Laboratório
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Científica

COMOD
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Relatório de Atividades

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1 INTRODUTION

Variational auto-encoder (VAE) model is a stochastic inference and learning algorithm based on variational Bayes (VB) inference proposed by **?**. This is a generative technique whose central idea is the development of representations in a low-dimensional latent space that can be mapped back into a realistic-looking image.

? introduced the β-VAE, a modification of the original VAE, that introduces an adjustable hyperparameter β to balance latent channel capacity and independence constraints with reconstruction accuracy. They demonstrate that with tuned values of β ($\beta > 1$) the β-VAE outperforms VAE ($\beta = 1$).

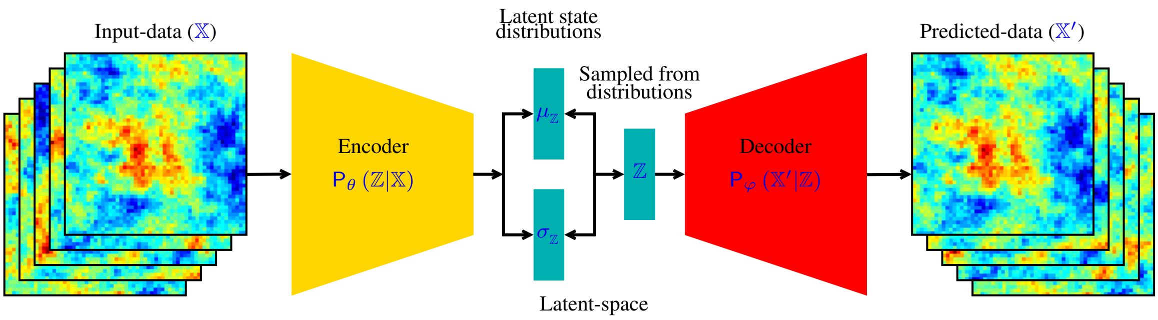
?????

2 ?

Variational auto-encoder (VAE) model is a generative network based on variational Bayes (VB) inference proposed by **?**.

? proposed a method to reconstruct porous media based on VAE and Fisher information with good quality and efficiency.

Consider the input data $\mathbb{X} = \left\{\boldsymbol{x}^{(i)}\right\}_{i=1}^{\mathsf{N}}$ consisting of N *i*ndependent and *i*dentically *d*istributed (*i.i.d.*) samples of the continuous (or discrete) variable $\boldsymbol{x}$.



$$\mathbb{Z} = \mu_\mathbb{Z} + \sigma_\mathbb{Z} \cdot \varepsilon, \quad \text{where} \quad \varepsilon \sim \mathbb{N}\left(0, 1\right) \tag{1}$$

The Kullback–Leibler divergence (also called relative entropy and I-divergence) is a measure of divergence between two distributions (**??**):

$$\mathcal{D}_{\mathsf{KL}}\left(f_\mathbb{P} || f_\mathbb{Q}\right) = \int_{-\infty}^{\infty} \tag{2}$$

? proposed a Fisher autoencoder

3 Fisher Information

Let $f(x;\theta)$ be the probability density function of the random variable $\mathbb{X}$ conditioned on the parameter θ. The Fisher information measures the amount of information that an observation of $\mathbb{X}$ carries about the unknown parameter θ. The partial derivative of the natural logarithm of the likelihood function is called **score** (S):

$$\mathsf{S}(x,\theta) = \frac{\partial}{\partial\theta}\log\left[f(x;\theta)\right]. \tag{3}$$

Fisher information is defined as the variance of the score S:

$$\begin{aligned}\mathcal{I}(\theta) = \mathsf{E}[\mathsf{S}^2|\theta] &= \mathsf{E}\left[\left(\frac{\partial}{\partial\theta}\log\left[f(x;\theta)\right]\right)^2\middle|\theta\right] \\ &= \int_{\mathbb{R}}\left(\frac{\partial}{\partial\theta}\log\left[f(x;\theta)\right]\right)^2 f(x;\theta)\,\mathrm{d}x\end{aligned}. \tag{4}$$

If $\log\left[f(x;\theta)\right]$ is twice differentiable with respect to θ and under certain regularuty conditions, Fisher information can be written as

$$\mathcal{I}(\theta) = \mathsf{E}\left[-\frac{\partial^2}{\partial\theta^2}\log\left[f(x;\theta)\right]\middle|\theta\right]. \tag{5}$$

Let $\mathbb{X}$ be a scalar Gaussian random variable *i.e.* $\mathbb{X}\sim\mathbb{N}\left(\mu,\sigma^2\right)$. Then the probability density function is parameterized by the parameters μ and σ:

$$f(x;\mu,\sigma) = \frac{1}{\sigma\sqrt{2\pi}}\exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right]. \tag{6}$$

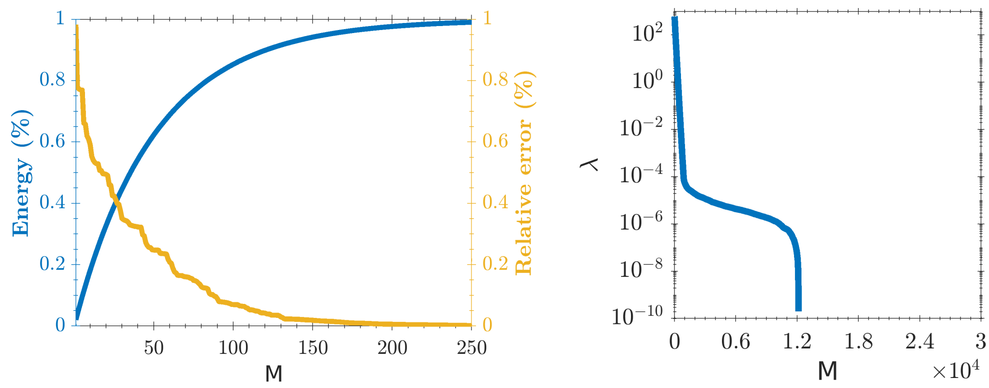
Substituting Eq. (6) in Eq. (5) where $\theta=\mu$ or σ we can compute de Fisher information for a Gaussian variable as:

3 Fisher Information

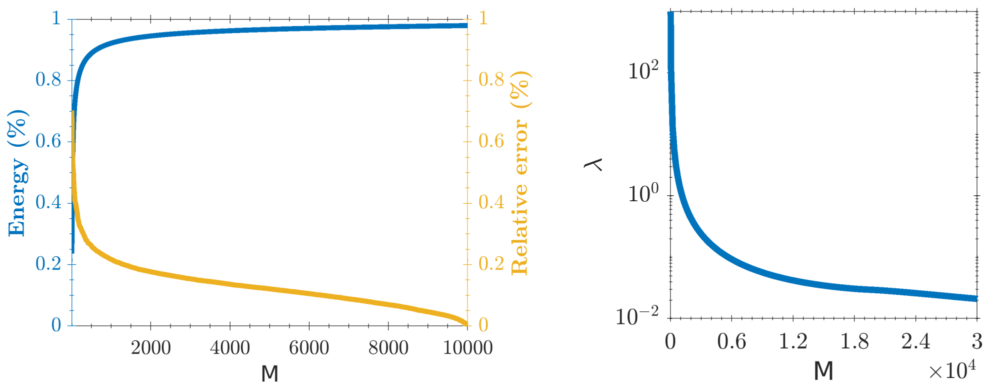
$$
\begin{aligned}
\mathcal{I}(\mu) &= \mathsf{E}\left[-\frac{\partial^2}{\partial\mu^2}\left[\log\left(\frac{1}{\sigma\sqrt{2\pi}}\right)-\frac{1}{2}\left(\frac{\mathbb{X}-\mu}{\sigma}\right)^2\right]\middle|\mu\right] \\
&= \mathsf{E}\left[-\frac{\partial}{\partial\mu}\left(-\frac{\mathbb{X}-\mu}{\sigma^2}\right)\middle|\mu\right] \\
&= \mathsf{E}\left[\frac{1}{\sigma^2}\right]=\frac{1}{\sigma^2}
\end{aligned} \tag{7}
$$

$$
\begin{aligned}
\mathcal{I}(\sigma) &= \mathsf{E}\left[-\frac{\partial}{\partial\sigma}\left[-\frac{1}{\sigma}+\frac{(\mathbb{X}-\mu)^2}{\sigma^3}\right]\middle|\sigma\right] \\
&= \mathsf{E}\left[-\frac{1}{\sigma^2}+\frac{3(\mathbb{X}-\mu)^2}{\sigma^4}\middle|\sigma\right] \\
&= -\frac{1}{\sigma^2}+\frac{3\sigma^2}{\sigma^4}=\frac{2}{\sigma^2}
\end{aligned} \tag{8}
$$

4 KLE



(a) Squared exponential covariance

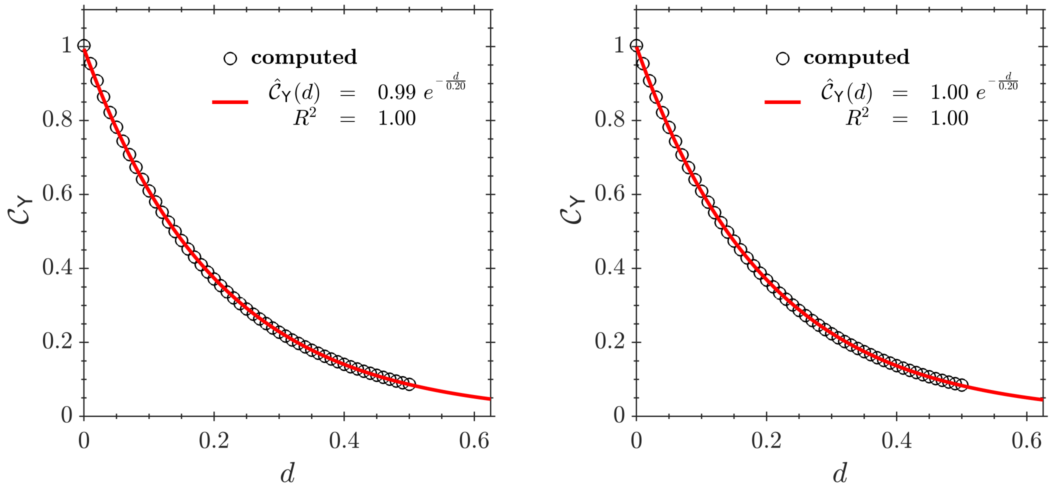


(b) Exponential covariance

Figure 1: Decay of eigenvalues and contained energy as a function of the number of terms in the expansion.

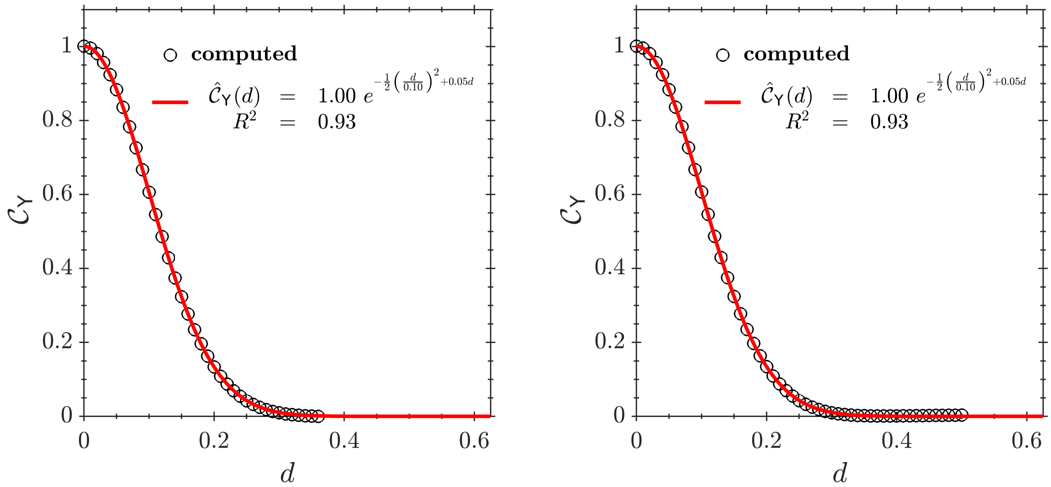
Table 1: Number of KL expansion terms needed to obtain an given energy level

Energy	M	
(%)	Exponential	Squared exp.
80	201	83
90	781	120
94	1928	148
96	3416	170
98	6111	208
100	∼30,000	∼



(a) x direction (b) y direction

Figure 2: Estimated covariance in a sample of $5,000$ fields generate by `KLE` with $30,000$ terms. Exponential case with $\ell = 0.2$.



(a) x direction (b) y direction

Figure 3: Estimated covariance in a sample of $5,000$ fields generate by `KLE` with $30,000$ terms. Squared exponential case with $\ell = 0.1$.

References

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