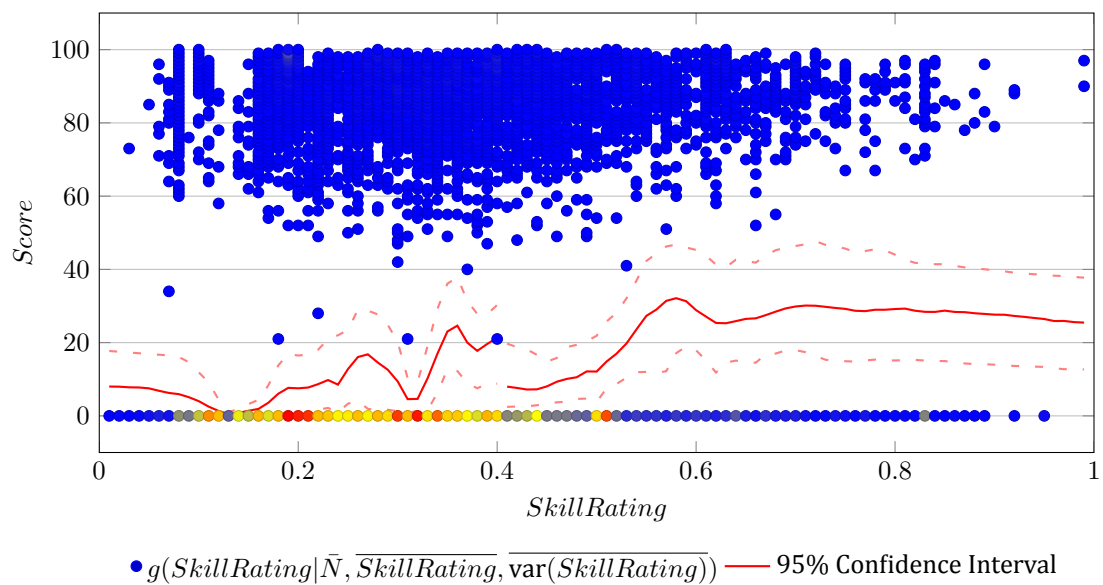


FIGURE 6
NONPARAMETRIC FIT USING MAX OF ALGORITHM AND DEVELOPMENT RATINGS AS ABILITY

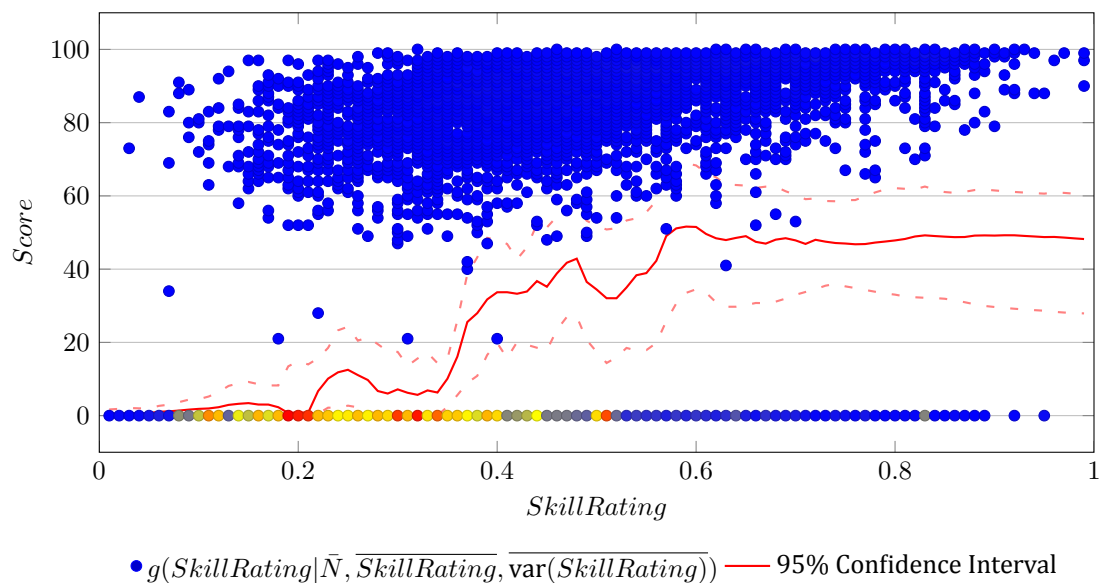


FIGURE 7
NONPARAMETRIC FIT WITH DISCONTINUITY USING MAX OF ALGORITHM AND DEVELOPMENT RATINGS AS ABILITY

