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Author(s): Christopher Harris and John Vickers
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Racing with Uncertainty

CHRISTOPHER HARRIS

and

JOHN VICKERS
Nuffield College, Oxford

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The paper presents two models of races in which there is both technological uncertainty and strategic interaction between competitors as the race unfolds. Most of the existing literature examines one or other of these features, but not the two combined. Our aim is to see how the efforts of competitors in a race vary with the intensity of rivalry between them. In our principal model, whch is of a one-dimensional race, it is shown that the leader in the race makes greater efforts than the follower, and efforts increase as the gap between competitors decreases. Under certain conditions the same results hold in our second, related model, which is of a two-dimensional race.

1. INTRODUCTION

A race is a competition in which a prize is awarded to the first competitor to achieve a given amount of progress. An important example in economics is a patent race, in which the first firm to acquire sufficient knowledge to make an innovation is granted a valuable patent.

There are at least two features that one would like to capture in a model of a race. First, there is *uncertainty* in the relationship between the efforts that a competitor makes and the progress towards success that he thereby achieves. Second, there is the *strategic interaction* between competitors as the race unfolds. Each competitor makes a series of decisions during the course of the race, each of which is made in the light of the progress that his rivals have made and with a view to influencing the efforts that they subsequently make. In particular one would like to ask whether the leader in a race makes greater efforts than a follower, whether efforts are greatest when the competitors are neck-and-neck, and so on.

Most existing models of this type of competition, however, focus on only one of these two features of races. The purpose of this paper is to present models of races that contain both features. Before considering models that attempt to combine both features, it is useful briefly to review the existing literature that has examined one or other of them.

The well-known models of R&D competition due to Loury (1979), Dasgupta and Stiglitz (1980), Lee and Wilde (1980), and Reinganum (1981, 1982) focus on uncertainty in the relationship between R&D efforts and progress towards success. They are not designed in a way that can reflect strategic interaction between competitors as the race unfolds. That is because in those models only one advance is needed for a firm to win the race, and technological uncertainty is exponential, i.e. the probability that z is sufficient knowledge to make the innovation is given as $1-\exp\{-z\}$. At any time the competition can be in one of only two states—either some firm has made an advance and hence secured victory (in which case the race is over) or else no firm has yet done so. With the assumption that technological uncertainty is exponential, it follows that the levels of knowledge acquired by firms as a result of their past R&D efforts are irrelevant to firms' current calculations and to the future of the race. There is no sense in which one can properly speak of one competitor being ahead of another, or of two competitors being neck-and-neck.

In the models of Loury, Dasgupta and Stiglitz, and Lee and Wilde, each firm chooses the level of its R&D efforts once-and-for-all, i.e. its strategy set is R_+. However, the stationary structure of these models is such that this involves no effective restriction, because at equilibrium no firm would wish to revise its rate of effort over time. In crude terms the reason is that if the race is still in progress at time $t>0$, the subgame starting from t is a carbon copy of the game commencing at time zero. The same is not true in Reinganum's model, where a firm's effort rate may depend upon time and upon rivals' knowledge levels, for two reasons. First, in her model there is a terminal date T at which all competition ceases. Second, there is an asymmetry in that the costs of R&D efforts are discounted over time, but the value of the prize at stake is not. The result is that firms in the model increase their R&D efforts as time progresses. If Reinganum's model is modified by discounting the value of the prize as well as costs—as might seem desirable—and if it is given an infinite horizon to eliminate end effects, then firms cease to vary their efforts with time. Indeed, the analysis of Reinganum's modified model shows that, even when the strategy spaces of Lee and Wilde's model are enlarged, equilibrium remains the same as originally determined by them. Thus, not only can Lee and Wilde's equilibrium be regarded as *an* equilibrium in a more general model, it can also be regarded as the *only* equilibrium. This provides some justification for employing the simpler strategy set.

The other feature of races—strategic interaction—has been examined in contexts containing no technological uncertainty by Fudenberg *et al.* (1983, Section 4) and Harris and Vickers (1985*a*, *b*). Their models yield stark results— for example that the behaviour of the winner of the race is often as if he faced no rivalry whatsoever, because his competitors know that he would rationally outdo any efforts that they made to beat him to the finishing line in the race; hence he has credible threats that deter them from making any efforts. Several factors combine to determine who is in that happy position, including the competitors' respective incentives to win the prize, their efficiency at R&D, their discount rates, and their initial distances from the finishing line. The stark nature of the results is largely due to the exclusion of uncertainty, which is a major shortcoming of the models.

The present paper is an attempt to develop two models in which there is technological uncertainty *and* strategic interaction between competitors. These models have their origins in the work of Vickers (1985), where they were analysed non-rigorously and with special functional forms. Our principal model is of a race between two players in which a prize is won by the first player to achieve a given lead over his rival. We label this a model of a *tug-of-war*. Progress in the competition is measured discretely. More precisely, the competition may be in any one of N active states. If the competition is in state n, where $1 \leqq n \leqq N$, and player A makes an advance, then it moves to state $n+1$, whereas if B makes an advance it moves to state $n-1$. The competition ends in victory for A (B) if state $N+1$ (0) is reached, and the victor is awarded a prize. (Note that if $N=1$ the model would be the two-player version of Lee and Wilde's (1980) model.) It is perhaps the simplest model with which to address the question of how effort rates vary with the intensity of rivalry, because the single state variable is a measure of the distance between the players. At symmetric equilibria of the model, which are shown to exist, the player that is ahead makes greater efforts than the follower; the follower's efforts fall if his deficit increases; the leader's efforts may or may not fall if his lead increases, but they continue to fall once they begin to do so.

The model of a tug-of-war is of interest in that it is the simplest framework in which to ask how efforts vary with intensity of rivalry. It also offers insights into the workings of our second model, which is of a *multi-stage race*, and in which the winner is the first of the two players to make a given number of advances. The current state of the race is described by a *pair* (m, n) where $m\,(n)$ is the number of advances that player $A\,(B)$ requires to achieve victory. Thus the race has two dimensions, whereas the tug-of-war is one-dimensional. At the unique equilibrium, the leader makes greater efforts than the follower if the leader is close to success in the sense of having no more than two stages to go. Subject to a similar proviso, efforts are greater when the gap between the competitors diminishes, i.e. as the follower catches up. These results hold generally in examples that we have solved explicitly.

In independent work other authors have also examined aspects of races with uncertainty, notably Grossman and Shapiro (1985) and Judd (1985).[1] Grossman and Shapiro characterise equilibrium in a model that is closely related to our second model. The main differences are that they have positive discounting, and they consider only a two-stage race, which is a subcase of our more general model. We show that some properties of the two-stage race hold more generally and that others do not. As regards discounting, we find it more illuminating (albeit in one sense less general) not to have positive discounting, because it is then possible to explore the nature of strategic interaction in its purest form. Grossman and Shapiro use their model to consider such issues as patent licensing and research joint ventures. Judd's (1985) model of a multistage race is also related to our second model. A competitor may make progress in his model in one of two ways—either by gradual jumps or by a leap to the finishing line. Progress depends stochastically upon competitors' efforts, distance is measured continuously, and cost functions are assumed to be square. It is shown that subgame perfect equilibrium exists if the value of the prize P is sufficiently small. Judd's results, which are concerned with welfare comparisons between optimal and equilibrium outcomes, are conditional upon P being small, because the analysis uses Taylor series approximations.

The plan of the paper is as follows. In the next section we briefly explain a version of the model of a (one state) R&D competition due to Lee and Wilde (1980). That model serves as a component part of each of the (multi-state) models that follow, and it is useful to gather some preliminary results in advance. Sections 3 and 4 respectively contain the models of a tug-of-war and of a multi-stage race described in the previous two paragraphs. Section 5 briefly describes the effect of introducing discounting into those two models.

2. THE SINGLE-STAGE GAME

Two players, A and B, are in a competition in which each strives to complete a single stage of research. If A completes first then he receives a prize of V^+ and B receives a prize of W^-. If B completes first then A's and B's respective prizes are V^- and W^+. If neither player completes than the prizes are 0. We assume that $V^+ > V^-$ and $W^+ > W^-$, so that each player prefers to complete before the other.

At the outset the players choose effort rates x and y respectively. The time T_A until A completes is then distributed exponentially with parameter x, if $x > 0$. If $x = 0$ then A never completes. In the same way, the time T_B until B completes is distributed exponentially with parameter y. Let $T = \min\{T_A, T_B\}$ be the time until the first completion. T is distributed exponentially with parameter $x+y$, if $x+y>0$. Until T, player A (B) incurs a flow cost $c_A(x)$ ($c_B(y)$). This cost ceases once T is reached. There is no discounting.

Let us calculate A's expected payoff on the assumption that $x+y>0$. It can be shown that the probability that A completes first is $x/(x+y)$, and B completes first with complementary probability $y/(x+y)$. Thus A's expected benefit is $(xV^+ + yV^-)/(x+y)$. Also the expected time until completion is $1/(x+y)$. Thus A's expected costs are $c_A(x)/(x+y)$. Overall, the expected payoff to A given that x and y are the effort rates is

$$\pi_A(x, y) = \frac{xV^+ + yV^- - c_A(x)}{x+y}.$$

B's expected payoff $\pi_B(x, y)$ can be calculated analogously.

We assume that players arrive at a Nash equilibrium. That is, (x, y) is an equilibrium if x is optimal given y, and vice versa. We have not allowed the players the option of revising their effort rates over time. Notice that if (x, y) is an equilibrium, neither player would want to do this even if he had the option. For the distributions of $T_A - t$ and $T_B - t$ conditional on $T>t$ are still exponential with parameters x and y respectively. Thus (x, y) can be regarded as a (stationary) perfect equilibrium.

We impose the following standing assumptions on the c_I:

Assumption C1. c_I is convex, strictly increasing, and $c_I(0)=0$.

Assumption C2. c_I is continuously differentiable, $c_I'(0)=0$, and $c_I'(z)\to\infty$ as $z\to\infty$.

In some cases we shall want the further condition:

Assumption C3. c_I is twice continuously differentiable, and $zc_I''(z)$ is non-decreasing in z.

These assumptions are not always the weakest necessary to guarantee our results, but they do simplify matters. We now state some results concerning the existence and characterisation of equilibrium in the single state game. We do no more than sketch their proofs; a more rigorous treatment may be found in Harris and Vickers (1986, Section 2).

Consider A's optimal choice of x when B chooses effort rate y. First, note that we can write $\pi_A(x, y)$ in the form

$$\pi_A(x, y) = V^- + \frac{x\alpha}{x+y} - \frac{c_A(x)}{x+y} \tag{1}$$

where $\alpha = V^+ - V^-$. (B's payoff can be expressed similarly in terms of $\beta = W^+ - W^-$.) Thus the optimal x depend only on y and α. Suppose that $y>0$. Now, $\partial\pi_A/\partial x = \alpha/y>0$ when $x=0$, and $\pi_A\to-\infty$ as $x\to\infty$. Hence there exist optimal x. These x are strictly positive. A necessary and sufficient condition for the maximisation of $\pi_A(x, y)$ is that x satisfies

$$y = \frac{xc_A'(x) - c_A(x)}{\alpha - c_A'(x)}. \tag{2}$$

We denote by $R_A(y)$ the set of optimal responses to y. R_A may be multivalued, but its inverse, which is given by

$$R_A^{-1}(x) = \frac{xc_A'(x) - c_A(x)}{\alpha - c_A'(x)} \tag{3}$$

is not. R_A is strictly monotonic.

If $y=0$ and $V^{+}>0$ (as it always will be in the models below), then there is no optimal x. For A is then certain to complete eventually provided that he makes strictly positive efforts. With no discounting, the time until completion is of no concern to him, and the smaller his (positive) effort rate the better. But his payoff from making no effort is zero, because in that case he never completes.

The analysis of B's optimal responses is analogous. Therefore, if there is an equilibrium, it must involve strictly positive effort rates by both players.

Proposition 2.1. *There exists an equilibrium.*

Proposition 2.2. *If Assumption* C3 *holds then equilibrium is unique.*

Figure 1 illustrates these propositions. A's inverse reaction function is continuous, has a zero derivative at the origin, and (due to the shape of c_A) goes to infinity as x approaches a limit $\bar{x}$. Similarly for B. Therefore it must be true that the inverse reaction functions cross at some strictly positive point. This establishes existence. If Assumption C3 holds, the inverse reaction functions are convex, and it is geometrically obvious that equilibrium is unique.

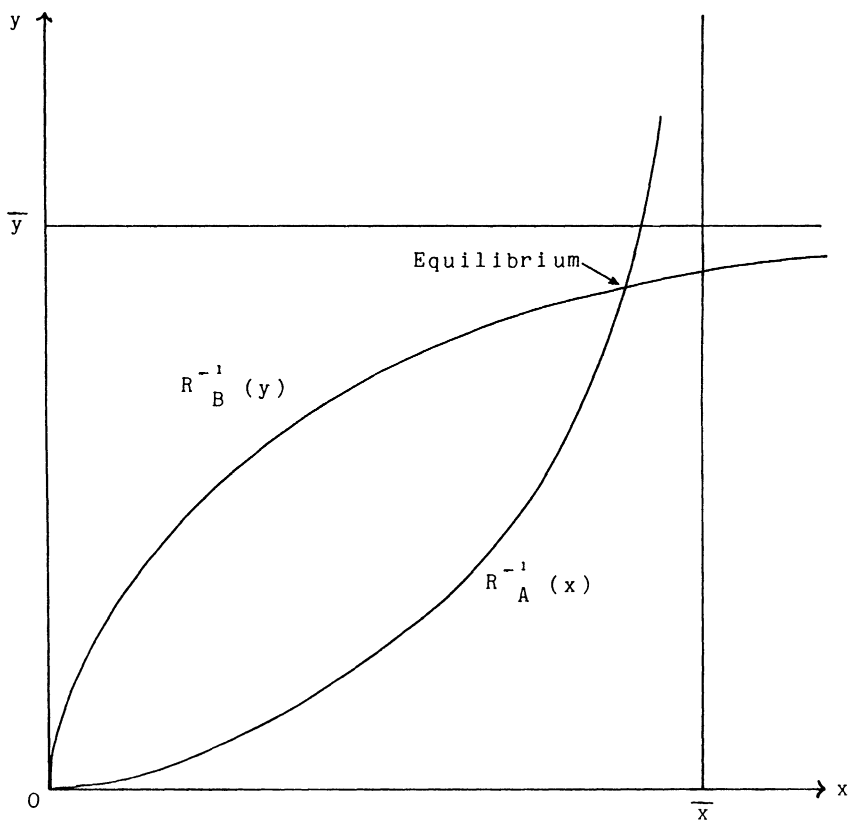

FIGURE 1

Inverse reaction functions in the single stage game. The functions have been drawn convex: see Proposition 2.2. As α increases, A's inverse reaction function pivots around the origin in a clockwise direction. As β increases, B's inverse reaction function pivots in an anticlockwise direction: see Property 2.4.

We now determine some of the properties of equilibrium. The first property is of interest in its own right.

Property 2.1. *Suppose that* $c_A \equiv c_B$. *Then* $x > y$ *at equilibrium iff* $\alpha > \beta$.

We may regard α and β as the incentives of A and B respectively. Thus Property 2.1 is that the player with the higher incentive works harder. Property 2.1 has an immediate corollary. While there may be multiple equilibria, all such equilibria are symmetric. Another immediate corollary of Property 2.1 is that $x > y$ at equilibrium iff $V^+ + W^- > V^- + W^+$. This follows from the definitions of α and β. That is, if cost functions are identical, A makes greater efforts than B if and only if joint payoffs in the event of A winning exceed those in the event of B winning.

Let V and W be the players expected payoffs in equilibrium. These quantities will be of particular interest in later sections, where we shall be using dynamic programming techniques.

Property 2.2. $V^+ > V > V^-$ *and* $W^+ > W > W^-$.

Property 2.3. *At equilibrium* $c'_A(x) = V^+ - V$ *and* $c'_B(y) = W^+ - W$.

In other words, the marginal cost of effort equals its marginal benefit. These equalities follow immediately by substituting for $c_A(x)$ and $c_B(y)$ in players' expected payoffs from their respective first order conditions.

Finally we have a simple comparative statics result.

Property 2.4. *Suppose that Assumption* C3 *holds. Then equilibrium x and y are both strictly increasing in each of* α *and* β.

3. A TUG-OF-WAR

In this section we consider a multistage version of the model of Section 2. In the new model the goal of each player is to achieve a lead of several stages of research over his rival. If player A completes a stage he moves one step closer to his goal, while B finds himself one step further from his. If A reaches his goal first then he obtains a prize of P_A and B gets nothing. Similarly, if B is first he wins a prize of P_B and A gets nothing. If neither player ever reaches his goal then neither player receives anything. It may be useful to think of this competition as a tug-of-war.

More concretely, there are N active states, numbered from 1 to N. There are also two dummy states, numbered 0 and $N+1$, which are to be thought of as player B's and player A's goals respectively. On arrival at any state n, $1 \leqq n \leqq N$, players must choose effort rates x_n and y_n. These effort rates determine the distributions of their times until completion of the next stage, and their flow costs, as before. If A completes first then play moves to state $n+1$, while if B completes first then play moves to state $n-1$. Once again there is no discounting.

Loosely speaking, each player's strategy specifies his effort rate for every possible "contingency", i.e. for every state that might be reached and for all possible paths to that state. More precisely, let us suppose that the probability with which the game begins at each state is given exogenously. Each player must decide, for all $k \geqq 1$, for all states that it is *a priori* possible to reach k-th, and for all paths by which it is *a priori* possible to reach them, on an effort rate to be employed in that contingency. For example, suppose that the sequence of states in a game happens to be 5, 6, 7, 6. . . . . Then the state that is reached fourth is state 6. Another possible way of reaching that state fourth would be 7, 8, 7, 6, ... Player A's strategy specifies his strategy in all these possible contingencies. (We do not allow for the possibility that he takes into account the time spent in each state.) A path is *a priori* possible if and only if any two consecutive states on the path are contiguous. For example, a path could not include a direct jump from state 5 to state 7. There is no requirement that the path can arise with positive probability. State n can be reached k-th if and only if there is an *a priori* possible path whose k-th state is n. Thus, if $N > 1$, any (active) state can be reached k-th.

Let us describe the players' choices for k by X_k and Y_k. Then X_k and Y_k are finite dimensional vectors whose dimensions increase with k. The players' strategies for the game as a whole will be denoted by $X = \langle X_k \mid k \geqq 1\rangle$ and $Y = \langle Y_k \mid k \geqq 1\rangle$. If a player's strategy depends only on the numbers of the states visited, and not on k or on the paths by which they are reached, then we say that it is stationary. We write stationary strategies for A and B as $x = \langle x_n \mid 1 \leqq n \leqq N\rangle$ and $y = \langle y_n \mid 1 \leqq n \leqq N\rangle$.

Intuitively it is clear that there is no loss of generality in assuming that players' strategies are bounded. Indeed, suppose that player A must choose an effort rate in state n. If the next state visited is $n+1$, then his continuation payoff cannot exceed P_A. If the next state is $n-1$ then his continuation payoff will be no less than 0. After all, the option of making no effort at all is always open to him. Overall, his effort rate will never exceed $\bar{x}$, where $\bar{x}$ satisfies $c'_A(\bar{x}) = P_A$. Similarly, B's efforts will never exceed a level of $\bar{y}$.

Formally we shall simply assume that player A's efforts are restricted to lie in the interval $[0, \bar{x}]$, and that B's lie in $[0, \bar{y}]$. This guarantees in particular that the random process corresponding to any given pair of strategies is "honest", i.e. that it does not undergo an infinite number of transitions in finite time with positive probability. It turns out that the implicit constraint on players' strategies does not bind in equilibrium.

Until future notice we fix a stationary strategy y for B, and examine the problem of finding an optimal response to y. We assume that $y_n > 0$ for all n. (If $y_n = 0$ for some n then it can be shown that A possesses no optimal response.) We note that A's strategy set is $[0, \bar{x}]^\infty$, and that this set is compact in the product topology. Furthermore:

Proposition 3.1. *Suppose the probabilities with which the game begins in each active state are given, and let B's strategy be fixed at the stationary and strictly positive value y. Then A's expected payoff is continuous in his strategy.*

The proof of Proposition 3.1 is given in the appendix, where the precise role that it plays in the analysis is also outlined. The Proposition should be plausible since there is a positive lower bound on the drift towards state 0—B's effort rates are bounded away from zero and A's are bounded above. But in view of the absence of discounting from the model, the weakness of the product topology, and the fact that the game takes place in continuous time, the Proposition probably should not be regarded as obvious.

It follows at once that A has a strategy that maximises his expected payoff in the class of his (possibly non-stationary) strategies. Pick any initial distribution that assigns positive probability to each state, and let X be a strategy that maximises A's expected payoff. Then X is an *optimal response* to y, in the sense that it maximises A's expected payoff irrespective of the initial distribution over states. Furthermore, this response can be chosen to be stationary. To see this, let $X^{(k)}$ be defined as follows. Let $X^{(k)}$ specify that whenever state n is visited before the k-th transition, the effort rate $(X_1)_n$ should be employed. Also, let $X^{(k)}$ specify that if the k-th transition is to state n then the strategy X should be employed henceforth in exactly the way it would be employed if the game had started there. Thus, if $n_1, n_2, \ldots, n_r$ are the $(k+1)$-th, $(k+2)$-th, $\ldots$, $(k+r)$-th states visited, then the effort rate employed in n_r is that specified by X when the r-th state reached is n_r and the path by which it is reached is $n_1, n_2, \ldots, n_r$. A straightforward induction shows that $X^{(k)}$ too is an optimal response. But $X^{(k)}$ converges to the stationary strategy $x = X_1$ as $k \to \infty$. Hence X is optimal.

For each n, $1 \leqq n \leqq N$, let V_n be the value of reaching state n. Let $V_0 = 0$ and $V_{N+1} = P_A$. Suppose that the initial state turns out to be n. If A employs effort rate a initially, but then reverts to the strategy x as soon as the first transition occurs, then his expected payoff will be

$$\frac{aV_{n+1} + y_n V_{n-1} - c_A(a)}{a + y_n}. \tag{4}$$

Since x is optimal in the class of non-stationary strategies, x_n maximises (4) as a function of a unconstrainedly. The resulting value of the expression is V_n:

$$V_n = \frac{x_n V_{n+1} + y_n V_{n-1} - c_A(x_n)}{x_n + y_n}. \tag{5}$$

Exploiting the facts that x_n maximises (4) and satisfies (5), one can show that $V_{n+1} > V_n$ for $N \geqq n \geqq 0$. Hence the observation that x_n maximises (4) can be characterised by the associated first order condition

$$y_n = \frac{x_n c'_A(x_n) - c_A(x_n)}{\alpha_n - c'_A(x_n)}, \tag{6}$$

where $\alpha_n = V_{n+1} - V_{n-1}$ is A's incentive at n.

It has been shown that for x to be an optimal response it is necessary that equation (6) be satisfied for all n, $1 \leqq n \leqq N$. It is also sufficient. This follows from an argument very similar to that which established that the optimal response X could be taken to be stationary. Indeed, let $\tilde{x}$ satisfy (6) for all n. Then it is optimal to employ $\tilde{x}$ before the first transition if the optimal strategy X will be employed thereafter. Iterating this argument one sees that it is optimal to employ $\tilde{x}$ up to the k-th transition. Taking the limit as $k \to \infty$, $\tilde{x}$ must be optimal. (In particular, $\tilde{x}$ satifies (5) for all n.)

Given the V_n, (6) for all n characterises the set of optimal stationary responses. Since the set of $\tilde{x}_n$ satisfying (6) is a closed interval, the set of $\tilde{x}$ satisfying (6) for all n is a product of closed intervals. In particular, it is closed and convex. This is the first of the ingredients we need for an application of Kakutani's fixed point theorem.

Next, note that (5) and (6) for all n imply

$$y_n = \frac{x_n c'_A(x_n) - c_A(x_n)}{c'_A(x_{n-1})} \quad \text{for all } n \geqq 2 \tag{7}$$

and

$$y_1 = \frac{x_1 c'_A(x_1) - c_A(x_1)}{P_A - \sum_{n=1}^{N} c'_A(x_n)}. \tag{8}$$

That is, (7) and (8) are necessary for the optimality of x.

For the converse, suppose that $\tilde{x}$ satisfies (7) and (8). Let $\tilde{V}_{N+1} = P_A$, let $\tilde{V}_n = V_{n+1} - c'_A(\tilde{x}_n)$ for $N \geqq n \geqq 1$, and let $\tilde{V}_0 = 0$. Then $\tilde{x}$ satisfies the analogues of (5) and (6) obtained by replacing the V_n with the $\tilde{V}_n$. It follows that the $\tilde{V}_n$ are the values associated with the strategy $\tilde{x}$, and that it is optimal to employ $\tilde{x}$ up to the first transition given that $\tilde{x}$ will be employed thereafter. Iterating this observation we obtain that the expected payoff from employing $\tilde{x}$ up to the k-th transition given that $\tilde{x}$ will be employed thereafter is at least as high as that from employing x up to the k-th transition given that $\tilde{x}$ will be employed thereafter. Letting $k \to \infty$ we conclude that the expected payoff from employing $\tilde{x}$ is at least that from employing x. Thus $\tilde{x}$ must be optimal.

Thus (7) and (8) characterise the set of optimal stationary responses to a strictly positive y. Since both sides of both equations are continuous in x and y, the optimal reaction correspondence must be upper hemicontinuous. This is the second ingredient necessary for Kakutani's fixed point theorem.

The only remaining obstacle to an application of Kakutani's theorem is the fact that the domain of the optimal response correspondence, namely $(0, \bar{y}]^N$, is not closed. We need to show that there exists some $\varepsilon > 0$ such that this domain can be further restricted to $[\varepsilon, \bar{y}]^N$ without loss of generality. This is done in the course of the proof of the following Proposition.

Proposition 3.2. *The model has a stationary perfect equilibrium. If the game is symmetric, then there exists a symmetric stationary perfect equilibrium.*

The game is symmetric iff $c_A = c_B$ and $P_A = P_B$. The proof of Proposition 3.2 is in the appendix.

Having established existence, our next concern is to characterise equilibrium. As a first step it is useful to define the "pivot" of the game as the state where joint expected payoffs are minimised. It will be clear from Property 3.1 why we use this terminology. More precisely, from state n play must move either to state $n+1$ or to state $n-1$. Before transition occurs, some costs will be incurred. Let $S_n = V_n + W_n$ be joint expected payoffs beginning from state n. Then $S_n < \max\{S_{n-1}, S_{n+1}\}$ for all n, $1 \leqq n \leqq N$. But this implies that there exists N_0 such that $\{S_n \mid n \geqq N_0\}$ is strictly increasing and $\{S_n \mid n \leqq N_0\}$ is strictly decreasing. We call N_0 the *pivot.* (It can occur at most once that two consecutive total values coincide. If this does not happen, N_0 is the value of n for which S_n is minimised. If it does, N_0 is the average of the two values of n for which S_n is minimised.) In order to derive interesting conclusions from this finding we need to assume that $c_A = c_B$.

Property 3.1. *Suppose that $c_A = c_B$. If $n > N_0$ then*

(i) $x_n > y_n$;
(ii) $y_{n+1} < y_n$;
(iii) *if* $x_n \leqq x_{n-1}$ *then* $x_{n+1} < x_n$.

The analogues of these results hold for $n < N_0$.

Notice that it is not assumed that $P_A = P_B$. N_0 is the pivot of the game in the sense that beyond N_0, A is ahead; before N_0, B is ahead. With this interpretation, (i) states that the leader works harder than the follower, (ii) that the follower slows down as he gets further behind, and (iii) that the leader continues to slow down once he begins to do so. We show below by means of examples that the leader may well continue to accelerate even after the pivot has been passed. The proof of Property 3.1 is in the appendix.

If the model is symmetric (with $c_A = c_B$ and $P_A = P_B$), and if equilibrium too is symmetric, then naturally $N_0 = (N+1)/2$. However, even if the model is symmetric it is not necessarily the case that the pivot $N_0 = (N+1)/2$. The reason is that equilibrium might be asymmetric, even though the players are symmetrically placed. Hence we must admit in principle the possibility that players' expectations about one another's strategies can also affect the position of the pivot. If, for example, B believes that A will fight hard, B may choose to make a lesser effort. But if A is faced with a small challenge, it may be optimal for him to respond by working disproportionately hard.

If $P_A = P_B$ and $N \geqq 2$ then the pivot must be strictly interior. For certainly $S_1 < S_0$ and $S_N < S_{N+1}$, which implies that $1 \leqq N_0 \leqq N$.

It is of some interest to relate the properties of players' efforts to properties of their values. Let us suppose that the model is symmetric. In view of Property 3.1 (ii) and (iii) we can find N_A such that x_n is strictly increasing for $n \leqq N_A$ and strictly decreasing for $n \geqq N_A$ (like N_0, N_A may have to be chosen midway between two states). Next, from Property 2.3, we have that $V_{n+1} - V_n = c'_A(x_n)$ for all n, $1 \leqq n \leqq N$. Hence V_n is concave for $N_A \leqq n \leqq N+1$ and convex for $1 \leqq n \leqq N_A$. Finally, $y_1 \geqq x_1$. For either $N = 1$, in which case $x_1 = y_1$ by Property 2.1, or $N \geqq 2$, in which case the pivot is strictly interior and $y_1 > x_1$ by Property 3.1(i). Hence

$$\begin{aligned} V_1 - V_0 &= (x_1 c'(x_1) - c_A(x_1))/y_1 \\ &\leqq (x_1 c'(x_1) - c_A(x_1))/x_1 \\ &< c'(x_1) = V_2 - V_1, \end{aligned}$$

and V_n is convex for $0 \leqq n \leqq N_A$. We summarise these findings in the following result.

Property 3.2. *Suppose that $P_A = P_B$ and that $c_A = c_B$. Let N_A be the value of n for which x_n is maximised. (If there are two such values then they must be contiguous, and we let N_A be their average.) Then V_n is convex for $0 \leqq n \leqq N_A$ and concave for $N_A \leqq n \leqq N+1$. If c_A is strictly convex, then this convexity (concavity) is strict. Analogous results hold for $\{W_n \mid 0 \leqq n \leqq N+1\}$.*

It can be shown by means of a parametric example that the leader does not necessarily slow down once the pivot is passed. Let us suppose that $c_A(z) = c_B(z) = z^\eta$ for some $\eta > 1$, that $P_A = P_B$, and that we are dealing with a symmetric equilibrium. If $y_1^\eta = \rho x_1^\eta$ then it can be shown that

$$y_n^\eta = \rho \zeta^{2n-2} x_n^\eta \tag{9}$$

for all n, and

$$y_n^{\eta-1} = \rho \zeta^{2n-3} x_{n-1}^{\eta-1} \tag{10}$$

for all $n \geqq 2$, where $\zeta = (\eta - 1)/\eta$.[2] Now $y_N^\eta = \rho \zeta^{2N-2} x_N^\eta$. Thus, by symmetry, $\rho \zeta^{2N-2} = \rho^{-1}$, and so $\rho = \zeta^{-(N-1)}$. Next, raising (9) to the power $\eta - 1$ and (10) to the power η we obtain

$$\rho^{\eta-1} \zeta^{(2n-2)(\eta-1)} x_n^{\eta(\eta-1)} = \rho^\eta \zeta^{(2n-3)\eta} x_{n-1}^{\eta(\eta-1)}.$$

Hence $x_{n-1} > x_n$ iff

$$\rho^{\eta-1} \zeta^{(2n-2)(\eta-1)} > \rho^\eta \zeta^{(2n-3)\eta}.$$

Substituting in the latter expression for ρ in terms of ζ and comparing the exponents of ζ (note that $\zeta<1$), it follows that $x_{n-1}>x_n$ iff $n>\eta/2+(N+1)/2$. Thus the larger η, the further past the midpoint before the leader begins to slow down.

Let us pursue this example a little further. Assume, to fix ideas, that $\eta=2$ and that N is even. We know that A's efforts will peak just past halfway, at $N_A=N/2+1$. We can also calculate that the ratio of his effort rates, x_n/x_{n-1}, is equal to $2^{\frac{1}{2}+(N/2+1)-n}$. Thus his effort rate at N_A is $\sqrt{2}$ times his effort rates at N_A-1 and N_A+1. Beyond N_A his effort rates fall much faster than exponentially—actually the ratio between consecutive effort rates falls exponentially. Next, the ratio between his effort rate and that of B, x_n/y_n, is $2^{n-(N+1)/2}$. As expected $x_n/y_n>1$ once the pivot, $N_0=(N+1)/2$ is passed. More interestingly, x_n/y_n rises exponentially. Thus, although A slows down very quickly as he approaches his goal, he works harder and harder relative to B. Since it is relative effort rates that determine the probability that A completes a stage first, we see that the probability that A wins a stage is larger, the closer that stage is to his goal. Finally, $y_n/y_{n-1}=2^{(N+1)/2-n}$. As expected, B's effort rate falls once the pivot is passed. Also, as can anyway be inferred from the previous discussion, the ratio y_n/y_{n-1} is exactly half of x_n/x_{n-1}— the follower B's effort rates tail off faster than those of the leader A.

We conclude this section with a discussion of the question of uniqueness. Our positive results all relate to the uniqueness of *symmetric* equilibrium. It would certainly be much more interesting to have conditions under which equilibrium must be symmetric, for under such conditions the pivot would always be $(N+1)/2$. Unfortunately, the parametric case of the previous paragraph is sufficiently rich to provide examples of asymmetric equilibria even when there are just two active states.

Proposition 3.3. *Suppose that $P_A=P_B$ and $c_A=c_B$. Then symmetric equilibrium is unique if one of the following conditions holds*:

(i) $c_A(z)=c_B(z)=z^\eta$ *for some* $\eta>1$;
(ii) $N\leqq 3$ *and Assumption* C3 holds;
(iii) $N\leqq 5$, $c_A=c_B$ *is twice continuously differentiable, and* $c_A''=c_B''$ *is monotonic non-decreasing.*

For a proof of Proposition 3.3 we refer the reader to Harris and Vickers (1986).

Note that when the cost functions are isoelastic as in (i), the problem is homogeneous in the sense that the set of equilibria is unchanged if the prizes and the cost functions are scaled up by the same factor. Thus there would be no gain in generality from allowing an extra multiplicative constant in (i). Also, the conditions on the cost functions in (iii) imply Assumption C3. They amount to a strengthening of the convexity conditions on the cost functions. It would appear that, in general, more and more convexity of the cost functions is required to guarantee uniqueness as N increases.

One would ideally like to have examples showing that uniqueness of symmetric equilibrium can break down. Unfortunately, as part (i) shows, the simplest parametric version of our model cannot provide such examples. The effort involved in finding such examples could therefore be considerable, and is certainly out of all proportion to the insight to be gained from doing so. We have not, consequently, tried to find any.

The simplest parametric version of one model can, however, be used to obtain a little insight into the question of uniqueness of unrestricted equilibrium. It can be shown, first, that equilibrium is unique if $\eta\geqq N$. Secondly, even in the case $N=2$ there exist asymmetric equilibria. Specifically, one can show that there exist asymmetric equilibria for η in a neighbourhood of 1. The method of proof used to show this does not generalise to the case $N>2$. Computer calculations suggest, however, that the existence of asymmetric equilibria does. Indeed, it would appear that, for suitable $\eta(N)$, the N state war has $2N-1$ equilibria ($N-1$ pairs of asymmetric equilibria and one symmetric equilibrium), and that there is no η for which it has more than this number. Overall, the cost functions must be "sufficiently convex" if uniqueness is to obtain, and the degree of convexity required increases with N. All of this accords well with expectation.

4. A MULTI-STAGE RACE

In the previous section we considered a model of a tug-of-war in which players' interests were directly opposed in the sense that a step forward for A was a step backward for B. The model was one-dimensional in the sense that the gap between A and B measured the state of competition between them. In that model it was possible to confirm analytically the intuitions that a leader works harder than a follower, and that a follower slows down as the gap between himself and the leader widens. In the present section we consider a second model in which players' interests are less directly opposed. Each player is striving to reach a given finishing line before his rival. Thus the model is two-dimensional in the sense that the state of competition consists of the pair of the two players' current distances from the finishing line. In this model we can only give a partial analytic confirmation of our intuitions. We have therefore conducted numerical simulations. The simulations are entirely consistent with the intuitions.

The second model is related to the model of a tug-of-war in three ways. First, it is a natural extension of that one-dimensional model to two dimensions. The second model is more complex to analyse, but it is a more appropriate representation of some economic situations than the first model. Secondly, the model of a tug-of-war provides many useful insights into the workings of the second model. Thirdly, we believe that there is a sense in which the solutions of the two models converge. At the end of this section we state a precise conjecture about this convergence result, and we provide some evidence for it.

A state in the new model is characterised by a pair (m, n) where m is A's distance from the finishing line and n is B's distance. As soon as a state is reached, each player chooses an effort rate which determines the probability distribution of the time until he completes and also his flow costs, as before. If A completes first, then he moves one step closer to the finishing line, but B's distance from the finishing line is unchanged. That is, play moves to state $(m-1, n)$. If A (B) reaches the finishing line first then he obtains a prize of P_A (P_B). There is no discounting.

A straightforward induction based on Proposition 2.1 and Property 2.2 establishes:

Proposition 4.1. *Equilibrium exists. If the model is symmetric, then equilibrium may be taken to be symmetric.*

Furthermore, repeated application of Proposition 2.2 gives:

Proposition 4.2. *If Assumption* C3 *holds then equilibrium is unique. In particular, if Assumption* C3 *holds and the model is symmetric then the unique equilibrium is symmetric.*

Two properties of players' values in equilibrium are easily derived:

Property 4.1. *For all* $m \geqq 1$:

(i) $V(m, n)$ *is strictly increasing in* n *and* $W(m, n)$ *is strictly decreasing in* n;
(ii) $V(m, n) \to P_A$ *and* $W(m, n) \to 0$ *as* $n \to \infty$.

Analogous results hold when n *is held fixed and* m *is varied.*

The further a player is from the finishing line, the worse off he is. Indeed, as his distance from the finishing line tends to infinity, his expected payoff from the game tends to 0. Similarly, the further his opponent is from the finishing line, the better off he is, and his expected payoff tends to the value of the prize to him as this distance tends to infinity.

We need some notation. Let $x(m, n)$ and $y(m, n)$ be the players' effort rates in equilibrium when the game is in state (m, n). Let $\alpha(m, n) = V(m-1, n) - V(m, n-1)$ and $\beta(m, n) = W(m, n-1) - W(m-1, n)$ be the players' incentives at (m, n). Finally, let $S(m, n) = V(m, n) + W(m, n)$ be the total value of (m, n).

Let us assume that the model is symmetric. There is then a natural interpretation of "being ahead"—A is ahead of B if $m < n$. Furthermore the intuition derived from our analysis of the tug-of-war suggests many conjectures.

Consider the diagonal perpendicular to the leading diagonal (on which $m = n$) given by the points $(k-r, r)$ as r varies from 0 to k. Each increment in r leaves A one step closer to the finishing line and B one step further away. Thus A's lead over B would seem to increase unambiguously with r. We might therefore expect to find that $S(k-r, r)$ is single troughed with its minimum at $r = k/2$. Correspondingly, $x(k-r, r)$ should exceed $y(k-r, r)$ for $r > k/2$. Further $x(k-r, r)$ should be single peaked, with its peak occurring at some $r \geqq k/2$. $\alpha(k-r, r)$ too should peak at or beyond $k/2$.

One could also consider the line parallel to A's finishing line consisting of the points (k, r) obtained for $r \geqq 0$. B's distance from the finishing line increases with r. Thus, even though A's distance remains constant, it seems reasonable to say that his effective lead increases. A's efforts should therefore peak for $r \geqq k$, as should his incentives. Similarly, B's efforts should peak for $r \leqq k$ along with his incentives.

We have only been able to establish a few of these results analytically. We begin with the question of whether the leader works harder than the follower. Note first that $S(0, k) > S(1, k-1)$ for all $k \geqq 2$. For the total expected payoff from the game is P_A, but there are expected costs too if it begins in an active state. It follows that $\alpha(1, k) > \beta(1, k)$ and so, by Property 2.1, that $x(1, k) > y(1, k)$. Similarly $y(k, 1) > x(k, 1)$ for all $k \geqq 2$. This finding can actually be extended to the case where $\min\{m, n\} \leqq 2$, if equilibrium is unique.

Property 4.2. *Suppose that the model is symmetric and that Assumption* C3 *holds. Then, for all* m, n *such that* $m < n$ *and* $m \leqq 2$, $x(m, n) > y(m, n)$. *Similarly,* $x(m, n) < y(m, n)$ *when* $n < m$ *and* $n \leqq 2$.

We note that in their independent work, Grossman and Shapiro established Property 4.2 for the special case $(m, n) \in \{(1, 2), (2, 1)\}$. This case is covered by the reasonably straightforward considerations of the previous paragraph. It is startling how much more difficult the proof of the more general case is. This proof is in the appendix. Two other properties of independent interest are established in its course:

Property 4.3. *Suppose that the model is symmetric. Then*

(i) *For all* $n \geqq 2$, $\alpha(2, n) > \alpha(1, n+1)$ *and* $\beta(2, n) > \beta(1, n+1)$;
(ii) *if, in addition, Assumption* C3 *holds, then, for all* $n \geqq 2$, $x(2, n) > x(1, n+1)$ *and* $y(2, n) > y(1, n+1)$.

Analogous results obtain for $m \geqq 2$.

Property 4.3(ii) is reminiscent of Properties 3.1(ii) and (iii). A is unambiguously further ahead at $(1, n+1)$ than he is at $(2, n)$. That the follower, B, goes more slowly at $(1, n+1)$ than he does at $(2, n)$ is therefore in complete accord with Property 3.1(ii). In the case of the leader, however, we were able to establish in the tug-of-war only that he continues to slow down once he begins to do so. Moreover, in some cases he never began to slow down. *Now* we have that the leader definitely slows near the finishing line. Similarly, Property 4.3(i) is reminiscent of Property 3.2.

The only other monotonicity result that we have established is the following. Once again the proof is in the appendix.

Property 4.4. *The efforts* $y(1, n)$ *and* $x(1, n)$ *are strictly decreasing in n.*

Notice that, in addition to the result expected for the follower, Property 4.4 tells us that the leader too increases his effort rate as the follower catches up. Notice too that the result does not depend on any assumptions of symmetry or uniqueness.

An example

In view of the apparent difficulty of obtaining further analytical results, we have subjected our intuitions concerning the model to a different kind of check. We have simulated numerically the special case of an isoelastic cost function $c(z) = z^{\eta}$ for values of η ranging from 1·25 to 12. Explicit tables for the case $\eta = 2$ are given in Harris and Vickers (1986). Here we simply summarise some of the findings verbally. All are consistent with the intuitions set out above.

The first finding is that the leader always makes greater efforts than the follower. That is, for $m < n$ we have $x(m, n) > y(m, n)$. Moreover, the ratio of the leader's efforts to those of the follower increases as the gap between them widens, in the sense that $x(k-r, r)/y(k-r, r)$ is strictly increasing in r.

Secondly, the follower speeds up as the gap between the players narrows. This remark is true in each of the following senses. If $n > m$ then

(i) $y(m, n) < y(m+1, n-1)$,
(ii) $y(m, n) < y(m, n-1)$,
(iii) $y(m, n) < y(m+1, n)$.

Note that (i) is implied by the conjunction of (ii) and (iii). Inequality (ii) says that the follower speeds up if he catches up by one step, and (iii) says that he slows down if the leader extends his lead.

Thirdly, it is not necessarily so that the leader speeds up as the gap between the players narrows. In the case $\eta = 2$, $x(4, 4)$ is less than both $x(4, 5)$ and $x(3, 4)$. Thus the leader's efforts may increase as his lead extends, although once they begin to decline as the gap widens, they continue to do so. Along diagonals such that $m + n = k$, $x(m, n)$ is single peaked, and its peak occurs for $n \geqq m$. Similarly, holding either m or n fixed, $x(m, n)$ peaks when $n \geqq m$.

Fourthly, $x(m, m)$ exhibits intriguing cyclical behaviour, which demonstrates that distance from the finishing line has complex influence. Although it is straightforward to show analytically (see Grossman and Shapiro (1985)) that $x(2,2) < x(1,1)$, the numerical simulations reveal that the result fails to generalise. Thus, in the case $\eta = 2$, $x(m, m)$ falls from 0·33333 at $m = 1$ to 0·14099 at $m = 4$, then rises to 0·15285 at $m = 9$ before falling to 0·15269 at $m = 10$.

Fifthly, $\alpha(m, n)$ is single-peaked, and once again the peak occurs for $m \geqq n$, though not necessarily for $n = m$. For instance $\alpha(3,5) > \alpha(4,4)$ when $\eta = 2$.

Sixthly, the race quickly approaches "monopoly" as the disparity between player's positions widens. When $\eta = 2$, $V(7, 10)$ is more than 95% of the payoff of a player facing no rivalry, and $V(6, 10)$ exceeds $99\frac{1}{2}$% of that payoff. This finding, however, is contingent on the value of η. The larger η, the greater the number of stages before monopoly is reached.

Seventhly, the joint expected payoff function has remarkable monotonicity properties. Not only does $S(m, n)$ fall steadily to its minimum at the middle of the $m + n = k$ diagonal (which of course implies that leader goes faster than follower) but it also attains its minimum at $m = n$ when either m or n is varied.

What is striking about these results for the multistage race is how closely they parallel the results obtained analytically for the model of a tug of war in Section 3. Indeed, we believe that there is a precise sense in which the solutions to the two models converge.

We elucidate this claim in the case in which players cost functions are isoelastic with parameter greater than unity. That is, we assume that $c(z) = z^{\eta}$ with $\eta > 1$.

First we adapt the notation in the tug-of-war. Suppose now that there is an odd number of states, numbered from $-N$ to N, with 0 as this state at which players are neck-and-neck. Let $x_n^{(N)}$ and $y_n^{(N)}$ be the players' effort rates at state n, $\alpha_n^{(N)}$ and $\beta_n^{(N)}$ their incentives, and $V_n^{(N)}$ and $W_n^{(N)}$ their values. It can be shown that all these variables have non-zero limits as $N \to \infty$. Let us call these limits x_n, y_n, α_n, β_n, V_n and W_n. They satisfy the equations

$$V_n = \frac{x_n V_{n+1} + y_n V_{n-1} - c(x_n)}{x_n + y_n} \tag{11}$$

and

$$c'(x_n) = V_{n+1} - V_n \tag{12}$$

for all n, and

$$\textstyle\sum_{n=-\infty}^{\infty} c'(x_n) = P, \tag{13}$$

as well as the analogous equations for y and W. They are also the only symmetric solution to these equations. Thus they may be thought of as the unique symmetric equilibrium of an infinite state tug-of-war with prize P.

Next, we can take diagonal cross-sections of the multistage race. The k-th cross-section consists of the variables $x(m, n)$, $y(m, n)$, $\alpha(m, n)$, $\beta(m, n)$, $V(m, n)$ and $W(m, n)$ for $m + n = 2k$ *and* $m + n = 2k - 1$. In such a cross-section the variables are arranged in a descending staircase. That is, they are are arranged in the order: $(2k-1, 1)$, $(2k-2, 1)$, $(2k-2, 2), \ldots, (k, k-1)$, (k, k), $(k-1, k), \ldots, (2, 2k-2)$, $(1, 2k-2)$, $(1, 2k-1)$. There are $4k-3$ points in all, and they can be regarded as being centered on the leading diagonal, on (k, k). Relabelling accordingly, we can write the x variables as $x_{-(2k-3)}^{[k]}$, $x_{-(2k-2)}^{[k]}, \ldots, x_0^{[k]}, \ldots, x_{2k-2}^{[k]}$, $x_{2k-3}^{[k]}$, and the other variables analogously. The similarity of this notation with that just adopted for the tug-of-war is deliberate. Indeed, we believe the following to be true.

Conjecture. *Suppose that the players have isoelastic cost functions. Then the k-th cross-section of the multi-stage race converges to the equilibrium of the infinite state tug-of-war with prize P as $k \to \infty$.*

We are unable to prove this conjecture at present. What we can show is that if the leader always works harder than the follower (as we believe he does), and if the cross-sections do converge to a limit, then that limit is the symmetric equilibrium of the infinite state tug-of-war. Also, subject to the same assumption concerning the leader and the follower, the cross-sections have limit points. Any such limit point is non-degenerate in the sense that it involves strictly positive efforts at all points. We emphasize that our simulations are consistent with the conjecture, and strikingly so for lower values of η (say $\eta \leqq 4$).

5. DISCOUNTING

Until this point we have assumed that competitors in the race do not discount the future. We have done this because our main aim is to see how the intensity of rivalry—i.e. the current state of the competition—affects the efforts that the players make. We therefore examined that parameter in isolation in order to see its influence as clearly as possible. However, it may be of interest to see how the previous results are modified when positive discounting is introduced. Grossman and Shapiro (1985) have positive discounting in their two-stage race. Their model is indeed a subcase of ours when we introduce positive discounting.

In the one-stage competition described in Section 2, the effect of introducing positive discounting is as follows. Recall that V^+ and W^+ are, respectively, the prizes that A and B receive if they win. When the interest rate $r > 0$, there is a tendency for A to make greater efforts than B if $V^+ > W^+$, with the opposite tendency if the inequality is reversed. The reason is that the player whose prize is more valuable has a more urgent desire to win. Instead of our earlier result (Property 2.3) that $x > y$ if and only if $\alpha > \beta$, we now have that $x > y$ if $\alpha \geqq \beta$ and $V^+ \geqq W^+$ (with either inequality strict) and a similar sufficient condition for $y > x$. If, on the other hand, $\alpha > \beta$ and $W^+ > V^+$, the relative magnitude of x and y is ambiguous.

Nearly all the results obtained for the model of a tug of war in Section 3 are unaffected when $r > 0$. Indeed, there is an additional reason why the leader goes faster than the follower in a symmetric equilibrium: because, in the terms of the previous paragraph, $V^+ > W^+$ when A is ahead. It also remains true that the follower makes smaller efforts if he falls further behind. In fact, discounting provides an additional reason for him to slow down. In the same way, discounting provides a reason why the leader should speed up as he approaches his goal—with $r > 0$ he is impatient to receive his prize. This effect combines with his tendency to slow down for strategic reasons, which was highlighted by the model without discounting, to produce an overall effect that is ambiguous.

The introduction of positive discounting into the model of a multi-stage race in Section 4 has more ambiguous effects. It does not alter the results when each player has two or fewer stages to complete (indeed Grossman and Shapiro (1985) demonstrate these results). Nor does it alter the results when at least one player is within one step of the finishing line. Further from the finishing line, however, the effects of competitors' impatience mingle with those due purely to intensity of rivalry to produce a picture that is by no means clear. We therefore repeat that our concern has been primarily to identify the effects upon competitors' efforts of the intensity of rivalry between them, not the combined effects of rivalry and impatience.

APPENDIX

Before giving the proof of Proposition 3.1, we describe briefly the role that it plays in the analysis in the text. It is used there, first, to demonstrate the existence of an optimal stationary strategy. Its role there is not essential. The existence of such a strategy could be obtained instead from Theorem 5.2 of Yushkevich (1977). The standing assumptions of Yushkevich's paper are easily checked. The only significant effort would be to justify condition (5.4) of that paper. This amounts to showing that the game ends with probability 1, which can be done via a significantly abbreviated version of the proof of Proposition 3.1. Proposition 3.1 is used, secondly, to demonstrate the sufficiency of equations (7) and (8), from which the upper hemicontinuity of the optimal reaction correspondence follows. Its role here is essential. It is not in general true that a stationary strategy that is optimal given that that same strategy will be followed in the future is optimal overall.

Proof of Proposition 3.1. The problem of showing that A's expected payoff is continuous in his strategy is, as it stands, one of comparing integrals of the same function relative to different measures. (The measures are, roughly speaking, measures on the space of step functions taking values in the set $\{0, 1, \ldots, N+1\}$. The function is A's payoff.) Viewed in this way it is rather intractable. We therefore construct a simple representation for our random processes that reduces this problem to one of comparing the integrals of different functions relative to the same measure. Continuity is then established by an application of Lebesgue's dominated convergence theorem.

For each $k \geqq 1$ let C_k and D_k be an independent pair of random variables, distributed uniformly on $[0, 1]$, and suppose that (C_{k_1}, D_{k_1}) and (C_{k_2}, D_{k_2}) are independent if $k_1 \neq k_2$. The random sequence $\{(C_k, D_k) \mid k \geqq 1\}$ is the basis for our representation. Let F_z denote the distribution function of the exponential distribution with parameter z, and let I_p take the value 1 on the set $[p, 1]$ and -1 on the set $[0, p)$ for all $p \in [0, 1]$. If U is uniformly distributed on $[0, 1]$ then $F_z^{-1}(U)$ is exponential with parameter z and $I_p(U)$ takes the value -1 with probability p and 1 with probability $1-p$.

Fix a contingency. Suppose that this contingency involves reaching n at the k-th stage. A's strategy specifies some effort rate z to be used in this contingency, while B's specifies y_n. We specify that the time spent in state n in this contingency is $F_{z+y_n}^{-1}(C_k)$ and that the transition is $I_{y_n/(z+y_n)}(D_k)$. In this way we obtain a representation for the process derived from X and y.

We must check that this representation preserves the feature of interest in our problem, namely A's expected payoff. It is not completely obvious that it does so. Our specification of the model made the joint distribution of time to transition and direction of transition for a given contingency independent of that for all other contingencies. By contrast, in our representation the joint distributions of two contingencies with the same k will be dependent. (On the other hand, for a given contingency, the time to transition and the direction of transition remain independent.)

To see that A's expected payoff is preserved, note that his payoff can be written in the form

$$\pi_A(X, y) = J_A P_A - \textstyle\sum_\gamma T_\gamma c_A(x_\gamma),$$

where J_A is the indicator function of the event that A eventually wins, T_γ is the time to transition in contingency γ, and x_γ is the effort rate employed by A in the same contingency. Thus his expected payoff is

$$E(\pi_A(X, y)) = P_A \Pr\{A \text{ wins}\} - \textstyle\sum_\gamma E(T_\gamma) c_A(x_\gamma). \tag{A1}$$

(That this equality holds follows from Lebesgue's monotone convergence theorem for extended valued integration. Both expressions may be negatively infinite for all we know at this point.)

Now the probability that A wins is the sum of the probabilities of the distinct paths by which he can win. Consider such a path, and suppose that it terminates at $N+1$ at the K-th transition. Then this path consists of K transitions, for $k=1$ to K. These transitions *are* independent. The probability of the path can therefore be written as the product of the probabilities of each transition of which it is composed. These probabilities are the same as those in the original problem. Thus the probability that A wins is unchanged. Next, let p_γ be the probability that contingency γ occurs. By the argument just given, this probability is unchanged from the previous problem. T_γ is zero with probability $1-p_\gamma$, and exponential with parameter $x_\gamma + y_n$ conditional on being positive, where n is the state of the game in γ. Thus the distribution of T_γ, too, is unchanged. In particular, $E(T_\gamma)$ is unchanged. Overall, then, the R.H.S. of (A1) is equal to A's payoff in the original problem, which yields the desired conclusion.

Suppose that we are given a sequence of strategies $\{X^r \mid r \geqq 1\}$ from G, and that $X^r \to X$ in the product topology. (This means that the effort rates employed in X^r converge pointwise to those employed in X.) In order to apply Lebesgue's dominated convergence theorem we need to construct a random variable dominating the $\pi_A(X^r, y)$, and show that $\pi_A(X^r, y) \to \pi_A(X, y)$ almost everywhere.

Now $|\pi_A(X^r, y)| \leqq P_A + c_A(\bar{x}) \sum_\gamma T_\gamma^r = P_A + c_A(\bar{x}) T^r$, where T_γ^r is the time to transition in contingency γ for the process involving X^r, and $T^r = \sum_\gamma T_\gamma^r$ is simply the time until the process stops. We therefore compare T^r with the times until stopping of three other processes, all of which are understood to be represented in the same way as X^r.

In our first process, the transition probabilities are unchanged, but the times until transition are all taken to be distributed exponentially with parameter $\underline{y}$, where $\underline{y} = \min\{y_n \mid 1 \leqq n \leqq N\}$. If $\tilde{T}$ is the time until this process stops, then $T^r \leqq \tilde{T}$, as follows from a contingency-by-contingency comparison of times until transition. The second process is the same as the first, except that now we change state N into a reflecting barrier. (Thus, if x_γ is A's effort rate for a contingency at state N, then the process moves to $N-1$ with probability $y_N/(x_\gamma + y_N)$ and stays at N with probability $x_\gamma/(x_\gamma + y_N)$.) If $\hat{T}$ is the time until stopping of this process, then $\tilde{T} \leqq \hat{T}$. For where the first process would have stopped, the second continues. The third and final process is adapted from the second by lowering all the probabilities of downward transition to $\bar{x}/(\bar{x} + \underline{y})$. At any given time, the third process is in a state at least as high as that occupied by the second. If its time until stopping is $\bar{T}$, then $\hat{T} \leqq \bar{T}$.

The third process is very well behaved. It is a continuous-time Markov chain. Imbedded within it is a discrete-time Markov chain with one reflecting barrier, one absorbing barrier, and a finite number of states. Let S be the number of transitions that this chain experiences before it hits the absorbing barrier. $E(S)$ is finite. (Indeed, S has finite variance, as follows from the fact that the recurrence times of a finite irreducible Markov chain have finite variance—see Feller (1968; p. 443).) Furthermore, $\bar{T}$ takes the form $\bar{T} = \sum_{k=1}^{S} \bar{T}_k$ where the $\bar{T}_k$ are mutually independent random variables distributed exponentially with parameter $\underline{y}$. Finally, S is independent of the T_k. It follows from Wald's identity that $E(\bar{T}) = E(T_1)E(S)$, and so that $E(\bar{T})$ is finite. $\bar{T}$ is the required dominating random variable for the $\pi_A(X^r, y)$ (and for $\pi_A(X, y)$, for that matter).

It remains only to show that $\pi_A(X^r, y) \to \pi_A(X, y)$ almost everywhere. In order to do this we write

$$\pi_A(X^r, y) = P_A \sum_\eta J_\eta^r - \sum_\gamma T_\gamma^r c_A(x_\gamma^r),$$

where η ranges over all paths by which state $N+1$ can be reached and J_η^r is the indicator function of η. At most one of the J_η^r can be 1. (For a given realisation of $\{(C_k, D_k)\}$, the occurrence or non-occurrence of η will depend on X^r. The superscript is intended to reflect this.) Similarly, we write

$$\pi_A(X, y) = P_A \sum_\eta J_\eta - \sum_\gamma T_\gamma c_A(x_\gamma)$$

for A's payoff from X. Next, for all γ, let $n(\gamma)$ be the state that has been reached in γ and suppose that it is reached $k(\gamma)$-th. Let $L_k = \{y_{n(\gamma)}/(x_\gamma + y_{n(\gamma)}) \mid k(\gamma) = k\}$. Let H_1 be the event that $D_k \in L_k$ for some k. H_1 has probability 0. Let H_2 be the event that $\bar{T} = \infty$. H_2, too, has zero probability.

Now, fix a realisation ω of $\{(C_k, D_k)\}$. Provided $\omega \notin H_1$, $\sum_\eta J_\eta^r$ converges finitely to $\sum_\eta J_\eta$. Also, for all γ, $T_\gamma^r = F^{-1}_{x_\gamma^r + y_{n(\gamma)}}(C_{k(\gamma)}) \to F^{-1}_{x_\gamma + y_{n(\gamma)}}(C_{k(\gamma)}) = T_\gamma$ as $r \to \infty$. Thus the terms of the series $\sum_\gamma T_\gamma^r c_A(x_\gamma^r)$ converge to those of the series $\sum_\gamma T_\gamma c_A(x_\gamma)$. But if $\omega \notin H_2$ then those series are dominated by the convergent series $\sum_\gamma \bar{T}_\gamma c_A(\bar{x})$. (In the case of this latter series, the number of contingencies covered by γ is greater.) Thus, outside H_2,

$$\sum_\gamma T_\gamma^r c_A(x_\gamma^r) \to \sum_\gamma T_\gamma c_A(x_\gamma) \quad \text{as } r \to \infty.$$

Overall, $\pi_A(X^r, y) \to \pi_A(X, y)$ outside the event $H_1 \cup H_2$, and so with probability 1. ||

Proof of Proposition 3.2. In order to prove the theorem it is sufficient to show that there exists a pair x and y such that x is an optimal response to y and vice versa, where "optimal response" is given the same sense as in the text.

Recall that if x is an optimal response to y then it satisfies (7) and (8). In particular, $\sum_n c_A'(x_n) < P_A$ and $x_n < \bar{x}$ for all n. Similarly, the upper bound on optimal responses y does not bind. We need to find lower bounds on x any y with the same property.

As a first step in the direction of such a bound for x, let $\underline{y} = \min\{y_n \mid 1 \leqq n \leqq N\}$. For all $n \geqq 2$, we have $x_n > \min\{x_{n-1}, \underline{y}\}$. For (A2) shows that

$$y_n c_A'(x_{n-1}) = x_n c_A'(x_n) - c_A(x_n).$$

Hence $\underline{y}c'_A(x_{n-1}) \leqq y_n c'_A(x_{n-1}) = x_n c'_A(x_n) - c_A(x_n) < x_n c'_A(x_n)$, and, if $x_n \leqq x_{n-1}$, $x_n > \underline{y}c'_A(x_{n-1})/c'_A(x_n) \geqq \underline{y}$. If, on the other hand, $x_n > x_{n-1}$, then there is nothing to prove. Using this relationship iteratively we get $x_n > \min\{x_1, \underline{y}\}$ for all $n \geqq 2$. Similarly, if $\underline{x} = \min\{x_n \mid 1 \leqq n \leqq N\}$ and y is an optimal response to x then $y_n > \min\{y_N, \underline{x}\}$ for all $n \leqq N-1$. Thus we need to bound x_1 (y_N) in terms of y (x).

Let $g_A(z) = zc'_A(z) - c_A(z)$. g_A is non-decreasing, continuous, and $g_A(0) = 0$. Also,

$$x_n \in g_A^{-1}(y_n c'_A(x_{n-1})) \leqq \max g_A^{-1}(\bar{y}c'_A(x_{n-1}))$$

for all $n \geqq 2$, and c'_A is continuous. Hence there exists $\varepsilon_A > 0$ such that $P_A - \sum_n c'_A(x_n) \geqq P_A/2$ if $x_1 \leqq 2\varepsilon_A$. Lowering ε_A if necessary, we may also assume that $c'_A(z) \leqq P_A/4$ for all $z \leqq 2\varepsilon_A$. One can find ε_B analogously. Let $\varepsilon = \min\{\varepsilon_A, \varepsilon_B\}$. If $y_1 \geqq \varepsilon$ then $x_1 \geqq 2\varepsilon$. For if not then we have, from (8), that

$$y_1 \leqq x_1 c'_A(x_1)/(P_A/2) < 2\varepsilon(P_A/4)/(P_A/2) = \varepsilon,$$

which is a contradiction. Similarly, if $x_N \geqq \varepsilon$ and y is an optimal response to x, then $y_N \geqq 2\varepsilon$.

Overall we conclude that if $\varepsilon \leqq y_n \leqq \bar{y}$ for all n and x is an optimal response to y, then $\varepsilon < x_n < \bar{x}$ for all n, and vice versa. Treating $[\varepsilon, \bar{y}]^N$ and $[\varepsilon, \bar{x}]^N$ as the domains of the optimal response correspondences, then, we obtain the required fixed point. As usual this fixed point may be taken to be symmetric if the game is symmetric. ||

Proof of Property 3.1. Part (i) follows immediately from the fact that $S_{n+1} > S_{n-1}$ when $n > N_0$. (Recall the corollary of Property 2.1 concerning joint payoffs.) Next, the analogy for B of (7) shows that

$$y_n c'_B(y_n) - c_B(y_n) = x_n c'_B(y_{n+1}).$$

Moreover $x_n > y_n$ by part (i). Hence

$$y_n c'_B(y_n) > y_n c'_B(y_n) - c_B(y_n) = x_n c'_B(y_{n+1}) > y_n c'_B(y_{n+1}),$$

or $c'_B(y_n) > c'_B(y_{n+1})$. It then follows from the monotonicity of c'_B that $y_n > y_{n+1}$. This establishes part (ii). Finally, for part (iii), recall the notation $g_A(z) = zc'_A(z) - c_A(z)$ from the proof of Proposition 3.2. Using (7) we have $y_n c'_A(x_{n-1}) = g_A(x_n)$ and $y_{n+1} c'_A(x_n) = g_A(x_{n+1})$. Hence

$$g_A(x_n)/g_A(x_{n+1}) = (y_n/y_{n+1})(c'_A(x_{n-1})/c'_A(x_n)) > 1$$

by part (ii) and by our assumption that $x_{n-1} \geqq x_n$. We deduce immediately from the monotonicity of g_A that $x_n > x_{n+1}$, as required. ||

Proof of Property 4.2. We established the result in the case $m = 1$ in the text, so we turn straight to the case $m = 2$. We consider only the case $m \leqq 2$ since the case $n \leqq 2$ is analogous. The proof is in three steps. The first two steps show, respectively, that $\alpha(2, n) > \alpha(1, n+1)$ for all $n \geqq 2$ and that $\beta(2, n) > \beta(1, n+1)$ for all $n \geqq 2$. The third step uses these results to derive the final conclusion.

Step (i). We have

$$\begin{aligned} \alpha(2, n) > \alpha(1, n+1) &\Leftrightarrow V(1, n) - V(2, n-1) > V(0, n+1) - V(1, n) \\ &\Leftrightarrow 2V(1, n) > V(0, n+1) + V(2, n-1) \\ &= P + V(2, n-1). \end{aligned}$$

But

$$\begin{aligned} 2V(1, n) &= 2\left[\frac{x(1, n)V(0, n) + y(1, n)V(1, n-1) - c(x(1, n))}{x(1, n) + y(1, n)}\right] \\ &> 2\left[\frac{x(1, n)V(0, n) + x(1, n)V(1, n-1) - c(x(1, n))}{2x(1, n)}\right] \end{aligned}$$

(since raising $y(1, n)$ to $x(1, n)$ makes A strictly worse off)

$$\begin{aligned} &= P + V(1, n-1) - c(x(1, n))/x(1, n) \\ &\geqq P + V(1, n-1) - c'(x(1, n)) \end{aligned}$$

(by convexity of c)

$$= P + V(2, n-1)$$

(by Property 2.3). (We have suppressed the subscripts on the prizes and the cost functions.) Thus $\alpha(2, n) > \alpha(1, n+1)$ as required.

Step (ii). We have

$$\beta(2, n) > \beta(1, n+1) \Leftrightarrow W(2, n-1) - W(1, n) > W(1, n) - W(0, n+1)$$
$$\Leftrightarrow W(2, n-1) > 2W(1, n).$$

But

$$2W(1, n) = 2\left[\frac{x(1, n)W(0, n) + y(1, n)W(1, n-1) - c(y(1, n))}{x(1, n) + y(1, n)}\right]$$
$$< W(0, n) + W(1, n-1) - c(y(1, n))/y(1, n)$$
$$< W(2, n-1).$$

The first inequality holds because $x(1, n) > y(1, n)$ and lowering $x(1, n)$ to $y(1, n)$ makes B strictly better off, and the second inequality holds because $W(0, n) = 0$ and $W(1, n-1) < W(2, n-1)$). Thus $\beta(2, n) > \beta(1, n+1)$ as required.

Step (iii). We are now in a position to show that $x(2, n) > y(2, n)$ for all $n \geqq 3$. The proof is by induction. First we establish the inductive step. Thus, suppose that $x(2, n) > y(2, n)$ for some $n \geqq 3$. We know that $x(2, n+1) > y(2, n+1)$ if and only if $S(1, n+1) > S(2, n)$. Now $S(1, n+1)$ is composed of expected benefits and expected costs. The expected benefits are

$$\frac{x(1, n+1)S(0, n+1) + y(1, n+1)S(1, n)}{x(1, n+1) + y(1, n+1)}.$$

But $S(0, n+1) > S(1, n)$ and $S(1, n) > S(2, n-1)$ because $x(1, n+1) > y(1, n+1)$ and $x(2, n) > y(2, n)$ respectively. Hence the expected benefits in $S(1, n+1)$ exceed those in $S(2, n)$, namely

$$\frac{x(2, n)S(1, n) + y(2, n)S(2, n-1)}{x(2, n) + y(2, n)}.$$

Next, the expected costs in $S(1, n+1)$ are

$$\frac{c(x(1, n+1)) + c(y(1, n+1)}{x(1, n+1) + y(1, n+1)}.$$

But $x(1, n+1) < x(2, n)$ and $y(1, n+1) < y(2, n)$ by the results of steps (i) and (ii) and by Property 2.4. Thus the expected costs in $S(1, n+1)$ are lower than those in $S(2, n)$, namely

$$\frac{c(x(2, n)) + c(y(2, n))}{x(2, n) + y(2, n)}.$$

Overall, $S(1, n+1) > S(2, n)$, and $x(2, n+1) > y(2, n+1)$ as required.

It remains to establish the inductive hypothesis in the case $n = 3$. The proof is identical to that given for the inductive step, except that we can no longer appeal to an inductive hypothesis to conclude that $S(1, n-1) > S(2, n-2)$. Rather, we note that, in view of the symmetry of equilibrium, $S(1, n-1) = S(1, 2) = S(2, 1) = S(2, n-2)$ (strict inequality at this step is unnecessary for the conclusion). ||

Proof of Property 4.4. Since $c'_A(x(1, n)) = P_A - V(1, n)$, $x(1, n)$ is strictly decreasing in n. Next, $\beta(1, n) = W(1, n-1)$ is strictly decreasing in n. Combined with the fact that $x(1, n)$ is strictly decreasing, this implies that $y(1, n)$ is strictly decreasing in n. ||

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NOTES

1. There is also a simple model of a two-stage patent race with uncertainty in Fudenberg *et al.* (1985, Section 3). Their model is intended to show that a player might compete at the first stage even though he is behind his rival, because there is a chance of leapfrogging. At the second stage, R&D is fixed intensity. Our models have a broader purpose, and have variable intensity of R&D effort, since a major objective is to see how effort rates vary as the race unfolds.

2. A brief outline of the derivation is as follows: (9) holds for $n = 1$; (9) with $n = k$ and the analogy of (7) for B implies (10) for $n = k+1$, finally, (10) for $n = k$ and (7) implies (9) for $n = k$.

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