RACES VS. TOURNAMENTS?

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1. Assumptions

Consider a set $N = \{1, 2, ..., n\}$ of risk-neutral agents. Agents ability is a random variable $a \in [0, 1]$ with a beta-family probability mass function $f_{[0,1]}(a; \alpha, \beta)$, that is iid across agents. Consider that the quality of a solution is $q \in [0, \infty)$ and the time to produce a solution of a given quality is represented by a continuous real-valued function

$$t(a, q) = a^{\gamma} \cdot q^{\delta}$$

with $\gamma < 0$ so that the partial derivative of time with respect to ability $t'(a) < 0$ if $q \neq 0$ and $\delta > 0$ so that the partial derivative of time with respect to quality $t'(q) > 0$. [1] Agents' cost of time of development is represented by strictly increasing function $c(t(a, q)) = C \cdot t(a, q)^{\kappa}$, and with no loss of generality we set $\kappa = 1$.

1.1. **Race.** The first agent to develop a solution that hits $q \geq \bar{q}$ is awarded a prize $v > 0$. Under these rules, an agent i's ex-post utility is:

$$U_i = \begin{cases} v - c(t(a_i, q_i)) & if q_i \geq \bar{q} \text{ } and \text{ } t(a_i, q_i) < t(a_j, q_j) \ \forall j \in N_i = \{k \in N/i : q_k \geq \bar{q}\} \\ -c(t(a_i, q_i)) & otherwise \end{cases}$$

At an interim stage, the agent learns his type a_i and has to decide how much quality to develop (or equivalently how much time to develop a solution of a given quality). The maximization problem is (forget about ties):

$$\max_{q_i \in A} \mathbf{1}_{(q_i \geq \bar{q})} \cdot \Pr(t(a_i, q_i) < \min\{t(a_j, q_j)\}_{j \in N_k} | a_i) \cdot v - c(t(a_i, q_i))$$

where $\mathbf{1}()$ is an indicator function. Note, fixed other agents' strategies, it is easy to show that the above problem reduces to a binary decision on $q \in \{0, \bar{q}\}$, as all other actions are (strictly) dominated. Indeed, the probability of winning is non-increasing in quality for any level $q_i > \bar{q}$ and recall that costs are strictly increasing in q_i. So this can be rewritten:

$$\max_{q_i \in \{0, \bar{q}\}} \mathbf{1}_{(q_i = \bar{q})} \cdot F_A(a_i > a_j)^{n-1} \cdot v - c(t(a_i, q_i))$$

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[1] Here the idea is that agents commit resources at the beginning of the race in order to deliver a solution of a given quality by a certain time. It's clear that if any of the opponents wins the race, this means that the others will have some free time ex-post. Our assumption throught the paper is that this extra-time is a cost that cannot cannot be reinvested. In a more general model, we can assume that ...

Furthermore, because the above expresion is monotone in a_i, we can simply look at a the marginal agent to pin down a threshold level of ability $\hat{a}$ above which all agents are going to bid $q^* = \bar{q}$ and below which they will bid $q^* = 0$. That is,

$$v \cdot \left(\frac{B(\hat{a}, \alpha, \beta)}{B(\alpha, \gamma)} \right)^{n-1} = C \cdot \hat{a}^{\gamma} \cdot \bar{q}^{\delta}$$

which simplifies to the following remarkably simple expression if we assume that abilities are distributed uniformly on the unit interval:

$$\hat{a} = \left(\frac{C \cdot \bar{q}^{\delta}}{v} \right)^{1/(n-1-\gamma)}$$

notice that when $\delta/(n-1-\gamma) > 1$, the threshold grows exponentially in $\bar{q}$, whereas when $\delta/(n-1-\gamma) < 1$ it grows logarithmically (or linear for the equal). This has a clear interpretation ...

For a principal, the problem is the following:

$$w_{race} = \Pr(\max\{q_i\}_{i \in N} = \bar{q}) \cdot (\bar{q} - v)$$

which for the uniform unit interval case becomes:

$$w_{race} = \left[1 - \left(\frac{C \cdot \bar{q}^{\delta}}{v} \right)^{n/(n-1-\gamma)} \right] \cdot (\bar{q} - v)$$

2. Equilibrium in the All-pay tournament

The problem here is the following:

$$\arg\max_{q} \qquad v \cdot \Pr\left(q_i > \max\{q_j\}_{j \neq i}\right) - c(t(a_i, q_i))$$

and by following M-S (for $n > 2$ and uniform unit interval) we have:

$$q^*_{allpay} = \left(\frac{n-1}{n-1+\gamma} v \cdot a^{n-1+\gamma} \right)^{1/\delta}$$

3. Which one is betterd?

We are in the shoes of a "principal" i.e., sponsor/crowdsourcing platform. First, suppose the principal does not care about time per se, but only as a mean to provide good incentives to participants or to save costs. Second, the principal wants to max the expected *max* quality. Third, the principal needs to optimally set the incentives. For example, in a race the problem of a principal is:

$$\max_{\rho \leq 1} \ \hat{w}^{races} = w/V = \left\{ 1 - [1 - F(\bar{x}(\rho))]^N \right\} \cdot (\rho - 1)$$

On the other hand, in a tournament the only parameter to change is V. First, given the bidding function, we want to know the expected min cost in the race:

$$F(\min(x, y) < k) = F(x < k) * F(y < k) = F(k)^2$$

Hence, for N=2,

$$\int_{t_m in}^{1} F(k)^2 dk = \int_{t_m}^{1} \frac{k - t_m}{1 - t_m} dk = \frac{1 - t_m}{2}$$

And the expected max value is

$$\hat{w}^{allpay} = w/V = \frac{log(2/(1-t_m))}{1-t_m} - 1$$

4. Description of the data